CALCULUS - EARLY TRANSCENDENTALS



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Calculus - Early Transcendentals

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Licensing

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CHAPTER OVERVIEW

1: Functions and Models

A general Calculus Textmap organized around the textbook

Calculus: Early Transcendentals

by James Stewart

I II III IV V VI VII VIII IX X XI XII XIII XIV XV XVI XVII

This Textmap is currently under construction... please be patient with us.

Template:HideTOC

Topic hierarchy

1.1: Four Ways to Represent a Function
1.2: Mathematical Models- A Catalog of Essential Functions
1.3: New Functions from Old Functions
1.4: Exponential Functions
1.5: Inverse Functions and Logarithms

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1.1: Four Ways to Represent a Function

Learning Objectives

- Determine whether a relation represents a function.
- Find the value of a function.
- Determine whether a function is one-to-one.
- Use the vertical line test to identify functions.
- Graph the functions listed in the library of functions.

A jetliner changes altitude as its distance from the starting point of a flight increases. The weight of a growing child increases with time. In each case, one quantity depends on another. There is a relationship between the two quantities that we can describe, analyze, and use to make predictions. In this section, we will analyze such relationships.

Determining Whether a Relation Represents a Function

A relation is a set of ordered pairs. The set of the first components of each ordered pair is called the domain and the set of the second components of each ordered pair is called the range. Consider the following set of ordered pairs. The first numbers in each pair are the first five natural numbers. The second number in each pair is twice that of the first.

$$\{(1,2),(2,4),(3,6),(4,8),(5,10)\} \tag{1.1.1}$$

The domain is $\{1,2,3,4,5\}$ The range is $\{2,4,6,8,10\}$

Note that each value in the domain is also known as an **input** value, or **independent variable**, and is often labeled with the lowercase letter x. Each value in the range is also known as an output value, or **dependent variable**, and is often labeled lowercase letter y.

A function f is a relation that assigns a single value in the range to each value in the domain. In other words, no x-values are repeated. For our example that relates the first five **natural numbers** to numbers double their values, this relation is a function because each element in the domain, {1, 2, 3, 4, 5}, is paired with exactly one element in the range, $\{2,4,6,8,10\}$

Now let's consider the set of ordered pairs that relates the terms "even" and "odd" to the first five natural numbers. It would appear as

$$\{(\text{odd},1),(\text{even},2),(\text{odd},3),(\text{even},4),(\text{odd},5)\} \tag{1.1.2}$$

Notice that each element in the domain, {even, odd} is not paired with exactly one element in the range, $\{1,2,3,4,5\}$ For example, the term "odd" corresponds to three values from the range, $\{1,3,5\}$,and the term "even" corresponds to two values from the range, $\{2,4\}$. This violates the definition of a function, so this relation is not a function.

Figure 1.1.1 compares relations that are functions and not functions.

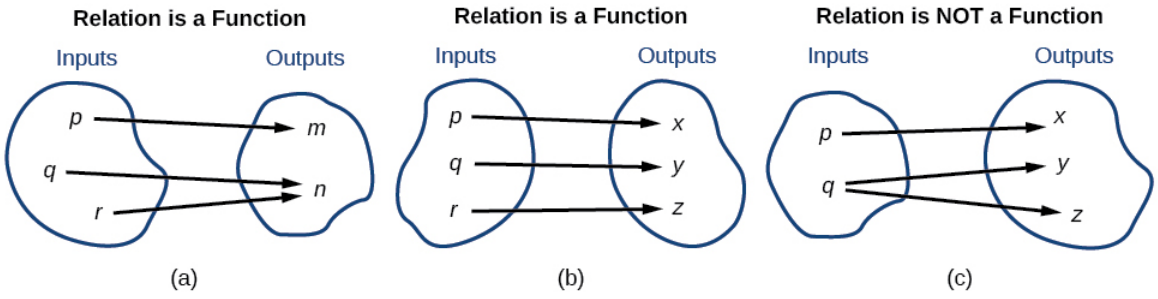


Figure 1.1.1: (a) This relationship is a function because each input is associated with a single output. Note that input q and r both give output n. (b) This relationship is also a function. In this case, each input is associated with a single output. (c) This relationship is not a function because input q is associated with two different outputs.

Function

A **function** is a relation in which each possible input value leads to exactly one output value. We say "the output is a function of the input."

The **input** values make up the **domain**, and the **output** values make up the **range**.

How To: Given a relationship between two quantities, determine whether the relationship is a function

1. Identify the input values.
2. Identify the output values.
3. If each input value leads to only one output value, classify the relationship as a function. If any input value leads to two or more outputs, do not classify the relationship as a function.

Example 1.1.1: Determining If Menu Price Lists Are Functions

The coffee shop menu, shown in Figure 1.1.2 consists of items and their prices.

a. Is price a function of the item?
b. Is the item a function of the price?

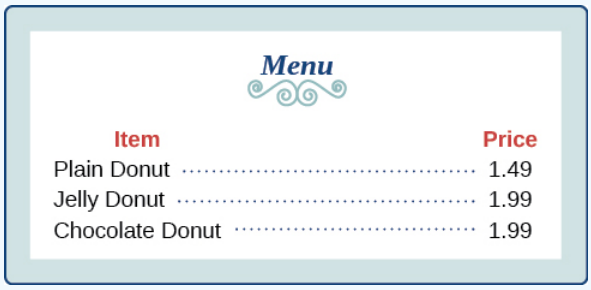


Figure 1.1.2: A menu of donut prices from a coffee shop where a plain donut is $1.49 and a jelly donut and chocolate donut are $1.99.

Solution

a. Let's begin by considering the input as the items on the menu. The output values are then the prices. See Figure 1.1.3.



Figure 1.1.3: A menu of donut prices from a coffee shop where a plain donut is $1.49 and a jelly donut and chocolate donut are $1.99.

Each item on the menu has only one price, so the price is a function of the item.

a. Two items on the menu have the same price. If we consider the prices to be the input values and the items to be the output, then the same input value could have more than one output associated with it. See Figure 1.1.4.

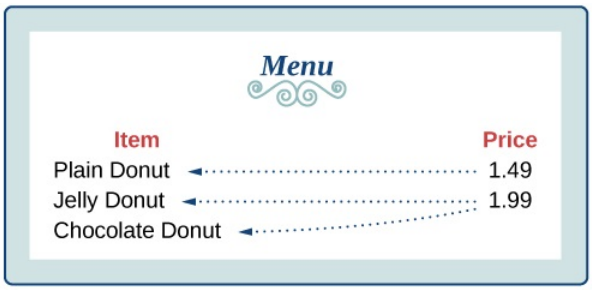


Figure 1.1.4: Association of the prices to the donuts.

Therefore, the item is a not a function of price.

Example 1.1.2: Determining If Class Grade Rules Are Functions

In a particular math class, the overall percent grade corresponds to a grade point average. Is grade point average a function of the percent grade? Is the percent grade a function of the grade point average? Table 1.1.1 shows a possible rule for assigning grade points.

Table 1.1.1: Class grade points.

Percent grade	0–56	57–61	62–66	67–71	72–77	78–86	87–91	92–100
Grade point average	0.0	1.0	1.5	2.0	2.5	3.0	3.5	4.0

Solution

For any percent grade earned, there is an associated grade point average, so the grade point average is a function of the percent grade. In other words, if we input the percent grade, the output is a specific grade point average.

In the grading system given, there is a range of percent grades that correspond to the same grade point average. For example, students who receive a grade point average of 3.0 could have a variety of percent grades ranging from 78 all the way to 86. Thus, percent grade is not a function of grade point average.

Exercise 1.1.2

Table 1.1.2 lists the five greatest baseball players of all time in order of rank.

Table 1.1.2: Five greatest baseball players.

Player	Rank
Babe Ruth	1
Willie Mays	2
Ty Cobb	3
Walter Johnson	4
Hank Aaron	5

a. Is the rank a function of the player name?
b. Is the player name a function of the rank?

Answer a

Yes

Answer b

yes. (Note: If two players had been tied for, say, 4th place, then the name would not have been a function of rank.)

Using Function Notation

Once we determine that a relationship is a function, we need to display and define the functional relationships so that we can understand and use them, and sometimes also so that we can program them into computers. There are various ways of representing functions. A standard function notation is one representation that facilitates working with functions.

To represent "height is a function of age," we start by identifying the descriptive variables h for height and a for age. The letters f, g,and h are often used to represent functions just as we use x, y,and z to represent numbers and A, B, and C to represent sets.

$$\begin{aligned} &h \text{ is } f \text{ of } a && \text{We name the function } f\text{; height is a function of age.} \\ &h = f(a) && \text{We use parentheses to indicate the function input.} \\ &f(a) && \text{We name the function } f \text{ ; the expression is read as "} f \text{ of } a\text{."} \end{aligned} \tag{1.1.1}$$

Remember, we can use any letter to name the function; the notation $h(a)$ shows us that h depends on a. The value a must be put into the function h to get a result. The parentheses indicate that age is input into the function; they do not indicate multiplication.

We can also give an algebraic expression as the input to a function. For example $f(a+b)$ means "first add a and b, and the result is the input for the function f." The operations must be performed in this order to obtain the correct result.

Function Notation

The notation $y = f(x)$ defines a function named f. This is read as "y is a function of x." The letter x represents the input value, or independent variable. The letter y, or $f(x)$, represents the output value, or dependent variable.

Example 1.1.3: Using Function Notation for Days in a Month

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month.

Solution

Using Function Notation for Days in a Month

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month.

The number of days in a month is a function of the name of the month, so if we name the function f, we write days $= f(\text{month})$ or $d = f(m)$. The name of the month is the input to a "rule" that associates a specific number (the output) with each input.

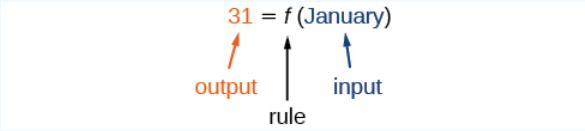


Figure 1.1.5: The function $31 = f(January)$ where 31 is the output, f is the rule, and January is the input.

For example, $f(\text{March}) = 31$, because March has 31 days. The notation $d = f(m)$ reminds us that the number of days, d (the output), is dependent on the name of the month, m (the input).

Analysis

Note that the inputs to a function do not have to be numbers; function inputs can be names of people, labels of geometric objects, or any other element that determines some kind of output. However, most of the functions we will work with in this book will have numbers as inputs and outputs.

Example 1.1.3B: Interpreting Function Notation

A function $N = f(y)$ gives the number of police officers, N, in a town in year y. What does $f(2005) = 300$ represent?

Solution

When we read $f(2005) = 300$, we see that the input year is 2005. The value for the output, the number of police officers (N), is 300. Remember, $N = f(y)$. The statement $f(2005) = 300$ tells us that in the year 2005 there were 300 police officers in the town.

Exercise 1.1.3

Use function notation to express the weight of a pig in pounds as a function of its age in days d.

Answer

$w = f(d)$

Q&A

Instead of a notation such as $y = f(x)$, could we use the same symbol for the output as for the function, such as $y = y(x)$, meaning "y is a function of x?"

Yes, this is often done, especially in applied subjects that use higher math, such as physics and engineering. However, in exploring math itself we like to maintain a distinction between a function such as f, which is a rule or procedure, and the output y we get by applying f to a particular input x. This is why we usually use notation such as $y = f(x)$, $P = W(d)$, and so on.

Representing Functions Using Tables

A common method of representing functions is in the form of a table. The table rows or columns display the corresponding input and output values. In some cases, these values represent all we know about the relationship; other times, the table provides a few select examples from a more complete relationship.

Table 1.1.3 lists the input number of each month ($\text{January} = 1$, $\text{February} = 2$, and so on) and the output value of the number of days in that month. This information represents all we know about the months and days for a given year (that is not a leap year). Note that, in this table, we define a days-in-a-month function f where $D = f(m)$ identifies months by an integer rather than by name.

Table 1.1.3: Months and number of days per month.

Month number, m (input)	1	2	3	4	5	6	7	8	9	10	11	12
Days in month, D (output)	31	28	31	30	31	30	31	31	30	31	30	31

Table 1.1.4 defines a function $Q = g(n)$ Remember, this notation tells us that g is the name of the function that takes the input n and gives the output Q.

Table 1.1.4: Function $Q = g(n)$

n	1	2	3	4	5
Q	8	6	7	6	8

Table 1.1.5 displays the age of children in years and their corresponding heights. This table displays just some of the data available for the heights and ages of children. We can see right away that this table does not represent a function because the same input value, 5 years, has two different output values, 40 in. and 42 in.

Table 1.1.5: Age of children and their corresponding heights.

Age in years, a (input)	5	5	6	7	8	9	10
Height in inches, h (output)	40	42	44	47	50	52	54

How To: Given a table of input and output values, determine whether the table represents a function

1. Identify the input and output values.
2. Check to see if each input value is paired with only one output value. If so, the table represents a function.

Example 1.1.5: Identifying Tables that Represent Functions

Which table, Table 1.1.6, Table 1.1.7, or Table 1.1.8, represents a function (if any)?

Table 1.1.6

Input	Output
2	1
5	3
8	6

Table 1.1.7

Input	Output
-3	5
0	1
4	5

Table 1.1.8

Input	Output
1	0
5	2
5	4

Solution

Table 1.1.6 and Table 1.1.7 define functions. In both, each input value corresponds to exactly one output value. Table 1.1.8 does not define a function because the input value of 5 corresponds to two different output values.

When a table represents a function, corresponding input and output values can also be specified using function notation.

The function represented by Table 1.1.6 can be represented by writing

$$f(2) = 1,\ f(5) = 3, \text{ and } f(8) = 6$$

Similarly, the statements

$$g(-3) = 5,\ g(0) = 1, \text{ and } g(4) = 5$$

represent the function in Table 1.1.7.

Table 1.1.8 cannot be expressed in a similar way because it does not represent a function.

Exercise 1.1.5

Does Table 1.1.9 represent a function?

Table 1.1.9

Input	Output
1	10
2	100
3	1000

Answer

yes

Finding Input and Output Values of a Function

When we know an input value and want to determine the corresponding output value for a function, we evaluate the function. Evaluating will always produce one result because each input value of a function corresponds to exactly one output value.

When we know an output value and want to determine the input values that would produce that output value, we set the output equal to the function's formula and solve for the input. Solving can produce more than one solution because different input values can produce the same output value.

Evaluation of Functions in Algebraic Forms

When we have a function in formula form, it is usually a simple matter to evaluate the function. For example, the function $f(x) = 5 - 3x^2$ can be evaluated by squaring the input value, multiplying by 3, and then subtracting the product from 5.

How To: Given the formula for a function, evaluate.

Given the formula for a function, evaluate.

1. Replace the input variable in the formula with the value provided.
2. Calculate the result.

Example 1.1.6A: Evaluating Functions at Specific Values

1. Evaluate $f(x) = x^2 + 3x - 4$ at
   a. 2
   b. a
   c. $a + h$
   d. Evaluate $\frac{f(a+h) - f(a)}{h}$

Solution

Replace the x in the function with each specified value.

a. Because the input value is a number, 2, we can use simple algebra to simplify.

$$\begin{aligned} f(2) &= 2^2 + 3(2) - 4 \\ &= 4 + 6 - 4 \\ &= 6 \end{aligned}$$

b. In this case, the input value is a letter so we cannot simplify the answer any further.

$$f(a) = a^2 + 3a - 4$$

c. With an input value of $a+h$, we must use the distributive property.

$$\begin{aligned} f(a+h) &= (a+h)^2 + 3(a+h) - 4 \\ &= a^2 + 2ah + h^2 + 3a + 3h - 4 \end{aligned}$$

d. In this case, we apply the input values to the function more than once, and then perform algebraic operations on the result. We already found that

$$f(a+h) = a^2 + 2ah + h^2 + 3a + 3h - 4$$

and we know that

$$f(a) = a^2 + 3a - 4$$

Now we combine the results and simplify.

$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= \frac{(a^2 + 2ah + h^2 + 3a + 3h - 4) - (a^2 + 3a - 4)}{h} \\ &= \frac{(2ah + h^2 + 3h)}{h} \\ &= \frac{h(2a + h + 3)}{h} && \text{Factor out h.} \\ &= 2a + h + 3 && \text{Simplify.} \end{aligned}$$

Example 1.1.6B: Evaluating Functions

Given the function $h(p) = p^2 + 2p$, evaluate $h(4)$.

Solution

To evaluate $h(4)$, we substitute the value 4 for the input variable p in the given function.

$$\begin{aligned} h(p) &= p^2 + 2p \\ h(4) &= (4)^2 + 2(4) \\ &= 16 + 8 \\ &= 24 \end{aligned}$$

Therefore, for an input of 4, we have an output of 24.

Exercise 1.1.6

Given the function $g(m) = \sqrt{m-4}$, evaluate $g(5)$.

Answer

$g(5) = 1$

Example 1.1.7: Solving Functions

Given the function $h(p) = p^2 + 2p$, solve for $h(p) = 3$.

Solution

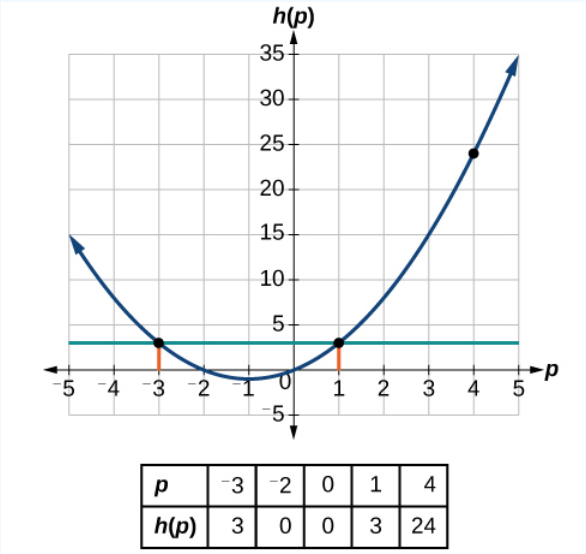
$$\begin{aligned} h(p) &= 3 \\ p^2+2p &= 3 \quad \text{Substitute the original function} \\ p^2+2p-3 &= 0 \quad \text{Subtract 3 from each side.} \\ (p+3)(p-1) &= 0 \quad \text{Factor.} \end{aligned}$$

If $(p+3)(p-1)=0$, either $(p+3)=0$ or $(p-1)=0$ (or both of them equal 0). We will set each factor equal to 0 and solve for p in each case.

$$(p+3)=0,\ p=-3$$

$$(p-1)=0,\ p=1$$

This gives us two solutions. The output $h(p)=3$ when the input is either $p=1$ or $p=-3$. We can also verify by graphing as in Figure 1.1.6. The graph verifies that $h(1)=h(-3)=3$ and $h(4)=24$.



p	−3	−2	0	1	4
h(p)	3	0	0	3	24

Figure 1.1.6: Graph of $h(p)=p^2+2p$

Exercise 1.1.7

Given the function $g(m)=\sqrt{m-4}$, solve $g(m)=2$.

Answer

$m=8$

Evaluating Functions Expressed in Formulas

Some functions are defined by mathematical rules or procedures expressed in **equation** form. If it is possible to express the function output with a formula involving the input quantity, then we can define a function in algebraic form. For example, the equation $2n+6p=12$ expresses a functional relationship between n and p. We can rewrite it to decide if p is a function of n.

How to: Given a function in equation form, write its algebraic formula.

1. Solve the equation to isolate the output variable on one side of the equal sign, with the other side as an expression that involves only the input variable.
2. Use all the usual algebraic methods for solving equations, such as adding or subtracting the same quantity to or from both sides, or multiplying or dividing both sides of the equation by the same quantity.

Example 1.1.8A: Finding an Equation of a Function

Express the relationship $2n+6p=12$ as a function $p=f(n)$, if possible.

Solution

To express the relationship in this form, we need to be able to write the relationship where p is a function of n, which means writing it as $p=[\text{expression involving } n]$.

$$\begin{aligned} 2n+6p &= 12 \\ 6p &= 12-2n && \text{Subtract 2n from both sides.} \\ p &= \frac{12-2n}{6} && \text{Divide both sides by 6 and simplify.} \\ p &= \frac{12}{6}-\frac{2n}{6} \\ p &= 2-\frac{1}{3}n \end{aligned}$$

Therefore, p as a function of n is written as

$$p=f(n)=2-\frac{1}{3}n$$

Analysis

It is important to note that not every relationship expressed by an equation can also be expressed as a function with a formula.

Example 1.1.8B: Expressing the Equation of a Circle as a Function

Does the equation $x^2+y^2=1$ represent a function with x as input and y as output? If so, express the relationship as a function $y=f(x)$.

Solution

First we subtract x^2 from both sides.

$$y^2=1-x^2$$

We now try to solve for y in this equation.

$$y=\pm\sqrt{1-x^2}$$

$$\text{so, } y=\sqrt{1-x^2} \text{ and } y=-\sqrt{1-x^2}$$

We get two outputs corresponding to the same input, so this relationship cannot be represented as a single function $y=f(x)$.

Exercise 1.1.8

If $x-8y^3=0$, express y as a function of x.

Answer

$$y=f(x)=\frac{\sqrt[3]{x}}{2}$$

Q & A

Are there relationships expressed by an equation that do represent a function but which still cannot be represented by an algebraic formula?

Yes, this can happen. For example, given the equation $x=y+2^y$, if we want to express y as a function of x, there is no simple algebraic formula involving only x that equals y. However, each x does determine a unique value for y, and there are

mathematical procedures by which y can be found to any desired accuracy. In this case, we say that the equation gives an implicit (implied) rule for y as a function of x, even though the formula cannot be written explicitly.

Evaluating a Function Given in Tabular Form

As we saw above, we can represent functions in tables. Conversely, we can use information in tables to write functions, and we can evaluate functions using the tables. For example, how well do our pets recall the fond memories we share with them? There is an urban legend that a goldfish has a memory of 3 seconds, but this is just a myth. Goldfish can remember up to 3 months, while the beta fish has a memory of up to 5 months. And while a puppy's memory span is no longer than 30 seconds, the adult dog can remember for 5 minutes. This is meager compared to a cat, whose memory span lasts for 16 hours.

The function that relates the type of pet to the duration of its memory span is more easily visualized with the use of a table (Table 1.1.10).

Table 1.1.10

Pet Memory	span in hours
Puppy	0.008
Adult Dog	0.083
Cat	3
Goldfish	2160
Beta Fish	3600

At times, evaluating a function in table form may be more useful than using equations. Here let us call the function P. The domain of the function is the type of pet and the range is a real number representing the number of hours the pet's memory span lasts. We can evaluate the function P at the input value of "goldfish." We would write $P(goldfish) = 2160$. Notice that, to evaluate the function in table form, we identify the input value and the corresponding output value from the pertinent row of the table. The tabular form for function P seems ideally suited to this function, more so than writing it in paragraph or function form.

How To: Given a function represented by a table, identify specific output and input values

1. Find the given input in the row (or column) of input values.
2. Identify the corresponding output value paired with that input value.
3. Find the given output values in the row (or column) of output values, noting every time that output value appears.
4. Identify the input value(s) corresponding to the given output value.

Example 1.1.9: Evaluating and Solving a Tabular Function

Using Table 1.1.11,

a. Evaluate $g(3)$.
b. Solve $g(n) = 6$.

Table 1.1.11

n	1	2	3	4	5
$g(n)$	8	6	7	6	8

Solution

a. Evaluating $g(3)$ means determining the output value of the function g for the input value of $n = 3$. The table output value corresponding to $n = 3$ is 7, so $g(3) = 7$.
b. Solving $g(n) = 6$ means identifying the input values, n,that produce an output value of 6. Table 1.1.12 shows two solutions: 2 and 4.

Table 1.1.12

n	1	2	3	4	5
$g(n)$	8	6	7	6	8

When we input 2 into the function g, our output is 6. When we input 4 into the function g, our output is also 6.

Exercise 1.1.1

Using Table 1.1.12, evaluate $g(1)$.

Answer

$g(1) = 8$

Finding Function Values from a Graph

Evaluating a function using a graph also requires finding the corresponding output value for a given input value, only in this case, we find the output value by looking at the graph. Solving a function equation using a graph requires finding all instances of the given output value on the graph and observing the corresponding input value(s).

Example 1.1.10: Reading Function Values from a Graph

Given the graph in Figure 1.1.7,

a. Evaluate $f(2)$.
b. Solve $f(x) = 4$.

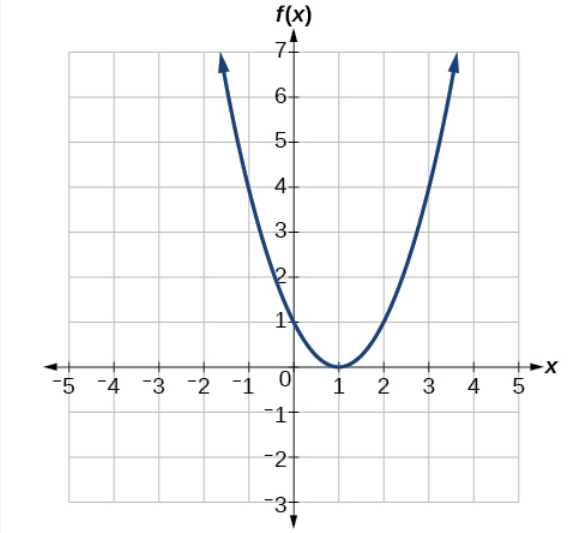
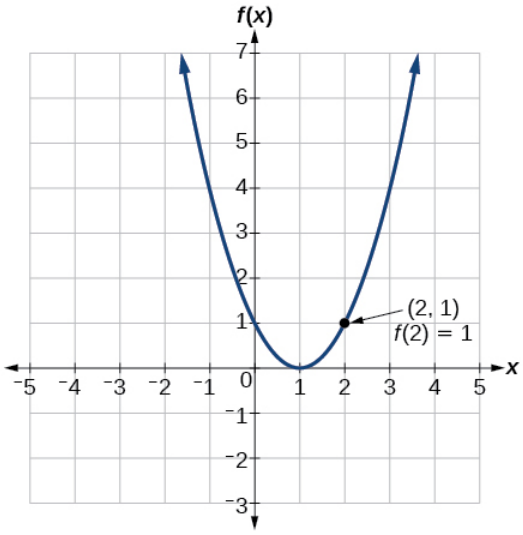


Figure 1.1.7: Graph of a positive parabola centered at $(1, 0)$.

Solution

To evaluate $f(2)$, locate the point on the curve where $x = 2$, then read the y-coordinate of that point. The point has coordinates $(2, 1)$, so $f(2) = 1$. See Figure 1.1.8.



1.1.8: Graph of a positive parabola centered at $(1,0)$ with the labeled point $(2,1)$ where $f(2)=1$.

To solve $f(x)=4$, we find the output value 4 on the vertical axis. Moving horizontally along the line $y=4$, we locate two points of the curve with output value 4: $(-1,4)$ and $(3,4)$. These points represent the two solutions to $f(x)=4$: -1 or 3. This means $f(-1)=4$ and $f(3)=4$, or when the input is -1 or 3, the output is 4. See Figure 1.1.9.

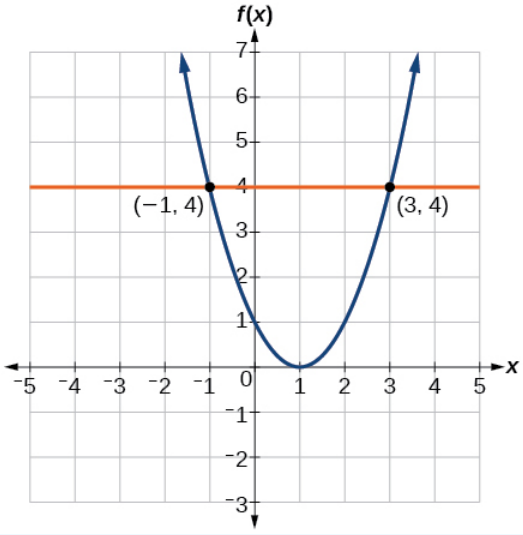


Figure 1.1.9: Graph of an upward-facing parabola with a vertex at $(0,1)$ and labeled points at $(-1,4)$ and $(3,4)$. A line at $y=4$ intersects the parabola at the labeled points.

Exercise 1.1.10

Given the graph in Figure 1.1.7, solve $f(x)=1$.

Answer

$x=0$ or $x=2$

Determining Whether a Function is One-to-One

Some functions have a given output value that corresponds to two or more input values. For example, in the stock chart shown in the Figure at the beginning of this chapter, the stock price was $1000 on five different dates, meaning that there were five different input values that all resulted in the same output value of $1000.

However, some functions have only one input value for each output value, as well as having only one output for each input. We call these functions one-to-one functions. As an example, consider a school that uses only letter grades and decimal equivalents, as listed in Table 1.1.13.

Table 1.1.13: Letter grades and decimal equivalents.

Letter Grade	Grade Point Average
A	4.0
B	3.0
C	2.0
D	1.0

This grading system represents a one-to-one function, because each letter input yields one particular grade point average output and each grade point average corresponds to one input letter.

To visualize this concept, let's look again at the two simple functions sketched in Figures 1.1.1*a* and 1.1.1*b*. The function in part (a) shows a relationship that is not a one-to-one function because inputs q and r both give output n. The function in part (b) shows a relationship that is a one-to-one function because each input is associated with a single output.

One-to-One Functions

A **one-to-one function** is a function in which each output value corresponds to exactly one input value.

Example 1.1.11: Determining Whether a Relationship Is a One-to-One Function

Is the area of a circle a function of its radius? If yes, is the function one-to-one?

Solution

A circle of radius r has a unique area measure given by $A = \pi r^2$, so for any input, r, there is only one output, A. The area is a function of radius$r$.

If the function is one-to-one, the output value, the area, must correspond to a unique input value, the radius. Any area measure A is given by the formula $A = \pi r^2$. Because areas and radii are positive numbers, there is exactly one solution: $\sqrt{\frac{A}{\pi}}$. So the area of a circle is a one-to-one function of the circle's radius.

Exercise 1.1.11A

a. Is a balance a function of the bank account number?
b. Is a bank account number a function of the balance?
c. Is a balance a one-to-one function of the bank account number?

Answer

a. yes, because each bank account has a single balance at any given time;

b. no, because several bank account numbers may have the same balance;

c. no, because the same output may correspond to more than one input.

Exercise 1.1.11B

Evaluate the following:

a. If each percent grade earned in a course translates to one letter grade, is the letter grade a function of the percent grade?
b. If so, is the function one-to-one?

Answer

a. Yes, letter grade is a function of percent grade;
b. No, it is not one-to-one. There are 100 different percent numbers we could get but only about five possible letter grades, so there cannot be only one percent number that corresponds to each letter grade.

Using the Vertical Line Test

As we have seen in some examples above, we can represent a function using a graph. Graphs display a great many input-output pairs in a small space. The visual information they provide often makes relationships easier to understand. By convention, graphs are typically constructed with the input values along the horizontal axis and the output values along the vertical axis.

The most common graphs name the input value x and the output y, and we say y is a function of x, or $y = f(x)$ when the function is named f. The graph of the function is the set of all points (x, y) in the plane that satisfies the equation $y = f(x)$. If the function is defined for only a few input values, then the graph of the function is only a few points, where the x-coordinate of each point is an input value and the y-coordinate of each point is the corresponding output value. For example, the black dots on the graph in Figure 1.1.10 tell us that $f(0) = 2$ and $f(6) = 1$. However, the set of all points (x, y) satisfying $y = f(x)$ is a curve. The curve shown includes $(0, 2)$ and $(6, 1)$ because the curve passes through those points

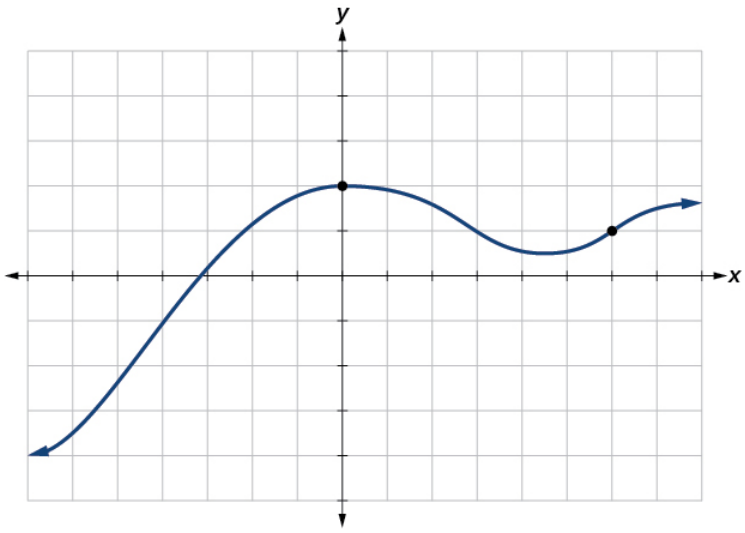


Figure 1.1.10: Graph of a polynomial.

The vertical line test can be used to determine whether a graph represents a function. If we can draw any vertical line that intersects a graph more than once, then the graph does not define a function because a function has only one output value for each input value. See Figure 1.1.11.

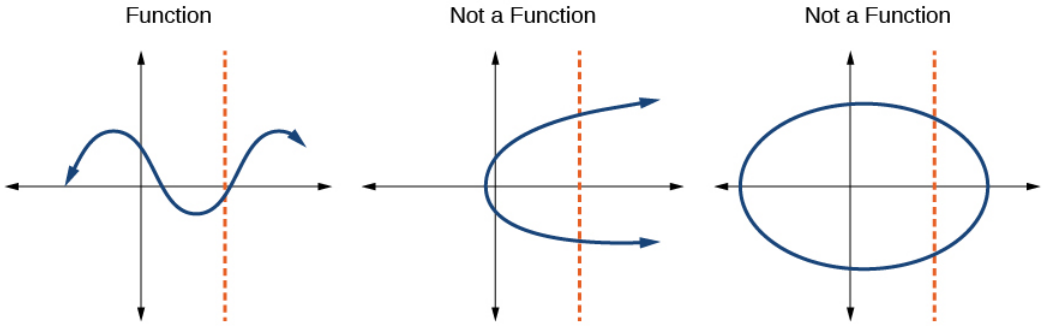


Figure 1.1.11: Three graphs visually showing what is and is not a function.

Howto: Given a graph, use the vertical line test to determine if the graph represents a function

1. Inspect the graph to see if any vertical line drawn would intersect the curve more than once.
2. If there is any such line, determine that the graph does not represent a function.

Example 1.1.12: Applying the Vertical Line Test

Which of the graphs in Figure 1.1.12 represent(s) a function $y = f(x)$?

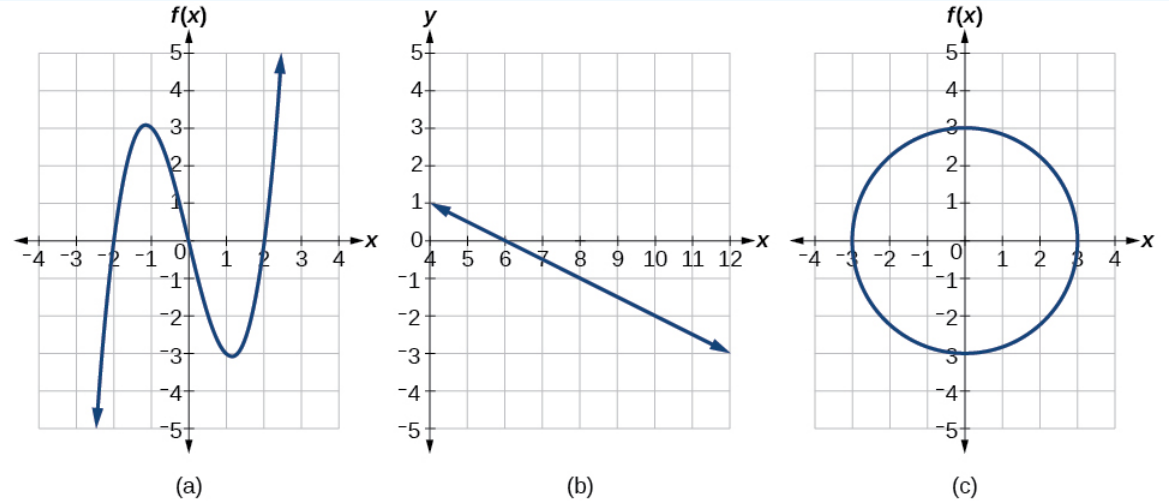


Figure 1.1.12: Graph of a polynomial (a), a downward-sloping line (b), and a circle (c).

Solution

If any vertical line intersects a graph more than once, the relation represented by the graph is not a function. Notice that any vertical line would pass through only one point of the two graphs shown in parts (a) and (b) of Figure 1.1.12. From this we can conclude that these two graphs represent functions. The third graph does not represent a function because, at most x-values, a vertical line would intersect the graph at more than one point, as shown in Figure 1.1.13.

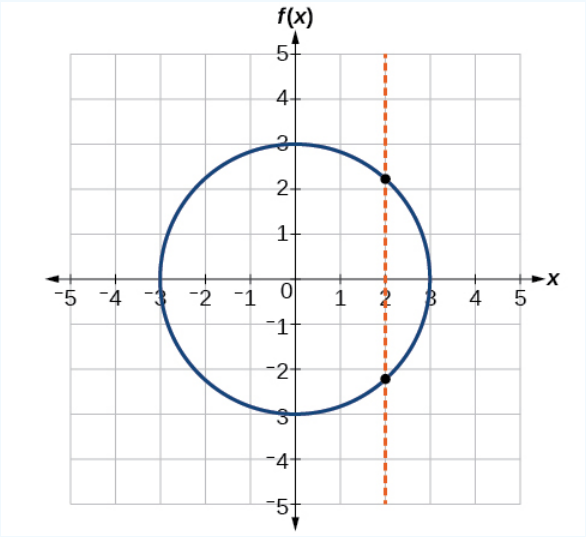


Figure 1.1.13: Graph of a circle.

Exercise 1.1.12

Does the graph in Figure 1.1.14 represent a function?

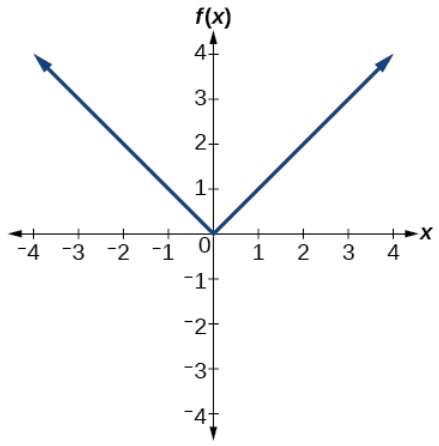


Figure 1.1.14: Graph of absolute value function.

Answer

yes

Using the Horizontal Line Test

Once we have determined that a graph defines a function, an easy way to determine if it is a one-to-one function is to use the **horizontal line test**. Draw horizontal lines through the graph. If any horizontal line intersects the graph more than once, then the graph does not represent a one-to-one function.

Howto: Given a graph of a function, use the horizontal line test to determine if the graph represents a one-to-one function

1. Inspect the graph to see if any horizontal line drawn would intersect the curve more than once.
2. If there is any such line, determine that the function is not one-to-one.

Example 1.1.13: Applying the Horizontal Line Test

Consider the functions shown in Figure $1.1.12a$ and Figure $1.1.12b$ Are either of the functions one-to-one?

Solution

The function in Figure $1.1.12a$ is not one-to-one. The horizontal line shown in Figure $1.1.15$ intersects the graph of the function at two points (and we can even find horizontal lines that intersect it at three points.)

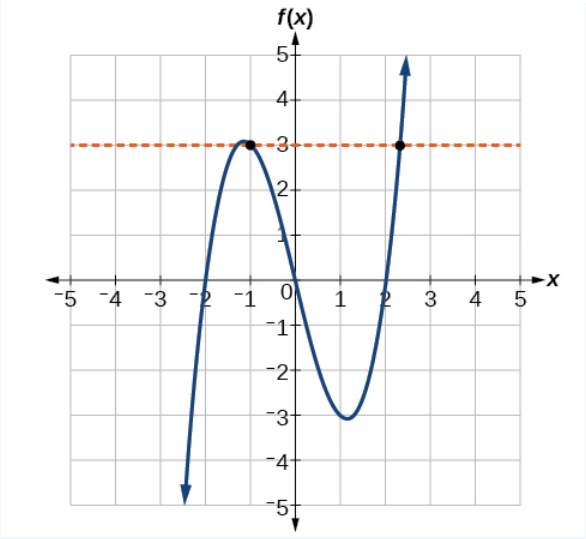


Figure 1.1.15: Graph of a polynomial with a horizontal line crossing through 2 points

The function in Figure $1.1.12b$is one-to-one. Any horizontal line will intersect a diagonal line at most once.

Exercise 1.1.13

Is the graph shown in Figure $1.1.13$one-to-one?

Answer

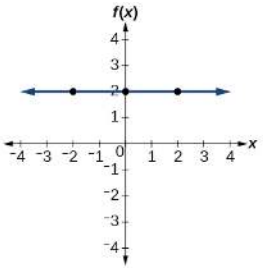
No, because it does not pass the horizontal line test.

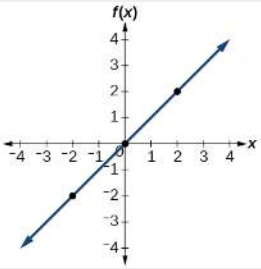
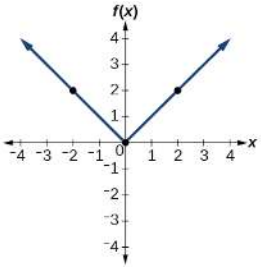
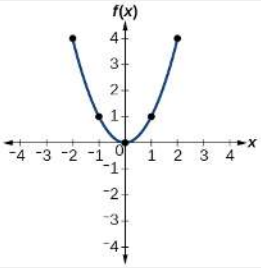
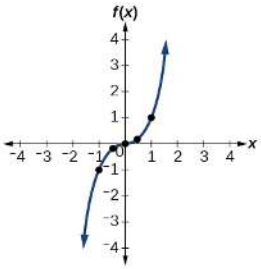
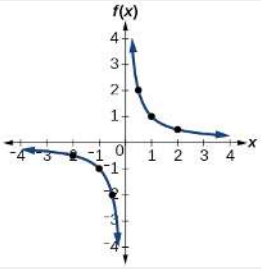
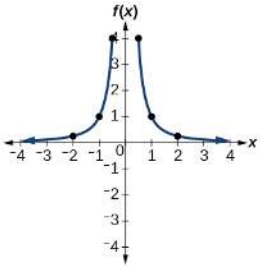
Identifying Basic Toolkit Functions

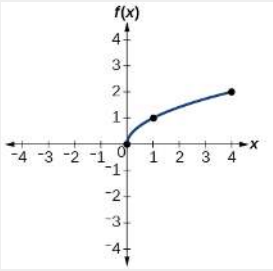
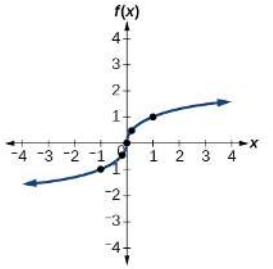
In this text, we will be exploring functions—the shapes of their graphs, their unique characteristics, their algebraic formulas, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of building-block elements. We call these our "toolkit functions," which form a set of basic named functions for which we know the graph, formula, and special properties. Some of these functions are programmed to individual buttons on many calculators. For these definitions we will use x as the input variable and $y = f(x)$ as the output variable.

We will see these toolkit functions, combinations of toolkit functions, their graphs, and their transformations frequently throughout this book. It will be very helpful if we can recognize these toolkit functions and their features quickly by name, formula, graph, and basic table properties. The graphs and sample table values are included with each function shown in Table $1.1.14$

Table 1.1.14: Toolkit Functions

Name	Function	Graph
Constant	$f(x) = c$ where c is a constant	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>2</td></tr><tr><td>0</td><td>2</td></tr><tr><td>2</td><td>2</td></tr></table>

Name	Function	Graph
Identity	$f(x) = x$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>−2</td></tr><tr><td>0</td><td>0</td></tr><tr><td>2</td><td>2</td></tr></table>
Absolute Value	$f(x) = \lvert x \rvert$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>2</td></tr><tr><td>0</td><td>0</td></tr><tr><td>2</td><td>2</td></tr></table>
Quadratic	$f(x) = x^2$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>4</td></tr><tr><td>−1</td><td>1</td></tr><tr><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>4</td></tr></table>
Cubic	$f(x) = x^3$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−1</td><td>−1</td></tr><tr><td>−0.5</td><td>−0.125</td></tr><tr><td>0</td><td>0</td></tr><tr><td>0.5</td><td>0.125</td></tr><tr><td>1</td><td>1</td></tr></table>
reciprocal	$f(x) = \frac{1}{x}$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>−0.5</td></tr><tr><td>−1</td><td>−1</td></tr><tr><td>−0.5</td><td>−2</td></tr><tr><td>0.5</td><td>2</td></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>0.5</td></tr></table>
Reciprocal squared	$f(x) = \frac{1}{x^2}$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>0.25</td></tr><tr><td>−1</td><td>1</td></tr><tr><td>−0.5</td><td>4</td></tr><tr><td>0.5</td><td>4</td></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>0.25</td></tr></table>

Name	Function	Graph
Square root	$f(x) = \sqrt{x}$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td></tr><tr><td>4</td><td>2</td></tr></table>
Cube root	$f(x) = \sqrt[3]{x}$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−1</td><td>−1</td></tr><tr><td>−0.125</td><td>−0.5</td></tr><tr><td>0</td><td>0</td></tr><tr><td>0.125</td><td>0.5</td></tr><tr><td>1</td><td>1</td></tr></table>

Key Equations

- Constant function $f(x) = c$, where c is a constant
- Identity function $f(x) = x$
- Absolute value function $f(x) = |x|$
- Quadratic function $f(x) = x^2$
- Cubic function $f(x) = x^3$
- Reciprocal function $f(x) = \dfrac{1}{x}$
- Reciprocal squared function $f(x) = \frac{1}{x^2}$
- Square root function $f(x) = \sqrt{x}$
- Cube root function $f(x) = 3\sqrt{x}$

Key Concepts

- A relation is a set of ordered pairs. A function is a specific type of relation in which each domain value, or input, leads to exactly one range value, or output.
- Function notation is a shorthand method for relating the input to the output in the form $y = f(x)$.
- In tabular form, a function can be represented by rows or columns that relate to input and output values.
- To evaluate a function, we determine an output value for a corresponding input value. Algebraic forms of a function can be evaluated by replacing the input variable with a given value.
- To solve for a specific function value, we determine the input values that yield the specific output value.
- An algebraic form of a function can be written from an equation.
- Input and output values of a function can be identified from a table.
- Relating input values to output values on a graph is another way to evaluate a function.
- A function is one-to-one if each output value corresponds to only one input value.
- A graph represents a function if any vertical line drawn on the graph intersects the graph at no more than one point.
- The graph of a one-to-one function passes the horizontal line test.

Footnotes

1 http://www.baseball-almanac.com/lege.../lisn100.shtml. Accessed 3/24/2014.
2 www.kgbanswers.com/how-long-i...y-span/4221590. Accessed 3/24/2014.

Glossary

dependent variable
an output variable

domain
the set of all possible input values for a relation

function
a relation in which each input value yields a unique output value

horizontal line test
a method of testing whether a function is one-to-one by determining whether any horizontal line intersects the graph more than once

independent variable
an input variable

input
each object or value in a domain that relates to another object or value by a relationship known as a function

one-to-one function
a function for which each value of the output is associated with a unique input value

output
each object or value in the range that is produced when an input value is entered into a function

range
the set of output values that result from the input values in a relation

relation
a set of ordered pairs

vertical line test
a method of testing whether a graph represents a function by determining whether a vertical line intersects the graph no more than once

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1.3: New Functions from Old Functions

Learning Objectives

- Combine functions using algebraic operations.
- Create a new function by composition of functions.
- Evaluate composite functions.
- Find the domain of a composite function.
- Decompose a composite function into its component functions.

Suppose we want to calculate how much it costs to heat a house on a particular day of the year. The cost to heat a house will depend on the average daily temperature, and in turn, the average daily temperature depends on the particular day of the year. Notice how we have just defined two relationships: The cost depends on the temperature, and the temperature depends on the day.

Using descriptive variables, we can notate these two functions. The function $C(T)$ gives the cost C of heating a house for a given average daily temperature in T degrees Celsius. The function $T(d)$ gives the average daily temperature on day d of the year. For any given day, $Cost = C(T(d))$ means that the cost depends on the temperature, which in turns depends on the day of the year. Thus, we can evaluate the cost function at the temperature $T(d)$. For example, we could evaluate $T(5)$ to determine the average daily temperature on the 5th day of the year. Then, we could evaluate the **cost function** at that temperature. We would write $C(T(5))$.

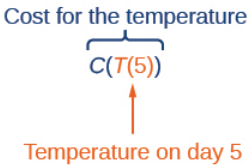


Figure 1.3.1: Explanation of $C(T(5))$, which is the cost for the temperature and $T(5)$ is the temperature on day 5.

By combining these two relationships into one function, we have performed function composition, which is the focus of this section.

Combining Functions Using Algebraic Operations

Function composition is only one way to combine existing functions. Another way is to carry out the usual algebraic operations on functions, such as addition, subtraction, multiplication and division. We do this by performing the operations with the function outputs, defining the result as the output of our new function.

Suppose we need to add two columns of numbers that represent a husband and wife's separate annual incomes over a period of years, with the result being their total household income. We want to do this for every year, adding only that year's incomes and then collecting all the data in a new column. If $w(y)$ is the wife's income and $h(y)$ is the husband's income in year y, and we want T to represent the total income, then we can define a new function.

$$T(y) = h(y) + w(y)$$

If this holds true for every year, then we can focus on the relation between the functions without reference to a year and write

$$T = h + w$$

Just as for this sum of two functions, we can define difference, product, and ratio functions for any pair of functions that have the same kinds of inputs (not necessarily numbers) and also the same kinds of outputs (which do have to be numbers so that the usual operations of algebra can apply to them, and which also must have the same units or no units when we add and subtract). In this way, we can think of adding, subtracting, multiplying, and dividing functions.

For two functions $f(x)$ and $g(x)$ with real number outputs, we define new functions $f+g$, $f-g$, fg, and $\frac{f}{g}$ by the relations.

$$
\begin{aligned}
(f+g)(x) &= f(x)+g(x) \\
(f-g)(x) &= f(x)-g(x) \\
(fg)(x) &= f(x)g(x) \\
\left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)}
\end{aligned}
$$

Example 1.3.1: Performing Algebraic Operations on Functions

Find and simplify the functions $(g-f)(x)$ and $\left(\frac{g}{f}\right)(x)$, given $f(x)=x-1$ and $g(x)=x^2-1$. Are they the same function?

Solution

Begin by writing the general form, and then substitute the given functions.

$$
\begin{aligned}
(g-f)(x) &= g(x)-f(x) \\
(g-f)(x) &= x^2-1-(x-1) \\
&= x^2-x \\
&= x(x-1)
\end{aligned}
$$

$$
\left(\frac{g}{f}\right)(x) = g(x)f(x)
$$

$$
\begin{aligned}
\left(\frac{g}{f}\right)(x) &= \frac{x^2-1}{x-1} \\
&= \frac{(x+1)(x-1)}{x-1} \\
&= x+1
\end{aligned}
$$

No, the functions are not the same.

Note: For $\left(\frac{g}{f}\right)(x)$, the condition $x \neq 1$ is necessary because when $x=1$, the denominator is equal to 0, which makes the function undefined.

Exercise 1.3.1

Find and simplify the functions $(fg)(x)$ and $(f-g)(x)$.

$$
f(x)=x-1
$$

and

$$
g(x)=x^2-1
$$

Are they the same function?

Answer

$(fg)(x)=f(x)g(x)=(x-1)(x2-1)=x^3-x^2-x+1$

$(f-g)(x)=f(x)-g(x)=(x-1)-(x^2-1)=x-x^2$

No, the functions are not the same.

Create a Function by Composition of Functions

Performing algebraic operations on functions combines them into a new function, but we can also create functions by composing functions. When we wanted to compute a heating cost from a day of the year, we created a new function that takes a day as input and yields a cost as output. The process of **combining functions** so that the output of one function becomes the input of another is known as a **composition of functions**. The resulting function is known as a **composite function**. We represent this combination by the following notation:

$$f \circ g(x) = f(g(x)) \tag{1.3.1}$$

We read the left-hand side as"f composed with g at x," and the right-hand side as"f of g of x."The two sides of the equation have the same mathematical meaning and are equal. The open circle symbol $\circ$ is called the composition operator. We use this operator mainly when we wish to emphasize the relationship between the functions themselves without referring to any particular input value. Composition is a binary operation that takes two functions and forms a new function, much as addition or multiplication takes two numbers and gives a new number. However, it is important not to confuse function composition with multiplication because, as we learned above, in most cases $f(g(x)) \neq f(x)g(x)$.

It is also important to understand the order of operations in evaluating a composite function. We follow the usual convention with parentheses by starting with the innermost parentheses first, and then working to the outside. In the equation above, the function g takes the input x first and yields an output $g(x)$. Then the function f takes $g(x)$ as an input and yields an output $f(g(x))$.

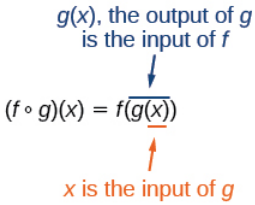


Figure 1.3.2: Explanation of the composite function.

In general, $f \circ g$ and $g \circ f$ are different functions. In other words, in many cases $f(g(x)) \neq g(f(x))$ for all x. We will also see that sometimes two functions can be composed only in one specific order.

For example, if $f(x) = x^2$ and $g(x) = x + 2$, then

$$\begin{aligned} f(g(x)) &= f(x+2) \\ &= (x+2)^2 \\ &= x^2 + 4x + 4 \end{aligned}$$

but

$$\begin{aligned} g(f(x)) &= g(x^2) \\ &= x^2 + 2 \end{aligned}$$

These expressions are not equal for all values of x, so the two functions are not equal. It is irrelevant that the expressions happen to be equal for the single input value $x = -\frac{1}{2}$.

Note that the range of the inside function (the first function to be evaluated) needs to be within the domain of the outside function. Less formally, the composition has to make sense in terms of inputs and outputs.

Composition of Functions

When the output of one function is used as the input of another, we call the entire operation a composition of functions. For any input x and functions f and g, this action defines a **composite function**, which we write as $f \circ g$ such that

$$(f \circ g)(x) = f(g(x)) \tag{1.3.2}$$

The domain of the composite function $f \circ g$ is all x such that x is in the domain of g and $g(x)$ is in the domain of f.

It is important to realize that the product of functions fg is not the same as the function composition $f(g(x))$, because, in general, $f(x)g(x) \neq f(g(x))$.

Example 1.3.2: Determining whether Composition of Functions is Commutative

Using the functions provided, find $f(g(x))$ and $g(f(x))$. Determine whether the composition of the functions is **commutative**.

$$f(x)=2x+1 \quad g(x)=3-x$$

Solution

Let's begin by substituting $g(x)$ into $f(x)$.

$$\begin{aligned} f(g(x)) &= 2(3-x)+1 \\ &= 6-2x+1 \\ &= 7-2x \end{aligned}$$

Now we can substitute $f(x)$ into $g(x)$.

$$\begin{aligned} g(f(x)) &= 3-(2x+1) \\ &= 3-2x-1 \\ &= 2-2x \end{aligned}$$

We find that $g(f(x))\neq f(g(x))$, so the operation of function composition is not commutative.

Example 1.3.3: Interpreting Composite Functions

The function $c(s)$ gives the number of calories burned completing s sit-ups, and $s(t)$ gives the number of sit-ups a person can complete in t minutes. Interpret $c(s(3))$.

Solution

The inside expression in the composition is $s(3)$. Because the input to the s-function is time, $t=3$ represents 3 minutes, and $s(3)$ is the number of sit-ups completed in 3 minutes.

Using $s(3)$ as the input to the function $c(s)$ gives us the number of calories burned during the number of sit-ups that can be completed in 3 minutes, or simply the number of calories burned in 3 minutes (by doing sit-ups).

Example 1.3.4: Investigating the Order of Function Composition

Suppose $f(x)$ gives miles that can be driven in x hours and $g(y)$ gives the gallons of gas used in driving y miles. Which of these expressions is meaningful: $f(g(y))$ or $g(f(x))$?

Solution

The function $y=f(x)$ is a function whose output is the number of miles driven corresponding to the number of hours driven.

$$\text{number of miles } = f(\text{number of hours})$$

The function $g(y)$ is a function whose output is the number of gallons used corresponding to the number of miles driven. This means:

$$\text{number of gallons } = g(\text{number of miles})$$

The expression $g(y)$ takes miles as the input and a number of gallons as the output. The function $f(x)$ requires a number of hours as the input. Trying to input a number of gallons does not make sense. The expression $f(g(y))$ is meaningless.

The expression $f(x)$ takes hours as input and a number of miles driven as the output. The function $g(y)$ requires a number of miles as the input. Using $f(x)$ (miles driven) as an input value for $g(y)$, where gallons of gas depends on miles driven, does make sense. The expression $g(f(x))$ makes sense, and will yield the number of gallons of gas used, g, driving a certain number of miles, $f(x)$, in x hours.

Question/Answer

Are there any situations where $f(g(y))$ and $g(f(x))$ would both be meaningful or useful expressions?

Yes. For many pure mathematical functions, both compositions make sense, even though they usually produce different new functions. In real-world problems, functions whose inputs and outputs have the same units also may give compositions that are meaningful in either order

Exercise 1.3.2

The gravitational force on a planet a distance r from the sun is given by the function $G(r)$. The acceleration of a planet subjected to any force F is given by the function $a(F)$. Form a meaningful composition of these two functions, and explain what it means.

Answer

A gravitational force is still a force, so $a(G(r))$ makes sense as the acceleration of a planet at a distance r from the Sun (due to gravity), but $G(a(F))$ does not make sense.

Evaluating Composite Functions

Once we compose a new function from two existing functions, we need to be able to evaluate it for any input in its domain. We will do this with specific numerical inputs for functions expressed as tables, graphs, and formulas and with variables as inputs to functions expressed as formulas. In each case, we evaluate the inner function using the starting input and then use the inner function's output as the input for the outer function.

Evaluating Composite Functions Using Tables

When working with functions given as tables, we read input and output values from the table entries and always work from the inside to the outside. We evaluate the inside function first and then use the output of the inside function as the input to the outside function.

Example 1.3.5: Using a Table to Evaluate a Composite Function

Using Table 1.3.1, evaluate $f(g(3))$ and $g(f(3))$.

Table 1.3.1

x	$f(x)$	$g(x)$
1	6	3
2	8	5
3	3	2
4	1	7

Solution

To evaluate $f(g(3))$, we start from the inside with the input value 3. We then evaluate the inside expression $g(3)$ using the table that defines the function g : $g(3) = 2$. We can then use that result as the input to the function f, so $g(3)$ is replaced by 2 and we get $f(2)$. Then, using the table that defines the function f, we find that $f(2) = 8$.

$$g(3) = 2$$

$$f(g(3)) = f(2) = 8$$

To evaluate $g(f(3))$, we first evaluate the inside expression $f(3)$ using the first table: $f(3) = 3$. Then, using the table for g, we can evaluate

$$g(f(3)) = g(3) = 2$$

Table 1.3.2 shows the composite functions $f \circ g$ and $g \circ f$ as tables.

Table 1.3.2

x	$g(x)$	$f(g(x))$	$f(x)$	$g(f(x))$
3	2	8	3	2

Exercise 1.3.3

Using Table 1.3.1, evaluate $f(g(1))$ and $g(f(4))$.

Answer

$f(g(1)) = f(3) = 3$ and $g(f(4)) = g(1) = 3$

Evaluating Composite Functions Using Graphs

When we are given individual functions as graphs, the procedure for evaluating composite functions is similar to the process we use for evaluating tables. We read the input and output values, but this time, from the x- and y-axes of the graphs.

How To ...

Given a composite function and graphs of its individual functions, evaluate it using the information provided by the graphs.

1. Locate the given input to the inner function on the x-axis of its graph.
2. Read off the output of the inner function from the y-axis of its graph.
3. Locate the inner function output on the x-axis of the graph of the outer function.
4. Read the output of the outer function from the y-axis of its graph. This is the output of the composite function.

Example 1.3.6: Using a Graph to Evaluate a Composite Function

Using Figure 1.3.3, evaluate $f(g(1))$.

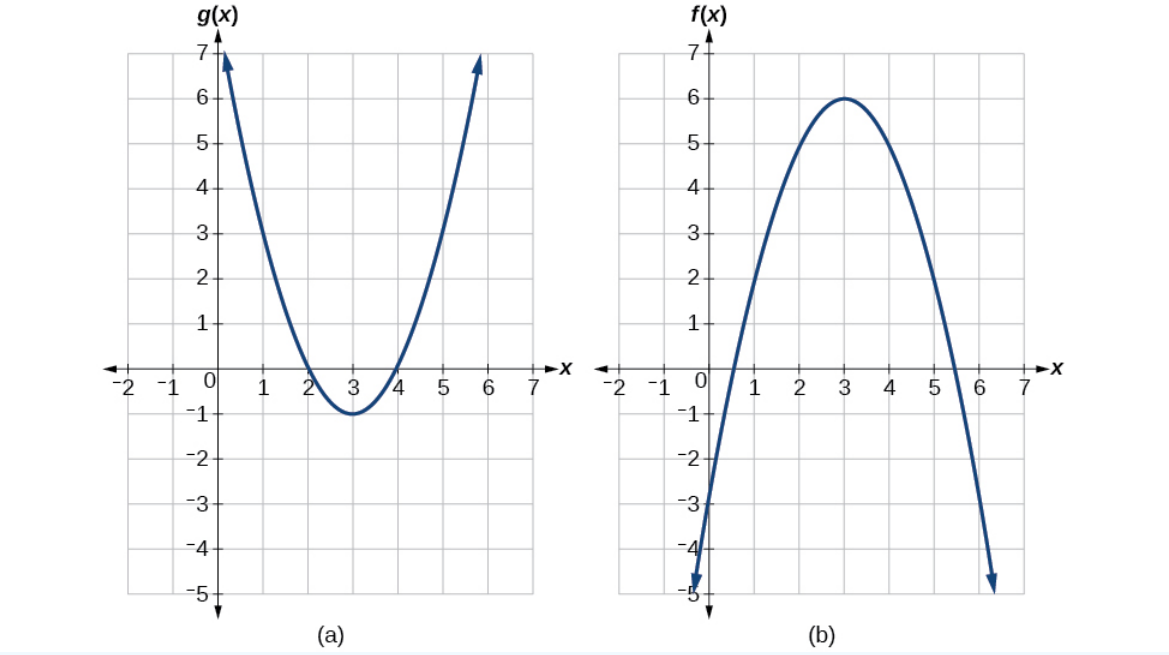


Figure 1.3.3: Two graphs of a positive and negative parabola.

Solution

To evaluate $f(g(1))$, we start with the inside evaluation. See Figure 1.3.4.

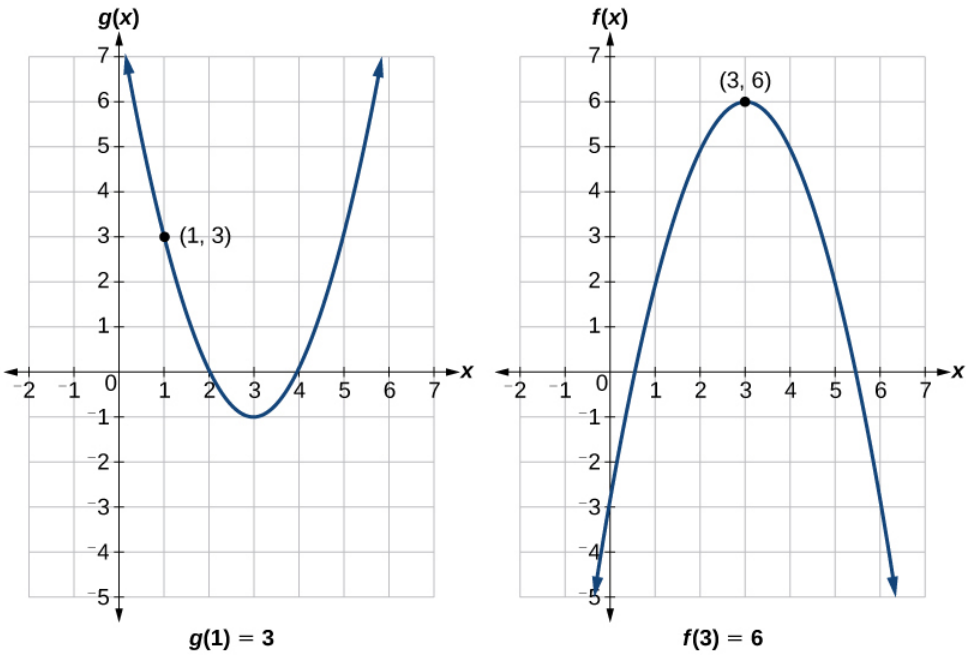


Figure 1.3.4: Two graphs of a positive parabola $g(x)$ and a negative parabola $f(x)$. The following points are plotted: $g(1) = 3$ and $f(3) = 6$.

We evaluate $g(1)$ using the graph of $g(x)$, finding the input of 1 on the x-axis and finding the output value of the graph at that input. Here, $g(1) = 3$. We use this value as the input to the function f.

$$f(g(1)) = f(3)$$

We can then evaluate the composite function by looking to the graph of $f(x)$, finding the input of 3 on the x-axis and reading the output value of the graph at this input. Here, $f(3) = 6$, so $f(g(1)) = 6$.

Analysis

Figure 1.3.5 shows how we can mark the graphs with arrows to trace the path from the input value to the output value.

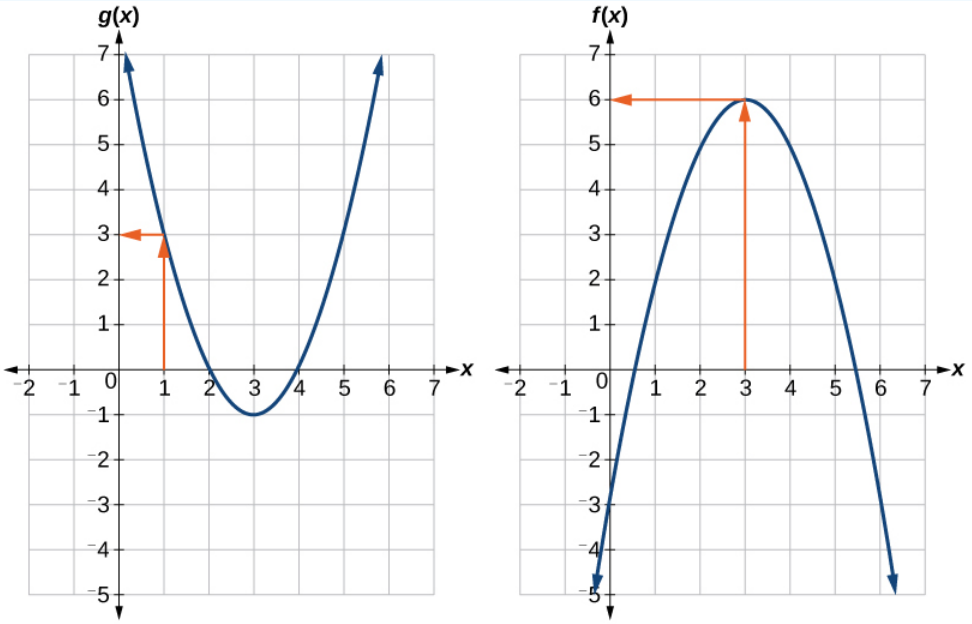


Figure 1.3.5: Two graphs of a positive and negative parabola.

Exercise 1.3.4

Using Figure 1.3.3, evaluate $g(f(2))$.

Answer

$g(f(2)) = g(5) = 3$

Evaluating Composite Functions Using Formulas

When evaluating a composite function where we have either created or been given formulas, the rule of working from the inside out remains the same. The input value to the outer function will be the output of the inner function, which may be a numerical value, a variable name, or a more complicated expression.

While we can compose the functions for each individual input value, it is sometimes helpful to find a single formula that will calculate the result of a composition $f(g(x))$. To do this, we will extend our idea of function evaluation. Recall that, when we evaluate a function like $f(t) = t^2 - t$, we substitute the value inside the parentheses into the formula wherever we see the input variable.

How To...

Given a formula for a composite function, evaluate the function.

1. Evaluate the inside function using the input value or variable provided.
2. Use the resulting output as the input to the outside function.

Example 1.3.7: Evaluating a Composition of Functions Expressed as Formulas with a Numerical Input

Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate $f(h(1))$.

Solution

Because the inside expression is $h(1)$, we start by evaluating $h(x)$ at 1.

$$h(1) = 3(1) + 2$$
$$h(1) = 5$$

Then $f(h(1)) = f(5)$, so we evaluate $f(t)$ at an input of 5.

$$f(h(1)) = f(5)$$
$$f(h(1)) = 5^2 - 5$$
$$f(h(1)) = 20$$

Analysis

It makes no difference what the input variables t and x were called in this problem because we evaluated for specific numerical values.

Exercise 1.3.5

Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate

a. $h(f(2))$
b. $h(f(-2))$

Answer a

8

Answer b

20

Finding the Domain of a Composite Function

As we discussed previously, the **domain of a composite function** such as $f \circ g$ is dependent on the domain of g and the domain of f. It is important to know when we can apply a composite function and when we cannot, that is, to know the domain of a function such as $f \circ g$. Let us assume we know the domains of the functions f and g separately. If we write the composite function for an input x as $f(g(x))$, we can see right away that x must be a member of the domain of g in order for the expression to be meaningful, because otherwise we cannot complete the inner function evaluation. However, we also see that $g(x)$ must be a member of the domain of f, otherwise the second function evaluation in $f(g(x))$ cannot be completed, and the expression is still undefined. Thus the domain of $f \circ g$ consists of only those inputs in the domain of g that produce outputs from g belonging to the domain of f. Note that the domain of f composed with g is the set of all x such that x is in the domain of g and g(x)\) is in the domain of f.

Definition: Domain of a Composite Function

The **domain of a composite function** $f(g(x))$ is the set of those inputs x in the domain of g for which $g(x)$ is in the domain of f.

How To...

Given a function composition $f(g(x))$, determine its domain.

1. Find the domain of g.
2. Find the domain of f.
3. Find those inputs x in the domain of g for which $g(x)$ is in the domain of f. That is, exclude those inputs x from the domain of g for which $g(x)$ is not in the domain of f. The resulting set is the domain of $f \circ g$.

Example 1.3.8A: Finding the Domain of a Composite Function

Find the domain of

$$(f \circ g)(x) \text{ where } f(x) = \frac{5}{x-1} \text{ and } g(x) = \frac{4}{3x-2}$$

Solution

The domain of $g(x)$ consists of all real numbers except $x = \frac{2}{3}$, since that input value would cause us to divide by 0. Likewise, the domain of f consists of all real numbers except 1. So we need to exclude from the domain of $g(x)$ that value of x for which $g(x) = 1$.

$$\begin{aligned} \frac{4}{3x-2} &= 1 \\ 4 &= 3x-2 \\ 6 &= 3x \\ x &= 2 \end{aligned}$$

So the domain of $f \circ g$ is the set of all real numbers except $\frac{2}{3}$ and 2. This means that

$$x \neq \frac{2}{3} \text{ or } x \neq 2$$

We can write this in interval notation as

$$\left(-\infty, \frac{2}{3}\right) \cup \left(\frac{2}{3}, 2\right) \cup (2, \infty)$$

Example 1.3.8B: Finding the Domain of a Composite Function Involving Radicals

Find the domain of

$$(f\circ g)(x) \text{ where } f(x)=\sqrt{x+2} \text{ and } g(x)=\sqrt{3-x}$$

Solution

Because we cannot take the square root of a negative number, the domain of g is $(-\infty, 3]$. Now we check the domain of the composite function

$$(f\circ g)(x)=\sqrt{\sqrt{3-x}+2}$$

For $(f\circ g)(x)=\sqrt{\sqrt{3-x}+2}$, $\sqrt{3-x}+2\geq 0$, since the radicand of a square root must be positive. Since square roots are positive, $\sqrt{3-x}\geq 0$,or, $3-x\geq 0$, which gives a domain of $(-\infty, 3]$.

Analysis

This example shows that knowledge of the range of functions (specifically the inner function) can also be helpful in finding the domain of a composite function. It also shows that the domain of $f\circ g$ can contain values that are not in the domain of f, though they must be in the domain of g.

Exercise 1.3.6

Find the domain of

$$(f\circ g)(x) \text{ where } f(x)=\frac{1}{x-2} \text{ and } g(x)=\sqrt{x+4}$$

Answer

$[-4, 0)\cup(0, \infty)$

Decomposing a Composite Function into its Component Functions

In some cases, it is necessary to decompose a complicated function. In other words, we can write it as a composition of two simpler functions. There may be more than one way to **decompose a composite function**, so we may choose the decomposition that appears to be most expedient.

Example 1.3.9: Decomposing a Function

Write $f(x)=\sqrt{5-x^2}$ as the composition of two functions.

Solution

We are looking for two functions, g and h, so $f(x)=g(h(x))$. To do this, we look for a function inside a function in the formula for $f(x)$. As one possibility, we might notice that the expression $5-x^2$ is the inside of the square root. We could then decompose the function as

$$h(x)=5-x^2 \text{ and } g(x)=\sqrt{x}$$

We can check our answer by recomposing the functions.

$$g(h(x))=g(5-x^2)=\sqrt{5-x^2}$$

Exercise 1.3.7

Write $f(x)=\dfrac{4}{3-\sqrt{4+x^2}}$ as the composition of two functions.

Answer

Possible answers:

$g(x) = \sqrt{4 + x^2}$

$h(x) = \dfrac{4}{3 - x}$

$f = h \circ g$

Access these online resources for additional instruction and practice with composite functions.

- Composite Functions (http://openstaxcollege.org/l/compfunction)
- Composite Function Notation Application (http://openstaxcollege.org/l/compfuncnot)
- Composite Functions Using Graphs (http://openstaxcollege.org/l/compfuncgraph)
- Decompose Functions (http://openstaxcollege.org/l/decompfunction)
- Composite Function Values (http://openstaxcollege.org/l/compfuncvalue)

Key Equation

- Composite function $(f \circ g)(x) = f(g(x))$

Key Concepts

- We can perform algebraic operations on functions. See Example.
- When functions are combined, the output of the first (inner) function becomes the input of the second (outer) function.
- The function produced by combining two functions is a composite function. See Example and Example.
- The order of function composition must be considered when interpreting the meaning of composite functions. See Example.
- A composite function can be evaluated by evaluating the inner function using the given input value and then evaluating the outer function taking as its input the output of the inner function.
- A composite function can be evaluated from a table. See Example.
- A composite function can be evaluated from a graph. See Example.
- A composite function can be evaluated from a formula. See Example.
- The domain of a composite function consists of those inputs in the domain of the inner function that correspond to outputs of the inner function that are in the domain of the outer function. See Example and Example.
- Just as functions can be combined to form a composite function, composite functions can be decomposed into simpler functions.
- Functions can often be decomposed in more than one way. See Example.

Glossary

composite function

the new function formed by function composition, when the output of one function is used as the input of another

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CHAPTER OVERVIEW

2: Limits and Derivatives

A general Calculus Textmap organized around the textbook

Calculus: Early Transcendentals

by James Stewart

I II III IV V VI VII VIII IX X XI XII XIII XIV XV XVI XVII

This Textmap is currently under construction... please be patient with us.

Template:HideTOC

Topic hierarchy

2.1: The Tangent and Velocity Problems
2.2: The Limit of a Function
2.3: Calculating Limits Using the Limit Laws
2.4: The Precise Definition of a Limit
2.5: Continuity
2.6: Limits at Infinity; Horizontal Asymptotes
2.7: Derivatives and Rates of Change
2.8: The Derivative as a Function

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2.1: The Tangent and Velocity Problems

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2.2: The Limit of a Function

Learning Objectives

- Using correct notation, describe the limit of a function.
- Use a table of values to estimate the limit of a function or to identify when the limit does not exist.
- Use a graph to estimate the limit of a function or to identify when the limit does not exist.
- Define one-sided limits and provide examples.
- Explain the relationship between one-sided and two-sided limits.
- Using correct notation, describe an infinite limit.
- Define a vertical asymptote.

The concept of a limit or limiting process, essential to the understanding of calculus, has been around for thousands of years. In fact, early mathematicians used a limiting process to obtain better and better approximations of areas of circles. Yet, the formal definition of a limit—as we know and understand it today—did not appear until the late 19th century. We therefore begin our quest to understand limits, as our mathematical ancestors did, by using an intuitive approach. At the end of this chapter, armed with a conceptual understanding of limits, we examine the formal definition of a limit.

We begin our exploration of limits by taking a look at the graphs of the functions

- $f(x) = \dfrac{x^2 - 4}{x - 2}$,
- $g(x) = \dfrac{|x - 2|}{x - 2}$, and
- $h(x) = \dfrac{1}{(x - 2)^2}$,

which are shown in Figure 2.2.1. In particular, let's focus our attention on the behavior of each graph at and around $x = 2$.

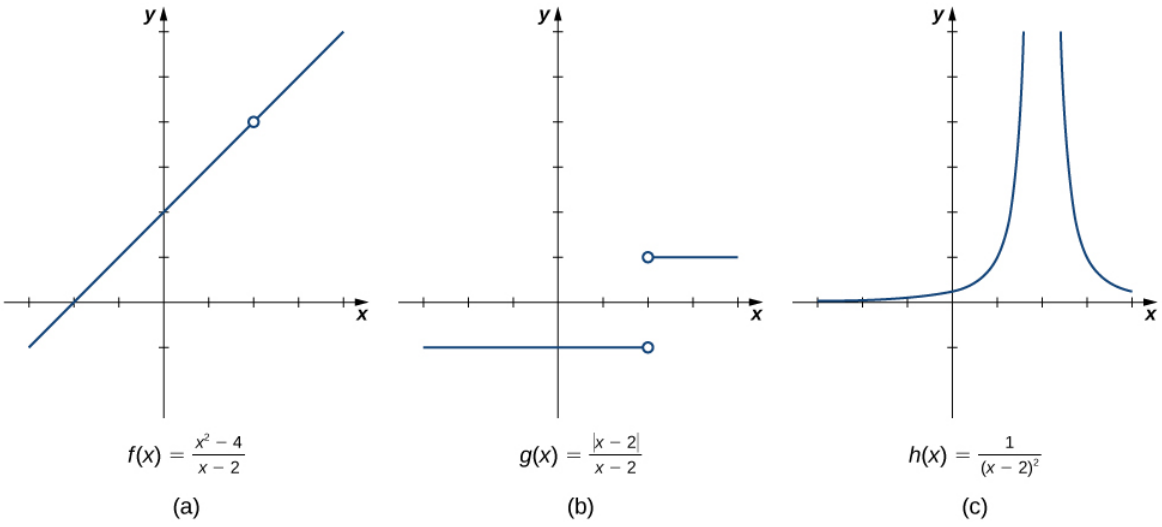


Figure 2.2.1: These graphs show the behavior of three different functions around $x = 2$.

Each of the three functions is undefined at $x = 2$, but if we make this statement and no other, we give a very incomplete picture of how each function behaves in the vicinity of $x = 2$. To express the behavior of each graph in the vicinity of 2 more completely, we need to introduce the concept of a limit.

Intuitive Definition of a Limit

Let's first take a closer look at how the function $f(x) = (x^2 - 4)/(x - 2)$ behaves around $x = 2$ in Figure 2.2.1. As the values of x approach 2 from either side of 2, the values of $y = f(x)$ approach 4. Mathematically, we say that the limit of $f(x)$ as x approaches 2 is 4. Symbolically, we express this limit as

$$\lim_{x \to 2} f(x) = 4$$

From this very brief informal look at one limit, let's start to develop an *intuitive definition of the limit*. We can think of the limit of a function at a number a as being the one real number L that the functional values approach as the x-values approach a, provided such a real number L exists. Stated more carefully, we have the following definition:

Definition (Intuitive): Limit

Let $f(x)$ be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number. If all values of the function $f(x)$ approach the real number L as the values of $x (\neq a)$ approach the number a, then we say that the limit of $f(x)$ as x approaches a is L. (More succinct, as x gets closer to a, $f(x)$ gets closer and stays close to L.) Symbolically, we express this idea as

$$\lim_{x \to a} f(x) = L. \tag{2.2.1}$$

We can estimate limits by constructing tables of functional values and by looking at their graphs. This process is described in the following Problem-Solving Strategy.

Problem-Solving Strategy: Evaluating a Limit Using a Table of Functional Values

1. To evaluate $\lim_{x \to a} f(x)$, we begin by completing a table of functional values. We should choose two sets of x-values—one set of values approaching a and less than a, and another set of values approaching a and greater than a. Table $2.2.1$ demonstrates what your tables might look like.

Table 2.2.1

x	$f(x)$	x	$f(x)$
$a - 0.1$	$f(a - 0.1)$	$a + 0.1$	$f(a + 0.1)$
$a - 0.01$	$f(a - 0.01)$	$a + 0.01$	$f(a + 0.01)$
$a - 0.001$	$f(a - 0.001)$	$a + 0.001$	$f(a + 0.001)$
$a - 0.0001$	$f(a - 0.0001)$	$a + 0.0001$	$f(a + 0.0001)$
Use additional values as necessary.		Use additional values as necessary.	

2. Next, let's look at the values in each of the $f(x)$ columns and determine whether the values seem to be approaching a single value as we move down each column. In our columns, we look at the sequence $f(a-0.1)$, $f(a-0.01)$, $f(a-0.001)$, $f(a-0.0001)$, and so on, and $f(a+0.1)$, $f(a+0.01)$, $f(a+0.001)$, $f(a+0.0001)$, and so on. (Note: Although we have chosen the x-values $a \pm 0.1$, $a \pm 0.01$, $a \pm 0.001$, $a \pm 0.0001$, and so forth, and these values will probably work nearly every time, on very rare occasions we may need to modify our choices.)

3. If both columns approach a common y-value L, we state $\lim_{x \to a} f(x) = L$. We can use the following strategy to confirm the result obtained from the table or as an alternative method for estimating a limit.

4. Using a graphing calculator or computer software that allows us graph functions, we can plot the function $f(x)$, making sure the functional values of $f(x)$ for x-values near a are in our window. We can use the trace feature to move along the graph of the function and watch the y-value readout as the x-values approach a. If the y-values approach L as our x-values approach a from both directions, then $\lim_{x \to a} f(x) = L$. We may need to zoom in on our graph and repeat this process several times.

We apply this Problem-Solving Strategy to compute a limit in Examples $2.2.1A$ and $2.2.1B$.

Example 2.2.1A: Evaluating a Limit Using a Table of Functional Values

Evaluate $\lim_{x\to 0}\dfrac{\sin x}{x}$ using a table of functional values.

Solution

We have calculated the values of $f(x)=\dfrac{\sin x}{x}$ for the values of x listed in Table 2.2.2.

Table 2.2.2

x	$\frac{\sin x}{x}$	x	$\frac{\sin x}{x}$
-0.1	0.998334166468	0.1	0.998334166468
-0.01	0.999983333417	0.01	0.999983333417
-0.001	0.999999833333	0.001	0.999999833333
-0.0001	0.999999998333	0.0001	0.999999998333

Note: The values in this table were obtained using a calculator and using all the places given in the calculator output.

As we read down each $\dfrac{\sin x}{x}$ column, we see that the values in each column appear to be approaching one. Thus, it is fairly reasonable to conclude that $\lim_{x\to 0}\dfrac{\sin x}{x}=1$. A calculator-or computer-generated graph of $f(x)=\dfrac{\sin x}{x}$ would be similar to that shown in Figure 2.2.2, and it confirms our estimate.

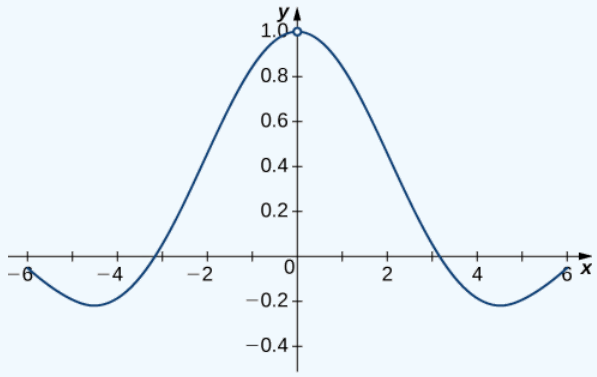


Figure 2.2.2: The graph of $f(x)=(\sin x)/x$ confirms the estimate from Table 2.2.2.

Example 2.2.1B: Evaluating a Limit Using a Table of Functional Values

Evaluate $\lim_{x\to 4}\dfrac{\sqrt{x}-2}{x-4}$ using a table of functional values.

Solution

As before, we use a table—in this case, Table 2.2.3—to list the values of the function for the given values of x.

Table 2.2.3

x	$\frac{\sqrt{x}-2}{x-4}$	x	$\frac{\sqrt{x}-2}{x-4}$
3.9	0.251582341869	4.1	0.248456731317
3.99	0.25015644562	4.01	0.24984394501
3.999	0.250015627	4.001	0.249984377
3.9999	0.250001563	4.0001	0.249998438
3.99999	0.25000016	4.00001	0.24999984

After inspecting this table, we see that the functional values less than 4 appear to be decreasing toward 0.25 whereas the functional values greater than 4 appear to be increasing toward 0.25. We conclude that $\lim\limits_{x\to 4}\dfrac{\sqrt{x}-2}{x-4}=0.25$. We confirm this estimate using the graph of $f(x)=\dfrac{\sqrt{x}-2}{x-4}$ shown in Figure 2.2.3.

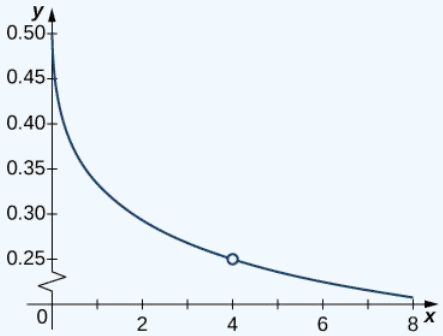


Figure 2.2.3: The graph of $\frac{\sqrt{x}-2}{x-4}$ confirms the estimate from Table 2.2.3.

? Exercise 2.2.1

Estimate $\lim\limits_{x\to 1}\dfrac{\frac{1}{x}-1}{x-1}$ using a table of functional values. Use a graph to confirm your estimate.

Hint

Use 0.9, 0.99, 0.999, 0.9999, 0.99999 and 1.1, 1.01, 1.001, 1.0001, 1.00001 as your table values.

Answer

$$\lim_{x\to 1}\frac{\frac{1}{x}-1}{x-1}=-1$$

At this point, we see from Examples 2.2.1A and 2.2.1b that it may be just as easy, if not easier, to estimate a limit of a function by inspecting its graph as it is to estimate the limit by using a table of functional values. In Example 2.2.2, we evaluate a limit exclusively by looking at a graph rather than by using a table of functional values.

✓ Example 2.2.2: Evaluating a Limit Using a Graph

For $g(x)$ shown in Figure 2.2.4, evaluate $\lim\limits_{x\to -1} g(x)$.

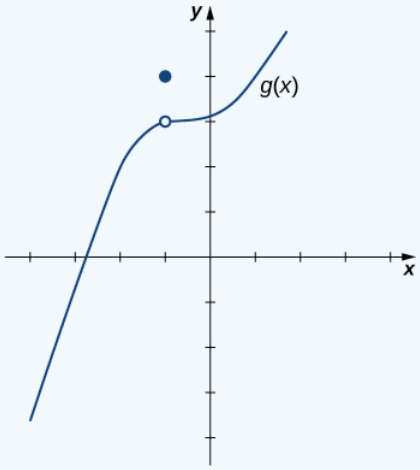


Figure 2.2.4: The graph of $g(x)$ includes one value not on a smooth curve.

Solution:

Despite the fact that $g(-1) = 4$, as the x-values approach -1 from either side, the $g(x)$ values approach 3. Therefore, $\lim_{x \to -1} g(x) = 3$. Note that we can determine this limit without even knowing the algebraic expression of the function.

Based on Example 2.2.2, we make the following observation: It is possible for the limit of a function to exist at a point, and for the function to be defined at this point, but the limit of the function and the value of the function at the point may be different.

Exercise 2.2.2

Use the graph of $h(x)$ in Figure 2.2.5 to evaluate $\lim_{x \to 2} h(x)$, if possible.

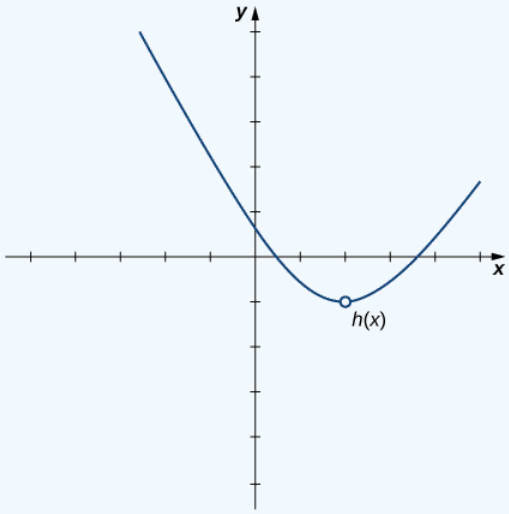


Figure 2.2.5:

Hint

What y-value does the function approach as the x-values approach 2?

Solution

$\lim_{x \to 2} h(x) = -1$.

Looking at a table of functional values or looking at the graph of a function provides us with useful insight into the value of the limit of a function at a given point. However, these techniques rely too much on guesswork. We eventually need to develop alternative methods of evaluating limits. These new methods are more algebraic in nature and we explore them in the next section; however, at this point we introduce two special limits that are foundational to the techniques to come.

Two Important Limits

Let a be a real number and c be a constant.

i. $\lim_{x \to a} x = a$
ii. $\lim_{x \to a} c = c$

We can make the following observations about these two limits.

i. For the first limit, observe that as x approaches a, so does $f(x)$, because $f(x) = x$. Consequently, $\lim_{x \to a} x = a$.
ii. For the second limit, consider Table 2.2.4.

Table 2.2.4

x	$f(x) = c$	x	$f(x) = c$
$a - 0.1$	c	$a + 0.1$	c

x	$f(x)=c$	x	$f(x)=c$
$a-0.01$	c	$a+0.01$	c
$a-0.001$	c	$a+0.001$	c
$a-0.0001$	c	$a+0.0001$	c

Observe that for all values of x (regardless of whether they are approaching a), the values $f(x)$ remain constant at c. We have no choice but to conclude $\lim\limits_{x\to a} c = c$.

The Existence of a Limit

As we consider the limit in the next example, keep in mind that for the limit of a function to exist at a point, the functional values must approach a single real-number value at that point. If the functional values do not approach a single value, then the limit does not exist.

Example 2.2.3: Evaluating a Limit That Fails to Exist

Evaluate $\lim\limits_{x\to 0} \sin(1/x)$ using a table of values.

Solution

Table 2.2.5 lists values for the function $\sin(1/x)$ for the given values of x.

Table 2.2.5

x	$\sin(1/x)$	x	$\sin(1/x)$
-0.1	0.544021110889	0.1	−0.544021110889
-0.01	0.50636564111	0.01	−0.50636564111
-0.001	−0.8268795405312	0.001	0.8268795405312
-0.0001	0.305614388888	0.0001	−0.305614388888
-0.00001	−0.035748797987	0.00001	0.035748797987
-0.000001	0.349993504187	0.000001	−0.349993504187

After examining the table of functional values, we can see that the y-values do not seem to approach any one single value. It appears the limit does not exist. Before drawing this conclusion, let's take a more systematic approach. Take the following sequence of x-values approaching 0:

$$\frac{2}{\pi}, \frac{2}{3\pi}, \frac{2}{5\pi}, \frac{2}{7\pi}, \frac{2}{9\pi}, \frac{2}{11\pi}, \ldots.$$

The corresponding y-values are

$$1, -1, 1, -1, 1, -1, \ldots.$$

At this point we can indeed conclude that $\lim\limits_{x\to 0} \sin(1/x)$ does not exist. (Mathematicians frequently abbreviate "does not exist" as DNE. Thus, we would write $\lim\limits_{x\to 0} \sin(1/x)$ DNE.) The graph of $f(x) = \sin(1/x)$ is shown in Figure 2.2.6 and it gives a clearer picture of the behavior of $\sin(1/x)$ as x approaches 0. You can see that $\sin(1/x)$ oscillates ever more wildly between -1 and 1 as x approaches 0.

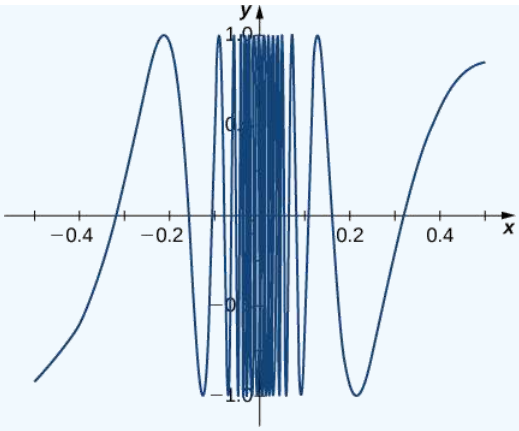


Figure 2.2.6: The graph of $f(x) = \sin(1/x)$ oscillates rapidly between -1 and 1 as x approaches 0.

? Exercise 2.2.3

Use a table of functional values to evaluate $\lim\limits_{x \to 2} \dfrac{|x^2 - 4|}{x - 2}$, if possible.

Hint

Use x-values 1.9, 1.99, 1.999, 1.9999, 1.99999 and 2.1, 2.01, 2.001, 2.0001, 2.00001 in your table.

Answer

$\lim\limits_{x \to 2} \dfrac{|x^2 - 4|}{x - 2}$ does not exist.

One-Sided Limits

Sometimes indicating that the limit of a function fails to exist at a point does not provide us with enough information about the behavior of the function at that particular point. To see this, we now revisit the function $g(x) = |x - 2|/(x - 2)$ introduced at the beginning of the section (see Figure 2.2.1(b)). As we pick values of x close to 2, $g(x)$ does not approach a single value, so the limit as x approaches 2 does not exist—that is, $\lim\limits_{x \to 2} g(x)$ DNE. However, this statement alone does not give us a complete picture of the behavior of the function around the x-value 2. To provide a more accurate description, we introduce the idea of a **one-sided limit**. For all values to the left of 2 (or the negative side of 2), $g(x) = -1$. Thus, as x approaches 2 from the left, $g(x)$ approaches -1. Mathematically, we say that the limit as x approaches 2 from the left is -1. Symbolically, we express this idea as

$$\lim_{x \to 2^-} g(x) = -1.$$

Similarly, as x approaches 2 from the right (or from the positive side), $g(x)$ approaches 1. Symbolically, we express this idea as

$$\lim_{x \to 2^+} g(x) = 1.$$

We can now present an informal definition of one-sided limits.

Definition: One-sided Limits

We define two types of one-sided limits.

Limit from the left:

Let $f(x)$ be a function defined at all values in an open interval of the form (z, a), and let L be a real number. If the values of the function $f(x)$ approach the real number L as the values of x (where $x < a$) approach the number a, then we say that L is the limit of $f(x)$ as x approaches a from the left. Symbolically, we express this idea as

$$\lim_{x \to a^-} f(x) = L.$$

Limit from the right:

Let $f(x)$ be a function defined at all values in an open interval of the form (a, c), and let L be a real number. If the values of the function $f(x)$ approach the real number L as the values of x (where $x > a$) approach the number a, then we say that L is the limit of $f(x)$ as x approaches a from the right. Symbolically, we express this idea as

$$\lim_{x \to a^+} f(x) = L.$$

Example 2.2.4: Evaluating One-Sided Limits

For the function $f(x) = \begin{cases} x+1, & \text{if } x < 2 \\ x^2 - 4, & \text{if } x \geq 2 \end{cases}$, evaluate each of the following limits.

a. $\lim_{x \to 2^-} f(x)$
b. $\lim_{x \to 2^+} f(x)$

Solution

We can use tables of functional values again. Observe in Table 2.2.6 that for values of x less than 2, we use $f(x) = x + 1$ and for values of x greater than 2, we use $f(x) = x^2 - 4$.

Table 2.2.6

x	$f(x) = x + 1$	x	$f(x) = x^2 - 4$
1.9	2.9	2.1	0.41
1.99	2.99	2.01	0.0401
1.999	2.999	2.001	0.004001
1.9999	2.9999	2.0001	0.00040001
1.99999	2.99999	2.00001	0.0000400001

Based on this table, we can conclude that a. $\lim_{x \to 2^-} f(x) = 3$ and b. $\lim_{x \to 2^+} f(x) = 0$. Therefore, the (two-sided) limit of $f(x)$ does not exist at $x = 2$. Figure 2.2.7 shows a graph of $f(x)$ and reinforces our conclusion about these limits.

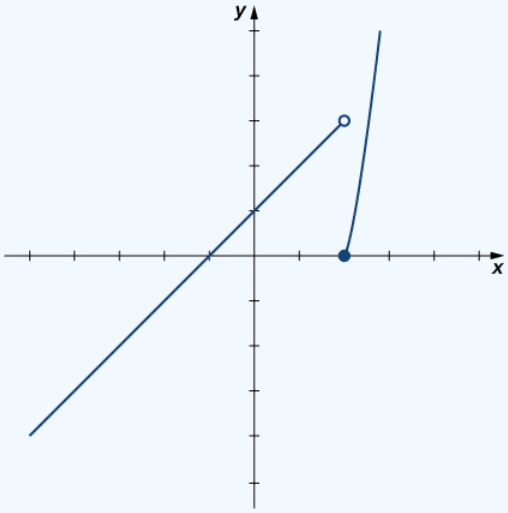


Figure 2.2.7: The graph of $f(x) = \begin{cases} x+1, & \text{if } x < 2 \\ x^2 - 4, & \text{if } x \geq 2 \end{cases}$ has a break at $x = 2$.

Exercise 2.2.4

Use a table of functional values to estimate the following limits, if possible.

a. $\lim\limits_{x\to 2^-} \dfrac{|x^2-4|}{x-2}$

b. $\lim\limits_{x\to 2^+} \dfrac{|x^2-4|}{x-2}$

Hint

Use x-values 1.9, 1.99, 1.999, 1.9999, 1.99999 to estimate $\lim\limits_{x\to 2^-} \dfrac{|x^2-4|}{x-2}$.

Use x-values 2.1, 2.01, 2.001, 2.0001, 2.00001 to estimate $\lim\limits_{x\to 2^+} \dfrac{|x^2-4|}{x-2}$.

(These tables are available from a previous Checkpoint problem.)

Solution a

a. $\lim\limits_{x\to 2^-} \dfrac{|x^2-4|}{x-2} = -4$

Solution b

$\lim\limits_{x\to 2^+} \dfrac{|x^2-4|}{x-2} = 4$

Let us now consider the relationship between the limit of a function at a point and the limits from the right and left at that point. It seems clear that if the limit from the right and the limit from the left have a common value, then that common value is the limit of the function at that point. Similarly, if the limit from the left and the limit from the right take on different values, the limit of the function does not exist. These conclusions are summarized in Note.

Relating One-Sided and Two-Sided Limits

Let $f(x)$ be a function defined at all values in an open interval containing a, with the possible exception of a itself, and let L be a real number. Then,

$$\lim_{x\to a} f(x) = L$$

if and only if $\lim\limits_{x\to a^-} f(x) = L$ and $\lim\limits_{x\to a^+} f(x) = L$.

Infinite Limits

Evaluating the limit of a function at a point or evaluating the limit of a function from the right and left at a point helps us to characterize the behavior of a function around a given value. As we shall see, we can also describe the behavior of functions that do not have finite limits.

We now turn our attention to $h(x) = 1/(x-2)^2$, the third and final function introduced at the beginning of this section (see Figure 2.2.1(c)). From its graph we see that as the values of x approach 2, the values of $h(x) = 1/(x-2)^2$ become larger and larger and, in fact, become infinite. Mathematically, we say that the limit of $h(x)$ as x approaches 2 is positive infinity. Symbolically, we express this idea as

$$\lim_{x\to 2} h(x) = +\infty.$$

More generally, we define *infinite limits* as follows:

Definitions: Infinite Limits

We define three types of *infinite limits*.

Infinite limits from the left: Let $f(x)$ be a function defined at all values in an open interval of the form (b, a).

i. If the values of $f(x)$ increase without bound as the values of x (where $x < a$) approach the number a, then we say that the limit as x approaches a from the left is positive infinity and we write

$$\lim_{x \to a^-} f(x) = +\infty.$$

ii. If the values of $f(x)$ decrease without bound as the values of x (where $x < a$) approach the number a, then we say that the limit as x approaches a from the left is negative infinity and we write

$$\lim_{x \to a^-} f(x) = -\infty.$$

Infinite limits from the right: Let $f(x)$ be a function defined at all values in an open interval of the form (a, c).

i. If the values of $f(x)$ increase without bound as the values of x (where $x > a$) approach the number a, then we say that the limit as x approaches a from the right is positive infinity and we write

$$\lim_{x \to a^+} f(x) = +\infty.$$

ii. If the values of $f(x)$ decrease without bound as the values of x (where $x > a$) approach the number a, then we say that the limit as x approaches a from the right is negative infinity and we write

$$\lim_{x \to a^+} f(x) = -\infty.$$

Two-sided infinite limit: Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a

i. If the values of $f(x)$ increase without bound as the values of x (where $x \neq a$) approach the number a, then we say that the limit as x approaches a is positive infinity and we write

$$\lim_{x \to a} f(x) = +\infty.$$

ii. If the values of $f(x)$ decrease without bound as the values of x (where $x \neq a$) approach the number a, then we say that the limit as x approaches a is negative infinity and we write

$$\lim_{x \to a} f(x) = -\infty.$$

It is important to understand that when we write statements such as $\lim_{x \to a} f(x) = +\infty$ or $\lim_{x \to a} f(x) = -\infty$ we are describing the behavior of the function, as we have just defined it. We are not asserting that a limit exists. For the limit of a function $f(x)$ to exist at a, it must approach a real number L as x approaches a. That said, if, for example, $\lim_{x \to a} f(x) = +\infty$, we always write $\lim_{x \to a} f(x) = +\infty$ rather than $\lim_{x \to a} f(x)$ DNE.

Example 2.2.5: Recognizing an Infinite Limit

Evaluate each of the following limits, if possible. Use a table of functional values and graph $f(x) = 1/x$ to confirm your conclusion.

a. $\lim_{x \to 0^-} \dfrac{1}{x}$

b. $\lim_{x \to 0^+} \dfrac{1}{x}$

c. $\lim_{x \to 0} \dfrac{1}{x}$

Solution

Begin by constructing a table of functional values.

Table 2.2.7

x	$\frac{1}{x}$	x	$\frac{1}{x}$
-0.1	-10	0.1	10
-0.01	-100	0.01	100
-0.001	-1000	0.001	1000
-0.0001	-10,000	0.0001	10,000
-0.00001	-100,000	0.00001	100,000
-0.000001	-1,000,000	0.000001	1,000,000

a. The values of $1/x$ decrease without bound as x approaches 0 from the left. We conclude that

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

b. The values of $1/x$ increase without bound as x approaches 0 from the right. We conclude that

$$\lim_{x\to 0^+} \frac{1}{x} = +\infty.$$

c. Since $\lim_{x\to 0^-} \frac{1}{x} = -\infty$ and $\lim_{x\to 0^+} \frac{1}{x} = +\infty$ have different values, we conclude that

$$\lim_{x\to 0} \frac{1}{x} \quad \text{DNE.}$$

The graph of $f(x) = 1/x$ in Figure 2.2.8 confirms these conclusions.

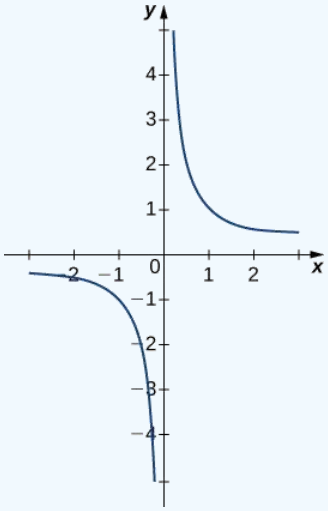


Figure 2.2.8: The graph of $f(x) = 1/x$ confirms that the limit as x approaches 0 does not exist.

Exercise 2.2.5

Evaluate each of the following limits, if possible. Use a table of functional values and graph $f(x) = 1/x^2$ to confirm your conclusion.

a. $\lim_{x\to 0^-} \frac{1}{x^2}$

b. $\lim_{x\to 0^+} \frac{1}{x^2}$

c. $\lim_{x\to 0} \frac{1}{x^2}$

Hint

Follow the procedures from Example 2.2.5

Answer

a. $\lim\limits_{x\to 0^-} \dfrac{1}{x^2} = +\infty$;

b. $\lim\limits_{x\to 0^+} \dfrac{1}{x^2} = +\infty$;

c. $\lim\limits_{x\to 0} \dfrac{1}{x^2} = +\infty$

It is useful to point out that functions of the form $f(x) = 1/(x-a)^n$, where n is a positive integer, have infinite limits as x approaches a from either the left or right (Figure 2.2.9). These limits are summarized in the above definitions.

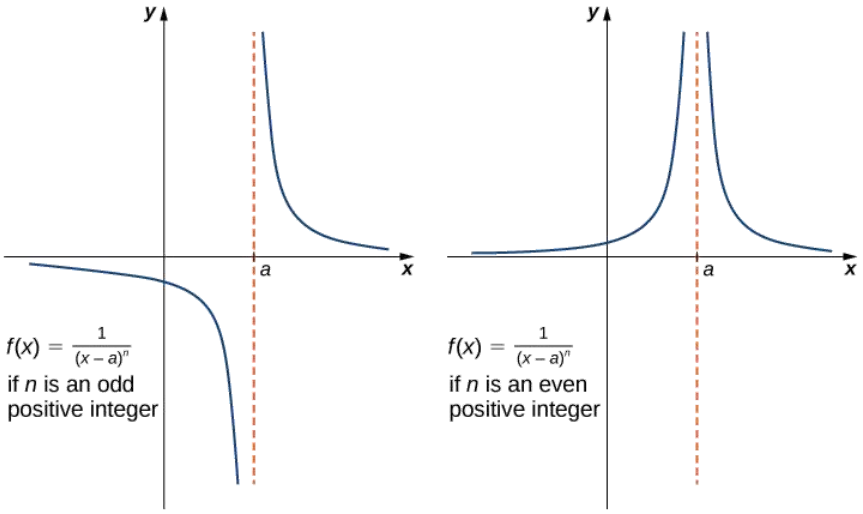


Figure 2.2.9: The function $f(x) = 1/(x-a)^n$ has infinite limits at a.

Infinite Limits from Positive Integers

If n is a positive even integer, then

$$\lim_{x\to a} \frac{1}{(x-a)^n} = +\infty. \tag{2.2.2}$$

If n is a positive odd integer, then

$$\lim_{x\to a^+} \frac{1}{(x-a)^n} = +\infty \tag{2.2.3}$$

and

$$\lim_{x\to a^-} \frac{1}{(x-a)^n} = -\infty. \tag{2.2.4}$$

We should also point out that in the graphs of $f(x) = 1/(x-a)^n$, points on the graph having x-coordinates very near to a are very close to the vertical line $x = a$. That is, as x approaches a, the points on the graph of $f(x)$ are closer to the line $x = a$. The line $x = a$ is called a vertical asymptote of the graph. We formally define a *vertical asymptote* as follows:

Definition: Vertical Asymptotes

Let $f(x)$ be a function. If any of the following conditions hold, then the line $x = a$ is a vertical asymptote of $f(x)$.

$$\lim_{x\to a^-} f(x) = +\infty$$

$$\lim_{x \to a^-} f(x) = -\infty$$

$$\lim_{x \to a^+} f(x) = +\infty$$

$$\lim_{x \to a^+} f(x) = -\infty$$

$$\lim_{x \to a} f(x) = +\infty$$

$$\lim_{x \to a} f(x) = -\infty$$

Example 2.2.6: Finding a Vertical Asymptote

Evaluate each of the following limits using Equations 2.2.2, 2.2.3, and 2.2.4 above. Identify any vertical asymptotes of the function $f(x) = 1/(x+3)^4$.

a. $\lim_{x \to -3^-} \dfrac{1}{(x+3)^4}$

b. $\lim_{x \to -3^+} \dfrac{1}{(x+3)^4}$

c. $\lim_{x \to -3} \dfrac{1}{(x+3)^4}$

Solution

We can use the above equations directly.

a. $\lim_{x \to -3^-} \dfrac{1}{(x+3)^4} = +\infty$

b. $\lim_{x \to -3^+} \dfrac{1}{(x+3)^4} = +\infty$

c. $\lim_{x \to -3} \dfrac{1}{(x+3)^4} = +\infty$

The function $f(x) = 1/(x+3)^4$ has a vertical asymptote of $x = -3$.

Exercise 2.2.6

Evaluate each of the following limits. Identify any vertical asymptotes of the function $f(x) = \dfrac{1}{(x-2)^3}$.

a. $\lim_{x \to 2^-} \dfrac{1}{(x-2)^3}$

b. $\lim_{x \to 2^+} \dfrac{1}{(x-2)^3}$

c. $\lim_{x \to 2} \dfrac{1}{(x-2)^3}$

Answer a

$$\lim_{x \to 2^-} \frac{1}{(x-2)^3} = -\infty$$

Answer b

$$\lim_{x \to 2^+} \frac{1}{(x-2)^3} = +\infty$$

Answer c

$\lim_{x \to 2} \dfrac{1}{(x-2)^3}$ DNE. The line $x = 2$ is the vertical asymptote of $f(x) = 1/(x-2)^3$.

In the next example we put our knowledge of various types of limits to use to analyze the behavior of a function at several different points.

Example 2.2.7: Behavior of a Function at Different Points

Use the graph of $f(x)$ in Figure 2.2.10 to determine each of the following values:

a. $\lim\limits_{x\to -4^-} f(x)$; $\lim\limits_{x\to -4^+} f(x)$; $\lim\limits_{x\to -4} f(x)$; $f(-4)$
b. $\lim\limits_{x\to -2^-} f(x)$; $\lim\limits_{x\to -2^+} f(x)$; $\lim\limits_{x\to -2} f(x)$; $f(-2)$
c. $\lim\limits_{x\to 1^-} f(x)$; $\lim\limits_{x\to 1^+} f(x)$; $\lim\limits_{x\to 1} f(x)$; $f(1)$
d. $\lim\limits_{x\to 3^-} f(x)$; $\lim\limits_{x\to 3^+} f(x)$; $\lim\limits_{x\to 3} f(x)$; $f(3)$

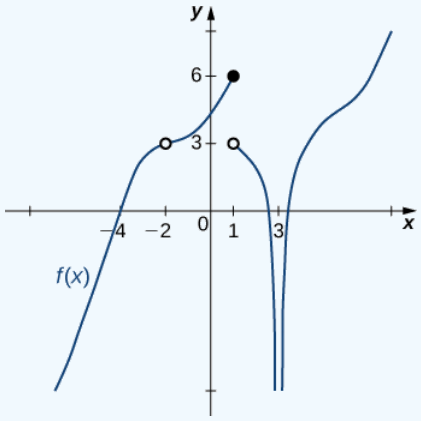


Figure 2.2.10: The graph shows $f(x)$.

Solution

Using the definitions above and the graph for reference, we arrive at the following values:

a. $\lim\limits_{x\to -4^-} f(x) = 0$; $\lim\limits_{x\to -4^+} f(x) = 0$; $\lim\limits_{x\to -4} f(x) = 0$; $f(-4) = 0$
b. $\lim\limits_{x\to -2^-} f(x) = 3$; $\lim\limits_{x\to -2^+} f(x) = 3$; $\lim\limits_{x\to -2} f(x) = 3$; $f(-2)$ is undefined
c. $\lim\limits_{x\to 1^-} f(x) = 6$; $\lim\limits_{x\to 1^+} f(x) = 3$; $\lim\limits_{x\to 1} f(x)$ DNE; $f(1) = 6$
d. $\lim\limits_{x\to 3^-} f(x) = -\infty$; $\lim\limits_{x\to 3^+} f(x) = -\infty$; $\lim\limits_{x\to 3} f(x) = -\infty$; $f(3)$ is undefined

Exercise 2.2.7

Evaluate $\lim\limits_{x\to 1} f(x)$ for $f(x)$ shown here:

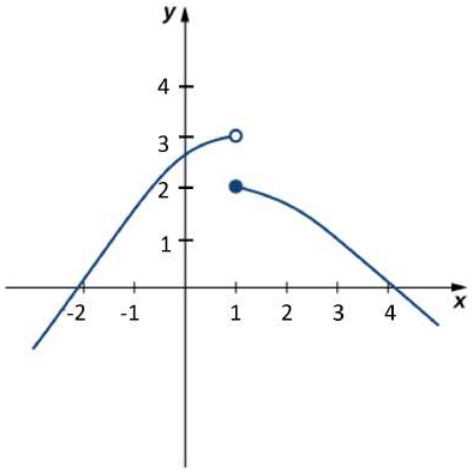


Figure 2.2.11. The graph of a piecewise function f.

Hint

Compare the limit from the right with the limit from the left.

Answer

$\lim_{x \to 1} f(x)$ does not exist

Example 2.2.8: Einstein's Equation

In the Chapter opener we mentioned briefly how Albert Einstein showed that a limit exists to how fast any object can travel. Given Einstein's equation for the mass of a moving object

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}},$$

what is the value of this bound?

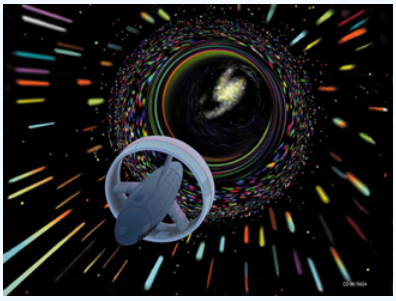


Figure 2.2.12. (Credit:NASA)

Solution

Our starting point is Einstein's equation for the mass of a moving object,

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}},$$

where m_0 is the object's mass at rest, v is its speed, and c is the speed of light. To see how the mass changes at high speeds, we can graph the ratio of masses m/m_0 as a function of the ratio of speeds, v/c (Figure 2.2.13).

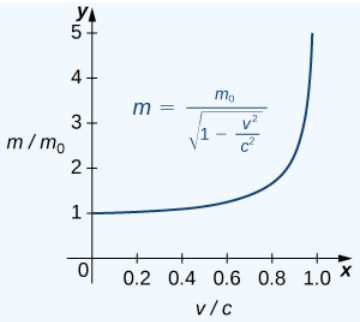


Figure 2.2.13: This graph shows the ratio of masses as a function of the ratio of speeds in Einstein's equation for the mass of a moving object.

We can see that as the ratio of speeds approaches 1—that is, as the speed of the object approaches the speed of light—the ratio of masses increases without bound. In other words, the function has a vertical asymptote at $v/c = 1$. We can try a few values of this ratio to test this idea.

Table 2.2.8

v/c	$\sqrt{1-\frac{v^2}{c^2}}$	m/m_o
0.99	0.1411	7.089
0.999	0.0447	22.37
0.9999	0.0141	70.7

Thus, according to Table 2.2.8:, if an object with mass 100 kg is traveling at 0.9999c, its mass becomes 7071 kg. Since no object can have an infinite mass, we conclude that no object can travel at or more than the speed of light.

Key Concepts

- A table of values or graph may be used to estimate a limit.
- If the limit of a function at a point does not exist, it is still possible that the limits from the left and right at that point may exist.
- If the limits of a function from the left and right exist and are equal, then the limit of the function is that common value.
- We may use limits to describe infinite behavior of a function at a point.

Key Equations

- **Intuitive Definition of the Limit**

  $$\lim_{x \to a} f(x) = L$$

- **Two Important Limits**

  $$\lim_{x \to a} x = a \qquad \lim_{x \to a} c = c$$

- **One-Sided Limits**

  $$\lim_{x \to a^-} f(x) = L \qquad \lim_{x \to a^+} f(x) = L$$

- **Infinite Limits from the Left**

  $$\lim_{x \to a^-} f(x) = +\infty \qquad \lim_{x \to a^-} f(x) = -\infty$$

- **Infinite Limits from the Right**

  $$\lim_{x \to a^+} f(x) = +\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

- **Two-Sided Infinite Limits**

  $$\lim_{x \to a} f(x) = +\infty \text{: } \lim_{x \to a^-} f(x) = +\infty \text{ and } \lim_{x \to a^+} f(x) = +\infty$$

$\lim_{x \to a} f(x) = -\infty$: $\lim_{x \to a^-} f(x) = -\infty$ and $\lim_{x \to a^+} f(x) = -\infty$

Glossary

infinite limit

A function has an infinite limit at a point a if it either increases or decreases without bound as it approaches a

intuitive definition of the limit

If all values of the function $f(x)$ approach the real number L as the values of $x(\neq a)$ approach a, $f(x)$ approaches L

one-sided limit

A one-sided limit of a function is a limit taken from either the left or the right

vertical asymptote

A function has a vertical asymptote at $x = a$ if the limit as x approaches a from the right or left is infinite

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2.3: Calculating Limits Using the Limit Laws

Learning Objectives

- Recognize the basic limit laws.
- Use the limit laws to evaluate the limit of a function.
- Evaluate the limit of a function by factoring.
- Use the limit laws to evaluate the limit of a polynomial or rational function.
- Evaluate the limit of a function by factoring or by using conjugates.
- Evaluate the limit of a function by using the squeeze theorem.

In the previous section, we evaluated limits by looking at graphs or by constructing a table of values. In this section, we establish laws for calculating limits and learn how to apply these laws. In the Student Project at the end of this section, you have the opportunity to apply these limit laws to derive the formula for the area of a circle by adapting a method devised by the Greek mathematician Archimedes. We begin by restating two useful limit results from the previous section. These two results, together with the limit laws, serve as a foundation for calculating many limits.

Evaluating Limits with the Limit Laws

The first two limit laws were stated previously and we repeat them here. These basic results, together with the other limit laws, allow us to evaluate limits of many algebraic functions.

Theorem 2.3.1: Basic Limit Results

For any real number a and any constant c,

I. $\lim\limits_{x \to a} x = a$
II. $\lim\limits_{x \to a} c = c$

Example 2.3.1: Evaluating a Basic Limit

Evaluate each of the following limits using "Basic Limit Results."

a. $\lim\limits_{x \to 2} x$
b. $\lim\limits_{x \to 2} 5$

Solution

a. The limit of x as x approaches a is a: $\lim\limits_{x \to 2} x = 2$.
b. The limit of a constant is that constant: $\lim\limits_{x \to 2} 5 = 5$.

We now take a look at the **limit laws**, the individual properties of limits. The proofs that these laws hold are omitted here.

Theorem 2.3.2: Limit Laws

Let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing a. Assume that L and M are real numbers such that $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$. Let c be a constant. Then, each of the following statements holds:

- **Sum law for limits**:

$$\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$$

- **Difference law for limits**:

$$\lim_{x \to a}(f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M$$

- **Constant multiple law for limits**:

$$\lim_{x\to a} cf(x) = c\cdot \lim_{x\to a} f(x) = cL$$

- **Product law for limits**:

$$\lim_{x\to a}(f(x)\cdot g(x)) = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x) = L\cdot M$$

- **Quotient law for limits**:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \frac{L}{M}$$

for $M\neq 0$.

- **Power law for limits**:

$$\lim_{x\to a}\big(f(x)\big)^n = \big(\lim_{x\to a} f(x)\big)^n = L^n$$

for every positive integer n.

- **Root law for limits**:

$$\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}$$

for all L if n is odd and for $L\geq 0$ if n is even.

We now practice applying these limit laws to evaluate a limit.

Example 2.3.2A: Evaluating a Limit Using Limit Laws

Use the limit laws to evaluate

$$\lim_{x\to -3}(4x+2).$$

Solution

Let's apply the limit laws one step at a time to be sure we understand how they work. We need to keep in mind the requirement that, at each application of a limit law, the new limits must exist for the limit law to be applied.

$$\begin{aligned}\lim_{x\to -3}(4x+2) &= \lim_{x\to -3} 4x + \lim_{x\to -3} 2 && \text{Apply the sum law.}\\ &= 4\cdot\lim_{x\to -3} x + \lim_{x\to -3} 2 && \text{Apply the constant multiple law.}\\ &= 4\cdot(-3)+2 = -10. && \text{Apply the basic limit results and simplify.}\end{aligned}$$

Example 2.3.2B: Using Limit Laws Repeatedly

Use the limit laws to evaluate

$$\lim_{x\to 2}\frac{2x^2-3x+1}{x^3+4}.$$

Solution

To find this limit, we need to apply the limit laws several times. Again, we need to keep in mind that as we rewrite the limit in terms of other limits, each new limit must exist for the limit law to be applied.

$$\lim_{x\to 2}\frac{2x^2-3x+1}{x^3+4}=\frac{\lim_{x\to 2}(2x^2-3x+1)}{\lim_{x\to 2}(x^3+4)}$$ Apply the quotient law, make sure that $(2)^3+4\neq 0$.

$$=\frac{2\cdot\lim_{x\to 2}x^2-3\cdot\lim_{x\to 2}x+\lim_{x\to 2}1}{\lim_{x\to 2}x^3+\lim_{x\to 2}4}$$ Apply the sum law and constant multiple law.

$$=\frac{2\cdot\left(\lim_{x\to 2}x\right)^2-3\cdot\lim_{x\to 2}x+\lim_{x\to 2}1}{\left(\lim_{x\to 2}x\right)^3+\lim_{x\to 2}4}$$ Apply the power law.

$$=\frac{2(4)-3(2)+1}{(2)^3+4}=\frac{1}{4}.$$ Apply the basic limit laws and simplify.

? Exercise 2.3.2

Use the limit laws to evaluate $\lim_{x\to 6}(2x-1)\sqrt{x+4}$. In each step, indicate the limit law applied.

Hint

Begin by applying the product law.

Answer

$11\sqrt{10}$

Limits of Polynomial and Rational Functions

By now you have probably noticed that, in each of the previous examples, it has been the case that $\lim_{x\to a}f(x)=f(a)$. This is not always true, but it does hold for all polynomials for any choice of a and for all rational functions at all values of a for which the rational function is defined.

Theorem 2.3.3: Limits of Polynomial and Rational Functions

Let $p(x)$ and $q(x)$ be polynomial functions. Let a be a real number. Then,

$$\lim_{x\to a}p(x)=p(a)$$

$$\lim_{x\to a}\frac{p(x)}{q(x)}=\frac{p(a)}{q(a)}$$

when $q(a)\neq 0$.

To see that this theorem holds, consider the polynomial

$$p(x)=c_nx^n+c_{n-1}x^{n-1}+\cdots+c_1x+c_0.$$

By applying the sum, constant multiple, and power laws, we end up with

$$\begin{aligned}\lim_{x\to a}p(x)&=\lim_{x\to a}(c_nx^n+c_{n-1}x^{n-1}+\cdots+c_1x+c_0)\\&=c_n\left(\lim_{x\to a}x\right)^n+c_{n-1}\left(\lim_{x\to a}x\right)^{n-1}+\cdots+c_1\left(\lim_{x\to a}x\right)+\lim_{x\to a}c_0\\&=c_na^n+c_{n-1}a^{n-1}+\cdots+c_1a+c_0\\&=p(a)\end{aligned}$$

It now follows from the quotient law that if $p(x)$ and $q(x)$ are polynomials for which $q(a)\neq 0$,

then

$$\lim_{x\to a}\frac{p(x)}{q(x)}=\frac{p(a)}{q(a)}.$$

Example 2.3.3: Evaluating a Limit of a Rational Function

Evaluate the $\lim_{x\to 3}\dfrac{2x^2-3x+1}{5x+4}$.

Solution

Since 3 is in the domain of the rational function $f(x)=\dfrac{2x^2-3x+1}{5x+4}$, we can calculate the limit by substituting 3 for x into the function. Thus,

$$\lim_{x\to 3}\frac{2x^2-3x+1}{5x+4}=\frac{10}{19}.$$

Exercise 2.3.3

Evaluate $\lim_{x\to -2}(3x^3-2x+7)$.

Hint

Use LIMITS OF POLYNOMIAL AND RATIONAL FUNCTIONS as reference

Answer

−13

Additional Limit Evaluation Techniques

As we have seen, we may evaluate easily the limits of polynomials and limits of some (but not all) rational functions by direct substitution. However, as we saw in the introductory section on limits, it is certainly possible for $\lim_{x\to a}f(x)$ to exist when $f(a)$ is undefined. The following observation allows us to evaluate many limits of this type:

If for all $x\neq a,\ f(x)=g(x)$ over some open interval containing a, then

$$\lim_{x\to a}f(x)=\lim_{x\to a}g(x).$$

To understand this idea better, consider the limit $\lim_{x\to 1}\dfrac{x^2-1}{x-1}$.

The function

$$f(x)=\frac{x^2-1}{x-1}=\frac{(x-1)(x+1)}{x-1}$$

and the function $g(x)=x+1$ are identical for all values of $x\neq 1$. The graphs of these two functions are shown in Figure 2.3.1.

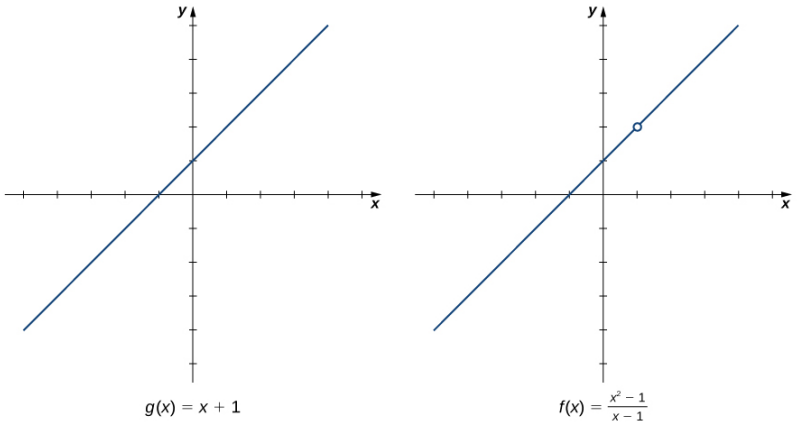


Figure 2.3.1: The graphs of $f(x)$ and $g(x)$ are identical for all $x \neq 1$. Their limits at 1 are equal.

We see that

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}\frac{(x-1)(x+1)}{x-1}=\lim_{x\to 1}(x+1)=2.$$

The limit has the form $\lim_{x\to a} f(x)/g(x)$, where $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$. (In this case, we say that $f(x)/g(x)$ has the indeterminate form $0/0$.) The following Problem-Solving Strategy provides a general outline for evaluating limits of this type.

Problem-Solving Strategy: Calculating a Limit When $f(x)/g(x)$ has the Indeterminate Form $0/0$

1. First, we need to make sure that our function has the appropriate form and cannot be evaluated immediately using the limit laws.
2. We then need to find a function that is equal to $h(x)=f(x)/g(x)$ for all $x \neq a$ over some interval containing a. To do this, we may need to try one or more of the following steps:
   a. If $f(x)$ and $g(x)$ are polynomials, we should factor each function and cancel out any common factors.
   b. If the numerator or denominator contains a difference involving a square root, we should try multiplying the numerator and denominator by the conjugate of the expression involving the square root.
   c. If $f(x)/g(x)$ is a complex fraction, we begin by simplifying it.
3. Last, we apply the limit laws.

The next examples demonstrate the use of this Problem-Solving Strategy. Example 2.3.4 illustrates the factor-and-cancel technique; Example 2.3.5 shows multiplying by a conjugate. In Example 2.3.6, we look at simplifying a complex fraction.

Example 2.3.4: Evaluating a Limit by Factoring and Canceling

Evaluate $\lim_{x\to 3}\dfrac{x^2-3x}{2x^2-5x-3}$.

Solution

Step 1. The function $f(x)=\dfrac{x^2-3x}{2x^2-5x-3}$ is undefined for $x=3$. In fact, if we substitute 3 into the function we get $0/0$, which is undefined. Factoring and canceling is a good strategy:

$$\lim_{x\to 3}\frac{x^2-3x}{2x^2-5x-3}=\lim_{x\to 3}\frac{x(x-3)}{(x-3)(2x+1)}$$

Step 2. For all $x \neq 3$, $\dfrac{x^2-3x}{2x^2-5x-3}=\dfrac{x}{2x+1}$. Therefore,

$$\lim_{x\to 3}\frac{x(x-3)}{(x-3)(2x+1)}=\lim_{x\to 3}\frac{x}{2x+1}.$$

Step 3. Evaluate using the limit laws:

$$\lim_{x\to 3}\frac{x}{2x+1}=\frac{3}{7}.$$

Exercise 2.3.4

Evaluate $\lim_{x\to -3}\dfrac{x^2+4x+3}{x^2-9}$.

Hint

Follow the steps in the Problem-Solving Strategy

Answer

$\dfrac{1}{3}$

Example 2.3.5: Evaluating a Limit by Multiplying by a Conjugate

Evaluate $\lim_{x\to -1}\dfrac{\sqrt{x+2}-1}{x+1}$.

Solution

Step 1. $\dfrac{\sqrt{x+2}-1}{x+1}$ has the form $0/0$ at -1. Let's begin by multiplying by $\sqrt{x+2}+1$, the conjugate of $\sqrt{x+2}-1$, on the numerator and denominator:

$$\lim_{x\to -1}\frac{\sqrt{x+2}-1}{x+1}=\lim_{x\to -1}\frac{\sqrt{x+2}-1}{x+1}\cdot\frac{\sqrt{x+2}+1}{\sqrt{x+2}+1}.$$

Step 2. We then multiply out the numerator. We don't multiply out the denominator because we are hoping that the $(x+1)$ in the denominator cancels out in the end:

$$=\lim_{x\to -1}\frac{x+1}{(x+1)(\sqrt{x+2}+1)}.$$

Step 3. Then we cancel:

$$=\lim_{x\to -1}\frac{1}{\sqrt{x+2}+1}.$$

Step 4. Last, we apply the limit laws:

$$\lim_{x\to -1}\frac{1}{\sqrt{x+2}+1}=\frac{1}{2}.$$

Exercise 2.3.5

Evaluate $\lim_{x\to 5}\dfrac{\sqrt{x-1}-2}{x-5}$.

Hint

Follow the steps in the Problem-Solving Strategy

Answer

$\dfrac{1}{4}$

Example 2.3.6: Evaluating a Limit by Simplifying a Complex Fraction

Evaluate $\lim\limits_{x\to 1}\dfrac{\dfrac{1}{x+1}-\dfrac{1}{2}}{x-1}$.

Solution

Step 1. $\dfrac{\dfrac{1}{x+1}-\dfrac{1}{2}}{x-1}$ has the form $0/0$ at 1. We simplify the algebraic fraction by multiplying by $2(x+1)/2(x+1)$:

$$\lim_{x\to 1}\frac{\dfrac{1}{x+1}-\dfrac{1}{2}}{x-1}=\lim_{x\to 1}\frac{\dfrac{1}{x+1}-\dfrac{1}{2}}{x-1}\cdot\frac{2(x+1)}{2(x+1)}.$$

Step 2. Next, we multiply through the numerators. Do not multiply the denominators because we want to be able to cancel the factor $(x-1)$:

$$=\lim_{x\to 1}\frac{2-(x+1)}{2(x-1)(x+1)}.$$

Step 3. Then, we simplify the numerator:

$$=\lim_{x\to 1}\frac{-x+1}{2(x-1)(x+1)}.$$

Step 4. Now we factor out -1 from the numerator:

$$=\lim_{x\to 1}\frac{-(x-1)}{2(x-1)(x+1)}.$$

Step 5. Then, we cancel the common factors of $(x-1)$:

$$=\lim_{x\to 1}\frac{-1}{2(x+1)}.$$

Step 6. Last, we evaluate using the limit laws:

$$\lim_{x\to 1}\frac{-1}{2(x+1)}=-\frac{1}{4}.$$

Exercise 2.3.6

Evaluate $\lim\limits_{x\to -3}\dfrac{\dfrac{1}{x+2}+1}{x+3}$.

Hint

Follow the steps in the Problem-Solving Strategy

Answer

-1

Example 2.3.7 does not fall neatly into any of the patterns established in the previous examples. However, with a little creativity, we can still use these same techniques.

Example 2.3.7: Evaluating a Limit When the Limit Laws Do Not Apply

Evaluate $\lim\limits_{x\to 0}\left(\dfrac{1}{x}+\dfrac{5}{x(x-5)}\right)$.

Solution:

Both $1/x$ and $5/x(x-5)$ fail to have a limit at zero. Since neither of the two functions has a limit at zero, we cannot apply the sum law for limits; we must use a different strategy. In this case, we find the limit by performing addition and then applying one of our previous strategies. Observe that

$$\frac{1}{x}+\frac{5}{x(x-5)}=\frac{x-5+5}{x(x-5)}=\frac{x}{x(x-5)}.$$

Thus,

$$\lim_{x\to 0}\left(\frac{1}{x}+\frac{5}{x(x-5)}\right)=\lim_{x\to 0}\frac{x}{x(x-5)}=\lim_{x\to 0}\frac{1}{x-5}=-\frac{1}{5}.$$

Exercise 2.3.7

Evaluate $\lim\limits_{x\to 3}\left(\dfrac{1}{x-3}-\dfrac{4}{x^2-2x-3}\right)$.

Hint

Use the same technique as Example 2.3.7 Don't forget to factor x^2-2x-3 before getting a common denominator.

Answer

$\dfrac{1}{4}$

Let's now revisit one-sided limits. Simple modifications in the limit laws allow us to apply them to one-sided limits. For example, to apply the limit laws to a limit of the form $\lim\limits_{x\to a^-}h(x)$, we require the function $h(x)$ to be defined over an open interval of the form (b,a); for a limit of the form $\lim\limits_{x\to a^+}h(x)$, we require the function $h(x)$ to be defined over an open interval of the form (a,c). Example 2.3.8A illustrates this point.

Example 2.3.8A: Evaluating a One-Sided Limit Using the Limit Laws

Evaluate each of the following limits, if possible.

a. $\lim\limits_{x\to 3^-}\sqrt{x-3}$

b. $\lim\limits_{x\to 3^+}\sqrt{x-3}$

Solution

Figure 2.3.2 illustrates the function $f(x)=\sqrt{x-3}$ and aids in our understanding of these limits.

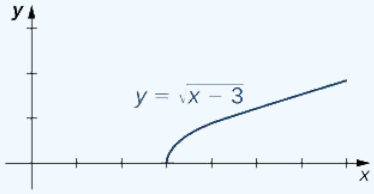


Figure 2.3.2: The graph shows the function $f(x)=\sqrt{x-3}$.

a. The function $f(x)=\sqrt{x-3}$ is defined over the interval $[3,+\infty)$. Since this function is not defined to the left of 3, we cannot apply the limit laws to compute $\lim\limits_{x\to 3^-}\sqrt{x-3}$. In fact, since $f(x)=\sqrt{x-3}$ is undefined to the left of 3, $\lim\limits_{x\to 3^-}\sqrt{x-3}$

does not exist.

b. Since $f(x)=\sqrt{x-3}$ is defined to the right of 3, the limit laws do apply to $\lim\limits_{x\to 3^+}\sqrt{x-3}$. By applying these limit laws we obtain $\lim\limits_{x\to 3^+}\sqrt{x-3}=0$.

In Example $2.3.8B$ we look at one-sided limits of a piecewise-defined function and use these limits to draw a conclusion about a two-sided limit of the same function.

Example $2.3.8B$: Evaluating a Two-Sided Limit Using the Limit Laws

For $f(x)=\begin{cases}4x-3, & \text{if } x<2\\ (x-3)^2, & \text{if } x\geq 2\end{cases}$, evaluate each of the following limits:

a. $\lim\limits_{x\to 2^-}f(x)$
b. $\lim\limits_{x\to 2^+}f(x)$
c. $\lim\limits_{x\to 2}f(x)$

Solution

Figure $2.3.3$ illustrates the function $f(x)$ and aids in our understanding of these limits.

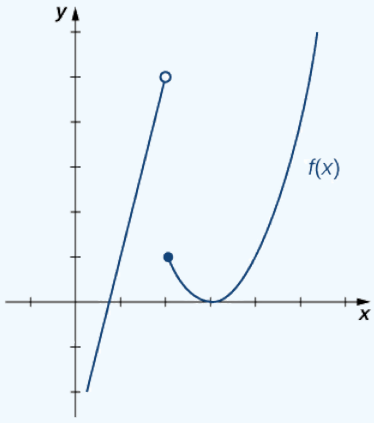


Figure 2.3.3: This graph shows a function $f(x)$.

a. Since $f(x)=4x-3$ for all x in $(-\infty,2)$, replace $f(x)$ in the limit with $4x-3$ and apply the limit laws:

$$\lim_{x\to 2^-}f(x)=\lim_{x\to 2^-}(4x-3)=5$$

b. Since $f(x)=(x-3)^2$ for all x in $(2,+\infty)$, replace $f(x)$ in the limit with $(x-3)^2$ and apply the limit laws:

$$\lim_{x\to 2^+}f(x)=\lim_{x\to 2^-}(x-3)^2=1.$$

c. Since $\lim\limits_{x\to 2^-}f(x)=5$ and $\lim\limits_{x\to 2^+}f(x)=1$, we conclude that $\lim\limits_{x\to 2}f(x)$ does not exist.

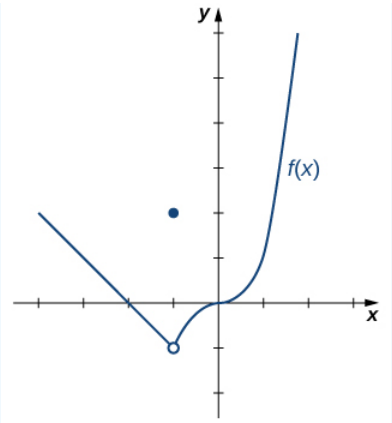
Exercise $2.3.8$

Graph $f(x)=\begin{cases}-x-2, & \text{if } x<-1\\ 2, & \text{if } x=-1\\ x^3, & \text{if } x>-1\end{cases}$ and evaluate $\lim\limits_{x\to -1^-}f(x)$.

Hint

Use the method in Example $2.3.8B$ to evaluate the limit.

Answer



$$\lim_{x \to -1^-} f(x) = -1$$

We now turn our attention to evaluating a limit of the form $\lim_{x \to a} \dfrac{f(x)}{g(x)}$, where $\lim_{x \to a} f(x) = K$, where $K \neq 0$ and $\lim_{x \to a} g(x) = 0$. That is, $f(x)/g(x)$ has the form $K/0, K \neq 0$ at a.

Example 2.3.9: Evaluating a Limit of the Form $K/0$, $K \neq 0$ Using the Limit Laws

Evaluate $\lim_{x \to 2^-} \dfrac{x-3}{x^2-2x}$.

Solution

Step 1. After substituting in $x = 2$, we see that this limit has the form $-1/0$. That is, as x approaches 2 from the left, the numerator approaches -1; and the denominator approaches 0. Consequently, the magnitude of $\dfrac{x-3}{x(x-2)}$ becomes infinite. To get a better idea of what the limit is, we need to factor the denominator:

$$\lim_{x \to 2^-} \frac{x-3}{x^2-2x} = \lim_{x \to 2^-} \frac{x-3}{x(x-2)}$$

Step 2. Since $x-2$ is the only part of the denominator that is zero when 2 is substituted, we then separate $1/(x-2)$ from the rest of the function:

$$= \lim_{x \to 2^-} \frac{x-3}{x} \cdot \frac{1}{x-2}$$

Step 3. Using the Limit Laws, we can write:

$$= \left(\lim_{x \to 2^-} \frac{x-3}{x} \right) \cdot \left(\lim_{x \to 2^-} \frac{1}{x-2} \right).$$

Step 4. $\lim_{x \to 2^-} \dfrac{x-3}{x} = -\dfrac{1}{2}$ and $\lim_{x \to 2^-} \dfrac{1}{x-2} = -\infty$. Therefore, the product of $(x-3)/x$ and $1/(x-2)$ has a limit of $+\infty$:

$$\lim_{x \to 2^-} \frac{x-3}{x^2-2x} = +\infty.$$

Exercise 2.3.9

Evaluate $\lim_{x \to 1} \dfrac{x+2}{(x-1)^2}$.

Solution

Use the methods from Example 2.3.9

Answer

$+\infty$

The Squeeze Theorem

The techniques we have developed thus far work very well for algebraic functions, but we are still unable to evaluate limits of very basic trigonometric functions. The next theorem, called the **squeeze theorem,** proves very useful for establishing basic trigonometric limits. This theorem allows us to calculate limits by "squeezing" a function, with a limit at a point a that is unknown, between two functions having a common known limit at a. Figure $2.3.4$ illustrates this idea.

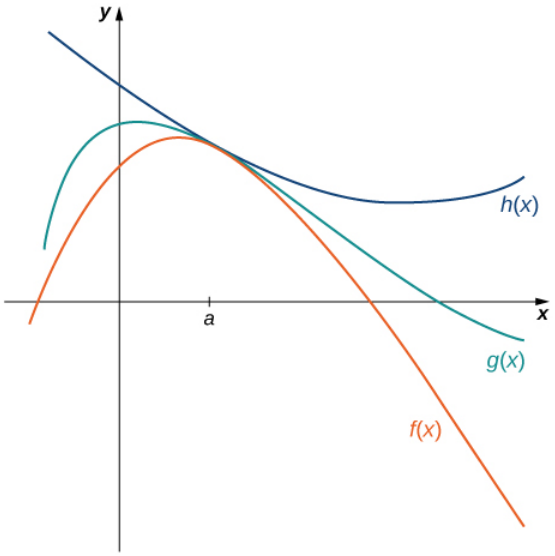


Figure $2.3.4$: The Squeeze Theorem applies when $f(x) \le g(x) \le h(x)$ and $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} h(x)$.

Theorem 2.3.4: The Squeeze Theorem

Let $f(x)$, $g(x)$, and $h(x)$ be defined for all $x \neq a$ over an open interval containing a. If

$$f(x) \le g(x) \le h(x)$$

for all $x \neq a$ in an open interval containing a and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$$

where L is a real number, then $\lim\limits_{x \to a} g(x) = L$.

Example 2.3.10: Applying the Squeeze Theorem

Apply the squeeze theorem to evaluate $\lim\limits_{x \to 0} x \cos x$.

Solution

Because $-1 \le \cos x \le 1$ for all x, we have $-x \le x \cos x \le x$ for $x \ge 0$ and $-x \ge x \cos x \ge x$ for $x \le 0$ (if x is negative the direction of the inequalities changes when we multiply). Since $\lim\limits_{x \to 0}(-x) = 0 = \lim\limits_{x \to 0} x$, from the squeeze theorem, we obtain $\lim\limits_{x \to 0} x \cos x = 0$. The graphs of $f(x) = -x$, $g(x) = x \cos x$, and $h(x) = x$ are shown in Figure $2.3.5$.

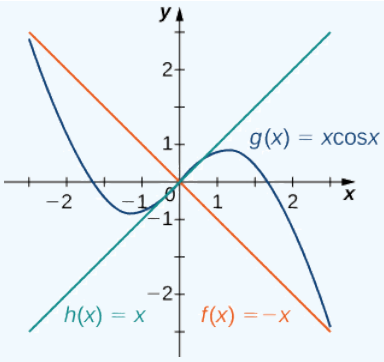


Figure 2.3.5: The graphs of $f(x)$, $g(x)$, and $h(x)$ are shown around the point $x = 0$.

? Exercise 2.3.10

Use the squeeze theorem to evaluate $\lim_{x\to 0} x^2 \sin\dfrac{1}{x}$.

Hint

Use the fact that $-x^2 \le x^2 \sin(1/x) \le x^2$ to help you find two functions such that $x^2 \sin(1/x)$ is squeezed between them.

Answer

0

We now use the squeeze theorem to tackle several very important limits. Although this discussion is somewhat lengthy, these limits prove invaluable for the development of the material in both the next section and the next chapter. The first of these limits is $\lim_{\theta\to 0} \sin\theta$. Consider the unit circle shown in Figure 2.3.6. In the figure, we see that $\sin\theta$ is the y-coordinate on the unit circle and it corresponds to the line segment shown in blue. The radian measure of angle θ is the length of the arc it subtends on the unit circle. Therefore, we see that for $0 < \theta < \dfrac{\pi}{2}$, we have $0 < \sin\theta < \theta$.

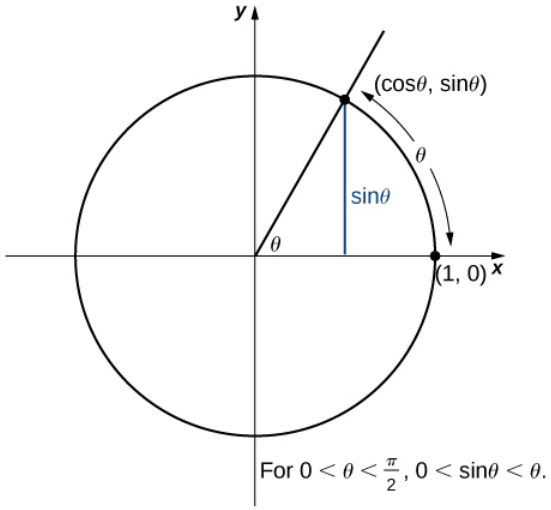


Figure 2.3.6: The sine function is shown as a line on the unit circle.

Because $\lim_{\theta\to 0^+} 0 = 0$ and $\lim_{\theta\to 0^+} \theta = 0$, by using the squeeze theorem we conclude that

$$\lim_{\theta\to 0^+} \sin\theta = 0.$$

To see that $\lim_{\theta\to 0^-} \sin\theta = 0$ as well, observe that for $-\dfrac{\pi}{2} < \theta < 0, 0 < -\theta < \dfrac{\pi}{2}$ and hence, $0 < \sin(-\theta) < -\theta$. Consequently, $0 < -\sin\theta < -\theta$. It follows that $0 > \sin\theta > \theta$. An application of the squeeze theorem produces the desired limit. Thus, since

$\lim_{\theta \to 0^+} \sin\theta = 0$ and $\lim_{\theta \to 0^-} \sin\theta = 0$,

$$\lim_{\theta \to 0} \sin\theta = 0$$

Next, using the identity $\cos\theta = \sqrt{1 - \sin^2\theta}$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, we see that

$$\lim_{\theta \to 0} \cos\theta = \lim_{\theta \to 0} \sqrt{1 - \sin^2\theta} = 1.$$

We now take a look at a limit that plays an important role in later chapters—namely, $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$. To evaluate this limit, we use the unit circle in Figure 2.3.7. Notice that this figure adds one additional triangle to Figure 2.3.6. We see that the length of the side opposite angle θ in this new triangle is $\tan\theta$. Thus, we see that for $0 < \theta < \frac{\pi}{2}$, we have $\sin\theta < \theta < \tan\theta$.

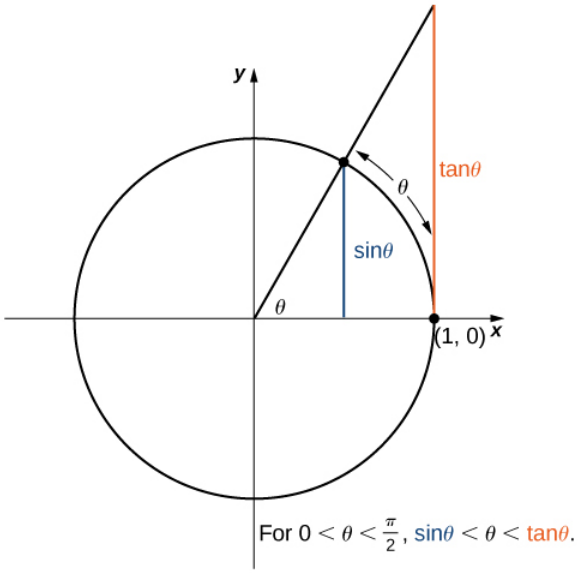


Figure 2.3.7: The sine and tangent functions are shown as lines on the unit circle.

By dividing by $\sin\theta$ in all parts of the inequality, we obtain

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Equivalently, we have

$$1 > \frac{\sin\theta}{\theta} > \cos\theta.$$

Since $\lim_{\theta \to 0^+} 1 = 1 = \lim_{\theta \to 0^+} \cos\theta$, we conclude that $\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1$, by the squeeze theorem. By applying a manipulation similar to that used in demonstrating that $\lim_{\theta \to 0^-} \sin\theta = 0$, we can show that $\lim_{\theta \to 0^-} \frac{\sin\theta}{\theta} = 1$. Thus,

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$$

In Example 2.3.11, we use this limit to establish $\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = 0$. This limit also proves useful in later chapters.

Example 2.3.11: Evaluating an Important Trigonometric Limit

Evaluate $\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta}$.

Solution

In the first step, we multiply by the conjugate so that we can use a trigonometric identity to convert the cosine in the numerator to a sine:

$$\begin{aligned}\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} &= \lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta}\\ &= \lim_{\theta\to 0}\frac{1-\cos^2\theta}{\theta(1+\cos\theta)}\\ &= \lim_{\theta\to 0}\frac{\sin^2\theta}{\theta(1+\cos\theta)}\\ &= \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\frac{\sin\theta}{1+\cos\theta}\\ &= \left(\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\right)\cdot\left(\lim_{\theta\to 0}\frac{\sin\theta}{1+\cos\theta}\right)\\ &= 1\cdot\frac{0}{2}=0.\end{aligned}$$

Therefore,

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}=0.$$

Exercise 2.3.11

Evaluate $\lim_{\theta\to 0}\dfrac{1-\cos\theta}{\sin\theta}$.

Hint

Multiply numerator and denominator by $1+\cos\theta$.

Answer

0

Deriving the Formula for the Area of a Circle

Some of the geometric formulas we take for granted today were first derived by methods that anticipate some of the methods of calculus. The Greek mathematician *Archimedes* (ca. 287−212; BCE) was particularly inventive, using polygons inscribed within circles to approximate the area of the circle as the number of sides of the polygon increased. He never came up with the idea of a limit, but we can use this idea to see what his geometric constructions could have predicted about the limit.

We can estimate the area of a circle by computing the area of an inscribed regular polygon. Think of the regular polygon as being made up of n triangles. By taking the limit as the vertex angle of these triangles goes to zero, you can obtain the area of the circle. To see this, carry out the following steps:

1. Express the height h and the base b of the isosceles triangle in Figure $2.3.8$ in terms of θ and r.

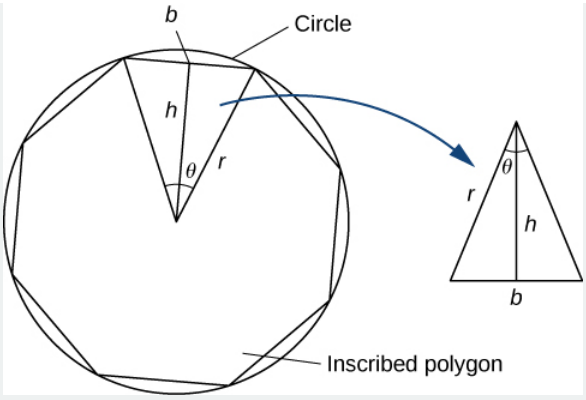


Figure 2.3.8

2. Using the expressions that you obtained in step 1, express the area of the isosceles triangle in terms of θ and r.

(Substitute $\frac{1}{2}\sin\theta$ for $\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$ in your expression.)

3. If an n-sided regular polygon is inscribed in a circle of radius r, find a relationship between θ and n. Solve this for n. Keep in mind there are 2π radians in a circle. (Use radians, not degrees.)

4. Find an expression for the area of the n-sided polygon in terms of r and θ.

5. To find a formula for the area of the circle, find the limit of the expression in step 4 as θ goes to zero. (Hint: $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$).

The technique of estimating areas of regions by using polygons is revisited in Introduction to Integration.

Key Concepts

- The limit laws allow us to evaluate limits of functions without having to go through step-by-step processes each time.
- For polynomials and rational functions,

$$\lim_{x\to a} f(x) = f(a).$$

- You can evaluate the limit of a function by factoring and canceling, by multiplying by a conjugate, or by simplifying a complex fraction.
- The squeeze theorem allows you to find the limit of a function if the function is always greater than one function and less than another function with limits that are known.

Key Equations

- **Basic Limit Results**

$$\lim_{x\to a} x = a \qquad \lim_{x\to a} c = c$$

- **Important Limits**

$$\lim_{\theta\to 0}\sin\theta = 0$$

$$\lim_{\theta\to 0}\cos\theta = 1$$

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = 0$$

Glossary

constant multiple law for limits

the limit law

$$\lim_{x \to a} cf(x) = c \cdot \lim_{x \to a} f(x) = cL$$

difference law for limits

the limit law

$$\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M$$

limit laws

the individual properties of limits; for each of the individual laws, let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing a; assume that L and M are real numbers so that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$; let c be a constant

power law for limits

the limit law

$$\lim_{x \to a} (f(x))^n = (\lim_{x \to a} f(x))^n = L^n$$

for every positive integer n

product law for limits

the limit law

$$\lim_{x \to a} (f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L \cdot M$$

quotient law for limits

the limit law $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \dfrac{L}{M}$ for M≠0

root law for limits

the limit law $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L}$ for all L if n is odd and for $L \geq 0$ if n is even

squeeze theorem

states that if $f(x) \leq g(x) \leq h(x)$ for all $x \neq a$ over an open interval containing a and $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$ where L is a real number, then $\lim_{x \to a} g(x) = L$

sum law for limits

The limit law $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$

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2.5: Continuity

Learning Objectives

- Explain the three conditions for continuity at a point.
- Describe three kinds of discontinuities.
- Define continuity on an interval.
- State the theorem for limits of composite functions.
- Provide an example of the intermediate value theorem.

Many functions have the property that their graphs can be traced with a pencil without lifting the pencil from the page. Such functions are called *continuous*. Other functions have points at which a break in the graph occurs, but satisfy this property over intervals contained in their domains. They are continuous on these intervals and are said to have a *discontinuity* at a point where a break occurs.

We begin our investigation of continuity by exploring what it means for a function to have *continuity at a point*. Intuitively, a function is continuous at a particular point if there is no break in its graph at that point.

Continuity at a Point

Before we look at a formal definition of what it means for a function to be continuous at a point, let's consider various functions that fail to meet our intuitive notion of what it means to be continuous at a point. We then create a list of conditions that prevent such failures.

Our first function of interest is shown in Figure 2.5.1. We see that the graph of $f(x)$ has a hole at a. In fact, $f(a)$ is undefined. At the very least, for $f(x)$ to be continuous at a, we need the following condition:

i. $f(a)$ is defined

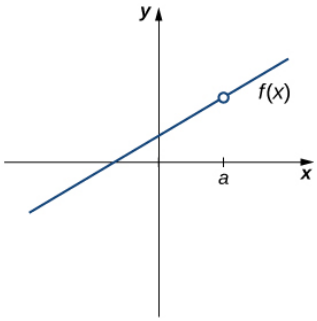


Figure 2.5.1: The function $f(x)$ is not continuous at a because $f(a)$ is undefined.

However, as we see in Figure 2.5.2, this condition alone is insufficient to guarantee continuity at the point a. Although $f(a)$ is defined, the function has a gap at a. In this example, the gap exists because $\lim_{x \to a} f(x)$ does not exist. We must add another condition for continuity at a—namely,

ii. $\lim_{x \to a} f(x)$ exists

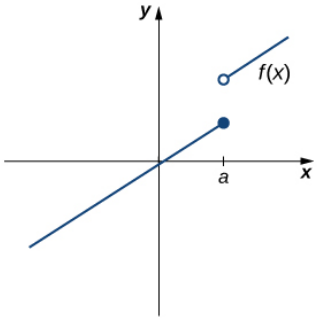


Figure 2.5.2: The function $f(x)$ is not continuous at a because $\lim_{x \to a} f(x)$ does not exist.

However, as we see in Figure 2.5.3, these two conditions by themselves do not guarantee continuity at a point. The function in this figure satisfies both of our first two conditions, but is still not continuous at a. We must add a third condition to our list:

iii. $\lim_{x \to a} f(x) = f(a)$

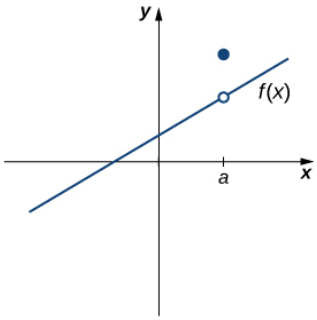


Figure 2.5.3: The function $f(x)$ is not continuous at a because $\lim_{x \to a} f(x) \neq f(a)$.

Now we put our list of conditions together and form a definition of continuity at a point.

Definition: Continuous at a Point

A function $f(x)$ is **continuous at a point a** if and only if the following three conditions are satisfied:

i. $f(a)$ is defined
ii. $\lim_{x \to a} f(x)$ exists
iii. $\lim_{x \to a} f(x) = f(a)$

A function is **discontinuous at a point a** if it fails to be continuous at a.

The following procedure can be used to analyze the continuity of a function at a point using this definition.

Problem-Solving Strategy: Determining Continuity at a Point

1. Check to see if $f(a)$ is defined. If $f(a)$ is undefined, we need go no further. The function is not continuous at a. If $f(a)$ is defined, continue to step 2.
2. Compute $\lim_{x \to a} f(x)$. In some cases, we may need to do this by first computing $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$. If $\lim_{x \to a} f(x)$ does not exist (that is, it is not a real number), then the function is not continuous at a and the problem is solved. If $\lim_{x \to a} f(x)$ exists, then continue to step 3.
3. Compare $f(a)$ and $\lim_{x \to a} f(x)$. If $\lim_{x \to a} f(x) \neq f(a)$, then the function is not continuous at a. If $\lim_{x \to a} f(x) = f(a)$, then the function is continuous at a.

The next three examples demonstrate how to apply this definition to determine whether a function is continuous at a given point. These examples illustrate situations in which each of the conditions for continuity in the definition succeed or fail.

Example 2.5.1A: Determining Continuity at a Point, Condition 1

Using the definition, determine whether the function $f(x) = \dfrac{x^2 - 4}{x - 2}$ is continuous at $x = 2$. Justify the conclusion.

Solution

Let's begin by trying to calculate $f(2)$. We can see that $f(2) = 0/0$, which is undefined. Therefore, $f(x) = \dfrac{x^2 - 4}{x - 2}$ is discontinuous at 2 because $f(2)$ is undefined. The graph of $f(x)$ is shown in Figure 2.5.4.

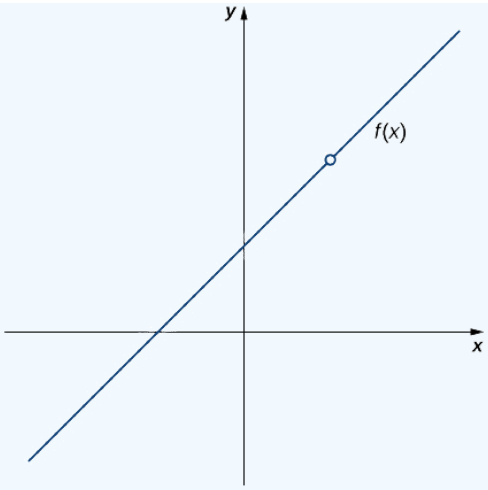


Figure 2.5.4: The function $f(x)$ is discontinuous at 2 because $f(2)$ is undefined.

Example 2.5.1B: Determining Continuity at a Point, Condition 2

Using the definition, determine whether the function $f(x) = \begin{cases} -x^2+4, & \text{if } x \leq 3 \\ 4x-8, & \text{if } x > 3 \end{cases}$ is continuous at $x = 3$. Justify the conclusion.

Solution

Let's begin by trying to calculate $f(3)$.

$$f(3) = -(3^2)+4 = -5\ .$$

Thus, $f(3)$ is defined. Next, we calculate $\lim\limits_{x\to 3} f(x)$. To do this, we must compute $\lim\limits_{x\to 3^-} f(x)$ and $\lim\limits_{x\to 3^+} f(x)$:

$$\lim_{x\to 3^-} f(x) = -(3^2)+4 = -5$$

and

$$\lim_{x\to 3^+} f(x) = 4(3)-8 = 4\ .$$

Therefore, $\lim\limits_{x\to 3} f(x)$ does not exist. Thus, $f(x)$ is not continuous at 3. The graph of $f(x)$ is shown in Figure 2.5.5.

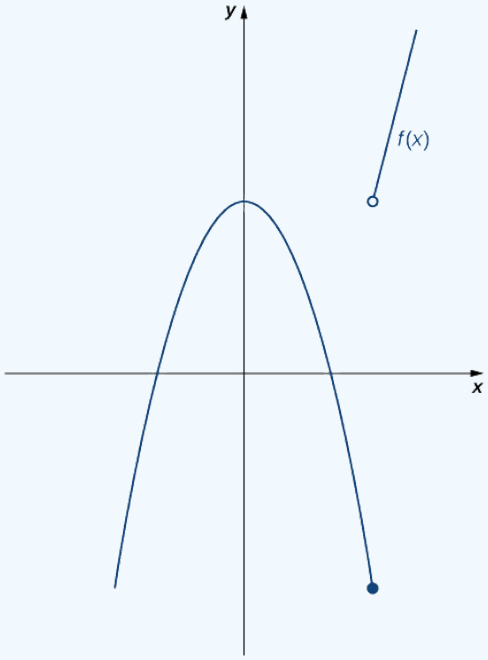


Figure 2.5.5: The function $f(x)$ is not continuous at 3 because $\lim\limits_{x\to 3} f(x)$ does not exist.

Example 2.5.1C: Determining Continuity at a Point, Condition 3

Using the definition, determine whether the function $f(x)=\begin{cases}\frac{\sin x}{x}, & \text{if } x\neq 0\\ 1, & \text{if } x=0\end{cases}$ is continuous at $x=0$.

Solution

First, observe that

$$f(0)=1$$

Next,

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}\frac{\sin x}{x}=1\,.$$

Last, compare $f(0)$ and $\lim_{x\to 0} f(x)$. We see that

$$f(0)=1=\lim_{x\to 0} f(x).$$

Since all three of the conditions in the definition of continuity are satisfied, $f(x)$ is continuous at $x=0$.

Exercise 2.5.1

Using the definition, determine whether the function $f(x)=\begin{cases}2x+1, & \text{if } x<1\\ 2, & \text{if } x=1\\ -x+4, & \text{if } x>1\end{cases}$ is continuous at $x=1$. If the function is not continuous at 1, indicate the condition for continuity at a point that fails to hold.

Hint

Check each condition of the definition.

Answer

f is not continuous at 1 because $f(1)=2\neq 3=\lim_{x\to 1} f(x)$.

By applying the definition of continuity and previously established theorems concerning the evaluation of limits, we can state the following theorem.

Theorem 2.5.1: Continuity of Polynomials and Rational Functions

Polynomials and rational functions are continuous at every point in their domains.

Proof

Previously, we showed that if $p(x)$ and $q(x)$ are polynomials, $\lim_{x\to a} p(x)=p(a)$ for every polynomial $p(x)$ and $\lim_{x\to a}\frac{p(x)}{q(x)}=\frac{p(a)}{q(a)}$ as long as $q(a)\neq 0$. Therefore, polynomials and rational functions are continuous on their domains.

□

We now apply Theorem 2.5.1 to determine the points at which a given rational function is continuous.

Example 2.5.2:Continuity of a Rational Function

For what values of x is $f(x)=\dfrac{x+1}{x-5}$ continuous?

Solution

The rational function $f(x)=\dfrac{x+1}{x-5}$ is continuous for every value of x except $x=5$.

Exercise 2.5.2

For what values of x is $f(x) = 3x^4 - 4x^2$ continuous?

Hint

Use the Continuity of Polynomials and Rational Functions stated above.

Answer

$f(x)$ is continuous at every real number.

Types of Discontinuities

As we have seen in Example $2.5.1A$ and Example $2.5.1B$, discontinuities take on several different appearances. We classify the types of discontinuities we have seen thus far as removable discontinuities, infinite discontinuities, or jump discontinuities. Intuitively, a **removable discontinuity** is a discontinuity for which there is a hole in the graph, a **jump discontinuity** is a noninfinite discontinuity for which the sections of the function do not meet up, and an **infinite discontinuity** is a discontinuity located at a vertical asymptote. Figure $2.5.6$ illustrates the differences in these types of discontinuities. Although these terms provide a handy way of describing three common types of discontinuities, keep in mind that not all discontinuities fit neatly into these categories.

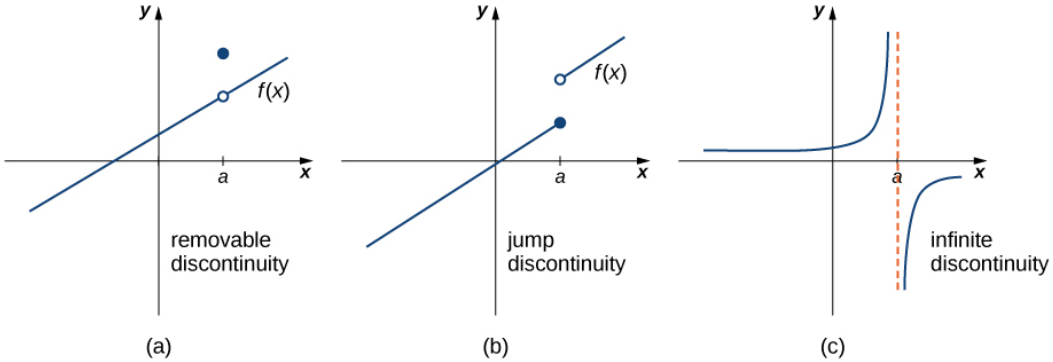


Figure 2.5.6: Discontinuities are classified as (a) removable, (b) jump, or (c) infinite.

These three discontinuities are formally defined as follows:

Definition

If $f(x)$ is discontinuous at a, then

1. f has a **removable discontinuity** at a if $\lim\limits_{x\to a} f(x)$ exists. (Note: When we state that $\lim\limits_{x\to a} f(x)$ exists, we mean that $\lim\limits_{x\to a} f(x) = L$, where L is a real number.)
2. f has a **jump discontinuity** at a if $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ both exist, but $\lim\limits_{x\to a^-} f(x) \neq \lim\limits_{x\to a^+} f(x)$. (Note: When we state that $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ both exist, we mean that both are real-valued and that neither take on the values $\pm\infty$.)
3. f has an **infinite discontinuity** at a if $\lim\limits_{x\to a^-} f(x) = \pm\infty$ or $\lim\limits_{x\to a^+} f(x) = \pm\infty$.

Example 2.5.3: Classifying a Discontinuity

In Example $2.5.1A$, we showed that $f(x) = \dfrac{x^2-4}{x-2}$ is discontinuous at $x = 2$. Classify this discontinuity as removable, jump, or infinite.

Solution

To classify the discontinuity at 2 we must evaluate $\lim\limits_{x\to 2} f(x)$:

$$\begin{aligned}\lim_{x\to 2} f(x) &= \lim_{x\to 2} \frac{x^2-4}{x-2} \\ &= \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} \\ &= \lim_{x\to 2} (x+2) \\ &= 4.\end{aligned}$$

Since f is discontinuous at 2 and $\lim_{x\to 2} f(x)$ exists, f has a removable discontinuity at $x=2$.

Example 2.5.4: Classifying a Discontinuity

In Example 2.5.1B, we showed that $f(x) = \begin{cases} -x^2+4, & \text{if } x \le 3 \\ 4x-8, & \text{if } x > 3 \end{cases}$ is discontinuous at $x=3$. Classify this discontinuity as removable, jump, or infinite.

Solution

Earlier, we showed that f is discontinuous at 3 because $\lim_{x\to 3} f(x)$ does not exist. However, since $\lim_{x\to 3^-} f(x) = -5$ and $\lim_{x\to 3^+} f(x) = 4$ both exist, we conclude that the function has a jump discontinuity at 3.

Example 2.5.5: Classifying a Discontinuity

Determine whether $f(x) = \dfrac{x+2}{x+1}$ is continuous at -1. If the function is discontinuous at -1, classify the discontinuity as removable, jump, or infinite.

Solution

The function value $f(-1)$ is undefined. Therefore, the function is not continuous at -1. To determine the type of discontinuity, we must determine the limit at -1. We see that $\lim_{x\to -1^-} \dfrac{x+2}{x+1} = -\infty$ and $\lim_{x\to -1^+} \dfrac{x+2}{x+1} = +\infty$. Therefore, the function has an infinite discontinuity at -1.

Exercise 2.5.3

For $f(x) = \begin{cases} x^2, & \text{if } x \ne 1 \\ 3, & \text{if } x = 1 \end{cases}$, decide whether f is continuous at 1. If f is not continuous at 1, classify the discontinuity as removable, jump, or infinite.

Hint

Consider the definitions of the various kinds of discontinuity stated above. If the function is discontinuous at 1, look at $\lim_{x\to 1} f(x)$

Answer

Discontinuous at 1; removable

Continuity over an Interval

Now that we have explored the concept of continuity at a point, we extend that idea to **continuity over an interval**. As we develop this idea for different types of intervals, it may be useful to keep in mind the intuitive idea that a function is continuous over an interval if we can use a pencil to trace the function between any two points in the interval without lifting the pencil from the paper. In preparation for defining continuity on an interval, we begin by looking at the definition of what it means for a function to be continuous from the right at a point and continuous from the left at a point.

Definition: Continuity from the Right and from the Left

A function $f(x)$ is said to be **continuous from the right** at a if $\lim_{x\to a^+} f(x) = f(a)$.

A function $f(x)$ is said to be **continuous from the left** at a if $\lim_{x\to a^-} f(x) = f(a)$

A function is continuous over an open interval if it is continuous at every point in the interval. A function $f(x)$ is continuous over a closed interval of the form $[a, b]$ if it is continuous at every point in (a, b) and is continuous from the right at a and is continuous from the left at b. Analogously, a function $f(x)$ is continuous over an interval of the form $(a, b]$ if it is continuous over (a, b) and is continuous from the left at b. Continuity over other types of intervals are defined in a similar fashion.

Requiring that $\lim_{x\to a^+} f(x) = f(a)$ and $\lim_{x\to b^-} f(x) = f(b)$ ensures that we can trace the graph of the function from the point $(a, f(a))$ to the point $(b, f(b))$ without lifting the pencil. If, for example, $\lim_{x\to a^+} f(x) \neq f(a)$, we would need to lift our pencil to jump from $f(a)$ to the graph of the rest of the function over $(a, b]$.

Example 2.5.6: Continuity on an Interval

State the interval(s) over which the function $f(x) = \dfrac{x-1}{x^2+2x}$ is continuous.

Solution

Since $f(x) = \dfrac{x-1}{x^2+2x}$ is a rational function, it is continuous at every point in its domain. The domain of $f(x)$ is the set $(-\infty, -2) \cup (-2, 0) \cup (0, +\infty)$. Thus, $f(x)$ is continuous over each of the intervals $(-\infty, -2)$, $(-2, 0)$, and $(0, +\infty)$.

Example 2.5.7: Continuity over an Interval

State the interval(s) over which the function $f(x) = \sqrt{4-x^2}$ is continuous.

Solution

From the limit laws, we know that $\lim_{x\to a} \sqrt{4-x^2} = \sqrt{4-a^2}$ for all values of a in $(-2, 2)$. We also know that $\lim_{x\to -2^+} \sqrt{4-x^2} = 0$ exists and $\lim_{x\to 2^-} \sqrt{4-x^2} = 0$ exists. Therefore, $f(x)$ is continuous over the interval $[-2, 2]$.

Exercise 2.5.4

State the interval(s) over which the function $f(x) = \sqrt{x+3}$ is continuous.

Hint

Use Example 2.5.7as a guide.

Answer

$[-3, +\infty)$

Theorem 2.5.2 allows us to expand our ability to compute limits. In particular, this theorem ultimately allows us to demonstrate that trigonometric functions are continuous over their domains.

Theorem 2.5.2: Composite Function Theorem

If $f(x)$ is continuous at L and $\lim_{x\to a} g(x) = L$, then

$$\lim_{x\to a} f\big(g(x)\big) = f\big(\lim_{x\to a} g(x)\big) = f(L).$$

Before we move on to Example 2.5.8, recall that earlier, in the section on limit laws, we showed $\lim_{x\to 0} \cos x = 1 = \cos(0)$. Consequently, we know that $f(x) = \cos x$ is continuous at 0. In Example 2.5.8, we see how to combine this result with the composite function theorem.

Example 2.5.8: Limit of a Composite Cosine Function

Evaluate $\lim_{x\to\pi/2}\cos\left(x-\frac{\pi}{2}\right)$.

Solution

The given function is a composite of $\cos x$ and $x-\frac{\pi}{2}$. Since $\lim_{x\to\pi/2}\left(x-\frac{\pi}{2}\right)=0$ and $\cos x$ is continuous at 0, we may apply the composite function theorem. Thus,

$$\lim_{x\to\pi/2}\cos\left(x-\frac{\pi}{2}\right)=\cos\left(\lim_{x\to\pi/2}\left(x-\frac{\pi}{2}\right)\right)=\cos(0)=1.$$

Exercise 2.5.4:

Evaluate $\lim_{x\to\pi}\sin(x-\pi)$.

Hint

$f(x)=\sin x$ is continuous at 0. Use Example 2.5.8as a guide.

Answer

0

The proof of the next theorem uses the composite function theorem as well as the continuity of $f(x)=\sin x$ and $g(x)=\cos x$ at the point 0 to show that trigonometric functions are continuous over their entire domains.

Theorem 2.5.3: Continuity of Trigonometric Functions

Trigonometric functions are continuous over their entire domains.

Proof

We begin by demonstrating that $\cos x$ is continuous at every real number. To do this, we must show that $\lim_{x\to a}\cos x=\cos a$ for all values of a.

$$\begin{aligned}
\lim_{x\to a}\cos x &= \lim_{x\to a}\cos((x-a)+a) && \text{Rewrite } x=x-a+a.\\
&= \lim_{x\to a}(\cos(x-a)\cos a-\sin(x-a)\sin a) && \text{Apply the identity for the cosine of the sum of two angles.}\\
&= \cos(\lim_{x\to a}(x-a))\cos a-\sin(\lim_{x\to a}(x-a))\sin a && \text{Since } \lim_{x\to a}(x-a)=0, \text{ and } \sin x \text{ and } \cos x \text{ are continuous at } 0.\\
&= \cos(0)\cos a-\sin(0)\sin a && \text{Evaluate } \cos(0) \text{ and } \sin(0) \text{ and simplify.}\\
&= 1\cdot\cos a-0\cdot\sin a=\cos a.
\end{aligned}$$

The proof that $\sin x$ is continuous at every real number is analogous. Because the remaining trigonometric functions may be expressed in terms of $\sin x$ and $\cos x$, their continuity follows from the quotient limit law.

□

As you can see, the composite function theorem is invaluable in demonstrating the continuity of trigonometric functions. As we continue our study of calculus, we revisit this theorem many times.

The Intermediate Value Theorem

Functions that are continuous over intervals of the form $[a,b]$, where a and b are real numbers, exhibit many useful properties. Throughout our study of calculus, we will encounter many powerful theorems concerning such functions. The first of these theorems is the **Intermediate Value Theorem.**

The Intermediate Value Theorem

Let f be continuous over a closed, bounded interval $[a, b]$. If z is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ satisfying $f(c) = z$ in Figure 2.5.7.

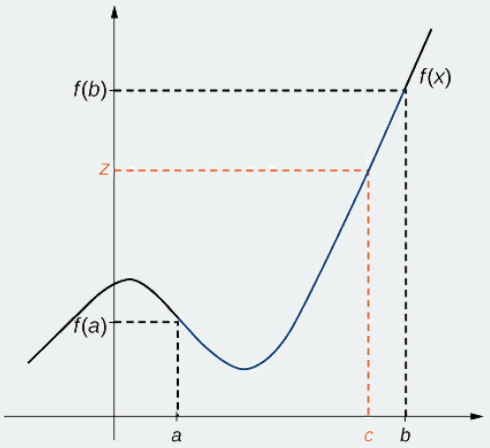


Figure 2.5.7: There is a number $c \in [a, b]$ that satisfies $f(c) = z$.

Example 2.5.9: Application of the Intermediate Value Theorem

Show that $f(x) = x - \cos x$ has at least one zero.

Solution

Since $f(x) = x - \cos x$ is continuous over $(-\infty, +\infty)$, it is continuous over any closed interval of the form $[a, b]$. If you can find an interval $[a, b]$ such that $f(a)$ and $f(b)$ have opposite signs, you can use the Intermediate Value Theorem to conclude there must be a real number c in (a, b) that satisfies $f(c) = 0$. Note that

$$f(0) = 0 - \cos(0) = -1 < 0$$

and

$$f(\tfrac{\pi}{2}) = \tfrac{\pi}{2} - \cos\tfrac{\pi}{2} = \tfrac{\pi}{2} > 0\ .$$

Using the Intermediate Value Theorem, we can see that there must be a real number c in $[0, \pi/2]$ that satisfies $f(c) = 0$. Therefore, $f(x) = x - \cos x$ has at least one zero.

Example 2.5.10: When Can You Apply the Intermediate Value Theorem?

If $f(x)$ is continuous over $[0, 2]$, $f(0) > 0$ and $f(2) > 0$, can we use the Intermediate Value Theorem to conclude that $f(x)$ has no zeros in the interval $[0, 2]$? Explain.

Solution

No. The Intermediate Value Theorem only allows us to conclude that we can find a value between $f(0)$ and $f(2)$; it doesn't allow us to conclude that we can't find other values. To see this more clearly, consider the function $f(x) = (x-1)^2$. It satisfies $f(0) = 1 > 0$, $f(2) = 1 > 0$, and $f(1) = 0$.

Example 2.5.11: When Can You Apply the Intermediate Value Theorem?

For $f(x) = 1/x$, $f(-1) = -1 < 0$ and $f(1) = 1 > 0$. Can we conclude that $f(x)$ has a zero in the interval $[-1, 1]$?

Solution

No. The function is not continuous over $[-1, 1]$. The Intermediate Value Theorem does not apply here.

Exercise 2.5.5

Show that $f(x) = x^3 - x^2 - 3x + 1$ has a zero over the interval $[0, 1]$.

Hint

Find $f(0)$ and $f(1)$. Apply the Intermediate Value Theorem.

Answer

$f(0) = 1 > 0,\ f(1) = -2 < 0;\ f(x)$ is continuous over $[0, 1]$. It must have a zero on this interval.

Key Concepts

- For a function to be continuous at a point, it must be defined at that point, its limit must exist at the point, and the value of the function at that point must equal the value of the limit at that point.
- Discontinuities may be classified as removable, jump, or infinite.
- A function is continuous over an open interval if it is continuous at every point in the interval. It is continuous over a closed interval if it is continuous at every point in its interior and is continuous at its endpoints.
- The composite function theorem states: If $f(x)$ is continuous at L and $\lim_{x \to a} g(x) = L$, then $\lim_{x \to a} f\big(g(x)\big) = f\big(\lim_{x \to a} g(x)\big) = f(L)$.
- The Intermediate Value Theorem guarantees that if a function is continuous over a closed interval, then the function takes on every value between the values at its endpoints.

Glossary

continuity at a point

A function $f(x)$ is continuous at a point a if and only if the following three conditions are satisfied: (1) $f(a)$ is defined, (2) $\lim_{x \to a} f(x)$ exists, and (3) $\lim x \to a f(x) = f(a)$

continuity from the left

A function is continuous from the left at b if $\lim_{x \to b^-} f(x) = f(b)$

continuity from the right

A function is continuous from the right at a if $\lim_{x \to a^+} f(x) = f(a)$

continuity over an interval

a function that can be traced with a pencil without lifting the pencil; a function is continuous over an open interval if it is continuous at every point in the interval; a function $f(x)$ is continuous over a closed interval of the form $[a, b]$ if it is continuous at every point in (a, b), and it is continuous from the right at a and from the left at b

discontinuity at a point

A function is discontinuous at a point or has a discontinuity at a point if it is not continuous at the point

infinite discontinuity

An infinite discontinuity occurs at a point a if $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$

Intermediate Value Theorem

Let f be continuous over a closed bounded interval $[a, b]$ if z is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ satisfying $f(c) = z$

jump discontinuity

A jump discontinuity occurs at a point a if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist, but $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

removable discontinuity

A removable discontinuity occurs at a point a if $f(x)$ is discontinuous at a, but $\lim_{x \to a} f(x)$ exists

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2.6: Limits at Infinity; Horizontal Asymptotes

In Definition 1 we stated that in the equation $\lim_{x\to c} f(x) = L$, both c and L were numbers. In this section we relax that definition a bit by considering situations when it makes sense to let c and/or L be "infinity."

As a motivating example, consider $f(x) = 1/x^2$, as shown in Figure 1.30. Note how, as x approaches 0, $f(x)$ grows very, very large. It seems appropriate, and descriptive, to state that

$$\lim_{x\to 0} \frac{1}{x^2} = \infty. \tag{2.6.1}$$

Also note that as x gets very large, $f(x)$ gets very, very small. We could represent this concept with notation such as

$$\lim_{x\to\infty} \frac{1}{x^2} = 0. \tag{2.6.2}$$

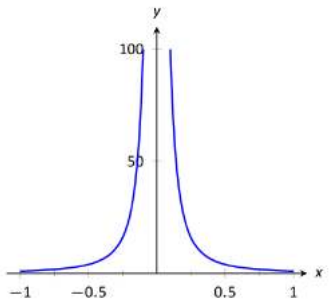


FIGURE 1.30: Graphing $f(x) = 1/x^2$ for values of x near 0.

We explore both types of use of ∞ in turn.

Definition 5: Limit of infinity

We say $\lim_{x\to c} f(x) = \infty$ if for every $M > 0$ there exists $\delta > 0$ such that for all $x \neq c$, if $|x - c| < \delta$, then $f(x) \geq M$.

This is just like the ϵ--δ definition from Section 1.2. In that definition, given any (small) value ϵ, if we let x get close enough to c (within δ units of c) then $f(x)$ is guaranteed to be within ϵ of $f(c)$. Here, given any (large) value M, if we let x get close enough to c (within δ units of c), then $f(x)$ will be at least as large as M. In other words, if we get close enough to c, then we can make $f(x)$ as large as we want. We can define limits equal to $-\infty$ in a similar way.

It is important to note that by saying $\lim_{x\to c} f(x) = \infty$ we are implicitly stating that \textit{the} limit of $f(x)$, as x approaches c, *does not exist*. A limit only exists when $f(x)$ approaches an actual numeric value. We use the concept of limits that approach infinity because it is helpful and descriptive.

Example 26: Evaluating limits involving infinity

Find $\lim_{x\to 1} \frac{1}{(x-1)^2}$ as shown in Figure 1.31.

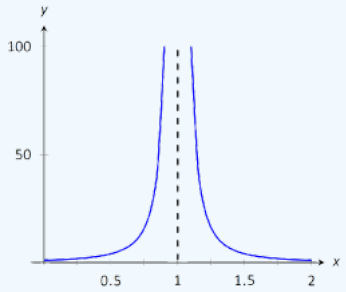


FIGURE 1.31: Observing infinite limit as $x \to 1$ in Example 26.

Solution

In Example 4 of Section 1.1, by inspecting values of x close to 1 we concluded that this limit does not exist. That is, it cannot equal any real number. But the limit could be infinite. And in fact, we see that the function does appear to be growing larger and larger, as $f(.99) = 10^4$, $f(.999) = 10^6$, $f(.9999) = 10^8$. A similar thing happens on the other side of 1. In general, let a "large" value M be given. Let $\delta = 1/\sqrt{M}$. If x is within δ of 1, i.e., if $|x-1| < 1/\sqrt{M}$, then:

$$\begin{aligned} |x-1| &< \frac{1}{\sqrt{M}} \\ (x-1)^2 &< \frac{1}{M} \\ \frac{1}{(x-1)^2} &> M, \end{aligned}$$

which is what we wanted to show. So we may say $\lim\limits_{x\to 1} 1/(x-1)^2 = \infty$.

Example 27: Evaluating limits involving infinity

Find $\lim\limits_{x\to 0} \frac{1}{x}$, as shown in Figure 1.32.

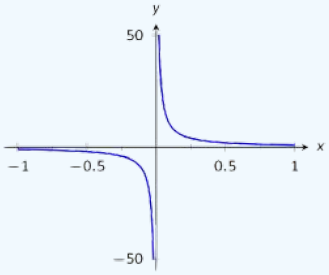


FIGURE 1.32: Evaluating $\lim\limits_{x\to 0} \frac{1}{x}$.

Solution

It is easy to see that the function grows without bound near 0, but it does so in different ways on different sides of 0. Since its behavior is not consistent, we cannot say that $\lim\limits_{x\to 0} \frac{1}{x} = \infty$. However, we can make a statement about one--sided limits. We can state that $\lim\limits_{x\to 0^+} \frac{1}{x} = \infty$ and $\lim\limits_{x\to 0^-} \frac{1}{x} = -\infty$.

Vertical Asymptotes

If the limit of $f(x)$ as x approaches c from either the left or right (or both) is ∞ or $-\infty$, we say the function has a **vertical asymptote** at c.

Example 28: Finding vertical asymptotes

Find the vertical asymptotes of $f(x) = \dfrac{3x}{x^2-4}$.

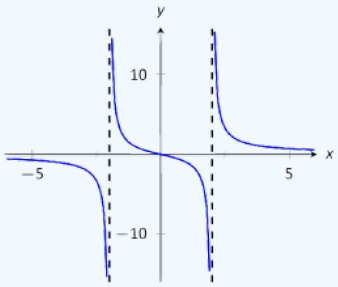


FIGURE 1.33: Graphing $f(x) = \frac{3x}{x^2-4}$.

Solution

Vertical asymptotes occur where the function grows without bound; this can occur at values of c where the denominator is 0. When x is near c, the denominator is small, which in turn can make the function take on large values. In the case of the given function, the denominator is 0 at $x = \pm 2$. Substituting in values of x close to 2 and -2 seems to indicate that the function tends toward ∞ or $-\infty$ at those points. We can graphically confirm this by looking at Figure 1.33. Thus the vertical asymptotes are at $x = \pm 2$.

When a rational function has a vertical asymptote at $x = c$, we can conclude that the denominator is 0 at $x = c$. However, just because the denominator is 0 at a certain point does not mean there is a vertical asymptote there. For instance, $f(x) = (x^2 - 1)/(x - 1)$ does not have a vertical asymptote at $x = 1$, as shown in Figure 1.34. While the denominator does get small near $x = 1$, the numerator gets small too, matching the denominator step for step. In fact, factoring the numerator, we get

$$f(x) = \frac{(x-1)(x+1)}{x-1}. \tag{2.6.3}$$

Canceling the common term, we get that $f(x) = x + 1$ for $x \neq 1$. So there is clearly no asymptote, rather a hole exists in the graph at $x = 1$.

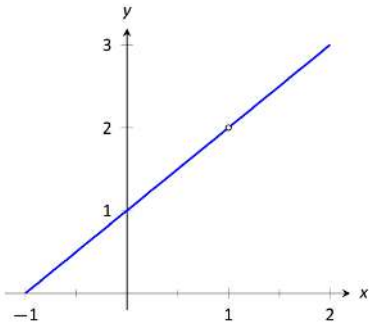


FIGURE 1.34: Graphically showing that $f(x) = \frac{x^2-1}{x-1}$ does not have an asymptote at $x = 1$.

The above example may seem a little contrived. Another example demonstrating this important concept is $f(x) = (\sin x)/x$. We have considered this function several times in the previous sections. We found that $\lim_{x\to 0} \frac{\sin x}{x} = 1$; i.e., there is no vertical asymptote. No simple algebraic cancellation makes this fact obvious; we used the Squeeze Theorem in Section 1.3 to prove this.

If the denominator is 0 at a certain point but the numerator is not, then there will usually be a vertical asymptote at that point. On the other hand, if the numerator and denominator are both zero at that point, then there may or may not be a vertical asymptote at that point. This case where the numerator and denominator are both zero returns us to an important topic.

Indeterminate Forms

We have seen how the limits

$$\lim_{x\to 0} \frac{\sin x}{x} \quad \text{and} \quad \lim_{x\to 1} \frac{x^2 - 1}{x - 1} \tag{2.6.4}$$

each return the indeterminate form "$0/0$" when we blindly plug in $x = 0$ and $x = 1$, respectively. However, $0/0$ is not a valid arithmetical expression. It gives no indication that the respective limits are 1 and 2.

With a little cleverness, one can come up $0/0$ expressions which have a limit of ∞, 0, or any other real number. That is why this expression is called *indeterminate*.

A key concept to understand is that such limits do not really return $0/0$. Rather, keep in mind that we are taking *limits*. What is really happening is that the numerator is shrinking to 0 while the denominator is also shrinking to 0. The respective rates at which they do this are very important and determine the actual value of the limit.

An indeterminate form indicates that one needs to do more work in order to compute the limit. That work may be algebraic (such as factoring and canceling) or it may require a tool such as the Squeeze Theorem. In a later section we will learn a technique called l'Hospital's Rule that provides another way to handle indeterminate forms.

Some other common indeterminate forms are $\infty - \infty$, $\infty \cdot 0$, ∞/∞, 0^0, ∞^0 and 1^∞. Again, keep in mind that these are the "blind" results of evaluating a limit, and each, in and of itself, has no meaning. The expression $\infty - \infty$ does not really mean "subtract infinity from infinity." Rather, it means "One quantity is subtracted from the other, but both are growing without bound." What is the result? It is possible to get every value between $-\infty$ and ∞

Note that $1/0$ and $\infty/0$ are not indeterminate forms, though they are not exactly valid mathematical expressions, either. In each, the function is growing without bound, indicating that the limit will be ∞, $-\infty$, or simply not exist if the left- and right-hand limits do not match.

Limits at Infinity and Horizontal Asymptotes

At the beginning of this section we briefly considered what happens to $f(x) = 1/x^2$ as x grew very large. Graphically, it concerns the behavior of the function to the "far right" of the graph. We make this notion more explicit in the following definition.

Definition 6: Limits at Infinity and Horizontal Asymptote

1. We say $\lim\limits_{x\to\infty} f(x) = L$ if for every $\epsilon > 0$ there exists $M > 0$ such that if $x \geq M$, then $|f(x) - L| < \epsilon$.
2. We say $\lim\limits_{x\to-\infty} f(x) = L$ if for every $\epsilon > 0$ there exists $M < 0$ such that if $x \leq M$, then $|f(x) - L| < \epsilon$.
3. If $\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to-\infty} f(x) = L$, we say that $y = L$ is a **horizontal asymptote** of f.

We can also define limits such as $\lim\limits_{x\to\infty} f(x) = \infty$ by combining this definition with Definition 5.

Example 29: Approximating horizontal asymptotes

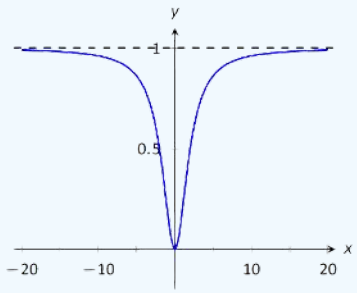
Approximate the horizontal asymptote(s) of $f(x) = \frac{x^2}{x^2+4}$.

Solution

We will approximate the horizontal asymptotes by approximating the limits

$$\lim_{x\to-\infty} \frac{x^2}{x^2+4} \quad \text{and} \quad \lim_{x\to\infty} \frac{x^2}{x^2+4}. \tag{2.6.5}$$

Figure 1.35(a) shows a sketch of f, and part (b) gives values of $f(x)$ for large magnitude values of x. It seems reasonable to conclude from both of these sources that f has a horizontal asymptote at $y = 1$.



(a)

x	$f(x)$
10	0.9615
100	0.9996
10000	0.999996
-10	0.9615
-100	0.9996
-10000	0.999996

(b)

FIGURE 1.35: Using a graph and a table to approximate a horizontal asymptote in Example 29.

Later, we will show how to determine this analytically.

Horizontal asymptotes can take on a variety of forms. Figure 1.36(a) shows that $f(x) = x/(x^2+1)$ has a horizontal asymptote of $y = 0$, where 0 is approached from both above and below.

Figure 1.36(b) shows that $f(x) = x/\sqrt{x^2+1}$ has two horizontal asymptotes; one at $y = 1$ and the other at $y = -1$.

Figure 1.36(c) shows that $f(x) = (\sin x)/x$ has even more interesting behavior than at just $x = 0$; as x approaches $\pm\infty$, $f(x)$ approaches 0, but oscillates as it does this.

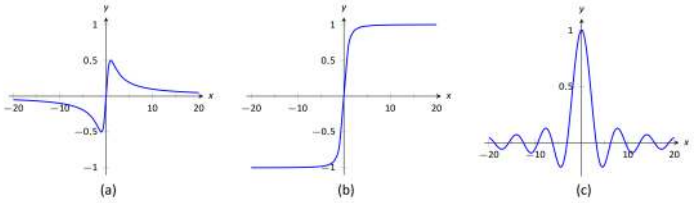


FIGURE 1.36: Considering different types of horizontal asymptotes.

We can analytically evaluate limits at infinity for rational functions once we understand $\lim_{x\to\infty} 1/x$. As x gets larger and larger, the $1/x$ gets smaller and smaller, approaching 0. We can, in fact, make $1/x$ as small as we want by choosing a large enough value of x. Given ϵ, we can make $1/x < \epsilon$ by choosing $x > 1/\epsilon$. Thus we have $\lim_{x\to\infty} 1/x = 0$.

It is now not much of a jump to conclude the following:

$$\lim_{x\to\infty} \frac{1}{x^n} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x^n} = 0 \tag{2.6.6}$$

Now suppose we need to compute the following limit:

$$\lim_{x\to\infty} \frac{x^3+2x+1}{4x^3-2x^2+9}. \tag{2.6.7}$$

A good way of approaching this is to divide through the numerator and denominator by x^3 (hence dividing by 1), which is the largest power of x to appear in the function. Doing this, we get

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^3+2x+1}{4x^3-2x^2+9} &= \lim_{x\to\infty} \frac{1/x^3}{1/x^3} \cdot \frac{x^3+2x+1}{4x^3-2x^2+9} \\
&= \lim_{x\to\infty} \frac{x^3/x^3+2x/x^3+1/x^3}{4x^3/x^3-2x^2/x^3+9/x^3} \\
&= \lim_{x\to\infty} \frac{1+2/x^2+1/x^3}{4-2/x+9/x^3}.
\end{aligned}$$

Then using the rules for limits (which also hold for limits at infinity), as well as the fact about limits of $1/x^n$, we see that the limit becomes

$$\frac{1+0+0}{4-0+0} = \frac{1}{4}. \tag{2.6.8}$$

This procedure works for any rational function. In fact, it gives us the following theorem.

Theorem 11: Limits of Rational Functions at Infinity

Let $f(x)$ be a rational function of the following form:

$$f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}, \tag{2.6.9}$$

where any of the coefficients may be 0 except for a_n and b_m.

1. If $n = m$, then $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to-\infty} f(x) = \frac{a_n}{b_m}$.
2. If $n < m$, then $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to-\infty} f(x) = 0$.
3. If $n > m$, then $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$ are both infinite.

We can see why this is true. If the highest power of x is the same in both the numerator and denominator (i.e. $n = m$), we will be in a situation like the example above, where we will divide by x^n and in the limit all the terms will approach 0 except for $a_n x^n/x^n$ and $b_m x^m/x^n$. Since $n = m$, this will leave us with the limit a_n/b_m. If $n < m$, then after dividing through by x^m, all the terms in the numerator will approach 0 in the limit, leaving us with $0/b_m$ or 0. If $n > m$, and we try dividing through by x^n, we end up with all the terms in the denominator tending toward 0, while the x^n term in the numerator does not approach 0. This is indicative of some sort of infinite limit.

Intuitively, as x gets very large, all the terms in the numerator are small in comparison to $a_n x^n$, and likewise all the terms in the denominator are small compared to $b_n x^m$. If $n = m$, looking only at these two important terms, we have $(a_n x^n)/(b_n x^m)$. This reduces to a_n/b_m. If $n < m$, the function behaves like $a_n/(b_m x^{m-n})$, which tends toward 0. If $n > m$, the function behaves like $a_n x^{n-m}/b_m$, which will tend to either ∞ or $-\infty$ depending on the values of n, m, a_n, b_m and whether you are looking for $\lim\limits_{x\to\infty} f(x)$ or $\lim\limits_{x\to-\infty} f(x)$.

With care, we can quickly evaluate limits at infinity for a large number of functions by considering the largest powers of x. For instance, consider again $\lim\limits_{x\to\pm\infty} \frac{x}{\sqrt{x^2+1}}$, graphed in Figure ???(b). When x is very large, $x^2 + 1 \approx x^2$. Thus

$$\sqrt{x^2+1} \approx \sqrt{x^2} = |x|, \quad \text{and} \quad \frac{x}{\sqrt{x^2+1}} \approx \frac{x}{|x|}. \tag{2.6.10}$$

This expression is 1 when x is positive and -1 when x is negative. Hence we get asymptotes of $y = 1$ and $y = -1$, respectively.

Example 30: Finding a limit of a rational function

Confirm analytically that $y = 1$ is the horizontal asymptote of $f(x) = \frac{x^2}{x^2+4}$, as approximated in Example 29.

Solution

Before using Theorem 11, let's use the technique of evaluating limits at infinity of rational functions that led to that theorem. The largest power of x in f is 2, so divide the numerator and denominator of f by x^2, then take limits.

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^2}{x^2+4} &= \lim_{x\to\infty} \frac{x^2/x^2}{x^2/x^2 + 4/x^2} \\
&= \lim_{x\to\infty} \frac{1}{1+4/x^2} \\
&= \frac{1}{1+0} \\
&= 1.
\end{aligned}$$

We can also use Theorem 11 directly; in this case $n = m$ so the limit is the ratio of the leading coefficients of the numerator and denominator, i.e., 1/1 = 1.

Example 31: Finding limits of rational functions

Use Theorem 11 to evaluate each of the following limits.

$$1.\ \lim_{x\to-\infty}\frac{x^2+2x-1}{x^3+1} \qquad 3.\ \lim_{x\to\infty}\frac{x^2-1}{3-x} \tag{2.6.11}$$

$$2.\ \lim_{x\to\infty}\frac{x^2+2x-1}{1-x-3x^2} \tag{2.6.12}$$

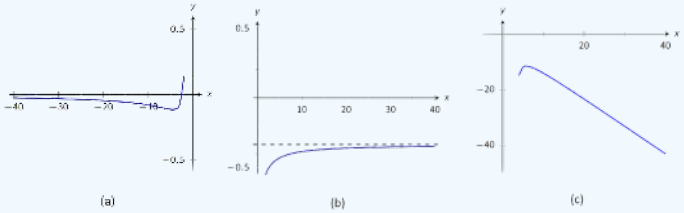


FIGURE 1.37: Visualizing the functions in Example 31.

Solution

1. The highest power of x is in the denominator. Therefore, the limit is 0; see Figure 1.37(a).
2. The highest power of x is x^2, which occurs in both the numerator and denominator. The limit is therefore the ratio of the coefficients of x^2, which is $-1/3$. See Figure 1.37(b).
3. The highest power of x is in the numerator so the limit will be ∞ or $-\infty$. To see which, consider only the dominant terms from the numerator and denominator, which are x^2 and $-x$. The expression in the limit will behave like $x^2/(-x) = -x$ for large values of x. Therefore, the limit is $-\infty$. See Figure 1.37(c).

Chapter Summary

In this chapter we:

- defined the limit,
- found accessible ways to approximate their values numerically and graphically,
- developed a not--so--easy method of proving the value of a limit ($\epsilon-\delta$ proofs),
- explored when limits do not exist,
- defined continuity and explored properties of continuous functions, and
- considered limits that involved infinity.

Why? Mathematics is famous for building on itself and calculus proves to be no exception. In the next chapter we will be interested in "dividing by 0." That is, we will want to divide a quantity by a smaller and smaller number and see what value the quotient approaches. In other words, we will want to find a limit. These limits will enable us to, among other things, determine *exactly* how fast something is moving when we are only given position information.

Later, we will want to add up an infinite list of numbers. We will do so by first adding up a finite list of numbers, then take a limit as the number of things we are adding approaches infinity. Surprisingly, this sum often is finite; that is, we can add up an infinite list of numbers and get, for instance, 42.

These are just two quick examples of why we are interested in limits. Many students dislike this topic when they are first introduced to it, but over time an appreciation is often formed based on the scope of its applicability.

Contributors and Attributions

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CHAPTER OVERVIEW

3: Differentiation Rules

A general Calculus Textmap organized around the textbook

Calculus: Early Transcendentals

by James Stewart

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This Textmap is currently under construction... please be patient with us.

Template:HideTOC

Topic hierarchy

3.1: Derivatives of Polynomials and Exponential Functions
3.2: The Product and Quotient Rules
3.3: Derivatives of Trigonometric Functions
3.4: The Chain Rule
3.5: Implicit Differentiation
3.6: Derivatives of Logarithmic Functions
3.7: Rates of Change in the Natural and Social Sciences
3.8: Exponential Growth and Decay
3.9: Related Rates
3.10: Linear Approximations and Differentials
3.11: Hyperbolic Functions

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3.2: The Product and Quotient Rules

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3.3: Derivatives of Trigonometric Functions

- 4.2: The Derivative of sin x - I
- 4.4: The Derivative of Sin x Part II
- 4.5: Derivatives of the Trigonometric Functions

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3.4: The Chain Rule

We have covered almost all of the derivative rules that deal with combinations of two (or more) functions. The operations of addition, subtraction, multiplication (including by a constant) and division led to the Sum and Difference rules, the Constant Multiple Rule, the Power Rule, the Product Rule and the Quotient Rule. To complete the list of differentiation rules, we look at the last way two (or more) functions can be combined: the process of composition (i.e. one function "inside" another).

One example of a composition of functions is $f(x) = \cos(x^2)$. We currently do not know how to compute this derivative. If forced to guess, one would likely guess $f'(x) = -\sin(2x)$,where we recognize $-\sin x$ as the derivative of $\cos x$ and $2x$ as the derivative of x^2. However, this is not the case; $f'(x) \neq -\sin(2x)$. In Example 62 we'll see the correct answer, which employs the new rule this section introduces, the **Chain Rule**.

Before we define this new rule, recall the notation for composition of functions. We write $(f \circ g)(x)$ or $f(g(x))$,read as "f of g of x," to denote composing f with g. In shorthand, we simply write $f \circ g$ or $f(g)$ and read it as "f of g." Before giving the corresponding differentiation rule, we note that the rule extends to multiple compositions like $f(g(h(x)))$ or $f(g(h(j(x))))$,etc.

To motivate the rule, let's look at three derivatives we can already compute.

Example 59: Exploring similar derivatives

Find the derivatives of

a. $F_1(x) = (1-x)^2$,
b. $F_2(x) = (1-x)^3$, and
c. $F_3(x) = (1-x)^4$.

We'll see later why we are using subscripts for different functions and an uppercase F.

Solution

In order to use the rules we already have, we must first expand each function as

a. $F_1(x) = 1 - 2x + x^2$,
b. $F_2(x) = 1 - 3x + 3x^2 - x^3$ and
c. $F_3(x) = 1 - 4x + 6x^2 - 4x^3 + x^4$.

It is not hard to see that:

$$\begin{aligned} F_1'(x) &= -2 + 2x \\ F_2'(x) &= -3 + 6x - 3x^2 \\ F_3'(x) &= -4 + 12x - 12x^2 + 4x^3. \end{aligned}$$

An interesting fact is that these can be rewritten as

$$F_1'(x) = -2(1-x), \quad F_2'(x) = -3(1-x)^2 \text{ and } F_3'(x) = -4(1-x)^3. \tag{3.4.1}$$

A pattern might jump out at you. Recognize that each of these functions is a composition, letting $g(x) = 1 - x$:

$$\begin{aligned} F_1(x) &= f_1(g(x)), \text{ where } f_1(x) = x^2, \\ F_2(x) &= f_2(g(x)), \text{ where } f_2(x) = x^3, \\ F_3(x) &= f_3(g(x)), \text{ where } f_3(x) = x^4. \end{aligned}$$

We'll come back to this example after giving the formal statements of the Chain Rule; for now, we are just illustrating a pattern.

Theorem 18: The Chain Rule

Let $y = f(u)$ be a differentiable function of u and let $u = g(x)$ be a differentiable function of x. Then $y = f(g(x))$ is a differentiable function of x,and

$$y' = f'(g(x)) \cdot g'(x). \tag{3.4.2}$$

To help understand the Chain Rule, we return to Example 59.

Example 60: Using the Chain Rule

Use the Chain Rule to find the derivatives of the following functions, as given in Example 59.

Solution

Example 59 ended with the recognition that each of the given functions was actually a composition of functions. To avoid confusion, we ignore most of the subscripts here.

$F_1(x) = (1-x)^2$:

We found that

$$y = (1-x)^2 = f(g(x)), \text{ where } f(x) = x^2 \text{ and } g(x) = 1-x. \tag{3.4.3}$$

To find y', we apply the Chain Rule. We need $f'(x) = 2x$ and $g'(x) = -1$.

Part of the Chain Rule uses $f'(g(x))$. This means substitute $g(x)$ for x in the equation for $f'(x)$. That is, $f'(x) = 2(1-x)$. Finishing out the Chain Rule we have

$$y' = f'(g(x)) \cdot g'(x) = 2(1-x) \cdot (-1) = -2(1-x) = 2x - 2. \tag{3.4.4}$$

$F_2(x) = (1-x)^3$:

Let $y = (1-x)^3 = f(g(x))$,where $f(x) = x^3$ and $g(x) = (1-x)$. We have $f'(x) = 3x^2$,so $f'(g(x)) = 3(1-x)^2$. The Chain Rule then states

$$y' = f'(g(x)) \cdot g'(x) = 3(1-x)^2 \cdot (-1) = -3(1-x)^2. \tag{3.4.5}$$

$F_3(x) = (1-x)^4$:

Finally, when $y = (1-x)^4$,we have $f(x) = x^4$ and $g(x) = (1-x)$. Thus $f'(x) = 4x^3$ and $f'(g(x)) = 4(1-x)^3$. Thus

$$y' = f'(g(x)) \cdot g'(x) = 4(1-x)^3 \cdot (-1) = -4(1-x)^3. \tag{3.4.6}$$

Example 60 demonstrated a particular pattern: when $f(x) = x^n$,then $y' = n \cdot (g(x))^{n-1} \cdot g'(x)$. This is called the Generalized Power Rule.

Theorem 19: Generalized Power Rule

Let $g(x)$ be a differentiable function and let $n \neq 0$ be an integer. Then

$$\frac{d}{dx}\left(g(x)^n\right) = n \cdot \left(g(x)\right)^{n-1} \cdot g'(x). \tag{3.4.7}$$

This allows us to quickly find the derivative of functions like $y = (3x^2 - 5x + 7 + \sin x)^{20}$. While it may look intimidating, the Generalized Power Rule states that

$$y' = 20(3x^2 - 5x + 7 + \sin x)^{19} \cdot (6x - 5 + \cos x). \tag{3.4.8}$$

Treat the derivative--taking process step--by--step. In the example just given, first multiply by 20, then rewrite the inside of the parentheses, raising it all to the 19th power. Then think about the derivative of the expression inside the parentheses, and multiply by that.

We now consider more examples that employ the Chain Rule.

Example 61: Using the Chain Rule

Find the derivatives of the following functions:

a. $y = \sin 2x$
b. $y = \ln(4x^3 - 2x^2)$
c. $y = e^{-x^2}$

Solution

a. Consider $y = \sin 2x$. Recognize that this is a composition of functions, where $f(x) = \sin x$ and $g(x) = 2x$. Thus

$$y' = f'(g(x)) \cdot g'(x) = \cos(2x) \cdot 2 = 2\cos 2x. \tag{3.4.9}$$

b. Recognize that $y = \ln(4x^3 - 2x^2)$ is the composition of $f(x) = \ln x$ and $g(x) = 4x^3 - 2x^2$. Also, recall that

$$\frac{d}{dx}\Big(\ln x\Big) = \frac{1}{x}. \tag{3.4.10}$$

This leads us to:

$$y' = \frac{1}{4x^3 - 2x^2} \cdot (12x^2 - 4x) = \frac{12x^2 - 4x}{4x^3 - 2x^2} = \frac{4x(3x-1)}{2x(2x^2 - x)} = \frac{2(3x-1)}{2x^2 - x}. \tag{3.4.11}$$

c. Recognize that $y = e^{-x^2}$ is the composition of $f(x) = e^x$ and $g(x) = -x^2$. Remembering that $f'(x) = e^x$,we have

$$y' = e^{-x^2} \cdot (-2x) = (-2x)e^{-x^2}. \tag{3.4.12}$$

Example 62: Using the Chain Rule to find a tangent line

Let $f(x) = \cos x^2$. Find the equation of the line tangent to the graph of f at $x = 1$.

Solution

The tangent line goes through the point $(1, f(1)) \approx (1, 0.54)$with slope $f'(1)$. To find f',we need the Chain Rule.

$f'(x) = -\sin(x^2) \cdot (2x) = -2x\sin x^2$. Evaluated at $x = 1$,we have $f'(1) = -2\sin 1 \approx -1.68$. Thus the equation of the tangent line is

$$y = -1.68(x-1) + 0.54. \tag{3.4.13}$$

The tangent line is sketched along with f in Figure 2.17.

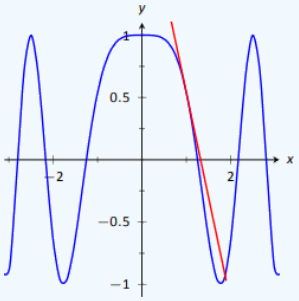


Figure 2.17: $f(x) = \cos x^2$ sketched along with its tangent line at $x = 1$.

The Chain Rule is used often in taking derivatives. Because of this, one can become familiar with the basic process and learn patterns that facilitate finding derivatives quickly. For instance,

$$\frac{d}{dx}\Big(\ln(\text{anything})\Big) = \frac{1}{\text{anything}} \cdot (\text{anything})' = \frac{(\text{anything})'}{\text{anything}}. \tag{3.4.14}$$

A concrete example of this is

$$\frac{d}{dx}\Big(\ln(3x^{15} - \cos x + e^x)\Big) = \frac{45x^{14} + \sin x + e^x}{3x^{15} - \cos x + e^x}. \tag{3.4.15}$$

While the derivative may look intimidating at first, look for the pattern. The denominator is the same as what was inside the natural log function; the numerator is simply its derivative.

This pattern recognition process can be applied to lots of functions. In general, instead of writing "anything", we use u as a generic function of x. We then say

$$\frac{d}{dx}\Big(\ln u\Big) = \frac{u'}{u}. \tag{3.4.16}$$

The following is a short list of how the Chain Rule can be quickly applied to familiar functions.

1. $\frac{d}{dx}\left(u^n\right) = n \cdot u^{n-1} \cdot u'$.
2. $\frac{d}{dx}\left(e^u\right) = u' \cdot e^u$.
3. $\frac{d}{dx}\left(\sin u\right) = u' \cdot \cos u$.
4. $\frac{d}{dx}\left(\cos u\right) = -u' \cdot \sin u$.
5. $\frac{d}{dx}\left(\tan u\right) = u' \cdot \sec^2 u$.

Of course, the Chain Rule can be applied in conjunction with any of the other rules we have already learned. We practice this next.

Example 63: Using the Product, Quotient and Chain Rules

Find the derivatives of the following functions.

a. $f(x) = x^5 \sin 2x^3$

b. $f(x) = \dfrac{5x^3}{e^{-x^2}}$.

Solution

a. We must use the Product and Chain Rules. Do not think that you must be able to "see" the whole answer immediately; rather, just proceed step--by--step.

$$f'(x) = x^5\left(6x^2 \cos 2x^3\right) + 5x^4\left(\sin 2x^3\right) = 6x^7 \cos 2x^3 + 5x^4 \sin 2x^3. \tag{3.4.17}$$

b. We must employ the Quotient Rule along with the Chain Rule. Again, proceed step--by--step.

$$f'(x) = \frac{e^{-x^2}\left(15x^2\right) - 5x^3\left((-2x)e^{-x^2}\right)}{\left(e^{-x^2}\right)^2} = \frac{e^{-x^2}\left(10x^4 + 15x^2\right)}{e^{-2x^2}}$$
$$= e^{x^2}\left(10x^4 + 15x^2\right).$$

A key to correctly working these problems is to break the problem down into smaller, more manageable pieces. For instance, when using the Product and Chain Rules together, just consider the first part of the Product Rule at first: $f(x)g'(x)$. Just rewrite $f(x)$,then find $g'(x)$. Then move on to the $f'(x)g(x)$ part. Don't attempt to figure out both parts at once.

Likewise, using the Quotient Rule, approach the numerator in two steps and handle the denominator after completing that. Only simplify afterward.

We can also employ the Chain Rule itself several times, as shown in the next example.

Example 64: Using the Chain Rule multiple times

Find the derivative of $y = \tan^5(6x^3 - 7x)$.

Solution

Recognize that we have the $g(x) = \tan(6x^3 - 7x)$ function "inside" the $f(x) = x^5$ function; that is, we have $y = \left(\tan(6x^3 - 7x)\right)^5$. We begin using the Generalized Power Rule; in this first step, we do not fully compute the derivative. Rather, we are approaching this step--by--step.

$$y' = 5\left(\tan(6x^3 - 7x)\right)^4 \cdot g'(x). \tag{3.4.18}$$

We now find $g'(x)$. We again need the Chain Rule;

$$g'(x) = \sec^2(6x^3 - 7x) \cdot (18x^2 - 7). \tag{3.4.19}$$

Combine this with what we found above to give

$$\begin{aligned} y' &= 5\left(\tan(6x^3 - 7x)\right)^4 \cdot \sec^2(6x^3 - 7x) \cdot (18x^2 - 7) \\ &= (90x^2 - 35)\sec^2(6x^3 - 7x)\tan^4(6x^3 - 7x). \end{aligned}$$

This function is frankly a ridiculous function, possessing no real practical value. It is very difficult to graph, as the tangent function has many vertical asymptotes and $6x^3 - 7x$ grows so very fast. The important thing to learn from this is that the

derivative can be found. In fact, it is not "hard;" one must take several simple steps and be careful to keep track of how to apply each of these steps.

It is a traditional mathematical exercise to find the derivatives of arbitrarily complicated functions just to demonstrate that it *can be done*. Just break everything down into smaller pieces.

Example 65: Using the Product, Quotient and Chain Rules

Find the derivative of $f(x) = \dfrac{x\cos(x^{-2}) - \sin^2(e^{4x})}{\ln(x^2+5x^4)}$.

Solution

This function likely has no practical use outside of demonstrating derivative skills. The answer is given below without simplification. It employs the Quotient Rule, the Product Rule, and the Chain Rule three times.

$$f'(x) = \frac{\Big(\ln(x^2+5x^4)\Big)\cdot\Big[\big(x\cdot(-\sin(x^{-2}))\cdot(-2x^{-3})+1\cdot\cos(x^{-2})\big) - 2\sin(e^{4x})\cdot\cos(e^{4x})\cdot(4e^{4x})\Big] - \Big(x\cos(x^{-2})-\sin^2(e^{4x})\Big)\cdot\dfrac{2x+20x^3}{x^2+5x^4}}{\Big(\ln(x^2+5x^4)\Big)^2}. \quad (3.4.20)$$

The reader is highly encouraged to look at each term and recognize why it is there. (I.e., the Quotient Rule is used; in the numerator, identify the "LOdHI" term, etc.) This example demonstrates that derivatives can be computed systematically, no matter how arbitrarily complicated the function is.

The Chain Rule also has theoretic value. That is, it can be used to find the derivatives of functions that we have not yet learned as we do in the following example.

Example 66: The Chain Rule and exponential functions

Use the Chain Rule to find the derivative of $y = a^x$ where $a > 0, a \neq 1$ is constant.

Solution

We only know how to find the derivative of one exponential function: $y = e^x$; this problem is asking us to find the derivative of functions such as $y = 2^x$.

This can be accomplished by rewriting a^x in terms of e. Recalling that e^x and $\ln x$ are inverse functions, we can write

$$a = e^{\ln a} \quad \text{and so} \quad y = a^x = e^{\ln(a^x)}.$$

By the exponent property of logarithms, we can "bring down" the power to get

$$y = a^x = e^{x(\ln a)}.$$

The function is now the composition $y = f(g(x))$,with $f(x) = e^x$ and $g(x) = x(\ln a)$. Since $f'(x) = e^x$ and $g'(x) = \ln a$, the Chain Rule gives

$$y' = e^{x(\ln a)}\cdot \ln a.$$

Recall that the $e^{x(\ln a)}$ term on the right hand side is just a^x,our original function. Thus, the derivative contains the original function itself. We have

$$y' = y\cdot \ln a = a^x \cdot \ln a.$$

The Chain Rule, coupled with the derivative rule of e^x,allows us to find the derivatives of all exponential functions.

The previous example produced a result worthy of its own "box."

Theorem 20: Derivatives of Exponential Functions

Let $f(x) = a^x$,for $a > 0, a \neq 1$. Then f is differentiable for all real numbers and

$$f'(x) = \ln a \cdot a^x.$$

Alternate Chain Rule Notation

It is instructive to understand what the Chain Rule "looks like" using "$\frac{dy}{dx}$" notation instead of y' notation. Suppose that $y = f(u)$ is a function of u,where $u = g(x)$ is a function of x,as stated in Theorem 18. Then, through the composition $f \circ g$,we can think of y as a function of x,as $y = f(g(x))$. Thus the derivative of y with respect to x makes sense; we can talk about $\frac{dy}{dx}$. This leads to an interesting progression of notation:

$$\begin{aligned} y' &= f'(g(x)) \cdot g'(x) \\ \frac{dy}{dx} &= y'(u) \cdot u'(x) \quad (\text{since } y = f(u) \text{ and } u = g(x)) \\ \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \quad (\text{using "fractional" notation for the derivative}) \end{aligned}$$

Here the "fractional" aspect of the derivative notation stands out. On the right hand side, it seems as though the "du" terms cancel out, leaving

$$\frac{dy}{dx} = \frac{dy}{dx}. \tag{3.4.21}$$

It is important to realize that we *are not* canceling these terms; the derivative notation of $\frac{dy}{dx}$ is one symbol. It is equally important to realize that this notation was chosen precisely because of this behavior. It makes applying the Chain Rule easy with multiple variables. For instance,

$$\frac{dy}{dt} = \frac{dy}{d\bigcirc} \cdot \frac{d\bigcirc}{d\triangle} \cdot \frac{d\triangle}{dt}. \tag{3.4.22}$$

where $\bigcirc$ and $\triangle$ are any variables you'd like to use.

One of the most common ways of "visualizing" the Chain Rule is to consider a set of gears, as shown in Figure 2.18. The gears have 36, 18, and 6 teeth, respectively. That means for every revolution of the x gear, the u gear revolves twice. That is, the rate at which the u gear makes a revolution is twice as fast as the rate at which the x gear makes a revolution. Using the terminology of calculus, the rate of u-change, with respect to x,is $\frac{du}{dx} = 2$.

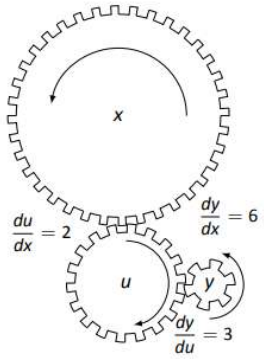


Figure 2.18: A series of gears to demonstrate the Chain Rule. Note how $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

Likewise, every revolution of u causes 3 revolutions of y: $\frac{dy}{du} = 3$. How does y change with respect to x? For each revolution of x,y revolves 6 times; that is,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2 \cdot 3 = 6. \tag{3.4.23}$$

We can then extend the Chain Rule with more variables by adding more gears to the picture.

It is difficult to overstate the importance of the Chain Rule. So often the functions that we deal with are compositions of two or more functions, requiring us to use this rule to compute derivatives. It is often used in practice when actual functions are unknown. Rather, through measurement, we can calculate $\frac{dy}{du}$ and $\frac{du}{dx}$. With our knowledge of the Chain Rule, finding $\frac{dy}{dx}$ is straightforward.

In the next section, we use the Chain Rule to justify another differentiation technique. There are many curves that we can draw in the plane that fail the "vertical line test." For instance, consider $x^2 + y^2 = 1$,which describes the unit circle. We may still be interested in finding slopes of tangent lines to the circle at various points. The next section shows how we can find $\frac{dy}{dx}$ without first "solving for y." While we can in this instance, in many other instances solving for y is impossible. In these situations, *implicit differentiation* is indispensable.

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3.5: Implicit Differentiation

In the previous sections we learned to find the derivative, $\frac{dy}{dx}$, or y', when y is given *explicitly* as a function of x. That is, if we know $y = f(x)$ for some function f, we can find y'. For example, given $y = 3x^2 - 7$, we can easily find $y' = 6x$. (Here we explicitly state how x and y are related. Knowing x, we can directly find y.)

Sometimes the relationship between y and x is not explicit; rather, it is *implicit*. For instance, we might know that $x^2 - y = 4$. This equality defines a relationship between x and y; if we know x, we could figure out y. Can we still find y'? In this case, sure; we solve for y to get $y = x^2 - 4$ (hence we now know y explicitly) and then differentiate to get $y' = 2x$.

Sometimes the *implicit* relationship between x and y is complicated. Suppose we are given $\sin(y) + y^3 = 6 - x^3$. A graph of this implicit function is given in Figure 2.19. In this case there is absolutely no way to solve for y in terms of elementary functions. The surprising thing is, however, that we can still find y' via a process known as **implicit differentiation**.

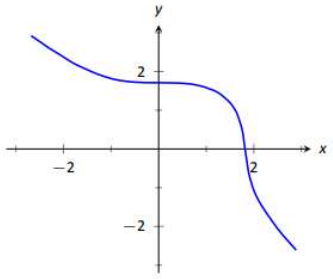


Figure 2.19: A graph of the implicit function $\sin(y) + y^3 = 6 - x^2$.

Implicit differentiation is a technique based on the Chain Rule that is used to find a derivative when the relationship between the variables is given implicitly rather than explicitly (solved for one variable in terms of the other).

We begin by reviewing the Chain Rule. Let f and g be functions of x. Then

$$\frac{d}{dx}\Big(f(g(x))\Big) = f'(g(x)) \cdot g'(x). \tag{3.5.1}$$

Suppose now that $y = g(x)$. We can rewrite the above as

$$\frac{d}{dx}\Big(f(y)\Big) = f'(y) \cdot y', \quad \text{or} \quad \frac{d}{dx}\Big(f(y)\Big) = f'(y) \cdot \frac{dy}{dx}. \tag{3.5.2}$$

These equations look strange; the key concept to learn here is that we can find y' even if we don't exactly know how y and x relate.

We demonstrate this process in the following example.

Example 67: Using Implicit Differentiation

Find y' given that $\sin(y) + y^3 = 6 - x^3$.

Solution

We start by taking the derivative of both sides (thus maintaining the equality.) We have :

$$\frac{d}{dx}\Big(\sin(y) + y^3\Big) = \frac{d}{dx}\Big(6 - x^3\Big). \tag{3.5.3}$$

The right hand side is easy; it returns $-3x^2$.

The left hand side requires more consideration. We take the derivative term--by--term. Using the technique derived from Equation 2.1 above, we can see that

$$\frac{d}{dx}\Big(\sin y\Big) = \cos y \cdot y'. \tag{3.5.4}$$

We apply the same process to the y^3 term.

$$\frac{d}{dx}\left(y^3\right) = \frac{d}{dx}\left((y)^3\right) = 3(y)^2 \cdot y'. \tag{3.5.5}$$

Putting this together with the right hand side, we have

$$\cos(y)y' + 3y^2y' = -3x^2. \tag{3.5.6}$$

Now solve for y'.

$$\begin{aligned} \cos(y)y' + 3y^2y' &= -3x^2. \\ \left(\cos y + 3y^2\right)y' &= -3x^2 \\ y' &= \frac{-3x^2}{\cos y + 3y^2} \end{aligned}$$

This equation for y' probably seems unusual for it contains both x and y terms. How is it to be used? We'll address that next.

Implicit functions are generally harder to deal with than explicit functions. With an explicit function, given an x value, we have an explicit formula for computing the corresponding y value. With an implicit function, one often has to find x and y values *at the same time* that satisfy the equation. It is much easier to demonstrate that a given point satisfies the equation than to actually find such a point.

For instance, we can affirm easily that the point $(\sqrt[3]{6}, 0)$ lies on the graph of the implicit function $\sin y + y^3 = 6 - x^3$. Plugging in 0 for y, we see the left hand side is 0. Setting $x = \sqrt[3]{6}$, we see the right hand side is also 0; the equation is satisfied. The following example finds the equation of the tangent line to this function at this point.

Example 68: Using Implicit Differentiation to find a tangent line

Find the equation of the line tangent to the curve of the implicitly defined function $\sin y + y^3 = 6 - x^3$ at the point $(\sqrt[3]{6}, 0)$.

Solution

In Example 67 we found that

$$y' = \frac{-3x^2}{\cos y + 3y^2}. \tag{3.5.7}$$

We find the slope of the tangent line at the point $(\sqrt[3]{6}, 0)$ by substituting $\sqrt[3]{6}$ for x and 0 for y. Thus at the point $(\sqrt[3]{6}, 0)$, we have the slope as

$$y' = \frac{-3(\sqrt[3]{6})^2}{\cos 0 + 3 \cdot 0^2} = \frac{-3\sqrt[3]{36}}{1} \approx -9.91. \tag{3.5.8}$$

Therefore the equation of the tangent line to the implicitly defined function $\sin y + y^3 = 6 - x^3$ at the point $(\sqrt[3]{6}, 0)$ is

$$y = -3\sqrt[3]{36}(x - \sqrt[3]{6}) + 0 \approx -9.91x + 18. \tag{3.5.9}$$

The curve and this tangent line are shown in Figure 2.20.

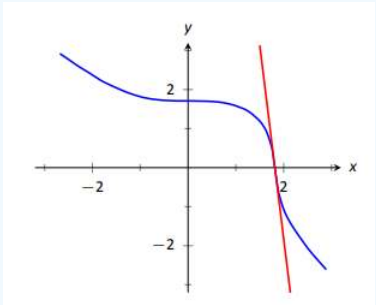


Figure 2.20: The function $\sin y + y^3 = 6 - x^2$ and its tangent line at the point $(\sqrt[3]{6}, 0)$.

This suggests a general method for implicit differentiation. For the steps below assume y is a function of x.

1. Take the derivative of each term in the equation. Treat the x terms like normal. When taking the derivatives of y terms, the usual rules apply except that, because of the Chain Rule, we need to multiply each term by y'.
2. Get all the y' terms on one side of the equal sign and put the remaining terms on the other side.
3. Factor out y'; solve for y' by dividing.

Practical Note: When working by hand, it may be beneficial to use the symbol $\frac{dy}{dx}$ instead of y', as the latter can be easily confused for y or y^1.

Example 69: Using Implicit Differentiation

Given the implicitly defined function $y^3 + x^2y^4 = 1 + 2x$, find y'.

Solution

We will take the implicit derivatives term by term. The derivative of y^3 is $3y^2y'$.

The second term, x^2y^4, is a little tricky. It requires the Product Rule as it is the product of two functions of x: x^2 and y^4. Its derivative is $x^2(4y^3y') + 2xy^4$. The first part of this expression requires a y' because we are taking the derivative of a y term. The second part does not require it because we are taking the derivative of x^2.

The derivative of the right hand side is easily found to be 2. In all, we get:

$$3y^2y' + 4x^2y^3y' + 2xy^4 = 2. \tag{3.5.10}$$

Move terms around so that the left side consists only of the y' terms and the right side consists of all the other terms:

$$3y^2y' + 4x^2y^3y' = 2 - 2xy^4. \tag{3.5.11}$$

Factor out y' from the left side and solve to get

$$y' = \frac{2 - 2xy^4}{3y^2 + 4x^2y^3}. \tag{3.5.12}$$

To confirm the validity of our work, let's find the equation of a tangent line to this function at a point. It is easy to confirm that the point $(0, 1)$ lies on the graph of this function. At this point, $y' = 2/3$. So the equation of the tangent line is $y = 2/3(x - 0) + 1$. The function and its tangent line are graphed in Figure 2.21.

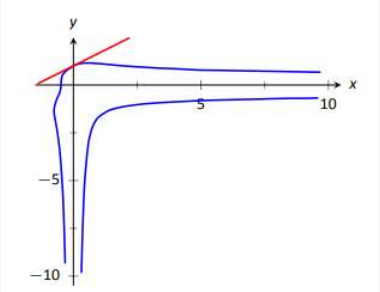


Figure 2.21: A graph of the implicitly defined function $y^3 + x^2y^4 = 1 + 2x$ along with its tangent line at the point $(0, 1)$.

Notice how our function looks much different than other functions we have seen. For one, it fails the vertical line test. Such functions are important in many areas of mathematics, so developing tools to deal with them is also important.

Example 70: Using Implicit Differentiation

Given the implicitly defined function $\sin(x^2y^2) + y^3 = x + y$, find y'.

Solution

Differentiating term by term, we find the most difficulty in the first term. It requires both the Chain and Product Rules.

$$\begin{aligned}\frac{d}{dx}\left(\sin(x^2y^2)\right) &= \cos(x^2y^2) \cdot \frac{d}{dx}\left(x^2y^2\right) \\ &= \cos(x^2y^2) \cdot \left(x^2(2yy') + 2xy^2\right) \\ &= 2(x^2yy' + xy^2)\cos(x^2y^2).\end{aligned}$$

We leave the derivatives of the other terms to the reader. After taking the derivatives of both sides, we have

$$2(x^2yy' + xy^2)\cos(x^2y^2) + 3y^2y' = 1 + y'. \tag{3.5.13}$$

We now have to be careful to properly solve for y', particularly because of the product on the left. It is best to multiply out the product. Doing this, we get

$$2x^2y\cos(x^2y^2)y' + 2xy^2\cos(x^2y^2) + 3y^2y' = 1 + y'. \tag{3.5.14}$$

From here we can safely move around terms to get the following:

$$2x^2y\cos(x^2y^2)y' + 3y^2y' - y' = 1 - 2xy^2\cos(x^2y^2). \tag{3.5.15}$$

Then we can solve for y' to get

$$y' = \frac{1 - 2xy^2\cos(x^2y^2)}{2x^2y\cos(x^2y^2) + 3y^2 - 1}. \tag{3.5.16}$$

A graph of this implicit function is given in Figure 2.22. It is easy to verify that the points $(0,0)$, $(0,1)$ and $(0,-1)$ all lie on the graph. We can find the slopes of the tangent lines at each of these points using our formula for y'.

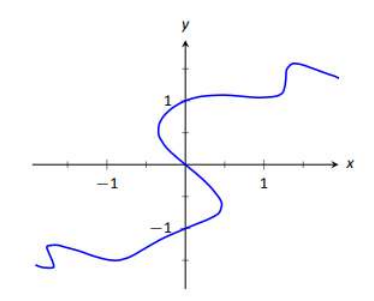


Figure 2.22: A graph of the implicitly defined function $\sin(x^2y^2) + y^3 = x + y$.

At $(0,0)$, the slope is -1.

At $(0,1)$, the slope is $1/2$.

At $(0,-1)$, the slope is also $1/2$.

The tangent lines have been added to the graph of the function in Figure 2.23.

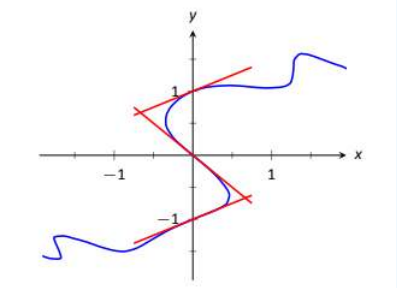


Figure 2.23: A graph of the implicitly defined function $\sin(x^2y^2) + y^3 = x + y$ and certain tangent lines.

Quite a few "famous" curves have equations that are given implicitly. We can use implicit differentiation to find the slope at various points on those curves. We investigate two such curves in the next examples.

Example 71: Finding slopes of tangent lines to a circle

Find the slope of the tangent line to the circle $x^2 + y^2 = 1$ at the point $(1/2, \sqrt{3}/2)$.

Solution

Taking derivatives, we get $2x + 2yy' = 0$. Solving for y' gives:

$$y' = \frac{-x}{y}. \tag{3.5.17}$$

This is a clever formula. Recall that the slope of the line through the origin and the point (x, y) on the circle will be y/x. We have found that the slope of the tangent line to the circle at that point is the opposite reciprocal of y/x, namely, $-x/y$. Hence these two lines are always perpendicular.

At the point $(1/2, \sqrt{3}/2)$, we have the tangent line's slope as

$$y' = \frac{-1/2}{\sqrt{3}/2} = \frac{-1}{\sqrt{3}} \approx -0.577. \tag{3.5.18}$$

A graph of the circle and its tangent line at $(1/2, \sqrt{3}/2)$ is given in Figure 2.24, along with a thin dashed line from the origin that is perpendicular to the tangent line. (It turns out that all normal lines to a circle pass through the center of the circle.)

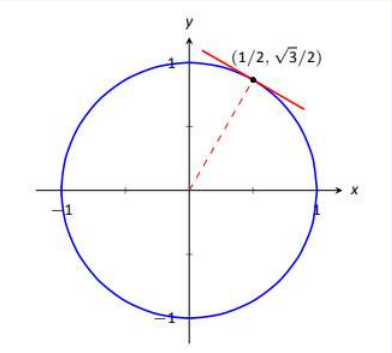


Figure 2.24: The unit circle with its tangent line at $(1/2, \sqrt{3}/2)$.

This section has shown how to find the derivatives of implicitly defined functions, whose graphs include a wide variety of interesting and unusual shapes. Implicit differentiation can also be used to further our understanding of "regular" differentiation.

One hole in our current understanding of derivatives is this: what is the derivative of the square root function? That is,

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{1/2}\right) = ? \tag{3.5.19}$$

We allude to a possible solution, as we can write the square root function as a power function with a rational (or, fractional) power. We are then tempted to apply the Power Rule and obtain

$$\frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}. \tag{3.5.20}$$

The trouble with this is that the Power Rule was initially defined only for positive integer powers, $n > 0$. While we did not justify this at the time, generally the Power Rule is proved using something called the Binomial Theorem, which deals only with positive integers. The Quotient Rule allowed us to extend the Power Rule to negative integer powers. Implicit Differentiation allows us to extend the Power Rule to rational powers, as shown below.

Let $y = x^{m/n}$, where m and n are integers with no common factors (so $m = 2$ and $n = 5$ is fine, but $m = 2$ and $n = 4$ is not). We can rewrite this explicit function implicitly as $y^n = x^m$. Now apply implicit differentiation.

$$
\begin{aligned}
y &= x^{m/n} \\
y^n &= x^m \\
\frac{d}{dx}\left(y^n\right) &= \frac{d}{dx}\left(x^m\right) \\
n \cdot y^{n-1} \cdot y' &= m \cdot x^{m-1} \\
y' &= \frac{m}{n}\frac{x^{m-1}}{y^{n-1}} \quad \text{(now substitute } x^{m/n} \text{ for } y\text{)} \\
&= \frac{m}{n}\frac{x^{m-1}}{(x^{m/n})^{n-1}} \quad \text{(apply lots of algebra)} \\
&= \frac{m}{n}x^{(m-n)/n} \\
&= \frac{m}{n}x^{m/n-1}.
\end{aligned}
$$

The above derivation is the key to the proof extending the Power Rule to rational powers. Using limits, we can extend this once more to include *all* powers, including irrational (even transcendental!) powers, giving the following theorem.

Theorem 21: Power Rule for Differentiation

Let $f(x) = x^n$, where $n \neq 0$ is a real number. Then f is a differentiable function, and $f'(x) = n \cdot x^{n-1}$.

This theorem allows us to say the derivative of x^π is $\pi x^{\pi-1}$.

We now apply this final version of the Power Rule in the next example, the second investigation of a "famous" curve.

Example 72: Using the Power Rule

Find the slope of $x^{2/3} + y^{2/3} = 8$ at the point $(8, 8)$.

Solution

This is a particularly interesting curve called an *astroid*. It is the shape traced out by a point on the edge of a circle that is rolling around inside of a larger circle, as shown in Figure 2.25.

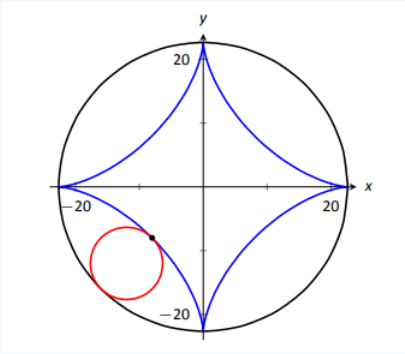


Figure 2.25: An astroid, traced out by a point on the smaller circle as it rolls inside the larger circle.

To find the slope of the astroid at the point $(8, 8)$, we take the derivative implicitly.

$$
\begin{aligned}
\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' &= 0 \\
\frac{2}{3}y^{-1/3}y' &= -\frac{2}{3}x^{-1/3} \\
y' &= -\frac{x^{-1/3}}{y^{-1/3}} \\
y' &= -\frac{y^{1/3}}{x^{1/3}} = -\sqrt[3]{\frac{y}{x}}.
\end{aligned}
$$

Plugging in $x = 8$ and $y = 8$, we get a slope of -1. The astroid, with its tangent line at $(8, 8)$, is shown in Figure 2.26.

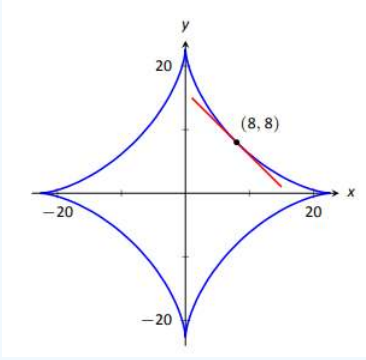


Figure 2.26: An astroid with a tangent line.

Implicit Differentiation and the Second Derivative

We can use implicit differentiation to find higher order derivatives. In theory, this is simple: first find $\frac{dy}{dx}$, then take its derivative with respect to x. In practice, it is not hard, but it often requires a bit of algebra. We demonstrate this in an example.

Example 73: Finding the second derivative

Given $x^2+y^2=1$, find $\frac{d^2y}{dx^2}=y''$.

Solution

We found that $y'=\frac{dy}{dx}=-x/y$ in Example 71. To find y'', we apply implicit differentiation to y'.

$$\begin{aligned} y'' &= \frac{d}{dx}(y') \\ &= \frac{d}{dx}\left(-\frac{x}{y}\right) \qquad \text{(Now use the Quotient Rule.)} \\ &= -\frac{y(1)-x(y')}{y^2} \end{aligned}$$

replace y' with $-x/y$:

$$\begin{aligned} &= -\frac{y-x(-x/y)}{y^2} \\ &= -\frac{y+x^2/y}{y^2}. \end{aligned}$$

While this is not a particularly simple expression, it is usable. We can see that $y''>0$ when $y<0$ and $y''<0$ when $y>0$. In Section 3.4, we will see how this relates to the shape of the graph.

Logarithmic Differentiation

Consider the function $y=x^x$; it is graphed in Figure 2.27. It is well--defined for $x>0$ and we might be interested in finding equations of lines tangent and normal to its graph. How do we take its derivative?

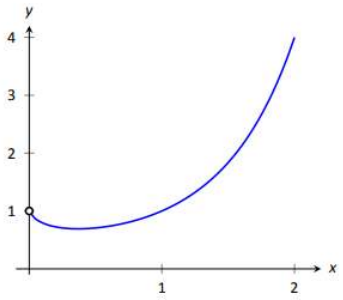


Figure 2.27: A plot of $y = x^x$.

The function is not a power function: it has a "power" of x, not a constant. It is not an exponential function: it has a "base" of x, not a constant.

A differentiation technique known as *logarithmic differentiation* becomes useful here. The basic principle is this: take the natural log of both sides of an equation $y = f(x)$, then use implicit differentiation to find y'. We demonstrate this in the following example.

Example 74: Using Logarithmic Differentiation

Given $y = x^x$, use logarithmic differentiation to find y'.

Solution

As suggested above, we start by taking the natural log of both sides then applying implicit differentiation.

$$\begin{aligned} y &= x^x \\ \ln(y) &= \ln(x^x) \text{(apply logarithm rule)} \\ \ln(y) &= x\ln x \text{(now use implicit differentiation)} \\ \frac{d}{dx}\Big(\ln(y)\Big) &= \frac{d}{dx}\Big(x\ln x\Big) \\ \frac{y'}{y} &= \ln x + x\cdot\frac{1}{x} \\ \frac{y'}{y} &= \ln x + 1 \\ y' &= y\big(\ln x + 1\big) \text{(substitute } y = x^x\text{)} \\ y' &= x^x\big(\ln x + 1\big). \end{aligned}$$

To "test" our answer, let's use it to find the equation of the tangent line at $x = 1.5$. The point on the graph our tangent line must pass through is $(1.5, 1.5^{1.5}) \approx (1.5, 1.837)$. Using the equation for y', we find the slope as

$$y' = 1.5^{1.5}\big(\ln 1.5 + 1\big) \approx 1.837(1.405) \approx 2.582. \tag{3.5.21}$$

Thus the equation of the tangent line is $y = 1.6833(x - 1.5) + 1.837$. Figure 2.28 graphs $y = x^x$ along with this tangent line.

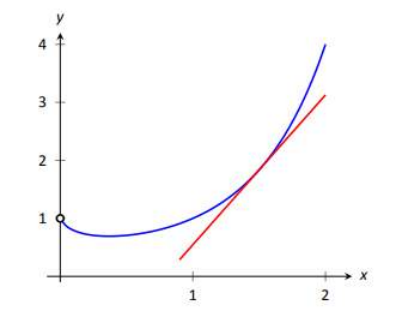


Figure 2.22: A graph of $y = x^x$ and its tangent line at $x = 1.5$.

Implicit differentiation proves to be useful as it allows us to find the instantaneous rates of change of a variety of functions. In particular, it extended the Power Rule to rational exponents, which we then extended to all real numbers. In the next section, implicit differentiation will be used to find the derivatives of *inverse* functions, such as $y = \sin^{-1} x$.

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3.6: Derivatives of Logarithmic Functions

As with the sine, we do not know anything about derivatives that allows us to compute the derivatives of the exponential and logarithmic functions without going back to basics. Let's do a little work with the definition again:

$$\begin{aligned}\frac{d}{dx}a^x &= \lim_{\Delta x\to 0}\frac{a^{x+\Delta x}-a^x}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{a^x a^{\Delta x}-a^x}{\Delta x}\\ &= \lim_{\Delta x\to 0}a^x\frac{a^{\Delta x}-1}{\Delta x}\\ &= a^x\lim_{\Delta x\to 0}\frac{a^{\Delta x}-1}{\Delta x}.\end{aligned}$$

There are two interesting things to note here: As in the case of the sine function we are left with a limit that involves Δx but not x, which means that whatever $\lim_{\Delta x\to 0}(a^{\Delta x}-1)/\Delta x$ is, we know that it is a number, that is, a constant. This means that a^x has a remarkable property: its derivative is a constant times itself.

We earlier remarked that the hardest limit we would compute is $\lim_{x\to 0}\sin x/x = 1$; we now have a limit that is just a bit too hard to include here. In fact the hard part is to see that $\lim_{\Delta x\to 0}(a^{\Delta x}-1)/\Delta x$ even exists---does this fraction really get closer and closer to some fixed value? Yes it does, but we will not prove this fact.

We can look at some examples. Consider $(2^x-1)/x$ for some small values of x: 1, 0.828427124, 0.756828460, 0.724061864, 0.70838051, $0.70070877$when x is 1, $1/2$, $1/4$, $1/8$, $1/16$, $1/32$, respectively. It looks like this is settling in around 0.7, which turns out to be true (but the limit is not exactly 0.7). Consider next $(3^x-1)/x$: 2, 1.464101616, 1.264296052, 1.177621520, 1.13720773, 1.11768854, at the same values of x. It turns out to be true that in the limit this is about 1.1.

Two examples don't establish a pattern, but if you do more examples you will find that the limit varies directly with the value of a: bigger a, bigger limit; smaller a, smaller limit. As we can already see, some of these limits will be less than 1 and some larger than 1. Somewhere between $a=2$ and $a=3$ the limit will be exactly 1; the value at which this happens is called e, so that

$$\lim_{\Delta x\to 0}\frac{e^{\Delta x}-1}{\Delta x} = 1.$$

As you might guess from our two examples, e is closer to 3 than to 2, and in fact $e\approx 2.718$.

Now we see that the function e^x has a truly remarkable property:

$$\begin{aligned}\frac{d}{dx}e^x &= \lim_{\Delta x\to 0}\frac{e^{x+\Delta x}-e^x}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{e^x e^{\Delta x}-e^x}{\Delta x}\\ &= \lim_{\Delta x\to 0}e^x\frac{e^{\Delta x}-1}{\Delta x}\\ &= e^x\lim_{\Delta x\to 0}\frac{e^{\Delta x}-1}{\Delta x}\\ &= e^x.\end{aligned}$$

That is, e^x is its own derivative, or in other words the slope of e^x is the same as its height, or the same as its second coordinate: The function $f(x)=e^x$ goes through the point (z,e^z) and has slope e^z there, no matter what z is. It is sometimes convenient to express the function e^x without an exponent, since complicated exponents can be hard to read. In such cases we use $\exp(x)$, e.g., $\exp(1+x^2)$ instead of e^{1+x^2}.

What about the logarithm function? This too is hard, but as the cosine function was easier to do once the sine was done, so the logarithm is easier to do now that we know the derivative of the exponential function. Let's start with $\log_e x$, which as you probably know is often abbreviated $\ln x$ and called the "natural logarithm" function.

Consider the relationship between the two functions, namely, that they are inverses, that one "undoes" the other. Graphically this means that they have the same graph except that one is "flipped" or "reflected" through the line $y = x$, as shown in Figure 3.6.1.

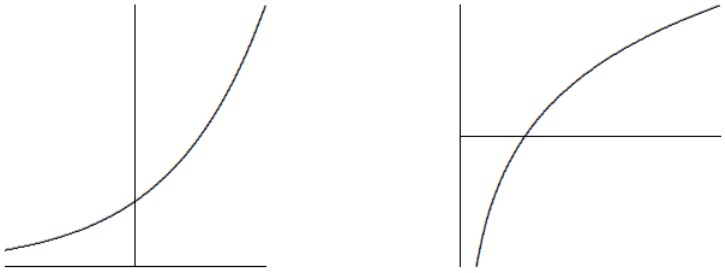


Figure 3.6.1: *The exponential (green) and logarithmic (blue) functions. As inverses of each other, their graphs are reflections of each other across the line $y = x$ (dashed).*

This means that the slopes of these two functions are closely related as well: For example, the slope of e^x is e at $x = 1$; at the corresponding point on the $\ln(x)$ curve, the slope must be $1/e$, because the "rise" and the "run" have been interchanged. Since the slope of e^x is e at the point $(1, e)$, the slope of $\ln(x)$ is $1/e$ at the point $(e, 1)$.

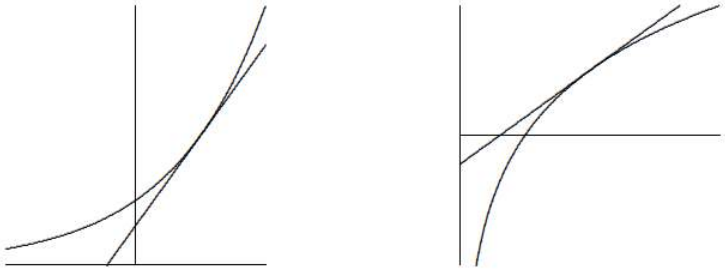


Figure 3.6.2: *The exponential (green) and logarithmic (blue) functions. The dashed lines indicate the slope of the respective functions at the points $(1, e)$ and $(e, 1)$. It is interesting to note that these lines intersect at the origin.*

More generally, we know that the slope of e^x is e^z at the point (z, e^z), so the slope of $\ln(x)$ is $1/e^z$ at (e^z, z), as indicated in Figure 3.6.2. In other words, the slope of $\ln x$ is the reciprocal of the first coordinate at any point; this means that the slope of $\ln x$ at $(x, \ln x)$ is $1/x$. The upshot is: $\frac{d}{dx} \ln x = \frac{1}{x}$. We have discussed this from the point of view of the graphs, which is easy to understand but is not normally considered a rigorous proof---it is too easy to be led astray by pictures that seem reasonable but that miss some hard point. It is possible to do this derivation without resorting to pictures, and indeed we will see an alternate approach soon.

Note that $\ln x$ is defined only for $x > 0$. It is sometimes useful to consider the function $\ln|x|$, a function defined for $x \neq 0$. When $x < 0$, $\ln|x| = \ln(-x)$ and

$$\frac{d}{dx} \ln|x| = \frac{d}{dx} \ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}.$$

Thus whether x is positive or negative, the derivative is the same.

What about the functions a^x and $\log_a x$? We know that the derivative of a^x is some constant times a^x itself, but what constant? Remember that "the logarithm is the exponent" and you will see that $a = e^{\ln a}$. Then $a^x = (e^{\ln a})^x = e^{x \ln a}$, and we can compute the derivative using the chain rule:

$$\frac{d}{dx} a^x = \frac{d}{dx} (e^{\ln a})^x = \frac{d}{dx} e^{x \ln a} = (\ln a) e^{x \ln a} = (\ln a) a^x.$$

The constant is simply $\ln a$. Likewise we can compute the derivative of the logarithm function $\log_a x$. Since $x = e^{\ln x}$ we can take the logarithm base a of both sides to get $\log_a(x) = \log_a(e^{\ln x}) = \ln x \log_a e$. Then

$$\frac{d}{dx} \log_a x = \frac{1}{x} \log_a e.$$

This is a perfectly good answer, but we can improve it slightly. Since

$$\begin{aligned} a &= e^{\ln a} \\ \log_a(a) &= \log_a(e^{\ln a}) = \ln a \log_a e \\ 1 &= \ln a \log_a e \\ \frac{1}{\ln a} &= \log_a e, \end{aligned}$$

we can replace $\log_a e$ to get $\frac{d}{dx}\log_a x = \frac{1}{x \ln a}$.

You may if you wish memorize the formulas

$$\frac{d}{dx}a^x = (\ln a)a^x \quad \text{and} \quad \frac{d}{dx}\log_a x = \frac{1}{x \ln a}.$$

Because the "trick" $a = e^{\ln a}$ is often useful, and sometimes essential, it may be better to remember the trick, not the formula.

Example 3.6.1

Compute the derivative of $f(x) = 2^x$.

Solution

$$\begin{aligned} \frac{d}{dx}2^x &= \frac{d}{dx}(e^{\ln 2})^x \\ &= \frac{d}{dx}e^{x \ln 2} \\ &= \left(\frac{d}{dx}x \ln 2\right) e^{x \ln 2} \\ &= (\ln 2)e^{x \ln 2} = 2^x \ln 2 \end{aligned}$$

Example 3.6.2

Compute the derivative of $f(x) = 2^{x^2} = 2^{(x^2)}$.

$$\begin{aligned} \frac{d}{dx}2^{x^2} &= \frac{d}{dx}e^{x^2 \ln 2} \\ &= \left(\frac{d}{dx}x^2 \ln 2\right) e^{x^2 \ln 2} \\ &= (2 \ln 2)xe^{x^2 \ln 2} \\ &= (2 \ln 2)x2^{x^2} \end{aligned}$$

Example 3.6.3

Compute the derivative of $f(x) = x^x$. At first this appears to be a new kind of function: it is not a constant power of x, and it does not seem to be an exponential function, since the base is not constant. But in fact it is no harder than the previous example.

$$\begin{aligned} \frac{d}{dx}x^x &= \frac{d}{dx}e^{x \ln x} \\ &= \left(\frac{d}{dx}x \ln x\right) e^{x \ln x} \\ &= (x\frac{1}{x} + \ln x)x^x \\ &= (1 + \ln x)x^x \end{aligned}$$

Example 3.6.4

Recall that we have not justified the power rule except when the exponent is a positive or negative integer. We can use the exponential function to take care of other exponents.

$$\begin{aligned}\frac{d}{dx}x^r &= \frac{d}{dx}e^{r\ln x}\\ &= \left(\frac{d}{dx}r\ln x\right)e^{r\ln x}\\ &= (r\frac{1}{x})x^r\\ &= rx^{r-1}\end{aligned}$$

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- Integrated by Justin Marshall.

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3.7: Rates of Change in the Natural and Social Sciences

Learning Objectives

- Determine a new value of a quantity from the old value and the amount of change.
- Calculate the average rate of change and explain how it differs from the instantaneous rate of change.
- Apply rates of change to displacement, velocity, and acceleration of an object moving along a straight line.
- Predict the future population from the present value and the population growth rate.
- Use derivatives to calculate marginal cost and revenue in a business situation.

In this section we look at some applications of the derivative by focusing on the interpretation of the derivative as the rate of change of a function. These applications include **acceleration** and velocity in physics, **population growth rates** in biology, and marginal functions in economics.

Amount of Change Formula

One application for derivatives is to estimate an unknown value of a function at a point by using a known value of a function at some given point together with its rate of change at the given point. If $f(x)$ is a function defined on an interval $[a, a+h]$, then the **amount of change** of $f(x)$ over the interval is the change in the y values of the function over that interval and is given by

$$f(a+h)-f(a).$$

The **average rate of change** of the function f over that same interval is the ratio of the amount of change over that interval to the corresponding change in the x values. It is given by

$$\frac{f(a+h)-f(a)}{h}.$$

As we already know, the instantaneous rate of change of $f(x)$ at a is its derivative

$$f\,'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

For small enough values of h, $f\,'(a)\approx\frac{f(a+h)-f(a)}{h}$. We can then solve for $f(a+h)$ to get the amount of change formula:

$$f(a+h)\approx f(a)+f\,'(a)h. \tag{3.7.1}$$

We can use this formula if we know only $f(a)$and $f\,'(a)$ and wish to estimate the value of $f(a+h)$. For example, we may use the current population of a city and the rate at which it is growing to estimate its population in the near future. As we can see in Figure 3.7.1, we are approximating $f(a+h)$ by the y coordinate at a+h on the line tangent to $f(x)$ at $x=a$. Observe that the accuracy of this estimate depends on the value of h as well as the value of $f\,'(a)$.

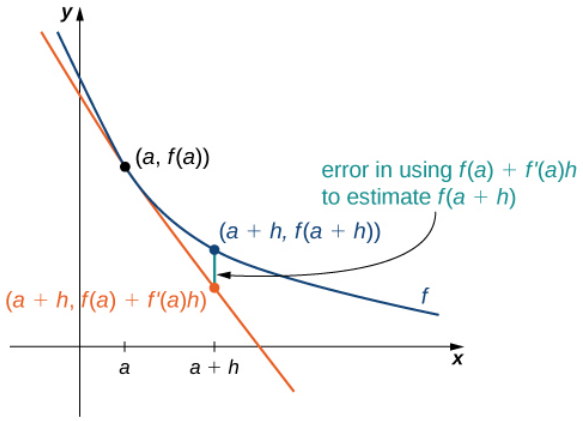


Figure 3.7.1: The new value of a changed quantity equals the original value plus the rate of change times the interval of change: $f(a+h)\approx f(a)+f\,'(a)h$.

Example 3.7.1: Estimating the Value of a Function

If $f(3) = 2$ and $f'(3) = 5$, estimate $f(3.2)$.

Solution

Begin by finding h. We have $h = 3.2 - 3 = 0.2$. Thus,

$$f(3.2) = f(3 + 0.2) \approx f(3) + (0.2)f'(3) = 2 + 0.2(5) = 3.$$

Exercise 3.7.1

Given $f(10) = -5$ and $f'(10) = 6$, estimate $f(10.1)$.

Hint

Use the same process as in the preceding example.

Answer

-4.4

Motion along a Line

Another use for the derivative is to analyze motion along a line. We have described velocity as the rate of change of position. If we take the derivative of the velocity, we can find the acceleration, or the rate of change of velocity. It is also important to introduce the idea of **speed**, which is the magnitude of velocity. Thus, we can state the following mathematical definitions.

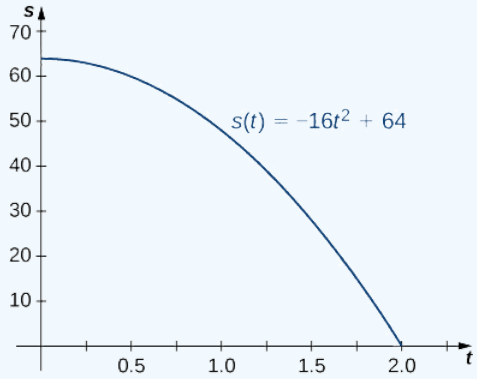
Definition

Let $s(t)$ be a function giving the position of an object at time t.

- The velocity of the object at time t is given by $v(t) = s'(t)$.
- The speed of the object at time t is given by $|v(t)|$.
- The acceleration of the object at t is given by $a(t) = v'(t) = s''(t)$.

Example 3.7.2: Comparing Instantaneous Velocity and Average Velocity

A ball is dropped from a height of 64 feet. Its height above ground (in feet) t seconds later is given by $s(t) = -16t^2 + 64$.



a. What is the instantaneous velocity of the ball when it hits the ground?
b. What is the average velocity during its fall?

Solution

The first thing to do is determine how long it takes the ball to reach the ground. To do this, set $s(t) = 0$. Solving $-16t^2 + 64 = 0$, we get $t = 2$, so it takes 2 seconds for the ball to reach the ground.

a. The instantaneous velocity of the ball as it strikes the ground is $v(2)$. Since $v(t) = s'(t) = -32t$, we obtain $v(t) = -64$ ft/s.

b. The average velocity of the ball during its fall is

$$v_{ave} = \frac{s(2)-s(0)}{2-0} = \frac{0-64}{2} = -32 \text{ ft/s.}$$

Example 3.7.3: Interpreting the Relationship between $v(t)$ and $a(t)$

A particle moves along a coordinate axis in the positive direction to the right. Its position at time t is given by $s(t) = t^3 - 4t + 2$. Find $v(1)$ and $a(1)$ and use these values to answer the following questions.

a. Is the particle moving from left to right or from right to left at time $t = 1$?

b. Is the particle speeding up or slowing down at time $t = 1$?

Solution

Begin by finding $v(t)$ and $a(t)$.

$v(t) = s'(t) = 3t^2 - 4$ and $a(t) = v'(t) = s''(t) = 6t$.

Evaluating these functions at $t = 1$, we obtain $v(1) = -1$ and $a(1) = 6$.

a. Because $v(1) < 0$, the particle is moving from right to left.

b. Because $v(1) < 0$ and $a(1) > 0$, velocity and acceleration are acting in opposite directions. In other words, the particle is being accelerated in the direction opposite the direction in which it is traveling, causing $|v(t)|$ to decrease. The particle is slowing down.

Example 3.7.4: Position and Velocity

The position of a particle moving along a coordinate axis is given by $s(t) = t^3 - 9t^2 + 24t + 4,\ t \geq 0$.

a. Find $v(t)$.

b. At what time(s) is the particle at rest?

c. On what time intervals is the particle moving from left to right? From right to left?

d. Use the information obtained to sketch the path of the particle along a coordinate axis.

Solution

a. The velocity is the derivative of the position function:

$$v(t) = s'(t) = 3t^2 - 18t + 24.$$

b. The particle is at rest when $v(t) = 0$, so set $3t^2 - 18t + 24 = 0$. Factoring the left-hand side of the equation produces $3(t-2)(t-4) = 0$. Solving, we find that the particle is at rest at $t = 2$ and $t = 4$.

c. The particle is moving from left to right when $v(t) > 0$ and from right to left when $v(t) < 0$. Figure 3.7.2 gives the analysis of the sign of $v(t)$ for $t \geq 0$, but it does not represent the axis along which the particle is moving.



Figure 3.7.2: The sign of $v(t)$ determines the direction of the particle.

- Since $3t^2 - 18t + 24 > 0$ on $[0, 2) \cup (4, +\infty)$, the particle is moving from left to right on these intervals.
- Since $3t^2 - 18t + 24 < 0$ on $(2, 4)$, the particle is moving from right to left on this interval.

d. Before we can sketch the graph of the particle, we need to know its position at the time it starts moving $(t = 0)$ and at the times that it changes direction $(t = 2, 4)$. We have $s(0) = 4$, $s(2) = 24$, and $s(4) = 20$. This means that the particle begins on the coordinate axis at 4 and changes direction at 24 and 20 on the coordinate axis. The path of the particle is shown on a coordinate axis in Figure 3.7.3.

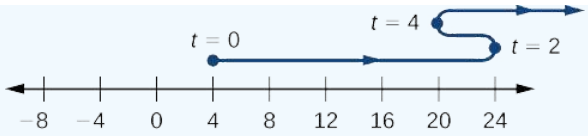


Figure 3.7.3: The path of the particle can be determined by analyzing $v(t)$.

Exercise 3.7.2

A particle moves along a coordinate axis. Its position at time t is given by $s(t) = t^2 - 5t + 1$. Is the particle moving from right to left or from left to right at time $t = 3$?

Hint

Find $v(3)$ and look at the sign.

Answer

left to right

Population Change

In addition to analyzing velocity, speed, acceleration, and position, we can use derivatives to analyze various types of populations, including those as diverse as bacteria colonies and cities. We can use a current population, together with a growth rate, to estimate the size of a population in the future. The population growth rate is the rate of change of a population and consequently can be represented by the derivative of the size of the population.

Definition

If $P(t)$ is the number of entities present in a population, then the population growth rate of $P(t)$ is defined to be $P'(t)$.

Example 3.7.5: Estimating a Population

The population of a city is tripling every 5 years. If its current population is 10,000, what will be its approximate population 2 years from now?

Solution

Let $P(t)$ be the population (in thousands) t years from now. Thus, we know that $P(0) = 10$ and based on the information, we anticipate $P(5) = 30$. Now estimate $P'(0)$, the current growth rate, using

$$P'(0) \approx \frac{P(5) - P(0)}{5 - 0} = \frac{30 - 10}{5} = 4 \, .$$

By applying Equation 3.7.1 to $P(t)$, we can estimate the population 2 years from now by writing

$$P(2) \approx P(0) + (2)P'(0) \approx 10 + 2(4) = 18 \, ;$$

thus, in 2 years the population will be 18,000.

Exercise 3.7.3

The current population of a mosquito colony is known to be 3,000; that is, $P(0) = 3,000$. If $P'(0) = 100$, estimate the size of the population in 3 days, where t is measured in days.

Hint

Use $P(3) \approx P(0) + 3P'(0)$

Answer

3,300

Changes in Cost and Revenue

In addition to analyzing motion along a line and population growth, derivatives are useful in analyzing changes in cost, revenue, and profit. The concept of a marginal function is common in the fields of business and economics and implies the use of derivatives. The marginal cost is the derivative of the cost function. The marginal revenue is the derivative of the revenue function. The marginal profit is the derivative of the profit function, which is based on the cost function and the revenue function.

Definition

- If $C(x)$ is the cost of producing x items, then the **marginal cost** $MC(x)$ is $MC(x) = C'(x)$.
- If $R(x)$ is the revenue obtained from selling x items, then the marginal revenue $MR(x)$ is $MR(x) = R'(x)$.
- If $P(x) = R(x) - C(x)$ is the profit obtained from selling x items, then the **marginal profit** $MP(x)$ is defined to be $MP(x) = P'(x) = MR(x) - MC(x) = R'(x) - C'(x)$.

We can roughly approximate

$$MC(x) = C'(x) = \lim_{h \to 0} \frac{C(x+h) - C(x)}{h}$$

by choosing an appropriate value for h. Since x represents objects, a reasonable and small value for h is 1. Thus, by substituting $h = 1$, we get the approximation $MC(x) = C'(x) \approx C(x+1) - C(x)$. Consequently, $C'(x)$ for a given value of x can be thought of as the change in cost associated with producing one additional item. In a similar way, $MR(x) = R'(x)$ approximates the revenue obtained by selling one additional item, and $MP(x) = P'(x)$ approximates the profit obtained by producing and selling one additional item.

Example 3.7.6: Applying Marginal Revenue

Assume that the number of barbeque dinners that can be sold, x, can be related to the price charged, p, by the equation $p(x) = 9 - 0.03x, 0 \le x \le 300$.

In this case, the revenue in dollars obtained by selling x barbeque dinners is given by

$$R(x) = xp(x) = x(9 - 0.03x) = -0.03x^2 + 9x \quad \text{for } 0 \le x \le 300\,.$$

Use the marginal revenue function to estimate the revenue obtained from selling the 101st barbeque dinner. Compare this to the actual revenue obtained from the sale of this dinner.

Solution

First, find the marginal revenue function: $MR(x) = R'(x) = -0.06x + 9$.

Next, use $R'(100)$ to approximate $R(101) - R(100)$, the revenue obtained from the sale of the 101st dinner. Since $R'(100) = 3$, the revenue obtained from the sale of the 101st dinner is approximately \$3.

The actual revenue obtained from the sale of the 101st dinner is

$$R(101) - R(100) = 602.97 - 600 = 2.97, \text{ or } \$2.97.$$

The marginal revenue is a fairly good estimate in this case and has the advantage of being easy to compute.

Exercise 3.7.4

Suppose that the profit obtained from the sale of x fish-fry dinners is given by $P(x) = -0.03x^2 + 8x - 50$. Use the marginal profit function to estimate the profit from the sale of the 101st fish-fry dinner.

Hint

Use $P'(100)$ to approximate $P(101) - P(100)$.

Answer

\$2

Key Concepts

- Using $f(a+h) \approx f(a) + f'(a)h$, it is possible to estimate $f(a+h)$ given $f'(a)$ and $f(a)$.
- The rate of change of position is velocity, and the rate of change of velocity is acceleration. Speed is the absolute value, or magnitude, of velocity.
- The population growth rate and the present population can be used to predict the size of a future population.
- Marginal cost, marginal revenue, and marginal profit functions can be used to predict, respectively, the cost of producing one more item, the revenue obtained by selling one more item, and the profit obtained by producing and selling one more item.

Glossary

acceleration

is the rate of change of the velocity, that is, the derivative of velocity

amount of change

the amount of a function $f(x)$ over an interval $[x, x+h] is f(x+h) - f(x)$

average rate of change

is a function $f(x)$ over an interval $[x, x+h]$ is $\frac{f(x+h)-f(a)}{b-a}$

marginal cost

is the derivative of the cost function, or the approximate cost of producing one more item

marginal revenue

is the derivative of the revenue function, or the approximate revenue obtained by selling one more item

marginal profit

is the derivative of the profit function, or the approximate profit obtained by producing and selling one more item

population growth rate

is the derivative of the population with respect to time

speed

is the absolute value of velocity, that is, $|v(t)|$ is the speed of an object at time t whose velocity is given by $v(t)$

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3.10: Linear Approximations and Differentials

Learning Objectives

- Describe the linear approximation to a function at a point.
- Write the linearization of a given function.
- Draw a graph that illustrates the use of differentials to approximate the change in a quantity.
- Calculate the relative error and percentage error in using a differential approximation.

We have just seen how derivatives allow us to compare related quantities that are changing over time. In this section, we examine another application of derivatives: the ability to approximate functions locally by linear functions. Linear functions are the easiest functions with which to work, so they provide a useful tool for approximating function values. In addition, the ideas presented in this section are generalized later in the text when we study how to approximate functions by higher-degree polynomials Introduction to Power Series and Functions.

Linear Approximation of a Function at a Point

Consider a function f that is differentiable at a point $x=a$. Recall that the tangent line to the graph of f at a is given by the equation

$$y=f(a)+f'(a)(x-a).$$

For example, consider the function $f(x)=\frac{1}{x}$ at $a=2$. Since f is differentiable at $x=2$ and $f'(x)=-\frac{1}{x^2}$, we see that $f'(2)=-\frac{1}{4}$. Therefore, the tangent line to the graph of f at $a=2$ is given by the equation

$$y=\frac{1}{2}-\frac{1}{4}(x-2).$$

Figure $3.10.1a$ shows a graph of $f(x)=\frac{1}{x}$ along with the tangent line to f at $x=2$. Note that for x near 2, the graph of the tangent line is close to the graph of f. As a result, we can use the equation of the tangent line to approximate $f(x)$ for x near 2. For example, if $x=2.1$, the y value of the corresponding point on the tangent line is

$$y=\frac{1}{2}-\frac{1}{4}(2.1-2)=0.475.$$

The actual value of $f(2.1)$ is given by

$$f(2.1)=\frac{1}{2.1}\approx 0.47619.$$

Therefore, the tangent line gives us a fairly good approximation of $f(2.1)$ (Figure $3.10.1b$). However, note that for values of x far from 2, the equation of the tangent line does not give us a good approximation. For example, if $x=10$, the y-value of the corresponding point on the tangent line is

$$y=\frac{1}{2}-\frac{1}{4}(10-2)=\frac{1}{2}-2=-1.5,$$

whereas the value of the function at $x=10$ is $f(10)=0.1$.

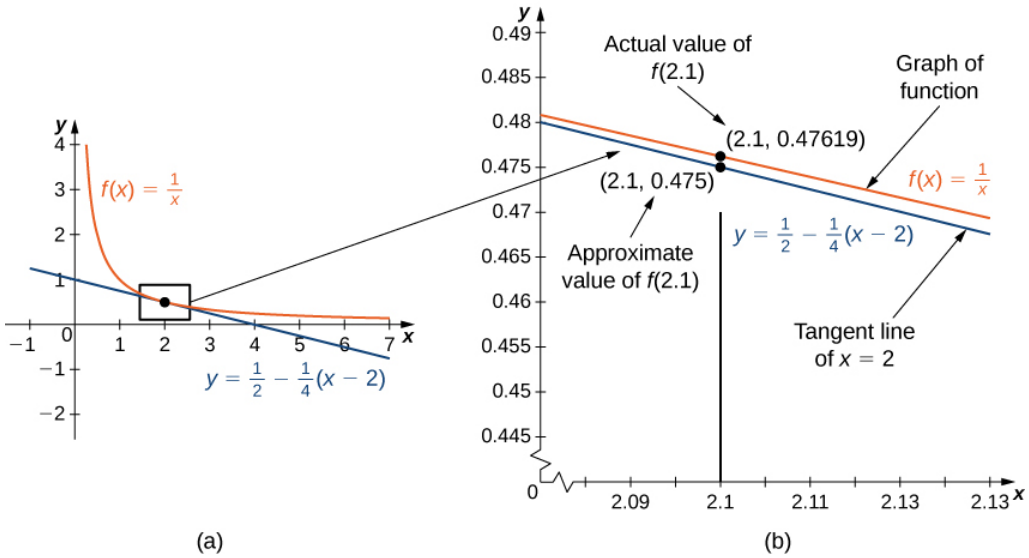


Figure 3.10.1: (a) The tangent line to $f(x) = 1/x$ at $x = 2$ provides a good approximation to f for x near 2. (b) At $x = 2.1$, the value of y on the tangent line to $f(x) = 1/x$ is 0.475. The actual value of $f(2.1)$ is $1/2.1$, which is approximately 0.47619.

In general, for a differentiable function f, the equation of the tangent line to f at $x = a$ can be used to approximate $f(x)$ for x near a. Therefore, we can write

$$f(x) \approx f(a) + f'(a)(x - a) \quad \text{for } x \text{ near } a.$$

We call the linear function

$$L(x) = f(a) + f'(a)(x - a) \tag{3.10.1}$$

the **linear approximation,** or **tangent line approximation**, of f at $x = a$. This function L is also known as the **linearization** of f at $x = a$.

To show how useful the linear approximation can be, we look at how to find the linear approximation for $f(x) = \sqrt{x}$ at $x = 9$.

Example 3.10.1: Linear Approximation of $\sqrt{x}$

Find the linear approximation of $f(x) = \sqrt{x}$ at $x = 9$ and use the approximation to estimate $\sqrt{9.1}$.

Solution

Since we are looking for the linear approximation at $x = 9$, using Equation 3.10.1 we know the linear approximation is given by

$$L(x) = f(9) + f'(9)(x - 9).$$

We need to find $f(9)$ and $f'(9)$.

$$f(x) = \sqrt{x} \Rightarrow f(9) = \sqrt{9} = 3$$
$$f'(x) = \tfrac{1}{2\sqrt{x}} \Rightarrow f'(9) = \tfrac{1}{2\sqrt{9}} = \tfrac{1}{6}$$

Therefore, the linear approximation is given by Figure 3.10.2

$$L(x) = 3 + \frac{1}{6}(x - 9)$$

Using the linear approximation, we can estimate $\sqrt{9.1}$ by writing

$$\sqrt{9.1} = f(9.1) \approx L(9.1) = 3 + \frac{1}{6}(9.1 - 9) \approx 3.0167.$$

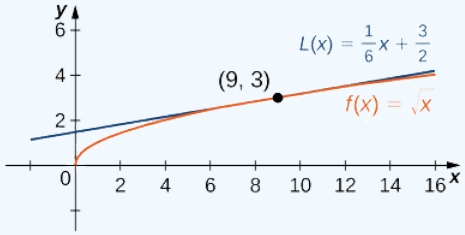


Figure 3.10.2: The local linear approximation to $f(x) = \sqrt{x}$ at $x = 9$ provides an approximation to f for x near 9.

Analysis

Using a calculator, the value of $\sqrt{9.1}$ to four decimal places is 3.0166. The value given by the linear approximation, 3.0167, is very close to the value obtained with a calculator, so it appears that using this linear approximation is a good way to estimate $\sqrt{x}$, at least for x near 9. At the same time, it may seem odd to use a linear approximation when we can just push a few buttons on a calculator to evaluate $\sqrt{9.1}$. However, how does the calculator evaluate $\sqrt{9.1}$? The calculator uses an approximation! In fact, calculators and computers use approximations all the time to evaluate mathematical expressions; they just use higher-degree approximations.

? Exercise 3.10.1

Find the local linear approximation to $f(x) = \sqrt[3]{x}$ at $x = 8$. Use it to approximate $\sqrt[3]{8.1}$ to five decimal places.

Hint

$L(x) = f(a) + f'(a)(x - a)$

Answer

$L(x) = 2 + \frac{1}{12}(x - 8)$; 2.00833

✔ Example 3.10.2: Linear Approximation of $\sin x$

Find the linear approximation of $f(x) = \sin x$ at $x = \frac{\pi}{3}$ and use it to approximate $\sin(62^\circ)$.

Solution

First we note that since $\frac{\pi}{3}$ rad is equivalent to 60°, using the linear approximation at $x = \pi/3$ seems reasonable. The linear approximation is given by

$$L(x) = f(\tfrac{\pi}{3}) + f'(\tfrac{\pi}{3})(x - \tfrac{\pi}{3}).$$

We see that

$$f(x) = \sin x \Rightarrow f(\tfrac{\pi}{3}) = \sin(\tfrac{\pi}{3}) = \tfrac{\sqrt{3}}{2}$$
$$f'(x) = \cos x \Rightarrow f'(\tfrac{\pi}{3}) = \cos(\tfrac{\pi}{3}) = \tfrac{1}{2}$$

Therefore, the linear approximation of f at $x = \pi/3$ is given by Figure 3.10.3

$$L(x) = \tfrac{\sqrt{3}}{2} + \tfrac{1}{2}(x - \tfrac{\pi}{3})$$

To estimate $\sin(62^\circ)$ using L, we must first convert 62° to radians. We have $62^\circ = \frac{62\pi}{180}$ radians, so the estimate for $\sin(62^\circ)$ is given by

$$\sin(62^\circ) = f(\tfrac{62\pi}{180}) \approx L(\tfrac{62\pi}{180}) = \tfrac{\sqrt{3}}{2} + \tfrac{1}{2}(\tfrac{62\pi}{180} - \tfrac{\pi}{3}) = \tfrac{\sqrt{3}}{2} + \tfrac{1}{2}(\tfrac{2\pi}{180}) = \tfrac{\sqrt{3}}{2} + \tfrac{\pi}{180} \approx 0.88348.$$

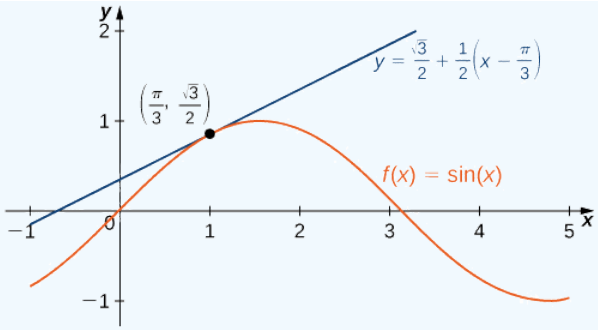


Figure 3.10.3: The linear approximation to $f(x) = \sin x$ at $x = \pi/3$ provides an approximation to $\sin x$ for x near $\pi/3$.

? Exercise 3.10.2

Find the linear approximation for $f(x) = \cos x$ at $x = \frac{\pi}{2}$.

Hint

$L(x) = f(a) + f'(a)(x - a)$

Answer

$L(x) = -x + \frac{\pi}{2}$

Linear approximations may be used in estimating roots and powers. In the next example, we find the linear approximation for $f(x) = (1+x)^n$ at $x = 0$, which can be used to estimate roots and powers for real numbers near 1. The same idea can be extended to a function of the form $f(x) = (m+x)^n$ to estimate roots and powers near a different number m.

✓ Example 3.10.3: Approximating Roots and Powers

Find the linear approximation of $f(x) = (1+x)^n$ at $x = 0$. Use this approximation to estimate $(1.01)^3$.

Solution

The linear approximation at $x = 0$ is given by

$$L(x) = f(0) + f'(0)(x - 0).$$

Because

$$f(x) = (1+x)^n \Rightarrow f(0) = 1$$
$$f'(x) = n(1+x)^{n-1} \Rightarrow f'(0) = n,$$

the linear approximation is given by Figure 3.10.4a

$$L(x) = 1 + n(x - 0) = 1 + nx$$

We can approximate $(1.01)^3$ by evaluating $L(0.01)$ when $n = 3$. We conclude that

$$(1.01)^3 = f(1.01) \approx L(1.01) = 1 + 3(0.01) = 1.03.$$

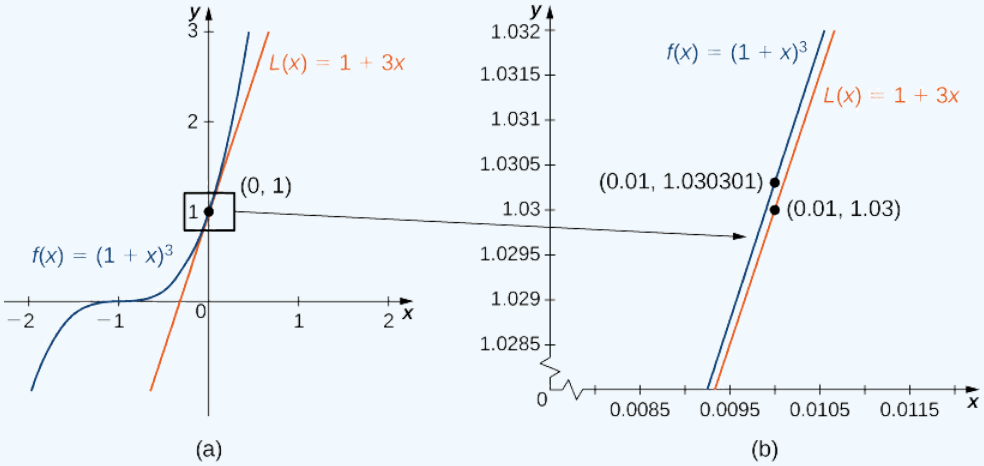


Figure 3.10.4: (a) The linear approximation of $f(x)$ at $x = 0$ is $L(x)$. (b) The actual value of 1.01^3 is 1.030301. The linear approximation of $f(x)$ at $x = 0$ estimates 1.01^3 to be 1.03.

? Exercise 3.10.3

Find the linear approximation of $f(x) = (1 + x)^4$ at $x = 0$ without using the result from the preceding example.

Hint

$f'(x) = 4(1 + x)^3$

Answer

$L(x) = 1 + 4x$

Differentials

We have seen that linear approximations can be used to estimate function values. They can also be used to estimate the amount a function value changes as a result of a small change in the input. To discuss this more formally, we define a related concept: **differentials**. Differentials provide us with a way of estimating the amount a function changes as a result of a small change in input values.

When we first looked at derivatives, we used the Leibniz notation dy/dx to represent the derivative of y with respect to x. Although we used the expressions dy and dx in this notation, they did not have meaning on their own. Here we see a meaning to the expressions dy and dx. Suppose $y = f(x)$ is a differentiable function. Let dx be an independent variable that can be assigned any nonzero real number, and define the dependent variable dy by

$$dy = f'(x)\,dx. \tag{3.10.2}$$

It is important to notice that dy is a function of both x and dx. The expressions dy and dx are called differentials. We can divide both sides of Equation 3.10.2 by dx, which yields

$$\frac{dy}{dx} = f'(x). \tag{3.10.3}$$

This is the familiar expression we have used to denote a derivative. Equation 3.10.3 is known as the **differential form** of Equation 3.10.2.

✓ Example 3.10.4: Computing Differentials

For each of the following functions, find dy and evaluate when $x = 3$ and $dx = 0.1$.

a. $y = x^2 + 2x$
b. $y = \cos x$

Solution

The key step is calculating the derivative. When we have that, we can obtain dy directly.

a. Since $f(x) = x^2 + 2x,$ we know $f'(x) = 2x + 2$, and therefore

$$dy = (2x + 2)\, dx.$$

When $x = 3$ and $dx = 0.1$,

$$dy = (2 \cdot 3 + 2)(0.1) = 0.8.$$

b. Since $f(x) = \cos x$, $f'(x) = -\sin(x)$. This gives us

$$dy = -\sin x\, dx.$$

When $x = 3$ and $dx = 0.1$,

$$dy = -\sin(3)(0.1) = -0.1 \sin(3).$$

? Exercise 3.10.4

For $y = e^{x^2}$, find dy.

Hint

$dy = f'(x)\, dx$

Answer

$dy = 2xe^{x^2} dx$

We now connect differentials to linear approximations. Differentials can be used to estimate the change in the value of a function resulting from a small change in input values. Consider a function f that is differentiable at point a. Suppose the input x changes by a small amount. We are interested in how much the output y changes. If x changes from a to $a + dx$, then the change in x is dx (also denoted Δx), and the change in y is given by

$$\Delta y = f(a + dx) - f(a).$$

Instead of calculating the exact change in y, however, it is often easier to approximate the change in y by using a linear approximation. For x near a, $f(x)$ can be approximated by the linear approximation (Equation 3.10.1)

$$L(x) = f(a) + f'(a)(x - a).$$

Therefore, if dx is small,

$$f(a + dx) \approx L(a + dx) = f(a) + f'(a)(a + dx - a).$$

That is,

$$f(a + dx) - f(a) \approx L(a + dx) - f(a) = f'(a)\, dx.$$

In other words, the actual change in the function f if x increases from a to $a + dx$ is approximately the difference between $L(a + dx)$ and $f(a)$, where $L(x)$ is the linear approximation of f at a. By definition of $L(x)$, this difference is equal to $f'(a)\, dx$. In summary,

$$\Delta y = f(a + dx) - f(a) \approx L(a + dx) - f(a) = f'(a)\, dx = dy.$$

Therefore, we can use the differential $dy = f'(a)\, dx$ to approximate the change in y if x increases from $x = a$ to $x = a + dx$. We can see this in the following graph.

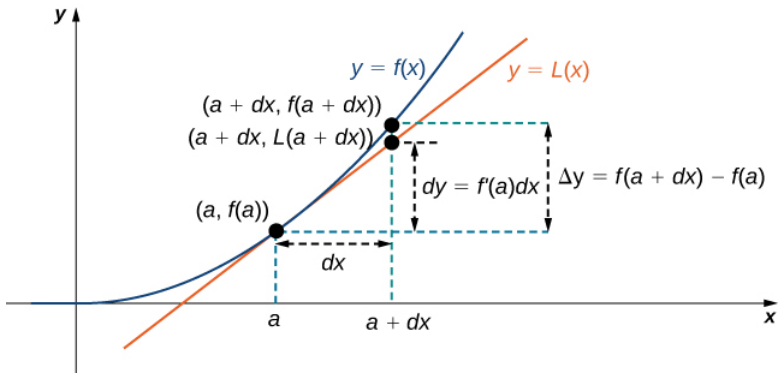


Figure 3.10.5: The differential $dy = f'(a)\,dx$ is used to approximate the actual change in y if x increases from a to $a + dx$.

We now take a look at how to use differentials to approximate the change in the value of the function that results from a small change in the value of the input. Note the calculation with differentials is much simpler than calculating actual values of functions and the result is very close to what we would obtain with the more exact calculation.

Example 3.10.5: Approximating Change with Differentials

Let $y = x^2 + 2x$. Compute Δy and dy at $x = 3$ if $dx = 0.1$.

Solution

The actual change in y if x changes from $x = 3$ to $x = 3.1$ is given by

$$\Delta y = f(3.1) - f(3) = [(3.1)^2 + 2(3.1)] - [3^2 + 2(3)] = 0.81.$$

The approximate change in y is given by $dy = f'(3)\,dx$. Since $f'(x) = 2x + 2$, we have

$$dy = f'(3)\,dx = (2(3) + 2)(0.1) = 0.8.$$

Exercise 3.10.5

For $y = x^2 + 2x$, find Δy and dy at $x = 3$ if $dx = 0.2$.

Hint

$dy = f'(3)\,dx,\ \Delta y = f(3.2) - f(3)$

Answer

$dy = 1.6,\ \Delta y = 1.64$

Calculating the Amount of Error

Any type of measurement is prone to a certain amount of error. In many applications, certain quantities are calculated based on measurements. For example, the area of a circle is calculated by measuring the radius of the circle. An error in the measurement of the radius leads to an error in the computed value of the area. Here we examine this type of error and study how differentials can be used to estimate the error.

Consider a function f with an input that is a measured quantity. Suppose the exact value of the measured quantity is a, but the measured value is $a + dx$. We say the measurement error is dx (or Δx). As a result, an error occurs in the calculated quantity $f(x)$. This type of error is known as a **propagated error** and is given by

$$\Delta y = f(a + dx) - f(a).$$

Since all measurements are prone to some degree of error, we do not know the exact value of a measured quantity, so we cannot calculate the propagated error exactly. However, given an estimate of the accuracy of a measurement, we can use differentials to approximate the propagated error Δy. Specifically, if f is a differentiable function at a,the propagated error is

$$\Delta y \approx dy = f'(a)\,dx.$$

Unfortunately, we do not know the exact value a. However, we can use the measured value $a+dx,$ and estimate

$$\Delta y \approx dy \approx f'(a+dx)\,dx.$$

In the next example, we look at how differentials can be used to estimate the error in calculating the volume of a box if we assume the measurement of the side length is made with a certain amount of accuracy.

Example 3.10.6: Volume of a Cube

Suppose the side length of a cube is measured to be 5 cm with an accuracy of 0.1 cm.

a. Use differentials to estimate the error in the computed volume of the cube.
b. Compute the volume of the cube if the side length is (i) 4.9 cm and (ii) 5.1 cm to compare the estimated error with the actual potential error.

Solution

a. The measurement of the side length is accurate to within ± 0.1 cm. Therefore,

$$-0.1 \le dx \le 0.1.$$

The volume of a cube is given by $V = x^3$, which leads to

$$dV = 3x^2 dx.$$

Using the measured side length of 5 cm, we can estimate that

$$-3(5)^2(0.1) \le dV \le 3(5)^2(0.1).$$

Therefore,

$$-7.5 \le dV \le 7.5.$$

b. If the side length is actually 4.9 cm, then the volume of the cube is

$$V(4.9) = (4.9)^3 = 117.649\text{cm}^3.$$

If the side length is actually 5.1 cm, then the volume of the cube is

$$V(5.1) = (5.1)^3 = 132.651\text{cm}^3.$$

Therefore, the actual volume of the cube is between $117.649$and 132.651. Since the side length is measured to be 5 cm, the computed volume is $V(5) = 5^3 = 125.$ Therefore, the error in the computed volume is

$$117.649 - 125 \le \Delta V \le 132.651 - 125.$$

That is,

$$-7.351 \le \Delta V \le 7.651.$$

We see the estimated error dV is relatively close to the actual potential error in the computed volume.

Exercise 3.10.6

Estimate the error in the computed volume of a cube if the side length is measured to be 6 cm with an accuracy of 0.2 cm.

Hint

$dV = 3x^2 dx$

Answer

The volume measurement is accurate to within 21.6 cm^3.

The measurement error dx $(= \Delta x)$ and the propagated error Δy are absolute errors. We are typically interested in the size of an error relative to the size of the quantity being measured or calculated. Given an absolute error Δq for a particular quantity, we define the **relative error** as $\frac{\Delta q}{q}$, where q is the actual value of the quantity. The **percentage error** is the relative error expressed as

a percentage. For example, if we measure the height of a ladder to be 63 in. when the actual height is 62 in., the absolute error is 1 in. but the relative error is $\frac{1}{62} = 0.016$, or 1.6%. By comparison, if we measure the width of a piece of cardboard to be 8.25 in. when the actual width is 8 in., our absolute error is $\frac{1}{4}$ in., whereas the relative error is $\frac{0.25}{8} = \frac{1}{32}$, or 3.1%. Therefore, the percentage error in the measurement of the cardboard is larger, even though 0.25 in. is less than 1 in.

Example 3.10.7: Relative and Percentage Error

An astronaut using a camera measures the radius of Earth as 4000 mi with an error of ± 80 mi. Let's use differentials to estimate the relative and percentage error of using this radius measurement to calculate the volume of Earth, assuming the planet is a perfect sphere.

Solution: If the measurement of the radius is accurate to within ± 80, we have

$$-80 \le dr \le 80.$$

Since the volume of a sphere is given by $V = (\frac{4}{3})\pi r^3$, we have

$$dV = 4\pi r^2 dr.$$

Using the measured radius of 4000 mi, we can estimate

$$-4\pi(4000)^2(80) \le dV \le 4\pi(4000)^2(80).$$

To estimate the relative error, consider $\dfrac{dV}{V}$. Since we do not know the exact value of the volume V, use the measured radius $r = 4000$ mi to estimate V. We obtain $V \approx (\frac{4}{3})\pi(4000)^3$. Therefore the relative error satisfies

$$\frac{-4\pi(4000)^2(80)}{4\pi(4000)^3/3} \le \frac{dV}{V} \le \frac{4\pi(4000)^2(80)}{4\pi(4000)^3/3},$$

which simplifies to

$$-0.06 \le \frac{dV}{V} \le 0.06.$$

The relative error is 0.06 and the percentage error is 6%.

Exercise 3.10.7

Determine the percentage error if the radius of Earth is measured to be 3950 mi with an error of ± 100 mi.

Hint

Use the fact that $dV = 4\pi r^2 dr$ to find dV/V.

Answer

7.6%

Key Concepts

- A differentiable function $y = f(x)$ can be approximated at a by the linear function
$$L(x) = f(a) + f'(a)(x - a).$$
- For a function $y = f(x)$, if x changes from a to $a + dx$, then
$$dy = f'(x)\,dx$$
is an approximation for the change in y. The actual change in y is
$$\Delta y = f(a + dx) - f(a).$$
- A measurement error dx can lead to an error in a calculated quantity $f(x)$. The error in the calculated quantity is known as the propagated error. The propagated error can be estimated by
$$dy \approx f'(x)\,dx.$$

- To estimate the relative error of a particular quantity q, we estimate $\frac{\Delta q}{q}$.

Key Equations

- **Linear approximation**

$L(x) = f(a) + f'(a)(x - a)$

- **A differential**

$dy = f'(x)\,dx$

Glossary

differential

the differential dx is an independent variable that can be assigned any nonzero real number; the differential dy is defined to be $dy = f'(x)\,dx$

differential form

given a differentiable function $y = f'(x),$ the equation $dy = f'(x)\,dx$ is the differential form of the derivative of y with respect to x

linear approximation

the linear function $L(x) = f(a) + f'(a)(x - a)$ is the linear approximation of f at $x = a$

percentage error

the relative error expressed as a percentage

propagated error

the error that results in a calculated quantity $f(x)$ resulting from a measurement error dx

relative error

given an absolute error Δq for a particular quantity, $\frac{\Delta q}{q}$ is the relative error.

tangent line approximation (linearization)

since the linear approximation of f at $x = a$ is defined using the equation of the tangent line, the linear approximation of f at $x = a$ is also known as the tangent line approximation to f at $x = a$

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3.11: Hyperbolic Functions

The hyperbolic functions appear with some frequency in applications, and are quite similar in many respects to the trigonometric functions. This is a bit surprising given our initial definitions.

Definition 4.11.1: Hyperbolic Cosines and Sines

The **hyperbolic cosine** is the function

$$\cosh x = \frac{e^x + e^{-x}}{2},$$

and the **hyperbolic sine** is the function

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

Notice that cosh is even (that is, $\cosh(-x) = \cosh(x)$) while sinh is odd ($\sinh(-x) = -\sinh(x)$), and $\cosh x + \sinh x = e^x$. Also, for all x, $\cosh x > 0$, while $\sinh x = 0$ if and only if $e^x - e^{-x} = 0$, which is true precisely when $x = 0$.

Lemma 4.11.2

The range of $\cosh x$ is $[1, \infty)$.

Proof

Let $y = \cosh x$. We solve for x:

$$\begin{aligned} y &= \frac{e^x + e^{-x}}{2} \\ 2y &= e^x + e^{-x} \\ 2ye^x &= e^{2x} + 1 \\ 0 &= e^{2x} - 2ye^x + 1 \\ e^x &= \frac{2y \pm \sqrt{4y^2 - 4}}{2} \\ e^x &= y \pm \sqrt{y^2 - 1} \end{aligned}$$

From the last equation, we see $y^2 \geq 1$, and since $y \geq 0$, it follows that $y \geq 1$.

Now suppose $y \geq 1$, so $y \pm \sqrt{y^2 - 1} > 0$. Then $x = \ln(y \pm \sqrt{y^2 - 1})$ is a real number, and $y = \cosh x$, so y is in the range of $\cosh(x)$.

□

Definition 4.11.3: Hyperbolic Tangent and Cotangent

The other hyperbolic functions are

$$\begin{aligned} \tanh x &= \frac{\sinh x}{\cosh x} \\ \coth x &= \frac{\cosh x}{\sinh x} \\ \operatorname{sech} x &= \frac{1}{\cosh x} \\ \operatorname{csch} x &= \frac{1}{\sinh x} \end{aligned}$$

The domain of coth and csch is $x \neq 0$ while the domain of the other hyperbolic functions is all real numbers. Graphs are shown in Figure 3.11.1

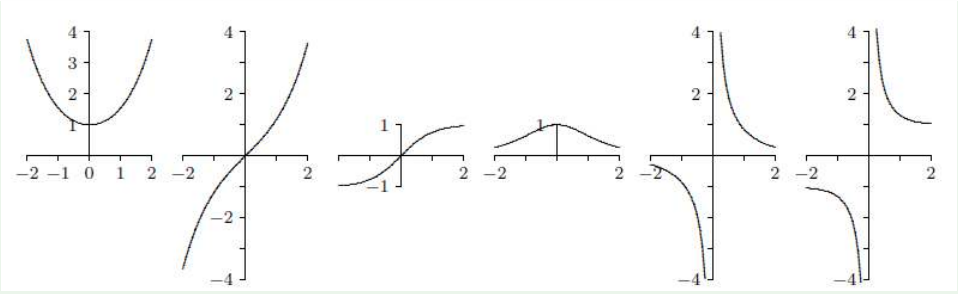


***Figure* 3.11.1:** *The hyperbolic functions.*

Certainly the hyperbolic functions do not closely resemble the trigonometric functions graphically. But they do have analogous properties, beginning with the following identity.

Theorem 4.11.4

For all x in $\mathbb{R}$, $\cosh^2 x - \sinh^2 x = 1$.

Proof

The proof is a straightforward computation:

$$\cosh^2 x - \sinh^2 x = \frac{(e^x + e^{-x})^2}{4} - \frac{(e^x - e^{-x})^2}{4} = \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4} = \frac{4}{4} = 1.$$

□

This immediately gives two additional identities:

$$1 - \tanh^2 x = \operatorname{sech}^2 x \qquad \text{and} \qquad \coth^2 x - 1 = \operatorname{csch}^2 x.$$

The identity of the theorem also helps to provide a geometric motivation. Recall that the graph of $x^2 - y^2 = 1$ is a hyperbola with asymptotes $x = \pm y$ whose x-intercepts are ± 1. If (x, y) is a point on the right half of the hyperbola, and if we let $x = \cosh t$, then $y = \pm\sqrt{x^2 - 1} = \pm\sqrt{\cosh^2 x - 1} = \pm \sinh t$. So for some suitable t, $\cosh t$ and $\sinh t$ are the coordinates of a typical point on the hyperbola. In fact, it turns out that t is twice the area shown in the first graph of Figure 3.11.2 Even this is analogous to trigonometry; $\cos t$ and $\sin t$ are the coordinates of a typical point on the unit circle, and t is twice the area shown in the second graph of Figure 3.11.2

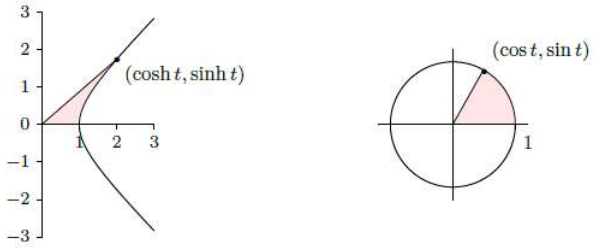


***Figure* 3.11.2:** *Geometric definitions of sin, cos, sinh, cosh: t is twice the shaded area in each figure.*

Given the definitions of the hyperbolic functions, finding their derivatives is straightforward. Here again we see similarities to the trigonometric functions.

Theorem 4.11.5

$\frac{d}{dx}\cosh x = \sinh x$ and \thmrdef{thm:hyperbolic derivatives} $\frac{d}{dx}\sinh x = \cosh x$.

Proof

$$\frac{d}{dx}\cosh x = \frac{d}{dx}\frac{e^x + e^{-x}}{2} = \frac{e^x - e^{-x}}{2} = \sinh x,$$

and

$$\frac{d}{dx}\sinh x = \frac{d}{dx}\frac{e^x - e^{-x}}{2} = \frac{e^x + e^{-x}}{2} = \cosh x.$$

□

Since $\cosh x > 0$, $\sinh x$ is increasing and hence injective, so $\sinh x$ has an inverse, $\operatorname{arcsinh} x$. Also, $\sinh x > 0$ when $x > 0$, so $\cosh x$ is injective on $[0, \infty)$ and has a (partial) inverse, $\operatorname{arccosh} x$. The other hyperbolic functions have inverses as well, though $\operatorname{arcsech} x$ is only a partial inverse. We may compute the derivatives of these functions as we have other inverse functions.

Theorem 4.11.6

$\frac{d}{dx}\operatorname{arcsinh} x = \frac{1}{\sqrt{1+x^2}}$.

Proof

Let $y = \operatorname{arcsinh} x$, so $\sinh y = x$. Then

$$\frac{d}{dx}\sinh y = \cosh(y) \cdot y' = 1,$$

and so

$$y' = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}.$$

□

The other derivatives are left to the exercises.

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CHAPTER OVERVIEW

4: Applications of Differentiation

A general Calculus Textmap organized around the textbook

Calculus: Early Transcendentals

by James Stewart

I II III IV V VI VII VIII IX X XI XII XIII XIV XV XVI XVII

This Textmap is currently under construction... please be patient with us.

Template:HideTOC

Topic hierarchy

4.1: Maximum and Minimum Values
4.2: The Mean Value Theorem
4.3: How Derivatives Affect the Shape of a Graph
4.4: Indeterminate Forms and l'Hospital's Rule
4.5: Summary of Curve Sketching
4.6: Graphing with Calculus and Calculators
4.7: Optimization Problems
4.8: Newton's Method
4.9: Antiderivatives

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4.1: Maximum and Minimum Values

Learning Objectives

- Define absolute extrema.
- Define local extrema.
- Explain how to find the critical points of a function over a closed interval.
- Describe how to use critical points to locate absolute extrema over a closed interval.

Given a particular function, we are often interested in determining the largest and smallest values of the function. This information is important in creating accurate graphs. Finding the maximum and minimum values of a function also has practical significance, because we can use this method to solve optimization problems, such as maximizing profit, minimizing the amount of material used in manufacturing an aluminum can, or finding the maximum height a rocket can reach. In this section, we look at how to use derivatives to find the largest and smallest values for a function.

Absolute Extrema

Consider the function $f(x) = x^2 + 1$ over the interval $(-\infty, \infty)$. As $x \to \pm\infty$, $f(x) \to \infty$. Therefore, the function does not have a largest value. However, since $x^2 + 1 \geq 1$ for all real numbers x and $x^2 + 1 = 1$ when $x = 0$, the function has a smallest value, 1, when $x = 0$. We say that 1 is the absolute minimum of $f(x) = x^2 + 1$ and it occurs at $x = 0$. We say that $f(x) = x^2 + 1$ does not have an absolute maximum (Figure 4.1.1).

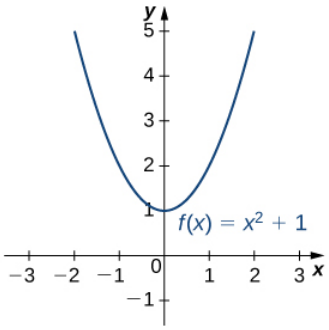


Figure 4.1.1: The given function has an absolute minimum of 1 at $x = 0$. The function does not have an absolute maximum.

Definition: Absolute Extrema

Let f be a function defined over an interval I and let $c \in I$. We say f has an *absolute maximum* on I at c if $f(c) \geq f(x)$ for all $x \in I$. We say f has an *absolute minimum* on I at c if $f(c) \leq f(x)$ for all $x \in I$. If f has an absolute maximum on I at c or an absolute minimum on I at c, we say f has *an absolute extremum* on I at c.

Before proceeding, let's note two important issues regarding this definition. First, the term absolute here does not refer to absolute value. An absolute extremum may be positive, negative, or zero. Second, if a function f has an absolute extremum over an interval I at c, the absolute extremum is $f(c)$. The real number c is a point in the domain at which the absolute extremum occurs. For example, consider the function $f(x) = 1/(x^2 + 1)$ over the interval $(-\infty, \infty)$. Since

$$f(0) = 1 \geq \frac{1}{x^2 + 1} = f(x)$$

for all real numbers x, we say f has an absolute maximum over $(-\infty, \infty)$ at $x = 0$. The absolute maximum is $f(0) = 1$. It occurs at $x = 0$, as shown in Figure 4.1.2(b).

A function may have both an absolute maximum and an absolute minimum, just one extremum, or neither. Figure 4.1.2 shows several functions and some of the different possibilities regarding absolute extrema. However, the following theorem, called the **Extreme Value Theorem**, guarantees that a continuous function f over a closed, bounded interval $[a, b]$ has both an absolute maximum and an absolute minimum.

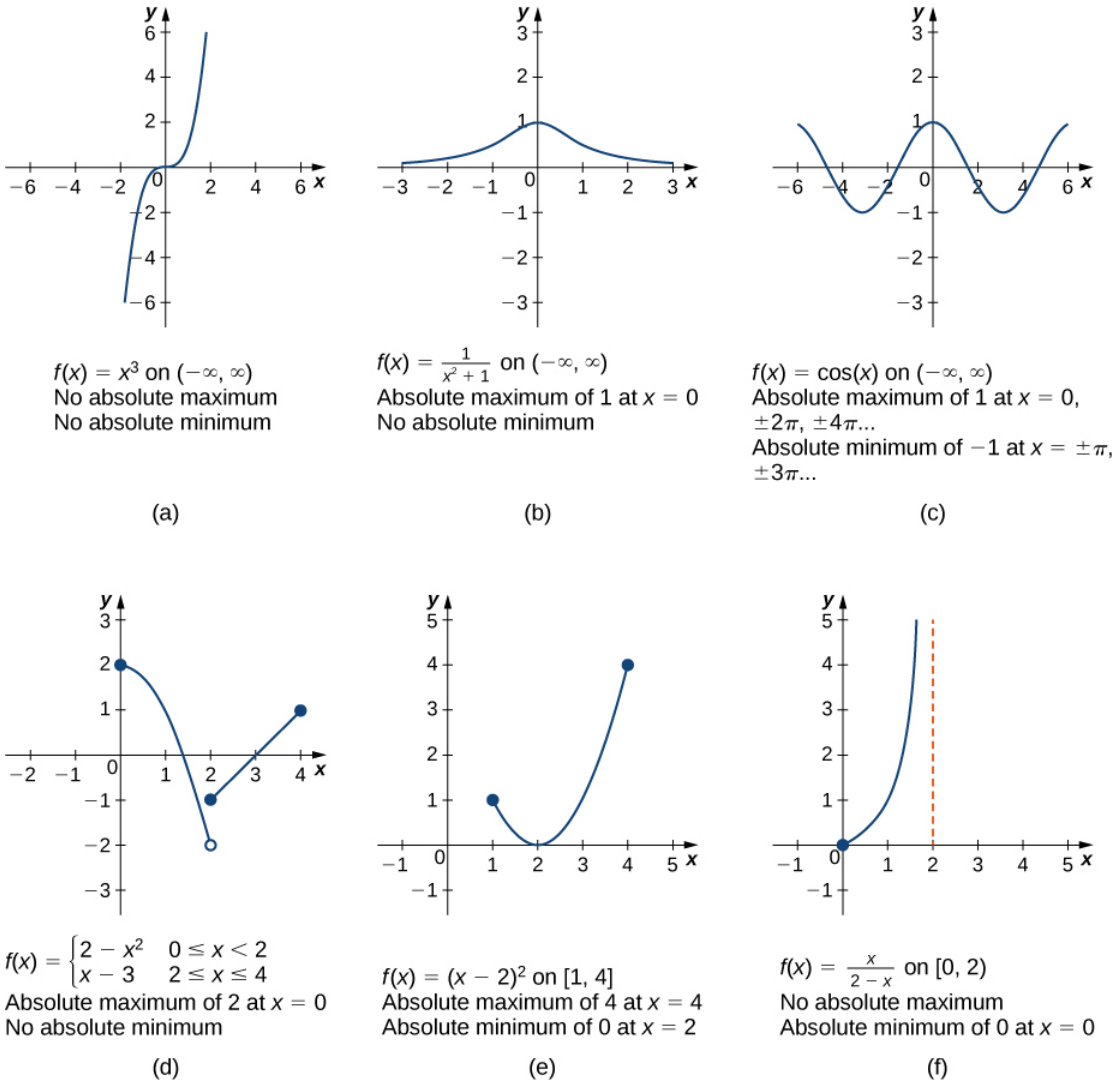


Figure 4.1.2: Graphs (a), (b), and (c) show several possibilities for absolute extrema for functions with a domain of $(-\infty, \infty)$. Graphs (d), (e), and (f) show several possibilities for absolute extrema for functions with a domain that is a bounded interval.

Theorem 4.1.1: Extreme Value Theorem

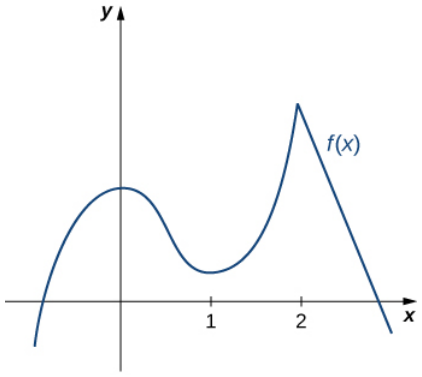
If f is a continuous function over the closed, bounded interval $[a, b]$, then there is a point in $[a, b]$ at which f has an absolute maximum over $[a, b]$ and there is a point in $[a, b]$ at which f has an absolute minimum over $[a, b]$.

The proof of the extreme value theorem is beyond the scope of this text. Typically, it is proved in a course on real analysis. There are a couple of key points to note about the statement of this theorem. For the extreme value theorem to apply, the function must be continuous over a closed, bounded interval. If the interval I is open or the function has even one point of discontinuity, the function may not have an absolute maximum or absolute minimum over I. For example, consider the functions shown in Figure $4.1.2$(d), (e), and (f). All three of these functions are defined over bounded intervals. However, the function in graph (e) is the only one that has both an absolute maximum and an absolute minimum over its domain. The extreme value theorem cannot be applied to the functions in graphs (d) and (f) because neither of these functions is continuous over a closed, bounded interval. Although the function in graph (d) is defined over the closed interval $[0, 4]$, the function is discontinuous at $x = 2$. The function has an absolute maximum over $[0, 4]$ but does not have an absolute minimum. The function in graph (f) is continuous over the half-open interval $[0, 2)$, but is not defined at $x = 2$, and therefore is not continuous over a closed, bounded interval. The function has an absolute minimum over $[0, 2)$, but does not have an absolute maximum over $[0, 2)$. These two graphs illustrate why a function over a bounded interval may fail to have an absolute maximum and/or absolute minimum.

Before looking at how to find absolute extrema, let's examine the related concept of local extrema. This idea is useful in determining where absolute extrema occur.

Local Extrema and Critical Points

Consider the function f shown in Figure 4.1.3. The graph can be described as two mountains with a valley in the middle. The absolute maximum value of the function occurs at the higher peak, at $x = 2$. However, $x = 0$ is also a point of interest. Although $f(0)$ is not the largest value of f, the value $f(0)$ is larger than $f(x)$ for all x near 0. We say f has a local maximum at $x = 0$. Similarly, the function f does not have an absolute minimum, but it does have a local minimum at $x = 1$ because $f(1)$ is less than $f(x)$ for x near 1.



$f(x)$ defined on $(-\infty, \infty)$
Local maxima at $x = 0$ and $x = 2$
Local minimum at $x = 1$

Figure 4.1.3: This function f has two local maxima and one local minimum. The local maximum at $x = 2$ is also the absolute maximum.

Definition: Local Extrema

A function f has a *local maximum* at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \geq f(x)$ for all $x \in I$. A function f has a *local minimum* at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \leq f(x)$ for all $x \in I$. A function f has a *local extremum* at c if f has a local maximum at c or f has a local minimum at c.

Note that if f has an absolute extremum at c and f is defined over an interval containing c, then $f(c)$ is also considered a local extremum. If an absolute extremum for a function f occurs at an endpoint, we do not consider that to be a local extremum, but instead refer to that as an endpoint extremum.

Given the graph of a function f, it is sometimes easy to see where a local maximum or local minimum occurs. However, it is not always easy to see, since the interesting features on the graph of a function may not be visible because they occur at a very small scale. Also, we may not have a graph of the function. In these cases, how can we use a formula for a function to determine where these extrema occur?

To answer this question, let's look at Figure 4.1.3 again. The local extrema occur at $x = 0, x = 1$, and $x = 2$. Notice that at $x = 0$ and $x = 1$, the derivative $f'(x) = 0$. At $x = 2$, the derivative $f'(x)$ does not exist, since the function f has a corner there. In fact, if f has a local extremum at a point $x = c$, the derivative $f'(c)$ must satisfy one of the following conditions: either $f'(c) = 0$ or $f'(c)$ is undefined. Such a value c is known as a critical point and it is important in finding extreme values for functions.

Definition: Critical Points

Let c be an interior point in the domain of f. We say that c is a *critical point* of f if $f'(c) = 0$ or $f'(c)$ is undefined.

As mentioned earlier, if f has a local extremum at a point $x = c$, then c must be a critical point of f. This fact is known as **Fermat's theorem.**

Theorem 4.1.2: Fermat's Theorem

If f has a local extremum at c and f is differentiable at c, then $f'(c) = 0$.

Proof

Suppose f has a local extremum at c and f is differentiable at c. We need to show that $f'(c) = 0$. To do this, we will show that $f'(c) \geq 0$ and $f'(c) \leq 0$, and therefore $f'(c) = 0$. Since f has a local extremum at c, f has a local maximum or local minimum at c. Suppose f has a local maximum at c. The case in which f has a local minimum at c can be handled similarly. There then exists an open interval I such that $f(c) \geq f(x)$ for all $x \in I$. Since f is differentiable at c, from the definition of the derivative, we know that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

Since this limit exists, both one-sided limits also exist and equal $f'(c)$. Therefore,

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c,} \tag{4.1.1}$$

and

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}.$$

Since $f(c)$ is a local maximum, we see that $f(x) - f(c) \leq 0$ for x near c. Therefore, for x near c, but $x > c$, we have $\frac{f(x)-f(c)}{x-c} \leq 0$. From Equation 4.1.1 we conclude that $f'(c) \leq 0$. Similarly, it can be shown that $f'(c) \geq 0$. Therefore, $f'(c) = 0$.

□

From Fermat's theorem, we conclude that if f has a local extremum at c, then either $f'(c) = 0$ or $f'(c)$ is undefined. In other words, local extrema can only occur at critical points.

Note this theorem does not claim that a function f must have a local extremum at a critical point. Rather, it states that critical points are candidates for local extrema. For example, consider the function $f(x) = x^3$. We have $f'(x) = 3x^2 = 0$ when $x = 0$. Therefore, $x = 0$ is a critical point. However, $f(x) = x^3$ is increasing over $(-\infty, \infty)$, and thus f does not have a local extremum at $x = 0$. In Figure 4.1.4, we see several different possibilities for critical points. In some of these cases, the functions have local extrema at critical points, whereas in other cases the functions do not. Note that these graphs do not show all possibilities for the behavior of a function at a critical point.

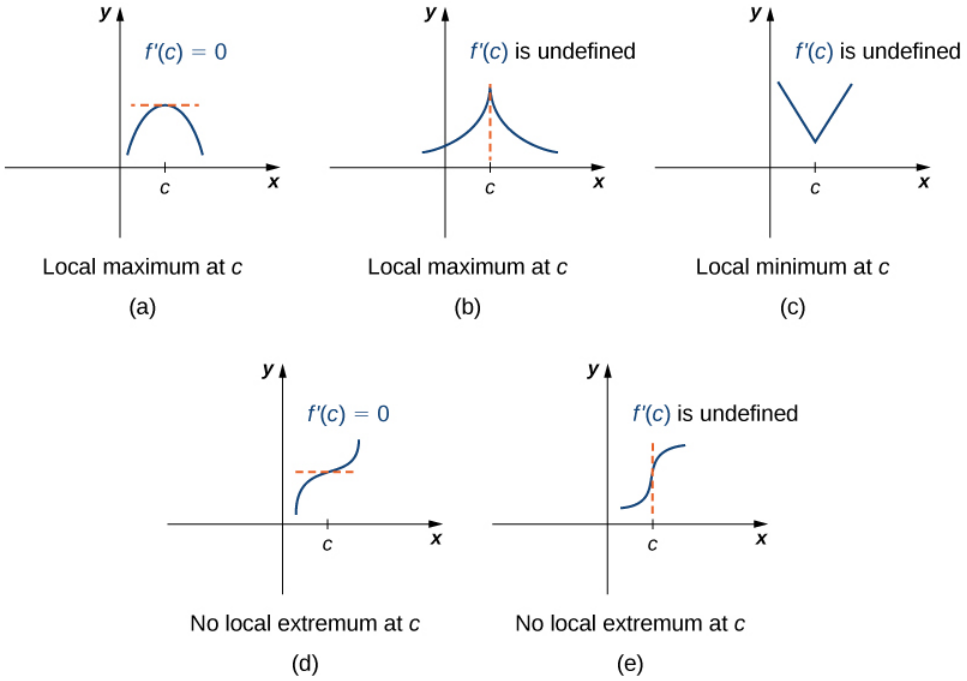


Figure 4.1.4: (a–e) A function f has a critical point at c if $f'(c) = 0$ or $f'(c)$ is undefined. A function may or may not have a local extremum at a critical point.

Later in this chapter we look at analytical methods for determining whether a function actually has a local extremum at a critical point. For now, let's turn our attention to finding critical points. We will use graphical observations to determine whether a critical point is associated with a local extremum.

Example 4.1.1: Locating Critical Points

For each of the following functions, find all critical points. Use a graphing utility to determine whether the function has a local extremum at each of the critical points.

a. $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x$
b. $f(x) = (x^2 - 1)^3$
c. $f(x) = \frac{4x}{1+x^2}$

Solution

a. The derivative $f'(x) = x^2 - 5x + 4$ is defined for all real numbers x. Therefore, we only need to find the values for x where $f'(x) = 0$. Since $f'(x) = x^2 - 5x + 4 = (x-4)(x-1)$, the critical points are $x = 1$ and $x = 4$. From the graph of f in Figure 4.1.5, we see that f has a local maximum at $x = 1$ and a local minimum at $x = 4$.

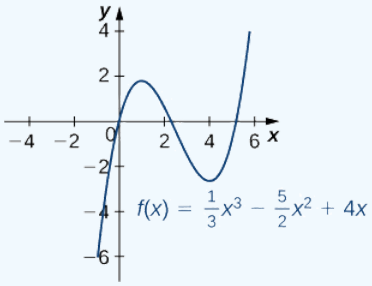


Figure 4.1.5: This function has a local maximum and a local minimum.

b. Using the chain rule, we see the derivative is

$$f'(x) = 3(x^2 - 1)^2(2x) = 6x(x^2 - 1)^2.$$

Therefore, f has critical points when $x = 0$ and when $x^2 - 1 = 0$. We conclude that the critical points are $x = 0, \pm 1$. From the graph of f in Figure 4.1.6, we see that f has a local (and absolute) minimum at $x = 0$, but does not have a local extremum

at $x = 1$ or $x = -1$.

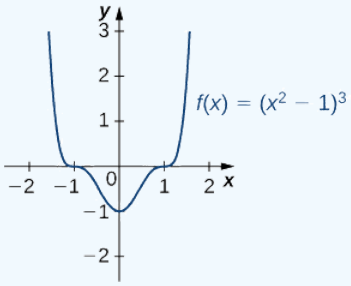


Figure 4.1.6: This function has three critical points: $x = 0$, $x = 1$, and $x = -1$. The function has a local (and absolute) minimum at $x = 0$, but does not have extrema at the other two critical points.

c. By the quotient rule, we see that the derivative is

$$f'(x) = \frac{4(1+x^2)-4x(2x)}{(1+x^2)^2} = \frac{4-4x^2}{(1+x^2)^2}\ .$$

The derivative is defined everywhere. Therefore, we only need to find values for x where $f'(x) = 0$. Solving $f'(x) = 0$, we see that $4 - 4x^2 = 0,$ which implies $x = \pm 1$. Therefore, the critical points are $x = \pm 1$. From the graph of f in Figure 4.1.7, we see that f has an absolute maximum at $x = 1$ and an absolute minimum at $x = -1$. Hence, f has a local maximum at $x = 1$ and a local minimum at $x = -1$. (Note that if f has an absolute extremum over an interval I at a point c that is not an endpoint of I, then f has a local extremum at c.)

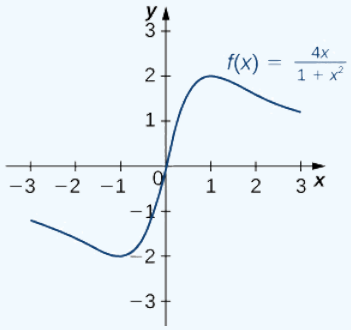


Figure 4.1.7: This function has an absolute maximum and an absolute minimum.

? Exercise 4.1.1

Find all critical points for $f(x) = x^3 - \frac{1}{2}x^2 - 2x + 1$.

Hint

Calculate $f'(x)$.

Answer

$x = \frac{-2}{3}, x = 1$

Locating Absolute Extrema

The extreme value theorem states that a continuous function over a closed, bounded interval has an absolute maximum and an absolute minimum. As shown in Figure 4.1.2, one or both of these absolute extrema could occur at an endpoint. If an absolute extremum does not occur at an endpoint, however, it must occur at an interior point, in which case the absolute extremum is a local extremum. Therefore, by Fermat's Theorem, the point c at which the local extremum occurs must be a critical point. We summarize this result in the following theorem.

Theorem 4.1.3: Location of Absolute Extrema

Let f be a continuous function over a closed, bounded interval I. The absolute maximum of f over I and the absolute minimum of f over I must occur at endpoints of I or at critical points of f in I.

With this idea in mind, let's examine a procedure for locating absolute extrema.

Problem-Solving Strategy: Locating Absolute Extrema over a Closed Interval

Consider a continuous function f defined over the closed interval $[a, b]$.

1. Evaluate f at the endpoints $x = a$ and $x = b$.
2. Find all critical points of f that lie over the interval (a, b) and evaluate f at those critical points.
3. Compare all values found in (1) and (2). From "Location of Absolute Extrema," the absolute extrema must occur at endpoints or critical points. Therefore, the largest of these values is the absolute maximum of f. The smallest of these values is the absolute minimum of f.

Now let's look at how to use this strategy to find the absolute maximum and absolute minimum values for continuous functions.

Example 4.1.2: Locating Absolute Extrema

For each of the following functions, find the absolute maximum and absolute minimum over the specified interval and state where those values occur.

a. $f(x) = -x^2 + 3x - 2$ over $[1, 3]$.
b. $f(x) = x^2 - 3x^{2/3}$ over $[0, 2]$.

Solution

a. Step 1. Evaluate f at the endpoints $x = 1$ and $x = 3$.

$$f(1) = 0 \text{ and } f(3) = -2$$

Step 2. Since $f'(x) = -2x + 3$, f' is defined for all real numbers x. Therefore, there are no critical points where the derivative is undefined. It remains to check where $f'(x) = 0$. Since $f'(x) = -2x + 3 = 0$ at $x = \frac{3}{2}$ and $\frac{3}{2}$ is in the interval $[1, 3]$, $f(\frac{3}{2})$ is a candidate for an absolute extremum of f over $[1, 3]$. We evaluate $f(\frac{3}{2})$ and find

$$f\left(\tfrac{3}{2}\right) = \tfrac{1}{4}.$$

Step 3. We set up the following table to compare the values found in steps 1 and 2.

x	$f(x)$	Conclusion
1	0	
$\frac{3}{2}$	$\frac{1}{4}$	Absolute maximum
3	-2	Absolute minimum

From the table, we find that the absolute maximum of f over the interval [1, 3] is $\frac{1}{4}$, and it occurs at $x = \frac{3}{2}$. The absolute minimum of f over the interval $[1, 3]$ is -2, and it occurs at $x = 3$ as shown in Figure 4.1.8.

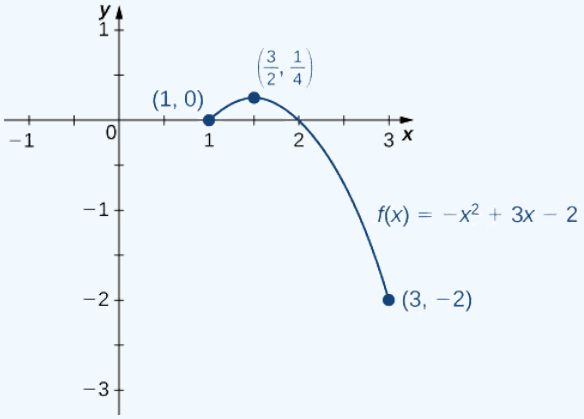


Figure 4.1.8: This function has both an absolute maximum and an absolute minimum.

b. Step 1. Evaluate f at the endpoints $x = 0$ and $x = 2$.

$$f(0) = 0 \text{ and } f(2) = 4 - 3(2)^{2/3} \approx -0.762$$

Step 2. The derivative of f is given by

$$f'(x) = 2x - \frac{2}{x^{1/3}} = \frac{2x^{4/3} - 2}{x^{1/3}}$$

for $x \neq 0$. The derivative is zero when $2x^{4/3} - 2 = 0$, which implies $x = \pm 1$. The derivative is undefined at $x = 0$. Therefore, the critical points of f are $x = 0, 1, -1$. The point $x = 0$ is an endpoint, so we already evaluated $f(0)$ in step 1. The point $x = -1$ is not in the interval of interest, so we need only evaluate $f(1)$. We find that

$$f(1) = -2.$$

Step 3. We compare the values found in steps 1 and 2, in the following table.

x	$f(x)$	Conclusion
0	0	Absolute maximum
1	-2	Absolute minimum
2	-0.762	

We conclude that the absolute maximum of f over the interval $[0, 2]$ is zero, and it occurs at $x = 0$. The absolute minimum is -2, and it occurs at $x = 1$ as shown in Figure 4.1.9.

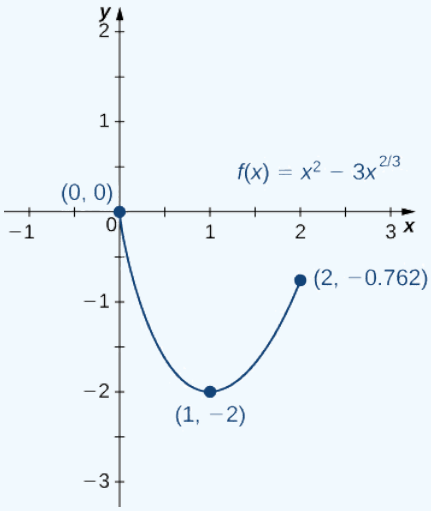


Figure 4.1.9: This function has an absolute maximum at an endpoint of the interval.

Exercise 4.1.2

Find the absolute maximum and absolute minimum of $f(x) = x^2 - 4x + 3$ over the interval $[1, 4]$.

Hint

Look for critical points. Evaluate f at all critical points and at the endpoints.

Answer

The absolute maximum is 3 and it occurs at $x = 4$. The absolute minimum is -1 and it occurs at $x = 2$.

At this point, we know how to locate absolute extrema for continuous functions over closed intervals. We have also defined local extrema and determined that if a function f has a local extremum at a point c, then c must be a critical point of f. However, c being a critical point is not a sufficient condition for f to have a local extremum at c. Later in this chapter, we show how to determine whether a function actually has a local extremum at a critical point. First, however, we need to introduce the Mean Value Theorem, which will help as we analyze the behavior of the graph of a function.

Key Concepts

- A function may have both an absolute maximum and an absolute minimum, have just one absolute extremum, or have no absolute maximum or absolute minimum.
- If a function has a local extremum, the point at which it occurs must be a critical point. However, a function need not have a local extremum at a critical point.
- A continuous function over a closed, bounded interval has an absolute maximum and an absolute minimum. Each extremum occurs at a critical point or an endpoint.

Glossary

absolute extremum

if f has an absolute maximum or absolute minimum at c, we say f has an absolute extremum at c

absolute maximum

if $f(c) \geq f(x)$ for all x in the domain of f, we say f has an absolute maximum at c

absolute minimum

if $f(c) \leq f(x)$ for all x in the domain of f, we say f has an absolute minimum at c

critical point

if $f'(c) = 0$ or $f'(c)$ is undefined, we say that c is a critical point of f

extreme value theorem

if f is a continuous function over a finite, closed interval, then f has an absolute maximum and an absolute minimum

Fermat's theorem

if f has a local extremum at c, then c is a critical point of f

local extremum

if f has a local maximum or local minimum at c, we say f has a local extremum at c

local maximum

if there exists an interval I such that $f(c) \geq f(x)$ for all $x \in I$, we say f has a local maximum at c

local minimum

if there exists an interval I such that $f(c) \leq f(x)$ for all $x \in I$, we say f has a local minimum at c

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4.2: The Mean Value Theorem

Learning Objectives

- Explain the meaning of Rolle's theorem.
- Describe the significance of the Mean Value Theorem.
- State three important consequences of the Mean Value Theorem.

The *Mean Value Theorem* is one of the most important theorems in calculus. We look at some of its implications at the end of this section. First, let's start with a special case of the Mean Value Theorem, called Rolle's theorem.

Rolle's Theorem

Informally, **Rolle's theorem** states that if the outputs of a differentiable function f are equal at the endpoints of an interval, then there must be an interior point c where $f'(c) = 0$. Figure $4.2.1$ illustrates this theorem.

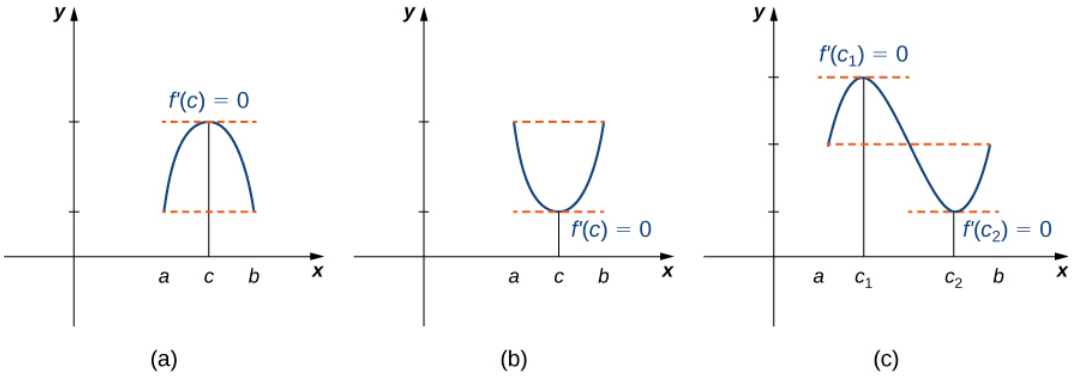


Figure 4.2.1: If a differentiable function f satisfies $f(a) = f(b)$, then its derivative must be zero at some point(s) between a and b.

Rolle's Theorem

Let f be a continuous function over the closed interval $[a, b]$ and differentiable over the open interval (a, b) such that $f(a) = f(b)$. There then exists at least one $c \in (a, b)$ such that $f'(c) = 0$.

Proof

Let $k = f(a) = f(b)$. We consider three cases:

1. $f(x) = k$ for all $x \in (a, b)$.
2. There exists $x \in (a, b)$ such that $f(x) > k$.
3. There exists $x \in (a, b)$ such that $f(x) < k$.

Case 1: If $f(x) = k$ for all $x \in (a, b)$, then $f'(x) = 0$ for all $x \in (a, b)$.

Case 2: Since f is a continuous function over the closed, bounded interval $[a, b]$, by the extreme value theorem, it has an absolute maximum. Also, since there is a point $x \in (a, b)$ such that $f(x) > k$, the absolute maximum is greater than k. Therefore, the absolute maximum does not occur at either endpoint. As a result, the absolute maximum must occur at an interior point $c \in (a, b)$. Because f has a maximum at an interior point c, and f is differentiable at c, by Fermat's theorem, $f'(c) = 0$.

Case 3: The case when there exists a point $x \in (a, b)$ such that $f(x) < k$ is analogous to case 2, with maximum replaced by minimum.

□

An important point about Rolle's theorem is that the differentiability of the function f is critical. If f is not differentiable, even at a single point, the result may not hold. For example, the function $f(x) = |x| - 1$ is continuous over $[-1, 1]$ and $f(-1) = 0 = f(1)$, but $f'(c) \neq 0$ for any $c \in (-1, 1)$ as shown in the following figure.

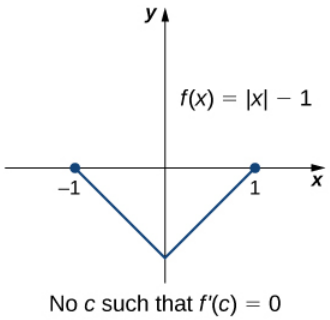


Figure 4.2.2: Since $f(x) = |x| - 1$ is not differentiable at $x = 0$, the conditions of Rolle's theorem are not satisfied. In fact, the conclusion does not hold here; there is no $c \in (-1, 1)$ such that $f'(c) = 0$.

Let's now consider functions that satisfy the conditions of Rolle's theorem and calculate explicitly the points c where $f'(c) = 0$.

Example 4.2.1: Using Rolle's Theorem

For each of the following functions, verify that the function satisfies the criteria stated in Rolle's theorem and find all values c in the given interval where $f'(c) = 0$.

a. $f(x) = x^2 + 2x$ over $[-2, 0]$
b. $f(x) = x^3 - 4x$ over $[-2, 2]$

Solution

a. Since f is a polynomial, it is continuous and differentiable everywhere. In addition, $f(-2) = 0 = f(0)$. Therefore, f satisfies the criteria of Rolle's theorem. We conclude that there exists at least one value $c \in (-2, 0)$ such that $f'(c) = 0$. Since $f'(x) = 2x + 2 = 2(x + 1)$, we see that $f'(c) = 2(c + 1) = 0$ implies $c = -1$ as shown in the following graph.

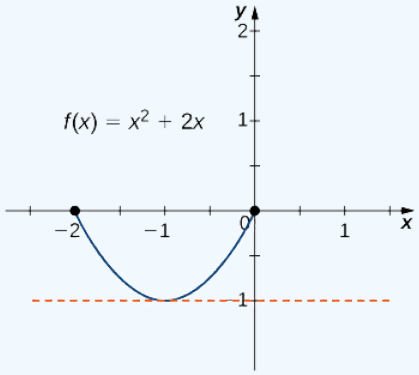


Figure 4.2.3: This function is continuous and differentiable over [−2,0], $f'(c) = 0$ when $c = -1$.

b. As in part a. f is a polynomial and therefore is continuous and differentiable everywhere. Also, $f(-2) = 0 = f(2)$. That said, f satisfies the criteria of Rolle's theorem. Differentiating, we find that $f'(x) = 3x^2 - 4$. Therefore, $f'(c) = 0$ when $x = \pm\frac{2}{\sqrt{3}}$. Both points are in the interval $[-2, 2]$, and, therefore, both points satisfy the conclusion of Rolle's theorem as shown in the following graph.

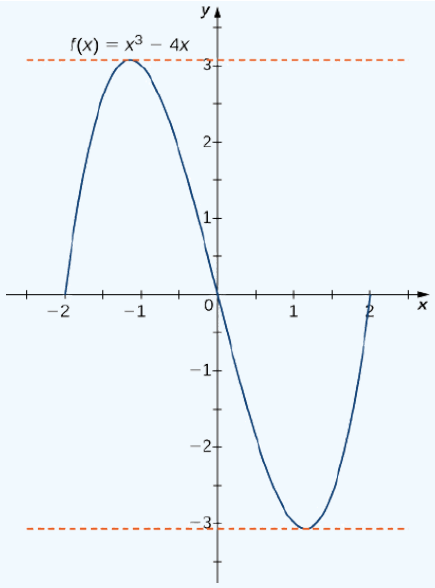


Figure 4.2.4: For this polynomial over $[-2, 2]$, $f'(c) = 0$ at $x = \pm 2/\sqrt{3}$.

Exercise 4.2.1

Verify that the function $f(x) = 2x^2 - 8x + 6$ defined over the interval $[1, 3]$ satisfies the conditions of Rolle's theorem. Find all points c guaranteed by Rolle's theorem.

Hint

Find all values c, where $f'(c) = 0$.

Answer

$c = 2$

The Mean Value Theorem and Its Meaning

Rolle's theorem is a special case of the Mean Value Theorem. In Rolle's theorem, we consider differentiable functions f that are zero at the endpoints. The Mean Value Theorem generalizes Rolle's theorem by considering functions that are not necessarily zero at the endpoints. Consequently, we can view the Mean Value Theorem as a slanted version of Rolle's theorem (Figure 4.2.5). The Mean Value Theorem states that if f is continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b), then there exists a point $c \in (a, b)$ such that the tangent line to the graph of f at c is parallel to the secant line connecting $(a, f(a))$ and $(b, f(b))$.

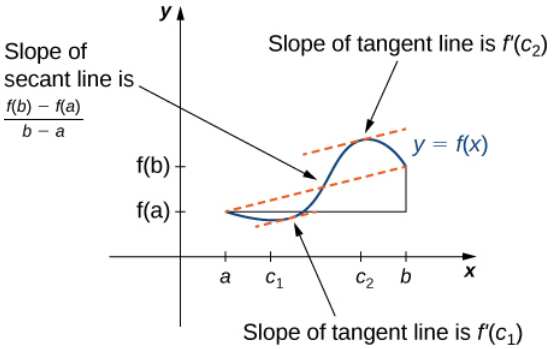


Figure 4.2.5: The Mean Value Theorem says that for a function that meets its conditions, at some point the tangent line has the same slope as the secant line between the ends. For this function, there are two values c_1 and c_2 such that the tangent line to f at c_1 and c_2 has the same slope as the secant line.

Mean Value Theorem

Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b). Then, there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof

The proof follows from Rolle's theorem by introducing an appropriate function that satisfies the criteria of Rolle's theorem. Consider the line connecting $(a, f(a))$ and $(b, f(b))$. Since the slope of that line is

$$\frac{f(b) - f(a)}{b - a}$$

and the line passes through the point $(a, f(a))$, the equation of that line can be written as

$$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a).$$

Let $g(x)$ denote the vertical difference between the point $(x, f(x))$ and the point (x, y) on that line. Therefore,

$$g(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right].$$

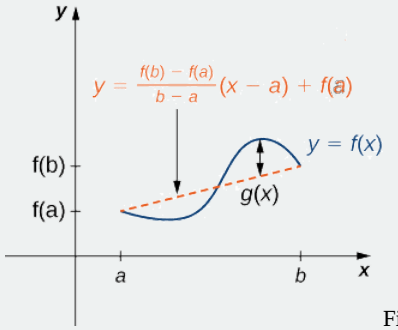


Figure 4.2.6: The value $g(x)$ is the vertical difference between the point $(x, f(x))$ and the point (x, y) on the secant line connecting $(a, f(a))$ and $(b, f(b))$.

Since the graph of f intersects the secant line when $x = a$ and $x = b$, we see that $g(a) = 0 = g(b)$. Since f is a differentiable function over (a, b), g is also a differentiable function over (a, b). Furthermore, since f is continuous over $[a, b]$, g is also continuous over $[a, b]$. Therefore, g satisfies the criteria of Rolle's theorem. Consequently, there exists a point $c \in (a, b)$ such that $g'(c) = 0$. Since

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$

we see that

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

Since $g'(c) = 0$, we conclude that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

□

In the next example, we show how the Mean Value Theorem can be applied to the function $f(x) = \sqrt{x}$ over the interval $[0, 9]$. The method is the same for other functions, although sometimes with more interesting consequences.

Example 4.2.2: Verifying that the Mean Value Theorem Applies

For $f(x) = \sqrt{x}$ over the interval $[0, 9]$, show that f satisfies the hypothesis of the Mean Value Theorem, and therefore there exists at least one value $c \in (0, 9)$ such that $f'(c)$ is equal to the slope of the line connecting $(0, f(0))$ and $(9, f(9))$. Find these values c guaranteed by the Mean Value Theorem.

Solution

We know that $f(x) = \sqrt{x}$ is continuous over $[0, 9]$ and differentiable over $(0, 9)$. Therefore, f satisfies the hypotheses of the Mean Value Theorem, and there must exist at least one value $c \in (0, 9)$ such that $f'(c)$ is equal to the slope of the line connecting $(0, f(0))$ and $(9, f(9))$ (Figure 4.2.7). To determine which value(s) of c are guaranteed, first calculate the derivative of f. The derivative $f'(x) = \frac{1}{(2\sqrt{x})}$. The slope of the line connecting $(0, f(0))$ and $(9, f(9))$ is given by

$$\frac{f(9) - f(0)}{9 - 0} = \frac{\sqrt{9} - \sqrt{0}}{9 - 0} = \frac{3}{9} = \frac{1}{3}.$$

We want to find c such that $f'(c) = \frac{1}{3}$. That is, we want to find c such that

$$\frac{1}{2\sqrt{c}} = \frac{1}{3}.$$

Solving this equation for c, we obtain $c = \frac{9}{4}$. At this point, the slope of the tangent line equals the slope of the line joining the endpoints.

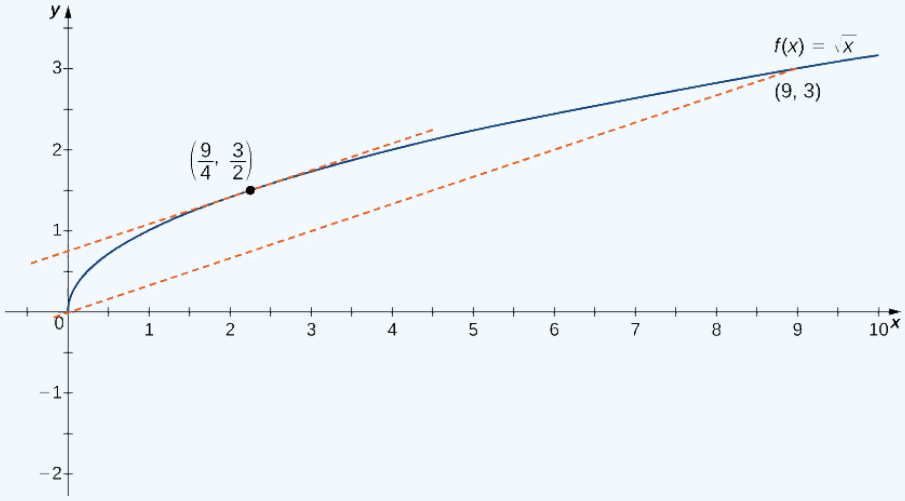


Figure 4.2.7: The slope of the tangent line at $c = 9/4$ is the same as the slope of the line segment connecting (0,0) and (9,3).

One application that helps illustrate the Mean Value Theorem involves velocity. For example, suppose we drive a car for 1 h down a straight road with an average velocity of 45 mph. Let $s(t)$ and $v(t)$ denote the position and velocity of the car, respectively, for $0 \leq t \leq 1$ h. Assuming that the position function $s(t)$ is differentiable, we can apply the Mean Value Theorem to conclude that, at some time $c \in (0, 1)$, the speed of the car was exactly

$$v(c) = s'(c) = \frac{s(1) - s(0)}{1 - 0} = 45 \text{ mph}.$$

Example 4.2.3: Mean Value Theorem and Velocity

If a rock is dropped from a height of 100 ft, its position t seconds after it is dropped until it hits the ground is given by the function $s(t) = -16t^2 + 100$.

a. Determine how long it takes before the rock hits the ground.
b. Find the average velocity v_{avg} of the rock for when the rock is released and the rock hits the ground.
c. Find the time t guaranteed by the Mean Value Theorem when the instantaneous velocity of the rock is v_{avg}.

Solution

a. When the rock hits the ground, its position is $s(t)=0$. Solving the equation $-16t^2+100=0$ for t, we find that $t=\pm\frac{5}{2}\,sec$. Since we are only considering $t\geq 0$, the ball will hit the ground $\frac{5}{2}$ sec after it is dropped.

b. The average velocity is given by

$$v_{avg}=\frac{s(5/2)-s(0)}{5/2-0}=\frac{0-100}{5/2}=-40\text{ ft/sec}.$$

c. The instantaneous velocity is given by the derivative of the position function. Therefore, we need to find a time t such that $v(t)=s'(t)=v_{avg}=-40$ ft/sec. Since $s(t)$ is continuous over the interval $[0,5/2]$ and differentiable over the interval $(0,5/2)$, by the Mean Value Theorem, there is guaranteed to be a point $c\in(0,5/2)$ such that

$$s'(c)=\frac{s(5/2)-s(0)}{5/2-0}=-40.$$

Taking the derivative of the position function $s(t)$, we find that $s'(t)=-32t$. Therefore, the equation reduces to $s'(c)=-32c=-40$. Solving this equation for c, we have $c=\frac{5}{4}$. Therefore, $\frac{5}{4}$ sec after the rock is dropped, the instantaneous velocity equals the average velocity of the rock during its free fall: -40 ft/sec.

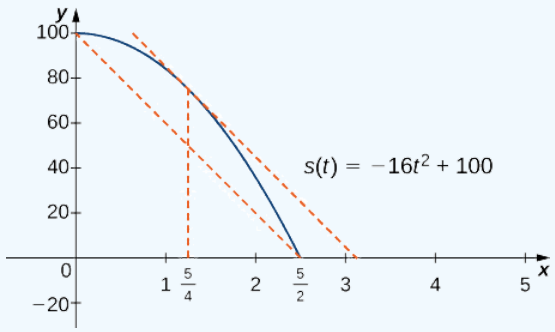


Figure 4.2.8: At time $t=5/4$ sec, the velocity of the rock is equal to its average velocity from the time it is dropped until it hits the ground.

Exercise 4.2.2

Suppose a ball is dropped from a height of 200 ft. Its position at time t is $s(t)=-16t^2+200$. Find the time t when the instantaneous velocity of the ball equals its average velocity.

Hint

First, determine how long it takes for the ball to hit the ground. Then, find the average velocity of the ball from the time it is dropped until it hits the ground.

Answer

$\frac{5}{2\sqrt{2}}$ sec

Corollaries of the Mean Value Theorem

Let's now look at three corollaries of the Mean Value Theorem. These results have important consequences, which we use in upcoming sections.

At this point, we know the derivative of any constant function is zero. The Mean Value Theorem allows us to conclude that the converse is also true. In particular, if $f'(x)=0$ for all x in some interval I, then $f(x)$ is constant over that interval. This result may seem intuitively obvious, but it has important implications that are not obvious, and we discuss them shortly.

Corollary 1: Functions with a Derivative of Zero

Let f be differentiable over an interval I. If $f'(x) = 0$ for all $x \in I$, then $f(x) =$ constant for all $x \in I$.

Proof

Since f is differentiable over I, f must be continuous over I. Suppose $f(x)$ is not constant for all x in I. Then there exist $a, b \in I$, where $a \neq b$ and $f(a) \neq f(b)$. Choose the notation so that $a < b$. Therefore,

$$\frac{f(b) - f(a)}{b - a} \neq 0.$$

Since f is a differentiable function, by the Mean Value Theorem, there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Therefore, there exists $c \in I$ such that $f'(c) \neq 0$, which contradicts the assumption that $f'(x) = 0$ for all $x \in I$.

□

From "Corollary 1: Functions with a Derivative of Zero," it follows that if two functions have the same derivative, they differ by, at most, a constant.

Corollary 2: Constant Difference Theorem

If f and g are differentiable over an interval I and $f'(x) = g'(x)$ for all $x \in I$, then $f(x) = g(x) + C$ for some constant C.

Proof

Let $h(x) = f(x) - g(x)$. Then, $h'(x) = f'(x) - g'(x) = 0$ for all $x \in I$. By Corollary 1, there is a constant C such that $h(x) = C$ for all $x \in I$. Therefore, $f(x) = g(x) + C$ for all $x \in I$.

□

The third corollary of the Mean Value Theorem discusses when a function is increasing and when it is decreasing. Recall that a function f is increasing over I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, whereas f is decreasing over I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$. Using the Mean Value Theorem, we can show that if the derivative of a function is positive, then the function is increasing; if the derivative is negative, then the function is decreasing (Figure 4.2.9). We make use of this fact in the next section, where we show how to use the derivative of a function to locate local maximum and minimum values of the function, and how to determine the shape of the graph.

This fact is important because it means that for a given function f, if there exists a function F such that $F'(x) = f(x)$; then, the only other functions that have a derivative equal to f are $F(x) + C$ for some constant C. We discuss this result in more detail later in the chapter.

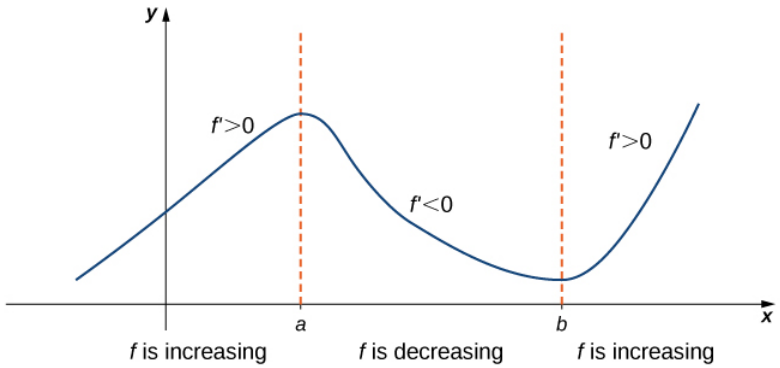


Figure 4.2.9: If a function has a positive derivative over some interval I, then the function increases over that interval I; if the derivative is negative over some interval I, then the function decreases over that interval I.

Corollary 3: Increasing and Decreasing Functions

Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b).

i. If $f'(x) > 0$ for all $x \in (a, b)$, then f is an increasing function over $[a, b]$.
ii. If $f'(x) < 0$ for all $x \in (a, b)$, then f is a decreasing function over $[a, b]$.

Proof

We will prove i.; the proof of ii. is similar. Suppose f is not an increasing function on I. Then there exist a and b in I such that $a < b$, but $f(a) \geq f(b)$. Since f is a differentiable function over I, by the Mean Value Theorem there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Since $f(a) \geq f(b)$, we know that $f(b) - f(a) \leq 0$. Also, $a < b$ tells us that $b - a > 0$. We conclude that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \leq 0.$$

However, $f'(x) > 0$ for all $x \in I$. This is a contradiction, and therefore f must be an increasing function over I.

□

Key Concepts

- If f is continuous over $[a, b]$ and differentiable over (a, b) and $f(a) = f(b)$, then there exists a point $c \in (a, b)$ such that $f'(c) = 0$. This is Rolle's theorem.
- If f is continuous over $[a, b]$ and differentiable over (a, b), then there exists a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

  This is the Mean Value Theorem.
- If $f'(x) = 0$ over an interval I, then f is constant over I.
- If two differentiable functions f and g satisfy $f'(x) = g'(x)$ over I, then $f(x) = g(x) + C$ for some constant C.
- If $f'(x) > 0$ over an interval I, then f is increasing over I. If $f'(x) < 0$ over I, then f is decreasing over I.

Glossary

mean value theorem

if f is continuous over $[a, b]$ and differentiable over (a, b), then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

rolle's theorem

if f is continuous over $[a, b]$ and differentiable over (a, b), and if $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$

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4.3: How Derivatives Affect the Shape of a Graph

Learning Objectives

- Explain how the sign of the first derivative affects the shape of a function's graph.
- State the first derivative test for critical points.
- Use concavity and inflection points to explain how the sign of the second derivative affects the shape of a function's graph.
- Explain the concavity test for a function over an open interval.
- Explain the relationship between a function and its first and second derivatives.
- State the second derivative test for local extrema.

Earlier in this chapter we stated that if a function f has a local extremum at a point c, then c must be a critical point of f. However, a function is not guaranteed to have a local extremum at a critical point. For example, $f(x) = x^3$ has a critical point at $x = 0$ since $f'(x) = 3x^2$ is zero at $x = 0$, but f does not have a local extremum at $x = 0$. Using the results from the previous section, we are now able to determine whether a critical point of a function actually corresponds to a local extreme value. In this section, we also see how the second derivative provides information about the shape of a graph by describing whether the graph of a function curves upward or curves downward.

The First Derivative Test

Corollary 3 of the Mean Value Theorem showed that if the derivative of a function is positive over an interval I then the function is increasing over I. On the other hand, if the derivative of the function is negative over an interval I, then the function is decreasing over I as shown in the following figure.

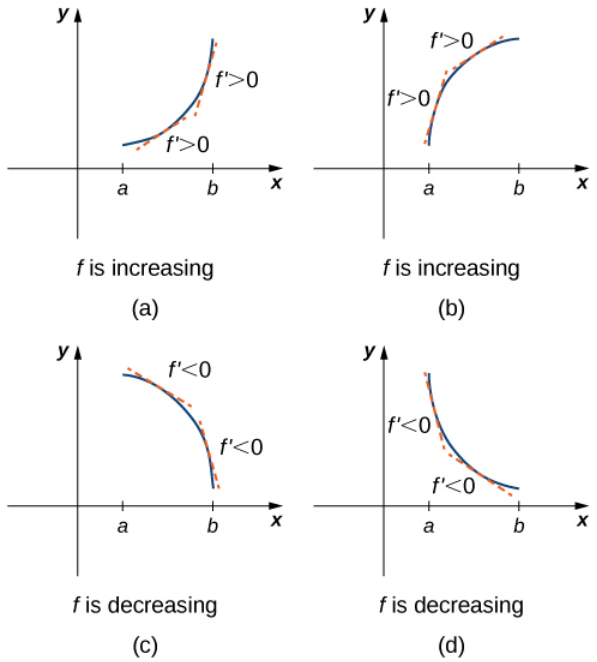


Figure 4.3.1: Both functions are increasing over the interval (a, b). At each point x, the derivative $f'(x) > 0$. Both functions are decreasing over the interval (a, b). At each point x, the derivative $f'(x) < 0$.

A continuous function f has a local maximum at point c if and only if f switches from increasing to decreasing at point c. Similarly, f has a local minimum at c if and only if f switches from decreasing to increasing at c. If f is a continuous function over an interval I containing c and differentiable over I, except possibly at c, the only way f can switch from increasing to decreasing (or vice versa) at point c is if f' changes sign as x increases through c. If f is differentiable at c, the only way that f' can change sign as x increases through c is if $f'(c) = 0$. Therefore, for a function f that is continuous over an interval I containing c and differentiable over I, except possibly at c, the only way f can switch from increasing to decreasing (or vice versa) is if $f'(c) = 0$ or $f'(c)$ is undefined. Consequently, to locate local extrema for a function f, we look for points c in the domain of f such that $f'(c) = 0$ or $f'(c)$ is undefined. Recall that such points are called critical points of f.

Note that f need not have a local extrema at a critical point. The critical points are candidates for local extrema only. In Figure 4.3.2, we show that if a continuous function f has a local extremum, it must occur at a critical point, but a function may not have a local extremum at a critical point. We show that if f has a local extremum at a critical point, then the sign of f' switches as x increases through that point.

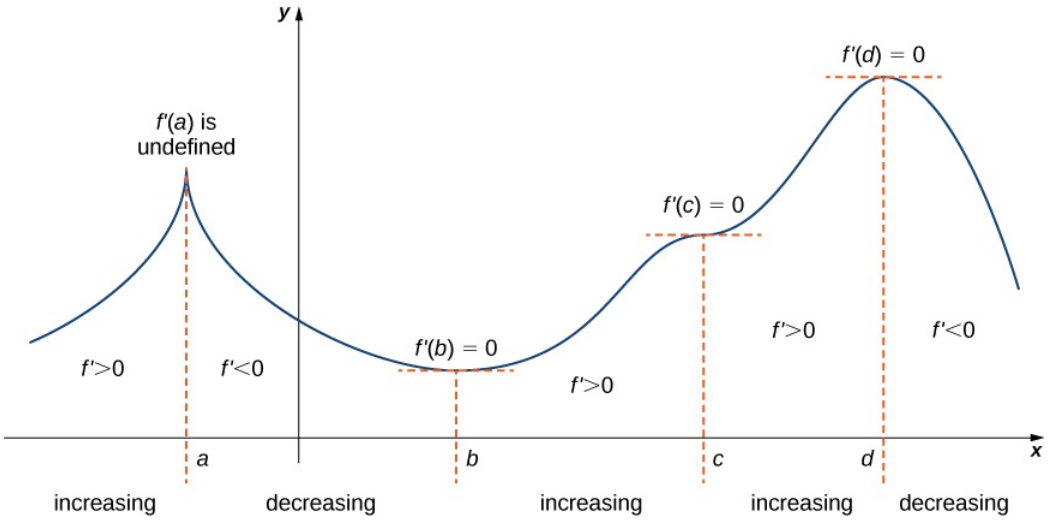


Figure 4.3.2: The function f has four critical points: a, b, c ,and d. The function f has local maxima at a and d, and a local minimum at b. The function f does not have a local extremum at c. The sign of f' changes at all local extrema.

Using Figure 4.3.2, we summarize the main results regarding local extrema.

- If a continuous function f has a local extremum, it must occur at a critical point c.
- The function has a local extremum at the critical point c if and only if the derivative f' switches sign as x increases through c.
- Therefore, to test whether a function has a local extremum at a critical point c, we must determine the sign of $f'(x)$ to the left and right of c.

This result is known as the **first derivative test.**

First Derivative Test

Suppose that f is a continuous function over an interval I containing a critical point c. If f is differentiable over I, except possibly at point c, then $f(c)$ satisfies one of the following descriptions:

i. If f' changes sign from positive when $x < c$ to negative when $x > c$, then $f(c)$ is a local maximum of f.
ii. If f' changes sign from negative when $x < c$ to positive when $x > c$, then $f(c)$ is a local minimum of f.
iii. If f' has the same sign for $x < c$ and $x > c$, then $f(c)$ is neither a local maximum nor a local minimum of f

Now let's look at how to use this strategy to locate all local extrema for particular functions.

Example 4.3.1: Using the First Derivative Test to Find Local Extrema

Use the first derivative test to find the location of all local extrema for $f(x) = x^3 - 3x^2 - 9x - 1$. Use a graphing utility to confirm your results.

Solution

Step 1. The derivative is $f'(x) = 3x^2 - 6x - 9$. To find the critical points, we need to find where $f'(x) = 0$. Factoring the polynomial, we conclude that the critical points must satisfy

$$3(x^2 - 2x - 3) = 3(x - 3)(x + 1) = 0.$$

Therefore, the critical points are $x = 3, -1$. Now divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, -1), (-1, 3)$ and $(3, \infty)$.

Step 2. Since f' is a continuous function, to determine the sign of $f'(x)$ over each subinterval, it suffices to choose a point over each of the intervals $(-\infty, -1), (-1, 3)$ and $(3, \infty)$ and determine the sign of f' at each of these points. For example, let's choose $x = -2$, $x = 0$, and $x = 4$ as test points.

Table: 4.3.1: First Derivative Test for $f(x) = x^3 - 3x^2 - 9x - 1$.

Interval	Test Point	Sign of $f'(x) = 3(x-3)(x+1)$ at Test Point	Conclusion
$(-\infty, -1)$	$x = -2$	(+)(−)(−)=+	f is increasing.
$(-1, 3)$	$x = 0$	(+)(−)(+)=-	f is decreasing.
$(3, \infty)$	$x = 4$	(+)(+)(+)=+	f is increasing.

Step 3. Since f' switches sign from positive to negative as x increases through -1, f has a local maximum at $x = -1$. Since f' switches sign from negative to positive as x increases through 3, f has a local minimum at $x = 3$. These analytical results agree with the following graph.

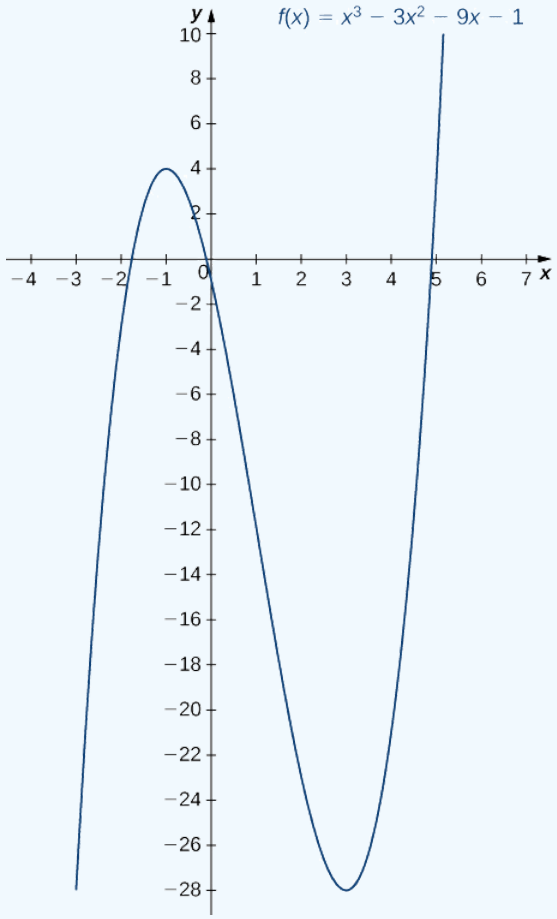


Figure 4.3.3: The function f has a maximum at $x = -1$ and a minimum at $x = 3$

? Exercise 4.3.1

Use the first derivative test to locate all local extrema for $f(x) = -x^3 + \frac{3}{2}x^2 + 18x$.

Hint

Find all critical points of f and determine the signs of $f'(x)$ over particular intervals determined by the critical points.

Answer

f has a local minimum at -2 and a local maximum at 3.

Example 4.3.2: Using the First Derivative Test

Use the first derivative test to find the location of all local extrema for $f(x) = 5x^{1/3} - x^{5/3}$. Use a graphing utility to confirm your results.

Solution

Step 1. The derivative is

$$f'(x) = \frac{5}{3}x^{-2/3} - \frac{5}{3}x^{2/3} = \frac{5}{3x^{2/3}} - \frac{5x^{2/3}}{3} = \frac{5 - 5x^{4/3}}{3x^{2/3}} = \frac{5(1 - x^{4/3})}{3x^{2/3}}.$$

The derivative $f'(x) = 0$ when $1 - x^{4/3} = 0$. Therefore, $f'(x) = 0$ at $x = \pm 1$. The derivative $f'(x)$ is undefined at $x = 0$. Therefore, we have three critical points: $x = 0$, $x = 1$, and $x = -1$. Consequently, divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, \infty)$.

Step 2: Since f' is continuous over each subinterval, it suffices to choose a test point x in each of the intervals from step 1 and determine the sign of f' at each of these points. The points $x = -2$, $x = -\frac{1}{2}$, $x = \frac{1}{2}$, and $x = 2$ are test points for these intervals.

Table: 4.3.2: First Derivative Test for $f(x) = 5x^{1/3} - x^{5/3}$.

Interval	Test Point	Sign of $f'(x) = \frac{5(1-x^{4/3})}{3x^{2/3}}$ at Test Point	Conclusion
$(-\infty, -1)$	$x = -2$	$\frac{(+)(-)}{+} = -$	f is decreasing.
$(-1, 0)$	$x = -\frac{1}{2}$	$\frac{(+)(+)}{+} = +$	f is increasing.
$(0, 1)$	$x = \frac{1}{2}$	$\frac{(+)(+)}{+} = +$	f is increasing.
$(1, \infty)$	$x = 2$	$\frac{(+)(-)}{+} = -$	f is decreasing.

Step 3: Since f is decreasing over the interval $(-\infty, -1)$ and increasing over the interval $(-1, 0)$, f has a local minimum at $x = -1$. Since f is increasing over the interval $(-1, 0)$ and the interval $(0, 1)$, f does not have a local extremum at $x = 0$. Since f is increasing over the interval $(0, 1)$ and decreasing over the interval $(1, \infty)$, f has a local maximum at $x = 1$. The analytical results agree with the following graph.

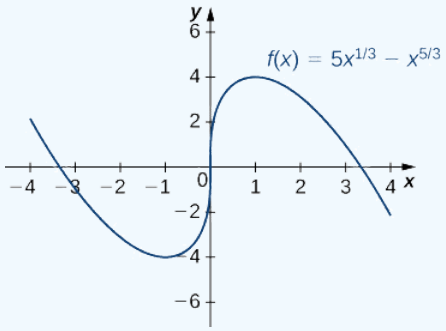


Figure 4.3.4: The function f has a local minimum at $x = -1$ and a local maximum at $x = 1$

Exercise 4.3.2

Use the first derivative test to find all local extrema for $f(x) = \dfrac{3}{x - 1}$.

Hint

The only critical point of f is $x = 1$.

Answer

f has no local extrema because f' does not change sign at $x = 1$.

Concavity and Points of Inflection

We now know how to determine where a function is increasing or decreasing. However, there is another issue to consider regarding the shape of the graph of a function. If the graph curves, does it curve upward or curve downward? This notion is called the **concavity** of the function.

Figure $4.3.5a$ shows a function f with a graph that curves upward. As x increases, the slope of the tangent line increases. Thus, since the derivative increases as x increases, f' is an increasing function. We say this function f is concave up. Figure $4.3.5b$ shows a function f that curves downward. As x increases, the slope of the tangent line decreases. Since the derivative decreases as x increases, f' is a decreasing function. We say this function f is concave down.

Definition: concavity test

Let f be a function that is differentiable over an open interval I. If f' is increasing over I, we say f is **concave up** over I. If f' is decreasing over I, we say f is **concave down** over I.

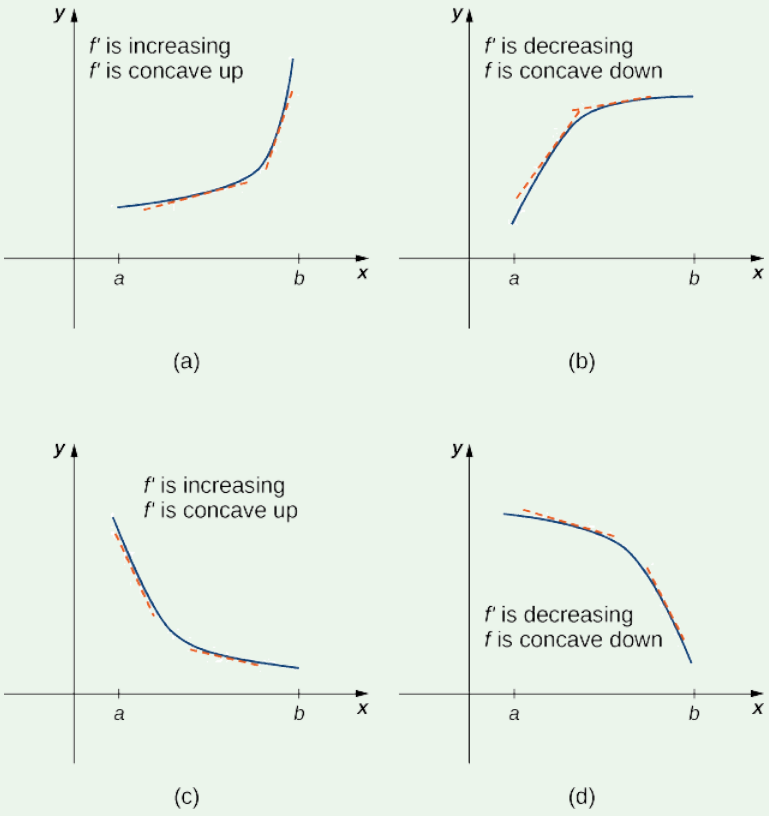


Figure 4.3.5: (a), (c) Since f' is increasing over the interval (a, b), we say f is concave up over (a, b). $(b), (d)$ Since f' is decreasing over the interval (a, b), we say f is concave down over (a, b).

In general, without having the graph of a function f, how can we determine its concavity? By definition, a function f is concave up if f' is increasing. From Corollary 3, we know that if f' is a differentiable function, then f' is increasing if its derivative $f''(x) > 0$. Therefore, a function f that is twice differentiable is concave up when $f''(x) > 0$. Similarly, a function f is concave down if f' is decreasing. We know that a differentiable function f' is decreasing if its derivative $f''(x) < 0$. Therefore, a twice-differentiable function f is concave down when $f''(x) < 0$. Applying this logic is known as the *concavity test*.

Test for Concavity

Let f be a function that is twice differentiable over an interval I.

i. If $f''(x) > 0$ for all $x \in I$, then f is concave up over I
ii. If $f''(x) < 0$ for all $x \in I$, then f is concave down over I.

We conclude that we can determine the concavity of a function f by looking at the second derivative of f. In addition, we observe that a function f can switch concavity (Figure 4.3.6). However, a continuous function can switch concavity only at a point x if $f''(x) = 0$ or $f''(x)$ is undefined. Consequently, to determine the intervals where a function f is concave up and concave down, we look for those values of x where $f''(x) = 0$ or $f''(x)$ is undefined. When we have determined these points, we divide the domain of f into smaller intervals and determine the sign of f'' over each of these smaller intervals. If f'' changes sign as we pass through a point x, then f changes concavity. It is important to remember that a function f may not change concavity at a point x even if $f''(x) = 0$ or $f''(x)$ is undefined. If, however, f does change concavity at a point a and f is continuous at a, we say the point $(a, f(a))$ is an **inflection point** of f.

Definition: inflection point

If f is continuous at a and f changes concavity at a, the point $(a,\ f(a))$ is an *inflection point* of f.

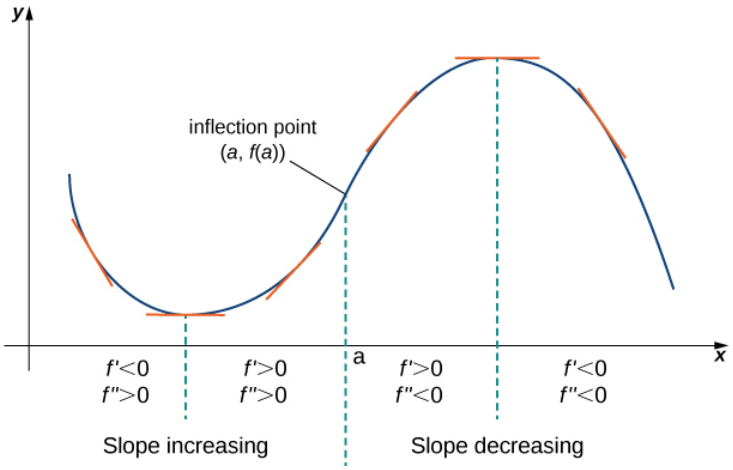


Figure 4.3.6: Since $f''(x) > 0$ for $x < a$, the function f is concave up over the interval $(-\infty, a)$. Since $f''(x) < 0$ for $x > a$, the function f is concave down over the interval (a, ∞). The point $(a, f(a))$ is an inflection point of f.

Example 4.3.3: Testing for Concavity

For the function $f(x) = x^3 - 6x^2 + 9x + 30$, determine all intervals where f is concave up and all intervals where f is concave down. List all inflection points for f. Use a graphing utility to confirm your results.

Solution

To determine concavity, we need to find the second derivative $f''(x)$. The first derivative is $f'(x) = 3x^2 - 12x + 9$, so the second derivative is $f''(x) = 6x - 12$. If the function changes concavity, it occurs either when $f''(x) = 0$ or $f''(x)$ is undefined. Since f'' is defined for all real numbers x, we need only find where $f''(x) = 0$. Solving the equation $6x - 12 = 0$, we see that $x = 2$ is the only place where f could change concavity. We now test points over the intervals $(-\infty, 2)$ and $(2, \infty)$ to determine the concavity of f. The points $x = 0$ and $x = 3$ are test points for these intervals.

Table: 4.3.3: Test for Concavity for $f(x) = x^3 - 6x^2 + 9x + 30$.

Interval	Test Point	Sign of $f''(x) = 6x - 12$ at Test Point	Conclusion
$(-\infty, 2)$	$x = 0$	$-$	f is concave down
$(2, \infty)$	$x = 3$	$+$	f is concave up

We conclude that f is concave down over the interval $(-\infty, 2)$ and concave up over the interval $(2, \infty)$. Since f changes concavity at $x = 2$, the point $(2, f(2)) = (2, 32)$ is an inflection point. Figure $4.3.7$ confirms the analytical results.

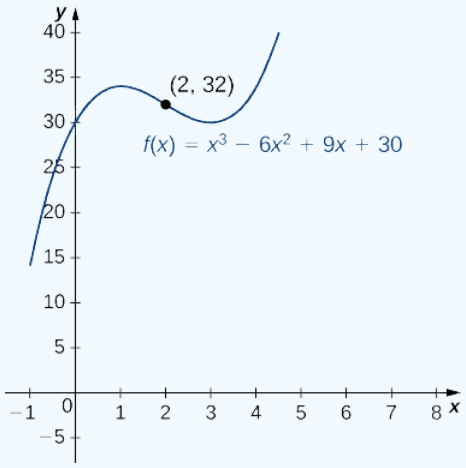


Figure $4.3.7$: The given function has a point of inflection at $(2, 32)$ where the graph changes concavity.

Exercise $4.3.3$

For $f(x) = -x^3 + \frac{3}{2}x^2 + 18x$, find all intervals where f is concave up and all intervals where f is concave down.

Hint

Find where $f''(x) = 0$

Answer

f is concave up over the interval $(-\infty, \frac{1}{2})$ and concave down over the interval $(\frac{1}{2}, \infty)$

We now summarize, in Table $4.3.4$, the information that the first and second derivatives of a function f provide about the graph of f, and illustrate this information in Figure $4.3.8$.

Table: $4.3.4$: What Derivatives Tell Us about Graphs

Sign of f'	Sign of f''	Is f increasing or decreasing?	Concavity
Positive	Positive	Increasing	Concave up
Positive	Negative	Increasing	Concave down
Negative	Positive	Decreasing	Concave up
Negative	Negative	Decreasing	Concave down

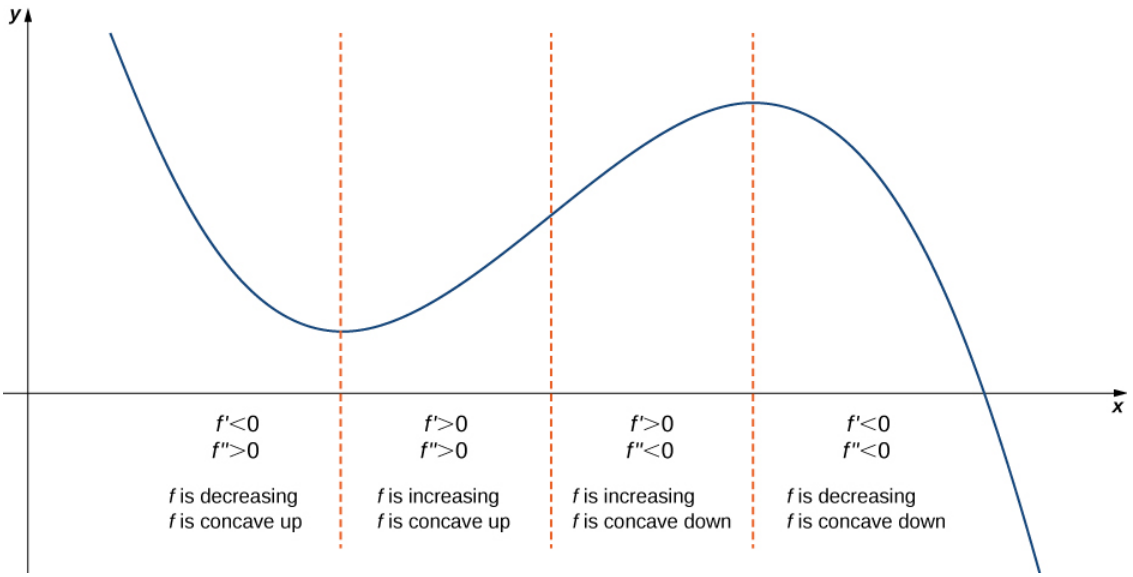


Figure 4.3.8:Consider a twice-differentiable function f over an open interval I. If $f'(x) > 0$ for all $x \in I$, the function is increasing over I. If $f'(x) < 0$ for all $x \in I$, the function is decreasing over I. If $f''(x) > 0$ for all $x \in I$, the function is concave up. If $f''(x) < 0$ for all $x \in I$, the function is concave down on I.

The Second Derivative Test

The first derivative test provides an analytical tool for finding local extrema, but the second derivative can also be used to locate extreme values. Using the second derivative can sometimes be a simpler method than using the first derivative.

We know that if a continuous function has a local extremum, it must occur at a critical point. However, a function need not have a local extremum at a critical point. Here we examine how the **second derivative test** can be used to determine whether a function has a local extremum at a critical point. Let f be a twice-differentiable function such that $f'(a) = 0$ and f'' is continuous over an open interval I containing a. Suppose $f''(a) < 0$. Since f'' is continuous over I, $f''(x) < 0$ for all $x \in I$ (Figure 4.3.9). Then, by Corollary 3, f' is a decreasing function over I. Since $f'(a) = 0$, we conclude that for all $x \in I$, $f'(x) > 0$ if $x < a$ and $f'(x) < 0$ if $x > a$. Therefore, by the first derivative test, f has a local maximum at $x = a$.

On the other hand, suppose there exists a point b such that $f'(b) = 0$ but $f''(b) > 0$. Since f'' is continuous over an open interval I containing b, then $f''(x) > 0$ for all $x \in I$ (Figure 4.3.9). Then, by Corollary 3, f' is an increasing function over I. Since $f'(b) = 0$, we conclude that for all $x \in I$, $f'(x) < 0$ if $x < b$ and $f'(x) > 0$ if $x > b$. Therefore, by the first derivative test, f has a local minimum at $x = b$.

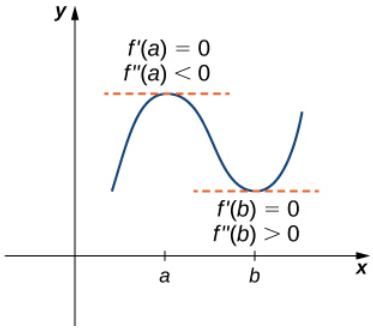


Figure 4.3.9: Consider a twice-differentiable function f such that f'' is continuous. Since $f'(a) = 0$ and $f''(a) < 0$, there is an interval I containing a such that for all x in I, f is increasing if $x < a$ and f is decreasing if $x > a$. As a result, f has a local maximum at $x = a$. Since $f'(b) = 0$ and $f''(b) > 0$, there is an interval I containing b such that for all x in I, f is decreasing if $x < b$ and f is increasing if $x > b$. As a result, f has a local minimum at $x = b$.

Second Derivative Test

Suppose $f'(c) = 0$ and f'' is continuous over an interval containing c.

i. If $f''(c) > 0$, then f has a local minimum at c.
ii. If $f''(c) < 0$, then f has a local maximum at c.
iii. If $f''(c) = 0$, then the test is inconclusive.

Note that for case iii. when $f''(c) = 0$, then f may have a local maximum, local minimum, or neither at c. For example, the functions $f(x) = x^3$, $f(x) = x^4$, and $f(x) = -x^4$ all have critical points at $x = 0$. In each case, the second derivative is zero at $x = 0$. However, the function $f(x) = x^4$ has a local minimum at $x = 0$ whereas the function $f(x) = -x^4$ has a local maximum at $x = 0$, and the function $f(x) = x^3$ does not have a local extremum at $x = 0$.

Let's now look at how to use the second derivative test to determine whether f has a local maximum or local minimum at a critical point c where $f'(c) = 0$.

Example 4.3.4: Using the Second Derivative Test

Use the second derivative to find the location of all local extrema for $f(x) = x^5 - 5x^3$.

Solution

To apply the second derivative test, we first need to find critical points c where $f'(c) = 0$. The derivative is $f'(x) = 5x^4 - 15x^2$. Therefore, $f'(x) = 5x^4 - 15x^2 = 5x^2(x^2 - 3) = 0$ when $x = 0, \pm\sqrt{3}$.

To determine whether f has a local extremum at any of these points, we need to evaluate the sign of f'' at these points. The second derivative is

$$f''(x) = 20x^3 - 30x = 10x(2x^2 - 3).$$

In the following table, we evaluate the second derivative at each of the critical points and use the second derivative test to determine whether f has a local maximum or local minimum at any of these points.

Table: 4.3.5: Second Derivative Test for $f(x) = x^5 - 5x^3$.

x	$f''(x)$	Conclusion
$-\sqrt{3}$	$-30\sqrt{3}$	Local maximum
0	0	Second derivative test is inconclusive
$\sqrt{3}$	$30\sqrt{3}$	Local minimum

By the second derivative test, we conclude that f has a local maximum at $x = -\sqrt{3}$ and f has a local minimum at $x = \sqrt{3}$. The second derivative test is inconclusive at $x = 0$. To determine whether f has a local extrema at $x = 0$, we apply the first derivative test. To evaluate the sign of $f'(x) = 5x^2(x^2 - 3)$ for $x \in (-\sqrt{3}, 0)$ and $x \in (0, \sqrt{3})$, let $x = -1$ and $x = 1$ be the two test points. Since $f'(-1) < 0$ and $f'(1) < 0$, we conclude that f is decreasing on both intervals and, therefore, f does not have a local extrema at $x = 0$ as shown in the following graph.

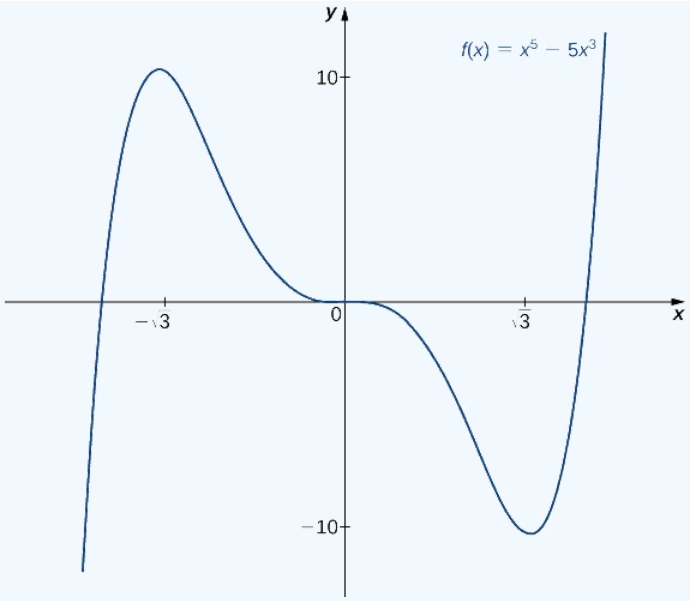


Figure 4.3.10:The function f has a local maximum at $x=-\sqrt{3}$ and a local minimum at $x=\sqrt{3}$

? Exercise 4.3.4

Consider the function $f(x)=x^3-(\frac{3}{2})x^2-18x$. The points $c=3,\ -2$ satisfy $f'(c)=0$. Use the second derivative test to determine whether f has a local maximum or local minimum at those points.

Hint

$f''(x)=6x-3$

Answer

f has a local maximum at -2 and a local minimum at 3.

We have now developed the tools we need to determine where a function is increasing and decreasing, as well as acquired an understanding of the basic shape of the graph. In the next section we discuss what happens to a function as $x\to\pm\infty$. At that point, we have enough tools to provide accurate graphs of a large variety of functions.

Key Concepts

- If c is a critical point of f and $f'(x)>0$ for $x<c$ and $f'(x)<0$ for $x>c$, then f has a local maximum at c.
- If c is a critical point of f and $f'(x)<0$ for $x<c$ and $f'(x)>0$ for $x>c$, then f has a local minimum at c.
- If $f''(x)>0$ over an interval I, then f is concave up over I.
- If $f''(x)<0$ over an interval I, then f is concave down over I.
- If $f'(c)=0$ and $f''(c)>0$, then f has a local minimum at c.
- If $f'(c)=0$ and $f''(c)<0$, then f has a local maximum at c.
- If $f'(c)=0$ and $f''(c)=0$, then evaluate $f'(x)$ at a test point x to the left of c and a test point x to the right of c, to determine whether f has a local extremum at c.

Glossary

concave down

if f is differentiable over an interval I and f' is decreasing over I, then f is concave down over I

concave up

if f is differentiable over an interval I and f' is increasing over I, then f is concave up over I

concavity

the upward or downward curve of the graph of a function

concavity test

suppose f is twice differentiable over an interval I; if $f'' > 0$ over I, then f is concave up over I; if $f'' <$ over I, then f is concave down over I

first derivative test

let f be a continuous function over an interval I containing a critical point c such that f is differentiable over I except possibly at c; if f' changes sign from positive to negative as x increases through c, then f has a local maximum at c; if f' changes sign from negative to positive as x increases through c, then f has a local minimum at c; if f' does not change sign as x increases through c, then f does not have a local extremum at c

inflection point

if f is continuous at c and f changes concavity at c, the point $(c, f(c))$ is an inflection point of f

second derivative test

suppose $f'(c) = 0$ and f'' is continuous over an interval containing c; if $f''(c) > 0$, then f has a local minimum at c; if $f''(c) < 0$, then f has a local maximum at c; if $f''(c) = 0$, then the test is inconclusive

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4.4: Indeterminate Forms and l'Hospital's Rule

Learning Objectives

- Recognize when to apply L'Hôpital's rule.
- Identify indeterminate forms produced by quotients, products, subtractions, and powers, and apply L'Hôpital's rule in each case.
- Describe the relative growth rates of functions.

In this section, we examine a powerful tool for evaluating limits. This tool, known as **L'Hôpital's rule,** uses derivatives to calculate limits. With this rule, we will be able to evaluate many limits we have not yet been able to determine. Instead of relying on numerical evidence to conjecture that a limit exists, we will be able to show definitively that a limit exists and to determine its exact value.

Applying L'Hôpital's Rule

L'Hôpital's rule can be used to evaluate limits involving the quotient of two functions. Consider

$$\lim_{x\to a}\frac{f(x)}{g(x)}.$$

If $\lim_{x\to a} f(x)=L_1$ and $\lim_{x\to a} g(x)=L_2\neq 0,$ then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{L_1}{L_2}.$$

However, what happens if $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$? We call this one of the **indeterminate forms,** of type $\frac{0}{0}$. This is considered an indeterminate form because we cannot determine the exact behavior of $\frac{f(x)}{g(x)}$ as $x\to a$ without further analysis. We have seen examples of this earlier in the text. For example, consider

$$\lim_{x\to 2}\frac{x^2-4}{x-2}$$

and

$$\lim_{x\to 0}\frac{\sin x}{x}.$$

For the first of these examples, we can evaluate the limit by factoring the numerator and writing

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x+2)(x-2)}{x-2}=\lim_{x\to 2}(x+2)=2+2=4.$$

For $\lim_{x\to 0}\frac{\sin x}{x}$ we were able to show, using a geometric argument, that

$$\lim_{x\to 0}\frac{\sin x}{x}=1.$$

Here we use a different technique for evaluating limits such as these. Not only does this technique provide an easier way to evaluate these limits, but also, and more importantly, it provides us with a way to evaluate many other limits that we could not calculate previously.

The idea behind L'Hôpital's rule can be explained using local linear approximations. Consider two differentiable functions f and g such that $\lim_{x\to a} f(x)=0=\lim_{x\to a} g(x)$ and such that $g'(a)\neq 0$ For x near a,we can write

$$f(x)\approx f(a)+f'(a)(x-a)$$

and

$$g(x) \approx g(a) + g'(a)(x-a).$$

Therefore,

$$\frac{f(x)}{g(x)} \approx \frac{f(a) + f'(a)(x-a)}{g(a) + g'(a)(x-a)}.$$

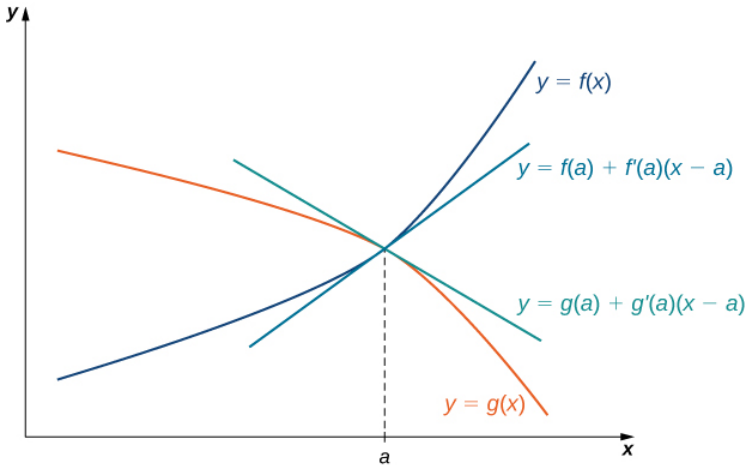


Figure 4.4.1: If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$, then the ratio $f(x)/g(x)$ is approximately equal to the ratio of their linear approximations near a.

Since f is differentiable at a, then f is continuous at a, and therefore $f(a) = \lim_{x \to a} f(x) = 0$. Similarly, $g(a) = \lim_{x \to a} g(x) = 0$. If we also assume that f' and g' are continuous at $x = a$, then $f'(a) = \lim_{x \to a} f'(x)$ and $g'(a) = \lim_{x \to a} g'(x)$. Using these ideas, we conclude that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)(x-a)}{g'(x)(x-a)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Note that the assumption that f' and g' are continuous at a and $g'(a) \neq 0$ can be loosened. We state L'Hôpital's rule formally for the indeterminate form $\frac{0}{0}$. Also note that the notation $\frac{0}{0}$ does not mean we are actually dividing zero by zero. Rather, we are using the notation $\frac{0}{0}$ to represent a quotient of limits, each of which is zero.

L'Hôpital's Rule (0/0 Case)

Suppose f and g are differentiable functions over an open interval containing a, except possibly at a. If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming the limit on the right exists or is ∞ or $-\infty$. This result also holds if we are considering one-sided limits, or if $a = \infty$ or $a = -\infty$.

Proof

We provide a proof of this theorem in the special case when f, g, f', and g' are all continuous over an open interval containing a. In that case, since $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$ and f and g are continuous at a, it follows that $f(a) = 0 = g(a)$. Therefore,

$$\begin{aligned}
\lim_{x\to a}\frac{f(x)}{g(x)} &= \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)} && \text{Since } f(a)=0=g(a)\\
&= \lim_{x\to a}\frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}} && \text{Multiply numerator and denominator by } \frac{1}{x-a}\\
&= \frac{\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}}{\lim\limits_{x\to a}\dfrac{g(x)-g(a)}{x-a}} && \text{The limit of a quotient is the quotient of the limits.}\\
&= \frac{f'(a)}{g'(a)} && \text{By the definition of the derivative}\\
&= \frac{\lim\limits_{x\to a}f'(x)}{\lim\limits_{x\to a}g'(x)} && \text{By the continuity of } f' \text{ and } g'\\
&= \lim_{x\to a}\frac{f'(x)}{g'(x)}. && \text{The limit of a quotient}
\end{aligned}$$

Note that L'Hôpital's rule states we can calculate the limit of a quotient $\frac{f}{g}$ by considering the limit of the quotient of the derivatives $\frac{f'}{g'}$. It is important to realize that we are not calculating the derivative of the quotient $\frac{f}{g}$.

□

Example 4.4.1: Applying L'Hôpital's Rule (0/0 Case)

Evaluate each of the following limits by applying L'Hôpital's rule.

a. $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x}$

b. $\lim\limits_{x\to 1}\dfrac{\sin(\pi x)}{\ln x}$

c. $\lim\limits_{x\to\infty}\dfrac{e^{1/x}-1}{1/x}$

d. $\lim\limits_{x\to 0}\dfrac{\sin x - x}{x^2}$

Solution

a. Since the numerator $1-\cos x\to 0$ and the denominator $x\to 0$, we can apply L'Hôpital's rule to evaluate this limit. We have

$$\lim_{x\to 0}\frac{1-\cos x}{x}=\lim_{x\to 0}\frac{\frac{d}{dx}(1-\cos x)}{\frac{d}{dx}(x)}=\lim_{x\to 0}\frac{\sin x}{1}=\frac{\lim\limits_{x\to 0}\sin x}{\lim\limits_{x\to 0}1}=\frac{0}{1}=0.$$

b. As $x\to 1$, the numerator $\sin(\pi x)\to 0$ and the denominator $\ln(x)\to 0$. Therefore, we can apply L'Hôpital's rule. We obtain

$$\begin{aligned}
\lim_{x\to 1}\frac{\sin(\pi x)}{\ln x} &= \lim_{x\to 1}\frac{\pi\cos(\pi x)}{1/x}\\
&= \lim_{x\to 1}(\pi x)\cos(\pi x)\\
&= (\pi\cdot 1)(-1)=-\pi.
\end{aligned}$$

c. As $x \to \infty$, the numerator $e^{1/x} - 1 \to 0$ and the denominator $\frac{1}{x} \to 0$. Therefore, we can apply L'Hôpital's rule. We obtain

$$\lim_{x\to\infty} \frac{e^{1/x}-1}{\frac{1}{x}} = \lim_{x\to\infty} \frac{e^{1/x}\left(\frac{-1}{x^2}\right)}{\left(\frac{-1}{x^2}\right)} = \lim_{x\to\infty} e^{1/x} = e^0 = 1.$$

d. As $x \to 0$, both the numerator and denominator approach zero. Therefore, we can apply L'Hôpital's rule. We obtain

$$\lim_{x\to 0} \frac{\sin x - x}{x^2} = \lim_{x\to 0} \frac{\cos x - 1}{2x}.$$

Since the numerator and denominator of this new quotient both approach zero as $x \to 0$, we apply L'Hôpital's rule again. In doing so, we see that

$$\lim_{x\to 0} \frac{\cos x - 1}{2x} = \lim_{x\to 0} \frac{-\sin x}{2} = 0.$$

Therefore, we conclude that

$$\lim_{x\to 0} \frac{\sin x - x}{x^2} = 0.$$

Exercise 4.4.1

Evaluate $\lim_{x\to 0} \frac{x}{\tan x}$.

Hint

$\frac{d}{dx}\left(\tan x\right) = \sec^2 x$

Answer

1

We can also use L'Hôpital's rule to evaluate limits of quotients $\frac{f(x)}{g(x)}$ in which $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$. Limits of this form are classified as indeterminate forms of type ∞/∞. Again, note that we are not actually dividing ∞ by ∞. Since ∞ is not a real number, that is impossible; rather, ∞/∞ is used to represent a quotient of limits, each of which is ∞ or $-\infty$.

L'Hôpital's Rule (∞/∞ Case)

Suppose f and g are differentiable functions over an open interval containing a, except possibly at a. Suppose $\lim_{x\to a} f(x) = \infty$ (or $-\infty$) and $\lim_{x\to a} g(x) = \infty$ (or $-\infty$). Then,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f\,'(x)}{g'(x)}$$

assuming the limit on the right exists or is ∞ or $-\infty$. This result also holds if the limit is infinite, if $a = \infty$ or $-\infty$, or the limit is one-sided.

Example 4.4.2: Applying L'Hôpital's Rule (∞/∞) Case

Evaluate each of the following limits by applying L'Hôpital's rule.

a. $\lim_{x\to\infty} \frac{3x+5}{2x+1}$

b. $\lim_{x\to 0^+} \frac{\ln x}{\cot x}$

Solution

a. Since $3x+5$ and $2x+1$ are first-degree polynomials with positive leading coefficients, $\lim_{x\to\infty}(3x+5)=\infty$ and $\lim_{x\to\infty}(2x+1)=\infty$. Therefore, we apply L'Hôpital's rule and obtain

$$\lim_{x\to\infty}\frac{3x+5}{2x+1}=\lim_{x\to\infty}\frac{3}{2}=\frac{3}{2}.$$

Note that this limit can also be calculated without invoking L'Hôpital's rule. Earlier in the chapter we showed how to evaluate such a limit by dividing the numerator and denominator by the highest power of x in the denominator. In doing so, we saw that

$$\lim_{x\to\infty}\frac{3x+5}{2x+1}=\lim_{x\to\infty}\frac{3+5/x}{2+1/x}=\frac{3}{2}.$$

L'Hôpital's rule provides us with an alternative means of evaluating this type of limit.

b. Here, $\lim_{x\to0^+}\ln x=-\infty$ and $\lim_{x\to0^+}\cot x=\infty$. Therefore, we can apply L'Hôpital's rule and obtain

$$\lim_{x\to0^+}\frac{\ln x}{\cot x}=\lim_{x\to0^+}\frac{1/x}{-\csc^2 x}=\lim_{x\to0^+}\frac{1}{-x\csc^2 x}.$$

Now as $x\to0^+$, $\csc^2 x\to\infty$. Therefore, the first term in the denominator is approaching zero and the second term is getting really large. In such a case, anything can happen with the product. Therefore, we cannot make any conclusion yet. To evaluate the limit, we use the definition of $\csc x$ to write

$$\lim_{x\to0^+}\frac{1}{-x\csc^2 x}=\lim_{x\to0^+}\frac{\sin^2 x}{-x}.$$

Now $\lim_{x\to0^+}\sin^2 x=0$ and $\lim_{x\to0^+}-x=0$, so we apply L'Hôpital's rule again. We find

$$\lim_{x\to0^+}\frac{\sin^2 x}{-x}=\lim_{x\to0^+}\frac{2\sin x\cos x}{-1}=\frac{0}{-1}=0.$$

We conclude that

$$\lim_{x\to0^+}\frac{\ln x}{\cot x}=0.$$

? Exercise 4.4.2

Evaluate $\lim_{x\to\infty}\frac{\ln x}{5x}$.

Hint

$\frac{d}{dx}(\ln x)=\frac{1}{x}$

Answer

0

As mentioned, L'Hôpital's rule is an extremely useful tool for evaluating limits. It is important to remember, however, that to apply L'Hôpital's rule to a quotient $\frac{f(x)}{g(x)}$, it is essential that the limit of $\frac{f(x)}{g(x)}$ be of the form $\frac{0}{0}$ or ∞/∞. Consider the following example.

Example 4.4.3: When L'Hôpital's Rule Does Not Apply

Consider $\lim_{x\to 1}\dfrac{x^2+5}{3x+4}$.

Show that the limit cannot be evaluated by applying L'Hôpital's rule.

Solution

Because the limits of the numerator and denominator are not both zero and are not both infinite, we cannot apply L'Hôpital's rule. If we try to do so, we get

$$\frac{d}{dx}(x^2+5)=2x$$

and

$$\frac{d}{dx}(3x+4)=3.$$

At which point we would conclude erroneously that

$$\lim_{x\to 1}\frac{x^2+5}{3x+4}=\lim_{x\to 1}\frac{2x}{3}=\frac{2}{3}.$$

However, since $\lim_{x\to 1}(x^2+5)=6$ and $\lim_{x\to 1}(3x+4)=7,$ we actually have

$$\lim_{x\to 1}\frac{x^2+5}{3x+4}=\frac{6}{7}.$$

We can conclude that

$$\lim_{x\to 1}\frac{x^2+5}{3x+4}\neq\lim_{x\to 1}\frac{\dfrac{d}{dx}(x^2+5)}{\dfrac{d}{dx}(3x+4)}.$$

Exercise 4.4.3

Explain why we cannot apply L'Hôpital's rule to evaluate $\lim_{x\to 0^+}\dfrac{\cos x}{x}$. Evaluate $\lim_{x\to 0^+}\dfrac{\cos x}{x}$ by other means.

Hint

Determine the limits of the numerator and denominator separately.

Answer

$\lim_{x\to 0^+}\cos x=1$. Therefore, we cannot apply L'Hôpital's rule. The limit of the quotient is ∞.

Other Indeterminate Forms

L'Hôpital's rule is very useful for evaluating limits involving the indeterminate forms $\dfrac{0}{0}$ and ∞/∞. However, we can also use L'Hôpital's rule to help evaluate limits involving other indeterminate forms that arise when evaluating limits. The expressions $0\cdot\infty, \infty-\infty, 1^\infty, \infty^0$, and 0^0 are all considered indeterminate forms. These expressions are not real numbers. Rather, they represent forms that arise when trying to evaluate certain limits. Next we realize why these are indeterminate forms and then understand how to use L'Hôpital's rule in these cases. The key idea is that we must rewrite the indeterminate forms in such a way that we arrive at the indeterminate form $\dfrac{0}{0}$ or ∞/∞.

Indeterminate Form of Type $0 \cdot \infty$

Suppose we want to evaluate $\lim_{x \to a}(f(x) \cdot g(x))$, where $f(x) \to 0$ and $g(x) \to \infty$ (or $-\infty$) as $x \to a$. Since one term in the product is approaching zero but the other term is becoming arbitrarily large (in magnitude), anything can happen to the product. We use the notation $0 \cdot \infty$ to denote the form that arises in this situation. The expression $0 \cdot \infty$ is considered indeterminate because we cannot determine without further analysis the exact behavior of the product $f(x)g(x)$ as $x \to \infty$. For example, let n be a positive integer and consider

$$f(x) = \frac{1}{(x^n + 1)} \text{ and } g(x) = 3x^2.$$

As $x \to \infty$, $f(x) \to 0$ and $g(x) \to \infty$. However, the limit as $x \to \infty$ of $f(x)g(x) = \dfrac{3x^2}{(x^n + 1)}$ varies, depending on n. If $n = 2$, then $\lim_{x \to \infty} f(x)g(x) = 3$. If $n = 1$, then $\lim_{x \to \infty} f(x)g(x) = \infty$. If $n = 3$, then $\lim_{x \to \infty} f(x)g(x) = 0$. Here we consider another limit involving the indeterminate form $0 \cdot \infty$ and show how to rewrite the function as a quotient to use L'Hôpital's rule.

Example 4.4.4: Indeterminate Form of Type $0 \cdot \infty$

Evaluate $\lim_{x \to 0^+} x \ln x$.

Solution

First, rewrite the function $x \ln x$ as a quotient to apply L'Hôpital's rule. If we write

$$x \ln x = \frac{\ln x}{1/x}$$

we see that $\ln x \to -\infty$ as $x \to 0^+$ and $\dfrac{1}{x} \to \infty$ as $x \to 0^+$. Therefore, we can apply L'Hôpital's rule and obtain

$$\lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(1/x)} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0.$$

We conclude that

$$\lim_{x \to 0^+} x \ln x = 0.$$

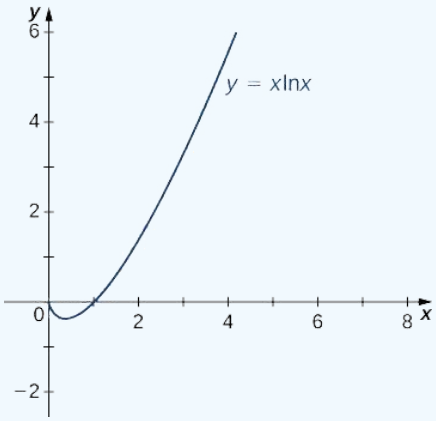


Figure 4.4.2: Finding the limit at $x = 0$ of the function $f(x) = x \ln x$.

Exercise 4.4.4

Evaluate $\lim_{x \to 0} x \cot x$.

Hint

Write $x \cot x = \dfrac{x \cos x}{\sin x}$

Answer

1

Indeterminate Form of Type ∞-∞

Another type of indeterminate form is $\infty - \infty$. Consider the following example. Let n be a positive integer and let $f(x) = 3x^n$ and $g(x) = 3x^2 + 5$. As $x \to \infty$, $f(x) \to \infty$ and $g(x) \to \infty$. We are interested in $\lim\limits_{x \to \infty} (f(x) - g(x))$. Depending on whether $f(x)$ grows faster, $g(x)$ grows faster, or they grow at the same rate, as we see next, anything can happen in this limit. Since $f(x) \to \infty$ and $g(x) \to \infty$, we write $\infty - \infty$ to denote the form of this limit. As with our other indeterminate forms, $\infty - \infty$ has no meaning on its own and we must do more analysis to determine the value of the limit. For example, suppose the exponent n in the function $f(x) = 3x^n$ is $n = 3$, then

$$\lim_{x \to \infty} (f(x) - g(x)) = \lim_{x \to \infty} (3x^3 - 3x^2 - 5) = \infty.$$

On the other hand, if $n = 2$, then

$$\lim_{x \to \infty} (f(x) - g(x)) = \lim_{x \to \infty} (3x^2 - 3x^2 - 5) = -5.$$

However, if $n = 1$, then

$$\lim_{x \to \infty} (f(x) - g(x)) = \lim_{x \to \infty} (3x - 3x^2 - 5) = -\infty.$$

Therefore, the limit cannot be determined by considering only $\infty - \infty$. Next we see how to rewrite an expression involving the indeterminate form $\infty - \infty$ as a fraction to apply L'Hôpital's rule.

Example 4.4.5: Indeterminate Form of Type $\infty - \infty$

Evaluate $\lim\limits_{x \to 0^+} \left(\dfrac{1}{x^2} - \dfrac{1}{\tan x} \right)$.

Solution

By combining the fractions, we can write the function as a quotient. Since the least common denominator is $x^2 \tan x$, we have

$$\frac{1}{x^2} - \frac{1}{\tan x} = \frac{(\tan x) - x^2}{x^2 \tan x}.$$

As $x \to 0^+$, the numerator $\tan x - x^2 \to 0$ and the denominator $x^2 \tan x \to 0$. Therefore, we can apply L'Hôpital's rule. Taking the derivatives of the numerator and the denominator, we have

$$\lim_{x \to 0^+} \frac{(\tan x) - x^2}{x^2 \tan x} = \lim_{x \to 0^+} \frac{(\sec^2 x) - 2x}{x^2 \sec^2 x + 2x \tan x}.$$

As $x \to 0^+$, $(\sec^2 x) - 2x \to 1$ and $x^2 \sec^2 x + 2x \tan x \to 0$. Since the denominator is positive as x approaches zero from the right, we conclude that

$$\lim_{x \to 0^+} \frac{(\sec^2 x) - 2x}{x^2 \sec^2 x + 2x \tan x} = \infty.$$

Therefore,

$$\lim_{x \to 0^+} \left(\frac{1}{x^2} - \frac{1}{\tan x} \right) = \infty.$$

? Exercise 4.4.5

Evaluate $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$.

Hint

Rewrite the difference of fractions as a single fraction.

Answer

0

Indeterminate Forms Involving Exponents

Another type of indeterminate form that arises when evaluating limits involves exponents. The expressions 0^0, ∞^0, and 1^∞ are all indeterminate forms. On their own, these expressions are meaningless because we cannot actually evaluate these expressions as we would evaluate an expression involving real numbers. Rather, these expressions represent forms that arise when finding limits. Now we examine how L'Hôpital's rule can be used to evaluate limits involving these indeterminate forms.

Since L'Hôpital's rule applies to quotients, we use the natural logarithm function and its properties to reduce a problem evaluating a limit involving exponents to a related problem involving a limit of a quotient. For example, suppose we want to evaluate $\lim_{x\to a} f(x)^{g(x)}$ and we arrive at the indeterminate form ∞^0. (The indeterminate forms 0^0 and 1^∞ can be handled similarly.) We proceed as follows. Let

$$y = f(x)^{g(x)}.$$

Then,

$$\ln y = \ln(f(x)^{g(x)}) = g(x)\ln(f(x)).$$

Therefore,

$$\lim_{x\to a}[\ln(y)] = \lim_{x\to a}[g(x)\ln(f(x))].$$

Since $\lim_{x\to a} f(x) = \infty$, we know that $\lim_{x\to a}\ln(f(x)) = \infty$. Therefore, $\lim_{x\to a} g(x)\ln(f(x))$ is of the indeterminate form $0\cdot\infty$, and we can use the techniques discussed earlier to rewrite the expression $g(x)\ln(f(x))$ in a form so that we can apply L'Hôpital's rule. Suppose $\lim_{x\to a} g(x)\ln(f(x)) = L$, where L may be ∞ or $-\infty$. Then

$$\lim_{x\to a}[\ln(y)] = L.$$

Since the natural logarithm function is continuous, we conclude that

$$\ln\left(\lim_{x\to a} y\right) = L,$$

which gives us

$$\lim_{x\to a} y = \lim_{x\to a} f(x)^{g(x)} = e^L.$$

✓ Example 4.4.6: Indeterminate Form of Type ∞^0

Evaluate $\lim_{x\to\infty} x^{1/x}$.

Solution

Let $y = x^{1/x}$.Then,

$$\ln(x^{1/x}) = \frac{1}{x}\ln x = \frac{\ln x}{x}.$$

We need to evaluate $\lim_{x\to\infty}\dfrac{\ln x}{x}$. Applying L'Hôpital's rule, we obtain

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{\ln x}{x}=\lim_{x\to\infty}\frac{1/x}{1}=0.$$

Therefore, $\lim_{x\to\infty}\ln y=0$. Since the natural logarithm function is continuous, we conclude that

$$\ln\left(\lim_{x\to\infty}y\right)=0,$$

which leads to

$$\lim_{x\to\infty}x^{1/x}=\lim_{x\to\infty}y=e^{\ln\left(\lim_{x\to\infty}y\right)}=e^0=1.$$

Hence,

$$\lim_{x\to\infty}x^{1/x}=1.$$

Exercise 4.4.6

Evaluate $\lim_{x\to\infty}x^{1/\ln(x)}$.

Hint

Let $y=x^{1/\ln(x)}$ and apply the natural logarithm to both sides of the equation.

Answer

e

Example 4.4.7: Indeterminate Form of Type 0^0

Evaluate $\lim_{x\to0^+}x^{\sin x}$.

Solution

Let

$$y=x^{\sin x}.$$

Therefore,

$$\ln y=\ln(x^{\sin x})=\sin x\ln x.$$

We now evaluate $\lim_{x\to0^+}\sin x\ln x$. Since $\lim_{x\to0^+}\sin x=0$ and $\lim_{x\to0^+}\ln x=-\infty$, we have the indeterminate form $0\cdot\infty$. To apply L'Hôpital's rule, we need to rewrite $\sin x\ln x$ as a fraction. We could write

$$\sin x\ln x=\frac{\sin x}{1/\ln x}$$

or

$$\sin x\ln x=\frac{\ln x}{1/\sin x}=\frac{\ln x}{\csc x}.$$

Let's consider the first option. In this case, applying L'Hôpital's rule, we would obtain

$$\lim_{x\to0^+}\sin x\ln x=\lim_{x\to0^+}\frac{\sin x}{1/\ln x}=\lim_{x\to0^+}\frac{\cos x}{-1/(x(\ln x)^2)}=\lim_{x\to0^+}(-x(\ln x)^2\cos x).$$

Unfortunately, we not only have another expression involving the indeterminate form $0 \cdot \infty$, but the new limit is even more complicated to evaluate than the one with which we started. Instead, we try the second option. By writing

$$\sin x \ln x = \frac{\ln x}{1/\sin x} = \frac{\ln x}{\csc x},$$

and applying L'Hôpital's rule, we obtain

$$\lim_{x \to 0^+} \sin x \ln x = \lim_{x \to 0^+} \frac{\ln x}{\csc x} = \lim_{x \to 0^+} \frac{1/x}{-\csc x \cot x} = \lim_{x \to 0^+} \frac{-1}{x \csc x \cot x}.$$

Using the fact that $\csc x = \dfrac{1}{\sin x}$ and $\cot x = \dfrac{\cos x}{\sin x}$, we can rewrite the expression on the right-hand side as

$$\lim_{x \to 0^+} \frac{-\sin^2 x}{x \cos x} = \lim_{x \to 0^+} \left[\frac{\sin x}{x} \cdot (-\tan x) \right] = \left(\lim_{x \to 0^+} \frac{\sin x}{x} \right) \cdot \left(\lim_{x \to 0^+} (-\tan x) \right) = 1 \cdot 0 = 0.$$

We conclude that $\lim\limits_{x \to 0^+} \ln y = 0$. Therefore, $\ln \left(\lim\limits_{x \to 0^+} y \right) = 0$ and we have

$$\lim_{x \to 0^+} y = \lim_{x \to 0^+} x^{\sin x} = e^0 = 1.$$

Hence,

$$\lim_{x \to 0^+} x^{\sin x} = 1.$$

? Exercise 4.4.7

Evaluate $\lim\limits_{x \to 0^+} x^x$.

Hint

Let $y = x^x$ and take the natural logarithm of both sides of the equation.

Answer

1

Growth Rates of Functions

Suppose the functions f and g both approach infinity as $x \to \infty$. Although the values of both functions become arbitrarily large as the values of x become sufficiently large, sometimes one function is growing more quickly than the other. For example, $f(x) = x^2$ and $g(x) = x^3$ both approach infinity as $x \to \infty$. However, as Table 4.4.1 shows, the values of x^3 are growing much faster than the values of x^2.

Table 4.4.1: Comparing the Growth Rates of x^2 and x^3

x	10	100	1000	10,000
$f(x) = x^2$	100	10,000	1,000,000	100,000,000
$g(x) = x^3$	1000	1,000,000	1,000,000,000	1,000,000,000,000

In fact,

$$\lim_{x \to \infty} \frac{x^3}{x^2} = \lim_{x \to \infty} x = \infty.$$

or, equivalently

$$\lim_{x \to \infty} \frac{x^2}{x^3} = \lim_{x \to \infty} \frac{1}{x} = 0.$$

As a result, we say x^3 is growing more rapidly than x^2 as $x \to \infty$. On the other hand, for $f(x) = x^2$ and $g(x) = 3x^2 + 4x + 1$, although the values of $g(x)$ are always greater than the values of $f(x)$ for $x > 0$, each value of $g(x)$ is roughly three times the corresponding value of $f(x)$ as $x \to \infty$, as shown in Table 4.4.2. In fact,

$$\lim_{x \to \infty} \frac{x^2}{3x^2 + 4x + 1} = \frac{1}{3}.$$

Table 4.4.2: Comparing the Growth Rates of x^2 and $3x^2 + 4x + 1$

x	10	100	1000	10,000
$f(x) = x^2$	100	10,000	1,000,000	100,000,000
$g(x) = 3x^2 + 4x + 1$	341	30,401	3,004,001	300,040,001

In this case, we say that x^2 and $3x^2 + 4x + 1$ are growing at the same rate as $x \to \infty$.

More generally, suppose f and g are two functions that approach infinity as $x \to \infty$. We say g grows more rapidly than f as $x \to \infty$ if

$$\lim_{x \to \infty} \frac{g(x)}{f(x)} = \infty \quad \text{or, equivalently,} \quad \lim_{x \to \infty} \frac{f(x)}{g(x)} = 0.$$

On the other hand, if there exists a constant $M \neq 0$ such that

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = M,$$

we say f and g grow at the same rate as $x \to \infty$.

Next we see how to use L'Hôpital's rule to compare the growth rates of power, exponential, and logarithmic functions.

Example 4.4.8: Comparing the Growth Rates of $\ln(x)$, x^2, and e^x

For each of the following pairs of functions, use L'Hôpital's rule to evaluate

$$\lim_{x \to \infty} \frac{f(x)}{g(x)}.$$

a. $f(x) = x^2$ and $g(x) = e^x$
b. $f(x) = \ln(x)$ and $g(x) = x^2$

Solution

a. Since $\lim_{x \to \infty} x^2 = \infty$ and $\lim_{x \to \infty} e^x = \infty$, we can use L'Hôpital's rule to evaluate $\lim_{x \to \infty} \left[\frac{x^2}{e^x}\right]$. We obtain

$$\lim_{x \to \infty} \frac{x^2}{e^x} = \lim_{x \to \infty} \frac{2x}{e^x}.$$

Since $\lim_{x \to \infty} 2x = \infty$ and $\lim_{x \to \infty} e^x = \infty$, we can apply L'Hôpital's rule again. Since

$$\lim_{x \to \infty} \frac{2x}{e^x} = \lim_{x \to \infty} \frac{2}{e^x} = 0,$$

we conclude that

$$\lim_{x \to \infty} \frac{x^2}{e^x} = 0.$$

Therefore, e^x grows more rapidly than x^2 as $x \to \infty$ (See Figure 4.4.3 and Table 4.4.3)

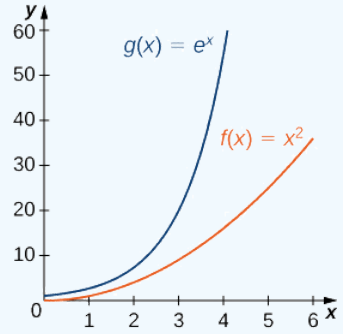


Figure 4.4.3: An exponential function grows at a faster rate than a power function.

Table 4.4.3: Growth rates of a power function and an exponential function.

x	5	10	15	20
x^2	25	100	225	400
e^x	148	22,026	3,269,017	485,165,195

b. Since $\lim_{x\to\infty} \ln x = \infty$ and $\lim_{x\to\infty} x^2 = \infty$, we can use L'Hôpital's rule to evaluate $\lim_{x\to\infty} \dfrac{\ln x}{x^2}$. We obtain

$$\lim_{x\to\infty} \frac{\ln x}{x^2} = \lim_{x\to\infty} \frac{1/x}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0.$$

Thus, x^2 grows more rapidly than $\ln x$ as $x \to \infty$ (see Figure 4.4.4 and Table 4.4.4).

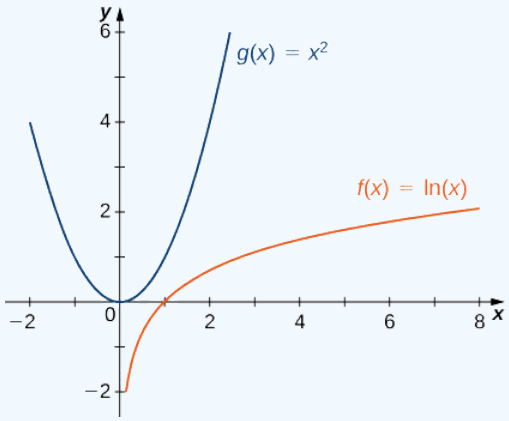


Figure 4.4.4: A power function grows at a faster rate than a logarithmic function.

Table 4.4.4: Growth rates of a power function and a logarithmic function

x	10	100	1000	10,000
$\ln(x)$	2.303	4.605	6.908	9.210
x^2	100	10,000	1,000,000	100,000,000

? Exercise 4.4.8

Compare the growth rates of x^{100} and 2^x.

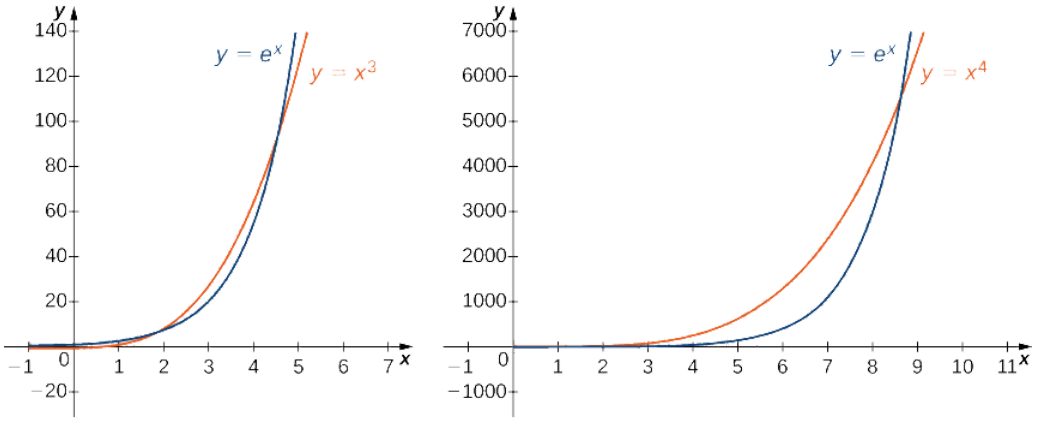
Hint

Apply L'Hôpital's rule to $x^{100}/2^x$.

Answer

The function 2^x grows faster than x^{100}.

Using the same ideas as in Example 4.4.8a, it is not difficult to show that e^x grows more rapidly than x^p for any $p > 0$. In Figure 4.4.5 and Table 4.4.5, we compare e^x with x^3 and x^4 as $x \to \infty$.



(a) (b)

Figure 4.4.5: The exponential function e^x grows faster than x^p for any $p > 0$. (a) A comparison of e^x with x^3. (b) A comparison of e^x with x^4.

Table 4.4.5: An exponential function grows at a faster rate than any power function

x	5	10	15	20
x^3	125	1000	3375	8000
x^4	625	10,000	50,625	160,000
e^x	148	22,026	3,269,017	485,165,195

Similarly, it is not difficult to show that x^p grows more rapidly than $\ln x$ for any $p > 0$. In Figure 4.4.6 and Table 4.4.6, we compare $\ln x$ with $\sqrt[3]{x}$ and $\sqrt{x}$.

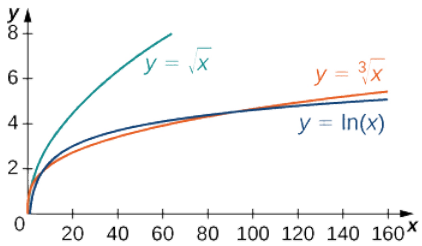


Figure 4.4.6: The function $y = \ln(x)$ grows more slowly than x^p for any $p > 0$ as $x \to \infty$.

Table 4.4.6: A logarithmic function grows at a slower rate than any root function

x	10	100	1000	10,000
$\ln(x)$	2.303	4.605	6.908	9.210
$\sqrt[3]{x}$	2.154	4.642	10	21.544
$\sqrt{x}$	3.162	10	31.623	100

Key Concepts

- L'Hôpital's rule can be used to evaluate the limit of a quotient when the indeterminate form $\frac{0}{0}$ or ∞/∞ arises.
- L'Hôpital's rule can also be applied to other indeterminate forms if they can be rewritten in terms of a limit involving a quotient that has the indeterminate form $\frac{0}{0}$ or ∞/∞.

- The exponential function e^x grows faster than any power function $x^p, p > 0$.
- The logarithmic function $\ln x$ grows more slowly than any power function $x^p, p > 0$.

Glossary

indeterminate forms

When evaluating a limit, the forms $\frac{0}{0}, \infty/\infty, 0 \cdot \infty, \infty - \infty, 0^0, \infty^0$, and 1^∞ are considered indeterminate because further analysis is required to determine whether the limit exists and, if so, what its value is.

L'Hôpital's rule

If f and g are differentiable functions over an interval a, except possibly at a, and $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$ or $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ are infinite, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, assuming the limit on the right exists or is ∞ or $-\infty$.

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4.5: Summary of Curve Sketching

Learning Objectives

- Explain how the sign of the first derivative affects the shape of a function's graph.
- State the first derivative test for critical points.
- Use concavity and inflection points to explain how the sign of the second derivative affects the shape of a function's graph.
- Explain the concavity test for a function over an open interval.
- Explain the relationship between a function and its first and second derivatives.
- State the second derivative test for local extrema.

Earlier in this chapter we stated that if a function f has a local extremum at a point c, then c must be a critical point of f. However, a function is not guaranteed to have a local extremum at a critical point. For example, $f(x) = x^3$ has a critical point at $x = 0$ since $f'(x) = 3x^2$ is zero at $x = 0$, but f does not have a local extremum at $x = 0$. Using the results from the previous section, we are now able to determine whether a critical point of a function actually corresponds to a local extreme value. In this section, we also see how the second derivative provides information about the shape of a graph by describing whether the graph of a function curves upward or curves downward.

The First Derivative Test

Corollary 3 of the Mean Value Theorem showed that if the derivative of a function is positive over an interval I then the function is increasing over I. On the other hand, if the derivative of the function is negative over an interval I, then the function is decreasing over I as shown in the following figure.

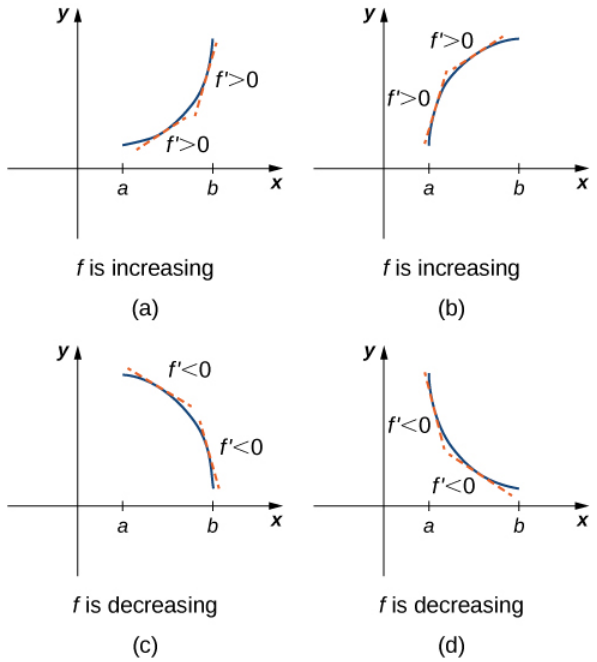


Figure 4.5.1: Both functions are increasing over the interval (a, b). At each point x, the derivative $f'(x) > 0$. Both functions are decreasing over the interval (a, b). At each point x, the derivative $f'(x) < 0$.

A continuous function f has a local maximum at point c if and only if f switches from increasing to decreasing at point c. Similarly, f has a local minimum at c if and only if f switches from decreasing to increasing at c. If f is a continuous function over an interval I containing c and differentiable over I, except possibly at c, the only way f can switch from increasing to decreasing (or vice versa) at point c is if f' changes sign as x increases through c. If f is differentiable at c, the only way that f' can change sign as x increases through c is if $f'(c) = 0$. Therefore, for a function f that is continuous over an interval I containing c and differentiable over I, except possibly at c, the only way f can switch from increasing to decreasing (or vice versa) is if $f'(c) = 0$ or $f'(c)$ is undefined. Consequently, to locate local extrema for a function f, we look for points c in the domain of f such that $f'(c) = 0$ or $f'(c)$ is undefined. Recall that such points are called critical points of f.

Note that f need not have a local extrema at a critical point. The critical points are candidates for local extrema only. In Figure 4.5.2, we show that if a continuous function f has a local extremum, it must occur at a critical point, but a function may not have a local extremum at a critical point. We show that if f has a local extremum at a critical point, then the sign of f' switches as x increases through that point.

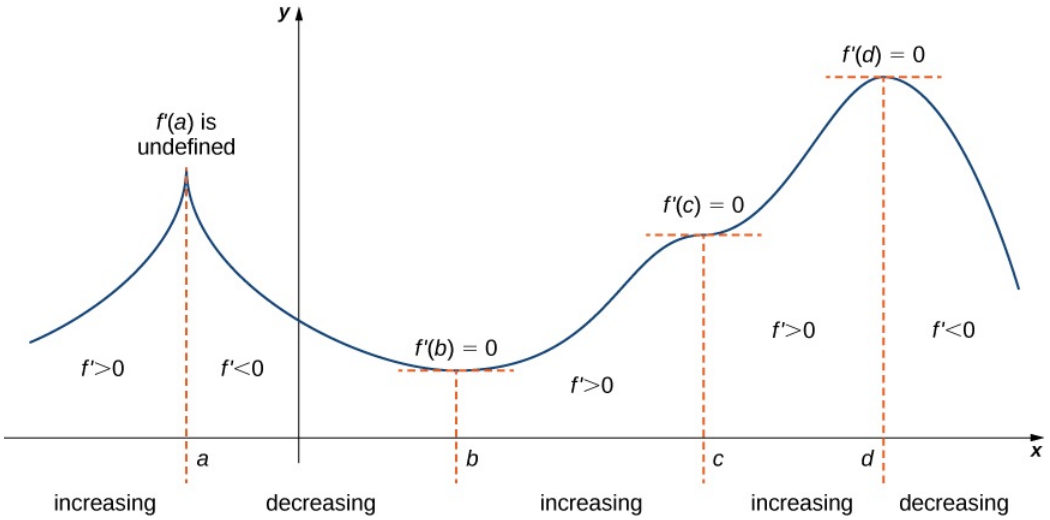


Figure 4.5.2: The function f has four critical points: a, b, c ,and d. The function f has local maxima at a and d, and a local minimum at b. The function f does not have a local extremum at c. The sign of f' changes at all local extrema.

Using Figure 4.5.2, we summarize the main results regarding local extrema.

- If a continuous function f has a local extremum, it must occur at a critical point c.
- The function has a local extremum at the critical point c if and only if the derivative f' switches sign as x increases through c.
- Therefore, to test whether a function has a local extremum at a critical point c, we must determine the sign of $f'(x)$ to the left and right of c.

This result is known as the **first derivative test.**

First Derivative Test

Suppose that f is a continuous function over an interval I containing a critical point c. If f is differentiable over I, except possibly at point c, then $f(c)$ satisfies one of the following descriptions:

i. If f' changes sign from positive when $x < c$ to negative when $x > c$, then $f(c)$ is a local maximum of f.
ii. If f' changes sign from negative when $x < c$ to positive when $x > c$, then $f(c)$ is a local minimum of f.
iii. If f' has the same sign for $x < c$ and $x > c$, then $f(c)$ is neither a local maximum nor a local minimum of f

Now let's look at how to use this strategy to locate all local extrema for particular functions.

Example 4.5.1: Using the First Derivative Test to Find Local Extrema

Use the first derivative test to find the location of all local extrema for $f(x) = x^3 - 3x^2 - 9x - 1$. Use a graphing utility to confirm your results.

Solution

Step 1. The derivative is $f'(x) = 3x^2 - 6x - 9$. To find the critical points, we need to find where $f'(x) = 0$. Factoring the polynomial, we conclude that the critical points must satisfy

$$3(x^2 - 2x - 3) = 3(x - 3)(x + 1) = 0.$$

Therefore, the critical points are $x = 3, -1$. Now divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, -1), (-1, 3)$ and $(3, \infty)$.

Step 2. Since f' is a continuous function, to determine the sign of $f'(x)$ over each subinterval, it suffices to choose a point over each of the intervals $(-\infty, -1), (-1, 3)$ and $(3, \infty)$ and determine the sign of f' at each of these points. For example, let's choose $x = -2$, $x = 0$, and $x = 4$ as test points.

Table: 4.5.1: First Derivative Test for $f(x) = x^3 - 3x^2 - 9x - 1$.

Interval	Test Point	Sign of $f'(x) = 3(x-3)(x+1)$ at Test Point	Conclusion
$(-\infty, -1)$	$x = -2$	(+)(−)(−)=+	f is increasing.
$(-1, 3)$	$x = 0$	(+)(−)(+)=-	f is decreasing.
$(3, \infty)$	$x = 4$	(+)(+)(+)=+	f is increasing.

Step 3. Since f' switches sign from positive to negative as x increases through -1, f has a local maximum at $x = -1$. Since f' switches sign from negative to positive as x increases through 3, f has a local minimum at $x = 3$. These analytical results agree with the following graph.

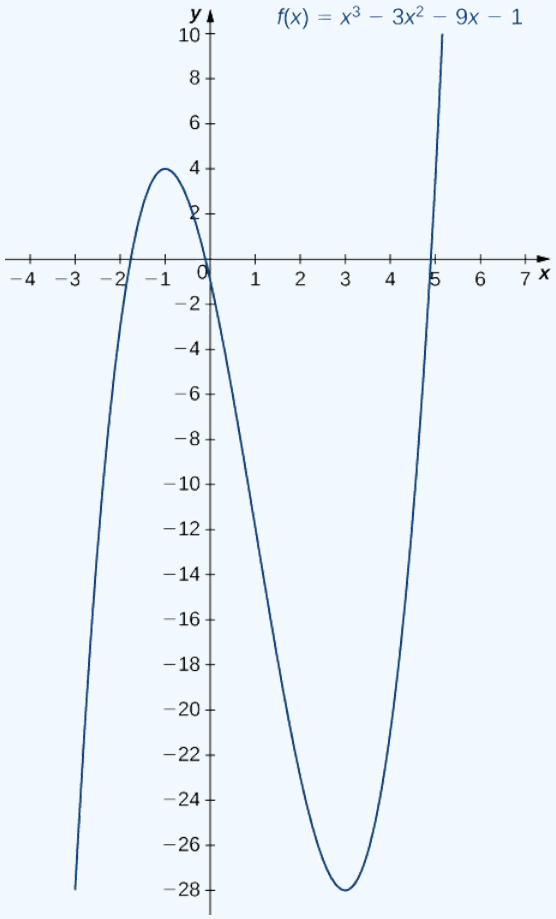


Figure 4.5.3: The function f has a maximum at $x = -1$ and a minimum at $x = 3$

? Exercise 4.5.1

Use the first derivative test to locate all local extrema for $f(x) = -x^3 + \frac{3}{2}x^2 + 18x$.

Hint

Find all critical points of f and determine the signs of $f'(x)$ over particular intervals determined by the critical points.

Answer

f has a local minimum at -2 and a local maximum at 3.

Example 4.5.2: Using the First Derivative Test

Use the first derivative test to find the location of all local extrema for $f(x) = 5x^{1/3} - x^{5/3}$. Use a graphing utility to confirm your results.

Solution

Step 1. The derivative is

$$f'(x) = \frac{5}{3}x^{-2/3} - \frac{5}{3}x^{2/3} = \frac{5}{3x^{2/3}} - \frac{5x^{2/3}}{3} = \frac{5 - 5x^{4/3}}{3x^{2/3}} = \frac{5(1 - x^{4/3})}{3x^{2/3}}.$$

The derivative $f'(x) = 0$ when $1 - x^{4/3} = 0$. Therefore, $f'(x) = 0$ at $x = \pm 1$. The derivative $f'(x)$ is undefined at $x = 0$. Therefore, we have three critical points: $x = 0$, $x = 1$, and $x = -1$. Consequently, divide the interval $(-\infty, \infty)$ into the smaller intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, \infty)$.

Step 2: Since f' is continuous over each subinterval, it suffices to choose a test point x in each of the intervals from step 1 and determine the sign of f' at each of these points. The points $x = -2$, $x = -\frac{1}{2}$, $x = \frac{1}{2}$, and $x = 2$ are test points for these intervals.

Table: 4.5.2: First Derivative Test for $f(x) = 5x^{1/3} - x^{5/3}$.

Interval	Test Point	Sign of $f'(x) = \frac{5(1-x^{4/3})}{3x^{2/3}}$ at Test Point	Conclusion
$(-\infty, -1)$	$x = -2$	$\frac{(+)(-)}{+} = -$	f is decreasing.
$(-1, 0)$	$x = -\frac{1}{2}$	$\frac{(+)(+)}{+} = +$	f is increasing.
$(0, 1)$	$x = \frac{1}{2}$	$\frac{(+)(+)}{+} = +$	f is increasing.
$(1, \infty)$	$x = 2$	$\frac{(+)(-)}{+} = -$	f is decreasing.

Step 3: Since f is decreasing over the interval $(-\infty, -1)$ and increasing over the interval $(-1, 0)$, f has a local minimum at $x = -1$. Since f is increasing over the interval $(-1, 0)$ and the interval $(0, 1)$, f does not have a local extremum at $x = 0$. Since f is increasing over the interval $(0, 1)$ and decreasing over the interval $(1, \infty)$, f has a local maximum at $x = 1$. The analytical results agree with the following graph.

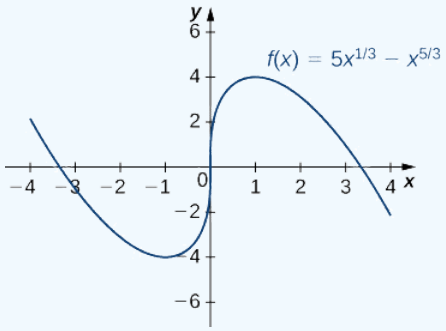


Figure 4.5.4: The function f has a local minimum at $x = -1$ and a local maximum at $x = 1$

Exercise 4.5.2

Use the first derivative test to find all local extrema for $f(x) = \dfrac{3}{x - 1}$.

Hint

The only critical point of f is $x = 1$.

Answer

f has no local extrema because f' does not change sign at $x = 1$.

Concavity and Points of Inflection

We now know how to determine where a function is increasing or decreasing. However, there is another issue to consider regarding the shape of the graph of a function. If the graph curves, does it curve upward or curve downward? This notion is called the **concavity** of the function.

Figure $4.5.5a$ shows a function f with a graph that curves upward. As x increases, the slope of the tangent line increases. Thus, since the derivative increases as x increases, f' is an increasing function. We say this function f is concave up. Figure $4.5.5b$ shows a function f that curves downward. As x increases, the slope of the tangent line decreases. Since the derivative decreases as x increases, f' is a decreasing function. We say this function f is concave down.

Definition: concavity test

Let f be a function that is differentiable over an open interval I. If f' is increasing over I, we say f is **concave up** over I. If f' is decreasing over I, we say f is **concave down** over I.

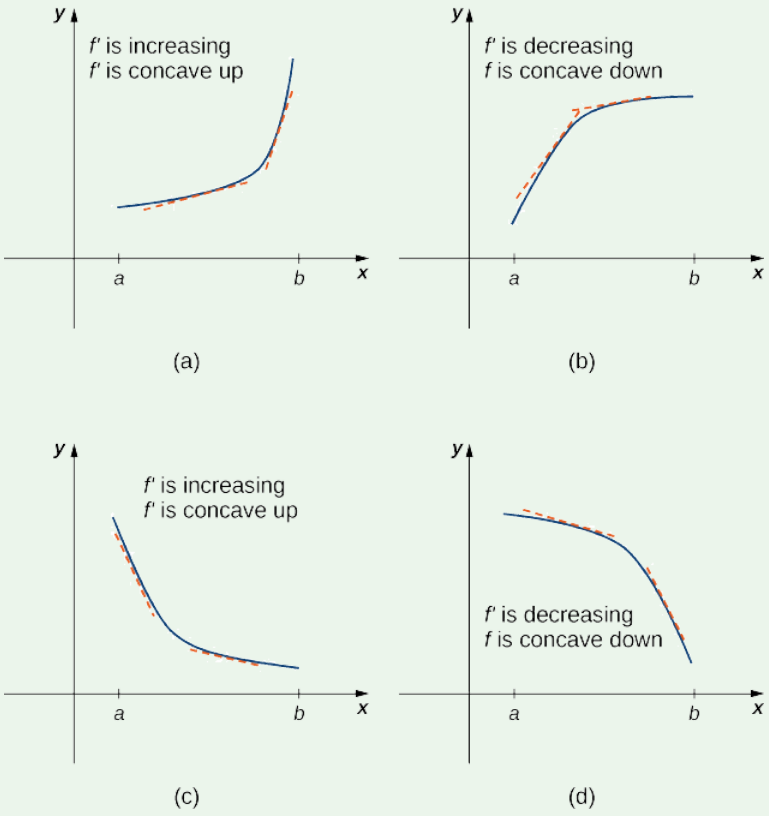


Figure 4.5.5: (a), (c) Since f' is increasing over the interval (a, b), we say f is concave up over (a, b). $(b), (d)$ Since f' is decreasing over the interval (a, b), we say f is concave down over (a, b).

In general, without having the graph of a function f, how can we determine its concavity? By definition, a function f is concave up if f' is increasing. From Corollary 3, we know that if f' is a differentiable function, then f' is increasing if its derivative $f''(x) > 0$. Therefore, a function f that is twice differentiable is concave up when $f''(x) > 0$. Similarly, a function f is concave down if f' is decreasing. We know that a differentiable function f' is decreasing if its derivative $f''(x) < 0$. Therefore, a twice-differentiable function f is concave down when $f''(x) < 0$. Applying this logic is known as the *concavity test*.

Test for Concavity

Let f be a function that is twice differentiable over an interval I.

i. If $f''(x) > 0$ for all $x \in I$, then f is concave up over I
ii. If $f''(x) < 0$ for all $x \in I$, then f is concave down over I.

We conclude that we can determine the concavity of a function f by looking at the second derivative of f. In addition, we observe that a function f can switch concavity (Figure 4.5.6). However, a continuous function can switch concavity only at a point x if $f''(x) = 0$ or $f''(x)$ is undefined. Consequently, to determine the intervals where a function f is concave up and concave down, we look for those values of x where $f''(x) = 0$ or $f''(x)$ is undefined. When we have determined these points, we divide the domain of f into smaller intervals and determine the sign of f'' over each of these smaller intervals. If f'' changes sign as we pass through a point x, then f changes concavity. It is important to remember that a function f may not change concavity at a point x even if $f''(x) = 0$ or $f''(x)$ is undefined. If, however, f does change concavity at a point a and f is continuous at a, we say the point $(a, f(a))$ is an **inflection point** of f.

Definition: inflection point

If f is continuous at a and f changes concavity at a, the point $(a,\ f(a))$ is an *inflection point* of f.

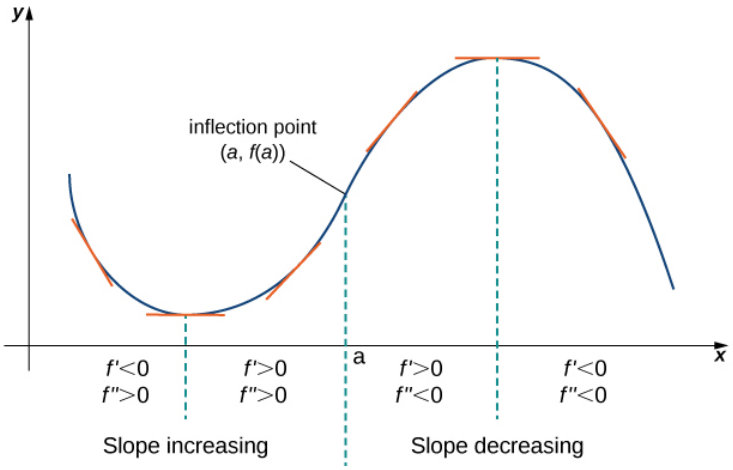


Figure 4.5.6: Since $f''(x) > 0$ for $x < a$, the function f is concave up over the interval $(-\infty, a)$. Since $f''(x) < 0$ for $x > a$, the function f is concave down over the interval (a, ∞). The point $(a, f(a))$ is an inflection point of f.

Example 4.5.3: Testing for Concavity

For the function $f(x) = x^3 - 6x^2 + 9x + 30$, determine all intervals where f is concave up and all intervals where f is concave down. List all inflection points for f. Use a graphing utility to confirm your results.

Solution

To determine concavity, we need to find the second derivative $f''(x)$. The first derivative is $f'(x) = 3x^2 - 12x + 9$, so the second derivative is $f''(x) = 6x - 12$. If the function changes concavity, it occurs either when $f''(x) = 0$ or $f''(x)$ is undefined. Since f'' is defined for all real numbers x, we need only find where $f''(x) = 0$. Solving the equation $6x - 12 = 0$, we see that $x = 2$ is the only place where f could change concavity. We now test points over the intervals $(-\infty, 2)$ and $(2, \infty)$ to determine the concavity of f. The points $x = 0$ and $x = 3$ are test points for these intervals.

Table: 4.5.3: Test for Concavity for $f(x) = x^3 - 6x^2 + 9x + 30$.

Interval	Test Point	Sign of $f''(x) = 6x - 12$ at Test Point	Conclusion
$(-\infty, 2)$	$x = 0$	$-$	f is concave down
$(2, \infty)$	$x = 3$	$+$	f is concave up

We conclude that f is concave down over the interval $(-\infty, 2)$ and concave up over the interval $(2, \infty)$. Since f changes concavity at $x = 2$, the point $(2, f(2)) = (2, 32)$ is an inflection point. Figure $4.5.7$ confirms the analytical results.

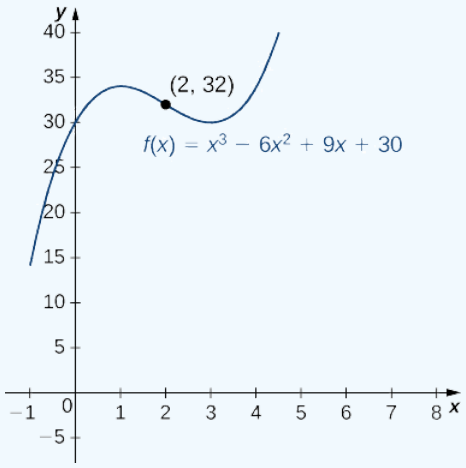


Figure $4.5.7$: The given function has a point of inflection at $(2, 32)$ where the graph changes concavity.

Exercise $4.5.3$

For $f(x) = -x^3 + \frac{3}{2}x^2 + 18x$, find all intervals where f is concave up and all intervals where f is concave down.

Hint

Find where $f''(x) = 0$

Answer

f is concave up over the interval $(-\infty, \frac{1}{2})$ and concave down over the interval $(\frac{1}{2}, \infty)$

We now summarize, in Table $4.5.4$, the information that the first and second derivatives of a function f provide about the graph of f, and illustrate this information in Figure $4.5.8$.

Table: $4.5.4$: What Derivatives Tell Us about Graphs

Sign of f'	Sign of f''	Is f increasing or decreasing?	Concavity
Positive	Positive	Increasing	Concave up
Positive	Negative	Increasing	Concave down
Negative	Positive	Decreasing	Concave up
Negative	Negative	Decreasing	Concave down

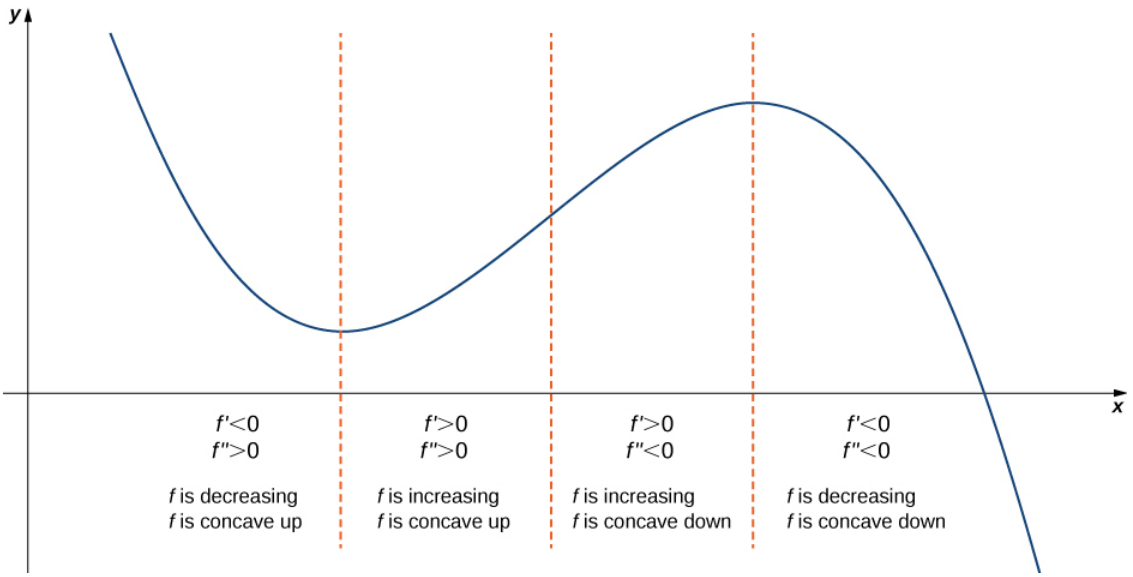


Figure 4.5.8:Consider a twice-differentiable function f over an open interval I. If $f'(x) > 0$ for all $x \in I$, the function is increasing over I. If $f'(x) < 0$ for all $x \in I$, the function is decreasing over I. If $f''(x) > 0$ for all $x \in I$, the function is concave up. If $f''(x) < 0$ for all $x \in I$, the function is concave down on I.

The Second Derivative Test

The first derivative test provides an analytical tool for finding local extrema, but the second derivative can also be used to locate extreme values. Using the second derivative can sometimes be a simpler method than using the first derivative.

We know that if a continuous function has a local extremum, it must occur at a critical point. However, a function need not have a local extremum at a critical point. Here we examine how the **second derivative test** can be used to determine whether a function has a local extremum at a critical point. Let f be a twice-differentiable function such that $f'(a) = 0$ and f'' is continuous over an open interval I containing a. Suppose $f''(a) < 0$. Since f'' is continuous over I, $f''(x) < 0$ for all $x \in I$ (Figure 4.5.9). Then, by Corollary 3, f' is a decreasing function over I. Since $f'(a) = 0$, we conclude that for all $x \in I$, $f'(x) > 0$ if $x < a$ and $f'(x) < 0$ if $x > a$. Therefore, by the first derivative test, f has a local maximum at $x = a$.

On the other hand, suppose there exists a point b such that $f'(b) = 0$ but $f''(b) > 0$. Since f'' is continuous over an open interval I containing b, then $f''(x) > 0$ for all $x \in I$ (Figure 4.5.9). Then, by Corollary 3, f' is an increasing function over I. Since $f'(b) = 0$, we conclude that for all $x \in I$, $f'(x) < 0$ if $x < b$ and $f'(x) > 0$ if $x > b$. Therefore, by the first derivative test, f has a local minimum at $x = b$.

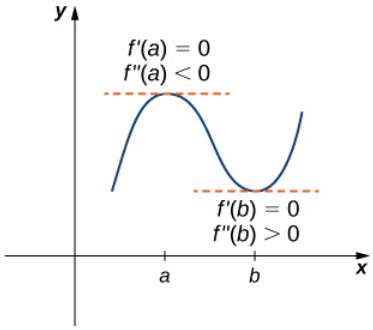


Figure 4.5.9: Consider a twice-differentiable function f such that f'' is continuous. Since $f'(a) = 0$ and $f''(a) < 0$, there is an interval I containing a such that for all x in I, f is increasing if $x < a$ and f is decreasing if $x > a$. As a result, f has a local maximum at $x = a$. Since $f'(b) = 0$ and $f''(b) > 0$, there is an interval I containing b such that for all x in I, f is decreasing if $x < b$ and f is increasing if $x > b$. As a result, f has a local minimum at $x = b$.

Second Derivative Test

Suppose $f'(c) = 0$ and f'' is continuous over an interval containing c.

i. If $f''(c) > 0$, then f has a local minimum at c.
ii. If $f''(c) < 0$, then f has a local maximum at c.
iii. If $f''(c) = 0$, then the test is inconclusive.

Note that for case iii. when $f''(c) = 0$, then f may have a local maximum, local minimum, or neither at c. For example, the functions $f(x) = x^3$, $f(x) = x^4$, and $f(x) = -x^4$ all have critical points at $x = 0$. In each case, the second derivative is zero at $x = 0$. However, the function $f(x) = x^4$ has a local minimum at $x = 0$ whereas the function $f(x) = -x^4$ has a local maximum at $x = 0$, and the function $f(x) = x^3$ does not have a local extremum at $x = 0$.

Let's now look at how to use the second derivative test to determine whether f has a local maximum or local minimum at a critical point c where $f'(c) = 0$.

Example 4.5.4: Using the Second Derivative Test

Use the second derivative to find the location of all local extrema for $f(x) = x^5 - 5x^3$.

Solution

To apply the second derivative test, we first need to find critical points c where $f'(c) = 0$. The derivative is $f'(x) = 5x^4 - 15x^2$. Therefore, $f'(x) = 5x^4 - 15x^2 = 5x^2(x^2 - 3) = 0$ when $x = 0, \pm\sqrt{3}$.

To determine whether f has a local extremum at any of these points, we need to evaluate the sign of f'' at these points. The second derivative is

$$f''(x) = 20x^3 - 30x = 10x(2x^2 - 3).$$

In the following table, we evaluate the second derivative at each of the critical points and use the second derivative test to determine whether f has a local maximum or local minimum at any of these points.

Table: 4.5.5: Second Derivative Test for $f(x) = x^5 - 5x^3$.

x	$f''(x)$	Conclusion
$-\sqrt{3}$	$-30\sqrt{3}$	Local maximum
0	0	Second derivative test is inconclusive
$\sqrt{3}$	$30\sqrt{3}$	Local minimum

By the second derivative test, we conclude that f has a local maximum at $x = -\sqrt{3}$ and f has a local minimum at $x = \sqrt{3}$. The second derivative test is inconclusive at $x = 0$. To determine whether f has a local extrema at $x = 0$, we apply the first derivative test. To evaluate the sign of $f'(x) = 5x^2(x^2 - 3)$ for $x \in (-\sqrt{3}, 0)$ and $x \in (0, \sqrt{3})$, let $x = -1$ and $x = 1$ be the two test points. Since $f'(-1) < 0$ and $f'(1) < 0$, we conclude that f is decreasing on both intervals and, therefore, f does not have a local extrema at $x = 0$ as shown in the following graph.

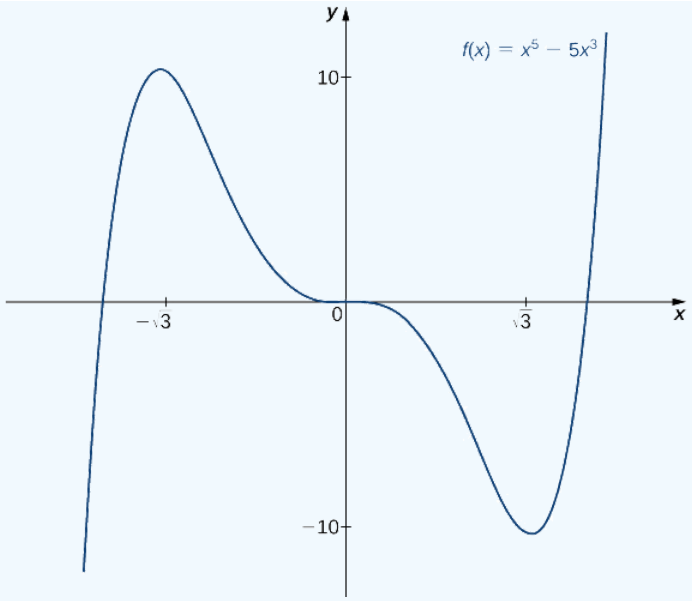


Figure 4.5.10:The function f has a local maximum at $x=-\sqrt{3}$ and a local minimum at $x=\sqrt{3}$

? Exercise 4.5.4

Consider the function $f(x)=x^3-(\frac{3}{2})x^2-18x$. The points $c=3,\ -2$ satisfy $f'(c)=0$. Use the second derivative test to determine whether f has a local maximum or local minimum at those points.

Hint

$f''(x)=6x-3$

Answer

f has a local maximum at -2 and a local minimum at 3.

We have now developed the tools we need to determine where a function is increasing and decreasing, as well as acquired an understanding of the basic shape of the graph. In the next section we discuss what happens to a function as $x\to\pm\infty$. At that point, we have enough tools to provide accurate graphs of a large variety of functions.

Key Concepts

- If c is a critical point of f and $f'(x)>0$ for $x<c$ and $f'(x)<0$ for $x>c$, then f has a local maximum at c.
- If c is a critical point of f and $f'(x)<0$ for $x<c$ and $f'(x)>0$ for $x>c$, then f has a local minimum at c.
- If $f''(x)>0$ over an interval I, then f is concave up over I.
- If $f''(x)<0$ over an interval I, then f is concave down over I.
- If $f'(c)=0$ and $f''(c)>0$, then f has a local minimum at c.
- If $f'(c)=0$ and $f''(c)<0$, then f has a local maximum at c.
- If $f'(c)=0$ and $f''(c)=0$, then evaluate $f'(x)$ at a test point x to the left of c and a test point x to the right of c, to determine whether f has a local extremum at c.

Glossary

concave down

if f is differentiable over an interval I and f' is decreasing over I, then f is concave down over I

concave up

if f is differentiable over an interval I and f' is increasing over I, then f is concave up over I

concavity

the upward or downward curve of the graph of a function

concavity test

suppose f is twice differentiable over an interval I; if $f'' > 0$ over I, then f is concave up over I; if $f'' <$ over I, then f is concave down over I

first derivative test

let f be a continuous function over an interval I containing a critical point c such that f is differentiable over I except possibly at c; if f' changes sign from positive to negative as x increases through c, then f has a local maximum at c; if f' changes sign from negative to positive as x increases through c, then f has a local minimum at c; if f' does not change sign as x increases through c, then f does not have a local extremum at c

inflection point

if f is continuous at c and f changes concavity at c, the point $(c, f(c))$ is an inflection point of f

second derivative test

suppose $f'(c) = 0$ and f'' is continuous over an interval containing c; if $f''(c) > 0$, then f has a local minimum at c; if $f''(c) < 0$, then f has a local maximum at c; if $f''(c) = 0$, then the test is inconclusive

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4.7: Optimization Problems

Learning Objectives

- Set up and solve optimization problems in several applied fields.

One common application of calculus is calculating the minimum or maximum value of a function. For example, companies often want to minimize production costs or maximize revenue. In manufacturing, it is often desirable to minimize the amount of material used to package a product with a certain volume. In this section, we show how to set up these types of minimization and maximization problems and solve them by using the tools developed in this chapter.

Solving Optimization Problems over a Closed, Bounded Interval

The basic idea of the optimization problems that follow is the same. We have a particular quantity that we are interested in maximizing or minimizing. However, we also have some auxiliary condition that needs to be satisfied. For example, in Example 4.7.1, we are interested in maximizing the area of a rectangular garden. Certainly, if we keep making the side lengths of the garden larger, the area will continue to become larger. However, what if we have some restriction on how much fencing we can use for the perimeter? In this case, we cannot make the garden as large as we like. Let's look at how we can maximize the area of a rectangle subject to some constraint on the perimeter.

Example 4.7.1: Maximizing the Area of a Garden

A rectangular garden is to be constructed using a rock wall as one side of the garden and wire fencing for the other three sides (Figure 4.7.1). Given 100 ft of wire fencing, determine the dimensions that would create a garden of maximum area. What is the maximum area?

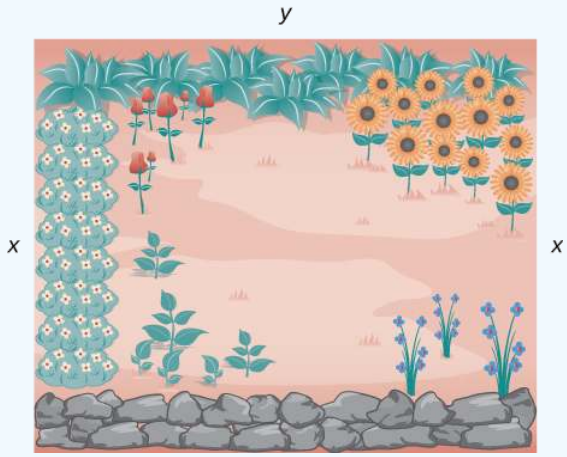


Figure 4.7.1: We want to determine the measurements x and y that will create a garden with a maximum area using 100 ft of fencing.

Solution

Let x denote the length of the side of the garden perpendicular to the rock wall and y denote the length of the side parallel to the rock wall. Then the area of the garden is

$$A = x \cdot y.$$

We want to find the maximum possible area subject to the constraint that the total fencing is 100 ft. From Figure 4.7.1, the total amount of fencing used will be $2x + y$. Therefore, the constraint equation is

$$2x + y = 100.$$

Solving this equation for y, we have $y = 100 - 2x$. Thus, we can write the area as

$$A(x) = x \cdot (100 - 2x) = 100x - 2x^2.$$

Before trying to maximize the area function $A(x) = 100x - 2x^2$, we need to determine the domain under consideration. To construct a rectangular garden, we certainly need the lengths of both sides to be positive. Therefore, we need $x > 0$ and $y > 0$. Since $y = 100 - 2x$, if $y > 0$, then $x < 50$. Therefore, we are trying to determine the maximum value of $A(x)$ for x over the open interval $(0, 50)$. We do not know that a function necessarily has a maximum value over an open interval. However, we do know that a continuous function has an absolute maximum (and absolute minimum) over a closed interval. Therefore, let's consider the function $A(x) = 100x - 2x^2$ over the closed interval $[0, 50]$. If the maximum value occurs at an interior point, then we have found the value x in the open interval $(0, 50)$ that maximizes the area of the garden.

Therefore, we consider the following problem:

Maximize $A(x) = 100x - 2x^2$ over the interval $[0, 50]$.

As mentioned earlier, since A is a continuous function on a closed, bounded interval, by the extreme value theorem, it has a maximum and a minimum. These extreme values occur either at endpoints or critical points. At the endpoints, $A(x) = 0$. Since the area is positive for all x in the open interval $(0, 50)$, the maximum must occur at a critical point. Differentiating the function $A(x)$, we obtain

$$A'(x) = 100 - 4x.$$

Therefore, the only critical point is $x = 25$ (Figure 4.7.2). We conclude that the maximum area must occur when $x = 25$.

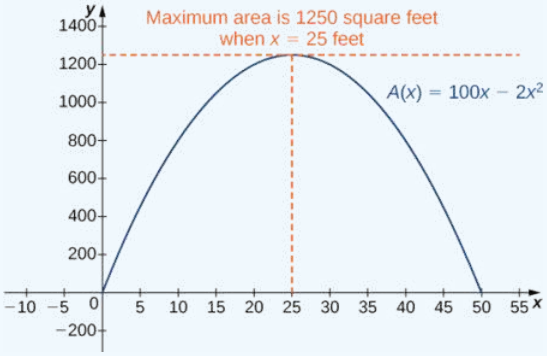


Figure 4.7.2: To maximize the area of the garden, we need to find the maximum value of the function $A(x) = 100x - 2x^2$.

Then we have $y = 100 - 2x = 100 - 2(25) = 50$. To maximize the area of the garden, let $x = 25$ ft and $y = 50$ ft. The area of this garden is 1250 ft^2.

Exercise 4.7.1

Determine the maximum area if we want to make the same rectangular garden as in Figure 4.7.2, but we have 200 ft of fencing.

Hint

We need to maximize the function $A(x) = 200x - 2x^2$ over the interval $[0, 100]$.

Answer

The maximum area is 5000 ft^2.

Now let's look at a general strategy for solving optimization problems similar to Example 4.7.1.

Problem-Solving Strategy: Solving Optimization Problems

1. Introduce all variables. If applicable, draw a figure and label all variables.
2. Determine which quantity is to be maximized or minimized, and for what range of values of the other variables (if this can be determined at this time).
3. Write a formula for the quantity to be maximized or minimized in terms of the variables. This formula may involve more than one variable.

4. Write any equations relating the independent variables in the formula from step 3. Use these equations to write the quantity to be maximized or minimized as a function of one variable.
5. Identify the domain of consideration for the function in step 4 based on the physical problem to be solved.
6. Locate the maximum or minimum value of the function from step 4. This step typically involves looking for critical points and evaluating a function at endpoints.

Now let's apply this strategy to maximize the volume of an open-top box given a constraint on the amount of material to be used.

Example 4.7.2: Maximizing the Volume of a Box

An open-top box is to be made from a 24 in. by 36 in. piece of cardboard by removing a square from each corner of the box and folding up the flaps on each side. What size square should be cut out of each corner to get a box with the maximum volume?

Solution

Step 1: Let x be the side length of the square to be removed from each corner (Figure $4.7.3$). Then, the remaining four flaps can be folded up to form an open-top box. Let V be the volume of the resulting box.

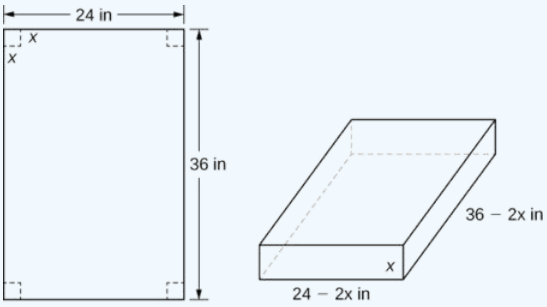


Figure $4.7.3$: A square with side length x inches is removed from each corner of the piece of cardboard. The remaining flaps are folded to form an open-top box.

Step 2: We are trying to maximize the volume of a box. Therefore, the problem is to maximize V.

Step 3: As mentioned in step 2, are trying to maximize the volume of a box. The volume of a box is

$$V = L \cdot W \cdot H,$$

where L, W, and H are the length, width, and height, respectively.

Step 4: From Figure $4.7.3$, we see that the height of the box is x inches, the length is $36 - 2x$ inches, and the width is $24 - 2x$ inches. Therefore, the volume of the box is

$$\begin{aligned} V(x) &= (36 - 2x)(24 - 2x)x \\ &= 4x^3 - 120x^2 + 864x \end{aligned}.$$

Step 5: To determine the domain of consideration, let's examine Figure $4.7.3$. Certainly, we need $x > 0$. Furthermore, the side length of the square cannot be greater than or equal to half the length of the shorter side, 24 in.; otherwise, one of the flaps would be completely cut off. Therefore, we are trying to determine whether there is a maximum volume of the box for x over the open interval $(0, 12)$. Since V is a continuous function over the closed interval $[0, 12]$, we know V will have an absolute maximum over the closed interval. Therefore, we consider V over the closed interval $[0, 12]$ and check whether the absolute maximum occurs at an interior point.

Step 6: Since $V(x)$ is a continuous function over the closed, bounded interval $[0, 12]$, V must have an absolute maximum (and an absolute minimum). Since $V(x) = 0$ at the endpoints and $V(x) > 0$ for $0 < x < 12$, the maximum must occur at a critical point. The derivative is

$$V'(x) = 12x^2 - 240x + 864.$$

To find the critical points, we need to solve the equation

$$12x^2 - 240x + 864 = 0.$$

Dividing both sides of this equation by 12, the problem simplifies to solving the equation

$$x^2 - 20x + 72 = 0.$$

Using the quadratic formula, we find that the critical points are

$$\begin{aligned} x &= \frac{20 \pm \sqrt{(-20)^2 - 4(1)(72)}}{2} \\ &= \frac{20 \pm \sqrt{112}}{2} \\ &= \frac{20 \pm 4\sqrt{7}}{2} \\ &= 10 \pm 2\sqrt{7} \end{aligned} .$$

Since $10 + 2\sqrt{7}$ is not in the domain of consideration, the only critical point we need to consider is $10 - 2\sqrt{7}$. Therefore, the volume is maximized if we let $x = 10 - 2\sqrt{7}\,\text{in}$. The maximum volume is

$$V(10 - 2\sqrt{7}) = 640 + 448\sqrt{7} \approx 1825\,\text{in}^3.$$

as shown in the following graph.

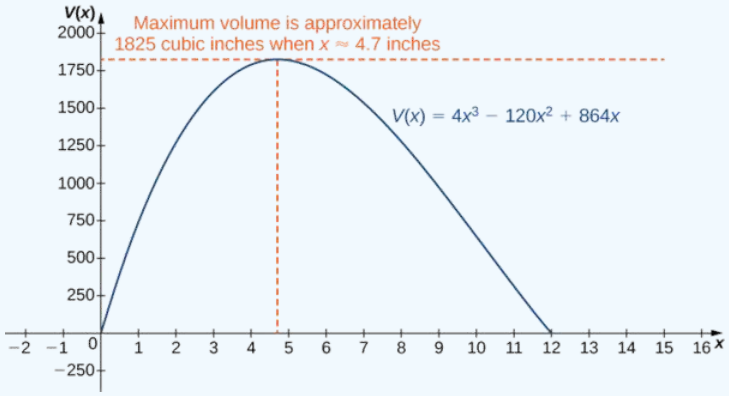


Figure 4.7.4: Maximizing the volume of the box leads to finding the maximum value of a cubic polynomial.

Exercise 4.7.2

Suppose the dimensions of the cardboard in Example 4.7.2 are 20 in. by 30 in. Let x be the side length of each square and write the volume of the open-top box as a function of x. Determine the domain of consideration for x.

Hint

The volume of the box is $L \cdot W \cdot H$.

Answer

$V(x) = x(20 - 2x)(30 - 2x)$. The domain is $[0, 10]$.

Example 4.7.3: Minimizing Travel Time

An island is 2 mi due north of its closest point along a straight shoreline. A visitor is staying at a cabin on the shore that is 6 mi west of that point. The visitor is planning to go from the cabin to the island. Suppose the visitor runs at a rate of 8 mph and swims at a rate of 3 mph. How far should the visitor run before swimming to minimize the time it takes to reach the island?

Solution

Step 1: Let x be the distance running and let y be the distance swimming (Figure 4.7.5). Let T be the time it takes to get from the cabin to the island.

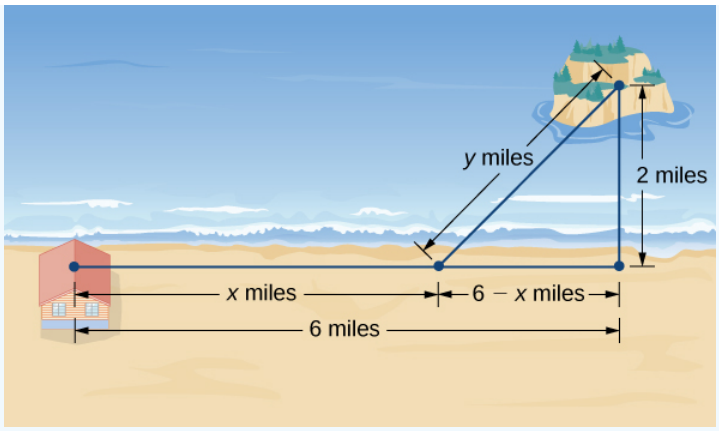


Figure 4.7.5: How can we choose x and y to minimize the travel time from the cabin to the island?

Step 2: The problem is to minimize T.

Step 3: To find the time spent traveling from the cabin to the island, add the time spent running and the time spent swimming. Since Distance = Rate × Time $(D = R \times T)$, the time spent running is

$$T_{running} = \frac{D_{running}}{R_{running}} = \frac{x}{8},$$

and the time spent swimming is

$$T_{swimming} = \frac{D_{swimming}}{R_{swimming}} = \frac{y}{3}.$$

Therefore, the total time spent traveling is

$$T = \frac{x}{8} + \frac{y}{3}.$$

Step 4: From Figure 4.7.5, the line segment of y miles forms the hypotenuse of a right triangle with legs of length 2 mi and $6 - x$ mi. Therefore, by the Pythagorean theorem, $2^2 + (6-x)^2 = y^2$, and we obtain $y = \sqrt{(6-x)^2 + 4}$. Thus, the total time spent traveling is given by the function

$$T(x) = \frac{x}{8} + \frac{\sqrt{(6-x)^2 + 4}}{3}.$$

Step 5: From Figure 4.7.5, we see that $0 \le x \le 6$. Therefore, $[0, 6]$ is the domain of consideration.

Step 6: Since $T(x)$ is a continuous function over a closed, bounded interval, it has a maximum and a minimum. Let's begin by looking for any critical points of T over the interval $[0, 6]$. The derivative is

$$\begin{aligned} T'(x) &= \frac{1}{8} - \frac{1}{2}\frac{[(6-x)^2 + 4]^{-1/2}}{3} \cdot 2(6-x) \\ &= \frac{1}{8} - \frac{(6-x)}{3\sqrt{(6-x)^2 + 4}} \end{aligned}$$

If $T'(x) = 0$,, then

$$\frac{1}{8} = \frac{6-x}{3\sqrt{(6-x)^2 + 4}} \tag{4.7.1}$$

Therefore,

$$3\sqrt{(6-x)^2 + 4} = 8(6-x). \tag{4.7.2}$$

Squaring both sides of this equation, we see that if x satisfies this equation, then x must satisfy

$$9[(6-x)^2 + 4] = 64(6-x)^2,$$

which implies

$$55(6-x)^2 = 36.$$

We conclude that if x is a critical point, then x satisfies

$$(x-6)^2 = \frac{36}{55}.$$

[Note that since we are squaring, $(x-6)^2 = (6-x)^2$.]

Therefore, the possibilities for critical points are

$$x = 6 \pm \frac{6}{\sqrt{55}}.$$

Since $x = 6 + 6/\sqrt{55}$ is not in the domain, it is not a possibility for a critical point. On the other hand, $x = 6 - 6/\sqrt{55}$ is in the domain. Since we squared both sides of Equation 4.7.2 to arrive at the possible critical points, it remains to verify that $x = 6 - 6/\sqrt{55}$ satisfies Equation 4.7.1. Since $x = 6 - 6/\sqrt{55}$ does satisfy that equation, we conclude that $x = 6 - 6/\sqrt{55}$ is a critical point, and it is the only one. To justify that the time is minimized for this value of x, we just need to check the values of $T(x)$ at the endpoints $x = 0$ and $x = 6$, and compare them with the value of $T(x)$ at the critical point $x = 6 - 6/\sqrt{55}$. We find that $T(0) \approx 2.108$ h and $T(6) \approx 1.417$ h, whereas

$$T(6 - 6/\sqrt{55}) \approx 1.368 \text{ h}.$$

Therefore, we conclude that T has a local minimum at $x \approx 5.19$ mi.

? Exercise 4.7.3

Suppose the island is 1 mi from shore, and the distance from the cabin to the point on the shore closest to the island is 15 mi. Suppose a visitor swims at the rate of 2.5 mph and runs at a rate of 6 mph. Let x denote the distance the visitor will run before swimming, and find a function for the time it takes the visitor to get from the cabin to the island.

Hint

The time $T = T_{running} + T_{swimming}$.

Answer

$$T(x) = \frac{x}{6} + \frac{\sqrt{(15-x)^2 + 1}}{2.5}$$

In business, companies are interested in **maximizing revenue.** In the following example, we consider a scenario in which a company has collected data on how many cars it is able to lease, depending on the price it charges its customers to rent a car. Let's use these data to determine the price the company should charge to maximize the amount of money it brings in.

✓ Example 4.7.4: Maximizing Revenue

Owners of a car rental company have determined that if they charge customers p dollars per day to rent a car, where $50 \le p \le 200$, the number of cars n they rent per day can be modeled by the linear function $n(p) = 1000 - 5p$. If they charge $\$50$ per day or less, they will rent all their cars. If they charge $\$200$ per day or more, they will not rent any cars. Assuming the owners plan to charge customers between $\$50$ per day and $\$200$ per day to rent a car, how much should they charge to maximize their revenue?

Solution

Step 1: Let p be the price charged per car per day and let n be the number of cars rented per day. Let R be the revenue per day.

Step 2: The problem is to maximize R.

Step 3: The revenue (per day) is equal to the number of cars rented per day times the price charged per car per day—that is, $R = n \times p$.

Step 4: Since the number of cars rented per day is modeled by the linear function $n(p) = 1000 - 5p$, the revenue R can be represented by the function

$$\begin{aligned} R(p) &= n \times p \\ &= (1000 - 5p)p \\ &= -5p^2 + 1000p. \end{aligned}$$

Step 5: Since the owners plan to charge between $\$50$ per car per day and $\$200$ per car per day, the problem is to find the maximum revenue $R(p)$ for p in the closed interval $[50, 200]$.

Step 6: Since R is a continuous function over the closed, bounded interval $[50, 200]$, it has an absolute maximum (and an absolute minimum) in that interval. To find the maximum value, look for critical points. The derivative is $R'(p) = -10p + 1000$. Therefore, the critical point is $p = 100$. When $p = 100, R(100) = \$50,000$. When $p = 50, R(p) = \$37,500$. When $p = 200, R(p) = \$0$.

Therefore, the absolute maximum occurs at $p = \$100$. The car rental company should charge $\$100$ per day per car to maximize revenue as shown in the following figure.

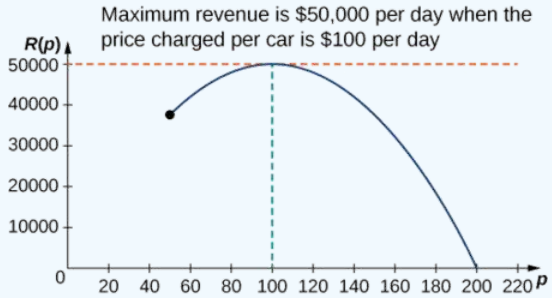


Figure 4.7.6: To maximize revenue, a car rental company has to balance the price of a rental against the number of cars people will rent at that price.

Exercise 4.7.4

A car rental company charges its customers p dollars per day, where $60 \leq p \leq 150$. It has found that the number of cars rented per day can be modeled by the linear function $n(p) = 750 - 5p$. How much should the company charge each customer to maximize revenue?

Hint

$R(p) = n \times p$, where n is the number of cars rented and p is the price charged per car.

Answer

The company should charge $\$75$ per car per day.

Example 4.7.5: Maximizing the Area of an Inscribed Rectangle

A rectangle is to be inscribed in the ellipse

$$\frac{x^2}{4} + y^2 = 1.$$

What should the dimensions of the rectangle be to maximize its area? What is the maximum area?

Solution

Step 1: For a rectangle to be inscribed in the ellipse, the sides of the rectangle must be parallel to the axes. Let L be the length of the rectangle and W be its width. Let A be the area of the rectangle.

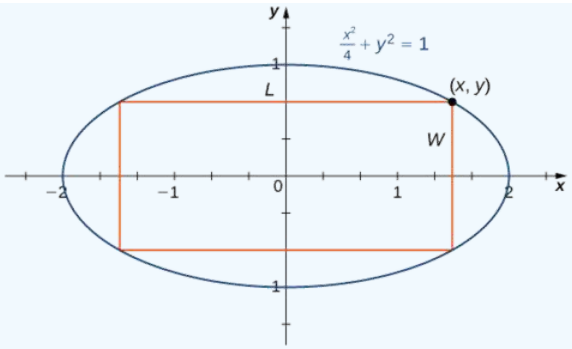


Figure 4.7.7: We want to maximize the area of a rectangle inscribed in an ellipse.

Step 2: The problem is to maximize A.

Step 3: The area of the rectangle is $A = LW$.

Step 4: Let (x, y) be the corner of the rectangle that lies in the first quadrant, as shown in Figure 4.7.7. We can write length $L = 2x$ and width $W = 2y$. Since $\frac{x^2}{4} + y^2 = 1$ and $y > 0$, we have $y = \sqrt{1 - \frac{x^2}{4}}$. Therefore, the area is

$$A = LW = (2x)(2y) = 4x\sqrt{1 - \frac{x^2}{4}} = 2x\sqrt{4 - x^2}$$

Step 5: From Figure 4.7.7, we see that to inscribe a rectangle in the ellipse, the x-coordinate of the corner in the first quadrant must satisfy $0 < x < 2$. Therefore, the problem reduces to looking for the maximum value of $A(x)$ over the open interval $(0, 2)$. Since $A(x)$ will have an absolute maximum (and absolute minimum) over the closed interval $[0, 2]$, we consider $A(x) = 2x\sqrt{4 - x^2}$ over the interval $[0, 2]$. If the absolute maximum occurs at an interior point, then we have found an absolute maximum in the open interval.

Step 6: As mentioned earlier, $A(x)$ is a continuous function over the closed, bounded interval $[0, 2]$. Therefore, it has an absolute maximum (and absolute minimum). At the endpoints $x = 0$ and $x = 2$, $A(x) = 0$. For $0 < x < 2$, $A(x) > 0$.

Therefore, the maximum must occur at a critical point. Taking the derivative of $A(x)$, we obtain

$$\begin{aligned} A'(x) &= 2\sqrt{4 - x^2} + 2x \cdot \frac{1}{2\sqrt{4 - x^2}}(-2x) \\ &= 2\sqrt{4 - x^2} - \frac{2x^2}{\sqrt{4 - x^2}} \\ &= \frac{8 - 4x^2}{\sqrt{4 - x^2}}. \end{aligned}$$

To find critical points, we need to find where $A'(x) = 0$. We can see that if x is a solution of

$$\frac{8 - 4x^2}{\sqrt{4 - x^2}} = 0, \tag{4.7.3}$$

then x must satisfy

$$8 - 4x^2 = 0.$$

Therefore, $x^2 = 2$. Thus, $x = \pm\sqrt{2}$ are the possible solutions of Equation 4.7.3. Since we are considering x over the interval $[0, 2]$, $x = \sqrt{2}$ is a possibility for a critical point, but $x = -\sqrt{2}$ is not. Therefore, we check whether $\sqrt{2}$ is a solution of Equation 4.7.3. Since $x = \sqrt{2}$ is a solution of Equation 4.7.3, we conclude that $\sqrt{2}$ is the only critical point of $A(x)$ in the interval $[0, 2]$.

Therefore, $A(x)$ must have an absolute maximum at the critical point $x = \sqrt{2}$. To determine the dimensions of the rectangle, we need to find the length L and the width W. If $x = \sqrt{2}$ then

$$y = \sqrt{1 - \frac{(\sqrt{2})^2}{4}} = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}.$$

Therefore, the dimensions of the rectangle are $L = 2x = 2\sqrt{2}$ and $W = 2y = \dfrac{2}{\sqrt{2}} = \sqrt{2}$. The area of this rectangle is $A = LW = (2\sqrt{2})(\sqrt{2}) = 4$.

? Exercise 4.7.5

Modify the area function A if the rectangle is to be inscribed in the unit circle $x^2 + y^2 = 1$. What is the domain of consideration?

Hint

If (x, y) is the vertex of the square that lies in the first quadrant, then the area of the square is $A = (2x)(2y) = 4xy$.

Answer

$A(x) = 4x\sqrt{1 - x^2}$. The domain of consideration is $[0, 1]$.

Solving Optimization Problems when the Interval Is Not Closed or Is Unbounded

In the previous examples, we considered functions on closed, bounded domains. Consequently, by the extreme value theorem, we were guaranteed that the functions had absolute extrema. Let's now consider functions for which the domain is neither closed nor bounded.

Many functions still have at least one absolute extrema, even if the domain is not closed or the domain is unbounded. For example, the function $f(x) = x^2 + 4$ over $(-\infty, \infty)$ has an absolute minimum of 4 at $x = 0$. Therefore, we can still consider functions over unbounded domains or open intervals and determine whether they have any absolute extrema. In the next example, we try to minimize a function over an unbounded domain. We will see that, although the domain of consideration is $(0, \infty)$, the function has an absolute minimum.

In the following example, we look at constructing a box of least surface area with a prescribed volume. It is not difficult to show that for a closed-top box, by symmetry, among all boxes with a specified volume, a cube will have the smallest surface area. Consequently, we consider the modified problem of determining which open-topped box with a specified volume has the smallest surface area.

✓ Example 4.7.6: Minimizing Surface Area

A rectangular box with a square base, an open top, and a volume of $216\ \text{in}^3$ is to be constructed. What should the dimensions of the box be to minimize the surface area of the box? What is the minimum surface area?

Solution

Step 1: Draw a rectangular box and introduce the variable x to represent the length of each side of the square base; let y represent the height of the box. Let S denote the surface area of the open-top box.

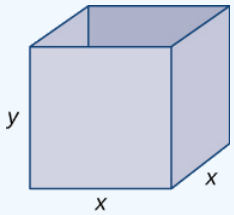


Figure 4.7.8: We want to minimize the surface area of a square-based box with a given volume.

Step 2: We need to minimize the surface area. Therefore, we need to minimize S.

Step 3: Since the box has an open top, we need only determine the area of the four vertical sides and the base. The area of each of the four vertical sides is $x \cdot y$. The area of the base is x^2. Therefore, the surface area of the box is

$$S = 4xy + x^2.$$

Step 4: Since the volume of this box is x^2y and the volume is given as $216\,\text{in}^3$, the constraint equation is

$$x^2y = 216.$$

Solving the constraint equation for y, we have $y = \dfrac{216}{x^2}$. Therefore, we can write the surface area as a function of x only:

$$S(x) = 4x\left(\frac{216}{x^2}\right) + x^2.$$

Therefore, $S(x) = \dfrac{864}{x} + x^2$.

Step 5: Since we are requiring that $x^2y = 216$, we cannot have $x = 0$. Therefore, we need $x > 0$. On the other hand, x is allowed to have any positive value. Note that as x becomes large, the height of the box y becomes correspondingly small so that $x^2y = 216$. Similarly, as x becomes small, the height of the box becomes correspondingly large. We conclude that the domain is the open, unbounded interval $(0, \infty)$. Note that, unlike the previous examples, we cannot reduce our problem to looking for an absolute maximum or absolute minimum over a closed, bounded interval. However, in the next step, we discover why this function must have an absolute minimum over the interval $(0, \infty)$.

Step 6: Note that as $x \to 0^+$, $S(x) \to \infty$. Also, as $x \to \infty$, $S(x) \to \infty$. Since S is a continuous function that approaches infinity at the ends, it must have an absolute minimum at some $x \in (0, \infty)$. This minimum must occur at a critical point of S. The derivative is

$$S'(x) = -\frac{864}{x^2} + 2x.$$

Therefore, $S'(x) = 0$ when $2x = \dfrac{864}{x^2}$. Solving this equation for x, we obtain $x^3 = 432$, so $x = \sqrt[3]{432} = 6\sqrt[3]{2}$. Since this is the only critical point of S, the absolute minimum must occur at $x = 6\sqrt[3]{2}$ (see Figure 4.7.9).

When $x = 6\sqrt[3]{2}$, $y = \dfrac{216}{(6\sqrt[3]{2})^2} = 3\sqrt[3]{2}\,\text{in}$. Therefore, the dimensions of the box should be $x = 6\sqrt[3]{2}\,\text{in}$. and $y = 3\sqrt[3]{2}\,\text{in}$. With these dimensions, the surface area is

$$S(6\sqrt[3]{2}) = \frac{864}{6\sqrt[3]{2}} + (6\sqrt[3]{2})^2 = 108\sqrt[3]{4}\,\text{in}^2$$

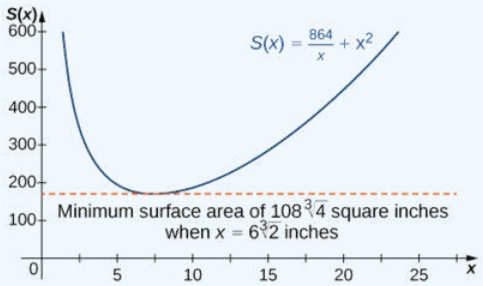


Figure 4.7.9: We can use a graph to determine the dimensions of a box of given the volume and the minimum surface area.

Exercise 4.7.6

Consider the same open-top box, which is to have volume $216\,\text{in}^3$. Suppose the cost of the material for the base is $20¢/\text{in}^2$ and the cost of the material for the sides is $30¢/\text{in}^2$ and we are trying to minimize the cost of this box. Write the cost as a function of the side lengths of the base. (Let x be the side length of the base and y be the height of the box.)

Hint

If the cost of one of the sides is $30¢/\text{in}^2$,the cost of that side is $0.30xy$dollars.

Answer

$$c(x) = \frac{259.2}{x} + 0.2x^2 \text{ dollars}$$

Key Concepts

- To solve an optimization problem, begin by drawing a picture and introducing variables.
- Find an equation relating the variables.
- Find a function of one variable to describe the quantity that is to be minimized or maximized.
- Look for critical points to locate local extrema.

Glossary

optimization problems

problems that are solved by finding the maximum or minimum value of a function

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4.8: Newton's Method

Learning Objectives

- Describe the steps of Newton's method.
- Explain what an iterative process means.
- Recognize when Newton's method does not work.
- Apply iterative processes to various situations.

In many areas of pure and applied mathematics, we are interested in finding solutions to an equation of the form $f(x)=0$. For most functions, however, it is difficult—if not impossible—to calculate their zeroes explicitly. In this section, we take a look at a technique that provides a very efficient way of approximating the **zeroes of functions**. This technique makes use of tangent line approximations and is behind the method used often by calculators and computers to find zeroes.

Describing Newton's Method

Consider the task of finding the solutions of $f(x)=0$. If f is the first-degree polynomial $f(x)=ax+b$, then the solution of $f(x)=0$ is given by the formula $x=-\frac{b}{a}$. If f is the second-degree polynomial $f(x)=ax^2+bx+c$, the solutions of $f(x)=0$ can be found by using the quadratic formula. However, for polynomials of degree **3** or more, finding roots of f becomes more complicated. Although formulas exist for third- and fourth-degree polynomials, they are quite complicated. Also, if f is a polynomial of degree **5** or greater, it is known that no such formulas exist. For example, consider the function

$$f(x)=x^5+8x^4+4x^3-2x-7.$$

No formula exists that allows us to find the solutions of $f(x)=0$. Similar difficulties exist for nonpolynomial functions. For example, consider the task of finding solutions of $\tan(x)-x=0$. No simple formula exists for the solutions of this equation. In cases such as these, we can use Newton's method to approximate the roots.

Newton's method makes use of the following idea to approximate the solutions of $f(x)=0$. By sketching a graph of f, we can estimate a root of $f(x)=0$. Let's call this estimate x_0. We then draw the tangent line to f at x_0. If $f'(x_0)\neq 0$, this tangent line intersects the x-axis at some point $(x_1,0)$. Now let x_1 be the next approximation to the actual root. Typically, x_1 is closer than x_0 to an actual root. Next we draw the tangent line to f at x_1. If $f'(x_1)\neq 0$, this tangent line also intersects the x-axis, producing another approximation, x_2. We continue in this way, deriving a list of approximations: x_0, x_1, x_2, ... Typically, the numbers x_0, x_1, x_2, ... quickly approach an actual root x^*, as shown in the following figure.

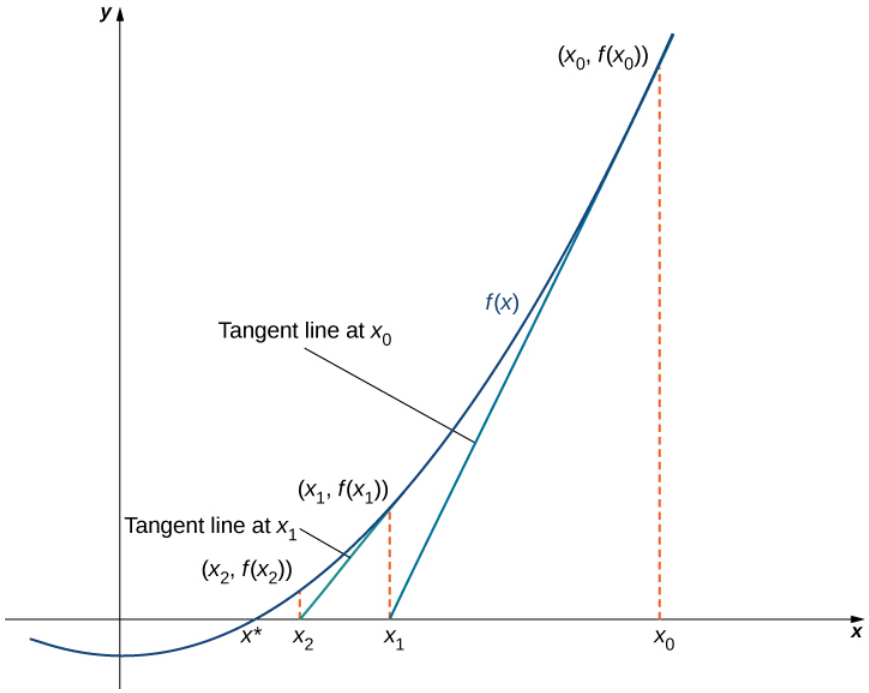


Figure 4.8.1:The approximations $x_0, x_1, x_2, \ldots$ approach the actual root x^*. The approximations are derived by looking at tangent lines to the graph of f.

Now let's look at how to calculate the approximations $x_0, x_1, x_2, \ldots$. If x_0 is our first approximation, the approximation x_1 is defined by letting $(x_1, 0)$ be the x-intercept of the tangent line to f at x_0. The equation of this tangent line is given by

$$y = f(x_0) + f'(x_0)(x - x_0).$$

Therefore, x_1 must satisfy

$$f(x_0) + f'(x_0)(x_1 - x_0) = 0.$$

Solving this equation for x_1, we conclude that

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Similarly, the point $(x_2, 0)$ is the x-intercept of the tangent line to f at x_1. Therefore, x_2 satisfies the equation

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

In general, for $n > 0$, x_n satisfies

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}. \tag{4.8.1}$$

Next we see how to make use of this technique to approximate the root of the polynomial $f(x) = x^3 - 3x + 1$.

Example 4.8.1: Finding a Root of a Polynomial

Use Newton's method to approximate a root of $f(x) = x^3 - 3x + 1$ in the interval $[1, 2]$. Let $x_0 = 2$ and find x_1, x_2, x_3, x_4, and x_5.

Solution

From Figure 4.8.2, we see that f has one root over the interval $[1, 2]$. Therefore $x_0 = 2$ seems like a reasonable first approximation. To find the next approximation, we use Equation 4.8.1. Since $f(x) = x^3 - 3x + 1$, the derivative is $f'(x) = 3x^2 - 3$. Using Equation 4.8.1 with $n = 1$ (and a calculator that displays 10 digits), we obtain

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{3}{9} \approx 1.666666667.$$

To find the next approximation, x_2, we use Equation 4.8.1 with $n = 2$ and the value of x_1 stored on the calculator. We find that

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 1.548611111.$$

Continuing in this way, we obtain the following results:

- $x_1 \approx 1.666666667$
- $x_2 \approx 1.548611111$
- $x_3 \approx 1.532390162$
- $x_4 \approx 1.532088989$
- $x_5 \approx 1.532088886$
- $x_6 \approx 1.532088886.$

We note that we obtained the same value for x_5 and x_6. Therefore, any subsequent application of Newton's method will most likely give the same value for x_n.

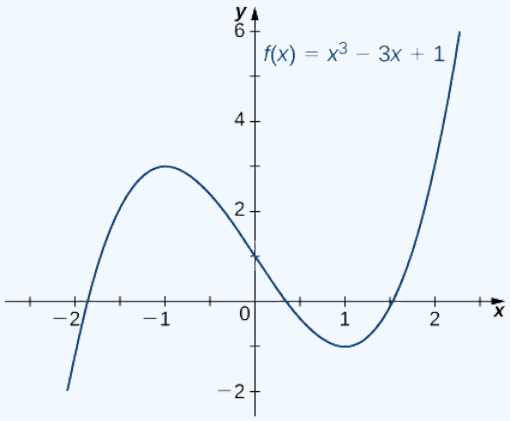


Figure 4.8.2: The function $f(x) = x^3 - 3x + 1$ has one root over the interval $[1, 2]$.

? Exercise 4.8.1

Letting $x_0 = 0$, let's use Newton's method to approximate the root of $f(x) = x^3 - 3x + 1$ over the interval $[0, 1]$ by calculating x_1 and x_2.

Hint

Use Equation 4.8.1.

Answer

$x_1 \approx 0.33333333$
$x_2 \approx 0.347222222$

Newton's method can also be used to approximate square roots. Here we show how to approximate $\sqrt{2}$. This method can be modified to approximate the square root of any positive number.

✔ Example 4.8.2: Finding a Square Root

Use Newton's method to approximate $\sqrt{2}$ (Figure 4.8.3). Let $f(x) = x^2 - 2$, let $x_0 = 2$, and calculate x_1, x_2, x_3, x_4, x_5. (We note that since $f(x) = x^2 - 2$ has a zero at $\sqrt{2}$, the initial value $x_0 = 2$ is a reasonable choice to approximate $\sqrt{2}$).

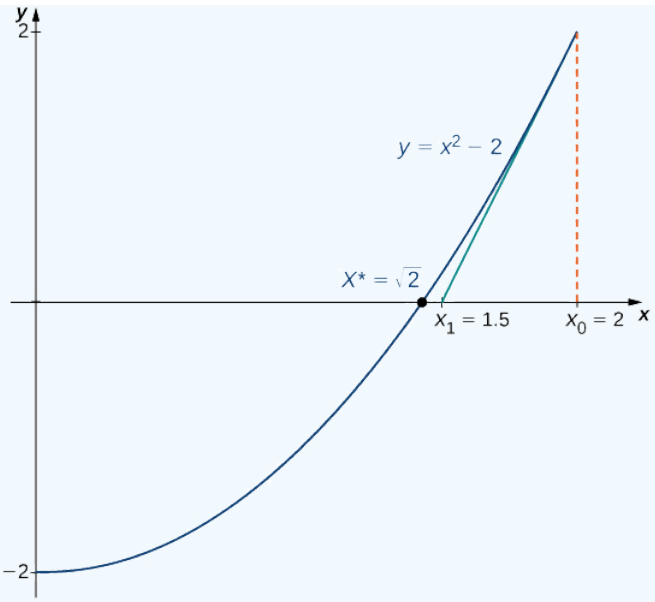


Figure 4.8.3: We can use Newton's method to find $\sqrt{2}$.

Solution

For $f(x)=x^2-2,\ f'(x)=2x$. From Equation 4.8.1, we know that

$$\begin{aligned} x_n &= x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \\ &= x_{n-1} - \frac{x_{n-1}^2 - 2}{2x_{n-1}} \\ &= \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}} \\ &= \frac{1}{2}\left(x_{n-1} + \frac{2}{x_{n-1}}\right). \end{aligned}$$

Therefore,

$$x_1 = \tfrac{1}{2}\left(x_0 + \tfrac{2}{x_0}\right) = \tfrac{1}{2}\left(2 + \tfrac{2}{2}\right) = 1.5$$

$$x_2 = \tfrac{1}{2}\left(x_1 + \tfrac{2}{x_1}\right) = \tfrac{1}{2}\left(1.5 + \tfrac{2}{1.5}\right) \approx 1.416666667.$$

Continuing in this way, we find that

$$\begin{aligned} x_1 &= 1.5 \\ x_2 &\approx 1.416666667 \\ x_3 &\approx 1.414215686 \\ x_4 &\approx 1.414213562 \\ x_5 &\approx 1.414213562. \end{aligned}$$

Since we obtained the same value for x_4 and x_5, it is unlikely that the value x_n will change on any subsequent application of Newton's method. We conclude that $\sqrt{2} \approx 1.414213562$.

? Exercise 4.8.2

Use Newton's method to approximate $\sqrt{3}$ by letting $f(x)=x^2-3$ and $x_0=3$. Find x_1 and x_2.

Hint

For $f(x) = x^2 - 3$, Equation 4.8.1reduces to $x_n = \frac{x_{n-1}}{2} + \frac{3}{2x_{n-1}}$.

Answer

$x_1 = 2$
$x_2 = 1.75$

When using Newton's method, each approximation after the initial guess is defined in terms of the previous approximation by using the same formula. In particular, by defining the function $F(x) = x - \left[\frac{f(x)}{f'(x)}\right]$, we can rewrite Equation 4.8.1 as $x_n = F(x_{n-1})$. This type of process, where each x_n is defined in terms of x_{n-1} by repeating the same function, is an example of an iterative process. Shortly, we examine other iterative processes. First, let's look at the reasons why Newton's method could fail to find a root.

Failures of Newton's Method

Typically, Newton's method is used to find roots fairly quickly. However, things can go wrong. Some reasons why Newton's method might fail include the following:

1. At one of the approximations x_n, the derivative f' is zero at x_n, but $f(x_n) \neq 0$. As a result, the tangent line of f at x_n does not intersect the x-axis. Therefore, we cannot continue the iterative process.
2. The approximations $x_0, x_1, x_2, \ldots$ may approach a different root. If the function f has more than one root, it is possible that our approximations do not approach the one for which we are looking, but approach a different root (see Figure 4.8.4). This event most often occurs when we do not choose the approximation x_0 close enough to the desired root.
3. The approximations may fail to approach a root entirely. In Example 4.8.3, we provide an example of a function and an initial guess x_0 such that the successive approximations never approach a root because the successive approximations continue to alternate back and forth between two values.

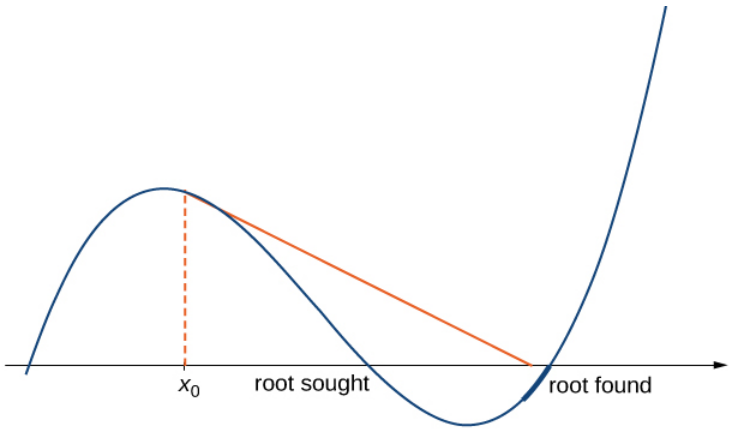


Figure 4.8.4: If the initial guess x_0 is too far from the root sought, it may lead to approximations that approach a different root.

Example 4.8.3: When Newton's Method Fails

Consider the function $f(x) = x^3 - 2x + 2$. Let $x_0 = 0$. Show that the sequence $x_1, x_2, \ldots$ fails to approach a root of f.

Solution

For $f(x) = x^3 - 2x + 2$, the derivative is $f'(x) = 3x^2 - 2$ .Therefore,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{f(0)}{f'(0)} = -\frac{2}{-2} = 1.$$

In the next step,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{1} = 0.$$

Consequently, the numbers $x_0, x_1, x_2, \ldots$ continue to bounce back and forth between 0 and 1 and never get closer to the root of f which is over the interval $[-2, -1]$ (Figure 4.8.5). Fortunately, if we choose an initial approximation x_0 closer to the actual root, we can avoid this situation.

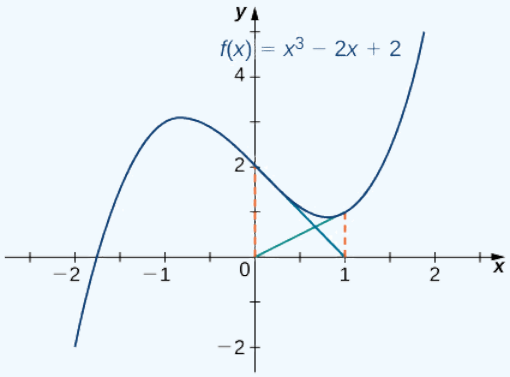


Figure 4.8.5: The approximations continue to alternate between 0 and 1 and never approach the root of f.

? Exercise 4.8.3

For $f(x) = x^3 - 2x + 2$, let $x_0 = -1.5$ and find x_1 and x_2.

Hint

Use Equation 4.8.1.

Answer

$x_1 \approx -1.842105263$
$x_2 \approx -1.772826920$

From Example 4.8.3, we see that Newton's method does not always work. However, when it does work, the sequence of approximations approaches the root very quickly. Discussions of how quickly the sequence of approximations approach a root found using Newton's method are included in texts on numerical analysis.

Other Iterative Processes

As mentioned earlier, Newton's method is a type of iterative process. We now look at an example of a different type of iterative process.

Consider a function F and an initial number x_0. Define the subsequent numbers x_n by the formula $x_n = F(x_{n-1})$. This process is an iterative process that creates a list of numbers $x_0, x_1, x_2, \ldots, x_n, \ldots$. This list of numbers may approach a finite number x^* as n gets larger, or it may not. In Example 4.8.4, we see an example of a function F and an initial guess x_0 such that the resulting list of numbers approaches a finite value.

✓ Example 4.8.4: Finding a Limit for an Iterative Process

Let $F(x) = \frac{1}{2}x + 4$ and let $x_0 = 0$. For all $n \geq 1$, let $x_n = F(x_{n-1})$. Find the values x_1, x_2, x_3, x_4, x_5. Make a conjecture about what happens to this list of numbers $x_1, x_2, x_3, \ldots, x_n, \ldots$ as $n \to \infty$. If the list of numbers $x_1, x_2, x_3, \ldots$ approaches a finite number x^*, then x^* satisfies $x^* = F(x^*)$, and x^* is called a fixed point of F.

Solution

If $x_0 = 0$, then

- $x_1 = \frac{1}{2}(0) + 4 = 4$
- $x_2 = \frac{1}{2}(4) + 4 = 6$
- $x_3 = \frac{1}{2}(6) + 4 = 7$
- $x_4 = \frac{1}{2}(7) + 4 = 7.5$

- $x_5 = \frac{1}{2}(7.5) + 4 = 7.75$
- $x_6 = \frac{1}{2}(7.75) + 4 = 7.875$
- $x_7 = \frac{1}{2}(7.875) + 4 = 7.9375$
- $x_8 = \frac{1}{2}(7.9375) + 4 = 7.96875$
- $x_9 = \frac{1}{2}(7.96875) + 4 = 7.984375.$

From this list, we conjecture that the values x_n approach 8.

Figure 4.8.6 provides a graphical argument that the values approach 8 as $n \to \infty$. Starting at the point (x_0, x_0), we draw a vertical line to the point $(x_0, F(x_0))$. The next number in our list is $x_1 = F(x_0)$. We use x_1 to calculate x_2. Therefore, we draw a horizontal line connecting (x_0, x_1) to the point (x_1, x_1) on the line $y = x$, and then draw a vertical line connecting (x_1, x_1) to the point $(x_1, F(x_1))$. The output $F(x_1)$ becomes x_2. Continuing in this way, we could create an infinite number of line segments. These line segments are trapped between the lines $F(x) = \frac{x}{2} + 4$ and $y = x$. The line segments get closer to the intersection point of these two lines, which occurs when $x = F(x)$. Solving the equation $x = \frac{x}{2} + 4$, we conclude they intersect at $x = 8$. Therefore, our graphical evidence agrees with our numerical evidence that the list of numbers $x_0, x_1, x_2, \ldots$ approaches $x^* = 8$ as $n \to \infty$.

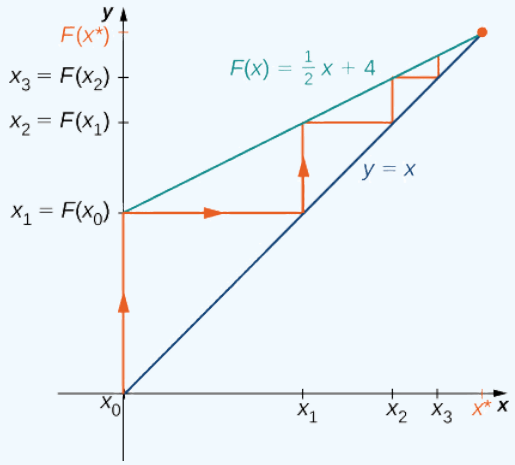


Figure 4.8.6: This iterative process approaches the value $x^* = 8$.

Exercise 4.8.4

Consider the function $F(x) = \frac{1}{3}x + 6$. Let $x_0 = 0$ and let $x_n = F(x_{n-1})$ for $n \geq 2$. Find x_1, x_2, x_3, x_4, x_5. Make a conjecture about what happens to the list of numbers $x_1, x_2, x_3, \ldots x_n, \ldots$ as $n \to \infty$.

Hint

Consider the point where the lines $y = x$ and $y = F(x)$ intersect.

Answer

$x_1 = 6,\ x_2 = 8,\ x_3 = \frac{26}{3},\ x_4 = \frac{80}{9},\ x_5 = \frac{242}{27};\ x^* = 9$

Iterative Processes and Chaos

Iterative processes can yield some very interesting behavior. In this section, we have seen several examples of iterative processes that converge to a fixed point. We also saw in Example 4.8.3 that the iterative process bounced back and forth between two values. We call this kind of behavior a 2-cycle. Iterative processes can converge to cycles with various periodicities, such as 2−cycles, 4−cycles (where the iterative process repeats a sequence of four values), 8-cycles, and so on.

Some iterative processes yield what mathematicians call chaos. In this case, the iterative process jumps from value to value in a seemingly random fashion and never converges or settles into a cycle. Although a complete exploration of chaos is beyond the scope of this text, in this project we look at one of the key properties of a chaotic iterative process: sensitive dependence on

initial conditions. This property refers to the concept that small changes in initial conditions can generate drastically different behavior in the iterative process.

Probably the best-known example of chaos is the **Mandelbrot set** (see Figure 4.8.7), named after Benoit Mandelbrot (1924–2010), who investigated its properties and helped popularize the field of chaos theory. The Mandelbrot set is usually generated by computer and shows fascinating details on enlargement, including self-replication of the set. Several colorized versions of the set have been shown in museums and can be found online and in popular books on the subject.

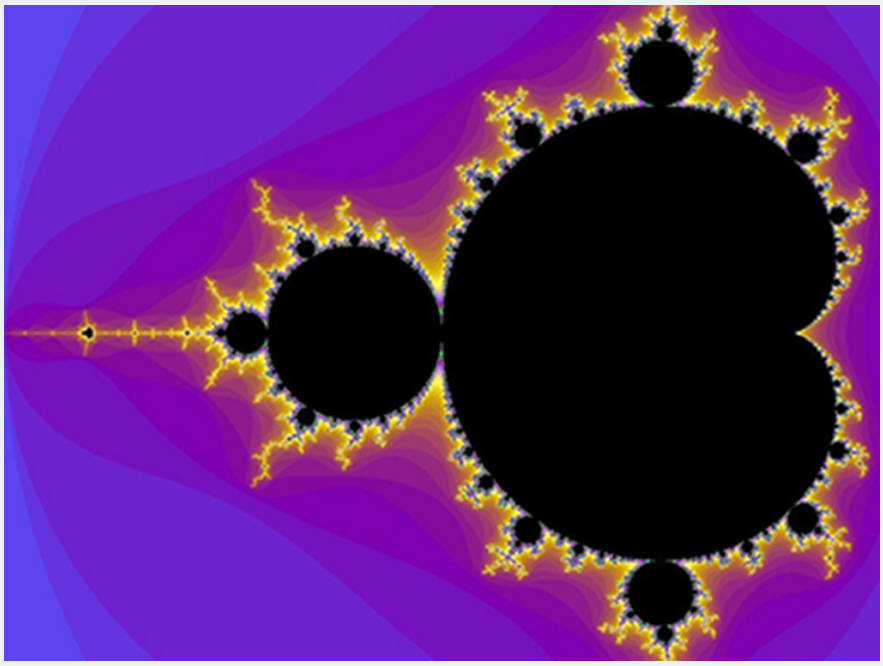


Figure 4.8.7: The Mandelbrot set is a well-known example of a set of points generated by the iterative chaotic behavior of a relatively simple function.

In this project we use the logistic map

$$f(x) = rx(1-x)$$

where $x \in [0, 1]$ and $r > 0$

as the function in our iterative process. The logistic map is a deceptively simple function; but, depending on the value of r, the resulting iterative process displays some very interesting behavior. It can lead to fixed points, cycles, and even chaos.

To visualize the long-term behavior of the iterative process associated with the logistic map, we will use a tool called a cobweb diagram. As we did with the iterative process we examined earlier in this section, we first draw a vertical line from the point $(x_0, 0)$ to the point $(x_0, f(x_0)) = (x_0, x_1)$. We then draw a horizontal line from that point to the point (x_1, x_1), then draw a vertical line to $(x_1, f(x_1)) = (x_1, x_2)$, and continue the process until the long-term behavior of the system becomes apparent. Figure 4.8.8 shows the long-term behavior of the logistic map when $r = 3.55$ and $x_0 = 0.2$. (The first 100 iterations are not plotted.) The long-term behavior of this iterative process is an 8-cycle.

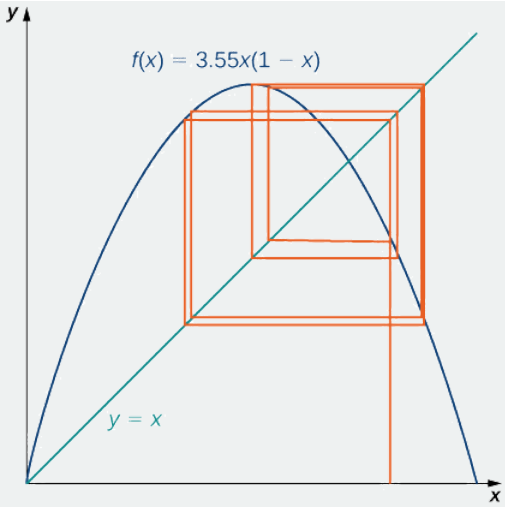


Figure 4.8.8: A cobweb diagram for $f(x) = 3.55x(1 - x)$ is presented here. The sequence of values results in an 8-cycle.

1. Let $r = 0.5$ and choose $x_0 = 0.2$. Either by hand or by using a computer, calculate the first 10 values in the sequence. Does the sequence appear to converge? If so, to what value? Does it result in a cycle? If so, what kind of cycle (for example, $2-$cycle, $4-$cycle.)?
2. What happens when $r = 2$?
3. For $r = 3.2$ and $r = 3.5$, calculate the first 100 sequence values. Generate a cobweb diagram for each iterative process. (Several free applets are available online that generate cobweb diagrams for the logistic map.) What is the long-term behavior in each of these cases?
4. Now let $r = 4$. Calculate the first 100 sequence values and generate a cobweb diagram. What is the long-term behavior in this case?
5. Repeat the process for $r = 4$, but let $x_0 = 0.201$. How does this behavior compare with the behavior for $x_0 = 0.2$?

Key Concepts

- Newton's method approximates roots of $f(x) = 0$ by starting with an initial approximation x_0, then uses tangent lines to the graph of f to create a sequence of approximations $x_1, x_2, x_3, \ldots$.
- Typically, Newton's method is an efficient method for finding a particular root. In certain cases, Newton's method fails to work because the list of numbers $x_0, x_1, x_2, \ldots$ does not approach a finite value or it approaches a value other than the root sought.
- Any process in which a list of numbers $x_0, x_1, x_2, \ldots$ is generated by defining an initial number x_0 and defining the subsequent numbers by the equation $x_n = F(x_{n-1})$ for some function F is an iterative process. Newton's method is an example of an iterative process, where the function $F(x) = x - \left[\frac{f(x)}{f'(x)}\right]$ for a given function f.

Glossary

iterative process

process in which a list of numbers $x_0, x_1, x_2, x_3 \ldots$ is generated by starting with a number x_0 and defining $x_n = F(x_{n-1})$ for $n \geq 1$

Newton's method

method for approximating roots of $f(x) = 0$; using an initial guess x_0; each subsequent approximation is defined by the equation $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$

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4.9: Antiderivatives

At this point, we have seen how to calculate derivatives of many functions and have been introduced to a variety of their applications. We now ask a question that turns this process around: Given a function f, how do we find a function with the derivative f and why would we be interested in such a function?

We answer the first part of this question by defining antiderivatives. The antiderivative of a function f is a function with a derivative f. Why are we interested in antiderivatives? The need for antiderivatives arises in many situations, and we look at various examples throughout the remainder of the text. Here we examine one specific example that involves rectilinear motion. In our examination in Derivatives of rectilinear motion, we showed that given a position function $s(t)$ of an object, then its velocity function $v(t)$ is the derivative of $s(t)$—that is, $v(t) = s'(t)$. Furthermore, the acceleration $a(t)$ is the derivative of the velocity $v(t)$—that is, $a(t) = v'(t) = s''(t)$. Now suppose we are given an acceleration function a, but not the velocity function v or the position function s. Since $a(t) = v'(t)$, determining the velocity function requires us to find an antiderivative of the acceleration function. Then, since $v(t) = s'(t)$, determining the position function requires us to find an antiderivative of the velocity function. Rectilinear motion is just one case in which the need for antiderivatives arises. We will see many more examples throughout the remainder of the text. For now, let's look at the terminology and notation for antiderivatives, and determine the antiderivatives for several types of functions. We examine various techniques for finding antiderivatives of more complicated functions later in the text (Introduction to Techniques of Integration).

The Reverse of Differentiation

At this point, we know how to find derivatives of various functions. We now ask the opposite question. Given a function f, how can we find a function with derivative f? If we can find a function F derivative f, we call F an antiderivative of f.

Definition: Antiderivative

A function F is an **antiderivative** of the function f if

$$F'(x) = f(x) \tag{4.9.1}$$

for all x in the domain of f.

Consider the function $f(x) = 2x$. Knowing the power rule of differentiation, we conclude that $F(x) = x^2$ is an antiderivative of f since $F'(x) = 2x$. Are there any other antiderivatives of f? Yes; since the derivative of any constant C is zero, $x^2 + C$ is also an antiderivative of $2x$. Therefore, $x^2 + 5$ and $x^2 - \sqrt{2}$ are also antiderivatives. Are there any others that are not of the form $x^2 + C$ for some constant C? The answer is no. From Corollary 2 of the Mean Value Theorem, we know that if F and G are differentiable functions such that $F'(x) = G'(x)$, then $F(x) - G(x) = C$ for some constant C. This fact leads to the following important theorem.

General Form of an Antiderivative

Let F be an antiderivative of f over an interval I. Then,

I. for each constant C, the function $F(x) + C$ is also an antiderivative of f over I;
II. if G is an antiderivative of f over I, there is a constant C for which $G(x) = F(x) + C$ over I.

In other words, the most general form of the antiderivative of f over I is $F(x) + C$.

We use this fact and our knowledge of derivatives to find all the antiderivatives for several functions.

Example 4.9.1: Finding Antiderivatives

For each of the following functions, find all antiderivatives.

a. $f(x) = 3x^2$
b. $f(x) = \dfrac{1}{x}$
c. $f(x) = \cos x$
d. $f(x) = e^x$

Solution:

a. Because

$$\frac{d}{dx}(x^3) = 3x^2$$

then $F(x) = x^3$ is an antiderivative of $3x^2$. Therefore, every antiderivative of $3x^2$ is of the form $x^3 + C$ for some constant C, and every function of the form $x^3 + C$ is an antiderivative of $3x^2$.

b. Let $f(x) = \ln|x|$. For $x > 0$, $f(x) = \ln(x)$ and

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

For $x < 0$, $f(x) = \ln(-x)$ and

$$\frac{d}{dx}(\ln(-x)) = -\frac{1}{-x} = \frac{1}{x}.$$

Therefore,

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}.$$

Thus, $F(x) = \ln|x|$ is an antiderivative of $\frac{1}{x}$. Therefore, every antiderivative of $\frac{1}{x}$ is of the form $\ln|x| + C$ for some constant C and every function of the form $\ln|x| + C$ is an antiderivative of $\frac{1}{x}$.

c. We have

$$\frac{d}{dx}(\sin x) = \cos x,$$

so $F(x) = \sin x$ is an antiderivative of $\cos x$. Therefore, every antiderivative of $\cos x$ is of the form $\sin x + C$ for some constant C and every function of the form $\sin x + C$ is an antiderivative of $\cos x$.

d. Since

$$\frac{d}{dx}(e^x) = e^x,$$

then $F(x) = e^x$ is an antiderivative of e^x. Therefore, every antiderivative of e^x is of the form $e^x + C$ for some constant C and every function of the form $e^x + C$ is an antiderivative of e^x.

Exercise 4.9.1

Find all antiderivatives of $f(x) = \sin x$.

Hint

What function has a derivative of $\sin x$?

Answer

$-\cos x + C$

Indefinite Integrals

We now look at the formal notation used to represent antiderivatives and examine some of their properties. These properties allow us to find antiderivatives of more complicated functions. Given a function f, we use the notation $f'(x)$ or $\frac{df}{dx}$ to denote the derivative of f. Here we introduce notation for antiderivatives. If F is an antiderivative of f, we say that $F(x) + C$ is the most general antiderivative of f and write

$$\int f(x)dx = F(x) + C. \tag{4.9.2}$$

The symbol $\int$ is called an integral sign, and $\int f(x)dx$ is called the indefinite integral of f.

Definition: Indefinite Integrals

Given a function f, the indefinite integral of f, denoted

$$\int f(x)dx, \tag{4.9.3}$$

is the most general antiderivative of f. If F is an antiderivative of f, then

$$\int f(x)dx = F(x) + C. \tag{4.9.4}$$

The expression $f(x)$ is called the integrand and the variable x is the variable of integration.

Given the terminology introduced in this definition, the act of finding the antiderivatives of a function f is usually referred to as integrating f.

For a function f and an antiderivative F, the functions $F(x)+C$, where C is any real number, is often referred to as the family of antiderivatives of f. For example, since x^2 is an antiderivative of $2x$ and any antiderivative of $2x$ is of the form x^2+C, we write

$$\int 2x\,dx = x^2 + C. \tag{4.9.5}$$

The collection of all functions of the form x^2+C, where C is any real number, is known as the family of antiderivatives of $2x$. Figure shows a graph of this family of antiderivatives.

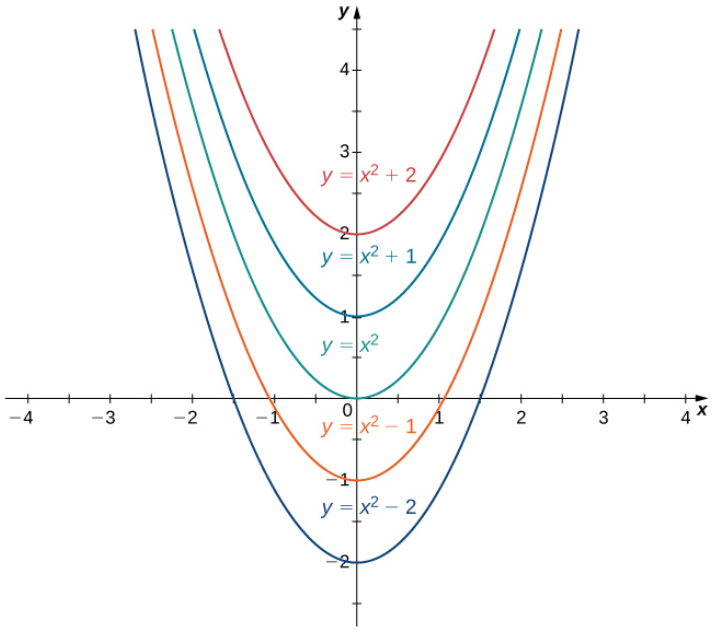


Figure* 4.9.1*: *The family of antiderivatives of $2x$ consists of all functions of the form x^2+C, where C is any real number.*

For some functions, evaluating indefinite integrals follows directly from properties of derivatives. For example, for $n \neq -1$,

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C,$$

which comes directly from

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = (n+1)\frac{x^n}{n+1} = x^n \ .$$

This fact is known as the power rule for integrals.

Power Rule for Integrals

For $n \neq -1$,

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C. \tag{4.9.6}$$

Evaluating indefinite integrals for some other functions is also a straightforward calculation. The following table lists the indefinite integrals for several common functions. A more complete list appears in Appendix B.

Table : Integration Formulas

Differentiation Formula	Indefinite Integral
$\frac{d}{dx}(k) = 0$	$\int k\,dx = \int kx^0\,dx = kx + C$
$\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n dn = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$
$\frac{d}{dx}(\ln\lvert x\rvert) = \frac{1}{x}$	$\int \frac{1}{x}dx = \ln\lvert x\rvert + C$
$\frac{d}{dx}(e^x) = e^x$	$\int e^x dx = e^x + C$

Differentiation Formula	Indefinite Integral
$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x dx = \sin x + C$
$\frac{d}{dx}(\cos x) = -\sin x$	$\int \sin x dx = -\cos x + C$
$\frac{d}{dx}(\tan x) = sec^2 x$	$\int sec^2 x dx = \tan x + C$
$\frac{d}{dx}(cscx) = -cscx \cot x$	$\int cscx \cot x dx = -cscx + C$
$\frac{d}{dx}(secx) = secx \tan x$	$\int secx \tan x dx = secx + C$
$\frac{d}{dx}(\cot x) = -csc^2 x$	$\int csc^2 x dx = -\cot x + C$
$\frac{d}{dx}(sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} = sin^{-1} x + C$
$\frac{d}{dx}(tan^{-1} x) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} dx = tan^{-1} x + C$
$\frac{d}{dx}(sec^{-1} \lvert x \rvert) = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}} dx = sec^{-1} \lvert x \rvert + C$

From the definition of indefinite integral of f, we know

$$\int f(x)dx = F(x) + C \tag{4.9.7}$$

if and only if F is an antiderivative of f. Therefore, when claiming that

$$\int f(x)dx = F(x) + C \tag{4.9.8}$$

it is important to check whether this statement is correct by verifying that $F'(x) = f(x)$.

Example 4.9.2: Verifying an Indefinite Integral

Each of the following statements is of the form $\int f(x)dx = F(x) + C$. Verify that each statement is correct by showing that $F'(x) = f(x)$.

a. $\int (x + e^x)dx = \frac{x^2}{2} + e^x + C$
b. $\int xe^x dx = xe^x - e^x + C$

Solution:

a. Since

$$\frac{d}{dx}(\frac{x^2}{2} + e^x + C) = x + e^x \ ,$$

the statement

$$\int (x + e^x)dx = \frac{x^2}{2} + e^x + C$$

is correct.

Note that we are verifying an indefinite integral for a sum. Furthermore, $\frac{x^2}{2}$ and e^x are antiderivatives of x and e^x, respectively, and the sum of the antiderivatives is an antiderivative of the sum. We discuss this fact again later in this section.

b. Using the product rule, we see that

$$\frac{d}{dx}(xe^x - e^x + C) = e^x + xe^x - e^x = xe^x.$$

Therefore, the statement

$$\int xe^x dx = xe^x - e^x + C$$

is correct.

Note that we are verifying an indefinite integral for a product. The antiderivative xex−ex is not a product of the antiderivatives. Furthermore, the product of antiderivatives, $x^2e^x/2$ is not an antiderivative of xe^x since

$$\frac{d}{dx}(\frac{x^2e^x}{2}) = xe^x + \frac{x^2e^x}{2} \neq xe^x \ .$$

In general, the product of antiderivatives is not an antiderivative of a product.

Exercise 4.9.2

Verify that

$$\int x\cos x\,dx = x\sin x + \cos x + C.$$

Hint

Calculate

$$\frac{d}{dx}(x\sin x + \cos x + C).$$

Answer

$$\frac{d}{dx}(x\sin x + \cos x + C) = \sin x + x\cos x - \sin x = x\cos x$$

In Table, we listed the indefinite integrals for many elementary functions. Let's now turn our attention to evaluating indefinite integrals for more complicated functions. For example, consider finding an antiderivative of a sum $f+g$. In Example a. we showed that an antiderivative of the sum $x+e^x$ is given by the sum $(\frac{x^2}{2})+e^x$ —that is, an antiderivative of a sum is given by a sum of antiderivatives. This result was not specific to this example. In general, if F and G are antiderivatives of any functions f and g, respectively, then

$$\frac{d}{dx}(F(x)+G(x)) = F'(x)+G'(x) = f(x)+g(x).$$

Therefore, $F(x)+G(x)$ is an antiderivative of $f(x)+g(x)$ and we have

$$\int (f(x)+g(x))dx = F(x)+G(x)+C.$$

Similarly,

$$\int (f(x)-g(x))dx = F(x)-G(x)+C.$$

In addition, consider the task of finding an antiderivative of $kf(x)$, where k is any real number. Since

$$\frac{d}{dx}(kf(x)) = k\frac{d}{dx}F(x) = kF'(x)$$

for any real number k, we conclude that

$$\int kf(x)dx = kF(x)+C.$$

These properties are summarized next.

Properties of Indefinite Integrals

Let F and G be antiderivatives of f and g, respectively, and let k be any real number.

Sums and Differences

$$\int (f(x)\pm g(x))dx = F(x)\pm G(x)+C$$

Constant Multiples

$$\int kf(x)dx = kF(x)+C$$

From this theorem, we can evaluate any integral involving a sum, difference, or constant multiple of functions with antiderivatives that are known. Evaluating integrals involving products, quotients, or compositions is more complicated (see Exampleb. for an example involving an antiderivative of a product.) We look at and address integrals involving these more complicated functions in Introduction to Integration. In the next example, we examine how to use this theorem to calculate the indefinite integrals of several functions.

Example 4.9.3: Evaluating Indefinite Integrals

Evaluate each of the following indefinite integrals:

a. $\int(5x^3-7x^2+3x+4)dx$
b. $\int \dfrac{x^2+4\sqrt[3]{x}}{x}dx$
c. $\int \dfrac{4}{1+x^2}dx$
d. $\int \tan x \cos x dx$

Solution:

a. Using Note, we can integrate each of the four terms in the integrand separately. We obtain

$$\int(5x^3-7x^2+3x+4)dx = \int 5x^3dx - \int 7x^2dx + \int 3xdx + \int 4dx.$$

From the second part of Note, each coefficient can be written in front of the integral sign, which gives

$$\int 5x^3dx - \int 7x^2dx + \int 3xdx + \int 4dx = 5\int x^3dx - 7\int x^2dx + 3\int xdx + 4\int 1dx.$$

Using the power rule for integrals, we conclude that

$$\int(5x^3-7x^2+3x+4)dx = \frac{5}{4}x^4 - \frac{7}{3}x^3 + \frac{3}{2}x^2 + 4x + C.$$

b. Rewrite the integrand as

$$\frac{x^2+4\sqrt[3]{x}}{x} = \frac{x^2}{x} + \frac{4\sqrt[3]{x}}{x} = 0.$$

Then, to evaluate the integral, integrate each of these terms separately. Using the power rule, we have

$$\begin{aligned}\int(x+\frac{4}{x^{2/3}})dx &= \int xdx + 4\int x^{-2/3}dx \\ &= \frac{1}{2}x^2 + 4\frac{1}{(\frac{-2}{3})+1}x^{(-2/3)+1}+C]) \\ &= \frac{1}{2}x^2 + 12x^{1/3} + C.\end{aligned}$$

c. Using Note, write the integral as

$$4\int \frac{1}{1+x^2}dx.$$

Then, use the fact that $tan^{-1}(x)$ is an antiderivative of $\dfrac{1}{(1+x^2)}$ to conclude that

$$\int \frac{4}{1+x^2}dx = 4tan^{-1}(x) + C.$$

d. Rewrite the integrand as

$$\tan x \cos x = \frac{\sin x}{\cos x}\cos x = \sin x.$$

Therefore,

$$\int \tan x \cos x = \int \sin x = -\cos x + C.$$

Exercise 4.9.3

Evaluate $\int(4x^3-5x^2+x-7)dx$.

Hint

Integrate each term in the integrand separately, making use of the power rule.

Answer

$$x^4 - \frac{5}{3}x^3 + \frac{1}{2}x^2 - 7x + C$$

Initial-Value Problems

We look at techniques for integrating a large variety of functions involving products, quotients, and compositions later in the text. Here we turn to one common use for antiderivatives that arises often in many applications: solving differential equations.

A differential equation is an equation that relates an unknown function and one or more of its derivatives. The equation

$$\frac{dy}{dx} = f(x)$$

is a simple example of a differential equation. Solving this equation means finding a function y with a derivative f. Therefore, the solutions of Equation are the antiderivatives of f. If F is one antiderivative of f, every function of the form $y = F(x) + C$ is a solution of that differential equation. For example, the solutions of

$$\frac{dy}{dx} = 6x^2$$

are given by

$$y = \int 6x^2 dx = 2x^3 + C \ .$$

Sometimes we are interested in determining whether a particular solution curve passes through a certain point (x_0, y_0) —that is, $y(x_0) = y_0$. The problem of finding a function y that satisfies a differential equation

$$\frac{dy}{dx} = f(x)$$

with the additional condition

$$y(x_0) = y_0$$

is an example of an initial-value problem. The condition $y(x_0) = y_0$ is known as an initial condition. For example, looking for a function y that satisfies the differential equation

$$\frac{dy}{dx} = 6x^2$$

and the initial condition

$$y(1) = 5$$

is an example of an initial-value problem. Since the solutions of the differential equation are $y = 2x^3 + C$, to find a function y that also satisfies the initial condition, we need to find C such that $y(1) = 2(1)^3 + C = 5$. From this equation, we see that $C = 3$, and we conclude that $y = 2x^3 + 3$ is the solution of this initial-value problem as shown in the following graph.

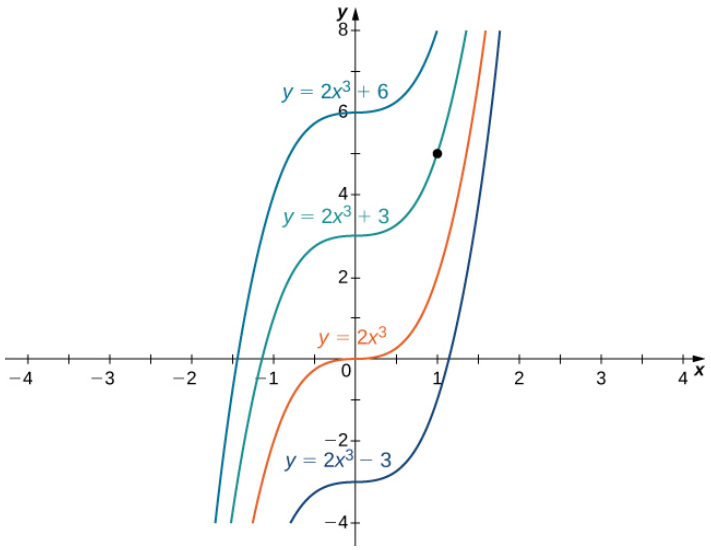


Figure 4.9.2: *Some of the solution curves of the differential equation* $\frac{dy}{dx} = 6x^2$ *are displayed. The function* $y = 2x^3 + 3$ *satisfies the differential equation and the initial condition* $y(1) = 5$.

Example 4.9.4: Solving an Initial-Value Problem

Solve the initial-value problem

$$\frac{dy}{dx} = \sin x, y(0) = 5. \tag{4.9.9}$$

Solution

First we need to solve the differential equation. If $\frac{dy}{dx} = \sin x$, then

$$y = \int sin(x)dx = -\cos x + C. \tag{4.9.10}$$

Next we need to look for a solution y that satisfies the initial condition. The initial condition y(0)=5 means we need a constant C such that $-\cos x + C = 5$. Therefore,

$$C = 5 + cos(0) = 6. \tag{4.9.11}$$

The solution of the initial-value problem is $y = -\cos x + 6$.

Exercise 4.9.4

Solve the initial value problem $\frac{dy}{dx} = 3x^{-2}, y(1) = 2$.

Hint

Find all antiderivatives of $f(x) = 3x^{-2.}$

Answer

$y = -\frac{3}{x} + 5$

Initial-value problems arise in many applications. Next we consider a problem in which a driver applies the brakes in a car. We are interested in how long it takes for the car to stop. Recall that the velocity function $v(t)$ is the derivative of a position function $s(t)$, and the acceleration $a(t)$ is the derivative of the velocity function. In earlier examples in the text, we could calculate the velocity from the position and then compute the acceleration from the velocity. In the next example we work the other way around. Given an acceleration function, we calculate the velocity function. We then use the velocity function to determine the position function.

Example 4.9.5:

A car is traveling at the rate of 88 ft/sec (60 mph) when the brakes are applied. The car begins decelerating at a constant rate of 15 ft/sec^2.

a. How many seconds elapse before the car stops?
b. How far does the car travel during that time?

Solution

a. First we introduce variables for this problem. Let t be the time (in seconds) after the brakes are first applied. Let $a(t)$ be the acceleration of the car (in feet per seconds squared) at time t. Let $v(t)$ be the velocity of the car (in feet per second) at time t. Let $s(t)$ be the car's position (in feet) beyond the point where the brakes are applied at time t.

The car is traveling at a rate of $88 ft/sec$. Therefore, the initial velocity is $v(0) = 88$ ft/sec. Since the car is decelerating, the acceleration is

$$a(t) = -15 ft/s^2.$$

The acceleration is the derivative of the velocity,

$$v'(t) = 15.$$

Therefore, we have an initial-value problem to solve:

$$v'(t) = -15, v(0) = 88.$$

Integrating, we find that

$$v(t) = -15t + C.$$

Since $v(0) = 88, C = 88$. Thus, the velocity function is

$$v(t) = -15t + 88.$$

To find how long it takes for the car to stop, we need to find the time t such that the velocity is zero. Solving $-15t+88=0,$ we obtain $t=\frac{88}{15}$ sec.

b. To find how far the car travels during this time, we need to find the position of the car after $\frac{88}{15}$ sec. We know the velocity $v(t)$ is the derivative of the position $s(t)$. Consider the initial position to be $s(0)=0$. Therefore, we need to solve the initial-value problem

$$s'(t)=-15t+88, s(0)=0.$$

Integrating, we have

$$s(t)=-\frac{15}{2}t^2+88t+C.$$

Since $s(0)=0$, the constant is $C=0$. Therefore, the position function is

$$s(t)=-\frac{15}{2}t^2+88t.$$

After $t=\frac{88}{15}$ sec, the position is $s(\frac{88}{15})\approx 258.133$ ft.

Exercise 4.9.5

Suppose the car is traveling at the rate of 44 ft/sec. How long does it take for the car to stop? How far will the car travel?

Hint

$v(t)=-15t+44.$

Answer

$2.93 sec, 64.5 ft$

Key Concepts

- If F is an antiderivative of f, then every antiderivative of f is of the form $F(x)+C$ for some constant C.
- Solving the initial-value problem

$$\frac{dy}{dx}=f(x), y(x_0)=y_0$$

- requires us first to find the set of antiderivatives of f and then to look for the particular antiderivative that also satisfies the initial condition.

Glossary

antiderivative

a function F such that $F'(x)=f(x)$ for all x in the domain of f is an antiderivative of f

indefinite integral

the most general antiderivative of $f(x)$ is the indefinite integral of f; we use the notation $\int f(x)dx$ to denote the indefinite integral of f

initial value problem

a problem that requires finding a function y that satisfies the differential equation $\frac{dy}{dx}=f(x)$ together with the initial condition $y(x_0)=y_0$

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CHAPTER OVERVIEW

5: Integrals

A general Calculus Textmap organized around the textbook

Calculus: Early Transcendentals

by James Stewart

I II III IV V VI VII VIII IX X XI XII XIII XIV XV XVI XVII

This Textmap is currently under construction... please be patient with us.

Template:HideTOC

Topic hierarchy

5.1: Areas and Distances
5.2: The Definite Integral
5.3: The Fundamental Theorem of Calculus
5.4: Indefinite Integrals and the Net Change Theorem
5.5: The Substitution Rule

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5.1: Areas and Distances

Learning Objectives

- Use sigma (summation) notation to calculate sums and powers of integers.
- Use the sum of rectangular areas to approximate the area under a curve.
- Use Riemann sums to approximate area.

Archimedes was fascinated with calculating the areas of various shapes—in other words, the amount of space enclosed by the shape. He used a process that has come to be known as the **method of exhaustion**, which used smaller and smaller shapes, the areas of which could be calculated exactly, to fill an irregular region and thereby obtain closer and closer approximations to the total area. In this process, an area bounded by curves is filled with rectangles, triangles, and shapes with exact area formulas. These areas are then summed to approximate the area of the curved region.

In this section, we develop techniques to approximate the area between a curve, defined by a function $f(x)$, and the x-axis on a closed interval $[a, b]$. Like Archimedes, we first approximate the area under the curve using shapes of known area (namely, rectangles). By using smaller and smaller rectangles, we get closer and closer approximations to the area. Taking a limit allows us to calculate the exact area under the curve.

Let's start by introducing some notation to make the calculations easier. We then consider the case when $f(x)$ is continuous and nonnegative. Later in the chapter, we relax some of these restrictions and develop techniques that apply in more general cases.

Sigma (Summation) Notation

As mentioned, we will use shapes of known area to approximate the area of an irregular region bounded by curves. This process often requires adding up long strings of numbers. To make it easier to write down these lengthy sums, we look at some new notation here, called **sigma notation** (also known as **summation notation**). The Greek capital letter Σ, sigma, is used to express long sums of values in a compact form. For example, if we want to add all the integers from 1 to 20 without sigma notation, we have to write

$$1+2+3+4+5+6+7+8+9+10+11+12+13+14+15+16+17+18+19+20.$$

We could probably skip writing a couple of terms and write

$$1+2+3+4+\cdots+19+20,$$

which is better, but still cumbersome. With sigma notation, we write this sum as

$$\sum_{i=1}^{20} i$$

which is much more compact. Typically, sigma notation is presented in the form

$$\sum_{i=1}^{n} a_i$$

where a_i describes the terms to be added, and the i is called the $index$. Each term is evaluated, then we sum all the values, beginning with the value when $i = 1$ and ending with the value when $i = n$. For example, an expression like $\sum_{i=2}^{7} s_i$ is interpreted as $s_2 + s_3 + s_4 + s_5 + s_6 + s_7$. Note that the index is used only to keep track of the terms to be added; it does not factor into the calculation of the sum itself. The index is therefore called a **dummy variable**. We can use any letter we like for the index. Typically, mathematicians use i, j, k, m, and n for indices.

Let's try a couple of examples of using sigma notation.

Example 5.1.1: Using Sigma Notation

a. Write in sigma notation and evaluate the sum of terms 3^i for $i = 1, 2, 3, 4, 5$.
b. Write the sum in sigma notation:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}.$$

Solution

a. Write

$$\sum_{i=1}^{5} 3^i = 3 + 3^2 + 3^3 + 3^4 + 3^5 = 363.$$

b. The denominator of each term is a perfect square. Using sigma notation, this sum can be written as $\sum_{i=1}^{5} \frac{1}{i^2}$.

Exercise 5.1.1

Write in sigma notation and evaluate the sum of terms 2^i for $i = 3, 4, 5, 6$.

Hint

Use the solving steps in Example 5.1.1as a guide.

Answer

$$\sum_{i=3}^{6} 2^i = 2^3 + 2^4 + 2^5 + 2^6 = 120$$

The properties associated with the summation process are given in the following rule.

Rule: Properties of Sigma Notation

Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ represent two sequences of terms and let c be a constant. The following properties hold for all positive integers n and for integers m, with $1 \le m \le n$.

i. $\sum_{i=1}^{n} c = nc$
ii. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$
iii. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$
iv. $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$
v. $\sum_{i=1}^{n} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i$

Proof

We prove properties (ii.) and (iii.) here, and leave proof of the other properties to the Exercises.

(ii.) We have

$$\sum_{i=1}^{n} ca_i = ca_1 + ca_2 + ca_3 + \cdots + ca_n = c(a_1 + a_2 + a_3 + \cdots + a_n) = c\sum_{i=1}^{n} a_i.$$

(iii.) We have

$$\sum_{i=1}^{n}(a_i+b_i) = (a_1+b_1)+(a_2+b_2)+(a_3+b_3)+\cdots+(a_n+b_n) \tag{5.1.1}$$

$$= (a_1+a_2+a_3+\cdots+a_n)+(b_1+b_2+b_3+\cdots+b_n) \tag{5.1.2}$$

$$= \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i. \tag{5.1.3}$$

□

A few more formulas for frequently found functions simplify the summation process further. These are shown in the next rule, for **sums and powers of integers**, and we use them in the next set of examples.

Rule: Sums and Powers of Integers

1. The sum of n integers is given by

$$\sum_{i=1}^{n} i = 1+2+\cdots+n = \frac{n(n+1)}{2}. \tag{5.1.4}$$

2. The sum of consecutive integers squared is given by

$$\sum_{i=1}^{n} i^2 = 1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}. \tag{5.1.5}$$

3. The sum of consecutive integers cubed is given by

$$\sum_{i=1}^{n} i^3 = 1^3+2^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4}. \tag{5.1.6}$$

Example 5.1.2: Evaluation Using Sigma Notation

Write using sigma notation and evaluate:

a. The sum of the terms $(i-3)^2$ for $i = 1, 2, \ldots, 200$.
b. The sum of the terms (i^3-i^2) for $i = 1, 2, 3, 4, 5, 6$

Solution

a. Multiplying out $(i-3)^2$, we can break the expression into three terms.

$$\begin{aligned}
\sum_{i=1}^{200}(i-3)^2 &= \sum_{i=1}^{200}(i^2-6i+9) \\
&= \sum_{i=1}^{200} i^2 - \sum_{i=1}^{200} 6i + \sum_{i=1}^{200} 9 \\
&= \sum_{i=1}^{200} i^2 - 6\sum_{i=1}^{200} i + \sum_{i=1}^{200} 9 \\
&= \frac{200(200+1)(400+1)}{6} - 6\left[\frac{200(200+1)}{2}\right] + 9(200) \\
&= 2,686,700 - 120,600 + 1800 \\
&= 2,567,900
\end{aligned}$$

b. Use sigma notation property iv. and the rules for the sum of squared terms and the sum of cubed terms.

$$\begin{aligned}\sum_{i=1}^{6}(i^3-i^2) &= \sum_{i=1}^{6} i^3 - \sum_{i=1}^{6} i^2 \\ &= \frac{6^2(6+1)^2}{4} - \frac{6(6+1)(2(6)+1)}{6} \\ &= \frac{1764}{4} - \frac{546}{6} \\ &= 350\end{aligned}$$

? Exercise 5.1.2

Find the sum of the values of $4+3i$ for $i=1,2,\ldots,100$.

Hint

Use the properties of sigma notation to solve the problem.

Answer

$15,550$

✓ Example 5.1.3: Finding the Sum of the Function Values

Find the sum of the values of $f(x)=x^3$ over the integers $1,2,3,\ldots,10$.

Solution

Using Equation 5.1.6, we have

$$\sum_{i=0}^{10} i^3 = \frac{(10)^2(10+1)^2}{4} = \frac{100(121)}{4} = 3025$$

? Exercise 5.1.3

Evaluate the sum indicated by the notation $\sum_{k=1}^{20}(2k+1)$.

Hint

Use the rule on sum and powers of integers (Equations 5.1.4–5.1.6).

Answer

440

Approximating Area

Now that we have the necessary notation, we return to the problem at hand: approximating the area under a curve. Let $f(x)$ be a continuous, nonnegative function defined on the closed interval $[a,b]$. We want to approximate the area A bounded by $f(x)$ above, the x-axis below, the line $x=a$ on the left, and the line $x=b$ on the right (Figure 5.1.1).

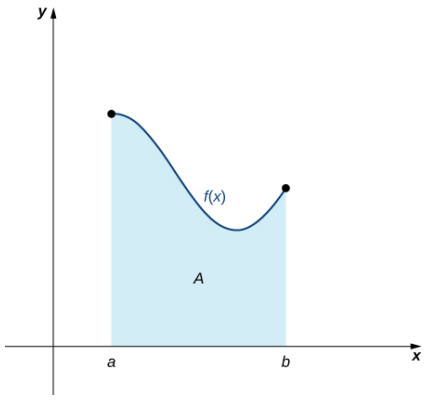


Figure 5.1.1: An area (shaded region) bounded by the curve $f(x)$ at top, the x-axis at bottom, the line $x = a$ to the left, and the line $x = b$ at right.

How do we approximate the area under this curve? The approach is a geometric one. By dividing a region into many small shapes that have known area formulas, we can sum these areas and obtain a reasonable estimate of the true area. We begin by dividing the interval $[a, b]$ into n subintervals of equal width, $\dfrac{b-a}{n}$. We do this by selecting equally spaced points $x_0, x_1, x_2, \ldots, x_n$ with $x_0 = a, x_n = b,$ and

$$x_i - x_{i-1} = \frac{b-a}{n}$$

for $i = 1, 2, 3, \ldots, n$.

We denote the width of each subinterval with the notation Δx, so $\Delta x = \frac{b-a}{n}$ and

$$x_i = x_0 + i\Delta x$$

for $i = 1, 2, 3, \ldots, n$. This notion of dividing an interval $[a, b]$ into subintervals by selecting points from within the interval is used quite often in approximating the area under a curve, so let's define some relevant terminology.

Definition: Partitions

A set of points $P = x_i$ for $i = 0, 1, 2, \ldots, n$ with $a = x_0 < x_1 < x_2 < \ldots < x_n = b$, which divides the interval $[a, b]$ into subintervals of the form $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ is called a **partition** of $[a, b]$. If the subintervals all have the same width, the set of points forms a **regular partition** (or uniform partition) of the interval $[a, b]$.

We can use this regular partition as the basis of a method for estimating the area under the curve. We next examine two methods: the left-endpoint approximation and the right-endpoint approximation.

Rule: Left-Endpoint Approximation

On each subinterval $[x_{i-1}, x_i]$ (for $i = 1, 2, 3, \ldots, n$), construct a rectangle with width Δx and height equal to $f(x_{i-1})$, which is the function value at the left endpoint of the subinterval. Then the area of this rectangle is $f(x_{i-1})\Delta x$. Adding the areas of all these rectangles, we get an approximate value for A (Figure 5.1.2). We use the notation L_n to denote that this is a **left-endpoint approximation** of A using n subintervals.

$$A \approx L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

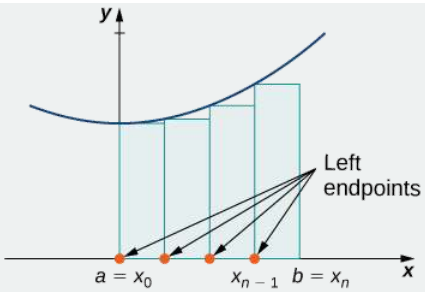


Figure 5.1.2: In the left-endpoint approximation of area under a curve, the height of each rectangle is determined by the function value at the left of each subinterval.

The second method for approximating area under a curve is the right-endpoint approximation. It is almost the same as the left-endpoint approximation, but now the heights of the rectangles are determined by the function values at the right of each subinterval.

Rule: Right-Endpoint Approximation

Construct a rectangle on each subinterval $[x_{i-1}, x_i]$, only this time the height of the rectangle is determined by the function value $f(x_i)$ at the right endpoint of the subinterval. Then, the area of each rectangle is $f(x_i)\,\Delta x$ and the approximation for A is given by

$$A \approx R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x.$$

The notation R_n indicates this is a **right-endpoint approximation** for A (Figure 5.1.3).

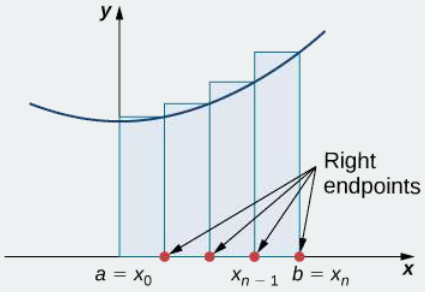


Figure 5.1.3: In the right-endpoint approximation of area under a curve, the height of each rectangle is determined by the function value at the right of each subinterval. Note that the right-endpoint approximation differs from the left-endpoint approximation in Figure 5.1.2.

The graphs in Figure 5.1.4 represent the curve $f(x) = \dfrac{x^2}{2}$. In Figure 5.1.4*b* we divide the region represented by the interval $[0, 3]$ into six subintervals, each of width 0.5. Thus, $\Delta x = 0.5$. We then form six rectangles by drawing vertical lines perpendicular to x_{i-1}, the left endpoint of each subinterval. We determine the height of each rectangle by calculating $f(x_{i-1})$ for $i = 1, 2, 3, 4, 5, 6$. The intervals are $[0, 0.5], [0.5, 1], [1, 1.5], [1.5, 2], [2, 2.5], [2.5, 3]$. We find the area of each rectangle by multiplying the height by the width. Then, the sum of the rectangular areas approximates the area between $f(x)$ and the x-axis. When the left endpoints are used to calculate height, we have a left-endpoint approximation. Thus,

$$\begin{aligned} A \approx L_6 &= \sum_{i=1}^{6} f(x_{i-1})\Delta x = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x \\ &= f(0)0.5 + f(0.5)0.5 + f(1)0.5 + f(1.5)0.5 + f(2)0.5 + f(2.5)0.5 \\ &= (0)0.5 + (0.125)0.5 + (0.5)0.5 + (1.125)0.5 + (2)0.5 + (3.125)0.5 \\ &= 0 + 0.0625 + 0.25 + 0.5625 + 1 + 1.5625 \\ &= 3.4375 \text{ units}^2 \end{aligned}$$

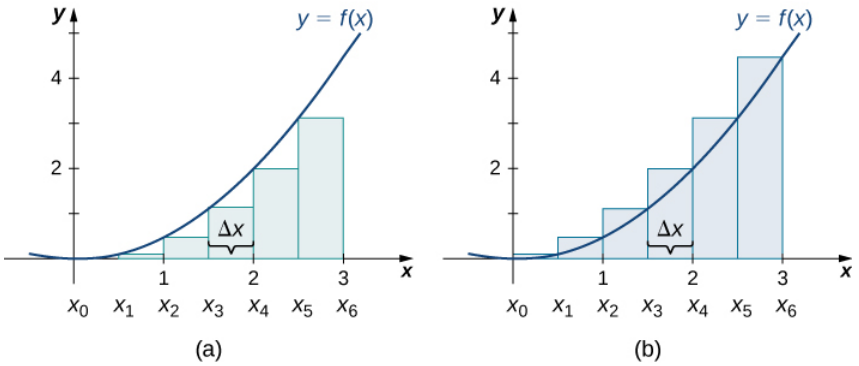


Figure 5.1.4: Methods of approximating the area under a curve by using (a) the left endpoints and (b) the right endpoints.

In Figure 5.1.4b, we draw vertical lines perpendicular to x_i such that x_i is the right endpoint of each subinterval, and calculate $f(x_i)$ for $i = 1, 2, 3, 4, 5, 6$. We multiply each $f(x_i)$ by Δx to find the rectangular areas, and then add them. This is a right-endpoint approximation of the area under $f(x)$. Thus,

$$\begin{aligned} A \approx R_6 &= \sum_{i=1}^{6} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x \\ &= f(0.5)0.5 + f(1)0.5 + f(1.5)0.5 + f(2)0.5 + f(2.5)0.5 + f(3)0.5 \\ &= (0.125)0.5 + (0.5)0.5 + (1.125)0.5 + (2)0.5 + (3.125)0.5 + (4.5)0.5 \\ &= 0.0625 + 0.25 + 0.5625 + 1 + 1.5625 + 2.25 \\ &= 5.6875 \text{ units}^2. \end{aligned}$$

Example 5.1.4: Approximating the Area Under a Curve

Use both left-endpoint and right-endpoint approximations to approximate the area under the curve of $f(x) = x^2$ on the interval $[0, 2]$; use $n = 4$.

Solution

First, divide the interval $[0, 2]$ into n equal subintervals. Using $n = 4$, $\Delta x = \dfrac{(2-0)}{4} = 0.5$. This is the width of each rectangle. The intervals $[0, 0.5], [0.5, 1], [1, 1.5], [1.5, 2]$ are shown in Figure 5.1.5. Using a left-endpoint approximation, the heights are $f(0) = 0$, $f(0.5) = 0.25$, $f(1) = 1$, and $f(1.5) = 2.25$. Then,

$$\begin{aligned} L_4 &= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x \\ &= 0(0.5) + 0.25(0.5) + 1(0.5) + 2.25(0.5) \\ &= 1.75 \text{ units}^2 \end{aligned}$$

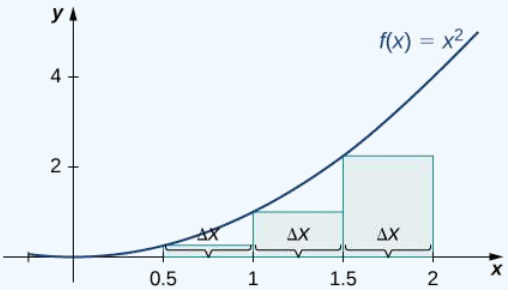


Figure 5.1.5: The graph shows the left-endpoint approximation of the area under $f(x) = x^2$ from 0 to 2.

The right-endpoint approximation is shown in Figure 5.1.6. The intervals are the same, $\Delta x = 0.5$, but now use the right endpoint to calculate the height of the rectangles. We have

$$\begin{aligned} R_4 &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \\ &= 0.25(0.5) + 1(0.5) + 2.25(0.5) + 4(0.5) \\ &= 3.75 \text{ units}^2 \end{aligned}$$

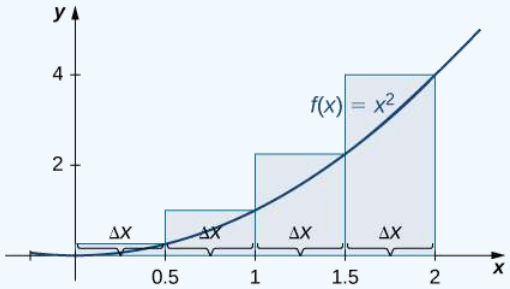


Figure 5.1.6: The graph shows the right-endpoint approximation of the area under $f(x) = x^2$ from 0 to 2.

The left-endpoint approximation is 1.75 units^2; the right-endpoint approximation is 3.75 units^2.

? Exercise 5.1.4

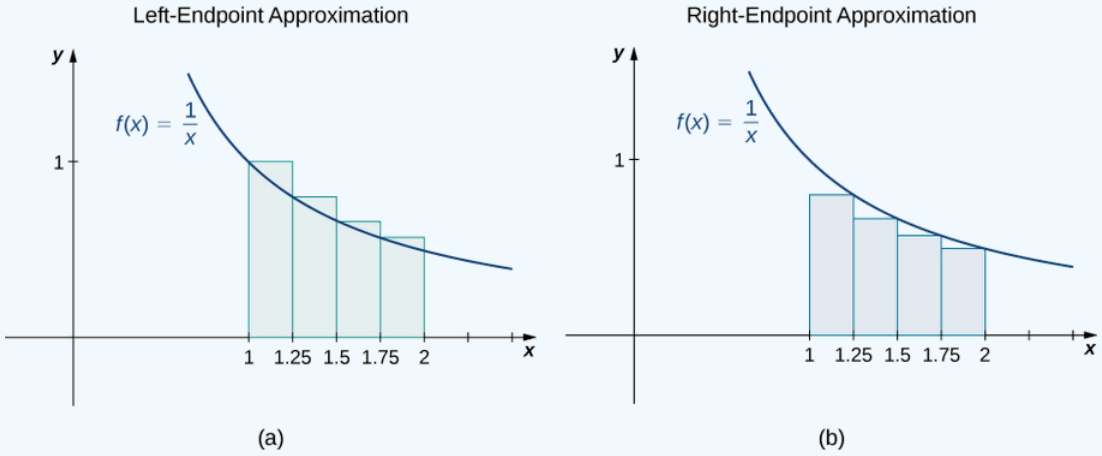
Sketch left-endpoint and right-endpoint approximations for $f(x) = \dfrac{1}{x}$ on $[1, 2]$; use $n = 4$. Approximate the area using both methods.

Hint

Follow the solving strategy in Example 5.1.4step-by-step.

Answer

The left-endpoint approximation is 0.7595 units^2. The right-endpoint approximation is 0.6345 units^2. See the below Media.



Looking at Figure 5.1.4 and the graphs in Example 5.1.4, we can see that when we use a small number of intervals, neither the left-endpoint approximation nor the right-endpoint approximation is a particularly accurate estimate of the area under the curve. However, it seems logical that if we increase the number of points in our partition, our estimate of A will improve. We will have more rectangles, but each rectangle will be thinner, so we will be able to fit the rectangles to the curve more precisely.

We can demonstrate the improved approximation obtained through smaller intervals with an example. Let's explore the idea of increasing n, first in a left-endpoint approximation with four rectangles, then eight rectangles, and finally 32 rectangles. Then, let's do the same thing in a right-endpoint approximation, using the same sets of intervals, of the same curved region. Figure 5.1.7 shows the area of the region under the curve $f(x) = (x-1)^3 + 4$ on the interval $[0, 2]$ using a left-endpoint approximation where $n = 4$. The width of each rectangle is

$$\Delta x = \frac{2-0}{4} = \frac{1}{2}.$$

The area is approximated by the summed areas of the rectangles, or

$$L_4 = f(0)(0.5) + f(0.5)(0.5) + f(1)(0.5) + f(1.5)0.5 = 7.5 \text{ units}^2$$

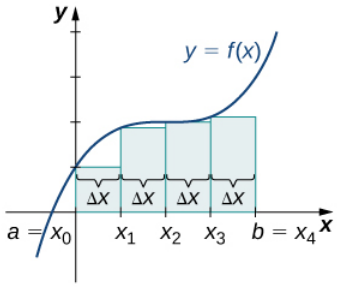


Figure 5.1.7: With a left-endpoint approximation and dividing the region from a to b into four equal intervals, the area under the curve is approximately equal to the sum of the areas of the rectangles.

Figure 5.1.8 shows the same curve divided into eight subintervals. Comparing the graph with four rectangles in Figure 5.1.7 with this graph with eight rectangles, we can see there appears to be less white space under the curve when $n = 8$. This white space is area under the curve we are unable to include using our approximation. The area of the rectangles is

$$L_8 = f(0)(0.25) + f(0.25)(0.25) + f(0.5)(0.25) + f(0.75)(0.25) + f(1)(0.25) + f(1.25)(0.25) + f(1.5)(0.25) + f(1.75)(0.25) = 7.75 \text{ units}^2$$

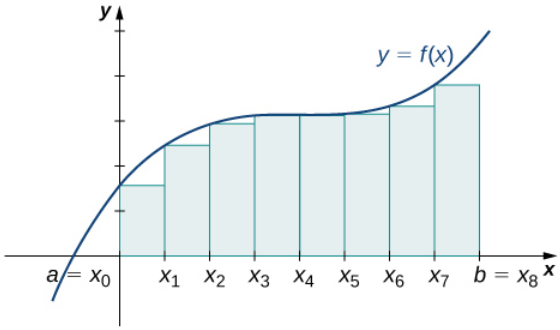


Figure 5.1.8: The region under the curve is divided into $n = 8$ rectangular areas of equal width for a left-endpoint approximation.

The graph in Figure 5.1.9 shows the same function with 32 rectangles inscribed under the curve. There appears to be little white space left. The area occupied by the rectangles is

$$L_{32} = f(0)(0.0625) + f(0.0625)(0.0625) + f(0.125)(0.0625) + \cdots + f(1.9375)(0.0625) = 7.9375 \text{ units}^2.$$

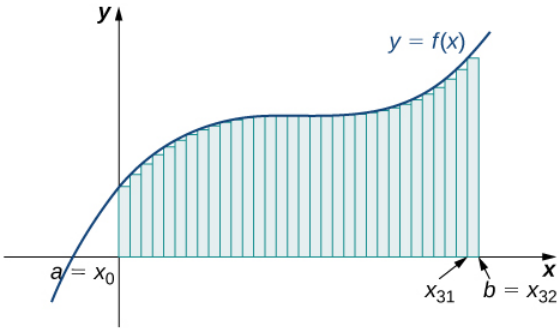


Figure 5.1.9: Here, 32 rectangles are inscribed under the curve for a left-endpoint approximation.

We can carry out a similar process for the right-endpoint approximation method. A right-endpoint approximation of the same curve, using four rectangles (Figure 5.1.10), yields an area

$$R_4 = f(0.5)(0.5) + f(1)(0.5) + f(1.5)(0.5) + f(2)(0.5) = 8.5 \text{ units}^2.$$

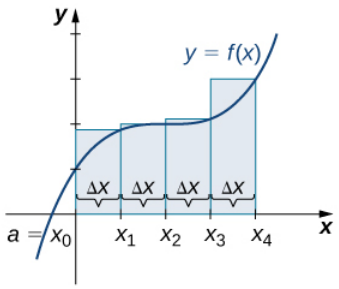


Figure 5.1.10: Now we divide the area under the curve into four equal subintervals for a right-endpoint approximation.

Dividing the region over the interval $[0, 2]$ into eight rectangles results in $\Delta x = \dfrac{2-0}{8} = 0.25$. The graph is shown in Figure 5.1.11. The area is

$$R_8 = f(0.25)(0.25) + f(0.5)(0.25) + f(0.75)(0.25) + f(1)(0.25) + f(1.25)(0.25) + f(1.5)(0.25) + f(1.75)(0.25) + f(2)(0.25) = 8.25 \text{ units}^2$$

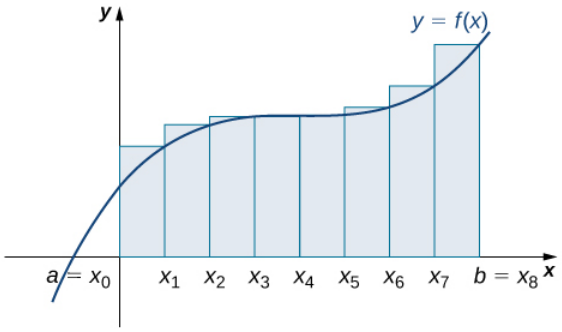


Figure 5.1.11: Here we use right-endpoint approximation for a region divided into eight equal subintervals.

Last, the right-endpoint approximation with $n = 32$ is close to the actual area (Figure 5.1.12). The area is approximately

$$R_{32} = f(0.0625)(0.0625) + f(0.125)(0.0625) + f(0.1875)(0.0625) + \cdots + f(2)(0.0625) = 8.0625 \text{ units}^2$$

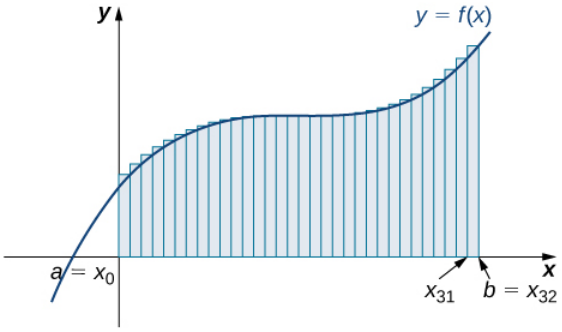


Figure 5.1.12: The region is divided into 32 equal subintervals for a right-endpoint approximation.

Based on these figures and calculations, it appears we are on the right track; the rectangles appear to approximate the area under the curve better as n gets larger. Furthermore, as n increases, both the left-endpoint and right-endpoint approximations appear to approach an area of 8 square units. Table 5.1.15 shows a numerical comparison of the left- and right-endpoint methods. The idea that the approximations of the area under the curve get better and better as n gets larger and larger is very important, and we now explore this idea in more detail.

Table 5.1.15: Converging Values of Left- and Right-Endpoint Approximations as n Increases

Value of n	Approximate Area L_n	Approximate Area R_n
$n = 4$	7.5	8.5
$n = 8$	7.75	8.25

Value of n	Approximate Area L_n	Approximate Area R_n
$n = 32$	7.94	8.06

Forming Riemann Sums

So far we have been using rectangles to approximate the area under a curve. The heights of these rectangles have been determined by evaluating the function at either the right or left endpoints of the subinterval $[x_{i-1}, x_i]$. In reality, there is no reason to restrict evaluation of the function to one of these two points only. We could evaluate the function at any point x_i^* in the subinterval $[x_{i-1}, x_i]$, and use $f(x_i^*)$ as the height of our rectangle. This gives us an estimate for the area of the form

$$A \approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$

A sum of this form is called a Riemann sum, named for the 19th-century mathematician Bernhard Riemann, who developed the idea.

Definition: Riemann sum

Let $f(x)$ be defined on a closed interval $[a, b]$ and let P be any partition of $[a, b]$. Let Δx_i be the width of each subinterval $[x_{i-1}, x_i]$ and for each i, let x_i^* be any point in $[x_{i-1},\ x_i]$. A Riemann sum is defined for $f(x)$ as

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i.$$

At this point, we'll choose a regular partition P, as we have in our examples above. This forces all Δx_i to be equal to $\Delta x = \dfrac{b-a}{n}$ for any natural number of intervals n.

Recall that with the left- and right-endpoint approximations, the estimates seem to get better and better as n get larger and larger. The same thing happens with Riemann sums. Riemann sums give better approximations for larger values of n. We are now ready to define the area under a curve in terms of Riemann sums.

Definition: Area Under the Curve

Let $f(x)$ be a continuous, nonnegative function on an interval $[a, b]$, and let $\displaystyle\sum_{i=1}^{n} f(x_i^*)\,\Delta x$ be a Riemann sum for $f(x)$ with a regular partition P. Then, the **area under the curve** $y = f(x)$ on $[a, b]$ is given by

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$

See a graphical demonstration of the construction of a Riemann sum.

Some subtleties here are worth discussing. First, note that taking the limit of a sum is a little different from taking the limit of a function $f(x)$ as x goes to infinity. Limits of sums are discussed in detail in the chapter on Sequences and Series; however, for now we can assume that the computational techniques we used to compute limits of functions can also be used to calculate limits of sums.

Second, we must consider what to do if the expression converges to different limits for different choices of x_i^*. Fortunately, this does not happen. Although the proof is beyond the scope of this text, it can be shown that if $f(x)$ is continuous on the closed interval $[a, b]$, then $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$ exists and is unique (in other words, it does not depend on the choice of x_i^*).

We look at some examples shortly. But, before we do, let's take a moment and talk about some specific choices for x_i^*. Although any choice for x_i^* gives us an estimate of the area under the curve, we don't necessarily know whether that estimate is too high

(overestimate) or too low (underestimate). If it is important to know whether our estimate is high or low, we can select our value for x_i^* to guarantee one result or the other.

If we want an overestimate, for example, we can choose x_i^* such that for $i = 1, 2, 3, \ldots, n, f(x_i^*) \geq f(x)$ for all $x \in [x_i - 1, x_i]$. In other words, we choose x_i^* so that for $i = 1, 2, 3, \ldots, n, f(x_i^*)$ is the maximum function value on the interval $[x_{i-1}, x_i]$. If we select x_i^* in this way, then the Riemann sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called an **upper sum**. Similarly, if we want an underestimate, we can choose x_i^* so that for $i = 1, 2, 3, \ldots, n, f(x_i^*)$ is the minimum function value on the interval $[x_{i-1}, x_i]$. In this case, the associated Riemann sum is called a **lower sum**. Note that if $f(x)$ is either increasing or decreasing throughout the interval $[a, b]$, then the maximum and minimum values of the function occur at the endpoints of the subintervals, so the upper and lower sums are just the same as the left- and right-endpoint approximations.

Example 5.1.5: Finding Lower and Upper Sums

Find a lower sum for $f(x) = 10 - x^2$ on $[1, 2]$; let $n = 4$ subintervals.

Solution

With $n = 4$ over the interval $[1, 2]$, $\Delta x = \frac{1}{4}$. We can list the intervals as $[1, 1.25]$, $[1.25, 1.5]$, $[1.5, 1.75]$, and $[1.75, 2]$. Because the function is decreasing over the interval $[1, 2]$, Figure shows that a lower sum is obtained by using the right endpoints.

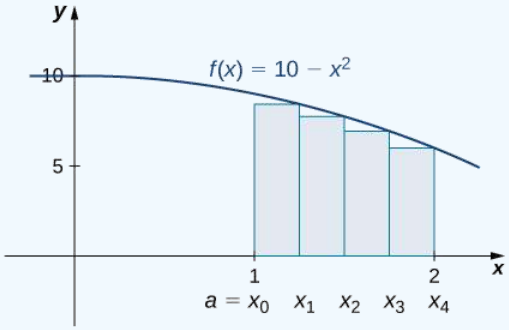


Figure 5.1.13: The graph of $f(x) = 10 - x^2$ is set up for a right-endpoint approximation of the area bounded by the curve and the x-axis on $[1, 2]$, and it shows a lower sum.

The Riemann sum is

$$\begin{aligned}\sum_{k=1}^{4}(10 - x^2)(0.25) &= 0.25[10 - (1.25)^2 + 10 - (1.5)^2 + 10 - (1.75)^2 + 10 - (2)^2] \\ &= 0.25[8.4375 + 7.75 + 6.9375 + 6] \\ &= 7.28 \text{ units}^2.\end{aligned}$$

The area of 7.28 units^2 is a lower sum and an underestimate.

Exercise 5.1.5

a. Find an upper sum for $f(x) = 10 - x^2$ on $[1, 2]$; let $n = 4$.
b. Sketch the approximation.

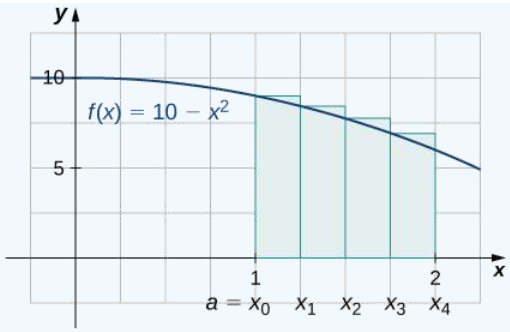
Hint

$f(x)$ is decreasing on $[1, 2]$, so the maximum function values occur at the left endpoints of the subintervals.

Answer

a. Upper sum=8.0313 units^2.

b.



Example 5.1.6: Finding Lower and Upper Sums for $f(x) = \sin x$

Find a lower sum for $f(x) = \sin x$ over the interval $[a, b] = \left[0, \frac{\pi}{2}\right]$; let $n = 6$.

Solution

Let's first look at the graph in Figure 5.1.14 to get a better idea of the area of interest.

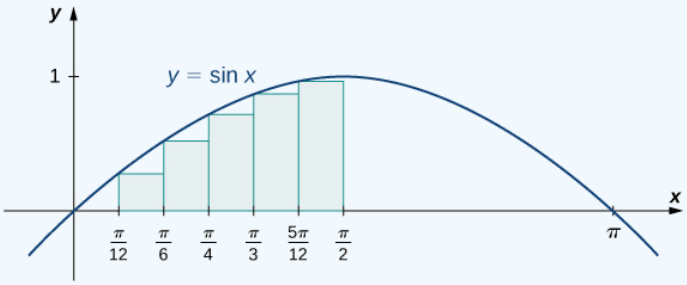


Figure 5.1.14: The graph of $y = \sin x$ is divided into six regions: $\Delta x = \dfrac{\pi/2}{6} = \dfrac{\pi}{12}$.

The intervals are $\left[0, \frac{\pi}{12}\right]$, $\left[\frac{\pi}{12}, \frac{\pi}{6}\right]$, $\left[\frac{\pi}{6}, \frac{\pi}{4}\right]$, $\left[\frac{\pi}{4}, \frac{\pi}{3}\right]$, $\left[\frac{\pi}{3}, \frac{5\pi}{12}\right]$, and $\left[\frac{5\pi}{12}, \frac{\pi}{2}\right]$. Note that $f(x) = \sin x$ is increasing on the interval $\left[0, \frac{\pi}{2}\right]$, so a left-endpoint approximation gives us the lower sum. A left-endpoint approximation is the Riemann sum $\sum_{i=0}^{5} \sin x_i \left(\frac{\pi}{12}\right)$. We have

$$A \approx \sin(0)\left(\tfrac{\pi}{12}\right) + \sin\left(\tfrac{\pi}{12}\right)\left(\tfrac{\pi}{12}\right) + \sin\left(\tfrac{\pi}{6}\right)\left(\tfrac{\pi}{12}\right) + \sin\left(\tfrac{\pi}{4}\right)\left(\tfrac{\pi}{12}\right) + \sin\left(\tfrac{\pi}{3}\right)\left(\tfrac{\pi}{12}\right) + \sin\left(\tfrac{5\pi}{12}\right)\left(\tfrac{\pi}{12}\right) \approx 0.863 \text{ units}^2.$$

Exercise 5.1.6

Using the function $f(x) = \sin x$ over the interval $\left[0, \frac{\pi}{2}\right]$, find an upper sum; let $n = 6$.

Hint

Follow the steps from Example 5.1.6

Answer

$A \approx 1.125 \text{ units}^2$

Key Concepts

- The use of sigma (summation) notation of the form $\sum_{i=1}^{n} a_i$ is useful for expressing long sums of values in compact form.
- For a continuous function defined over an interval $[a, b]$, the process of dividing the interval into n equal parts, extending a rectangle to the graph of the function, calculating the areas of the series of rectangles, and then summing the areas yields an approximation of the area of that region.
- When using a regular partition, the width of each rectangle is $\Delta x = \dfrac{b-a}{n}$.

- Riemann sums are expressions of the form $\sum_{i=1}^{n} f(x_i^*)\Delta x$, and can be used to estimate the area under the curve $y = f(x)$. Left- and right-endpoint approximations are special kinds of Riemann sums where the values of x_i^* are chosen to be the left or right endpoints of the subintervals, respectively.
- Riemann sums allow for much flexibility in choosing the set of points x_i^* at which the function is evaluated, often with an eye to obtaining a lower sum or an upper sum.

Key Equations

- **Properties of Sigma Notation**

$$\sum_{i=1}^{n} c = nc$$

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n}(a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i$$

- **Sums and Powers of Integers**

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=0}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

- **Left-Endpoint Approximation**

$$A \approx L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

- **Right-Endpoint Approximation**

$$A \approx R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x$$

Glossary

left-endpoint approximation

an approximation of the area under a curve computed by using the left endpoint of each subinterval to calculate the height of the vertical sides of each rectangle

lower sum

a sum obtained by using the minimum value of $f(x)$ on each subinterval

partition

a set of points that divides an interval into subintervals

regular partition

a partition in which the subintervals all have the same width

riemann sum

an estimate of the area under the curve of the form $A \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$

right-endpoint approximation

the right-endpoint approximation is an approximation of the area of the rectangles under a curve using the right endpoint of each subinterval to construct the vertical sides of each rectangle

sigma notation

(also, **summation notation**) the Greek letter sigma (Σ) indicates addition of the values; the values of the index above and below the sigma indicate where to begin the summation and where to end it

upper sum

a sum obtained by using the maximum value of $f(x)$ on each subinterval

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5.4: Indefinite Integrals and the Net Change Theorem

Learning Objectives

- Apply the basic integration formulas.
- Explain the significance of the net change theorem.
- Use the net change theorem to solve applied problems.
- Apply the integrals of odd and even functions.

In this section, we use some basic integration formulas studied previously to solve some key applied problems. It is important to note that these formulas are presented in terms of indefinite integrals. Although definite and indefinite integrals are closely related, there are some key differences to keep in mind. A definite integral is either a number (when the limits of integration are constants) or a single function (when one or both of the limits of integration are variables). An indefinite integral represents a family of functions, all of which differ by a constant. As you become more familiar with integration, you will get a feel for when to use definite integrals and when to use indefinite integrals. You will naturally select the correct approach for a given problem without thinking too much about it. However, until these concepts are cemented in your mind, think carefully about whether you need a definite integral or an indefinite integral and make sure you are using the proper notation based on your choice.

Basic Integration Formulas

Recall the integration formulas given in the section on Antiderivatives and the properties of definite integrals. Let's look at a few examples of how to apply these formulas and properties.

Example 5.4.1: Integrating a Function Using the Power Rule

Use the power rule to integrate the function $\int_1^4 \sqrt{t}(1+t)\,dt$.

Solution

The first step is to rewrite the function and simplify it so we can apply the power rule:

$$\begin{aligned}\int_1^4 \sqrt{t}(1+t)\,dt &= \int_1^4 t^{1/2}(1+t)\,dt \\ &= \int_1^4 (t^{1/2}+t^{3/2})\,dt.\end{aligned}$$

Now apply the power rule:

$$\begin{aligned}\int_1^4 (t^{1/2}+t^{3/2})\,dt &= \left(\frac{2}{3}t^{3/2}+\frac{2}{5}t^{5/2}\right)\Big|_1^4 \\ &= \left[\frac{2}{3}(4)^{3/2}+\frac{2}{5}(4)^{5/2}\right]-\left[\frac{2}{3}(1)^{3/2}+\frac{2}{5}(1)^{5/2}\right] \\ &= \frac{256}{15}.\end{aligned}$$

Exercise 5.4.1

Find the definite integral of $f(x)=x^2-3x$ over the interval $[1,3]$.

Hint

Follow the process from Example 5.4.1to solve the problem.

Answer

$$\int_1^3 \left(x^2 - 3x\right)\, dx = -\frac{10}{3}$$

The Net Change Theorem

The **net change theorem** considers the integral of a **rate of change**. It says that when a quantity changes, the new value equals the initial value plus the integral of the rate of change of that quantity. The formula can be expressed in two ways. The second is more familiar; it is simply the definite integral.

Net Change Theorem

The new value of a changing quantity equals the initial value plus the integral of the rate of change:

$$F(b) = F(a) + \int_a^b F'(x)dx \tag{5.4.1}$$

or

$$\int_a^b F'(x)dx = F(b) - F(a). \tag{5.4.2}$$

Subtracting $F(a)$ from both sides of the Equation 5.4.1 yields Equation 5.4.2. Since they are equivalent formulas, which one we use depends on the application.

The significance of the net change theorem lies in the results. Net change can be applied to area, distance, and volume, to name only a few applications. Net change accounts for negative quantities automatically without having to write more than one integral. To illustrate, let's apply the net change theorem to a **velocity** function in which the result is **displacement.**

We looked at a simple example of this in The Definite Integral section. Suppose a car is moving due north (the positive direction) at 40 mph between 2 p.m. and 4 p.m., then the car moves south at 30 mph between 4 p.m. and 5 p.m. We can graph this motion as shown in Figure 5.4.1.

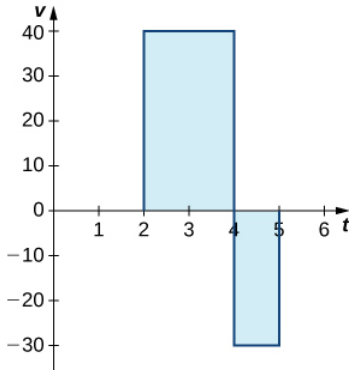


Figure 5.4.1: The graph shows speed versus time for the given motion of a car.

Just as we did before, we can use definite integrals to calculate the net displacement as well as the total distance traveled. The **net displacement** is given by

$$\int_2^5 v(t)\, dt = \int_2^4 40\, dt + \int_4^5 -30\, dt = 80 - 30 = 50.$$

Thus, at 5 p.m. the car is 50 mi north of its starting position. The **total distance** traveled is given by

$$\int_2^5 |v(t)|\, dt = \int_2^4 40\, dt + \int_4^5 30\, dt = 80 + 30 = 110.$$

Therefore, between 2 p.m. and 5 p.m., the car traveled a total of 110 mi.

To summarize, net displacement may include both positive and negative values. In other words, the velocity function accounts for both forward distance and backward distance. To find net displacement, integrate the velocity function over the interval. Total distance traveled, on the other hand, is always positive. To find the total distance traveled by an object, regardless of direction, we need to integrate the absolute value of the velocity function.

Example 5.4.2: Finding Net Displacement

Given a velocity function $v(t) = 3t - 5$ (in meters per second) for a particle in motion from time $t = 0$ to time $t = 3$, find the net displacement of the particle.

Solution

Applying the net change theorem, we have

$$\int_0^3 (3t-5)\,dt = \left(\frac{3t^2}{2} - 5t\right)\Big|_0^3 = \left[\frac{3(3)^2}{2} - 5(3)\right] - 0 = \frac{27}{2} - 15 = \frac{27}{2} - \frac{30}{2} = -\frac{3}{2}.$$

The net displacement is $-\frac{3}{2}$ m (Figure 5.4.2).

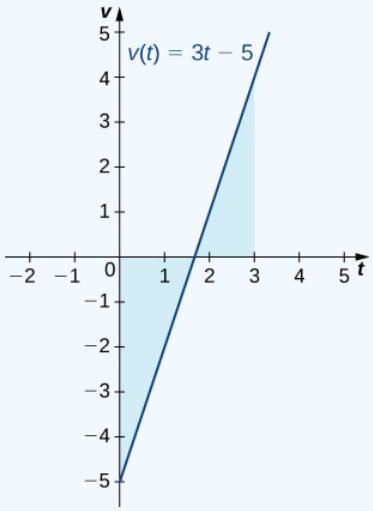


Figure 5.4.2: The graph shows velocity versus time for a particle moving with a linear velocity function.

Example 5.4.3: Finding the Total Distance Traveled

Use Example 5.4.2 to find the total distance traveled by a particle according to the velocity function $v(t) = 3t - 5$ m/sec over a time interval $[0, 3]$.

Solution

The total distance traveled includes both the positive and the negative values. Therefore, we must integrate the absolute value of the velocity function to find the total distance traveled.

To continue with the example, use two integrals to find the total distance. First, find the t-intercept of the function, since that is where the division of the interval occurs. Set the equation equal to zero and solve for t. Thus,

$$\begin{aligned} 3t - 5 &= 0 \\ 3t &= 5 \\ t &= \frac{5}{3}. \end{aligned}$$

The two subintervals are $\left[0, \frac{5}{3}\right]$ and $\left[\frac{5}{3}, 3\right]$. To find the total distance traveled, integrate the absolute value of the function. Since the function is negative over the interval $\left[0, \frac{5}{3}\right]$, we have $|v(t)| = -v(t)$ over that interval. Over $\left[\frac{5}{3}, 3\right]$, the function is positive, so $|v(t)| = v(t)$. Thus, we have

$$\begin{aligned}\int_0^3 |v(t)|\,dt &= \int_0^{5/3} -v(t)\,dt + \int_{5/3}^3 v(t)\,dt \\ &= \int_0^{5/3} 5-3t\,dt + \int_{5/3}^3 3t-5\,dt \\ &= \left(5t-\frac{3t^2}{2}\right)\Big|_0^{5/3} + \left(\frac{3t^2}{2}-5t\right)\Big|_{5/3}^3 \\ &= \left[5(\frac{5}{3})-\frac{3(5/3)^2}{2}\right]-0+\left[\frac{27}{2}-15\right]-\left[\frac{3(5/3)^2}{2}-\frac{25}{3}\right] \\ &= \frac{25}{3}-\frac{25}{6}+\frac{27}{2}-15-\frac{25}{6}+\frac{25}{3} \\ &= \frac{41}{6}\end{aligned}$$

So, the total distance traveled is $\frac{41}{6}$ m.

? Exercise 5.4.2

Find the net displacement and total distance traveled in meters given the velocity function $f(t)=\frac{1}{2}e^t-2$ over the interval $[0,2]$.

Hint

Follow the procedures from Examples 5.4.2and 5.4.3 Note that $f(t)\le 0$ for $t\le \ln 4$ and $f(t)\ge 0$ for $t\ge \ln 4$.

Answer

Net displacement: $\frac{e^2-9}{2}\approx -0.8055$ m; total distance traveled: $4\ln 4-7.5+\frac{e^2}{2}\approx 1.740$ m.

Applying the Net Change Theorem

The net change theorem can be applied to the flow and consumption of fluids, as shown in Example 5.4.4.

✓ Example 5.4.4: How Many Gallons of Gasoline Are Consumed?

If the motor on a motorboat is started at $t=0$ and the boat consumes gasoline at the rate of $5-t^3$ gal/hr, how much gasoline is used in the first 2 hours?

Solution

Express the problem as a definite integral, integrate, and evaluate using the Fundamental Theorem of Calculus. The limits of integration are the endpoints of the interval [0,2]. We have

$$\begin{aligned}\int_0^2 \left(5-t^3\right)\,dt &= \left(5t-\frac{t^4}{4}\right)\Big|_0^2 \\ &= \left[5(2)-\frac{(2)^4}{4}\right]-0 \\ &= 10-\frac{16}{4} \\ &= 6.\end{aligned}$$

Thus, the motorboat uses 6 gal of gas in 2 hours.

Example 5.4.5: Chapter Opener: Iceboats

As we saw at the beginning of the chapter, top **iceboat** racers can attain speeds of up to five times the wind speed. Andrew is an intermediate iceboater, though, so he attains speeds equal to only twice the wind speed.



Figure 5.4.3: (credit: modification of work by Carter Brown, Flickr)

Suppose Andrew takes his iceboat out one morning when a light 5-mph breeze has been blowing all morning. As Andrew gets his iceboat set up, though, the wind begins to pick up. During his first half hour of iceboating, the wind speed increases according to the function $v(t) = 20t + 5$. For the second half hour of Andrew's outing, the wind remains steady at 15 mph. In other words, the wind speed is given by

$$v(t) = \begin{cases} 20t+5, & \text{for } 0 \le t \le \frac{1}{2} \\ 15, & \text{for } \frac{1}{2} \le t \le 1 \end{cases}$$

Recalling that Andrew's iceboat travels at twice the wind speed, and assuming he moves in a straight line away from his starting point, how far is Andrew from his starting point after 1 hour?

Solution

To figure out how far Andrew has traveled, we need to integrate his velocity, which is twice the wind speed. Then

$$\text{Distance} = \int_0^1 2v(t)\,dt.$$

Substituting the expressions we were given for $v(t)$, we get

$$\begin{aligned} \int_0^1 2v(t)\,dt &= \int_0^{1/2} 2v(t)\,dt + \int_{1/2}^1 2v(t)\,dt \\ &= \int_0^{1/2} 2(20t+5)\,dt + \int_{1/3}^1 2(15)\,dt \\ &= \int_0^{1/2} (40t+10)\,dt + \int_{1/2}^1 30\,dt \\ &= \left[20t^2+10t\right]\Big|_0^{1/2} + \left[30t\right]\Big|_{1/2}^1 \\ &= \left(\frac{20}{4}+5\right) - 0 + (30-15) \\ &= 25. \end{aligned}$$

Andrew is 25 mi from his starting point after 1 hour.

Exercise 5.4.3

Suppose that, instead of remaining steady during the second half hour of Andrew's outing, the wind starts to die down according to the function $v(t) = -10t + 15$. In other words, the wind speed is given by

$$v(t) = \begin{cases} 20t + 5, & \text{for } 0 \le t \le \frac{1}{2} \\ -10t + 15, & \text{for } \frac{1}{2} \le t \le 1 \end{cases}.$$

Under these conditions, how far from his starting point is Andrew after 1 hour?

Hint

Don't forget that Andrew's iceboat moves twice as fast as the wind.

Answer

17.5mi

Integrating Even and Odd Functions

We saw in Functions and Graphs that an **even function** is a function in which $f(-x) = f(x)$ for all x in the domain—that is, the graph of the curve is unchanged when x is replaced with $-x$. The graphs of even functions are symmetric about the y-axis. An **odd function** is one in which $f(-x) = -f(x)$ for all x in the domain, and the graph of the function is symmetric about the origin.

Integrals of even functions, when the limits of integration are from $-a$ to a, involve two equal areas, because they are symmetric about the y-axis. Integrals of odd functions, when the limits of integration are similarly $[-a, a]$, evaluate to zero because the areas above and below the x-axis are equal.

Integrals of Even and Odd Functions

For continuous even functions such that $f(-x) = f(x)$,

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

For continuous odd functions such that $f(-x) = -f(x)$,

$$\int_{-a}^{a} f(x)\,dx = 0.$$

Example 5.4.6: Integrating an Even Function

Integrate the even function $\int_{-2}^{2} (3x^8 - 2)\,dx$ and verify that the integration formula for even functions holds.

Solution

The symmetry appears in the graphs in Figure 5.4.4. Graph (a) shows the region below the curve and above the x-axis. We have to zoom in to this graph by a huge amount to see the region. Graph (b) shows the region above the curve and below the x-axis. The signed area of this region is negative. Both views illustrate the symmetry about the y-axis of an even function. We have

$$\begin{aligned}\int_{-2}^{2}(3x^8-2)\,dx &= \left(\frac{x^9}{3}-2x\right)\Big|_{-2}^{2}\\ &= \left[\frac{(2)^9}{3}-2(2)\right]-\left[\frac{(-2)^9}{3}-2(-2)\right]\\ &= \left(\frac{512}{3}-4\right)-\left(-\frac{512}{3}+4\right)\\ &= \frac{1000}{3}.\end{aligned}$$

To verify the integration formula for even functions, we can calculate the integral from 0 to 2 and double it, then check to make sure we get the same answer.

$$\int_{0}^{2}(3x^8-2)\,dx = \left(\frac{x^9}{3}-2x\right)\Big|_{0}^{2} = \frac{512}{3}-4 = \frac{500}{3}$$

Since $2\cdot\frac{500}{3}=\frac{1000}{3}$, we have verified the formula for even functions in this particular example.

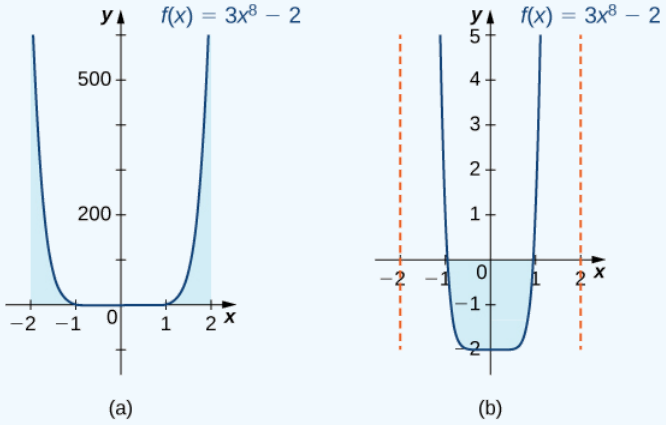


Figure 5.4.4: Graph (a) shows the positive area between the curve and the x-axis, whereas graph (b) shows the negative area between the curve and the x-axis. Both views show the symmetry about the y-axis.

Example 5.4.7: Integrating an Odd Function

Evaluate the definite integral of the odd function $-5\sin x$ over the interval $[-\pi,\pi]$.

Solution

The graph is shown in Figure 5.4.5. We can see the symmetry about the origin by the positive area above the x-axis over $[-\pi,0]$, and the negative area below the x-axis over $[0,\pi]$. we have

$$\begin{aligned}\int_{-\pi}^{\pi}-5\sin x\,dx &= -5(-\cos x)\Big|_{-\pi}^{\pi}\\ &= 5\cos x\Big|_{-\pi}^{\pi}\\ &= [5\cos\pi]-[5\cos(-\pi)]\\ &= -5-(-5)=0.\end{aligned}$$

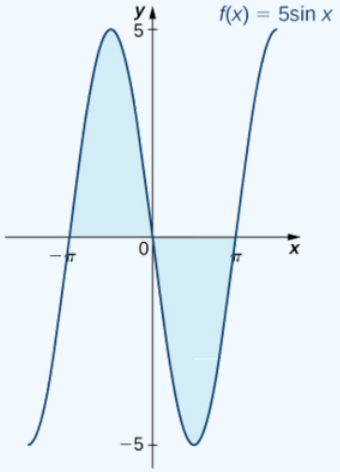


Figure 5.4.5:The graph shows areas between a curve and the x-axis for an odd function.

? Exercise 5.4.4

Integrate the function $\int_{-2}^{2} x^4\,dx$.

Hint

Integrate an even function.

Answer

$\frac{64}{5}$

Key Concepts

- The net change theorem states that when a quantity changes, the final value equals the initial value plus the integral of the rate of change. Net change can be a positive number, a negative number, or zero.
- The area under an even function over a symmetric interval can be calculated by doubling the area over the positive x-axis. For an odd function, the integral over a symmetric interval equals zero, because half the area is negative.

Key Equations

- Net Change Theorem

$$F(b) = F(a) + \int_a^b F'(x)\,dx$$

or

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Glossary

net change theorem

if we know the rate of change of a quantity, the net change theorem says the future quantity is equal to the initial quantity plus the integral of the rate of change of the quantity

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5.5: The Substitution Rule

Learning Objectives

- Use substitution to evaluate indefinite integrals.
- Use substitution to evaluate definite integrals.

The Fundamental Theorem of Calculus gave us a method to evaluate integrals without using Riemann sums. The drawback of this method, though, is that we must be able to find an antiderivative, and this is not always easy. In this section we examine a technique, called *integration by substitution*, to help us find antiderivatives. Specifically, this method helps us find antiderivatives when the integrand is the result of a chain-rule derivative.

At first, the approach to the substitution procedure may not appear very obvious. However, it is primarily a visual task—that is, the integrand shows you what to do; it is a matter of recognizing the form of the function. So, what are we supposed to see? We are looking for an integrand of the form $f\big[g(x)\big]\,g'(x)\,dx$. For example, in the integral

$$\int (x^2-3)^3\,2x\,dx. \tag{5.5.1}$$

we have

$$f(x)=x^3$$

and

$$g(x)=x^2-3.$$

Then

$$g'(x)=2x.$$

and

$$f[g(x)]g'(x)=(x^2-3)^3(2x),$$

and we see that our integrand is in the correct form. The method is called substitution because we substitute part of the integrand with the variable u and part of the integrand with du. It is also referred to as *change of variables* because we are changing variables to obtain an expression that is easier to work with for applying the integration rules.

Substitution with Indefinite Integrals

Let $u=g(x)$, where $g'(x)$ is continuous over an interval, let $f(x)$ be continuous over the corresponding range of g, and let $F(x)$ be an antiderivative of $f(x)$. Then,

$$\begin{aligned}\int f[g(x)]g'(x)\,dx &= \int f(u)\,du \\ &= F(u)+C \\ &= F(g(x))+C\end{aligned}$$

Proof

Let f, g, u, and F be as specified in the theorem. Then

$$\frac{d}{dx}\big[F(g(x))\big]=F'(g(x))g'(x)=f[g(x)]g'(x).$$

Integrating both sides with respect to x, we see that

$$\int f[g(x)]g'(x)\,dx=F(g(x))+C.$$

If we now substitute $u = g(x)$, and $du = g'(x)\,dx$, we get

$$\int f[g(x)]g'(x)\,dx = \int f(u)\,du = F(u) + C = F(g(x)) + C.$$

□

Returning to the problem we looked at originally, we let $u = x^2 - 3$ and then $du = 2x\,dx$.

Rewrite the integral (Equation 5.5.1) in terms of u:

$$\int (x^2 - 3)^3 (2x\,dx) = \int u^3\,du.$$

Using the power rule for integrals, we have

$$\int u^3\,du = \frac{u^4}{4} + C.$$

Substitute the original expression for x back into the solution:

$$\frac{u^4}{4} + C = \frac{(x^2 - 3)^4}{4} + C.$$

We can generalize the procedure in the following Problem-Solving Strategy.

Problem-Solving Strategy: Integration by Substitution

1. Look carefully at the integrand and select an expression $g(x)$ within the integrand to set equal to u. Let's select $g(x)$. such that $g'(x)$ is also part of the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)dx$. into the integral.
3. We should now be able to evaluate the integral with respect to u. If the integral can't be evaluated we need to go back and select a different expression to use as u.
4. Evaluate the integral in terms of u.
5. Write the result in terms of x and the expression $g(x)$.

Example 5.5.1: Using Substitution to Find an Antiderivative

Use substitution to find the antiderivative of $\int 6x(3x^2 + 4)^4\,dx$.

Solution

The first step is to choose an expression for u. We choose $u = 3x^2 + 4$ because then $du = 6x\,dx$ and we already have du in the integrand. Write the integral in terms of u:

$$\int 6x(3x^2 + 4)^4\,dx = \int u^4\,du.$$

Remember that du is the derivative of the expression chosen for u, regardless of what is inside the integrand. Now we can evaluate the integral with respect to u:

$$\int u^4\,du = \frac{u^5}{5} + C = \frac{(3x^2 + 4)^5}{5} + C.$$

Analysis

We can check our answer by taking the derivative of the result of integration. We should obtain the integrand. Picking a value for C of 1, we let $y = \frac{1}{5}(3x^2 + 4)^5 + 1$. We have

$$y = \frac{1}{5}(3x^2 + 4)^5 + 1,$$

so

$$y' = \left(\frac{1}{5}\right) 5(3x^2+4)^4 6x$$
$$= 6x(3x^2+4)^4.$$

This is exactly the expression we started with inside the integrand.

? Exercise 5.5.1

Use substitution to find the antiderivative of $\int 3x^2(x^3-3)^2\,dx$.

Hint

Let $u = x^3 - 3$.

Answer

$$\int 3x^2(x^3-3)^2\,dx = \frac{1}{3}(x^3-3)^3 + C$$

Sometimes we need to adjust the constants in our integral if they don't match up exactly with the expressions we are substituting.

✓ Example 5.5.2: Using Substitution with Alteration

Use substitution to find the antiderivative of

$$\int z\sqrt{z^2-5}\,dz.$$

Solution

Rewrite the integral as $\int z(z^2-5)^{1/2}\,dz$. Let $u = z^2-5$ and $du = 2z\,dz$. Now we have a problem because $du = 2z\,dz$ and the original expression has only $z\,dz$. We have to alter our expression for du or the integral in u will be twice as large as it should be. If we multiply both sides of the du equation by $\frac{1}{2}$. we can solve this problem. Thus,

$$u = z^2 - 5$$
$$du = 2z\,dz$$
$$\frac{1}{2}du = \frac{1}{2}(2z)\,dz = z\,dz.$$

Write the integral in terms of u, but pull the $\frac{1}{2}$ outside the integration symbol:

$$\int z(z^2-5)^{1/2}\,dz = \frac{1}{2}\int u^{1/2}\,du.$$

Integrate the expression in u:

$$\frac{1}{2}\int u^{1/2}\,du = \left(\frac{1}{2}\right)\frac{u^{3/2}}{\frac{3}{2}} + C$$
$$= \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)u^{3/2} + C$$
$$= \frac{1}{3}u^{3/2} + C$$
$$= \frac{1}{3}(z^2-5)^{3/2} + C$$

Exercise 5.5.2

Use substitution to find the antiderivative of $\int x^2(x^3+5)^9\,dx$.

Hint

Multiply the du equation by $\frac{1}{3}$.

Answer

$$\int x^2(x^3+5)^9\,dx = \frac{(x^3+5)^{10}}{30} + C$$

Example 5.5.3: Using Substitution with Integrals of Trigonometric Functions

Use substitution to evaluate the integral $\int \frac{\sin t}{\cos^3 t}\,dt$.

Solution

We know the derivative of $\cos t$ is $-\sin t$, so we set $u = \cos t$. Then $du = -\sin t\,dt$.

Substituting into the integral, we have

$$\int \frac{\sin t}{\cos^3 t}\,dt = -\int \frac{du}{u^3}.$$

Evaluating the integral, we get

$$-\int \frac{du}{u^3} = -\int u^{-3}\,du = -\left(-\frac{1}{2}\right)u^{-2} + C.$$

Putting the answer back in terms of t, we get

$$\int \frac{\sin t}{\cos^3 t}\,dt = \frac{1}{2u^2} + C = \frac{1}{2\cos^2 t} + C.$$

Exercise 5.5.3

Use substitution to evaluate the integral $\int \frac{\cos t}{\sin^2 t}\,dt$.

Hint

Use the process from Example 5.5.3to solve the problem.

Answer

$$\int \frac{\cos t}{\sin^2 t}\,dt = -\frac{1}{\sin t} + C$$

Exercise 5.5.4

Use substitution to evaluate the indefinite integral $\int \cos^3 t \sin t\,dt$.

Hint

Use the process from Example 5.5.3to solve the problem.

Answer

$$\int \cos^3 t \sin t\,dt = -\frac{\cos^4 t}{4} + C$$

Sometimes we need to manipulate an integral in ways that are more complicated than just multiplying or dividing by a constant. We need to eliminate all the expressions within the integrand that are in terms of the original variable. When we are done, u should be the only variable in the integrand. In some cases, this means solving for the original variable in terms of u. This technique should become clear in the next example.

Example 5.5.4: Finding an Antiderivative Using u-Substitution

Use substitution to find the antiderivative of

$$\int \frac{x}{\sqrt{x-1}}\,dx.$$

Solution

If we let $u = x - 1$, then $du = dx$. But this does not account for the x in the numerator of the integrand. We need to express x in terms of u. If $u = x - 1$, then $x = u + 1$. Now we can rewrite the integral in terms of u :

$$\int \frac{x}{\sqrt{x-1}}\,dx = \int \frac{u+1}{\sqrt{u}}\,du = \int \left(\sqrt{u} + \frac{1}{\sqrt{u}}\right)\,du = \int \left(u^{1/2} + u^{-1/2}\right)\,du.$$

Then we integrate in the usual way, replace u with the original expression, and factor and simplify the result. Thus,

$$\begin{aligned}
\int (u^{1/2} + u^{-1/2})\,du &= \frac{2}{3}u^{3/2} + 2u^{1/2} + C \\
&= \frac{2}{3}(x-1)^{3/2} + 2(x-1)^{1/2} + C \\
&= (x-1)^{1/2}\left[\frac{2}{3}(x-1) + 2\right] + C \\
&= (x-1)^{1/2}\left(\frac{2}{3}x - \frac{2}{3} + \frac{6}{3}\right) \\
&= (x-1)^{1/2}\left(\frac{2}{3}x + \frac{4}{3}\right) \\
&= \frac{2}{3}(x-1)^{1/2}(x+2) + C.
\end{aligned}$$

Substitution for Definite Integrals

Substitution can be used with definite integrals, too. However, using substitution to evaluate a definite integral requires a change to the limits of integration. If we change variables in the integrand, the limits of integration change as well.

Substitution with Definite Integrals

Let $u = g(x)$ and let g' be continuous over an interval $[a, b]$, and let f be continuous over the range of $u = g(x)$. Then,

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Although we will not formally prove this theorem, we justify it with some calculations here. From the substitution rule for indefinite integrals, if $F(x)$ is an antiderivative of $f(x)$, we have

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C.$$

Then

$$\begin{aligned}\int_a^b f[g(x)]g'(x)\,dx &= F(g(x))\Big|_{x=a}^{x=b} \\ &= F(g(b)) - F(g(a)) \\ &= F(u)\Big|_{u=g(a)}^{u=g(b)} \\ &= \int_{g(a)}^{g(b)} f(u)\,du\end{aligned}$$

and we have the desired result.

Example 5.5.5: Using Substitution to Evaluate a Definite Integral

Use substitution to evaluate

$$\int_0^1 x^2(1+2x^3)^5\,dx.$$

Solution

Let $u = 1 + 2x^3$, so $du = 6x^2\,dx$. Since the original function includes one factor of x^2 and $du = 6x^2\,dx$, multiply both sides of the du equation by $1/6$. Then,

$$du = 6x^2\,dx$$

$$\text{becomes} \quad \frac{1}{6}du = x^2\,dx.$$

To adjust the limits of integration, note that when $x = 0, u = 1 + 2(0) = 1,$ and when $x = 1,\ u = 1 + 2(1) = 3.$

Then

$$\int_0^1 x^2(1+2x^3)^5 dx = \frac{1}{6}\int_1^3 u^5\,du.$$

Evaluating this expression, we get

$$\begin{aligned}\frac{1}{6}\int_1^3 u^5\,du &= \left(\frac{1}{6}\right)\left(\frac{u^6}{6}\right)\Big|_1^3 \\ &= \frac{1}{36}\left[(3)^6 - (1)^6\right] \\ &= \frac{182}{9}.\end{aligned}$$

Exercise 5.5.5

Use substitution to evaluate the definite integral $\int_{-1}^{0} y(2y^2-3)^5\,dy$.

Hint

Use the steps from Example 5.5.5to solve the problem.

Answer

$$\int_{-1}^{0} y(2y^2-3)^5\,dy = \frac{91}{3}$$

Exercise 5.5.6

Use substitution to evaluate $\int_{0}^{1} x^2\cos\left(\frac{\pi}{2}x^3\right)\,dx$.

Hint

Use the process from Example 5.5.5to solve the problem.

Answer

$$\int_{0}^{1} x^2\cos\left(\frac{\pi}{2}x^3\right)\,dx = \frac{2}{3\pi} \approx 0.2122$$

Example 5.5.6: Using Substitution with an Exponential Function

Use substitution to evaluate

$$\int_{0}^{1} xe^{4x^2+3}\,dx.$$

Solution

Let $u = 4x^3+3$. Then, $du = 8x\,dx$. To adjust the limits of integration, we note that when $x=0$, $u=3$, and when $x=1$, $u=7$. So our substitution gives

$$\begin{aligned}\int_{0}^{1} xe^{4x^2+3}\,dx &= \frac{1}{8}\int_{3}^{7} e^u\,du \\ &= \frac{1}{8}e^u\Big|_3^7 \\ &= \frac{e^7-e^3}{8} \\ &\approx 134.568\end{aligned}$$

Substitution may be only one of the techniques needed to evaluate a definite integral. All of the properties and rules of integration apply independently, and trigonometric functions may need to be rewritten using a trigonometric identity before we can apply substitution. Also, we have the option of replacing the original expression for u after we find the antiderivative, which means that we do not have to change the limits of integration. These two approaches are shown in Example 5.5.7.

Example 5.5.7: Using Substitution to Evaluate a Trigonometric Integral

Use substitution to evaluate

$$\int_0^{\pi/2} \cos^2\theta\, d\theta.$$

Solution

Let us first use a trigonometric identity to rewrite the integral. The trig identity $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$ allows us to rewrite the integral as

$$\int_0^{\pi/2} \cos^2\theta\, d\theta = \int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta.$$

Then,

$$\begin{aligned}\int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right) d\theta &= \int_0^{\pi/2} \left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right) d\theta \\ &= \frac{1}{2}\int_0^{\pi/2} d\theta + \int_0^{\pi/2} \cos 2\theta\, d\theta.\end{aligned}$$

We can evaluate the first integral as it is, but we need to make a substitution to evaluate the second integral. Let $u = 2\theta$. Then, $du = 2\, d\theta$, or $\dfrac{1}{2}\, du = d\theta$. Also, when $\theta = 0$, $u = 0$, and when $\theta = \pi/2$, $u = \pi$. Expressing the second integral in terms of u, we have

$$\begin{aligned}\frac{1}{2}\int_0^{\pi/2} d\theta + \frac{1}{2}\int_0^{\pi/2} \cos 2\theta\, d\theta &= \frac{1}{2}\int_0^{\pi/2} d\theta + \frac{1}{2}\left(\frac{1}{2}\right)\int_0^{\pi} \cos u\, du \\ &= \frac{\theta}{2}\Big|_{\theta=0}^{\theta=\pi/2} + \frac{1}{4}\sin u\Big|_{u=0}^{u=\pi} \\ &= \left(\frac{\pi}{4}-0\right)+(0-0) = \frac{\pi}{4}\end{aligned}$$

Key Concepts

- Substitution is a technique that simplifies the integration of functions that are the result of a chain-rule derivative. The term 'substitution' refers to changing variables or substituting the variable u and du for appropriate expressions in the integrand.
- When using substitution for a definite integral, we also have to change the limits of integration.

Key Equations

- **Substitution with Indefinite Integrals**

$$\int f[g(x)]g'(x)\, dx = \int f(u)\, du = F(u)+C = F(g(x))+C$$

- **Substitution with Definite Integrals**

$$\int_a^b f(g(x))g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Glossary

change of variables

the substitution of a variable, such as u, for an expression in the integrand

integration by substitution

a technique for integration that allows integration of functions that are the result of a chain-rule derivative

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CHAPTER OVERVIEW

6: Applications of Integration

A general Calculus Textmap organized around the textbook

Calculus: Early Transcendentals

by James Stewart

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6.1: Areas Between Curves

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6.3: Volumes by Cylindrical Shells

Learning Objectives

- Calculate the volume of a solid of revolution by using the method of cylindrical shells.
- Compare the different methods for calculating a volume of revolution.

In this section, we examine the method of cylindrical shells, the final method for finding the volume of a solid of revolution. We can use this method on the same kinds of solids as the disk method or the washer method; however, with the disk and washer methods, we integrate along the coordinate axis parallel to the axis of revolution. With the method of cylindrical shells, we integrate along the coordinate *axis perpendicular* to the axis of revolution. The ability to choose which variable of integration we want to use can be a significant advantage with more complicated functions. Also, the specific geometry of the solid sometimes makes the method of using cylindrical shells more appealing than using the washer method. In the last part of this section, we review all the methods for finding volume that we have studied and lay out some guidelines to help you determine which method to use in a given situation.

The Method of Cylindrical Shells

Again, we are working with a solid of revolution. As before, we define a region R, bounded above by the graph of a function $y = f(x)$, below by the $\boldsymbol{x}$**-axis**, and on the left and right by the lines $x = a$ and $x = b$, respectively, as shown in Figure $6.3.1a$. We then revolve this region around the y-axis, as shown in Figure $6.3.1b$. Note that this is different from what we have done before. Previously, regions defined in terms of functions of x were revolved around the $\boldsymbol{x}$**-axis** or a line parallel to it.

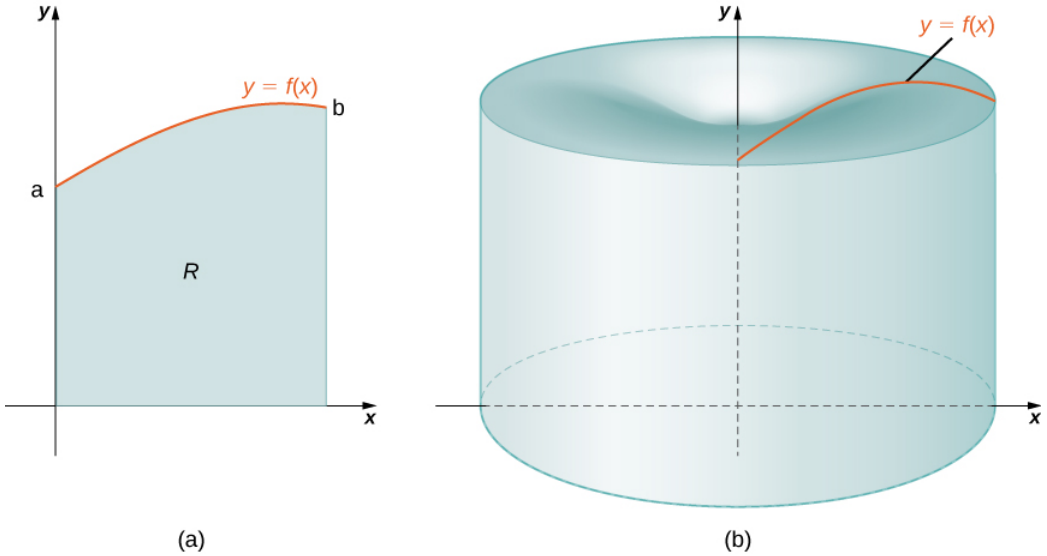


Figure $6.3.1$: (a) A region bounded by the graph of a function of x. (b) The solid of revolution formed when the region is revolved around the $\boldsymbol{y}$**-axis.**

As we have done many times before, partition the interval $[a, b]$ using a regular partition, $P = x_0, x_1, \ldots, x_n$ and, for $i = 1, 2, \ldots, n$, choose a point $x_i^* \in [x_{i-1}, x_i]$. Then, construct a rectangle over the interval $[x_{i-1}, x_i]$ of height $f(x_i^*)$ and width Δx. A representative rectangle is shown in Figure $6.3.2a$. When that rectangle is revolved around the y-axis, instead of a disk or a washer, we get a cylindrical shell, as shown in Figure $6.3.2$.

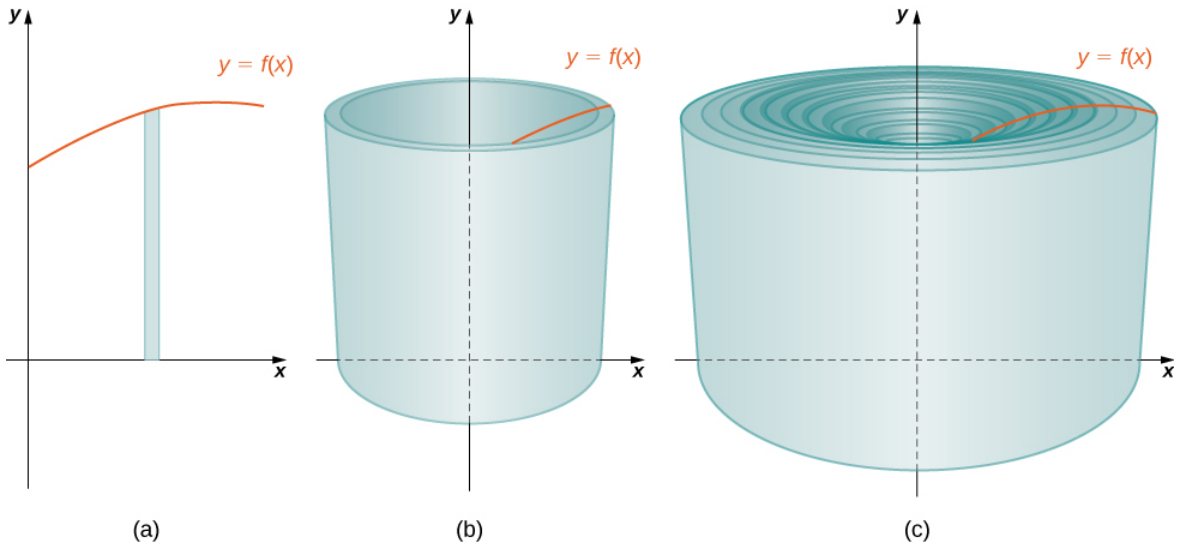


Figure 6.3.2: (a) A representative rectangle. (b) When this rectangle is revolved around the ***y*-axis**, the result is a cylindrical shell. (c) When we put all the shells together, we get an approximation of the original solid.

To calculate the volume of this shell, consider Figure 6.3.3.

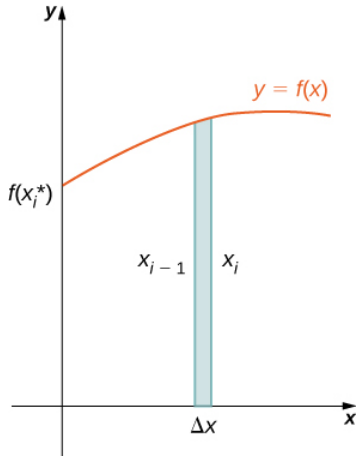


Figure 6.3.3: Calculating the volume of the shell.

The shell is a cylinder, so its volume is the cross-sectional area multiplied by the height of the cylinder. The cross-sections are annuli (ring-shaped regions—essentially, circles with a hole in the center), with outer radius x_i and inner radius x_{i-1}. Thus, the cross-sectional area is $\pi x_i^2 - \pi x_{i-1}^2$. The height of the cylinder is $f(x_i^*)$. Then the volume of the shell is

$$\begin{aligned} V_{shell} &= f(x_i^*)(\pi\, x_i^2 - \pi\, x_{i-1}^2) \\ &= \pi\, f(x_i^*)(x_i^2 - x_{i-1}^2) \\ &= \pi\, f(x_i^*)(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= 2\pi\, f(x_i^*)\left(\frac{x_i + x_{i-1}}{2}\right)(x_i - x_{i-1}). \end{aligned}$$

Note that $x_i - x_{i-1} = \Delta x,$ so we have

$$V_{shell} = 2\pi\, f(x_i^*)\left(\frac{x_i + x_{i-1}}{2}\right)\Delta x.$$

Furthermore, $\dfrac{x_i + x_{i-1}}{2}$ is both the midpoint of the interval $[x_{i-1}, x_i]$ and the average radius of the shell, and we can approximate this by x_i^*. We then have

$$V_{shell} \approx 2\pi\, f(x_i^*) x_i^*\, \Delta x.$$

Another way to think of this is to think of making a vertical cut in the shell and then opening it up to form a flat plate (Figure 6.3.4).

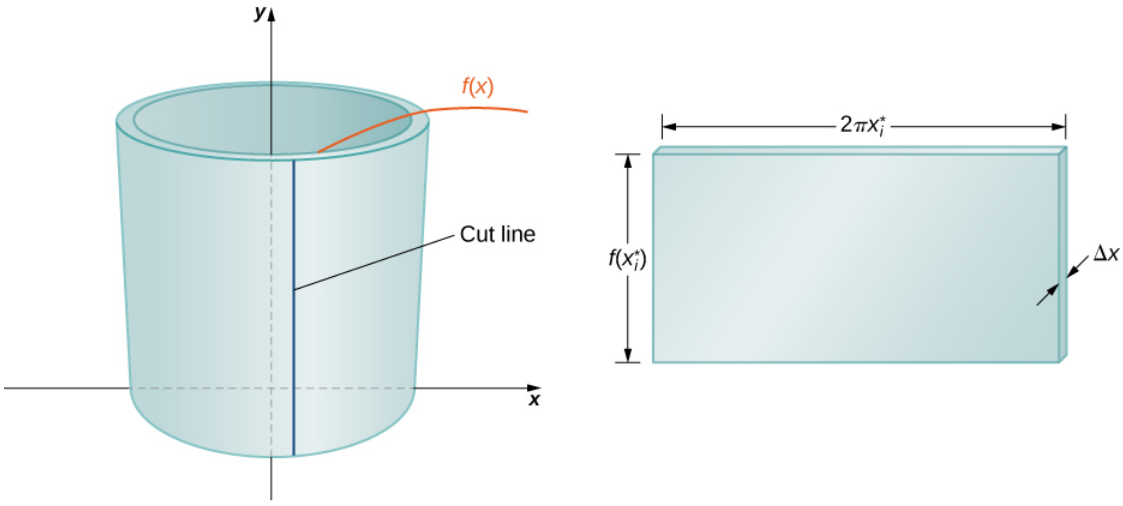


Figure 6.3.4: (a) Make a vertical cut in a representative shell. (b) Open the shell up to form a flat plate.

In reality, the outer radius of the shell is greater than the inner radius, and hence the back edge of the plate would be slightly longer than the front edge of the plate. However, we can approximate the flattened shell by a flat plate of height $f(x_i^*)$, width $2\pi x_i^*$, and thickness Δx (Figure). The volume of the shell, then, is approximately the volume of the flat plate. Multiplying the height, width, and depth of the plate, we get

$$V_{shell} \approx f(x_i^*)(2\pi\, x_i^*)\,\Delta x,$$

which is the same formula we had before.

To calculate the volume of the entire solid, we then add the volumes of all the shells and obtain

$$V \approx \sum_{i=1}^{n} (2\pi\, x_i^* f(x_i^*)\,\Delta x).$$

Here we have another Riemann sum, this time for the function $2\pi\, x\, f(x)$.Taking the limit as $n \to \infty$ gives us

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} (2\pi\, x_i^* f(x_i^*)\,\Delta x) = \int_a^b (2\pi\, x\, f(x))\, dx.$$

This leads to the following rule for the *method of cylindrical shells*.

Rule: The Method of Cylindrical Shells

Let $f(x)$ be continuous and nonnegative. Define R as the region bounded above by the graph of $f(x)$, below by the **x-axis,** on the left by the line $x = a$, and on the right by the line $x = b$. Then the volume of the solid of revolution formed by revolving R around the y-axis is given by

$$V = \int_a^b (2\pi\, x\, f(x))\, dx.$$

Now let's consider an example.

Example 6.3.1: The Method of Cylindrical Shells I

Define R as the region bounded above by the graph of $f(x) = 1/x$ and below by the **x-axis** over the interval $[1, 3]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.

Solution

First we must graph the region R and the associated solid of revolution, as shown in Figure $6.3.5$.

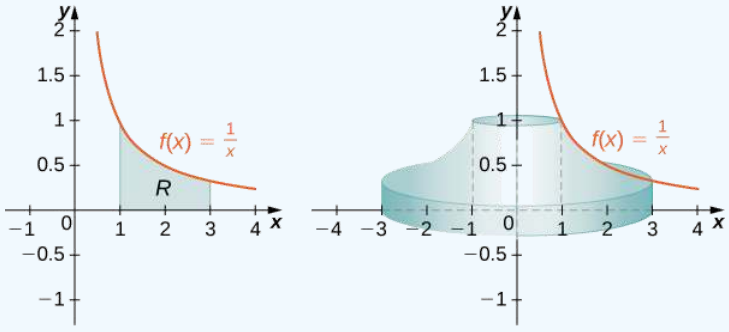


Figure $6.3.5$: (a) The region R under the graph of $f(x) = 1/x$ over the interval $[1, 3]$. (b) The solid of revolution generated by revolving R about the y-axis.

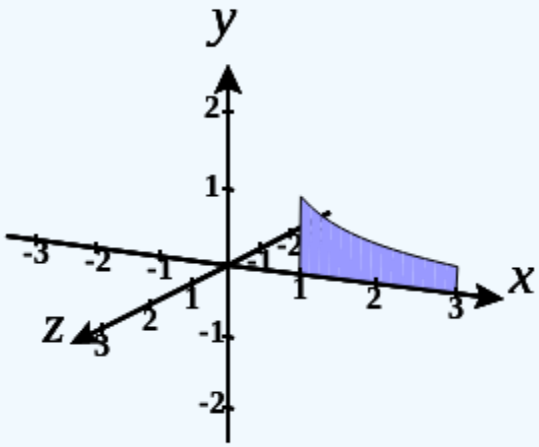


Figure $6.3.5$ (c) Visualizing the solid of revolution with CalcPlot3D.

Then the volume of the solid is given by

$$V = \int_a^b (2\pi\, x\, f(x))\, dx$$
$$= \int_1^3 \left(2\pi\, x \left(\frac{1}{x}\right)\right) dx$$
$$= \int_1^3 2\pi\, dx$$
$$= 2\pi\, x\Big|_1^3 = 4\pi \text{ units}^3.$$

? Exercise 6.3.1

Define R as the region bounded above by the graph of $f(x) = x^2$ and below by the x-axis over the interval $[1, 2]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.

Hint

Use the procedure from Example 6.3.1

Answer

$\frac{15\pi}{2}$ units^3

✓ Example 6.3.2: The Method of Cylindrical Shells II

Define R as the region bounded above by the graph of $f(x) = 2x - x^2$ and below by the x-axis over the interval $[0, 2]$. Find the volume of the solid of revolution formed by revolving R around the **y-axis.**

Solution

First graph the region R and the associated solid of revolution, as shown in Figure 6.3.6.

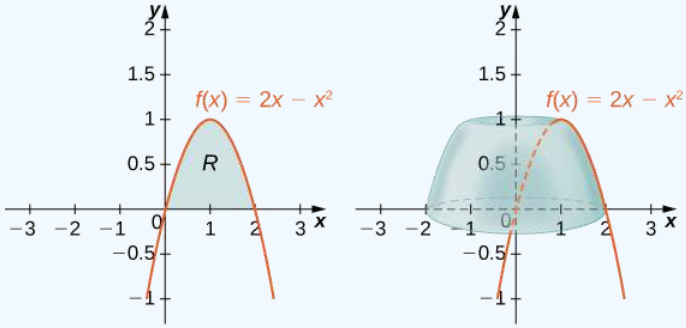


Figure 6.3.6: (a) The region R *under the graph of* $f(x) = 2x - x^2$ *over the interval* $[0, 2]$. *(b) The volume of revolution obtained by revolving* R *about the* y*-axis.*

Then the volume of the solid is given by

$$
\begin{aligned}
V &= \int_a^b (2\pi\, x\, f(x))\, dx \\
&= \int_0^2 (2\pi\, x(2x - x^2))\, dx \\
&= 2\pi \int_0^2 (2x^2 - x^3)\, dx \\
&= 2\pi \left[\frac{2x^3}{3} - \frac{x^4}{4} \right] \Big|_0^2 \\
&= \frac{8\pi}{3} \text{ units}^3
\end{aligned}
$$

Exercise 6.3.2

Define R as the region bounded above by the graph of $f(x) = 3x - x^2$ and below by the x-axis over the interval $[0, 2]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.

Hint

Use the process from Example 6.3.2

Answer

$8\pi \text{ units}^3$

As with the disk method and the washer method, we can use the method of cylindrical shells with solids of revolution, revolved around the x-axis, when we want to integrate with respect to y. The analogous rule for this type of solid is given here.

Rule: The Method of Cylindrical Shells for Solids of Revolution around the x-axis

Let $g(y)$ be continuous and nonnegative. Define Q as the region bounded on the right by the graph of $g(y)$, on the left by the y-axis, below by the line $y = c$, and above by the line $y = d$. Then, the volume of the solid of revolution formed by revolving Q around the x-axis is given by

$$V = \int_c^d (2\pi\, y\, g(y))\, dy.$$

Example 6.3.3: The Method of Cylindrical Shells for a Solid Revolved around the x-axis

Define Q as the region bounded on the right by the graph of $g(y) = 2\sqrt{y}$ and on the left by the y-axis for $y \in [0, 4]$. Find the volume of the solid of revolution formed by revolving Q around the x-axis.

Solution

First, we need to graph the region Q and the associated solid of revolution, as shown in Figure 6.3.7.

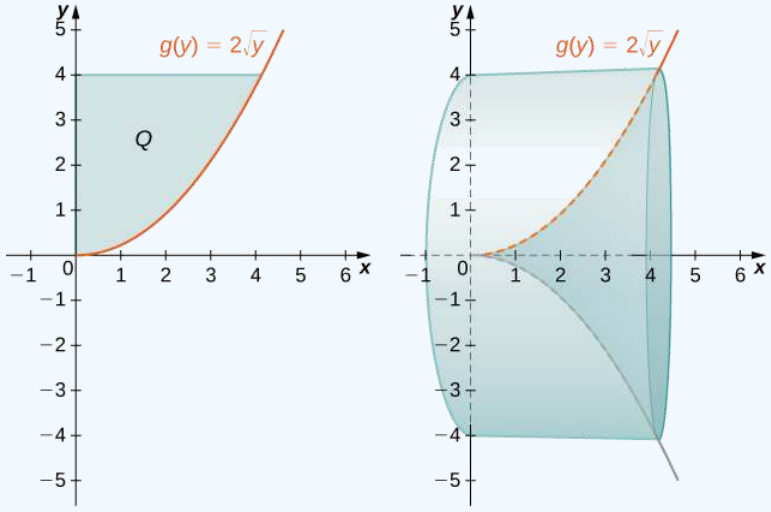


Figure 6.3.7: (a) The region Q to the left of the function $g(y)$ over the interval $[0,4]$. (b) The solid of revolution generated by revolving Q around the x-axis.

Label the shaded region Q. Then the volume of the solid is given by

$$\begin{aligned} V &= \int_c^d (2\pi\, y\, g(y))\, dy \\ &= \int_0^4 (2\pi\, y(2\sqrt{y}))\, dy \\ &= 4\pi \int_0^4 y^{3/2}\, dy \\ &= 4\pi \left[\frac{2y^{5/2}}{5}\right]\Bigg|_0^4 \\ &= \frac{256\pi}{5} \text{ units}^3 \end{aligned}$$

? Exercise 6.3.3

Define Q as the region bounded on the right by the graph of $g(y) = 3/y$ and on the left by the y-axis for $y \in [1,3]$. Find the volume of the solid of revolution formed by revolving Q around the x-axis.

Hint

Use the process from Example 6.3.3

Answer

12π units^3

For the next example, we look at a solid of revolution for which the graph of a function is revolved around a line other than one of the two coordinate axes. To set this up, we need to revisit the development of the method of cylindrical shells. Recall that we found the volume of one of the shells to be given by

$$\begin{aligned} V_{shell} &= f(x_i^*)(\pi\, x_i^2 - \pi\, x_{i-1}^2) \\ &= \pi\, f(x_i^*)(x_i^2 - x_{i-1}^2) \\ &= \pi\, f(x_i^*)(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= 2\pi\, f(x_i^*)\left(\frac{x_i + x_{i-1}}{2}\right)(x_i - x_{i-1}). \end{aligned}$$

This was based on a shell with an outer radius of x_i and an inner radius of x_{i-1}. If, however, we rotate the region around a line other than the y-axis, we have a different outer and inner radius. Suppose, for example, that we rotate the region around the line $x = -k$, where k is some positive constant. Then, the outer radius of the shell is $x_i + k$ and the inner radius of the shell is $x_{i-1} + k$. Substituting these terms into the expression for volume, we see that when a plane region is rotated around the line $x = -k$, the volume of a shell is given by

$$\begin{aligned} V_{shell} &= 2\pi\, f(x_i^*)\left(\frac{(x_i + k) + (x_{i-1} + k)}{2}\right)((x_i + k) - (x_{i-1} + k)) \\ &= 2\pi\, f(x_i^*)\left(\left(\frac{x_i + x_{i-2}}{2}\right) + k\right)\Delta x. \end{aligned}$$

As before, we notice that $\dfrac{x_i + x_{i-1}}{2}$ is the midpoint of the interval $[x_{i-1}, x_i]$ and can be approximated by x_i^*. Then, the approximate volume of the shell is

$$V_{shell} \approx 2\pi(x_i^* + k)f(x_i^*)\Delta x.$$

The remainder of the development proceeds as before, and we see that

$$V = \int_a^b (2\pi(x + k)f(x))dx.$$

We could also rotate the region around other horizontal or vertical lines, such as a vertical line in the right half plane. In each case, the volume formula must be adjusted accordingly. Specifically, the x-term in the integral must be replaced with an expression representing the radius of a shell. To see how this works, consider the following example.

Example 6.3.4: A Region of Revolution Revolved around a Line

Define R as the region bounded above by the graph of $f(x) = x$ and below by the x-axis over the interval $[1, 2]$. Find the volume of the solid of revolution formed by revolving R around the line $x = -1$.

Solution

First, graph the region R and the associated solid of revolution, as shown in Figure 6.3.8.

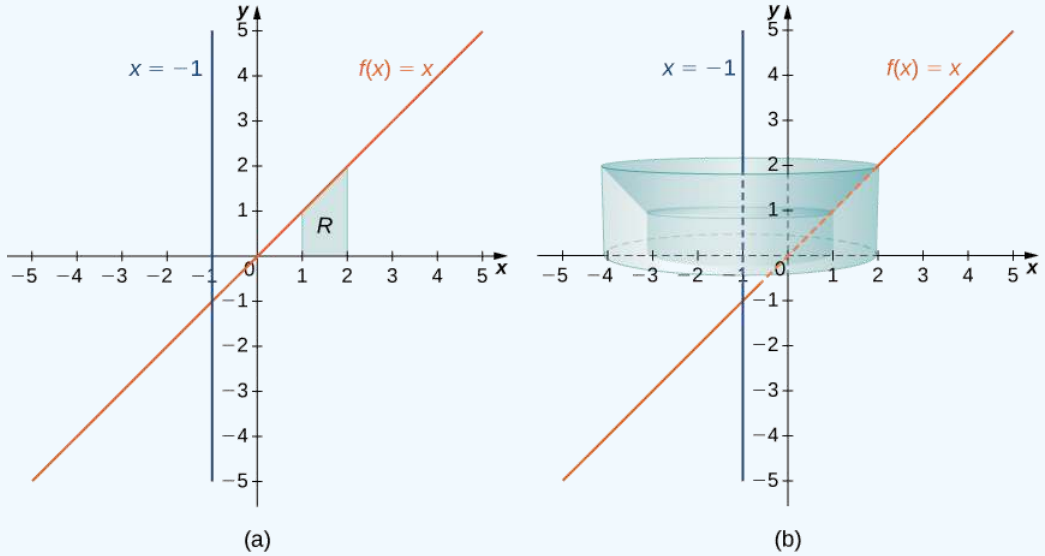


Figure 6.3.8: (a) The region R between the graph of $f(x)$ and the x-axis over the interval $[1,2]$. (b) The solid of revolution generated by revolving R around the line $x=-1$.

Note that the radius of a shell is given by $x+1$. Then the volume of the solid is given by

$$\begin{aligned} V &= \int_1^2 2\pi(x+1)f(x)\,dx \\ &= \int_1^2 2\pi(x+1)x\,dx = 2\pi\int_1^2 x^2+x\,dx \\ &= 2\pi\left[\frac{x^3}{3}+\frac{x^2}{2}\right]\Big|_1^2 \\ &= \frac{23\pi}{3}\text{ units}^3 \end{aligned}$$

Exercise 6.3.4

Define R as the region bounded above by the graph of $f(x)=x^2$ and below by the x-axis over the interval $[0,1]$. Find the volume of the solid of revolution formed by revolving R around the line $x=-2$.

Hint

Use the process from Example 6.3.4

Answer

$\frac{11\pi}{6}$ units^3

For our final example in this section, let's look at the volume of a solid of revolution for which the region of revolution is bounded by the graphs of two functions.

Example 6.3.5: A Region of Revolution Bounded by the Graphs of Two Functions

Define R as the region bounded above by the graph of the function $f(x)=\sqrt{x}$ and below by the graph of the function $g(x)=1/x$ over the interval $[1,4]$. Find the volume of the solid of revolution generated by revolving R around the y-axis.

Solution

First, graph the region R and the associated solid of revolution, as shown in Figure 6.3.9.

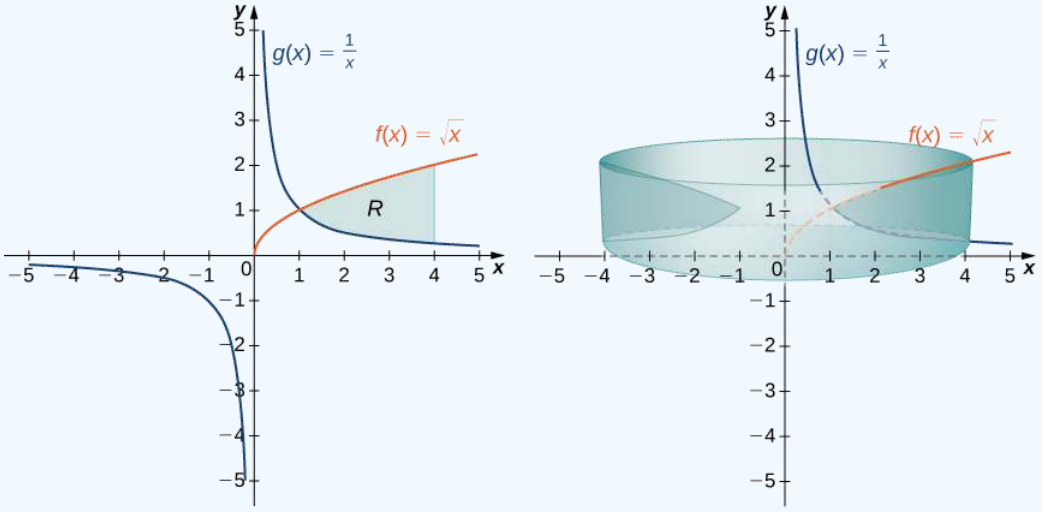


Figure 6.3.9: (a) The region R between the graph of $f(x)$ and the graph of $g(x)$ over the interval $[1,4]$. (b) The solid of revolution generated by revolving R around the y-axis.

Note that the axis of revolution is the y-axis, so the radius of a shell is given simply by x. We don't need to make any adjustments to the x-term of our integrand. The height of a shell, though, is given by $f(x)-g(x)$, so in this case we need to adjust the $f(x)$ term of the integrand. Then the volume of the solid is given by

$$V = \int_1^4 (2\pi\, x(f(x)-g(x)))\, dx$$

$$= \int_1^4 (2\pi\, x(\sqrt{x} - \frac{1}{x}))\, dx = 2\pi \int_1^4 (x^{3/2} - 1) dx$$

$$= 2\pi \left[\frac{2x^{5/2}}{5} - x\right]\Big|_1^4 = \frac{94\pi}{5} \text{ units}^3.$$

? Exercise 6.3.5

Define R as the region bounded above by the graph of $f(x) = x$ and below by the graph of $g(x) = x^2$ over the interval $[0,1]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.

Hint

Hint: Use the process from Example 6.3.5

Answer

$\frac{\pi}{6}$ units^3

Which Method Should We Use?

We have studied several methods for finding the volume of a solid of revolution, but how do we know which method to use? It often comes down to a choice of which integral is easiest to evaluate. Figure 6.3.10 describes the different approaches for solids of revolution around the x-axis. It's up to you to develop the analogous table for solids of revolution around the y-axis.

Comparing the Methods for Finding the Volume of a Solid Revolution around the x-axis

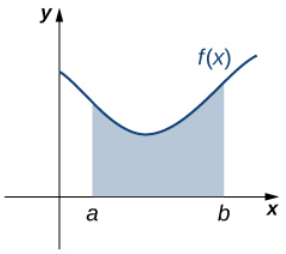
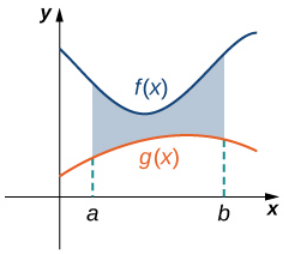
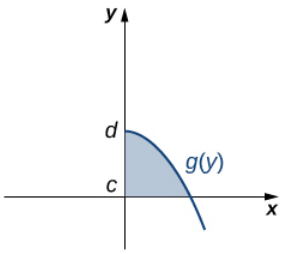
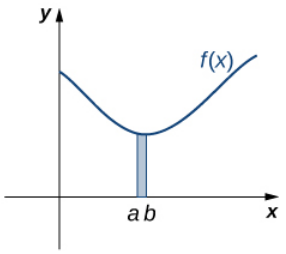
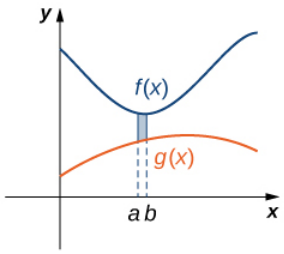
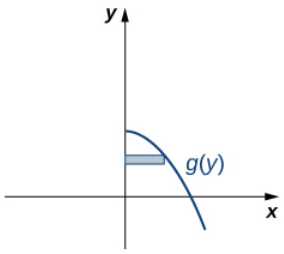
Compare	Disk Method	Washer Method	Shell Method
Volume formula	$V = \int_a^b \pi[f(x)]^2\,dx$	$V = \int_a^b \pi\left[(f(x))^2 - (g(x))^2\right]dx$	$V = \int_c^d 2\pi y\, g(y)\,dy$
Solid	No cavity in the center	Cavity in the center	With or without a cavity in the center
Interval to partition	$[a, b]$ on x-axis	$[a, b]$ on x-axis	$[c, d]$ on y-axis
Rectangle	Vertical	Vertical	Horizontal
Typical region			
Typical element			

Figure 6.3.10

Let's take a look at a couple of additional problems and decide on the best approach to take for solving them.

Example 6.3.6: Selecting the Best Method

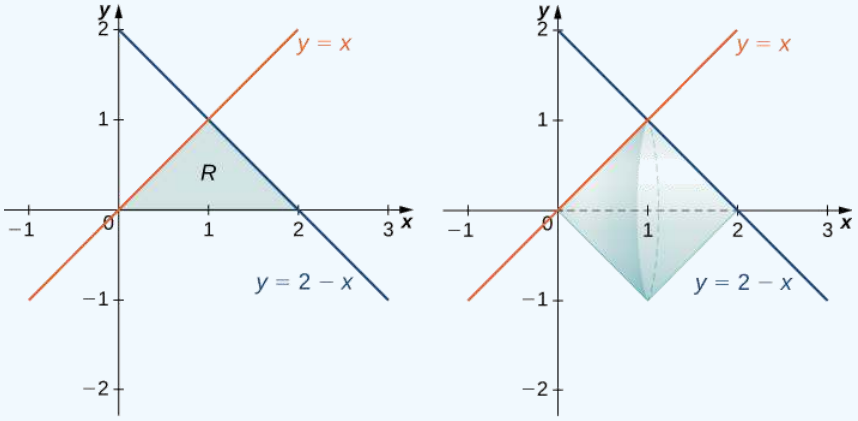
For each of the following problems, select the best method to find the volume of a solid of revolution generated by revolving the given region around the x-axis, and set up the integral to find the volume (do not evaluate the integral).

a. The region bounded by the graphs of $y = x, y = 2 - x,$ and the x-axis.
b. The region bounded by the graphs of $y = 4x - x^2$ and the x-axis.

Solution

a.

First, sketch the region and the solid of revolution as shown.



(a) (b)

Figure 6.3.11: (a) The region R bounded by two lines and the x-axis. (b) The solid of revolution generated by revolving R about the x-axis.

Looking at the region, if we want to integrate with respect to x, we would have to break the integral into two pieces, because we have different functions bounding the region over $[0, 1]$ and $[1, 2]$. In this case, using the disk method, we would have

$$V = \int_0^1 \pi x^2 \, dx + \int_1^2 \pi (2 - x)^2 \, dx.$$

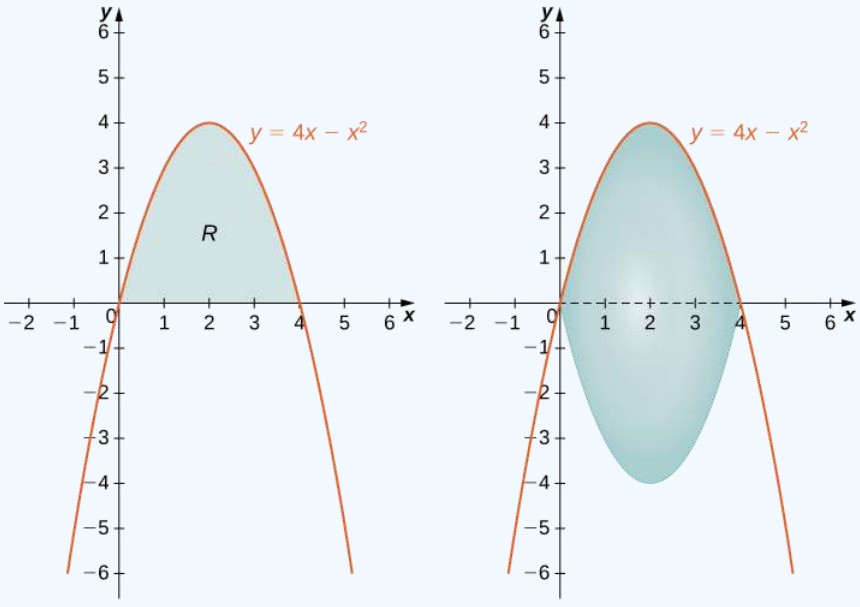
If we used the shell method instead, we would use functions of y to represent the curves, producing

$$V = \int_0^1 2\pi y[(2 - y) - y] \, dy = \int_0^1 2\pi y[2 - 2y] \, dy.$$

Neither of these integrals is particularly onerous, but since the shell method requires only one integral, and the integrand requires less simplification, we should probably go with the shell method in this case.

b.

First, sketch the region and the solid of revolution as shown.



(a) (b)

Figure 6.3.12: (a) The region R between the curve and the x-axis. (b) The solid of revolution generated by revolving R about the x-axis.

Looking at the region, it would be problematic to define a horizontal rectangle; the region is bounded on the left and right by the same function. Therefore, we can dismiss the method of shells. The solid has no cavity in the middle, so we can use the method of disks. Then

$$V = \int_0^4 \pi \left(4x - x^2\right)^2 \, dx$$

? Exercise 6.3.6

Select the best method to find the volume of a solid of revolution generated by revolving the given region around the x-axis, and set up the integral to find the volume (do not evaluate the integral): the region bounded by the graphs of $y = 2 - x^2$ and $y = x^2$.

Hint

Sketch the region and use Figure 6.3.12to decide which integral is easiest to evaluate.

Answer

Use the method of washers;

$$V = \int_{-1}^1 \pi \left[\left(2 - x^2\right)^2 - \left(x^2\right)^2 \right] \, dx$$

Key Concepts

- The method of cylindrical shells is another method for using a definite integral to calculate the volume of a solid of revolution. This method is sometimes preferable to either the method of disks or the method of washers because we integrate with respect to the other variable. In some cases, one integral is substantially more complicated than the other.
- The geometry of the functions and the difficulty of the integration are the main factors in deciding which integration method to use.

Key Equations

- **Method of Cylindrical Shells**

$$V = \int_a^b \left(2\pi \, x \, f(x)\right) \, dx$$

Glossary

method of cylindrical shells

a method of calculating the volume of a solid of revolution by dividing the solid into nested cylindrical shells; this method is different from the methods of disks or washers in that we integrate with respect to the opposite variable

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6.4: Work

Learning Objectives

In this section, we strive to understand the ideas generated by the following important questions:

- How do we measure the work accomplished by a varying force that moves an object a certain distance?
- What is the total force exerted by water against a dam?
- How are both of the above concepts and their corresponding use of definite integrals similar to problems we have encountered in the past involving formulas such as “distance equals rate times time” and “mass equals density times volume”?

In our work to date with the definite integral, we have seen several different circumstances where the integral enables us to measure the accumulation of a quantity that varies, provided the quantity is approximately constant over small intervals. For instance, based on the fact that the area of a rectangle is $A = l \cdot w$, if we wish to find the area bounded by a nonnegative curve $y = f(x)$ and the x-axis on an interval $[a, b]$, a representative slice of width Δx has area $A_{\text{slice}} = f(x)\Delta x$, and thus as we let the width of the representative slice tend to zero, we find that the exact area of the region is

$$A = \int_a^b f(x)dx. \tag{6.4.1}$$

In a similar way, if we know that the velocity of a moving object is given by the function $y = v(t)$, and we wish to know the distance the object travels on an interval $[a, b]$ where $v(t)$ is nonnegative, we can use a definite integral to generalize the fact that $d = r \cdot t$ when the rate, r, is constant. More specifically, on a short time interval Δt, $v(t)$ is roughly constant, and hence for a small slice of time, $d_{\text{slice}} = v(t)\Delta t$, and so as the width of the time interval Δt tends to zero, the exact distance traveled is given by the definite integral

$$d = \int_a^b v(t)dt. \tag{6.4.2}$$

Finally, when we recently learned about the mass of an object of non-constant density, we saw that since $M = D \cdot V$ (mass equals density times volume, provided that density is constant), if we can consider a small slice of an object on which the density is approximately constant, a definite integral may be used to determine the exact mass of the object. For instance, if we have a thin rod whose cross sections have constant density, but whose density is distributed along the x axis according to the function $y = \rho(x)$, it follows that for a small slice of the rod that is Δx thick, $M_{\text{slice}} = \rho(x)\Delta x$. In the limit as $\Delta x \to 0$, we then find that the total mass is given by

$$M = \int_a^b \rho(x)dx. \tag{6.4.3}$$

Note that all three of these situations are similar in that we have a basic rule ($A = l \cdot w, d = r \cdot t, M = D \cdot V$) where one of the two quantities being multiplied is no longer constant; in each, we consider a small interval for the other variable in the formula, calculate the approximate value of the desired quantity (area, distance, or mass) over the small interval, and then use a definite integral to sum the results as the length of the small intervals is allowed to approach zero. It should be apparent that this approach will work effectively for other situations where we have a quantity of interest that varies. We next turn to the notion of work: from physics, a basic principal is that work is the product of force and distance. For example, if a person exerts a force of 20 pounds to lift a 20-pound weight 4 feet off the ground, the total work accomplished is

$$W = F \cdot d \tag{6.4.4}$$

$$= 20 \cdot 4 \tag{6.4.5}$$

$$= 80 \text{ foot-pounds}. \tag{6.4.6}$$

If force and distance are measured in English units (pounds and feet), then the units on work are *foot-pounds*. If instead we work in metric units, where forces are measured in Newtons and distances in meters, the units on work are Newton-meters.

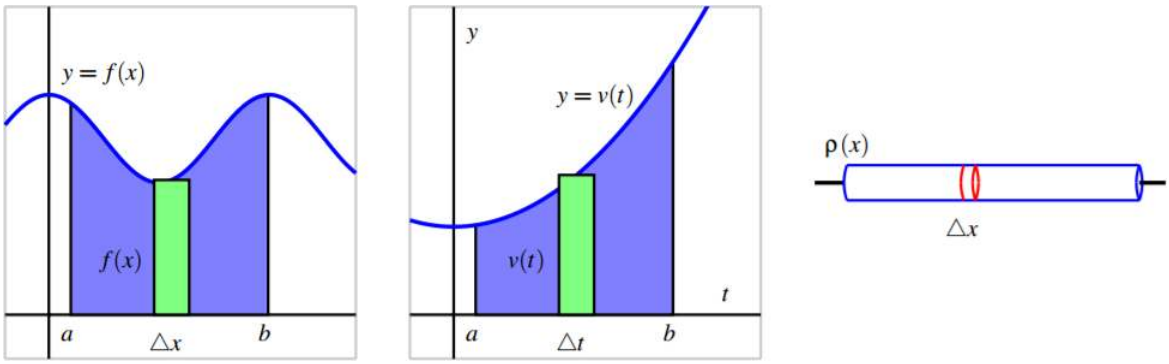


Figure 6.14: *Three settings where we compute the accumulation of a varying quantity: the area under $y = f(x)$, the distance traveled by an object with velocity $y = v(t)$, and the mass of a bar with density function $y = \rho(x)$.*

Of course, the formula $W = F \cdot d$ only applies when the force is constant while it is exerted over the distance d. In Preview Activity 6.4, we explore one way that we can use a definite integral to compute the total work accomplished when the force exerted varies.

Preview Activity 6.4.1

A bucket is being lifted from the bottom of a 50-foot deep well; its weight (including the water), B, in pounds at a height h feet above the water is given by the function $B(h)$. When the bucket leaves the water, the bucket and water together weigh $B(0) = 20$ pounds, and when the bucket reaches the top of the well, $B(50) = 12$ pounds. Assume that the bucket loses water at a constant rate (as a function of height, h) throughout its journey from the bottom to the top of the well.

a. Find a formula for $B(h)$.
b. Compute the value of the product $B(5)\Delta h$, where $\Delta h = 2$ feet. Include units on your answer. Explain why this product represents the approximate work it took to move the bucket of water from $h = 5$ to $h = 7$.
c. Is the value in (b) an over- or under-estimate of the actual amount of work it took to move the bucket from $h = 5$ to $h = 7$? Why?
d. Compute the value of the product $B(22)\Delta h$, where $\Delta h = 0.25$ feet. Include units on your answer. What is the meaning of the value you found?
e. More generally, what does the quantity $W_{\text{slice}} = B(h)\Delta h$ measure for a given value of h and a small positive value of Δh?
f. Evaluate the definite integral $\int^5 0_0 B(h)dh$. What is the meaning of the value you find? Why?

Work

Because work is calculated by the rule $W = F \cdot d$, whenever the force F is constant, it follows that we can use a definite integral to compute the work accomplished by a varying force. For example, suppose that in a setting similar to the problem posed in Preview Activity 6.4, we have a bucket being lifted in a 50-foot well whose weight at height h is given by

$$B(h) = 12 + 8e^{-0.1h}. \tag{6.4.7}$$

In contrast to the problem in the preview activity, this bucket is not leaking at a constant rate; but because the weight of the bucket and water is not constant, we have to use a definite integral to determine the total work that results from lifting the bucket. Observe that at a height h above the water, the approximate work to move the bucket a small distance Δh is

$$W_{\text{slice}} = B(h)\Delta h = (12 + 8e^{-0.1h})\Delta h. \tag{6.4.8}$$

Hence, if we let Δh tend to 0 and take the sum of all of the slices of work accomplished on these small intervals, it follows that the total work is given by

$$W = \int_0^{50} B(h)\, dh = \int_0^{50} (12 + 8e^{-0.1h})dh. \tag{6.4.9}$$

While is a straightforward exercise to evaluate this integral exactly using the First Fundamental Theorem of Calculus, in applied settings such as this one we will typically use computing technology to find accurate approximations of integrals that are of interest to us. Here, it turns out that

$$W = \int_0^{50} (12 + 8e^{-0.1h}) dh \approx 679.461 \text{ foot-pounds.} \tag{6.4.10}$$

Our work in Preview Activity 6.1 and in the most recent example above employs the following important general principle.

For an object being moved in the positive direction along an axis, x, by a force $F(x)$, the total work to move the object from a to b is given by

$$W = \int_a^b F(x) dx. \tag{6.4.11}$$

Activity 6.4.1

Consider the following situations in which a varying force accomplishes work.

a. Suppose that a heavy rope hangs over the side of a cliff. The rope is 200 feet long and weighs 0.3 pounds per foot; initially the rope is fully extended. How much work is required to haul in the entire length of the rope? (Hint: set up a function $F(h)$ whose value is the weight of the rope remaining over the cliff after h feet have been hauled in.)

b. A leaky bucket is being hauled up from a 100 foot deep well. When lifted from the water, the bucket and water together weigh 40 pounds. As the bucket is being hauled upward at a constant rate, the bucket leaks water at a constant rate so that it is losing weight at a rate of 0.1 pounds per foot. What function $B(h)$ tells the weight of the bucket after the bucket has been lifted h feet? What is the total amount of work accomplished in lifting the bucket to the top of the well?

c. Now suppose that the bucket in (b) does not leak at a constant rate, but rather that its weight at a height h feet above the water is given by $B(h) = 25 + 15e^{-0.05h}$. What is the total work required to lift the bucket 100 feet? What is the average force exerted on the bucket on the interval $h = 0$ to $h = 100$?

d. From physics, *Hooke's Law* for springs states that the amount of force required to hold a spring that is compressed (or extended) to a particular length is proportionate to the distance the spring is compressed (or extended) from its natural length. That is, the force to compress (or extend) a spring x units from its natural length is $F(x) = kx$ for some constant k (which is called the spring constant.) For springs, we choose to measure the force in pounds and the distance the spring is compressed in feet. Suppose that a force of 5 pounds extends a particular spring 4 inches (1/3 foot) beyond its natural length.

   i. Use the given fact that $F(1/3) = 5$ to find the spring constant k.
   ii. Find the work done to extend the spring from its natural length to 1 foot beyond its natural length.
   iii. Find the work required to extend the spring from 1 foot beyond its natural length to 1.5 feet beyond its natural length.

Work: Pumping Liquid from a Tank

In certain geographic locations where the water table is high, residential homes with basements have a peculiar feature: in the basement, one finds a large hole in the floor, and in the hole, there is water. For example, in Figure 6.15 where we see a *sump crock*.



Figure 6.15: A sump crock. Image credit to www.warreninspect.com/basement-moisture.

Essentially, a sump crock provides an outlet for water that may build up beneath the basement floor; of course, as that water rises, it is imperative that the water not flood the basement. Hence, in the crock we see the presence of a floating pump that sits on the surface of the water: this pump is activated by elevation, so when the water level reaches a particular height, the pump turns on and pumps a certain portion of the water out of the crock, hence relieving the water buildup beneath the foundation. One of the questions we'd like to answer is: how much work does a sump pump accomplish? To that end, let's suppose that we have a sump crock that has the shape of a frustum of a cone, as pictured in Figure 6.16. Assume that the crock has a diameter of 3 feet at its surface, a diameter of 1.5 feet at its base, and a depth of 4 feet. In addition, suppose that the sump pump is set up so that it pumps the water vertically up a pipe to a drain that is located at ground level just outside a basement window. To accomplish this, the pump must send the water to a location 9 feet above the surface of the sump crock.

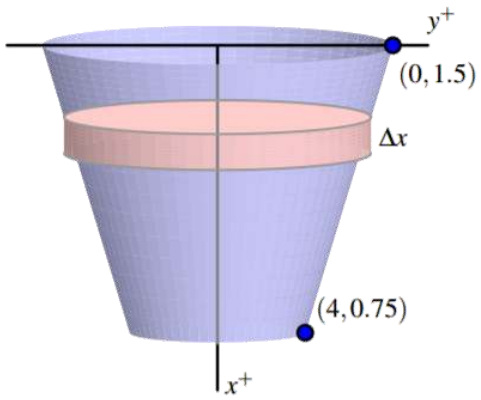


Figure 6.16: A sump crock with approximately cylindrical cross-sections that is 4 feet deep, 1.5 feet in diameter at its base, and 3 feet in diameter at its top.

It turns out to be advantageous to think of the depth below the surface of the crock as being the independent variable, so, in problems such as this one we typically let the positive x-axis point down, and the positive y-axis to the right, as pictured in the figure. As we think about the work that the pump does, we first realize that the pump sits on the surface of the water, so it makes sense to think about the pump moving the water one "slice" at a time, where it takes a thin slice from the surface, pumps it out of the tank, and then proceeds to pump the next slice below. For the sump crock described in this example, each slice of water is cylindrical in shape. We see that the radius of each approximately cylindrical slice varies according to the linear function $y = f(x)$ that passes through the points (0, 1.5) and (4, 0.75), where x is the depth of the particular slice in the tank; it is a straightforward exercise to find that $f(x) = 1.5 - 0.1875x$. Now we are prepared to think about the overall problem in several steps:

a. determining the volume of a typical slice;
b. finding the weight (We assume that the weight density of water is 62.4 pounds per cubic foot) of a typical slice (and thus the force that must be exerted on it)
c. deciding the distance that a typical slice moves; and
d. computing the work to move a representative slice. Once we know the work it takes to move one slice, we use a definite integral over an appropriate interval to find the total work.

Consider a representative cylindrical slice that sits on the surface of the water at a depth of x feet below the top of the crock. It follows that the approximate volume of that slice is given by

$$V_{\text{slice}} = \pi f(x)^2 \Delta x = \pi(1.5 - 0.1875x)^2 \Delta x$$

Since water weighs 62.4 lb/ft3 , it follows that the approximate weight of a representative slice, which is also the approximate force the pump must exert to move the slice, is

$$F_{\text{slice}} = 62.4 \cdot V_{\text{slice}} = 62.4\pi(1.5 - 0.1875x)^2 \Delta x.$$

Because the slice is located at a depth of x feet below the top of the crock, the slice being moved by the pump must move x feet to get to the level of the basement floor, and then, as stated in the problem description, be moved another 9 feet to reach the drain at ground level outside a basement window. Hence, the total distance a representative slice travels is

$$d_{\text{slice}} = x + 9\,.$$

Finally, we note that the work to move a representative slice is given by

$$W_{\text{slice}} = F_{\text{slice}} \cdot d_{\text{slice}} = 62.4\pi(1.5 - 0.1875x)^2 \Delta x \cdot (x + 9),$$

since the force to move a particular slice is constant. We sum the work required to move slices throughout the tank (from $x = 0$ to $x = 4$), let $\Delta x \to 0$, and hence

$$W = \int_0^4 62.4\pi(1.5 - 0.1875x)^2(x + 9)dx,$$

which, when evaluated using appropriate technology, shows that the total work is $W = 10970.5\pi$ foot-pounds.

The preceding example demonstrates the standard approach to finding the work required to empty a tank filled with liquid. The main task in each such problem is to determine the volume of a representative slice, followed by the force exerted on the slice, as well as the distance such a slice moves. In the case where the units are metric, there is one key difference: in the metric setting, rather than weight, we normally first find the mass of a slice. For instance, if distance is measured in meters, the mass density of water is 1000 kg/m^3 . In that setting, we can find the mass of a typical slice (in kg). To determine the force required to move it, we use F = ma, where m is the object's mass and a is the gravitational constant 9.81 N/kg^3 . That is, in metric units, the weight density of water is 9810 N/m^3 .

Activity 6.4.2

In each of the following problems, determine the total work required to accomplish the described task. In parts (b) and (c), a key step is to find a formula for a function that describes the curve that forms the side boundary of the tank.

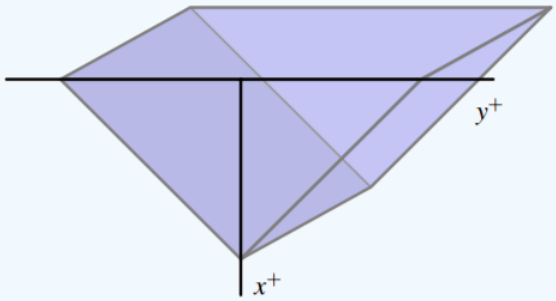


Figure 6.17: A trough with triangular ends, as described in Activity 6.11, part (c).

a. Consider a vertical cylindrical tank of radius 2 meters and depth 6 meters. Suppose the tank is filled with 4 meters of water of mass density 1000 kg/m^3 , and the top 1 meter of water is pumped over the top of the tank.
b. Consider a hemispherical tank with a radius of 10 feet. Suppose that the tank is full to a depth of 7 feet with water of weight density 62.4 pounds/ft^3, and the top 5 feet of water are pumped out of the tank to a tanker truck whose height is 5 feet above the top of the tank.
c. Consider a trough with triangular ends, as pictured in Figure 6.17, where the tank is 10 feet long, the top is 5 feet wide, and the tank is 4 feet deep. Say that the trough is full to within 1 foot of the top with water of weight density 62.4 pounds/ft^3, and a pump is used to empty the tank until the water remaining in the tank is 1 foot deep.

Force due to Hydrostatic Pressure

When a dam is built, it is imperative to for engineers to understand how much force water will exert against the face of the dam. The first thing we realize is the force exerted by the fluid is related to the natural concept of pressure. The pressure a force exerts on a region is measured in units of force per unit of area: for example, the air pressure in a tire is often measured in pounds per square inch (PSI). Hence, we see that the general relationship is given by

$$P = \frac{F}{A}, \text{ or } F = P \cdot A,$$

where P represents pressure, F represents force, and A the area of the region being considered. Of course, in the equation F = PA, we assume that the pressure is constant over the entire region A.

Most people know from experience that the deeper one dives underwater while swimming, the greater the pressure that is exerted by the water. This is due to the fact that the deeper one dives, the more water there is right on top of the swimmer: it is the force

that “column” of water exerts that determines the pressure the swimmer experiences. To get water pressure measured in its standard units (pounds per square foot), we say that the total water pressure is found by computing the total weight of the column of water that lies above a region of area 1 square foot at a fixed depth. Such a rectangular column with a 1 × 1 base and a depth of d feet has volume V = 1 · 1 · d ft^3, and thus the corresponding weight of the water overhead is 62.4d. Since this is also the amount of force being exerted on a 1 square foot region at a depth d feet underwater, we see that P = 62.4d (lbs/ft^2) is the pressure exerted by water at depth d.

The understanding that P = 62.4d will tell us the pressure exerted by water at a depth of d, along with the fact that F = PA, will now enable us to compute the total force that water exerts on a dam, as we see in the following example.

Example 6.4.3

Consider a trapezoid-shaped dam that is 60 feet wide at its base and 90 feet wide at its top, and assume the dam is 25 feet tall with water that rises to within 5 feet of the top of its face. Water weighs 62.5 pounds per cubic foot. How much force does the water exert against the dam?

Solution

First, we sketch a picture of the dam, as shown in Figure 6.18. Note that, as in problems involving the work to pump out a tank, we let the positive x-axis point down.

It is essential to use the fact that pressure is constant at a fixed depth. Hence, we consider a slice of water at constant depth on the face, such as the one shown in the figure. First, the approximate area of this slice is the area of the pictured rectangle. Since the width of that rectangle depends on the variable x (which represents the how far the slice lies from the top of the dam), we find a formula for the function $y = f(x)$ that determines one side of the face of the dam. Since f is linear, it is straightforward to find that $y = f(x) = 45 - \frac{3}{5}x$. Hence, the approximate area of a representative slice is

$$A_{\text{slice}} = 2f(x)\Delta x = 2(45 - \frac{3}{5}x)\Delta x.$$

At any point on this slice, the depth is approximately constant, and thus the pressure can be considered constant. In particular, we note that since x measures the distance to the top of the dam, and because the water rises to within 5 feet of the top of the dam, the depth of any point on the representative slice is approximately $(x - 5)$. Now, since pressure

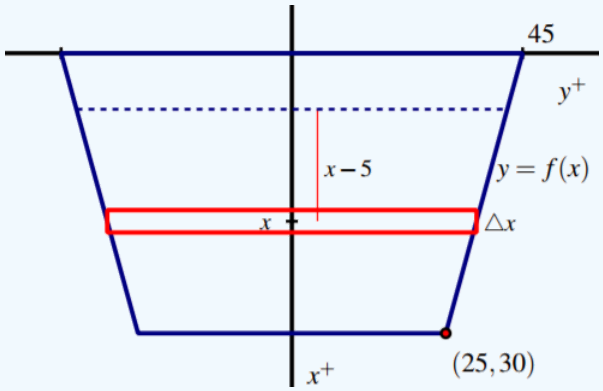


Figure 6.18: A trapezoidal dam that is 25 feet tall, 60 feet wide at its base, 90 feet wide at its top, with the water line 5 feet down from the top of its face.

is given by $P = 62.4d$, we have that at any point on the representative slice

$$P_{\text{slice}} = 62.4(x - 5).$$

Knowing both the pressure and area, we can find the force the water exerts on the slice. Using $F = PA$, it follows that

$$F_{\text{slice}} = P_{\text{slice}} \cdot A_{\text{slice}} = 62.4(x - 5) \cdot 2(45 - \frac{3}{5}x)\Delta x \ .$$

Finally, we use a definite integral to sum the forces over the appropriate range of x-values. Since the water rises to within 5 feet of the top of the dam, we start at $x = 5$ and slice all the way to the bottom of the dam, where $x = 30$. Hence,

$$F = \int_{x=5}^{x=30} 62.4(x-5) \cdot 2(45 - \frac{3}{5}x)dx\,.$$

Using technology to evaluate the integral, we find F ≈ 1.248 × 10^6 pounds.

Activity 6.4.4

In each of the following problems, determine the total force exerted by water against the surface that is described.

a. Consider a rectangular dam that is 100 feet wide and 50 feet tall, and suppose that water presses against the dam all the way to the top.
b. Consider a semicircular dam with a radius of 30 feet. Suppose that the water rises to within 10 feet of the top of the dam.
c. Consider a trough with triangular ends, as pictured in Figure 6.17, where the tank is 10 feet long, the top is 5 feet wide, and the tank is 4 feet deep. Say that the trough is full to within 1 foot of the top with water of weight density 62.4 pounds/ft^3. How much force does the water exert against one of the triangular ends?

While there are many different formulas that we use in solving problems involving work, force, and pressure, it is important to understand that the fundamental ideas behind these problems are similar to several others that we've encountered in applications of the definite integral. In particular, the basic idea is to take a difficult problem and somehow slice it into more manageable pieces that we understand, and then use a definite integral to add up these simpler pieces.

Summary

In this section, we encountered the following important ideas:

- To measure the work accomplished by a varying force that moves an object, we subdivide the problem into pieces on which we can use the formula W = F · d, and then use a definite integral to sum the work accomplished on each piece.
- To find the total force exerted by water against a dam, we use the formula F = P · A to measure the force exerted on a slice that lies at a fixed depth, and then use a definite integral to sum the forces across the appropriate range of depths.
- Because work is computed as the product of force and distance (provided force is constant), and the force water exerts on a dam can be computed as the product of pressure and area (provided pressure is constant), problems involving these concepts are similar to earlier problems we did using definite integrals to find distance (via "distance equals rate times time") and mass ("mass equals density times volume").

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6.5: Average Value of a Function

The average of some finite set of values is a familiar concept. If, for example, the class scores on a quiz are 10, 9, 10, 8, 7, 5, 7, 6, 3, 2, 7, 8, then the average score is the sum of these numbers divided by the size of the class:

$$\text{average score} = \frac{10+9+10+8+7+5+7+6+3+2+7+8}{12} = \frac{82}{12} \approx 6.83.$$

Suppose that between $t=0$ and $t=1$ the speed of an object is $\sin(\pi t)$. What is the average speed of the object over that time? The question sounds as if it must make sense, yet we can't merely add up some number of speeds and divide, since the speed is changing continuously over the time interval.

To make sense of "average" in this context, we fall back on the idea of approximation. Consider the speed of the object at tenth of a second intervals: $\sin 0$, $\sin(0.1\pi)$, $\sin(0.2\pi)$, $\sin(0.3\pi)$,..., $\sin(0.9\pi)$. The average speed "should" be fairly close to the average of these ten speeds:

$$\frac{1}{10}\sum_{i=0}^{9}\sin(\pi i/10) \approx \frac{1}{10}6.3 = 0.63.$$

Of course, if we compute more speeds at more times, the average of these speeds should be closer to the "real" average. If we take the average of n speeds at evenly spaced times, we get:

$$\frac{1}{n}\sum_{i=0}^{n-1}\sin(\pi i/n).$$

\ Here the individual times are $t_i = i/n$, so rewriting slightly we have

$$\frac{1}{n}\sum_{i=0}^{n-1}\sin(\pi t_i).$$

This is almost the sort of sum that we know turns into an integral; what's apparently missing is Δt---but in fact, $\Delta t = 1/n$, the length of each subinterval. So rewriting again:

$$\sum_{i=0}^{n-1}\sin(\pi t_i)\frac{1}{n} = \sum_{i=0}^{n-1}\sin(\pi t_i)\Delta t.$$

Now this has exactly the right form, so that in the limit we get

$$\text{average speed} = \int_0^1 \sin(\pi t)\,dt = -\frac{\cos(\pi t)}{\pi}\bigg|_0^1 = -\frac{\cos(\pi)}{\pi} + \frac{\cos(0)}{\pi} = \frac{2}{\pi} \approx 0.6366 \approx 0.64.$$

It's not entirely obvious from this one simple example how to compute such an average in general. Let's look at a somewhat more complicated case. Suppose that the velocity of an object is $16t^2+5$ feet per second. What is the average velocity between $t=1$ and $t=3$? Again we set up an approximation to the average:

$$\frac{1}{n}\sum_{i=0}^{n-1}16t_i^2+5,$$

where the values t_i are evenly spaced times between 1 and 3. Once again we are "missing" Δt, and this time $1/n$ is not the correct value. What is Δt in general? It is the length of a subinterval; in this case we take the interval $[1,3]$ and divide it into n subintervals, so each has length $(3-1)/n = 2/n = \Delta t$. Now with the usual "multiply and divide by the same thing" trick we can rewrite the sum:

$$\frac{1}{n}\sum_{i=0}^{n-1}16t_i^2+5 = \frac{1}{3-1}\sum_{i=0}^{n-1}(16t_i^2+5)\frac{3-1}{n} = \frac{1}{2}\sum_{i=0}^{n-1}(16t_i^2+5)\frac{2}{n} = \frac{1}{2}\sum_{i=0}^{n-1}(16t_i^2+5)\Delta t.$$

In the limit this becomes

$$\frac{1}{2}\int_1^3 16t^2+5\,dt = \frac{1}{2}\frac{446}{3} = \frac{223}{3}.$$

Does this seem reasonable? Let's picture it: in figure 9.4.1 is the velocity function together with the horizontal line $y = 223/3 \approx 74.3$. Certainly the height of the horizontal line looks at least plausible for the average height of the curve.

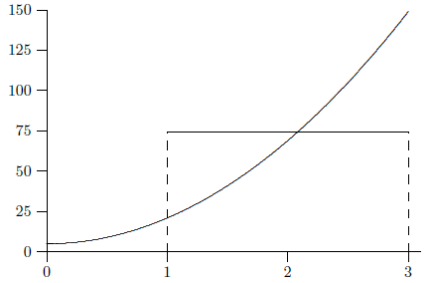


Figure 9.4.1. Average velocity.

Here's another way to interpret "average" that may make our computation appear even more reasonable. The object of our example goes a certain distance between $t = 1$ and $t = 3$. If instead the object were to travel at the average speed over the same time, it should go the same distance. At an average speed of $223/3$ feet per second for two seconds the object would go $446/3$ feet. How far does it actually go? We know how to compute this:

$$\int_1^3 v(t)\,dt = \int_1^3 16t^2+5\,dt = \frac{446}{3}.$$

So now we see that another interpretation of the calculation

$$\frac{1}{2}\int_1^3 16t^2+5\,dt = \frac{1}{2}\frac{446}{3} = \frac{223}{3}$$

is: total distance traveled divided by the time in transit, namely, the usual interpretation of average speed.

In the case of speed, or more properly velocity, we can always interpret "average" as total (net) distance divided by time. But in the case of a different sort of quantity this interpretation does not obviously apply, while the approximation approach always does. We might interpret the same problem geometrically: what is the average height of $16x^2+5$ on the interval $[1,3]$? We approximate this in exactly the same way, by adding up many sample heights and dividing by the number of samples. In the limit we get the same result:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} 16x_i^2+5 = \frac{1}{2}\int_1^3 16x^2+5\,dx = \frac{1}{2}\frac{446}{3} = \frac{223}{3}.$$

We can interpret this result in a slightly different way. The area under $y = 16x^2+5$ above $[1,3]$ is

$$\int_1^3 16t^2+5\,dt = \frac{446}{3}.$$

The area under $y = 223/3$ over the same interval $[1,3]$ is simply the area of a rectangle that is 2 by $223/3$ with area $446/3$. So the average height of a function is the height of the horizontal line that produces the same area over the given interval.

Contributors and Attributions

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CHAPTER OVERVIEW

7: Techniques of Integration

A general Calculus Textmap organized around the textbook

Calculus: Early Transcendentals

by James Stewart

I II III IV V VI VII VIII IX X XI XII XIII XIV XV XVI XVII

This Textmap is currently under construction... please be patient with us.

Template:HideTOC

Topic hierarchy

7.1: Integration by Parts
7.2: Trigonometric Integrals
7.3: Trigonometric Substitution
7.4: Integration of Rational Functions by Partial Fractions
7.5: Strategy for Integration
7.6: Integration Using Tables and Computer Algebra Systems
7.7: Approximate Integration
7.8: Improper Integrals

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7.1: Integration by Parts

Learning Objectives

- Recognize when to use integration by parts.
- Use the integration-by-parts formula to solve integration problems.
- Use the integration-by-parts formula for definite integrals.

By now we have a fairly thorough procedure for how to evaluate many basic integrals. However, although we can integrate $\int x\sin(x^2)\,dx$ by using the substitution, $u=x^2$, something as simple looking as $\int x\sin x\;dx$ defies us. Many students want to know whether there is a product rule for integration. There is not, but there is a technique based on the product rule for differentiation that allows us to exchange one integral for another. We call this technique **integration by parts.**

The Integration-by-Parts Formula

If, $h(x)=f(x)g(x)$, then by using the product rule, we obtain

$$h'(x)=f'(x)g(x)+g'(x)f(x). \tag{7.1.1}$$

Although at first it may seem counterproductive, let's now integrate both sides of Equation 7.1.1:

$$\int h'(x)\;dx=\int (g(x)f'(x)+f(x)g'(x))\;dx.$$

This gives us

$$h(x)=f(x)g(x)=\int g(x)f'(x)\,dx+\int f(x)g'(x)\;dx.$$

Now we solve for $\int f(x)g'(x)\;dx$:

$$\int f(x)g'(x)\,dx=f(x)g(x)-\int g(x)f'(x)\;dx.$$

By making the substitutions $u=f(x)$ and $v=g(x)$, which in turn make $du=f'(x)\,dx$ and $dv=g'(x)\,dx$, we have the more compact form

$$\int u\,dv=uv-\int v\,du.$$

Integration by Parts

Let $u=f(x)$ and $v=g(x)$ be functions with continuous derivatives. Then, the integration-by-parts formula for the integral involving these two functions is:

$$\int u\,dv=uv-\int v\,du. \tag{7.1.2}$$

The advantage of using the integration-by-parts formula is that we can use it to exchange one integral for another, possibly easier, integral. The following example illustrates its use.

Example 7.1.1: Using Integration by Parts

Use integration by parts with $u=x$ and $dv=\sin x\;dx$ to evaluate

$$\int x\sin x\;dx.$$

Solution

By choosing $u=x$, we have $du=1\;dx$. Since $dv=\sin x\;dx$, we get

$$v = \int \sin x \; dx = -\cos x.$$

It is handy to keep track of these values as follows:

- $u = x$
- $dv = \sin x \; dx$
- $du = 1 \, dx$
- $v = \int \sin x \; dx = -\cos x.$

Applying the integration-by-parts formula (Equation 7.1.2) results in

$$\begin{aligned} \int x \sin x \; dx &= (x)(-\cos x) - \int (-\cos x)(1 \; dx) && \text{(Substitute)} \\ &= -x \cos x + \int \cos x \; dx && \text{(Simplify)} \end{aligned}$$

Then use

$$\int \cos x \; dx = \sin x + C.$$

to obtain

$$\int x \sin x \; dx = -x \cos x + \sin x + C.$$

Analysis

At this point, there are probably a few items that need clarification. First of all, you may be curious about what would have happened if we had chosen $u = \sin x$ and $dv = x \, dx$. If we had done so, then we would have $du = \cos x$ and $v = \frac{1}{2}x^2$. Thus, after applying integration by parts (Equation 7.1.2), we have

$$\int x \sin x \; dx = \frac{1}{2}x^2 \sin x - \int \frac{1}{2}x^2 \cos x \; dx.$$

Unfortunately, with the new integral, we are in no better position than before. It is important to keep in mind that when we apply integration by parts, we may need to try several choices for u and dv before finding a choice that works.

Second, you may wonder why, when we find $v = \int \sin x \; dx = -\cos x$, we do not use $v = -\cos x + K$. To see that it makes no difference, we can rework the problem using $v = -\cos x + K$:

$$\begin{aligned} \int x \sin x \; dx &= (x)(-\cos x + K) - \int (-\cos x + K)(1 \; dx) \\ &= -x \cos x + Kx + \int \cos x \; dx - \int K \; dx \\ &= -x \cos x + Kx + \sin x - Kx + C \\ &= -x \cos x + \sin x + C. \end{aligned}$$

As you can see, it makes no difference in the final solution.

Last, we can check to make sure that our antiderivative is correct by differentiating $-x \cos x + \sin x + C$:

$$\begin{aligned} \frac{d}{dx}(-x \cos x + \sin x + C) &= \cancel{(-1)\cos x} + (-x)(-\sin x) + \cancel{\cos x} \\ &= x \sin x \end{aligned}$$

Therefore, the antiderivative checks out.

? Exercise 7.1.1

Evaluate $\int xe^{2x}\,dx$ using the integration-by-parts formula (Equation 7.1.2) with $u = x$ and $dv = e^{2x}\,dx$.

Hint

Find du and v, and use the previous example as a guide.

Answer

$$\int xe^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

The natural question to ask at this point is: How do we know how to choose u and dv? Sometimes it is a matter of trial and error; however, the acronym **LIATE** can often help to take some of the guesswork out of our choices. This acronym stands for **L**ogarithmic Functions, **I**nverse Trigonometric Functions, **A**lgebraic Functions, **T**rigonometric Functions, and **E**xponential Functions. This mnemonic serves as an aid in determining an appropriate choice for u. The type of function in the integral that appears first in the list should be our first choice of u.

For example, if an integral contains a **logarithmic function** and an **algebraic function,** we should choose u to be the logarithmic function, because L comes before A in LIATE. The integral in Example 7.1.1 has a trigonometric function ($\sin x$) and an algebraic function (x). Because A comes before T in LIATE, we chose u to be the algebraic function. When we have chosen u, dv is selected to be the remaining part of the function to be integrated, together with dx.

Why does this mnemonic work? Remember that whatever we pick to be dv must be something we can integrate. Since we do not have integration formulas that allow us to integrate simple logarithmic functions and inverse trigonometric functions, it makes sense that they should not be chosen as values for dv. Consequently, they should be at the head of the list as choices for u. Thus, we put LI at the beginning of the mnemonic. (We could just as easily have started with IL, since these two types of functions won't appear together in an integration-by-parts problem.) The exponential and trigonometric functions are at the end of our list because they are fairly easy to integrate and make good choices for dv. Thus, we have TE at the end of our mnemonic. (We could just as easily have used ET at the end, since when these types of functions appear together it usually doesn't really matter which one is u and which one is dv.) Algebraic functions are generally easy both to integrate and to differentiate, and they come in the middle of the mnemonic.

✓ Example 7.1.2: Using Integration by Parts

Evaluate

$$\int \frac{\ln x}{x^3}\,dx.$$

Solution

Begin by rewriting the integral:

$$\int \frac{\ln x}{x^3}\,dx = \int x^{-3}\ln x\,dx.$$

Since this integral contains the algebraic function x^{-3} and the logarithmic function $\ln x$, choose $u = \ln x$, since L comes before A in LIATE. After we have chosen $u = \ln x$, we must choose $dv = x^{-3}\,dx$.

Next, since $u = \ln x$, we have $du = \frac{1}{x}\,dx$. Also, $v = \int x^{-3}\,dx = -\frac{1}{2}x^{-2}$. Summarizing,

- $u = \ln x$
- $du = \frac{1}{x}\,dx$
- $dv = x^{-3}\,dx$
- $v = \int x^{-3}\,dx = -\frac{1}{2}x^{-2}$.

Substituting into the integration-by-parts formula (Equation 7.1.2) gives

$$\begin{aligned}\int \frac{\ln x}{x^3}\,dx &= \int x^{-3}\ln x\,dx = (\ln x)(-\frac{1}{2}x^{-2}) - \int (-\frac{1}{2}x^{-2})(\frac{1}{x}\,dx)\\ &= -\frac{1}{2}x^{-2}\ln x + \int \frac{1}{2}x^{-3}\,dx\\ &= -\frac{1}{2}x^{-2}\ln x - \frac{1}{4}x^{-2} + C\\ &= -\frac{1}{2x^2}\ln x - \frac{1}{4x^2} + C\end{aligned}$$

? Exercise 7.1.2

Evaluate

$$\int x\ln x\,\,dx.$$

Hint

Use $u = \ln x$ and $dv = x\,\,dx$.

Answer

$$\int x\ln x\,\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$$

In some cases, as in the next two examples, it may be necessary to apply integration by parts more than once.

✓ Example 7.1.3A: Applying Integration by Parts More Than Once

Evaluate

$$\int x^2 e^{3x}\,dx.$$

Solution

Using LIATE, choose $u = x^2$ and $dv = e^{3x}\,dx$. Thus, $du = 2x\,dx$ and $v = \int e^{3x}\,dx = \left(\frac{1}{3}\right)e^{3x}$. Therefore,

- $u = x^2$
- $du = 2x\,dx$
- $dv = e^{3x}\,dx$
- $v = \int e^{3x}\,dx = \frac{1}{3}e^{3x}$.

Substituting into Equation 7.1.2 produces

$$\int x^2 e^{3x}\,dx = \frac{1}{3}x^2 e^{3x} - \int \frac{2}{3}xe^{3x}\,dx. \tag{7.1.3}$$

We still cannot integrate $\int \frac{2}{3}xe^{3x}\,dx$ directly, but the integral now has a lower power on x. We can evaluate this new integral by using integration by parts again. To do this, choose

$$u = x$$

and

$$dv = \frac{2}{3}e^{3x}\,dx.$$

Thus,

$$du = dx$$

and

$$v = \int \left(\frac{2}{3}\right) e^{3x}\, dx = \left(\frac{2}{9}\right) e^{3x}.$$

Now we have

- $u = x$
- $du = dx$
- $dv = \frac{2}{3} e^{3x}\, dx$
- $v = \int \frac{2}{3} e^{3x}\, dx = \frac{2}{9} e^{3x}.$

Substituting back into Equation 7.1.3 yields

$$\int x^2 e^{3x}\, dx = \frac{1}{3} x^2 e^{3x} - \left(\frac{2}{9} x e^{3x} - \int \frac{2}{9} e^{3x}\, dx\right).$$

After evaluating the last integral and simplifying, we obtain

$$\int x^2 e^{3x}\, dx = \frac{1}{3} x^2 e^{3x} - \frac{2}{9} x e^{3x} + \frac{2}{27} e^{3x} + C.$$

Example 7.1.3B: Applying Integration by Parts When LIATE Does not Quite Work

Evaluate

$$\int t^3 e^{t^2}\, dt.$$

Solution

If we use a strict interpretation of the mnemonic LIATE to make our choice of u, we end up with $u = t^3$ and $dv = e^{t^2}\, dt$. Unfortunately, this choice won't work because we are unable to evaluate $\int e^{t^2}\, dt$. However, since we can evaluate $\int t e^{t^2}\, dx$, we can try choosing $u = t^2$ and $dv = t e^{t^2}\, dt$. With these choices we have

- $u = t^2$
- $du = 2t dt$
- $dv = t e^{t^2}\, dt$
- $v = \int t e^{t^2}\, dt = \frac{1}{2} e^{t^2}.$

Thus, we obtain

$$\begin{aligned}\int t^3 e^{t^2}\, dt &= \frac{1}{2} t^2 e^{t^2} - \int \frac{1}{2} e^{t^2} 2t\, dt \\ &= \frac{1}{2} t^2 e^{t^2} - \frac{1}{2} e^{t^2} + C.\end{aligned}$$

Example 7.1.3C: Applying Integration by Parts More Than Once

Evaluate

$$\int \sin(\ln x)\, dx.$$

Solution

This integral appears to have only one function—namely, $\sin(\ln x)$—however, we can always use the constant function 1 as the other function. In this example, let's choose $u = \sin(\ln x)$ and $dv = 1\, dx$. (The decision to use $u = \sin(\ln x)$ is easy. We

can't choose $dv = \sin(\ln x)\,dx$ because if we could integrate it, we wouldn't be using integration by parts in the first place!) Consequently, $du = (1/x)\cos(\ln x)\,dx$ and $v = \int 1\,dx = x$. After applying integration by parts to the integral and simplifying, we have

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - \int \cos(\ln x)\,dx.$$

Unfortunately, this process leaves us with a new integral that is very similar to the original. However, let's see what happens when we apply integration by parts again. This time let's choose $u = \cos(\ln x)$ and $dv = 1\,dx$, making $du = -(1/x)\sin(\ln x)\,dx$ and $v = \int 1\,dx = x$.

Substituting, we have

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - (x\cos(\ln x) - \int -\sin(\ln x)\,dx).$$

After simplifying, we obtain

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\,dx.$$

The last integral is now the same as the original. It may seem that we have simply gone in a circle, but now we can actually evaluate the integral. To see how to do this more clearly, substitute $I = \int \sin(\ln x)\,dx$. Thus, the equation becomes

$$I = x\sin(\ln x) - x\cos(\ln x) - I.$$

First, add I to both sides of the equation to obtain

$$2I = x\sin(\ln x) - x\cos(\ln x).$$

Next, divide by 2:

$$I = \frac{1}{2}x\sin(\ln x) - \frac{1}{2}x\cos(\ln x).$$

Substituting $I = \int \sin(\ln x)\,dx$ again, we have

$$\int \sin(\ln x)\,dx = \frac{1}{2}x\sin(\ln x) - \frac{1}{2}x\cos(\ln x).$$

From this we see that $(1/2)x\sin(\ln x) - (1/2)x\cos(\ln x)$ is an antiderivative of $\sin(\ln x)\,dx$. For the most general antiderivative, add $+C$:

$$\int \sin(\ln x)\,dx = \frac{1}{2}x\sin(\ln x) - \frac{1}{2}x\cos(\ln x) + C.$$

Analysis

If this method feels a little strange at first, we can check the answer by differentiation:

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{2}x\sin(\ln x) - \frac{1}{2}x\cos(\ln x)\right) &= \frac{1}{2}(\sin(\ln x)) + \cos(\ln x)\cdot\frac{1}{x}\cdot\frac{1}{2}x - \left(\frac{1}{2}\cos(\ln x) - \sin(\ln x)\cdot\frac{1}{x}\cdot\frac{1}{2}x\right)\\ &= \sin(\ln x).\end{aligned}$$

? Exercise 7.1.3

Evaluate

$$\int x^2 \sin x\,dx.$$

Hint

This is similar to Examples $7.1.3A$- $7.1.3C$.

Answer

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

Integration by Parts for Definite Integrals

Now that we have used integration by parts successfully to evaluate *indefinite integrals*, we turn our attention to definite integrals. The integration technique is really the same, only we add a step to evaluate the integral at the upper and lower limits of integration.

Integration by Parts for Definite Integrals

Let $u = f(x)$ and $v = g(x)$ be functions with continuous derivatives on $[a, b]$. Then

$$\int_a^b u \, dv = uv\Big|_a^b - \int_a^b v \, du$$

Example $7.1.4A$: Finding the Area of a Region

Find the area of the region bounded above by the graph of $y = \tan^{-1} x$ and below by the x-axis over the interval $[0, 1]$.

Solution

This region is shown in Figure $7.1.1$. To find the area, we must evaluate

$$\int_0^1 \tan^{-1} x \; dx.$$

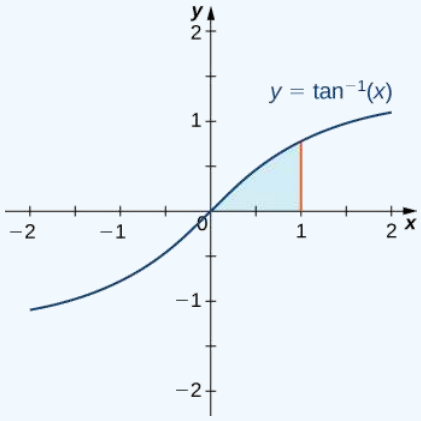


Figure $7.1.1$: To find the area of the shaded region, we have to use integration by parts.

For this integral, let's choose $u = tan^{-1} x$ and $dv = dx$, thereby making $du = \dfrac{1}{x^2+1} dx$ and $v = x$. After applying the integration-by-parts formula (Equation 7.1.2) we obtain

$$\text{Area} = x \tan^{-1} x\Big|_0^1 - \int_0^1 \frac{x}{x^2+1} \, dx.$$

Use u-substitution to obtain

$$\int_0^1 \frac{x}{x^2+1} \, dx = \frac{1}{2} \ln\left(x^2+1\right)\Big|_0^1.$$

Thus,

$$\text{Area} = x\tan^{-1}x\Big|_0^1 - \frac{1}{2}\ln(x^2+1)\Big|_0^1 = \left(\frac{\pi}{4} - \frac{1}{2}\ln 2\right) \text{ units}^2.$$

At this point it might not be a bad idea to do a "reality check" on the reasonableness of our solution. Since $\frac{\pi}{4} - \frac{1}{2}\ln 2 \approx 0.4388 \text{ units}^2$, and from Figure 7.1.1 we expect our area to be slightly less than 0.5 units^2, this solution appears to be reasonable.

Example 7.1.4B: Finding a Volume of Revolution

Find the volume of the solid obtained by revolving the region bounded by the graph of $f(x) = e^{-x}$, the x-axis, the y-axis, and the line $x = 1$ about the y-axis.

Solution

The best option to solving this problem is to use the shell method. Begin by sketching the region to be revolved, along with a typical rectangle (Figure 7.1.2).

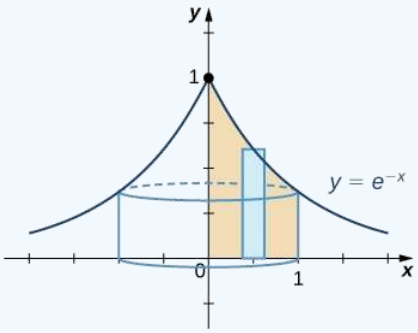


Figure 7.1.2: We can use the shell method to find a volume of revolution.

To find the volume using shells, we must evaluate

$$2\pi \int_0^1 xe^{-x}\,dx. \tag{7.1.4}$$

To do this, let $u = x$ and $dv = e^{-x}$. These choices lead to $du = dx$ and $v = \int e^{-x}\,dx = -e^{-x}$. Using the Shell Method formula, we obtain

$$\begin{aligned}
\text{Volume} &= 2\pi \int_0^1 xe^{-x}\,dx \\
&= 2\pi\left(-xe^{-x}\Big|_0^1 + \int_0^1 e^{-x}\,dx\right) && \text{(Use integration by parts)} \\
&= 2\pi\left(-e^{-1} + 0 - e^{-x}\Big|_0^1\right) \\
&= 2\pi\left(-e^{-1} - e^{-1} + 1\right) \\
&= 2\pi\left(1 - \frac{2}{e}\right) \text{ units}^3. && \text{(Evaluate and simplify)}
\end{aligned}$$

Analysis

Again, it is a good idea to check the reasonableness of our solution. We observe that the solid has a volume slightly less than that of a cylinder of radius 1 and height of $1/e$ added to the volume of a cone of base radius 1 and height of $1 - \frac{1}{e}$. Consequently, the solid should have a volume a bit less than

$$\pi(1)^2\frac{1}{e} + \left(\frac{\pi}{3}\right)(1)^2\left(1 - \frac{1}{e}\right) = \frac{2\pi}{3e} + \frac{\pi}{3} \approx 1.8177 \text{ units}^3.$$

Since $2\pi - \frac{4\pi}{e} \approx 1.6603$, we see that our calculated volume is reasonable.

? Exercise 7.1.4

Evaluate

$$\int_0^{\pi/2} x \cos x \, dx.$$

Hint

Use Equation 7.1.2with $u = x$ and $dv = \cos x \, dx$.

Answer

$$\int_0^{\pi/2} x \cos x \, dx = \frac{\pi}{2} - 1$$

Key Concepts

- The integration-by-parts formula (Equation 7.1.2) allows the exchange of one integral for another, possibly easier, integral.
- Integration by parts applies to both definite and indefinite integrals.

Key Equations

- **Integration by parts formula**

  $$\int u \, dv = uv - \int v \, du$$

- **Integration by parts for definite integrals**

  $$\int_a^b u \, dv = uv\Big|_a^b - \int_a^b v \, du$$

Glossary

integration by parts

a technique of integration that allows the exchange of one integral for another using the formula $\int u \, dv = uv - \int v \, du$

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7.2: Trigonometric Integrals

Learning Objectives

- Solve integration problems involving products and powers of $\sin x$ and $\cos x$.
- Solve integration problems involving products and powers of $\tan x$ and $\sec x$.
- Use reduction formulas to solve trigonometric integrals.

In this section we look at how to integrate a variety of products of trigonometric functions. These integrals are called **trigonometric integrals**. They are an important part of the integration technique called *trigonometric substitution*, which is featured in Trigonometric Substitution. This technique allows us to convert algebraic expressions that we may not be able to integrate into expressions involving trigonometric functions, which we may be able to integrate using the techniques described in this section. In addition, these types of integrals appear frequently when we study polar, cylindrical, and spherical coordinate systems later. Let's begin our study with products of $\sin x$ and $\cos x$.

Integrating Products and Powers of sin x and cos x

A key idea behind the strategy used to integrate combinations of products and powers of $\sin x$ and $\cos x$ involves rewriting these expressions as sums and differences of integrals of the form $\int \sin^j x \cos x\, dx$ or $\int \cos^j x \sin x\, dx$. After rewriting these integrals, we evaluate them using u-substitution. Before describing the general process in detail, let's take a look at the following examples.

Example 7.2.1: Integrating $\int \cos^j x \sin x\, dx$

Evaluate $\int \cos^3 x \sin x\, dx$.

Solution

Use u-substitution and let $u = \cos x$. In this case, $du = -\sin x\, dx$.

Thus,

$$\int \cos^3 x \sin x\, dx = -\int u^3\, du = -\frac{1}{4}u^4 + C = -\frac{1}{4}\cos^4 x + C.$$

Exercise 7.2.1

Evaluate $\int \sin^4 x \cos x\, dx$.

Hint

Let $u = \sin x$.

Answer

$$\int \sin^4 x \cos x\, dx = \frac{1}{5}\sin^5 x + C$$

Example 7.2.2: A Preliminary Example: Integrating $\int \cos^j x \sin^k x\, dx$ where k is Odd

Evaluate $\int \cos^2 x \sin^3 x\, dx$.

Solution

To convert this integral to integrals of the form $\int \cos^j x \sin x\, dx$, rewrite $\sin^3 x = \sin^2 x \sin x$ and make the substitution $\sin^2 x = 1 - \cos^2 x$.

Thus,

$$\int \cos^2 x \sin^3 x\, dx = \int \cos^2 x (1-\cos^2 x)\sin x\, dx \qquad \text{Let } u = \cos x;\ \text{then } du = -\sin x\, dx.$$

$$= -\int u^2(1-u^2)\, du$$

$$= \int (u^4 - u^2)\, du$$

$$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C$$

$$= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C.$$

? Exercise 7.2.2

Evaluate $\int \cos^3 x \sin^2 x\, dx$.

Hint

Write $\cos^3 x = \cos^2 x \cos x = (1-\sin^2 x)\cos x$ and let $u = \sin x$.

Answer

$$\int \cos^3 x \sin^2 x\, dx = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$$

In the next example, we see the strategy that must be applied when there are only even powers of $\sin x$ and $\cos x$. For integrals of this type, the identities

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x) = \frac{1-\cos(2x)}{2}$$

and

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x) = \frac{1+\cos(2x)}{2}$$

are invaluable. These identities are sometimes known as **power-reducing identities** and they may be derived from the double-angle identity $\cos(2x) = \cos^2 x - \sin^2 x$ and the Pythagorean identity $\cos^2 x + \sin^2 x = 1$.

✓ Example 7.2.3: Integrating an Even Power of $\sin x$

Evaluate $\int \sin^2 x\, dx$.

Solution

To evaluate this integral, let's use the trigonometric identity $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$. Thus,

$$\int \sin^2 x\, dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right) dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C.$$

? Exercise 7.2.3

Evaluate $\int \cos^2 x\, dx$.

Hint

$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$

Answer

$$\int \cos^2 x\, dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$$

The general process for integrating products of powers of $\sin x$ and $\cos x$ is summarized in the following set of guidelines.

Problem-Solving Strategy: Integrating Products and Powers of $\sin x$ and $\cos x$

To integrate $\int \cos^j x \sin^k x\, dx$ use the following strategies:

1. If k is odd, rewrite $\sin^k x = \sin^{k-1} x \sin x$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to rewrite $\sin^{k-1} x$ in terms of $\cos x$. Integrate using the substitution $u = \cos x$. This substitution makes $du = -\sin x\, dx$.

2. If j is odd, rewrite $\cos^j x = \cos^{j-1} x \cos x$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to rewrite $\cos^{j-1} x$ in terms of $\sin x$. Integrate using the substitution $u = \sin x$. This substitution makes $du = \cos x\, dx$. (Note: If both j and k are odd, either strategy 1 or strategy 2 may be used.)

3. If both j and k are even, use $\sin^2 x = \dfrac{1 - \cos(2x)}{2}$ and $\cos^2 x = \dfrac{1 + \cos(2x)}{2}$. After applying these formulas, simplify and reapply strategies 1 through 3 as appropriate.

Example 7.2.4: Integrating $\int \cos^j x \sin^k x\, dx$ where k is Odd

Evaluate $\int \cos^8 x \sin^5 x\, dx$.

Solution

Since the power on $\sin x$ is odd, use strategy 1. Thus,

$$\begin{aligned}
\int \cos^8 x \sin^5 x\, dx &= \int \cos^8 x \sin^4 x \sin x\, dx && \text{Break off } \sin x. \\
&= \int \cos^8 x (\sin^2 x)^2 \sin x\, dx && \text{Rewrite } \sin^4 x = (\sin^2 x)^2. \\
&= \int \cos^8 x (1 - \cos^2 x)^2 \sin x\, dx && \text{Substitute } \sin^2 x = 1 - \cos^2 x. \\
&= \int u^8 (1 - u^2)^2 (-du) && \text{Let } u = \cos x \text{ and } du = -\sin x\, dx. \\
&= \int (-u^8 + 2u^{10} - u^{12}) du && \text{Expand.} \\
&= -\frac{1}{9}u^9 + \frac{2}{11}u^{11} - \frac{1}{13}u^{13} + C && \text{Evaluate the integral.} \\
&= -\frac{1}{9}\cos^9 x + \frac{2}{11}\cos^{11} x - \frac{1}{13}\cos^{13} x + C && \text{Substitute } u = \cos x.
\end{aligned}$$

Example 7.2.5: Integrating $\int \cos^j x \sin^k x\, dx$ where k and j are Even

Evaluate $\int \sin^4 x\, dx$.

Solution: Since the power on $\sin x$ is even $(k = 4)$ and the power on $\cos x$ is even $(j = 0)$, we must use strategy 3. Thus,

$$\int \sin^4 x\,dx = \int \left(\sin^2 x\right)^2 dx \qquad \text{Rewrite } \sin^4 x = \left(\sin^2 x\right)^2.$$

$$= \int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)^2 dx \qquad \text{Substitute } \sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x).$$

$$= \int \left(\frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{4}\cos^2(2x)\right) dx \qquad \text{Expand } \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)^2.$$

$$= \int \left(\frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\cos(4x)\right)\right) dx \qquad \text{Use Power Reduction Formula on } \cos^2(2x).$$

$$= \int \left(\frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)\right) dx \qquad \text{Simplify.}$$

$$= \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C \qquad \text{Evaluate the integral.}$$

? Exercise 7.2.4

Evaluate $\int \cos^3 x\,dx$.

Hint

Use strategy 2. Write $\cos^3 x = \cos^2 x \cos x$ and substitute $\cos^2 x = 1 - \sin^2 x$.

Answer

$$\int \cos^3 x\,dx = \sin x - \frac{1}{3}\sin^3 x + C$$

? Exercise 7.2.5

Evaluate $\int \cos^2(3x)\,dx$.

Hint

Use strategy 3. Substitute $\cos^2(3x) = \frac{1}{2} + \frac{1}{2}\cos(6x)$

Answer

$$\int \cos^2(3x)\,dx = \frac{1}{2}x + \frac{1}{12}\sin(6x) + C$$

Product to Sum Formulas

In some areas of physics, such as quantum mechanics, signal processing, and the computation of Fourier series, it is often necessary to integrate products that include $sin(ax), sin(bx), cos(ax)$, and $cos(bx)$. These integrals are evaluated by applying trigonometric identities, as outlined in the following rule.

Rule: Integrating Products of Sines and Cosines of Different Angles

To integrate products involving $\sin(ax)$, $\sin(bx)$, $\cos(ax)$, and $\cos(bx)$, use the substitutions

$$\sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x)$$

$$\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x)$$

$$\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x)$$

These formulas may be derived from the sum-of-angle formulas for sine and cosine.

Example 7.2.6: Evaluating $\int \sin(ax)\cos(bx)\,dx$

Evaluate $\int \sin(5x)\cos(3x)\,dx$.

Solution: Apply the identity $\sin(5x)\cos(3x) = \frac{1}{2}\sin(2x) + \frac{1}{2}\sin(8x)$. Thus,

$$\int \sin(5x)\cos(3x)\,dx = \int \frac{1}{2}\sin(2x) + \frac{1}{2}\sin(8x)\,dx = -\frac{1}{4}\cos(2x) - \frac{1}{16}\cos(8x) + C.$$

Exercise 7.2.6

Evaluate $\int \cos(6x)\cos(5x)\,dx$.

Hint

Substitute $\cos(6x)\cos(5x) = \frac{1}{2}\cos x + \frac{1}{2}\cos(11x)$.

Answer

$$\int \cos(6x)\cos(5x)\,dx = \frac{1}{2}\sin x + \frac{1}{22}\sin(11x) + C$$

Integrating Products and Powers of tan x and sec x

Before discussing the integration of products and powers of $\tan x$ and $\sec x$, it is useful to recall the integrals involving $\tan x$ and $\sec x$ we have already learned:

1. $\int \sec^2 x\,dx = \tan x + C$
2. $\int \sec x \tan x\,dx = \sec x + C$
3. $\int \tan x\,dx = \ln|\sec x| + C$
4. $\int \sec x\,dx = \ln|\sec x + \tan x| + C.$

For most integrals of products and powers of $\tan x$ and $\sec x$, we rewrite the expression we wish to integrate as the sum or difference of integrals of the form $\int \tan^j x \sec^2 x\,dx$ or $\int \sec^j x \tan x\,dx$. As we see in the following example, we can evaluate these new integrals by using u-substitution.

Example 7.2.7: Evaluating $\int \sec^j x \tan x\,dx$

Evaluate $\int \sec^5 x \tan x\,dx$.

Solution: Start by rewriting $\sec^5 x \tan x$ as $\sec^4 x \sec x \tan x$.

$$\begin{aligned}\int \sec^5 x \tan x\, dx &= \int \sec^4 x \sec x \tan x\, dx \\ &= \int u^4\, du && \text{Let } u = \sec x;\ \text{then, } du = \sec x \tan x\, dx. \\ &= \tfrac{1}{5}u^5 + C && \text{Evaluate the integral.} \\ &= \tfrac{1}{5}\sec^5 x + C && \text{Substitute } \sec x = u.\end{aligned}$$

You can read some interesting information at this website to learn about a common integral involving the secant.

Exercise 7.2.7

Evaluate $\int \tan^5 x \sec^2 x\, dx$.

Hint

Let $u = \tan x$ and $du = \sec^2 x$.

Answer

$$\int \tan^5 x \sec^2 x\, dx = \tfrac{1}{6}\tan^6 x + C$$

We now take a look at the various strategies for integrating products and powers of $\sec x$ and $\tan x$.

Problem-Solving Strategy: Integrating $\int \tan^k x \sec^j x\, dx$

To integrate $\int \tan^k x \sec^j x\, dx$, use the following strategies:

1. If j is even and $j \geq 2$, rewrite $\sec^j x = \sec^{j-2} x \sec^2 x$ and use $\sec^2 x = \tan^2 x + 1$ to rewrite $\sec^{j-2} x$ in terms of $\tan x$. Let $u = \tan x$ and $du = \sec^2 x$.

2. If k is odd and $j \geq 1$, rewrite $\tan^k x \sec^j x = \tan^{k-1} x \sec^{j-1} x \sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to rewrite $\tan^{k-1} x$ in terms of $\sec x$. Let $u = \sec x$ and $du = \sec x \tan x\, dx$. (Note: If j is even and k is odd, then either strategy 1 or strategy 2 may be used.)

3. If k is odd where $k \geq 3$ and $j = 0$, rewrite $\tan^k x = \tan^{k-2} x \tan^2 x = \tan^{k-2} x(\sec^2 x - 1) = \tan^{k-2} x \sec^2 x - \tan^{k-2} x$. It may be necessary to repeat this process on the $\tan^{k-2} x$ term.

4. If k is even and j is odd, then use $\tan^2 x = \sec^2 x - 1$ to express $\tan^k x$ in terms of $\sec x$. Use integration by parts to integrate odd powers of $\sec x$.

Example 7.2.8: Integrating $\int \tan^k x \sec^j x\, dx$ when j is Even

Evaluate $\int \tan^6 x \sec^4 x\, dx$.

Solution

Since the power on $\sec x$ is even, rewrite $\sec^4 x = \sec^2 x \sec^2 x$ and use $\sec^2 x = \tan^2 x + 1$ to rewrite the first $\sec^2 x$ in terms of $\tan x$. Thus,

$$\begin{aligned}
\int \tan^6 x \sec^4 x\, dx &= \int \tan^6 x(\tan^2 x+1)\sec^2 x\, dx \\
&= \int u^6(u^2+1)\, du && \text{Let } u=\tan x \text{ and } du=\sec^2 x. \\
&= \int (u^8+u^6)\, du && \text{Expand.} \\
&= \frac{1}{9}u^9+\frac{1}{7}u^7+C && \text{Evaluate the integral.} \\
&= \frac{1}{9}\tan^9 x+\frac{1}{7}\tan^7 x+C. && \text{Substitute } \tan x=u.
\end{aligned}$$

Example 7.2.9: Integrating $\int \tan^k x \sec^j x\, dx$ when k is Odd

Evaluate $\int \tan^5 x \sec^3 x\, dx$.

Solution

Since the power on $\tan x$ is odd, begin by rewriting $\tan^5 x \sec^3 x = \tan^4 x \sec^2 x \sec x \tan x$. Thus,

$$\begin{aligned}
\int \tan^5 x \sec^3 x\, dx &= \int \tan^4 x \sec^2 x \sec x \tan x\, dx. \\
&= \int (\tan^2 x)^2 \sec^2 x \sec x \tan x\, dx && \text{Write } \tan^4 x=(\tan^2 x)^2. \\
&= \int (\sec^2 x-1)^2 \sec^2 x \sec x \tan x\, dx && \text{Use } \tan^2 x=\sec^2 x-1. \\
&= \int (u^2-1)^2 u^2\, du && \text{Let } u=\sec x \text{ and } du=\sec x\tan x\, dx \\
&= \int (u^6-2u^4+u^2)du && \text{Expand.} \\
&= \frac{1}{7}u^7-\frac{2}{5}u^5+\frac{1}{3}u^3+C && \text{Integrate.} \\
&= \frac{1}{7}\sec^7 x-\frac{2}{5}\sec^5 x+\frac{1}{3}\sec^3 x+C && \text{Substitute } \sec x=u.
\end{aligned}$$

Example 7.2.10: Integrating $\int \tan^k x\, dx$ where k is Odd and $k \geq 3$

Evaluate $\int \tan^3 x\, dx$.

Solution

Begin by rewriting $\tan^3 x = \tan x \tan^2 x = \tan x(\sec^2 x-1) = \tan x \sec^2 x - \tan x$. Thus,

$$\begin{aligned}
\int \tan^3 x\, dx &= \int (\tan x \sec^2 x - \tan x)\, dx \\
&= \int \tan x \sec^2 x\, dx - \int \tan x\, dx \\
&= \frac{1}{2}\tan^2 x - \ln|\sec x| + C.
\end{aligned}$$

For the first integral, use the substitution $u=\tan x$. For the second integral, use the formula.

Example 7.2.11: Integrating $\int \sec^3 x\,dx$

Integrate $\int \sec^3 x\,dx$.

Solution

This integral requires integration by parts. To begin, let $u = \sec x$ and $dv = \sec^2 x$. These choices make $du = \sec x \tan x$ and $v = \tan x$. Thus,

$$\begin{aligned}
\int \sec^3 x\,dx &= \sec x \tan x - \int \tan x \sec x \tan x\,dx \\
&= \sec x \tan x - \int \tan^2 x \sec x\,dx && \text{Simplify.} \\
&= \sec x \tan x - \int (\sec^2 x - 1)\sec x\,dx && \text{Substitute } \tan^2 x = \sec^2 x - 1. \\
&= \sec x \tan x + \int \sec x\,dx - \int \sec^3 x\,dx && \text{Rewrite.} \\
&= \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x\,dx. && \text{Evaluate } \int \sec x\,dx.
\end{aligned}$$

We now have

$$\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x\,dx.$$

Since the integral $\int \sec^3 x\,dx$ has reappeared on the right-hand side, we can solve for $\int \sec^3 x\,dx$ by adding it to both sides. In doing so, we obtain

$$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x|.$$

Dividing by 2, we arrive at

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$$

Exercise 7.2.8

Evaluate $\int \tan^3 x \sec^7 x\,dx$.

Hint

Use Example 7.2.9as a guide.

Answer

$$\int \tan^3 x \sec^7 x\,dx = \frac{1}{9}\sec^9 x - \frac{1}{7}\sec^7 x + C$$

Reduction Formulas

Evaluating $\int \sec^n x\,dx$ for values of n where n is odd requires integration by parts. In addition, we must also know the value of $\int \sec^{n-2} x\,dx$ to evaluate $\int \sec^n x\,dx$. The evaluation of $\int \tan^n x\,dx$ also requires being able to integrate $\int \tan^{n-2} x\,dx$. To

make the process easier, we can derive and apply the following power reduction formulas. These rules allow us to replace the integral of a power of $\sec x$ or $\tan x$ with the integral of a lower power of $\sec x$ or $\tan x$.

Rule: Reduction Formulas for $\int \sec^n x\,dx$ and $\int \tan^n x\,dx$

$$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$$

$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx$$

The first power reduction rule may be verified by applying integration by parts. The second may be verified by following the strategy outlined for integrating odd powers of $\tan x$.

Example 7.2.12: Revisiting $\int \sec^3 x\,dx$

Apply a reduction formula to evaluate $\int \sec^3 x\,dx$.

Solution: By applying the first reduction formula, we obtain

$$\begin{aligned}\int \sec^3 x\,dx &= \frac{1}{2}\sec x \tan x + \frac{1}{2}\int \sec x\,dx \\ &= \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.\end{aligned}$$

Example 7.2.13: Using a Reduction Formula

Evaluate $\int \tan^4 x\,dx$.

Solution: Applying the reduction formula for $\int \tan^4 x\,dx$ we have

$$\begin{aligned}\int \tan^4 x\,dx &= \frac{1}{3}\tan^3 x - \int \tan^2 x\,dx \\ &= \frac{1}{3}\tan^3 x - (\tan x - \int \tan^0 x\,dx) && \text{Apply the reduction formula to } \int \tan^2 x\,dx. \\ &= \frac{1}{3}\tan^3 x - \tan x + \int 1\,dx && \text{Simplify.} \\ &= \frac{1}{3}\tan^3 x - \tan x + x + C && \text{Evaluate } \int 1\,dx\end{aligned}$$

Exercise 7.2.9

Apply the reduction formula to $\int \sec^5 x\,dx$.

Hint

Use reduction formula 1 and let $n = 5$.

Answer

$$\int \sec^5 x\,dx = \frac{1}{4}\sec^3 x \tan x + \frac{3}{4}\int \sec^3 x$$

Key Concepts

Integrals of trigonometric functions can be evaluated by the use of various strategies. These strategies include

1. Applying trigonometric identities to rewrite the integral so that it may be evaluated by u-substitution
2. Using integration by parts
3. Applying trigonometric identities to rewrite products of sines and cosines with different arguments as the sum of individual sine and cosine functions
4. Applying reduction formulas

Key Equations

To integrate products involving $\sin(ax)$, $\sin(bx)$, $\cos(ax)$,and $\cos(bx)$, use the substitutions.

- **Sine Products**

  $$\sin(ax)\sin(bx) = \tfrac{1}{2}\cos((a-b)x) - \tfrac{1}{2}\cos((a+b)x)$$

- **Sine and Cosine Products**

  $$\sin(ax)\cos(bx) = \tfrac{1}{2}\sin((a-b)x) + \tfrac{1}{2}\sin((a+b)x)$$

- **Cosine Products**

  $$\cos(ax)\cos(bx) = \tfrac{1}{2}\cos((a-b)x) + \tfrac{1}{2}\cos((a+b)x)$$

- **Power Reduction Formula**

  $$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2} x\tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$$

- **Power Reduction Formula**

  $$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx$$

Glossary

power reduction formula
a rule that allows an integral of a power of a trigonometric function to be exchanged for an integral involving a lower power

trigonometric integral
an integral involving powers and products of trigonometric functions

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7.3: Trigonometric Substitution

Learning Objectives

- Solve integration problems involving the square root of a sum or difference of two squares.

In this section, we explore integrals containing expressions of the form $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, and $\sqrt{x^2-a^2}$, where the values of a are positive. We have already encountered and evaluated integrals containing some expressions of this type, but many still remain inaccessible. The technique of *trigonometric substitution* comes in very handy when evaluating these integrals. This technique uses substitution to rewrite these integrals as trigonometric integrals.

Integrals Involving $\sqrt{a^2-x^2}$

Before developing a general strategy for integrals containing $\sqrt{a^2-x^2}$, consider the integral $\int \sqrt{9-x^2}\,dx$. This integral cannot be evaluated using any of the techniques we have discussed so far. However, if we make the substitution $x=3\sin\theta$, we have $dx=3\cos\theta\,d\theta$. After substituting into the integral, we have

$$\int \sqrt{9-x^2}\,dx=\int \sqrt{9-(3\sin\theta)^2}\cdot 3\cos\theta\,d\theta.$$

After simplifying, we have

$$\int \sqrt{9-x^2}\,dx=\int 9\sqrt{1-\sin^2\theta}\cdot\cos\theta\,d\theta.$$

Letting $1-\sin^2\theta=\cos^2\theta$, we now have

$$\int \sqrt{9-x^2}\,dx=\int 9\sqrt{\cos^2\theta}\cos\theta\,d\theta.$$

Assuming that $\cos\theta\geq 0$, we have

$$\int \sqrt{9-x^2}\,dx=\int 9\cos^2\theta\,d\theta.$$

At this point, we can evaluate the integral using the techniques developed for integrating powers and products of trigonometric functions. Before completing this example, let's take a look at the general theory behind this idea.

To evaluate integrals involving $\sqrt{a^2-x^2}$, we make the substitution $x=a\sin\theta$ and $dx=a\cos\theta$. To see that this actually makes sense, consider the following argument: The domain of $\sqrt{a^2-x^2}$ is $[-a,a]$. Thus,

$$-a\leq x\leq a.$$

Consequently,

$$-1\leq\frac{x}{a}\leq 1.$$

Since the range of $\sin x$ over $[-(\pi/2),\pi/2]$is $[-1,1]$, there is a unique angle θ satisfying $-(\pi/2)\leq\theta\leq\pi/2$ so that $\sin\theta=x/a$, or equivalently, so that $x=a\sin\theta$. If we substitute $x=a\sin\theta$ into $\sqrt{a^2-x^2}$, we get

$$\begin{aligned}
\sqrt{a^2-x^2} &= \sqrt{a^2-(a\sin\theta)^2} && \text{Let } x = a\sin\theta \text{ where } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.\\
& && \text{Simplify.}\\
&= \sqrt{a^2-a^2\sin^2\theta} && \text{Factor out } a^2.\\
&= \sqrt{a^2(1-\sin^2\theta)} && \text{Substitute } 1-\sin^2 x = \cos^2 x.\\
&= \sqrt{a^2\cos^2\theta} && \text{Take the square root.}\\
&= |a\cos\theta|\\
&= a\cos\theta
\end{aligned}$$

Since $\cos x \ge 0$ on $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $a > 0$, $|a\cos\theta| = a\cos\theta$. We can see, from this discussion, that by making the substitution $x = a\sin\theta$, we are able to convert an integral involving a radical into an integral involving trigonometric functions. After we evaluate the integral, we can convert the solution back to an expression involving x. To see how to do this, let's begin by assuming that $0 < x < a$. In this case, $0 < \theta < \frac{\pi}{2}$. Since $\sin\theta = \frac{x}{a}$, we can draw the reference triangle in Figure 7.3.1 to assist in expressing the values of $\cos\theta$, $\tan\theta$, and the remaining trigonometric functions in terms of x. It can be shown that this triangle actually produces the correct values of the trigonometric functions evaluated at θ for all θ satisfying $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. It is useful to observe that the expression $\sqrt{a^2-x^2}$ actually appears as the length of one side of the triangle. Last, should θ appear by itself, we use $\theta = \sin^{-1}\left(\frac{x}{a}\right)$.

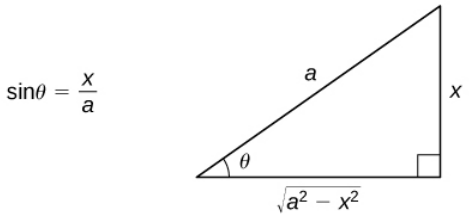


Figure 7.3.1: A reference triangle can help express the trigonometric functions evaluated at θ in terms of x.

The essential part of this discussion is summarized in the following problem-solving strategy.

Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2-x^2}$

1. It is a good idea to make sure the integral cannot be evaluated easily in another way. For example, although this method can be applied to integrals of the form $\int \frac{1}{\sqrt{a^2-x^2}}\,dx$, $\int \frac{x}{\sqrt{a^2-x^2}}\,dx$, and $\int x\sqrt{a^2-x^2}\,dx$, they can each be integrated directly either by formula or by a simple u-substitution.
2. Make the substitution $x = a\sin\theta$ and $dx = a\cos\theta\,d\theta$. Note: This substitution yields $\sqrt{a^2-x^2} = a\cos\theta$.
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangle from Figure 1 to rewrite the result in terms of x. You may also need to use some trigonometric identities and the relationship $\theta = \sin^{-1}\left(\frac{x}{a}\right)$.

The following example demonstrates the application of this problem-solving strategy.

Example 7.3.1: Integrating an Expression Involving $\sqrt{a^2-x^2}$

Evaluate

$$\int \sqrt{9-x^2}\,dx.$$

Solution

Begin by making the substitutions $x = 3\sin\theta$ and $dx = 3\cos\theta\, d\theta$. Since $\sin\theta = \dfrac{x}{3}$, we can construct the reference triangle shown in Figure 2.

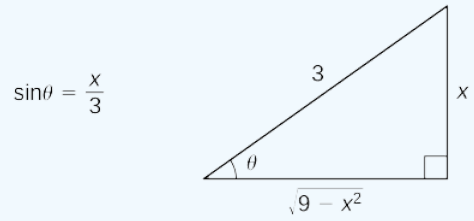


Figure 7.3.2: A reference triangle can be constructed for Example 7.3.1.

Thus,

$$\begin{aligned}
\int \sqrt{9-x^2}\, dx &= \int \sqrt{9-(3\sin\theta)^2}\, 3\cos\theta\, d\theta \quad \text{Substitute } x = 3\sin\theta \text{ and } dx = 3\cos\theta\, d\theta. \\
&= \int \sqrt{9(1-\sin^2\theta)} \cdot 3\cos\theta\, d\theta \quad \text{Simplify.} \\
&= \int \sqrt{9\cos^2\theta} \cdot 3\cos\theta\, d\theta \quad \text{Substitute } \cos^2\theta = 1-\sin^2\theta\,. \\
&= \int 3|\cos\theta| 3\cos\theta\, d\theta \quad \text{Take the square root.} \\
&= \int 9\cos^2\theta\, d\theta \quad \text{Simplify. Since } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \cos\theta \ge 0 \text{ and } |\cos\theta| = \cos\theta. \\
&= \int 9\left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right) d\theta \quad \text{Use the strategy for integrating an even power of } \cos\theta. \\
&= \frac{9}{2}\theta + \frac{9}{4}\sin(2\theta) + C \quad \text{Evaluate the integral.} \\
&= \frac{9}{2}\theta + \frac{9}{4}(2\sin\theta\cos\theta) + C \quad \text{Substitute } \sin(2\theta) = 2\sin\theta\cos\theta\,. \\
&= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} + C \quad \text{Substitute } \sin^{-1}\left(\frac{x}{3}\right) = \theta \text{ and } \sin\theta = \tfrac{x}{3}\,. \\
&\qquad \text{Use the reference triangle to see that } \cos\theta = \frac{\sqrt{9-x^2}}{3} \text{ and make this substitution. Simplify.} \\
&= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{x\sqrt{9-x^2}}{2} + C. \quad \text{Simplify.}
\end{aligned}$$

Example 7.3.2: Integrating an Expression Involving $\sqrt{a^2 - x^2}$

Evaluate

$$\int \frac{\sqrt{4-x^2}}{x}\, dx.$$

Solution

First make the substitutions $x = 2\sin\theta$ and $dx = 2\cos\theta\, d\theta$. Since $\sin\theta = \dfrac{x}{2}$, we can construct the reference triangle shown in Figure 7.3.3.

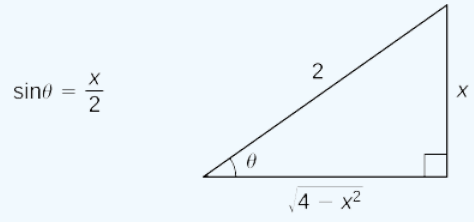


Figure 7.3.3: A reference triangle can be constructed for Example 7.3.2.

Thus,

$$\int \frac{\sqrt{4-x^2}}{x}dx = \int \frac{\sqrt{4-(2\sin\theta)^2}}{2\sin\theta}2\cos\theta\,d\theta \text{ Substitute } x=2\sin\theta \text{ and } dx=2\cos\theta\,d\theta.$$

$$= \int \frac{2\cos^2\theta}{\sin\theta}\,d\theta \text{ Substitute } \cos^2\theta = 1-\sin^2\theta \text{ and simplify.}$$

$$= \int \frac{2(1-\sin^2\theta)}{\sin\theta}\,d\theta \text{ Substitute } \cos^2\theta = 1-\sin^2\theta\,.$$

$$= \int (2\csc\theta - 2\sin\theta)\,d\theta \text{ Separate the numerator, simplify, and use } \csc\theta = \frac{1}{\sin\theta}\,.$$

$$= 2\ln|\csc\theta - \cot\theta| + 2\cos\theta + C \text{ Evaluate the integral.}$$

$$= 2\ln\left|\frac{2}{x} - \frac{\sqrt{4-x^2}}{x}\right| + \sqrt{4-x^2} + C. \text{ Use the reference triangle to rewrite the expression in terms of } x \text{ and simplify.}$$

In the next example, we see that we sometimes have a choice of methods.

Example 7.3.3: Integrating an Expression Involving $\sqrt{a^2-x^2}$ Two Ways

Evaluate $\int x^3\sqrt{1-x^2}\,dx$ two ways: first by using the substitution $u=1-x^2$ and then by using a trigonometric substitution.

Method 1

Let $u=1-x^2$ and hence $x^2=1-u$. Thus, $du=-2x\,dx$. In this case, the integral becomes

$$\int x^3\sqrt{1-x^2}\,dx = -\frac{1}{2}\int x^2\sqrt{1-x^2}\,(-2x\,dx) \text{ Make the substitution.}$$

$$= -\frac{1}{2}\int (1-u)\sqrt{u}\,du \text{ Expand the expression.}$$

$$= -\frac{1}{2}\int (u^{1/2}-u^{3/2})\,du \text{ Evaluate the integral.}$$

$$= -\frac{1}{2}\left(\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right) + C \text{ Rewrite in terms of x.}$$

$$= -\frac{1}{3}(1-x^2)^{3/2} + \frac{1}{5}(1-x^2)^{5/2} + C.$$

Method 2

Let $x=\sin\theta$. In this case, $dx=\cos\theta\,d\theta$. Using this substitution, we have

$$\int x^3\sqrt{1-x^2}\,dx = \int \sin^3\theta\cos^2\theta\,d\theta$$

$$= \int (1-\cos^2\theta)\cos^2\theta\sin\theta\,d\theta \text{ Let } u=\cos\theta. \text{ Thus,} du=-\sin\theta\,d\theta.$$

$$= \int (u^4-u^2)\,du$$

$$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C \text{ Substitute } \cos\theta = u.$$

$$= \frac{1}{5}\cos^5\theta - \frac{1}{3}\cos^3\theta + C \text{ Use a reference triangle to see that } \cos\theta = \sqrt{1-x^2}\,.$$

$$= \frac{1}{5}(1-x^2)^{5/2} - \frac{1}{3}(1-x^2)^{3/2} + C.$$

Exercise 7.3.1

Rewrite the integral $\displaystyle\int \frac{x^3}{\sqrt{25-x^2}}\,dx$ using the appropriate trigonometric substitution (do not evaluate the integral).

Hint

Substitute $x = 5\sin\theta$ and $dx = 5\cos\theta\, d\theta$.

Answer

$$\int 125\sin^3\theta\, d\theta$$

Integrating Expressions Involving $\sqrt{a^2 + x^2}$

For integrals containing $\sqrt{a^2+x^2}$,let's first consider the domain of this expression. Since $\sqrt{a^2+x^2}$ is defined for all real values of x, we restrict our choice to those trigonometric functions that have a range of all real numbers. Thus, our choice is restricted to selecting either $x = a\tan\theta$ or $x = a\cot\theta$. Either of these substitutions would actually work, but the standard substitution is $x = a\tan\theta$ or, equivalently, $\tan\theta = x/a$. With this substitution, we make the assumption that $-(\pi/2) < \theta < \pi/2$, so that we also have $\theta = \tan^{-1}(x/a)$. The procedure for using this substitution is outlined in the following problem-solving strategy.

Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2 + x^2}$

1. Check to see whether the integral can be evaluated easily by using another method. In some cases, it is more convenient to use an alternative method.
2. Substitute $x = a\tan\theta$ and $dx = a\sec^2\theta\, d\theta$. This substitution yields $\sqrt{a^2+x^2} = \sqrt{a^2+(a\tan\theta)^2} = \sqrt{a^2(1+\tan^2\theta)} = \sqrt{a^2 sec^2\theta} = |a\sec\theta| = a\sec\theta$. (Since $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$ and $\sec\theta > 0$ over this interval, $|a\sec\theta| = a\sec\theta$.)
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangle from Figure 7.3.4 to rewrite the result in terms of x. You may also need to use some trigonometric identities and the relationship $\theta = \tan^{-1}\left(\dfrac{x}{a}\right)$. (Note: The reference triangle is based on the assumption that $x > 0$; however, the trigonometric ratios produced from the reference triangle are the same as the ratios for which $x \le 0$.)

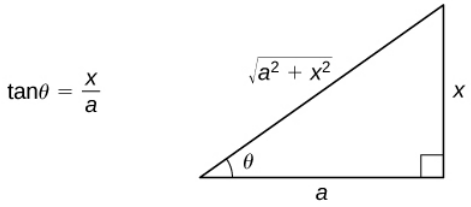


Figure 7.3.4: A reference triangle can be constructed to express the trigonometric functions evaluated at θ in terms of x.

Example 7.3.4: Integrating an Expression Involving $\sqrt{a^2 + x^2}$

Evaluate $\displaystyle\int \frac{dx}{\sqrt{1+x^2}}$ and check the solution by differentiating.

Solution

Begin with the substitution $x = \tan\theta$ and $dx = sec^2\theta\, d\theta$. Since $\tan\theta = x$, draw the reference triangle in Figure 7.3.5.

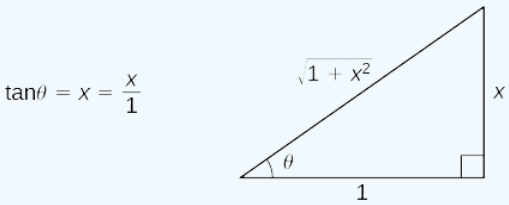


Figure 7.3.5: The reference triangle for Example 7.3.4.

Thus,

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\sec^2\theta}{\sec\theta}\,d\theta \qquad \text{Substitute } x=\tan\theta \text{ and } dx=\sec^2\theta\,d\theta.$$

$$\text{This substitution makes } \sqrt{1+x^2}=\sec\theta. \text{ Simplify.}$$

$$= \int \sec\theta\,d\theta \qquad \text{Evaluate the integral.}$$

$$= \ln|\sec\theta+\tan\theta|+C \qquad \text{Use the reference triangle to express the result in terms of } x.$$

$$= \ln|\sqrt{1+x^2}+x|+C$$

To check the solution, differentiate:

$$\frac{d}{dx}\left(\ln|\sqrt{1+x^2}+x|\right) = \frac{1}{\sqrt{1+x^2}+x}\cdot\left(\frac{x}{\sqrt{1+x^2}}+1\right) = \frac{1}{\sqrt{1+x^2}+x}\cdot\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}}.$$

Since $\sqrt{1+x^2}+x>0$ for all values of x, we could rewrite $\ln|\sqrt{1+x^2}+x|+C=\ln(\sqrt{1+x^2}+x)+C$, if desired.

Example 7.3.5: Evaluating $\int \frac{dx}{\sqrt{1+x^2}}$ Using a Different Substitution

Use the substitution $x=\sinh\theta$ to evaluate $\int \frac{dx}{\sqrt{1+x^2}}$.

Solution

Because $\sinh\theta$ has a range of all real numbers, and $1+\sinh^2\theta=\cosh^2\theta$, we may also use the substitution $x=\sinh\theta$ to evaluate this integral. In this case, $dx=\cosh\theta\,d\theta$. Consequently,

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\cosh\theta}{\sqrt{1+\sinh^2\theta}}\,d\theta \qquad \text{Substitute } x=\sinh\theta \text{ and } dx=\cosh\theta\,d\theta.$$

$$\text{Substitute } 1+\sinh^2\theta=\cosh^2\theta.$$

$$= \int \frac{\cosh\theta}{\sqrt{\cosh^2\theta}}\,d\theta \qquad \text{Since } \sqrt{\cosh^2\theta}=|\cosh\theta|$$

$$= \int \frac{\cosh\theta}{|\cosh\theta|}\,d\theta \qquad |\cosh\theta|=\cosh\theta \text{ since } \cosh\theta>0 \text{ for all } \theta.$$

$$= \int \frac{\cosh\theta}{\cosh\theta}\,d\theta \qquad \text{Simplify.}$$

$$= \int 1\,d\theta \qquad \text{Evaluate the integral.}$$

$$= \theta+C \qquad \text{Since } x=\sinh\theta, \text{ we know } \theta=\sinh^{-1}x.$$

$$= \sinh^{-1}x+C.$$

Analysis

This answer looks quite different from the answer obtained using the substitution $x=\tan\theta$. To see that the solutions are the same, set $y=\sinh^{-1}x$. Thus, $\sinh y=x$. From this equation we obtain:

$$\frac{e^y-e^{-y}}{2}=x.$$

After multiplying both sides by $2e^y$ and rewriting, this equation becomes:

$$e^{2y}-2xe^y-1=0.$$

Use the quadratic equation to solve for e^y:

$$e^y=\frac{2x\pm\sqrt{4x^2+4}}{2}.$$

Simplifying, we have:

$$e^y = x \pm \sqrt{x^2+1}.$$

Since $x - \sqrt{x^2+1} < 0$, it must be the case that $e^y = x + \sqrt{x^2+1}$. Thus,

$$y = \ln(x + \sqrt{x^2+1}).$$

Last, we obtain

$$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}).$$

After we make the final observation that, since $x + \sqrt{x^2+1} > 0$,

$$\ln(x + \sqrt{x^2+1}) = \ln|\sqrt{1+x^2} + x|,$$

we see that the two different methods produced equivalent solutions.

Example 7.3.6: Finding an Arc Length

Find the length of the curve $y = x^2$ over the interval $[0, \frac{1}{2}]$.

Solution

Because $\frac{dy}{dx} = 2x$, the arc length is given by

$$\int_0^{1/2} \sqrt{1+(2x)^2}\,dx = \int_0^{1/2} \sqrt{1+4x^2}\,dx.$$

To evaluate this integral, use the substitution $x = \frac{1}{2}\tan\theta$ and $dx = \frac{1}{2}\sec^2\theta\,d\theta$. We also need to change the limits of integration. If $x = 0$, then $\theta = 0$ and if $x = \frac{1}{2}$, then $\theta = \frac{\pi}{4}$. Thus,

$\int_0^{1/2} \sqrt{1+4x^2}\,dx = \int_0^{\pi/4} \sqrt{1+\tan^2\theta} \cdot \frac{1}{2}\sec^2\theta\,d\theta$ After substitution, $\sqrt{1+4x^2} = \sec\theta$. (Substitute $1+\tan^2\theta = \sec^2\theta$ and simplify.)

$= \frac{1}{2}\int_0^{\pi/4} \sec^3\theta\,d\theta$ We derived this integral in the previous section.

$= \frac{1}{2}\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta|\right)\Big|_0^{\pi/4}$ Evaluate and simplify.

$= \frac{1}{4}(\sqrt{2} + \ln(\sqrt{2}+1)).$

Exercise 7.3.2

Rewrite $\int x^3\sqrt{x^2+4}\,dx$ by using a substitution involving $\tan\theta$.

Hint

Use $x = 2\tan\theta$ and $dx = 2\sec^2\theta\,d\theta$.

Answer

$$\int 32\tan^3\theta\sec^3\theta\,d\theta$$

Integrating Expressions Involving $\sqrt{x^2 - a^2}$

The domain of the expression $\sqrt{x^2 - a^2}$ is $(-\infty, -a] \cup [a, +\infty)$. Thus, either $x \leq -a$ or $x \geq a$. Hence, $\frac{x}{a} \leq -1$ or $\frac{x}{a} \geq 1$. Since these intervals correspond to the range of $\sec\theta$ on the set $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$, it makes sense to use the substitution $\sec\theta = \frac{x}{a}$ or, equivalently, $x = a\sec\theta$, where $0 \leq \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \leq \pi$. The corresponding substitution for dx is $dx = a\sec\theta\tan\theta\, d\theta$. The procedure for using this substitution is outlined in the following problem-solving strategy.

Problem-Solving Strategy: Integrals Involving $\sqrt{x^2 - a^2}$

1. Check to see whether the integral cannot be evaluated using another method. If so, we may wish to consider applying an alternative technique.
2. Substitute $x = a\sec\theta$ and $dx = a\sec\theta\tan\theta\, d\theta$. This substitution yields

$$\sqrt{x^2 - a^2} = \sqrt{(a\sec\theta)^2 - a^2} = \sqrt{a^2(\sec^2\theta - 1)} = \sqrt{a^2\tan^2\theta} = |a\tan\theta|.$$

   For $x \geq a$, $|a\tan\theta| = a\tan\theta$ and for $x \leq -a$, $|a\tan\theta| = -a\tan\theta$.
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangles from Figure 7.3.6 to rewrite the result in terms of x.
6. You may also need to use some trigonometric identities and the relationship $\theta = \sec^{-1}\left(\frac{x}{a}\right)$. (Note: We need both reference triangles, since the values of some of the trigonometric ratios are different depending on whether $x > a$ or $x < -a$.)

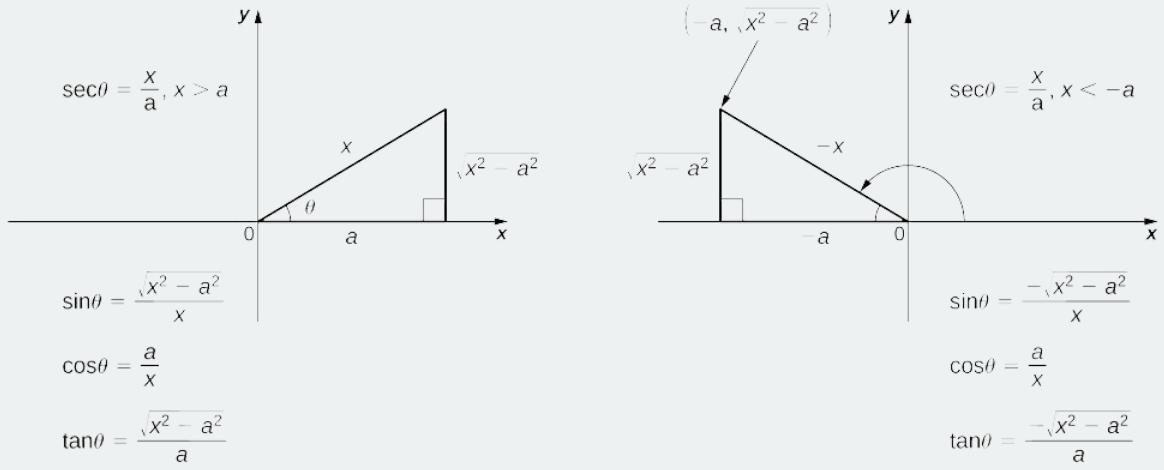


Figure 7.3.6: Use the appropriate reference triangle to express the trigonometric functions evaluated at θ in terms of x.

Example 7.3.7: Finding the Area of a Region

Find the area of the region between the graph of $f(x) = \sqrt{x^2 - 9}$ and the x-axis over the interval $[3, 5]$.

Solution

First, sketch a rough graph of the region described in the problem, as shown in the following figure.

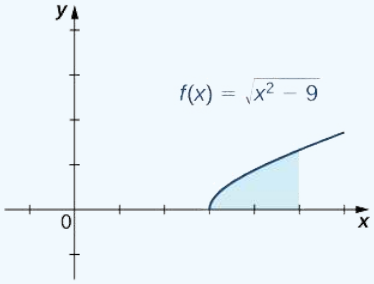


Figure 7.3.7: Calculating the area of the shaded region requires evaluating an integral with a trigonometric substitution.

We can see that the area is $A = \int_3^5 \sqrt{x^2-9}\,dx$. To evaluate this definite integral, substitute $x = 3\sec\theta$ and $dx = 3\sec\theta\tan\theta\,d\theta$. We must also change the limits of integration. If $x = 3$, then $3 = 3\sec\theta$ and hence $\theta = 0$. If $x = 5$, then $\theta = \sec^{-1}(\frac{5}{3})$. After making these substitutions and simplifying, we have

$$\begin{aligned}
\textbf{Area} &= \int_3^5 \sqrt{x^2-9}\,dx \\
&= \int_0^{\sec^{-1}(5/3)} 9\tan^2\theta\sec\theta\,d\theta \quad \text{Use } \tan^2\theta = \sec^2\theta - 1. \\
&= \int_0^{\sec^{-1}(5/3)} 9(\sec^2\theta - 1)\sec\theta\,d\theta \quad \text{Expand.} \\
&= \int_0^{\sec^{-1}(5/3)} 9(\sec^3\theta - \sec\theta)\,d\theta \quad \text{Evaluate the integral.} \\
&= \left(\frac{9}{2}\ln|\sec\theta+\tan\theta| + \frac{9}{2}\sec\theta\tan\theta\right) - 9\ln|\sec\theta+\tan\theta|\Big|_0^{\sec^{-1}(5/3)} \quad \text{Simplify.} \\
&= \frac{9}{2}\sec\theta\tan\theta - \frac{9}{2}\ln|\sec\theta+\tan\theta|\Big|_0^{\sec^{-1}(5/3)} \quad \text{Evaluate. Use } \sec(\sec^{-1}\tfrac{5}{3}) = \tfrac{5}{3} \text{ and } \tan(\sec^{-1}\tfrac{5}{3}) = \tfrac{4}{3}. \\
&= \frac{9}{2}\cdot\frac{5}{3}\cdot\frac{4}{3} - \frac{9}{2}\ln\left|\frac{5}{3}+\frac{4}{3}\right| - \left(\frac{9}{2}\cdot 1\cdot 0 - \frac{9}{2}\ln|1+0|\right) \\
&= 10 - \frac{9}{2}\ln 3
\end{aligned}$$

Exercise 7.3.3

Evaluate

$$\int \frac{dx}{\sqrt{x^2-4}}.$$

Assume that $x > 2$.

Hint

Substitute $x = 2\sec\theta$ and $dx = 2\sec\theta\tan\theta\,d\theta$.

Answer

$$\ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + C$$

Key Concepts

- For integrals involving $\sqrt{a^2-x^2}$, use the substitution $x = a\sin\theta$ and $dx = a\cos\theta\,d\theta$.
- For integrals involving $\sqrt{a^2+x^2}$, use the substitution $x = a\tan\theta$ and $dx = a\sec^2\theta\,d\theta$.
- For integrals involving $\sqrt{x^2-a^2}$, substitute $x = a\sec\theta$ and $dx = a\sec\theta\tan\theta\,d\theta$.

Glossary

trigonometric substitution

an integration technique that converts an algebraic integral containing expressions of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$ into a trigonometric integral

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7.4: Integration of Rational Functions by Partial Fractions

Learning Objectives

- Integrate a rational function using the method of partial fractions.
- Recognize simple linear factors in a rational function.
- Recognize repeated linear factors in a rational function.
- Recognize quadratic factors in a rational function.

We have seen some techniques that allow us to integrate specific rational functions. For example, we know that

$$\int \frac{du}{u} = \ln|u| + C$$

and

$$\int \frac{du}{u^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C.$$

However, we do not yet have a technique that allows us to tackle arbitrary quotients of this type. Thus, it is not immediately obvious how to go about evaluating

$$\int \frac{3x}{x^2 - x - 2}\, dx.$$

However, we know from material previously developed that

$$\int \left(\frac{1}{x+1} + \frac{2}{x-2}\right) dx = \ln|x+1| + 2\ln|x-2| + C.$$

In fact, by getting a common denominator, we see that

$$\frac{1}{x+1} + \frac{2}{x-2} = \frac{3x}{x^2 - x - 2}.$$

Consequently,

$$\int \frac{3x}{x^2 - x - 2}\, dx = \int \left(\frac{1}{x+1} + \frac{2}{x-2}\right) dx.$$

In this section, we examine the method of **partial fraction decomposition**, which allows us to decompose **rational functions** into sums of simpler, more easily integrated rational functions. Using this method, we can rewrite an expression such as:

$$\frac{3x}{x^2 - x - 2}$$

as an expression such as

$$\frac{1}{x+1} + \frac{2}{x-2}.$$

The key to the method of partial fraction decomposition is being able to anticipate the form that the decomposition of a rational function will take. As we shall see, this form is both predictable and highly dependent on the factorization of the denominator of the rational function. It is also extremely important to keep in mind that partial fraction decomposition can be applied to a rational function $\frac{P(x)}{Q(x)}$ only if $deg(P(x)) < deg(Q(x))$. In the case when $deg(P(x)) \geq deg(Q(x))$, we must first perform long division to rewrite the quotient $\frac{P(x)}{Q(x)}$ in the form $A(x) + \frac{R(x)}{Q(x)}$, where $deg(R(x)) < deg(Q(x))$. We then do a partial fraction

decomposition on $\frac{R(x)}{Q(x)}$. The following example, although not requiring partial fraction decomposition, illustrates our approach to integrals of rational functions of the form $\int \frac{P(x)}{Q(x)}\,dx$, where $deg(P(x)) \geq deg(Q(x))$.

Example 7.4.1: Integrating $\int \frac{P(x)}{Q(x)}\,dx$, where $deg(P(x)) \geq deg(Q(x))$

Evaluate

$$\int \frac{x^2+3x+5}{x+1}\,dx.$$

Solution

Since $deg(x^2+3x+5) \geq deg(x+1)$, we perform long division to obtain

$$\frac{x^2+3x+5}{x+1} = x+2+\frac{3}{x+1}.$$

Thus,

$$\int \frac{x^2+3x+5}{x+1}\,dx = \int \left(x+2+\frac{3}{x+1}\right)\,dx = \frac{1}{2}x^2+2x+3\ln|x+1|+C.$$

Visit this website for a review of long division of polynomials.

Exercise 7.4.1

Evaluate

$$\int \frac{x-3}{x+2}\,dx.$$

Hint

Use long division to obtain $\frac{x-3}{x+2} = 1-\frac{5}{x+2}$.

Answer

$$x-5\ln|x+2|+C$$

To integrate $\int \frac{P(x)}{Q(x)}\,dx$, where $deg(P(x)) < deg(Q(x))$, we must begin by factoring $Q(x)$.

Nonrepeated Linear Factors

If $Q(x)$ can be factored as $(a_1x+b_1)(a_2x+b_2)\ldots(a_nx+b_n)$, where each linear factor is distinct, then it is possible to find constants $A_1, A_2, \ldots A_n$ satisfying

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1}+\frac{A_2}{a_2x+b_2}+\cdots+\frac{A_n}{a_nx+b_n}. \tag{7.4.1}$$

The proof that such constants exist is beyond the scope of this course.

In this next example, we see how to use partial fractions to integrate a rational function of this type.

Example 7.4.2: Partial Fractions with Nonrepeated Linear Factors

Evaluate $\int \frac{3x+2}{x^3-x^2-2x}\,dx$.

Solution

Since $deg(3x+2) < deg(x^3-x^2-2x)$, we begin by factoring the denominator of $\frac{3x+2}{x^3-x^2-2x}$. We can see that $x^3-x^2-2x = x(x-2)(x+1)$. Thus, there are constants A, B, and C satisfying Equation 7.4.1 such that

$$\frac{3x+2}{x(x-2)(x+1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1}.$$

We must now find these constants. To do so, we begin by getting a common denominator on the right. Thus,

$$\frac{3x+2}{x(x-2)(x+1)} = \frac{A(x-2)(x+1)+Bx(x+1)+Cx(x-2)}{x(x-2)(x+1)}.$$

Now, we set the numerators equal to each other, obtaining

$$3x+2 = A(x-2)(x+1)+Bx(x+1)+Cx(x-2). \tag{7.4.2}$$

There are two different strategies for finding the coefficients A, B, and C. We refer to these as the *method of equating coefficients* and the *method of strategic substitution*.

Strategy one: Method of Equating Coefficients

Rewrite Equation 7.4.2 in the form

$$3x+2 = (A+B+C)x^2+(-A+B-2C)x+(-2A).$$

Equating coefficients produces the system of equations

$$\begin{aligned} A+B+C &= 0 \\ -A+B-2C &= 3 \\ -2A &= 2. \end{aligned}$$

To solve this system, we first observe that $-2A = 2 \Rightarrow A = -1$. Substituting this value into the first two equations gives us the system

$$\begin{aligned} B+C &= 1 \\ B-2C &= 2\,. \end{aligned}$$

Multiplying the second equation by -1 and adding the resulting equation to the first produces

$$-3C = 1,$$

which in turn implies that $C = -\frac{1}{3}$. Substituting this value into the equation $B+C=1$ yields $B = \frac{4}{3}$. Thus, solving these equations yields $A = -1, B = \frac{4}{3}$, and $C = -\frac{1}{3}$.

It is important to note that the system produced by this method is consistent if and only if we have set up the decomposition correctly. If the system is inconsistent, there is an error in our decomposition.

Strategy two: Method of Strategic Substitution

The method of strategic substitution is based on the assumption that we have set up the decomposition correctly. If the decomposition is set up correctly, then there must be values of A, B, and C that satisfy Equation 7.4.2 for all values of x. That is, this equation must be true for any value of x we care to substitute into it. Therefore, by choosing values of x carefully and substituting them into the equation, we may find A, B, and C easily. For example, if we substitute $x = 0$, the equation reduces to $2 = A(-2)(1)$. Solving for A yields $A = -1$. Next, by substituting $x = 2$, the equation reduces to $8 = B(2)(3)$, or

equivalently $B=4/3$. Last, we substitute $x=-1$ into the equation and obtain $-1=C(-1)(-3)$. Solving, we have $C=-\frac{1}{3}$.

It is important to keep in mind that if we attempt to use this method with a decomposition that has not been set up correctly, we are still able to find values for the constants, but these constants are meaningless. If we do opt to use the method of strategic substitution, then it is a good idea to check the result by recombining the terms algebraically.

Now that we have the values of A, B, and C, we rewrite the original integral:

$$\int \frac{3x+2}{x^3-x^2-2x}\,dx=\int\left(-\frac{1}{x}+\frac{4}{3}\cdot\frac{1}{x-2}-\frac{1}{3}\cdot\frac{1}{x+1}\right)dx.$$

Evaluating the integral gives us

$$\int \frac{3x+2}{x^3-x^2-2x}\,dx=-\ln|x|+\frac{4}{3}\ln|x-2|-\frac{1}{3}\ln|x+1|+C.$$

In the next example, we integrate a rational function in which the degree of the numerator is not less than the degree of the denominator.

Example 7.4.3: Dividing before Applying Partial Fractions

Evaluate $\int \frac{x^2+3x+1}{x^2-4}\,dx$.

Solution

Since $deg(x^2+3x+1)\geq deg(x^2-4)$, we must perform long division of polynomials. This results in

$$\frac{x^2+3x+1}{x^2-4}=1+\frac{3x+5}{x^2-4}$$

Next, we perform partial fraction decomposition on $\frac{3x+5}{x^2-4}=\frac{3x+5}{(x+2)(x-2)}$. We have

$$\frac{3x+5}{(x-2)(x+2)}=\frac{A}{x-2}+\frac{B}{x+2}.$$

Thus,

$$3x+5=A(x+2)+B(x-2).$$

Solving for A and B using either method, we obtain $A=11/4$ and $B=1/4$.

Rewriting the original integral, we have

$$\int \frac{x^2+3x+1}{x^2-4}\,dx=\int\left(1+\frac{11}{4}\cdot\frac{1}{x-2}+\frac{1}{4}\cdot\frac{1}{x+2}\right)dx.$$

Evaluating the integral produces

$$\int \frac{x^2+3x+1}{x^2-4}\,dx=x+\frac{11}{4}\ln|x-2|+\frac{1}{4}\ln|x+2|+C.$$

As we see in the next example, it may be possible to apply the technique of partial fraction decomposition to a nonrational function. The trick is to convert the nonrational function to a rational function through a substitution.

Example 7.4.4: Applying Partial Fractions after a Substitution

Evaluate $\int \frac{\cos x}{\sin^2 x-\sin x}\,dx$.

Solution

Let's begin by letting $u = \sin x$. Consequently, $du = \cos x\, dx$. After making these substitutions, we have

$$\int \frac{\cos x}{\sin^2 x - \sin x}\, dx = \int \frac{du}{u^2 - u} = \int \frac{du}{u(u-1)}.$$

Applying partial fraction decomposition to $\dfrac{1}{u(u-1)}$ gives $\dfrac{1}{u(u-1)} = -\dfrac{1}{u} + \dfrac{1}{u-1}$.

Thus,

$$\int \frac{\cos x}{\sin^2 x - \sin x}\, dx = -\ln|u| + \ln|u-1| + C = -\ln|\sin x| + \ln|\sin x - 1| + C.$$

? Exercise 7.4.2

Evaluate $\displaystyle\int \frac{x+1}{(x+3)(x-2)}\, dx$.

Hint

$$\frac{x+1}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$$

Answer

$$\frac{2}{5}\ln|x+3| + \frac{3}{5}\ln|x-2| + C$$

Repeated Linear Factors

For some applications, we need to integrate rational expressions that have denominators with repeated linear factors—that is, rational functions with at least one factor of the form $(ax+b)^n$, where n is a positive integer greater than or equal to 2. If the denominator contains the repeated linear factor $(ax+b)^n$, then the decomposition must contain

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}. \tag{7.4.3}$$

As we see in our next example, the basic technique used for solving for the coefficients is the same, but it requires more algebra to determine the numerators of the partial fractions.

✓ Example 7.4.5: Partial Fractions with Repeated Linear Factors

Evaluate $\displaystyle\int \frac{x-2}{(2x-1)^2(x-1)}\, dx$.

Solution

We have $deg(x-2) < deg((2x-1)^2(x-1))$, so we can proceed with the decomposition. Since $(2x-1)^2$ is a repeated linear factor, include

$$\frac{A}{2x-1} + \frac{B}{(2x-1)^2}$$

in the decomposition in Equation 7.4.3. Thus,

$$\frac{x-2}{(2x-1)^2(x-1)} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} + \frac{C}{x-1}.$$

After getting a common denominator and equating the numerators, we have

$$x - 2 = A(2x-1)(x-1) + B(x-1) + C(2x-1)^2. \tag{7.4.4}$$

We then use the method of equating coefficients to find the values of A, B, and C.

$$x-2=(2A+4C)x^2+(-3A+B-4C)x+(A-B+C).$$

Equating coefficients yields $2A+4C=0$, $-3A+B-4C=1$, and $A-B+C=-2$. Solving this system yields $A=2, B=3,$ and $C=-1$.

Alternatively, we can use the method of strategic substitution. In this case, substituting $x=1$ and $x=1/2$ into Equation 7.4.4 easily produces the values $B=3$ and $C=-1$. At this point, it may seem that we have run out of good choices for x, however, since we already have values for B and C, we can substitute in these values and choose any value for x not previously used. The value $x=0$ is a good option. In this case, we obtain the equation $-2=A(-1)(-1)+3(-1)+(-1)(-1)^2$ or, equivalently, $A=2$.

Now that we have the values for A, B, and C, we rewrite the original integral and evaluate it:

$$\begin{aligned}\int \frac{x-2}{(2x-1)^2(x-1)}\,dx &= \int\left(\frac{2}{2x-1}+\frac{3}{(2x-1)^2}-\frac{1}{x-1}\right)dx\\ &= \ln|2x-1|-\frac{3}{2(2x-1)}-\ln|x-1|+C.\end{aligned}$$

? Exercise 7.4.3

Set up the partial fraction decomposition for

$$\int \frac{x+2}{(x+3)^3(x-4)^2}\,dx.$$

(Do not solve for the coefficients or complete the integration.)

Hint

Use the problem-solving method of Example 7.4.5for guidance.

Answer

$$\frac{x+2}{(x+3)^3(x-4)^2}=\frac{A}{x+3}+\frac{B}{(x+3)^2}+\frac{C}{(x+3)^3}+\frac{D}{(x-4)}+\frac{E}{(x-4)^2}$$

The General Method

Now that we are beginning to get the idea of how the technique of partial fraction decomposition works, let's outline the basic method in the following problem-solving strategy.

Problem-Solving Strategy: Partial Fraction Decomposition

To decompose the rational function $P(x)/Q(x)$, use the following steps:

1. Make sure that $deg(P(x))<deg(Q(x))$. If not, perform long division of polynomials.
2. Factor $Q(x)$ into the product of linear and irreducible quadratic factors. An irreducible quadratic is a quadratic that has no real zeros.
3. Assuming that $deg(P(x))<deg(Q(x)$, the factors of $Q(x)$ determine the form of the decomposition of $P(x)/Q(x)$.
   a. If $Q(x)$ can be factored as $(a_1x+b_1)(a_2x+b_2)\ldots(a_nx+b_n)$, where each linear factor is distinct, then it is possible to find constants $A_1, A_2, \ldots A_n$ satisfying

$$\frac{P(x)}{Q(x)}=\frac{A_1}{a_1x+b_1}+\frac{A_2}{a_2x+b_2}+\cdots+\frac{A_n}{a_nx+b_n}.$$

   b. If $Q(x)$ contains the repeated linear factor $(ax+b)^n$, then the decomposition must contain

$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_n}{(ax+b)^n}.$$

c. For each irreducible quadratic factor ax^2+bx+c that $Q(x)$ contains, the decomposition must include

$$\frac{Ax+B}{ax^2+bx+c}.$$

d. For each repeated irreducible quadratic factor $(ax^2+bx+c)^n$, the decomposition must include

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}.$$

e. After the appropriate decomposition is determined, solve for the constants.

f. Last, rewrite the integral in its decomposed form and evaluate it using previously developed techniques or integration formulas.

Simple Quadratic Factors

Now let's look at integrating a rational expression in which the denominator contains an irreducible quadratic factor. Recall that the quadratic ax^2+bx+c is irreducible if $ax^2+bx+c=0$ has no real zeros—that is, if $b^2-4ac<0$.

Example 7.4.6: Rational Expressions with an Irreducible Quadratic Factor

Evaluate

$$\int\frac{2x-3}{x^3+x}\,dx.$$

Solution

Since $deg(2x-3)<deg(x^3+x)$, factor the denominator and proceed with partial fraction decomposition. Since $x^3+x=x(x^2+1)$ contains the irreducible quadratic factor x^2+1, include $\frac{Ax+B}{x^2+1}$ as part of the decomposition, along with $\frac{C}{x}$ for the linear term x. Thus, the decomposition has the form

$$\frac{2x-3}{x(x^2+1)}=\frac{Ax+B}{x^2+1}+\frac{C}{x}.$$

After getting a common denominator and equating the numerators, we obtain the equation

$$2x-3=(Ax+B)x+C(x^2+1).$$

Solving for A, B, and C, we get $A=3$, $B=2$, and $C=-3$.

Thus,

$$\frac{2x-3}{x^3+x}=\frac{3x+2}{x^2+1}-\frac{3}{x}.$$

Substituting back into the integral, we obtain

$$\begin{aligned}\int\frac{2x-3}{x^3+x}\,dx &= \int\left(\frac{3x+2}{x^2+1}-\frac{3}{x}\right)dx \\ &= 3\int\frac{x}{x^2+1}\,dx+2\int\frac{1}{x^2+1}\,dx-3\int\frac{1}{x}\,dx && \text{Split up the integral} \\ &= \frac{3}{2}\ln|x^2+1|+2\tan^{-1}x-3\ln|x|+C. && \text{Evaluate each integral}\end{aligned}$$

Note: We may rewrite $\ln|x^2+1|=\ln(x^2+1)$, if we wish to do so, since $x^2+1>0$.

Example 7.4.7: Partial Fractions with an Irreducible Quadratic Factor

Evaluate $\int \frac{dx}{x^3-8}$.

Solution: We can start by factoring $x^3-8=(x-2)(x^2+2x+4)$. We see that the quadratic factor x^2+2x+4 is irreducible since $2^2-4(1)(4)=-12<0$. Using the decomposition described in the problem-solving strategy, we get

$$\frac{1}{(x-2)(x^2+2x+4)}=\frac{A}{x-2}+\frac{Bx+C}{x^2+2x+4}.$$

After obtaining a common denominator and equating the numerators, this becomes

$$1=A(x^2+2x+4)+(Bx+C)(x-2).$$

Applying either method, we get $A=\frac{1}{12}, B=-\frac{1}{12},$ and $C=-\frac{1}{3}$.

Rewriting $\int \frac{dx}{x^3-8}$, we have

$$\int \frac{dx}{x^3-8}=\frac{1}{12}\int \frac{1}{x-2}\,dx-\frac{1}{12}\int \frac{x+4}{x^2+2x+4}\,dx.$$

We can see that

$$\int \frac{1}{x-2}\,dx=\ln|x-2|+C,$$

but

$$\int \frac{x+4}{x^2+2x+4}\,dx$$

requires a bit more effort. Let's begin by **completing the square** on x^2+2x+4 to obtain

$$x^2+2x+4=(x+1)^2+3.$$

By letting $u=x+1$ and consequently $du=dx$, we see that

$$\begin{aligned}
\int \frac{x+4}{x^2+2x+4}\,dx &= \int \frac{x+4}{(x+1)^2+3}\,dx && \text{Complete the square on the denominator}\\
&= \int \frac{u+3}{u^2+3}\,du && \text{Substitute } u=x+1,\ x=u-1,\ \text{and } du=dx\\
&= \int \frac{u}{u^2+3}du+\int \frac{3}{u^2+3}du && \text{Split the numerator apart}\\
&= \frac{1}{2}\ln|u^2+3|+\frac{3}{\sqrt{3}}\tan^{-1}\frac{u}{\sqrt{3}}+C && \text{Evaluate each integral}\\
&= \frac{1}{2}\ln|x^2+2x+4|+\sqrt{3}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)+C && \text{Rewrite in terms of } x \text{ and simplify}
\end{aligned}$$

Substituting back into the original integral and simplifying gives

$$\int \frac{dx}{x^3-8}=\frac{1}{12}\ln|x-2|-\frac{1}{24}\ln|x^2+2x+4|-\frac{\sqrt{3}}{12}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)+C.$$

Here again, we can drop the absolute value if we wish to do so, since $x^2+2x+4>0$ for all x.

Example 7.4.8: Finding a Volume

Find the volume of the solid of revolution obtained by revolving the region enclosed by the graph of $f(x)=\dfrac{x^2}{(x^2+1)^2}$ and the x-axis over the interval $[0,1]$ about the y-axis.

Solution

Let's begin by sketching the region to be revolved (see Figure 7.4.1). From the sketch, we see that the shell method is a good choice for solving this problem.

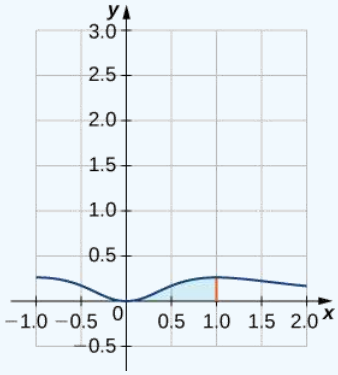


Figure 7.4.1: We can use the shell method to find the volume of revolution obtained by revolving the region shown about the y-axis.

The volume is given by

$$V=2\pi\int_0^1 x\cdot\frac{x^2}{(x^2+1)^2}\,dx=2\pi\int_0^1\frac{x^3}{(x^2+1)^2}\,dx.$$

Since $deg((x^2+1)^2)=4>3=deg(x^3),$ we can proceed with partial fraction decomposition. Note that $(x^2+1)^2$ is a repeated irreducible quadratic. Using the decomposition described in the problem-solving strategy, we get

$$\frac{x^3}{(x^2+1)^2}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{(x^2+1)^2}.$$

Finding a common denominator and equating the numerators gives

$$x^3=(Ax+B)(x^2+1)+Cx+D.$$

Solving, we obtain $A=1, B=0, C=-1,$ and $D=0$. Substituting back into the integral, we have

$$V=2\pi\int_0^1\frac{x^3}{(x^2+1)^2}\,dx=2\pi\int_0^1\left(\frac{x}{x^2+1}-\frac{x}{(x^2+1)^2}\right)dx=2\pi\left(\frac{1}{2}\ln(x^2+1)+\frac{1}{2}\cdot\frac{1}{x^2+1}\right)\Big|_0^1=$$
$$\pi\left(\ln 2-\tfrac{1}{2}\right).$$

Exercise 7.4.4

Set up the partial fraction decomposition for

$$\int\frac{x^2+3x+1}{(x+2)(x-3)^2(x^2+4)^2}\,dx.$$

Hint

Use the problem-solving strategy.

Answer

$$\frac{x^2+3x+1}{(x+2)(x-3)^2(x^2+4)^2}=\frac{A}{x+2}+\frac{B}{x-3}+\frac{C}{(x-3)^2}+\frac{Dx+E}{x^2+4}+\frac{Fx+G}{(x^2+4)^2}$$

Key Concepts

- Partial fraction decomposition is a technique used to break down a rational function into a sum of simple rational functions that can be integrated using previously learned techniques.
- When applying partial fraction decomposition, we must make sure that the degree of the numerator is less than the degree of the denominator. If not, we need to perform long division before attempting partial fraction decomposition.
- The form the decomposition takes depends on the type of factors in the denominator. The types of factors include nonrepeated linear factors, repeated linear factors, nonrepeated irreducible quadratic factors, and repeated irreducible quadratic factors.

Glossary

partial fraction decomposition

a technique used to break down a rational function into the sum of simple rational functions

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7.5: Strategy for Integration

Learning Objectives

- Use a table of integrals to solve integration problems.
- Use a computer algebra system (CAS) to solve integration problems.

In addition to the techniques of integration we have already seen, several other tools are widely available to assist with the process of integration. Among these tools are **integration tables**, which are readily available in many books, including the appendices to this one. Also widely available are **computer algebra systems (CAS)**, which are found on calculators and in many campus computer labs, and are free online.

Tables of Integrals

Integration tables, if used in the right manner, can be a handy way either to evaluate or check an integral quickly. Keep in mind that when using a table to check an answer, it is possible for two completely correct solutions to look very different. For example, in Trigonometric Substitution, we found that, by using the substitution $x = \tan\theta$, we can arrive at

$$\int \frac{dx}{\sqrt{1+x^2}} = \ln\left|x+\sqrt{x^2+1}\right| + C.$$

However, using $x = \sinh\theta$, we obtained a different solution—namely,

$$\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x + C.$$

We later showed algebraically that the two solutions are equivalent. That is, we showed that $\sinh^{-1} x = \ln\left|x+\sqrt{x^2+1}\right|$. In this case, the two antiderivatives that we found were actually equal. This need not be the case. However, as long as the difference in the two antiderivatives is a constant, they are equivalent.

Example 7.5.1: Using a Formula from a Table to Evaluate an Integral

Use the table formula

$$\int \frac{\sqrt{a^2-u^2}}{u^2}du = -\frac{\sqrt{a^2-u^2}}{u} - \sin^{-1}\frac{u}{a} + C$$

to evaluate $\displaystyle\int \frac{\sqrt{16-e^{2x}}}{e^x}dx$.

Solution

If we look at integration tables, we see that several formulas contain expressions of the form $\sqrt{a^2-u^2}$. This expression is actually similar to $\sqrt{16-e^{2x}}$, where $a=4$ and $u=e^x$. Keep in mind that we must also have $du=e^x$. Multiplying the numerator and the denominator of the given integral by e^x should help to put this integral in a useful form. Thus, we now have

$$\int \frac{\sqrt{16-e^{2x}}}{e^x}dx = \int \frac{\sqrt{16-e^{2x}}}{e^{2x}}e^x dx.$$

Substituting $u=e^x$ and $du=e^x\,dx$ produces $\displaystyle\int \frac{\sqrt{a^2-u^2}}{u^2}du$. From the integration table (#88 in Appendix A),

$$\int \frac{\sqrt{a^2-u^2}}{u^2}du = -\frac{\sqrt{a^2-u^2}}{u} - \sin^{-1}\frac{u}{a} + C.$$

Thus,

$$\begin{aligned}\int \frac{\sqrt{16-e^{2x}}}{e^x}dx &= \int \frac{\sqrt{16-e^{2x}}}{e^{2x}}e^x dx \text{ Substitute } u=e^x \text{ and } du=e^x dx.\\ &= \int \frac{\sqrt{4^2-u^2}}{u^2}du \text{ Apply the formula using } a=4.\end{aligned}$$

$$= -\frac{\sqrt{4^2-u^2}}{u} - \sin^{-1}\frac{u}{4} + C \quad \text{Substitute } u = e^x.$$
$$= -\frac{\sqrt{16-e^{2x}}}{e^x} - \sin^{-1}\left(\frac{e^x}{4}\right) + C$$

Computer Algebra Systems

If available, a CAS is a faster alternative to a table for solving an integration problem. Many such systems are widely available and are, in general, quite easy to use.

Example 7.5.2: Using a Computer Algebra System to Evaluate an Integral

Use a computer algebra system to evaluate $\int \frac{dx}{\sqrt{x^2-4}}$. Compare this result with $\ln\left|\frac{\sqrt{x^2-4}}{2} + \frac{x}{2}\right| + C$, a result we might have obtained if we had used trigonometric substitution.

Solution

Using Wolfram Alpha, we obtain

$$\int \frac{dx}{\sqrt{x^2-4}} = \ln\left|\sqrt{x^2-4} + x\right| + C.$$

Notice that

$$\ln\left|\frac{\sqrt{x^2-4}}{2} + \frac{x}{2}\right| + C = \ln\left|\frac{\sqrt{x^2-4}+x}{2}\right| + C = \ln\left|\sqrt{x^2-4} + x\right| - \ln 2 + C.$$

Since these two antiderivatives differ by only a constant, the solutions are equivalent. We could have also demonstrated that each of these antiderivatives is correct by differentiating them.

You can access an integral calculator for more examples.

Example 7.5.3: Using a CAS to Evaluate an Integral

Evaluate $\int \sin^3 x\, dx$ using a CAS. Compare the result to $\frac{1}{3}\cos^3 x - \cos x + C$, the result we might have obtained using the technique for integrating odd powers of $\sin x$ discussed earlier in this chapter.

Solution

Using Wolfram Alpha, we obtain

$$\int \sin^3 x\, dx = \frac{1}{12}(\cos(3x) - 9\cos x) + C.$$

This looks quite different from $\frac{1}{3}\cos^3 x - \cos x + C$. To see that these antiderivatives are equivalent, we can make use of a few trigonometric identities:

$$\begin{aligned}
\frac{1}{12}(\cos(3x) - 9\cos x) &= \frac{1}{12}(\cos(x+2x) - 9\cos x) \\
&= \frac{1}{12}(\cos(x)\cos(2x) - \sin(x)\sin(2x) - 9\cos x) \\
&= \frac{1}{12}(\cos x(2\cos^2 x - 1) - \sin x(2\sin x\cos x) - 9\cos x) \\
&= \frac{1}{12}(2\cos^3 x - \cos x - 2\cos x(1-\cos^2 x) - 9\cos x) \\
&= \frac{1}{12}(4\cos^3 x - 12\cos x)
\end{aligned}$$

$$= \frac{1}{3}\cos^3 x - \cos x.$$

Thus, the two antiderivatives are identical.

We may also use a CAS to compare the graphs of the two functions, as shown in the following figure.

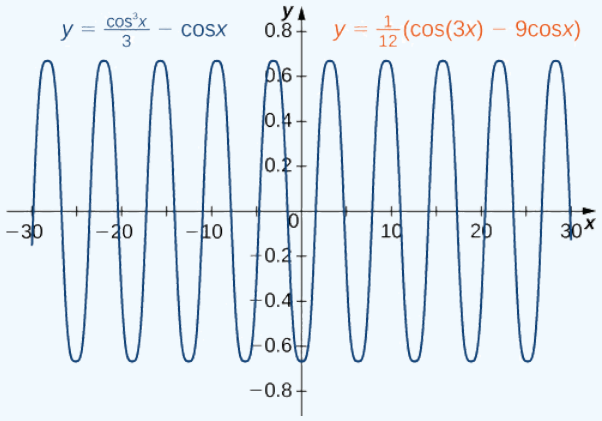


Figure 7.5.1: The graphs of $y = \frac{1}{3}\cos^3 x - \cos x$ and $y = \frac{1}{12}(\cos(3x) - 9\cos x)$ are identical.

Exercise 7.5.1

Use a CAS to evaluate $\int \frac{dx}{\sqrt{x^2+4}}$.

Hint

Answers may vary.

Answer

Possible solutions include $\sinh^{-1}\left(\frac{x}{2}\right) + C$ and $\ln\left|\sqrt{x^2+4} + x\right| + C$.

Key Concepts

- An integration table may be used to evaluate indefinite integrals.
- A CAS (or computer algebra system) may be used to evaluate indefinite integrals.
- It may require some effort to reconcile equivalent solutions obtained using different methods.

Glossary

computer algebra system (CAS)

technology used to perform many mathematical tasks, including integration

integration table

a table that lists integration formulas

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7.6: Integration Using Tables and Computer Algebra Systems

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7.7: Approximate Integration

Learning Objectives

- Approximate the value of a definite integral by using the midpoint and trapezoidal rules.
- Determine the absolute and relative error in using a numerical integration technique.
- Estimate the absolute and relative error using an error-bound formula.
- Recognize when the midpoint and trapezoidal rules over- or underestimate the true value of an integral.
- Use Simpson's rule to approximate the value of a definite integral to a given accuracy.

The antiderivatives of many functions either cannot be expressed or cannot be expressed easily in closed form (that is, in terms of known functions). Consequently, rather than evaluate definite integrals of these functions directly, we resort to various techniques of **numerical integration** to approximate their values. In this section, we explore several of these techniques. In addition, we examine the process of estimating the error in using these techniques.

The Midpoint Rule

Earlier in this text we defined the definite integral of a function over an interval as the limit of **Riemann sums**. In general, any Riemann sum of a function $f(x)$ over an interval $[a, b]$ may be viewed as an estimate of $\int_a^b f(x)\,dx$. Recall that a Riemann sum of a function $f(x)$ over an interval $[a, b]$ is obtained by selecting a partition

$$P = \{x_0, x_1, x_2, \ldots, x_n\}$$

where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$

and a set

$$S = \{x_1^*, x_2^*, \ldots, x_n^*\}$$

where $x_{i-1} \le x_i^* \le x_i \quad$ for all i.

The Riemann sum corresponding to the partition P and the set S is given by $\sum_{i=1}^{n} f(x_i^*)\Delta x_i$, where $\Delta x_i = x_i - x_{i-1}$, the length of the ith subinterval.

The midpoint rule for estimating a definite integral uses a Riemann sum with subintervals of equal width and the midpoints, m_i, of each subinterval in place of x_i^*. Formally, we state a theorem regarding the convergence of the midpoint rule as follows.

The Midpoint Rule

Assume that $f(x)$ is continuous on $[a, b]$. Let n be a positive integer and $\Delta x = \dfrac{b-a}{n}$. If $[a, b]$ is divided into n subintervals, each of length Δx, and m_i is the midpoint of the ith subinterval, set

$$M_n = \sum_{i=1}^{n} f(m_i)\Delta x.$$

Then $\lim_{n\to\infty} M_n = \int_a^b f(x)\,dx$.

As we can see in Figure 7.7.1, if $f(x) \ge 0$ over $[a, b]$, then $\sum_{i=1}^{n} f(m_i)\Delta x$ corresponds to the sum of the areas of rectangles approximating the area between the graph of $f(x)$ and the x-axis over $[a, b]$. The graph shows the rectangles corresponding to M_4 for a nonnegative function over a closed interval $[a, b]$.

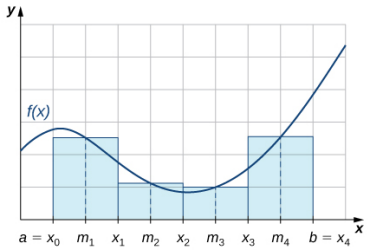


Figure 7.7.1: The midpoint rule approximates the area between the graph of $f(x)$ and the x-axis by summing the areas of rectangles with midpoints that are points on $f(x)$.

Example 7.7.1: Using the Midpoint Rule with M_4

Use the midpoint rule to estimate $\int_0^1 x^2\,dx$ using four subintervals. Compare the result with the actual value of this integral.

Solution: Each subinterval has length $\Delta x = \dfrac{1-0}{4} = \dfrac{1}{4}$. Therefore, the subintervals consist of

$$\left[0, \tfrac{1}{4}\right], \left[\tfrac{1}{4}, \tfrac{1}{2}\right], \left[\tfrac{1}{2}, \tfrac{3}{4}\right], \text{ and } \left[\tfrac{3}{4}, 1\right].$$

The midpoints of these subintervals are $\left\{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\right\}$. Thus,

$$\begin{aligned} M_4 &= \frac{1}{4}\cdot f\left(\frac{1}{8}\right)+\frac{1}{4}\cdot f\left(\frac{3}{8}\right)+\frac{1}{4}\cdot f\left(\frac{5}{8}\right)+\frac{1}{4}\cdot f\left(\frac{7}{8}\right) \\ &= \frac{1}{4}\cdot\frac{1}{64}+\frac{1}{4}\cdot\frac{9}{64}+\frac{1}{4}\cdot\frac{25}{64}+\frac{1}{4}\cdot\frac{49}{64} \\ &= \frac{21}{64}=0.328125. \end{aligned}$$

Since

$$\int_0^1 x^2\,dx=\frac{1}{3},$$

the absolute error in this approximation is:

$$\left|\frac{1}{3}-\frac{21}{64}\right|=\frac{1}{192}\approx 0.0052,$$

and we see that the midpoint rule produces an estimate that is somewhat close to the actual value of the definite integral.

Example 7.7.2: Using the Midpoint Rule with M_6

Use M_6 to estimate the length of the curve $y=\frac{1}{2}x^2$ on $[1,4]$.

Solution: The length of $y=\frac{1}{2}x^2$ on $[1,4]$ is

$$s=\int_1^4\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

Since $\dfrac{dy}{dx}=x$, this integral becomes $\displaystyle\int_1^4\sqrt{1+x^2}\,dx$.

If $[1,4]$ is divided into six subintervals, then each subinterval has length $\Delta x=\dfrac{4-1}{6}=\dfrac{1}{2}$ and the midpoints of the subintervals are $\left\{\frac{5}{4},\frac{7}{4},\frac{9}{4},\frac{11}{4},\frac{13}{4},\frac{15}{4}\right\}$. If we set $f(x)=\sqrt{1+x^2}$,

$$\begin{aligned} M_6 &= \tfrac{1}{2}\cdot f\left(\frac{5}{4}\right)+\tfrac{1}{2}\cdot f\left(\frac{7}{4}\right)+\frac{1}{2}\cdot f\left(\frac{9}{4}\right)+\frac{1}{2}\cdot f\left(\frac{11}{4}\right)+\frac{1}{2}\cdot f\left(\frac{13}{4}\right)+\frac{1}{2}\cdot f\left(\frac{15}{4}\right) \\ &\approx \frac{1}{2}(1.6008+2.0156+2.4622+2.9262+3.4004+3.8810)=8.1431 \text{ units.} \end{aligned}$$

Exercise 7.7.1

Use the midpoint rule with $n=2$ to estimate $\displaystyle\int_1^2\frac{1}{x}\,dx$.

Hint

$\Delta x=\frac{1}{2}$, $m_1=\frac{5}{4}$, and $m_2=\frac{7}{4}$.

Answer

$\dfrac{24}{35}\approx 0.685714$

The Trapezoidal Rule

We can also approximate the value of a definite integral by using trapezoids rather than rectangles. In Figure 7.7.2, the area beneath the curve is approximated by trapezoids rather than by rectangles.

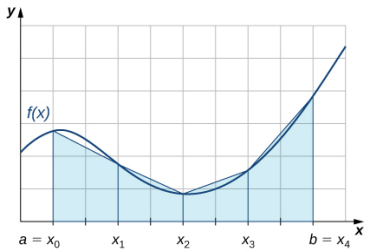


Figure 7.7.2: Trapezoids may be used to approximate the area under a curve, hence approximating the definite integral.

The **trapezoidal rule** for estimating definite integrals uses trapezoids rather than rectangles to approximate the area under a curve. To gain insight into the final form of the rule, consider the trapezoids shown in Figure 7.7.2. We assume that the length of each subinterval is given by Δx. First, recall that the area of a trapezoid with a height of h and bases of length b_1 and b_2 is given by $\text{Area}=\frac{1}{2}h(b_1+b_2)$. We see that the first trapezoid has a height Δx and parallel bases of length $f(x_0)$ and $f(x_1)$. Thus, the area of the first trapezoid in Figure 7.7.2 is

$$\frac{1}{2}\Delta x\Big(f(x_0)+f(x_1)\Big).$$

The areas of the remaining three trapezoids are

$$\frac{1}{2}\Delta x\Big(f(x_1)+f(x_2)\Big),\ \frac{1}{2}\Delta x\Big(f(x_2)+f(x_3)\Big),\text{ and }\frac{1}{2}\Delta x\Big(f(x_3)+f(x_4)\Big).$$

Consequently,

$$\int_a^b f(x)\,dx \approx \frac{1}{2}\Delta x\Big(f(x_0)+f(x_1)\Big)+\frac{1}{2}\Delta x\Big(f(x_1)+f(x_2)\Big)+\frac{1}{2}\Delta x\Big(f(x_2)+f(x_3)\Big)+\frac{1}{2}\Delta x\Big(f(x_3)+f(x_4)\Big).$$

After taking out a common factor of $\frac{1}{2}\Delta x$ and combining like terms, we have

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\Big[f(x_0)+2\,f(x_1)+2\,f(x_2)+2\,f(x_3)+f(x_4)\Big].$$

Generalizing, we formally state the following rule.

The Trapezoidal Rule

Assume that $f(x)$ is continuous over $[a,b]$. Let n be a positive integer and $\Delta x = \dfrac{b-a}{n}$. Let $[a,b]$ be divided into n subintervals, each of length Δx, with endpoints at $P=\{x_0,x_1,x_2\ldots,x_n\}$.

Set

$$T_n = \frac{\Delta x}{2}\Big[f(x_0)+2\,f(x_1)+2\,f(x_2)+\cdots+2\,f(x_{n-1})+f(x_n)\Big].$$

Then, $\lim\limits_{n\to+\infty} T_n = \int_a^b f(x)\,dx$.

Before continuing, let's make a few observations about the trapezoidal rule. First of all, it is useful to note that

$$T_n = \frac{1}{2}(L_n+R_n) \text{ where } L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x \text{ and } R_n = \sum_{i=1}^{n} f(x_i)\Delta x.$$

That is, L_n and R_n approximate the integral using the left-hand and right-hand endpoints of each subinterval, respectively. In addition, a careful examination of Figure 7.7.3 leads us to make the following observations about using the trapezoidal rules and midpoint rules to estimate the definite integral of a nonnegative function. The trapezoidal rule tends to overestimate the value of a definite integral systematically over intervals where the function is concave up and to underestimate the value of a definite integral systematically over intervals where the function is concave down. On the other hand, the midpoint rule tends to average out these errors somewhat by partially overestimating and partially underestimating the value of the definite integral over these same types of intervals. This leads us to hypothesize that, in general, the midpoint rule tends to be more accurate than the trapezoidal rule.

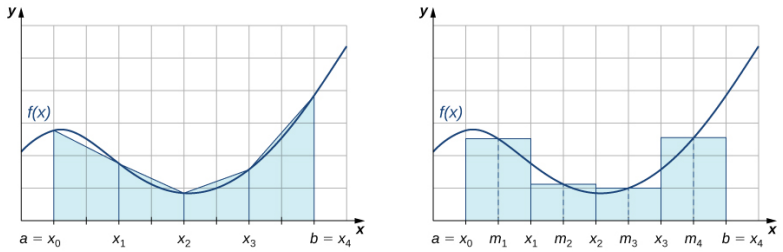


Figure 7.7.3:The trapezoidal rule tends to be less accurate than the midpoint rule.

Example 7.7.3: Using the Trapezoidal Rule

Use the trapezoidal rule to estimate $\int_0^1 x^2\,dx$ using four subintervals.

Solution

The endpoints of the subintervals consist of elements of the set $P=\left\{0,\frac{1}{4},\frac{1}{2},\frac{3}{4},1\right\}$ and $\Delta x = \frac{1-0}{4}=\frac{1}{4}$. Thus,

$$\begin{aligned}\int_0^1 x^2dx &\approx \frac{1}{2}\cdot\frac{1}{4}\Big[f(0)+2\,f\left(\tfrac{1}{4}\right)+2\,f\left(\tfrac{1}{2}\right)+2\,f\left(\tfrac{3}{4}\right)+f(1)\Big]\\ &= \tfrac{1}{8}\left(0+2\cdot\tfrac{1}{16}+2\cdot\tfrac{1}{4}+2\cdot\tfrac{9}{16}+1\right)\\ &= \frac{11}{32}=0.34375\end{aligned}$$

Exercise 7.7.2

Use the trapezoidal rule with $n=2$ to estimate $\int_1^2 \frac{1}{x}\,dx$.

Hint

Set $\Delta x = \dfrac{1}{2}$. The endpoints of the subintervals are the elements of the set $P=\left\{1,\frac{3}{2},2\right\}$.

Answer

$\dfrac{17}{24}\approx 0.708333$

Absolute and Relative Error

An important aspect of using these numerical approximation rules consists of calculating the error in using them for estimating the value of a definite integral. We first need to define absolute error and relative error.

Definition: absolute and relative error

If B is our estimate of some quantity having an actual value of A, then the **absolute error** is given by $|A-B|$.

The **relative error** is the error as a percentage of the actual value and is given by

$$\left|\frac{A-B}{A}\right| \cdot 100\%.$$

Example 7.7.4: Calculating Error in the Midpoint Rule

Calculate the absolute and relative error in the estimate of $\int_0^1 x^2\,dx$ using the midpoint rule, found in Example 7.7.1.

Solution: The calculated value is $\int_0^1 x^2\,dx = \frac{1}{3}$ and our estimate from the example is $M_4 = \frac{21}{64}$. Thus, the absolute error is given by $\left|\frac{1}{3} - \frac{21}{64}\right| = \frac{1}{192} \approx 0.0052$.

The relative error is

$$\frac{1/192}{1/3} = \frac{1}{64} \approx 0.015625 \approx 1.6\%.$$

Example 7.7.5: Calculating Error in the Trapezoidal Rule

Calculate the absolute and relative error in the estimate of $\int_0^1 x^2\,dx$ using the trapezoidal rule, found in Example 7.7.3.

Solution: The calculated value is $\int_0^1 x^2\,dx = \frac{1}{3}$ and our estimate from the example is $T_4 = \frac{11}{32}$. Thus, the absolute error is given by $\left|\frac{1}{3} - \frac{11}{32}\right| = \frac{1}{96} \approx 0.0104$.

The relative error is given by

$$\frac{1/96}{1/3} = 0.03125 \approx 3.1\%.$$

Exercise 7.7.3

In an earlier checkpoint, we estimated $\int_1^2 \frac{1}{x}\,dx$ to be $\frac{24}{35}$ using M_2. The actual value of this integral is $\ln 2$. Using $\frac{24}{35} \approx 0.6857$ and $\ln 2 \approx 0.6931$, calculate the absolute error and the relative error.

Hint

Use the previous examples as a guide.

Answer

absolute error ≈ 0.0074,and relative error $\approx 1.1\%$

Error Bounds on the Midpoint and Trapezoidal Rules

In the two previous examples, we were able to compare our estimate of an integral with the actual value of the integral; however, we do not typically have this luxury. In general, if we are approximating an integral, we are doing so because we cannot compute the exact value of the integral itself easily. Therefore, it is often helpful to be able to determine an upper bound for the error in an approximation of an integral. The following theorem provides error bounds for the midpoint and trapezoidal rules. The theorem is stated without proof.

Error Bounds for the Midpoint and Trapezoidal Rules

Let $f(x)$ be a continuous function over $[a,b]$, having a second derivative $f''(x)$ over this interval. If M is the maximum value of $|f''(x)|$ over $[a,b]$, then the upper bounds for the error in using M_n and T_n to estimate $\int_a^b f(x)\,dx$ are

$$\text{Error in } M_n \le \frac{M(b-a)^3}{24n^2} \tag{7.7.1}$$

and

$$\text{Error in } T_n \le \frac{M(b-a)^3}{12n^2}$$

.

We can use these bounds to determine the value of n necessary to guarantee that the error in an estimate is less than a specified value.

Example 7.7.6: Determining the Number of Intervals to Use

What value of n should be used to guarantee that an estimate of $\int_0^1 e^{x^2}\,dx$ is accurate to within 0.01 if we use the midpoint rule?

Solution

We begin by determining the value of M, the maximum value of $|f''(x)|$ over $[0,1]$ for $f(x) = e^{x^2}$. Since $f'(x) = 2xe^{x^2}$, we have

$$f''(x) = 2e^{x^2} + 4x^2e^{x^2}.$$

Thus,

$$|f''(x)| = 2e^{x^2}(1+2x^2) \le 2 \cdot e \cdot 3 = 6e.$$

From the error-bound Equation 7.7.1, we have

$$\text{Error in } M_n \le \frac{M(b-a)^3}{24n^2} \le \frac{6e(1-0)^3}{24n^2} = \frac{6e}{24n^2}.$$

Now we solve the following inequality for n:

$$\frac{6e}{24n^2} \le 0.01.$$

Thus, $n \ge \sqrt{\frac{600e}{24}} \approx 8.24$. Since n must be an integer satisfying this inequality, a choice of $n = 9$ would guarantee that

$$\left| \int_0^1 e^{x^2}\,dx - M_n \right| < 0.01.$$

Analysis

We might have been tempted to round 8.24 down and choose $n = 8$, but this would be incorrect because we must have an integer greater than or equal to 8.24. We need to keep in mind that the error estimates provide an upper bound only for the error. The actual estimate may, in fact, be a much better approximation than is indicated by the error bound.

? Exercise 7.7.4

Use Equation 7.7.1 to find an upper bound for the error in using M_4 to estimate $\int_0^1 x^2\,dx$.

Hint

$f''(x) = 2,$ so $M = 2.$

Answer

$\frac{1}{192}$

Simpson's Rule

With the midpoint rule, we estimated areas of regions under curves by using rectangles. In a sense, we approximated the curve with piecewise constant functions. With the trapezoidal rule, we approximated the curve by using piecewise linear functions. What if we were, instead, to approximate a curve using piecewise quadratic functions? With **Simpson's rule,** we do just this. We partition the interval into an even number of subintervals, each of equal width. Over the first pair of subintervals we approximate $\int_{x_0}^{x_2} f(x)\,dx$ with $\int_{x_0}^{x_2} p(x)\,dx$, where $p(x) = Ax^2 + Bx + C$ is the quadratic function passing through $(x_0, f(x_0))$, $(x_1, f(x_1))$, and $(x_2, f(x_2))$ (Figure 7.7.4). Over the next pair of subintervals we approximate $\int_{x_2}^{x_4} f(x)\,dx$ with the integral of another quadratic function passing through $(x_2, f(x_2))$, $(x_3, f(x_3))$, and $(x_4, f(x_4))$. This process is continued with each successive pair of subintervals.

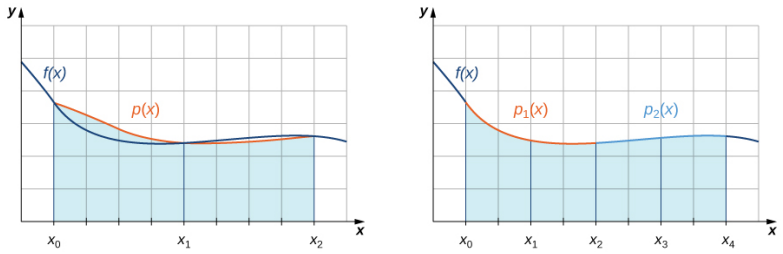


Figure 7.7.4: With Simpson's rule, we approximate a definite integral by integrating a piecewise quadratic function.

To understand the formula that we obtain for Simpson's rule, we begin by deriving a formula for this approximation over the first two subintervals. As we go through the derivation, we need to keep in mind the following relationships:

$$f(x_0) = p(x_0) = Ax_0^2 + Bx_0 + C$$

$$f(x_1) = p(x_1) = Ax_1^2 + Bx_1 + C$$

$$f(x_2) = p(x_2) = Ax_2^2 + Bx_2 + C$$

$x_2 - x_0 = 2\Delta x$, where Δx is the length of a subinterval.

$$x_2 + x_0 = 2x_1, \text{ since } x_1 = \frac{(x_2 + x_0)}{2}.$$

Thus,

$$\int_{x_0}^{x_2} f(x)\,dx \approx \int_{x_0}^{x_2} p(x)\,dx$$

$$= \int_{x_0}^{x_2} (Ax^2+Bx+C)\,dx$$

$$= \left(\frac{A}{3}x^3+\frac{B}{2}x^2+Cx\right)\Big|_{x_0}^{x_2} \qquad \text{Find the antiderivative.}$$

$$= \frac{A}{3}(x_2^3-x_0^3)+\frac{B}{2}(x_2^2-x_0^2)+C(x_2-x_0) \qquad \text{Evaluate the antiderivative.}$$

$$= \frac{A}{3}(x_2-x_0)(x_2^2+x_2x_0+x_0^2)+\frac{B}{2}(x_2-x_0)(x_2+x_0)+C(x_2-x_0)$$

$$= \frac{x_2-x_0}{6}\left(2A(x_2^2+x_2x_0+x_0^2)+3B(x_2+x_0)+6C\right) \qquad \text{Factor out } \frac{x_2-x_0}{6}.$$

$$= \frac{\Delta x}{3}\left((Ax_2^2+Bx_2+C)+(Ax_0^2+Bx_0+C)+A(x_2^2+2x_2x_0+x_0^2)+2B(x_2+x_0)+4C\right) \qquad \text{Rearrange the terms. Note: } \Delta x = \frac{x_2-x_0}{2}$$

$$= \frac{\Delta x}{3}\left(f(x_2)+f(x_0)+A(x_2+x_0)^2+2B(x_2+x_0)+4C\right) \qquad \text{Factor and substitute:}$$

$$f(x_2) = Ax_2^2+Bx_2+C \text{ and } f(x_0) = Ax_0^2+Bx_0+C.$$

$$= \frac{\Delta x}{3}\left(f(x_2)+f(x_0)+A(2x_1)^2+2B(2x_1)+4C\right) \qquad \text{Substitute } x_2+x_0 = 2x_1.$$

$$\text{Note: } x_1 = \frac{x_2+x_0}{2} \text{ is the midpoint.}$$

$$= \frac{\Delta x}{3}\left(f(x_2)+4f(x_1)+f(x_0)\right). \qquad \text{Expand and substitute } f(x_1) = Ax_1^2+Bx_1+C.$$

If we approximate $\int_{x_2}^{x_4} f(x)\,dx$ using the same method, we see that we have

$$\int_{x_2}^{x_4} f(x)\,dx \approx \frac{\Delta x}{3}(f(x_4)+4\,f(x_3)+f(x_2)).$$

Combining these two approximations, we get

$$\int_{x_0}^{x_4} f(x)\,dx \approx \frac{\Delta x}{3}(f(x_0)+4\,f(x_1)+2\,f(x_2)+4\,f(x_3)+f(x_4)).$$

The pattern continues as we add pairs of subintervals to our approximation. The general rule may be stated as follows.

Simpson's Rule

Assume that $f(x)$ is continuous over $[a,b]$. Let n be a positive even integer and $\Delta x = \frac{b-a}{n}$. Let $[a,b]$ be divided into n subintervals, each of length Δx, with endpoints at $P = \{x_0, x_1, x_2, \ldots, x_n\}$. Set

$$S_n = \frac{\Delta x}{3}\Big[f(x_0)+4\,f(x_1)+2\,f(x_2)+4\,f(x_3)+2\,f(x_4)+\cdots+2\,f(x_{n-2})+4\,f(x_{n-1})+f(x_n)\Big].$$

Then,

$$\lim_{n\to+\infty} S_n = \int_a^b f(x)\,dx.$$

Just as the trapezoidal rule is the average of the left-hand and right-hand rules for estimating definite integrals, Simpson's rule may be obtained from the midpoint and trapezoidal rules by using a weighted average. It can be shown that $S_{2n} = \left(\frac{2}{3}\right) M_n + \left(\frac{1}{3}\right) T_n$.

It is also possible to put a bound on the error when using Simpson's rule to approximate a definite integral. The bound in the error is given by the following rule:

Rule: Error Bound for Simpson's Rule

Let $f(x)$ be a continuous function over $[a,b]$ having a fourth derivative, $f^{(4)}(x)$, over this interval. If M is the maximum value of $\left|f^{(4)}(x)\right|$ over $[a,b]$, then the upper bound for the error in using S_n to estimate $\int_a^b f(x)\,dx$ is given by

$$\text{Error in } S_n \le \frac{M(b-a)^5}{180n^4}.$$

Example 7.7.7: Applying Simpson's Rule 1

Use S_2 to approximate $\int_0^1 x^3\,dx$. Estimate a bound for the error in S_2.

Solution

Since $[0,1]$ is divided into two intervals, each subinterval has length $\Delta x = \frac{1-0}{2} = \frac{1}{2}$. The endpoints of these subintervals are $\left\{0, \frac{1}{2}, 1\right\}$. If we set $f(x) = x^3$, then

$$S_2 = \frac{1}{3}\cdot\frac{1}{2}(f(0)+4\,f(\tfrac{1}{2})+f(1)) = \frac{1}{6}(0+4\cdot\frac{1}{8}+1) = \frac{1}{4}.$$

Since $f^{(4)}(x) = 0$ and consequently $M = 0$, we see that

$$\text{Error in } S_2 \le \frac{0(1)^5}{180 \cdot 2^4} = 0.$$

This bound indicates that the value obtained through Simpson's rule is exact. A quick check will verify that, in fact, $\int_0^1 x^3\,dx = \frac{1}{4}$.

Example 7.7.8: Applying Simpson's Rule 2

Use S_6 to estimate the length of the curve $y = \frac{1}{2}x^2$ over $[1, 4]$.

Solution

The length of $y = \frac{1}{2}x^2$ over $[1, 4]$ is $\int_1^4 \sqrt{1+x^2}\,dx$. If we divide $[1, 4]$ into six subintervals, then each subinterval has length $\Delta x = \frac{4-1}{6} = \frac{1}{2}$, and the endpoints of the subintervals are $\{1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4\}$. Setting $f(x) = \sqrt{1+x^2}$,

$$S_6 = \frac{1}{3} \cdot \frac{1}{2}(f(1) + 4f(\frac{3}{2}) + 2f(2) + 4f(\frac{5}{2}) + 2f(3) + 4f(\frac{7}{2}) + f(4)).$$

After substituting, we have

$$S_6 = \frac{1}{6}(1.4142 + 4 \cdot 1.80278 + 2 \cdot 2.23607 + 4 \cdot 2.69258 + 2 \cdot 3.16228 + 4 \cdot 3.64005 + 4.12311) \approx 8.14594 \text{ units}.$$

Exercise 7.7.5

Use S_2 to estimate $\int_1^2 \frac{1}{x}\,dx$.

Hint

$$S_2 = \frac{1}{3}\Delta x\,(f(x_0) + 4f(x_1) + f(x_2))$$

Answer

$\frac{25}{36} \approx 0.694444$

Key Concepts

- We can use numerical integration to estimate the values of definite integrals when a closed form of the integral is difficult to find or when an approximate value only of the definite integral is needed.
- The most commonly used techniques for numerical integration are the midpoint rule, trapezoidal rule, and Simpson's rule.
- The midpoint rule approximates the definite integral using rectangular regions whereas the trapezoidal rule approximates the definite integral using trapezoidal approximations.
- Simpson's rule approximates the definite integral by first approximating the original function using piecewise quadratic functions.

Key Equations

- **Midpoint rule**

  $M_n = \sum_{i=1}^{n} f(m_i)\Delta x$

- **Trapezoidal rule**

  $T_n = \frac{\Delta x}{2}\left[f(x_0) + 2\,f(x_1) + 2\,f(x_2) + \cdots + 2\,f(x_{n-1}) + f(x_n)\right]$

- **Simpson's rule**

  $S_n = \frac{\Delta x}{3}\left[f(x_0) + 4\,f(x_1) + 2\,f(x_2) + 4\,f(x_3) + 2\,f(x_4) + 4\,f(x_5) + \cdots + 2\,f(x_{n-2}) + 4\,f(x_{n-1}) + f(x_n)\right]$

- **Error bound for midpoint rule**

  Error in $M_n \le \frac{M(b-a)^3}{24n^2}$, where M is the maximum value of $|f''(x)|$ over $[a, b]$.

- **Error bound for trapezoidal rule**

  Error in $T_n \le \frac{M(b-a)^3}{12n^2}$, where M is the maximum value of $|f''(x)|$ over $[a, b]$.

- **Error bound for Simpson's rule**

  Error in $S_n \le \frac{M(b-a)^5}{180n^4}$, where M is the maximum value of $\left|f^{(4)}(x)\right|$ over $[a, b]$.

Glossary

absolute error

if B is an estimate of some quantity having an actual value of A, then the absolute error is given by $|A - B|$

midpoint rule

a rule that uses a Riemann sum of the form $M_n = \sum_{i=1}^{n} f(m_i)\Delta x$, where m_i is the midpoint of the ith subinterval to approximate $\int_a^b f(x)\,dx$

numerical integration

the variety of numerical methods used to estimate the value of a definite integral, including the midpoint rule, trapezoidal rule, and Simpson's rule

relative error

error as a percentage of the actual value, given by

$$\text{relative error} = \left| \frac{A - B}{A} \right| \cdot 100\%$$

Simpson's rule

a rule that approximates $\int_a^b f(x)\,dx$ using the area under a piecewise quadratic function.

The approximation S_n to $\int_a^b f(x)\,dx$ is given by

$$S_n = \frac{\Delta x}{3}\big(f(x_0) + 4\,f(x_1) + 2\,f(x_2) + 4\,f(x_3) + 2\,f(x_4) + \cdots + 2\,f(x_{n-2}) + 4\,f(x_{n-1}) + f(x_n)\big).$$

trapezoidal rule

a rule that approximates $\int_a^b f(x)\,dx$ using the area of trapezoids.

The approximation T_n to $\int_a^b f(x)\,dx$ is given by

$$T_n = \frac{\Delta x}{2}\big(f(x_0) + 2\,f(x_1) + 2\,f(x_2) + \cdots + 2\,f(x_{n-1}) + f(x_n)\big).$$

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- Edited by Paul Seeburger (Monroe Community College). Notes added to development of area under a parabola and typos fixed in original text.

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7.8: Improper Integrals

Learning Objectives

- Evaluate an integral over an infinite interval.
- Evaluate an integral over a closed interval with an infinite discontinuity within the interval.
- Use the comparison theorem to determine whether a definite integral is convergent.

Is the area between the graph of $f(x) = \dfrac{1}{x}$ and the x-axis over the interval $[1, +\infty)$ finite or infinite? If this same region is revolved about the x-axis, is the volume finite or infinite? Surprisingly, the area of the region described is infinite, but the volume of the solid obtained by revolving this region about the x-axis is finite.

In this section, we define integrals over an infinite interval as well as integrals of functions containing a discontinuity on the interval. Integrals of these types are called improper integrals. We examine several techniques for evaluating improper integrals, all of which involve taking limits.

Integrating over an Infinite Interval

How should we go about defining an integral of the type $\displaystyle\int_a^{+\infty} f(x)\,dx$? We can integrate $\displaystyle\int_a^t f(x)\,dx$ for any value of t, so it is reasonable to look at the behavior of this integral as we substitute larger values of t. Figure 7.8.1 shows that $\displaystyle\int_a^t f(x)\,dx$ may be interpreted as area for various values of t. In other words, we may define an improper integral as a limit, taken as one of the limits of integration increases or decreases without bound.

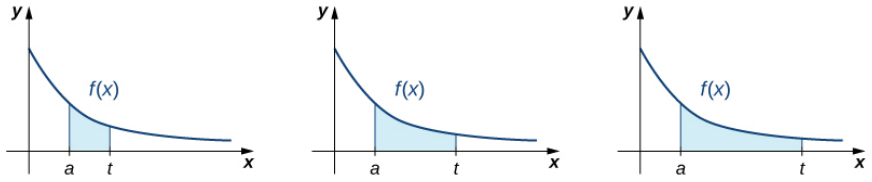


Figure 7.8.1: To integrate a function over an infinite interval, we consider the limit of the integral as the upper limit increases without bound.

Definition: Improper Integral

1. Let $f(x)$ be continuous over an interval of the form $[a, +\infty)$. Then

$$\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty} \int_a^t f(x)\,dx, \tag{7.8.1}$$

   provided this limit exists.

2. Let $f(x)$ be continuous over an interval of the form $(-\infty, b]$. Then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx, \tag{7.8.2}$$

   provided this limit exists.

In each case, if the limit exists, then the *improper integral* is said to converge. If the limit does not exist, then the improper integral is said to diverge.

3. Let $f(x)$ be continuous over $(-\infty, +\infty)$. Then

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^{+\infty} f(x)\,dx, \tag{7.8.3}$$

   provided that $\displaystyle\int_{-\infty}^0 f(x)\,dx$ and $\displaystyle\int_0^{+\infty} f(x)\,dx$ both converge.

   If either of these two integrals diverge, then $\displaystyle\int_{-\infty}^{+\infty} f(x)\,dx$ diverges.

(It can be shown that, in fact, $\displaystyle\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^{+\infty} f(x)\,dx$ for any value of a.).

In our first example, we return to the question we posed at the start of this section: Is the area between the graph of $f(x) = \dfrac{1}{x}$ and the x-axis over the interval $[1, +\infty)$ finite or infinite?

Example 7.8.1: Finding an Area

Determine whether the area between the graph of $f(x) = \dfrac{1}{x}$ and the x-axis over the interval $[1, +\infty)$ is finite or infinite.

Solution

We first do a quick sketch of the region in question, as shown in Figure 7.8.2.

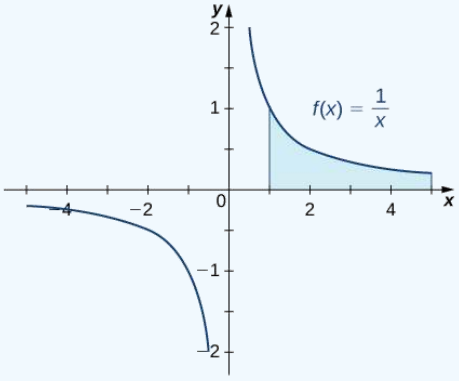


Figure 7.8.2: We can find the area between the curve $f(x) = 1/x$ and the x-axis on an infinite interval.

We can see that the area of this region is given by

$$A = \int_1^{\infty} \frac{1}{x}\, dx.$$

which can be evaluated using Equation 7.8.1:

$$\begin{aligned} A &= \int_1^{\infty} \frac{1}{x}\, dx \\ &= \lim_{t\to+\infty} \int_1^{t} \frac{1}{x}\, dx && \text{Rewrite the improper integral as a limit} \\ &= \lim_{t\to+\infty} \ln|x|\Big|_1^{t} && \text{Find the antiderivative} \\ &= \lim_{t\to+\infty} (\ln|t| - \ln 1) && \text{Evaluate the antiderivative} \\ &= +\infty. && \text{Evaluate the limit.} \end{aligned}$$

Since the improper integral diverges to $+\infty$, the area of the region is infinite.

Example 7.8.2: Finding a Volume

Find the volume of the solid obtained by revolving the region bounded by the graph of $f(x) = \dfrac{1}{x}$ and the x-axis over the interval $[1, +\infty)$ about the x-axis.

Solution

The solid is shown in Figure 7.8.3. Using the disk method, we see that the volume V is

$$V = \pi \int_1^{+\infty} \frac{1}{x^2}\, dx.$$

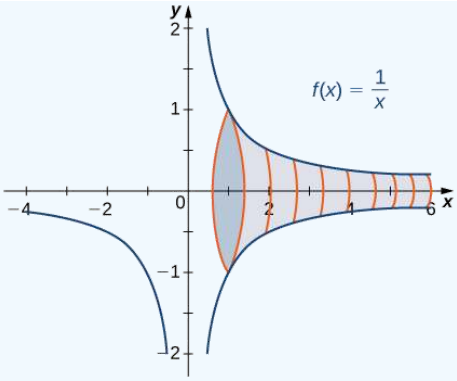


Figure 7.8.3: The solid of revolution can be generated by rotating an infinite area about the x-axis.

Then we have

$$
\begin{aligned}
V &= \pi \int_1^{+\infty} \frac{1}{x^2}\,dx \\
&= \pi \lim_{t\to+\infty} \int_1^t \frac{1}{x^2}\,dx && \text{Rewrite as a limit.} \\
&= \pi \lim_{t\to+\infty} -\frac{1}{x}\Big|_1^t && \text{Find the antiderivative.} \\
&= \pi \lim_{t\to+\infty} \left(-\frac{1}{t}+1\right) && \text{Evaluate the antiderivative.} \\
&= \pi \lim_{t\to+\infty} \left(-\frac{1}{\cancel{t}^{\,0}}+1\right) && \text{Evaluate the antiderivative.} \\
&= \pi
\end{aligned}
$$

The improper integral converges to π. Therefore, the volume of the solid of revolution is π.

In conclusion, although the area of the region between the x-axis and the graph of $f(x) = 1/x$ over the interval $[1, +\infty)$ is infinite, the volume of the solid generated by revolving this region about the x-axis is finite. The solid generated is known as Gabriel's Horn.

Note: Gabriel's horn (also called Torricelli's trumpet) is a geometric figure which has *infinite* surface area, but *finite* volume. The name refers to the tradition identifying the Archangel Gabriel as the angel who blows the horn to announce Judgment Day, associating the divine, or infinite, with the finite. The properties of this figure were first studied by Italian physicist and mathematician Evangelista Torricelli in the 17th century.

Example 7.8.3: Traffic Accidents in a City

Suppose that at a busy intersection, traffic accidents occur at an average rate of one every three months. After residents complained, changes were made to the traffic lights at the intersection. It has now been eight months since the changes were made and there have been no accidents. Were the changes effective or is the 8-month interval without an accident a result of chance?



Figure 7.8.4: Modification of work by David McKelvey, Flickr.

Probability theory tells us that if the average time between events is k, the *probability* that X, the time between events, is between a and b is given by

$$(P(a \le x \le b) = \int_a^b f(x)\,dx$$

where

$$f(x)=\begin{cases}0, & \text{if } x<0\\ ke^{-kx}, & \text{if } x\geq 0\end{cases}.$$

Thus, if accidents are occurring at a rate of one every 3 months, then the probability that X, the time between accidents, is between a and b is given by

$$P(a\leq x\leq b)=\int_a^b f(x)\,dx$$

where

$$f(x)=\begin{cases}0, & \text{if } x<0\\ 3e^{-3x}, & \text{if } x\geq 0\end{cases}.$$

To answer the question, we must compute $P(X\geq 8)=\int_8^{+\infty} 3e^{-3x}\,dx$ and decide whether it is likely that 8 months could have passed without an accident if there had been no improvement in the traffic situation.

Solution

We need to calculate the probability as an improper integral:

$$\begin{aligned}P(X\geq 8)&=\int_8^{+\infty} 3e^{-3x}\,dx\\ &=\lim_{t\to+\infty}\int_8^t 3e^{-3x}\,dx\\ &=\lim_{t\to+\infty} -e^{-3x}\Big|_8^t\\ &=\lim_{t\to+\infty}\left(-e^{-3t}+e^{-24}\right)\\ &\approx 3.8\times 10^{-11}.\end{aligned}$$

The value 3.8×10^{-11} represents the probability of no accidents in 8 months under the initial conditions. Since this value is very, very small, it is reasonable to conclude the changes were effective.

Example 7.8.4: Evaluating an Improper Integral over an Infinite Interval

Evaluate $\int_{-\infty}^0 \frac{1}{x^2+4}\,dx$. State whether the improper integral converges or diverges.

Solution

Begin by rewriting $\int_{-\infty}^0 \frac{1}{x^2+4}\,dx$ as a limit using Equation 7.8.2 from the definition. Thus,

$$\begin{aligned}\int_{-\infty}^0 \frac{1}{x^2+4}\,dx&=\lim_{t\to-\infty}\int_t^0 \frac{1}{x^2+4}\,dx && \text{Rewrite as a limit.}\\ &=\lim_{t\to-\infty}\frac{1}{2}\tan^{-1}\frac{x}{2}\Big|_t^0 && \text{Find the antiderivative.}\\ &=\lim_{t\to-\infty}\left(\frac{1}{2}\tan^{-1}0-\frac{1}{2}\tan^{-1}\frac{t}{2}\right) && \text{Evaluate the antiderivative.}\\ &=\frac{\pi}{4}. && \text{Evaluate the limit and simplify.}\end{aligned}$$

The improper integral converges to $\frac{\pi}{4}$.

Example 7.8.5: Evaluating an Improper Integral on $(-\infty,+\infty)$

Evaluate $\int_{-\infty}^{+\infty} xe^x\,dx$. State whether the improper integral converges or diverges.

Solution

Start by splitting up the integral:

$$\int_{-\infty}^{+\infty} xe^x\,dx = \int_{-\infty}^{0} xe^x\,dx + \int_{0}^{+\infty} xe^x\,dx.$$

If either $\int_{-\infty}^{0} xe^x\,dx$ or $\int_{0}^{+\infty} xe^x\,dx$ diverges, then $\int_{-\infty}^{+\infty} xe^x\,dx$ diverges.

Compute each integral separately.

For the first integral,

$$\begin{aligned}
\int_{-\infty}^{0} xe^x\,dx &= \lim_{t\to-\infty}\int_{t}^{0} xe^x\,dx && \text{Rewrite as a limit.}\\
&= \lim_{t\to-\infty}(xe^x - e^x)\Big|_{t}^{0} && \text{Use integration by parts to find the antiderivative. (Here } u = x \text{ and } dv = e^x\,dx.\text{)}\\
&= \lim_{t\to-\infty}(-1 - te^t + e^t) && \text{Evaluate the antiderivative.}\\
&= -1.
\end{aligned}$$

Evaluate the limit. Note: $\lim_{t\to-\infty} te^t$ is indeterminate of the form $0\cdot\infty$.Thus, $\lim_{t\to-\infty} te^t = \lim_{t\to-\infty}\frac{t}{e^{-t}} = \lim_{t\to-\infty}\frac{-1}{e^{-t}} = \lim_{t\to-\infty} -e^t = 0$ by L'Hôpital's Rule.

The first improper integral converges. For the second integral,

$$\begin{aligned}
\int_{0}^{+\infty} xe^x\,dx &= \lim_{t\to+\infty}\int_{0}^{t} xe^x\,dx && \text{Rewrite as a limit.}\\
&= \lim_{t\to+\infty}(xe^x - e^x)\Big|_{0}^{t} && \text{Find the antiderivative.}\\
&= \lim_{t\to+\infty}(te^t - e^t + 1) && \text{Evaluate the antiderivative.}\\
&= \lim_{t\to+\infty}((t-1)e^t + 1) && \text{Rewrite. } (te^t - e^t \text{ is indeterminate.})\\
&= +\infty. && \text{Evaluate the limit.}
\end{aligned}$$

Thus, $\int_{0}^{+\infty} xe^x\,dx$ diverges. Since this integral diverges, $\int_{-\infty}^{+\infty} xe^x\,dx$ diverges as well.

? Exercise 7.8.1

Evaluate $\int_{-3}^{+\infty} e^{-x}\,dx$. State whether the improper integral converges or diverges.

Hint

$$\int_{-3}^{+\infty} e^{-x}\,dx = \lim_{t\to+\infty}\int_{-3}^{t} e^{-x}\,dx$$

Answer

It converges to e^3.

Integrating a Discontinuous Integrand

Now let's examine integrals of functions containing an infinite discontinuity in the interval over which the integration occurs. Consider an integral of the form $\int_a^b f(x)\,dx$, where $f(x)$ is continuous over $[a,b)$ and discontinuous at b. Since the function $f(x)$ is continuous over $[a,t]$ for all values of t satisfying $a \le t < b$, the integral $\int_a^t f(x)\,dx$ is defined for all such values of t. Thus, it makes sense to consider the values of $\int_a^t f(x)\,dx$ as t approaches b for $a \le t < b$. That is, we define $\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$, provided this limit exists. Figure 7.8.5 illustrates $\int_a^t f(x)\,dx$ as areas of regions for values of t approaching b.

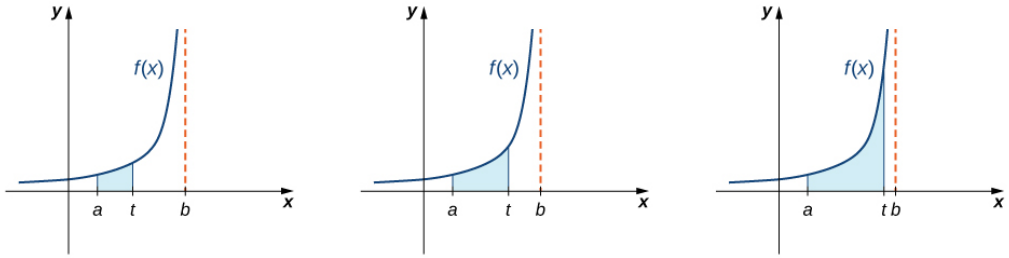


Figure 7.8.5: As t approaches b from the left, the value of the area from a to t approaches the area from a to b.

We use a similar approach to define $\int_a^b f(x)\,dx$, where $f(x)$ is continuous over $(a, b]$ and discontinuous at a. We now proceed with a formal definition.

Definition: Converging and Diverging Improper Integral

1. Let $f(x)$ be continuous over $[a, b)$. Then,

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx. \tag{7.8.4}$$

2. Let $f(x)$ be continuous over $(a, b]$. Then,

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx. \tag{7.8.5}$$

In each case, if the limit exists, then the improper integral is said to converge. If the limit does not exist, then the improper integral is said to diverge.

3. If $f(x)$ is continuous over $[a, b]$ except at a point c in (a, b), then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \tag{7.8.6}$$

provided both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ converge. If either of these integrals diverges, then $\int_a^b f(x)\,dx$ diverges.

The following examples demonstrate the application of this definition.

Example 7.8.6: Integrating a Discontinuous Integrand

Evaluate $\int_0^4 \frac{1}{\sqrt{4-x}}\,dx$, if possible. State whether the integral converges or diverges.

Solution

The function $f(x) = \frac{1}{\sqrt{4-x}}$ is continuous over $[0, 4)$ and discontinuous at 4. Using Equation 7.8.4 from the definition, rewrite $\int_0^4 \frac{1}{\sqrt{4-x}}\,dx$ as a limit:

$$\begin{aligned}
\int_0^4 \frac{1}{\sqrt{4-x}}\,dx &= \lim_{t\to 4^-} \int_0^t \frac{1}{\sqrt{4-x}}\,dx && \text{Rewrite as a limit.}\\
&= \lim_{t\to 4^-} \left(-2\sqrt{4-x}\right)\Big|_0^t && \text{Find the antiderivative.}\\
&= \lim_{t\to 4^-} \left(-2\sqrt{4-t}+4\right) && \text{Evaluate the antiderivative.}\\
&= 4. && \text{Evaluate the limit.}
\end{aligned}$$

The improper integral converges.

Example 7.8.7: Integrating a Discontinuous Integrand

Evaluate $\int_0^2 x \ln x \, dx$. State whether the integral converges or diverges.

Solution

Since $f(x) = x \ln x$ is continuous over $(0, 2]$ and is discontinuous at zero, we can rewrite the integral in limit form using Equation 7.8.5:

$$\int_0^2 x \ln x \, dx = \lim_{t \to 0^+} \int_t^2 x \ln x \, dx \qquad \text{Rewrite as a limit.}$$

$$= \lim_{t \to 0^+} \left(\frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 \right) \Big|_t^2 \qquad \text{Evaluate } \int x \ln x \, dx \text{ using integration by parts with } u = \ln x \text{ and } dv = x.$$

$$= \lim_{t \to 0^+} \left(2 \ln 2 - 1 - \frac{1}{2} t^2 \ln t + \frac{1}{4} t^2 \right). \qquad \text{Evaluate the antiderivative.}$$

$$= 2 \ln 2 - 1. \qquad \text{Evaluate the limit. Note that } \lim_{t \to 0^+} t^2 \ln t \text{ is indeterminate.}$$

To evaluate it, rewrite it as a quotient and apply L'Hôpital's rule.

The improper integral converges.

Example 7.8.8: Integrating a Discontinuous Integrand

Evaluate $\int_{-1}^1 \frac{1}{x^3} \, dx$. State whether the improper integral converges or diverges.

Solution

Since $f(x) = 1/x^3$ is discontinuous at zero, using Equation 7.8.6, we can write

$$\int_{-1}^1 \frac{1}{x^3} \, dx = \int_{-1}^0 \frac{1}{x^3} \, dx \quad + \quad \int_0^1 \frac{1}{x^3} \, dx.$$

If either of the two integrals diverges, then the original integral diverges. Begin with $\int_{-1}^0 \frac{1}{x^3} \, dx$:

$$\int_{-1}^0 \frac{1}{x^3} \, dx = \lim_{t \to 0^-} \int_{-1}^t \frac{1}{x^3} \, dx \qquad \text{Rewrite as a limit.}$$

$$= \lim_{t \to 0^-} \left(-\frac{1}{2x^2} \right) \Big|_{-1}^t \qquad \text{Find the antiderivative.}$$

$$= \lim_{t \to 0^-} \left(-\frac{1}{2t^2} + \frac{1}{2} \right) \qquad \text{Evaluate the antiderivative.}$$

$$= +\infty. \qquad \text{Evaluate the limit.}$$

Therefore, $\int_{-1}^0 \frac{1}{x^3} \, dx$ diverges.

Since $\int_{-1}^0 \frac{1}{x^3} \, dx$ diverges, $\int_{-1}^1 \frac{1}{x^3} \, dx$ also diverges.

Exercise 7.8.2

Evaluate $\int_0^2 \frac{1}{x} \, dx$. State whether the integral converges or diverges.

Hint

Write $\int_0^2 \frac{1}{x} \, dx$ in limit form using Equation 7.8.5

Answer

$+\infty$, It diverges.

A Comparison Theorem

It is not always easy or even possible to evaluate an improper integral directly; however, by comparing it with another carefully chosen integral, it may be possible to determine its convergence or divergence. To see this, consider two continuous functions $f(x)$ and $g(x)$ satisfying $0 \le f(x) \le g(x)$ for $x \ge a$ (Figure 7.8.6). In this case, we may view integrals of these functions over intervals of the form $[a, t]$ as areas, so we have the relationship

$$0 \le \int_a^t f(x)\,dx \le \int_a^t g(x)\,dx$$

for $t \ge a$.

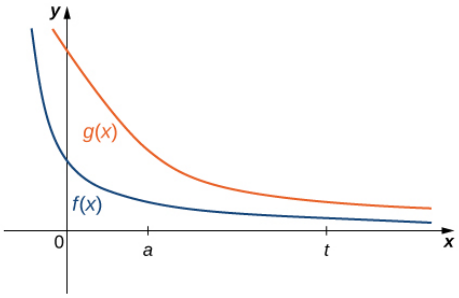


Figure 7.8.6: If $0 \le f(x) \le g(x)$ for $x \ge a$, then for $t \ge a$, $\int_a^t f(x)\,dx \le \int_a^t g(x)\,dx$.

Thus, if

$$\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty} \int_a^t f(x)\,dx = +\infty,$$

then

$$\int_a^{+\infty} g(x)\,dx = \lim_{t\to+\infty} \int_a^t g(x)\,dx = +\infty$$

as well. That is, if the area of the region between the graph of $f(x)$ and the x-axis over $[a, +\infty)$ is infinite, then the area of the region between the graph of $g(x)$ and the x-axis over $[a, +\infty)$ is infinite too.

On the other hand, if

$$\int_a^{+\infty} g(x)\,dx = \lim_{t\to+\infty} \int_a^t g(x)\,dx = L$$

for some real number L, then

$$\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty} \int_a^t f(x)\,dx$$

must converge to some value less than or equal to L, since $\int_a^t f(x)\,dx$ increases as t increases and $\int_a^t f(x)\,dx \le L$ for all $t \ge a$.

If the area of the region between the graph of $g(x)$ and the x-axis over $[a, +\infty)$ is finite, then the area of the region between the graph of $f(x)$ and the x-axis over $[a, +\infty)$ is also finite.

These conclusions are summarized in the following theorem.

A Comparison Theorem

Let $f(x)$ and $g(x)$ be continuous over $[a, +\infty)$. Assume that $0 \le f(x) \le g(x)$ for $x \ge a$.

i. If

$$\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty} \int_a^t f(x)\,dx = +\infty,$$

then

$$\int_a^{+\infty} g(x)\,dx = \lim_{t\to+\infty} \int_a^t g(x)\,dx = +\infty.$$

ii. If

$$\int_a^{+\infty} g(x)\,dx = \lim_{t\to+\infty}\int_a^t g(x)\,dx = L,$$

where L is a real number, then

$$\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty}\int_a^t f(x)\,dx = M$$

for some real number $M \le L$.

Example 7.8.9: Applying the Comparison Theorem

Use a comparison to show that $\int_1^{+\infty} \frac{1}{xe^x}\,dx$.

converges.

Solution

We can see that $0 \le \frac{1}{xe^x} \le \frac{1}{e^x} = e^{-x}$,

so if $\int_1^{+\infty} e^{-x}\,dx$ converges, then so does $\int_1^{+\infty} \frac{1}{xe^x}\,dx$.

To evaluate $\int_1^{+\infty} e^{-x}\,dx$, first rewrite it as a limit:

$$\begin{aligned}\int_1^{+\infty} e^{-x}\,dx &= \lim_{t\to+\infty}\int_1^t e^{-x}\,dx \\ &= \lim_{t\to+\infty}(-e^{-x})\Big|_1^t \\ &= \lim_{t\to+\infty}(-e^{-t}+e^{-1}) \\ &= e^{-1}.\end{aligned}$$

Since $\int_1^{+\infty} e^{-x}\,dx$ converges, so does $\int_1^{+\infty} \frac{1}{xe^x}\,dx$.

Example 7.8.10: Applying the Comparison Theorem

Use the comparison theorem to show that $\int_1^{+\infty} \frac{1}{x^p}\,dx$ diverges for all $p < 1$.

Solution

For $p < 1$, $\frac{1}{x} \le \frac{1}{x^p}$ over $[1, +\infty)$.

In Example 7.8.1, we showed that $\int_1^{+\infty} \frac{1}{x}\,dx = +\infty$.

Therefore, $\int_1^{+\infty} \frac{1}{x^p}\,dx$ diverges for all $p < 1$.

Exercise 7.8.3

Use a comparison to show that $\int_e^{+\infty} \frac{\ln x}{x}\,dx$ diverges.

Hint

$\frac{1}{x} \le \frac{\ln x}{x}$ on $[e, +\infty)$

Answer

Since $\int_e^{+\infty} \frac{1}{x}\,dx = +\infty$, $\int_e^{+\infty} \frac{\ln x}{x}\,dx$ diverges.

Laplace Transforms

Laplace Transforms

In the last few chapters, we have looked at several ways to use integration for solving real-world problems. For this next project, we are going to explore a more advanced application of integration: integral transforms. Specifically, we describe the Laplace transform and some of its properties. The **Laplace transform** is used in engineering and physics to simplify the computations needed to solve some problems. It takes functions expressed in terms of time and *transforms* them to functions expressed in terms of frequency. It turns out that, in many cases, the computations needed to solve problems in the frequency domain are much simpler than those required in the time domain.

The Laplace transform is defined in terms of an integral as

$$\mathscr{L}\big(f(t)\big) = F(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Note that the input to a Laplace transform is a function of time, $f(t)$, and the output is a function of frequency, $F(s)$. Although many real-world examples require the use of complex numbers (involving the imaginary number $i = \sqrt{-1}$), in this project we limit ourselves to functions of real numbers.

Let's start with a simple example. Here we calculate the Laplace transform of $f(t) = t$. We have

$$\mathscr{L}(t) = \int_0^\infty te^{-st}\, dt.$$

This is an improper integral, so we express it in terms of a limit, which gives

$$\mathscr{L}(t) = \int_0^\infty te^{-st}\, dt = \lim_{z\to\infty} \int_0^z te^{-st}\, dt.$$

Now we use integration by parts to evaluate the integral. Note that we are integrating with respect to t, so we treat the variable s as a constant. We have

$$\begin{aligned} u &= t & dv &= e^{-st}\, dt \\ du &= dt & v &= -\tfrac{1}{s}e^{-st} \end{aligned}$$

Then we obtain

$$\begin{aligned} \lim_{z\to\infty} \int_0^z te^{-st}\, dt &= \lim_{z\to\infty} \left[-\frac{t}{s}e^{-st}\Big|_0^z + \frac{1}{s}\int_0^z e^{-st}\, dt \right] \\ &= \lim_{z\to\infty} \left[\left[-\frac{z}{s}e^{-sz} + \frac{0}{s}e^{-0s} \right] + \frac{1}{s}\int_0^z e^{-st}\, dt \right] \\ &= \lim_{z\to\infty} \left[\left[-\frac{z}{s}e^{-sz} + 0 \right] - \frac{1}{s}\left[\frac{e^{-st}}{s} \right] \Big|_0^z \right] \\ &= \lim_{z\to\infty} \left[-\frac{z}{s}e^{-sz} - \frac{1}{s^2}\left[e^{-sz} - 1 \right] \right] \\ &= \lim_{z\to\infty} \left[-\frac{z}{se^{sz}} \right] - \lim_{z\to\infty} \frac{1}{s^2e^{sz}} + \lim_{z\to\infty} \frac{1}{s^2} \\ &= 0 - 0 + \frac{1}{s^2} \qquad \text{Note that the first limit requires L'Hôpital's rule. See work below.} \\ &= \frac{1}{s^2}. \end{aligned}$$

As noted above, $\lim\limits_{z\to\infty} \left[-\dfrac{z}{se^{sz}} \right]$ is indeterminate of the form $-\dfrac{\infty}{\infty}$ and requires L'Hôpital's rule to evaluate. Using L'Hôpital's rule,

$$\lim_{z\to\infty} \left[-\frac{z}{se^{sz}} \right] = \lim_{z\to\infty} \left[-\frac{1}{s^2e^{sz}} \right] = 0.$$

1. Calculate the Laplace transform of $f(t) = 1$.
2. Calculate the Laplace transform of $f(t) = e^{-3t}$.
3. Calculate the Laplace transform of $f(t) = t^2$. (Note, you will have to integrate by parts twice.)

Laplace transforms are often used to solve differential equations. Differential equations are not covered in detail until later in this book; but, for now, let's look at the relationship between the Laplace transform of a function and the Laplace transform of its derivative.

Let's start with the definition of the Laplace transform. We have

$$\mathscr{L}\big(f(t)\big) = \int_0^{\infty} e^{-st} f(t)\, dt = \lim_{z\to\infty} \int_0^{z} e^{-st} f(t)\, dt.$$

Use integration by parts to evaluate $\lim_{z\to\infty} \int_0^{z} e^{-st} f(t)\, dt$. (Let $u = f(t)$ and $dv = e^{-st}\, dt$.)

After integrating by parts and evaluating the limit, you should see that

$$\mathscr{L}\big(f(t)\big) = \frac{f(0)}{s} + \frac{1}{s}\big[\mathscr{L}\big(f'(t)\big)\big].$$

Then,

$$\mathscr{L}\big(f'(t)\big) = s\mathscr{L}\big(f(t)\big) - f(0).$$

Thus, differentiation in the time domain simplifies to multiplication by s in the frequency domain.

The final thing we look at in this project is how the Laplace transforms of $f(t)$ and its antiderivative are related. Let $g(t) = \int_0^{t} f(u)\, du$.

Then,

$$\mathscr{L}\big(g(t)\big) = \int_0^{\infty} e^{-st} g(t)\, dt = \lim_{z\to\infty} \int_0^{z} e^{-st} g(t)\, dt.$$

Use integration by parts to evaluate $\lim_{z\to\infty} \int_0^{z} e^{-st} g(t)\, dt$. (Let $u = g(t)$ and $dv = e^{-st}\, dt$. Note, by the way that we have defined $g(t)$, $du = f(t)\, dt$.)

As you might expect, you should see that

$$\mathscr{L}\big(g(t)\big) = \frac{1}{s} \cdot \mathscr{L}\big(f(t)\big).$$

Integration in the time domain simplifies to division by s in the frequency domain.

Key Concepts

- Integrals of functions over infinite intervals are defined in terms of limits.
- Integrals of functions over an interval for which the function has a discontinuity at an endpoint may be defined in terms of limits.
- The convergence or divergence of an improper integral may be determined by comparing it with the value of an improper integral for which the convergence or divergence is known.

Key Equations

- **Improper integrals**

$$\int_a^{+\infty} f(x)\, dx = \lim_{t\to+\infty} \int_a^{t} f(x)\, dx$$

$$\int_{-\infty}^{b} f(x)\, dx = \lim_{t\to-\infty} \int_t^{b} f(x)\, dx$$

$$\int_{-\infty}^{+\infty} f(x)\, dx = \int_{-\infty}^{0} f(x)\, dx + \int_0^{+\infty} f(x)\, dx$$

Glossary

improper integral

an integral over an infinite interval or an integral of a function containing an infinite discontinuity on the interval; an improper integral is defined in terms of a limit. The improper integral converges if this limit is a finite real number; otherwise, the improper integral diverges

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CHAPTER OVERVIEW

8: Further Applications of Integration

A general Calculus Textmap organized around the textbook

Calculus: Early Transcendentals

by James Stewart

I II III IV V VI VII VIII IX X XI XII XIII XIV XV XVI XVII

This Textmap is currently under construction... please be patient with us.

Template:HideTOC

Topic hierarchy

8.1: Arc Length
8.2: Area of a Surface of Revolution
8.3: Applications to Physics and Engineering
8.4: Applications to Economics and Biology
8.5: Probability

Thumbnail: Volume of a solid of revolution can be calculated via integration techniques. Image used with permission (GPLC, Matt Boelkins, David Austin, and Steve Schlicker, Grand Valley State University).

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8.1: Arc Length

Learning Objectives

- Determine the length of a curve, $y = f(x)$, between two points.
- Determine the length of a curve, $x = g(y)$, between two points.
- Find the surface area of a solid of revolution.

In this section, we use definite integrals to find the arc length of a curve. We can think of **arc length** as the distance you would travel if you were walking along the path of the curve. Many real-world applications involve arc length. If a rocket is launched along a parabolic path, we might want to know how far the rocket travels. Or, if a curve on a map represents a road, we might want to know how far we have to drive to reach our destination.

We begin by calculating the arc length of curves defined as functions of x, then we examine the same process for curves defined as functions of y. (The process is identical, with the roles of x and y reversed.) The techniques we use to find arc length can be extended to find the surface area of a surface of revolution, and we close the section with an examination of this concept.

Arc Length of the Curve y = f(x)

In previous applications of integration, we required the function $f(x)$ to be integrable, or at most continuous. However, for calculating arc length we have a more stringent requirement for $f(x)$. Here, we require $f(x)$ to be differentiable, and furthermore we require its derivative, $f'(x)$, to be continuous. Functions like this, which have continuous derivatives, are called **smooth.** (This property comes up again in later chapters.)

Let $f(x)$ be a smooth function defined over $[a, b]$. We want to calculate the length of the curve from the point $(a, f(a))$ to the point $(b, f(b))$. We start by using line segments to approximate the length of the curve. For $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of $[a, b]$. Then, for $i = 1, 2, \ldots, n$, construct a line segment from the point $(x_{i-1}, f(x_{i-1}))$ to the point $(x_i, f(x_i))$. Although it might seem logical to use either horizontal or vertical line segments, we want our line segments to approximate the curve as closely as possible. Figure 8.1.1 depicts this construct for $n = 5$.

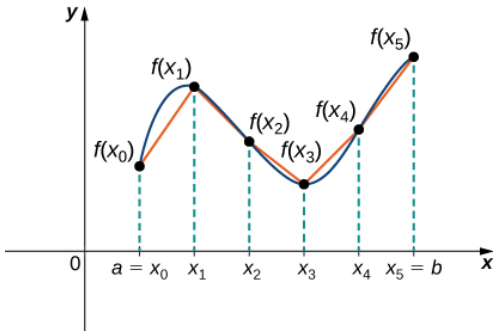


Figure 8.1.1: We can approximate the length of a curve by adding line segments.

To help us find the length of each line segment, we look at the change in vertical distance as well as the change in horizontal distance over each interval. Because we have used a regular partition, the change in horizontal distance over each interval is given by Δx. The change in vertical distance varies from interval to interval, though, so we use $\Delta y_i = f(x_i) - f(x_{i-1})$ to represent the change in vertical distance over the interval $[x_{i-1}, x_i]$, as shown in Figure 8.1.2. Note that some (or all) Δy_i may be negative.

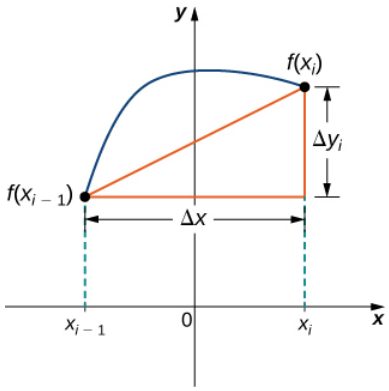


Figure 8.1.2: A representative line segment approximates the curve over the interval $[x_{i-1}, x_i]$.

By the Pythagorean theorem, the length of the line segment is

$$\sqrt{(\Delta x)^2 + (\Delta y_i)^2}.$$

We can also write this as

$$\Delta x \sqrt{1 + ((\Delta y_i)/(\Delta x))^2}.$$

Now, by the Mean Value Theorem, there is a point $x_i^* \in [x_{i-1}, x_i]$ such that $f'(x_i^*) = (\Delta y_i)/(\Delta x)$. Then the length of the line segment is given by

$$\Delta x \sqrt{1 + [f'(x_i^*)]^2}.$$

Adding up the lengths of all the line segments, we get

$$\text{Arc Length} \approx \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\,\Delta x.$$

This is a Riemann sum. Taking the limit as $n \to \infty$, we have

$$\begin{aligned} \text{Arc Length} &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\,\Delta x \\ &= \int_a^b \sqrt{1 + [f'(x)]^2}\,dx. \end{aligned}$$

We summarize these findings in the following theorem.

Arc Length for $y = f(x)$

Let $f(x)$ be a smooth function over the interval $[a, b]$. Then the arc length of the portion of the graph of $f(x)$ from the point $(a, f(a))$ to the point $(b, f(b))$ is given by

$$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx.$$

Note that we are integrating an expression involving $f'(x)$, so we need to be sure $f'(x)$ is integrable. This is why we require $f(x)$ to be smooth. The following example shows how to apply the theorem.

Example 8.1.1: Calculating the Arc Length of a Function of x

Let $f(x) = 2x^{3/2}$. Calculate the arc length of the graph of $f(x)$ over the interval $[0, 1]$. Round the answer to three decimal places.

Solution

We have $f'(x) = 3x^{1/2}$, so $[f'(x)]^2 = 9x$. Then, the arc length is

$$\text{Arc Length} = \int_a^b \sqrt{1+[f'(x)]^2}dx$$
$$= \int_0^1 \sqrt{1+9x}\, dx.$$

Substitute $u = 1+9x$. Then, $du = 9\, dx$. When $x = 0$, then $u = 1$, and when $x = 1$, then $u = 10$. Thus,

$$\text{Arc Length} = \int_0^1 \sqrt{1+9x}\, dx$$
$$= \frac{1}{9}\int_0^1 \sqrt{1+9x}\cdot 9\, dx$$
$$= \frac{1}{9}\int_1^{10} \sqrt{u}\, du$$
$$= \frac{1}{9}\cdot\frac{2}{3}u^{3/2}\Big|_1^{10} = \frac{2}{27}[10\sqrt{10}-1]$$
$$\approx 2.268 \text{ units}.$$

Exercise 8.1.1

Let $f(x) = (4/3)x^{3/2}$. Calculate the arc length of the graph of $f(x)$ over the interval $[0, 1]$. Round the answer to three decimal places.

Hint

Use the process from the previous example. Don't forget to change the limits of integration.

Answer

$$\frac{1}{6}(5\sqrt{5}-1) \approx 1.697 \text{ units}$$

Although it is nice to have a formula for calculating arc length, this particular theorem can generate expressions that are difficult to integrate. We study some techniques for integration in Introduction to Techniques of Integration. In some cases, we may have to use a computer or calculator to approximate the value of the integral.

Example 8.1.2: Using a Computer or Calculator to Determine the Arc Length of a Function of x

Let $f(x) = x^2$. Calculate the arc length of the graph of $f(x)$ over the interval $[1, 3]$.

Solution

We have $f'(x) = 2x$, so $[f'(x)]^2 = 4x^2$. Then the arc length is given by

$$\text{Arc Length} = \int_a^b \sqrt{1+[f'(x)]^2}\, dx$$
$$= \int_1^3 \sqrt{1+4x^2}\, dx.$$

Using a computer to approximate the value of this integral, we get

$$\int_1^3 \sqrt{1+4x^2}\, dx \approx 8.26815 \text{ units}.$$

Exercise 8.1.2

Let $f(x) = \sin x$. Calculate the arc length of the graph of $f(x)$ over the interval $[0, \pi]$. Use a computer or calculator to approximate the value of the integral.

Hint

Use the process from the previous example.

Answer

$$\text{Arc Length} \approx 3.8202 \text{ units}$$

Arc Length of the Curve x = g(y)

We have just seen how to approximate the length of a curve with line segments. If we want to find the arc length of the graph of a function of y, we can repeat the same process, except we partition the y-axis instead of the x-axis. Figure 8.1.3 shows a representative line segment.

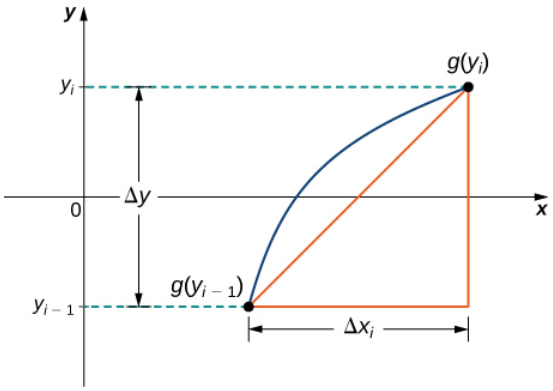


Figure 8.1.3: A representative line segment over the interval $[y_{i-1}, y_i]$.

Then the length of the line segment is

$$\sqrt{(\Delta y)^2 + (\Delta x_i)^2},$$

which can also be written as

$$\Delta y \sqrt{1 + \left(\frac{\Delta x_i}{\Delta y}\right)^2}.$$

If we now follow the same development we did earlier, we get a formula for arc length of a function $x = g(y)$.

Arc Length for $x = g(y)$

Let $g(y)$ be a smooth function over an interval $[c, d]$. Then, the arc length of the graph of $g(y)$ from the point $(c, g(c))$ to the point $(d, g(d))$ is given by

$$\text{Arc Length} = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy.$$

Example 8.1.3: Calculating the Arc Length of a Function of y

Let $g(y) = 3y^3$. Calculate the arc length of the graph of $g(y)$ over the interval $[1, 2]$.

Solution

We have $g'(y) = 9y^2$, so $[g'(y)]^2 = 81y^4$. Then the arc length is

$$\text{Arc Length} = \int_c^d \sqrt{1+[g'(y)]^2}\, dy$$
$$= \int_1^2 \sqrt{1+81y^4}\, dy.$$

Using a computer to approximate the value of this integral, we obtain

$$\int_1^2 \sqrt{1+81y^4} dy \approx 21.0277 \text{ units}.$$

? Exercise 8.1.3

Let $g(y) = 1/y$. Calculate the arc length of the graph of $g(y)$ over the interval $[1, 4]$. Use a computer or calculator to approximate the value of the integral.

Hint

Use the process from the previous example.

Answer

$$\text{Arc Length} = 3.15018 \text{ units}$$

Area of a Surface of Revolution

The concepts we used to find the arc length of a curve can be extended to find the surface area of a surface of revolution. **Surface area** is the total area of the outer layer of an object. For objects such as cubes or bricks, the surface area of the object is the sum of the areas of all of its faces. For curved surfaces, the situation is a little more complex. Let $f(x)$ be a nonnegative smooth function over the interval $[a, b]$. We wish to find the surface area of the surface of revolution created by revolving the graph of $y = f(x)$ around the x-axis as shown in the following figure.

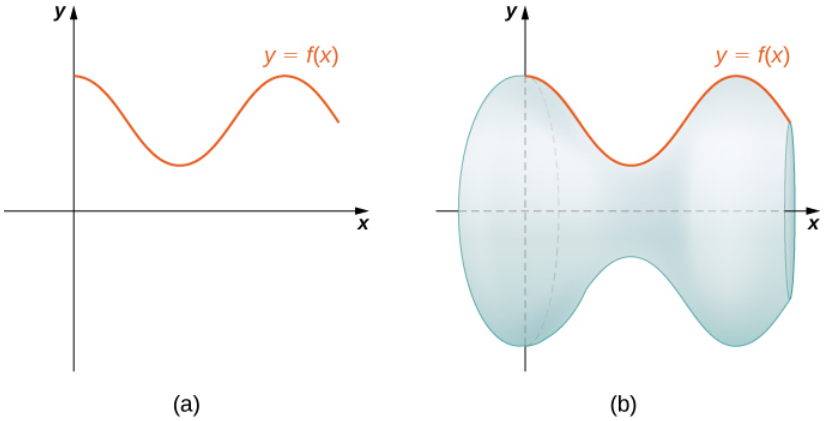


Figure 8.1.4: (a) A curve representing the function $f(x)$. (b) The surface of revolution formed by revolving the graph of $f(x)$ around the $x-axis$.

As we have done many times before, we are going to partition the interval $[a, b]$ and approximate the surface area by calculating the surface area of simpler shapes. We start by using line segments to approximate the curve, as we did earlier in this section. For $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of $[a, b]$. Then, for $i = 1, 2, \ldots, n$, construct a line segment from the point $(x_{i-1}, f(x_{i-1}))$ to the point $(x_i, f(x_i))$. Now, revolve these line segments around the x-axis to generate an approximation of the surface of revolution as shown in the following figure.

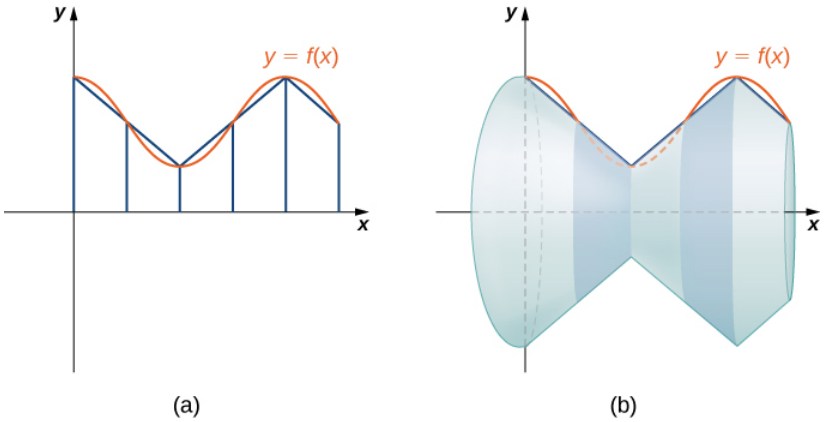


Figure 8.1.5: (a) Approximating $f(x)$ with line segments. (b) The surface of revolution formed by revolving the line segments around the x-axis.

Notice that when each line segment is revolved around the axis, it produces a band. These bands are actually pieces of cones (think of an ice cream cone with the pointy end cut off). A piece of a cone like this is called a **frustum** of a cone.

To find the surface area of the band, we need to find the lateral surface area, S, of the frustum (the area of just the slanted outside surface of the frustum, not including the areas of the top or bottom faces). Let r_1 and r_2 be the radii of the wide end and the narrow end of the frustum, respectively, and let l be the slant height of the frustum as shown in the following figure.

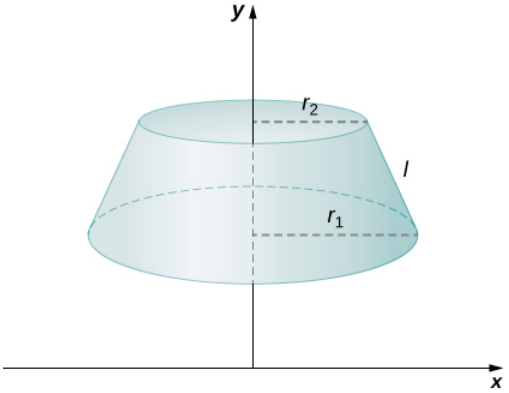


Figure 8.1.6: A frustum of a cone can approximate a small part of surface area.

We know the lateral surface area of a cone is given by

$$\text{Lateral Surface Area } = \pi r s,$$

where r is the radius of the base of the cone and s is the **slant height** (Figure 8.1.7).

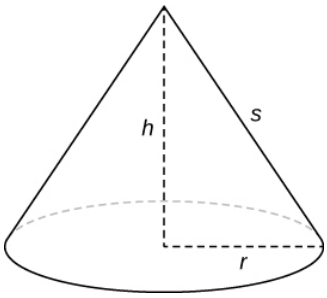


Figure 8.1.7: The lateral surface area of the cone is given by $\pi r s$.

Since a frustum can be thought of as a piece of a cone, the lateral surface area of the frustum is given by the lateral surface area of the whole cone less the lateral surface area of the smaller cone (the pointy tip) that was cut off (Figure 8.1.8).

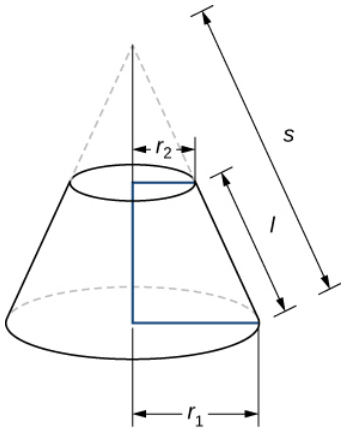


Figure 8.1.8: Calculating the lateral surface area of a frustum of a cone.

The cross-sections of the small cone and the large cone are similar triangles, so we see that

$$\frac{r_2}{r_1} = \frac{s-l}{s}$$

Solving for s, we get

$$\begin{aligned} \frac{r_2}{r_1} &= \frac{s-l}{s} \\ r_2 s &= r_1(s-l) \\ r_2 s &= r_1 s - r_1 l \\ r_1 l &= r_1 s - r_2 s \\ r_1 l &= (r_1 - r_2)s \\ \frac{r_1 l}{r_1 - r_2} &= s \end{aligned}$$

Then the lateral surface area (SA) of the frustum is

$$\begin{aligned} S &= (\text{Lateral SA of large cone}) - (\text{Lateral SA of small cone}) \\ &= \pi r_1 s - \pi r_2 (s-l) \\ &= \pi r_1 \left(\frac{r_1 l}{r_1 - r_2}\right) - \pi r_2 \left(\frac{r_1 l}{r_1 - r_2} - l\right) \\ &= \frac{\pi r_1^2 l}{r^1 - r^2} - \frac{\pi r_1 r_2 l}{r_1 - r_2} + \pi r_2 l \\ &= \frac{\pi r_1^2 l}{r_1 - r_2} - \frac{\pi r_1 r 2_l}{r_1 - r_2} + \frac{\pi r_2 l (r_1 - r_2)}{r_1 - r_2} \\ &= \frac{\pi r_1^2}{l r_1 - r_2} - \frac{\pi r_1 r_2 l}{r_1 - r_2} + \frac{\pi r_1 r_2 l}{r_1 - r_2} - \frac{\pi r_2^2 l}{r_1 - r_3} \\ &= \frac{\pi(r_1^2 - r_2^2)l}{r_1 - r_2} = \frac{\pi(r_1 - r + 2)(r1 + r2)l}{r_1 - r_2} \\ &= \pi(r_1 + r_2)l. \end{aligned}$$

Let's now use this formula to calculate the surface area of each of the bands formed by revolving the line segments around the x-axis. A representative band is shown in the following figure.

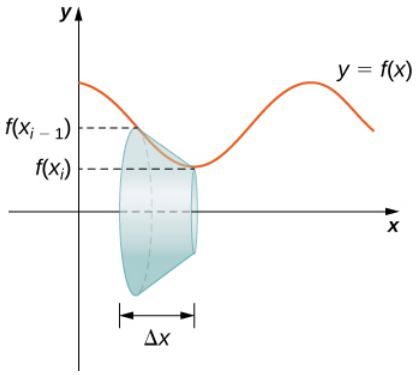


Figure 8.1.9: A representative band used for determining surface area.

Note that the slant height of this frustum is just the length of the line segment used to generate it. So, applying the surface area formula, we have

$$\begin{aligned} S &= \pi(r_1 + r_2)l \\ &= \pi\big(f(x_{i-1}) + f(x_i)\big)\sqrt{\Delta x^2 + (\Delta y_i)^2} \\ &= \pi\big(f(x_{i-1}) + f(x_i)\big)\Delta x\sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2} \end{aligned}$$

Now, as we did in the development of the arc length formula, we apply the Mean Value Theorem to select $x_i^* \in [x_{i-1}, x_i]$ such that $f\,'(x_i^*) = (\Delta y_i)/\Delta x$. This gives us

$$S = \pi\big(f(x_{i-1}) + f(x_i)\big)\Delta x\sqrt{1 + \big(f\,'(x_i^*)\big)^2}$$

Furthermore, since $f(x)$ is continuous, by the *Intermediate Value Theorem*, there is a point $x_i^{**} \in [x_{i-1}, x_i]$ such that $f(x_i^{**}) = \frac{1}{2}\left[f(x_{i-1}) + f(x_i)\right]$,

so we get

$$S = 2\pi f(x_i^{**})\Delta x\sqrt{1 + \big(f\,'(x_i^*)\big)^2}.$$

Then the approximate surface area of the whole surface of revolution is given by

$$\text{Surface Area} \approx \sum_{i=1}^{n} 2\pi f(x_i^{**})\Delta x\sqrt{1 + \big(f\,'(x_i^*)\big)^2}.$$

This almost looks like a Riemann sum, except we have functions evaluated at two different points, x_i^* and x_i^{**}, over the interval $[x_{i-1}, x_i]$. Although we do not examine the details here, it turns out that because $f(x)$ is smooth, if we let $n \to \infty$, the limit works the same as a Riemann sum even with the two different evaluation points. This makes sense intuitively. Both x_i^* and x_i^{**} are in the interval $[x_{i-1}, x_i]$, so it makes sense that as $n \to \infty$, both x_i^* and x_i^{**} approach x Those of you who are interested in the details should consult an advanced calculus text.

Taking the limit as $n \to \infty$, we get

$$\begin{aligned} \text{Surface Area} &= \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi f(x_i^{**})\Delta x\sqrt{1 + \big(f\,'(x_i^*)\big)^2} \\ &= \int_a^b \left(2\pi f(x)\sqrt{1 + \big(f\,'(x)\big)^2}\right) dx \end{aligned}$$

As with arc length, we can conduct a similar development for functions of y to get a formula for the surface area of surfaces of revolution about the y-axis. These findings are summarized in the following theorem.

Surface Area of a Surface of Revolution

Let $f(x)$ be a nonnegative smooth function over the interval $[a, b]$. Then, the surface area of the surface of revolution formed by revolving the graph of $f(x)$ around the x-axis is given by

$$\text{Surface Area} = \int_a^b \left(2\pi f(x)\sqrt{1+\left(f'(x)\right)^2}\right) dx$$

Similarly, let $g(y)$ be a nonnegative smooth function over the interval $[c, d]$. Then, the surface area of the surface of revolution formed by revolving the graph of $g(y)$ around the y-axis is given by

$$\text{Surface Area} = \int_c^d \left(2\pi g(y)\sqrt{1+\left(g'(y)\right)^2}\right) dy$$

Example 8.1.4: Calculating the Surface Area of a Surface of Revolution 1.

Let $f(x) = \sqrt{x}$ over the interval $[1, 4]$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the x-axis. Round the answer to three decimal places.

Solution

The graph of $f(x)$ and the surface of rotation are shown in Figure 8.1.10

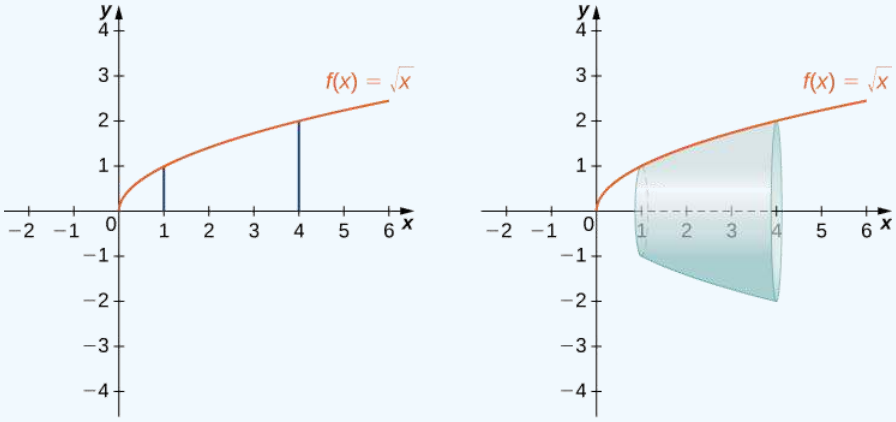


Figure 8.1.10: (a) The graph of $f(x)$. (b) The surface of revolution.

We have $f(x) = \sqrt{x}$. Then, $f'(x) = \frac{1}{2\sqrt{x}}$ and $\left(f'(x)\right)^2 = \frac{1}{4x}$. Then,

$$\begin{aligned}\text{Surface Area} &= \int_a^b \left(2\pi f(x)\sqrt{1+\left(f'(x)\right)^2}\right) dx \\ &= \int_1^4 \left(2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}\right) dx \\ &= \int_1^4 \left(2\pi\sqrt{x+\frac{1}{4}}\right) dx.\end{aligned}$$

Let $u = x + 1/4$. Then, $du = dx$. When $x = 1,\ u = 5/4$, and when $x = 4,\ u = 17/4$. This gives us

$$\begin{aligned}\int_0^1 \left(2\pi\sqrt{x+\frac{1}{4}}\right) dx &= \int_{5/4}^{17/4} 2\pi\sqrt{u}\, du \\ &= 2\pi\left[\frac{2}{3}u^{3/2}\right]\Big|_{5/4}^{17/4} \\ &= \frac{\pi}{6}[17\sqrt{17} - 5\sqrt{5}] \approx 30.846 \text{ units}^2\end{aligned}$$

Exercise 8.1.4

Let $f(x) = \sqrt{1-x}$ over the interval $[0, 1/2]$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the x-axis. Round the answer to three decimal places.

Hint

Use the process from the previous example.

Answer

$$\frac{\pi}{6}(5\sqrt{5} - 3\sqrt{3}) \approx 3.133 \text{ units}^2$$

Example 8.1.5: Calculating the Surface Area of a Surface of Revolution 2

Let $f(x) = y = \sqrt[3]{3x}$. Consider the portion of the curve where $0 \le y \le 2$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the y-axis.

Solution

Notice that we are revolving the curve around the y-axis, and the interval is in terms of y, so we want to rewrite the function as a function of y. We get $x = g(y) = (1/3)y^3$. The graph of $g(y)$ and the surface of rotation are shown in the following figure.

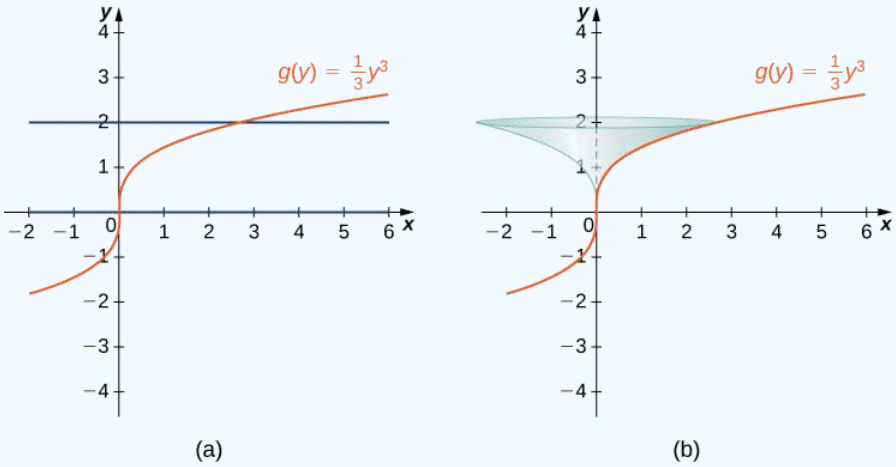


Figure 8.1.11: (a) The graph of $g(y)$. (b) The surface of revolution.

We have $g(y) = (1/3)y^3$, so $g'(y) = y^2$ and $\left(g'(y)\right)^2 = y^4$. Then

$$\begin{aligned} \text{Surface Area} &= \int_c^d \left(2\pi g(y)\sqrt{1+\left(g'(y)\right)^2}\right) dy \\ &= \int_0^2 \left(2\pi\left(\frac{1}{3}y^3\right)\sqrt{1+y^4}\right) dy \\ &= \frac{2\pi}{3}\int_0^2 \left(y^3\sqrt{1+y^4}\right) dy. \end{aligned}$$

Let $u = y^4 + 1$. Then $du = 4y^3\, dy$. When $y = 0$, $u = 1$, and when $y = 2$, $u = 17$. Then

$$\begin{aligned} \frac{2\pi}{3}\int_0^2 \left(y^3\sqrt{1+y^4}\right) dy &= \frac{2\pi}{3}\int_1^{17} \frac{1}{4}\sqrt{u}\, du \\ &= \frac{\pi}{6}\left[\frac{2}{3}u^{3/2}\right]\Big|_1^{17} = \frac{\pi}{9}\left[(17)^{3/2} - 1\right] \approx 24.118 \text{ units}^2. \end{aligned}$$

Exercise 8.1.5

Let $g(y) = \sqrt{9 - y^2}$ over the interval $y \in [0, 2]$. Find the surface area of the surface generated by revolving the graph of $g(y)$ around the y-axis.

Hint

Use the process from the previous example.

Answer

12π units^2

Key Concepts

- The arc length of a curve can be calculated using a definite integral.
- The arc length is first approximated using line segments, which generates a Riemann sum. Taking a limit then gives us the definite integral formula. The same process can be applied to functions of y.
- The concepts used to calculate the arc length can be generalized to find the surface area of a surface of revolution.
- The integrals generated by both the arc length and surface area formulas are often difficult to evaluate. It may be necessary to use a computer or calculator to approximate the values of the integrals.

Key Equations

- **Arc Length of a Function of x**

$$\textbf{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

- **Arc Length of a Function of y**

$$\textbf{Arc Length} = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$$

- **Surface Area of a Function of x**

$$\textbf{Surface Area} = \int_a^b \left(2\pi f(x)\sqrt{1 + \left(f'(x)\right)^2}\right) dx$$

Glossary

arc length

the arc length of a curve can be thought of as the distance a person would travel along the path of the curve

frustum

a portion of a cone; a frustum is constructed by cutting the cone with a plane parallel to the base

surface area

the surface area of a solid is the total area of the outer layer of the object; for objects such as cubes or bricks, the surface area of the object is the sum of the areas of all of its faces

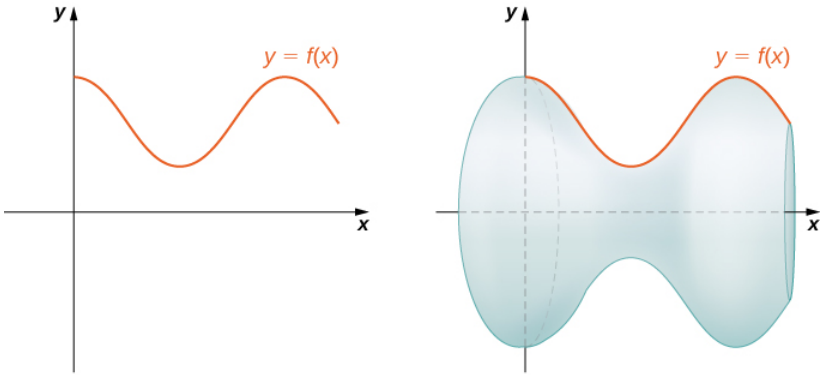
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8.2: Area of a Surface of Revolution

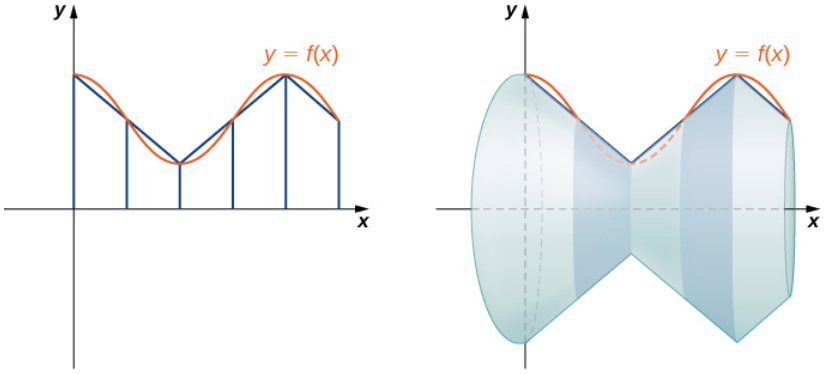
The concepts we used to find the arc length of a curve can be extended to find the surface area of a surface of revolution. **Surface area** is the total area of the outer layer of an object. For objects such as cubes or bricks, the surface area of the object is the sum of the areas of all of its faces. For curved surfaces, the situation is a little more complex. Let $f(x)$ be a nonnegative smooth function over the interval $[a, b]$. We wish to find the surface area of the surface of revolution created by revolving the graph of $y = f(x)$ around the x-axis as shown in the following figure.



(a) (b)

Figure 8.2.4: (a) A curve representing the function $f(x)$. (b) The surface of revolution formed by revolving the graph of $f(x)$ around the $x - axis$.

As we have done many times before, we are going to partition the interval $[a, b]$ and approximate the surface area by calculating the surface area of simpler shapes. We start by using line segments to approximate the curve, as we did earlier in this section. For $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of $[a, b]$. Then, for $i = 1, 2, \ldots, n$, construct a line segment from the point $(x_{i-1}, f(x_{i-1}))$ to the point $(x_i, f(x_i))$. Now, revolve these line segments around the x-axis to generate an approximation of the surface of revolution as shown in the following figure.



(a) (b)

Figure 8.2.5: (a) Approximating $f(x)$ with line segments. (b) The surface of revolution formed by revolving the line segments around the x-axis.

Notice that when each line segment is revolved around the axis, it produces a band. These bands are actually pieces of cones (think of an ice cream cone with the pointy end cut off). A piece of a cone like this is called a **frustum** of a cone.

To find the surface area of the band, we need to find the lateral surface area, S, of the frustum (the area of just the slanted outside surface of the frustum, not including the areas of the top or bottom faces). Let r_1 and r_2 be the radii of the wide end and the narrow end of the frustum, respectively, and let l be the slant height of the frustum as shown in the following figure.

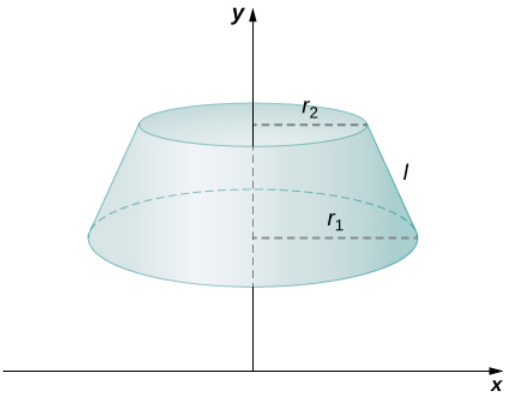


Figure 8.2.6: A frustum of a cone can approximate a small part of surface area.

We know the lateral surface area of a cone is given by

$$\text{Lateral Surface Area } = \pi r s,$$

where r is the radius of the base of the cone and s is the **slant height** (Figure 8.2.7).

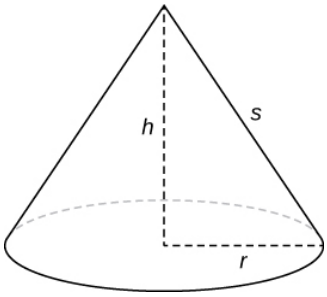


Figure 8.2.7: The lateral surface area of the cone is given by $\pi r s$.

Since a frustum can be thought of as a piece of a cone, the lateral surface area of the frustum is given by the lateral surface area of the whole cone less the lateral surface area of the smaller cone (the pointy tip) that was cut off (Figure 8.2.8).

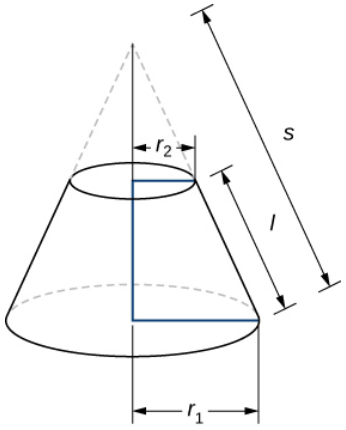


Figure 8.2.8: Calculating the lateral surface area of a frustum of a cone.

The cross-sections of the small cone and the large cone are similar triangles, so we see that

$$\frac{r_2}{r_1} = \frac{s-l}{s}$$

Solving for s, we get

$$\begin{aligned}\frac{r_2}{r_1} &= \frac{s-l}{s} \\ r_2 s &= r_1(s-l) \\ r_2 s &= r_1 s - r_1 l \\ r_1 l &= r_1 s - r_2 s \\ r_1 l &= (r_1 - r_2)s \\ \frac{r_1 l}{r_1 - r_2} &= s\end{aligned}$$

Then the lateral surface area (SA) of the frustum is

$$\begin{aligned}S &= (\text{Lateral SA of large cone}) - (\text{Lateral SA of small cone}) \\ &= \pi r_1 s - \pi r_2 (s-l) \\ &= \pi r_1 \left(\frac{r_1 l}{r_1 - r_2}\right) - \pi r_2 \left(\frac{r_1 l}{r_1 - r_2} - l\right) \\ &= \frac{\pi r_1^2 l}{r^1 - r^2} - \frac{\pi r_1 r_2 l}{r_1 - r_2} + \pi r_2 l \\ &= \frac{\pi r_1^2 l}{r_1 - r_2} - \frac{\pi r_1 r 2_l}{r_1 - r_2} + \frac{\pi r_2 l (r_1 - r_2)}{r_1 - r_2} \\ &= \frac{\pi r_1^2}{l r_1 - r_2} - \frac{\pi r_1 r_2 l}{r_1 - r_2} + \frac{\pi r_1 r_2 l}{r_1 - r_2} - \frac{\pi r_2^2 l}{r_1 - r_3} \\ &= \frac{\pi(r_1^2 - r_2^2)l}{r_1 - r_2} = \frac{\pi(r_1 - r + 2)(r1 + r2)l}{r_1 - r_2} \\ &= \pi(r_1 + r_2)l.\end{aligned}$$

Let's now use this formula to calculate the surface area of each of the bands formed by revolving the line segments around the x-axis. A representative band is shown in the following figure.

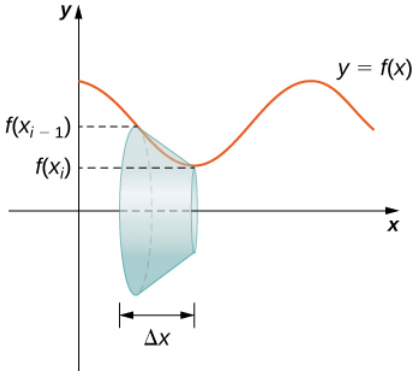


Figure 8.2.9: A representative band used for determining surface area.

Note that the slant height of this frustum is just the length of the line segment used to generate it. So, applying the surface area formula, we have

$$\begin{aligned}S &= \pi(r_1 + r_2)l \\ &= \pi\big(f(x_{i-1}) + f(x_i)\big)\sqrt{\Delta x^2 + (\Delta y_i)^2} \\ &= \pi\big(f(x_{i-1}) + f(x_i)\big)\Delta x\sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}\end{aligned}$$

Now, as we did in the development of the arc length formula, we apply the Mean Value Theorem to select $x_i^* \in [x_{i-1}, x_i]$ such that $f\,'(x_i^*) = (\Delta y_i)/\Delta x$. This gives us

$$S = \pi\big(f(x_{i-1}) + f(x_i)\big)\Delta x\sqrt{1 + \big(f\,'(x_i^*)\big)^2}$$

Furthermore, since $f(x)$ is continuous, by the *Intermediate Value Theorem*, there is a point $x_i^{**} \in [x_{i-1}, x_i]$ such that $f(x_i^{**}) = \frac{1}{2}\left[f(x_{i-1}) + f(x_i)\right]$,

so we get

$$S = 2\pi f(x_i^{**})\Delta x\sqrt{1 + \big(f\,'(x_i^*)\big)^2}.$$

Then the approximate surface area of the whole surface of revolution is given by

$$\text{Surface Area} \approx \sum_{i=1}^{n} 2\pi f(x_i^{**})\Delta x\sqrt{1 + \big(f\,'(x_i^*)\big)^2}.$$

This almost looks like a Riemann sum, except we have functions evaluated at two different points, x_i^* and x_i^{**}, over the interval $[x_{i-1}, x_i]$. Although we do not examine the details here, it turns out that because $f(x)$ is smooth, if we let $n \to \infty$, the limit works the same as a Riemann sum even with the two different evaluation points. This makes sense intuitively. Both x_i^* and x_i^{**} are in the interval $[x_{i-1}, x_i]$, so it makes sense that as $n \to \infty$, both x_i^* and x_i^{**} approach x Those of you who are interested in the details should consult an advanced calculus text.

Taking the limit as $n \to \infty$, we get

$$\begin{aligned}\text{Surface Area} &= \lim_{n\to\infty}\sum_{i=1}^{n} 2\pi f(x_i^{**})\Delta x\sqrt{1 + \big(f\,'(x_i^*)\big)^2} \\ &= \int_a^b \left(2\pi f(x)\sqrt{1 + \big(f\,'(x)\big)^2}\right) dx\end{aligned}$$

As with arc length, we can conduct a similar development for functions of y to get a formula for the surface area of surfaces of revolution about the y-axis. These findings are summarized in the following theorem.

Surface Area of a Surface of Revolution

Let $f(x)$ be a nonnegative smooth function over the interval $[a, b]$. Then, the surface area of the surface of revolution formed by revolving the graph of $f(x)$ around the x-axis is given by

$$\text{Surface Area} = \int_a^b \left(2\pi f(x)\sqrt{1 + \big(f\,'(x)\big)^2}\right) dx$$

Similarly, let $g(y)$ be a nonnegative smooth function over the interval $[c, d]$. Then, the surface area of the surface of revolution formed by revolving the graph of $g(y)$ around the y-axis is given by

$$\text{Surface Area} = \int_c^d \left(2\pi g(y)\sqrt{1 + \big(g'(y)\big)^2}\right) dy$$

Example 8.2.4: Calculating the Surface Area of a Surface of Revolution 1.

Let $f(x) = \sqrt{x}$ over the interval $[1, 4]$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the x-axis. Round the answer to three decimal places.

Solution

The graph of $f(x)$ and the surface of rotation are shown in Figure 8.2.10

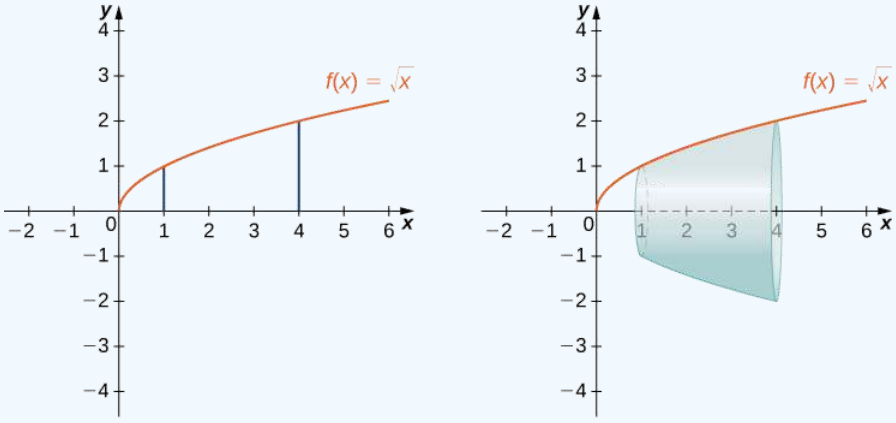


Figure 8.2.10: (a) The graph of $f(x)$. (b) The surface of revolution.

We have $f(x)=\sqrt{x}$. Then, $f'(x)=\frac{1}{2\sqrt{x}}$ and $\left(f'(x)\right)^2=\frac{1}{4x}$. Then,

$$\begin{aligned}\text{Surface Area} &= \int_a^b \left(2\pi f(x)\sqrt{1+\left(f'(x)\right)^2}\right)dx \\ &= \int_1^4 \left(2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}\right)dx \\ &= \int_1^4 \left(2\pi\sqrt{x+\frac{1}{4}}\right)dx.\end{aligned}$$

Let $u=x+1/4$. Then, $du=dx$. When $x=1$, $u=5/4$, and when $x=4$, $u=17/4$. This gives us

$$\begin{aligned}\int_0^1 \left(2\pi\sqrt{x+\frac{1}{4}}\right)dx &= \int_{5/4}^{17/4} 2\pi\sqrt{u}\,du \\ &= 2\pi\left[\frac{2}{3}u^{3/2}\right]\Big|_{5/4}^{17/4} \\ &= \frac{\pi}{6}[17\sqrt{17}-5\sqrt{5}] \approx 30.846 \text{ units}^2\end{aligned}$$

Exercise 8.2.4

Let $f(x)=\sqrt{1-x}$ over the interval $[0,1/2]$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the x-axis. Round the answer to three decimal places.

Hint

Use the process from the previous example.

Answer

$$\frac{\pi}{6}\left(5\sqrt{5}-3\sqrt{3}\right) \approx 3.133 \text{ units}^2$$

Example 8.2.5: Calculating the Surface Area of a Surface of Revolution 2

Let $f(x)=y=\sqrt[3]{3x}$. Consider the portion of the curve where $0\le y\le 2$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the y-axis.

Solution

Notice that we are revolving the curve around the y-axis, and the interval is in terms of y, so we want to rewrite the function as a function of y. We get $x = g(y) = (1/3)y^3$. The graph of $g(y)$ and the surface of rotation are shown in the following figure.

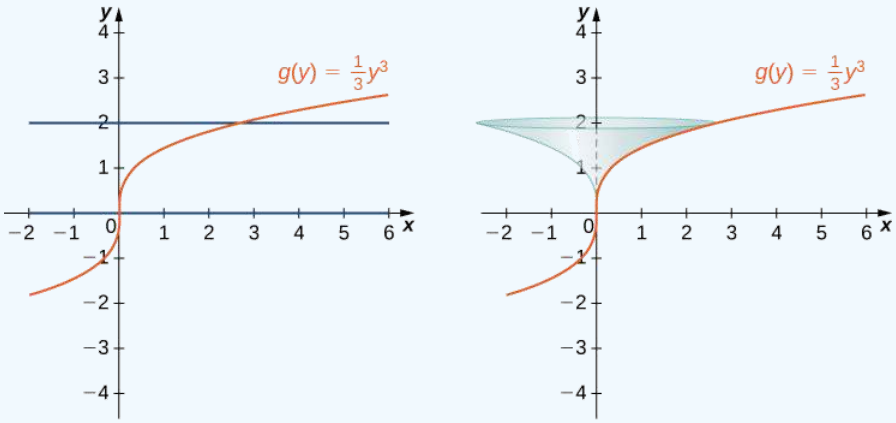


Figure 8.2.11: (a) The graph of $g(y)$. (b) The surface of revolution.

We have $g(y) = (1/3)y^3$, so $g'(y) = y^2$ and $\left(g'(y)\right)^2 = y^4$. Then

$$\begin{aligned}\text{Surface Area} &= \int_c^d \left(2\pi g(y)\sqrt{1+\left(g'(y)\right)^2}\right)\,dy \\ &= \int_0^2 \left(2\pi\left(\frac{1}{3}y^3\right)\sqrt{1+y^4}\right)\,dy \\ &= \frac{2\pi}{3}\int_0^2 \left(y^3\sqrt{1+y^4}\right)\,dy.\end{aligned}$$

Let $u = y^4 + 1$. Then $du = 4y^3\,dy$. When $y = 0$, $u = 1$, and when $y = 2$, $u = 17$. Then

$$\begin{aligned}\frac{2\pi}{3}\int_0^2 \left(y^3\sqrt{1+y^4}\right)\,dy &= \frac{2\pi}{3}\int_1^{17}\frac{1}{4}\sqrt{u}\,du \\ &= \frac{\pi}{6}\left[\frac{2}{3}u^{3/2}\right]\Big|_1^{17} = \frac{\pi}{9}\left[(17)^{3/2}-1\right] \approx 24.118 \text{ units}^2.\end{aligned}$$

Exercise 8.2.5

Let $g(y) = \sqrt{9-y^2}$ over the interval $y \in [0, 2]$. Find the surface area of the surface generated by revolving the graph of $g(y)$ around the y-axis.

Hint

Use the process from the previous example.

Answer

$12\pi \text{ units}^2$

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8.3: Applications to Physics and Engineering

Learning Objectives

In this section, we strive to understand the ideas generated by the following important questions:

- How do we measure the work accomplished by a varying force that moves an object a certain distance?
- What is the total force exerted by water against a dam?
- How are both of the above concepts and their corresponding use of definite integrals similar to problems we have encountered in the past involving formulas such as "distance equals rate times time" and "mass equals density times volume"?

In our work to date with the definite integral, we have seen several different circumstances where the integral enables us to measure the accumulation of a quantity that varies, provided the quantity is approximately constant over small intervals. For instance, based on the fact that the area of a rectangle is $A = l \cdot w$, if we wish to find the area bounded by a nonnegative curve $y = f(x)$ and the x-axis on an interval $[a, b]$, a representative slice of width Δx has area $A_{\text{slice}} = f(x)\Delta x$, and thus as we let the width of the representative slice tend to zero, we find that the exact area of the region is

$$A = \int_a^b f(x)dx. \tag{8.3.1}$$

In a similar way, if we know that the velocity of a moving object is given by the function $y = v(t)$, and we wish to know the distance the object travels on an interval $[a, b]$ where $v(t)$ is nonnegative, we can use a definite integral to generalize the fact that $d = r \cdot t$ when the rate, r, is constant. More specifically, on a short time interval Δt, $v(t)$ is roughly constant, and hence for a small slice of time, $d_{\text{slice}} = v(t)\Delta t$, and so as the width of the time interval Δt tends to zero, the exact distance traveled is given by the definite integral

$$d = \int_a^b v(t)dt. \tag{8.3.2}$$

Finally, when we recently learned about the mass of an object of non-constant density, we saw that since $M = D \cdot V$ (mass equals density times volume, provided that density is constant), if we can consider a small slice of an object on which the density is approximately constant, a definite integral may be used to determine the exact mass of the object. For instance, if we have a thin rod whose cross sections have constant density, but whose density is distributed along the x axis according to the function $y = \rho(x)$, it follows that for a small slice of the rod that is Δx thick, $M_{\text{slice}} = \rho(x)\Delta x$. In the limit as $\Delta x \to 0$, we then find that the total mass is given by

$$M = \int_a^b \rho(x)dx. \tag{8.3.3}$$

Note that all three of these situations are similar in that we have a basic rule ($A = l \cdot w, d = r \cdot t, M = D \cdot V$) where one of the two quantities being multiplied is no longer constant; in each, we consider a small interval for the other variable in the formula, calculate the approximate value of the desired quantity (area, distance, or mass) over the small interval, and then use a definite integral to sum the results as the length of the small intervals is allowed to approach zero. It should be apparent that this approach will work effectively for other situations where we have a quantity of interest that varies. We next turn to the notion of work: from physics, a basic principal is that work is the product of force and distance. For example, if a person exerts a force of 20 pounds to lift a 20-pound weight 4 feet off the ground, the total work accomplished is

$$W = F \cdot d \tag{8.3.4}$$

$$= 20 \cdot 4 \tag{8.3.5}$$

$$= 80 \text{ foot-pounds}. \tag{8.3.6}$$

If force and distance are measured in English units (pounds and feet), then the units on work are *foot-pounds*. If instead we work in metric units, where forces are measured in Newtons and distances in meters, the units on work are Newton-meters.

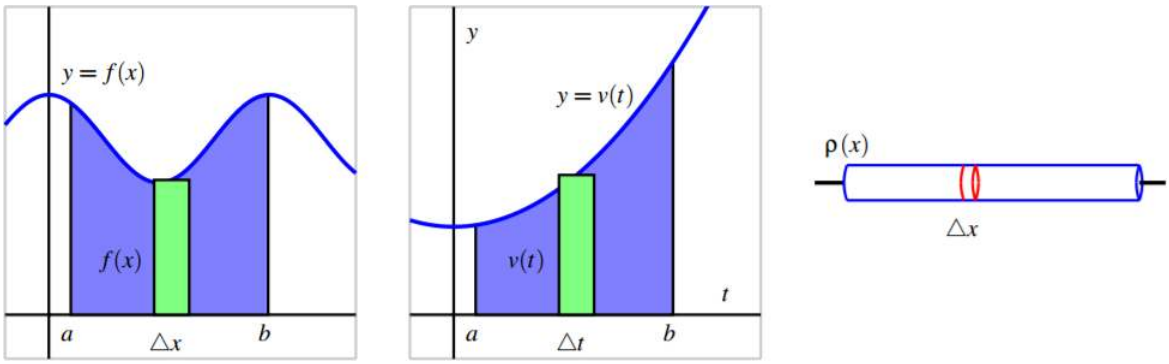


Figure 6.14: *Three settings where we compute the accumulation of a varying quantity: the area under $y = f(x)$, the distance traveled by an object with velocity $y = v(t)$, and the mass of a bar with density function $y = \rho(x)$.*

Of course, the formula $W = F \cdot d$ only applies when the force is constant while it is exerted over the distance d. In Preview Activity 6.4, we explore one way that we can use a definite integral to compute the total work accomplished when the force exerted varies.

Preview Activity 8.3.1

A bucket is being lifted from the bottom of a 50-foot deep well; its weight (including the water), B, in pounds at a height h feet above the water is given by the function $B(h)$. When the bucket leaves the water, the bucket and water together weigh $B(0) = 20$ pounds, and when the bucket reaches the top of the well, $B(50) = 12$ pounds. Assume that the bucket loses water at a constant rate (as a function of height, h) throughout its journey from the bottom to the top of the well.

a. Find a formula for $B(h)$.
b. Compute the value of the product $B(5)\Delta h$, where $\Delta h = 2$ feet. Include units on your answer. Explain why this product represents the approximate work it took to move the bucket of water from $h = 5$ to $h = 7$.
c. Is the value in (b) an over- or under-estimate of the actual amount of work it took to move the bucket from $h = 5$ to $h = 7$? Why?
d. Compute the value of the product $B(22)\Delta h$, where $\Delta h = 0.25$ feet. Include units on your answer. What is the meaning of the value you found?
e. More generally, what does the quantity $W_{\text{slice}} = B(h)\Delta h$ measure for a given value of h and a small positive value of Δh?
f. Evaluate the definite integral $\int^5 0_0 B(h)dh$. What is the meaning of the value you find? Why?

Work

Because work is calculated by the rule $W = F \cdot d$, whenever the force F is constant, it follows that we can use a definite integral to compute the work accomplished by a varying force. For example, suppose that in a setting similar to the problem posed in Preview Activity 6.4, we have a bucket being lifted in a 50-foot well whose weight at height h is given by

$$B(h) = 12 + 8e^{-0.1h}. \tag{8.3.7}$$

In contrast to the problem in the preview activity, this bucket is not leaking at a constant rate; but because the weight of the bucket and water is not constant, we have to use a definite integral to determine the total work that results from lifting the bucket. Observe that at a height h above the water, the approximate work to move the bucket a small distance Δh is

$$W_{\text{slice}} = B(h)\Delta h = (12 + 8e^{-0.1h})\Delta h. \tag{8.3.8}$$

Hence, if we let Δh tend to 0 and take the sum of all of the slices of work accomplished on these small intervals, it follows that the total work is given by

$$W = \int_0^{50} B(h)\, dh = \int_0^{50} (12 + 8e^{-0.1h})dh. \tag{8.3.9}$$

While is a straightforward exercise to evaluate this integral exactly using the First Fundamental Theorem of Calculus, in applied settings such as this one we will typically use computing technology to find accurate approximations of integrals that are of interest to us. Here, it turns out that

$$W = \int_0^{50} (12 + 8e^{-0.1h}) dh \approx 679.461 \text{ foot-pounds.} \tag{8.3.10}$$

Our work in Preview Activity 6.1 and in the most recent example above employs the following important general principle.

For an object being moved in the positive direction along an axis, x, by a force $F(x)$, the total work to move the object from a to b is given by

$$W = \int_a^b F(x) dx. \tag{8.3.11}$$

Activity 8.3.1

Consider the following situations in which a varying force accomplishes work.

a. Suppose that a heavy rope hangs over the side of a cliff. The rope is 200 feet long and weighs 0.3 pounds per foot; initially the rope is fully extended. How much work is required to haul in the entire length of the rope? (Hint: set up a function $F(h)$ whose value is the weight of the rope remaining over the cliff after h feet have been hauled in.)

b. A leaky bucket is being hauled up from a 100 foot deep well. When lifted from the water, the bucket and water together weigh 40 pounds. As the bucket is being hauled upward at a constant rate, the bucket leaks water at a constant rate so that it is losing weight at a rate of 0.1 pounds per foot. What function $B(h)$ tells the weight of the bucket after the bucket has been lifted h feet? What is the total amount of work accomplished in lifting the bucket to the top of the well?

c. Now suppose that the bucket in (b) does not leak at a constant rate, but rather that its weight at a height h feet above the water is given by $B(h) = 25 + 15e^{-0.05h}$. What is the total work required to lift the bucket 100 feet? What is the average force exerted on the bucket on the interval $h = 0$ to $h = 100$?

d. From physics, *Hooke's Law* for springs states that the amount of force required to hold a spring that is compressed (or extended) to a particular length is proportionate to the distance the spring is compressed (or extended) from its natural length. That is, the force to compress (or extend) a spring x units from its natural length is $F(x) = kx$ for some constant k (which is called the spring constant.) For springs, we choose to measure the force in pounds and the distance the spring is compressed in feet. Suppose that a force of 5 pounds extends a particular spring 4 inches (1/3 foot) beyond its natural length.

   i. Use the given fact that $F(1/3) = 5$ to find the spring constant k.
   ii. Find the work done to extend the spring from its natural length to 1 foot beyond its natural length.
   iii. Find the work required to extend the spring from 1 foot beyond its natural length to 1.5 feet beyond its natural length.

Work: Pumping Liquid from a Tank

In certain geographic locations where the water table is high, residential homes with basements have a peculiar feature: in the basement, one finds a large hole in the floor, and in the hole, there is water. For example, in Figure 6.15 where we see a *sump crock*.



Figure 6.15: A sump crock. Image credit to www.warreninspect.com/basement-moisture.

Essentially, a sump crock provides an outlet for water that may build up beneath the basement floor; of course, as that water rises, it is imperative that the water not flood the basement. Hence, in the crock we see the presence of a floating pump that sits on the surface of the water: this pump is activated by elevation, so when the water level reaches a particular height, the pump turns on and pumps a certain portion of the water out of the crock, hence relieving the water buildup beneath the foundation. One of the questions we'd like to answer is: how much work does a sump pump accomplish? To that end, let's suppose that we have a sump crock that has the shape of a frustum of a cone, as pictured in Figure 6.16. Assume that the crock has a diameter of 3 feet at its surface, a diameter of 1.5 feet at its base, and a depth of 4 feet. In addition, suppose that the sump pump is set up so that it pumps the water vertically up a pipe to a drain that is located at ground level just outside a basement window. To accomplish this, the pump must send the water to a location 9 feet above the surface of the sump crock.

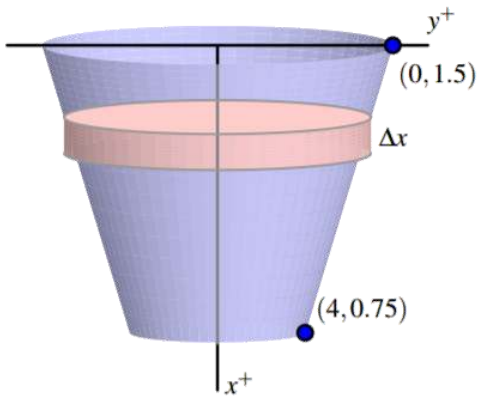


Figure 6.16: A sump crock with approximately cylindrical cross-sections that is 4 feet deep, 1.5 feet in diameter at its base, and 3 feet in diameter at its top.

It turns out to be advantageous to think of the depth below the surface of the crock as being the independent variable, so, in problems such as this one we typically let the positive x-axis point down, and the positive y-axis to the right, as pictured in the figure. As we think about the work that the pump does, we first realize that the pump sits on the surface of the water, so it makes sense to think about the pump moving the water one "slice" at a time, where it takes a thin slice from the surface, pumps it out of the tank, and then proceeds to pump the next slice below. For the sump crock described in this example, each slice of water is cylindrical in shape. We see that the radius of each approximately cylindrical slice varies according to the linear function $y = f(x)$ that passes through the points (0, 1.5) and (4, 0.75), where x is the depth of the particular slice in the tank; it is a straightforward exercise to find that $f(x) = 1.5 - 0.1875x$. Now we are prepared to think about the overall problem in several steps:

a. determining the volume of a typical slice;
b. finding the weight (We assume that the weight density of water is 62.4 pounds per cubic foot) of a typical slice (and thus the force that must be exerted on it)
c. deciding the distance that a typical slice moves; and
d. computing the work to move a representative slice. Once we know the work it takes to move one slice, we use a definite integral over an appropriate interval to find the total work.

Consider a representative cylindrical slice that sits on the surface of the water at a depth of x feet below the top of the crock. It follows that the approximate volume of that slice is given by

$$V_{\text{slice}} = \pi f(x)^2 \Delta x = \pi(1.5 - 0.1875x)^2 \Delta x$$

Since water weighs 62.4 lb/ft3 , it follows that the approximate weight of a representative slice, which is also the approximate force the pump must exert to move the slice, is

$$F_{\text{slice}} = 62.4 \cdot V_{\text{slice}} = 62.4\pi(1.5 - 0.1875x)^2 \Delta x.$$

Because the slice is located at a depth of x feet below the top of the crock, the slice being moved by the pump must move x feet to get to the level of the basement floor, and then, as stated in the problem description, be moved another 9 feet to reach the drain at ground level outside a basement window. Hence, the total distance a representative slice travels is

$$d_{\text{slice}} = x + 9\,.$$

Finally, we note that the work to move a representative slice is given by

$$W_{\text{slice}} = F_{\text{slice}} \cdot d_{\text{slice}} = 62.4\pi(1.5 - 0.1875x)^2 \Delta x \cdot (x + 9),$$

since the force to move a particular slice is constant. We sum the work required to move slices throughout the tank (from $x = 0$ to $x = 4$), let $\Delta x \to 0$, and hence

$$W = \int_0^4 62.4\pi(1.5 - 0.1875x)^2(x + 9)dx,$$

which, when evaluated using appropriate technology, shows that the total work is $W = 10970.5\pi$ foot-pounds.

The preceding example demonstrates the standard approach to finding the work required to empty a tank filled with liquid. The main task in each such problem is to determine the volume of a representative slice, followed by the force exerted on the slice, as well as the distance such a slice moves. In the case where the units are metric, there is one key difference: in the metric setting, rather than weight, we normally first find the mass of a slice. For instance, if distance is measured in meters, the mass density of water is 1000 kg/m^3 . In that setting, we can find the mass of a typical slice (in kg). To determine the force required to move it, we use F = ma, where m is the object's mass and a is the gravitational constant 9.81 N/kg^3 . That is, in metric units, the weight density of water is 9810 N/m^3 .

Activity 8.3.2

In each of the following problems, determine the total work required to accomplish the described task. In parts (b) and (c), a key step is to find a formula for a function that describes the curve that forms the side boundary of the tank.

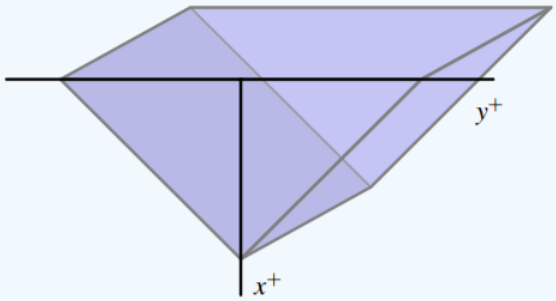


Figure 6.17: A trough with triangular ends, as described in Activity 6.11, part (c).

a. Consider a vertical cylindrical tank of radius 2 meters and depth 6 meters. Suppose the tank is filled with 4 meters of water of mass density 1000 kg/m^3 , and the top 1 meter of water is pumped over the top of the tank.
b. Consider a hemispherical tank with a radius of 10 feet. Suppose that the tank is full to a depth of 7 feet with water of weight density 62.4 pounds/ft^3, and the top 5 feet of water are pumped out of the tank to a tanker truck whose height is 5 feet above the top of the tank.
c. Consider a trough with triangular ends, as pictured in Figure 6.17, where the tank is 10 feet long, the top is 5 feet wide, and the tank is 4 feet deep. Say that the trough is full to within 1 foot of the top with water of weight density 62.4 pounds/ft^3, and a pump is used to empty the tank until the water remaining in the tank is 1 foot deep.

Force due to Hydrostatic Pressure

When a dam is built, it is imperative to for engineers to understand how much force water will exert against the face of the dam. The first thing we realize is the force exerted by the fluid is related to the natural concept of pressure. The pressure a force exerts on a region is measured in units of force per unit of area: for example, the air pressure in a tire is often measured in pounds per square inch (PSI). Hence, we see that the general relationship is given by

$$P = \frac{F}{A}, \text{ or } F = P \cdot A,$$

where P represents pressure, F represents force, and A the area of the region being considered. Of course, in the equation F = PA, we assume that the pressure is constant over the entire region A.

Most people know from experience that the deeper one dives underwater while swimming, the greater the pressure that is exerted by the water. This is due to the fact that the deeper one dives, the more water there is right on top of the swimmer: it is the force

that “column” of water exerts that determines the pressure the swimmer experiences. To get water pressure measured in its standard units (pounds per square foot), we say that the total water pressure is found by computing the total weight of the column of water that lies above a region of area 1 square foot at a fixed depth. Such a rectangular column with a 1 × 1 base and a depth of d feet has volume V = 1 · 1 · d ft^3, and thus the corresponding weight of the water overhead is 62.4d. Since this is also the amount of force being exerted on a 1 square foot region at a depth d feet underwater, we see that P = 62.4d (lbs/ft^2) is the pressure exerted by water at depth d.

The understanding that P = 62.4d will tell us the pressure exerted by water at a depth of d, along with the fact that F = PA, will now enable us to compute the total force that water exerts on a dam, as we see in the following example.

Example 8.3.3

Consider a trapezoid-shaped dam that is 60 feet wide at its base and 90 feet wide at its top, and assume the dam is 25 feet tall with water that rises to within 5 feet of the top of its face. Water weighs 62.5 pounds per cubic foot. How much force does the water exert against the dam?

Solution

First, we sketch a picture of the dam, as shown in Figure 6.18. Note that, as in problems involving the work to pump out a tank, we let the positive x-axis point down.

It is essential to use the fact that pressure is constant at a fixed depth. Hence, we consider a slice of water at constant depth on the face, such as the one shown in the figure. First, the approximate area of this slice is the area of the pictured rectangle. Since the width of that rectangle depends on the variable x (which represents the how far the slice lies from the top of the dam), we find a formula for the function $y = f(x)$ that determines one side of the face of the dam. Since f is linear, it is straightforward to find that $y = f(x) = 45 - \frac{3}{5}x$. Hence, the approximate area of a representative slice is

$$A_{\text{slice}} = 2f(x)\Delta x = 2(45 - \frac{3}{5}x)\Delta x.$$

At any point on this slice, the depth is approximately constant, and thus the pressure can be considered constant. In particular, we note that since x measures the distance to the top of the dam, and because the water rises to within 5 feet of the top of the dam, the depth of any point on the representative slice is approximately $(x - 5)$. Now, since pressure

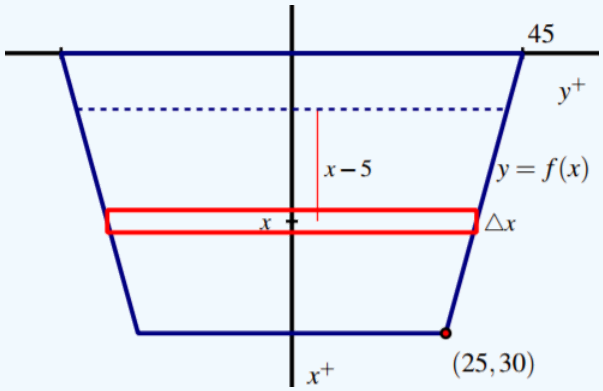


Figure 6.18: A trapezoidal dam that is 25 feet tall, 60 feet wide at its base, 90 feet wide at its top, with the water line 5 feet down from the top of its face.

is given by $P = 62.4d$, we have that at any point on the representative slice

$$P_{\text{slice}} = 62.4(x - 5).$$

Knowing both the pressure and area, we can find the force the water exerts on the slice. Using $F = PA$, it follows that

$$F_{\text{slice}} = P_{\text{slice}} \cdot A_{\text{slice}} = 62.4(x - 5) \cdot 2(45 - \frac{3}{5}x)\Delta x \ .$$

Finally, we use a definite integral to sum the forces over the appropriate range of x-values. Since the water rises to within 5 feet of the top of the dam, we start at $x = 5$ and slice all the way to the bottom of the dam, where $x = 30$. Hence,

$$F = \int_{x=5}^{x=30} 62.4(x-5) \cdot 2(45 - \frac{3}{5}x)dx\,.$$

Using technology to evaluate the integral, we find F ≈ 1.248 × 10^6 pounds.

Activity 8.3.4

In each of the following problems, determine the total force exerted by water against the surface that is described.

a. Consider a rectangular dam that is 100 feet wide and 50 feet tall, and suppose that water presses against the dam all the way to the top.
b. Consider a semicircular dam with a radius of 30 feet. Suppose that the water rises to within 10 feet of the top of the dam.
c. Consider a trough with triangular ends, as pictured in Figure 6.17, where the tank is 10 feet long, the top is 5 feet wide, and the tank is 4 feet deep. Say that the trough is full to within 1 foot of the top with water of weight density 62.4 pounds/ft^3. How much force does the water exert against one of the triangular ends?

While there are many different formulas that we use in solving problems involving work, force, and pressure, it is important to understand that the fundamental ideas behind these problems are similar to several others that we've encountered in applications of the definite integral. In particular, the basic idea is to take a difficult problem and somehow slice it into more manageable pieces that we understand, and then use a definite integral to add up these simpler pieces.

Summary

In this section, we encountered the following important ideas:

- To measure the work accomplished by a varying force that moves an object, we subdivide the problem into pieces on which we can use the formula W = F · d, and then use a definite integral to sum the work accomplished on each piece.
- To find the total force exerted by water against a dam, we use the formula F = P · A to measure the force exerted on a slice that lies at a fixed depth, and then use a definite integral to sum the forces across the appropriate range of depths.
- Because work is computed as the product of force and distance (provided force is constant), and the force water exerts on a dam can be computed as the product of pressure and area (provided pressure is constant), problems involving these concepts are similar to earlier problems we did using definite integrals to find distance (via "distance equals rate times time") and mass ("mass equals density times volume").

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8.4: Applications to Economics and Biology

Learning Objectives

- Use the exponential growth model in applications, including population growth and compound interest.
- Explain the concept of doubling time.
- Use the exponential decay model in applications, including radioactive decay and Newton's law of cooling.
- Explain the concept of half-life.

One of the most prevalent applications of exponential functions involves growth and decay models. Exponential growth and decay show up in a host of natural applications. From population growth and continuously compounded interest to radioactive decay and Newton's law of cooling, exponential functions are ubiquitous in nature. In this section, we examine exponential growth and decay in the context of some of these applications.

Exponential Growth Model

Many systems exhibit exponential growth. These systems follow a model of the form $y = y_0 e^{kt}$, where y_0 represents the initial state of the system and k is a positive constant, called the growth constant. Notice that in an exponential growth model, we have

$$y' = ky_0 e^{kt} = ky. \tag{8.4.1}$$

That is, the rate of growth is proportional to the current function value. This is a key feature of exponential growth. Equation 8.4.1 involves derivatives and is called *a differential equation.*

Exponential Growth

Systems that exhibit *exponential growth* increase according to the mathematical model

$$y = y_0 e^{kt}$$

where y_0 represents the initial state of the system and $k > 0$ is a constant, called the growth constant.

Population growth is a common example of exponential growth. Consider a population of bacteria, for instance. It seems plausible that the rate of population growth would be proportional to the size of the population. After all, the more bacteria there are to reproduce, the faster the population grows. Figure 8.4.1 and Table 8.4.1 represent the growth of a population of bacteria with an initial population of 200 bacteria and a growth constant of 0.02. Notice that after only 2 hours (120 minutes), the population is 10 times its original size!

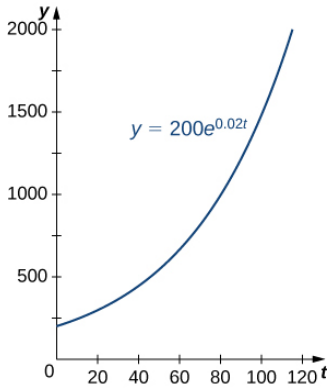


Figure 8.4.1: An example of exponential growth for bacteria.

Table 8.4.1: Exponential Growth of a Bacterial Population

Time(min)	Population Size (no. of bacteria)
10	244
20	298

Time(min)	Population Size (no. of bacteria)
30	364
40	445
50	544
60	664
70	811
80	991
90	1210
100	1478
110	1805
120	2205

Note that we are using a continuous function to model what is inherently discrete behavior. At any given time, the real-world population contains a whole number of bacteria, although the model takes on noninteger values. When using exponential growth models, we must always be careful to interpret the function values in the context of the phenomenon we are modeling.

Example 8.4.1: Population Growth

Consider the population of bacteria described earlier. This population grows according to the function $f(t) = 200e^{0.02t}$, where t is measured in minutes. How many bacteria are present in the population after 5 hours (300 minutes)? When does the population reach $100,000$ bacteria?

Solution

We have $f(t) = 200e^{0.02t}$. Then

$$f(300) = 200e^{0.02(300)} \approx 80,686.$$

There are $80,686$ bacteria in the population after 5 hours.

To find when the population reaches $100,000$ bacteria, we solve the equation

$$\begin{aligned} 100,000 &= 200e^{0.02t} \\ 500 &= e^{0.02t} \\ \ln 500 &= 0.02t \\ t &= \frac{\ln 500}{0.02} \approx 310.73. \end{aligned}$$

The population reaches $100,000$ bacteria after 310.73 minutes.

Exercise 8.4.1

Consider a population of bacteria that grows according to the function $f(t) = 500e^{0.05t}$, where t is measured in minutes. How many bacteria are present in the population after 4 hours? When does the population reach 100 million bacteria?

Answer

Use the process from the previous example.

Answer

There are $81,377,396$ bacteria in the population after 4 hours. The population reaches 100 million bacteria after 244.12 minutes.

Let's now turn our attention to a financial application: **compound interest**. Interest that is not compounded is called simple interest. ***Simple interest*** is paid once, at the end of the specified time period (usually 1 year). So, if we put $\$1000$ in a savings account earning 2 simple interest per year, then at the end of the year we have

$$1000(1+0.02)=\$1020.$$

Compound interest is paid multiple times per year, depending on the compounding period. Therefore, if the bank compounds the interest every 6 months, it credits half of the year's interest to the account after 6 months. During the second half of the year, the account earns interest not only on the initial $\$1000$, but also on the interest earned during the first half of the year. Mathematically speaking, at the end of the year, we have

$$1000\left(1+\frac{0.02}{2}\right)^2=\$1020.10.$$

Similarly, if the interest is compounded every 4 months, we have

$$1000\left(1+\frac{0.02}{3}\right)^3=\$1020.13,$$

and if the interest is compounded daily (365 times per year), we have $\$1020.20$ If we extend this concept, so that the interest is compounded continuously, after t years we have

$$1000\lim_{n\to\infty}\left(1+\frac{0.02}{n}\right)^{nt}.$$

Now let's manipulate this expression so that we have an exponential growth function. Recall that the number e can be expressed as a limit:

$$e=\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m.$$

Based on this, we want the expression inside the parentheses to have the form $(1+1/m)$. Let $n=0.02m$. Note that as $n\to\infty, m\to\infty$ as well. Then we get

$$1000\lim_{n\to\infty}\left(1+\frac{0.02}{n}\right)^{nt}=1000\lim_{m\to\infty}\left(1+\frac{0.02}{0.02m}\right)^{0.02mt}=1000\left[\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right]^{0.02t}.$$

We recognize the limit inside the brackets as the number e. So, the balance in our bank account after t years is given by $1000e^{0.02t}$. Generalizing this concept, we see that if a bank account with an initial balance of $\$P$ earns interest at a rate of r, compounded continuously, then the balance of the account after t years is

$$\text{Balance }=Pe^{rt}.$$

Example 8.4.2: Compound Interest

A 25-year-old student is offered an opportunity to invest some money in a retirement account that pays 5 annual interest compounded continuously. How much does the student need to invest today to have $\$1$ million when she retires at age 65? What if she could earn 6 annual interest compounded continuously instead?

Solution

We have

$$1,000,000=Pe^{0.05(40)}$$

$$P=135,335.28.$$

She must invest $\$135,335.28$at 5 interest.

If, instead, she is able to earn 6 then the equation becomes

$$1,000,000 = Pe^{0.06(40)}$$

$$P = 90,717.95.$$

In this case, she needs to invest only $\$90,717.95$.This is roughly two-thirds the amount she needs to invest at 5. The fact that the interest is compounded continuously greatly magnifies the effect of the 1 increase in interest rate.

? Exercise 8.4.2

Suppose instead of investing at age $25\sqrt{b^2-4ac}$, the student waits until age 35. How much would she have to invest at 5? At 6?

Hint

Use the process from the previous example.

Answer

At 5 interest, she must invest $\$223,130.16$At 6 interest, she must invest $\$165,298.89$.

If a quantity grows exponentially, the time it takes for the quantity to double remains constant. In other words, it takes the same amount of time for a population of bacteria to grow from 100 to 200 bacteria as it does to grow from $10,000$ to $20,000$ bacteria. This time is called the doubling time. To calculate the doubling time, we want to know when the quantity reaches twice its original size. So we have

$$\begin{aligned} 2y_0 &= y_0 e^{kt} \\ 2 &= e^{kt} \\ \ln 2 &= kt \\ t &= \frac{\ln 2}{k}. \end{aligned}$$

Definition: Doubling Time

If a quantity grows exponentially, the **doubling time** is the amount of time it takes the quantity to double. It is given by

$$\text{Doubling time} = \frac{\ln 2}{k}.$$

Example 8.4.3: Using the Doubling Time

Assume a population of fish grows exponentially. A pond is stocked initially with 500 fish. After 6 months, there are 1000 fish in the pond. The owner will allow his friends and neighbors to fish on his pond after the fish population reaches $10,000$. When will the owner's friends be allowed to fish?

Solution

We know it takes the population of fish 6 months to double in size. So, if t represents time in months, by the doubling-time formula, we have $6 = (\ln 2)/k$. Then, $k = (\ln 2)/6$. Thus, the population is given by $y = 500e^{((\ln 2)/6)t}$. To figure out when the population reaches $10,000$ fish, we must solve the following equation:

$$\begin{aligned} 10,000 &= 500e^{(\ln 2/6)t} \\ 20 &= e^{(\ln 2/6)t} \\ \ln 20 &= \left(\frac{\ln 2}{6}\right) t \\ t &= \frac{6(\ln 20)}{\ln 2} \\ &\approx 25.93. \end{aligned}$$

The owner's friends have to wait 25.93 months (a little more than 2 years) to fish in the pond.

Exercise 8.4.3

Suppose it takes 9 months for the fish population in Example $8.4.3$ to reach 1000 fish. Under these circumstances, how long do the owner's friends have to wait?

Hint

Use the process from the previous example.

Answer

38.90 months

Exponential Decay Model

Exponential functions can also be used to model populations that shrink (from disease, for example), or chemical compounds that break down over time. We say that such systems exhibit exponential decay, rather than exponential growth. The model is nearly the same, except there is a negative sign in the exponent. Thus, for some positive constant k, we have

$$y = y_0 e^{-kt}.$$

As with exponential growth, there is a differential equation associated with exponential decay. We have

$$y' = -ky_0 e^{-kt} = -ky.$$

Exponential Decay

Systems that exhibit exponential decay behave according to the model

$$y = y_0 e^{-kt},$$

where y_0 represents the initial state of the system and $k > 0$ is a constant, called the decay constant.

Figure $8.4.2$ shows a graph of a representative exponential decay function.

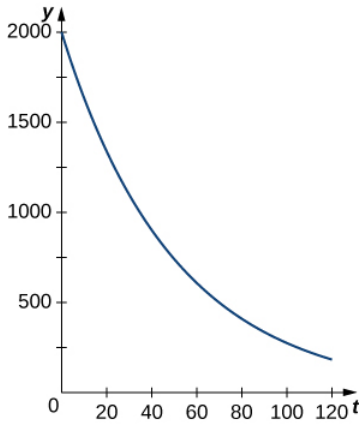


Figure 8.4.2: An example of exponential decay.

Let's look at a physical application of exponential decay. *Newton's law of cooling* says that an object cools at a rate proportional to the difference between the temperature of the object and the temperature of the surroundings. In other words, if T represents the temperature of the object and T_a represents the ambient temperature in a room, then

$$T' = -k(T - T_a).$$

Note that this is not quite the right model for exponential decay. We want the derivative to be proportional to the function, and this expression has the additional T_a term. Fortunately, we can make a change of variables that resolves this issue. Let $y(t) = T(t) - T_a$. Then $y'(t) = T'(t) - 0 = T'(t)$, and our equation becomes

$$y' = -ky.$$

From our previous work, we know this relationship between y and its derivative leads to exponential decay. Thus,

$$y = y_0 e^{-kt},$$

and we see that

$$T - T_a = (T_0 - T_a)e^{-kt}$$

$$T = (T_0 - T_a)e^{-kt} + T_a$$

where T_0 represents the initial temperature. Let's apply this formula in the following example.

Example 8.4.4: Newton's Law of Cooling

According to experienced baristas, the optimal temperature to serve coffee is between $155^\circ F$ and $175^\circ F$. Suppose coffee is poured at a temperature of $200^\circ F$, and after 2 minutes in a $70^\circ F$ room it has cooled to $180^\circ F$. When is the coffee first cool enough to serve? When is the coffee too cold to serve? Round answers to the nearest half minute.

Solution

We have

$$\begin{aligned} T &= (T_0 - T_a)e^{-kt} + T_a \\ 180 &= (200-70)e^{-k(2)} + 70 \\ 110 &= 130e^{-2k} \\ \frac{11}{13} &= e^{-2k} \\ \ln\frac{11}{13} &= -2k \\ \ln 11 - \ln 13 &= -2k \\ k &= \frac{\ln 13 - \ln 11}{2} \end{aligned}$$

Then, the model is

$$T = 130e^{(\ln 11 - \ln 13/2)t} + 70.$$

The coffee reaches $175^\circ F$ when

$$\begin{aligned} 175 &= 130e^{(\ln 11 - \ln 13/2)t} + 70 \\ 105 &= 130e^{(\ln 11 - \ln 13/2)t} \\ \frac{21}{26} &= e^{(\ln 11 - \ln 13/2)t} \\ \ln\frac{21}{26} &= \frac{\ln 11 - \ln 13}{2}t \\ \ln 21 - \ln 26 &= \left(\frac{\ln 11 - \ln 13}{2}\right)t \\ t &= \frac{2(\ln 21 - \ln 26)}{\ln 11 - \ln 13} \\ &\approx 2.56. \end{aligned}$$

The coffee can be served about 2.5 minutes after it is poured. The coffee reaches $155^\circ F$ at

$$\begin{aligned} 155 &= 130e^{(\ln 11 - \ln 13/2)t} + 70 \\ 85 &= 130e^{(\ln 11 - \ln 13)t} \\ \frac{17}{26} &= e^{(\ln 11 - \ln 13)t} \\ \ln 17 - \ln 26 &= \left(\frac{\ln 11 - \ln 13}{2}\right)t \\ t &= \frac{2(\ln 17 - \ln 26)}{\ln 11 - \ln 13} \\ &\approx 5.09. \end{aligned}$$

The coffee is too cold to be served about 5 minutes after it is poured.

? Exercise 8.4.4

Suppose the room is warmer $(75^\circ F)$ and, after 2 minutes, the coffee has cooled only to $185^\circ F$. When is the coffee first cool enough to serve? When is the coffee be too cold to serve? Round answers to the nearest half minute.

Hint

Use the process from the previous example.

Answer

The coffee is first cool enough to serve about 3.5 minutes after it is poured. The coffee is too cold to serve about 7 minutes after it is poured.

Just as systems exhibiting exponential growth have a constant doubling time, systems exhibiting exponential decay have a constant half-life. To calculate the half-life, we want to know when the quantity reaches half its original size. Therefore, we have

$$\begin{aligned} \frac{y_0}{2} &= y_0 e^{-kt} \\ \frac{1}{2} &= e^{-kt} \\ -\ln 2 &= -kt \\ t &= \frac{\ln 2}{k}. \end{aligned}$$

Note: This is the same expression we came up with for doubling time.

Definition: Half-Life

If a quantity decays exponentially, the half-life is the amount of time it takes the quantity to be reduced by half. It is given by

$$\text{Half-life} = \frac{\ln 2}{k}.$$

Example 8.4.5: Radiocarbon Dating

One of the most common applications of an exponential decay model is carbon dating. Carbon-14 decays (emits a radioactive particle) at a regular and consistent exponential rate. Therefore, if we know how much carbon-14 was originally present in an object and how much carbon-14 remains, we can determine the age of the object. The half-life of carbon-14 is approximately 5730 years—meaning, after that many years, half the material has converted from the original carbon-14 to the new nonradioactive nitrogen-14. If we have 100 g carbon-14 today, how much is left in 50 years? If an artifact that originally contained 100 g of carbon-14 now contains 10 g of carbon-14, how old is it? Round the answer to the nearest hundred years.

Solution

We have

$$\begin{aligned} 5730 &= \frac{\ln 2}{k} \\ k &= \frac{\ln 2}{5730}. \end{aligned}$$

So, the model says

$$y = 100e^{-(\ln 2/5730)t}.$$

In 50 years, we have

$$y = 100e^{-(\ln 2/5730)(50)} \approx 99.40$$

Therefore, in 50 years, 99.40 g of carbon-14 remains.

To determine the age of the artifact, we must solve

$$\begin{aligned} 10 &= 100e^{-(\ln 2/5730)t} \\ \frac{1}{10} &= e^{-(\ln 2/5730)t} \\ t &\approx 19035. \end{aligned}$$

The artifact is about $19,000$ years old.

Exercise 8.4.5: Carbon-14 Decay

If we have 100 g of carbon-14 , how much is left after 500 years? If an artifact that originally contained 100 g of carbon-14 now contains 20 g of carbon-14, how old is it? Round the answer to the nearest hundred years.

Hint

Use the process from the previous example.

Answer

A total of 94.13 g of carbon-14 remains after 500 years. The artifact is approximately 13,300 years old.

Key Concepts

- Exponential growth and exponential decay are two of the most common applications of exponential functions.
- Systems that exhibit exponential growth follow a model of the form $y = y_0 e^{kt}$.
- In exponential growth, the rate of growth is proportional to the quantity present. In other words, $y' = ky$.
- Systems that exhibit exponential growth have a constant doubling time, which is given by $(\ln 2)/k$.
- Systems that exhibit exponential decay follow a model of the form $y = y_0 e^{-kt}$.
- Systems that exhibit exponential decay have a constant half-life, which is given by $(\ln 2)/k$.

Glossary

doubling time

if a quantity grows exponentially, the doubling time is the amount of time it takes the quantity to double, and is given by $(\ln 2)/k$

exponential decay

systems that exhibit exponential decay follow a model of the form $y = y_0 e^{-kt}$

exponential growth

systems that exhibit exponential growth follow a model of the form $y = y_0 e^{kt}$

half-life

if a quantity decays exponentially, the half-life is the amount of time it takes the quantity to be reduced by half. It is given by $(\ln 2)/k$

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8.5: Probability

In this section we will briefly discuss some applications of multiple integrals in the field of probability theory. In particular we will see ways in which multiple integrals can be used to calculate *probabilities* and *expected values.*

Probability

Suppose that you have a standard six-sided (fair) die, and you let a variable X represent the value rolled. Then the *probability* of rolling a 3, written as $P(X=3)$, is 1 6 , since there are six sides on the die and each one is equally likely to be rolled, and hence in particular the 3 has a one out of six chance of being rolled. Likewise the probability of rolling *at most* a 3, written as $P(X \leq 3)$, is $\frac{3}{6} = \frac{1}{2}$, since of the six numbers on the die, there are three equally likely numbers (1, 2, and 3) that are less than or equal to 3. Note that:

$$P(X \leq 3) = P(X=1) + P(X=2) + P(X=3). \tag{8.5.1}$$

We call X a *discrete random variable* on the *sample space* (or *probability space*) Ω consisting of all possible outcomes. In our case, $\Omega = 1, 2, 3, 4, 5, 6$. An *event* A is a subset of the sample space. For example, in the case of the die, the event $X \leq 3$ is the set $1, 2, 3$.

Now let X be a variable representing a random real number in the interval $(0, 1)$. Note that the set of all real numbers between 0 and 1 is not a discrete (or *countable*) set of values, i.e. it can not be put into a one-to-one correspondence with the set of positive integers. In this case, for any real number x in $(0, 1)$, it makes no sense to consider $P(X = x)$ since it *must* be 0 (why?). Instead, we consider the probability $P(X \leq x)$, which is given by $P(X \leq x) = x$. The reasoning is this: the interval $(0, 1)$ has length 1, and for x in $(0, 1)$ the interval $(0, x)$ has length x. So since X represents a *random* number in $(0, 1)$, and hence is *uniformly distributed* over $(0, 1)$, then

$$P(X \leq x) = \frac{\text{length of } (0, x)}{\text{length of } (0, 1)} = \frac{x}{1} = x \tag{8.5.2}$$

We call X a *continuous random variable* on the *sample space* $\Omega = (0, 1)$. An event A is a subset of the sample space. For example, in our case the event $X \leq x$ is the set $(0, x)$.

In the case of a discrete random variable, we saw how the probability of an event was the *sum* of the probabilities of the individual outcomes comprising that event (e.g. $P(X \leq 3) = P(X=1) + P(X=2) + P(X=3)$ in the die example). For a continuous random variable, the probability of an event will instead be the *integral* of a function, which we will now describe.

Let X be a continuous real-valued random variable on a sample space Ω in $\mathbb{R}$. For simplicity, let $\Omega = (a, b)$. Define the *distribution function* F of X as

$$F(x) = P(X \leq x), \quad \text{for } -\infty < x < \infty \tag{8.5.3}$$

$$= \begin{cases} 1, & \text{for } x \geq b \\ P(X \leq x), & \text{for } a < x < b \\ 0, & \text{for } x \leq a. \end{cases} \tag{8.5.4}$$

Suppose that there is a nonnegative, continuous real-valued function f on $\mathbb{R}$ such that

$$F(x) = \int_{-\infty}^{x} f(y)\, dy, \quad \text{for } -\infty < x < \infty, \tag{8.5.5}$$

and

$$\int_{-\infty}^{\infty} f(x)\, dx = 1 \tag{8.5.6}$$

Then we call f the *probability density function* (or $p.d.f.$ for short) for X. We thus have

$$P(X \leq x) = \int_{a}^{x} f(y)\, dy, \quad \text{for } a < x < b \tag{8.5.7}$$

Also, by the Fundamental Theorem of Calculus, we have

$$F'(x) = f(x), \quad \text{for } -\infty < x < \infty. \tag{8.5.8}$$

Example 8.5.1: Uniform Distribution

Let X represent a randomly selected real number in the interval $(0, 1)$. We say that X has the *uniform distribution* on $(0, 1)$, with distribution function

$$F(x) = P(X \le x) = \begin{cases} 1, & \text{for } x \ge 1 \\ x, & \text{for } 0 < x < 1 \\ 0, & \text{for } x \le 0, \end{cases} \tag{8.5.9}$$

and probability density function

$$f(x) = F'(x) = \begin{cases} 1, & \text{for } 0 < x < 1 \\ 0, & \text{elsewhere.} \end{cases} \tag{8.5.10}$$

In general, if X represents a randomly selected real number in an interval (a, b), then X has the uniform distribution function

$$F(x) = P(X \le x) = \begin{cases} 1, & \text{\text{for }x \ge b} \\ \dfrac{x}{b-a}, & \text{\text{for }a<x<b} \\ 0, & \text{\text{for }x \le a} \end{cases} \tag{8.5.11}$$

$$f(x) = F'(x) = \begin{cases} \dfrac{1}{b-a}, & \text{\text{for }a<x<b} \\ 0, & \text{elsewhere.} \end{cases} \tag{8.5.12}$$

Example 8.5.2: Standard Normal Distribution

A famous distribution function is given by the *standard normal distribution*, whose probability density function f is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad \text{for } -\infty < x < \infty \tag{8.5.13}$$

This is often called a "bell curve", and is used widely in statistics. Since we are claiming that f is a $p.d.f.$, we *should* have

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = 1 \tag{8.5.14}$$

by Equation 8.5.6, which is equivalent to

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}. \tag{8.5.15}$$

We can use a double integral in polar coordinates to verify this integral. First,

$$\begin{aligned} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\, dx\, dy &= \int_{-\infty}^{\infty} e^{-y^2/2} \left(\int_{-\infty}^{\infty} e^{-x^2/2}\, dx \right) dy \\ &= \left(\int_{-\infty}^{\infty} e^{-x^2/2}\, dx \right) \left(\int_{-\infty}^{\infty} e^{-y^2/2}\, dy \right) \\ &= \left(\int_{-\infty}^{\infty} e^{-x^2/2}\, dx \right)^2 \end{aligned}$$

since the same function is being integrated twice in the middle equation, just with different variables. But using polar coordinates, we see that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\,dx\,dy = \int_0^{2\pi}\int_0^{\infty} e^{-r^2/2}\,r\,dr\,d\theta$$
$$= \int_0^{2\pi}\left(-e^{-r^2/2}\Big|_{r=0}^{r=\infty}\right)\,d\theta$$
$$= \int_0^{2\pi}(0-(-e^0))\,d\theta = \int_0^{2\pi} 1\,d\theta = 2\pi,$$

and so

$$\left(\int_{-\infty}^{\infty} e^{-x^2/2}\,dx\right)^2 = 2\pi, \text{ and hence}$$
$$\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$$

In addition to individual random variables, we can consider *jointly distributed* random variables. For this, we will let X, Y and Z be three real-valued continuous random variables defined on the same sample space Ω in $\mathbb{R}$ (the discussion for two random variables is similar). Then the *joint distribution function* F of X, Y and Z is given by

$$F(x,y,z) = P(X \le x, Y \le y, Z \le z), \text{ for } -\infty < x, y, z < \infty. \tag{8.5.16}$$

If there is a nonnegative, continuous real-valued function f on $\mathbb{R}^3$ such that

$$F(x,y,z) = \int_{-\infty}^{z}\int_{-\infty}^{y}\int_{-\infty}^{x} f(u,v,w)\,du\,dv\,dw, \text{ for } -\infty < x, y, z < \infty \tag{8.5.17}$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\,dx\,dy\,dz = 1, \tag{8.5.18}$$

then we call f the *joint probability density function* (or *joint p.d.f.* for short) for X, Y and Z. In general, for $a_1 < b_1, a_2 < b_2, a_3 < b_3$, we have

$$P(a_1 < X \le b_1, a_2 < Y \le b_2, a_3 < Z \le b_3) = \int_{a_3}^{b_3}\int_{a_2}^{b_2}\int_{a_1}^{b_1} f(x,y,z)\,dx\,dy\,dz, \tag{8.5.19}$$

with the $\le$ and $<$ symbols interchangeable in any combination. A triple integral, then, can be thought of as representing a probability (for a function f which is a $p.d.f.$).

Example 8.5.3

Let a, b, and c be real numbers selected randomly from the interval $(0,1)$. What is the probability that the equation $ax^2 + bx + c = 0$ has at least one real solution x?

Solution

We know by the quadratic formula that there is at least one real solution if $b^2 - 4ac \ge 0$. So we need to calculate $P(b^2 - 4ac \ge 0)$. We will use three jointly distributed random variables to do this. First, since $0 < a, b, c < 1$, we have

$$b^2 - 4ac \ge 0 \Leftrightarrow 0 < 4ac \le b^2 < 1 \Leftrightarrow 0 < 2\sqrt{a}\sqrt{c} \le b < 1,$$

where the last relation holds for all $0 < a, c < 1$ such that

$$0 < 4ac < 1 \Leftrightarrow 0 < c < \frac{1}{4a}$$

Considering a, b and c as real variables, the region R in the ac-plane where the above relation holds is given by $R = (a,c) : 0 < a < 1, 0 < c < 1, 0 < c < \frac{1}{4a}$, which we can see is a union of two regions R_1 and R_2, as in Figure 8.5.1.

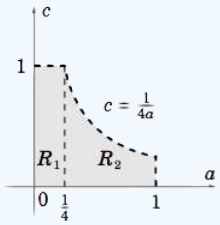


Figure 8.5.1: Region $R = R_1 \cup R_2$

Now let X, Y and Z be continuous random variables, each representing a randomly selected real number from the interval $(0,1)$ (think of X, Y and Z representing a, b and c, respectively). Then, similar to how we showed that $f(x) = 1$ is the $p.d.f.$ of the uniform distribution on $(0,1)$, it can be shown that $f(x,y,z) = 1$ for x, y, z in $(0,1)$ (0 elsewhere) is the joint $p.d.f.$ of X, Y and Z. Now,

$$P(b^2 - 4ac \geq 0) = P((a,c) \in R, 2\sqrt{a}\sqrt{c} \leq b < 1),$$

so this probability is the triple integral of $f(a,b,c) = 1$ as b varies from $2\sqrt{a}\sqrt{c}$ to 1 and as (a,c) varies over the region R. Since R can be divided into two regions R_1 and R_2, then the required triple integral can be split into a sum of two triple integrals, using vertical slices in R:

$$\begin{aligned}
P(b^2 - 4ac \geq 0) &= \underbrace{\int_0^{1/4}\int_0^1}_{R_1} \int_{2\sqrt{a}\sqrt{c}}^1 1 \, db\, dc\, da + \underbrace{\int_{1/4}^1\int_0^{1/4a}}_{R_2} \int_{2\sqrt{a}\sqrt{c}}^1 1 \, db\, dc\, da \\
&= \int_0^{1/4}\int_0^1 (1 - 2\sqrt{a}\sqrt{c}) dc\, da + \int_{1/4}^1\int_0^{1/4a} (1 - 2\sqrt{a}\sqrt{c}) dc\, da \\
&= \int_0^{1/4} \left(c - \frac{4}{3}\sqrt{a}c^{3/2}\Big|_{c=0}^{c=1} \right) da + \int_{1/4}^1 \left(c - \frac{4}{3}\sqrt{a}c^{3/2}\Big|_{c=0}^{c=1/4a} \right) da \\
&= \int_0^{1/4} \left(1 - \frac{4}{3}\sqrt{a} \right) da + \int_{1/4}^1 \frac{1}{12a}\, da \\
&= a - \frac{8}{9}a^{3/2}\Big|_0^{1/4} + \frac{1}{12}\ln a\Big|_{1/4}^1 \\
&= \left(\frac{1}{4} - \frac{1}{9}\right) + \left(0 - \frac{1}{12}\ln\frac{1}{4}\right) = \frac{5}{36} + \frac{1}{12}\ln 4 \\
P(b^2 - 4ac \geq 0) &= \frac{5 + 3\ln 4}{36} \approx 0.2544
\end{aligned}$$

In other words, the equation $ax^2 + bx + c = 0$ has about a 25% chance of being solved!

Expectation Value

The *expectation value* (or expected value) EX of a random variable X can be thought of as the "average" value of X as it varies over its sample space. If X is a discrete random variable, then

$$EX = \sum_x xP(X = x), \tag{8.5.20}$$

with the sum being taken over all elements x of the sample space. For example, if X represents the number rolled on a six-sided die, then

$$EX = \sum_{x=1}^6 xP(X = x) = \sum_{x=1}^6 x\frac{1}{6} = 3.5 \tag{8.5.21}$$

is the expected value of X, which is the average of the integers 1−6.

If X is a real-valued continuous random variable with p.d.f. f, then

$$EX = \int_{-\infty}^{\infty} x f(x)\, dx \tag{8.5.22}$$

For example, if X has the uniform distribution on the interval $(0,1)$, then its p.d.f. is

$$f(x) = \begin{cases} 1, & \text{for } 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases} \tag{8.5.23}$$

and so

$$EX = \int_{-\infty}^{\infty} x f(x)\, dx = \int_0^1 x\, dx = \frac{1}{2} \tag{8.5.24}$$

For a pair of jointly distributed, real-valued continuous random variables X and Y with joint p.d.f. $f(x,y)$, the expected values of X and Y are given by

$$EX = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)\, dx\, dy \text{ and } EY = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f(x,y)\, dx\, dy \tag{8.5.25}$$

respectively.

Example 8.5.4

If you were to pick $n > 2$ random real numbers from the interval $(0,1)$, what are the expected values for the smallest and largest of those numbers?

Solution

Let $U_1, \ldots, U_n$ be n continuous random variables, each representing a randomly selected real number from $(0,1)$, i.e. each has the uniform distribution on $(0,1)$. Define random variables X and Y by

$$X = \min(U_1, \ldots, U_n) \text{ and } Y = \max(U_1, \ldots, U_n).$$

Then it can be shown that the joint p.d.f. of X and Y is

$$f(x,y) = \begin{cases} n(n-1)(y-x)^{n-2}, & \text{for } 0 \le x \le y \le 1 \\ 0, & \text{elsewhere.} \end{cases} \tag{8.5.26}$$

Thus, the expected value of X is

$$\begin{aligned} EX &= \int_0^1 \int_x^1 n(n-1)x(y-x)^{n-2}\, dy\, dx \\ &= \int_0^1 \left(nx(y-x)^{n-1}\Big|_{y=x}^{y=1} \right) dx \\ &= \int_0^1 nx(1-x)^{n-1}\, dx, \text{ so integration by parts yields} \\ &= -x(1-x)^n - \frac{1}{n+1}(1-x)^{n+1}\Big|_0^1 \\ EX &= \frac{1}{n+1}, \end{aligned}$$

and similarly (see Exercise 3) it can be shown that

$$EY = \int_0^1 \int_0^y n(n-1)y(y-x)^{n-2}\, dx\, dy = \frac{n}{n+1}.$$

So, for example, if you were to repeatedly take samples of $n = 3$ random real numbers from $(0,1)$, and each time store the minimum and maximum values in the sample, then the average of the minimums would approach $\frac{1}{4}$ and the average of the

maximums would approach $\frac{3}{4}$ as the number of samples grows. It would be relatively simple (see Exercise 4) to write a computer program to test this.

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CHAPTER OVERVIEW

9: Differential Equations

A general Calculus Textmap organized around the textbook

Calculus: Early Transcendentals

by James Stewart

I II III IV V VI VII VIII IX X XI XII XIII XIV XV XVI XVII

This Textmap is currently under construction... please be patient with us.

Template:HideTOC

Topic hierarchy

9.1: Modeling with Differential Equations
9.2: Direction Fields and Euler's Method
9.3: Separable Equations
9.4: Models for Population Growth
9.5: Linear Equations
9.6: Predator-Prey Systems

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9.1: Modeling with Differential Equations

Learning Objectives

- Identify the order of a differential equation.
- Explain what is meant by a solution to a differential equation.
- Distinguish between the general solution and a particular solution of a differential equation.
- Identify an initial-value problem.
- Identify whether a given function is a solution to a differential equation or an initial-value problem.

Calculus is the mathematics of change, and rates of change are expressed by derivatives. Thus, one of the most common ways to use calculus is to set up an equation containing an unknown function $y = f(x)$ and its derivative, known as a differential equation. Solving such equations often provides information about how quantities change and frequently provides insight into how and why the changes occur.

Techniques for solving differential equations can take many different forms, including direct solution, use of graphs, or computer calculations. We introduce the main ideas in this chapter and describe them in a little more detail later in the course. In this section we study what differential equations are, how to verify their solutions, some methods that are used for solving them, and some examples of common and useful equations.

General Differential Equations

Consider the equation $y' = 3x^2$, which is an example of a differential equation because it includes a derivative. There is a relationship between the variables x and y : y is an unknown function of x. Furthermore, the left-hand side of the equation is the derivative of y. Therefore we can interpret this equation as follows: Start with some function $y = f(x)$ and take its derivative. The answer must be equal to $3x^2$. What function has a derivative that is equal to $3x^2$? One such function is $y = x^3$, so this function is considered a *solution* to a differential equation.

Definition: differential equation

A *differential equation* is an equation involving an unknown function $y = f(x)$ and one or more of its derivatives. A solution to a differential equation is a function $y = f(x)$ that satisfies the differential equation when f and its derivatives are substituted into the equation.

Go to this website to explore more on this topic.

Some examples of differential equations and their solutions appear in Table 9.1.1.

Table 9.1.1: Examples of Differential Equations and Their Solutions

Equation	Solution
$y' = 2x$	$y = x^2$
$y' + 3y = 6x + 11$	$y = e^{-3x} + 2x + 3$
$y'' - 3y' + 2y = 24e^{-2x}$	$y = 3e^x - 4e^{2x} + 2e^{-2x}$

Note that a solution to a differential equation is not necessarily unique, primarily because the derivative of a constant is zero. For example, $y = x^2 + 4$ is also a solution to the first differential equation in Table 9.1.1. We will return to this idea a little bit later in this section. For now, let's focus on what it means for a function to be a solution to a differential equation.

Example 9.1.1: Verifying Solutions of Differential Equations

Verify that the function $y = e^{-3x} + 2x + 3$ is a solution to the differential equation $y' + 3y = 6x + 11$.

Solution

To verify the solution, we first calculate y' using the chain rule for derivatives. This gives $y' = -3e^{-3x} + 2$. Next we substitute y and y' into the left-hand side of the differential equation:

$$(-3e^{-2x} + 2) + 3(e^{-2x} + 2x + 3).$$

The resulting expression can be simplified by first distributing to eliminate the parentheses, giving

$$-3e^{-2x} + 2 + 3e^{-2x} + 6x + 9.$$

Combining like terms leads to the expression $6x + 11$, which is equal to the right-hand side of the differential equation. This result verifies that $y = e^{-3x} + 2x + 3$ is a solution of the differential equation.

? Exercise 9.1.1

Verify that $y = 2e^{3x} - 2x - 2$ is a solution to the differential equation $y' - 3y = 6x + 4$.

Hint

First calculate y' then substitute both y' and y into the left-hand side.

It is convenient to define characteristics of differential equations that make it easier to talk about them and categorize them. The most basic characteristic of a differential equation is its order.

Definition: order of a differential equation

The *order of a differential equation* is the highest order of any derivative of the unknown function that appears in the equation.

✓ Example 9.1.2: Identifying the Order of a Differential Equation

The highest derivative in the equation is y',

What is the order of each of the following differential equations?

a. $y' - 4y = x^2 - 3x + 4$
b. $x^2y''' - 3xy'' + xy' - 3y = \sin x$
c. $\frac{4}{x}y^{(4)} - \frac{6}{x^2}y'' + \frac{12}{x^4}y = x^3 - 3x^2 + 4x - 12$

Solution

a. The highest derivative in the equation is y',so the order is 1.
b. The highest derivative in the equation is y''', so the order is 3.
c. The highest derivative in the equation is $y^{(4)}$, so the order is 4.

? Exercise 9.1.2

What is the order of the following differential equation?

$$(x^4 - 3x)y^{(5)} - (3x^2 + 1)y' + 3y = \sin x \cos x$$

Hint

What is the highest derivative in the equation?

Answer

5

General and Particular Solutions

We already noted that the differential equation $y' = 2x$ has at least two solutions: $y = x^2$ and $y = x^2 + 4$. The only difference between these two solutions is the last term, which is a constant. What if the last term is a different constant? Will this expression

still be a solution to the differential equation? In fact, any function of the form $y = x^2 + C$, where C represents any constant, is a solution as well. The reason is that the derivative of $x^2 + C$ is $2x$, regardless of the value of C. It can be shown that any solution of this differential equation must be of the form $y = x^2 + C$. This is an example of a **general solution** to a differential equation. A graph of some of these solutions is given in Figure 9.1.1. (Note: in this graph we used even integer values for C ranging between -4 and 4. In fact, there is no restriction on the value of C; it can be an integer or not.)

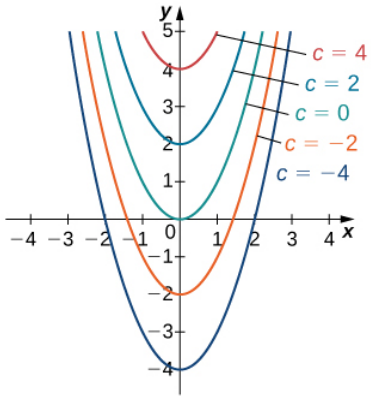


Figure 9.1.1: Family of solutions to the differential equation $y' = 2x$.

In this example, we are free to choose any solution we wish; for example, $y = x^2 - 3$ is a member of the family of solutions to this differential equation. This is called a particular solution to the differential equation. A particular solution can often be uniquely identified if we are given additional information about the problem.

Example 9.1.3: Finding a Particular Solution

Find the particular solution to the differential equation $y' = 2x$ passing through the point $(2, 7)$.

Solution

Any function of the form $y = x^2 + C$ is a solution to this differential equation. To determine the value of C, we substitute the values $x = 2$ and $y = 7$ into this equation and solve for C:

$$\begin{aligned} y &= x^2 + C \\ 7 &= 2^2 + C \\ &= 4 + C \\ C &= 3. \end{aligned}$$

Therefore the particular solution passing through the point $(2, 7)$ is $y = x^2 + 3$.

Exercise 9.1.3

Find the particular solution to the differential equation

$$y' = 4x + 3$$

passing through the point $(1, 7)$, given that $y = 2x^2 + 3x + C$ is a general solution to the differential equation.

Hint

First substitute $x = 1$ and $y = 7$ into the equation, then solve for C.

Answer

$$y = 2x^2 + 3x + 2$$

Initial-Value Problems

Usually a given differential equation has an infinite number of solutions, so it is natural to ask which one we want to use. To choose one solution, more information is needed. Some specific information that can be useful is an *initial value*, which is an ordered pair that is used to find a particular solution.

A differential equation together with one or more initial values is called an *initial-value problem*. The general rule is that the number of initial values needed for an initial-value problem is equal to the order of the differential equation. For example, if we have the differential equation $y' = 2x$, then $y(3) = 7$ is an initial value, and when taken together, these equations form an initial-value problem. The differential equation $y'' - 3y' + 2y = 4e^x$ is second order, so we need two initial values. With initial-value problems of order greater than one, the same value should be used for the independent variable. An example of initial values for this second-order equation would be $y(0) = 2$ and $y'(0) = -1$. These two initial values together with the differential equation form an initial-value problem. These problems are so named because often the independent variable in the unknown function is t, which represents time. Thus, a value of $t = 0$ represents the beginning of the problem.

Example 9.1.4: Verifying a Solution to an Initial-Value Problem

Verify that the function $y = 2e^{-2t} + e^t$ is a solution to the initial-value problem

$$y' - y = -6e^{-2t}, \quad y(0) = 3.$$

Solution

For a function to satisfy an initial-value problem, it must satisfy both the differential equation and the initial condition. To show that y satisfies the differential equation, we start by calculating y'. This gives $y' = -4e^{-2t} + e^t$. Next we substitute both y and y' into the left-hand side of the differential equation and simplify:

$$\begin{aligned} y' - y &= (-4e^{-2t} + e^t) - (2e^{-2t} + e^t) \\ &= -4e^{-2t} + e^t - 2e^{-2t} - e^t = -6e^{-2t}. \end{aligned}$$

This is equal to the right-hand side of the differential equation, so $y = 2e^{-2t} + e^t$ solves the differential equation. Next we calculate $y(0)$:

$$y(0) = 2e^{-2(0)} + e^0 = 2 + 1 = 3.$$

This result verifies the initial value. Therefore the given function satisfies the initial-value problem.

Exercise 9.1.4

Verify that $y = 3e^{2t} + 4\sin t$ is a solution to the initial-value problem

$$y' - 2y = 4\cos t - 8\sin t, y(0) = 3.$$

Hint

First verify that y solves the differential equation. Then check the initial value.

In Example 9.1.4, the initial-value problem consisted of two parts. The first part was the differential equation $y' - y = -6e^{-2t}$, and the second part was the initial value $y(0) = 3$. These two equations together formed the initial-value problem.

The same is true in general. An initial-value problem will consists of two parts: the differential equation and the initial condition. The differential equation has a family of solutions, and the initial condition determines the value of C. The family of solutions to the differential equation in Example 9.1.4 is given by $y = 2e^{-2t} + Ce^t$. This family of solutions is shown in Figure 9.1.2, with the particular solution $y = 2e^{-2t} + e^t$ labeled.

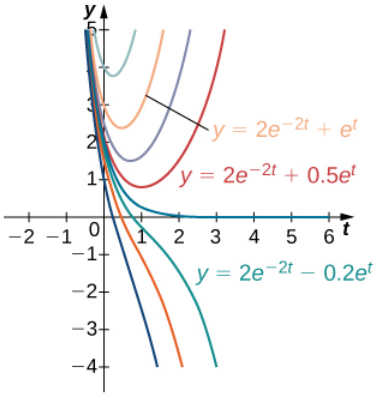


Figure 9.1.2: A family of solutions to the differential equation $y' - y = -6e^{-2t}$. The particular solution $y = 2e^{-2t} + e^t$ is labeled.

Example 9.1.5: Solving an Initial-value Problem

Solve the following initial-value problem:

$$y' = 3e^x + x^2 - 4, y(0) = 5.$$

Solution

The first step in solving this initial-value problem is to find a general family of solutions. To do this, we find an antiderivative of both sides of the differential equation

$$\int y'\,dx = \int (3e^x + x^2 - 4)\,dx,$$

namely,

$$y + C_1 = 3e^x + \tfrac{1}{3}x^3 - 4x + C_2 \quad .$$

We are able to integrate both sides because the y term appears by itself. Notice that there are two integration constants: C_1 and C_2. Solving this equation for y gives

$$y = 3e^x + \tfrac{1}{3}x^3 - 4x + C_2 - C_1.$$

Because C_1 and C_2 are both constants, $C_2 - C_1$ is also a constant. We can therefore define $C = C_2 - C_1$, which leads to the equation

$$y = 3e^x + \tfrac{1}{3}x^3 - 4x + C.$$

Next we determine the value of C. To do this, we substitute $x = 0$ and $y = 5$ into this equation and solve for C:

$$\begin{aligned} 5 &= 3e^0 + \frac{1}{3}0^3 - 4(0) + C \\ 5 &= 3 + C \\ C &= 2 \end{aligned} \quad .$$

Now we substitute the value $C = 2$ into the general equation. The solution to the initial-value problem is $y = 3e^x + \tfrac{1}{3}x^3 - 4x + 2$.

Analysis

The difference between a general solution and a particular solution is that a general solution involves a family of functions, either explicitly or implicitly defined, of the independent variable. The initial value or values determine which particular solution in the family of solutions satisfies the desired conditions.

Exercise 9.1.5

Solve the initial-value problem

$$y' = x^2 - 4x + 3 - 6e^x, y(0) = 8.$$

Hint

First take the antiderivative of both sides of the differential equation. Then substitute $x = 0$ and $y = 8$ into the resulting equation and solve for C.

Answer

$y = \frac{1}{3}x^3 - 2x^2 + 3x - 6e^x + 14$

Applications

In physics and engineering applications, we often consider the forces acting upon an object, and use this information to understand the resulting motion that may occur. For example, if we start with an object at Earth's surface, the primary force acting upon that object is gravity. Physicists and engineers can use this information, along with Newton's **second law of motion** (in equation form $F = ma$, where F represents force, m represents mass, and a represents acceleration), to derive an equation that can be solved.

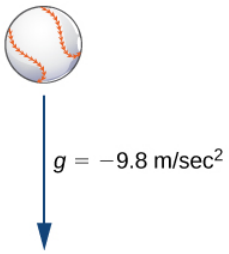


Figure 9.1.3: For a baseball falling in air, the only force acting on it is gravity (neglecting air resistance).

In Figure 9.1.3 we assume that the only force acting on a baseball is the force of gravity. This assumption ignores air resistance. (The force due to air resistance is considered in a later discussion.) The acceleration due to gravity at Earth's surface, g, is approximately $9.8\ \text{m/s}^2$. We introduce a frame of reference, where Earth's surface is at a height of 0 meters. Let $v(t)$ represent the velocity of the object in meters per second. If $v(t) > 0$, the ball is rising, and if $v(t) < 0$, the ball is falling (Figure).



Figure 9.1.4: Possible velocities for the rising/falling baseball.

Our goal is to solve for the velocity $v(t)$ at any time t. To do this, we set up an initial-value problem. Suppose the mass of the ball is m, where m is measured in kilograms. We use Newton's second law, which states that the force acting on an object is equal to its mass times its acceleration $(F = ma)$. Acceleration is the derivative of velocity, so $a(t) = v'(t)$. Therefore the force acting on the baseball is given by $F = mv'(t)$. However, this force must be equal to the force of gravity acting on the object, which (again using Newton's second law) is given by $F_g = -mg$, since this force acts in a downward direction. Therefore we obtain the equation $F = F_g$, which becomes $mv'(t) = -mg$. Dividing both sides of the equation by m gives the equation

$$v'(t) = -g.$$

Notice that this differential equation remains the same regardless of the mass of the object.

We now need an initial value. Because we are solving for velocity, it makes sense in the context of the problem to assume that we know the **initial velocity**, or the velocity at time $t = 0$. This is denoted by $v(0) = v_0$.

Example 9.1.6: Velocity of a Moving Baseball

A baseball is thrown upward from a height of 3 meters above Earth's surface with an initial velocity of 10 m/s, and the only force acting on it is gravity. The ball has a mass of 0.15 kg at Earth's surface.

a. Find the velocity $v(t)$ of the basevall at time t.
b. What is its velocity after 2 seconds?

Solution

a. From the preceding discussion, the differential equation that applies in this situation is

$$v'(t) = -g,$$

where $g = 9.8\,\text{m/s}^2$. The initial condition is $v(0) = v_0$, where $v_0 = 10$ m/s. Therefore the initial-value problem is $v'(t) = -9.8\,\text{m/s}^2$, $v(0) = 10$ m/s.

The first step in solving this initial-value problem is to take the antiderivative of both sides of the differential equation. This gives

$$\int v'(t)\,dt = \int -9.8\,dt$$

$$v(t) = -9.8t + C.$$

The next step is to solve for C. To do this, substitute $t = 0$ and $v(0) = 10$:

$$\begin{aligned} v(t) &= -9.8t + C \\ v(0) &= -9.8(0) + C \\ 10 &= C. \end{aligned}$$

Therefore $C = 10$ and the velocity function is given by $v(t) = -9.8t + 10$.

b. To find the velocity after 2 seconds, substitute $t = 2$ into $v(t)$.

$$\begin{aligned} v(t) &= -9.8t + 10 \\ v(2) &= -9.8(2) + 10 \\ v(2) &= -9.6 \end{aligned}$$

The units of velocity are meters per second. Since the answer is negative, the object is falling at a speed of 9.6 m/s.

Exercise 9.1.6

Suppose a rock falls from rest from a height of 100 meters and the only force acting on it is gravity. Find an equation for the velocity $v(t)$ as a function of time, measured in meters per second.

Hint

What is the initial velocity of the rock? Use this with the differential equation in Example 9.1.6 to form an initial-value problem, then solve for $v(t)$.

Answer

$v(t) = -9.8t$

A natural question to ask after solving this type of problem is how high the object will be above Earth's surface at a given point in time. Let $s(t)$ denote the height above Earth's surface of the object, measured in meters. Because velocity is the derivative of position (in this case height), this assumption gives the equation $s'(t) = v(t)$. An initial value is necessary; in this case the initial height of the object works well. Let the initial height be given by the equation $s(0) = s_0$. Together these assumptions give the initial-value problem

$$s'(t) = v(t), s(0) = s_0.$$

If the velocity function is known, then it is possible to solve for the position function as well.

Example 9.1.7: Height of a Moving Baseball

A baseball is thrown upward from a height of 3 meters above Earth's surface with an initial velocity of $10 m/s$, and the only force acting on it is gravity. The ball has a mass of 0.15 kilogram at Earth's surface.

a. Find the position $s(t)$ of the baseball at time t.
b. What is its height after 2 seconds?

Solution

We already know the velocity function for this problem is $v(t) = -9.8t + 10$. The initial height of the baseball is 3 meters, so $s_0 = 3$. Therefore the initial-value problem for this example is

To solve the initial-value problem, we first find the antiderivatives:

$$\int s'(t)\,dt = \int (-9.8t + 10)\,dt$$

$$s(t) = -4.9t^2 + 10t + C.$$

Next we substitute $t = 0$ and solve for C:

$$s(t) = -4.9t^2 + 10t + C$$

$$s(0) = -4.9(0)^2 + 10(0) + C$$

$$3 = C.$$

Therefore the position function is $s(t) = -4.9t^2 + 10t + 3$.

b. The height of the baseball after 2 sec is given by $s(2)$:

$$s(2) = -4.9(2)^2 + 10(2) + 3 = -4.9(4) + 23 = 3.4.$$

Therefore the baseball is 3.4 meters above Earth's surface after 2 seconds. It is worth noting that the mass of the ball cancelled out completely in the process of solving the problem.

Key Concepts

- A differential equation is an equation involving a function $y = f(x)$ and one or more of its derivatives. A solution is a function $y = f(x)$ that satisfies the differential equation when f and its derivatives are substituted into the equation.
- The order of a differential equation is the highest order of any derivative of the unknown function that appears in the equation.
- A differential equation coupled with an initial value is called an initial-value problem. To solve an initial-value problem, first find the general solution to the differential equation, then determine the value of the constant. Initial-value problems have many applications in science and engineering.

Glossary

differential equation

an equation involving a function $y = y(x)$ and one or more of its derivatives

general solution (or family of solutions)

the entire set of solutions to a given differential equation

initial value(s)
a value or set of values that a solution of a differential equation satisfies for a fixed value of the independent variable

initial velocity
the velocity at time $t = 0$

initial-value problem
a differential equation together with an initial value or values

order of a differential equation
the highest order of any derivative of the unknown function that appears in the equation

particular solution
member of a family of solutions to a differential equation that satisfies a particular initial condition

solution to a differential equation
a function $y = f(x)$ that satisfies a given differential equation

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9.5: Linear Equations

Learning Objectives

- Write a first-order linear differential equation in standard form.
- Find an integrating factor and use it to solve a first-order linear differential equation.
- Solve applied problems involving first-order linear differential equations.

Earlier, we studied an application of a first-order differential equation that involved solving for the velocity of an object. In particular, if a ball is thrown upward with an initial velocity of v_0 ft/s, then an initial-value problem that describes the velocity of the ball after t seconds is given by

$$\frac{dv}{dt} = -32$$

with $v(0) = v_0$.

This model assumes that the only force acting on the ball is gravity. Now we add to the problem by allowing for the possibility of **air resistance** acting on the ball.

Air resistance always acts in the direction opposite to motion. Therefore if an object is rising, air resistance acts in a downward direction. If the object is falling, air resistance acts in an upward direction (Figure 9.5.1). There is no exact relationship between the velocity of an object and the air resistance acting on it. For very small objects, air resistance is proportional to velocity; that is, the force due to air resistance is numerically equal to some constant k times v. For larger (e.g., baseball-sized) objects, depending on the shape, air resistance can be approximately proportional to the square of the velocity. In fact, air resistance may be proportional to $v^{1.5}$, or $v^{0.9}$, or some other power of v.

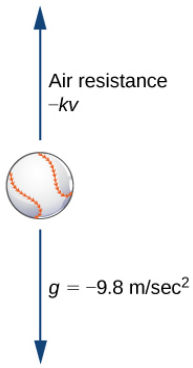


Figure 9.5.1: Forces acting on a moving baseball: gravity acts in a downward direction and air resistance acts in a direction opposite to the direction of motion.

We will work with the linear approximation for air resistance. If we assume $k > 0$, then the expression for the force F_A due to air resistance is given by $F_A = -kv$. Therefore the sum of the forces acting on the object is equal to the sum of the gravitational force and the force due to air resistance. This, in turn, is equal to the mass of the object multiplied by its acceleration at time t(Newton's second law). This gives us the differential equation

$$m\frac{dv}{dt} = -kv - mg.$$

Finally, we impose an initial condition $v(0) = v_0$, where v_0 is the initial velocity measured in meters per second. This makes $g = 9.8m/s^2$. The initial-value problem becomes

$$m\frac{dv}{dt} = -kv - mg$$

with $v(0) = v_0$.

The differential equation in this initial-value problem is an example of a first-order linear differential equation. (Recall that a differential equation is first-order if the highest-order derivative that appears in the equation is 1.) In this section, we study first-

order linear equations and examine a method for finding a general solution to these types of equations, as well as solving initial-value problems involving them.

Definition: Linear first-order differential equation

A first-order differential equation is *linear* if it can be written in the form

$$a(x)y'+b(x)y=c(x), \tag{9.5.1}$$

where $a(x), b(x),$ and $c(x)$ are arbitrary functions of x.

Remember that the unknown function y depends on the variable x; that is, x is the independent variable and y is the dependent variable. Some examples of first-order linear differential equations are

$$\begin{aligned}(3x^2-4)y'+(x-3)y &= \sin x\\ (\sin x)y'-(\cos x)y &= \cot x\\ 4xy'+(3\ln x)y &= x^3-4x.\end{aligned}$$

Examples of first-order nonlinear differential equations include

$$\begin{aligned}(y')^4-(y')^3 &= (3x-2)(y+4)\\ 4y'+3y^3 &= 4x-5\\ (y')^2 &= \sin y+\cos x.\end{aligned}$$

These equations are nonlinear because of terms like $(y')^4, y^3$, etc. Due to these terms, it is impossible to put these equations into the same form as Equation 9.5.1.

Standard Form

Consider the differential equation

$$(3x^2-4)y'+(x-3)y=\sin x.$$

Our main goal in this section is to derive a solution method for equations of this form. It is useful to have the coefficient of y' be equal to 1. To make this happen, we divide both sides by $3x^2-4$.

$$y'+\left(\frac{x-3}{3x^2-4}\right)y=\frac{\sin x}{3x^2-4}$$

This is called the **standard form** of the differential equation. We will use it later when finding the solution to a general first-order linear differential equation. Returning to Equation 9.5.1, we can divide both sides of the equation by $a(x)$. This leads to the equation

$$y'+\frac{b(x)}{a(x)}y=\frac{c(x)}{a(x)}. \tag{9.5.2}$$

Now define

$$p(x)=\frac{b(x)}{a(x)}$$

and

$$q(x)=\frac{c(x)}{a(x)}$$

Then Equation 9.5.2 becomes

$$y'+p(x)y=q(x).$$

We can write any first-order linear differential equation in this form, and this is referred to as the standard form for a first-order linear differential equation.

Example 9.5.1: Writing First-Order Linear Equations in Standard Form

Put each of the following first-order linear differential equations into standard form. Identify $p(x)$ and $q(x)$ for each equation.

a. $y' = 3x - 4y$

b. $\dfrac{3xy'}{4y-3} = 2$ (here $x > 0$)

c. $y = 3y' - 4x^2 + 5$

Solution

a. Add $4y$ to both sides:

$$y' + 4y = 3x.$$

In this equation, $p(x) = 4$ and $q(x) = 3x$.

b. Multiply both sides by $4y - 3$, then subtract $8y$ from each side:

$$\begin{aligned} \frac{3xy'}{4y-3} &= 2 \\ 3xy' &= 2(4y-3) \\ 3xy' &= 8y - 6 \\ 3xy' - 8y &= -6. \end{aligned}$$

Finally, divide both sides by $3x$ to make the coefficient of y' equal to 1:

$$y' - \frac{8}{3x}y = -\frac{2}{3x}.$$

This is allowable because in the original statement of this problem we assumed that $x > 0$. (If $x = 0$ then the original equation becomes $0 = 2$, which is clearly a false statement.)

In this equation, $p(x) = -\dfrac{8}{3x}$ and $q(x) = -\dfrac{2}{3x}$.

c. Subtract y from each side and add $4x^2 - 5$:

$$3y' - y = 4x^2 - 5.$$

Next divide both sides by 3:

$$y' - \frac{1}{3}y = \frac{4}{3}x^2 - \frac{5}{3}.$$

In this equation, $p(x) = -\dfrac{1}{3}$ and $q(x) = \dfrac{4}{3}x^2 - \dfrac{5}{3}$.

Exercise 9.5.1

Put the equation $\dfrac{(x+3)y'}{2x - 3y - 4} = 5$ into standard form and identify $p(x)$ and $q(x)$.

Hint

Multiply both sides by the common denominator, then collect all terms involving y on one side.

Answer

$$y' + \frac{15}{x+3}y = \frac{10x - 20}{x+3}$$

$$p(x) = \frac{15}{x+3}$$

and

$$q(x) = \frac{10x-20}{x+3}$$

Integrating Factors

We now develop a solution technique for any first-order linear differential equation. We start with the standard form of a first-order linear differential equation:

$$y' + p(x)y = q(x). \tag{9.5.3}$$

The first term on the left-hand side of Equation 9.5.3 is the derivative of the unknown function, and the second term is the product of a known function with the unknown function. This is somewhat reminiscent of the power rule. If we multiply Equation 9.5.3 by a yet-to-be-determined function $\mu(x)$, then the equation becomes

$$\mu(x)y' + \mu(x)p(x)y = \mu(x)q(x). \tag{9.5.4}$$

The left-hand side Equation 9.5.4 can be matched perfectly to the product rule:

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

Matching term by term gives $y = f(x), g(x) = \mu(x)$, and $g'(x) = \mu(x)p(x)$. Taking the derivative of $g(x) = \mu(x)$ and setting it equal to the right-hand side of $g'(x) = \mu(x)p(x)$ leads to

$$\mu'(x) = \mu(x)p(x).$$

This is a first-order, separable differential equation for $\mu(x)$. We know $p(x)$ because it appears in the differential equation we are solving. Separating variables and integrating yields

$$\begin{aligned}
\frac{\mu'(x)}{\mu(x)} &= p(x) \\
\int \frac{\mu'(x)}{\mu(x)}dx &= \int p(x)dx \\
\ln|\mu(x)| &= \int p(x)dx + C \\
e^{\ln|\mu(x)|} &= e^{\int p(x)dx + C} \\
|\mu(x)| &= C_1 e^{\int p(x)dx} \\
\mu(x) &= C_2 e^{\int p(x)dx}.
\end{aligned}$$

Here C_2 can be an arbitrary (positive or negative) constant. This leads to a general method for solving a first-order linear differential equation. We first multiply both sides of Equation 9.5.3 by this integrating factor by **integrating factor** $\mu(x)$. This gives

$$\mu(x)y' + \mu(x)p(x)y = \mu(x)q(x). \tag{9.5.5}$$

Note that, since the non-zero constant C_2 can be divided out of this equation (as it will appear in every term on both sides), we do not need to use it in this process. That is, we can simply use

$$\mu(x) = e^{\int p(x)dx}.$$

The left-hand side of Equation 9.5.5 can be rewritten as $\frac{d}{dx}(\mu(x)y)$.

$$\frac{d}{dx}(\mu(x)y) = \mu(x)q(x). \tag{9.5.6}$$

Next integrate both sides of Equation 9.5.6 with respect to x.

$$\int \frac{d}{dx}(\mu(x)y)dx = \int \mu(x)q(x)dx$$

$$\mu(x)y = \int \mu(x)q(x)dx$$

Divide both sides of Equation 9.5.6 by $\mu(x)$:

$$y = \frac{1}{\mu(x)}\left[\int \mu(x)q(x)dx + C\right]. \tag{9.5.7}$$

Since $\mu(x)$ was previously calculated, we are now finished. An important note about the integrating constant C: It may seem that we are inconsistent in the usage of the integrating constant. However, the integral involving $p(x)$ is necessary in order to find an integrating factor for Equation 9.5.3. Only one integrating factor is needed in order to solve the equation; therefore, it is safe to assign a value for C for this integral. We chose $C = 0$. When calculating the integral inside the brackets in Equation 9.5.7, it is necessary to keep our options open for the value of the integrating constant, because our goal is to find a general family of solutions to Equation 9.5.3. This integrating factor guarantees just that.

Problem-Solving Strategy: Solving a First-order Linear Differential Equation

1. Put the equation into standard form and identify $p(x)$ and $q(x)$.
2. Calculate the integrating factor

$$\mu(x) = e^{\int p(x)dx}. \tag{9.5.8}$$

3. Multiply both sides of the differential equation by $\mu(x)$.
4. Integrate both sides of the equation obtained in step 3, and divide both sides by $\mu(x)$.
5. If there is an initial condition, determine the value of C.

Example 9.5.2: Solving a First-order Linear Equation

Find a general solution for the differential equation $xy' + 3y = 4x^2 - 3x$. Assume $x > 0$.

Solution

1. To put this differential equation into standard form, divide both sides by x:

$$y' + \frac{3}{x}y = 4x - 3.$$

Therefore $p(x) = \dfrac{3}{x}$ and $q(x) = 4x - 3$.

2. The integrating factor is $\mu(x) = e^{\int (3/x)dx} = e^{3\ln x} = x^3$.

3. Multiplying both sides of the differential equation by $\mu(x)$ gives us

$$\begin{aligned} x^3y' + x^3\left(\frac{3}{x}\right)y &= x^3(4x-3) \\ x^3y' + 3x^2y &= 4x^4 - 3x^3 \\ \frac{d}{dx}(x^3y) &= 4x^4 - 3x^3. \end{aligned}$$

4. Integrate both sides of the equation.

$$\int \frac{d}{dx}(x^3 y)dx = \int 4x^4 - 3x^3 dx$$

$$x^3 y = \frac{4x^5}{5} - \frac{3x^4}{4} + C$$

$$y = \frac{4x^2}{5} - \frac{3x}{4} + Cx^{-3}.$$

5. There is no initial value, so the problem is complete.

Analysis

You may have noticed the condition that was imposed on the differential equation; namely, $x > 0$. For any nonzero value of C, the general solution is not defined at $x = 0$. Furthermore, when $x < 0$, the integrating factor changes. The integrating factor is given by Equation 9.5.8 as $\mu(x) = e^{\int p(x)dx}$. For this $p(x)$ we get

$$e^{\int p(x)dx} = e^{\int (3/x)dx}$$

$$= e^{3\ln|x|}$$

$$= |x|^3$$

since $x < 0$. The behavior of the general solution changes at $x = 0$ largely due to the fact that $p(x)$ is not defined there.

Exercise 9.5.2

Find the general solution to the differential equation $(x-2)y' + y = 3x^2 + 2x$. Assume $x > 2$.

Hint

Use the method outlined in the problem-solving strategy for first-order linear differential equations.

Answer

$$y = \frac{x^3 + x^2 + C}{x - 2}$$

Now we use the same strategy to find the solution to an initial-value problem.

Example 9.5.3: A First-order Linear Initial-Value Problem

Solve the initial-value problem

$$y' + 3y = 2x - 1, y(0) = 3.$$

Solution

1. This differential equation is already in standard form with $p(x) = 3$ and $q(x) = 2x - 1$.

2. The integrating factor is $\mu(x) = e^{\int 3dx} = e^{3x}$.

3. Multiplying both sides of the differential equation by $\mu(x)$ gives

$$e^{3x}y' + 3e^{3x}y = (2x-1)e^{3x}$$

$$\frac{d}{dx}[ye^{3x}] = (2x-1)e^{3x}.$$

Integrate both sides of the equation:

$$\int \frac{d}{dx}[ye^{3x}]dx = \int (2x-1)e^{3x}dx$$

$$ye^{3x} = \frac{e^{3x}}{3}(2x-1) - \int \frac{2}{3}e^{3x}dx$$

$$ye^{3x} = \frac{e^{3x}(2x-1)}{3} - \frac{2e^{3x}}{9} + C$$

$$y = \frac{2x-1}{3} - \frac{2}{9} + Ce^{-3x}$$

$$y = \frac{2x}{3} - \frac{5}{9} + Ce^{-3x}.$$

4. Now substitute $x = 0$ and $y = 3$ into the general solution and solve for C:

$$y = \frac{2}{3}x - \frac{5}{9} + Ce^{-3x}$$

$$3 = \frac{2}{3}(0) - \frac{5}{9} + Ce^{-3(0)}$$

$$3 = -\frac{5}{9} + C$$

$$C = \frac{32}{9}.$$

Therefore the solution to the initial-value problem is

$$y = \frac{2}{3}x - \frac{5}{9} + \frac{32}{9}e^{-3x}.$$

Example 9.5.4:

Solve the initial-value problem

$$y' - 2y = 4x + 3 y(0) = -2.$$

Solution

$$y = -2x - 4 + 2e^{2x}$$

Applications of First-order Linear Differential Equations

We look at two different applications of first-order linear differential equations. The first involves air resistance as it relates to objects that are rising or falling; the second involves an electrical circuit. Other applications are numerous, but most are solved in a similar fashion.

Free fall with air resistance

We discussed air resistance at the beginning of this section. The next example shows how to apply this concept for a ball in vertical motion. Other factors can affect the force of air resistance, such as the size and shape of the object, but we ignore them here.

Example 9.5.5: A Ball with Air Resistance

A racquetball is hit straight upward with an initial velocity of $2\ \text{m/s}$. The mass of a racquetball is approximately $0.0427\ \text{kg}$. Air resistance acts on the ball with a force numerically equal to $0.5v$, where v represents the velocity of the ball at time t.

a. Find the velocity of the ball as a function of time.
b. How long does it take for the ball to reach its maximum height?
c. If the ball is hit from an initial height of 1 meter, how high will it reach?

Solution

a. The mass $m = 0.0427\ kg, k = 0.5,$ and $g = 9.8\ m/s^2$. The initial velocity is $v_0 = 2\ m/s$. Therefore the initial-value problem is

$$0.0427\frac{dv}{dt} = -0.5v - 0.0427(9.8), v_0 = 2.$$

Dividing the differential equation by 0.0427 gives

$$\frac{dv}{dt} = -11.7096v - 9.8, v_0 = 2.$$

The differential equation is linear. Using the problem-solving strategy for linear differential equations:

Step 1. Rewrite the differential equation as $\frac{dv}{dt} + 11.7096v = -9.8$. This gives $p(t) = 11.7096$ and $q(t) = -9.8$

Step 2. The integrating factor is $\mu(t) = e^{\int 11.7096dt} = e^{11.7096t}$.

Step 3. Multiply the differential equation by $\mu(t)$:

$$e^{11.7096t}\frac{dv}{dt} + 11.7096ve^{11.7096t} = -9.8e^{11.7096t}$$

$$\frac{d}{dt}[ve^{11.7096t}] = -9.8e^{11.7096t}.$$

Step 4. Integrate both sides:

$$\int \frac{d}{dt}[ve^{11.7096t}]dt = \int -9.8e^{11.7096t}dt$$

$$ve^{11.7096t} = \frac{-9.8}{11.7096}e^{11.7096t} + C$$

$$v(t) = -0.8369 + Ce^{-11.7096t}.$$

Step 5. Solve for C using the initial condition $v_0 = v(0) = 2$:

$$v(t) = -0.8369 + Ce^{-11.7096t}$$

$$v(0) = -0.8369 + Ce^{-11.7096(0)}$$

$$2 = -0.8369 + C$$

$$C = 2.8369.$$

Therefore the solution to the initial-value problem is

$$v(t) = 2.8369e^{-11.7096t} - 0.8369.$$

b. The ball reaches its maximum height when the velocity is equal to zero. The reason is that when the velocity is positive, it is rising, and when it is negative, it is falling. Therefore when it is zero, it is neither rising nor falling, and is at its maximum height:

$$2.8369e^{-11.7096t} - 0.8369 = 0$$

$$2.8369e^{-11.7096t} = 0.8369$$

$$e^{-11.7096t} = \frac{0.8369}{2.8369} \approx 0.295$$

$$\ln e^{-11.7096t} = \ln 0.295 \approx -1.221$$

$$-11.7096t = -1.221$$

$$t \approx 0.104.$$

Therefore it takes approximately 0.104 second to reach maximum height.

c. To find the height of the ball as a function of time, use the fact that the derivative of position is velocity, i.e., if $h(t)$ represents the height at time t, then $h'(t) = v(t)$. Because we know $v(t)$ and the initial height, we can form an initial-value problem:

$$h'(t) = 2.8369e^{-11.7096t} - 0.8369,\ h(0) = 1.$$

Integrating both sides of the differential equation with respect to t gives

$$\begin{aligned}\int h'(t)dt &= \int 2.8369e^{-11.7096t} - 0.8369\ dt \\ h(t) &= -\frac{2.8369}{11.7096}e^{-11.7096t} - 0.8369t + C \\ h(t) &= -0.2423e^{-11.7096t} - 0.8369t + C.\end{aligned}$$

Solve for C by using the initial condition:

$$\begin{aligned}h(t) &= -0.2423e^{-11.7096t} - 0.8369t + C \\ h(0) &= -0.2423e^{-11.7096(0)} - 0.8369(0) + C \\ 1 &= -0.2423 + C \\ C &= 1.2423.\end{aligned}$$

Therefore

$$h(t) = -0.2423e^{-11.7096t} - 0.8369t + 1.2423.$$

After 0.104 second, the height is given by

$$\begin{aligned}h(0.2) &= -0.2423e^{-11.7096t} - 0.8369t + 1.2423 \\ &\approx 1.0836 \text{ meter.}\end{aligned}$$

? Exercise 9.5.3

The weight of a penny is 2.5 grams (United States Mint, "Coin Specifications," accessed April 9, 2015, www.usmint.gov/about_the_mint...specifications), and the upper observation deck of the Empire State Building is 369 meters above the street. Since the penny is a small and relatively smooth object, air resistance acting on the penny is actually quite small. We assume the air resistance is numerically equal to $0.0025u$ Furthermore, the penny is dropped with no initial velocity imparted to it.

a. Set up an initial-value problem that represents the falling penny.
b. Solve the problem for $v(t)$.
c. What is the terminal velocity of the penny (i.e., calculate the limit of the velocity as t approaches infinity)?

Hint

Set up the differential equation the same way as Example ? 9.5.5 Remember to convert from grams to kilograms.

Answer

a. $\dfrac{dv}{dt} = -v - 9.8\ \ v(0) = 0$

b. $v(t) = 9.8(e^{-t} - 1)$

c. $\lim\limits_{t\to\infty} v(t) = \lim\limits_{t\to\infty}(9.8(e^{-t} - 1)) = -9.8 \text{ m/s} \approx -21.922 \text{ mph}$

Electrical Circuits

A source of electromotive force (e.g., a battery or generator) produces a flow of current in a closed circuit, and this current produces a voltage drop across each resistor, inductor, and capacitor in the circuit. Kirchhoff's Loop Rule states that the sum of the voltage drops across resistors, inductors, and capacitors is equal to the total electromotive force in a closed circuit. We have the following three results:

1. The voltage drop across a resistor is given by

$$E_R = Ri,$$

where R is a constant of proportionality called the resistance, and i is the current.

2. The voltage drop across an inductor is given by

$$E_L = Li',$$

where L is a constant of proportionality called the inductance, and i again denotes the current.

3. The voltage drop across a capacitor is given by

$$E_C = \frac{1}{C}q,$$

where C is a constant of proportionality called the capacitance, and q is the instantaneous charge on the capacitor. The relationship between i and q is $i = q'$.

We use units of volts (V) to measure voltage E, amperes (A) to measure current i, coulombs (C) to measure charge q, ohms (Ω) to measure resistance R, henrys (H) to measure inductance L, and farads (F) to measure capacitance C. Consider the circuit in Figure 9.5.2.

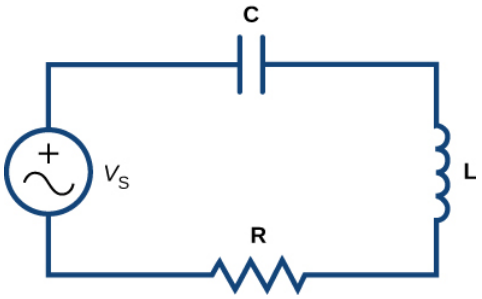


Figure 9.5.2: A typical electric circuit, containing a voltage generator (V_S), capacitor (C), inductor (L), and resistor (R).

Applying Kirchhoff's Loop Rule to this circuit, we let E denote the electromotive force supplied by the voltage generator. Then

$$E_L + E_R + E_C = E \ .$$

Substituting the expressions for E_L, E_R, and E_C into this equation, we obtain

$$Li' + Ri + \frac{1}{C}q = E. \tag{9.5.9}$$

If there is no capacitor in the circuit, then the equation becomes

$$Li' + Ri = E. \tag{9.5.10}$$

This is a first-order differential equation in i. The circuit is referred to as an **LR circuit**.

Next, suppose there is no inductor in the circuit, but there is a capacitor and a resistor, so $L = 0,\ R \neq 0,$ and $C \neq 0$. Then, since $q' = i$, Equation 9.5.9 can be rewritten as

$$Rq' + \frac{1}{C}q = E, \tag{9.5.11}$$

which is a first-order linear differential equation in q. This is referred to as an **RC circuit**. In either case, we can set up and solve an initial-value problem.

Electric Circuit

A circuit has in series an electromotive force given by $E = 50\sin 20tV$, a resistor of 5Ω, and an inductor of $0.4H$. If the initial current is 0, find the current at time $t > 0$.

Solution

We have a resistor and an inductor in the circuit, so we have an LR circuit and can use Equation 9.5.10. The voltage drop across the resistor is given by $E_R = Ri = 5i$. The voltage drop across the inductor is given by $E_L = Li' = 0.4i'$. The electromotive force becomes the right-hand side of Equation 9.5.10. Therefore Equation 9.5.10 becomes

$$0.4i' + 5i = 50\sin 20t.$$

Dividing both sides by 0.4 gives the equation

$$i' + 12.5i = 125\sin 20t.$$

Since the initial current is 0, this result gives an initial condition of $i(0) = 0$. We can solve this initial-value problem using the five-step strategy for solving first-order differential equations.

Step 1. Rewrite the differential equation as $i' + 12.5i = 125\sin 20t$. This gives $p(t) = 12.5$ and $q(t) = 125\sin 20t$.

Step 2. The integrating factor is $\mu(t) = e^{\int 12.5dt} = e^{12.5t}$.

Step 3. Multiply the differential equation by $\mu(t)$:

$$\begin{aligned} e^{12.5t}i' + 12.5e^{12.5t}i &= 125e^{12.5t}\sin 20t \\ \frac{d}{dt}[ie^{12.5t}] &= 125e^{12.5t}\sin 20t. \end{aligned}$$

Step 4. Integrate both sides:

$$\begin{aligned} \int \frac{d}{dt}[ie^{12.5t}]dt &= \int 125e^{12.5t}\sin 20t\, dt \\ ie^{12.5t} &= \left(\frac{250\sin 20t - 400\cos 20t}{89}\right)e^{12.5t} + C \\ i(t) &= \frac{250\sin 20t - 400\cos 20t}{89} + Ce^{-12.5t}. \end{aligned}$$

Step 5. Solve for C using the initial condition $i(0) = 0$:

$$\begin{aligned} i(t) &= \frac{250\sin 20t - 400\cos 20t}{89} + Ce^{-12.5t} \\ i(0) &= \frac{250\sin 20(0) - 400\cos 20(0)}{89} + Ce^{-12.5(0)} \\ 0 &= -\frac{400}{89} + C \\ C &= \frac{400}{89}. \end{aligned}$$

Therefore the solution to the initial-value problem is

$$i(t) = \frac{250\sin 20t - 400\cos 20t + 400e^{-12.5t}}{89} = \frac{250\sin 20t - 400\cos 20t}{89} + \frac{400e^{-12.5t}}{89}.$$

The first term can be rewritten as a single cosine function. First, multiply and divide by $\sqrt{250^2 + 400^2} = 50\sqrt{89}$:

$$\frac{250\sin 20t - 400\cos 20t}{89} = \frac{50\sqrt{89}}{89}\left(\frac{250\sin 20t - 400\cos 20t}{50\sqrt{89}}\right) = -\frac{50\sqrt{89}}{89}\left(\frac{8\cos 20t}{\sqrt{89}} - \frac{5\sin 20t}{\sqrt{89}}\right).$$

Next, define φ to be an acute angle such that $\cos\varphi = \frac{8}{\sqrt{89}}$. Then $\sin\varphi = \frac{5}{\sqrt{89}}$ and

$$-\frac{50\sqrt{89}}{89}\left(\frac{8\cos 20t}{\sqrt{89}}-\frac{5\sin 20t}{\sqrt{89}}\right)=-\frac{50\sqrt{89}}{89}(\cos\varphi\cos 20t-\sin\varphi\sin 20t)=-\frac{50\sqrt{89}}{89}\cos(20t+\varphi).$$

Therefore the solution can be written as

$$i(t)=-\frac{50\sqrt{89}}{89}cos(20t+\varphi)+\frac{400e^{-12.5t}}{89}\ .$$

The second term is called the attenuation term, because it disappears rapidly as t grows larger. The phase shift is given by φ, and the amplitude of the steady-state current is given by $\frac{50\sqrt{89}}{89}$. The graph of this solution appears in Figure 9.5.3:

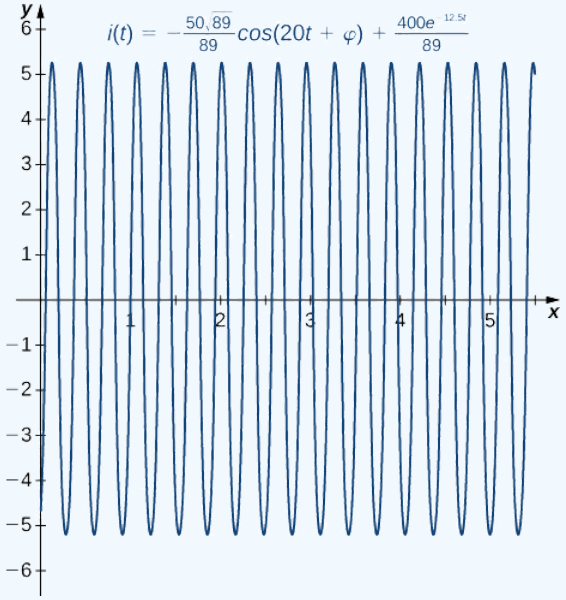


Figure 9.5.3.

? Exercise 9.5.4

A circuit has in series an electromotive force given by $E=20\sin 5t$ V, a capacitor with capacitance $0.02F$, and a resistor of 8Ω. If the initial charge is $4C$, find the charge at time $t>0$.

Hint

Use Equation 9.5.11 for an RC circuit to set up an initial-value problem.

Answer

Initial-value problem:

$$8q'+\frac{1}{0.02}q=20\sin 5t, q(0)=4$$

$$q(t)=\frac{10\sin 5t-8\cos 5t+172e^{-6.25t}}{41}$$

Key Concepts

- Any first-order linear differential equation can be written in the form $y'+p(x)y=q(x)$.
- We can use a five-step problem-solving strategy for solving a first-order linear differential equation that may or may not include an initial value.
- Applications of first-order linear differential equations include determining motion of a rising or falling object with air resistance and finding current in an electrical circuit.

Key Equations

- **standard form**

  $$y' + p(x)y = q(x)$$

- **integrating factor**

  $$\mu(x) = e^{\int p(x)dx}$$

Glossary

integrating factor
any function $f(x)$ that is multiplied on both sides of a differential equation to make the side involving the unknown function equal to the derivative of a product of two functions

linear
description of a first-order differential equation that can be written in the form $a(x)y' + b(x)y = c(x)$

standard form
the form of a first-order linear differential equation obtained by writing the differential equation in the form $y' + p(x)y = q(x)$

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9.6: Predator-Prey Systems

Learning Objectives

- Describe the concept of environmental carrying capacity in the logistic model of population growth.
- Draw a direction field for a logistic equation and interpret the solution curves.
- Solve a logistic equation and interpret the results.

Differential equations can be used to represent the size of a population as it varies over time. We saw this in an earlier chapter in the section on exponential growth and decay, which is the simplest model. A more realistic model includes other factors that affect the growth of the population. In this section, we study the logistic differential equation and see how it applies to the study of population dynamics in the context of biology.

Population Growth and Carrying Capacity

To model population growth using a differential equation, we first need to introduce some variables and relevant terms. The variable t. will represent time. The units of time can be hours, days, weeks, months, or even years. Any given problem must specify the units used in that particular problem. The variable P will represent population. Since the population varies over time, it is understood to be a function of time. Therefore we use the notation $P(t)$ for the population as a function of time. If $P(t)$ is a differentiable function, then the first derivative $\frac{dP}{dt}$ represents the instantaneous rate of change of the population as a function of time.

In Exponential Growth and Decay, we studied the exponential growth and decay of populations and radioactive substances. An example of an exponential growth function is $P(t) = P_0 e^{rt}$. In this function, $P(t)$ represents the population at time t, P_0 represents the **initial population** (population at time $t = 0$), and the constant $r > 0$ is called the **growth rate**. Figure $9.6.1$ shows a graph of $P(t) = 100e^{0.03t}$. Here $P_0 = 100$ and $r = 0.03$.

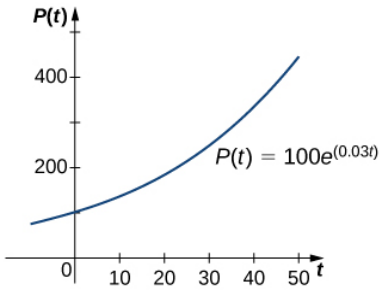


Figure 9.6.1: An exponential growth model of population.

We can verify that the function $P(t) = P_0 e^{rt}$ satisfies the initial-value problem

$$\frac{dP}{dt} = rP$$

with $P(0) = P_0$.

This differential equation has an interesting interpretation. The left-hand side represents the rate at which the population increases (or decreases). The right-hand side is equal to a positive constant multiplied by the current population. Therefore the differential equation states that the rate at which the population increases is proportional to the population at that point in time. Furthermore, it states that the constant of proportionality never changes.

One problem with this function is its prediction that as time goes on, the population grows without bound. This is unrealistic in a real-world setting. Various factors limit the rate of growth of a particular population, including birth rate, death rate, food supply, predators, and so on. The growth constant r usually takes into consideration the birth and death rates but none of the other factors, and it can be interpreted as a net (birth minus death) percent growth rate per unit time. A natural question to ask is whether the population growth rate stays constant, or whether it changes over time. Biologists have found that in many biological systems, the population grows until a certain steady-state population is reached. This possibility is not taken into account with exponential growth. However, the concept of carrying capacity allows for the possibility that in a given area, only a certain number of a given organism or animal can thrive without running into resource issues.

Definition: Carrying Capacity

The *carrying capacity* of an organism in a given environment is defined to be the maximum population of that organism that the environment can sustain indefinitely.

We use the variable K to denote the carrying capacity. The growth rate is represented by the variable r. Using these variables, we can define the logistic differential equation.

Definition: Logistic Differential Equation

Let K represent the carrying capacity for a particular organism in a given environment, and let r be a real number that represents the growth rate. The function $P(t)$ represents the population of this organism as a function of time t, and the constant P_0 represents the initial population (population of the organism at time $t = 0$). Then the *logistic differential equation* is

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right). \tag{9.6.1}$$

The logistic equation was first published by Pierre Verhulst in 1845. This differential equation can be coupled with the initial condition $P(0) = P_0$ to form an initial-value problem for $P(t)$.

Suppose that the initial population is small relative to the carrying capacity. Then $\frac{P}{K}$ is small, possibly close to zero. Thus, the quantity in parentheses on the right-hand side of Equation 9.6.1 is close to 1, and the right-hand side of this equation is close to rP. If $r > 0$, then the population grows rapidly, resembling exponential growth.

However, as the population grows, the ratio $\frac{P}{K}$ also grows, because K is constant. If the population remains below the carrying capacity, then $\frac{P}{K}$ is less than 1, so $1 - \frac{P}{K} > 0$. Therefore the right-hand side of Equation 9.6.1 is still positive, but the quantity in parentheses gets smaller, and the growth rate decreases as a result. If $P = K$ then the right-hand side is equal to zero, and the population does not change.

Now suppose that the population starts at a value higher than the carrying capacity. Then $\frac{P}{K} > 1$, and $1 - \frac{P}{K} < 0$. Then the right-hand side of Equation 9.6.1 is negative, and the population decreases. As long as $P > K$, the population decreases. It never actually reaches K because $\frac{dP}{dt}$ will get smaller and smaller, but the population approaches the carrying capacity as t approaches infinity. This analysis can be represented visually by way of a phase line. A **phase line** describes the general behavior of a solution to an autonomous differential equation, depending on the initial condition. For the case of a carrying capacity in the logistic equation, the phase line is as shown in Figure 9.6.2.



Figure 9.6.2: A phase line for the differential equation $\frac{dP}{dt} = rP(1 - \frac{P}{K})$.

This phase line shows that when P is less than zero or greater than K, the population decreases over time. When P is between 0 and K, the population increases over time.

Example 9.6.1: Examining the Carrying Capacity of a Deer Population

Let's consider the population of white-tailed deer (*Odocoileus virginianus*) in the state of Kentucky. The Kentucky Department of Fish and Wildlife Resources (KDFWR) sets guidelines for hunting and fishing in the state. Before the hunting season of 2004, it estimated a population of 900,000 deer. Johnson notes: "A deer population that has plenty to eat and is not hunted by humans or other predators will double every three years." (George Johnson, "The Problem of Exploding Deer Populations Has No Attractive Solutions," January 12,2001, accessed April 9, 2015)



Figure 9.6.3: (credit: modification of work by Rachel Kramer, Flickr)

This observation corresponds to a rate of increase $r = \dfrac{\ln(2)}{3} = 0.2311$, so the approximate growth rate is 23.11% per year. (This assumes that the population grows exponentially, which is reasonable—at least in the short term—with plentiful food supply and no predators.) The KDFWR also reports deer population densities for 32 counties in Kentucky, the average of which is approximately 27 deer per square mile. Suppose this is the deer density for the whole state (39,732 square miles). The carrying capacity K is 39,732 square miles times 27 deer per square mile, or 1,072,764 deer.

a. For this application, we have $P_0 = 900,000$, $K = 1,072,764$, and $r = 0.2311$. Substitute these values into Equation 9.6.1 and form the initial-value problem.
b. Solve the initial-value problem from part a.
c. According to this model, what will be the population in 3 years? Recall that the doubling time predicted by Johnson for the deer population was 3 years. How do these values compare?

Suppose the population managed to reach **1,200,000** What does the logistic equation predict will happen to the population in this scenario?

Solution

a. The initial value problem is

$$\frac{dP}{dt} = 0.2311P\left(1 - \frac{P}{1,072,764}\right), \quad P(0) = 900,000.$$

b. The logistic equation is an autonomous differential equation, so we can use the method of separation of variables.

Step 1: Setting the right-hand side equal to zero gives $P = 0$ and $P = 1,072,764$. This means that if the population starts at zero it will never change, and if it starts at the carrying capacity, it will never change.

Step 2: Rewrite the differential equation and multiply both sides by:

$$\frac{dP}{dt} = 0.2311P\left(\frac{1,072,764 - P}{1,072,764}\right)$$

$$dP = 0.2311P\left(\frac{1,072,764 - P}{1,072,764}\right)dt$$

$$\frac{dP}{P(1,072,764 - P)} = \frac{0.2311}{1,072,764}dt.$$

Step 3: Integrate both sides of the equation using partial fraction decomposition:

$$\int \frac{dP}{P(1,072,764-P)} = \int \frac{0.2311}{1,072,764} dt$$

$$\frac{1}{1,072,764} \int \left(\frac{1}{P} + \frac{1}{1,072,764-P} \right) dP = \frac{0.2311t}{1,072,764} + C$$

$$\frac{1}{1,072,764} (\ln|P| - \ln|1,072,764-P|) = \frac{0.2311t}{1,072,764} + C.$$

Step 4: Multiply both sides by 1,072,764 and use the quotient rule for logarithms:

$$\ln\left|\frac{P}{1,072,764-P}\right| = 0.2311t + C_1.$$

Here $C_1 = 1,072,764C$. Next exponentiate both sides and eliminate the absolute value:

$$e^{\ln\left|\frac{P}{1,072,764-P}\right|} = e^{0.2311t+C_1}$$

$$\left|\frac{P}{1,072,764-P}\right| = C_2 e^{0.2311t}$$

$$\frac{P}{1,072,764-P} = C_2 e^{0.2311t}.$$

Here $C_2 = e^{C_1}$ but after eliminating the absolute value, it can be negative as well. Now solve for:

$$P = C_2 e^{0.2311t}(1,072,764-P)$$

$$P = 1,072,764 C_2 e^{0.2311t} - C_2 P e^{0.2311t}$$

$$P + C_2 P e^{0.2311t} = 1,072,764 C_2 e^{0.2311t}$$

$$P(1 + C_2 e^{0.2311t} = 1,072,764 C_2 e^{0.2311t}$$

$$P(t) = \frac{1,072,764 C_2 e^{0.2311t}}{1 + C_2 e^{0.2311t}}.$$

Step 5: To determine the value of C_2, it is actually easier to go back a couple of steps to where C_2 was defined. In particular, use the equation

$$\frac{P}{1,072,764-P} = C_2 e^{0.2311t}.$$

The initial condition is $P(0) = 900,000$. Replace P with $900,000$ and t with zero:

$$\frac{P}{1,072,764-P} = C_2 e^{0.2311t}$$

$$\frac{900,000}{1,072,764-900,000} = C_2 e^{0.2311(0)}$$

$$\frac{900,000}{172,764} = C_2$$

$$C_2 = \frac{25,000}{4,799}$$

$$\approx 5.209.$$

Therefore

$$P(t) = \frac{1,072,764\left(\frac{25000}{4799}\right)e^{0.2311t}}{1+(250004799)e^{0.2311t}}$$
$$= \frac{1,072,764(25000)e^{0.2311t}}{4799+25000e^{0.2311t}}.$$

Dividing the numerator and denominator by **25,000** gives

$$P(t) = \frac{1,072,764e^{0.2311t}}{0.19196+e^{0.2311t}}.$$

Figure is a graph of this equation.

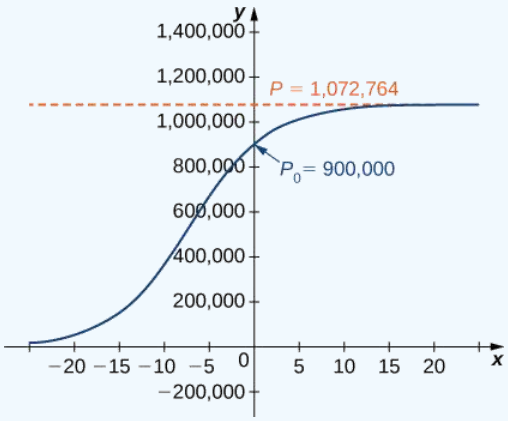


Figure 9.6.4: Logistic curve for the deer population with an initial population of **900,000** deer.

c. Using this model we can predict the population in **3** years.

$$P(3) = \frac{1,072,764e^{0.2311(3)}}{0.19196+e^{0.2311(3)}} \approx 978,830\, deer$$

This is far short of twice the initial population of $900,000$. Remember that the doubling time is based on the assumption that the growth rate never changes, but the logistic model takes this possibility into account.

d. If the population reached **1,200,000** deer, then the new initial-value problem would be

$$\frac{dP}{dt} = 0.2311P\left(1-\frac{P}{1,072,764}\right),\ P(0) = 1,200,000.$$

The general solution to the differential equation would remain the same.

$$P(t) = \frac{1,072,764C_2e^{0.2311t}}{1+C_2e^{0.2311t}}$$

To determine the value of the constant, return to the equation

$$\frac{P}{1,072,764-P} = C_2e^{0.2311t}.$$

Substituting the values $t = 0$ and $P = 1,200,000$, you get

$$C_2e^{0.2311(0)} = \frac{1,200,000}{1,072,764-1,200,000}$$
$$C_2 = -\frac{100,000}{10,603} \approx -9.431.$$

Therefore

$$P(t) = \frac{1,072,764C_2e^{0.2311t}}{1+C_2e^{0.2311t}}$$

$$= \frac{1,072,764\left(-\frac{100,000}{10,603}\right)e^{0.2311t}}{1+\left(-\frac{100,000}{10,603}\right)e^{0.2311t}}$$

$$= -\frac{107,276,400,000e^{0.2311t}}{100,000e^{0.2311t}-10,603}$$

$$\approx \frac{10,117,551e^{0.2311t}}{9.43129e^{0.2311t}-1}$$

This equation is graphed in Figure 9.6.5.

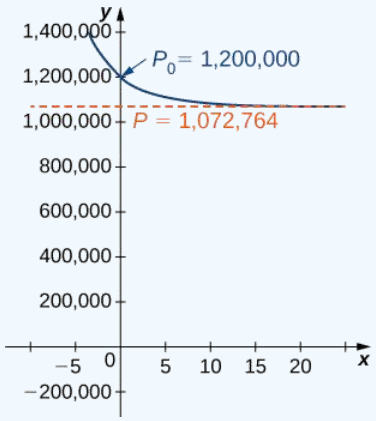


Figure 9.6.5: Logistic curve for the deer population with an initial population of **1,200,000** deer.

Solving the Logistic Differential Equation

The logistic differential equation is an autonomous differential equation, so we can use separation of variables to find the general solution, as we just did in Example 9.6.1.

Step 1: Setting the right-hand side equal to zero leads to $P = 0$ and $P = K$ as constant solutions. The first solution indicates that when there are no organisms present, the population will never grow. The second solution indicates that when the population starts at the carrying capacity, it will never change.

Step 2: Rewrite the differential equation in the form

$$\frac{dP}{dt} = \frac{rP(K-P)}{K}.$$

Then multiply both sides by dt and divide both sides by $P(K-P)$. This leads to

$$\frac{dP}{P(K-P)} = \frac{r}{K}dt.$$

Multiply both sides of the equation by K and integrate:

$$\int \frac{K}{P(K-P)}dP = \int rdt. \tag{9.6.2}$$

The left-hand side of this equation can be integrated using partial fraction decomposition. We leave it to you to verify that

$$\frac{K}{P(K-P)} = \frac{1}{P} + \frac{1}{K-P}.$$

Then the Equation 9.6.2 becomes

$$\int \frac{1}{P} + \frac{1}{K-P} dP = \int r dt$$

$$\ln|P| - \ln|K-P| = rt + C$$

$$\ln\left|\frac{P}{K-P}\right| = rt + C.$$

Now exponentiate both sides of the equation to eliminate the natural logarithm:

$$e^{\ln\left|\frac{P}{K-P}\right|} = e^{rt+C}$$

$$\left|\frac{P}{K-P}\right| = e^C e^{rt}.$$

We define $C_1 = e^c$ so that the equation becomes

$$\frac{P}{K-P} = C_1 e^{rt}. \tag{9.6.3}$$

To solve this equation for $P(t)$, first multiply both sides by $K-P$ and collect the terms containing P on the left-hand side of the equation:

$$P = C_1 e^{rt}(K-P)$$

$$= C_1 K e^{rt} - C_1 P e^{rt}$$

$$P + C_1 P e^{rt} = C_1 K e^{rt}.$$

Next, factor P from the left-hand side and divide both sides by the other factor:

$$P(1 + C_1 e^{rt}) = C_1 K e^{rt}$$

$$P(t) = \frac{C_1 K e^{rt}}{1 + C_1 e^{rt}}.$$

The last step is to determine the value of C_1. The easiest way to do this is to substitute $t = 0$ and P_0 in place of P in Equation and solve for C_1:

$$\frac{P}{K-P} = C_1 e^{rt}$$

$$\frac{P_0}{K-P_0} = C_1 e^{r(0)}$$

$$C_1 = \frac{P_0}{K-P_0}.$$

Finally, substitute the expression for C_1 into Equation 9.6.3:

$$P(t) = \frac{C_1 K e^{rt}}{1 + C_1 e^{rt}} = \frac{\frac{P_0}{K-P_0} K e^{rt}}{1 + \frac{P_0}{K-P_0} e^{rt}}$$

Now multiply the numerator and denominator of the right-hand side by $(K-P_0)$ and simplify:

$$P(t) = \frac{\frac{P_0}{K-P_0}Ke^{rt}}{1+\frac{P_0}{K-P_0}e^{rt}}$$

$$= \frac{\frac{P_0}{K-P_0}Ke^{rt}}{1+\frac{P_0}{K-P_0}e^{rt}} \cdot \frac{K-P_0}{K-P_0} = \frac{P_0Ke^{rt}}{(K-P_0)+P_0e^{rt}}.$$

We state this result as a theorem.

Solution of the Logistic Differential Equation

Consider the logistic differential equation subject to an initial population of P_0 with carrying capacity K and growth rate r. The solution to the corresponding initial-value problem is given by

$$P(t) = \frac{P_0Ke^{rt}}{(K-P_0)+P_0e^{rt}}$$

.

Now that we have the solution to the initial-value problem, we can choose values for P_0, r, and K and study the solution curve. For example, in Example we used the values $r = 0.2311, K = 1,072,764,$ and an initial population of $900,000$ deer. This leads to the solution

$$P(t) = \frac{P_0Ke^{rt}}{(K-P_0)+P_0e^{rt}}$$

$$= \frac{900,000(1,072,764)e^{0.2311t}}{(1,072,764-900,000)+900,000e^{0.2311t}}$$

$$= \frac{900,000(1,072,764)e^{0.2311t}}{172,764+900,000e^{0.2311t}}.$$

Dividing top and bottom by $900,000$ gives

$$P(t) = \frac{1,072,764e^{0.2311t}}{0.19196+e^{0.2311t}}.$$

This is the same as the original solution. The graph of this solution is shown again in blue in Figure $9.6.6$, superimposed over the graph of the exponential growth model with initial population $900,000$ and growth rate 0.2311 (appearing in green). The red dashed line represents the carrying capacity, and is a horizontal asymptote for the solution to the logistic equation.

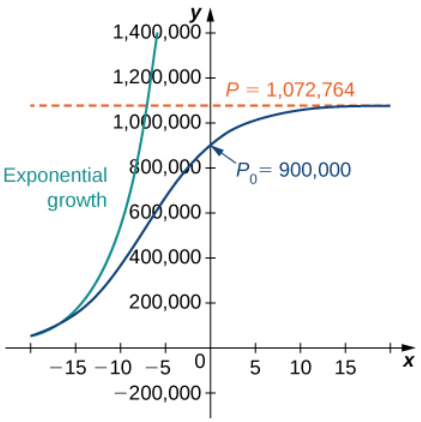


Figure $9.6.6$: A comparison of exponential versus logistic growth for the same initial population of $900,000$ organisms and growth rate of 23.11

Working under the assumption that the population grows according to the logistic differential equation, this graph predicts that approximately 20 years earlier (1984), the growth of the population was very close to exponential. The net growth rate at that time would have been around 23.1 per year. As time goes on, the two graphs separate. This happens because the population increases, and the logistic differential equation states that the growth rate decreases as the population increases. At the time the population was measured (2004), it was close to carrying capacity, and the population was starting to level off.

The solution to the logistic differential equation has a point of inflection. To find this point, set the second derivative equal to zero:

$$P(t) = \frac{P_0 K e^{rt}}{(K - P_0) + P_0 e^{rt}}$$

$$P'(t) = \frac{rP_0 K(K - P0)e^{rt}}{((K - P_0) + P_0 e^{rt})^2}$$

$$\begin{aligned} P''(t) &= \frac{r^2 P_0 K(K - P_0)^2 e^{rt} - r^2 P_0^2 K(K - P_0) e^{2rt}}{((K - P_0) + P_0 e^{rt})^3} \\ &= \frac{r^2 P_0 K(K - P_0) e^{rt}((K - P_0) - P_0 e^{rt})}{((K - P_0) + P_0 e^{rt})^3}. \end{aligned}$$

Setting the numerator equal to zero,

$$r^2 P_0 K(K - P_0) e^{rt}((K - P_0) - P_0 e^{rt}) = 0.$$

As long as $P_0 \neq K$, the entire quantity before and including $e^{rt}$is nonzero, so we can divide it out:

$$(K - P_0) - P_0 e^{rt} = 0.$$

Solving for t,

$$P_0 e^{rt} = K - P_0$$

$$e^{rt} = \frac{K - P_0}{P_0}$$

$$\ln e^{rt} = \ln \frac{K - P_0}{P_0}$$

$$rt = \ln \frac{K - P_0}{P_0}$$

$$t = \frac{1}{r} \ln \frac{K - P_0}{P_0}.$$

Notice that if $P_0 > K$, then this quantity is undefined, and the graph does not have a point of inflection. In the logistic graph, the point of inflection can be seen as the point where the graph changes from concave up to concave down. This is where the “leveling off” starts to occur, because the net growth rate becomes slower as the population starts to approach the carrying capacity.

? Exercise 9.6.1

A population of rabbits in a meadow is observed to be 200 rabbits at time $t = 0$. After a month, the rabbit population is observed to have increased by 4. Using an initial population of 200 and a growth rate of 0.04, with a carrying capacity of 750 rabbits,

a. Write the logistic differential equation and initial condition for this model.
b. Draw a slope field for this logistic differential equation, and sketch the solution corresponding to an initial population of 200 rabbits.
c. Solve the initial-value problem for $P(t)$.
d. Use the solution to predict the population after 1 year.

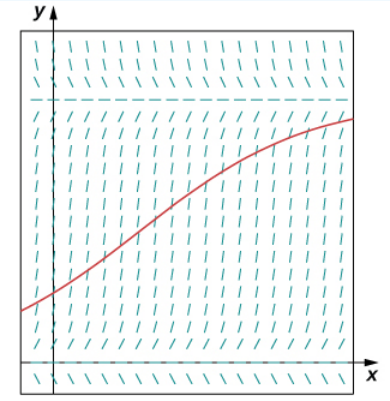
Hint

First determine the values of $r, K,$ and P_0. Then create the initial-value problem, draw the direction field, and solve the problem.

Answer

a. $\frac{dP}{dt} = 0.04(1 - \frac{P}{750}), P(0) = 200$

b.



c. $P(t) = \frac{3000e^{.04t}}{11 + 4e^{.04t}}$

d. After 12 months, the population will be $P(12) \approx 278$ rabbits.

Student Project: Logistic Equation with a Threshold Population

An improvement to the logistic model includes a **threshold population.** The threshold population is defined to be the minimum population that is necessary for the species to survive. We use the variable T to represent the threshold population. A differential equation that incorporates both the threshold population T and carrying capacity K is

$$\frac{dP}{dt} = -rP\left(1 - \frac{P}{K}\right)\left(1 - \frac{P}{T}\right)$$

where r represents the growth rate, as before.

1. The threshold population is useful to biologists and can be utilized to determine whether a given species should be placed on the endangered list. A group of Australian researchers say they have determined the threshold population for any species to survive: 5000 adults. (Catherine Clabby, “A Magic Number,” American Scientist 98(1): 24, doi:10.1511/2010.82.24. accessed April 9, 2015, www.americanscientist.org/iss...a-magic-number). Therefore we use $T = 5000$ as the threshold population in this project. Suppose that the environmental carrying capacity in Montana for elk is $25,000$. Set up Equation using the carrying capacity of $25,000$ and threshold population of 5000. Assume an annual net growth rate of 18%.
2. Draw the direction field for the differential equation from step 1, along with several solutions for different initial populations. What are the constant solutions of the differential equation? What do these solutions correspond to in the original population model (i.e., in a biological context)?
3. What is the limiting population for each initial population you chose in step 2? (Hint: use the slope field to see what happens for various initial populations, i.e., look for the horizontal asymptotes of your solutions.)
4. This equation can be solved using the method of separation of variables. However, it is very difficult to get the solution as an explicit function of t. Using an initial population of $18,000$ elk, solve the initial-value problem and express the solution as an implicit function of t, or solve the general initial-value problem, finding a solution in terms of $r, K, T,$ and P_0.

Key Concepts

- When studying population functions, different assumptions—such as exponential growth, logistic growth, or threshold population—lead to different rates of growth.

- The logistic differential equation incorporates the concept of a carrying capacity. This value is a limiting value on the population for any given environment.
- The logistic differential equation can be solved for any positive growth rate, initial population, and carrying capacity.

Key Equations

- **Logistic differential equation and initial-value problem**

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right), \quad P(0) = P_0$$

- **Solution to the logistic differential equation/initial-value problem**

$$P(t) = \frac{P_0 K e^{rt}}{(K - P_0) + P_0 e^{rt}}$$

- **Threshold population model**

$$\frac{dP}{dt} = -rP\left(1 - \frac{P}{K}\right)\left(1 - \frac{P}{T}\right)$$

Glossary

carrying capacity
the maximum population of an organism that the environment can sustain indefinitely

growth rate
the constant $r > 0$ in the exponential growth function $P(t) = P_0 e^{rt}$

initial population
the population at time $t = 0$

logistic differential equation
a differential equation that incorporates the carrying capacity K and growth rate rr into a population model

phase line
a visual representation of the behavior of solutions to an autonomous differential equation subject to various initial conditions

threshold population
the minimum population that is necessary for a species to survive

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CHAPTER OVERVIEW

10: Parametric Equations And Polar Coordinates

A general Calculus Textmap organized around the textbook

Calculus: Early Transcendentals

by James Stewart

I II III IV V VI VII VIII IX X XI XII XIII XIV XV XVI XVII

This Textmap is currently under construction... please be patient with us.

Template:HideTOC

Topic hierarchy

10.1: Curves Defined by Parametric Equations
10.2: Calculus with Parametric Curves
10.3: Polar Coordinates
10.4: Areas and Lengths in Polar Coordinates
10.5: Conic Sections
10.6: Conic Sections in Polar Coordinates

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10.1: Curves Defined by Parametric Equations

Learning Objectives

- Plot a curve described by parametric equations.
- Convert the parametric equations of a curve into the form $y = f(x)$.
- Recognize the parametric equations of basic curves, such as a line and a circle.
- Recognize the parametric equations of a cycloid.

In this section we examine parametric equations and their graphs. In the two-dimensional coordinate system, parametric equations are useful for describing curves that are not necessarily functions. The parameter is an independent variable that both x and y depend on, and as the parameter increases, the values of x and y trace out a path along a plane curve. For example, if the parameter is t (a common choice), then t might represent time. Then x and y are defined as functions of time, and $(x(t), y(t))$ can describe the position in the plane of a given object as it moves along a curved path.

Parametric Equations and Their Graphs

Consider the orbit of Earth around the Sun. Our year lasts approximately 365.25 days, but for this discussion we will use 365 days. On January 1 of each year, the physical location of Earth with respect to the Sun is nearly the same, except for leap years, when the lag introduced by the extra $\frac{1}{4}$ day of orbiting time is built into the calendar. We call January 1 “day 1” of the year. Then, for example, day 31 is January 31, day 59 is February 28, and so on.

The number of the day in a year can be considered a variable that determines Earth’s position in its orbit. As Earth revolves around the Sun, its physical location changes relative to the Sun. After one full year, we are back where we started, and a new year begins. According to Kepler’s laws of planetary motion, the shape of the orbit is elliptical, with the Sun at one focus of the ellipse. We study this idea in more detail in Conic Sections.

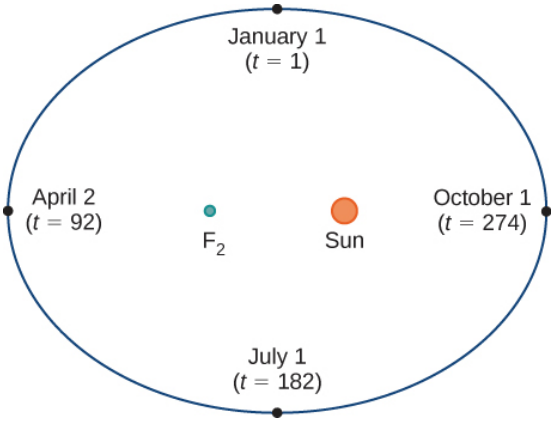


Figure 10.1.1: Earth’s orbit around the Sun in one year.

Figure 10.1.1 depicts **Earth’s orbit** around the Sun during one year. The point labeled F_2 is one of the foci of the ellipse; the other focus is occupied by the Sun. If we superimpose coordinate axes over this graph, then we can assign ordered pairs to each point on the ellipse (Figure 10.1.2). Then each x value on the graph is a value of position as a function of time, and each y value is also a value of position as a function of time. Therefore, each point on the graph corresponds to a value of Earth’s position as a function of time.

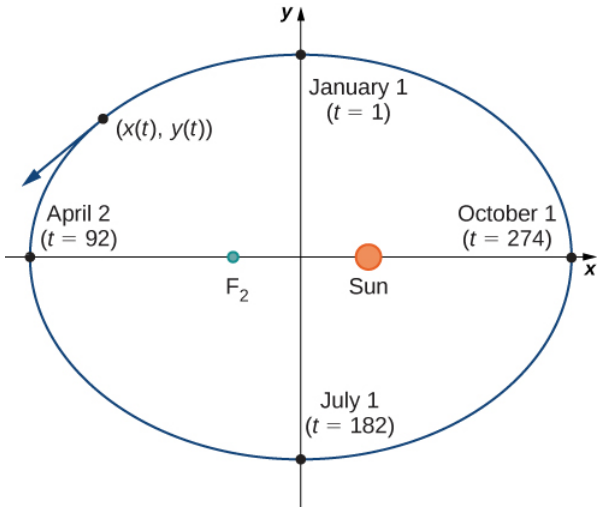


Figure 10.1.2: Coordinate axes superimposed on the orbit of Earth.

We can determine the functions for $x(t)$ and $y(t)$, thereby parameterizing the orbit of Earth around the Sun. The variable t is called an independent parameter and, in this context, represents time relative to the beginning of each year.

A curve in the (x, y) plane can be represented parametrically. The equations that are used to define the curve are called *parametric equations.*

Definition: Parametric Equations

If x and y are continuous functions of t on an interval I, then the equations

$$x = x(t)$$

and

$$y = y(t)$$

are called parametric equations and t is called the parameter. The set of points (x, y) obtained as t varies over the interval I is called the graph of the parametric equations. The graph of parametric equations is called a *parametric curve* or plane curve, and is denoted by C.

Notice in this definition that x and y are used in two ways. The first is as functions of the independent variable t. As t varies over the interval I, the functions $x(t)$ and $y(t)$ generate a set of ordered pairs (x, y). This set of ordered pairs generates the graph of the parametric equations. In this second usage, to designate the ordered pairs, x and y are variables. It is important to distinguish the variables x and y from the functions $x(t)$ and $y(t)$.

Example 10.1.1: Graphing a Parametrically Defined Curve

Sketch the curves described by the following parametric equations:

a. $x(t) = t - 1, \quad y(t) = 2t + 4, \quad \text{for } -3 \le t \le 2$
b. $x(t) = t^2 - 3, \quad y(t) = 2t + 1, \quad \text{for } -2 \le t \le 3$
c. $x(t) = 4\cos t, \quad y(t) = 4\sin t, \quad \text{for } 0 \le t \le 2\pi$

Solution a

To create a graph of this curve, first set up a table of values. Since the independent variable in both $x(t)$ and $y(t)$ is t, let t appear in the first column. Then $x(t)$ and $y(t)$ will appear in the second and third columns of the table.

t	$x(t)$	$y(t)$
−3	−4	−2
−2	−3	0

t	$x(t)$	$y(t)$
−1	−2	2
0	−1	4
1	0	6
2	1	8

The second and third columns in this table provide a set of points to be plotted. The graph of these points appears in Figure 10.1.3 The arrows on the graph indicate the **orientation** of the graph, that is, the direction that a point moves on the graph as t varies from −3 to 2.

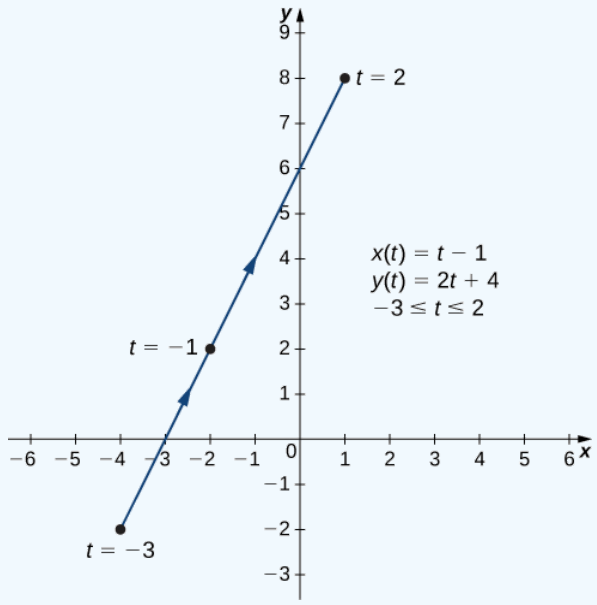


Figure 10.1.3: Graph of the plane curve described by the parametric equations in part a.

Solution b

To create a graph of this curve, again set up a table of values.

t	$x(t)$	$y(t)$
−2	1	−3
−1	−2	−1
0	−3	1
1	−2	3
2	1	5
3	6	7

The second and third columns in this table give a set of points to be plotted (Figure 10.1.4). The first point on the graph (corresponding to $t = -2$) has coordinates $(1, -3)$, and the last point (corresponding to $t = 3$) has coordinates $(6, 7)$. As t progresses from -2 to 3, the point on the curve travels along a parabola. The direction the point moves is again called the orientation and is indicated on the graph.

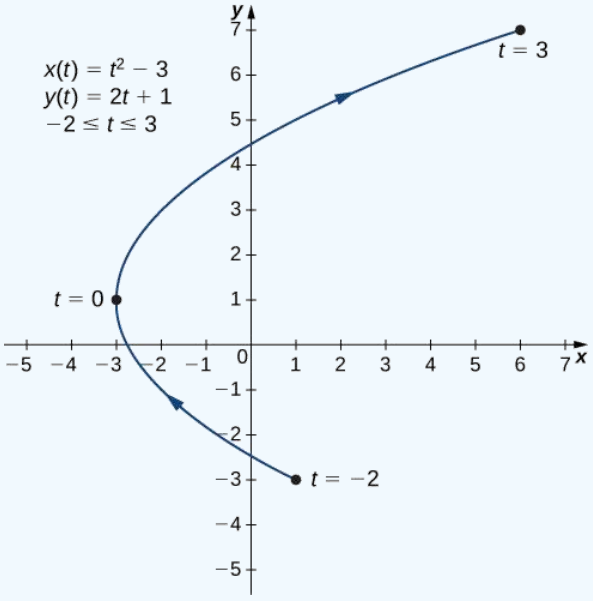


Figure 10.1.4: Graph of the plane curve described by the parametric equations in part b.

Solution c

In this case, use multiples of $\pi/6$ for t and create another table of values:

t	$x(t)$	$y(t)$	t	$x(t)$	$y(t)$
0	4	0	$\frac{7\pi}{6}$	$-2\sqrt{3} \approx -3.5$	-2
$\frac{\pi}{6}$	$2\sqrt{3} \approx 3.5$	2	$\frac{4\pi}{3}$	-2	$-2\sqrt{3} \approx -3.5$
$\frac{\pi}{3}$	2	$2\sqrt{3} \approx 3.5$	$\frac{3\pi}{2}$	0	-4
$\frac{\pi}{2}$	0	4	$\frac{5\pi}{3}$	2	$-2\sqrt{3} \approx -3.5$
$\frac{2\pi}{3}$	-2	$2\sqrt{3} \approx 3.5$	$\frac{11\pi}{6}$	$2\sqrt{3} \approx 3.5$	-2
$\frac{5\pi}{6}$	$-2\sqrt{3} \approx -3.5$	2	2π	4	0
π	-4	0			

The graph of this plane curve appears in the following graph.

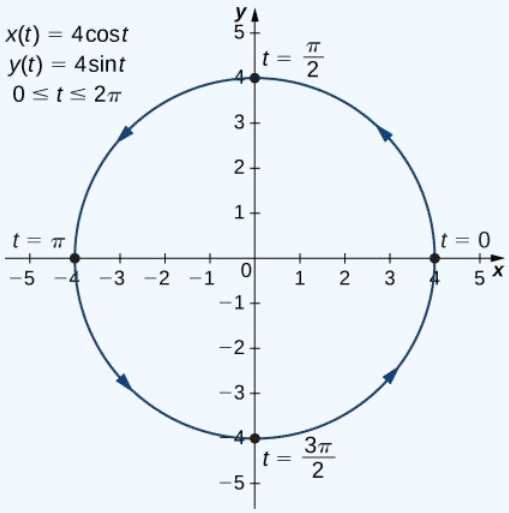


Figure 10.1.5: Graph of the plane curve described by the parametric equations in part c.

This is the graph of a circle with radius 4 centered at the origin, with a counterclockwise orientation. The starting point and ending points of the curve both have coordinates $(4, 0)$.

? Exercise 10.1.1

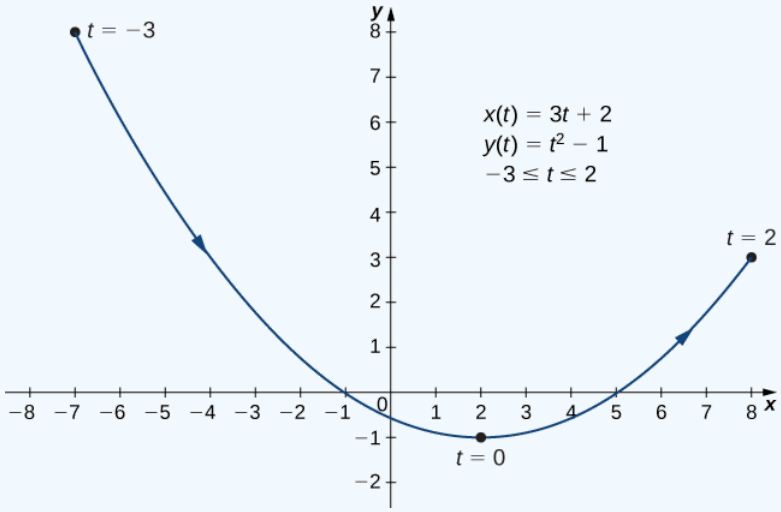
Sketch the curve described by the parametric equations

$$x(t) = 3t + 2, \quad y(t) = t^2 - 1, \quad \text{for } -3 \le t \le 2.$$

Hint

Make a table of values for $x(t)$ and $y(t)$ using t values from -3 to 2.

Answer



Eliminating the Parameter

To better understand the graph of a curve represented parametrically, it is useful to rewrite the two equations as a single equation relating the variables x and y. Then we can apply any previous knowledge of equations of curves in the plane to identify the curve. For example, the equations describing the plane curve in Example $10.1.1b$ are

$$x(t) = t^2 - 3 \tag{10.1.1}$$

$$y(t) = 2t + 1 \tag{10.1.2}$$

over the region $-2 \le t \le 3$.

Solving Equation 10.1.2 for t gives

$$t = \frac{y-1}{2}.$$

This can be substituted into Equation 10.1.1:

$$x = \left(\frac{y-1}{2}\right)^2 - 3 \tag{10.1.3}$$

$$= \frac{y^2 - 2y + 1}{4} - 3 \tag{10.1.4}$$

$$= \frac{y^2 - 2y - 11}{4}. \tag{10.1.5}$$

Equation 10.1.5 describes x as a function of y. These steps give an example of eliminating the parameter. The graph of this function is a parabola opening to the right (Figure 10.1.4). Recall that the plane curve started at $(1, -3)$ and ended at $(6, 7)$. These terminations were due to the restriction on the parameter t.

Example 10.1.2: Eliminating the Parameter

Eliminate the parameter for each of the plane curves described by the following parametric equations and describe the resulting graph.

a. $x(t) = \sqrt{2t+4}, \quad y(t) = 2t+1, \quad \text{for } -2 \le t \le 6$
b. $x(t) = 4\cos t, \quad y(t) = 3\sin t, \quad \text{for } 0 \le t \le 2\pi$

Solution a

To eliminate the parameter, we can solve either of the equations for t. For example, solving the first equation for t gives

$$\begin{aligned} x &= \sqrt{2t+4} \\ x^2 &= 2t+4 \\ x^2 - 4 &= 2t \\ t &= \frac{x^2-4}{2}. \end{aligned}$$

Note that when we square both sides it is important to observe that $x \ge 0$. Substituting $t = \dfrac{x^2-4}{2}$ into $y(t)$ yields

$$\begin{aligned} y(t) &= 2t+1 \\ y &= 2\left(\frac{x^2-4}{2}\right)+1 \\ y &= x^2-4+1 \\ y &= x^2-3. \end{aligned}$$

This is the equation of a parabola opening upward. There is, however, a domain restriction because of the limits on the parameter t. When $t = -2$, $x = \sqrt{2(-2)+4} = 0$, and when $t = 6$, $x = \sqrt{2(6)+4} = 4$. The graph of this plane curve follows.

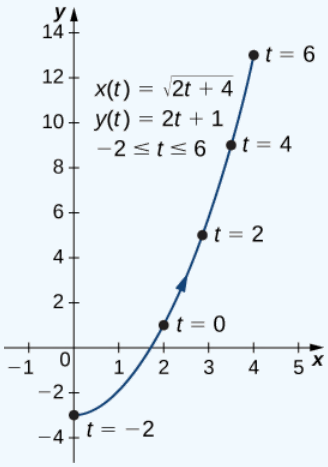


Figure 10.1.6: Graph of the plane curve described by the parametric equations in part a.

Solution b

Sometimes it is necessary to be a bit creative in eliminating the parameter. The parametric equations for this example are

$$x(t) = 4\cos t$$

and

$$y(t) = 3\sin t$$

Solving either equation for t directly is not advisable because sine and cosine are not one-to-one functions. However, dividing the first equation by 4 and the second equation by 3 (and suppressing the t) gives us

$$\cos t = \frac{x}{4}$$

and

$$\sin t = \frac{y}{3}.$$

Now use the Pythagorean identity $\cos^2 t + \sin^2 t = 1$ and replace the expressions for $\sin t$ and $\cos t$ with the equivalent expressions in terms of x and y. This gives

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

$$\frac{x^2}{16} + \frac{y^2}{9} = 1.$$

This is the equation of a horizontal ellipse centered at the origin, with semi-major axis 4 and semi-minor axis 3 as shown in the following graph.

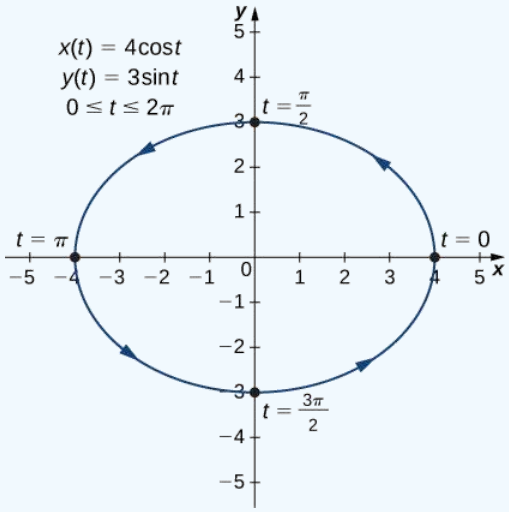


Figure 10.1.7: Graph of the plane curve described by the parametric equations in part b.

As t progresses from 0 to 2π, a point on the curve traverses the ellipse once, in a counterclockwise direction. Recall from the section opener that the orbit of Earth around the Sun is also elliptical. This is a perfect example of using parameterized curves to model a real-world phenomenon.

Exercise 10.1.2

Eliminate the parameter for the plane curve defined by the following parametric equations and describe the resulting graph.

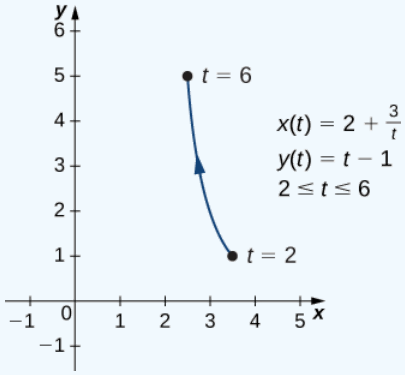
$$x(t) = 2 + \frac{3}{t}, \quad y(t) = t - 1, \quad \text{for } 2 \le t \le 6$$

Hint

Solve one of the equations for t and substitute into the other equation.

Answer

$x = 2 + \frac{3}{y+1}$, or $y = -1 + \frac{3}{x-2}$. This equation describes a portion of a rectangular hyperbola centered at $(2, -1)$.



So far we have seen the method of eliminating the parameter, assuming we know a set of parametric equations that describe a plane curve. What if we would like to start with the equation of a curve and determine a pair of parametric equations for that curve? This is certainly possible, and in fact it is possible to do so in many different ways for a given curve. The process is known as **parameterization of a curve.**

Example 10.1.3: Parameterizing a Curve

Find two different pairs of parametric equations to represent the graph of $y = 2x^2 - 3$.

Solution

First, it is always possible to parameterize a curve by defining $x(t) = t$, then replacing x with t in the equation for $y(t)$. This gives the parameterization

$$x(t) = t, \quad y(t) = 2t^2 - 3.$$

Since there is no restriction on the domain in the original graph, there is no restriction on the values of t.

We have complete freedom in the choice for the second parameterization. For example, we can choose $x(t) = 3t - 2$. The only thing we need to check is that there are no restrictions imposed on x; that is, the range of $x(t)$ is all real numbers. This is the case for $x(t) = 3t - 2$. Now since $y = 2x^2 - 3$, we can substitute $x(t) = 3t - 2$ for x. This gives

$$y(t) = 2(3t-2)^2 - 2 = 2(9t^2 - 12t + 4) - 2 = 18t^2 - 24t + 8 - 2 = 18t^2 - 24t + 6.$$

Therefore, a second parameterization of the curve can be written as

$$x(t) = 3t - 2 \text{ and } y(t) = 18t^2 - 24t + 6.$$

Exercise 10.1.3

Find two different sets of parametric equations to represent the graph of $y = x^2 + 2x$.

Hint

Follow the steps in Example 10.1.3 Remember we have freedom in choosing the parameterization for $x(t)$.

Answer

One possibility is $x(t) = t, \quad y(t) = t^2 + 2t.$ Another possibility is $x(t) = 2t - 3, \quad y(t) = (2t-3)^2 + 2(2t-3) = 4t^2 - 8t + 3.$ There are, in fact, an infinite number of possibilities.

Cycloids and Other Parametric Curves

Imagine going on a bicycle ride through the country. The tires stay in contact with the road and rotate in a predictable pattern. Now suppose a very determined ant is tired after a long day and wants to get home. So he hangs onto the side of the tire and gets a free ride. The path that this ant travels down a straight road is called a *cycloid* (Figure 10.1.8). A cycloid generated by a circle (or bicycle wheel) of radius a is given by the parametric equations

$$x(t) = a(t - \sin t), \quad y(t) = a(1 - \cos t).$$

To see why this is true, consider the path that the center of the wheel takes. The center moves along the x-axis at a constant height equal to the radius of the wheel. If the radius is a, then the coordinates of the center can be given by the equations

$$x(t) = at, \quad y(t) = a$$

for any value of t. Next, consider the ant, which rotates around the center along a circular path. If the bicycle is moving from left to right then the wheels are rotating in a clockwise direction. A possible parameterization of the circular motion of the ant (relative to the center of the wheel) is given by

$$x(t) = -a \sin t$$
$$y(t) = -a \cos t.$$

(The negative sign is needed to reverse the orientation of the curve. If the negative sign were not there, we would have to imagine the wheel rotating counterclockwise.) Adding these equations together gives the equations for the cycloid.

$$x(t) = a(t - \sin t)$$
$$y(t) = a(1 - \cos t)$$

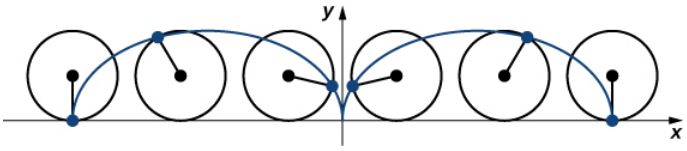


Figure 10.1.8: A wheel traveling along a road without slipping; the point on the edge of the wheel traces out a cycloid.

Now suppose that the bicycle wheel doesn't travel along a straight road but instead moves along the inside of a larger wheel, as in Figure 10.1.9. In this graph, the green circle is traveling around the blue circle in a counterclockwise direction. A point on the edge of the green circle traces out the red graph, which is called a *hypocycloid*.

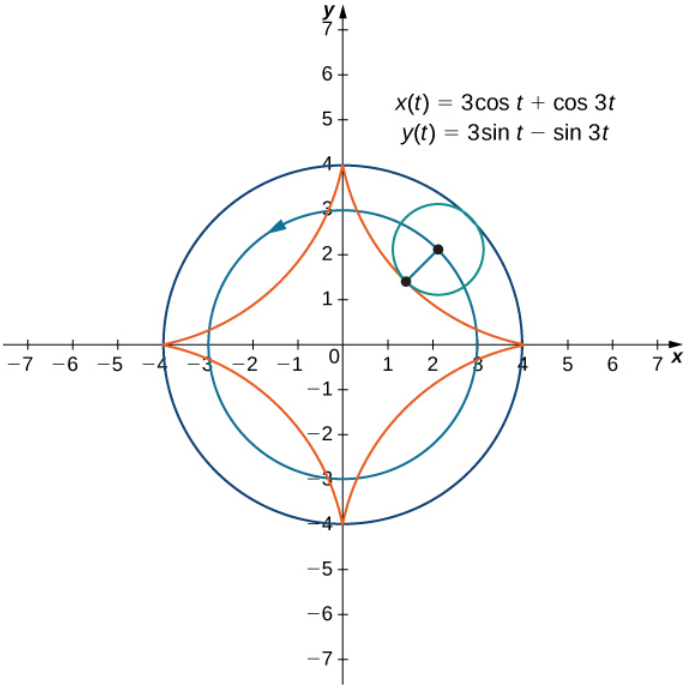


Figure 10.1.9: Graph of the hypocycloid described by the parametric equations shown.

The general parametric equations for a hypocycloid are

$$x(t) = (a - b)\cos t + b\cos\left(\frac{a - b}{b}\right)t$$

$$y(t) = (a-b)\sin t - b\sin(\frac{a-b}{b})t.$$

These equations are a bit more complicated, but the derivation is somewhat similar to the equations for the cycloid. In this case we assume the radius of the larger circle is a and the radius of the smaller circle is b. Then the center of the wheel travels along a circle of radius $a-b$. This fact explains the first term in each equation above. The period of the second trigonometric function in both $x(t)$ and $y(t)$ is equal to $\dfrac{2\pi b}{a-b}$.

The ratio $\dfrac{a}{b}$ is related to the number of **cusps** on the graph (cusps are the corners or pointed ends of the graph), as illustrated in Figure 10.1.10 This ratio can lead to some very interesting graphs, depending on whether or not the ratio is rational. Figure 10.1.9 corresponds to $a = 4$ and $b = 1$. The result is a hypocycloid with four cusps. Figure 10.1.10 shows some other possibilities. The last two hypocycloids have irrational values for $\dfrac{a}{b}$. In these cases the hypocycloids have an infinite number of cusps, so they never return to their starting point. These are examples of what are known as **space-filling curves.**

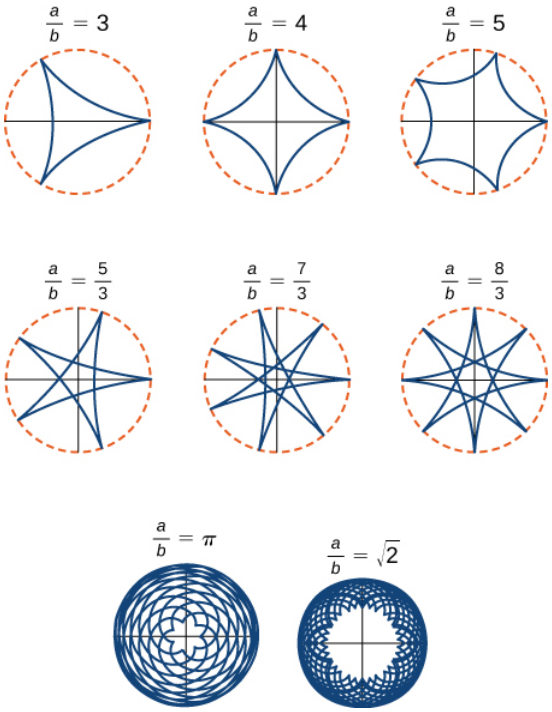


Figure 10.1.10: Graph of various hypocycloids corresponding to different values of a/b.

The Witch of Agnesi

Many plane curves in mathematics are named after the people who first investigated them, like the folium of Descartes or the spiral of Archimedes. However, perhaps the strangest name for a curve is the **witch of Agnesi.** Why a witch?

Maria Gaetana Agnesi (1718–1799) was one of the few recognized women mathematicians of eighteenth-century Italy. She wrote a popular book on analytic geometry, published in 1748, which included an interesting curve that had been studied by Fermat in 1630. The mathematician Guido Grandi showed in 1703 how to construct this curve, which he later called the "versoria," a Latin term for a rope used in sailing. Agnesi used the Italian term for this rope, "versiera," but in Latin, this same word means a "female goblin." When Agnesi's book was translated into English in 1801, the translator used the term "witch" for the curve, instead of rope. The name "witch of Agnesi" has stuck ever since.

The witch of Agnesi is a curve defined as follows: Start with a circle of radius a so that the points $(0,0)$ and $(0,2a)$ are points on the circle (Figure 10.1.11). Let O denote the origin. Choose any other point A on the circle, and draw the secant line OA. Let B denote the point at which the line OA intersects the horizontal line through $(0,2a)$. The vertical line through B intersects the horizontal line through A at the point P. As the point A varies, the path that the point P travels is the witch of Agnesi curve for the given circle.

Witch of Agnesi curves have applications in physics, including modeling water waves and distributions of spectral lines. In probability theory, the curve describes the probability density function of the *Cauchy distribution*. In this project you will parameterize these curves.

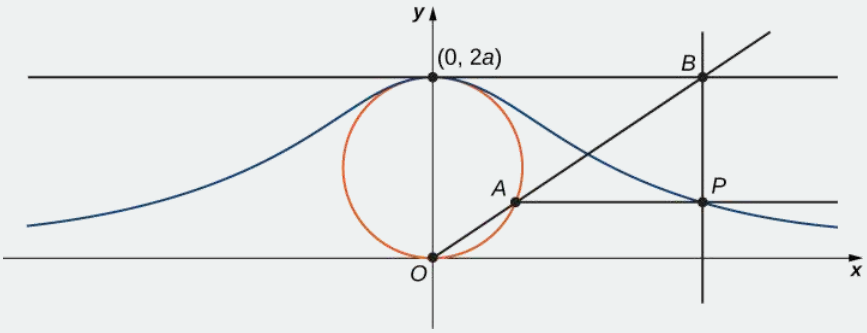


Figure 10.1.11: As the point A moves around the circle, the point P traces out the witch of Agnesi curve for the given circle.

1. On the figure, label the following points, lengths, and angle:
   a. C is the point on the x-axis with the same x-coordinate as A.
   b. x is the x-coordinate of P, and y is the y-coordinate of P.
   c. E is the point $(0, a)$.
   d. F is the point on the line segment OA such that the line segment EF is perpendicular to the line segment OA.
   e. b is the distance from O to F.
   f. c is the distance from F to A.
   g. d is the distance from O to B.
   h. θ is the measure of angle $\angle COA$.

The goal of this project is to parameterize the witch using θ as a parameter. To do this, write equations for x and y in terms of only θ.

2. Show that $d = \dfrac{2a}{\sin\theta}$.
3. Note that $x = d\cos\theta$. Show that $x = 2a\cot\theta$. When you do this, you will have parameterized the x-coordinate of the curve with respect to θ. If you can get a similar equation for y, you will have parameterized the curve.
4. In terms of θ, what is the angle $\angle EOA$?
5. Show that $b + c = 2a\cos\left(\frac{\pi}{2} - \theta\right)$.
6. Show that $y = 2a\cos\left(\frac{\pi}{2} - \theta\right)\sin\theta$.
7. Show that $y = 2a\sin^2\theta$. You have now parameterized the y-coordinate of the curve with respect to θ.
8. Conclude that a parameterization of the given witch curve is

$$x = 2a\cot\theta, \quad y = 2a\sin^2\theta, \quad \text{for } -\infty < \theta < \infty.$$

9. Use your parameterization to show that the given witch curve is the graph of the function $f(x) = \dfrac{8a^3}{x^2 + 4a^2}$.

Travels with My Ant: The Curtate and Prolate Cycloids

Earlier in this section, we looked at the parametric equations for a cycloid, which is the path a point on the edge of a wheel traces as the wheel rolls along a straight path. In this project we look at two different variations of the cycloid, called the curtate and prolate cycloids.

First, let's revisit the derivation of the parametric equations for a cycloid. Recall that we considered a tenacious ant trying to get home by hanging onto the edge of a bicycle tire. We have assumed the ant climbed onto the tire at the very edge, where the tire touches the ground. As the wheel rolls, the ant moves with the edge of the tire (Figure 10.1.12).

As we have discussed, we have a lot of flexibility when parameterizing a curve. In this case we let our parameter t represent the angle the tire has rotated through. Looking at Figure 10.1.12, we see that after the tire has rotated through an angle of t, the position of the center of the wheel, $C = (x_C, y_C)$, is given by

$$x_C = at \text{ and } y_C = a.$$

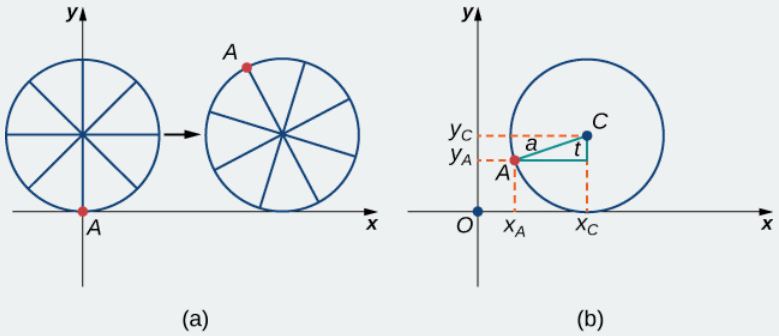
Furthermore, letting $A = (x_A, y_A)$ denote the position of the ant, we note that

$$x_C - x_A = a\sin t \text{ and } y_C - y_A = a\cos t$$

Then

$$x_A = x_C - a\sin t = at - a\sin t = a(t - \sin t)$$

$$y_A = y_C - a\cos t = a - a\cos t = a(1 - \cos t).$$

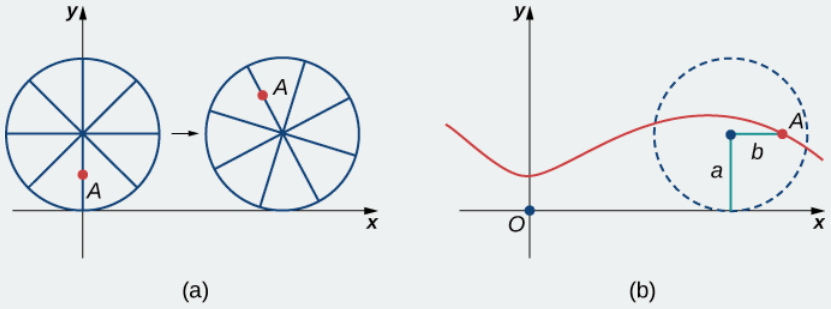


(a) (b)

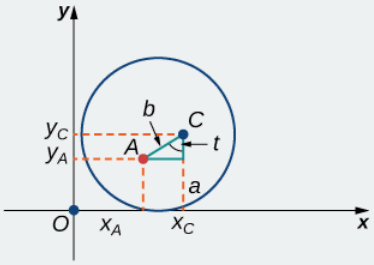
Figure 10.1.12: (a) The ant clings to the edge of the bicycle tire as the tire rolls along the ground. (b) Using geometry to determine the position of the ant after the tire has rotated through an angle of t.

Note that these are the same parametric representations we had before, but we have now assigned a physical meaning to the parametric variable t.

After a while the ant is getting dizzy from going round and round on the edge of the tire. So he climbs up one of the spokes toward the center of the wheel. By climbing toward the center of the wheel, the ant has changed his path of motion. The new path has less up-and-down motion and is called a **curtate cycloid** (Figure 10.1.13). As shown in the figure, we let b denote the distance along the spoke from the center of the wheel to the ant. As before, we let t represent the angle the tire has rotated through. Additionally, we let $C = (x_C, y_C)$ represent the position of the center of the wheel and $A = (x_A, y_A)$ represent the position of the ant.



(a) (b)



(c)

Figure 10.1.13: (a) The ant climbs up one of the spokes toward the center of the wheel. (b) The ant's path of motion after he climbs closer to the center of the wheel. This is called a curtate cycloid. (c) The new setup, now that the ant has moved closer to the center of the wheel.

1. What is the position of the center of the wheel after the tire has rotated through an angle of t?

2. Use geometry to find expressions for $x_C - x_A$ and for $y_C - y_A$.
3. On the basis of your answers to parts 1 and 2, what are the parametric equations representing the curtate cycloid?

Once the ant's head clears, he realizes that the bicyclist has made a turn, and is now traveling away from his home. So he drops off the bicycle tire and looks around. Fortunately, there is a set of train tracks nearby, headed back in the right direction. So the ant heads over to the train tracks to wait. After a while, a train goes by, heading in the right direction, and he manages to jump up and just catch the edge of the train wheel (without getting squished!).

The ant is still worried about getting dizzy, but the train wheel is slippery and has no spokes to climb, so he decides to just hang on to the edge of the wheel and hope for the best. Now, train wheels have a flange to keep the wheel running on the tracks. So, in this case, since the ant is hanging on to the very edge of the flange, the distance from the center of the wheel to the ant is actually greater than the radius of the wheel (Figure 10.1.14).

The setup here is essentially the same as when the ant climbed up the spoke on the bicycle wheel. We let b denote the distance from the center of the wheel to the ant, and we let t represent the angle the tire has rotated through. Additionally, we let $C = (x_C, y_C)$ represent the position of the center of the wheel and $A = (x_A, y_A)$ represent the position of the ant (Figure 10.1.14).

When the distance from the center of the wheel to the ant is greater than the radius of the wheel, his path of motion is called a prolate cycloid. A graph of a **prolate cycloid** is shown in the figure.

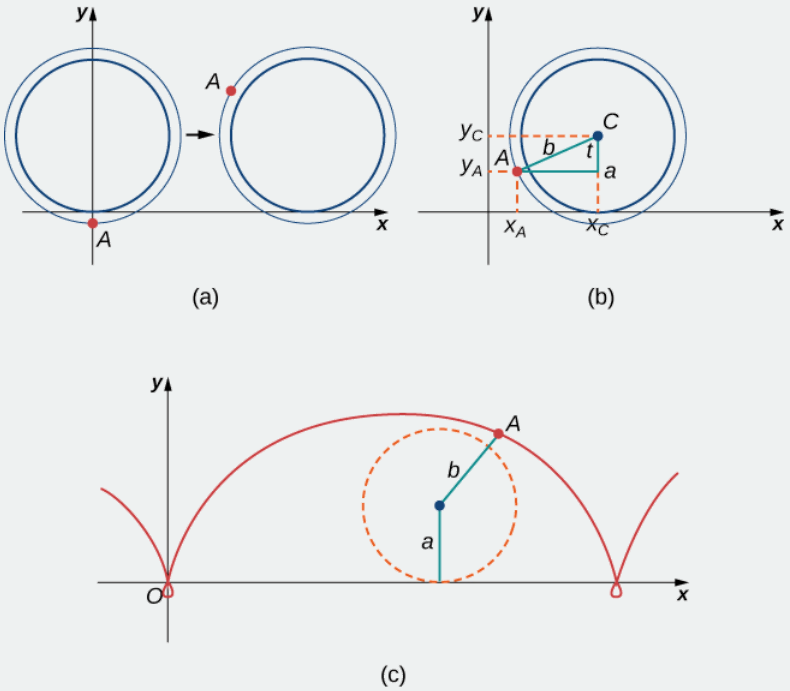


Figure 10.1.14: (a) The ant is hanging onto the flange of the train wheel. (b) The new setup, now that the ant has jumped onto the train wheel. (c) The ant travels along a prolate cycloid.

4. Using the same approach you used in parts 1– 3, find the parametric equations for the path of motion of the ant.
5. What do you notice about your answer to part 3 and your answer to part 4?

Notice that the ant is actually traveling backward at times (the "loops" in the graph), even though the train continues to move forward. He is probably going to be really dizzy by the time he gets home!

Key Concepts

- Parametric equations provide a convenient way to describe a curve. A parameter can represent time or some other meaningful quantity.
- It is often possible to eliminate the parameter in a parameterized curve to obtain a function or relation describing that curve.
- There is always more than one way to parameterize a curve.

- Parametric equations can describe complicated curves that are difficult or perhaps impossible to describe using rectangular coordinates.

Glossary

cycloid
the curve traced by a point on the rim of a circular wheel as the wheel rolls along a straight line without slippage

cusp
a pointed end or part where two curves meet

orientation
the direction that a point moves on a graph as the parameter increases

parameter
an independent variable that both x and y depend on in a parametric curve; usually represented by the variable t

parametric curve
the graph of the parametric equations $x(t)$ and $y(t)$ over an interval $a \leq t \leq b$ combined with the equations

parametric equations
the equations $x = x(t)$ and $y = y(t)$ that define a parametric curve

parameterization of a curve
rewriting the equation of a curve defined by a function $y = f(x)$ as parametric equations

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10.2: Calculus with Parametric Curves

Learning Objectives

- Determine derivatives and equations of tangents for parametric curves.
- Find the area under a parametric curve.
- Use the equation for arc length of a parametric curve.
- Apply the formula for surface area to a volume generated by a parametric curve.

Now that we have introduced the concept of a parameterized curve, our next step is to learn how to work with this concept in the context of calculus. For example, if we know a parameterization of a given curve, is it possible to calculate the slope of a tangent line to the curve? How about the arc length of the curve? Or the area under the curve?

Another scenario: Suppose we would like to represent the location of a baseball after the ball leaves a pitcher's hand. If the position of the baseball is represented by the plane curve $(x(t), y(t))$ then we should be able to use calculus to find the speed of the ball at any given time. Furthermore, we should be able to calculate just how far that ball has traveled as a function of time.

Derivatives of Parametric Equations

We start by asking how to calculate the slope of a line tangent to a parametric curve at a point. Consider the plane curve defined by the parametric equations

$$x(t) = 2t + 3 \tag{10.2.1}$$
$$y(t) = 3t - 4 \tag{10.2.2}$$

within $-2 \leq t \leq 3$.

The graph of this curve appears in Figure 10.2.1. It is a line segment starting at $(-1, -10)$ and ending at $(9, 5)$.

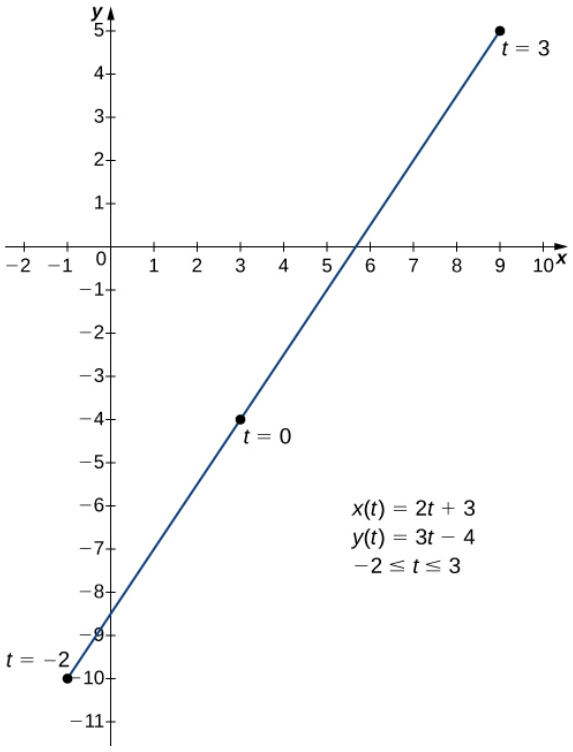


Figure 10.2.1: Graph of the line segment described by the given parametric equations.

We can eliminate the parameter by first solving Equation 10.2.1 for t:

$$x(t) = 2t + 3$$
$$x - 3 = 2t$$

$$t = \frac{x-3}{2}.$$

Substituting this into $y(t)$ (Equation 10.2.2), we obtain

$$y(t) = 3t - 4$$
$$y = 3\left(\frac{x-3}{2}\right) - 4$$
$$y = \frac{3x}{2} - \frac{9}{2} - 4$$
$$y = \frac{3x}{2} - \frac{17}{2}.$$

The slope of this line is given by $\frac{dy}{dx} = \frac{3}{2}$. Next we calculate $x'(t)$ and $y'(t)$. This gives $x'(t) = 2$ and $y'(t) = 3$. Notice that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3}{2}.$$

This is no coincidence, as outlined in the following theorem.

Derivative of Parametric Equations

Consider the plane curve defined by the parametric equations $x = x(t)$ and $y = y(t)$. Suppose that $x'(t)$ and $y'(t)$ exist, and assume that $x'(t) \neq 0$. Then the derivative $\frac{dy}{dx}$ is given by

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}. \tag{10.2.3}$$

Proof

This theorem can be proven using the Chain Rule. In particular, assume that the parameter t can be eliminated, yielding a differentiable function $y = F(x)$. Then $y(t) = F(x(t))$. Differentiating both sides of this equation using the Chain Rule yields

$$y'(t) = F'\big(x(t)\big)x'(t),$$

so

$$F'\big(x(t)\big) = \frac{y'(t)}{x'(t)}.$$

But $F'\big(x(t)\big) = \frac{dy}{dx}$, which proves the theorem.

□

Equation 10.2.3 can be used to calculate derivatives of plane curves, as well as critical points. Recall that a critical point of a differentiable function $y = f(x)$ is any point $x = x_0$ such that either $f'(x_0) = 0$ or $f'(x_0)$ does not exist. Equation 10.2.3 gives a formula for the slope of a tangent line to a curve defined parametrically regardless of whether the curve can be described by a function $y = f(x)$ or not.

Example 10.2.1: Finding the Derivative of a Parametric Curve

Calculate the derivative $\frac{dy}{dx}$ for each of the following parametrically defined plane curves, and locate any critical points on their respective graphs.

a. $x(t) = t^2 - 3, \quad y(t) = 2t - 1, \quad \text{for } -3 \leq t \leq 4$
b. $x(t) = 2t + 1, \quad y(t) = t^3 - 3t + 4, \quad \text{for } -2 \leq t \leq 2$
c. $x(t) = 5\cos t, \quad y(t) = 5\sin t, \quad \text{for } 0 \leq t \leq 2\pi$

Solution

a. To apply Equation 10.2.3, first calculate $x'(t)$ and $y'(t)$:

$$x'(t) = 2t$$
$$y'(t) = 2.$$

Next substitute these into the equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$
$$\frac{dy}{dx} = \frac{2}{2t}$$
$$\frac{dy}{dx} = \frac{1}{t}.$$

This derivative is undefined when $t = 0$. Calculating $x(0)$ and $y(0)$ gives $x(0) = (0)^2 - 3 = -3$ and $y(0) = 2(0) - 1 = -1$, which corresponds to the point $(-3, -1)$ on the graph. The graph of this curve is a parabola opening to the right, and the point $(-3, -1)$ is its vertex as shown.

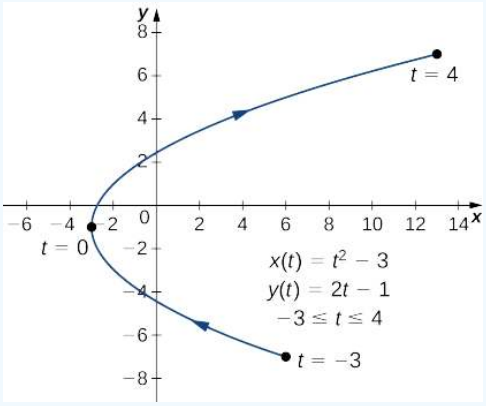


Figure 10.2.2: Graph of the parabola described by parametric equations in part a.

b. To apply Equation 10.2.3, first calculate $x'(t)$ and $y'(t)$:

$$x'(t) = 2$$
$$y'(t) = 3t^2 - 3\,.$$

Next substitute these into the equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$
$$\frac{dy}{dx} = \frac{3t^2 - 3}{2}.$$

This derivative is zero when $t = \pm 1$. When $t = -1$ we have

$$x(-1) = 2(-1) + 1 = -1 \text{ and } y(-1) = (-1)^3 - 3(-1) + 4 = -1 + 3 + 4 = 6\,,$$

which corresponds to the point $(-1, 6)$ on the graph. When $t = 1$ we have

$$x(1) = 2(1) + 1 = 3 \text{ and } y(1) = (1)^3 - 3(1) + 4 = 1 - 3 + 4 = 2,$$

which corresponds to the point $(3, 2)$ on the graph. The point $(3, 2)$ is a relative minimum and the point $(-1, 6)$ is a relative maximum, as seen in the following graph.

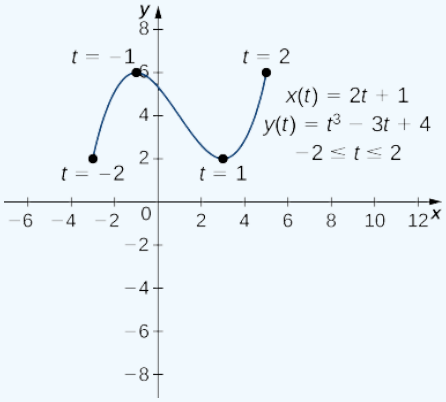


Figure 10.2.3: Graph of the curve described by parametric equations in part b.

c. To apply Equation 10.2.3, first calculate $x'(t)$ and $y'(t)$:

$$x'(t) = -5\sin t$$
$$y'(t) = 5\cos t.$$

Next substitute these into the equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$
$$\frac{dy}{dx} = \frac{5\cos t}{-5\sin t}$$
$$\frac{dy}{dx} = -\cot t.$$

This derivative is zero when $\cos t = 0$ and is undefined when $\sin t = 0$. This gives $t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π as critical points for t. Substituting each of these into $x(t)$ and $y(t)$, we obtain

t	$x(t)$	$y(t)$
0	5	0
$\frac{\pi}{2}$	0	5
π	−5	0
$\frac{3\pi}{2}$	0	−5
2π	5	0

These points correspond to the sides, top, and bottom of the circle that is represented by the parametric equations (Figure 10.2.4). On the left and right edges of the circle, the derivative is undefined, and on the top and bottom, the derivative equals zero.

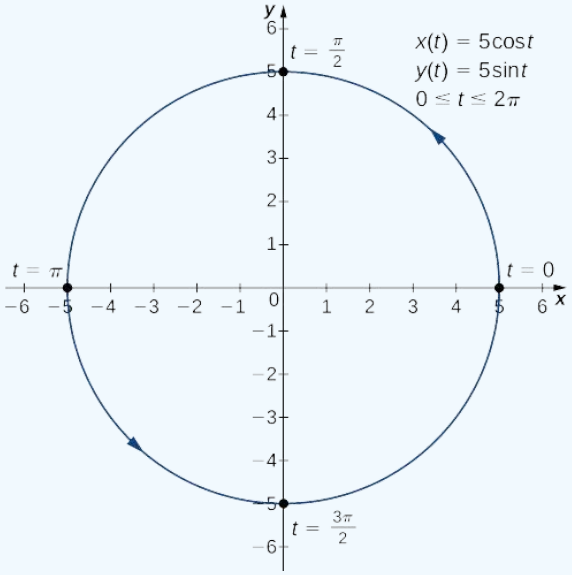


Figure 10.2.4: Graph of the curve described by parametric equations in part c.

Exercise 10.2.1

Calculate the derivative dy/dx for the plane curve defined by the equations

$$x(t) = t^2 - 4t, \quad y(t) = 2t^3 - 6t, \quad \text{for } -2 \le t \le 3$$

and locate any critical points on its graph.

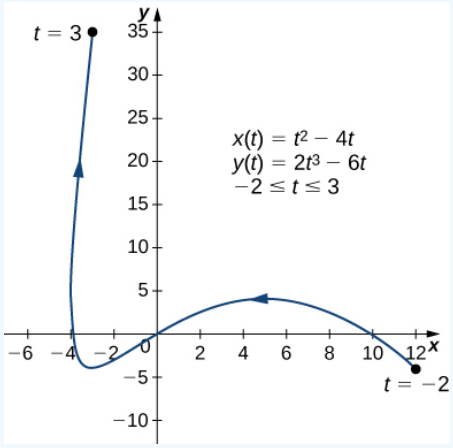
Hint

Calculate $x'(t)$ and $y'(t)$ and use Equation 10.2.3

Answer

$x'(t) = 2t - 4$ and $y'(t) = 6t^2 - 6$, so $\dfrac{dy}{dx} = \dfrac{6t^2 - 6}{2t - 4} = \dfrac{3t^2 - 3}{t - 2}$.

This expression is undefined when $t = 2$ and equal to zero when $t = \pm 1$.



Example 10.2.2: Finding a Tangent Line

Find the equation of the tangent line to the curve defined by the equations

$$x(t) = t^2 - 3, \quad y(t) = 2t - 1, \quad \text{for } -3 \le t \le 4$$

when $t = 2$.

Solution

First find the slope of the tangent line using Equation 10.2.3, which means calculating $x'(t)$ and $y'(t)$:

$$x'(t) = 2t$$
$$y'(t) = 2.$$

Next substitute these into the equation:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$
$$\frac{dy}{dx} = \frac{2}{2t}$$
$$\frac{dy}{dx} = \frac{1}{t}.$$

When $t = 2$, $\frac{dy}{dx} = \frac{1}{2}$, so this is the slope of the tangent line. Calculating $x(2)$ and $y(2)$ gives

$$x(2) = (2)^2 - 3 = 1 \text{ and } y(2) = 2(2) - 1 = 3,$$

which corresponds to the point $(1, 3)$ on the graph (Figure 10.2.5). Now use the point-slope form of the equation of a line to find the equation of the tangent line:

$$y - y_0 = m(x - x_0)$$
$$y - 3 = \frac{1}{2}(x - 1)$$
$$y - 3 = \frac{1}{2}x - \frac{1}{2}$$
$$y = \frac{1}{2}x + \frac{5}{2}.$$

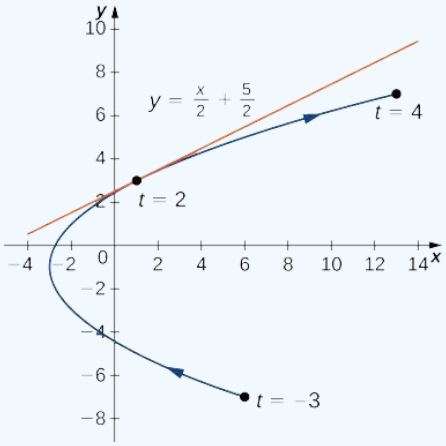


Figure 10.2.5: Tangent line to the parabola described by the given parametric equations when $t = 2$.

? Exercise 10.2.2

Find the equation of the tangent line to the curve defined by the equations

$$x(t) = t^2 - 4t, \quad y(t) = 2t^3 - 6t, \quad \text{for } -2 \le t \le 6 \quad \text{when } t = 5.$$

Hint

Calculate $x'(t)$ and $y'(t)$ and use Equation 10.2.3

Answer

The equation of the tangent line is $y = 24x + 100$.

Second-Order Derivatives

Our next goal is to see how to take the second derivative of a function defined parametrically. The second derivative of a function $y = f(x)$ is defined to be the derivative of the first derivative; that is,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right]. \tag{10.2.4}$$

Since

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt},$$

we can replace the y on both sides of Equation 10.2.4 with $\frac{dy}{dx}$. This gives us

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{(d/dt)(dy/dx)}{dx/dt}. \tag{10.2.5}$$

If we know dy/dx as a function of t, then this formula is straightforward to apply

Example 10.2.3: Finding a Second Derivative

Calculate the second derivative d^2y/dx^2 for the plane curve defined by the parametric equations $x(t) = t^2 - 3, \quad y(t) = 2t - 1, \quad \text{for } -3 \le t \le 4.$

Solution

From Example 10.2.1 we know that $\frac{dy}{dx} = \frac{2}{2t} = \frac{1}{t}$. Using Equation 10.2.5, we obtain

$$\frac{d^2y}{dx^2} = \frac{(d/dt)(dy/dx)}{dx/dt} = \frac{(d/dt)(1/t)}{2t} = \frac{-t^{-2}}{2t} = -\frac{1}{2t^3}.$$

Exercise 10.2.3

Calculate the second derivative d^2y/dx^2 for the plane curve defined by the equations

$$x(t) = t^2 - 4t, \quad y(t) = 2t^3 - 6t, \quad \text{for } -2 \le t \le 3$$

and locate any critical points on its graph.

Hint

Start with the solution from the previous exercise, and use Equation 10.2.5

Answer

$\frac{d^2y}{dx^2} = \frac{3t^2 - 12t + 3}{2(t-2)^3}$. Critical points $(5, 4)$, $(-3, -4)$, and $(-4, 6)$.

Integrals Involving Parametric Equations

Now that we have seen how to calculate the derivative of a plane curve, the next question is this: How do we find the area under a curve defined parametrically? Recall the cycloid defined by these parametric equations

$$\begin{aligned} x(t) &= t - \sin t \\ y(t) &= 1 - \cos t. \end{aligned}$$

Suppose we want to find the area of the shaded region in the following graph.

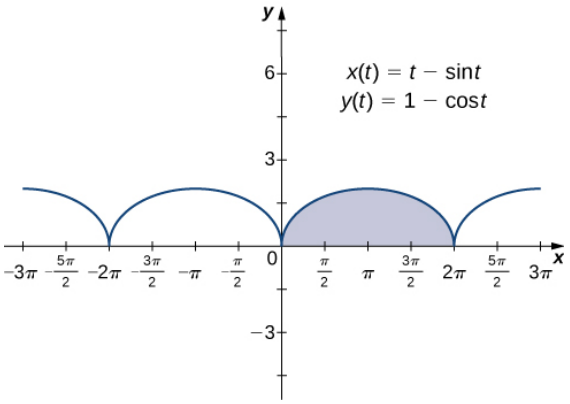


Figure 10.2.6: Graph of a cycloid with the arch over $[0, 2\pi]$ highlighted.

To derive a formula for the area under the curve defined by the functions

$$x = x(t)$$
$$y = y(t)$$

where $a \le t \le b$.

We assume that $x(t)$ is differentiable and start with an equal partition of the interval $a \le t \le b$. Suppose $t_0 = a < t_1 < t_2 < \cdots < t_n = b$ and consider the following graph.

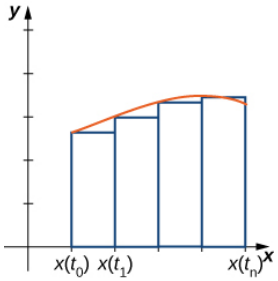


Figure 10.2.7: Approximating the area under a parametrically defined curve.

We use rectangles to approximate the area under the curve. The height of a typical rectangle in this parametrization is $y(x(\bar{t}_i))$ for some value $\bar{t}_i$ in the ith subinterval, and the width can be calculated as $x(t_i) - x(t_{i-1})$. Thus the area of the ith rectangle is given by

$$A_i = y(x(\bar{t}_i))(x(t_i) - x(t_{i-1})).$$

Then a Riemann sum for the area is

$$A_n = \sum_{i=1}^{n} y(x(\bar{t}_i))(x(t_i) - x(t_{i-1})).$$

Multiplying and dividing each area by $t_i - t_{i-1}$ gives

$$\begin{aligned} A_n &= \sum_{i=1}^{n} y(x(\bar{t}_i)) \left(\frac{x(t_i) - x(t_{i-1})}{t_i - t_{i-1}} \right) (t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} y(x(\bar{t}_i)) \left(\frac{x(t_i) - x(t_{i-1})}{\Delta t} \right) \Delta t. \end{aligned}$$

Taking the limit as n approaches infinity gives

$$A = \lim_{n \to \infty} A_n = \int_a^b y(t) x'(t)\, dt.$$

This leads to the following theorem.

Area under a Parametric Curve

Consider the non-self-intersecting plane curve defined by the parametric equations

$$x = x(t), \quad y = y(t), \quad \text{for } a \le t \le b$$

and assume that $x(t)$ is differentiable. The area under this curve is given by

$$A = \int_a^b y(t)x'(t)\,dt. \tag{10.2.6}$$

Example 10.2.4: Finding the Area under a Parametric Curve

Find the area under the curve of the cycloid defined by the equations

$$x(t) = t - \sin t, \quad y(t) = 1 - \cos t, \quad \text{for } 0 \le t \le 2\pi.$$

Solution

Using Equation 10.2.6, we have

$$\begin{aligned} A &= \int_a^b y(t)x'(t)\,dt \\ &= \int_0^{2\pi} (1-\cos t)(1-\cos t)\,dt \\ &= \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\,dt \\ &= \int_0^{2\pi} \left(1 - 2\cos t + \frac{1+\cos(2t)}{2}\right) dt \\ &= \int_0^{2\pi} \left(\frac{3}{2} - 2\cos t + \frac{\cos(2t)}{2}\right) dt \\ &= \frac{3t}{2} - 2\sin t + \frac{\sin(2t)}{4}\Big|_0^{2\pi} \\ &= 3\pi \end{aligned}$$

Exercise 10.2.4

Find the area under the curve of the hypocycloid defined by the equations

$$x(t) = 3\cos t + \cos(3t), \quad y(t) = 3\sin t - \sin(3t), \quad \text{for } 0 \le t \le \pi.$$

Hint

Use Equation 10.2.6 along with the identities $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$ and $\sin^2 t = \frac{1-\cos(2t)}{2}$.

Answer

$A = 3\pi$ (Note that the integral formula actually yields a negative answer. This is due to the fact that $x(t)$ is a decreasing function over the interval $[0, \pi]$; that is, the curve is traced from right to left.)

Arc Length of a Parametric Curve

In addition to finding the area under a parametric curve, we sometimes need to find the arc length of a parametric curve. In the case of a line segment, arc length is the same as the distance between the endpoints. If a particle travels from point A to point B along a curve, then the distance that particle travels is the arc length. To develop a formula for arc length, we start with an approximation by line segments as shown in the following graph.

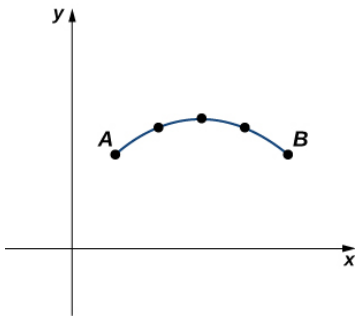


Figure 10.2.7: Approximation of a curve by line segments.

Given a plane curve defined by the functions $x = x(t), \quad y = y(t), \quad$ for $a \le t \le b$, we start by partitioning the interval $[a, b]$ into n equal subintervals: $t_0 = a < t_1 < t_2 < \cdots < t_n = b$. The width of each subinterval is given by $\Delta t = (b-a)/n$. We can calculate the length of each line segment:

$$d_1 = \sqrt{(x(t_1) - x(t_0))^2 + (y(t_1) - y(t_0))^2}$$

$$d_2 = \sqrt{(x(t_2) - x(t_1))^2 + (y(t_2) - y(t_1))^2}$$

etc.

Then add these up. We let s denote the exact arc length and s_n denote the approximation by n line segments:

$$s \approx \sum_{k=1}^{n} s_k = \sum_{k=1}^{n} \sqrt{(x(t_k) - x(t_{k-1}))^2 + (y(t_k) - y(t_{k-1}))^2} \, . \tag{10.2.7}$$

If we assume that $x(t)$ and $y(t)$ are differentiable functions of t, then the Mean Value Theorem applies, so in each subinterval $[t_{k-1}, t_k]$ there exist $\hat{t_k}$ and $\tilde{t_k}$ such that

$$x(t_k) - x(t_{k-1}) = x'(\hat{t_k})(t_k - t_{k-1}) = x'(\hat{t_k})\,\Delta t$$

$$y(t_k) - y(t_{k-1}) = y'(\tilde{t_k})(t_k - t_{k-1}) = y'(\tilde{t_k})\,\Delta t.$$

Therefore Equation 10.2.7 becomes

$$\begin{aligned} s \approx \sum_{k=1}^{n} s_k &= \sum_{k=1}^{n} \sqrt{(x'(\hat{t_k})\Delta t)^2 + (y'(\tilde{t_k})\Delta t)^2} \\ &= \sum_{k=1}^{n} \sqrt{(x'(\hat{t_k}))^2(\Delta t)^2 + (y'(\tilde{t_k}))^2(\Delta t)^2} \\ &= \sum_{k=1}^{n} \sqrt{(x'(\hat{t_k}))^2 + (y'(\tilde{t_k}))^2}\,\Delta t. \end{aligned}$$

This is a Riemann sum that approximates the arc length over a partition of the interval $[a, b]$. If we further assume that the derivatives are continuous and let the number of points in the partition increase without bound, the approximation approaches the exact arc length. This gives

$$\begin{aligned} s &= \lim_{n\to\infty} \sum_{k=1}^{n} s_k \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{(x'(\hat{t_k}))^2 + (y'(\tilde{t_k}))^2}\,\Delta t \\ &= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt. \end{aligned}$$

When taking the limit, the values of $\hat{t_k}$ and $\tilde{t_k}$ are both contained within the same ever-shrinking interval of width Δt, so they must converge to the same value.

We can summarize this method in the following theorem.

Arc Length of a Parametric Curve

Consider the plane curve defined by the parametric equations

$$x = x(t), \quad y = y(t), \quad \text{for } t_1 \le t \le t_2$$

and assume that $x(t)$ and $y(t)$ are differentiable functions of t. Then the arc length of this curve is given by

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \tag{10.2.8}$$

At this point a side derivation leads to a previous formula for arc length. In particular, suppose the parameter can be eliminated, leading to a function $y = F(x)$. Then $y(t) = F(x(t))$ and the Chain Rule gives

$$y'(t) = F'\big(x(t)\big)x'(t).$$

Substituting this into Equation 10.2.8 gives

$$\begin{aligned} s &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(F'(x)\frac{dx}{dt}\right)^2}\, dt \\ &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 \left(1 + (F'(x))^2\right)}\, dt \\ &= \int_{t_1}^{t_2} x'(t)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dt. \end{aligned}$$

Here we have assumed that $x'(t) > 0$, which is a reasonable assumption. The Chain Rule gives $dx = x'(t)\, dt$, and letting $a = x(t_1)$ and $b = x(t_2)$ we obtain the formula

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

which is the formula for arc length obtained in the Introduction to the Applications of Integration.

Example 10.2.5: Finding the Arc Length of a Parametric Curve

Find the arc length of the semicircle defined by the equations

$$x(t) = 3\cos t, \quad y(t) = 3\sin t, \quad \text{for } 0 \le t \le \pi.$$

Solution

The values $t = 0$ to $t = \pi$ trace out the blue curve in Figure 10.2.8. To determine its length, use Equation 10.2.8:

$$\begin{aligned} s &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_0^{\pi} \sqrt{(-3\sin t)^2 + (3\cos t)^2}\, dt \\ &= \int_0^{\pi} \sqrt{9\sin^2 t + 9\cos^2 t}\, dt \\ &= \int_0^{\pi} \sqrt{9(\sin^2 t + \cos^2 t)}\, dt \\ &= \int_0^{\pi} 3\, dt = 3t\Big|_0^{\pi} \\ &= 3\pi \text{ units}. \end{aligned}$$

Note that the formula for the arc length of a semicircle is πr and the radius of this circle is 3. This is a great example of using calculus to derive a known formula of a geometric quantity.

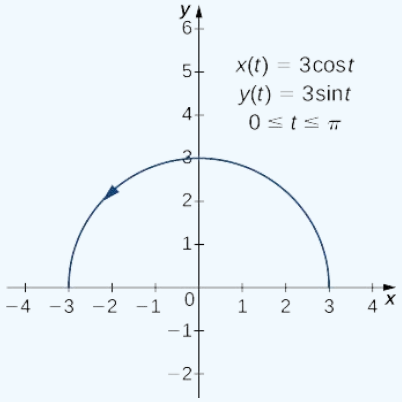


Figure 10.2.8: The arc length of the semicircle is equal to its radius times π.

? Exercise 10.2.5

Find the arc length of the curve defined by the equations

$$x(t) = 3t^2, \quad y(t) = 2t^3, \quad \text{for } 1 \le t \le 3.$$

Hint

Use Equation 10.2.8

Answer

$s = 2(10^{3/2} - 2^{3/2}) \approx 57.589$ units

We now return to the problem posed at the beginning of the section about a baseball leaving a pitcher's hand. Ignoring the effect of air resistance (unless it is a curve ball!), the ball travels a parabolic path. Assuming the pitcher's hand is at the origin and the ball travels left to right in the direction of the positive x-axis, the parametric equations for this curve can be written as

$$x(t) = 140t, \quad y(t) = -16t^2 + 2t$$

where t represents time. We first calculate the distance the ball travels as a function of time. This distance is represented by the arc length. We can modify the arc length formula slightly. First rewrite the functions $x(t)$ and $y(t)$ using v as an independent variable, so as to eliminate any confusion with the parameter t:

$$x(v) = 140v, \quad y(v) = -16v^2 + 2v.$$

Then we write the arc length formula as follows:

$$\begin{aligned} s(t) &= \int_0^t \sqrt{\left(\frac{dx}{dv}\right)^2 + \left(\frac{dy}{dv}\right)^2}\, dv \\ &= \int_0^t \sqrt{140^2 + (-32v+2)^2}\, dv \end{aligned}$$

The variable v acts as a dummy variable that disappears after integration, leaving the arc length as a function of time t. To integrate this expression we can use a formula from Appendix A,

$$\int \sqrt{a^2+u^2}\, du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left|\, u + \sqrt{a^2+u^2}\,\right| + C.$$

We set $a = 140$ and $u = -32v + 2$. This gives $du = -32\, dv$, so $dv = -\dfrac{1}{32}\, du$. Therefore

$$\begin{aligned} \int \sqrt{140^2 + (-32v+2)^2}\, dv &= -\frac{1}{32}\int \sqrt{a^2+u^2}\, du \\ &= -\frac{1}{32}\left[\frac{(-32v+2)}{2}\sqrt{140^2+(-32v+2)^2} + \frac{140^2}{2}\ln\left|\,(-32v+2) + \sqrt{140^2+(-32v+2)^2}\,\right| + C\right] \end{aligned}$$

and

$$\begin{aligned} s(t) &= -\frac{1}{32}\left[\frac{(-32t+2)}{2}\sqrt{140^2+(-32t+2)^2}+\frac{140^2}{2}\ln\left|(-32t+2)+\sqrt{140^2+(-32t+2)^2}\right|\right] \\ &\quad +\frac{1}{32}\left[\sqrt{140^2+2^2}+\frac{140^2}{2}\ln\left|2+\sqrt{140^2+2^2}\right|\right] \\ &= \left(\frac{t}{2}-\frac{1}{32}\right)\sqrt{1024t^2-128t+19604}-\frac{1225}{4}\ln\left|(-32t+2)+\sqrt{1024t^2-128t+19604}\right|+\frac{\sqrt{19604}}{32}+\frac{1225}{4}\ln \\ &\quad (2+\sqrt{19604}) \end{aligned}$$

This function represents the distance traveled by the ball as a function of time. To calculate the speed, take the derivative of this function with respect to t. While this may seem like a daunting task, it is possible to obtain the answer directly from the Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_a^x f(u)\,du = f(x).$$

Therefore

$$\begin{aligned} s'(t) &= \frac{d}{dt}\left[s(t)\right] \\ &= \frac{d}{dt}\left[\int_0^t \sqrt{140^2+(-32v+2)^2}\,dv\right] \\ &= \sqrt{140^2+(-32t+2)^2} \\ &= \sqrt{1024t^2-128t+19604} \\ &= 2\sqrt{256t^2-32t+4901}. \end{aligned}$$

One third of a second after the ball leaves the pitcher's hand, the distance it travels is equal to

$$\begin{aligned} s\left(\frac{1}{3}\right) &= \left(\frac{1/3}{2}-\frac{1}{32}\right)\sqrt{1024\left(\frac{1}{3}\right)^2-128\left(\frac{1}{3}\right)+19604} \\ &\quad -\frac{1225}{4}\ln\left|\left(-32\left(\frac{1}{3}\right)+2\right)+\sqrt{1024\left(\frac{1}{3}\right)^2-128\left(\frac{1}{3}\right)+19604}\right| \\ &\quad +\frac{\sqrt{19604}}{32}+\frac{1225}{4}\ln(2+\sqrt{19604}) \\ &\approx 46.69 \text{ feet}. \end{aligned}$$

This value is just over three quarters of the way to home plate. The speed of the ball is

$$s'\left(\tfrac{1}{3}\right) = 2\sqrt{256\left(\tfrac{1}{3}\right)^2-32\left(\tfrac{1}{3}\right)+4901} \approx 140.27 \text{ ft/s}.$$

This speed translates to approximately 95 mph—a major-league fastball.

Surface Area Generated by a Parametric Curve

Recall the problem of finding the surface area of a volume of revolution. In Curve Length and Surface Area, we derived a formula for finding the surface area of a volume generated by a function $y=f(x)$ from $x=a$ to $x=b$, revolved around the x-axis:

$$S = 2\pi\int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx.$$

We now consider a volume of revolution generated by revolving a parametrically defined curve $x=x(t), \quad y=y(t), \quad$ for $a\le t\le b$ around the x-axis as shown in Figure 10.2.9

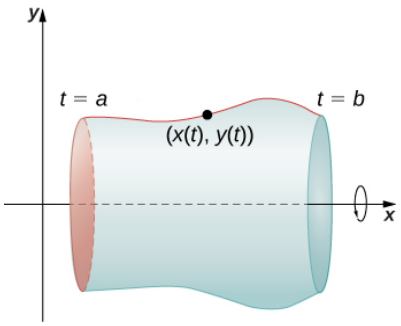


Figure 10.2.9: A surface of revolution generated by a parametrically defined curve.

The analogous formula for a parametrically defined curve is

$$S = 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt \tag{10.2.9}$$

provided that $y(t)$ is not negative on $[a, b]$.

Example 10.2.6: Finding Surface Area

Find the surface area of a sphere of radius r centered at the origin.

Solution

We start with the curve defined by the equations

$$x(t) = r\cos t, \quad y(t) = r\sin t, \quad \text{for } 0 \le t \le \pi.$$

This generates an upper semicircle of radius r centered at the origin as shown in the following graph.

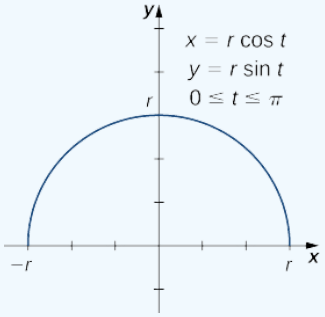


Figure 10.2.10: A semicircle generated by parametric equations.

When this curve is revolved around the x-axis, it generates a sphere of radius r. To calculate the surface area of the sphere, we use Equation 10.2.9:

$$
\begin{aligned}
S &= 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt \\
&= 2\pi \int_0^{\pi} r\sin t \sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt \\
&= 2\pi \int_0^{\pi} r\sin t \sqrt{r^2\sin^2 t + r^2\cos^2 t}\, dt \\
&= 2\pi \int_0^{\pi} r\sin t \sqrt{r^2(\sin^2 t + \cos^2 t)}\, dt \\
&= 2\pi \int_0^{\pi} r^2 \sin t\, dt \\
&= 2\pi r^2 \left(-\cos t\Big|_0^{\pi}\right) \\
&= 2\pi r^2(-\cos\pi + \cos 0) \\
&= 4\pi r^2 \text{ units}^2.
\end{aligned}
$$

This is, in fact, the formula for the surface area of a sphere.

? Exercise 10.2.6

Find the surface area generated when the plane curve defined by the equations

$$x(t) = t^3, \quad y(t) = t^2, \quad \text{for } 0 \le t \le 1$$

is revolved around the x-axis.

Hint

Use Equation 10.2.9 When evaluating the integral, use a u-substitution.

Answer

$$A = \frac{\pi(494\sqrt{13} + 128)}{1215} \text{ units}^2$$

Key Concepts

- The derivative of the parametrically defined curve $x = x(t)$ and $y = y(t)$ can be calculated using the formula $\dfrac{dy}{dx} = \dfrac{y'(t)}{x'(t)}$. Using the derivative, we can find the equation of a tangent line to a parametric curve.
- The area between a parametric curve and the x-axis can be determined by using the formula $A = \displaystyle\int_{t_1}^{t_2} y(t)x'(t)\, dt$.
- The arc length of a parametric curve can be calculated by using the formula

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

- The surface area of a volume of revolution revolved around the x-axis is given by

$$S = 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

- If the curve is revolved around the y-axis, then the formula is

$$S = 2\pi \int_a^b x(t)\sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

Key Equations

- **Derivative of parametric equations**

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}$$

- **Second-order derivative of parametric equations**

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{(d/dt)(dy/dx)}{dx/dt}$$

- **Area under a parametric curve**

$$A = \int_a^b y(t)x'(t)\,dt$$

- **Arc length of a parametric curve**

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

- **Surface area generated by a parametric curve**

$$S = 2\pi \int_a^b y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

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10.3: Polar Coordinates

Learning Objectives

- Locate points in a plane by using polar coordinates.
- Convert points between rectangular and polar coordinates.
- Sketch polar curves from given equations.
- Convert equations between rectangular and polar coordinates.
- Identify symmetry in polar curves and equations.

The rectangular coordinate system (or Cartesian plane) provides a means of mapping points to ordered pairs and ordered pairs to points. This is called a *one-to-one mapping* from points in the plane to ordered pairs. The polar coordinate system provides an alternative method of mapping points to ordered pairs. In this section we see that in some circumstances, polar coordinates can be more useful than rectangular coordinates.

Defining Polar Coordinates

To find the coordinates of a point in the polar coordinate system, consider Figure 10.3.1. The point P has Cartesian coordinates (x, y). The line segment connecting the origin to the point P measures the distance from the origin to P and has length r. The angle between the positive x-axis and the line segment has measure θ. This observation suggests a natural correspondence between the coordinate pair (x, y) and the values r and θ. This correspondence is the basis of the **polar coordinate system**. Note that every point in the Cartesian plane has two values (hence the term ordered pair) associated with it. In the polar coordinate system, each point also has two values associated with it: r and θ.

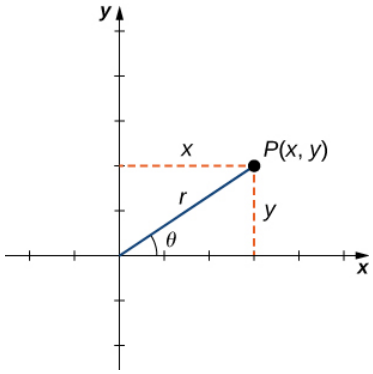


Figure 10.3.1: An arbitrary point in the Cartesian plane.

Using right-triangle trigonometry, the following equations are true for the point P:

$$\cos\theta = \frac{x}{r} \text{ so } x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \text{ so } y = r\sin\theta.$$

Furthermore,

$$r^2 = x^2 + y^2$$

and

$$\tan\theta = \frac{y}{x}.$$

Each point (x, y) in the Cartesian coordinate system can therefore be represented as an ordered pair (r, θ) in the polar coordinate system. The first coordinate is called the **radial coordinate** and the second coordinate is called the **angular coordinate**. Every point in the plane can be represented in this form.

Note that the equation $\tan\theta = y/x$ has an infinite number of solutions for any ordered pair (x, y). However, if we restrict the solutions to values between 0 and 2π then we can assign a unique solution to the quadrant in which the original point (x, y) is

located. Then the corresponding value of r is positive, so $r^2 = x^2 + y^2$.

Converting Points between Coordinate Systems

Given a point P in the plane with Cartesian coordinates (x, y) and polar coordinates (r, θ), the following conversion formulas hold true:

$$x = r\cos\theta \tag{10.3.1}$$

$$y = r\sin\theta \tag{10.3.2}$$

and

$$r^2 = x^2 + y^2 \tag{10.3.3}$$

$$\tan\theta = \frac{y}{x} \tag{10.3.4}$$

These formulas can be used to convert from rectangular to polar or from polar to rectangular coordinates. Notice that Equation 10.3.3 is the *Pythagorean theorem*. (Figure 10.3.1).

Example 10.3.1: Converting between Rectangular and Polar Coordinates

Convert each of the following points into polar coordinates.

a. $(1, 1)$
b. $(-3, 4)$
c. $(0, 3)$
d. $(5\sqrt{3}, -5)$

Convert each of the following points into rectangular coordinates.

e. $(3, \pi/3)$
f. $(2, 3\pi/2)$
g. $(6, -5\pi/6)$

Solution

a. Use $x = 1$ and $y = 1$ in Equation 10.3.3:

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= 1^2 + 1^2 \\ r &= \sqrt{2} \end{aligned}$$

and via Equation 10.3.4

$$\begin{aligned} \tan\theta &= \frac{y}{x} = \frac{1}{1} = 1 \\ \theta &= \frac{\pi}{4}. \end{aligned}$$

Therefore this point can be represented as $(\sqrt{2}, \frac{\pi}{4})$ in polar coordinates.

b. Use $x = -3$ and $y = 4$ in Equation 10.3.3:

$$\begin{aligned} r^2 &= x^2 + y^2 = (-3)^2 + (4)^2 \\ r &= 5 \end{aligned}$$

and via Equation 10.3.4

$$\tan\theta = \frac{y}{x} = -\frac{4}{3}$$

$$\theta = \arctan(-\frac{4}{3}) + \pi \approx 2.21.$$

Therefore this point can be represented as $(5, 2.21)$ in polar coordinates.

c. Use $x = 0$ and $y = 3$ in Equation 10.3.3:

$$r^2 = x^2 + y^2 = (3)^2 + (0)^2 = 9 + 0 \quad r = 3$$

and via Equation 10.3.4

$$\tan\theta = \frac{y}{x} = \frac{3}{0}\ .$$

Direct application of the second equation leads to division by zero. Graphing the point $(0, 3)$ on the rectangular coordinate system reveals that the point is located on the positive y-axis. The angle between the positive x-axis and the positive y-axis is $\frac{\pi}{2}$. Therefore this point can be represented as $(3, \frac{\pi}{2})$ in polar coordinates.

d. Use $x = 5\sqrt{3}$ and $y = -5$ in Equation 10.3.3:

$$r^2 = x^2 + y^2 = (5\sqrt{3})^2 + (-5)^2 = 75 + 25$$

$$r = 10$$

and via Equation 10.3.4

$$\tan\theta = \frac{y}{x} = \frac{-5}{5\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

$$\theta = -\frac{\pi}{6}\ .$$

Therefore this point can be represented as $(10, -\frac{\pi}{6})$ in polar coordinates.

e. Use $r = 3$ and $\theta = \frac{\pi}{3}$ in Equation 10.3.1:

$$x = r\cos\theta = 3\cos(\frac{\pi}{3}) = 3(\frac{1}{2}) = \frac{3}{2}$$

and

$$y = r\sin\theta = 3\sin(\frac{\pi}{3}) = 3(\frac{\sqrt{3}}{2}) = \frac{3\sqrt{3}}{2}\ .$$

Therefore this point can be represented as $(\frac{3}{2}, \frac{3\sqrt{3}}{2})$ in rectangular coordinates.

f. Use $r = 2$ and $\theta = \frac{3\pi}{2}$ in Equation 10.3.1:

$$x = r\cos\theta = 2\cos(\frac{3\pi}{2}) = 2(0) = 0$$

and

$$y = r\sin\theta = 2\sin(\frac{3\pi}{2}) = 2(-1) = -2.$$

Therefore this point can be represented as $(0, -2)$ in rectangular coordinates.

g. Use $r = 6$ and $\theta = -\frac{5\pi}{6}$ in Equation 10.3.1:

$$x = r\cos\theta = 6\cos(-\frac{5\pi}{6}) = 6(-\frac{\sqrt{3}}{2}) = -3\sqrt{3}$$

and

$$y = r\sin\theta = 6\sin(-\frac{5\pi}{6}) = 6(-\frac{1}{2}) = -3\ .$$

Therefore this point can be represented as $(-3\sqrt{3}, -3)$ in rectangular coordinates.

Exercise 10.3.1

Convert $(-8, -8)$ into polar coordinates and $(4, \frac{2\pi}{3})$ into rectangular coordinates.

Hint

Use Equation 10.3.3and Equation 10.3.1. Make sure to check the quadrant when calculating θ.

Answer

$(8\sqrt{2}, \frac{5\pi}{4})$ and $(-2, 2\sqrt{3})$

The polar representation of a point is not unique. For example, the polar coordinates $(2, \frac{\pi}{3})$ and $(2, \frac{7\pi}{3})$ both represent the point $(1, \sqrt{3})$ in the rectangular system. Also, the value of r can be negative. Therefore, the point with polar coordinates $(-2, \frac{4\pi}{3})$ also represents the point $(1, \sqrt{3})$ in the rectangular system, as we can see by using Equation 10.3.1:

$$x = r\cos\theta = -2\cos(\frac{4\pi}{3}) = -2(-\frac{1}{2}) = 1$$

and

$$y = r\sin\theta = -2\sin(\frac{4\pi}{3}) = -2(-\frac{\sqrt{3}}{2}) = \sqrt{3}.$$

Every point in the plane has an infinite number of representations in polar coordinates. However, each point in the plane has only one representation in the rectangular coordinate system.

Note that the polar representation of a point in the plane also has a visual interpretation. In particular, r is the directed distance that the point lies from the origin, and θ measures the angle that the line segment from the origin to the point makes with the positive x-axis. Positive angles are measured in a counterclockwise direction and negative angles are measured in a clockwise direction. The polar coordinate system appears in Figure 10.3.2

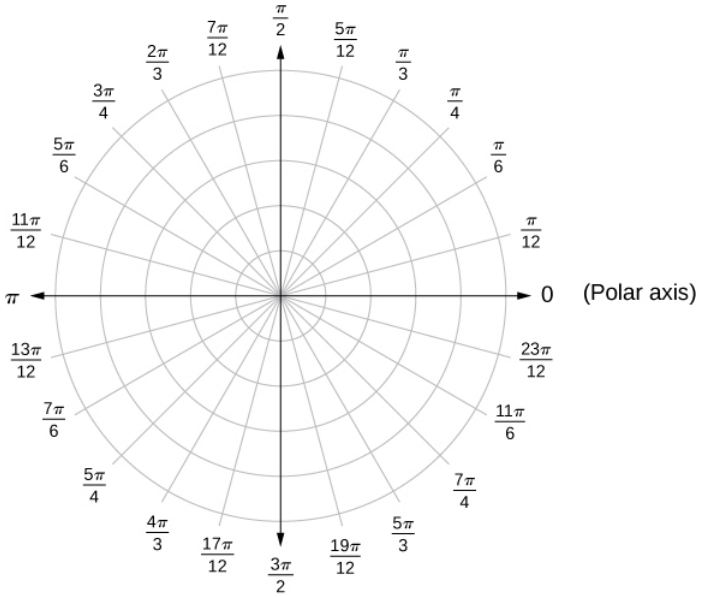


Figure 10.3.2: The polar coordinate system.

The line segment starting from the center of the graph going to the right (called the positive x-axis in the Cartesian system) is the **polar axis.** The center point is the **pole**, or origin, of the coordinate system, and corresponds to $r = 0$. The innermost circle shown in Figure 10.3.2contains all points a distance of 1 unit from the pole, and is represented by the equation $r = 1$. Then $r = 2$ is the

set of points 2 units from the pole, and so on. The line segments emanating from the pole correspond to fixed angles. To plot a point in the polar coordinate system, start with the angle. If the angle is positive, then measure the angle from the polar axis in a counterclockwise direction. If it is negative, then measure it clockwise. If the value of r is positive, move that distance along the terminal ray of the angle. If it is negative, move along the ray that is opposite the terminal ray of the given angle.

Example 10.3.2: Plotting Points in the Polar Plane

Plot each of the following points on the polar plane.

a. $(2, \frac{\pi}{4})$

b. $(-3, \frac{2\pi}{3})$

c. $(4, \frac{5\pi}{4})$

Solution

The three points are plotted in Figure $10.3.3$

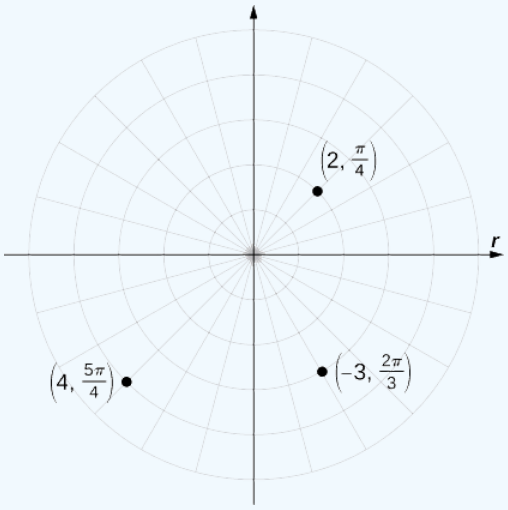


Figure 10.3.3: Three points plotted in the polar coordinate system.

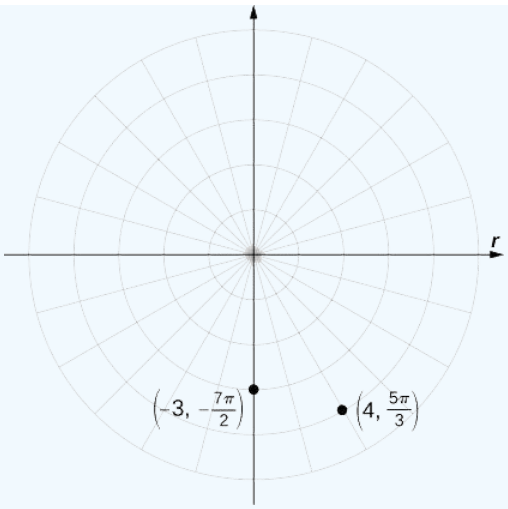
Exercise 10.3.2

Plot $(4, \frac{5\pi}{3})$ and $(-3, -\frac{7\pi}{2})$ on the polar plane.

Hint

Start with θ, then use r.

Answer



Polar Curves

Now that we know how to plot points in the polar coordinate system, we can discuss how to plot curves. In the rectangular coordinate system, we can graph a function $y = f(x)$ and create a curve in the Cartesian plane. In a similar fashion, we can graph a curve that is generated by a function $r = f(\theta)$.

The general idea behind graphing a function in polar coordinates is the same as graphing a function in rectangular coordinates. Start with a list of values for the independent variable (θ in this case) and calculate the corresponding values of the dependent variable r. This process generates a list of ordered pairs, which can be plotted in the polar coordinate system. Finally, connect the points, and take advantage of any patterns that may appear. The function may be periodic, for example, which indicates that only a limited number of values for the independent variable are needed.

Problem-Solving Strategy: Plotting a Curve in Polar Coordinates

1. Create a table with two columns. The first column is for θ, and the second column is for r.
2. Create a list of values for θ.
3. Calculate the corresponding r values for each θ.
4. Plot each ordered pair (r, θ) on the coordinate axes.
5. Connect the points and look for a pattern.

Example 10.3.3: Graphing a Function in Polar Coordinates

Graph the curve defined by the function $r = 4\sin\theta$. Identify the curve and rewrite the equation in rectangular coordinates.

Solution

Because the function is a multiple of a sine function, it is periodic with period 2π, so use values for θ between 0 and 2π. The result of steps 1–3 appear in the following table. Figure $10.3.4$shows the graph based on this table.

Table 10.3.1: Table of values for this polar function.

θ	$r = 4\sin\theta$	θ	$r = 4\sin\theta$
0	0	π	0
$\frac{\pi}{6}$	2	$\frac{7\pi}{6}$	-2
$\frac{\pi}{4}$	$2\sqrt{2} \approx 2.8$	$\frac{5\pi}{4}$	$-2\sqrt{2} \approx -2.8$
$\frac{\pi}{3}$	$2\sqrt{3} \approx 3.4$	$\frac{4\pi}{3}$	$-2\sqrt{3} \approx -3.4$

θ	$r=4\sin\theta$	θ	$r=4\sin\theta$
$\frac{\pi}{2}$	4	$\frac{3\pi}{2}$	-4
$\frac{2\pi}{3}$	$2\sqrt{3}\approx 3.4$	$\frac{5\pi}{3}$	$-2\sqrt{3}\approx -3.4$
$\frac{3\pi}{4}$	$2\sqrt{2}\approx 2.8$	$\frac{7\pi}{4}$	$-2\sqrt{2}\approx -2.8$
$\frac{5\pi}{6}$	2	$\frac{11\pi}{6}$	-2
		2π	0

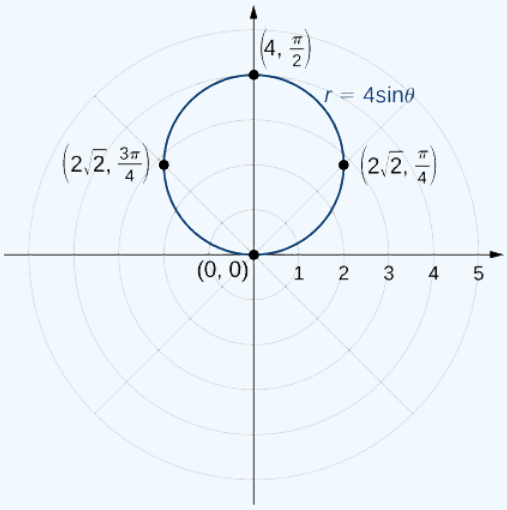


Figure 10.3.4: The graph of the function $r=4\sin\theta$ is a circle.

This is the graph of a circle. The equation $r=4\sin\theta$ can be converted into rectangular coordinates by first multiplying both sides by r. This gives the equation $r^2=4r\sin\theta$. Next use the facts that $r^2=x^2+y^2$ and $y=r\sin\theta$. This gives $x^2+y^2=4y$. To put this equation into standard form, subtract $4y$ from both sides of the equation and complete the square:

$$\begin{aligned} x^2+y^2-4y &= 0 \\ x^2+(y^2-4y) &= 0 \\ x^2+(y^2-4y+4) &= 0+4 \\ x^2+(y-2)^2 &= 4 \end{aligned}$$

This is the equation of a circle with radius 2 and center $(0,2)$ in the rectangular coordinate system.

? Exercise 10.3.3

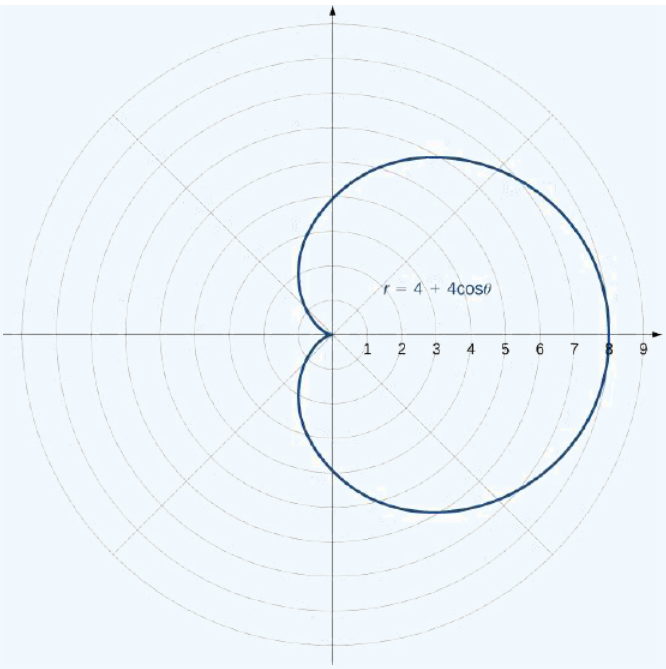
Create a graph of the curve defined by the function $r=4+4\cos\theta$.

Hint

Follow the problem-solving strategy for creating a graph in polar coordinates.

Answer

The name of this shape is a cardioid, which we will study further later in this section.



The graph in Example $10.3.3$ was that of a circle. The equation of the circle can be transformed into rectangular coordinates using the coordinate transformation formulas in Equation 10.3.1. Example $10.3.4$ gives some more examples of functions for transforming from polar to rectangular coordinates.

Example 10.3.4: Transforming Polar Equations to Rectangular Coordinates

Rewrite each of the following equations in rectangular coordinates and identify the graph.

a. $\theta = \dfrac{\pi}{3}$
b. $r = 3$
c. $r = 6\cos\theta - 8\sin\theta$

Solution:

a. Take the tangent of both sides. This gives $\tan\theta = \tan(\pi/3) = \sqrt{3}$. Since $\tan\theta = y/x$ we can replace the left-hand side of this equation by y/x. This gives $y/x = \sqrt{3}$, which can be rewritten as $y = x\sqrt{3}$. This is the equation of a straight line passing through the origin with slope $\sqrt{3}$. In general, any polar equation of the form $\theta = K$ represents a straight line through the pole with slope equal to $\tan K$.

b. First, square both sides of the equation. This gives $r^2 = 9$. Next replace r^2 with $x^2 + y^2$. This gives the equation $x^2 + y^2 = 9$, which is the equation of a circle centered at the origin with radius 3. In general, any polar equation of the form $r = k$ where *k* is a positive constant represents a circle of radius *k* centered at the origin. (Note: when squaring both sides of an equation it is possible to introduce new points unintentionally. This should always be taken into consideration. However, in this case we do not introduce new points. For example, $(-3, \dfrac{\pi}{3})$ is the same point as $(3, \dfrac{4\pi}{3})$.)

c. Multiply both sides of the equation by r. This leads to $r^2 = 6r\cos\theta - 8r\sin\theta$. Next use the formulas

$$r^2 = x^2 + y^2,\ x = r\cos\theta,\ y = r\sin\theta.$$

This gives

$$r^2 = 6(r\cos\theta) - 8(r\sin\theta)$$
$$x^2 + y^2 = 6x - 8y.$$

To put this equation into standard form, first move the variables from the right-hand side of the equation to the left-hand side, then complete the square.

$$x^2+y^2=6x-8y$$
$$x^2-6x+y^2+8y=0$$
$$(x^2-6x)+(y^2+8y)=0$$
$$(x^2-6x+9)+(y^2+8y+16)=9+16$$
$$(x-3)^2+(y+4)^2=25.$$

This is the equation of a circle with center at $(3,-4)$ and radius 5. Notice that the circle passes through the origin since the center is 5 units away.

Exercise 10.3.4

Rewrite the equation $r=\sec\theta\tan\theta$ in rectangular coordinates and identify its graph.

Hint

Convert to sine and cosine, then multiply both sides by cosine.

Answer

$y=x^2$, which is the equation of a parabola opening upward.

We have now seen several examples of drawing graphs of curves defined by **polar equations.** A summary of some common curves is given in the tables below. In each equation, *a* and *b* are arbitrary constants.

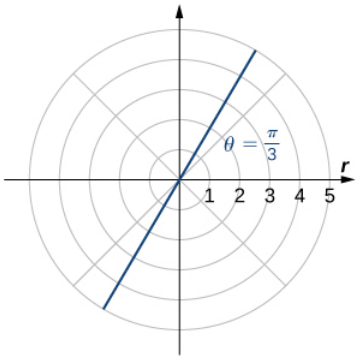
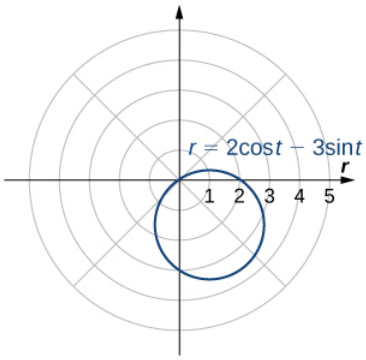
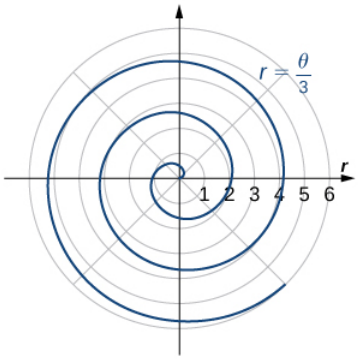
Name	Equation	Example
Line passing through the pole with slope tan K	$\theta = K$	
Circle	$r = a\cos\theta + b\sin\theta$	
Spiral	$r = a + b\theta$	

Figure 10.3.5

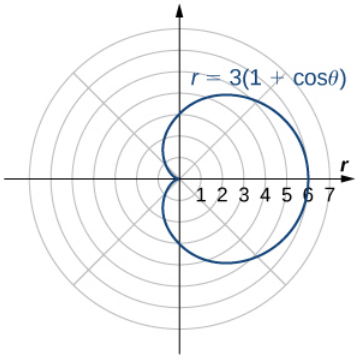
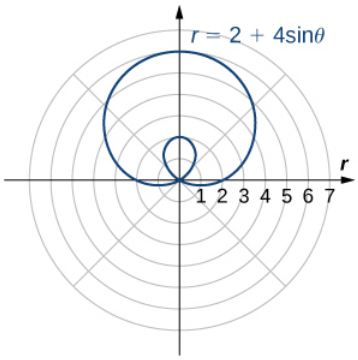
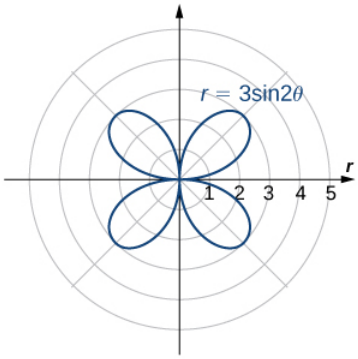
Name	Equation	Example
Cardioid	$r = a(1 + \cos\theta)$ $r = a(1 - \cos\theta)$ $r = a(1 + \sin\theta)$ $r = a(1 - \sin\theta)$	
Limaçon	$r = a\cos\theta + b$ $r = a\sin\theta + b$	
Rose	$r = a\cos(b\theta)$ $r = a\sin(b\theta)$	

Figure 10.3.6

A **cardioid** is a special case of a **limaçon** (pronounced “lee-mah-son”), in which $a = b$ or $a = -b$. The **rose i**s a very interesting curve. Notice that the graph of $r = 3\sin 2\theta$ has four petals. However, the graph of $r = 3\sin 3\theta$ has three petals as shown.

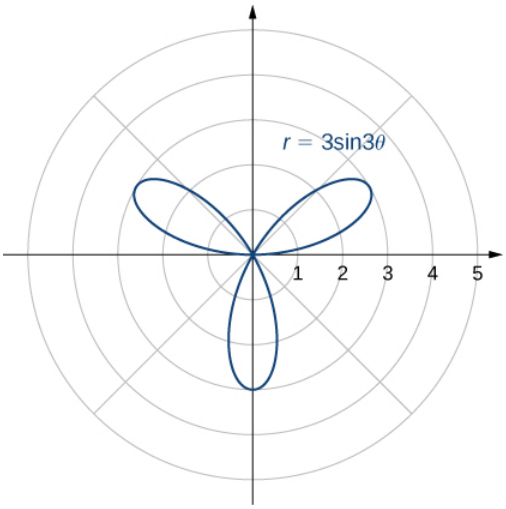


Figure 10.3.7: Graph of $r = 3\sin 3\theta$.

If the coefficient of θ is even, the graph has twice as many petals as the coefficient. If the coefficient of θ is odd, then the number of petals equals the coefficient. You are encouraged to explore why this happens. Even more interesting graphs emerge when the coefficient of θ is not an integer. For example, if it is rational, then the curve is closed; that is, it eventually ends where it started (Figure 10.3.8*a*). However, if the coefficient is irrational, then the curve never closes (Figure 10.3.8*b*). Although it may appear that the curve is closed, a closer examination reveals that the petals just above the positive x axis are slightly thicker. This is because the petal does not quite match up with the starting point.

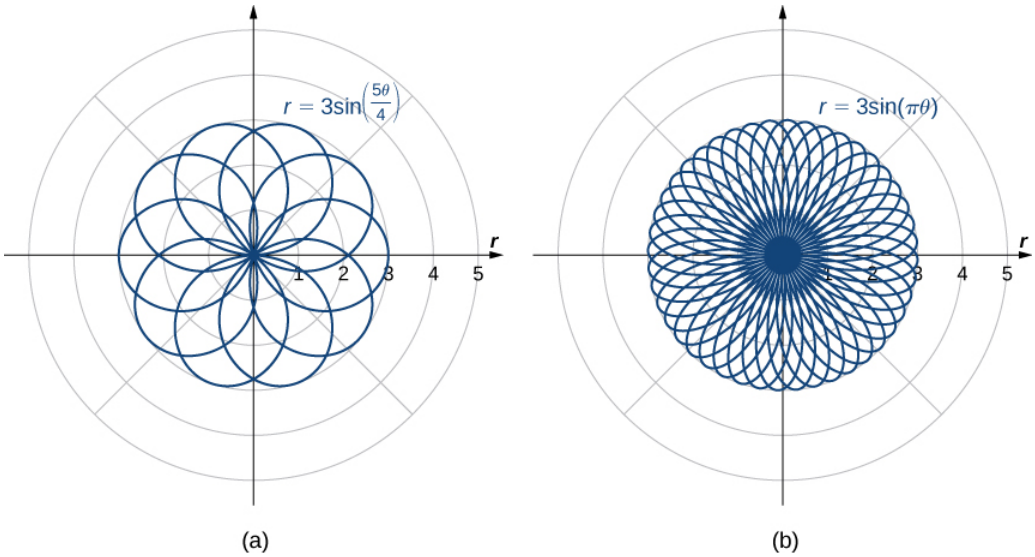


Figure 10.3.8: Polar rose graphs of functions with (a) rational coefficient and (b) irrational coefficient. Note that the rose in part (b) would actually fill the entire circle if plotted in full.

Since the curve defined by the graph of $r = 3\sin(\pi\theta)$ never closes, the curve depicted in Figure 10.3.8*b* is only a partial depiction. In fact, this is an example of a **space-filling curve.** A space-filling curve is one that in fact occupies a two-dimensional subset of the real plane. In this case the curve occupies the circle of radius 3 centered at the origin.

Example 10.3.5: Describing a Spiral

Recall the **chambered nautilus** introduced in the chapter prelude. This creature displays a spiral when half the outer shell is cut away. It is possible to describe a spiral using rectangular coordinates. Figure 10.3.9 shows a spiral in rectangular coordinates. How can we describe this curve mathematically?

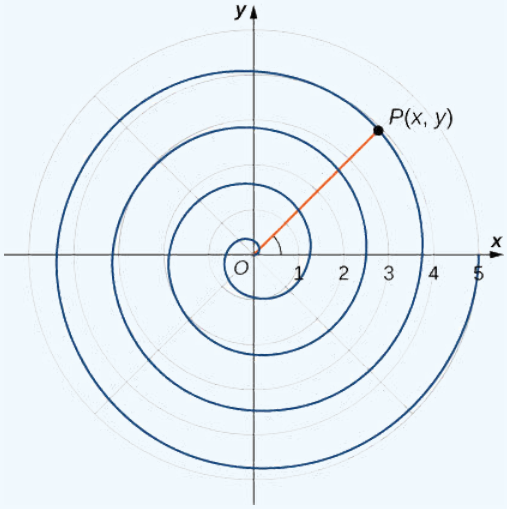


Figure 10.3.9: How can we describe a spiral graph mathematically?

Solution

As the point P travels around the spiral in a counterclockwise direction, its distance d from the origin increases. Assume that the distance d is a constant multiple k of the angle θ that the line segment OP makes with the positive x-axis. Therefore $d(P,O)=k\theta$, where O is the origin. Now use the distance formula and some trigonometry:

$$d(P,O)=k\theta$$

$$\sqrt{(x-0)^2+(y-0)^2}=k\arctan(\frac{y}{x})$$

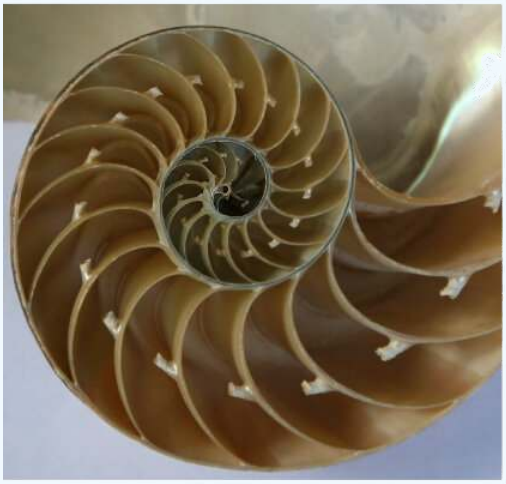
$$\sqrt{x^2+y^2}=k\arctan(\frac{y}{x})$$

$$\arctan(\frac{y}{x})=\frac{\sqrt{x^2+y^2}}{k}$$

$$y=x\tan(\frac{\sqrt{x^2+y^2}}{k}).$$

Although this equation describes the spiral, it is not possible to solve it directly for either x or y. However, if we use polar coordinates, the equation becomes much simpler. In particular, $d(P,O)=r$, and θ is the second coordinate. Therefore the equation for the spiral becomes $r=k\theta$. Note that when $\theta=0$ we also have $r=0$, so the spiral emanates from the origin. We can remove this restriction by adding a constant to the equation. Then the equation for the spiral becomes $r=a+k\theta$ for arbitrary constants a and k. This is referred to as an **Archimedean spiral**, after the Greek mathematician Archimedes.

Another type of spiral is the logarithmic spiral, described by the function $r=a\cdot b^{\theta}$. A graph of the function $r=1.2(1.25^{\theta})$ is given in Figure 10.3.10 This spiral describes the shell shape of the chambered nautilus.



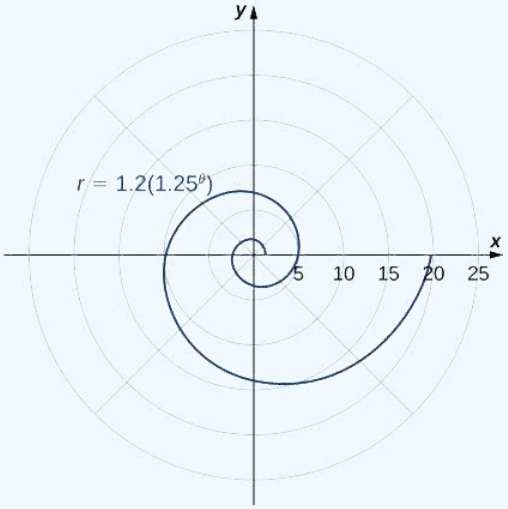


Figure 10.3.10: A logarithmic spiral is similar to the shape of the chambered nautilus shell. (credit: modification of work by Jitze Couperus, Flickr)

Suppose a curve is described in the polar coordinate system via the function $r = f(\theta)$. Since we have conversion formulas from polar to rectangular coordinates given by

$$x = r\cos\theta$$

$$y = r\sin\theta$$

,

it is possible to rewrite these formulas using the function

$$x = f(\theta)\cos\theta$$

$$y = f(\theta)\sin\theta.$$

This step gives a parameterization of the curve in rectangular coordinates using θ as the parameter. For example, the spiral formula $r = a + b\theta$ from Figure becomes

$$x = (a + b\theta)\cos\theta$$

$$y = (a + b\theta)\sin\theta.$$

Letting θ range from $-\infty$ to ∞ generates the entire spiral.

Symmetry in Polar Coordinates

When studying **symmetry** of functions in rectangular coordinates (i.e., in the form $y = f(x)$), we talk about symmetry with respect to the *y*-axis and symmetry with respect to the origin. In particular, if $f(-x) = f(x)$ for all x in the domain of f, then f is an even function and its graph is symmetric with respect to the *y*-axis. If $f(-x) = -f(x)$ for all *x* in the domain of f, then f is an odd function and its graph is symmetric with respect to the origin. By determining which types of symmetry a graph exhibits, we can learn more about the shape and appearance of the graph. Symmetry can also reveal other properties of the function that generates the graph. Symmetry in polar curves works in a similar fashion.

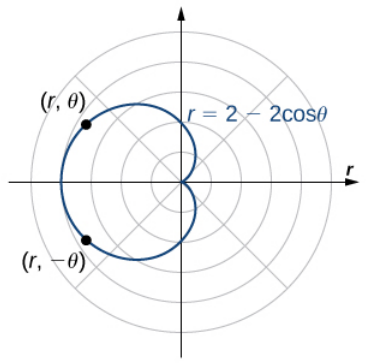
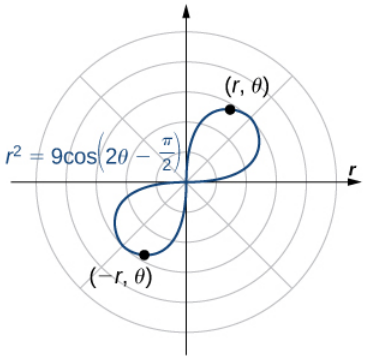
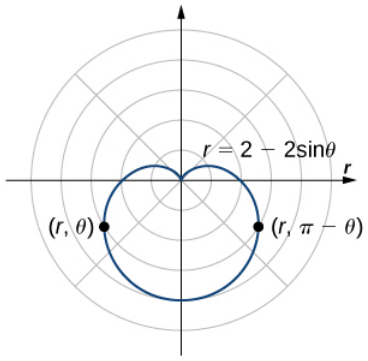
Symmetry in Polar Curves and Equations

Consider a curve generated by the function $r = f(\theta)$ in polar coordinates.

i. The curve is symmetric about the polar axis if for every point (r, θ) on the graph, the point $(r, -\theta)$ is also on the graph. Similarly, the equation $r = f(\theta)$ is unchanged by replacing θ with $-\theta$.
ii. The curve is symmetric about the pole if for every point (r, θ) on the graph, the point $(r, \pi + \theta)$ is also on the graph. Similarly, the equation $r = f(\theta)$ is unchanged when replacing r with $-r$, or θ with $\pi + \theta$.

iii. The curve is symmetric about the vertical line $\theta = \frac{\pi}{2}$ if for every point (r, θ) on the graph, the point $(r, \pi - \theta)$ is also on the graph. Similarly, the equation $r = f(\theta)$ is unchanged when θ is replaced by $\pi - \theta$.

The following table shows examples of each type of symmetry.

Symmetry with respect to the polar axis: For every point (r, θ) on the graph, there is also a point reflected directly across the horizontal (polar) axis.	
Symmetry with respect to the pole: For every point (r, θ) on the graph, there is also a point on the graph that is reflected through the pole as well.	
Symmetry with respect to the vertical line $\theta = \frac{\pi}{2}$: For every point (r, θ) on the graph, there is also a point reflected directly across the vertical axis.	

Example 10.3.6: Using Symmetry to Graph a Polar Equation

Find the symmetry of the rose defined by the equation $r = 3\sin(2\theta)$ and create a graph.

Solution

Suppose the point (r, θ) is on the graph of $r = 3\sin(2\theta)$.

i. To test for symmetry about the polar axis, first try replacing θ with $-\theta$. This gives $r = 3\sin(2(-\theta)) = -3\sin(2\theta)$. Since this changes the original equation, this test is not satisfied. However, returning to the original equation and replacing r with $-r$ and θ with $\pi - \theta$ yields

$$\begin{aligned} -r &= 3\sin(2(\pi-\theta)) \\ -r &= 3\sin(2\pi-2\theta) \\ -r &= 3\sin(-2\theta) \\ -r &= -3\sin 2\theta. \end{aligned}$$

Multiplying both sides of this equation by -1 gives $r=3\sin 2\theta$, which is the original equation. This demonstrates that the graph is symmetric with respect to the polar axis.

ii. To test for symmetry with respect to the pole, first replace r with $-r$, which yields $-r=3\sin(2\theta)$. Multiplying both sides by -1 gives $r=-3\sin(2\theta)$, which does not agree with the original equation. Therefore the equation does not pass the test for this symmetry. However, returning to the original equation and replacing θ with $\theta+\pi$ gives

$$\begin{aligned} r &= 3\sin(2(\theta+\pi)) \\ &= 3\sin(2\theta+2\pi) \\ &= 3(\sin 2\theta\cos 2\pi+\cos 2\theta\sin 2\pi) \\ &= 3\sin 2\theta. \end{aligned}$$

Since this agrees with the original equation, the graph is symmetric about the pole.

iii. To test for symmetry with respect to the vertical line $\theta=\dfrac{\pi}{2}$, first replace both r with $-r$ and θ with $-\theta$.

$$\begin{aligned} -r &= 3\sin(2(-\theta)) \\ -r &= 3\sin(-2\theta) \\ -r &= -3\sin 2\theta. \end{aligned}$$

Multiplying both sides of this equation by -1 gives $r=3\sin 2\theta$, which is the original equation. Therefore the graph is symmetric about the vertical line $\theta=\dfrac{\pi}{2}$.

This graph has symmetry with respect to the polar axis, the origin, and the vertical line going through the pole. To graph the function, tabulate values of θ between 0 and $\pi/2$ and then reflect the resulting graph.

Table 10.3.2: Table of values for this polar function.

θ	$r=3\sin(2\theta)$
0	0
$\dfrac{\pi}{6}$	$\dfrac{3\sqrt{3}}{2}\approx 2.6$
$\dfrac{\pi}{4}$	3
$\dfrac{\pi}{3}$	$\dfrac{3\sqrt{3}}{2}\approx 2.6$
$\dfrac{\pi}{2}$	0

This gives one petal of the rose, as shown in the following graph.

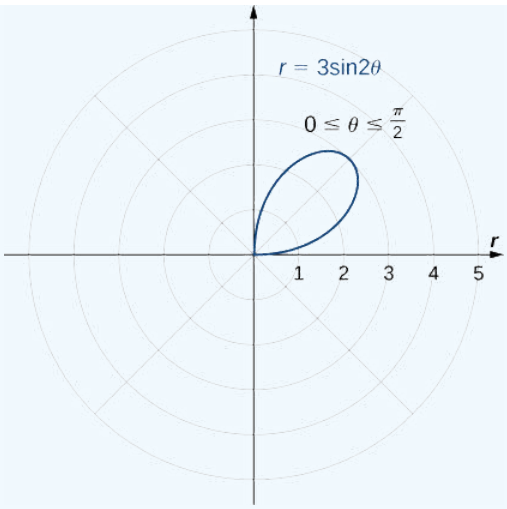


Figure 10.3.11: The graph of the equation between $\theta = 0$ and $\theta = \pi/2$.

Reflecting this image into the other three quadrants gives the entire graph as shown.

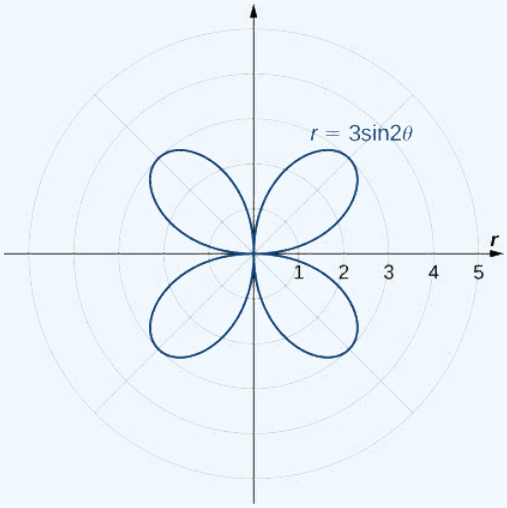


Figure 10.3.12: The entire graph of the equation is called a four-petaled rose.

? Exercise 10.3.5Symmetry

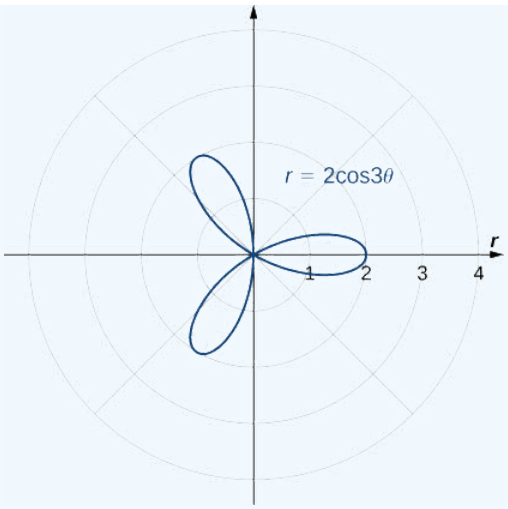
Determine the symmetry of the graph determined by the equation $r = 2\cos(3\theta)$ and create a graph.

Hint

Use Note.

Answer

Symmetric with respect to the polar axis.



Key Concepts

- The polar coordinate system provides an alternative way to locate points in the plane.
- Convert points between rectangular and polar coordinates using the formulas

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

and

$$r = \sqrt{x^2 + y^2} \text{ and } \tan\theta = \frac{y}{x}.$$

- To sketch a polar curve from a given polar function, make a table of values and take advantage of periodic properties.
- Use the conversion formulas to convert equations between rectangular and polar coordinates.
- Identify symmetry in polar curves, which can occur through the pole, the horizontal axis, or the vertical axis.

Glossary

angular coordinate

θ the angle formed by a line segment connecting the origin to a point in the polar coordinate system with the positive radial x-axis, measured counterclockwise

cardioid

a plane curve traced by a point on the perimeter of a circle that is rolling around a fixed circle of the same radius; the equation of a cardioid is $r = a(1 + \sin\theta)$ or $r = a(1 + \cos\theta)$

limaçon

the graph of the equation $r = a + b\sin\theta$ or $r = a + b\cos\theta$. If $a = b$ then the graph is a cardioid

polar axis

the horizontal axis in the polar coordinate system corresponding to $r \geq 0$

polar coordinate system

a system for locating points in the plane. The coordinates are r, the radial coordinate, and θ, the angular coordinate

polar equation

an equation or function relating the radial coordinate to the angular coordinate in the polar coordinate system

pole

the central point of the polar coordinate system, equivalent to the origin of a Cartesian system

radial coordinate

r the coordinate in the polar coordinate system that measures the distance from a point in the plane to the pole

rose

graph of the polar equation $r = a\cos 2\theta$ or $r = a\sin 2\theta$ for a positive constant a

space-filling curve

a curve that completely occupies a two-dimensional subset of the real plane

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10.4: Areas and Lengths in Polar Coordinates

Learning Objectives

- Apply the formula for area of a region in polar coordinates.
- Determine the arc length of a polar curve.

In the rectangular coordinate system, the definite integral provides a way to calculate the area under a curve. In particular, if we have a function $y = f(x)$ defined from $x = a$ to $x = b$ where $f(x) > 0$ on this interval, the area between the curve and the x-axis is given by

$$A = \int_a^b f(x)dx.$$

This fact, along with the formula for evaluating this integral, is summarized in the Fundamental Theorem of Calculus. Similarly, the arc length of this curve is given by

$$L = \int_a^b \sqrt{1 + (f'(x))^2}dx.$$

In this section, we study analogous formulas for area and arc length in the polar coordinate system.

Areas of Regions Bounded by Polar Curves

We have studied the formulas for area under a curve defined in rectangular coordinates and parametrically defined curves. Now we turn our attention to deriving a formula for the area of a region bounded by a polar curve. Recall that the proof of the Fundamental Theorem of Calculus used the concept of a Riemann sum to approximate the area under a curve by using rectangles. For polar curves we use the Riemann sum again, but the rectangles are replaced by sectors of a circle.

Consider a curve defined by the function $r = f(\theta)$, where $\alpha \leq \theta \leq \beta$. Our first step is to partition the interval $[\alpha, \beta]$ into n equal-width subintervals. The width of each subinterval is given by the formula $\Delta\theta = (\beta - \alpha)/n$, and the ith partition point θ_i is given by the formula $\theta_i = \alpha + i\Delta\theta$. Each partition point $\theta = \theta_i$ defines a line with slope $\tan\theta_i$ passing through the pole as shown in the following graph.

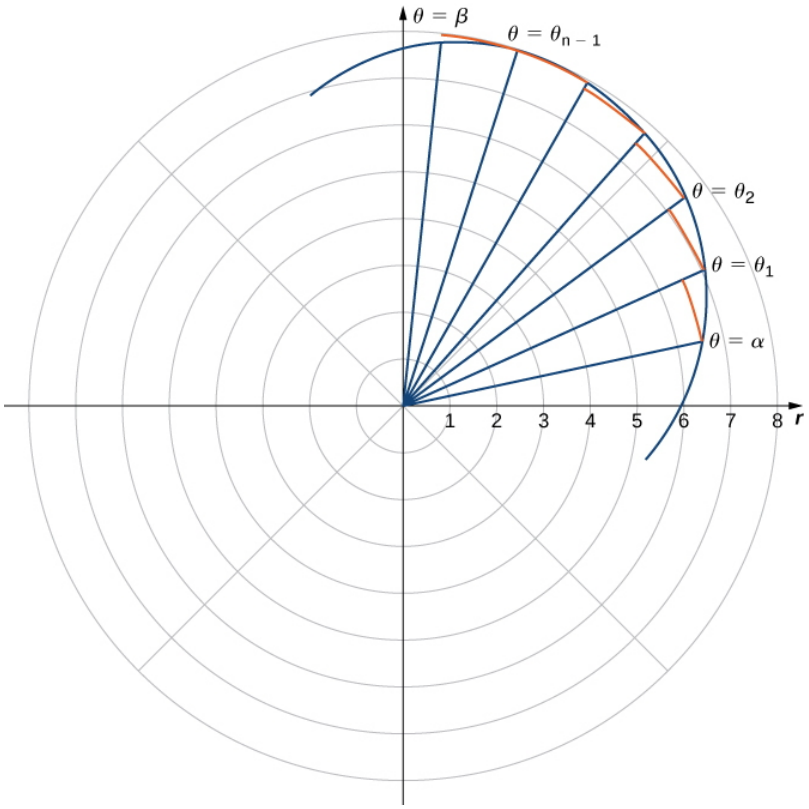


Figure 10.4.1: A partition of a typical curve in polar coordinates.

The line segments are connected by arcs of constant radius. This defines sectors whose areas can be calculated by using a geometric formula. The area of each sector is then used to approximate the area between successive line segments. We then sum the areas of the sectors to approximate the total area. This approach gives a Riemann sum approximation for the total area. The formula for the area of a sector of a circle is illustrated in the following figure.

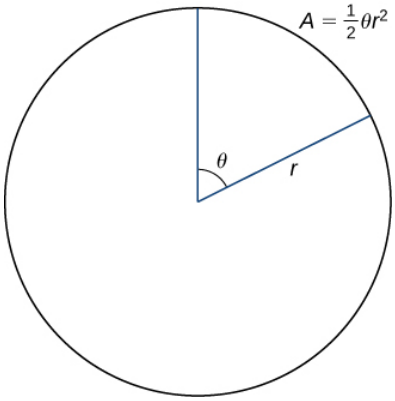


Figure 10.4.2: *The area of a sector of a circle is given by* $A = \frac{1}{2}\theta r^2$.

Recall that the area of a circle is $A = \pi r^2$. When measuring angles in radians, 360 degrees is equal to 2π radians. Therefore a fraction of a circle can be measured by the central angle θ. The fraction of the circle is given by $\frac{\theta}{2\pi}$, so the area of the sector is this fraction multiplied by the total area:

$$A = (\frac{\theta}{2\pi})\pi r^2 = \frac{1}{2}\theta r^2.$$

Since the radius of a typical sector in Figure 10.4.1is given by $r_i = f(\theta_i)$, the area of the *i*th sector is given by

$$A_i = \frac{1}{2}(\Delta\theta)(f(\theta_i))^2.$$

Therefore a Riemann sum that approximates the area is given by

$$A_n = \sum_{i=1}^{n} A_i \approx \sum_{i=1}^{n} \frac{1}{2}(\Delta\theta)(f(\theta_i))^2.$$

We take the limit as $n \to \infty$ to get the exact area:

$$A = \lim_{n\to\infty} A_n = \frac{1}{2}\int_{\alpha}^{\beta} (f(\theta))^2 d\theta.$$

This gives the following theorem.

Area of a Region Bounded by a Polar Curve

Suppose f is continuous and nonnegative on the interval $\alpha \le \theta \le \beta$ with $0 < \beta - \alpha \le 2\pi$. The area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 d\theta \tag{10.4.1}$$

$$= \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta. \tag{10.4.2}$$

Example 10.4.1: Finding an Area of a Polar Region

Find the area of one petal of the rose defined by the equation $r = 3\sin(2\theta)$.

Solution

The graph of $r = 3\sin(2\theta)$ follows.

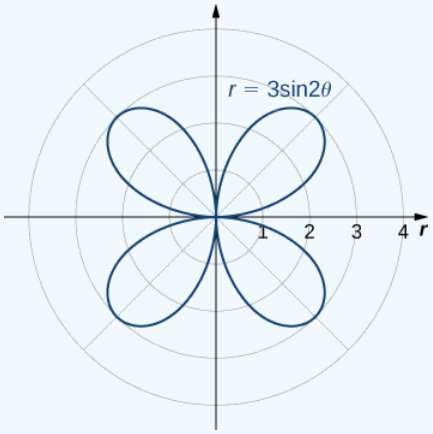


Figure 10.4.3: The graph of $r = 3\sin(2\theta)$.

When $\theta = 0$ we have $r = 3\sin(2(0)) = 0$. The next value for which $r = 0$ is $\theta = \pi/2$. This can be seen by solving the equation $3\sin(2\theta) = 0$ for θ. Therefore the values $\theta = 0$ to $\theta = \pi/2$ trace out the first petal of the rose. To find the area inside this petal, use Equation 10.4.2 with $f(\theta) = 3\sin(2\theta), \alpha = 0,$ and $\beta = \pi/2$:

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 d\theta$$

$$= \frac{1}{2}\int_{0}^{\pi/2} [3\sin(2\theta)]^2 d\theta$$

$$= \frac{1}{2}\int_{0}^{\pi/2} 9\sin^2(2\theta) d\theta.$$

To evaluate this integral, use the formula $\sin^2\alpha = (1 - \cos(2\alpha))/2$ with $\alpha = 2\theta$:

$$
\begin{aligned}
A &= \frac{1}{2}\int_0^{\pi/2} 9\sin^2(2\theta)d\theta \\
&= \frac{9}{2}\int_0^{\pi/2} \frac{(1-\cos(4\theta))}{2}d\theta \\
&= \frac{9}{4}(\int_0^{\pi/2} 1-\cos(4\theta)d\theta) \\
&= \frac{9}{4}(\theta - \frac{\sin(4\theta)}{4}\Big|_0^{\pi/2} \\
&= \frac{9}{4}(\frac{\pi}{2} - \frac{\sin 2\pi}{4}) - \frac{9}{4}(0 - \frac{\sin 4(0)}{4}) \\
&= \frac{9\pi}{8}
\end{aligned}
$$

? Exercise 10.4.1

Find the area inside the cardioid defined by the equation $r = 1 - \cos\theta$.

Hint

Use Equation 10.4.2 Be sure to determine the correct limits of integration before evaluating.

Answer

$A = 3\pi/2$

Example 10.4.1 involved finding the area inside one curve. We can also use Equation 10.4.2 to find the area between two polar curves. However, we often need to find the points of intersection of the curves and determine which function defines the outer curve or the inner curve between these two points.

✓ Example 10.4.2: Finding the Area between Two Polar Curves

Find the area outside the cardioid $r = 2 + 2\sin\theta$ and inside the circle $r = 6\sin\theta$.

Solution

First draw a graph containing both curves as shown.

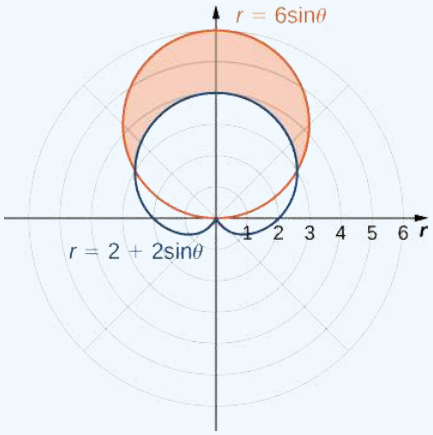


Figure 10.4.4: *The region between the curves* $r = 2 + 2\sin\theta$ *and* $r = 6\sin\theta$.

To determine the limits of integration, first find the points of intersection by setting the two functions equal to each other and solving for θ:

$$6\sin\theta = 2 + 2\sin\theta$$
$$4\sin\theta = 2$$
$$\sin\theta = \frac{1}{2}.$$

This gives the solutions $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$, which are the limits of integration. The circle $r = 3\sin\theta$ is the red graph, which is the outer function, and the cardioid $r = 2 + 2\sin\theta$ is the blue graph, which is the inner function. To calculate the area between the curves, start with the area inside the circle between $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$, then subtract the area inside the cardioid between $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$:

$$\begin{aligned} A &= \text{circle} - \text{cardioid} \\ &= \frac{1}{2}\int_{\pi/6}^{5\pi/6}[6\sin\theta]^2 d\theta - \frac{1}{2}\int_{\pi/6}^{5\pi/6}[2+2\sin\theta]^2 d\theta \\ &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} 36\sin^2\theta\, d\theta - \frac{1}{2}\int_{\pi/6}^{5\pi/6} 4 + 8\sin\theta + 4\sin^2\theta\, d\theta \\ &= 18\int_{\pi/6}^{5\pi/6} \frac{1-\cos(2\theta)}{2} d\theta - 2\int_{\pi/6}^{5\pi/6} 1 + 2\sin\theta + \frac{1-\cos(2\theta)}{2} d\theta \\ &= 9[\theta - \frac{\sin(2\theta)}{2}]_{\pi/6}^{5\pi/6} - 2[\frac{3\theta}{2} - 2\cos\theta - \frac{\sin(2\theta)}{4}]_{\pi/6}^{5\pi/6} \\ &= 9(\frac{5\pi}{6} - \frac{\sin(10\pi/6)}{2}) - 9(\frac{\pi}{6} - \frac{\sin(2\pi/6)}{2}) - (3(\frac{5\pi}{6}) - 4\cos\frac{5\pi}{6} - \frac{\sin(10\pi/6)}{2}) \\ &+ (3(\frac{\pi}{6}) - 4\cos\frac{\pi}{6} - \frac{\sin(2\pi/6)}{2}) \\ &= 4\pi. \end{aligned}$$

? Exercise 10.4.2

Find the area inside the circle $r = 4\cos\theta$ and outside the circle $r = 2$.

Hint

Use Equation 10.4.2 and take advantage of symmetry.

Answer

$$A = \frac{4\pi}{3} + 2\sqrt{3}$$

In Example 10.4.2 we found the area inside the circle and outside the cardioid by first finding their intersection points. Notice that solving the equation directly for θ yielded two solutions: $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$. However, in the graph there are three intersection points. The third intersection point is the origin. The reason why this point did not show up as a solution is because the origin is on both graphs but for different values of θ. For example, for the cardioid we get

$$2 + 2\sin\theta = 0$$
$$\sin\theta = -1,$$

so the values for θ that solve this equation are $\theta = \frac{3\pi}{2} + 2n\pi$, where n is any integer. For the circle we get

$$6\sin\theta = 0.$$

The solutions to this equation are of the form $\theta = n\pi$ for any integer value of n. These two solution sets have no points in common. Regardless of this fact, the curves intersect at the origin. This case must always be taken into consideration.

Arc Length in Polar Curves

Here we derive a formula for the arc length of a curve defined in polar coordinates. In rectangular coordinates, the arc length of a parameterized curve $(x(t), y(t))$ for $a \le t \le b$ is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

In polar coordinates we define the curve by the equation $r = f(\theta)$, where $\alpha \le \theta \le \beta$. In order to adapt the arc length formula for a polar curve, we use the equations

$$x = r\cos\theta = f(\theta)\cos\theta$$

and

$$y = r\sin\theta = f(\theta)\sin\theta,$$

and we replace the parameter t by θ. Then

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta$$

$$\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta.$$

We replace dt by $d\theta$, and the lower and upper limits of integration are α and β, respectively. Then the arc length formula becomes

$$\begin{aligned} L &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \\ &= \int_\alpha^\beta \sqrt{(f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2}\, d\theta \\ &= \int_\alpha^\beta \sqrt{(f'(\theta))^2(\cos^2\theta + \sin^2\theta) + (f(\theta))^2(\cos^2\theta + \sin^2\theta)}\, d\theta \\ &= \int_\alpha^\beta \sqrt{(f'(\theta))^2 + (f(\theta))^2}\, d\theta \\ &= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \end{aligned}$$

This gives us the following theorem.

Arc Length of a Curve Defined by a Polar Function

Let f be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The arc length of the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$L = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta \tag{10.4.3}$$

$$= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta. \tag{10.4.4}$$

Example 10.4.3: Finding the Arc Length of a cardioid

Find the arc length of the cardioid $r = 2 + 2\cos\theta$.

Solution

When $\theta = 0, r = 2 + 2\cos 0 = 4$. Furthermore, as θ goes from 0 to 2π, the cardioid is traced out exactly once. Therefore these are the limits of integration. Using $f(\theta) = 2 + 2\cos\theta, \alpha = 0,$ and $\beta = 2\pi$, Equation 10.4.3 becomes

$$\begin{aligned} L &= \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{[2 + 2\cos\theta]^2 + [-2\sin\theta]^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{4 + 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}\, d\theta \\ &= \int_0^{2\pi} \sqrt{4 + 8\cos\theta + 4(\cos^2\theta + \sin^2\theta)}\, d\theta \\ &= \int_0^{2\pi} \sqrt{8 + 8\cos\theta}\, d\theta \\ &= 2\int_0^{2\pi} \sqrt{2 + 2\cos\theta}\, d\theta. \end{aligned}$$

Next, using the identity $\cos(2\alpha) = 2\cos^2\alpha - 1,$ add 1 to both sides and multiply by 2. This gives $2 + 2\cos(2\alpha) = 4\cos^2\alpha$. Substituting $\alpha = \theta/2$ gives $2 + 2\cos\theta = 4\cos^2(\theta/2)$, so the integral becomes

$$\begin{aligned} L &= 2\int_0^{2\pi} \sqrt{2 + 2\cos\theta}\, d\theta \\ &= 2\int_0^{2\pi} \sqrt{4\cos^2(\frac{\theta}{2})}\, d\theta \\ &= 4\int_0^{2\pi} |\cos(\frac{\theta}{2})|\, d\theta. \end{aligned}$$

The absolute value is necessary because the cosine is negative for some values in its domain. To resolve this issue, change the limits from 0 to π and double the answer. This strategy works because cosine is positive between 0 and $\frac{\pi}{2}$. Thus,

$$\begin{aligned} L &= 4\int_0^{2\pi} |\cos(\frac{\theta}{2})|\, d\theta \\ &= 8\int_0^{\pi} \cos(\frac{\theta}{2})\, d\theta \\ &= 8(2\sin(\frac{\theta}{2})\Big|_0^{\pi} \\ &= 16 \end{aligned}$$

Exercise 10.4.3

Find the total arc length of the graph of $r = 3\sin\theta$.

Hint

Use Equation 10.4.3 To determine the correct limits, make a table of values.

Answer

$s = 3\pi$

Key Concepts

- The area of a region in polar coordinates defined by the equation $r = f(\theta)$ with $\alpha \leq \theta \leq \beta$ is given by the integral $A = \frac{1}{2}\int_{\alpha}^{\beta}[f(\theta)]^2 d\theta$.
- To find the area between two curves in the polar coordinate system, first find the points of intersection, then subtract the corresponding areas.
- The arc length of a polar curve defined by the equation $r = f(\theta)$ with $\alpha \leq \theta \leq \beta$ is given by the integral $L = \int_{\alpha}^{\beta}\sqrt{[f(\theta)]^2 + [f\,'(\theta)]^2}\,d\theta = \int_{\alpha}^{\beta}\sqrt{r^2 + (\frac{dr}{d\theta})^2}\,d\theta$.

Key Equations

- **Area of a region bounded by a polar curve**

$$A = \frac{1}{2}\int_{\alpha}^{\beta}[f(\theta)]^2 d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$$

- **Arc length of a polar curve**

$$L = \int_{\alpha}^{\beta}\sqrt{[f(\theta)]^2 + [f\,'(\theta)]^2}\,d\theta = \int_{\alpha}^{\beta}\sqrt{r^2 + (\frac{dr}{d\theta})^2}\,d\theta$$

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10.5: Conic Sections

Learning Objectives

- Identify the equation of a parabola in standard form with given focus and directrix.
- Identify the equation of an ellipse in standard form with given foci.
- Identify the equation of a hyperbola in standard form with given foci.
- Recognize a parabola, ellipse, or hyperbola from its eccentricity value.
- Write the polar equation of a conic section with eccentricity e.
- Identify when a general equation of degree two is a parabola, ellipse, or hyperbola.

Conic sections have been studied since the time of the ancient Greeks, and were considered to be an important mathematical concept. As early as 320 BCE, such Greek mathematicians as Menaechmus, Appollonius, and Archimedes were fascinated by these curves. Appollonius wrote an entire eight-volume treatise on conic sections in which he was, for example, able to derive a specific method for identifying a conic section through the use of geometry. Since then, important applications of conic sections have arisen (for example, in astronomy), and the properties of conic sections are used in radio telescopes, satellite dish receivers, and even architecture. In this section we discuss the three basic conic sections, some of their properties, and their equations.

Conic sections get their name because they can be generated by intersecting a plane with a cone. A cone has two identically shaped parts called **nappes.** One nappe is what most people mean by "cone," having the shape of a party hat. A right circular cone can be generated by revolving a line passing through the origin around the y-axis as shown in Figure 10.5.1.

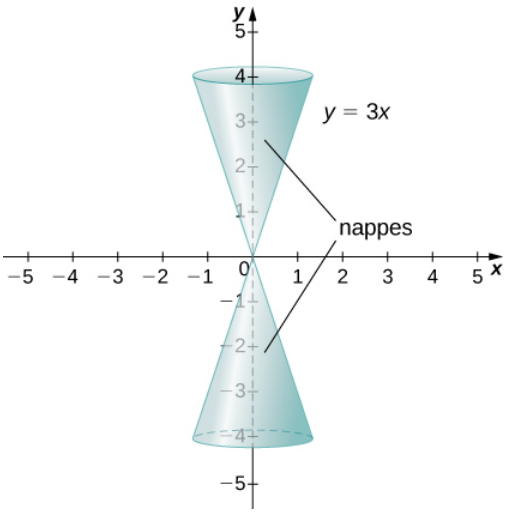


Figure 10.5.1: A cone generated by revolving the line $y = 3x$ around the y-axis.

Conic sections are generated by the intersection of a plane with a cone (Figure 10.5.2). If the plane is parallel to the axis of revolution (the y-axis), then the **conic section** is a hyperbola. If the plane is parallel to the generating line, the conic section is a parabola. If the plane is perpendicular to the axis of revolution, the conic section is a circle. If the plane intersects one nappe at an angle to the axis (other than **90°**), then the conic section is an ellipse.

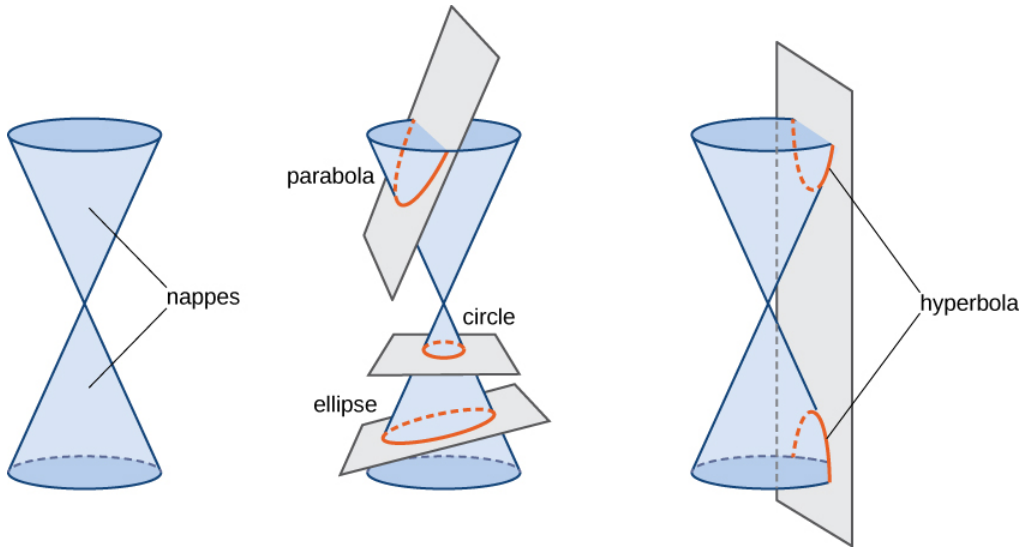


Figure 10.5.2: The four conic sections. Each conic is determined by the angle the plane makes with the axis of the cone.

Parabolas

A parabola is generated when a plane intersects a cone parallel to the generating line. In this case, the plane intersects only one of the nappes. A parabola can also be defined in terms of distances.

Definitions: The Focus, Directrix and Vertex

A parabola is the set of all points whose distance from a fixed point, called the *focus*, is equal to the distance from a fixed line, called the *directrix*. The point halfway between the focus and the directrix is called the *vertex* of the parabola.

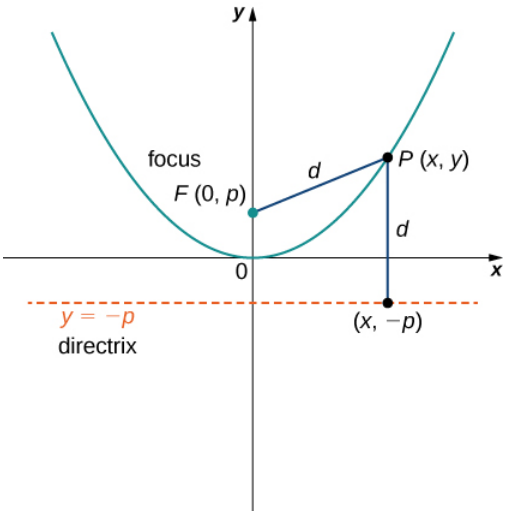


Figure 10.5.3: A typical parabola in which the distance from the focus to the vertex is represented by the variable p.

A graph of a typical parabola appears in Figure 10.5.3. Using this diagram in conjunction with the distance formula, we can derive an equation for a parabola. Recall the distance formula: Given point P with coordinates (x_1, y_1) and point Q with coordinates (x_2, y_2), the distance between them is given by the formula

$$d(P,Q) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\ .$$

Then from the definition of a parabola and Figure 10.5.3, we get

$$d(F,P) = d(P,Q)$$

$$\sqrt{(0-x)^2+(p-y)^2} = \sqrt{(x-x)^2+(-p-y)^2}.$$

Squaring both sides and simplifying yields

$$x^2+(p-y)^2 = 0^2+(-p-y)^2 \tag{10.5.1}$$
$$x^2+p^2-2py+y^2 = p^2+2py+y^2 \tag{10.5.2}$$
$$x^2-2py = 2py \tag{10.5.3}$$
$$x^2 = 4py. \tag{10.5.4}$$

Now suppose we want to relocate the vertex. We use the variables (h,k) to denote the coordinates of the vertex. Then if the focus is directly above the vertex, it has coordinates $(h,\ k+p)$ and the directrix has the equation $y=k-p$. Going through the same derivation yields the formula $(x-h)^2=4p(y-k)$. Solving this equation for y leads to the following theorem.

Equations for Parabolas: standard form

Given a parabola opening upward with vertex located at (h,k) and focus located at $(h,k+p)$, where p is a constant, the equation for the parabola is given by

$$y = \frac{1}{4p}(x-h)^2+k. \tag{10.5.5}$$

This is the *standard form* of a parabola.

We can also study the cases when the parabola opens down or to the left or the right. The equation for each of these cases can also be written in standard form as shown in the following graphs.

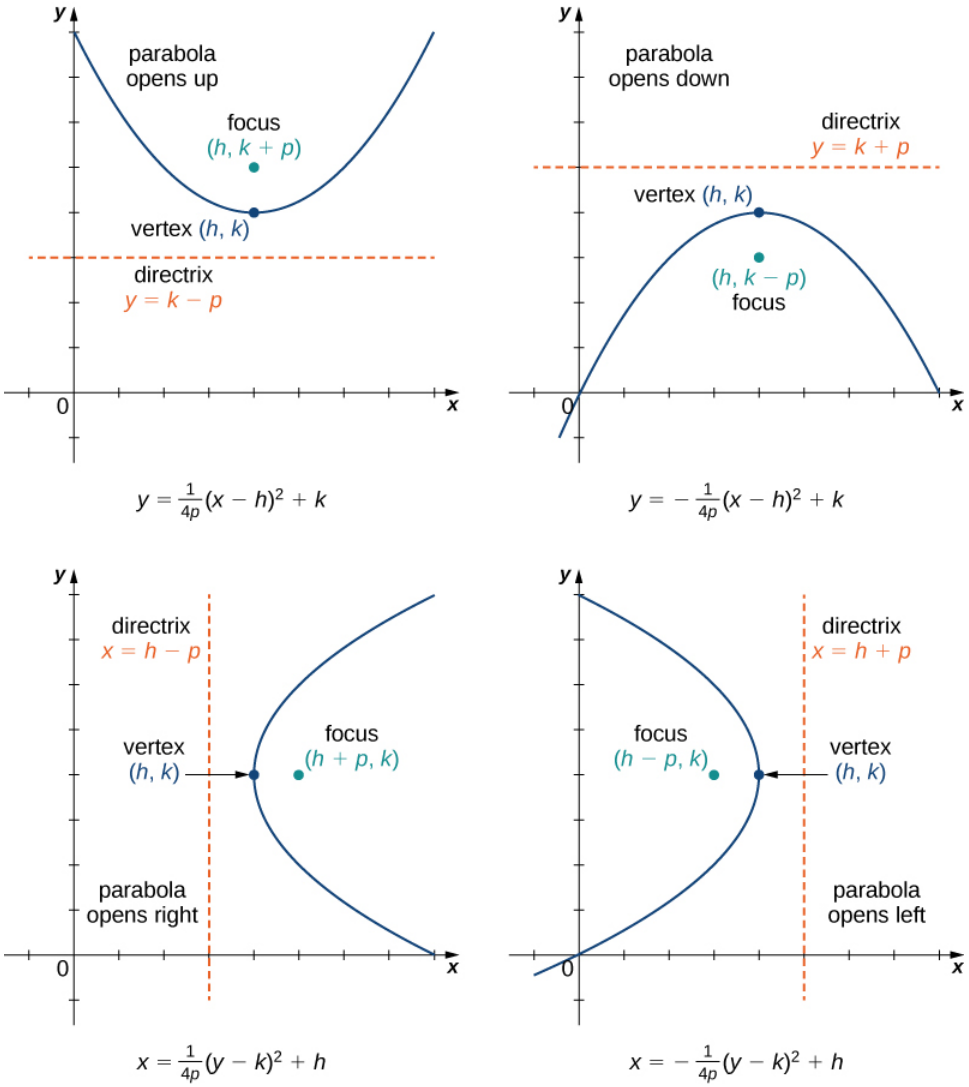


Figure 10.5.4: Four parabolas, opening in various directions, along with their equations in standard form.

In addition, the equation of a parabola can be written in the **general form**, though in this form the values of h, k, and p are not immediately recognizable. The general form of a parabola is written as

$$ax^2 + bx + cy + d = 0 \tag{10.5.6}$$

or

$$ay^2 + bx + cy + d = 0. \tag{10.5.7}$$

Equation 10.5.6 represents a parabola that opens either up or down. Equation 10.5.7 represents a parabola that opens either to the left or to the right. To put the equation into standard form, use the method of completing the square.

Example 10.5.1: Converting the Equation of a Parabola from General into Standard Form

Put the equation

$$x^2 - 4x - 8y + 12 = 0$$

into standard form and graph the resulting parabola.

Solution

Since y is not squared in this equation, we know that the parabola opens either upward or downward. Therefore we need to solve this equation for y, which will put the equation into standard form. To do that, first add $8y$ to both sides of the equation:

$$8y = x^2 - 4x + 12.$$

The next step is to complete the square on the right-hand side. Start by grouping the first two terms on the right-hand side using parentheses:

$$8y = (x^2 - 4x) + 12.$$

Next determine the constant that, when added inside the parentheses, makes the quantity inside the parentheses a perfect square trinomial. To do this, take half the coefficient of x and square it. This gives $\left(\frac{-4}{2}\right)^2 = 4$. Add 4 inside the parentheses and subtract 4 outside the parentheses, so the value of the equation is not changed:

$$8y = (x^2 - 4x + 4) + 12 - 4.$$

Now combine like terms and factor the quantity inside the parentheses:

$$8y = (x - 2)^2 + 8.$$

Finally, divide by 8:

$$y = \frac{1}{8}(x - 2)^2 + 1.$$

This equation is now in standard form. Comparing this to Equation 10.5.5 gives $h = 2$, $k = 1$, and $p = 2$. The parabola opens up, with vertex at $(2, 1)$, focus at $(2, 3)$, and directrix $y = -1$. The graph of this parabola appears as follows.

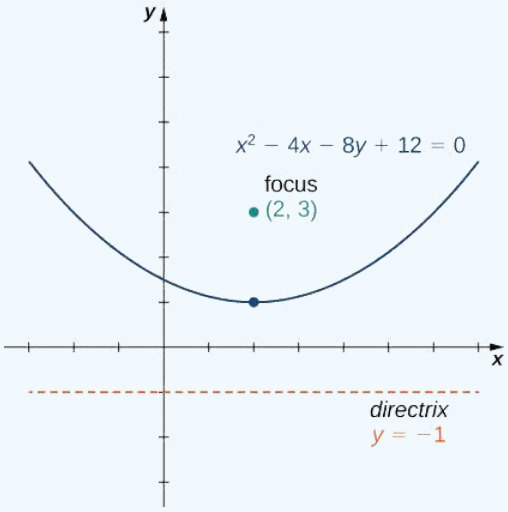


Figure 10.5.5: The parabola in Example 10.5.1.

Exercise 10.5.1

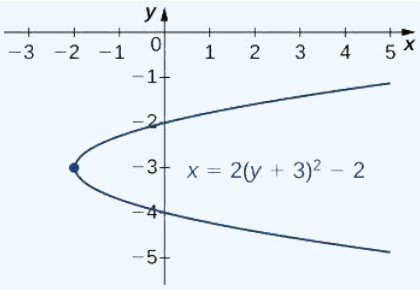
Put the equation $2y^2 - x + 12y + 16 = 0$ into standard form and graph the resulting parabola.

Hint

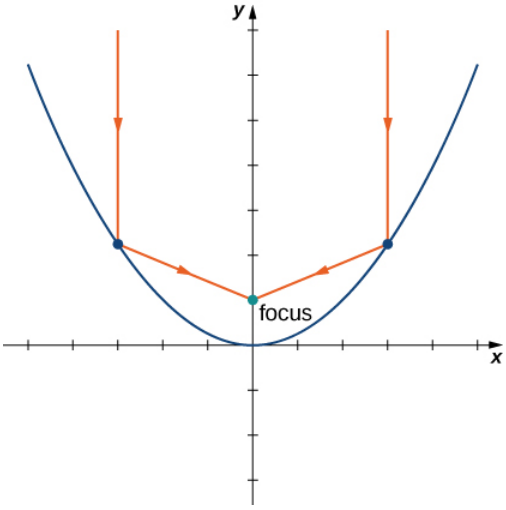
Solve for x. Check which direction the parabola opens.

Answer

$$x = 2(y + 3)^2 - 2$$



The axis of symmetry of a vertical (opening up or down) parabola is a vertical line passing through the vertex. The parabola has an interesting reflective property. Suppose we have a satellite dish with a parabolic cross section. If a beam of electromagnetic waves, such as light or radio waves, comes into the dish in a straight line from a satellite (parallel to the axis of symmetry), then the waves reflect off the dish and collect at the focus of the parabola as shown.



Consider a parabolic dish designed to collect signals from a satellite in space. The dish is aimed directly at the satellite, and a receiver is located at the focus of the parabola. Radio waves coming in from the satellite are reflected off the surface of the parabola to the receiver, which collects and decodes the digital signals. This allows a small receiver to gather signals from a wide angle of sky. Flashlights and headlights in a car work on the same principle, but in reverse: the source of the light (that is, the light bulb) is located at the focus and the reflecting surface on the parabolic mirror focuses the beam straight ahead. This allows a small light bulb to illuminate a wide angle of space in front of the flashlight or car.

Ellipses

An ellipse can also be defined in terms of distances. In the case of an ellipse, there are two foci (plural of focus), and two directrices (plural of directrix). We look at the directrices in more detail later in this section.

Definition: Ellipse

An ellipse is the set of all points for which the sum of their distances from two fixed points (the foci) is constant.

A graph of a typical ellipse is shown in Figure 10.5.6. In this figure the foci are labeled as F and F'. Both are the same fixed distance from the origin, and this distance is represented by the variable c. Therefore the coordinates of F are $(c, 0)$ and the coordinates of F' are $(-c, 0)$. The points P and P' are located at the ends of the **major axis** of the ellipse, and have coordinates $(a, 0)$ and $(-a, 0)$, respectively. The major axis is always the longest distance across the ellipse, and can be horizontal or vertical. Thus, the length of the major axis in this ellipse is $2a$. Furthermore, P and P' are called the vertices of the ellipse. The points Q and Q' are located at the ends of the **minor axis** of the ellipse, and have coordinates $(0, b)$ and $(0, -b)$, respectively. The minor axis is the shortest distance across the ellipse. The minor axis is perpendicular to the major axis.

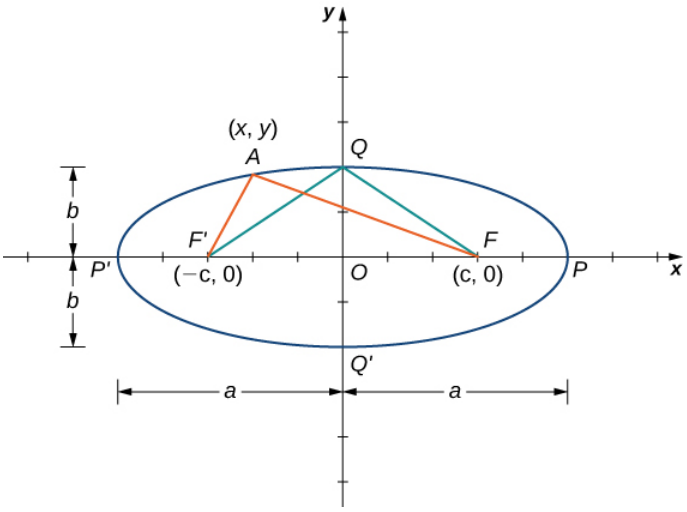


Figure 10.5.6: A typical ellipse in which the sum of the distances from any point on the ellipse to the foci is constant.

According to the definition of the ellipse, we can choose any point on the ellipse and the sum of the distances from this point to the two foci is constant. Suppose we choose the point P. Since the coordinates of point P are $(a, 0)$, the sum of the distances is

$$d(P, F) + d(P, F') = (a - c) + (a + c) = 2a.$$

Therefore the sum of the distances from an arbitrary point A with coordinates (x, y) is also equal to $2a$. Using the distance formula, we get

$$d(A, F) + d(A, F') = 2a.$$

$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a$$

Subtract the second radical from both sides and square both sides:

$$\sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}$$

$$(x-c)^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2$$

$$x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2cx + c^2 + y^2$$

$$-2cx = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + 2cx.$$

Now isolate the radical on the right-hand side and square again:

$$-2cx = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + 2cx$$

$$4a\sqrt{(x+c)^2 + y^2} = 4a^2 + 4cx$$

$$\sqrt{(x+c)^2 + y^2} = a + \frac{cx}{a}$$

$$(x+c)^2 + y^2 = a^2 + 2cx + \frac{c^2x^2}{a^2}$$

$$x^2 + 2cx + c^2 + y^2 = a^2 + 2cx + \frac{c^2x^2}{a^2}$$

$$x^2 + c^2 + y^2 = a^2 + \frac{c^2x^2}{a^2}.$$

Isolate the variables on the left-hand side of the equation and the constants on the right-hand side:

$$x^2 - \frac{c^2x^2}{a^2} + y^2 = a^2 - c^2$$
$$\frac{(a^2 - c^2)x^2}{a^2} + y^2 = a^2 - c^2.$$

Divide both sides by $a^2 - c^2$. This gives the equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

If we refer back to Figure 10.5.6, then the length of each of the two green line segments is equal to a. This is true because the sum of the distances from the point Q to the foci F and F' is equal to $2a$, and the lengths of these two line segments are equal. This line segment forms a right triangle with hypotenuse length a and leg lengths b and c. From the Pythagorean theorem, $b^2 + c^2 = a^2$ and $b^2 = a^2 - c^2$. Therefore the equation of the ellipse becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Finally, if the center of the ellipse is moved from the origin to a point (h, k), we have the following standard form of an ellipse.

Equation of an Ellipse in Standard Form

Consider the ellipse with center (h, k), a horizontal major axis with length $2a$, and a vertical minor axis with length $2b$. Then the equation of this ellipse in standard form is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \tag{10.5.8}$$

and the foci are located at $(h \pm c, k)$, where $c^2 = a^2 - b^2$. The equations of the directrices are $x = h \pm \frac{a^2}{c}$.

If the major axis is vertical, then the equation of the ellipse becomes

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1 \tag{10.5.9}$$

and the foci are located at $(h, k \pm c)$, where $c^2 = a^2 - b^2$. The equations of the directrices in this case are $y = k \pm \frac{a^2}{c}$.

If the major axis is horizontal, then the ellipse is called horizontal, and if the major axis is vertical, then the ellipse is called vertical. The equation of an ellipse is in general form if it is in the form

$$Ax^2 + By^2 + Cx + Dy + E = 0,$$

where *A* and *B* are either both positive or both negative. To convert the equation from general to standard form, use the method of **completing the square**.

Example 10.5.2: Finding the Standard Form of an Ellipse

Put the equation

$$9x^2 + 4y^2 - 36x + 24y + 36 = 0$$

into standard form and graph the resulting ellipse.

Solution

First subtract 36 from both sides of the equation:

$$9x^2 + 4y^2 - 36x + 24y = -36.$$

Next group the x terms together and the y terms together, and factor out the common factor:

$$(9x^2-36x)+(4y^2+24y)=-36$$

$$9(x^2-4x)+4(y^2+6y)=-36.$$

We need to determine the constant that, when added inside each set of parentheses, results in a perfect square. In the first set of parentheses, take half the coefficient of x and square it. This gives $\left(\frac{-4}{2}\right)^2=4$. In the second set of parentheses, take half the coefficient of y and square it. This gives $\left(\frac{6}{2}\right)^2=9$. Add these inside each pair of parentheses. Since the first set of parentheses has a 9 in front, we are actually adding 36 to the left-hand side. Similarly, we are adding 36 to the second set as well. Therefore the equation becomes

$$9(x^2-4x+4)+4(y^2+6y+9)=-36+36+36$$

$$9(x^2-4x+4)+4(y^2+6y+9)=36.$$

Now factor both sets of parentheses and divide by 36:

$$9(x-2)^2+4(y+3)^2=36$$

$$\frac{9(x-2)^2}{36}+\frac{4(y+3)^2}{36}=1$$

$$\frac{(x-2)^2}{4}+\frac{(y+3)^2}{9}=1.$$

The equation is now in standard form. Comparing this to Equation 10.5.9 gives $h=2, k=-3, a=3,$ and $b=2$. This is a vertical ellipse with center at $(2,-3)$, major axis 6, and minor axis 4. The graph of this ellipse appears as follows.

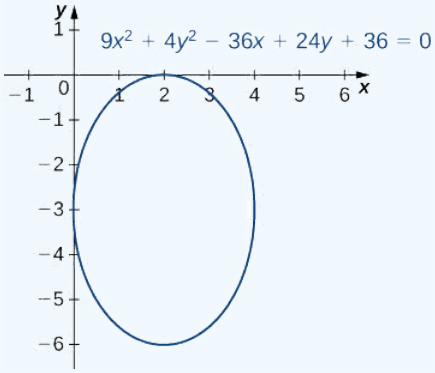


Figure 10.5.7: The ellipse in Example 10.5.2.

Exercise 10.5.2

Put the equation

$$9x^2+16y^2+18x-64y-71=0$$

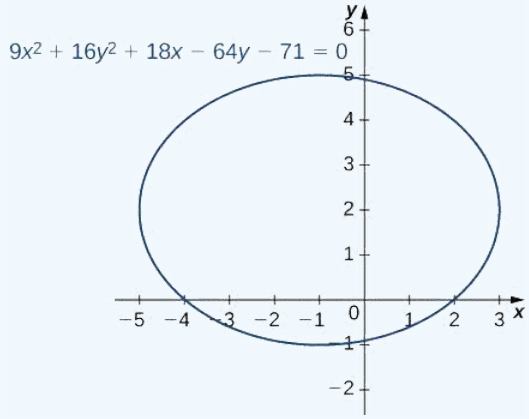
into standard form and graph the resulting ellipse.

Hint

Move the constant over and complete the square.

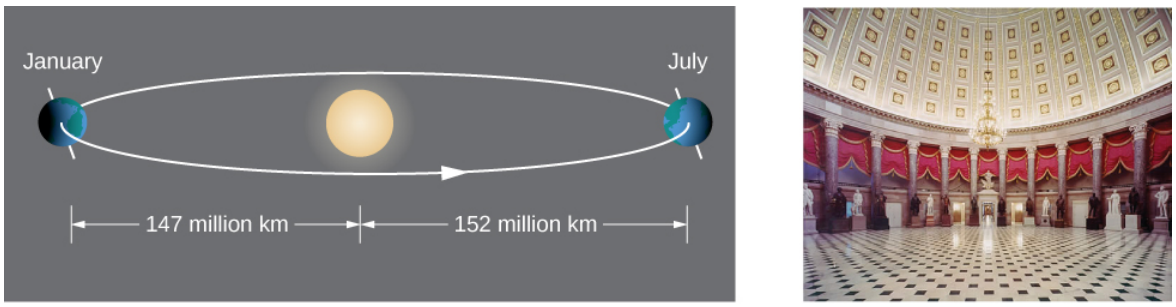
Answer

$$\frac{(x+1)^2}{16}+\frac{(y-2)^2}{9}=1$$



According to Kepler's first law of planetary motion, the orbit of a planet around the Sun is an ellipse with the Sun at one of the foci as shown in Figure $10.5.8A$. Because Earth's orbit is an ellipse, the distance from the Sun varies throughout the year. A commonly held misconception is that Earth is closer to the Sun in the summer. In fact, in summer for the northern hemisphere, Earth is farther from the Sun than during winter. The difference in season is caused by the tilt of Earth's axis in the orbital plane. Comets that orbit the Sun, such as Halley's Comet, also have elliptical orbits, as do moons orbiting the planets and satellites orbiting Earth.

Ellipses also have interesting reflective properties: A light ray emanating from one focus passes through the other focus after mirror reflection in the ellipse. The same thing occurs with a sound wave as well. The National Statuary Hall in the U.S. Capitol in Washington, DC, is a famous room in an elliptical shape as shown in Figure $10.5.8B$. This hall served as the meeting place for the U.S. House of Representatives for almost fifty years. The location of the two foci of this semi-elliptical room are clearly identified by marks on the floor, and even if the room is full of visitors, when two people stand on these spots and speak to each other, they can hear each other much more clearly than they can hear someone standing close by. Legend has it that John Quincy Adams had his desk located on one of the foci and was able to eavesdrop on everyone else in the House without ever needing to stand. Although this makes a good story, it is unlikely to be true, because the original ceiling produced so many echoes that the entire room had to be hung with carpets to dampen the noise. The ceiling was rebuilt in 1902 and only then did the now-famous whispering effect emerge. Another famous whispering gallery—the site of many marriage proposals—is in Grand Central Station in New York City.



(a) (b)

Figure 10.5.8: (a) Earth's orbit around the Sun is an ellipse with the Sun at one focus. (b) Statuary Hall in the U.S. Capitol is a whispering gallery with an elliptical cross section.

Hyperbolas

A hyperbola can also be defined in terms of distances. In the case of a hyperbola, there are two foci and two directrices. Hyperbolas also have two asymptotes.

Definition: hyperbola

A hyperbola is the set of all points where the difference between their distances from two fixed points (the foci) is constant.

A graph of a typical hyperbola appears as follows.

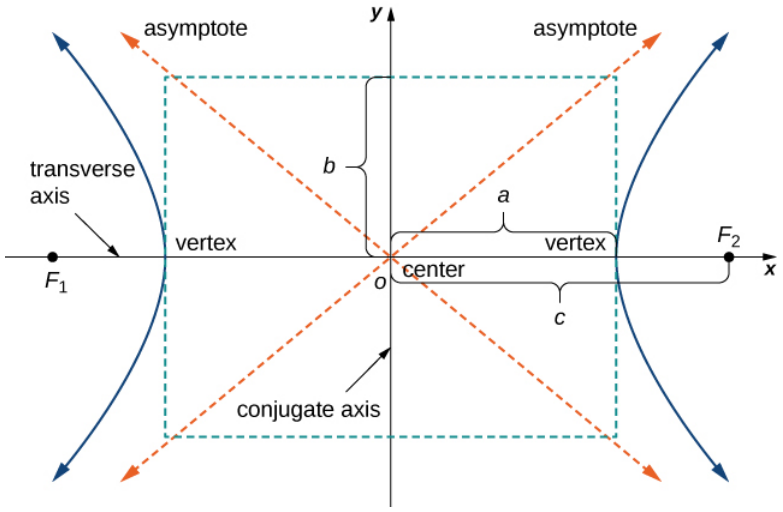


Figure 10.5.9: A typical hyperbola in which the difference of the distances from any point on the hyperbola to the foci is constant. The transverse axis is also called the major axis, and the conjugate axis is also called the minor axis.

The derivation of the equation of a hyperbola in standard form is virtually identical to that of an ellipse. One slight hitch lies in the definition: The difference between two numbers is always positive. Let P be a point on the hyperbola with coordinates (x, y). Then the definition of the hyperbola gives $|d(P, F_1) - d(P, F_2)| = constant$. To simplify the derivation, assume that P is on the right branch of the hyperbola, so the absolute value bars drop. If it is on the left branch, then the subtraction is reversed. The vertex of the right branch has coordinates $(a, 0)$, so

$$d(P, F_1) - d(P, F_2) = (c + a) - (c - a) = 2a.$$

This equation is therefore true for any point on the hyperbola. Returning to the coordinates (x, y) for P:

$$d(P, F_1) - d(P, F_2) = 2a$$

$$\sqrt{(x + c)^2 + y^2} - \sqrt{(x - c)^2 + y^2} = 2a.$$

Isolate the second radical and square both sides:

$$\sqrt{(x - c)^2 + y^2} = -2a + \sqrt{(x + c)^2 + y^2}$$

$$(x - c)^2 + y^2 = 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + (x + c)^2 + y^2$$

$$x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + x^2 + 2cx + c^2 + y^2$$

$$-2cx = 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + 2cx.$$

Now isolate the radical on the right-hand side and square again:

$$-2cx = 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + 2cx$$

$$-4a\sqrt{(x + c)^2 + y^2} = -4a^2 - 4cx$$

$$-\sqrt{(x + c)^2 + y^2} = -a - \frac{cx}{a}$$

$$(x + c)^2 + y^2 = a^2 + 2cx + \frac{c^2x^2}{a^2}$$

$$x^2 + 2cx + c^2 + y^2 = a^2 + 2cx + \frac{c^2x^2}{a^2}$$

$$x^2 + c^2 + y^2 = a^2 + \frac{c^2x^2}{a^2} \quad .$$

Isolate the variables on the left-hand side of the equation and the constants on the right-hand side:

$$x^2 - \frac{c^2x^2}{a^2} + y^2 = a^2 - c^2$$
$$\frac{(a^2 - c^2)x^2}{a^2} + y^2 = a^2 - c^2.$$

Finally, divide both sides by $a^2 - c^2$. This gives the equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

We now define b so that $b^2 = c^2 - a^2$. This is possible because $c > a$. Therefore the equation of the hyperbola becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Finally, if the center of the hyperbola is moved from the origin to the point (h, k), we have the following standard form of a hyperbola.

Equation of a Hyperbola in Standard Form

Consider the hyperbola with center (h, k), a horizontal major axis, and a vertical minor axis. Then the equation of this hyperbola is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \tag{10.5.10}$$

and the foci are located at $(h \pm c, k)$, where $c^2 = a^2 + b^2$. The equations of the asymptotes are given by $y = k \pm \frac{b}{a}(x - h)$. The equations of the directrices are

$$x = h \pm \frac{a^2}{\sqrt{a^2 + b^2}} = h \pm \frac{a^2}{c}$$

If the major axis is vertical, then the equation of the hyperbola becomes

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

and the foci are located at $(h, k \pm c)$, where $c^2 = a^2 + b^2$. The equations of the asymptotes are given by $y = k \pm \frac{a}{b}(x - h)$. The equations of the directrices are

$$y = k \pm \frac{a^2}{\sqrt{a^2 + b^2}} = k \pm \frac{a^2}{c}.$$

If the major axis (transverse axis) is horizontal, then the hyperbola is called horizontal, and if the major axis is vertical then the hyperbola is called vertical. The equation of a hyperbola is in general form if it is in the form

$$Ax^2 + By^2 + Cx + Dy + E = 0,$$

where A and B have opposite signs. In order to convert the equation from general to standard form, use the method of completing the square.

Example 10.5.3: Finding the Standard Form of a Hyperbola

Put the equation $9x^2 - 16y^2 + 36x + 32y - 124 = 0$ into standard form and graph the resulting hyperbola. What are the equations of the asymptotes?

Solution

First add 124 to both sides of the equation:

$$9x^2 - 16y^2 + 36x + 32y = 124.$$

Next group the x terms together and the y terms together, then factor out the common factors:

$$(9x^2+36x)-(16y^2-32y)=124$$
$$9(x^2+4x)-16(y^2-2y)=124\ .$$

We need to determine the constant that, when added inside each set of parentheses, results in a perfect square. In the first set of parentheses, take half the coefficient of x and square it. This gives $\left(\frac{4}{2}\right)^2=4$. In the second set of parentheses, take half the coefficient of y and square it. This gives $\left(\frac{-2}{2}\right)^2=1$. Add these inside each pair of parentheses. Since the first set of parentheses has a 9 in front, we are actually adding 36 to the left-hand side. Similarly, we are subtracting 16 from the second set of parentheses. Therefore the equation becomes

$$9(x^2+4x+4)-16(y^2-2y+1)=124+36-16$$
$$9(x^2+4x+4)-16(y^2-2y+1)=144.$$

Next factor both sets of parentheses and divide by 144:

$$9(x+2)^2-16(y-1)^2=144$$
$$\frac{9(x+2)^2}{144}-\frac{16(y-1)^2}{144}=1$$
$$\frac{(x+2)^2}{16}-\frac{(y-1)^2}{9}=1.$$

The equation is now in standard form. Comparing this to Equation 10.5.10gives $h=-2,\ k=1,\ a=4,$ and $b=3$. This is a horizontal hyperbola with center at $(-2,1)$ and asymptotes given by the equations $y=1\pm\frac{3}{4}(x+2)$. The graph of this hyperbola appears in Figure 10.5.10

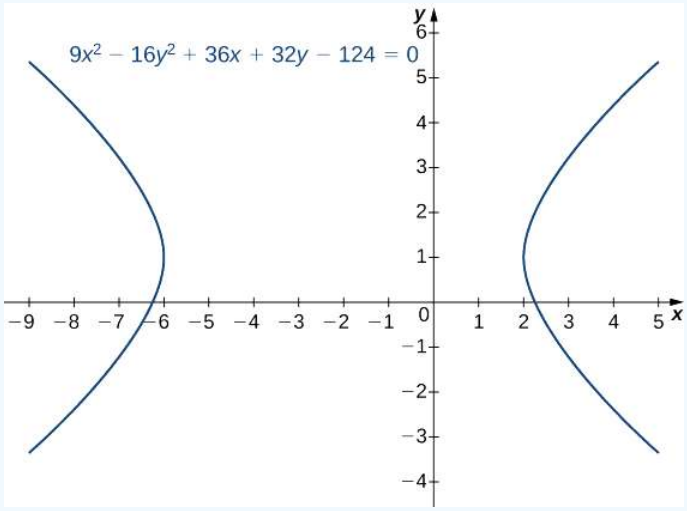


Figure 10.5.10: Graph of the hyperbola in Example 10.5.3.

? Exercise 10.5.3

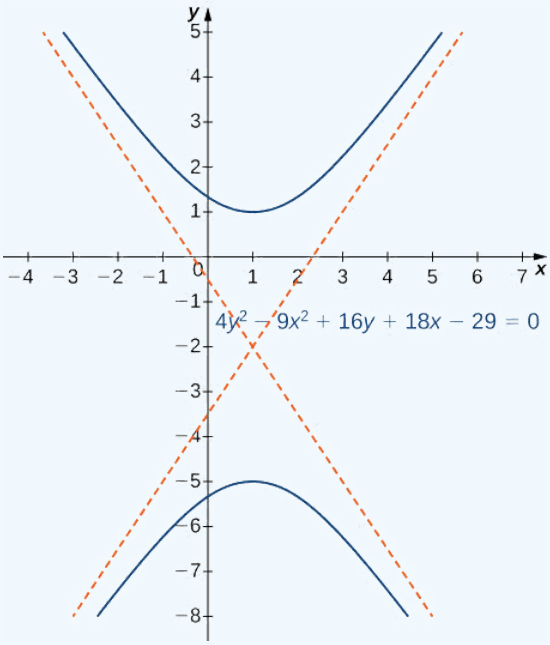
Put the equation $4y^2-9x^2+16y+18x-29=0$ into standard form and graph the resulting hyperbola. What are the equations of the asymptotes?

Hint

Move the constant over and complete the square. Check which direction the hyperbola opens

Answer

$\frac{(y+2)^2}{9} - \frac{(x-1)^2}{4} = 1$. This is a vertical hyperbola. Asymptotes $y = -2 \pm \frac{3}{2}(x-1)$.



Hyperbolas also have interesting reflective properties. A ray directed toward one focus of a hyperbola is reflected by a hyperbolic mirror toward the other focus. This concept is illustrated in Figure 10.5.11

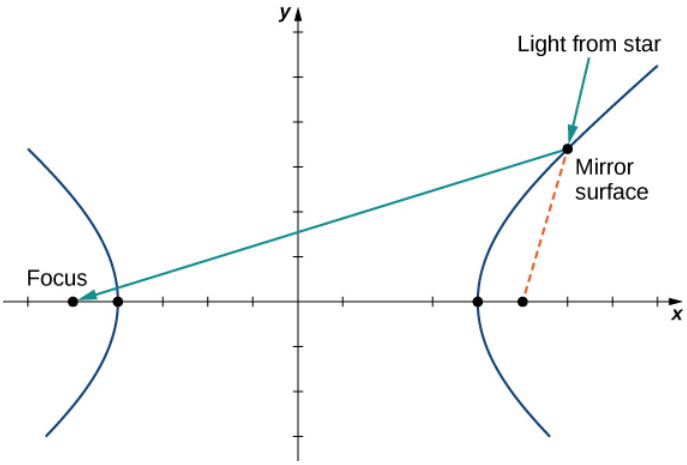


Figure 10.5.11: A hyperbolic mirror used to collect light from distant stars.

This property of the hyperbola has important applications. It is used in radio direction finding (since the difference in signals from two towers is constant along hyperbolas), and in the construction of mirrors inside telescopes (to reflect light coming from the parabolic mirror to the eyepiece). Another interesting fact about hyperbolas is that for a comet entering the solar system, if the speed is great enough to escape the Sun's gravitational pull, then the path that the comet takes as it passes through the solar system is hyperbolic.

Eccentricity and Directrix

An alternative way to describe a conic section involves the directrices, the foci, and a new property called eccentricity. We will see that the value of the eccentricity of a conic section can uniquely define that conic.

Definition: Eccentricity and Directrices

The *eccentricity* e of a conic section is defined to be the distance from any point on the conic section to its focus, divided by the perpendicular distance from that point to the nearest directrix. This value is constant for any conic section, and can define the conic section as well:

1. If $e = 1$, the conic is a parabola.
2. If $e < 1$, it is an ellipse.
3. If $e > 1$, it is a hyperbola.

The eccentricity of a circle is zero. The *directrix* of a conic section is the line that, together with the point known as the focus, serves to define a conic section. Hyperbolas and noncircular ellipses have two foci and two associated directrices. Parabolas have one focus and one directrix.

The three conic sections with their directrices appear in Figure 10.5.12

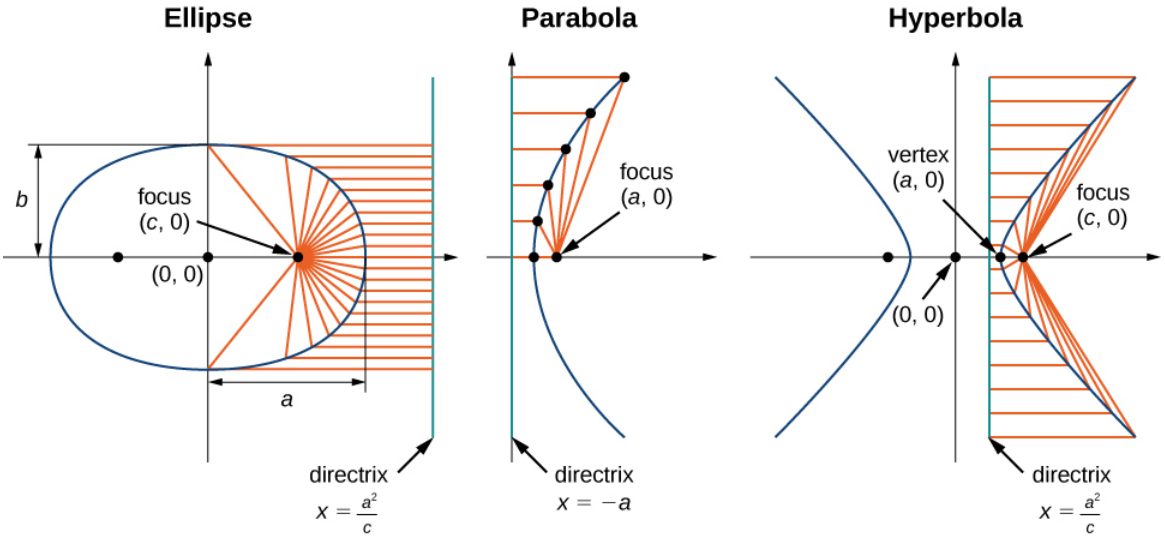


Figure 10.5.12: The three conic sections with their foci and directrices.

Recall from the definition of a parabola that the distance from any point on the parabola to the focus is equal to the distance from that same point to the directrix. Therefore, by definition, the eccentricity of a parabola must be 1. The equations of the directrices of a horizontal ellipse are $x = \pm\dfrac{a^2}{c}$. The right vertex of the ellipse is located at $(a, 0)$ and the right focus is $(c, 0)$. Therefore the distance from the vertex to the focus is $a - c$ and the distance from the vertex to the right directrix is $\dfrac{a^2}{c} - c$. This gives the eccentricity as

$$e = \frac{a-c}{\frac{a^2}{c} - a} = \frac{c(a-c)}{a^2 - ac} = \frac{c(a-c)}{a(a-c)} = \frac{c}{a}.$$

Since $c < a$, this step proves that the eccentricity of an ellipse is less than 1. The directrices of a horizontal hyperbola are also located at $x = \pm\dfrac{a^2}{c}$, and a similar calculation shows that the eccentricity of a hyperbola is also $e = \dfrac{c}{a}$. However in this case we have $c > a$, so the eccentricity of a hyperbola is greater than 1.

Example 10.5.4: Determining Eccentricity of a Conic Section

Determine the eccentricity of the ellipse described by the equation

$$\frac{(x-3)^2}{16} + \frac{(y+2)^2}{25} = 1.$$

Solution

From the equation we see that $a=5$ and $b=4$. The value of c can be calculated using the equation $a^2=b^2+c^2$ for an ellipse. Substituting the values of a and b and solving for c gives $c=3$. Therefore the eccentricity of the ellipse is $e=\dfrac{c}{a}=\dfrac{3}{5}=0.6$.

Exercise 10.5.4

Determine the eccentricity of the hyperbola described by the equation

$$\frac{(y-3)^2}{49}-\frac{(x+2)^2}{25}=1.$$

Hint

First find the values of a and b, then determine c using the equation $c^2=a^2+b^2$.

Answer

$$e=\frac{c}{a}=\frac{\sqrt{74}}{7}\approx 1.229$$

Polar Equations of Conic Sections

Sometimes it is useful to write or identify the equation of a conic section in polar form. To do this, we need the concept of the focal parameter. The **focal parameter** of a conic section p is defined as the distance from a focus to the nearest directrix. The following table gives the focal parameters for the different types of conics, where a is the length of the semi-major axis (i.e., half the length of the major axis), c is the distance from the origin to the focus, and e is the eccentricity. In the case of a parabola, a represents the distance from the vertex to the focus.

Table 10.5.1: Eccentricities and Focal Parameters of the Conic Sections

Conic	e	p
Ellipse	$0<e<1$	$\dfrac{a^2-c^2}{c}=\dfrac{a(1-e^2)}{c}$
Parabola	$e=1$	$2a$
Hyperbola	$e>1$	$\dfrac{c^2-a^2}{c}=\dfrac{a(e^2-1)}{c}$

Using the definitions of the focal parameter and eccentricity of the conic section, we can derive an equation for any conic section in polar coordinates. In particular, we assume that one of the foci of a given conic section lies at the pole. Then using the definition of the various conic sections in terms of distances, it is possible to prove the following theorem.

Polar Equation of Conic Sections

The polar equation of a conic section with focal parameter p is given by

$$r=\frac{ep}{1\pm e\cos\theta} \text{ or } r=\frac{ep}{1\pm e\sin\theta}.$$

In the equation on the left, the major axis of the conic section is horizontal, and in the equation on the right, the major axis is vertical. To work with a conic section written in polar form, first make the constant term in the denominator equal to 1. This can be done by dividing both the numerator and the denominator of the fraction by the constant that appears in front of the plus or minus in the denominator. Then the coefficient of the sine or cosine in the denominator is the eccentricity. This value identifies the conic. If cosine appears in the denominator, then the conic is horizontal. If sine appears, then the conic is vertical. If both appear then the axes are rotated. The center of the conic is not necessarily at the origin. The center is at the origin only if the conic is a circle (i.e., $e=0$).

Example 10.5.5: Graphing a Conic Section in Polar Coordinates

Identify and create a graph of the conic section described by the equation

$$r = \frac{3}{1+2\cos\theta}.$$

Solution

The constant term in the denominator is 1, so the eccentricity of the conic is 2. This is a hyperbola. The focal parameter p can be calculated by using the equation $ep = 3$. Since $e = 2$, this gives $p = \frac{3}{2}$. The cosine function appears in the denominator, so the hyperbola is horizontal. Pick a few values for θ and create a table of values. Then we can graph the hyperbola (Figure 10.5.13).

θ	r	θ	r
0	1	π	-3
$\frac{\pi}{4}$	$\frac{3}{1+\sqrt{2}} \approx 1.2426$	$\frac{5\pi}{4}$	$\frac{3}{1-\sqrt{2}} \approx -7.2426$
$\frac{\pi}{2}$	3	$\frac{3\pi}{2}$	3
$\frac{3\pi}{4}$	$\frac{3}{1-\sqrt{2}} \approx -7.2426$	$\frac{7\pi}{4}$	$\frac{3}{1+\sqrt{2}} \approx 1.2426$

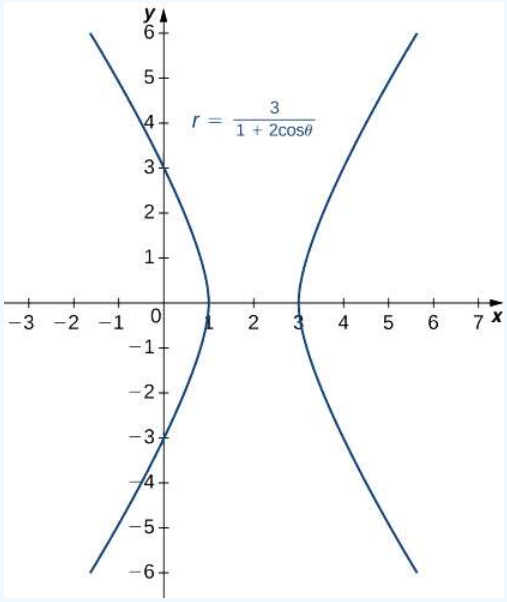


Figure 10.5.13: Graph of the hyperbola described in Example 10.5.5.

Exercise 10.5.5

Identify and create a graph of the conic section described by the equation

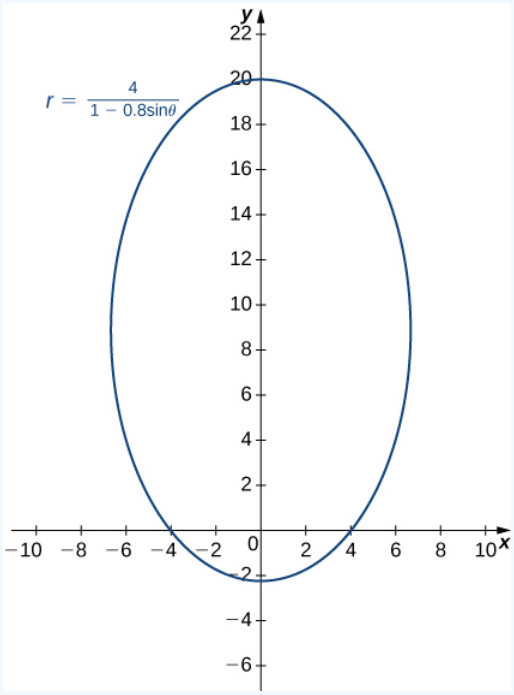
$$r = \frac{4}{1-0.8\sin\theta}.$$

Hint

First find the values of e and p, and then create a table of values.

Answer

Here $e = 0.8$ and $p = 5$. This conic section is an ellipse.



General Equations of Degree Two

A general equation of degree two can be written in the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

The graph of an equation of this form is a conic section. If $B \neq 0$ then the coordinate axes are rotated. To identify the conic section, we use the discriminant of the conic section $4AC - B^2$.

Identifying the Conic Section

One of the following cases must be true:

1. $4AC - B^2 > 0$. If so, the graph is an ellipse.
2. $4AC - B^2 = 0$. If so, the graph is a parabola.
3. $4AC - B^2 < 0$. If so, the graph is a hyperbola.

The simplest example of a second-degree equation involving a cross term is $xy = 1$. This equation can be solved for y to obtain $y = \dfrac{1}{x}$. The graph of this function is called a rectangular hyperbola as shown.

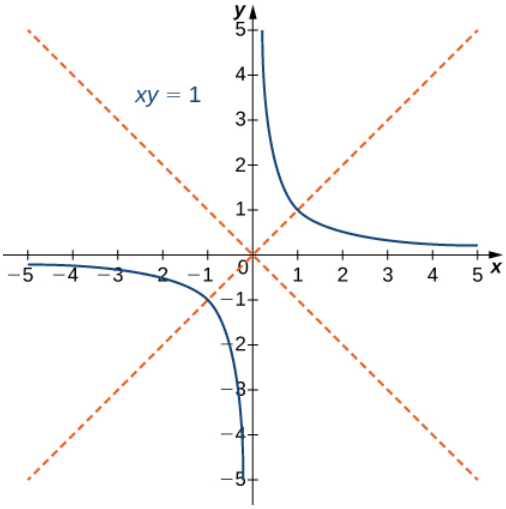


Figure 10.5.14: Graph of the equation $xy = 1$; The red lines indicate the rotated axes.

The asymptotes of this hyperbola are the x and y coordinate axes. To determine the angle θ of rotation of the conic section, we use the formula $\cot 2\theta = \frac{A-C}{B}$. In this case $A = C = 0$ and $B = 1$, so $\cot 2\theta = (0-0)/1 = 0$ and $\theta = 45^\circ$. The method for graphing a conic section with rotated axes involves determining the coefficients of the conic in the rotated coordinate system. The new coefficients are labeled A', B', C', D', E',and F', and are given by the formulas

$$\begin{aligned} A' &= A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta \\ B' &= 0 \\ C' &= A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta \\ D' &= D\cos\theta + E\sin\theta \\ E' &= -D\sin\theta + E\cos\theta \\ F' &= F. \end{aligned}$$

Procedure: graphing a rotated conic

The procedure for graphing a rotated conic is the following:

1. Identify the conic section using the discriminant $4AC - B^2$.
2. Determine θ using the formula

$$\cot 2\theta = \frac{A-C}{B}. \tag{10.5.11}$$

3. Calculate A', B', C', D', E', and F'.
4. Rewrite the original equation using A', B', C', D', E', and F'.
5. Draw a graph using the rotated equation.

Example 10.5.6: Identifying a Rotated Conic

Identify the conic and calculate the angle of rotation of axes for the curve described by the equation

$$13x^2 - 6\sqrt{3}xy + 7y^2 - 256 = 0.$$

Solution

In this equation, $A = 13$, $B = -6\sqrt{3}$, $C = 7$, $D = 0$, $E = 0$, and $F = -256$. The discriminant of this equation is

$$4AC - B^2 = 4(13)(7) - (-6\sqrt{3})^2 = 364 - 108 = 256.$$

Therefore this conic is an ellipse.

To calculate the angle of rotation of the axes, use Equation 10.5.11

$$\cot 2\theta = \frac{A-C}{B}.$$

This gives

$$\cot 2\theta = \frac{A-C}{B} = \frac{13-7}{-6\sqrt{3}} = -\frac{\sqrt{3}}{3}\ .$$

Therefore $2\theta = 120^o$ and $\theta = 60^o$, which is the angle of the rotation of the axes.

To determine the rotated coefficients, use the formulas given above:

$$\begin{aligned}
A' &= A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta \\
&= 13\cos^2 60 + (-6\sqrt{3})\cos 60\sin 60 + 7\sin^2 60 \\
&= 13\left(\frac{1}{2}\right)^2 - 6\sqrt{3}\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + 7\left(\frac{\sqrt{3}}{2}\right)^2 \\
&= 4, \\
B' &= 0 \\
C' &= A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta \\
&= 13\sin^2 60 + (6\sqrt{3})\sin 60\cos 60 + 7\cos^2 60 \\
&= 13\left(\frac{\sqrt{3}}{2}\right)^2 + 6\sqrt{3}\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) + 7\left(\frac{1}{2}\right)^2 \\
&= 16, \\
D' &= D\cos\theta + E\sin\theta \\
&= (0)\cos 60 + (0)\sin 60 \\
&= 0, \\
E' &= -D\sin\theta + E\cos\theta \\
&= -(0)\sin 60 + (0)\cos 60 \\
&= 0 \\
F' &= F \\
&= -256.
\end{aligned}$$

The equation of the conic in the rotated coordinate system becomes

$$\begin{aligned}
4(x')^2 + 16(y')^2 &= 256 \\
\frac{(x')^2}{64} + \frac{(y')^2}{16} &= 1\ .
\end{aligned}$$

A graph of this conic section appears as follows.

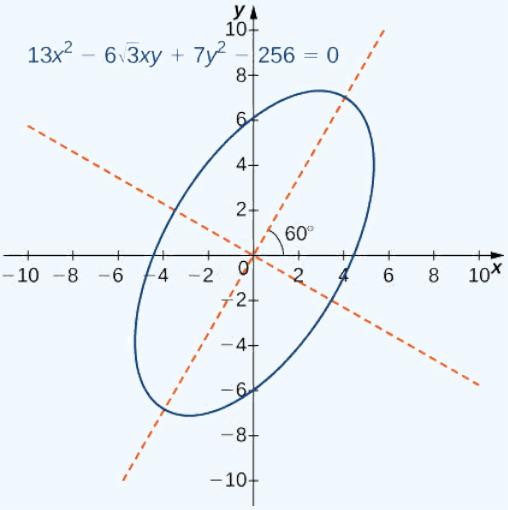


Figure 10.5.15: Graph of the ellipse described by the equation $13x^2 - 6\sqrt{3}xy + 7y^2 - 256 = 0$. The axes are rotated 60°. The red dashed lines indicate the rotated axes.

? Exercise 10.5.6

Identify the conic and calculate the angle of rotation of axes for the curve described by the equation

$$3x^2 + 5xy - 2y^2 - 125 = 0.$$

Hint

Follow steps 1 and 2 of the five-step method outlined above

Answer

The conic is a hyperbola and the angle of rotation of the axes is $\theta = 22.5^\circ$.

Key Concepts

- The equation of a vertical parabola in standard form with given focus and directrix is $y = \frac{1}{4p}(x-h)^2 + k$ where p is the distance from the vertex to the focus and (h, k) are the coordinates of the vertex.
- The equation of a horizontal ellipse in standard form is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ where the center has coordinates (h, k), the major axis has length $2a$, the minor axis has length $2b$, and the coordinates of the foci are $(h \pm c, k)$, where $c^2 = a^2 - b^2$.
- The equation of a horizontal hyperbola in standard form is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ where the center has coordinates (h, k), the vertices are located at $(h \pm a, k)$, and the coordinates of the foci are $(h \pm c, k)$, where $c^2 = a^2 + b^2$.
- The eccentricity of an ellipse is less than 1, the eccentricity of a parabola is equal to 1, and the eccentricity of a hyperbola is greater than 1. The eccentricity of a circle is 0.
- The polar equation of a conic section with eccentricity e is $r = \frac{ep}{1 \pm e\cos\theta}$ or $r = \frac{ep}{1 \pm e\sin\theta}$, where p represents the focal parameter.
- To identify a conic generated by the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$,first calculate the discriminant $D = 4AC - B^2$. If $D > 0$ then the conic is an ellipse, if $D = 0$ then the conic is a parabola, and if $D < 0$ then the conic is a hyperbola.

Glossary

conic section

a conic section is any curve formed by the intersection of a plane with a cone of two nappes

directrix

a directrix (plural: directrices) is a line used to construct and define a conic section; a parabola has one directrix; ellipses and hyperbolas have two

discriminant

the value $4AC - B^2$, which is used to identify a conic when the equation contains a term involving xy, is called a discriminant

focus

a focus (plural: foci) is a point used to construct and define a conic section; a parabola has one focus; an ellipse and a hyperbola have two

eccentricity

the eccentricity is defined as the distance from any point on the conic section to its focus divided by the perpendicular distance from that point to the nearest directrix

focal parameter

the focal parameter is the distance from a focus of a conic section to the nearest directrix

general form

an equation of a conic section written as a general second-degree equation

major axis

the major axis of a conic section passes through the vertex in the case of a parabola or through the two vertices in the case of an ellipse or hyperbola; it is also an axis of symmetry of the conic; also called the transverse axis

minor axis

the minor axis is perpendicular to the major axis and intersects the major axis at the center of the conic, or at the vertex in the case of the parabola; also called the conjugate axis

nappe

a nappe is one half of a double cone

standard form

an equation of a conic section showing its properties, such as location of the vertex or lengths of major and minor axes

vertex

a vertex is an extreme point on a conic section; a parabola has one vertex at its turning point. An ellipse has two vertices, one at each end of the major axis; a hyperbola has two vertices, one at the turning point of each branch

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10.6: Conic Sections in Polar Coordinates

Learning Objectives

- Identify a conic in polar form.
- Graph the polar equations of conics.
- Define conics in terms of a focus and a directrix.

Most of us are familiar with orbital motion, such as the motion of a planet around the sun or an electron around an atomic nucleus. Within the planetary system, orbits of planets, asteroids, and comets around a larger celestial body are often elliptical. Comets, however, may take on a parabolic or hyperbolic orbit instead. And, in reality, the characteristics of the planets' orbits may vary over time. Each orbit is tied to the location of the celestial body being orbited and the distance and direction of the planet or other object from that body. As a result, we tend to use polar coordinates to represent these orbits.

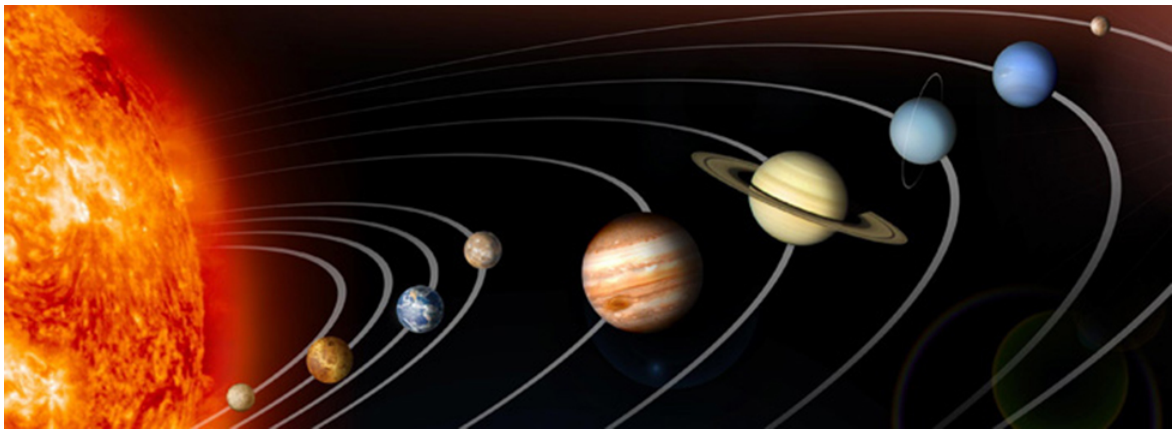


Figure 10.6.1: Planets orbiting the sun follow elliptical paths. (credit: NASA Blueshift, Flickr)

In an elliptical orbit, the **periapsis** is the point at which the two objects are closest, and the **apoapsis** is the point at which they are farthest apart. Generally, the velocity of the orbiting body tends to increase as it approaches the periapsis and decrease as it approaches the apoapsis. Some objects reach an escape velocity, which results in an infinite orbit. These bodies exhibit either a parabolic or a hyperbolic orbit about a body; the orbiting body breaks free of the celestial body's gravitational pull and fires off into space. Each of these orbits can be modeled by a conic section in the polar coordinate system.

Identifying a Conic in Polar Form

Any conic may be determined by three characteristics: a single **focus**, a fixed line called the **directrix**, and the ratio of the distances of each to a point on the graph. Consider the **parabola** $x = 2 + y^2$ shown in Figure $10.6.2$

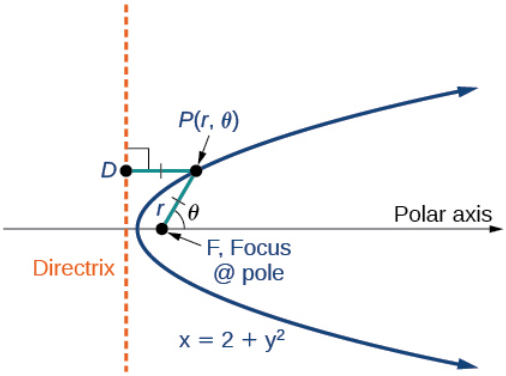


Figure 10.6.2

We previously learned how a parabola is defined by the focus (a fixed point) and the directrix (a fixed line). In this section, we will learn how to define any conic in the polar coordinate system in terms of a fixed point, the focus $P(r, \theta)$ at the pole, and a line, the directrix, which is perpendicular to the polar axis.

If F is a fixed point, the focus, and D is a fixed line, the directrix, then we can let e be a fixed positive number, called the **eccentricity**, which we can define as the ratio of the distances from a point on the graph to the focus and the point on the graph to the directrix. Then the set of all points P such that $e = \dfrac{PF}{PD}$ is a conic. In other words, we can define a conic as the set of all points P with the property that the ratio of the distance from P to F to the distance from P to D is equal to the constant e.

For a conic with eccentricity e,

- if $0 \le e < 1$, the conic is an ellipse
- if $e = 1$, the conic is a parabola
- if $e > 1$, the conic is an hyperbola

With this definition, we may now define a conic in terms of the directrix, $x = \pm p$, the eccentricity e, and the angle θ. Thus, each conic may be written as a **polar equation**, an equation written in terms of r and θ.

THE POLAR EQUATION FOR A CONIC

For a conic with a focus at the origin, if the directrix is $x = \pm p$, where p is a positive real number, and the eccentricity is a positive real number e, the conic has a polar equation

$$r = \frac{ep}{1 \pm e\cos\theta} \tag{10.6.1}$$

For a conic with a focus at the origin, if the directrix is $y = \pm p$, where p is a positive real number, and the eccentricity is a positive real number e, the conic has a polar equation

$$r = \frac{ep}{1 \pm e\sin\theta} \tag{10.6.2}$$

How to: Given the polar equation for a conic, identify the type of conic, the directrix, and the eccentricity.

1. Multiply the numerator and denominator by the reciprocal of the constant in the denominator to rewrite the equation in standard form.
2. Identify the eccentricity e as the coefficient of the trigonometric function in the denominator.
3. Compare e with 1 to determine the shape of the conic.
4. Determine the directrix as $x = p$ if cosine is in the denominator and $y = p$ if sine is in the denominator. Set ep equal to the numerator in standard form to solve for x or y.

Example 10.6.1: Identifying a Conic Given the Polar Form

For each of the following equations, identify the conic with focus at the origin, the directrix, and the eccentricity.

a. $r = \dfrac{6}{3 + 2\sin\theta}$

b. $r = \dfrac{12}{4 + 5\cos\theta}$

c. $r = \dfrac{7}{2 - 2\sin\theta}$

Solution

For each of the three conics, we will rewrite the equation in standard form. Standard form has a 1 as the constant in the denominator. Therefore, in all three parts, the first step will be to multiply the numerator and denominator by the reciprocal of the constant of the original equation, $\dfrac{1}{c}$, where c is that constant.

a. Multiply the numerator and denominator by $\dfrac{1}{3}$.

$$r=\frac{6}{3+2\sin\theta}\cdot\frac{\left(\frac{1}{3}\right)}{\left(\frac{1}{3}\right)}=\frac{6\left(\frac{1}{3}\right)}{3\left(\frac{1}{3}\right)+2\left(\frac{1}{3}\right)\sin\theta}=\frac{2}{1+\frac{2}{3}\sin\theta}$$

Because $\sin\theta$ is in the denominator, the directrix is $y=p$. Comparing to standard form, note that $e=\frac{2}{3}$ .Therefore, from the numerator,

$$\begin{aligned}2&=ep\\2&=\frac{2}{3}p\\\left(\frac{3}{2}\right)2&=\left(\frac{3}{2}\right)\frac{2}{3}p\\3&=p\end{aligned}$$

Since $e<1$, the conic is an **ellipse**. The eccentricity is $e=\frac{2}{3}$ and the directrix is $y=3$.

b. Multiply the numerator and denominator by $\frac{1}{4}$.

$$\begin{aligned}r&=\frac{12}{4+5\cos\theta}\cdot\frac{\left(\frac{1}{4}\right)}{\left(\frac{1}{4}\right)}\\r&=\frac{12\left(\frac{1}{4}\right)}{4\left(\frac{1}{4}\right)+5\left(\frac{1}{4}\right)\cos\theta}\\r&=\frac{3}{1+\frac{5}{4}\cos\theta}\end{aligned}$$

Because $\cos\theta$ is in the denominator, the directrix is $x=p$. Comparing to standard form, $e=\frac{5}{4}$. Therefore, from the numerator,

$$\begin{aligned}3&=ep\\3&=\frac{5}{4}p\\\left(\frac{4}{5}\right)3&=\left(\frac{4}{5}\right)\frac{5}{4}p\\\frac{12}{5}&=p\end{aligned}$$

Since $e>1$, the conic is a **hyperbola**. The eccentricity is $e=\frac{5}{4}$ and the directrix is $x=\frac{12}{5}=2.4$.

c. Multiply the numerator and denominator by $\frac{1}{2}$.

$$r=\frac{7}{2-2\sin\theta}\cdot\frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)}$$

$$r=\frac{7\left(\frac{1}{2}\right)}{2\left(\frac{1}{2}\right)-2\left(\frac{1}{2}\right)\sin\theta}$$

$$r=\frac{\frac{7}{2}}{1-\sin\theta}$$

Because sine is in the denominator, the directrix is $y=-p$. Comparing to standard form, $e=1$. Therefore, from the numerator,

$$\frac{7}{2}=ep$$
$$\frac{7}{2}=(1)p$$
$$\frac{7}{2}=p$$

Because $e=1$, the conic is a parabola. The eccentricity is $e=1$ and the directrix is $y=-\frac{7}{2}=-3.5$.

? Exercise 10.6.1

Identify the conic with focus at the origin, the directrix, and the eccentricity for $r=\frac{2}{3-\cos\theta}$.

Answer

ellipse; $e=\frac{1}{3}$; $x=-2$

Graphing the Polar Equations of Conics

When graphing in Cartesian coordinates, each conic section has a unique equation. This is not the case when graphing in polar coordinates. We must use the eccentricity of a conic section to determine which type of curve to graph, and then determine its specific characteristics. The first step is to rewrite the conic in standard form as we have done in the previous example. In other words, we need to rewrite the equation so that the denominator begins with 1. This enables us to determine e and, therefore, the shape of the curve. The next step is to substitute values for θ and solve for r to plot a few key points. Setting θ equal to 0, $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$ provides the vertices so we can create a rough sketch of the graph.

✓ Example 10.6.2A: Graphing a Parabola in Polar Form

Graph $r=\frac{5}{3+3\cos\theta}$.

Solution

First, we rewrite the conic in standard form by multiplying the numerator and denominator by the reciprocal of 3, which is $\frac{1}{3}$.

$$r=\frac{5}{3+3\cos\theta}=\frac{5\left(\frac{1}{3}\right)}{3\left(\frac{1}{3}\right)+3\left(\frac{1}{3}\right)\cos\theta}$$

$$r=\frac{\frac{5}{3}}{1+\cos\theta}$$

Because $e=1$,we will graph a **parabola** with a focus at the origin. The function has a $\cos\theta$, and there is an addition sign in the denominator, so the directrix is $x=p$.

$$\frac{5}{3}=ep$$
$$\frac{5}{3}=(1)p$$
$$\frac{5}{3}=p$$

The directrix is $x=\frac{5}{3}$.

Plotting a few key points as in Table 10.6.1 will enable us to see the vertices. See Figure 10.6.3

Table 10.6.1

	A	B	C	D
θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r=\frac{5}{3+3\cos\theta}$	$\frac{5}{6}\approx 0.83$	$\frac{5}{3}\approx 1.67$	undefined	$\frac{5}{3}\approx 1.67$

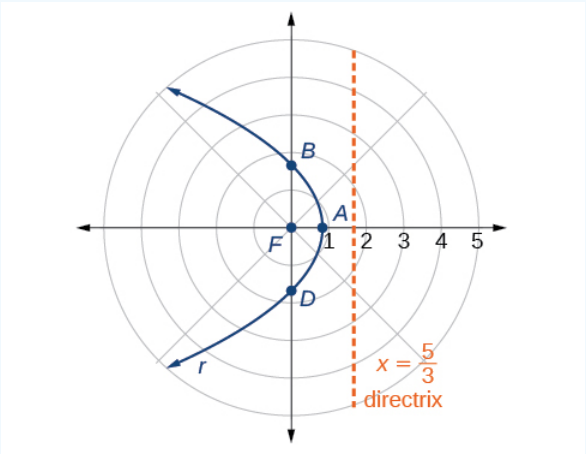


Figure 10.6.3

We can check our result with a graphing utility. See Figure 10.6.4

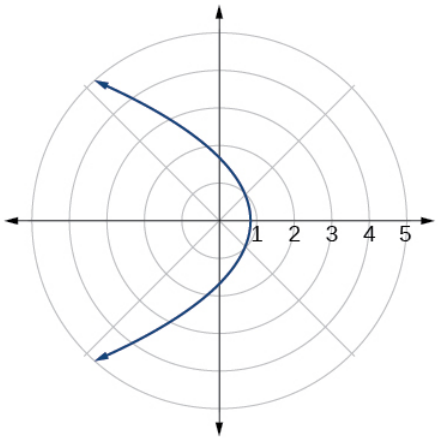


Figure 10.6.4

Example $10.6.2B$: Graphing a Hyperbola in Polar Form

Graph $r = \dfrac{8}{2-3\sin\theta}$.

Solution

First, we rewrite the conic in standard form by multiplying the numerator and denominator by the reciprocal of 2, which is $\frac{1}{2}$.

$$r = \frac{8}{2-3\sin\theta} = \frac{8\left(\frac{1}{2}\right)}{2\left(\frac{1}{2}\right)-3\left(\frac{1}{2}\right)\sin\theta}$$

$$r = \frac{4}{1-\frac{3}{2}\sin\theta}$$

Because $e = \frac{3}{2}$, $e > 1$, so we will graph a hyperbola with a focus at the origin. The function has a $\sin\theta$ term and there is a subtraction sign in the denominator, so the directrix is $y = -p$.

$$4 = ep$$
$$4 = \left(\frac{3}{2}\right)p$$
$$4\left(\frac{2}{3}\right) = p$$
$$\frac{8}{3} = p$$

The directrix is $y = -\frac{8}{3}$.

Plotting a few key points as in Table $10.6.2$ will enable us to see the vertices. See Figure $10.6.5$.

Table 10.6.2

	A	B	C	D
θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$r = \frac{8}{2-3\sin\theta}$	4	-8	4	$\frac{8}{5} = 1.6$

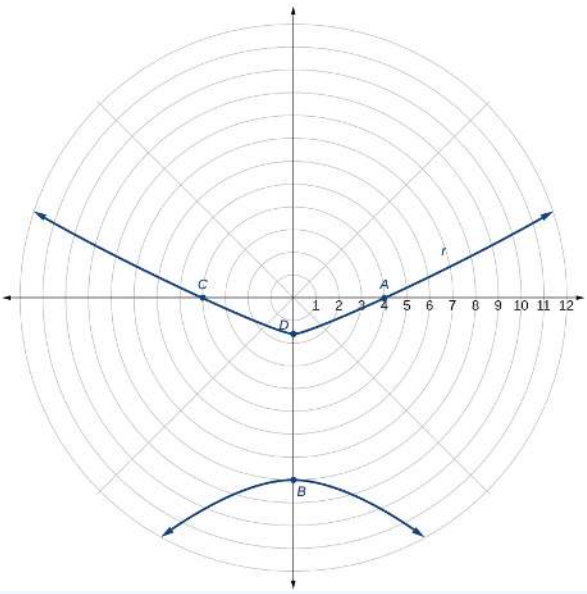


Figure 10.6.5

Example 10.6.2C: Graphing an Ellipse in Polar Form

Graph $r = \dfrac{10}{5 - 4\cos\theta}$.

Solution

First, we rewrite the conic in standard form by multiplying the numerator and denominator by the reciprocal of 5, which is $\frac{1}{5}$.

$$r = \frac{10}{5 - 4\cos\theta} = \frac{10\left(\frac{1}{5}\right)}{5\left(\frac{1}{5}\right) - 4\left(\frac{1}{5}\right)\cos\theta}$$

$$r = \frac{2}{1 - \frac{4}{5}\cos\theta}$$

Because $e = \frac{4}{5}$, $e < 1$, so we will graph an **ellipse** with a **focus** at the origin. The function has a $\cos\theta$, and there is a subtraction sign in the denominator, so the **directrix** is $x = -p$.

$$2 = ep$$
$$2 = \left(\frac{4}{5}\right)p$$
$$2\left(\frac{5}{4}\right) = p$$
$$\frac{5}{2} = p$$

The directrix is $x = -\frac{5}{2}$.

Plotting a few key points as in Table 10.6.3 will enable us to see the vertices. See Figure 10.6.6

Table 10.6.3

	A	B	C	D
θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$

	A	B	C	D
$r = \dfrac{10}{5 - 4\cos\theta}$	10	2	$\dfrac{10}{9} \approx 1.1$	2

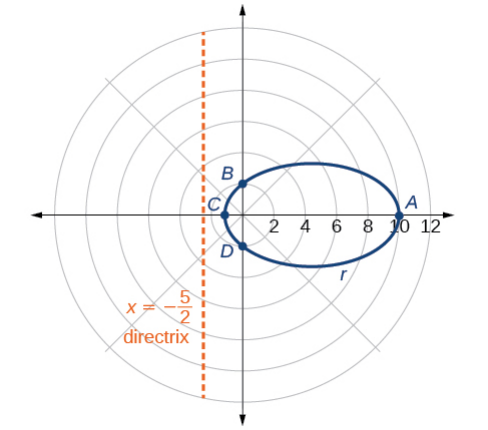


Figure 10.6.6

Analysis

We can check our result using a graphing utility. See Figure 10.6.7.

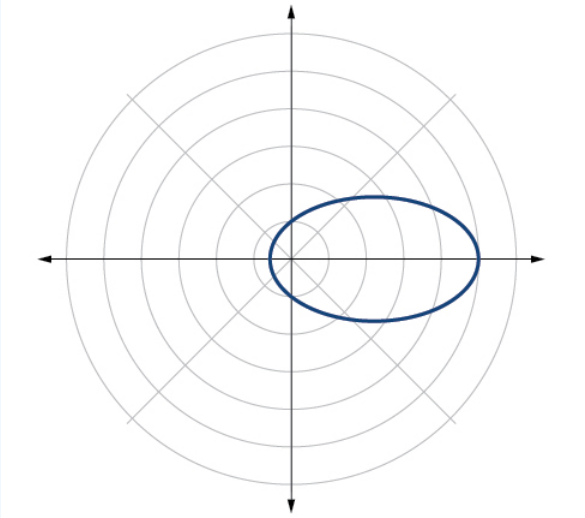


Figure 10.6.6: $r = \dfrac{10}{5 - 4\cos\theta}$ graphed on a viewing window of $[-3, 12, 1]$ by $[-4, 4, 1]$, $\theta min = 0$ and $\theta max = 2\pi$.

? Exercise 10.6.2

Graph $r = \dfrac{2}{4 - \cos\theta}$.

Answer

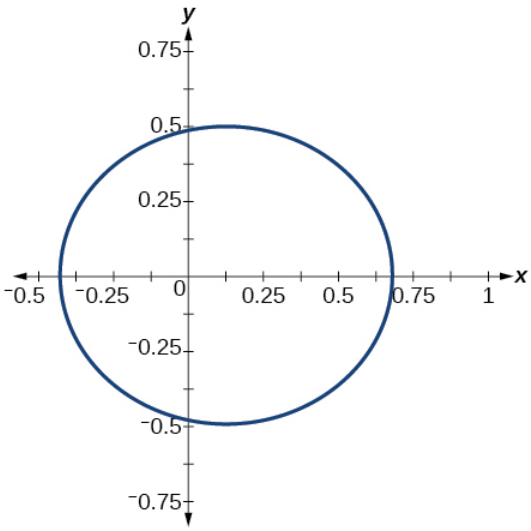


Figure 10.6.7

Defining Conics in Terms of a Focus and a Directrix

So far we have been using polar equations of conics to describe and graph the curve. Now we will work in reverse; we will use information about the origin, eccentricity, and directrix to determine the polar equation.

How to: Given the focus, eccentricity, and directrix of a conic, determine the polar equation

1. Determine whether the directrix is horizontal or vertical. If the directrix is given in terms of y, we use the general polar form in terms of sine. If the directrix is given in terms of x, we use the general polar form in terms of cosine.
2. Determine the sign in the denominator. If $p < 0$, use subtraction. If $p > 0$, use addition.
3. Write the coefficient of the trigonometric function as the given eccentricity.
4. Write the absolute value of p in the numerator, and simplify the equation.

Example $10.6.3A$: Finding the Polar Form of a Vertical Conic Given a Focus at the Origin and the Eccentricity and Directrix

Find the polar form of the conic given a focus at the origin, $e = 3$ and directrix $y = -2$.

Solution

The directrix is $y = -p$, so we know the trigonometric function in the denominator is sine.

Because $y = -2, -2 < 0$, so we know there is a subtraction sign in the denominator. We use the standard form of

$$r = \frac{ep}{1 - e\sin\theta}$$

and $e = 3$ and $|-2| = 2 = p$.

Therefore,

$$r = \frac{(3)(2)}{1 - 3\sin\theta}$$
$$r = \frac{6}{1 - 3\sin\theta}$$

Example $10.6.3B$: Finding the Polar Form of a Horizontal Conic Given a Focus at the Origin and the Eccentricity and Directrix

Find the polar form of a conic given a focus at the origin, $e = \frac{3}{5}$, and directrix $x = 4$.

Solution

Because the directrix is $x = p$, we know the function in the denominator is cosine. Because $x = 4$, $4 > 0$, so we know there is an addition sign in the denominator. We use the standard form of

$$r = \frac{ep}{1 + e\cos\theta}$$

and $e = \frac{3}{5}$ and $|4| = 4 = p$.

Therefore,

$$r = \frac{\left(\frac{3}{5}\right)(4)}{1 + \frac{3}{5}\cos\theta}$$

$$r = \frac{\frac{12}{5}}{1 + \frac{3}{5}\cos\theta}$$

$$r = \frac{\frac{12}{5}}{1\left(\frac{5}{5}\right) + \frac{3}{5}\cos\theta}$$

$$r = \frac{\frac{12}{5}}{\frac{5}{5} + \frac{3}{5}\cos\theta}$$

$$r = \frac{12}{5} \cdot \frac{5}{5 + 3\cos\theta}$$

$$r = \frac{12}{5 + 3\cos\theta}$$

Exercise 10.6.3

Find the polar form of the conic given a focus at the origin, $e = 1$, and directrix $x = -1$.

Answer

$$r = \frac{1}{1 - \cos\theta}$$

Example 10.6.4: Converting a Conic in Polar Form to Rectangular Form

Convert the conic $r = \frac{1}{5 - 5\sin\theta}$ to rectangular form.

Solution

We will rearrange the formula to use the identities $r = \sqrt{x^2 + y^2}$, $x = r\cos\theta$,and $y = r\sin\theta$.

$$
\begin{aligned}
r &= \frac{1}{5-5\sin\theta} \\
r\cdot(5-5\sin\theta) &= \frac{1}{5-5\sin\theta}\cdot(5-5\sin\theta) && \text{Eliminate the fraction.} \\
5r-5r\sin\theta &= 1 && \text{Distribute.} \\
5r &= 1+5r\sin\theta && \text{Isolate } 5r. \\
25r^2 &= (1+5r\sin\theta)^2 && \text{Square both sides.} \\
25(x^2+y^2) &= (1+5y)^2 && \text{Substitute } r=\sqrt{x^2+y^2} \text{ and } y=r\sin\theta. \\
25x^2+25y^2 &= 1+10y+25y^2 && \text{Distribute and use FOIL.} \\
25x^2-10y &= 1 && \text{Rearrange terms and set equal to 1.}
\end{aligned}
$$

Exercise 10.6.4

Convert the conic $r=\dfrac{2}{1+2\cos\theta}$ to rectangular form.

Answer

$4-8x+3x^2-y^2=0$

Media

Access these online resources for additional instruction and practice with conics in polar coordinates.

- Polar Equations of Conic Sections
- Graphing Polar Equations of Conics - 1
- Graphing Polar Equations of Conics - 2

Visit this website for additional practice questions from Learningpod.

Key Concepts

- Any conic may be determined by a single focus, the corresponding eccentricity, and the directrix. We can also define a conic in terms of a fixed point, the focus $P(r,\theta)$ at the pole, and a line, the directrix, which is perpendicular to the polar axis.
- A conic is the set of all points $e=\dfrac{PF}{PD}$, where eccentricity e is a positive real number. Each conic may be written in terms of its polar equation. See Example 10.6.1.
- The polar equations of conics can be graphed. See Example 10.6.2, Example 10.6.3, and Example 10.6.4.
- Conics can be defined in terms of a focus, a directrix, and eccentricity. See Example 10.6.5 and Example 10.6.6.
- We can use the identities $r=\sqrt{x^2+y^2}$, $x=r\cos\theta$, and $y=r\sin\theta$ to convert the equation for a conic from polar to rectangular form. See Example 10.6.7.

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CHAPTER OVERVIEW

11: Infinite Sequences And Series

A general Calculus Textmap organized around the textbook

Calculus: Early Transcendentals

by James Stewart

I II III IV V VI VII VIII IX X XI XII XIII XIV XV XVI XVII

This Textmap is currently under construction... please be patient with us.

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11.2: Series
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11.4: The Comparison Tests
11.5: Alternating Series
11.6: Absolute Convergence and the Ratio and Root Test
11.7: Strategy for Testing Series
11.8: Power Series
11.9: Representations of Functions as Power Series
11.10: Taylor and Maclaurin Series
11.11: Applications of Taylor Polynomials

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11.1: Sequences

While the idea of a sequence of numbers, $a_1, a_2, a_3, \ldots$ is straightforward, it is useful to think of a sequence as a function. We have up until now dealt with functions whose domains are the real numbers, or a subset of the real numbers, like $f(x) = \sin x$. A sequence is a function with domain the natural numbers $N = \{1, 2, 3, \ldots\}$ or the non-negative integers, $Z^{\geq 0} = \{0, 1, 2, 3, \ldots\}$. The range of the function is still allowed to be the real numbers; in symbols, we say that a sequence is a function $f\colon N \to R$. Sequences are written in a few different ways, all equivalent; these all mean the same thing:

$$a_1, a_2, a_3, \ldots$$
$$\{a_n\}_{n=1}^{\infty}$$
$$\{f(n)\}_{n=1}^{\infty}$$

As with functions on the real numbers, we will most often encounter sequences that can be expressed by a formula. We have already seen the sequence $a_i = f(i) = 1 - 1/2^i$, and others are easy to come by:

$$f(i) = \frac{i}{i+1}$$
$$f(n) = \frac{1}{2^n}$$
$$f(n) = \sin(n\pi/6)$$
$$f(i) = \frac{(i-1)(i+2)}{2^i}.$$

Frequently these formulas will make sense if thought of either as functions with domain $\mathcal{R}$ or $\mathcal{N}$, though occasionally one will make sense only for integer values.

Faced with a sequence we are interested in the limit $\lim_{i\to\infty} f(i) = \lim_{i\to\infty} a_i$. We already understand $\lim_{x\to\infty} f(x)$ when x is a real valued variable; now we simply want to restrict the "input" values to be integers. No real difference is required in the definition of limit, except that we specify, perhaps implicitly, that the variable is an integer. Compare this definition to definition 4.10.2.

Definition 11.1.1: Converging and Diverging Sequences

Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence. We say that $\lim_{n\to\infty} a_n = L$ if for every $\epsilon > 0$ there is an $N > 0$ so that whenever $n > N$, $|a_n - L| < \epsilon$. If $\lim_{n\to\infty} a_n = L$ we say that the sequence *converges*, otherwise it *diverges*.

If $f(i)$ defines a sequence, and $f(x)$ makes sense, and $\lim\limits_{x\to\infty} f(x) = L$, then it is clear that $\lim_{i\to\infty} f(i) = L$ as well, but it is important to note that the converse of this statement is not true. For example, since $\lim_{x\to\infty}(1/x) = 0$, it is clear that also $\lim_{i\to\infty}(1/i) = 0$, that is, the numbers

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$$

get closer and closer to 0. Consider this, however: Let $f(n) = \sin(n\pi)$.

This is the sequence

$$\sin(0\pi), \sin(1\pi), \sin(2\pi), \sin(3\pi), \ldots = 0, 0, 0, 0, \ldots$$

since

$$\sin(n\pi) = 0$$

when n is an integer. Thus $\lim_{n\to\infty} f(n) = 0$. But $\lim_{x\to\infty} f(x)$, when x is real, does not exist: as x gets bigger and bigger, the values $\sin(x\pi)$ do not get closer and closer to a single value, but take on all values between -1 and 1 over and over. In general, whenever you want to know $\lim_{n\to\infty} f(n)$ you should first attempt to compute $\lim_{x\to\infty} f(x)$, since if the latter exists it is also equal to the first limit. But if for some reason $\lim_{x\to\infty} f(x)$ does not exist, it may still be true that $\lim_{n\to\infty} f(n)$ exists, but you'll have to figure out another way to compute it.

It is occasionally useful to think of the graph of a sequence. Since the function is defined only for integer values, the graph is just a sequence of dots. In figure 11.1.1 we see the graphs of two sequences and the graphs of the corresponding real functions.

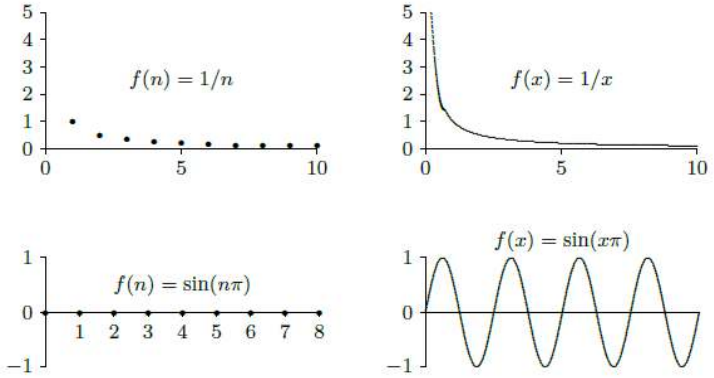


Figure 11.1.1. *Graphs of sequences and their corresponding real functions.*

Not surprisingly, the properties of limits of real functions translate into properties of sequences quite easily. Theorem 2.3.6 about limits becomes

Definition 11.1.2

Suppose that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$ and k is some constant. Then

$$\lim_{n\to\infty} ka_n = k \lim_{n\to\infty} a_n = kL$$
$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = L + M$$
$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = L - M$$
$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n = LM$$
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{L}{M}, \text{ if } M \text{ is not } 0.$$

Likewise the **Squeeze Theorem** (4.3.1) becomes

Theorem 11.1.3

Suppose that

$$a_n \le b_n \le c_n$$

for all $n > N$, for some N. If

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L,$$

then

$$\lim_{n\to\infty} b_n = L.$$

And a final useful fact:

Theorem 11.1.4

$$\lim_{n\to\infty} |a_n| = 0$$

if and only if

$$\lim_{n\to\infty} a_n = 0.$$

This theorem says simply that the size of a_n gets close to zero if and only if a_n gets close to zero.

Example 11.1.5

Determine whether $\left\{\frac{n}{n+1}\right\}_{n=0}^{\infty}$ converges or diverges. If it converges, compute the limit.

Solution

Since this makes sense for real numbers we consider

$$\lim_{x\to\infty} \frac{x}{x+1} = \lim_{x\to\infty} 1 - \frac{1}{x+1} = 1 - 0 = 1.$$

Thus the sequence converges to 1.

Example 11.1.6

Determine whether $\left\{\frac{\ln n}{n}\right\}_{n=1}^{\infty}$ converges or diverges. If it converges, compute the limit.

Solution

We compute $\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0,$ using L'Hôpital's Rule. Thus the sequence converges to 0.

Example 11.1.7

Determine whether $\{(-1)^n\}_{n=0}^{\infty}$ converges or diverges. If it converges, compute the limit.

Solution

This does not make sense for all real exponents, but the sequence is easy to understand: it is $1, -1, 1, -1, 1 \ldots$ and clearly diverges.

Example 11.1.8

Determine whether $\{(-1/2)^n\}_{n=0}^{\infty}$ converges or diverges. If it converges, compute the limit.

Solution

We consider the sequence

$$\{|(-1/2)^n|\}_{n=0}^{\infty} = \{(1/2)^n\}_{n=0}^{\infty}.$$

Then

$$\lim_{x\to\infty} \left(\frac{1}{2}\right)^x = \lim_{x\to\infty} \frac{1}{2^x} = 0,$$

so by theorem 11.1.4 the sequence converges to 0.

Example 11.1.9

Determine whether $\{(\sin n)/\sqrt{n}\}_{n=1}^{\infty}$ converges or diverges. If it converges, compute the limit.

Solution

Since $|\sin n| \le 1,\ 0 \le |\sin n/\sqrt{n}| \le 1/\sqrt{n}$ and we can use theorem 11.1.3 with $a_n = 0$ and $c_n = 1/\sqrt{n}$. Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = 0$, $\lim_{n\to\infty} \sin n/\sqrt{n} = 0$ and the sequence converges to 0.

Example 11.1.10

A particularly common and useful sequence is $\{r^n\}_{n=0}^{\infty}$, for various values of r. Some are quite easy to understand: If $r=1$ the sequence converges to 1 since every term is 1, and likewise if $r=0$ the sequence converges to 0. If $r=-1$ this is the sequence of example 11.1.7 and diverges. If $r>1$ or $r<-1$ the terms r^n get large without limit, so the sequence diverges. If $0<r<1$ then the sequence converges to 0. If $-1<r<0$ then $|r^n|=|r|^n$ and $0<|r|<1$, so the sequence $\{|r|^n\}_{n=0}^{\infty}$ converges to 0, so also $\{r^n\}_{n=0}^{\infty}$ converges to 0. converges. In summary, $\{r^n\}$ converges precisely when $-1<r\leq 1$ in which case $\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1<r<1 \\ 1 & \text{if } r=1. \end{cases}$

Sometimes we will not be able to determine the limit of a sequence, but we still would like to know whether it converges. In some cases we can determine this even without being able to compute the limit.

A sequence is called **increasing** or sometimes **strictly increasing** if $a_i < a_{i+1}$ for all i. It is called **non-decreasing** or sometimes (unfortunately) **increasing** if $a_i \leq a_{i+1}$ for all i. Similarly a sequence is **decreasing** if $a_i > a_{i+1}$ for all i and **non-increasing** if $a_i \geq a_{i+1}$ for all i. If a sequence has any of these properties it is called **monotonic**.

Example 11.1.11

The sequence

$$\left\{\frac{2^i-1}{2^i}\right\}_{i=1}^{\infty} = \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots$$

is increasing,

and

$$\left\{\frac{n+1}{n}\right\}_{i=1}^{\infty} = \frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$$

is decreasing.

A sequence is **bounded above** if there is some number N such that $a_n \leq N$ for every n, and **bounded below** if there is some number N such that $a_n \geq N$ for every n. If a sequence is bounded above and bounded below it is **bounded**. If a sequence $\{a_n\}_{n=0}^{\infty}$ is increasing or non-decreasing it is bounded below (by a_0), and if it is decreasing or non-increasing it is bounded above (by a_0). Finally, with all this new terminology we can state an important theorem.

Theorem 11.1.12

If a sequence is bounded and monotonic, then it converges.

We will not prove this; the proof appears in many calculus books. It is not hard to believe: suppose that a sequence is increasing and bounded, so each term is larger than the one before, yet never larger than some fixed value N. The terms must then get closer and closer to some value between a_0 and N. It need not be N, since N may be a "too-generous" upper bound; the limit will be the smallest number that is above all of the terms a_i.

Example 11.1.13

All of the terms $(2^i-1)/2^i$ are less than 2, and the sequence is increasing. As we have seen, the limit of the sequence is 1---1 is the smallest number that is bigger than all the terms in the sequence. Similarly, all of the terms $(n+1)/n$ are bigger than $1/2$, and the limit is 1---1 is the largest number that is smaller than the terms of the sequence.

We do not actually need to know that a sequence is monotonic to apply this theorem---it is enough to know that the sequence is "eventually" monotonic, that is, that at some point it becomes increasing or decreasing. For example, the sequence $10, 9, 8, 15, 3, 21, 4, 3/4, 7/8, 15/16, 31/32, \ldots$ is not increasing, because among the first few terms it is not. But starting with the term $3/4$ it is

increasing, so the theorem tells us that the sequence $3/4, 7/8, 15/16, 31/32, \ldots$ converges. Since convergence depends only on what happens as n gets large, adding a few terms at the beginning can't turn a convergent sequence into a divergent one.

Example 11.1.14

Show that $\{n^{1/n}\}$ converges.

Solution

We first show that this sequence is decreasing, that is, that $n^{1/n} > (n+1)^{1/(n+1)}$. Consider the real function $f(x) = x^{1/x}$ when $x \geq 1$. We can compute the derivative, $f'(x) = x^{1/x}(1 - \ln x)/x^2$, and note that when $x \geq 3$ this is negative. Since the function has negative slope, $n^{1/n} > (n+1)^{1/(n+1)}$ when $n \geq 3$. Since all terms of the sequence are positive, the sequence is decreasing and bounded when $n \geq 3$, and so the sequence converges. (As it happens, we can compute the limit in this case, but we know it converges even without knowing the limit; see exercise 1.)

Example 11.1.15

Show that $\{n!/n^n\}$ converges.

Solution

Again we show that the sequence is decreasing, and since each term is positive the sequence converges. We can't take the derivative this time, as $x!$ doesn't make sense for x real. But we note that if $a_{n+1}/a_n < 1$ then $a_{n+1} < a_n$, which is what we want to know. So we look at

$$a_{n+1}/a_n : \frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} = \frac{(n+1)!}{n!} \frac{n^n}{(n+1)^{n+1}} = \frac{n+1}{n+1} \left(\frac{n}{n+1}\right)^n = \left(\frac{n}{n+1}\right)^n < 1.$$

(Again it is possible to compute the limit; see exercise 2.)

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11.2: Series

While much more can be said about sequences, we now turn to our principal interest, series. Recall that a series, roughly speaking, is the sum of a sequence: if $\{a_n\}_{n=0}^{\infty}$ is a sequence then the associated series is

$$\sum_{i=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots$$

Associated with a series is a second sequence, called the **sequence of partial sums**:

$$\{s_n\}_{n=0}^{\infty}$$

with

$$s_n = \sum_{i=0}^{n} a_i.$$

So

$$s_0 = a_0, \quad s_1 = a_0 + a_1, \quad s_2 = a_0 + a_1 + a_2, \quad \ldots$$

A series converges if the sequence of partial sums converges, and otherwise the series diverges.

Example 11.2.1: Geometric Series

If $a_n = kx^n$, then

$$\sum_{n=0}^{\infty} a_n$$

is called a **geometric series**. A typical partial sum is

$$s_n = k + kx + kx^2 + kx^3 + \cdots + kx^n = k(1 + x + x^2 + x^3 + \cdots + x^n).$$

We note that

$$\begin{aligned} s_n(1-x) &= k(1 + x + x^2 + x^3 + \cdots + x^n)(1-x) \\ &= k(1 + x + x^2 + x^3 + \cdots + x^n)1 - k(1 + x + x^2 + x^3 + \cdots + x^{n-1} + x^n)x \\ &= k(1 + x + x^2 + x^3 + \cdots + x^n - x - x^2 - x^3 - \cdots - x^n - x^{n+1}) \\ &= k(1 - x^{n+1}) \end{aligned}$$

so

$$\begin{aligned} s_n(1-x) &= k(1 - x^{n+1}) \\ s_n &= k\frac{1 - x^{n+1}}{1-x}. \end{aligned}$$

If $|x| < 1$, $\lim_{n\to\infty} x^n = 0$ so

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} k\frac{1 - x^{n+1}}{1-x} = k\frac{1}{1-x}.$$

Thus, when $|x| < 1$ the geometric series converges to $k/(1-x)$. When, for example, $k = 1$ and $x = 1/2$:

$$s_n = \frac{1 - (1/2)^{n+1}}{1 - 1/2} = \frac{2^{n+1} - 1}{2^n} = 2 - \frac{1}{2^n} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1 - 1/2} = 2.$$

We began the chapter with the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$, namely, the geometric series without the first term 1. Each partial sum of this series is 1 less than the corresponding partial sum for the geometric series, so of course the limit is also one less than the value of the geometric series, that is,

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

It is not hard to see that the following theorem follows from theorem 11.1.2.

Theorem 11.2.2

Suppose that $\sum a_n$ and $\sum b_n$ are convergent series, and c is a constant. Then

1. $\sum ca_n$ is convergent and $\sum ca_n = c\sum a_n$
2. $\sum(a_n + b_n)$ is convergent and $\sum(a_n + b_n) = \sum a_n + \sum b_n$.

The two parts of this theorem are subtly different. Suppose that $\sum a_n$ diverges; does $\sum ca_n$ also diverge if c is non-zero? Yes: suppose instead that $\sum ca_n$ converges; then by the theorem, $\sum(1/c)ca_n$ converges, but this is the same as $\sum a_n$, which by assumption diverges. Hence $\sum ca_n$ also diverges. Note that we are applying the theorem with a_n replaced by ca_n and c replaced by $(1/c)$.

Now suppose that $\sum a_n$ and $\sum b_n$ diverge; does $\sum(a_n + b_n)$ also diverge? Now the answer is no: Let $a_n = 1$ and $b_n = -1$, so certainly $\sum a_n$ and $\sum b_n$ diverge. But

$$\sum(a_n + b_n) = \sum(1 + -1) = \sum 0 = 0.$$

Of course, sometimes $\sum(a_n + b_n)$ will also diverge, for example, if $a_n = b_n = 1$, then

$$\sum(a_n + b_n) = \sum(1 + 1) = \sum 2$$

diverges.

In general, the sequence of partial sums s_n is harder to understand and analyze than the sequence of terms a_n, and it is difficult to determine whether series converge and if so to what. Sometimes things are relatively simple, starting with the following.

Theorem 11.2.3

If

$$\sum a_n$$

converges then

$$\lim_{n\to\infty} a_n = 0.$$

Proof.

Since $\sum a_n$ converges, $\lim_{n\to\infty} s_n = L$ and $\lim_{n\to\infty} s_{n-1} = L$, because this really says the same thing but "renumbers" the terms. By theorem 11.1.2,

$$\lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = L - L = 0.$$

But

$$s_n - s_{n-1} = (a_0 + a_1 + a_2 + \cdots + a_n) - (a_0 + a_1 + a_2 + \cdots + a_{n-1}) = a_n,$$

so as desired $\lim_{n\to\infty} a_n = 0$.

This theorem presents an easy divergence test: if given a series $\sum a_n$ the limit $\lim_{n\to\infty} a_n$ does not exist or has a value other than zero, the series diverges. Note well that the converse is *not* true: If $\lim_{n\to\infty} a_n = 0$ then the series does not necessarily converge.

Example 11.2.4

Show that

$$\sum_{n=1}^{\infty} \frac{n}{n+1}$$

diverges.

Solution

We compute the limit:

$$\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0.$$

Looking at the first few terms perhaps makes it clear that the series has no chance of converging:

$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\cdots$$

will just get larger and larger; indeed, after a bit longer the series starts to look very much like $\cdots+1+1+1+1+\cdots$, and of course if we add up enough 1's we can make the sum as large as we desire.

Example 11.2.5: Harmonic Series

Show that

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

Solution

Here the theorem does not apply: $\lim_{n\to\infty} 1/n = 0$, so it looks like perhaps the series converges. Indeed, if you have the fortitude (or the software) to add up the first 1000 terms you will find that

$$\sum_{n=1}^{1000} \frac{1}{n} \approx 7.49,$$

so it might be reasonable to speculate that the series converges to something in the neighborhood of 10. But in fact the partial sums do go to infinity; they just get big very, very slowly. Consider the following:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4} > 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4} = 1+\frac{1}{2}+\frac{1}{2}$$

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} > 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} = 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}$$

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{16} > 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{16} = 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}$$

and so on. By swallowing up more and more terms we can always manage to add at least another $1/2$ to the sum, and by adding enough of these we can make the partial sums as big as we like. In fact, it's not hard to see from this pattern that

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^n} > 1+\frac{n}{2},$$

so to make sure the sum is over 100, for example, we'd add up terms until we get to around $1/2^{198}$, that is, about $4\cdot 10^{59}$ terms. This series, $\sum(1/n)$, is called the **harmonic series**.

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11.3: The Integral Test and Estimates of Sums

It is generally quite difficult, often impossible, to determine the value of a series exactly. In many cases it is possible at least to determine whether or not the series converges, and so we will spend most of our time on this problem.

If all of the terms a_n in a series are non-negative, then clearly the sequence of partial sums s_n is non-decreasing. This means that if we can show that the sequence of partial sums is bounded, the series must converge. We know that if the series converges, the terms a_n approach zero, but this does not mean that $a_n \geq a_{n+1}$ for every n. Many useful and interesting series do have this property, however, and they are among the easiest to understand. Let's look at an example.

Example 11.3.1

Show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \tag{11.3.1}$$

converges.

Solution

The terms $1/n^2$ are positive and decreasing, and since $\lim_{x\to\infty} 1/x^2 = 0$, the terms $1/n^2$ approach zero. We seek an upper bound for all the partial sums, that is, we want to find a number N so that $s_n \leq N$ for every n. The upper bound is provided courtesy of integration, and is inherent in figure 11.3.1.

The figure shows the graph of $y = 1/x^2$ together with some rectangles that lie completely below the curve and that all have base length one. Because the heights of the rectangles are determined by the height of the curve, the areas of the rectangles are $1/1^2$, $1/2^2$, $1/3^2$, and so on---in other words, exactly the terms of the series. The partial sum s_n is simply the sum of the areas of the first n rectangles. Because the rectangles all lie between the curve and the x-axis, any sum of rectangle areas is less than the corresponding area under the curve, and so of course any sum of rectangle areas is less than the area under the entire curve, that is, all the way to infinity. There is a bit of trouble at the left end, where there is an asymptote, but we can work around that easily. Here it is:

$$s_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 1 + \int_1^n \frac{1}{x^2}\,dx < 1 + \int_1^{\infty} \frac{1}{x^2}\,dx = 1 + 1 = 2, \tag{11.3.2}$$

recalling that we computed this improper integral in section 9.7. Since the sequence of partial sums s_n is increasing and bounded above by 2, we know that $\lim_{n\to\infty} s_n = L < 2$, and so the series converges to some number less than 2. In fact, it is possible, though difficult, to show that $L = \pi^2/6 \approx 1.6$.

We already know that $\sum 1/n$ diverges. What goes wrong if we try to apply this technique to it? Here's the calculation:

$$s_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < 1 + \int_1^n \frac{1}{x}\,dx < 1 + \int_1^{\infty} \frac{1}{x}\,dx = 1 + \infty. \tag{11.3.3}$$

The problem is that the improper integral doesn't converge. Note well that this does *not* prove that $\sum 1/n$ diverges, just that this particular calculation fails to prove that it converges. A slight modification, however, allows us to prove in a second way that $\sum 1/n$ diverges.

Example

Consider a slightly altered version of figure 11.3.1, shown in figure 11.3.2.

Solution

The rectangles this time are above the curve, that is, each rectangle completely contains the corresponding area under the curve. This means that

[(s_n = {1\over 1}+{1\over 2}+{1\over 3}+\cdots+{1\over n} > \int_1^{n+1} {1\over x}\,dx = \ln x\Big|_1^{n+1}=\ln(n+1).\]

As n gets bigger, $\ln(n+1)$ goes to infinity, so the sequence of partial sums s_n must also go to infinity, so the harmonic series diverges.

The important fact that clinches this example is that $\lim_{n\to\infty}\int_1^{n+1}\frac{1}{x}\,dx=\infty$, which we can rewrite as $\int_1^{\infty}\frac{1}{x}\,dx=\infty$. So these two examples taken together indicate that we can prove that a series converges or prove that it diverges with a single calculation of an improper integral. This is known as the **integral test**, which we state as a theorem.

Theorem 11.3.3: The Integral Test

Suppose that $f(x)>0$ and is decreasing on the infinite interval $[k,\infty)$ (for some $k\geq 1$) and that $a_n=f(n)$. Then the series

$$\sum_{n=1}^{\infty}a_n \tag{11.3.4}$$

converges if and only if the improper integral

$$\int_1^{\infty}f(x)\,dx \tag{11.3.5}$$

converges.

The two examples we have seen are called p-series; a p-series is any series of the form $\sum 1/n^p$. If $p\leq 0$, $\lim_{n\to\infty}1/n^p\neq 0$, so the series diverges. For positive values of \)p\) we can determine precisely which series converge.

Theorem 11.3.4

A p-series with $p>0$ converges if and only if $p>1$.

Proof

We use the integral test; we have already done $p=1$, so assume that $p\neq 1$.

$$\int_1^{\infty}\frac{1}{x^p}\,dx=\lim_{D\to\infty}\frac{x^{1-p}}{1-p}\Bigg|_1^D=\lim_{D\to\infty}\frac{D^{1-p}}{1-p}-\frac{1}{1-p}. \tag{11.3.6}$$

If $p>1$ then $1-p<0$ and $\lim_{D\to\infty}D^{1-p}=0$, so the integral converges. If $0<p<1$ then $1-p>0$ and $\lim_{D\to\infty}D^{1-p}=\infty$, so the integral diverges.

Example 11.3.5

Show that

$$\sum_{n=1}^{\infty}\frac{1}{n^3} \tag{11.3.7}$$

converges.

Solution

We could of course use the integral test, but now that we have the theorem we may simply note that this is a p-series with $p>1$.

Example 11.3.6

Show that

$$\sum_{n=1}^{\infty}\frac{5}{n^4} \tag{11.3.8}$$

converges.

Solution

We know that if

$$\sum_{n=1}^{\infty} 1/n^4 \tag{11.3.9}$$

converges then

$$\sum_{n=1}^{\infty} 5/n^4 \tag{11.3.10}$$

also converges, by theorem 11.2.2. Since $\sum_{n=1}^{\infty} 1/n^4$ is a convergent p-series, then $\sum_{n=1}^{\infty} 5/n^4$ converges also.

Example 11.3.7

Show that

$$\sum_{n=1}^{\infty} \frac{5}{\sqrt{n}} \tag{11.3.11}$$

diverges.

Solution

This also follows from theorem 11.2.2: Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series with $p = 1/2 < 1$, it diverges, and so does $\sum_{n=1}^{\infty} \frac{5}{\sqrt{n}}$.

Since it is typically difficult to compute the value of a series exactly, a good approximation is frequently required. In a real sense, a good approximation is only as good as we know it is, that is, while an approximation may in fact be good, it is only valuable in practice if we can guarantee its accuracy to some degree. This guarantee is usually easy to come by for series with decreasing positive terms.

Example 11.3.8

Approximate

$$\sum 1/n^2 \tag{11.3.12}$$

to two decimal places.

Solution

Referring to figure 11.3.1, if we approximate the sum by $\sum_{n=1}^{N} 1/n^2$, the error we make is the total area of the remaining rectangles, all of which lie under the curve $1/x^2$ from \)x=N\) out to infinity. So we know the true value of the series is larger than the approximation, and no bigger than the approximation plus the area under the curve from N to infinity. Roughly, then, we need to find N so that

$$\int_N^{\infty} \frac{1}{x^2}\, dx < 1/100. \tag{11.3.13}$$

We can compute the integral: $\int_N^{\infty} \frac{1}{x^2}\, dx = \frac{1}{N}$, so $N = 100$ is a good starting point. Adding up the first 100 terms gives approximately 1.634983900, and that plus $1/100$ is 1.644983900, so approximating the series by the value halfway between these will be at most $1/200 = 0.005$ in error. The midpoint is 1.639983900, but while this is correct to ± 0.005, we can't tell if the correct two-decimal approximation is 1.63 or 1.64.

We need to make N big enough to reduce the guaranteed error, perhaps to around 0.004 to be safe, so we would need $1/N \approx 0.008$, or $N = 125$. Now the sum of the first 125 terms is approximately 1.636965982, and that plus 0.008 is 1.644965982 and the point halfway between them is 1.640965982 The true value is then 1.640965982 ± 0.004 and all

numbers in this range round to 1.64, so 1.64 is correct to two decimal places. We have mentioned that the true value of this series can be shown to be $\pi^2/6 \approx 1.644934068$which rounds down to 1.64 (just barely) and is indeed below the upper bound of 1.644965982 again just barely. Frequently approximations will be even better than the "guaranteed" accuracy, but not always, as this example demonstrates.

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11.4: The Comparison Tests

As we begin to compile a list of convergent and divergent series, new ones can sometimes be analyzed by comparing them to ones that we already understand.

Example 11.5.1: Identifying if a Sum Converges

Does the following sum converge?

$$\sum_{n=2}^{\infty} \frac{1}{n^2 \ln n}$$

Solution

The obvious first approach, based on what we know, is the integral test. Unfortunately, we cannot compute the required antiderivative. But looking at the series, it would appear that it must converge, because the terms we are adding are smaller than the terms of a p-series, that is,

$$\frac{1}{n^2 \ln n} < \frac{1}{n^2},$$

when $n \geq 3$. Since adding up the terms $1/n^2$ doesn't get "too big", the new series "should" also converge. Let's make this more precise.

The series

$$\sum_{n=2}^{\infty} \frac{1}{n^2 \ln n}$$

converges if and only if

$$\sum_{n=3}^{\infty} \frac{1}{n^2 \ln n}$$

converges---all we've done is dropped the initial term. We know that

$$\sum_{n=3}^{\infty} \frac{1}{n^2}$$

converges. Looking at two typical partial sums:

\s_n={1\over 3^2\ln 3}+{1\over 4^2\ln 4}+{1\over 5^2\ln 5}+\cdots+ {1\over n^2\ln n} < {1\over 3^2}+{1\over 4^2}+ {1\over 5^2}+\cdots+{1\over n^2}=t_n. \nonumber \]

Since the p-series converges, say to L, and since the terms are positive, $t_n < L$. Since the terms of the new series are positive, the s_n form an increasing sequence and $s_n < t_n < L$ for all n. Hence the sequence $\{s_n\}$ is bounded and so converges.

Sometimes, even when the integral test applies, comparison to a known series is easier, so it's generally a good idea to think about doing a comparison before doing the integral test.

Example 11.5.2: Identifying if a Sum Converges

Does the following sum converge?

$$\sum_{n=2}^{\infty} \frac{|\sin n|}{n^2}$$

Solution

We cannot apply the integral test here, because the terms of this series are not decreasing. Just as in the previous example, however,

$$\frac{|\sin n|}{n^2} \le \frac{1}{n^2},$$

because $|\sin n| \le 1$. Once again the partial sums are non-decreasing and bounded above by

$$\sum 1/n^2 = L$$

so the new series converges.

Like the integral test, the comparison test can be used to show both convergence and divergence. In the case of the integral test, a single calculation will confirm whichever is the case. To use the comparison test we must first have a good idea as to convergence or divergence and pick the sequence for comparison accordingly.

Example 11.5.3: Identifying if a Sum Converges

Does the following sum converge?

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-3}}$$

Solution

We observe that the -3 should have little effect compared to the n^2 inside the square root, and therefore guess that the terms are enough like $1/\sqrt{n^2} = 1/n$ that the series should diverge. We attempt to show this by comparison to the harmonic series. We note that

$$\frac{1}{\sqrt{n^2-3}} > \frac{1}{\sqrt{n^2}} = \frac{1}{n},$$

so that

$$s_n = \frac{1}{\sqrt{2^2-3}} + \frac{1}{\sqrt{3^2-3}} + \cdots + \frac{1}{\sqrt{n^2-3}} > \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = t_n,$$

where t_n is 1 less than the corresponding partial sum of the harmonic series (because we start at $n=2$ instead of $n=1$). Since $\lim_{n\to\infty} t_n = \infty$, $\lim_{n\to\infty} s_n = \infty$ as well.

So the general approach is this: If you believe that a new series is convergent, attempt to find a convergent series whose terms are larger than the terms of the new series; if you believe that a new series is divergent, attempt to find a divergent series whose terms are smaller than the terms of the new series.

Example 11.5.4: Identifying if a Sum Converges

Does the following sum converge?

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+3}}$$

Solution

Just as in the last example, we guess that this is very much like the harmonic series and so diverges. Unfortunately,

$$\frac{1}{\sqrt{n^2+3}} < \frac{1}{n},$$

so we cannot compare the series directly to the harmonic series. A little thought leads us to

$$\frac{1}{\sqrt{n^2+3}} > \frac{1}{\sqrt{n^2+3n^2}} = \frac{1}{2n},$$

so if $\sum 1/(2n)$ diverges then the given series diverges. But since $\sum 1/(2n) = (1/2)\sum 1/n$, theorem 11.2.2 implies that it does indeed diverge.

For reference we summarize the comparison test in a theorem.

Theorem 11.5.5: The Comparison Test

Suppose that a_n and b_n are non-negative for all n and that $a_n \leq b_n$ when $n \geq N$, for some N.

Proof

If $\sum_{n=0}^{\infty} b_n$ converges, so does $\sum_{n=0}^{\infty} a_n$. If $\sum_{n=0}^{\infty} a_n$ diverges, so does $\sum_{n=0}^{\infty} b_n$.

□

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11.5: Alternating Series

Next we consider series with both positive and negative terms, but in a regular pattern: they alternate, as in the **alternating harmonic series** for example:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \frac{1}{1} + \frac{-1}{2} + \frac{1}{3} + \frac{-1}{4} + \cdots = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$$

In this series the sizes of the terms decrease, that is, $|a_n|$ forms a decreasing sequence, but this is not required in an alternating series. As with positive term series, however, when the terms do have decreasing sizes it is easier to analyze the series, much easier, in fact, than positive term series. Consider pictorially what is going on in the alternating harmonic series, shown in Figure 11.4.1. Because the sizes of the terms a_n are decreasing, the partial sums s_1, s_3, s_5, and so on, form a decreasing sequence that is bounded below by s_2, so this sequence must converge. Likewise, the partial sums s_2, s_4, s_6, and so on, form an increasing sequence that is bounded above by s_1, so this sequence also converges. Since all the even numbered partial sums are less than all the odd numbered ones, and since the "jumps" (that is, the a_i terms) are getting smaller and smaller, the two sequences must converge to the same value, meaning the entire sequence of partial sums $s_1, s_2, s_3, \ldots$ converges as well.

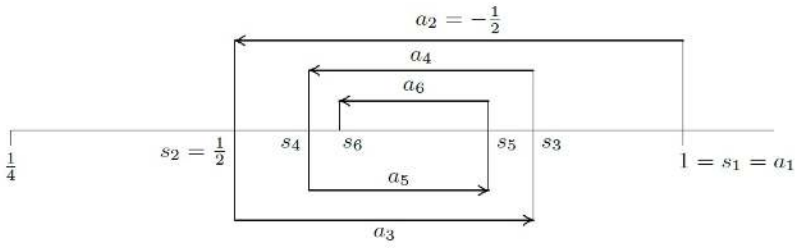


Figure 11.4.1. *The alternating harmonic series.*

There's nothing special about the alternating harmonic series---the same argument works for any alternating sequence with decreasing size terms. The alternating series test is worth calling a theorem.

Theorem 11.4.1: The Alternating Series Test

Suppose that $\{a_n\}_{n=1}^{\infty}$ is a non-increasing sequence of positive numbers and $\lim_{n\to\infty} a_n = 0$. Then the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ converges.

Proof

The odd numbered partial sums, s_1, s_3, s_5, and so on, form a non-increasing sequence, because $s_{2k+3} = s_{2k+1} - a_{2k+2} + a_{2k+3} \le s_{2k+1}$, since $a_{2k+2} \ge a_{2k+3}$. This sequence is bounded below by s_2, so it must converge, say $\lim_{k\to\infty} s_{2k+1} = L$. Likewise, the partial sums s_2, s_4, s_6, and so on, form a non-decreasing sequence that is bounded above by s_1, so this sequence also converges, say $\lim_{k\to\infty} s_{2k} = M$. Since $\lim_{n\to\infty} a_n = 0$ and $s_{2k+1} = s_{2k} + a_{2k+1}$,

$$L = \lim_{k\to\infty} s_{2k+1} = \lim_{k\to\infty} (s_{2k} + a_{2k+1}) = \lim_{k\to\infty} s_{2k} + \lim_{k\to\infty} a_{2k+1} = M + 0 = M,$$

so $L = M$, the two sequences of partial sums converge to the same limit, and this means the entire sequence of partial sums also converges to L.

□

Another useful fact is implicit in this discussion. Suppose that $L = \sum_{n=1}^{\infty}(-1)^{n-1}a_n$ and that we approximate L by a finite part of this sum, say $L \approx \sum_{n=1}^{N}(-1)^{n-1}a_n$. Because the terms are decreasing in size, we know that the true value of L must be between this approximation and the next one, that is, between $\sum_{n=1}^{N}(-1)^{n-1}a_n$ and $\sum_{n=1}^{N+1}(-1)^{n-1}a_n$. Depending on whether N is odd or even, the second will be larger or smaller than the first.

Example 11.4.2

Approximate the alternating harmonic series to one decimal place.

Solution

We need to go roughly to the point at which the next term to be added or subtracted is $1/10$. Adding up the first nine and the first ten terms we get approximately 0.746 and 0.646. These are $1/10$ apart, but it is not clear how the correct value would be rounded. It turns out that we are able to settle the question by computing the sums of the first eleven and twelve terms, which give 0.737 and 0.653, so correct to one place the value is 0.7.

We have considered alternating series with first index 1, and in which the first term is positive, but a little thought shows this is not crucial. The same test applies to any similar series, such as $\sum_{n=0}^{\infty}(-1)^n a_n$, $\sum_{n=1}^{\infty}(-1)^n a_n$, $\sum_{n=17}^{\infty}(-1)^n a_n$, etc.

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11.6: Absolute Convergence and the Ratio and Root Test

Roughly speaking there are two ways for a series to converge: As in the case of $\sum 1/n^2$, the individual terms get small very quickly, so that the sum of all of them stays finite, or, as in the case of $\sum(-1)^{n-1}/n$, the terms do not get small fast enough ($\sum 1/n$ diverges), but a mixture of positive and negative terms provides enough cancellation to keep the sum finite. You might guess from what we've seen that if the terms get small fast enough to do the job, then whether or not some terms are negative and some positive the series converges.

Theorem 11.6.1

If $\sum_{n=0}^{\infty}|a_n|$ converges, then $\sum_{n=0}^{\infty}a_n$ converges.

Proof.

Note that $0\le a_n+|a_n|\le 2|a_n|$ so by the comparison test $\sum_{n=0}^{\infty}(a_n+|a_n|)$ converges. Now

$$\sum_{n=0}^{\infty}(a_n+|a_n|)-\sum_{n=0}^{\infty}|a_n|=\sum_{n=0}^{\infty}a_n+|a_n|-|a_n|=\sum_{n=0}^{\infty}a_n$$

converges by theorem 11.2.2.

□

So given a series $\sum a_n$ with both positive and negative terms, you should first ask whether $\sum|a_n|$ converges. This may be an easier question to answer, because we have tests that apply specifically to terms with non-negative terms. If $\sum|a_n|$ converges then you know that $\sum a_n$ converges as well. If $\sum|a_n|$ diverges then it still may be true that $\sum a_n$ converges---you will have to do more work to decide the question. Another way to think of this result is: it is (potentially) easier for $\sum a_n$ to converge than for $\sum|a_n|$ to converge, because the latter series cannot take advantage of cancellation.

If $\sum|a_n|$ converges we say that $\sum a_n$ converges **absolutely**; to say that $\sum a_n$ converges absolutely is to say that any cancellation that happens to come along is not really needed, as the terms already get small so fast that convergence is guaranteed by that alone. If $\sum a_n$ converges but $\sum|a_n|$ does not, we say that $\sum a_n$ converges **conditionally**. For example $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}$ converges absolutely, while $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ converges conditionally.

Example 11.6.2

Does

$$\sum_{n=2}^{\infty}\frac{\sin n}{n^2}$$

converge?

Solution

In example 11.5.2 we saw that

$$\sum_{n=2}^{\infty}\frac{|\sin n|}{n^2}$$

converges, so the given series converges absolutely.

Example 11.6.3

Does $\sum_{n=0}^{\infty}(-1)^n\frac{3n+4}{2n^2+3n+5}$ converge?

Solution

Taking the absolute value,

$$\sum_{n=0}^{\infty} \frac{3n+4}{2n^2+3n+5}$$

diverges by comparison to

$$\sum_{n=1}^{\infty} \frac{3}{10n},$$

so if the series converges it does so conditionally. It is true that

$$\lim_{n\to\infty} (3n+4)/(2n^2+3n+5) = 0,$$

so to apply the alternating series test we need to know whether the terms are decreasing. If we let

$$f(x) = (3x+4)/(2x^2+3x+5)$$

then

$$f'(x) = -(6x^2+16x-3)/(2x^2+3x+5)^2,$$

and it is not hard to see that this is negative for $x \geq 1$, so the series is decreasing and by the alternating series test it converges.

Contributors

•

Does the series $\sum_{n=0}^{\infty} \frac{n^5}{5^n}$ converge? It is possible, but a bit unpleasant, to approach this with the integral test or the comparison test, but there is an easier way. Consider what happens as we move from one term to the next in this series:

$$\cdots + \frac{n^5}{5^n} + \frac{(n+1)^5}{5^{n+1}} + \cdots$$

The denominator goes up by a factor of 5, $5^{n+1} = 5 \cdot 5^n$, but the numerator goes up by much less:

$$(n+1)^5 = n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1,$$

which is much less than $5n^5$ when n is large, because $5n^4$ is much less than n^5. So we might guess that in the long run it begins to look as if each term is $1/5$ of the previous term. We have seen series that behave like this:

$$\sum_{n=0}^{\infty} \frac{1}{5^n} = \frac{5}{4},$$

a geometric series. So we might try comparing the given series to some variation of this geometric series. This is possible, but a bit messy. We can in effect do the same thing, but bypass most of the unpleasant work.

The key is to notice that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^5}{5^{n+1}} \frac{5^n}{n^5} = \lim_{n\to\infty} \frac{(n+1)^5}{n^5} \frac{1}{5} = 1 \cdot \frac{1}{5} = \frac{1}{5}.$$

This is really just what we noticed above, done a bit more officially: in the long run, each term is one fifth of the previous term. Now pick some number between $1/5$ and 1, say $1/2$. Because

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{5},$$

then when n is big enough, say $n \geq N$ for some N,

$$\frac{a_{n+1}}{a_n} < \frac{1}{2} \quad \text{and} \quad a_{n+1} < \frac{a_n}{2}.$$

So $a_{N+1} < a_N/2$, $a_{N+2} < a_{N+1}/2 < a_N/4$, $a_{N+3} < a_{N+2}/2 < a_{N+1}/4 < a_N/8$, and so on. The general form is $a_{N+k} < a_N/2^k$. So if we look at the series

$$\sum_{k=0}^{\infty} a_{N+k} = a_N + a_{N+1} + a_{N+2} + a_{N+3} + \cdots + a_{N+k} + \cdots,$$

its terms are less than or equal to the terms of the sequence

$$a_N + \frac{a_N}{2} + \frac{a_N}{4} + \frac{a_N}{8} + \cdots + \frac{a_N}{2^k} + \cdots = \sum_{k=0}^{\infty} \frac{a_N}{2^k} = 2a_N.$$

So by the comparison test, $\sum_{k=0}^{\infty} a_{N+k}$ converges, and this means that $\sum_{n=0}^{\infty} a_n$ converges, since we've just added the fixed number $a_0 + a_1 + \cdots + a_{N-1}$.

Under what circumstances could we do this? What was crucial was that the limit of a_{n+1}/a_n, say L, was less than 1 so that we could pick a value r so that $L < r < 1$. The fact that $L < r$ ($1/5 < 1/2$ in our example) means that we can compare the series $\sum a_n$ to $\sum r^n$, and the fact that $r < 1$ guarantees that $\sum r^n$ converges. That's really all that is required to make the argument work. We also made use of the fact that the terms of the series were positive; in general we simply consider the absolute values of the terms and we end up testing for absolute convergence.

Theroem 11.7.1: The Ratio Test

Suppose that

$$\lim_{n\to\infty} |a_{n+1}/a_n| = L.$$

If

$$L < 1$$

the series $\sum a_n$ converges **absolutely**, if $L > 1$ the series diverges, and if $L = 1$ this test gives no information.

Proof.
The example above essentially proves the first part of this, if we simply replace $1/5$ by L and $1/2$ by r. Suppose that $L > 1$, and pick r so that $1 < r < L$. Then for $n \geq N$, for some N,

$$\frac{|a_{n+1}|}{|a_n|} > r \quad \text{and} \quad |a_{n+1}| > r|a_n|.$$

This implies that

$$|a_{N+k}| > r^k |a_N|$$

, but since $r > 1$ this means that

$$\lim_{k\to\infty} |a_{N+k}| \neq 0$$

, which means also that

$$\lim_{n\to\infty} a_n \neq 0$$

. By the divergence test, the series diverges.

□

To see that we get no information when $L = 1$, we need to exhibit two series with $L = 1$, one that converges and one that diverges. It is easy to see that $\sum 1/n^2$ and $\sum 1/n$ do the job.

Example 11.7.2

The ratio test is particularly useful for series involving the factorial function. Consider $\sum_{n=0}^{\infty} 5^n/n!$.

$$\lim_{n\to\infty} \frac{5^{n+1}}{(n+1)!}\frac{n!}{5^n} = \lim_{n\to\infty} \frac{5^{n+1}}{5^n}\frac{n!}{(n+1)!} = \lim_{n\to\infty} 5\frac{1}{(n+1)} = 0.$$

Since $0 < 1$, the series converges.

A similar argument, which we will not do, justifies a similar test that is occasionally easier to apply.

Theroem 11.7.3: The Root Test

Suppose that

$$\lim_{n\to\infty} |a_n|^{1/n} = L.$$

If $L < 1$,the series $\sum a_n$ converges absolutely, if $L > 1$ the series diverges, and if $L = 1$ this test gives no information.

The proof of the root test is actually easier than that of the ratio test, and is a good exercise.

Example 11.7.4

Analyze $\sum_{n=0}^{\infty} \frac{5^n}{n^n}$.

Solution

The ratio test turns out to be a bit difficult on this series (try it). Using the root test:

$$\lim_{n\to\infty} \left(\frac{5^n}{n^n}\right)^{1/n} = \lim_{n\to\infty} \frac{(5^n)^{1/n}}{(n^n)^{1/n}} = \lim_{n\to\infty} \frac{5}{n} = 0.$$

Since $0 < 1$, the series converges.

The root test is frequently useful when n appears as an exponent in the general term of the series.

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11.7: Strategy for Testing Series

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11.8: Power Series

We have seen that some functions can be represented as series, which may give valuable information about the function. So far, we have seen only those examples that result from manipulation of our one fundamental example, the geometric series. We would like to start with a given function and produce a series to represent it, if possible.

Suppose that $f(x)=\sum_{n=0}^{\infty} a_n x^n$ on some interval of convergence. Then we know that we can compute derivatives of f by taking derivatives of the terms of the series. Let's look at the first few in general:

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots \\ f''(x) &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + 3\cdot 2 a_3 x + 4\cdot 3 a_4 x^2 + \cdots \\ f'''(x) &= \sum_{n=3}^{\infty} n(n-1)(n-2) a_n x^{n-3} = 3\cdot 2 a_3 + 4\cdot 3\cdot 2 a_4 x + \cdots \end{aligned}$$

By examining these it's not hard to discern the general pattern. The kth derivative must be

$$\begin{aligned} f^{(k)}(x) &= \sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-k+1) a_n x^{n-k} \\ &= k(k-1)(k-2)\cdots(2)(1)a_k + (k+1)(k)\cdots(2)a_{k+1}x + \\ &\quad + (k+2)(k+1)\cdots(3)a_{k+2}x^2 + \cdots \end{aligned}$$

We can shrink this quite a bit by using factorial notation:

$$f^{(k)}(x) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k} = k! a_k + (k+1)! a_{k+1} x + \frac{(k+2)!}{2!} a_{k+2} x^2 + \cdots$$

Now substitute $x=0$:

$$f^{(k)}(0) = k! a_k + \sum_{n=k+1}^{\infty} \frac{n!}{(n-k)!} a_n 0^{n-k} = k! a_k,$$

and solve for a_k:

$$a_k = \frac{f^{(k)}(0)}{k!}.$$

Note the special case, obtained from the series for f itself, that gives $f(0)=a_0$.

So if a function f can be represented by a series, we know just what series it is. Given a function f, the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

is called the **Maclaurin series** for f.

Example 11.10.1: Maclaurin series

Find the Maclaurin series for $f(x)=1/(1-x)$.

Solution

We need to compute the derivatives of f (and hope to spot a pattern).

$$\begin{aligned} f(x) &= (1-x)^{-1} \\ f'(x) &= (1-x)^{-2} \\ f''(x) &= 2(1-x)^{-3} \\ f'''(x) &= 6(1-x)^{-4} \\ f^{(4)}(x) &= 4!(1-x)^{-5} \\ &\vdots \\ f^{(n)}(x) &= n!(1-x)^{-n-1} \end{aligned}$$

So

$$\frac{f^{(n)}(0)}{n!} = \frac{n!(1-0)^{-n-1}}{n!} = 1$$

and the Maclaurin series is

$$\sum_{n=0}^{\infty} 1 \cdot x^n = \sum_{n=0}^{\infty} x^n,$$

the geometric series.

A warning is in order here. Given a function f we may be able to compute the Maclaurin series, but that does not mean we have found a series representation for f. We still need to know where the series converges, and if, where it converges, it converges to $f(x)$. While for most commonly encountered functions the Maclaurin series does indeed converge to f on some interval, this is not true of all functions, so care is required.

As a practical matter, if we are interested in using a series to approximate a function, we will need some finite number of terms of the series. Even for functions with messy derivatives we can compute these using computer software like Sage. If we want to know the whole series, that is, a typical term in the series, we need a function whose derivatives fall into a pattern that we can discern. A few of the most important functions are fortunately very easy.

Example 11.10.2: Maclaurin series

Find the Maclaurin series for $\sin x$.

Solution

The derivatives are quite easy: $f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$, $f^{(4)}(x) = \sin x$, and then the pattern repeats. We want to know the derivatives at zero: 1, 0, -1, 0, 1, 0, -1, 0,…, and so the Maclaurin series is

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

We should always determine the radius of convergence:

$$\lim_{n\to\infty} \frac{|x|^{2n+3}}{(2n+3)!} \frac{(2n+1)!}{|x|^{2n+1}} = \lim_{n\to\infty} \frac{|x|^2}{(2n+3)(2n+2)} = 0,$$

so the series converges for every x. Since it turns out that this series does indeed converge to $\sin x$ everywhere, we have a series representation for $\sin x$ for every x.

Sometimes the formula for the nth derivative of a function f is difficult to discover, but a combination of a known Maclaurin series and some algebraic manipulation leads easily to the Maclaurin series for f.

Example 11.10.3: Maclaurin series

Find the Maclaurin series for $x\sin(-x)$.

Solution

To get from $\sin x$ to $x\sin(-x)$ we substitute $-x$ for x and then multiply by x. We can do the same thing to the series for $\sin x$:

$$x\sum_{n=0}^{\infty}(-1)^n\frac{(-x)^{2n+1}}{(2n+1)!}=x\sum_{n=0}^{\infty}(-1)^n(-1)^{2n+1}\frac{x^{2n+1}}{(2n+1)!}=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^{2n+2}}{(2n+1)!}.$$

As we have seen, a general power series can be centered at a point other than zero, and the method that produces the Maclaurin series can also produce such series.

Taylor series

Find a series centered at -2 for $1/(1-x)$.

Solution

If the series is

$$\sum_{n=0}^{\infty}a_n(x+2)^n$$

then looking at the kth derivative:

$$k!(1-x)^{-k-1}=\sum_{n=k}^{\infty}\frac{n!}{(n-k)!}a_n(x+2)^{n-k}$$

and substituting $x=-2$ we get

$$k!3^{-k-1}=k!a_k$$

and

$$a_k=3^{-k-1}=1/3^{k+1},$$

so the series is

$$\sum_{n=0}^{\infty}\frac{(x+2)^n}{3^{n+1}}.$$

We've already seen this, in Section 11.8. Such a series is called the **Taylor series** for the function, and the general term has the form

$$\frac{f^{(n)}(a)}{n!}(x-a)^n.$$

A Maclaurin series is simply a Taylor series with $a=0$.

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11.9: Representations of Functions as Power Series

Now we know that some functions can be expressed as power series, which look like infinite polynomials. Since calculus, that is, computation of derivatives and antiderivatives, is easy for polynomials, the obvious question is whether the same is true for infinite series. The answer is yes:

Theorem 11.9.1

Suppose the power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n \tag{11.9.1}$$

has radius of convergence R. Then

$$\begin{aligned} f'(x) &= \sum_{n=0}^{\infty} n a_n (x-a)^{n-1}, \\ \int f(x)\,dx &= C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-a)^{n+1}, \end{aligned} \tag{11.9.2}$$

and these two series have radius of convergence R as well.

Example 11.9.2

$$\begin{aligned} \frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \\ \int \frac{1}{1-x}\,dx &= -\ln|1-x| = \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} \\ \ln|1-x| &= \sum_{n=0}^{\infty} -\frac{1}{n+1} x^{n+1} \end{aligned} \tag{11.9.3}$$

when $|x| < 1$. The series does not converge when $x = 1$ but does converge when $x = -1$ or $1 - x = 2$. The interval of convergence is $[-1, 1)$, or $0 < 1 - x \leq 2$, so we can use the series to represent $\ln(x)$ when $0 < x \leq 2$.

For example

$$\ln(3/2) = \ln(1 - -1/2) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} \frac{1}{2^{n+1}} \tag{11.9.4}$$

and so

$$\ln(3/2) \approx \frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \frac{1}{64} + \frac{1}{160} - \frac{1}{384} + \frac{1}{896} = \frac{909}{2240} \approx 0.406. \tag{11.9.5}$$

Because this is an alternating series with decreasing terms, we know that the true value is between $909/2240$ and $909/2240 - 1/2048 = 29053/71680 \approx .4053$, so correct to two decimal places the value is 0.41.

What about $\ln(9/4)$? Since $9/4$ is larger than 2 we cannot use the series directly, but $\ln(9/4) = \ln((3/2)^2) = 2\ln(3/2) \approx 0.82$, so in fact we get a lot more from this one calculation than first meets the eye. To estimate the true value accurately we actually need to be a bit more careful. When we multiply by two we know that the true value is between 0.8106 and 0.812, so rounded to two decimal places the true value is 0.81.

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11.10: Taylor and Maclaurin Series

Learning Objectives

- Describe the procedure for finding a Taylor polynomial of a given order for a function.
- Explain the meaning and significance of Taylor's theorem with remainder.
- Estimate the remainder for a Taylor series approximation of a given function.

In the previous two sections we discussed how to find power series representations for certain types of functions—specifically, functions related to geometric series. Here we discuss power series representations for other types of functions. In particular, we address the following questions: Which functions can be represented by power series and how do we find such representations? If we can find a power series representation for a particular function f and the series converges on some interval, how do we prove that the series actually converges to f?

Overview of Taylor/Maclaurin Series

Consider a function f that has a power series representation at $x=a$. Then the series has the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots. \tag{11.10.1}$$

What should the coefficients be? For now, we ignore issues of convergence, but instead focus on what the series should be, if one exists. We return to discuss convergence later in this section. If the series Equation 11.10.1is a representation for f at $x=a$, we certainly want the series to equal $f(a)$ at $x=a$. Evaluating the series at $x=a$, we see that

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(a-a) + c_2(a-a)^2 + \cdots = c_0. \tag{11.10.2}$$

Thus, the series equals $f(a)$ if the coefficient $c_0 = f(a)$. In addition, we would like the first derivative of the power series to equal $f'(a)$ at $x=a$. Differentiating Equation 11.10.2term-by-term, we see that

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots. \tag{11.10.3}$$

Therefore, at $x=a$, the derivative is

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = c_1 + 2c_2(a-a) + 3c_3(a-a)^2 + \cdots = c_1. \tag{11.10.4}$$

Therefore, the derivative of the series equals $f'(a)$ if the coefficient $c_1 = f'(a)$. Continuing in this way, we look for coefficients c_n such that all the derivatives of the power series Equation 11.10.4will agree with all the corresponding derivatives of f at $x=a$. The second and third derivatives of Equation 11.10.3are given by

$$\frac{d^2}{dx^2}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = 2c_2 + 3\cdot 2c_3(x-a) + 4\cdot 3c_4(x-a)^2 + \ldots \tag{11.10.5}$$

and

$$\frac{d^3}{dx^3}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(x-a) + 5\cdot 4\cdot 3c_5(x-a)^2 + \cdots. \tag{11.10.6}$$

Therefore, at $x=a$, the second and third derivatives

$$\frac{d^2}{dx^2}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = 2c_2 + 3\cdot 2c_3(a-a) + 4\cdot 3c_4(a-a)^2 + \cdots = 2c_2 \tag{11.10.7}$$

and

$$\frac{d^3}{dx^3}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(a-a) + 5\cdot 4\cdot 3c_5(a-a)^2 + \cdots = 3\cdot 2c_3 \tag{11.10.8}$$

equal $f''(a)$ and $f'''(a)$, respectively, if $c_2 = \dfrac{f''(a)}{2}$ and $c_3 = \dfrac{f'''(a)}{3\cdot 2}$. More generally, we see that if f has a power series representation at $x=a$, then the coefficients should be given by $c_n = \dfrac{f^{(n)}(a)}{n!}$. That is, the series should be

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

This power series for f is known as the Taylor series for f at a. If $a = 0$, then this series is known as the Maclaurin series for f.

Definition 11.10.1: Maclaurin and Taylor series

If f has derivatives of all orders at $x = a$, then the *Taylor series* for the function f at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

The Taylor series for f at 0 is known as the *Maclaurin series* for f.

Later in this section, we will show examples of finding Taylor series and discuss conditions under which the Taylor series for a function will converge to that function. Here, we state an important result. Recall that power series representations are unique. Therefore, if a function f has a power series at a, then it must be the Taylor series for f at a.

Theorem 11.10.1: Uniqueness of Taylor Series

If a function f has a power series at a that converges to f on some open interval containing a, then that power series is the Taylor series for f at a.

The proof follows directly from that discussed previously.

To determine if a Taylor series converges, we need to look at its sequence of partial sums. These partial sums are finite polynomials, known as **Taylor polynomials.**

Taylor Polynomials

The nth partial sum of the Taylor series for a function f at a is known as the nth-degree Taylor polynomial. For example, the 0th, 1st, 2nd, and 3rd partial sums of the Taylor series are given by

$$\begin{aligned} p_0(x) &= f(a) \\ p_1(x) &= f(a) + f'(a)(x-a) \\ p_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\ p_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \end{aligned}$$

respectively. These partial sums are known as the 0th, 1st, 2nd, and 3rd degree Taylor polynomials of f at a, respectively. If $a = 0$, then these polynomials are known as *Maclaurin polynomials* for f. We now provide a formal definition of Taylor and Maclaurin polynomials for a function f.

Definition 11.10.2: Maclaurin polynomial

If f has n derivatives at $x = a$, then the nth-degree Taylor polynomial of f at a is

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The nth-degree Taylor polynomial for f at 0 is known as the nth-degree Maclaurin polynomial for f.

We now show how to use this definition to find several Taylor polynomials for $f(x) = \ln x$ at $x = 1$.

Example 11.10.1: Finding Taylor Polynomials

Find the Taylor polynomials p_0, p_1, p_2 and p_3 for $f(x) = \ln x$ at $x = 1$. Use a graphing utility to compare the graph of f with the graphs of p_0, p_1, p_2 and p_3.

Solution

To find these Taylor polynomials, we need to evaluate f and its first three derivatives at $x = 1$.

$$f(x)=\ln x \qquad f(1)=0$$
$$f'(x)=\frac{1}{x} \qquad f'(1)=1$$
$$f''(x)=-\frac{1}{x^2} \qquad f''(1)=-1$$
$$f'''(x)=\frac{2}{x^3} \qquad f'''(1)=2$$

Therefore,

$$\begin{aligned}
p_0(x) &= f(1)=0,\\
p_1(x) &= f(1)+f'(1)(x-1)=x-1,\\
p_2(x) &= f(1)+f'(1)(x-1)+\frac{f''(1)}{2}(x-1)^2=(x-1)-\frac{1}{2}(x-1)^2\\
p_3(x) &= f(1)+f'(1)(x-1)+\frac{f''(1)}{2}(x-1)^2+\frac{f'''(1)}{3!}(x-1)^3=(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3
\end{aligned}$$

The graphs of $y=f(x)$ and the first three Taylor polynomials are shown in Figure 11.10.1

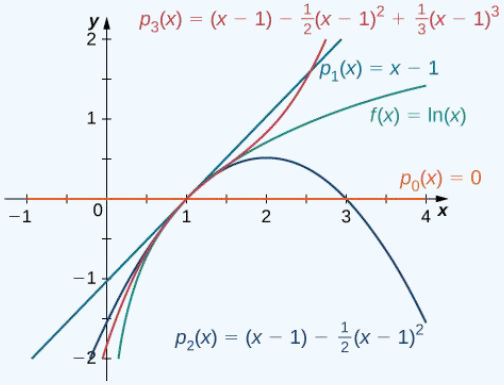


Figure 11.10.1: The function $y=\ln x$ and the Taylor polynomials p_0, p_1, p_2 and p_3 at $x=1$ are plotted on this graph.

? Exercise 11.10.1

Find the Taylor polynomials p_0, p_1, p_2 and p_3 for $f(x)=\dfrac{1}{x^2}$ at $x=1$.

Hint

Find the first three derivatives of f and evaluate them at $x=1$.

Answer

$$\begin{aligned}
p_0(x) &= 1\\
p_1(x) &= 1-2(x-1)\\
p_2(x) &= 1-2(x-1)+3(x-1)^2\\
p_3(x) &= 1-2(x-1)+3(x-1)^2-4(x-1)^3
\end{aligned}$$

We now show how to find Maclaurin polynomials for e^x, $\sin x$, and $\cos x$. As stated above, Maclaurin polynomials are Taylor polynomials centered at zero.

✓ Example 11.10.2: Finding Maclaurin Polynomials

For each of the following functions, find formulas for the Maclaurin polynomials p_0, p_1, p_2 and p_3. Find a formula for the nth-degree Maclaurin polynomial and write it using sigma notation. Use a graphing utility to compare the graphs of p_0, p_1, p_2 and p_3 with f.

a. $f(x)=e^x$
b. $f(x)=\sin x$
c. $f(x)=\cos x$

Solution

Since $f(x)=e^x$,we know that $f(x)=f'(x)=f''(x)=\cdots=f^{(n)}(x)=e^x$ for all positive integers n. Therefore,

$$f(0)=f'(0)=f''(0)=\cdots=f^{(n)}(0)=1$$

for all positive integers n. Therefore, we have

$$p_0(x)=f(0)=1,$$
$$p_1(x)=f(0)+f'(0)x=1+x,$$
$$p_2(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2=1+x+\frac{1}{2}x^2,$$
$$p_3(x)=f(0)+f'(0)x+\frac{f''(0)}{2}x^2+\frac{f'''(0)}{3!}x^3=1+x+\frac{1}{2}x^2+\frac{1}{3!}x^3,$$

$$\begin{aligned}p_n(x)&=f(0)+f'(0)x+\frac{f''(0)}{2}x^2+\frac{f'''(0)}{3!}x^3+\cdots+\frac{f^{(n)}(0)}{n!}x^n\\&=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}\\&=\sum_{k=0}^{n}\frac{x^k}{k!}\end{aligned}.$$

The function and the first three Maclaurin polynomials are shown in Figure 11.10.2

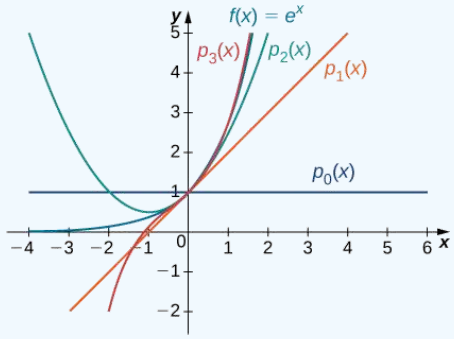


Figure 11.10.2: *The graph shows the function $y=e^x$ and the Maclaurin polynomials p_0, p_1, p_2 and p_3.*

b. For $f(x)=\sin x$, the values of the function and its first four derivatives at $x=0$ are given as follows:

$$\begin{aligned}f(x)&=\sin x & f(0)&=0\\f'(x)&=\cos x & f'(0)&=1\\f''(x)&=-\sin x & f''(0)&=0\\f'''(x)&=-\cos x & f'''(0)&=-1\\f^{(4)}(x)&=\sin x & f^{(4)}(0)&=0.\end{aligned}$$

Since the fourth derivative is $\sin x$, the pattern repeats. That is, $f^{(2m)}(0)=0$ and $f^{(2m+1)}(0)=(-1)^m$ for $m\geq 0$. Thus, we have

$$p_0(x)=0,$$
$$p_1(x)=0+x=x,$$
$$p_2(x)=0+x+0=x,$$
$$p_3(x)=0+x+0-\frac{1}{3!}x^3=x-\frac{x^3}{3!},$$
$$p_4(x)=0+x+0-\frac{1}{3!}x^3+0=x-\frac{x^3}{3!},$$
$$p_5(x)=0+x+0-\frac{1}{3!}x^3+0+\frac{1}{5!}x^5=x-\frac{x^3}{3!}+\frac{x^5}{5!},$$

and for $m\geq 0$,

$$p_{2m+1}(x) = p_{2m+2}(x) \quad = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^m \frac{x^{2m+1}}{(2m+1)!}$$

$$= \sum_{k=0}^{m} (-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$

Graphs of the function and its Maclaurin polynomials are shown in Figure 11.10.3

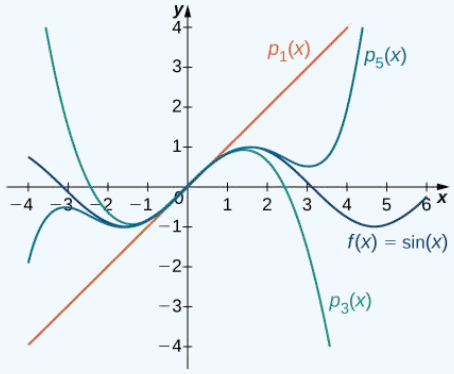


Figure 11.10.3: *The graph shows the function $y = \sin x$ and the Maclaurin polynomials p_1, p_3 and p_5.*

c. For $f(x) = \cos x$, the values of the function and its first four derivatives at $x = 0$ are given as follows:

$$\begin{aligned} f(x) &= \cos x & f(0) &= 1 \\ f'(x) &= -\sin x & f'(0) &= 0 \\ f''(x) &= -\cos x & f''(0) &= -1 \\ f'''(x) &= \sin x & f'''(0) &= 0 \\ f^{(4)}(x) &= \cos x & f^{(4)}(0) &= 1. \end{aligned}$$

Since the fourth derivative is $\sin x$, the pattern repeats. In other words, $f^{(2m)}(0) = (-1)^m$ and $f^{(2m+1)} = 0$ for $m \geq 0$. Therefore,

$$p_0(x) = 1,$$

$$p_1(x) = 1 + 0 = 1,$$

$$p_2(x) = 1 + 0 - \frac{1}{2!}x^2 = 1 - \frac{x^2}{2!},$$

$$p_3(x) = 1 + 0 - \frac{1}{2!}x^2 + 0 = 1 - \frac{x^2}{2!},$$

$$p_4(x) = 1 + 0 - \frac{1}{2!}x^2 + 0 + \frac{1}{4!}x^4 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!},$$

$$p_5(x) = 1 + 0 - \frac{1}{2!}x^2 + 0 + \frac{1}{4!}x^4 + 0 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!},$$

and for $n \geq 0$,

$$p_{2m}(x) = p_{2m+1}(x)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^m \frac{x^{2m}}{(2m)!}$$

$$= \sum_{k=0}^{m} (-1)^k \frac{x^{2k}}{(2k)!}.$$

Graphs of the function and the Maclaurin polynomials appear in Figure 11.10.4

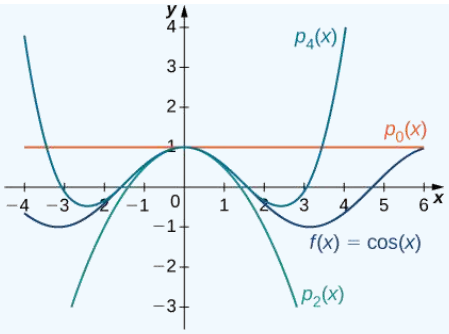


Figure 11.10.4: *The function $y = \cos x$ and the Maclaurin polynomials p_0, p_2 and p_4 are plotted on this graph.*

? Exercise 11.10.2

Find formulas for the Maclaurin polynomials p_0, p_1, p_2 and p_3 for $f(x) = \dfrac{1}{1+x}$.

Find a formula for the nth-degree Maclaurin polynomial. Write your answer using sigma notation.

Hint

Evaluate the first four derivatives of f and look for a pattern.

Answer

$$p_0(x) = 1;\ p_1(x) = 1 - x;\ p_2(x) = 1 - x + x^2;\ p_3(x) = 1 - x + x^2 - x^3;\ p_n(x) = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n = \sum_{k=0}^{n} (-1)^k x^k$$

Taylor's Theorem with Remainder

Recall that the nth-degree Taylor polynomial for a function f at a is the nth partial sum of the Taylor series for f at a. Therefore, to determine if the Taylor series converges, we need to determine whether the sequence of Taylor polynomials p_n converges. However, not only do we want to know if the sequence of Taylor polynomials converges, we want to know if it converges to f. To answer this question, we define the remainder $R_n(x)$ as

$$R_n(x) = f(x) - p_n(x).$$

For the sequence of Taylor polynomials to converge to f, we need the remainder R_n to converge to zero. To determine if R_n converges to zero, we introduce **Taylor's theorem with remainder**. Not only is this theorem useful in proving that a Taylor series converges to its related function, but it will also allow us to quantify how well the nth-degree Taylor polynomial approximates the function.

Here we look for a bound on $|R_n|$. Consider the simplest case: $n = 0$. Let p_0 be the 0th Taylor polynomial at a for a function f. The remainder R_0 satisfies

$$R_0(x) = f(x) - p_0(x) = f(x) - f(a).$$

If f is differentiable on an interval I containing a and x, then by the Mean Value Theorem there exists a real number c between a and x such that $f(x) - f(a) = f'(c)(x - a)$. Therefore,

$$R_0(x) = f'(c)(x - a).$$

Using the Mean Value Theorem in a similar argument, we can show that if f is n times differentiable on an interval I containing a and x, then the nth remainder R_n satisfies

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - a)^{n+1}$$

for some real number c between a and x. It is important to note that the value c in the numerator above is not the center a, but rather an unknown value c between a and x. This formula allows us to get a bound on the remainder R_n. If we happen to know that $\left|f^{(n+1)}(x)\right|$ is bounded by some real number M on this interval I, then

$$|R_n(x)| \le \frac{M}{(n+1)!}|x - a|^{n+1}$$

for all x in the interval I.

We now state Taylor's theorem, which provides the formal relationship between a function f and its nth-degree Taylor polynomial $p_n(x)$. This theorem allows us to bound the error when using a Taylor polynomial to approximate a function value, and will be important in proving that a Taylor series for f converges to f.

Theorem 11.10.2: Taylor's Theorem with Remainder

Let f be a function that can be differentiated $n+1$ times on an interval I containing the real number a. Let p_n be the nth-degree Taylor polynomial of f at a and let

$$R_n(x) = f(x) - p_n(x)$$

be the nth remainder. Then for each x in the interval I, there exists a real number c between a and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

.

If there exists a real number M such that $\left|f^{(n+1)}(x)\right| \leq M$ for all $x \in I$, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

for all x in I.

Proof

Fix a point $x \in I$ and introduce the function g such that

$$g(t) = f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \cdots - \frac{f^{(n)}(t)}{n!}(x-t)^n - R_n(x)\frac{(x-t)^{n+1}}{(x-a)^{n+1}}.$$

We claim that g satisfies the criteria of Rolle's theorem. Since g is a polynomial function (in t), it is a differentiable function. Also, g is zero at $t=a$ and $t=x$ because

$$\begin{aligned} g(a) &= f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n - R_n(x) \\ &= f(x) - p_n(x) - R_n(x) \\ &= 0, \\ g(x) &= f(x) - f(x) - 0 - \cdots - 0 \\ &= 0. \end{aligned}$$

Therefore, g satisfies Rolle's theorem, and consequently, there exists c between a and x such that $g'(c) = 0$. We now calculate g'. Using the product rule, we note that

$$\frac{d}{dt}\left[\frac{f^{(n)}(t)}{n!}(x-t)^n\right] = -\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} + \frac{f^{(n+1)}(t)}{n!}(x-t)^n.$$

Consequently,

$$\begin{aligned} g'(t) &= -f'(t) + [f'(t) - f''(t)(x-t)] + \left[f''(t)(x-t) - \frac{f'''(t)}{2!}(x-t)^2\right] + \cdots \\ &\quad + \left[\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!}(x-t)^n\right] + (n+1)R_n(x)\frac{(x-t)^n}{(x-a)^{n+1}} \end{aligned}$$

.

Notice that there is a telescoping effect. Therefore,

$$g'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n + (n+1)R_n(x)\frac{(x-t)^n}{(x-a)^{n+1}}$$

.

By Rolle's theorem, we conclude that there exists a number c between a and x such that $g'(c) = 0$. Since

$$g'(c) = -\frac{f^{(n+1)}(c)}{n!}(x-c)^n + (n+1)R_n(x)\frac{(x-c)^n}{(x-a)^{n+1}}$$

we conclude that

$$-\frac{f^{(n+1)}(c)}{n!}(x-c)^n+(n+1)R_n(x)\frac{(x-c)^n}{(x-a)^{n+1}}=0.$$

Adding the first term on the left-hand side to both sides of the equation and dividing both sides of the equation by $n+1$, we conclude that

$$R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

as desired. From this fact, it follows that if there exists M such that $\left|f^{(n+1)}(x)\right|\leq M$ for all x in I, then

$$|R_n(x)|\leq\frac{M}{(n+1)!}|x-a|^{n+1}$$

.

□

Not only does Taylor's theorem allow us to prove that a Taylor series converges to a function, but it also allows us to estimate the accuracy of Taylor polynomials in approximating function values. We begin by looking at linear and quadratic approximations of $f(x)=\sqrt[3]{x}$ at $x=8$ and determine how accurate these approximations are at estimating $\sqrt[3]{11}$.

Example 11.10.3: Using Linear and Quadratic Approximations to Estimate Function Values

Consider the function $f(x)=\sqrt[3]{x}$.

a. Find the first and second Taylor polynomials for f at $x=8$. Use a graphing utility to compare these polynomials with f near $x=8$.
b. Use these two polynomials to estimate $\sqrt[3]{11}$.
c. Use Taylor's theorem to bound the error.

Solution:

a. For $f(x)=\sqrt[3]{x}$, the values of the function and its first two derivatives at $x=8$ are as follows:

$$f(x)=\sqrt[3]{x}, \qquad f(8)=2$$
$$f'(x)=\frac{1}{3x^{2/3}}, \qquad f'(8)=\frac{1}{12}$$
$$f''(x)=\frac{-2}{9x^{5/3}}, \qquad f''(8)=-\frac{1}{144}.$$

Thus, the first and second Taylor polynomials at $x=8$ are given by

$$\begin{aligned}p_1(x)&=f(8)+f'(8)(x-8)\\&=2+\frac{1}{12}(x-8)\end{aligned}$$

$$\begin{aligned}p_2(x)&=f(8)+f'(8)(x-8)+\frac{f''(8)}{2!}(x-8)^2\\&=2+\frac{1}{12}(x-8)-\frac{1}{288}(x-8)^2.\end{aligned}$$

The function and the Taylor polynomials are shown in Figure 11.10.5

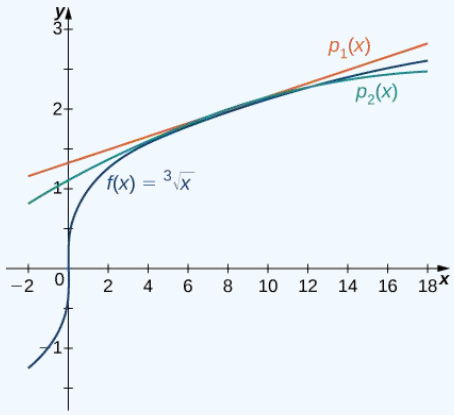


Figure 11.10.5: *The graphs of $f(x)=\sqrt[3]{x}$ and the linear and quadratic approximations $p_1(x)$ and $p_2(x)$*

b. Using the first Taylor polynomial at $x=8$, we can estimate

$$\sqrt[3]{11} \approx p_1(11) = 2 + \frac{1}{12}(11-8) = 2.25.$$

Using the second Taylor polynomial at $x=8$, we obtain

$$\sqrt[3]{11} \approx p_2(11) = 2 + \frac{1}{12}(11-8) - \frac{1}{288}(11-8)^2 = 2.21875.$$

c. By Theorem 11.10.2 there exists a c in the interval $(8,11)$ such that the remainder when approximating $\sqrt[3]{11}$ by the first Taylor polynomial satisfies

$$R_1(11) = \frac{f''(c)}{2!}(11-8)^2.$$

We do not know the exact value of c, so we find an upper bound on $R_1(11)$ by determining the maximum value of f'' on the interval $(8,11)$. Since $f''(x) = -\dfrac{2}{9x^{5/3}}$, the largest value for $|f''(x)|$ on that interval occurs at $x=8$. Using the fact that $f''(8) = -\dfrac{1}{144}$, we obtain

$$|R_1(11)| \le \frac{1}{144 \cdot 2!}(11-8)^2 = 0.03125.$$

Similarly, to estimate $R_2(11)$, we use the fact that

$$R_2(11) = \frac{f'''(c)}{3!}(11-8)^3.$$

Since $f'''(x) = \dfrac{10}{27x^{8/3}}$, the maximum value of f''' on the interval $(8,11)$ is $f'''(8) \approx 0.0014468$. Therefore, we have

$$|R_2(11)| \le \frac{0.0011468}{3!}(11-8)^3 \approx 0.0065104.$$

? Exercise 11.10.3:

Find the first and second Taylor polynomials for $f(x) = \sqrt{x}$ at $x=4$. Use these polynomials to estimate $\sqrt{6}$. Use Taylor's theorem to bound the error.

Hint

Evaluate $f(4)$, $f'(4)$,and $f''(4)$.

Answer

$p_1(x) = 2 + \dfrac{1}{4}(x-4); p_2(x) = 2 + \dfrac{1}{4}(x-4) - \dfrac{1}{64}(x-4)^2; p_1(6) = 2.5; p_2(6) = 2.4375;$

$|R_1(6)| \le 0.0625; |R_2(6)| \le 0.015625$

✓ Example 11.10.4: Approximating $\sin x$ Using Maclaurin Polynomials

From Example 11.10.2b, the Maclaurin polynomials for $\sin x$ are given by

$$p_{2m+1}(x) = p_{2m+2}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^m \frac{x^{2m+1}}{(2m+1)!}$$

for $m = 0, 1, 2, \ldots$.

a. Use the fifth Maclaurin polynomial for $\sin x$ to approximate $\sin\left(\dfrac{\pi}{18}\right)$ and bound the error.

b. For what values of x does the fifth Maclaurin polynomial approximate $\sin x$ to within 0.0001?

Solution

a.

The fifth Maclaurin polynomial is

$$p_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

.

Using this polynomial, we can estimate as follows:

$$\sin\left(\frac{\pi}{18}\right) \approx p_5\left(\frac{\pi}{18}\right) = \frac{\pi}{18} - \frac{1}{3!}\left(\frac{\pi}{18}\right)^3 + \frac{1}{5!}\left(\frac{\pi}{18}\right)^5 \approx 0.173648.$$

To estimate the error, use the fact that the sixth Maclaurin polynomial is $p_6(x) = p_5(x)$ and calculate a bound on $R_6(\frac{\pi}{18})$. By Theorem 11.10.2 the remainder is

$$R_6\left(\frac{\pi}{18}\right) = \frac{f^{(7)}(c)}{7!}\left(\frac{\pi}{18}\right)^7$$

for some c between 0 and $\frac{\pi}{18}$. Using the fact that $\left|f^{(7)}(x)\right| \leq 1$ for all x, we find that the magnitude of the error is at most

$$\frac{1}{7!} \cdot \left(\frac{\pi}{18}\right)^7 \leq 9.8 \times 10^{-10}.$$

b.

We need to find the values of x such that

$$\frac{1}{7!}|x|^7 \leq 0.0001.$$

Solving this inequality for x, we have that the fifth Maclaurin polynomial gives an estimate to within 0.0001 as long as $|x| < 0.907$.

? Exercise 11.10.4

Use the fourth Maclaurin polynomial for $\cos x$ to approximate $\cos\left(\frac{\pi}{12}\right)$.

Hint

The fourth Maclaurin polynomial is $p_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$.

Answer

0.96593

Now that we are able to bound the remainder $R_n(x)$, we can use this bound to prove that a Taylor series for f at a converges to f.

Representing Functions with Taylor and Maclaurin Series

We now discuss issues of convergence for Taylor series. We begin by showing how to find a Taylor series for a function, and how to find its interval of convergence.

✓ Example 11.10.5: Finding a Taylor Series

Find the Taylor series for $f(x) = \frac{1}{x}$ at $x = 1$. Determine the interval of convergence.

Solution

For $f(x) = \frac{1}{x}$, the values of the function and its first four derivatives at $x = 1$ are

$$f(x) = \frac{1}{x} \qquad f(1) = 1$$

$$f'(x) = -\frac{1}{x^2} \qquad f'(1) = -1$$

$$f''(x) = \frac{2}{x^3} \qquad f''(1) = 2!$$

$$f'''(x) = -\frac{3 \cdot 2}{x^4} \qquad f'''(1) = -3!$$

$$f^{(4)}(x) = \frac{4 \cdot 3 \cdot 2}{x^5} \qquad f^{(4)}(1) = 4!.$$

That is, we have $f^{(n)}(1) = (-1)^n n!$ for all $n \geq 0$. Therefore, the Taylor series for f at $x = 1$ is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^n .$$

To find the interval of convergence, we use the ratio test. We find that

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|(-1)^{n+1}(x-1)n^{+1}|}{|(-1)^n(x-1)^n|} = |x-1|\,.$$

Thus, the series converges if $|x-1|<1$. That is, the series converges for $0<x<2$. Next, we need to check the endpoints. At $x=2$, we see that

$$\sum_{n=0}^{\infty}(-1)^n(2-1)^n = \sum_{n=0}^{\infty}(-1)^n$$

diverges by the divergence test. Similarly, at $x=0$,

$$\sum_{n=0}^{\infty}(-1)^n(0-1)^n = \sum_{n=0}^{\infty}(-1)^{2n} = \sum_{n=0}^{\infty}1$$

diverges. Therefore, the interval of convergence is $(0,2)$.

? Exercise 11.10.5

Find the Taylor series for $f(x)=\dfrac{1}{2}$ at $x=2$ and determine its interval of convergence.

Hint

$f^{(n)}(2)=\dfrac{(-1)^n n!}{2^{n+1}}$

Answer

$\dfrac{1}{2}\displaystyle\sum_{n=0}^{\infty}\left(\frac{2-x}{2}\right)^n$. The interval of convergence is $(0,4)$.

We know that the Taylor series found in this example converges on the interval $(0,2)$, but how do we know it actually converges to f? We consider this question in more generality in a moment, but for this example, we can answer this question by writing

$$f(x)=\frac{1}{x}=\frac{1}{1-(1-x)}.$$

That is, f can be represented by the geometric series $\displaystyle\sum_{n=0}^{\infty}(1-x)^n$. Since this is a geometric series, it converges to $\dfrac{1}{x}$ as long as $|1-x|<1$. Therefore, the Taylor series found in Example 11.10.5does converge to $f(x)=\dfrac{1}{x}$ on $(0,2)$.

We now consider the more general question: if a Taylor series for a function f converges on some interval, how can we determine if it actually converges to f? To answer this question, recall that a series converges to a particular value if and only if its sequence of partial sums converges to that value. Given a Taylor series for f at a, the nth partial sum is given by the nth-degree Taylor polynomial p_n. Therefore, to determine if the Taylor series converges to f, we need to determine whether

$$\lim_{n\to\infty}p_n(x)=f(x).$$

Since the remainder $R_n(x)=f(x)-p_n(x)$, the Taylor series converges to f if and only if

$$\lim_{n\to\infty}R_n(x)=0.$$

We now state this theorem formally.

Theorem 11.10.3: Convergence of Taylor Series

Suppose that f has derivatives of all orders on an interval I containing a. Then the Taylor series

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

converges to $f(x)$ for all x in I if and only if

$$\lim_{n\to\infty}R_n(x)=0$$

for all x in I.

With this theorem, we can prove that a Taylor series for f at a converges to f if we can prove that the remainder $R_n(x) \to 0$. To prove that $R_n(x) \to 0$, we typically use the bound

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

from Taylor's theorem with remainder.

In the next example, we find the Maclaurin series for e^x and $\sin x$ and show that these series converge to the corresponding functions for all real numbers by proving that the remainders $R_n(x) \to 0$ for all real numbers x.

Example 11.10.6: Finding Maclaurin Series

For each of the following functions, find the Maclaurin series and its interval of convergence. Use Theorem 11.10.2to prove that the Maclaurin series for f converges to f on that interval.

a. e^x
b. $\sin x$

Solution

a. Using the nth-degree Maclaurin polynomial for e^x found in Example 11.10.2a, we find that the Maclaurin series for e^x is given by

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

To determine the interval of convergence, we use the ratio test. Since

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x|^n} = \frac{|x|}{n+1},$$

we have

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0$$

for all x. Therefore, the series converges absolutely for all x, and thus, the interval of convergence is $(-\infty, \infty)$. To show that the series converges to e^x for all x, we use the fact that $f^{(n)}(x) = e^x$ for all $n \geq 0$ and e^x is an increasing function on $(-\infty, \infty)$. Therefore, for any real number b, the maximum value of e^x for all $|x| \leq b$ is e^b. Thus,

$$|R_n(x)| \leq \frac{e^b}{(n+1)!}|x|^{n+1}.$$

Since we just showed that

$$\sum_{n=0}^{\infty} \frac{|x|^n}{n!}$$

converges for all x, by the divergence test, we know that

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

for any real number x. By combining this fact with the squeeze theorem, the result is $\lim_{n\to\infty} R_n(x) = 0$.

b. Using the nth-degree Maclaurin polynomial for $\sin x$ found in Example 11.10.2b, we find that the Maclaurin series for $\sin x$ is given by

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

In order to apply the ratio test, consider

$$\begin{aligned} \frac{|a_{n+1}|}{|a_n|} &= \frac{|x|^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{|x|^{2n+1}} \\ &= \frac{|x|^2}{(2n+3)(2n+2)} \end{aligned}.$$

Since

$$\lim_{n\to\infty} \frac{|x|^2}{(2n+3)(2n+2)} = 0$$

for all x, we obtain the interval of convergence as $(-\infty, \infty)$. To show that the Maclaurin series converges to $\sin x$, look at $R_n(x)$. For each x there exists a real number c between 0 and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}.$$

Since $\left|f^{(n+1)}(c)\right| \leq 1$ for all integers n and all real numbers c, we have

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$$

for all real numbers x. Using the same idea as in part a., the result is $\lim_{n\to\infty} R_n(x) = 0$ for all x, and therefore, the Maclaurin series for $\sin x$ converges to $\sin x$ for all real x.

Exercise 11.10.6

Find the Maclaurin series for $f(x) = \cos x$. Use the ratio test to show that the interval of convergence is $(-\infty, \infty)$. Show that the Maclaurin series converges to $\cos x$ for all real numbers x.

Hint

Use the Maclaurin polynomials for $\cos x$.

Answer

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

By the ratio test, the interval of convergence is $(-\infty, \infty)$. Since $|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$, the series converges to $\cos x$ for all real x.

Project: Proving that e is Irrational

In this project, we use the Maclaurin polynomials for e^x to prove that e is irrational. The proof relies on supposing that e is rational and arriving at a contradiction. Therefore, in the following steps, we suppose $e = r/s$ for some integers r and s where $s \neq 0$.

1. Write the Maclaurin polynomials $p_0(x), p_1(x), p_2(x), p_3(x), p_4(x)$for e^x. Evaluate $p_0(1), p_1(1), p_2(1), p_3(1), p_4(1)$to estimate e.
2. Let $R_n(x)$ denote the remainder when using $p_n(x)$ to estimate e^x. Therefore, $R_n(x) = e^x - p_n(x)$, and $R_n(1) = e - p_n(1)$. Assuming that $e = \frac{r}{s}$ for integers r and s, evaluate $R_0(1), R_1(1), R_2(1), R_3(1), R_4(1)$.
3. Using the results from part 2, show that for each remainder $R_0(1), R_1(1), R_2(1), R_3(1), R_4(1)$, we can find an integer k such that $kR_n(1)$ is an integer for $n = 0, 1, 2, 3, 4$.
4. Write down the formula for the nth-degree Maclaurin polynomial $p_n(x)$ for e^x and the corresponding remainder $R_n(x)$. Show that $sn!R_n(1)$ is an integer.
5. Use Taylor's theorem to write down an explicit formula for $R_n(1)$. Conclude that $R_n(1) \neq 0$, and therefore, $sn!R_n(1) \neq 0$.
6. Use Taylor's theorem to find an estimate on $R_n(1)$. Use this estimate combined with the result from part 5 to show that $|sn!R_n(1)| < \frac{se}{n+1}$. Conclude that if n is large enough, then $|sn!R_n(1)| < 1$. Therefore, $sn!R_n(1)$ is an integer with magnitude less than 1. Thus, $sn!R_n(1) = 0$. But from part 5, we know that $sn!R_n(1) \neq 0$. We have arrived at a contradiction, and consequently, the original supposition that e is rational must be false.

Key Concepts

- Taylor polynomials are used to approximate functions near a value $x = a$. Maclaurin polynomials are Taylor polynomials at $x = 0$.
- The nth-degree Taylor polynomials for a function f are the partial sums of the Taylor series for f.
- If a function f has a power series representation at $x = a$, then it is given by its Taylor series at $x = a$.
- A Taylor series for f converges to f if and only if $\lim_{n\to\infty} R_n(x) = 0$ where $R_n(x) = f(x) - p_n(x)$.
- The Taylor series for e^x, $\sin x$, and $\cos x$ converge to the respective functions for all real x.

Key Equations

- **Taylor series for the function f at the point $x = a$**

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

Glossary

Maclaurin polynomial

a Taylor polynomial centered at 0; the nth-degree Taylor polynomial for f at 0 is the nth-degree Maclaurin polynomial for f

Maclaurin series

a Taylor series for a function f at $x = 0$ is known as a Maclaurin series for f

Taylor polynomials

the nth-degree Taylor polynomial for f at $x = a$ is $p_n(x) = f(a) + f\,'(a)(x-a) + \dfrac{f''(a)}{2!}(x-a)^2 + \cdots + \dfrac{f^{(n)}(a)}{n!}(x-a)^n$

Taylor series

a power series at a that converges to a function f on some open interval containing a.

Taylor's theorem with remainder

for a function f and the nth-degree Taylor polynomial for f at $x = a$, the remainder $R_n(x) = f(x) - p_n(x)$ satisfies $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

for some$c$ between x and a; if there exists an interval I containing a and a real number M such that $\left|f^{(n+1)}(x)\right| \leq M$ for all x in I, then $|R_n(x)| \leq \dfrac{M}{(n+1)!}|x-a|^{n+1}$

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11.11: Applications of Taylor Polynomials

Learning Objectives

- Write the terms of the binomial series.
- Recognize the Taylor series expansions of common functions.
- Recognize and apply techniques to find the Taylor series for a function.
- Use Taylor series to solve differential equations.
- Use Taylor series to evaluate non-elementary integrals.

In the preceding section, we defined Taylor series and showed how to find the Taylor series for several common functions by explicitly calculating the coefficients of the Taylor polynomials. In this section we show how to use those Taylor series to derive Taylor series for other functions. We then present two common applications of power series. First, we show how power series can be used to solve differential equations. Second, we show how power series can be used to evaluate integrals when the antiderivative of the integrand cannot be expressed in terms of elementary functions. In one example, we consider $\int e^{-x^2}\,dx$, an integral that arises frequently in probability theory.

The Binomial Series

Our first goal in this section is to determine the Maclaurin series for the function $f(x)=(1+x)^r$ for all real numbers r. The Maclaurin series for this function is known as the **binomial series.** We begin by considering the simplest case: r is a nonnegative integer. We recall that, for $r=0,\ 1,\ 2,\ 3,\ 4,\ f(x)=(1+x)^r$ can be written as

$$\begin{aligned} f(x) &= (1+x)^0 = 1, \\ f(x) &= (1+x)^1 = 1+x, \\ f(x) &= (1+x)^2 = 1+2x+x^2, \\ f(x) &= (1+x)^3 = 1+3x+3x^2+x^3 \\ f(x) &= (1+x)^4 = 1+4x+6x^2+4x^3+x^4. \end{aligned}$$

The expressions on the right-hand side are known as binomial expansions and the coefficients are known as binomial coefficients. More generally, for any nonnegative integer r, the binomial coefficient of x^n in the binomial expansion of $(1+x)^r$ is given by

$$\binom{r}{n} = \frac{r!}{n!(r-n)!} \tag{11.11.1}$$

and

$$\begin{aligned} f(x) &= (1+x)^r \\ &= \binom{r}{0} + \binom{r}{1}x + \binom{r}{2}x^2 + \binom{r}{3}x^3 + \cdots + \binom{r}{r-1}x^{r-1} + \binom{r}{r}x^r \\ &= \sum_{n=0}^{r}\binom{r}{n}x^n. \end{aligned} \tag{11.11.2}$$

For example, using this formula for $r=5$, we see that

$$\begin{aligned} f(x) &= (1+x)^5 \\ &= \binom{5}{0}1 + \binom{5}{1}x + \binom{5}{2}x^2 + \binom{5}{3}x^3 + \binom{5}{4}x^4 + \binom{5}{5}x^5 \\ &= \frac{5!}{0!5!}1 + \frac{5!}{1!4!}x + \frac{5!}{2!3!}x^2 + \frac{5!}{3!2!}x^3 + \frac{5!}{4!1!}x^4 + \frac{5!}{5!0!}x^5 \\ &= 1+5x+10x^2+10x^3+5x^4+x^5. \end{aligned}$$

We now consider the case when the exponent r is any real number, not necessarily a nonnegative integer. If r is not a nonnegative integer, then $f(x)=(1+x)^r$ cannot be written as a finite polynomial. However, we can find a power series for f. Specifically, we look for the Maclaurin series for f. To do this, we find the derivatives of f and evaluate them at $x=0$.

$$\begin{aligned} f(x) &= (1+x)^r & f(0) &= 1 \\ f'(x) &= r(1+x)^{r-1} & f'(0) &= r \\ f''(x) &= r(r-1)(1+x)^{r-2} & f''(0) &= r(r-1) \\ f'''(x) &= r(r-1)(r-2)(1+x)^{r-3} & f'''(0) &= r(r-1)(r-2) \\ f^{(n)}(x) &= r(r-1)(r-2)\cdots(r-n+1)(1+x)^{r-n} & f^{(n)}(0) &= r(r-1)(r-2)\cdots(r-n+1) \end{aligned}$$

We conclude that the coefficients in the binomial series are given by

$$\frac{f^{(n)}(0)}{n!}=\frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}. \tag{11.11.3}$$

We note that if r is a nonnegative integer, then the $(r+1)^{\text{st}}$ derivative $f^{(r+1)}$ is the zero function, and the series terminates. In addition, if r is a nonnegative integer, then Equation 11.11.3 for the coefficients agrees with Equation 11.11.1 for the coefficients, and the formula for the binomial series agrees with Equation 11.11.2 for the finite binomial expansion. More generally, to denote the binomial coefficients for any real number r, we define

$$\binom{r}{n}=\frac{(r-1)(r-2)\cdots(r-n+1)}{n!}.$$

With this notation, we can write the binomial series for $(1+x)^r$ as

$$\sum_{n=0}^{\infty}\binom{r}{n}x^n=1+rx+\frac{r(r-1)}{2!}x^2+\cdots+\frac{r(r-1)\cdots(r-n+1)}{n!}x^n+\cdots. \tag{11.11.4}$$

We now need to determine the interval of convergence for the binomial series Equation 11.11.4 We apply the ratio test. Consequently, we consider

$$\begin{aligned} \frac{|a_{n+1}|}{|a_n|} &= \frac{|r(r-1)(r-2)\cdots(r-n)||x||^{n+1}}{(n+1)!}\cdot\frac{n!}{|r(r-1)(r-2)\cdots(r-n+1)||x|^n} \\ &= \frac{|r-n||x|}{|n+1|} \end{aligned}$$

.

Since

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=|x|<1$$

if and only if $|x|<1$, we conclude that the interval of convergence for the binomial series is $(-1,1)$. The behavior at the endpoints depends on r. It can be shown that for $r\geq 0$ the series converges at both endpoints; for $-1<r<0$, the series converges at $x=1$ and diverges at $x=-1$; and for $r<-1$, the series diverges at both endpoints. The binomial series does converge to $(1+x)^r$ in $(-1,1)$ for all real numbers r, but proving this fact by showing that the remainder $R_n(x)\to 0$ is difficult.

Definition: Binomial Series

For any real number r, the Maclaurin series for $f(x)=(1+x)^r$ is the binomial series. It converges to f for $|x|<1$, and we write

$$(1+x)^r=\sum_{n=0}^{\infty}\binom{r}{n}x^n=1+rx+\frac{r(r-1)}{2!}x^2+\cdots+r\frac{(r-1)\cdots(r-n+1)}{n!}x^n+\cdots$$

for $|x|<1$.

We can use this definition to find the binomial series for $f(x)=\sqrt{1+x}$ and use the series to approximate $\sqrt{1.5}$.

Example 11.11.1: Finding Binomial Series

a. Find the binomial series for $f(x)=\sqrt{1+x}$.
b. Use the third-order Maclaurin polynomial $p_3(x)$ to estimate $\sqrt{1.5}$. Use Taylor's theorem to bound the error. Use a graphing utility to compare the graphs of f and p_3.

Solution

a. Here $r=\dfrac{1}{2}$. Using the definition for the binomial series, we obtain

$$\begin{aligned}\sqrt{1+x} &= 1+\frac{1}{2}x+\frac{(1/2)(-1/2)}{2!}x^2+\frac{(1/2)(-1/2)(-3/2)}{3!}x^3+\cdots \\ &= 1+\frac{1}{2}x-\frac{1}{2!}\frac{1}{2^2}x^2+\frac{1}{3!}\frac{1\cdot 3}{2^3}x^3-\cdots+\frac{(-1)^{n+1}}{n!}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n}x^n+\cdots \\ &= 1+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n}x^n.\end{aligned}$$

b. From the result in part a. the third-order Maclaurin polynomial is

$$p_3(x)=1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3 \quad .$$

Therefore,

$$\sqrt{1.5}=\sqrt{1+0.5}\approx 1+\frac{1}{2}(0.5)-\frac{1}{8}(0.5)^2+\frac{1}{16}(0.5)^3\approx 1.2266.$$

From Taylor's theorem, the error satisfies

$$R_3(0.5)=\frac{f^{(4)}(c)}{4!}(0.5)^4$$

for some c between 0 and 0.5. Since $f^{(4)}(x)=-\dfrac{15}{2^4(1+x)^{7/2}}$, and the maximum value of $\left|f^{(4)}(x)\right|$ on the interval $(0,0.5)$ occurs at $x=0$, we have

$$|R_3(0.5)|\le\frac{15}{4!2^4}(0.5)^4\approx 0.00244.$$

The function and the Maclaurin polynomial p_3 are graphed in Figure 11.11.1.

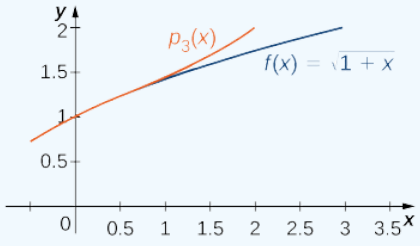


Figure 11.11.1: The third-order Maclaurin polynomial $p_3(x)$ provides a good approximation for $f(x)=\sqrt{1+x}$ for x near zero.

Exercise 11.11.1

Find the binomial series for $f(x)=\dfrac{1}{(1+x)^2}$.

Hint

Use the definition of binomial series for $r=-2$.

Answer

$$\sum_{n=0}^{\infty}(-1)^n(n+1)x^n$$

Common Functions Expressed as Taylor Series

At this point, we have derived Maclaurin series for exponential, trigonometric, and logarithmic functions, as well as functions of the form $f(x)=(1+x)^r$. In Table 11.11.1, we summarize the results of these series. We remark that the convergence of the Maclaurin series for $f(x)=\ln(1+x)$ at the endpoint $x=1$ and the Maclaurin series for $f(x)=\tan^{-1}x$ at the endpoints $x=1$ and $x=-1$ relies on a more advanced theorem than we present here. (Refer to Abel's theorem for a discussion of this more technical point.)

Table 11.11.1: Maclaurin Series for Common Functions

Function	Maclaurin Series	Interval of Convergence
$f(x)=\dfrac{1}{1-x}$	$\displaystyle\sum_{n=0}^{\infty}x^n$	$-1<x<1$
$f(x)=e^x$	$\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$	$-\infty<x<\infty$
$f(x)=\sin x$	$\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$	$-\infty<x<\infty$
$f(x)=\cos x$	$\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$	$-\infty<x<\infty$
$f(x)=\ln(1+x)$	$\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^n}{n}$	$-1<x<1$
$f(x)=\tan^{-1}x$	$\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$	$-1<x<1$
$f(x)=(1+x)^r$	$\displaystyle\sum_{n=0}^{\infty}\binom{r}{n}x^n$	$-1<x<1$

Earlier in the chapter, we showed how you could combine power series to create new power series. Here we use these properties, combined with the Maclaurin series in Table 11.11.1, to create Maclaurin series for other functions.

Example 11.11.2: Deriving Maclaurin Series from Known Series

Find the Maclaurin series of each of the following functions by using one of the series listed in Table 11.11.1.

a. $f(x)=\cos\sqrt{x}$
b. $f(x)=\sinh x$

Solution

a. Using the Maclaurin series for $\cos x$ we find that the Maclaurin series for $\cos\sqrt{x}$ is given by

$$\sum_{n=0}^{\infty}\frac{(-1)^n(\sqrt{x})^{2n}}{(2n)!}=\sum_{n=0}^{\infty}\frac{(-1)^nx^n}{(2n)!}=1-\frac{x}{2!}+\frac{x^2}{4!}-\frac{x^3}{6!}+\frac{x^4}{8!}-\cdots.$$

This series converges to $\cos\sqrt{x}$ for all x in the domain of $\cos\sqrt{x}$; that is, for all $x\geq 0$.

b. To find the Maclaurin series for $\sinh x$, we use the fact that

$$\sinh x=\frac{e^x-e^{-x}}{2}.$$

Using the Maclaurin series for e^x, we see that the nth term in the Maclaurin series for $\sinh x$ is given by

$$\frac{x^n}{n!} - \frac{(-x)^n}{n!}.$$

For n even, this term is zero. For n odd, this term is $\frac{2x^n}{n!}$. Therefore, the Maclaurin series for $\sinh x$ has only odd-order terms and is given by

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots.$$

? Exercise 11.11.2

Find the Maclaurin series for $\sin(x^2)$.

Hint

Use the Maclaurin series for $\sin x$.

Answer

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

We also showed previously in this chapter how power series can be differentiated term by term to create a new power series. In Example 11.11.3 we differentiate the binomial series for $\sqrt{1+x}$ term by term to find the binomial series for $\frac{1}{\sqrt{1+x}}$. Note that we could construct the binomial series for $\frac{1}{\sqrt{1+x}}$ directly from the definition, but differentiating the binomial series for $\sqrt{1+x}$ is an easier calculation.

✓ Example 11.11.3: Differentiating a Series to Find a New Series

Use the binomial series for $\sqrt{1+x}$ to find the binomial series for $\frac{1}{\sqrt{1+x}}$.

Solution

The two functions are related by

$$\frac{d}{dx}\sqrt{1+x} = \frac{1}{2\sqrt{1+x}},$$

so the binomial series for $\frac{1}{\sqrt{1+x}}$ is given by

$$\frac{1}{\sqrt{1+x}} = 2\frac{d}{dx}\sqrt{1+x} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} x^n.$$

? Exercise 11.11.3

Find the binomial series for $f(x) = \frac{1}{(1+x)^{3/2}}$

Hint

Differentiate the series for $\frac{1}{\sqrt{1+x}}$

Answer

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} x^n$$

In this example, we differentiated a known Taylor series to construct a Taylor series for another function. The ability to differentiate power series term by term makes them a powerful tool for solving differential equations. We now show how this is accomplished.

Solving Differential Equations with Power Series

Consider the differential equation

$$y'(x) = y.$$

Recall that this is a first-order separable equation and its solution is $y = Ce^x$. This equation is easily solved using techniques discussed earlier in the text. For most differential equations, however, we do not yet have analytical tools to solve them. Power series are an extremely useful tool for solving many types of differential equations. In this technique, we look for a solution of the form $y = \sum_{n=0}^{\infty} c_n x^n$ and determine what the coefficients would need to be. In the next example, we consider an initial-value problem involving $y' = y$ to illustrate the technique.

Example 11.11.4: Power Series Solution of a Differential Equation

Use power series to solve the initial-value problem $y' = y, \quad y(0) = 3.$

Solution

Suppose that there exists a power series solution

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots.$$

Differentiating this series term by term, we obtain

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots.$$

If y satisfies the differential equation, then

$$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots = c_1 + 2c_2 x + 3c_3 x^2 + 4c_3 x^3 + \cdots.$$

Using the uniqueness of power series representations, we know that these series can only be equal if their coefficients are equal. Therefore,

$$\begin{aligned} c_0 &= c_1, \\ c_1 &= 2c_2, \\ c_2 &= 3c_3, \\ c_3 &= 4c_4, \\ &\vdots \end{aligned}$$

Using the initial condition $y(0) = 3$ combined with the power series representation

$$y(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots \ ,$$

we find that $c_0 = 3$. We are now ready to solve for the rest of the coefficients. Using the fact that $c_0 = 3$, we have

$$
\begin{aligned}
c_1 &= c_0 = 3 = \frac{3}{1!},\\
c_2 &= \frac{c_1}{2} = \frac{3}{2} = \frac{3}{2!},\\
c_3 &= \frac{c_2}{3} = \frac{3}{3\cdot 2} = \frac{3}{3!},\\
c_4 &= \frac{c_3}{4} = \frac{3}{4\cdot 3\cdot 2} = \frac{3}{4!}.
\end{aligned}
$$

Therefore,

$$
y = 3\left[1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3\frac{1}{4!}x^4 + \cdots\right] = 3\sum_{n=0}^{\infty}\frac{x^n}{n!}.
$$

You might recognize

$$
\sum_{n=0}^{\infty}\frac{x^n}{n!}
$$

as the Taylor series for e^x. Therefore, the solution is $y = 3e^x$.

? Exercise 11.11.4

Use power series to solve $y' = 2y, \quad y(0) = 5.$

Hint

The equations for the first several coefficients c_n will satisfy $c_0 = 2c_1$, $c_1 = 2\cdot 2c_2$, $c_2 = 2\cdot 3c_3$, ... In general, for all $n \geq 0$, $c_n = 2(n+1)C_{n+1}$.

Answer

$y = 5e^{2x}$

We now consider an example involving a differential equation that we cannot solve using previously discussed methods. This differential equation

$$
y'' - xy = 0
$$

is known as **Airy's equation**. It has many applications in mathematical physics, such as modeling the diffraction of light. Here we show how to solve it using power series.

✓ Example 11.11.5: Power Series Solution of Airy's Equation

Use power series to solve $y'' - xy = 0$ with the initial conditions $y(0) = a$ and $y'(0) = b$.

Solution

We look for a solution of the form

$$
y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots
$$

Differentiating this function term by term, we obtain

$$
\begin{aligned}
y' &= c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots,\\
y'' &= 2\cdot 1c_2 + 3\cdot 2c_3 x + 4\cdot 3c_4 x^2 + \cdots.
\end{aligned}
$$

If y satisfies the equation $y'' = xy$, then

$$2\cdot 1c_2+3\cdot 2c_3x+4\cdot 3c_4x^2+\cdots=x(c_0+c_1x+c_2x^2+c_3x^3+\cdots).$$

Using the Uniqueness of Power Series Theorem (from an earlier Section) on the uniqueness of power series representations, we know that coefficients of the same degree must be equal. Therefore,

$$\begin{aligned}2\cdot 1c_2&=0,\\3\cdot 2c_3&=c_0,\\4\cdot 3c_4&=c_1,\\5\cdot 4c_5&=c_2,\\&\vdots\end{aligned}$$

More generally, for $n\geq 3$, we have $n\cdot(n-1)c_n=c_{n-3}$. In fact, all coefficients can be written in terms of c_0 and c_1. To see this, first note that $c_2=0$. Then

$$\begin{aligned}c_3&=\frac{c_0}{3\cdot 2},\\c_4&=\frac{c_1}{4\cdot 3}.\end{aligned}$$

For c_5, c_6, c_7, we see that

$$\begin{aligned}c_5&=\frac{c_2}{5\cdot 4}=0,\\c_6&=\frac{c_3}{6\cdot 5}=\frac{c_0}{6\cdot 5\cdot 3\cdot 2},\\c_7&=\frac{c_4}{7\cdot 6}=\frac{c_1}{7\cdot 6\cdot 4\cdot 3}.\end{aligned}$$

Therefore, the series solution of the differential equation is given by

$$y=c_0+c_1x+0\cdot x^2+\frac{c_0}{3\cdot 2}x^3+\frac{c_1}{4\cdot 3}x^4+0\cdot x^5+\frac{c_0}{6\cdot 5\cdot 3\cdot 2}x^6+\frac{c_1}{7\cdot 6\cdot 4\cdot 3}x^7+\cdots.$$

The initial condition $y(0)=a$ implies $c_0=a$. Differentiating this series term by term and using the fact that $y'(0)=b$, we conclude that $c_1=b$.

Therefore, the solution of this initial-value problem is

$$y=a\left(1+\frac{x^3}{3\cdot 2}+\frac{x^6}{6\cdot 5\cdot 3\cdot 2}+\cdots\right)+b\left(x+\frac{x^4}{4\cdot 3}+\frac{x^7}{7\cdot 6\cdot 4\cdot 3}+\cdots\right).$$

? Exercise 11.11.5

Use power series to solve $y''+x^2y=0$ with the initial condition $y(0)=a$ and $y'(0)=b$.

Hint

The coefficients satisfy $c_0=a$, $c_1=b$, $c_2=0$, $c_3=0$, and for $n\geq 4$, $n(n-1)c_n=-c_{n-4}$.

Answer

$$y=a\left(1-\frac{x^4}{3\cdot 4}+\frac{x^8}{3\cdot 4\cdot 7\cdot 8}-\cdots\right)+b\left(x-\frac{x^5}{4\cdot 5}+\frac{x^9}{4\cdot 5\cdot 8\cdot 9}-\cdots\right)$$

Evaluating Non-elementary Integrals

Solving differential equations is one common application of power series. We now turn to a second application. We show how power series can be used to evaluate integrals involving functions whose antiderivatives cannot be expressed using elementary functions.

One integral that arises often in applications in probability theory is $\int e^{-x^2}\,dx$. Unfortunately, the antiderivative of the integrand e^{-x^2} is not an elementary function. By elementary function, we mean a function that can be written using a finite number of algebraic combinations or compositions of exponential, logarithmic, trigonometric, or power functions. We remark that the term "elementary function" is not synonymous with noncomplicated function. For example, the function $f(x)=\sqrt{x^2-3x}+e^{x^3}-\sin(5x+4)$ is an elementary function, although not a particularly simple-looking function. Any integral of the form $\int f(x)\,dx$ where the antiderivative of f cannot be written as an elementary function is considered a **non-elementary integral.**

Non-elementary integrals cannot be evaluated using the basic integration techniques discussed earlier. One way to evaluate such integrals is by expressing the integrand as a power series and integrating term by term. We demonstrate this technique by considering $\int e^{-x^2}\,dx$.

Example 11.11.6: Using Taylor Series to Evaluate a Definite Integral

a. Express $\int e^{-x^2}\,dx$ as an infinite series.

b. Evaluate $\int_0^1 e^{-x^2}\,dx$ to within an error of 0.01.

Solution

a. The Maclaurin series for e^{-x^2} is given by

$$\begin{aligned} e^{-x^2} &= \sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!} \\ &= 1-x^2+\frac{x^4}{2!}-\frac{x^6}{3!}+\cdots+(-1)^n\frac{x^{2n}}{n!}+\cdots \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{n!}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int e^{-x^2}\,dx &= \int\left(1-x^2+\frac{x^4}{2!}-\frac{x^6}{3!}+\cdots+(-1)^n\frac{x^{2n}}{n!}+\cdots\right)dx \\ &= C+x-\frac{x^3}{3}+\frac{x^5}{5.2!}-\frac{x^7}{7.3!}+\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)n!}+\cdots. \end{aligned}$$

b. Using the result from part a. we have

$$\int_0^1 e^{-x^2}\,dx = 1-\frac{1}{3}+\frac{1}{10}-\frac{1}{42}+\frac{1}{216}-\cdots.$$

The sum of the first four terms is approximately 0.74. By the alternating series test, this estimate is accurate to within an error of less than $\frac{1}{216}\approx 0.0046296<0.01$.

Exercise 11.11.6

Express $\int \cos\sqrt{x}\,dx$ as an infinite series. Evaluate $\int_0^1 \cos\sqrt{x}\,dx$ to within an error of 0.01.

Hint

Use the series found in Example $11.11.6$

Answer

$C + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n(2n-2)!}$ The definite integral is approximately 0.514 to within an error of 0.01.

As mentioned above, the integral $\int e^{-x^2}\, dx$ arises often in probability theory. Specifically, it is used when studying data sets that are normally distributed, meaning the data values lie under a bell-shaped curve. For example, if a set of data values is normally distributed with mean μ and standard deviation σ, then the probability that a randomly chosen value lies between $x = a$ and $x = b$ is given by

$$\frac{1}{\sigma\sqrt{2\pi}} \int_a^b e^{-(x-\mu)^2/(2\sigma^2)}\, dx. \tag{11.11.5}$$

(See Figure 11.11.2)

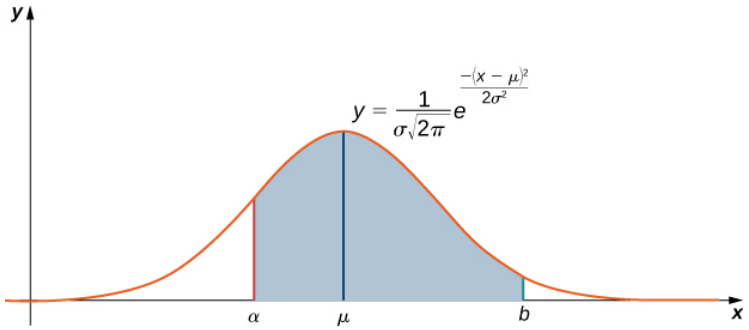


Figure 11.11.2: If data values are normally distributed with mean μ and standard deviation σ, the probability that a randomly selected data value is between a and b is the area under the curve $y = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$ between $x = a$ and $x = b$.

To simplify this integral, we typically let $z = \frac{x-\mu}{\sigma}$. This quantity z is known as the z score of a data value. With this simplification, integral Equation 11.11.5 becomes

$$\frac{1}{\sqrt{2\pi}} \int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} e^{-z^2/2}\, dz.$$

In Example 11.11.7, we show how we can use this integral in calculating probabilities.

Example 11.11.7: Using Maclaurin Series to Approximate a Probability

Suppose a set of standardized test scores are normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 50$. Use Equation 11.11.5 and the first six terms in the Maclaurin series for $e^{-x^2/2}$ to approximate the probability that a randomly selected test score is between $x = 100$ and $x = 200$. Use the alternating series test to determine how accurate your approximation is.

Solution

Since $\mu = 100, \sigma = 50$, and we are trying to determine the area under the curve from $a = 100$ to $b = 200$, integral Equation 11.11.5 becomes

$$\frac{1}{\sqrt{2\pi}} \int_0^2 e^{-z^2/2}\, dz. \tag{11.11.6}$$

The Maclaurin series for $e^{-x^2/2}$ is given by

$$e^{-x^2/2} = \sum_{n=0}^{\infty} \frac{\left(-\frac{x^2}{2}\right)^n}{n!}$$
$$= 1 - \frac{x^2}{2^1 \cdot 1!} + \frac{x^4}{2^2 \cdot 2!} - \frac{x^6}{2^3 \cdot 3!} + \cdots + (-1)^n \frac{x^{2n}}{2^n \cdot n}! + \cdots$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^n \cdot n!}.$$

Therefore,

$$\frac{1}{\sqrt{2\pi}} \int e^{-z^2/2}\, dz = \frac{1}{\sqrt{2\pi}} \int \left(1 - \frac{z^2}{2^1 \cdot 1!} + \frac{z^4}{2^2 \cdot 2!} - \frac{z^6}{2^3 \cdot 3!} + \cdots + (-1)^n \frac{z^{2n}}{2^n \cdot n!} + \cdots\right) dz$$
$$= \frac{1}{\sqrt{2\pi}} \left(C + z - \frac{z^3}{3 \cdot 2^1 \cdot 1!} + \frac{z^5}{5 \cdot 2^2 \cdot 2!} - \frac{z^7}{7 \cdot 2^3 \cdot 3!} + \cdots + (-1)^n \frac{z^{2n+1}}{(2n+1) 2^n \cdot n!} + \cdots\right)$$

We now use this result to evaluate the definite integral from Equation 11.11.6:

$$\frac{1}{\sqrt{2\pi}} \int_0^2 e^{-z^2/2}\, dz = \frac{1}{\sqrt{2\pi}} \left(2 - \frac{8}{6} + \frac{32}{40} - \frac{128}{336} + \frac{512}{3456} - \frac{2^{11}}{11 \cdot 2^5 \cdot 5!} + \cdots\right)$$

Using the first five terms, we estimate that the probability is approximately **0.4922.** By the alternating series test, we see that this estimate is accurate to within

$$\frac{1}{\sqrt{2\pi}} \frac{2^{13}}{13 \cdot 2^6 \cdot 6!} \approx 0.00546.$$

Analysis

If you are familiar with probability theory, you may know that the probability that a data value is within two standard deviations of the mean is approximately 95%. Here we calculated the probability that a data value is between the mean and two standard deviations above the mean, so the estimate should be around 47.5%. The estimate, combined with the bound on the accuracy, falls within this range.

? Exercise 11.11.7

Use the first five terms of the Maclaurin series for $e^{-x^2/2}$ to estimate the probability that a randomly selected test score is between 100 and 150. Use the alternating series test to determine the accuracy of this estimate.

Hint

Evaluate $\int_0^1 e^{-z^2/2}\, dz$ using the first five terms of the Maclaurin series for $e^{-z^2/2}$.

Answer

The estimate is approximately 0.3414.This estimate is accurate to within 0.0000094.

Another application in which a non-elementary integral arises involves the period of a pendulum. The integral is

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

.

An integral of this form is known as an **elliptic integral** of the first kind. Elliptic integrals originally arose when trying to calculate the arc length of an ellipse. We now show how to use power series to approximate this integral.

Example 11.11.8: Period of a Pendulum

The period of a pendulum is the time it takes for a pendulum to make one complete back-and-forth swing. For a pendulum with length L that makes a maximum angle θ_{max} with the vertical, its period T is given by

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

where g is the acceleration due to gravity and $k = \sin\left(\frac{\theta_{max}}{2}\right)$ (see Figure 11.11.3). (We note that this formula for the period arises from a non-linearized model of a pendulum. In some cases, for simplification, a linearized model is used and $\sin\theta$ is approximated by θ.)

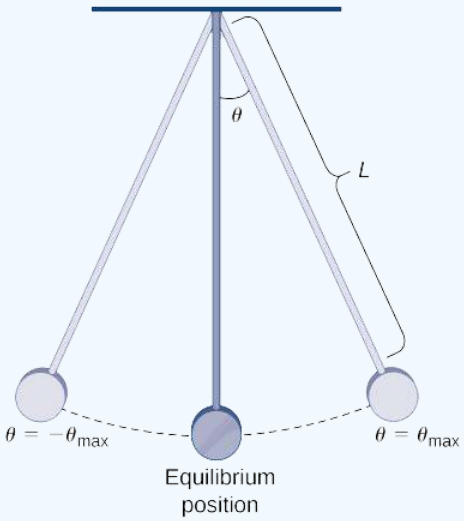


Figure 11.11.3: This pendulum has length L and makes a maximum angle θ_{max} with the vertical.

Use the binomial series

$$\frac{1}{\sqrt{1+x}} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}x^n$$

to estimate the period of this pendulum. Specifically, approximate the period of the pendulum if

a. you use only the first term in the binomial series, and
b. you use the first two terms in the binomial series.

Solution

We use the binomial series, replacing x with $-k^2\sin^2\theta$. Then we can write the period as

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left(1+\frac{1}{2}k^2\sin^2\theta + \frac{1\cdot 3}{2!2^2}k^4\sin^4\theta + \cdots\right)d\theta.$$

a. Using just the first term in the integrand, the first-order estimate is

$$T \approx 4\sqrt{\frac{L}{g}}\int_0^{\pi/2} d\theta = 2\pi\sqrt{\frac{L}{g}}.$$

If θ_{max} is small, then $k = \sin\left(\frac{\theta_{max}}{2}\right)$ is small. We claim that when k is small, this is a good estimate. To justify this claim, consider

$$\int_0^{\pi/2}\left(1+\frac{1}{2}k^2\sin^2\theta + \frac{1\cdot 3}{2!2^2}k^4\sin^4\theta + \cdots\right)d\theta.$$

Since $|\sin x| \leq 1$, this integral is bounded by

$$\int_0^{\pi/2} \left(\frac{1}{2}k^2 + \frac{1.3}{2!2^2}k^4 + \cdots\right) d\theta < \frac{\pi}{2}\left(\frac{1}{2}k^2 + \frac{1\cdot 3}{2!2^2}k^4 + \cdots\right).$$

Furthermore, it can be shown that each coefficient on the right-hand side is less than 1 and, therefore, that this expression is bounded by

$$\frac{\pi k^2}{2}(1 + k^2 + k^4 + \cdots) = \frac{\pi k^2}{2} \cdot \frac{1}{1-k^2} \quad ,$$

which is small for k small.

b. For larger values of θ_{max}, we can approximate T by using more terms in the integrand. By using the first two terms in the integral, we arrive at the estimate

$$T \approx 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \left(1 + \frac{1}{2}k^2 \sin^2\theta\right) d\theta = 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{k^2}{4}\right).$$

The applications of Taylor series in this section are intended to highlight their importance. In general, Taylor series are useful because they allow us to represent known functions using polynomials, thus providing us a tool for approximating function values and estimating complicated integrals. In addition, they allow us to define new functions as power series, thus providing us with a powerful tool for solving differential equations.

Key Concepts

- The binomial series is the Maclaurin series for $f(x) = (1+x)^r$. It converges for $|x| < 1$.
- Taylor series for functions can often be derived by algebraic operations with a known Taylor series or by differentiating or integrating a known Taylor series.
- Power series can be used to solve differential equations.
- Taylor series can be used to help approximate integrals that cannot be evaluated by other means.

Glossary

binomial series

the Maclaurin series for $f(x) = (1+x)^r$; it is given by

$$(1+x)^r = \sum_{n=0}^{\infty} \binom{r}{n} x^n = 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + \frac{r(r-1)\cdots(r-n+1)}{n!}x^n + \cdots \qquad \text{for } |x| < 1$$

non-elementary integral

an integral for which the antiderivative of the integrand cannot be expressed as an elementary function

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CHAPTER OVERVIEW

12: Vectors and The Geometry of Space

A general Calculus Textmap organized around the textbook

Calculus: Early Transcendentals

by James Stewart

I II III IV V VI VII VIII IX X XI XII XIII XIV XV XVI XVII

This Textmap is currently under construction... please be patient with us.

Template:HideTOC

Topic hierarchy

12.1: Three-Dimensional Coordinate Systems
12.2: Vectors
12.3: The Dot Product
12.4: The Cross Product
12.5: Equations of Lines and Planes
12.6: Cylinders and Quadric Surfaces

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12.1: Three-Dimensional Coordinate Systems

In single-variable calculus, the functions that one encounters are functions of a variable (usually x or t) that varies over some subset of the real number line (which we denote by $\mathbb{R}$). For such a function, say, $y = f(x)$, the **graph** of the function f consists of the points $(x, y) = (x, f(x))$. These points lie in the **Euclidean plane**, which, in the **Cartesian** or **rectangular** coordinate system, consists of all ordered pairs of real numbers (a, b). We use the word ``Euclidean" to denote a system in which all the usual rules of Euclidean geometry hold. We denote the Euclidean plane by $\mathbb{R}^2$; the "2" represents the number of *dimensions* of the plane. The Euclidean plane has two perpendicular **coordinate axes**: the x-axis and the y-axis.

In vector (or multivariable) calculus, we will deal with functions of two or three variables (usually x, y or x, y, z, respectively). The graph of a function of two variables, say, $z = f(x, y)$, lies in **Euclidean space**, which in the Cartesian coordinate system consists of all ordered triples of real numbers (a, b, c). Since Euclidean space is 3-dimensional, we denote it by $\mathbb{R}^3$. The graph of f consists of the points $(x, y, z) = (x, y, f(x, y))$. The 3-dimensional coordinate system of Euclidean space can be represented on a flat surface, such as this page or a blackboard, only by giving the illusion of three dimensions, in the manner shown in Figure 12.1.1 . Euclidean space has three mutually perpendicular coordinate axes (x, y and z), and three mutually perpendicular coordinate planes\index{plane!coordinate}: the xy-plane, yz-plane and xz-plane (Figure 12.1.2).

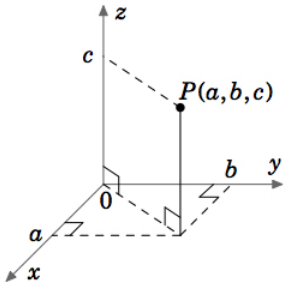


Figure 12.1.1

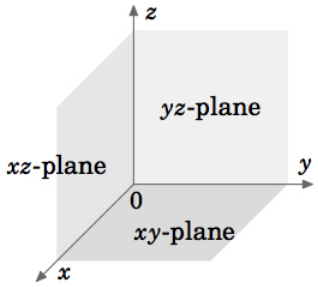


Figure 12.1.2

The coordinate system shown in Figure 12.1.1 is known as a **right-handed coordinate system**, because it is possible, using the right hand, to point the index finger in the positive direction of the x-axis, the middle finger in the positive direction of the y-axis, and the thumb in the positive direction of the z-axis, as in Figure 12.1.3

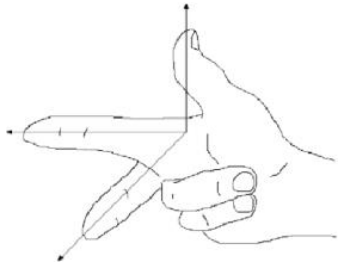


Fig 12.1.3 : *Right-handed coordinate system.*

An equivalent way of defining a right-handed system is if you can point your thumb upwards in the positive z-axis direction while using the remaining four fingers to rotate the x-axis towards the y-axis. Doing the same thing with the left hand is what defines a **left-handed coordinate system**. Notice that switching the x- and y-axes in a right-handed system results in a left-handed system, and that rotating either type of system does not change its ``handedness". Throughout the book we will use a right-handed system.

For functions of three variables, the graphs exist in 4-dimensional space (i.e. $\mathbb{R}^4$), which we can not see in our 3-dimensional space, let alone simulate in 2-dimensional space. So we can only think of 4-dimensional space abstractly. For an entertaining discussion of this subject, see the book by ABBOT.

So far, we have discussed the *position* of an object in 2-dimensional or 3-dimensional space. But what about something such as the velocity of the object, or its acceleration? Or the gravitational force acting on the object? These phenomena all seem to involve motion and *direction* in some way. This is where the idea of a *vector* comes in.

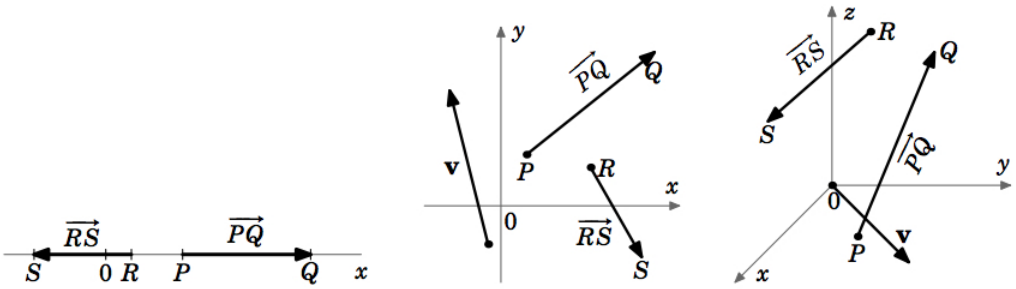
You have already dealt with velocity and acceleration in single-variable calculus. For example, for motion along a straight line, if $y = f(t)$ gives the displacement of an object after time t, then $dy/dt = f'(t)$ is the velocity of the object at time t. The derivative $f'(t)$ is just a number, which is positive if the object is moving in an agreed-upon "positive" direction, and negative if it moves in the opposite of that direction. So you can think of that number, which was called the velocity of the object, as having two components: a *magnitude*, indicated by a nonnegative number, preceded by a *direction*, indicated by a plus or minus symbol (representing motion in the positive direction or the negative direction, respectively), i.e. $f'(t) = \pm a$ for some number $a \geq 0$. Then a is the magnitude of the velocity (normally called the *speed* of the object), and the $\pm$ represents the direction of the velocity (though the $+$ is usually omitted for the positive direction).

For motion along a straight line, i.e. in a 1-dimensional space, the velocities are also contained in that 1-dimensional space, since they are just numbers. For general motion along a curve in 2- or 3-dimensional space, however, velocity will need to be represented by a multidimensional object which should have both a magnitude and a direction. A geometric object which has those features is an arrow, which in elementary geometry is called a ``directed line segment". This is the motivation for how we will define a vector.

Definition 12.1.1

A (nonzero) **vector** is a directed line segment drawn from a point P (called its **initial point**) to a point Q (called its **terminal point**), with P and Q being distinct points. The vector is denoted by $\overrightarrow{PQ}$. Its **magnitude** is the length of the line segment, denoted by $\|\overrightarrow{PQ}\|$, and its **direction** is the same as that of the directed line segment. The **zero vector** is just a point, and it is denoted by $\mathbf{0}$.

To indicate the direction of a vector, we draw an arrow from its initial point to its terminal point. We will often denote a vector by a single bold-faced letter (e.g. $\mathbf{v}$) and use the terms ``magnitude" and ``length" interchangeably. Note that our definition could apply to systems with any number of dimensions (Figure 1.1.4 (a)-(c)).



(a) One dimension **(b)** Two dimensions **(c)** Three dimensions

Figure 12.1.4 *Vectors in different dimensions*

A few things need to be noted about the zero vector. Our motivation for what a vector is included the notions of magnitude and direction. What is the magnitude of the zero vector? We define it to be zero, i.e. $\|\mathbf{0}\| = 0$. This agrees with the definition of the zero vector as just a point, which has zero length. What about the direction of the zero vector? A single point really has no well-defined direction. Notice that we were careful to only define the direction of a *nonzero* vector, which is well-defined since the initial and terminal points are distinct. Not everyone agrees on the direction of the zero vector. Some contend that the zero vector has *arbitrary* direction (i.e. can take any direction), some say that it has *indeterminate* direction (i.e. the direction cannot be determined), while others say that it has *no* direction. Our definition of the zero vector, however, does not require it to have a direction, and we will leave it at that.

Now that we know what a vector is, we need a way of determining when two vectors are equal. This leads us to the following definition.

Definition 12.1.2

Two nonzero vectors are **equal** if they have the same magnitude and the same direction. Any vector with zero magnitude is equal to the zero vector.

By this definition, vectors with the same magnitude and direction but with different initial points would be equal. For example, in Figure 1.1.5 the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ all have the same magnitude $\sqrt{5}$ (by the Pythagorean Theorem). And we see that $\mathbf{u}$ and $\mathbf{w}$ are parallel, since they lie on lines having the same slope $\frac{1}{2}$, and they point in the same direction. So $\mathbf{u} = \mathbf{w}$, even though they have different initial points. We also see that $\mathbf{v}$ is parallel to $\mathbf{u}$ but points in the opposite direction. So $\mathbf{u} \neq \mathbf{v}$.

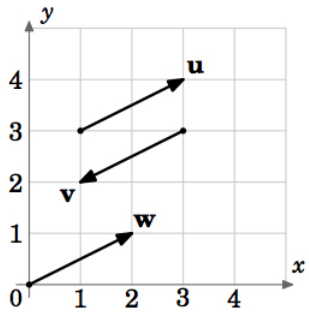


Figure 12.1.5

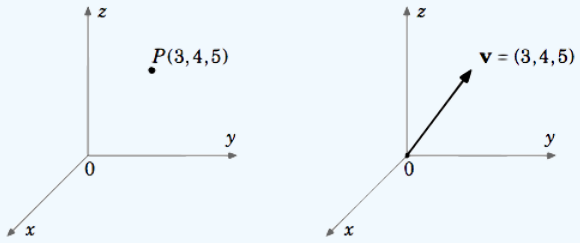
So we can see that there are an infinite number of vectors for a given magnitude and direction, those vectors all being equal and differing only by their initial and terminal points. Is there a single vector which we can choose to represent all those equal vectors? The answer is yes, and is suggested by the vector $\mathbf{w}$ in Figure 12.1.5

> Unless otherwise indicated, when speaking of "the vector" with a given magnitude and direction, we will mean the one whose initial point is at the origin of the coordinate system.

Thinking of vectors as starting from the origin provides a way of dealing with vectors in a standard way, since every coordinate system has an origin. But there will be times when it is convenient to consider a different initial point for a vector (for example, when adding vectors, which we will do in the next section). Another advantage of using the origin as the initial point is that it provides an easy correspondence between a vector and its terminal point.

Example 12.1.1

Let $\mathbf{v}$ be the vector in $\mathbb{R}^3$ whose initial point is at the origin and whose terminal point is $(3, 4, 5)$. Though the *point* $(3, 4, 5)$ and the vector $\mathbf{v}$ are different objects, it is convenient to write $\mathbf{v} = (3, 4, 5)$. When doing this, it is understood that the initial point of $\mathbf{v}$ is at the origin $(0, 0, 0)$ and the terminal point is $(3, 4, 5)$.



(a) The point (3,4,5) **(b)** The vector (3,4,5)

Figure 12.1.6 Correspondence between points and vectors

Unless otherwise stated, when we refer to vectors as $\mathbf{v} = (a, b)$ in $\mathbb{R}^2$ or $\mathbf{v} = (a, b, c)$ in $\mathbb{R}^3$, we mean vectors in Cartesian coordinates starting at the origin. Also, we will write the zero vector $\mathbf{0}$ in $\mathbb{R}^2$ and $\mathbb{R}^3$ as $(0, 0)$ and $(0, 0, 0)$, respectively.

The point-vector correspondence provides an easy way to check if two vectors are equal, without having to determine their magnitude and direction. Similar to seeing if two points are the same, you are now seeing if the terminal points of vectors starting at the origin are the same. For each vector, find the (unique!) vector it equals whose initial point is the origin. Then compare the coordinates of the terminal points of these ``new" vectors: if those coordinates are the same, then the original vectors are equal. To get the ``new" vectors starting at the origin, you *translate* each vector to start at the origin by subtracting the coordinates of the

original initial point from the original terminal point. The resulting point will be the terminal point of the ``new" vector whose initial point is the origin. Do this for each original vector then compare.

Example 12.1.2

Consider the vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ in $\mathbb{R}^3$, where $P=(2,1,5), Q=(3,5,7), R=(1,-3,-2)$ and $S=(2,1,0)$. Does $\overrightarrow{PQ}=\overrightarrow{RS}$?

The vector $\overrightarrow{PQ}$ is equal to the vector $\mathbf{v}$ with initial point $(0,0,0)$ and terminal point $Q-P=(3,5,7)-(2,1,5)=(3-2,5-1,7-5)=(1,4,2)$.

Similarly, $\overrightarrow{RS}$ is equal to the vector $\mathbf{w}$ with initial point $(0,0,0)$ and terminal point $S-R=(2,1,0)-(1,-3,-2)=(2-1,1-(-3),0-(-2))=(1,4,2)$.

So $\overrightarrow{PQ}=\mathbf{v}=(1,4,2)$ and $\overrightarrow{RS}=\mathbf{w}=(1,4,2)$.

$\therefore \overrightarrow{PQ}=\overrightarrow{RS}$

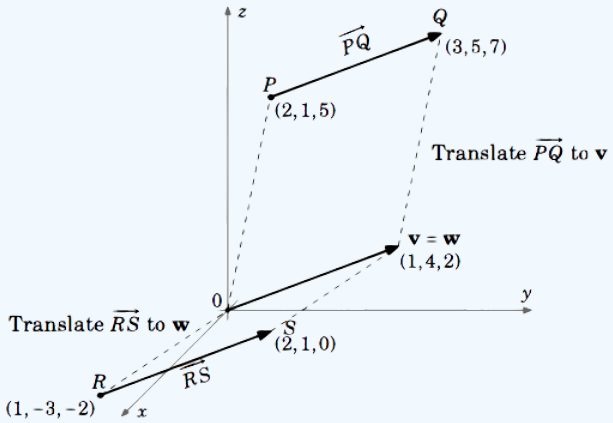


Figure 12.1.7

Recall the distance formula for points in the Euclidean plane:

For points $P=(x_1,y_1)$, $Q=(x_2,y_2)$ in $\mathbb{R}^2$, the distance d between P and Q is:

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \tag{12.1.1}$$

By this formula, we have the following result:

Note

For a vector $\overrightarrow{PQ}$ in $\mathbb{R}^2$ with initial point $P=(x_1,y_1)$ and terminal point $Q=(x_2,y_2)$, the magnitude of $\overrightarrow{PQ}$ is:

$$\|\overrightarrow{PQ}\|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \tag{12.1.2}$$

Finding the magnitude of a vector $\mathbf{v}=(a,b)$ in $\mathbb{R}^2$ is a special case of the above formula with $P=(0,0)$ and $Q=(a,b)$:

For a vector $\mathbf{v}=(a,b)$ in $\mathbb{R}^2$, the magnitude of $\mathbf{v}$ is:

$$\|\mathbf{v}\|=\sqrt{a^2+b^2} \tag{12.1.3}$$

To calculate the magnitude of vectors in $\mathbb{R}^3$, we need a distance formula for points in Euclidean space (we will postpone the proof until the next section):

Theorem 12.1.1

The distance d between points $P=(x_1,y_1,z_1)$ and $Q=(x_2,y_2,z_2)$ in $\mathbb{R}^3$ is:

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2} \tag{12.1.4}$$

The proof will use the following result:

Theorem 12.1.2

For a vector $\mathbf{v}=(a,b,c)$ in $\mathbb{R}^3$, the magnitude of $\mathbf{v}$ is:

$$\|\mathbf{v}\|=\sqrt{a^2+b^2+c^2} \tag{12.1.5}$$

Proof: There are four cases to consider:

Case 1: $a=b=c=0$. Then $\mathbf{v}=\mathbf{0}$, so $\|\mathbf{v}\|=0=\sqrt{0^2+0^2+0^2}=\sqrt{a^2+b^2+c^2}$.

Case 2: exactly two of a,b,c are 0. Without loss of generality, we assume that $a=b=0$ and $c\neq 0$ (the other two possibilities are handled in a similar manner). Then $\mathbf{v}=(0,0,c)$, which is a vector of length $|c|$ along the z-axis. So $\|\mathbf{v}\|=|c|=\sqrt{c^2}=\sqrt{0^2+0^2+c^2}=\sqrt{a^2+b^2+c^2}$.

Case 3: exactly one of a,b,c is 0. Without loss of generality, we assume that $a=0$, $b\neq 0$ and $c\neq 0$ (the other two possibilities are handled in a similar manner). Then $\mathbf{v}=(0,b,c)$, which is a vector in the yz-plane, so by the Pythagorean Theorem we have $\|\mathbf{v}\|=\sqrt{b^2+c^2}=\sqrt{0^2+b^2+c^2}=\sqrt{a^2+b^2+c^2}$.

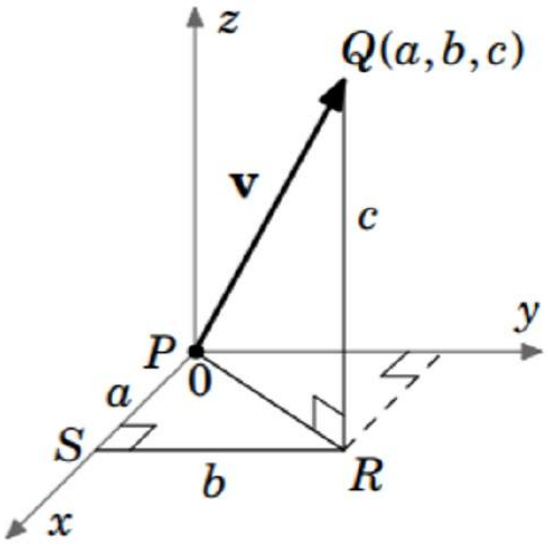


Figure 12.1.8

Case 4: none of a,b,c are 0. Without loss of generality, we can assume that a,b,c are all positive (the other seven possibilities are handled in a similar manner). Consider the points $P=(0,0,0)$, $Q=(a,b,c)$, $R=(a,b,0)$, and $S=(a,0,0)$, as shown in Figure 1.1.8. Applying the Pythagorean Theorem to the right triangle $\triangle PSR$ gives $|PR|^2=a^2+b^2$. A second application of the Pythagorean Theorem, this time to the right triangle $\triangle PQR$, gives $\|\mathbf{v}\|=|PQ|=\sqrt{|PR|^2+|QR|^2}=\sqrt{a^2+b^2+c^2}$. This proves the theorem.

(QED)

Example 1.3

Calculate the following:

1. The magnitude of the vector $\overrightarrow{PQ}$ in $\mathbb{R}^2$ with $P=(-1,2)$ and $Q=(5,5)$.
   Solution: By formula (1.2), $\|\overrightarrow{PQ}\| = \sqrt{(5-(-1))^2+(5-2)^2} = \sqrt{36+9} = \sqrt{45} = 3\sqrt{5}$.
2. The magnitude of the vector $\mathbf{v}=(8,3)$ in $\mathbb{R}^2$.
   Solution: By formula (1.3), $\|\mathbf{v}\| = \sqrt{8^2+3^2} = \sqrt{73}$.
3. The distance between the points $P=(2,-1,4)$ and $Q=(4,2,-3)$ in $\mathbb{R}^2$.
   Solution: By formula (1.4), the distance $d = \sqrt{(4-2)^2+(2-(-1))^2+(-3-4)^2} = \sqrt{4+9+49} = \sqrt{62}$.
4. The magnitude of the vector $\mathbf{v}=(5,8,-2)$ in $\mathbb{R}^3$.
   Solution: By formula (1.5), $\|\mathbf{v}\| = \sqrt{5^2+8^2+(-2)^2} = \sqrt{25+64+4} = \sqrt{93}$.

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12.2: Vectors

Learning Objectives

- Describe three-dimensional space mathematically.
- Locate points in space using coordinates.
- Write the distance formula in three dimensions.
- Write the equations for simple planes and spheres.
- Perform vector operations in $\mathbb{R}^3$.

Vectors are useful tools for solving two-dimensional problems. Life, however, happens in three dimensions. To expand the use of vectors to more realistic applications, it is necessary to create a framework for describing three-dimensional space. For example, although a two-dimensional map is a useful tool for navigating from one place to another, in some cases the topography of the land is important. Does your planned route go through the mountains? Do you have to cross a river? To appreciate fully the impact of these geographic features, you must use three dimensions. This section presents a natural extension of the two-dimensional Cartesian coordinate plane into three dimensions.

Three-Dimensional Coordinate Systems

As we have learned, the two-dimensional rectangular coordinate system contains two perpendicular axes: the horizontal x-axis and the vertical y-axis. We can add a third dimension, the z-axis, which is perpendicular to both the x-axis and the y-axis. We call this system the three-dimensional rectangular coordinate system. It represents the three dimensions we encounter in real life.

Definition: Three-dimensional Rectangular Coordinate System

The three-dimensional rectangular coordinate system consists of three perpendicular axes: the x-axis, the y-axis, and the z-axis. Because each axis is a number line representing all real numbers in $\mathbb{R}$, the three-dimensional system is often denoted by $\mathbb{R}^3$.

In Figure $12.2.1a$, the positive z-axis is shown above the plane containing the x- and y-axes. The positive x-axis appears to the left and the positive y-axis is to the right. A natural question to ask is: How was this arrangement determined? The system displayed follows the **right-hand rule**. If we take our right hand and align the fingers with the positive x-axis, then curl the fingers so they point in the direction of the positive y-axis, our thumb points in the direction of the positive z-axis (Figure $12.2.1b$). In this text, we always work with coordinate systems set up in accordance with the right-hand rule. Some systems do follow a left-hand rule, but the right-hand rule is considered the standard representation.

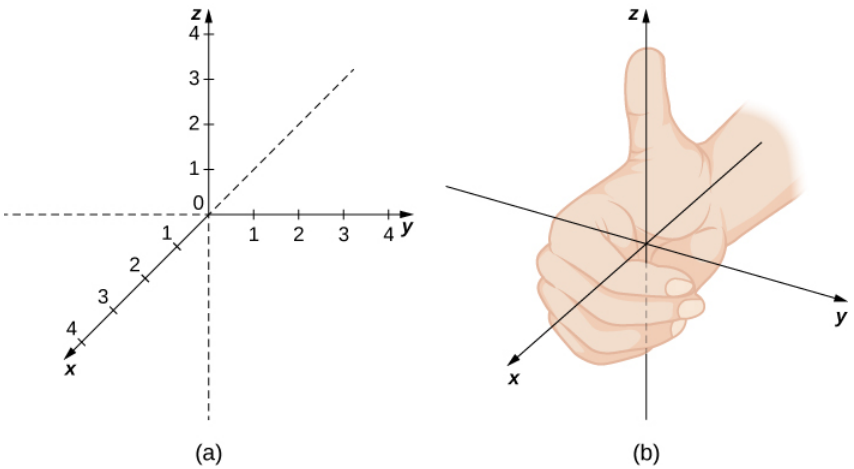


Figure 12.2.1: (a) We can extend the two-dimensional rectangular coordinate system by adding a third axis, the z-axis, that is perpendicular to both the x-axis and the y-axis. (b) The right-hand rule is used to determine the placement of the coordinate axes in the standard Cartesian plane.

In two dimensions, we describe a point in the plane with the coordinates (x, y). Each coordinate describes how the point aligns with the corresponding axis. In three dimensions, a new coordinate, z, is appended to indicate alignment with the z-axis: (x, y, z).

A point in space is identified by all three coordinates (Figure 12.2.2). To plot the point (x, y, z), go x units along the x-axis, then y units in the direction of the y-axis, then z units in the direction of the z-axis.

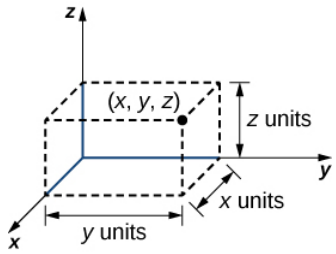


Figure 12.2.2: To plot the point (x, y, z) go x units along the x-axis, then y units in the direction of the y-axis, then z units in the direction of the z-axis.

Example 12.2.1: Locating Points in Space

Sketch the point $(1, -2, 3)$ in three-dimensional space.

Solution

To sketch a point, start by sketching three sides of a rectangular prism along the coordinate axes: one unit in the positive x direction, 2 units in the negative y direction, and 3 units in the positive z direction. Complete the prism to plot the point (Figure 12.2.3).

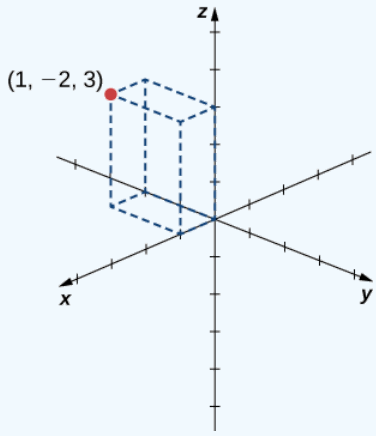


Figure 12.2.3: Sketching the point $(1, -2, 3)$.

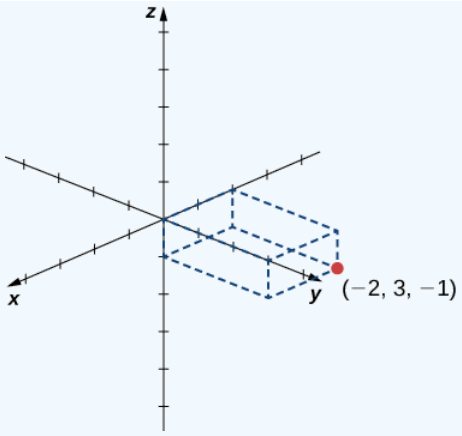
Exercise 12.2.1

Sketch the point $(-2, 3, -1)$ in three-dimensional space.

Hint

Start by sketching the coordinate axes. e.g., Figure 12.2.3 Then sketch a rectangular prism to help find the point in space.

Answer



In two-dimensional space, the coordinate plane is defined by a pair of perpendicular axes. These axes allow us to name any location within the plane. In three dimensions, we define **coordinate planes** by the coordinate axes, just as in two dimensions. There are three axes now, so there are three intersecting pairs of axes. Each pair of axes forms a coordinate plane: the xy-plane, the xz-plane, and the yz-plane (Figure 12.2.4). We define the xy-plane formally as the following set: $\{(x, y, 0) : x, y \in \mathbb{R}\}$. Similarly, the xz-plane and the yz-plane are defined as $\{(x, 0, z) : x, z \in \mathbb{R}\}$ and $\{(0, y, z) : y, z \in \mathbb{R}\}$, respectively.

To visualize this, imagine you're building a house and are standing in a room with only two of the four walls finished. (Assume the two finished walls are adjacent to each other.) If you stand with your back to the corner where the two finished walls meet, facing out into the room, the floor is the xy-plane, the wall to your right is the xz-plane, and the wall to your left is the yz-plane.

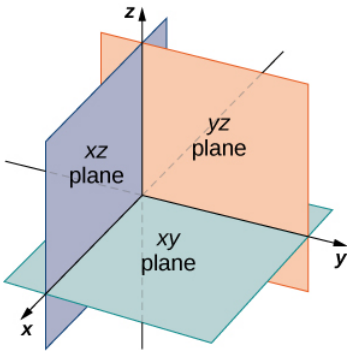


Figure 12.2.4: The plane containing the x- *and* y-axes is called the xy-plane. The plane containing the x- and z-axes is called the xz-plane, and the y- and z-axes define the yz-plane.

In two dimensions, the coordinate axes partition the plane into four quadrants. Similarly, the coordinate planes divide space between them into eight regions about the origin, called **octants.** The octants fill $\mathbb{R}^3$ in the same way that quadrants fill $\mathbb{R}^2$, as shown in Figure 12.2.5.

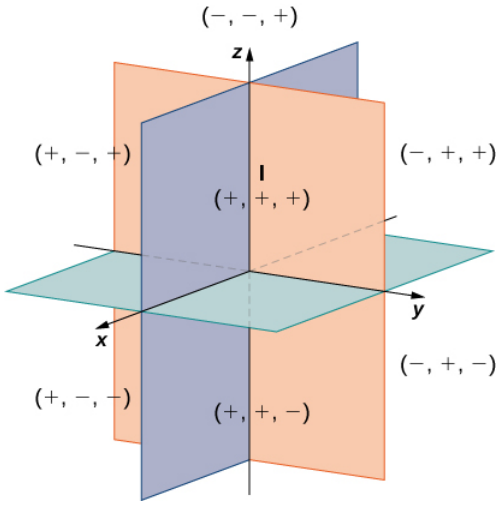


Figure 12.2.5: Points that lie in octants have three nonzero coordinates.

Most work in three-dimensional space is a comfortable extension of the corresponding concepts in two dimensions. In this section, we use our knowledge of circles to describe spheres, then we expand our understanding of vectors to three dimensions. To accomplish these goals, we begin by adapting the distance formula to three-dimensional space.

If two points lie in the same coordinate plane, then it is straightforward to calculate the distance between them. We know that the distance d between two points (x_1, y_1) and (x_2, y_2) in the xy-coordinate plane is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\,.$$

The formula for the distance between two points in space is a natural extension of this formula.

The Distance between Two Points in Space

The distance d between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\,. \tag{12.2.1}$$

The proof of this theorem is left as an exercise. (Hint: First find the distance d_1 between the points (x_1, y_1, z_1) and (x_2, y_2, z_1) as shown in Figure 12.2.6)

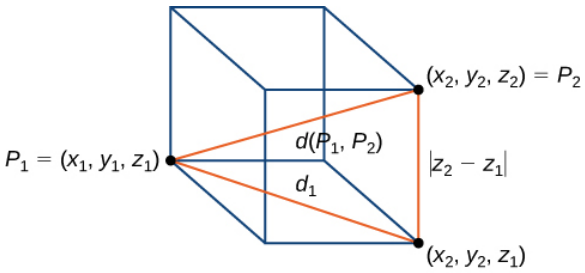


Figure 12.2.6: The distance between P_1 and P_2 is the length of the diagonal of the rectangular prism having P_1 and P_2 as opposite corners.

Example 12.2.2: Distance in Space

Find the distance between points $P_1 = (3, -1, 5)$ and $P_2 = (2, 1, -1)$.

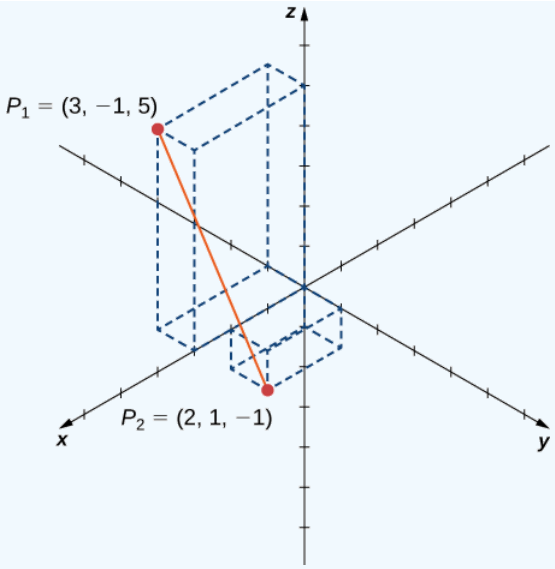


Figure 12.2.7: Find the distance between the two points.

Solution

Substitute values directly into the distance formula (Equation 12.2.1):

$$\begin{aligned} d(P_1, P_2) &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ &= \sqrt{(2-3)^2 + (1-(-1))^2 + (-1-5)^2} \\ &= \sqrt{(-1)^2 + 2^2 + (-6)^2} \\ &= \sqrt{41}. \end{aligned}$$

? Exercise 12.2.2

Find the distance between points $P_1 = (1, -5, 4)$ and $P_2 = (4, -1, -1)$.

Hint

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Answer

$$5\sqrt{2}$$

Before moving on to the next section, let's get a feel for how $\mathbb{R}^3$ differs from $\mathbb{R}^2$. For example, in $\mathbb{R}^2$, lines that are not parallel must always intersect. This is not the case in $\mathbb{R}^3$. For example, consider the lines shown in Figure 12.2.8 These two lines are not parallel, nor do they intersect.

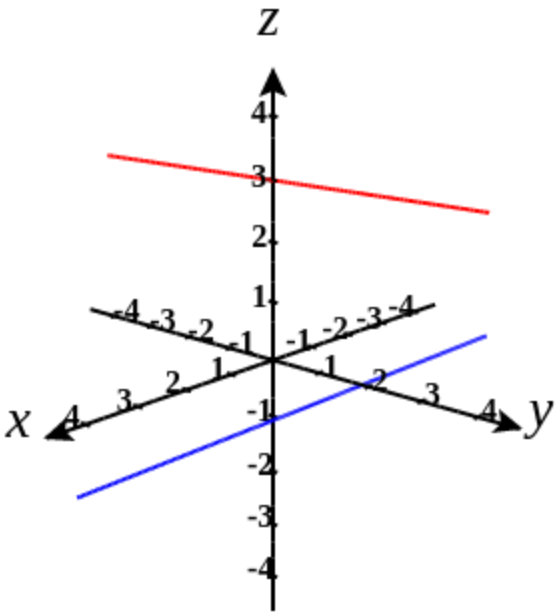


Figure 12.2.8: These two lines are not parallel, but still do not intersect.

You can also have circles that are interconnected but have no points in common, as in Figure 12.2.9.

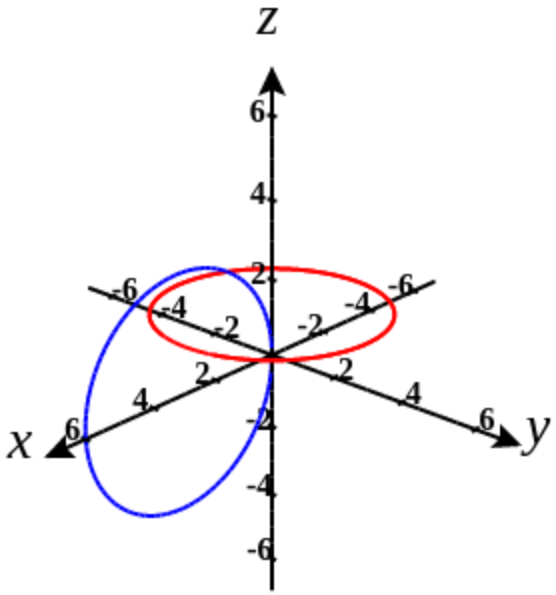


Figure 12.2.9: These circles are interconnected, but have no points in common.

We have a lot more flexibility working in three dimensions than we do if we stuck with only two dimensions.

Writing Equations in $\mathbb{R}^3$

Now that we can represent points in space and find the distance between them, we can learn how to write equations of geometric objects such as lines, planes, and curved surfaces in $\mathbb{R}^3$. First, we start with a simple equation. Compare the graphs of the equation $x = 0$ in $\mathbb{R}$, $\mathbb{R}^2$,and $\mathbb{R}^3$ (Figure 12.2.10). From these graphs, we can see the same equation can describe a point, a line, or a plane.

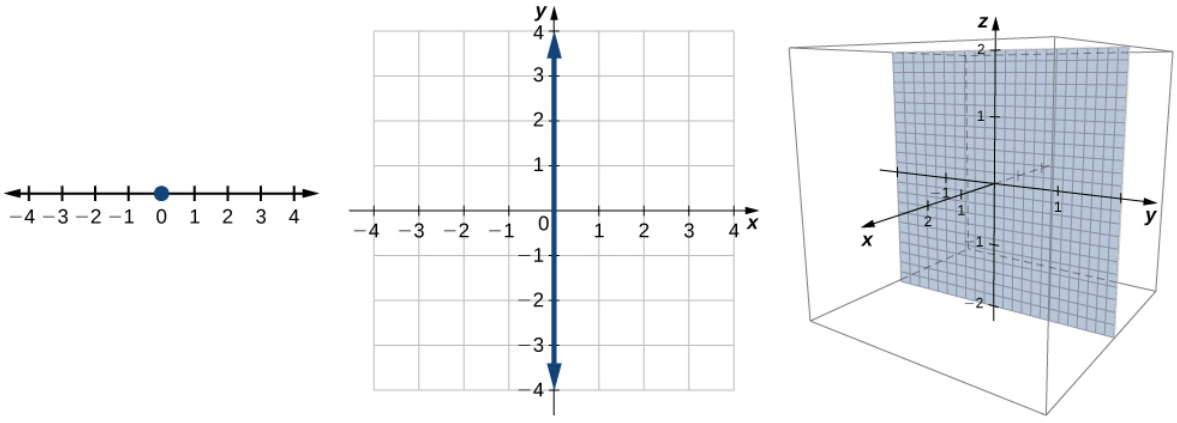


Figure 12.2.10: (a) In $\mathbb{R}$, the equation $x = 0$ describes a single point. (b) In $\mathbb{R}^2$, the equation $x = 0$ describes a line, the y-axis. (c) In $\mathbb{R}^3$, the equation $x = 0$ describes a plane, the yz-plane.

In space, the equation $x = 0$ describes all points $(0, y, z)$. This equation defines the yz-plane. Similarly, the xy-plane contains all points of the form $(x, y, 0)$. The equation $z = 0$ defines the xy-plane and the equation $y = 0$ describes the xz-plane (Figure 12.2.11).

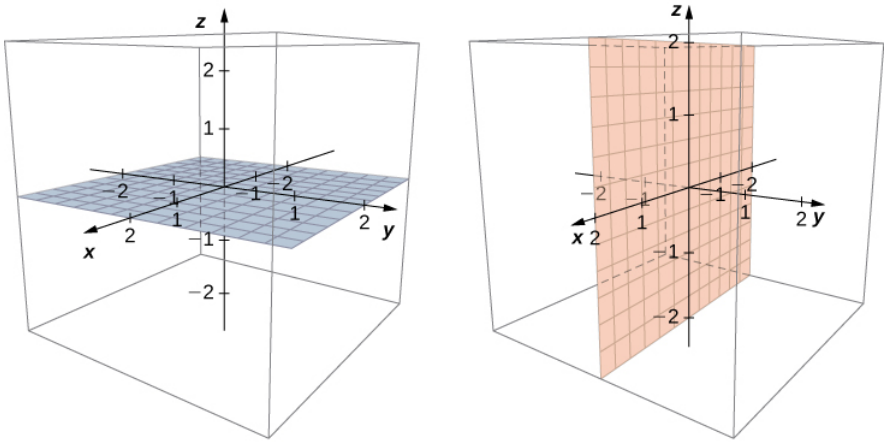


Figure 12.2.11: (a) In space, the equation $z = 0$ describes the xy-plane. (b) All points in the xz-plane satisfy the equation $y = 0$.

Understanding the equations of the coordinate planes allows us to write an equation for any plane that is parallel to one of the coordinate planes. When a plane is parallel to the xy-plane, for example, the z-coordinate of each point in the plane has the same constant value. Only the x- and y-coordinates of points in that plane vary from point to point.

Equations of Planes Parallel to Coordinate Planes

1. The plane in space that is parallel to the xy-plane and contains point (a, b, c) can be represented by the equation $z = c$.
2. The plane in space that is parallel to the xz-plane and contains point (a, b, c) can be represented by the equation $y = b$.
3. The plane in space that is parallel to the yz-plane and contains point (a, b, c) can be represented by the equation $x = a$.

Example 12.2.3: Writing Equations of Planes Parallel to Coordinate Planes

a. Write an equation of the plane passing through point $(3, 11, 7)$ that is parallel to the yz-plane.
b. Find an equation of the plane passing through points $(6, -2, 9)$, $(0, -2, 4)$,and $(1, -2, -3)$.

Solution

a. When a plane is parallel to the yz-plane, only the y- and z-coordinates may vary. The x-coordinate has the same constant value for all points in this plane, so this plane can be represented by the equation $x = 3$.
b. Each of the points $(6, -2, 9)$, $(0, -2, 4)$,and $(1, -2, -3)$ has the same y-coordinate. This plane can be represented by the equation $y = -2$.

Exercise 12.2.3

Write an equation of the plane passing through point $(1, -6, -4)$ that is parallel to the xy-plane.

Hint

If a plane is parallel to the xy-plane, the z-coordinates of the points in that plane do not vary.

Answer

$$z = -4$$

As we have seen, in $\mathbb{R}^2$ the equation $x = 5$ describes the vertical line passing through point $(5, 0)$. This line is parallel to the y-axis. In a natural extension, the equation $x = 5$ in $\mathbb{R}^3$ describes the plane passing through point $(5, 0, 0)$, which is parallel to the yz-plane. Another natural extension of a familiar equation is found in the equation of a sphere.

Definition: Sphere

A sphere is the set of all points in space equidistant from a fixed point, the center of the sphere (Figure 12.2.12), just as the set of all points in a plane that are equidistant from the center represents a circle. In a sphere, as in a circle, the distance from the center to a point on the sphere is called the *radius.*

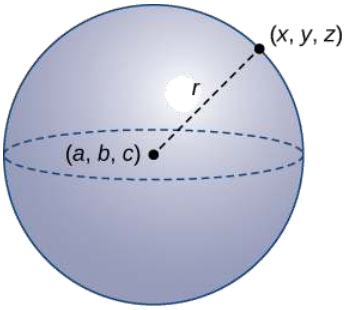


Figure 12.2.12: Each point (x, y, z) on the surface of a sphere is r units away from the center (a, b, c).

The equation of a circle is derived using the distance formula in two dimensions. In the same way, the equation of a sphere is based on the three-dimensional formula for distance.

Standard Equation of a Sphere

The sphere with center (a, b, c) and radius r can be represented by the equation

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2.$$

This equation is known as the *standard equation of a sphere*.

Example 12.2.4: Finding an Equation of a Sphere

Find the standard equation of the sphere with center $(10, 7, 4)$ and point $(-1, 3, -2)$, as shown in Figure 12.2.13

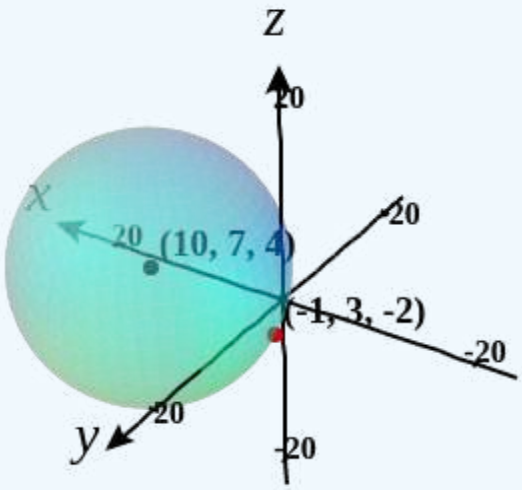


Figure 12.2.13 The sphere centered at $(10, 7, 4)$ containing point $(-1, 3, -2)$.

Solution

Use the distance formula to find the radius r of the sphere:

$$\begin{aligned} r &= \sqrt{(-1-10)^2+(3-7)^2+(-2-4)^2} \\ &= \sqrt{(-11)^2+(-4)^2+(-6)^2} \\ &= \sqrt{173} \end{aligned}$$

The standard equation of the sphere is

$$(x-10)^2+(y-7)^2+(z-4)^2 = 173.$$

? Exercise 12.2.4

Find the standard equation of the sphere with center $(-2, 4, -5)$ containing point $(4, 4, -1)$.

Hint

First use the distance formula to find the radius of the sphere.

Answer

$$(x+2)^2+(y-4)^2+(z+5)^2 = 52$$

Example 12.2.5: Finding the Equation of a Sphere

Let $P=(-5,2,3)$ and $Q=(3,4,-1)$, and suppose line segment $\overline{PQ}$ forms the diameter of a sphere (Figure 12.2.14). Find the equation of the sphere.

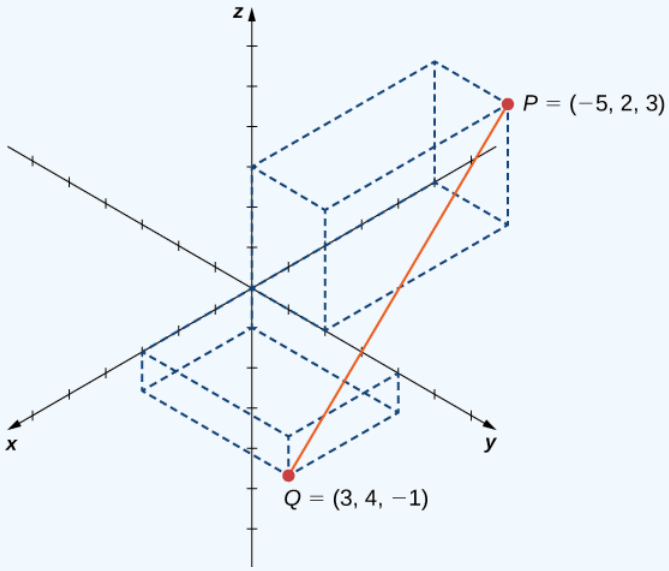


Figure 12.2.14: Line segment $\overline{PQ}$.

Solution:

Since $\overline{PQ}$ is a diameter of the sphere, we know the center of the sphere is the midpoint of $\overline{PQ}$.Then,

$$C=\left(\frac{-5+3}{2},\frac{2+4}{2},\frac{3+(-1)}{2}\right)=(-1,3,1).$$

Furthermore, we know the radius of the sphere is half the length of the diameter. This gives

$$\begin{aligned} r &= \frac{1}{2}\sqrt{(-5-3)^2+(2-4)^2+(3-(-1))^2} \\ &= \frac{1}{2}\sqrt{64+4+16} \\ &= \sqrt{21} \end{aligned}$$

Then, the equation of the sphere is $(x+1)^2+(y-3)^2+(z-1)^2=21$.

Exercise 12.2.5

Find the equation of the sphere with diameter $\overline{PQ}$, where $P=(2,-1,-3)$ and $Q=(-2,5,-1)$.

Hint

Find the midpoint of the diameter first.

Answer

$$x^2+(y-2)^2+(z+2)^2=14$$

Example 12.2.6: Graphing Other Equations in Three Dimensions

Describe the set of points that satisfies $(x-4)(z-2)=0$, and graph the set.

Solution

We must have either $x-4=0$ or $z-2=0$, so the set of points forms the two planes $x=4$ and $z=2$ (Figure 12.2.15).

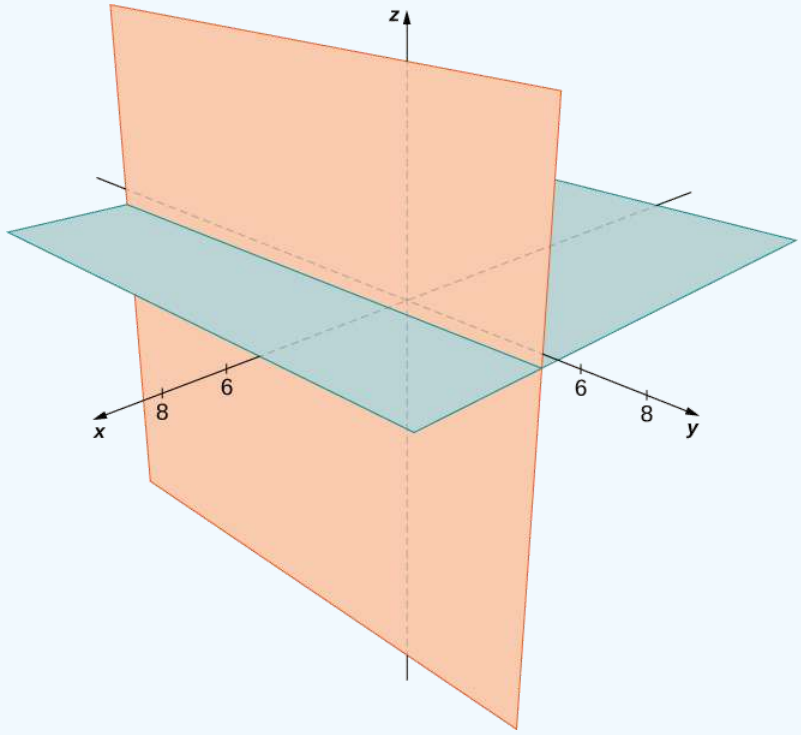


Figure 12.2.15: The set of points satisfying $(x-4)(z-2)=0$ forms the two planes $x=4$ and $z=2$.

Exercise 12.2.6

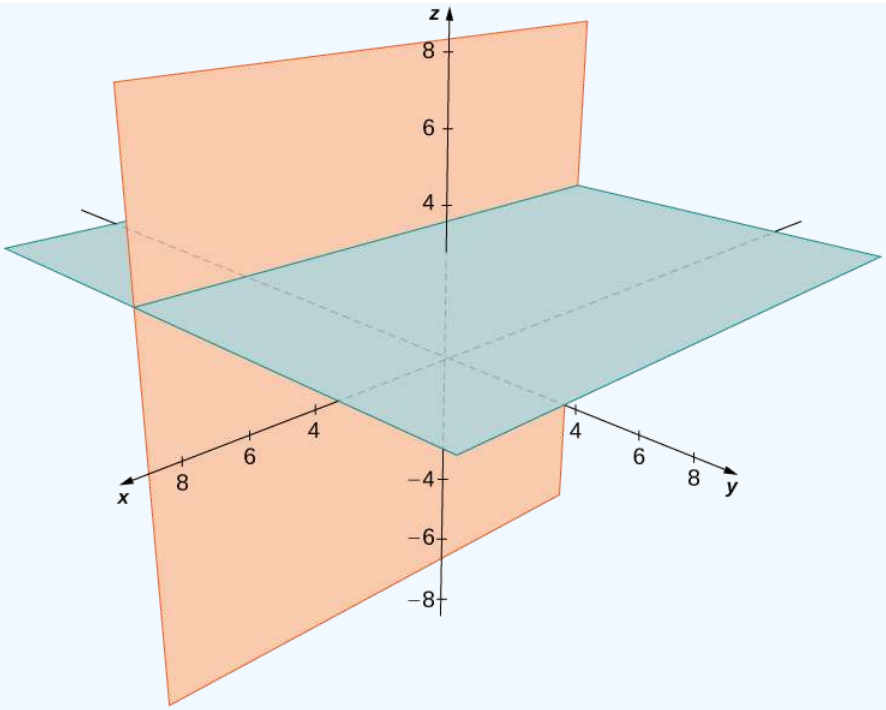
Describe the set of points that satisfies $(y+2)(z-3)=0,$ and graph the set.

Hint

One of the factors must be zero.

Answer

The set of points forms the two planes $y=-2$ and $z=3$.



Example 12.2.7: Graphing Other Equations in Three Dimensions

Describe the set of points in three-dimensional space that satisfies $(x-2)^2+(y-1)^2=4$, and graph the set.

Solution

The x- and y-coordinates form a circle in the xy-plane of radius 2, centered at $(2,1)$. Since there is no restriction on the z-coordinate, the three-dimensional result is a circular cylinder of radius 2 centered on the line with $x=2$ and $y=1$. The cylinder extends indefinitely in the z-direction (Figure 12.2.16).

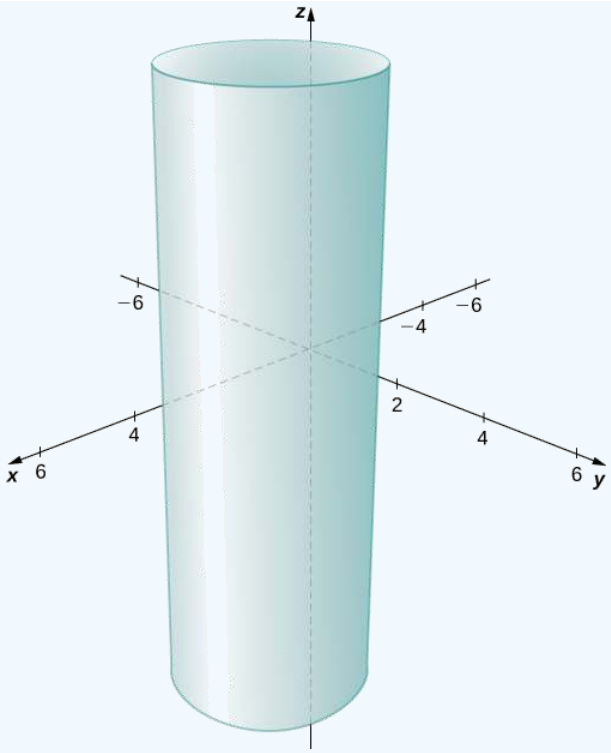


Figure 12.2.16: The set of points satisfying $(x-2)^2+(y-1)^2=4$. This is a cylinder of radius 2 centered on the line with $x=2$ and $y=1$.

Exercise 12.2.7

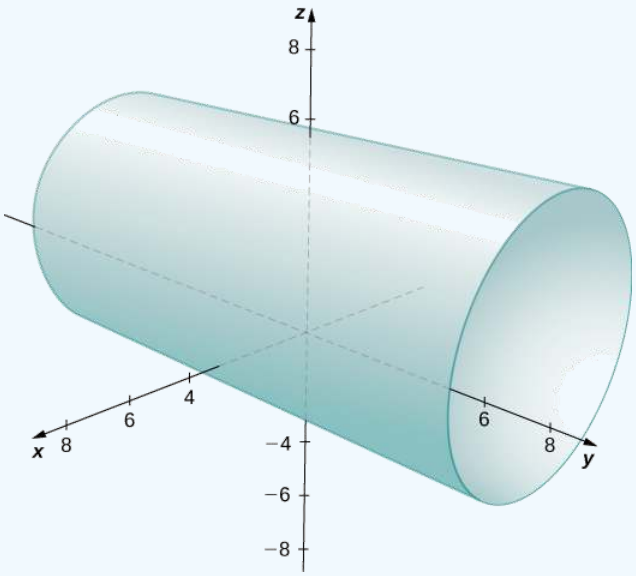
Describe the set of points in three dimensional space that satisfies $x^2+(z-2)^2=16$, and graph the surface.

Hint

Think about what happens if you plot this equation in two dimensions in the xz-plane.

Answer

A cylinder of radius 4 centered on the line with $x=0$ and $z=2$.



Working with Vectors in $\mathbb{R}^3$

Just like two-dimensional vectors, three-dimensional vectors are quantities with both magnitude and direction, and they are represented by directed line segments (arrows). With a three-dimensional vector, we use a three-dimensional arrow.

Three-dimensional vectors can also be represented in component form. The notation $\vec{\mathbf{v}} = \langle x, y, z \rangle$ is a natural extension of the two-dimensional case, representing a vector with the initial point at the origin, $(0, 0, 0)$, and terminal point (x, y, z). The zero vector is $\vec{\mathbf{0}} = \langle 0, 0, 0 \rangle$. So, for example, the three dimensional vector $\vec{\mathbf{v}} = \langle 2, 4, 1 \rangle$ is represented by a directed line segment from point $(0, 0, 0)$ to point $(2, 4, 1)$ (Figure 12.2.17).

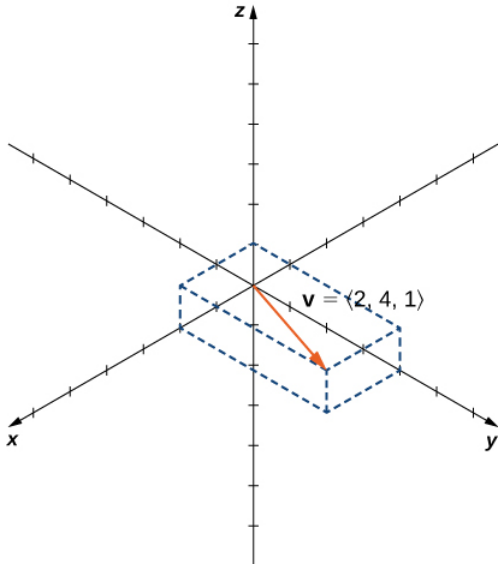


Figure 12.2.17: Vector $\vec{\mathbf{v}} = \langle 2, 4, 1 \rangle$ is represented by a directed line segment from point $(0, 0, 0)$ to point $(2, 4, 1)$.

Vector addition and scalar multiplication are defined analogously to the two-dimensional case. If $\vec{\mathbf{v}} = \langle x_1, y_1, z_1 \rangle$ and $\vec{\mathbf{w}} = \langle x_2, y_2, z_2 \rangle$ are vectors, and k is a scalar, then

$$\vec{\mathbf{v}} + \vec{\mathbf{w}} = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$$

and

$$k\vec{\mathbf{v}} = \langle kx_1, ky_1, kz_1 \rangle.$$

If $k = -1$, then $k\vec{\mathbf{v}} = (-1)\vec{\mathbf{v}}$ is written as $-\vec{\mathbf{v}}$, and vector subtraction is defined by $\vec{\mathbf{v}} - \vec{\mathbf{w}} = \vec{\mathbf{v}} + (-\vec{\mathbf{w}}) = \vec{\mathbf{v}} + (-1)\vec{\mathbf{w}}$.

The standard unit vectors extend easily into three dimensions as well, $\hat{\mathbf{i}} = \langle 1, 0, 0 \rangle$, $\hat{\mathbf{j}} = \langle 0, 1, 0 \rangle$, and $\hat{\mathbf{k}} = \langle 0, 0, 1 \rangle$, and we use them in the same way we used the standard unit vectors in two dimensions. Thus, we can represent a vector in $\mathbb{R}^3$ in the following ways:

$$\vec{\mathbf{v}} = \langle x, y, z \rangle = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

.

Example 12.2.8: Vector Representations

Let $\overrightarrow{PQ}$ be the vector with initial point $P = (3, 12, 6)$ and terminal point $Q = (-4, -3, 2)$ as shown in Figure 12.2.18 Express $\overrightarrow{PQ}$ in both component form and using standard unit vectors.

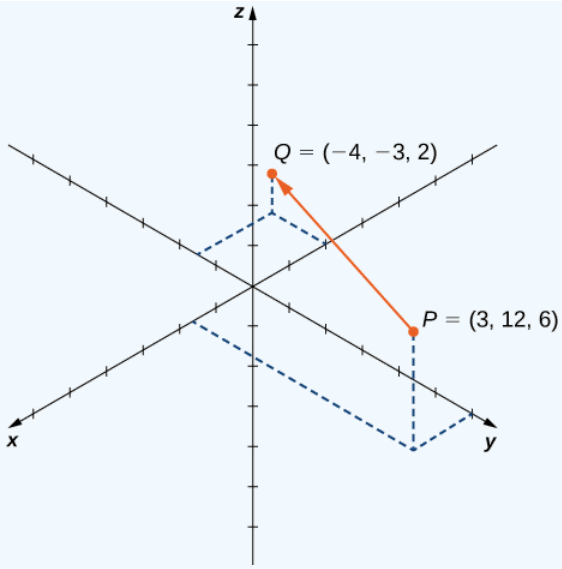


Figure 12.2.18: The vector with initial point $P = (3, 12, 6)$ and terminal point $Q = (-4, -3, 2)$.

Solution

In component form,

$$\begin{aligned}\overrightarrow{PQ} &= \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle \\ &= \langle -4 - 3, -3 - 12, 2 - 6 \rangle \\ &= \langle -7, -15, -4 \rangle.\end{aligned}$$

In standard unit form,

$$\overrightarrow{PQ} = -7\hat{\mathbf{i}} - 15\hat{\mathbf{j}} - 4\hat{\mathbf{k}}.$$

Exercise 12.2.8

Let $S = (3, 8, 2)$ and $T = (2, -1, 3)$. Express $\overrightarrow{ST}$ in component form and in standard unit form.

Hint

Write $\overrightarrow{ST}$ in component form first. T is the terminal point of $\overrightarrow{ST}$.

Answer

$\overrightarrow{ST} = \langle -1, -9, 1 \rangle = -\hat{\mathbf{i}} - 9\hat{\mathbf{j}} + \hat{\mathbf{k}}$

As described earlier, vectors in three dimensions behave in the same way as vectors in a plane. The geometric interpretation of vector addition, for example, is the same in both two- and three-dimensional space (Figure 12.2.19).

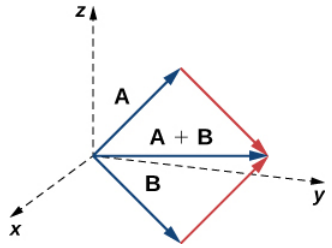


Figure 12.2.19: To add vectors in three dimensions, we follow the same procedures we learned for two dimensions.

We have already seen how some of the algebraic properties of vectors, such as vector addition and scalar multiplication, can be extended to three dimensions. Other properties can be extended in similar fashion. They are summarized here for our reference.

Properties of Vectors in Space

Let $\vec{\mathbf{v}} = \langle x_1, y_1, z_1 \rangle$ and $\vec{\mathbf{w}} = \langle x_2, y_2, z_2 \rangle$ be vectors, and let k be a scalar.

- **Scalar multiplication**:

$$k\vec{\mathbf{v}} = \langle kx_1, ky_1, kz_1 \rangle$$

- **Vector addition**:

$$\vec{\mathbf{v}} + \vec{\mathbf{w}} = \langle x_1, y_1, z_1 \rangle + \langle x_2, y_2, z_2 \rangle = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$$

- **Vector subtraction**:

$$\vec{\mathbf{v}} - \vec{\mathbf{w}} = \langle x_1, y_1, z_1 \rangle - \langle x_2, y_2, z_2 \rangle = \langle x_1 - x_2, y_1 - y_2, z_1 - z_2 \rangle$$

- **Vector magnitude**:

$$\|\vec{\mathbf{v}}\| = \sqrt{x_1^2 + y_1^2 + z_1^2}$$

- **Unit vector in the direction of $\vec{\mathbf{v}}$**:

$$\frac{1}{\|\vec{\mathbf{v}}\|}\vec{\mathbf{v}} = \frac{1}{\|\vec{\mathbf{v}}\|}\langle x_1, y_1, z_1 \rangle = \langle \frac{x_1}{\|\vec{\mathbf{v}}\|}, \frac{y_1}{\|\vec{\mathbf{v}}\|}, \frac{z_1}{\|\vec{\mathbf{v}}\|} \rangle, \quad \text{if } \vec{\mathbf{v}} \neq \vec{\mathbf{0}}$$

We have seen that vector addition in two dimensions satisfies the commutative, associative, and additive inverse properties. These properties of vector operations are valid for three-dimensional vectors as well. Scalar multiplication of vectors satisfies the distributive property, and the zero vector acts as an additive identity. The proofs to verify these properties in three dimensions are straightforward extensions of the proofs in two dimensions.

Example 12.2.9: Vector Operations in Three Dimensions

Let $\vec{\mathbf{v}} = \langle -2, 9, 5 \rangle$ and $\vec{\mathbf{w}} = \langle 1, -1, 0 \rangle$ (Figure 12.2.20). Find the following vectors.

a. $3\vec{\mathbf{v}} - 2\vec{\mathbf{w}}$
b. $5\|\vec{\mathbf{w}}\|$
c. $\|5\vec{\mathbf{w}}\|$
d. A unit vector in the direction of $\vec{\mathbf{v}}$

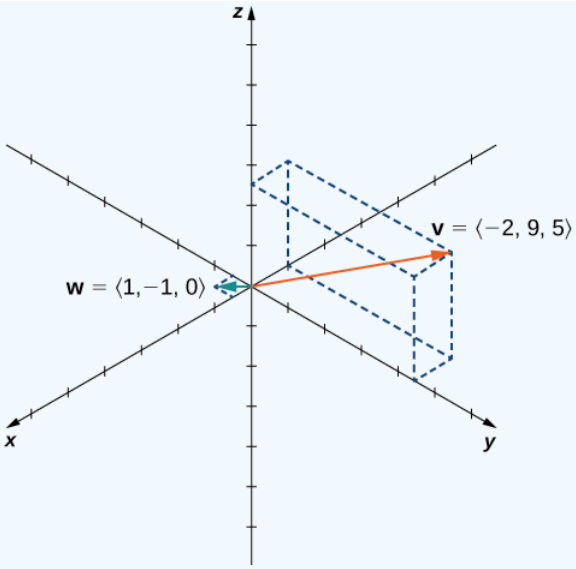


Figure 12.2.20: The vectors $\vec{\mathbf{v}} = \langle -2, 9, 5 \rangle$ and $\vec{\mathbf{w}} = \langle 1, -1, 0 \rangle$.

Solution

a. First, use scalar multiplication of each vector, then subtract:

$$\begin{aligned} 3\vec{\mathbf{v}} - 2\vec{\mathbf{w}} &= 3\langle -2, 9, 5 \rangle - 2\langle 1, -1, 0 \rangle \\ &= \langle -6, 27, 15 \rangle - \langle 2, -2, 0 \rangle \\ &= \langle -6 - 2, 27 - (-2), 15 - 0 \rangle \\ &= \langle -8, 29, 15 \rangle. \end{aligned}$$

b. Write the equation for the magnitude of the vector, then use scalar multiplication:

$$5\|\vec{\mathbf{w}}\| = 5\sqrt{1^2 + (-1)^2 + 0^2} = 5\sqrt{2}.$$

c. First, use scalar multiplication, then find the magnitude of the new vector. Note that the result is the same as for part b.:

$$\|5\vec{\mathbf{w}}\| = \| \langle 5, -5, 0 \rangle \| = \sqrt{5^2 + (-5)^2 + 0^2} = \sqrt{50} = 5\sqrt{2}$$

d. Recall that to find a unit vector in two dimensions, we divide a vector by its magnitude. The procedure is the same in three dimensions:

$$\begin{aligned} \frac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|} &= \frac{1}{\|\vec{\mathbf{v}}\|}\langle -2, 9, 5 \rangle \\ &= \frac{1}{\sqrt{(-2)^2 + 9^2 + 5^2}}\langle -2, 9, 5 \rangle \\ &= \frac{1}{\sqrt{110}}\langle -2, 9, 5 \rangle \\ &= \left\langle \frac{-2}{\sqrt{110}}, \frac{9}{\sqrt{110}}, \frac{5}{\sqrt{110}} \right\rangle. \end{aligned}$$

? Exercise 12.2.9:

Let $\vec{\mathbf{v}} = \langle -1, -1, 1 \rangle$ and $\vec{\mathbf{w}} = \langle 2, 0, 1 \rangle$. Find a unit vector in the direction of $5\vec{\mathbf{v}} + 3\vec{\mathbf{w}}$.

Hint

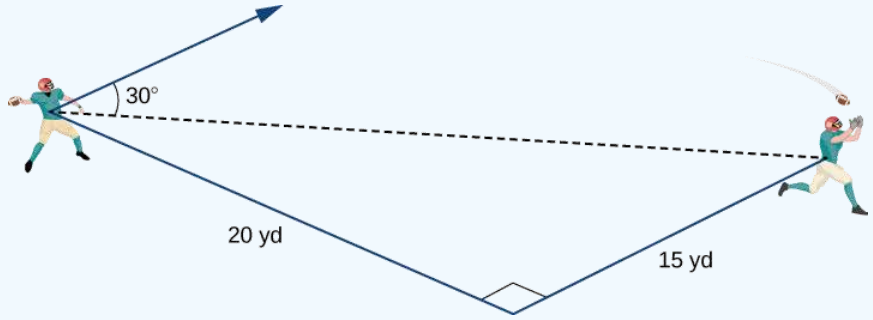
Start by writing $5\vec{\mathbf{v}} + 3\vec{\mathbf{w}}$ in component form.

Answer

$$\left\langle \frac{1}{3\sqrt{10}}, -\frac{5}{3\sqrt{10}}, \frac{8}{3\sqrt{10}} \right\rangle$$

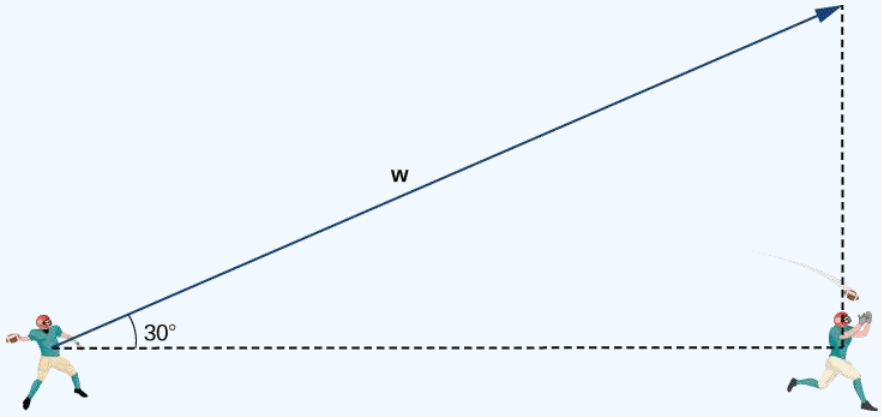
Example 12.2.10: Throwing a Forward Pass

A quarterback is standing on the football field preparing to throw a pass. His receiver is standing 20 yd down the field and 15 yd to the quarterback's left. The quarterback throws the ball at a velocity of 60 mph toward the receiver at an upward angle of 30° (see the following figure). Write the initial velocity vector of the ball, $\vec{\mathbf{v}}$, in component form.



Solution

The first thing we want to do is find a vector in the same direction as the velocity vector of the ball. We then scale the vector appropriately so that it has the right magnitude. Consider the vector $\vec{\mathbf{w}}$ extending from the quarterback's arm to a point directly above the receiver's head at an angle of 30° (see the following figure). This vector would have the same direction as $\vec{\mathbf{v}}$, but it may not have the right magnitude.



The receiver is 20 yd down the field and 15 yd to the quarterback's left. Therefore, the straight-line distance from the quarterback to the receiver is

$$\textbf{Dist from QB to receiver} = \sqrt{15^2 + 20^2} = \sqrt{225 + 400} = \sqrt{625} = 25 \text{ yd.}$$

We have $\dfrac{25}{\|\vec{\mathbf{w}}\|} = \cos 30^\circ$. Then the magnitude of $\vec{\mathbf{w}}$ is given by

$$\|\vec{\mathbf{w}}\| = \frac{25}{\cos 30^\circ} = \frac{25 \cdot 2}{\sqrt{3}} = \frac{50}{\sqrt{3}} \text{ yd}$$

and the vertical distance from the receiver to the terminal point of $\vec{\mathbf{w}}$ is

$$\textbf{Vert dist from receiver to terminal point of } \vec{\mathbf{w}} = \|\vec{\mathbf{w}}\| \sin 30^\circ = \frac{50}{\sqrt{3}} \cdot \frac{1}{2} = \frac{25}{\sqrt{3}} \text{ yd.}$$

Then $\vec{\mathbf{w}} = \langle 20, 15, \frac{25}{\sqrt{3}} \rangle$, and has the same direction as $\vec{\mathbf{v}}$.

Recall, though, that we calculated the magnitude of $\vec{\mathbf{w}}$ to be $\|\vec{\mathbf{w}}\| = \frac{50}{\sqrt{3}}$ yd, and $\vec{\mathbf{v}}$ has magnitude 60 mph. So, we need to multiply vector $\vec{\mathbf{w}}$ by an appropriate constant, k. We want to find a value of k so that $\|k\vec{\mathbf{w}}\| = 60$ mph*. We have

$$\|k\vec{\mathbf{w}}\| = k\|\vec{\mathbf{w}}\| = k\frac{50}{\sqrt{3}} \text{ yd},$$

so we want

$$k\left(\frac{50}{\sqrt{3}} \text{ yd}\right) = 60 \text{ mph}$$

$$k = \frac{60\sqrt{3}}{50} \text{ mph / yd}$$

$$k = \frac{6\sqrt{3}}{5} \text{ mph / yd}.$$

Then

$$\vec{\mathbf{v}} = k\vec{\mathbf{w}} = k\langle 20, 15, \frac{25}{\sqrt{3}} \rangle = \frac{6\sqrt{3}}{5} \langle 20, 15, \frac{25}{\sqrt{3}} \rangle = \langle 24\sqrt{3}, 18\sqrt{3}, 30 \rangle.$$

Let's double-check that $\|\vec{\mathbf{v}}\| = 60$ mph. We have

$$\|\vec{\mathbf{v}}\| = \sqrt{(24\sqrt{3})^2 + (18\sqrt{3})^2 + (30)^2} = \sqrt{1728 + 972 + 900} = \sqrt{3600} = 60 \text{ mph}.$$

So, we have found the correct components for $\vec{\mathbf{v}}$.

Note *

Readers who have been watching the units of measurement may be wondering what exactly is going on at this point: haven't we just mixed yards and miles per hour? We haven't, but the reason is subtle. One way to understand it is to realize that there are really two parallel coordinate systems in this problem: one gives positions down the field, across the field, and up into the air in units of yards; the other gives speeds down the field, across the field, and up into the air in units of miles per hour. The vector $\vec{\mathbf{w}}$ is calculated in the position coordinate system; vector $\vec{\mathbf{v}}$ will be in the speed system. Because corresponding axes in each system are parallel, directions in the two systems are also parallel, so the claim that $\vec{\mathbf{w}}$ and $\vec{\mathbf{v}}$ point in the same direction is correct. The constant k that we're looking for is a conversion factor between the magnitudes of these two vectors, converting from the position system to the speed one in the process. And as seen above, our calculation of k produces the right units for such a conversion, namely miles per hour per yard.

Exercise 12.2.10

Assume the quarterback and the receiver are in the same place as in the previous example. This time, however, the quarterback throws the ball at velocity of 40 mph and an angle of 45°. Write the initial velocity vector of the ball, $\vec{\mathbf{v}}$, in component form.

Hint

Follow the process used in the previous example.

Answer

$v = \langle 16\sqrt{2}, 12\sqrt{2}, 20\sqrt{2} \rangle$

Key Concepts

- The three-dimensional coordinate system is built around a set of three axes that intersect at right angles at a single point, the origin. Ordered triples (x, y, z) are used to describe the location of a point in space.
- The distance d between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\ .$$

- In three dimensions, the equations $x = a,\ y = b,$ and $z = c$ describe planes that are parallel to the coordinate planes.
- The standard equation of a sphere with center (a, b, c) and radius r is

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

- In three dimensions, as in two, vectors are commonly expressed in component form, $\vec{\mathbf{v}} = \langle x, y, z \rangle$, or in terms of the standard unit vectors, $\vec{\mathbf{v}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$.
- Properties of vectors in space are a natural extension of the properties for vectors in a plane. Let $\vec{\mathbf{v}} = \langle x_1, y_1, z_1 \rangle$ and $\vec{\mathbf{w}} = \langle x_2, y_2, z_2 \rangle$ be vectors, and let k be a scalar.

  Scalar multiplication:

$$k\vec{\mathbf{v}} = \langle kx_1, ky_1, kz_1 \rangle$$

  Vector addition:

$$\vec{\mathbf{v}} + \vec{\mathbf{w}} = \langle x_1, y_1, z_1 \rangle + \langle x_2, y_2, z_2 \rangle = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$$

  Vector subtraction:

$$\vec{\mathbf{v}} - \vec{\mathbf{w}} = \langle x_1, y_1, z_1 \rangle - \langle x_2, y_2, z_2 \rangle = \langle x_1 - x_2, y_1 - y_2, z_1 - z_2 \rangle$$

  Vector magnitude:

$$\|\vec{\mathbf{v}}\| = \sqrt{x_1^2 + y_1^2 + z_1^2}$$

  Unit vector in the direction of $\vec{\mathbf{v}}$:

$$\frac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|} = \frac{1}{\|\vec{\mathbf{v}}\|}\langle x_1, y_1, z_1 \rangle = \langle \frac{x_1}{\|\vec{\mathbf{v}}\|}, \frac{y_1}{\|\vec{\mathbf{v}}\|}, \frac{z_1}{\|\vec{\mathbf{v}}\|} \rangle,\ \vec{\mathbf{v}} \neq \vec{\mathbf{0}}$$

Key Equations

Distance between two points in space:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Sphere with center (a, b, c) and radius r:

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

Glossary

coordinate plane
a plane containing two of the three coordinate axes in the three-dimensional coordinate system, named by the axes it contains: the xy-plane, xz-plane, or the yz-plane

right-hand rule
a common way to define the orientation of the three-dimensional coordinate system; when the right hand is curved around the z-axis in such a way that the fingers curl from the positive x-axis to the positive y-axis, the thumb points in the direction of the positive z-axis

octants
the eight regions of space created by the coordinate planes

sphere
the set of all points equidistant from a given point known as the *center*

standard equation of a sphere

$(x-a)^2+(y-b)^2+(z-c)^2=r^2$ describes a sphere with center (a,b,c) and radius r

three-dimensional rectangular coordinate system

a coordinate system defined by three lines that intersect at right angles; every point in space is described by an ordered triple (x,y,z) that plots its location relative to the defining axes

Contributors

Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

Example 12.2.10has been modified by Doug Baldwin and Paul Seeburger to clarify the units of measurement that it uses and how it uses them.

Paul Seeburger also created dynamic versions of Figures 12.2.8, 12.2.9and 12.2.13using CalcPlot3D.

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12.3: The Dot Product

Learning Objectives

- Calculate the dot product of two given vectors.
- Determine whether two given vectors are perpendicular.
- Find the direction cosines of a given vector.
- Explain what is meant by the vector projection of one vector onto another vector, and describe how to compute it.
- Calculate the work done by a given force.

If we apply a force to an object so that the object moves, we say that work is done by the force. Previously, we looked at a constant force and we assumed the force was applied in the direction of motion of the object. Under those conditions, work can be expressed as the product of the force acting on an object and the distance the object moves. In this chapter, however, we have seen that both force and the motion of an object can be represented by vectors.

In this section, we develop an operation called the dot product, which allows us to calculate work in the case when the force vector and the motion vector have different directions. The dot product essentially tells us how much of the force vector is applied in the direction of the motion vector. The dot product can also help us measure the angle formed by a pair of vectors and the position of a vector relative to the coordinate axes. It even provides a simple test to determine whether two vectors meet at a right angle.

The Dot Product and Its Properties

We have already learned how to add and subtract vectors. In this chapter, we investigate two types of vector multiplication. The first type of vector multiplication is called the dot product, based on the notation we use for it, and it is defined as follows:

Definition: dot product

The *dot product* of vectors $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ is given by the sum of the products of the components

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Note that if u and v are two-dimensional vectors, we calculate the dot product in a similar fashion. Thus, if $\vec{\mathbf{u}} = \langle u_1, u_2 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2 \rangle$, then

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2.$$

When two vectors are combined under addition or subtraction, the result is a vector. When two vectors are combined using the dot product, the result is a scalar. For this reason, the dot product is often called the *scalar product.* It may also be called the *inner product.*

Example 12.3.1: Calculating Dot Products

a. Find the dot product of $\vec{\mathbf{u}} = \langle 3, 5, 2 \rangle$ and $\vec{\mathbf{v}} = \langle -1, 3, 0 \rangle$.

b. Find the scalar product of $\vec{\mathbf{p}} = 10\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$ and $\vec{\mathbf{q}} = -2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 6\hat{\mathbf{k}}$.

Solution:

a. Substitute the vector components into the formula for the dot product:

$$\begin{aligned} \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} &= u_1 v_1 + u_2 v_2 + u_3 v_3 \\ &= 3(-1) + 5(3) + 2(0) \\ &= -3 + 15 + 0 \\ &= 12. \end{aligned}$$

b. The calculation is the same if the vectors are written using standard unit vectors. We still have three components for each vector to substitute into the formula for the dot product:

$$\begin{aligned}\vec{\mathbf{p}}\cdot\vec{\mathbf{q}} &= p_1q_1+p_2q_2+p_3q_3\\ &= 10(-2)+(-4)(1)+(7)(6)\\ &= -20-4+42\\ &= 18.\end{aligned}$$

Exercise 12.3.1

Find $\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}$, where $\vec{\mathbf{u}}=\langle 2,9,-1\rangle$ and $\vec{\mathbf{v}}=\langle -3,1,-4\rangle$.

Hint

Multiply corresponding components and then add their products.

Answer

7

Like vector addition and subtraction, the dot product has several algebraic properties. We prove three of these properties and leave the rest as exercises.

Properties of the Dot Product

Let $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ be vectors, and let c be a scalar.

i. *Commutative property*

$$\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}=\vec{\mathbf{v}}\cdot\vec{\mathbf{u}}$$

ii. *Distributive property*

$$\vec{\mathbf{u}}\cdot(\vec{\mathbf{v}}+\vec{\mathbf{w}})=\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}+\vec{\mathbf{u}}\cdot\vec{\mathbf{w}}$$

iii. *Associative property*

$$c(\vec{\mathbf{u}}\cdot\vec{\mathbf{v}})=(c\vec{\mathbf{u}})\cdot\vec{\mathbf{v}}=\vec{\mathbf{u}}\cdot(c\vec{\mathbf{v}})$$

iv. *Property of magnitude*

$$\vec{\mathbf{v}}\cdot\vec{\mathbf{v}}=\|\vec{\mathbf{v}}\|^2$$

Proof

Let $\vec{\mathbf{u}}=\langle u_1,u_2,u_3\rangle$ and $\vec{\mathbf{v}}=\langle v_1,v_2,v_3\rangle$. Then

$$\begin{aligned}\vec{\mathbf{u}}\cdot\vec{\mathbf{v}} &= \langle u_1,u_2,u_3\rangle\cdot\langle v_1,v_2,v_3\rangle\\ &= u_1v_1+u_2v_2+u_3v_3\\ &= v_1u_1+v_2u_2+v_3u_3\\ &= \langle v_1,v_2,v_3\rangle\cdot\langle u_1,u_2,u_3\rangle\\ &= \vec{\mathbf{v}}\cdot\vec{\mathbf{u}}.\end{aligned}$$

The associative property looks like the associative property for real-number multiplication, but pay close attention to the difference between scalar and vector objects:

$$\begin{aligned} c(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}) &= c(u_1v_1 + u_2v_2 + u_3v_3) \\ &= c(u_1v_1) + c(u_2v_2) + c(u_3v_3) \\ &= (cu_1)v_1 + (cu_2)v_2 + (cu_3)v_3 \\ &= \langle cu_1, cu_2, cu_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle \\ &= c\langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle \\ &= (c\vec{\mathbf{u}}) \cdot \vec{\mathbf{v}}. \end{aligned}$$

The proof that $c(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}) = \vec{\mathbf{u}} \cdot (c\vec{\mathbf{v}})$ is similar.

The fourth property shows the relationship between the magnitude of a vector and its dot product with itself:

$$\begin{aligned} \vec{\mathbf{v}} \cdot \vec{\mathbf{v}} &= \langle v_1, v_2, v_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle \\ &= (v_1)^2 + (v_2)^2 + (v_3)^2 \\ &= \left[\sqrt{(v_1)^2 + (v_2)^2 + (v_3)^2} \right]^2 \\ &= \|\vec{\mathbf{v}}\|^2. \end{aligned}$$

□

Note that the definition of the dot product yields $\vec{\mathbf{0}} \cdot \vec{\mathbf{v}} = 0$. By property iv. if $\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = 0$, then $\vec{\mathbf{v}} = \vec{\mathbf{0}}$.

Example 12.3.2: Using Properties of the Dot Product

Let $\vec{\mathbf{a}} = \langle 1, 2, -3 \rangle$, $\vec{\mathbf{b}} = \langle 0, 2, 4 \rangle$, and $\vec{\mathbf{c}} = \langle 5, -1, 3 \rangle$.

Find each of the following products.

a. $(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})\vec{\mathbf{c}}$
b. $\vec{\mathbf{a}} \cdot (2\vec{\mathbf{c}})$
c. $\|\vec{\mathbf{b}}\|^2$

Solution

a. Note that this expression asks for the scalar multiple of $\vec{\mathbf{c}}$ by $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}$:

$$\begin{aligned} (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})\vec{\mathbf{c}} &= (\langle 1, 2, -3 \rangle \cdot \langle 0, 2, 4 \rangle)\langle 5, -1, 3 \rangle \\ &= (1(0) + 2(2) + (-3)(4))\langle 5, -1, 3 \rangle \\ &= -8\langle 5, -1, 3 \rangle \\ &= \langle -40, 8, -24 \rangle. \end{aligned}$$

b. This expression is a dot product of vector $\vec{\mathbf{a}}$ and scalar multiple $2\vec{\mathbf{c}}$:

$$\begin{aligned} \vec{\mathbf{a}} \cdot (2\vec{\mathbf{c}}) &= 2(\vec{\mathbf{a}} \cdot \vec{\mathbf{c}}) \\ &= 2(\langle 1, 2, -3 \rangle \cdot \langle 5, -1, 3 \rangle) \\ &= 2(1(5) + 2(-1) + (-3)(3)) \\ &= 2(-6) = -12. \end{aligned}$$

c. Simplifying this expression is a straightforward application of the dot product:

$$\begin{aligned}\|\vec{\mathbf{b}}\|^2 &= \vec{\mathbf{b}} \cdot \vec{\mathbf{b}} \\ &= \langle 0,2,4\rangle \cdot \langle 0,2,4\rangle \\ &= 0^2+2^2+4^2 \\ &= 0+4+16 \\ &= 20.\end{aligned}$$

Exercise 12.3.2

Find the following products for $\vec{\mathbf{p}} = \langle 7,0,2\rangle$, $\vec{\mathbf{q}} = \langle -2,2,-2\rangle$, and $\vec{\mathbf{r}} = \langle 0,2,-3\rangle$.

a. $(\vec{\mathbf{r}} \cdot \vec{\mathbf{p}})\vec{\mathbf{q}}$
b. $\|\vec{\mathbf{p}}\|^2$

Hint

$\vec{\mathbf{r}} \cdot \vec{\mathbf{p}}$ is a scalar.

Answer

a. $(\vec{\mathbf{r}} \cdot \vec{\mathbf{p}})\vec{\mathbf{q}} = \langle 12,-12,12\rangle$; *b.* $\|\vec{\mathbf{p}}\|^2 = 53$

Using the Dot Product to Find the Angle between Two Vectors

When two nonzero vectors are placed in standard position, whether in two dimensions or three dimensions, they form an angle between them (Figure 12.3.1). The dot product provides a way to find the measure of this angle. This property is a result of the fact that we can express the dot product in terms of the cosine of the angle formed by two vectors.

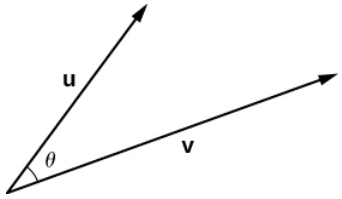


Figure 12.3.1: Let θ be the angle between two nonzero vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ such that $0 \le \theta \le \pi$.

Evaluating a Dot Product

The dot product of two vectors is the product of the magnitude of each vector and the cosine of the angle between them:

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\| \cos\theta. \tag{12.3.1}$$

Proof

Place vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ in standard position and consider the vector $\vec{\mathbf{v}} - \vec{\mathbf{u}}$ (Figure 12.3.2). These three vectors form a triangle with side lengths $\|\vec{\mathbf{u}}\|$, $\|\vec{\mathbf{v}}\|$, and $\|\vec{\mathbf{v}} - \vec{\mathbf{u}}\|$.

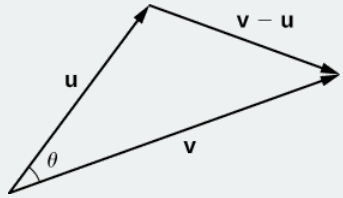


Figure 12.3.2: The lengths of the sides of the triangle are given by the magnitudes of the vectors that form the triangle.

Recall from trigonometry that the law of cosines describes the relationship among the side lengths of the triangle and the angle θ. Applying the law of cosines here gives

$$\|\vec{\mathbf{v}} - \vec{\mathbf{u}}\|^2 = \|\vec{\mathbf{u}}\|^2 + \|\vec{\mathbf{v}}\|^2 - 2\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\| \cos\theta. \tag{12.3.2}$$

The dot product provides a way to rewrite the left side of Equation 12.3.2:

$$\begin{aligned}\|\vec{\mathbf{v}}-\vec{\mathbf{u}}\|^2 &= (\vec{\mathbf{v}}-\vec{\mathbf{u}})\cdot(\vec{\mathbf{v}}-\vec{\mathbf{u}})\\ &= (\vec{\mathbf{v}}-\vec{\mathbf{u}})\cdot\vec{\mathbf{v}}-(\vec{\mathbf{v}}-\vec{\mathbf{u}})\cdot\vec{\mathbf{u}}\\ &= \vec{\mathbf{v}}\cdot\vec{\mathbf{v}}-\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}-\vec{\mathbf{v}}\cdot\vec{\mathbf{u}}+\vec{\mathbf{u}}\cdot\vec{\mathbf{u}}\\ &= \vec{\mathbf{v}}\cdot\vec{\mathbf{v}}-\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}-\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}+\vec{\mathbf{u}}\cdot\vec{\mathbf{u}}\\ &= \|\vec{\mathbf{v}}\|^2-2\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}+\|\vec{\mathbf{u}}\|^2.\end{aligned}$$

Substituting into the law of cosines yields

$$\begin{aligned}\|\vec{\mathbf{v}}-\vec{\mathbf{u}}\|^2 &= \|\vec{\mathbf{u}}\|^2+\|\vec{\mathbf{v}}\|^2-2\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|\cos\theta\\ \|\vec{\mathbf{v}}\|^2-2\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}+\|\vec{\mathbf{u}}\|^2 &= \|\vec{\mathbf{u}}\|^2+\|\vec{\mathbf{v}}\|^2-2\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|\cos\theta\\ -2\vec{\mathbf{u}}\cdot\vec{\mathbf{v}} &= -2\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|\cos\theta\\ \vec{\mathbf{u}}\cdot\vec{\mathbf{v}} &= \|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|\cos\theta.\end{aligned}$$

□

We can use the form of the dot product in Equation 12.3.1 to find the measure of the angle between two nonzero vectors by rearranging Equation 12.3.1 to solve for the cosine of the angle:

$$\cos\theta=\frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|}. \tag{12.3.3}$$

Using this equation, we can find the cosine of the angle between two nonzero vectors. Since we are considering the smallest angle between the vectors, we assume $0^\circ\le\theta\le180^\circ$ (or $0\le\theta\le\pi$ if we are working in radians). The inverse cosine is unique over this range, so we are then able to determine the measure of the angle θ.

Example 12.3.3: Finding the Angle between Two Vectors

Find the measure of the angle between each pair of vectors.

a. $\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and $2\hat{\mathbf{i}}-\hat{\mathbf{j}}-3\hat{\mathbf{k}}$
b. $\langle 2,5,6\rangle$ and $\langle -2,-4,4\rangle$

Solution

a. To find the cosine of the angle formed by the two vectors, substitute the components of the vectors into Equation 12.3.3:

$$\begin{aligned}\cos\theta &= \frac{(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})\cdot(2\hat{\mathbf{i}}-\hat{\mathbf{j}}-3\hat{\mathbf{k}})}{\left\|\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}\right\|\cdot\left\|2\hat{\mathbf{i}}-\hat{\mathbf{j}}-3\hat{\mathbf{k}}\right\|}\\ &= \frac{1(2)+(1)(-1)+(1)(-3)}{\sqrt{1^2+1^2+1^2}\sqrt{2^2+(-1)^2+(-3)^2}}\\ &= \frac{-2}{\sqrt{3}\sqrt{14}}=\frac{-2}{\sqrt{42}}.\end{aligned}$$

Therefore, $\theta=\arccos\dfrac{-2}{\sqrt{42}}$ rad.

b. Start by finding the value of the cosine of the angle between the vectors:

$$\cos\theta = \frac{\langle 2,5,6\rangle \cdot \langle -2,-4,4\rangle}{\|\langle 2,5,6\rangle\| \cdot \|\langle -2,-4,4\rangle\|}$$
$$= \frac{2(-2)+(5)(-4)+(6)(4)}{\sqrt{2^2+5^2+6^2}\sqrt{(-2)^2+(-4)^2+4^2}}$$
$$= \frac{0}{\sqrt{65}\sqrt{36}} = 0.$$

Now, $\cos\theta = 0$ and $0 \le \theta \le \pi$, so $\theta = \pi/2$.

Exercise 12.3.3

Find the measure of the angle, in radians, formed by vectors $\vec{\mathbf{a}} = \langle 1,2,0\rangle$ and $\vec{\mathbf{b}} = \langle 2,4,1\rangle$. Round to the nearest hundredth.

Hint

Use the Equation 12.3.3

Answer

$\theta \approx 0.22$ rad

The angle between two vectors can be acute $(0 < \cos\theta < 1)$, obtuse $(-1 < \cos\theta < 0)$, or straight $(\cos\theta = -1)$. If $\cos\theta = 1$, then both vectors have the same direction. If $\cos\theta = 0$, then the vectors, when placed in standard position, form a right angle (Figure 12.3.3). We can formalize this result into a theorem regarding orthogonal (perpendicular) vectors.

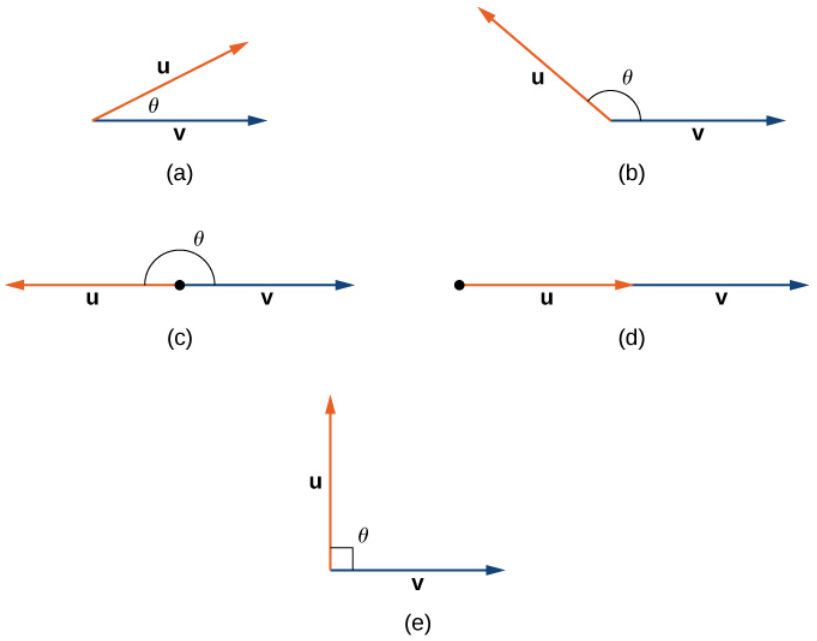


Figure 12.3.3: (a) An acute angle has $0 < \cos\theta < 1$. (b) An obtuse angle has $-1 < \cos\theta < 0$. (c) A straight line has $\cos\theta = -1$. (d) If the vectors have the same direction, $\cos\theta = 1$. (e) If the vectors are orthogonal (perpendicular), $\cos\theta = 0$.

Orthogonal Vectors

The nonzero vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are **orthogonal vectors** if and only if $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0$.

Proof

Let $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ be nonzero vectors, and let θ denote the angle between them. First, assume $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0$. Then

$$\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\| \cos\theta = 0.$$

However, $\|\vec{\mathbf{u}}\| \neq 0$ and $\|\vec{\mathbf{v}}\| \neq 0$, so we must have $\cos\theta = 0$. Hence, $\theta = 90^\circ$, and the vectors are orthogonal.

Now assume $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are orthogonal. Then $\theta = 90^\circ$ and we have

$$\begin{aligned} \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} &= \|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\| \cos\theta \\ &= \|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\| \cos 90^\circ \\ &= \|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\| (0) \\ &= 0. \end{aligned}$$

□

The terms **orthogonal, perpendicular,** and **normal** each indicate that mathematical objects are intersecting at right angles. The use of each term is determined mainly by its context. We say that vectors are orthogonal and lines are perpendicular. The term *normal* is used most often when measuring the angle made with a plane or other surface.

Example 12.3.4: Identifying Orthogonal Vectors

Determine whether $\vec{\mathbf{p}} = \langle 1, 0, 5 \rangle$ and $\vec{\mathbf{q}} = \langle 10, 3, -2 \rangle$ are orthogonal vectors.

Solution

Using the definition, we need only check the dot product of the vectors:

$$\vec{\mathbf{p}} \cdot \vec{\mathbf{q}} = 1(10) + (0)(3) + (5)(-2) = 10 + 0 - 10 = 0.$$

Because $\vec{\mathbf{p}} \cdot \vec{\mathbf{q}} = 0,$ the vectors are orthogonal (Figure 12.3.4).

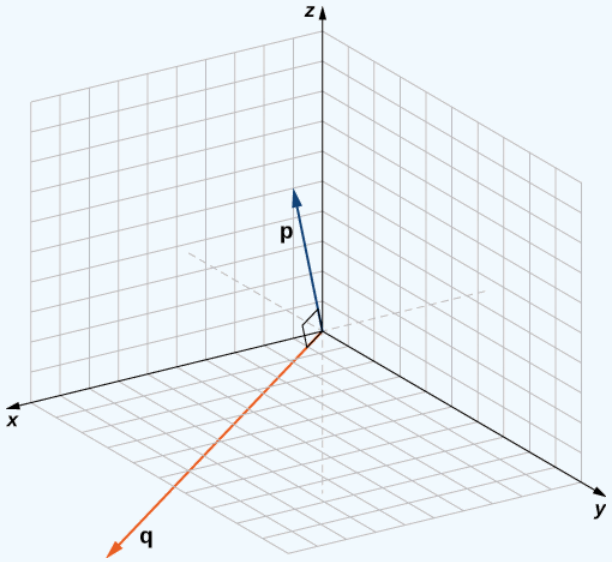


Figure 12.3.4: Vectors $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ form a right angle when their initial points are aligned.

Exercise 12.3.4

For which value of x is $\vec{\mathbf{p}} = \langle 2, 8, -1 \rangle$ orthogonal to $\vec{\mathbf{q}} = \langle x, -1, 2 \rangle$?

Hint

Vectors $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ are orthogonal if and only if $\vec{\mathbf{p}} \cdot \vec{\mathbf{q}} = 0$.

Answer

$x = 5$

Example 12.3.5: Measuring the Angle Formed by Two Vectors

Let $\vec{\mathbf{v}} = \langle 2, 3, 3 \rangle$. Find the measures of the angles formed by the following vectors.

a. $\vec{\mathbf{v}}$ and $\hat{\mathbf{i}}$
b. $\vec{\mathbf{v}}$ and $\hat{\mathbf{j}}$
c. $\vec{\mathbf{v}}$ and $\hat{\mathbf{k}}$

Solution

a.Let α be the angle formed by $\vec{\mathbf{v}}$ and $\hat{\mathbf{i}}$:

$$\cos\alpha = \frac{\vec{\mathbf{v}} \cdot \hat{\mathbf{i}}}{\|\vec{\mathbf{v}}\| \cdot \|\hat{\mathbf{i}}\|} = \frac{\langle 2, 3, 3 \rangle \cdot \langle 1, 0, 0 \rangle}{\sqrt{2^2 + 3^2 + 3^2}\sqrt{1}} = \frac{2}{\sqrt{22}}$$

$$\alpha = \arccos \frac{2}{\sqrt{22}} \approx 1.130 \text{ rad.}$$

b. Let β represent the angle formed by $\vec{\mathbf{v}}$ and $\hat{\mathbf{j}}$:

$$\cos\beta = \frac{\vec{\mathbf{v}} \cdot \hat{\mathbf{j}}}{\|\vec{\mathbf{v}}\| \cdot \|\hat{\mathbf{j}}\|} = \frac{\langle 2, 3, 3 \rangle \cdot \langle 0, 1, 0 \rangle}{\sqrt{2^2 + 3^2 + 3^2}\sqrt{1}} = \frac{3}{\sqrt{22}}$$

$$\beta = \arccos \frac{3}{\sqrt{22}} \approx 0.877 \text{ rad.}$$

c. Let γ represent the angle formed by $\vec{\mathbf{v}}$ and $\hat{\mathbf{k}}$:

$$\cos\gamma = \frac{\vec{\mathbf{v}} \cdot \hat{\mathbf{k}}}{\|\vec{\mathbf{v}}\| \cdot \|\hat{\mathbf{k}}\|} = \frac{\langle 2, 3, 3 \rangle \cdot \langle 0, 0, 1 \rangle}{\sqrt{2^2 + 3^2 + 3^2}\sqrt{1}} = \frac{3}{\sqrt{22}}$$

$$\gamma = \arccos \frac{3}{\sqrt{22}} \approx 0.877 \text{ rad.}$$

Exercise 12.3.5

Let $\vec{\mathbf{v}} = \langle 3, -5, 1 \rangle$. Find the measure of the angles formed by each pair of vectors.

a. $\vec{\mathbf{v}}$ and $\hat{\mathbf{i}}$
b. $\vec{\mathbf{v}}$ and $\hat{\mathbf{j}}$
c. $\vec{\mathbf{v}}$ and $\hat{\mathbf{k}}$

Hint

$\hat{\mathbf{i}} = \langle 1, 0, 0 \rangle$, $\hat{\mathbf{j}} = \langle 0, 1, 0 \rangle$, and $\hat{\mathbf{k}} = \langle 0, 0, 1 \rangle$

Answer

a. $\alpha \approx 1.04$ rad; b. $\beta \approx 2.58$ rad; c. $\gamma \approx 1.40$ rad

The angle a vector makes with each of the coordinate axes, called a direction angle, is very important in practical computations, especially in a field such as engineering. For example, in astronautical engineering, the angle at which a rocket is launched must be determined very precisely. A very small error in the angle can lead to the rocket going hundreds of miles off course. Direction angles are often calculated by using the dot product and the cosines of the angles, called the direction cosines. Therefore, we define both these angles and their cosines.

Definition: direction angles

The angles formed by a nonzero vector and the coordinate axes are called the **direction angles** for the vector (Figure 12.3.5). The cosines for these angles are called the **direction cosines.**

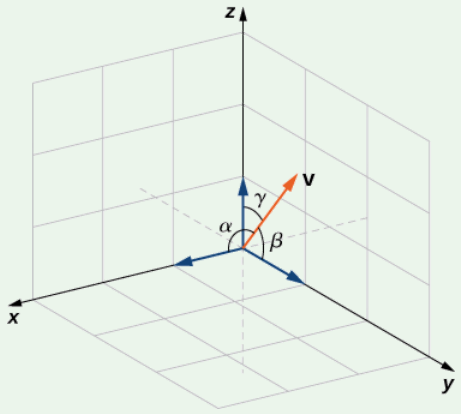


Figure 12.3.5: Angle α is formed by vector $\vec{\mathbf{v}}$ and unit vector $\hat{\mathbf{i}}$. Angle β is formed by vector $\vec{\mathbf{v}}$ and unit vector $\hat{\mathbf{j}}$. Angle γ is formed by vector $\vec{\mathbf{v}}$ and unit vector $\hat{\mathbf{k}}$.

In Example 12.3.5, the direction cosines of $\vec{\mathbf{v}} = \langle 2, 3, 3 \rangle$ are $\cos\alpha = \dfrac{2}{\sqrt{22}}, \cos\beta = \dfrac{3}{\sqrt{22}}$, and $\cos\gamma = \dfrac{3}{\sqrt{22}}$. The direction angles of $\vec{\mathbf{v}}$ are $\alpha = 1.130$ rad, $\beta = 0.877$ rad, and $\gamma = 0.877$ rad.

So far, we have focused mainly on vectors related to force, movement, and position in three-dimensional physical space. However, vectors are often used in more abstract ways. For example, suppose a fruit vendor sells apples, bananas, and oranges. On a given day, he sells 30 apples, 12 bananas, and 18 oranges. He might use a quantity vector, $\vec{\mathbf{q}} = \langle 30, 12, 18 \rangle$, to represent the quantity of fruit he sold that day. Similarly, he might want to use a price vector, $\vec{\mathbf{p}} = \langle 0.50, 0.25, 1 \rangle$, to indicate that he sells his apples for 50¢ each, bananas for 25¢ each, and oranges for $1 apiece. In this example, although we could still graph these vectors, we do not interpret them as literal representations of position in the physical world. We are simply using vectors to keep track of particular pieces of information about apples, bananas, and oranges.

This idea might seem a little strange, but if we simply regard vectors as a way to order and store data, we find they can be quite a powerful tool. Going back to the fruit vendor, let's think about the dot product, $\vec{\mathbf{q}} \cdot \vec{\mathbf{p}}$. We compute it by multiplying the number of apples sold (30) by the price per apple (50¢), the number of bananas sold by the price per banana, and the number of oranges sold by the price per orange. We then add all these values together. So, in this example, the dot product tells us how much money the fruit vendor had in sales on that particular day.

When we use vectors in this more general way, there is no reason to limit the number of components to three. What if the fruit vendor decides to start selling grapefruit? In that case, he would want to use four-dimensional quantity and price vectors to represent the number of apples, bananas, oranges, and grapefruit sold, and their unit prices. As you might expect, to calculate the dot product of four-dimensional vectors, we simply add the products of the components as before, but the sum has four terms instead of three.

Example 12.3.6: Using Vectors in an Economic Context

AAA Party Supply Store sells invitations, party favors, decorations, and food service items such as paper plates and napkins. When AAA buys its inventory, it pays 25¢ per package for invitations and party favors. Decorations cost AAA 50¢ each, and food service items cost 20¢ per package. AAA sells invitations for $2.50 per package and party favors for $1.50 per package. Decorations sell for $4.50 each and food service items for $1.25 per package.

During the month of May, AAA Party Supply Store sells 1258 invitations, 342 party favors, 2426 decorations, and 1354 food service items. Use vectors and dot products to calculate how much money AAA made in sales during the month of May. How much did the store make in profit?

Solution

The cost, price, and quantity vectors are

$$\vec{\mathbf{c}} = \langle 0.25, 0.25, 0.50, 0.20 \rangle$$
$$\vec{\mathbf{p}} = \langle 2.50, 1.50, 4.50, 1.25 \rangle$$
$$\vec{\mathbf{q}} = \langle 1258, 342, 2426, 1354 \rangle.$$

AAA sales for the month of May can be calculated using the dot product $\vec{\mathbf{p}} \cdot \vec{\mathbf{q}}$. We have

$$\begin{aligned} \vec{\mathbf{p}} \cdot \vec{\mathbf{q}} &= \langle 2.50, 1.50, 4.50, 1.25 \rangle \cdot \langle 1258, 342, 2426, 1354 \rangle \\ &= 3145 + 513 + 10917 + 1692.5 \\ &= 16267.5. \end{aligned}$$

So, AAA took in \$16,267.50 during the month of May. To calculate the profit, we must first calculate how much AAA paid for the items sold. We use the dot product $\vec{\mathbf{c}} \cdot \vec{\mathbf{q}}$ to get

$$\begin{aligned} \vec{\mathbf{c}} \cdot \vec{\mathbf{q}} &= \langle 0.25, 0.25, 0.50, 0.20 \rangle \cdot \langle 1258, 342, 2426, 1354 \rangle \\ &= 314.5 + 85.5 + 1213 + 270.8 \\ &= 1883.8. \end{aligned}$$

So, AAA paid \$1,883.80 for the items they sold. Their profit, then, is given by

$$\vec{\mathbf{p}} \cdot \vec{\mathbf{q}} - \vec{\mathbf{c}} \cdot \vec{\mathbf{q}} = 16267.5 - 1883.8 = 14383.7.$$

Therefore, AAA Party Supply Store made \$14,383.70 in May.

? Exercise 12.3.6

On June 1, AAA Party Supply Store decided to increase the price they charge for party favors to \$2 per package. They also changed suppliers for their invitations, and are now able to purchase invitations for only 10¢ per package. All their other costs and prices remain the same. If AAA sells 1408 invitations, 147 party favors, 2112 decorations, and 1894 food service items in the month of June, use vectors and dot products to calculate their total sales and profit for June.

Hint

Use four-dimensional vectors for cost, price, and quantity sold.

Answer

Sales = \$15,685.50; profit = \$14,073.15

Projections

As we have seen, addition combines two vectors to create a resultant vector. But what if we are given a vector and we need to find its component parts? We use vector projections to perform the opposite process; they can break down a vector into its components. The magnitude of a vector projection is a scalar projection. For example, if a child is pulling the handle of a wagon at a 55° angle, we can use projections to determine how much of the force on the handle is actually moving the wagon forward (12.3.6). We return to this example and learn how to solve it after we see how to calculate projections.

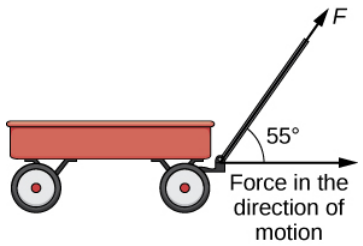


Figure 12.3.6: When a child pulls a wagon, only the horizontal component of the force propels the wagon forward.

Definition: Vector and Projection

The **vector projection** of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$ is the vector labeled $\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}$ in Figure 12.3.7. It has the same initial point as $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ and the same direction as $\vec{\mathbf{u}}$, and represents the component of $\vec{\mathbf{v}}$ that acts in the direction of $\vec{\mathbf{u}}$. If θ represents the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, then, by properties of triangles, we know the length of $\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}$ is $\|\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}\| = \|\vec{\mathbf{v}}\|\cos\theta$. When expressing $\cos\theta$ in terms of the dot product, this becomes

$$\|\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}\| = \|\vec{\mathbf{v}}\|\cos\theta = \|\vec{\mathbf{v}}\|\left(\frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\|}\right) = \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|}.$$

We now multiply by a unit vector in the direction of $\vec{\mathbf{u}}$ to get $\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}$:

$$\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} = \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|}\left(\frac{1}{\|\vec{\mathbf{u}}\|}\vec{\mathbf{u}}\right) = \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2}\vec{\mathbf{u}}.$$

The length of this vector is also known as the **scalar projection** of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$ and is denoted by

$$\|\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}\| = \text{comp}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} = \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|}.$$

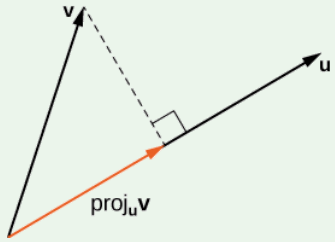


Figure 12.3.7: The projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$ shows the component of vector $\vec{\mathbf{v}}$ in the direction of $\vec{\mathbf{u}}$.

Example 12.3.7: Finding Projections

Find the projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$.

a. $\vec{\mathbf{v}} = \langle 3, 5, 1\rangle$ and $\vec{\mathbf{u}} = \langle -1, 4, 3\rangle$
b. $\vec{\mathbf{v}} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}}$ and $\vec{\mathbf{u}} = \hat{\mathbf{i}} + 6\hat{\mathbf{j}}$

Solution

a. Substitute the components of $\vec{\mathbf{v}}$ and $\vec{\mathbf{u}}$ into the formula for the projection:

$$\begin{aligned}\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} &= \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2}\vec{\mathbf{u}}\\ &= \frac{\langle -1,4,3\rangle\cdot\langle 3,5,1\rangle}{\|\langle -1,4,3\rangle\|^2}\langle -1,4,3\rangle\\ &= \frac{-3+20+3}{(-1)^2+4^2+3^2}\langle -1,4,3\rangle\\ &= \frac{20}{26}\langle -1,4,3\rangle\\ &= \langle -\frac{10}{13}, \frac{40}{13}, \frac{30}{13}\rangle.\end{aligned}$$

b. To find the two-dimensional projection, simply adapt the formula to the two-dimensional case:

$$\begin{aligned}\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} &= \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2}\vec{\mathbf{u}} \\ &= \frac{(\hat{\mathbf{i}}+6\hat{\mathbf{j}})\cdot(3\hat{\mathbf{i}}-2\hat{\mathbf{j}})}{\left\|\hat{\mathbf{i}}+6\hat{\mathbf{j}}\right\|^2}(\hat{\mathbf{i}}+6\hat{\mathbf{j}}) \\ &= \frac{1(3)+6(-2)}{1^2+6^2}(\hat{\mathbf{i}}+6\hat{\mathbf{j}}) \\ &= -\frac{9}{37}(\hat{\mathbf{i}}+6\hat{\mathbf{j}}) \\ &= -\frac{9}{37}\hat{\mathbf{i}}-\frac{54}{37}\hat{\mathbf{j}}.\end{aligned}$$

Sometimes it is useful to decompose vectors—that is, to break a vector apart into a sum. This process is called the **resolution of a vector into components.** Projections allow us to identify two orthogonal vectors having a desired sum. For example, let $\vec{\mathbf{v}}=\langle 6,-4\rangle$ and let $\vec{\mathbf{u}}=\langle 3,1\rangle$. We want to decompose the vector $\vec{\mathbf{v}}$ into orthogonal components such that one of the component vectors has the same direction as $\vec{\mathbf{u}}$.

We first find the component that has the same direction as $\vec{\mathbf{u}}$ by projecting $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$. Let $\vec{\mathbf{p}}=\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}$. Then, we have

$$\begin{aligned}\vec{\mathbf{p}} &= \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2}\vec{\mathbf{u}} \\ &= \frac{18-4}{9+1}\vec{\mathbf{u}} \\ = \frac{7}{5}\vec{\mathbf{u}} = \frac{7}{5}\langle 3,1\rangle &= \langle\frac{21}{5},\frac{7}{5}\rangle.\end{aligned}$$

Now consider the vector $\vec{\mathbf{q}}=\vec{\mathbf{v}}-\vec{\mathbf{p}}$. We have

$$\begin{aligned}\vec{\mathbf{q}} &= \vec{\mathbf{v}}-\vec{\mathbf{p}} \\ &= \langle 6,-4\rangle-\langle\frac{21}{5},\frac{7}{5}\rangle \\ &= \langle\frac{9}{5},-\frac{27}{5}\rangle.\end{aligned}$$

Clearly, by the way we defined $\vec{\mathbf{q}}$, we have $\vec{\mathbf{v}}=\vec{\mathbf{q}}+\vec{\mathbf{p}}$, and

$$\begin{aligned}\vec{\mathbf{q}}\cdot\vec{\mathbf{p}} &= \langle\frac{9}{5},-\frac{27}{5}\rangle\cdot\langle\frac{21}{5},\frac{7}{5}\rangle \\ &= \frac{9(21)}{25}+-\frac{27(7)}{25} \\ &= \frac{189}{25}-\frac{189}{25}=0.\end{aligned}$$

Therefore, $\vec{\mathbf{q}}$ and $\vec{\mathbf{p}}$ are orthogonal.

Example 12.3.8: Resolving Vectors into Components

Express $\vec{\mathbf{v}}=\langle 8,-3,-3\rangle$ as a sum of orthogonal vectors such that one of the vectors has the same direction as $\vec{\mathbf{u}}=\langle 2,3,2\rangle$.

Solution

Let $\vec{\mathbf{p}}$ represent the projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$:

$$\begin{aligned}\vec{\mathbf{p}} &= \text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}} \\ &= \frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2}\vec{\mathbf{u}} \\ &= \frac{\langle 2,3,2\rangle\cdot\langle 8,-3,-3\rangle}{\|\langle 2,3,2\rangle\|^2}\langle 2,3,2\rangle \\ &= \frac{16-9-6}{2^2+3^2+2^2}\langle 2,3,2\rangle \\ &= \frac{1}{17}\langle 2,3,2\rangle \\ &= \langle \frac{2}{17},\frac{3}{17},\frac{2}{17}\rangle.\end{aligned}$$

Then,

$$\begin{aligned}\vec{\mathbf{q}} &= \vec{\mathbf{v}}-\vec{\mathbf{p}} = \langle 8,-3,-3\rangle - \langle \frac{2}{17},\frac{3}{17},\frac{2}{17}\rangle \\ &= \langle \frac{134}{17},-\frac{54}{17},-\frac{53}{17}\rangle.\end{aligned}$$

To check our work, we can use the dot product to verify that $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ are orthogonal vectors:

$$\begin{aligned}\vec{\mathbf{p}}\cdot\vec{\mathbf{q}} &= \langle \frac{2}{17},\frac{3}{17},\frac{2}{17}\rangle\cdot\langle \frac{134}{17},-\frac{54}{17},-\frac{53}{17}\rangle \\ &= \frac{268}{289}-\frac{162}{289}-\frac{106}{289}=0.\end{aligned}$$

Then,

$$\vec{\mathbf{v}} = \vec{\mathbf{p}}+\vec{\mathbf{q}} = \langle \frac{2}{17},\frac{3}{17},\frac{2}{17}\rangle + \langle \frac{134}{17},-\frac{54}{17},-\frac{53}{17}\rangle.$$

Exercise 12.3.7

Express $\vec{\mathbf{v}} = 5\hat{\mathbf{i}}-\hat{\mathbf{j}}$ as a sum of orthogonal vectors such that one of the vectors has the same direction as $\vec{\mathbf{u}} = 4\hat{\mathbf{i}}+2\hat{\mathbf{j}}$.

Hint

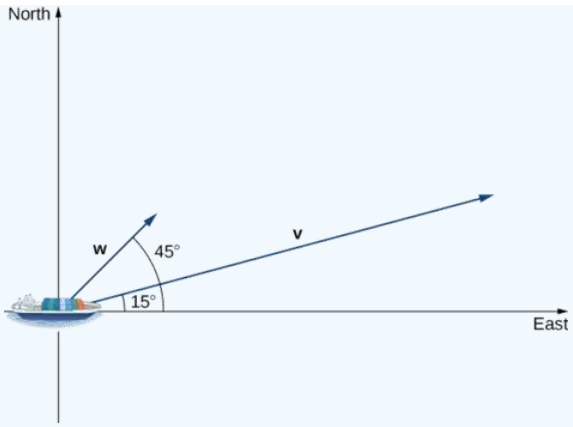
Start by finding the projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$.

Answer

$\vec{\mathbf{v}} = \vec{\mathbf{p}}+\vec{\mathbf{q}}$, where $\vec{\mathbf{p}} = \frac{18}{5}\hat{\mathbf{i}}+\frac{9}{5}\hat{\mathbf{j}}$ and $\vec{\mathbf{q}} = \frac{7}{5}\hat{\mathbf{i}}-\frac{14}{5}\hat{\mathbf{j}}$

Example 12.3.9: Scalar Projection of Velocity

A container ship leaves port traveling 15° north of east. Its engine generates a speed of 20 knots along that path (see the following figure). In addition, the ocean current moves the ship northeast at a speed of 2 knots. Considering both the engine and the current, how fast is the ship moving in the direction 15° north of east? Round the answer to two decimal places.



Solution

Let $\vec{\mathbf{v}}$ be the velocity vector generated by the engine, and let $\vec{\mathbf{w}}$ be the velocity vector of the current. We already know $\|\vec{\mathbf{v}}\| = 20$ along the desired route. We just need to add in the scalar projection of $\vec{\mathbf{w}}$ onto $\vec{\mathbf{v}}$. We get

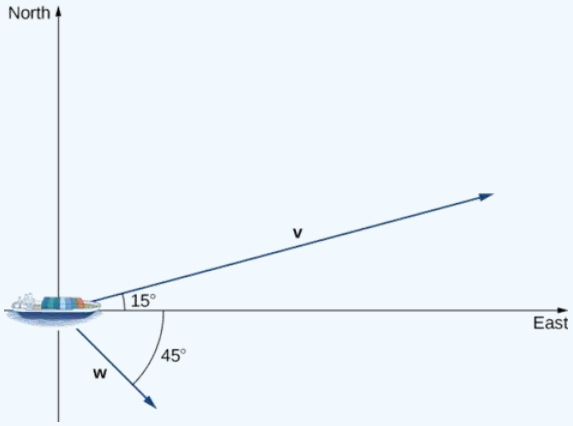
$$\text{comp}_{\vec{\mathbf{v}}}\vec{\mathbf{w}} = \frac{\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}}{\|\vec{\mathbf{v}}\|}$$

$$= \frac{\|\vec{\mathbf{v}}\|\|\vec{\mathbf{w}}\| \cos(30^\circ)}{\|\vec{\mathbf{v}}\|} = \|\vec{\mathbf{w}}\| \cos(30^\circ) = 2\frac{\sqrt{3}}{2} = \sqrt{3} \approx 1.73 \text{ knots.}$$

The ship is moving at 21.73 knots in the direction 15° north of east.

? Exercise 12.3.8

Repeat the previous example, but assume the ocean current is moving southeast instead of northeast, as shown in the following figure.



Hint

Compute the scalar projection of $\vec{\mathbf{w}}$ onto $\vec{\mathbf{v}}$.

Answer

21 knots

Work

Now that we understand dot products, we can see how to apply them to real-life situations. The most common application of the dot product of two vectors is in the calculation of work.

From physics, we know that work is done when an object is moved by a force. When the force is constant and applied in the same direction the object moves, then we define the work done as the product of the force and the distance the object travels: $W = Fd$. We saw several examples of this type in earlier chapters. Now imagine the direction of the force is different from the direction of motion, as with the example of a child pulling a wagon. To find the work done, we need to multiply the component of the force that acts in the direction of the motion by the magnitude of the displacement. The dot product allows us to do just that. If we represent an applied force by a vector $\vec{\mathbf{F}}$ and the displacement of an object by a vector $\vec{\mathbf{s}}$, then the **work done by the force** is the dot product of $\vec{\mathbf{F}}$ and $\vec{\mathbf{s}}$.

Definition: Constant Force

When a constant force is applied to an object so the object moves in a straight line from point P to point Q, the work W done by the force $\vec{\mathbf{F}}$, acting at an angle θ from the line of motion, is given by

$$W = \vec{\mathbf{F}} \cdot \overrightarrow{PQ} = \| \vec{\mathbf{F}} \| \| \overrightarrow{PQ} \| \cos\theta.$$

Let's revisit the problem of the child's wagon introduced earlier. Suppose a child is pulling a wagon with a force having a magnitude of 8 lb on the handle at an angle of 55°. If the child pulls the wagon 50 ft, find the work done by the force (Figure 12.3.8).

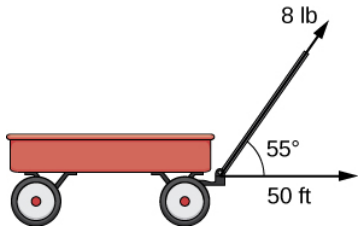


Figure 12.3.8: The horizontal component of the force is the projection of $\vec{\mathbf{F}}$ onto the positive x-axis.

We have

$$W = \| \vec{\mathbf{F}} \| \| \overrightarrow{PQ} \| \cos\theta = 8(50)(\cos(55^\circ)) \approx 229 \text{ ft·lb}.$$

In U.S. standard units, we measure the magnitude of force $\left\| \vec{\mathbf{F}} \right\|$ in pounds. The magnitude of the displacement vector $\left\| \overrightarrow{PQ} \right\|$ tells us how far the object moved, and it is measured in feet. The customary unit of measure for work, then, is the foot-pound. One foot-pound is the amount of work required to move an object weighing 1 lb a distance of 1 ft straight up. In the metric system, the unit of measure for force is the newton (N), and the unit of measure of magnitude for work is a newton-meter (N·m), or a joule (J).

Example 12.3.10: Calculating Work

A conveyor belt generates a force $\vec{\mathbf{F}} = 5\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$ that moves a suitcase from point $(1, 1, 1)$ to point $(9, 4, 7)$ along a straight line. Find the work done by the conveyor belt. The distance is measured in meters and the force is measured in newtons.

Solution

The displacement vector $\overrightarrow{PQ}$ has initial point $(1, 1, 1)$ and terminal point $(9, 4, 7)$:

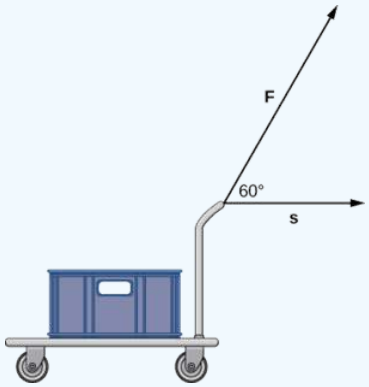
$$\overrightarrow{PQ} = \langle 9-1, 4-1, 7-1 \rangle = \langle 8, 3, 6 \rangle = 8\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}.$$

Work is the dot product of force and displacement:

$$
\begin{aligned}
W &= \vec{\mathbf{F}} \cdot \overrightarrow{PQ} \\
&= (5\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}) \cdot (8\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) \\
&= 5(8) + (-3)(3) + 1(6) \\
&= 37\,\text{N}\cdot\text{m} \\
&= 37\,\text{J}
\end{aligned}
$$

? Exercise 12.3.9

A constant force of 30 lb is applied at an angle of 60° to pull a handcart 10 ft across the ground. What is the work done by this force?



Hint

Use the definition of work as the dot product of force and distance.

Answer

150 ft-lb

Key Concepts

- The dot product, or scalar product, of two vectors $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ is $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2 + u_3 v_3$.
- The dot product satisfies the following properties:
  - $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \vec{\mathbf{v}} \cdot \vec{\mathbf{u}}$
  - $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} + \vec{\mathbf{u}} \cdot \vec{\mathbf{w}}$
  - $c(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}) = (c\vec{\mathbf{u}}) \cdot \vec{\mathbf{v}} = \vec{\mathbf{u}} \cdot (c\vec{\mathbf{v}})$
  - $\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = \|\vec{\mathbf{v}}\|^2$
- The dot product of two vectors can be expressed, alternatively, as $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \|\vec{\mathbf{u}}\| \|\vec{\mathbf{v}}\| \cos\theta$. This form of the dot product is useful for finding the measure of the angle formed by two vectors.
- Vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are orthogonal if $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0$.
- The angles formed by a nonzero vector and the coordinate axes are called the *direction angles* for the vector. The cosines of these angles are known as the *direction cosines.*
- The vector projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$ is the vector $\text{proj}_{\vec{\mathbf{u}}} \vec{\mathbf{v}} = \dfrac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2} \vec{\mathbf{u}}$. The magnitude of this vector is known as the *scalar projection* of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$, given by $\text{comp}_{\vec{\mathbf{u}}} \vec{\mathbf{v}} = \dfrac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|}$.
- Work is done when a force is applied to an object, causing displacement. When the force is represented by the vector $\vec{\mathbf{F}}$ and the displacement is represented by the vector $\vec{\mathbf{s}}$, then the work done W is given by the formula $W = \vec{\mathbf{F}} \cdot \vec{\mathbf{s}} = \| \vec{\mathbf{F}} \| \, \| \vec{\mathbf{s}} \| \cos\theta$.

Key Equations

- **Dot product of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$**

  $$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2 + u_3 v_3 = \|\vec{\mathbf{u}}\| \|\vec{\mathbf{v}}\| \cos\theta$$

- **Cosine of the angle formed by $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$**

  $$\cos\theta = \frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\| \|\vec{\mathbf{v}}\|}$$

- **Vector projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$**

  $$\text{proj}_{\vec{\mathbf{u}}} \vec{\mathbf{v}} = \frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|^2} \vec{\mathbf{u}}$$

- **Scalar projection of $\vec{\mathbf{v}}$ onto $\vec{\mathbf{u}}$**

  $$\text{comp}_{\vec{\mathbf{u}}} \vec{\mathbf{v}} = \frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\|\vec{\mathbf{u}}\|}$$

- **Work done by a force $\vec{\mathbf{F}}$ to move an object through displacement vector $\overrightarrow{PQ}$**

  $$W = \vec{\mathbf{F}} \cdot \overrightarrow{PQ} = \| \vec{\mathbf{F}} \| \| \overrightarrow{PQ} \| \cos\theta$$

Glossary

direction angles
the angles formed by a nonzero vector and the coordinate axes

direction cosines
the cosines of the angles formed by a nonzero vector and the coordinate axes

dot product or scalar product
$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2 + u_3 v_3$ where $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$

scalar projection
the magnitude of the vector projection of a vector

orthogonal vectors
vectors that form a right angle when placed in standard position

vector projection
the component of a vector that follows a given direction

work done by a force
work is generally thought of as the amount of energy it takes to move an object; if we represent an applied force by a vector $\vec{\mathbf{F}}$ and the displacement of an object by a vector $\vec{\mathbf{s}}$, then the work done by the force is the dot product of $\vec{\mathbf{F}}$ and $\vec{\mathbf{s}}$.

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- edited for vector notation by Paul Seeburger

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12.4: The Cross Product

Learning Objectives

- Calculate the cross product of two given vectors.
- Use determinants to calculate a cross product.
- Find a vector orthogonal to two given vectors.
- Determine areas and volumes by using the cross product.
- Calculate the torque of a given force and position vector.

Imagine a mechanic turning a wrench to tighten a bolt. The mechanic applies a force at the end of the wrench. This creates rotation, or torque, which tightens the bolt. We can use vectors to represent the force applied by the mechanic, and the distance (radius) from the bolt to the end of the wrench. Then, we can represent torque by a vector oriented along the axis of rotation. Note that the torque vector is orthogonal to both the force vector and the radius vector.

In this section, we develop an operation called the **cross product**, which allows us to find a vector orthogonal to two given vectors. Calculating torque is an important application of cross products, and we examine torque in more detail later in the section.

The Cross Product and Its Properties

The dot product is a multiplication of two vectors that results in a scalar. In this section, we introduce a product of two vectors that generates a third vector orthogonal to the first two. Consider how we might find such a vector. Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ be nonzero vectors. We want to find a vector $\vec{\mathbf{w}} = \langle w_1, w_2, w_3 \rangle$ orthogonal to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$—that is, we want to find $\vec{\mathbf{w}}$ such that $\vec{\mathbf{u}} \cdot \vec{\mathbf{w}} = 0$ and $\vec{\mathbf{v}} \cdot \vec{\mathbf{w}} = 0$. Therefore, w_1, w_2, and w_3 must satisfy

$$u_1 w_1 + u_2 w_2 + u_3 w_3 = 0 \tag{12.4.1}$$

$$v_1 w_1 + v_2 w_2 + v_3 w_3 = 0. \tag{12.4.2}$$

If we multiply the top equation by v_3 and the bottom equation by u_3 and subtract, we can eliminate the variable w_3, which gives

$$(u_1 v_3 - v_1 u_3) w_1 + (u_2 v_3 - v_2 u_3) w_2 = 0.$$

If we select

$$\begin{aligned} w_1 &= u_2 v_3 - u_3 v_2 \\ w_2 &= -(u_1 v_3 - u_3 v_1), \end{aligned}$$

we get a possible solution vector. Substituting these values back into the original equations (Equations 12.4.1 and 12.4.2) gives

$$w_3 = u_1 v_2 - u_2 v_1.$$

That is, vector

$$\vec{\mathbf{w}} = \langle u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1 \rangle$$

is orthogonal to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, which leads us to define the following operation, called the *cross product*.

Definition: Cross Product

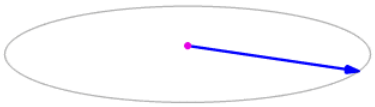
Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$. Then, the *cross product* $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is vector

$$\begin{aligned} \vec{\mathbf{u}} \times \vec{\mathbf{v}} &= (u_2 v_3 - u_3 v_2)\hat{\mathbf{i}} - (u_1 v_3 - u_3 v_1)\hat{\mathbf{j}} + (u_1 v_2 - u_2 v_1)\hat{\mathbf{k}} \\ &= \langle u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1 \rangle. \end{aligned} \tag{12.4.3}$$

From the way we have developed $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$, it should be clear that the cross product is orthogonal to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$. However, it never hurts to check. To show that $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is orthogonal to $\vec{\mathbf{u}}$, we calculate the dot product of $\vec{\mathbf{u}}$ and $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$.

$$\begin{aligned}\vec{\mathbf{u}}\cdot(\vec{\mathbf{u}}\times\vec{\mathbf{v}}) &= \langle u_1,u_2,u_3\rangle\cdot\langle u_2v_3-u_3v_2,-u_1v_3+u_3v_1,u_1v_2-u_2v_1\rangle\\ &= u_1(u_2v_3-u_3v_2)+u_2(-u_1v_3+u_3v_1)+u_3(u_1v_2-u_2v_1)\\ &= u_1u_2v_3-u_1u_3v_2-u_1u_2v_3+u_2u_3v_1+u_1u_3v_2-u_2u_3v_1\\ &= (u_1u_2v_3-u_1u_2v_3)+(-u_1u_3v_2+u_1u_3v_2)+(u_2u_3v_1-u_2u_3v_1)\\ &= 0\end{aligned}$$

In a similar manner, we can show that the cross product is also orthogonal to $\vec{\mathbf{v}}$.



The cross product $\vec{\mathbf{a}}\times\vec{\mathbf{b}}$ (vertical, in pink) changes as the angle between the vectors $\vec{\mathbf{a}}$ (blue) and $\vec{\mathbf{b}}$ (red) changes. The cross product (purple) is always perpendicular to both vectors, and has magnitude zero when the vectors are parallel and maximum magnitude $\|\vec{\mathbf{a}}\|\|\vec{\mathbf{b}}\|$ when they are perpendicular. (Public Domain; LucasVB).

Example 12.4.1: Finding a Cross Product

Let $\vec{\mathbf{p}}=\langle -1,2,5\rangle$ and $\vec{\mathbf{q}}=\langle 4,0,-3\rangle$ (Figure 12.4.1). Find $\vec{\mathbf{p}}\times\vec{\mathbf{q}}$.

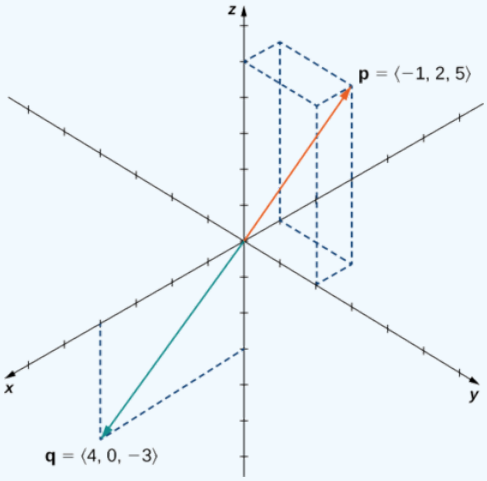


Figure 12.4.1: Finding a cross product to two given vectors.

Solution

Substitute the components of the vectors into Equation 12.4.3:

$$\begin{aligned}\vec{\mathbf{p}}\times\vec{\mathbf{q}} &= \langle -1,2,5\rangle\times\langle 4,0,-3\rangle\\ &= \langle p_2q_3-p_3q_2,-(p_1q_3-p_3q_1),p_1q_2-p_2q_1\rangle\\ &= \langle 2(-3)-5(0),-(-1)(-3)+5(4),(-1)(0)-2(4)\rangle\\ &= \langle -6,17,-8\rangle.\end{aligned}$$

Exercise 12.4.1

Find $\vec{\mathbf{p}} \times \vec{\mathbf{q}}$ for $\vec{\mathbf{p}} = \langle 5, 1, 2 \rangle$ and $\vec{\mathbf{q}} = \langle -2, 0, 1 \rangle$. Express the answer using standard unit vectors.

Hint

Use the formula $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (u_2 v_3 - u_3 v_2)\hat{\mathbf{i}} - (u_1 v_3 - u_3 v_1)\hat{\mathbf{j}} + (u_1 v_2 - u_2 v_1)\hat{\mathbf{k}}$.

Answer

$\vec{\mathbf{p}} \times \vec{\mathbf{q}} = \hat{\mathbf{i}} - 9\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

Although it may not be obvious from Equation 12.4.3, the direction of $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is given by the right-hand rule. If we hold the right hand out with the fingers pointing in the direction of $\vec{\mathbf{u}}$, then curl the fingers toward vector $\vec{\mathbf{v}}$, the thumb points in the direction of the cross product, as shown in Figure 12.4.2

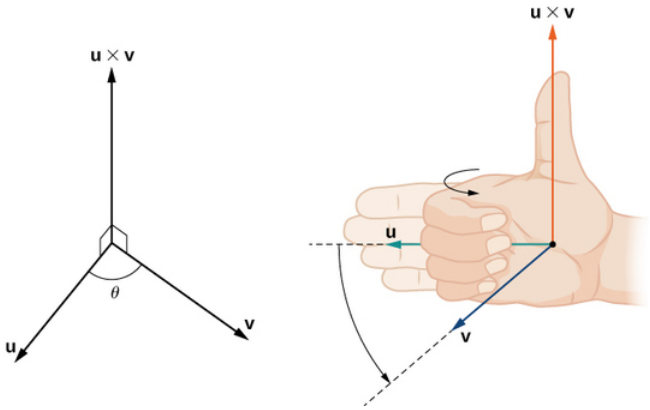


Figure 12.4.2: The direction of $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is determined by the right-hand rule.

Notice what this means for the direction of $\vec{\mathbf{v}} \times \vec{\mathbf{u}}$. If we apply the right-hand rule to $\vec{\mathbf{v}} \times \vec{\mathbf{u}}$, we start with our fingers pointed in the direction of $\vec{\mathbf{v}}$, then curl our fingers toward the vector $\vec{\mathbf{u}}$. In this case, the thumb points in the opposite direction of $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$. (Try it!)

Example 12.4.2: Anticommutativity of the Cross Product

Let $\vec{\mathbf{u}} = \langle 0, 2, 1 \rangle$ and $\vec{\mathbf{v}} = \langle 3, -1, 0 \rangle$. Calculate $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ and $\vec{\mathbf{v}} \times \vec{\mathbf{u}}$ and graph them.

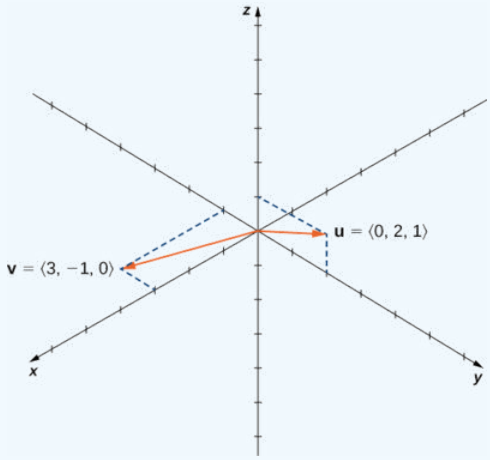


Figure 12.4.3: Are the cross products $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ and $\vec{\mathbf{v}} \times \vec{\mathbf{u}}$ in the same direction?

Solution

We have

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \langle (0+1), -(0-3), (0-6) \rangle = \langle 1, 3, -6 \rangle$$

$$\vec{\mathbf{v}} \times \vec{\mathbf{u}} = \langle (-1-0), -(3-0), (6-0) \rangle = \langle -1, -3, 6 \rangle.$$

We see that, in this case, $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = -(\vec{\mathbf{v}} \times \vec{\mathbf{u}})$ (Figure 12.4.4). We prove this in general later in this section.

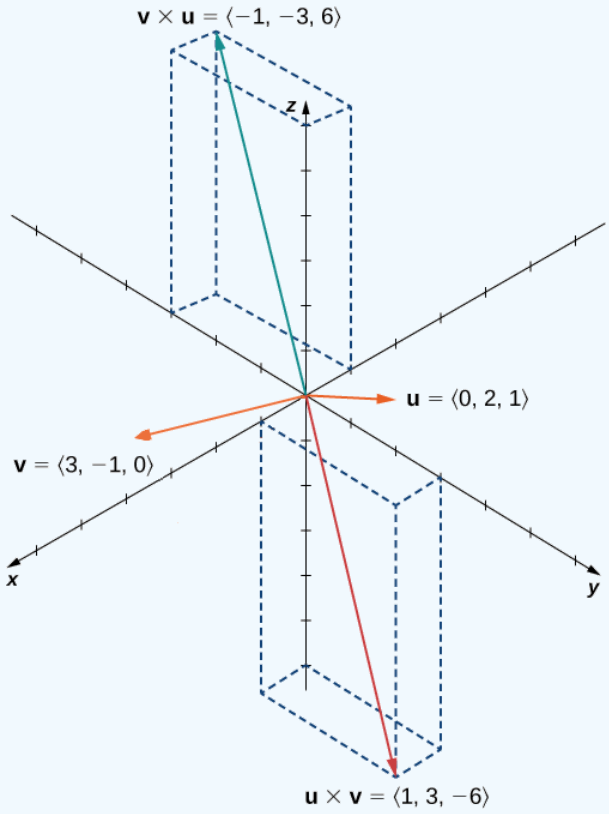


Figure 12.4.4: The cross products $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ and $\vec{\mathbf{v}} \times \vec{\mathbf{u}}$ are both orthogonal to $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, but in opposite directions.

? Exercise 12.4.2

Suppose vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ lie in the xy-plane (the z-component of each vector is zero). Now suppose the x- and y-components of $\vec{\mathbf{u}}$ and the y-component of $\vec{\mathbf{v}}$ are all positive, whereas the x-component of $\vec{\mathbf{v}}$ is negative. Assuming the coordinate axes are oriented in the usual positions, in which direction does $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ point?

Hint

Remember the right-hand rule (Figure 12.4.2).

Answer

Up (the positive z-direction)

The cross products of the standard unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ can be useful for simplifying some calculations, so let's consider these cross products. A straightforward application of the definition shows that

$$\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = \vec{\mathbf{0}}.$$

(The cross product of two vectors is a vector, so each of these products results in the zero vector, not the scalar 0.) It's up to you to verify the calculations on your own.

Furthermore, because the cross product of two vectors is orthogonal to each of these vectors, we know that the cross product of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ is parallel to $\hat{\mathbf{k}}$. Similarly, the vector product of $\hat{\mathbf{i}}$ and $\hat{\mathbf{k}}$ is parallel to $\hat{\mathbf{j}}$, and the vector product of $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ is parallel to $\hat{\mathbf{i}}$.

We can use the right-hand rule to determine the direction of each product. Then we have

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$$
$$\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}}$$

$$\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}$$
$$\hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}}$$

$$\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$$
$$\hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}.$$

These formulas come in handy later.

Example 12.4.3: Cross Product of Standard Unit Vectors

Find $\hat{\mathbf{i}} \times (\hat{\mathbf{j}} \times \hat{\mathbf{k}})$.

Solution

We know that $\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}$. Therefore, $\hat{\mathbf{i}} \times (\hat{\mathbf{j}} \times \hat{\mathbf{k}}) = \hat{\mathbf{i}} \times \hat{\mathbf{i}} = \vec{\mathbf{0}}$.

Exercise 12.4.3

Find $(\hat{\mathbf{i}} \times \hat{\mathbf{j}}) \times (\hat{\mathbf{k}} \times \hat{\mathbf{i}})$.

Hint

Remember the right-hand rule (Figure 12.4.2).

Answer

$-\hat{\mathbf{i}}$

As we have seen, the dot product is often called the *scalar product* because it results in a scalar. The cross product results in a vector, so it is sometimes called the **vector product.** These operations are both versions of vector multiplication, but they have very different properties and applications. Let's explore some properties of the cross product. We prove only a few of them. Proofs of the other properties are left as exercises.

Properties of the Cross Product

Let $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ be vectors in space, and let c be a scalar.

i. Anticommutative property:

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = -(\vec{\mathbf{v}} \times \vec{\mathbf{u}})$$

ii. Distributive property:

$$\vec{\mathbf{u}} \times (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \times \vec{\mathbf{v}} + \vec{\mathbf{u}} \times \vec{\mathbf{w}}$$

iii. Multiplication by a constant:

$$c(\vec{\mathbf{u}} \times \vec{\mathbf{v}}) = (c\vec{\mathbf{u}}) \times \vec{\mathbf{v}} = \vec{\mathbf{u}} \times (c\vec{\mathbf{v}})$$

iv. Cross product of the zero vector:

$$\vec{\mathbf{u}} \times \vec{\mathbf{0}} = \vec{\mathbf{0}} \times \vec{\mathbf{u}} = \vec{\mathbf{0}}$$

v. Cross product of a vector with itself:

$$\vec{\mathbf{v}} \times \vec{\mathbf{v}} = \vec{\mathbf{0}}$$

vi. Triple scalar product:

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) \cdot \vec{\mathbf{w}}$$

vii. Triple cross product:

$$\vec{\mathbf{u}} \times (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = (\vec{\mathbf{u}} \cdot \vec{\mathbf{w}})\, \vec{\mathbf{v}} - (\vec{\mathbf{u}} \cdot \vec{\mathbf{v}})\, \vec{\mathbf{w}}$$

Proof

For property i, we want to show $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = -(\vec{\mathbf{v}} \times \vec{\mathbf{u}})$. We have

$$\begin{aligned} \vec{\mathbf{u}} \times \vec{\mathbf{v}} &= \langle u_1, u_2, u_3 \rangle \times \langle v_1, v_2, v_3 \rangle \\ &= \langle u_2 v_3 - u_3 v_2, -u_1 v_3 + u_3 v_1, u_1 v_2 - u_2 v_1 \rangle \\ &= -\langle u_3 v_2 - u_2 v_3, -u_3 v_1 + u_1 v_3, u_2 v_1 - u_1 v_2 \rangle \\ &= -\langle v_1, v_2, v_3 \rangle \times \langle u_1, u_2, u_3 \rangle \\ &= -(\vec{\mathbf{v}} \times \vec{\mathbf{u}}). \end{aligned}$$

Unlike most operations we've seen, the cross product is not commutative. This makes sense if we think about the right-hand rule.

For property $iv.$, this follows directly from the definition of the cross product. We have

$$\vec{\mathbf{u}} \times \vec{\mathbf{0}} = \langle u_2(0) - u_3(0), -(u_1(0) - u_3(0)), u_1(0) - u_2(0) \rangle = \langle 0, 0, 0 \rangle = \vec{\mathbf{0}}.$$

Then, by property i., $\vec{\mathbf{0}} \times \vec{\mathbf{u}} = \vec{\mathbf{0}}$ as well. Remember that the dot product of a vector and the zero vector is the *scalar* 0, whereas the cross product of a vector with the zero vector is the *vector* $\vec{\mathbf{0}}$.

Property $vi.$ looks like the associative property, but note the change in operations:

$$\begin{aligned} \vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) &= u \cdot \langle v_2 w_3 - v_3 w_2, -v_1 w_3 + v_3 w_1, v_1 w_2 - v_2 w_1 \rangle \\ &= u_1(v_2 w_3 - v_3 w_2) + u_2(-v_1 w_3 + v_3 w_1) + u_3(v_1 w_2 - v_2 w_1) \\ &= u_1 v_2 w_3 - u_1 v_3 w_2 - u_2 v_1 w_3 + u_2 v_3 w_1 + u_3 v_1 w_2 - u_3 v_2 w_1 \\ &= (u_2 v_3 - u_3 v_2) w_1 + (u_3 v_1 - u_1 v_3) w_2 + (u_1 v_2 - u_2 v_1) w_3 \\ &= \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle \cdot \langle w_1, w_2, w_3 \rangle = (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) \cdot \vec{\mathbf{w}}. \end{aligned}$$

□

Example 12.4.4: Using the Properties of the Cross Product

Use the cross product properties to calculate $(2\hat{\mathbf{i}} \times 3\hat{\mathbf{j}}) \times \hat{\mathbf{j}}$.

Solution

$$\begin{aligned} (2\hat{\mathbf{i}} \times 3\hat{\mathbf{j}}) \times \hat{\mathbf{j}} &= 2(\hat{\mathbf{i}} \times 3\hat{\mathbf{j}}) \times \hat{\mathbf{j}} \\ &= 2(3)(\hat{\mathbf{i}} \times \hat{\mathbf{j}}) \times \hat{\mathbf{j}} \\ &= (6\hat{\mathbf{k}}) \times \hat{\mathbf{j}} \\ &= 6(\hat{\mathbf{k}} \times \hat{\mathbf{j}}) \\ &= 6(-\hat{\mathbf{i}}) = -6\hat{\mathbf{i}}. \end{aligned}$$

Exercise 12.4.4

Use the properties of the cross product to calculate $(\hat{\mathbf{i}} \times \hat{\mathbf{k}}) \times (\hat{\mathbf{k}} \times \hat{\mathbf{j}})$.

Hint

$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = -(\vec{\mathbf{v}} \times \vec{\mathbf{u}})$

Answer

$-\hat{\mathbf{k}}$

So far in this section, we have been concerned with the direction of the vector $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$, but we have not discussed its magnitude. It turns out there is a simple expression for the magnitude of $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ involving the magnitudes of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, and the sine of the angle between them.

Magnitude of the Cross Product

Let $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ be vectors, and let θ be the angle between them. Then, $\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\| = \|\vec{\mathbf{u}}\| \cdot \|\vec{\mathbf{v}}\| \cdot \sin\theta$.

Proof

Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ be vectors, and let θ denote the angle between them. Then

$$\begin{aligned}
\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\|^2 &= (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2 \\
&= u_2^2v_3^2 - 2u_2u_3v_2v_3 + u_3^2v_2^2 + u_3^2v_1^2 - 2u_1u_3v_1v_3 + u_1^2v_3^2 + u_1^2v_2^2 - 2u_1u_2v_1v_2 + u_2^2v_1^2 \\
&= u_1^2v_1^2 + u_1^2v_2^2 + u_1^2v_3^2 + u_2^2v_1^2 + u_2^2v_2^2 + u_2^2v_3^2 + u_3^2v_1^2 + u_3^2v_2^2 + u_3^2v_3^2 \\
&\quad - (u_1^2v_1^2 + u_2^2v_2^2 + u_3^2v_3^2 + 2u_1u_2v_1v_2 + 2u_1u_3v_1v_3 + 2u_2u_3v_2v_3) \\
&= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2 \\
&= \|\vec{\mathbf{u}}\|^2\|\vec{\mathbf{v}}\|^2 - (\vec{\mathbf{u}} \cdot \vec{\mathbf{v}})^2 \\
&= \|\vec{\mathbf{u}}\|^2\|\vec{\mathbf{v}}\|^2 - \|\vec{\mathbf{u}}\|^2\|\vec{\mathbf{v}}\|^2 \cos^2\theta \\
&= \|\vec{\mathbf{u}}\|^2\|\vec{\mathbf{v}}\|^2(1 - \cos^2\theta) \\
&= \|\vec{\mathbf{u}}\|^2\|\vec{\mathbf{v}}\|^2(\sin^2\theta).
\end{aligned}$$

Taking square roots and noting that $\sqrt{\sin^2\theta} = \sin\theta$ for $0 \leq \theta \leq 180^\circ$, we have the desired result:

$$\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\| = \|\vec{\mathbf{u}}\|\|\vec{\mathbf{v}}\| \sin\theta.$$

□

This definition of the cross product allows us to visualize or interpret the product geometrically. It is clear, for example, that the cross product is defined only for vectors in three dimensions, not for vectors in two dimensions. In two dimensions, it is impossible to generate a vector simultaneously orthogonal to two nonparallel vectors.

Example 12.4.5: Calculating the Cross Product

Use "Magnitude of the Cross Product" to find the magnitude of the cross product of $\vec{\mathbf{u}} = \langle 0, 4, 0 \rangle$ and $\vec{\mathbf{v}} = \langle 0, 0, -3 \rangle$.

Solution

We have

$$\begin{aligned}\|\vec{\mathbf{u}}\times\vec{\mathbf{v}}\| &= \|\vec{\mathbf{u}}\|\cdot\|\vec{\mathbf{v}}\|\cdot\sin\theta \\ &= \sqrt{0^2+4^2+0^2}\cdot\sqrt{0^2+0^2+(-3)^2}\cdot\sin\frac{\pi}{2} \\ &= 4(3)(1)=12\end{aligned}$$

? Exercise 12.4.5

Use "Magnitude of the Cross Product" to find the magnitude of $\vec{\mathbf{u}}\times\vec{\mathbf{v}}$, where $\vec{\mathbf{u}}=\langle -8,0,0\rangle$ and $\vec{\mathbf{v}}=\langle 0,2,0\rangle$.

Hint

Vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are orthogonal.

Answer

16

Determinants and the Cross Product

Using Equation 12.4.3 to find the cross product of two vectors is straightforward, and it presents the cross product in the useful component form. The formula, however, is complicated and difficult to remember. Fortunately, we have an alternative. We can calculate the cross product of two vectors using determinant notation.

A 2×2 determinant is defined by

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - b_1a_2.$$

For example,

$$\begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix} = 3(1)-5(-2)=3+10=13.$$

A 3×3 determinant is defined in terms of 2×2 determinants as follows:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}. \tag{12.4.4}$$

Equation 12.4.4 is referred to as the expansion of the determinant along the first row. Notice that the multipliers of each of the 2×2 determinants on the right side of this expression are the entries in the first row of the 3×3 determinant. Furthermore, each of the 2×2 determinants contains the entries from the 3×3 determinant that would remain if you crossed out the row and column containing the multiplier. Thus, for the first term on the right, a_1 is the multiplier, and the 2×2 determinant contains the entries that remain if you cross out the first row and first column of the 3×3 determinant. Similarly, for the second term, the multiplier is a_2, and the 2×2 determinant contains the entries that remain if you cross out the first row and second column of the 3×3 determinant. Notice, however, that the coefficient of the second term is negative. The third term can be calculated in similar fashion.

✓ Example 12.4.6: Using Expansion Along the First Row to Compute a 3×3 Determinant

Evaluate the determinant $\begin{vmatrix} 2 & 5 & -1 \\ -1 & 1 & 3 \\ -2 & 3 & 4 \end{vmatrix}$.

Solution

We have

$$\begin{vmatrix} 2 & 5 & -1 \\ -1 & 1 & 3 \\ -2 & 3 & 4 \end{vmatrix} = 2\begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} - 5\begin{vmatrix} -1 & 3 \\ -2 & 4 \end{vmatrix} - 1\begin{vmatrix} -1 & 1 \\ -2 & 3 \end{vmatrix}$$
$$= 2(4-9) - 5(-4+6) - 1(-3+2)$$
$$= 2(-5) - 5(2) - 1(-1) = -10 - 10 + 1$$
$$= -19$$

? Exercise 12.4.6

Evaluate the determinant $\begin{vmatrix} 1 & -2 & -1 \\ 3 & 2 & -3 \\ 1 & 5 & 4 \end{vmatrix}$.

Hint

Expand along the first row. Don't forget the second term is negative!

Answer

40

Technically, determinants are defined only in terms of arrays of real numbers. However, the determinant notation provides a useful mnemonic device for the cross product formula.

Rule: Cross Product Calculated by a Determinant

Let $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ be vectors. Then the cross product $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ is given by

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \hat{\mathbf{k}}.$$

✓ Example 12.4.7: Using Determinant Notation to find $\vec{\mathbf{p}} \times \vec{\mathbf{q}}$

Let $\vec{\mathbf{p}} = \langle -1, 2, 5 \rangle$ and $\vec{\mathbf{q}} = \langle 4, 0, -3 \rangle$. Find $\vec{\mathbf{p}} \times \vec{\mathbf{q}}$.

Solution

We set up our determinant by putting the standard unit vectors across the first row, the components of $\vec{\mathbf{u}}$ in the second row, and the components of $\vec{\mathbf{v}}$ in the third row. Then, we have

$$\vec{\mathbf{p}} \times \vec{\mathbf{q}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 2 & 5 \\ 4 & 0 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 5 \\ 0 & -3 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} -1 & 5 \\ 4 & -3 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} -1 & 2 \\ 4 & 0 \end{vmatrix} \hat{\mathbf{k}}$$
$$= (-6-0)\hat{\mathbf{i}} - (3-20)\hat{\mathbf{j}} + (0-8)\hat{\mathbf{k}}$$
$$= -6\hat{\mathbf{i}} + 17\hat{\mathbf{j}} - 8\hat{\mathbf{k}}.$$

Notice that this answer confirms the calculation of the cross product in Example 12.4.1.

? Exercise 12.4.7

Use determinant notation to find $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$, where $\vec{\mathbf{a}} = \langle 8, 2, 3 \rangle$ and $\vec{\mathbf{b}} = \langle -1, 0, 4 \rangle$.

Hint

Calculate the determinant $\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 8 & 2 & 3 \\ -1 & 0 & 4 \end{vmatrix}$.

Answer

$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = 8\hat{\mathbf{i}} - 35\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

Using the Cross Product

The cross product is very useful for several types of calculations, including finding a vector orthogonal to two given vectors, computing areas of triangles and parallelograms, and even determining the volume of the three-dimensional geometric shape made of parallelograms known as a *parallelepiped*. The following examples illustrate these calculations.

Example 12.4.8: Finding a Unit Vector Orthogonal to Two Given Vectors

Let $\vec{\mathbf{a}} = \langle 5, 2, -1 \rangle$ and $\vec{\mathbf{b}} = \langle 0, -1, 4 \rangle$. Find a unit vector orthogonal to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

Solution

The cross product $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is orthogonal to both vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$. We can calculate it with a determinant:

$$\begin{aligned} \vec{\mathbf{a}} \times \vec{\mathbf{b}} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 5 & 2 & -1 \\ 0 & -1 & 4 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} 5 & -1 \\ 0 & 4 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} 5 & 2 \\ 0 & -1 \end{vmatrix} \hat{\mathbf{k}} \\ &= (8-1)\hat{\mathbf{i}} - (20-0)\hat{\mathbf{j}} + (-5-0)\hat{\mathbf{k}} \\ &= 7\hat{\mathbf{i}} - 20\hat{\mathbf{j}} - 5\hat{\mathbf{k}}. \end{aligned}$$

Normalize this vector to find a unit vector in the same direction:

$$\|\vec{\mathbf{a}} \times \vec{\mathbf{b}}\| = \sqrt{(7)^2 + (-20)^2 + (-5)^2} = \sqrt{474}\,.$$

Thus, $\left\langle \frac{7}{\sqrt{474}}, \frac{-20}{\sqrt{474}}, \frac{-5}{\sqrt{474}} \right\rangle$ is a unit vector orthogonal to $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.

Simplified, this vector becomes $\left\langle \frac{7\sqrt{474}}{474}, \frac{-10\sqrt{474}}{237}, \frac{-5\sqrt{474}}{474} \right\rangle$.

Exercise 12.4.8

Find a unit vector orthogonal to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, where $\vec{\mathbf{a}} = \langle 4, 0, 3 \rangle$ and $\vec{\mathbf{b}} = \langle 1, 1, 4 \rangle$.

Hint

Normalize the cross product.

Answer

$\left\langle \frac{-3}{\sqrt{194}}, \frac{-13}{\sqrt{194}}, \frac{4}{\sqrt{194}} \right\rangle$ or, simplified as $\left\langle \frac{-3\sqrt{194}}{194}, \frac{-13\sqrt{194}}{194}, \frac{2\sqrt{194}}{97} \right\rangle$

To use the cross product for calculating areas, we state and prove the following theorem.

Area of a Parallelogram

If we locate vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ such that they form adjacent sides of a parallelogram, then the area of the parallelogram is given by $\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\|$ (Figure 12.4.5).

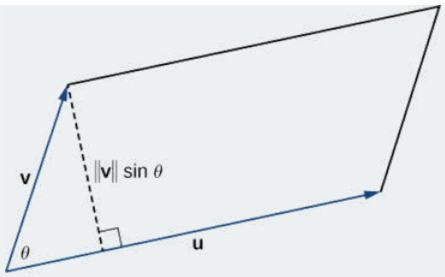


Figure 12.4.5: The parallelogram with adjacent sides $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ has base $\|\vec{\mathbf{u}}\|$ and height $\|\vec{\mathbf{v}}\| \sin\theta$.

Proof

We show that the magnitude of the cross product is equal to the base times height of the parallelogram.

$$\begin{aligned} \text{Area of a parallelogram} &= \text{base} \times \text{height} \\ &= \|\vec{\mathbf{u}}\|(\|\vec{\mathbf{v}}\| \sin\theta) \\ &= \|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\| \end{aligned}$$

□

Example 12.4.9: Finding the Area of a Triangle

Let $P = (1, 0, 0), Q = (0, 1, 0)$, and $R = (0, 0, 1)$ be the vertices of a triangle (Figure 12.4.6). Find its area.

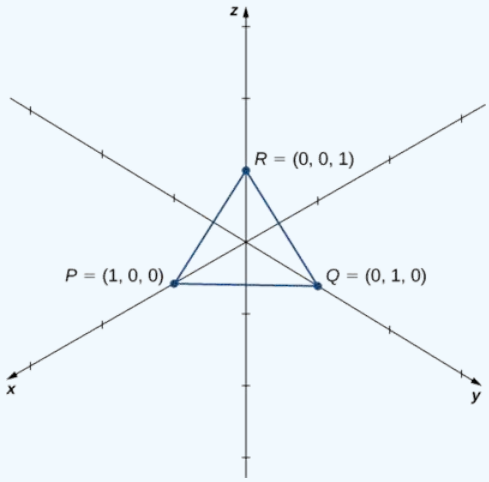


Figure 12.4.6: Finding the area of a triangle by using the cross product.

Solution

We have $\overrightarrow{PQ} = \langle 0-1, 1-0, 0-0\rangle = \langle -1, 1, 0\rangle$ and $\overrightarrow{PR} = \langle 0-1, 0-0, 1-0\rangle = \langle -1, 0, 1\rangle$. The area of the parallelogram with adjacent sides $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is given by $\left\|\overrightarrow{PQ} \times \overrightarrow{PR}\right\|$:

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$
$$= (1-0)\hat{\mathbf{i}} - (-1-0)\hat{\mathbf{j}} + (0-(-1))\hat{\mathbf{k}}$$
$$= \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$$

$$\left\| \overrightarrow{PQ} \times \overrightarrow{PR} \right\| = \| \langle 1, 1, 1 \rangle \|$$
$$= \sqrt{1^2 + 1^2 + 1^2}$$
$$= \sqrt{3}.$$

The area of ΔPQR is half the area of the parallelogram or $\sqrt{3}/2$ units^2.

Exercise 12.4.9

Find the area of the parallelogram $PQRS$ with vertices $P(1, 1, 0)$, $Q(7, 1, 0)$, $R(9, 4, 2)$, and $S(3, 4, 2)$.

Hint

Sketch the parallelogram and identify two vectors that form adjacent sides of the parallelogram.

Answer

$6\sqrt{13}$ units^2

The Triple Scalar Product

Because the cross product of two vectors is a vector, it is possible to combine the dot product and the cross product. The dot product of a vector with the cross product of two other vectors is called the **triple scalar product** because the result is a scalar.

Definition: Triple Scalar Product

The triple scalar product of vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ is

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}).$$

Calculating a Triple Scalar Product

The triple scalar product of vectors

$$\vec{\mathbf{u}} = u_1 \hat{\mathbf{i}} + u_2 \hat{\mathbf{j}} + u_3 \hat{\mathbf{k}}$$
$$\vec{\mathbf{v}} = v_1 \hat{\mathbf{i}} + v_2 \hat{\mathbf{j}} + v_3 \hat{\mathbf{k}}$$

and

$$\vec{\mathbf{w}} = w_1 \hat{\mathbf{i}} + w_2 \hat{\mathbf{j}} + w_3 \hat{\mathbf{k}}$$

is the determinant of the 3×3 matrix formed by the components of the vectors:

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}. \tag{12.4.5}$$

Proof

The calculation is straightforward.

$$\begin{aligned}\vec{\mathbf{u}}\cdot(\vec{\mathbf{v}}\times\vec{\mathbf{w}}) &= \langle u_1,u_2,u_3\rangle\cdot\langle v_2w_3-v_3w_2,-v_1w_3+v_3w_1,v_1w_2-v_2w_1\rangle\\ &= u_1(v_2w_3-v_3w_2)+u_2(-v_1w_3+v_3w_1)+u_3(v_1w_2-v_2w_1)\\ &= u_1(v_2w_3-v_3w_2)-u_2(v_1w_3-v_3w_1)+u_3(v_1w_2-v_2w_1)\\ &= \begin{vmatrix} u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\\ w_1 & w_2 & w_3\end{vmatrix}.\end{aligned}$$

□

Example 12.4.10: Calculating the Triple Scalar Product

Let $\vec{\mathbf{u}}=\langle 1,3,5\rangle$, $\vec{\mathbf{v}}=\langle 2,-1,0\rangle$ and $\vec{\mathbf{w}}=\langle -3,0,-1\rangle$. Calculate the triple scalar product $\vec{\mathbf{u}}\cdot(\vec{\mathbf{v}}\times\vec{\mathbf{w}})$.

Solution

Apply Equation 12.4.5 directly:

$$\begin{aligned}\vec{\mathbf{u}}\cdot(\vec{\mathbf{v}}\times\vec{\mathbf{w}}) &= \begin{vmatrix} 1 & 3 & 5\\ 2 & -1 & 0\\ -3 & 0 & -1\end{vmatrix}\\ &= 1\begin{vmatrix} -1 & 0\\ 0 & -1\end{vmatrix}-3\begin{vmatrix} 2 & 0\\ -3 & -1\end{vmatrix}+5\begin{vmatrix} 2 & -1\\ -3 & 0\end{vmatrix}\\ &= (1-0)-3(-2-0)+5(0-3)\\ &= 1+6-15=-8.\end{aligned}$$

Exercise 12.4.10

Calculate the triple scalar product $\vec{\mathbf{a}}\cdot(\vec{\mathbf{b}}\times\vec{\mathbf{c}})$, where $\vec{\mathbf{a}}=\langle 2,-4,1\rangle$, $\vec{\mathbf{b}}=\langle 0,3,-1\rangle$, and $\vec{\mathbf{c}}=\langle 5,-3,3\rangle$.

Hint

Place the vectors as the rows of a 3×3 matrix, then calculate the determinant.

Answer

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When we create a matrix from three vectors, we must be careful about the order in which we list the vectors. If we list them in a matrix in one order and then rearrange the rows, the absolute value of the determinant remains unchanged. However, each time two rows switch places, the determinant changes sign:

$$\begin{vmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}=d \qquad \begin{vmatrix} b_1 & b_2 & b_3\\ a_1 & a_2 & a_3\\ c_1 & c_2 & c_3\end{vmatrix}=-d \qquad \begin{vmatrix} b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\\ a_1 & a_2 & a_3\end{vmatrix}=d \qquad \begin{vmatrix} c_1 & c_2 & c_3\\ b_1 & b_2 & b_3\\ a_1 & a_2 & a_3\end{vmatrix}=-d$$

Verifying this fact is straightforward, but rather messy. Let's take a look at this with an example:

$$\begin{aligned}\begin{vmatrix} 1 & 2 & 1\\ -2 & 0 & 3\\ 4 & 1 & -1\end{vmatrix} &= \begin{vmatrix} 0 & 3\\ 1 & -1\end{vmatrix}-2\begin{vmatrix} -2 & 3\\ 4 & -1\end{vmatrix}+\begin{vmatrix} -2 & 0\\ 4 & 1\end{vmatrix}\\ &= (0-3)-2(2-12)+(-2-0)\\ &= -3+20-2=15.\end{aligned}$$

Switching the top two rows we have

$$\begin{vmatrix} -2 & 0 & 3 \\ 1 & 2 & 1 \\ 4 & 1 & -1 \end{vmatrix} = -2\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix}$$
$$= -2(-2-1) + 3(1-8)$$
$$= 6 - 21 = -15.$$

Rearranging vectors in the triple products is equivalent to reordering the rows in the matrix of the determinant. Let $\vec{\mathbf{u}} = u_1\,\hat{\mathbf{i}} + u_2\hat{\mathbf{j}} + u_3\hat{\mathbf{k}}$, $\vec{\mathbf{v}} = v_1\,\hat{\mathbf{i}} + v_2\hat{\mathbf{j}} + v_3\hat{\mathbf{k}}$, and $\vec{\mathbf{w}} = w_1\,\hat{\mathbf{i}} + w_2\hat{\mathbf{j}} + w_3\hat{\mathbf{k}}$. Applying Calculating a Triple Scalar Product, we have

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

and

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{w}} \times \vec{\mathbf{v}}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

We can obtain the determinant for calculating $\vec{\mathbf{u}} \cdot (\vec{\mathbf{w}} \times \vec{\mathbf{v}})$ by switching the bottom two rows of $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})$. Therefore, $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = -\vec{\mathbf{u}} \cdot (\vec{\mathbf{w}} \times \vec{\mathbf{v}})$.

Following this reasoning and exploring the different ways we can interchange variables in the triple scalar product lead to the following identities:

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = -\vec{\mathbf{u}} \cdot (\vec{\mathbf{w}} \times \vec{\mathbf{v}}) \tag{12.4.6}$$

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = \vec{\mathbf{v}} \cdot (\vec{\mathbf{w}} \times \vec{\mathbf{u}}) = \vec{\mathbf{w}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}}). \tag{12.4.7}$$

Let $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ be two vectors in standard position. If $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are not scalar multiples of each other, then these vectors form adjacent sides of a parallelogram. We saw in Area of a Parallelogram that the area of this parallelogram is $\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\|$. Now suppose we add a third vector $\vec{\mathbf{w}}$ that does not lie in the same plane as $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ but still shares the same initial point. Then these vectors form three edges of a **parallelepiped**, a three-dimensional prism with six faces that are each parallelograms, as shown in Figure 12.4.7. The volume of this prism is the product of the figure's height and the area of its base. The triple scalar product of $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ provides a simple method for calculating the volume of the parallelepiped defined by these vectors.

Volume of a Parallelepiped

The volume of a parallelepiped with adjacent edges given by the vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ is the absolute value of the triple scalar product (Figure 12.4.7):

$$V = |\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})|.$$

Note that, as the name indicates, the triple scalar product produces a scalar. The volume formula just presented uses the absolute value of a scalar quantity.

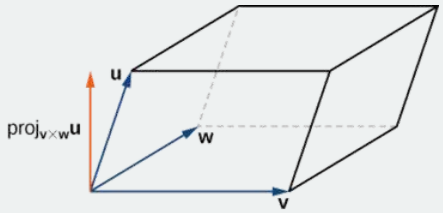


Figure 12.4.7: The height of the parallelepiped is given by $\|\text{proj}_{\vec{\mathbf{v}} \times \vec{\mathbf{w}}}\vec{\mathbf{u}}\|$.

Proof

The area of the base of the parallelepiped is given by $\|\vec{\mathbf{v}} \times \vec{\mathbf{w}}\|$. The height of the figure is given by $\|\text{proj}_{\vec{\mathbf{v}} \times \vec{\mathbf{w}}} \vec{\mathbf{u}}\|$. The volume of the parallelepiped is the product of the height and the area of the base, so we have

$$\begin{aligned} V &= \| \text{proj}_{\vec{\mathbf{v}} \times \vec{\mathbf{w}}} \vec{\mathbf{u}} \| \, \|\vec{\mathbf{v}} \times \vec{\mathbf{w}}\| \\ &= \left| \frac{\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})}{\|\vec{\mathbf{v}} \times \vec{\mathbf{w}}\|} \right| \|\vec{\mathbf{v}} \times \vec{\mathbf{w}}\| \\ &= |\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})|. \end{aligned}$$

□

Example 12.4.11: Calculating the Volume of a Parallelepiped

Let $\vec{\mathbf{u}} = \langle -1, -2, 1 \rangle$, $\vec{\mathbf{v}} = \langle 4, 3, 2 \rangle$, and $\vec{\mathbf{w}} = \langle 0, -5, -2 \rangle$. Find the volume of the parallelepiped with adjacent edges $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ (Figure 12.4.8).

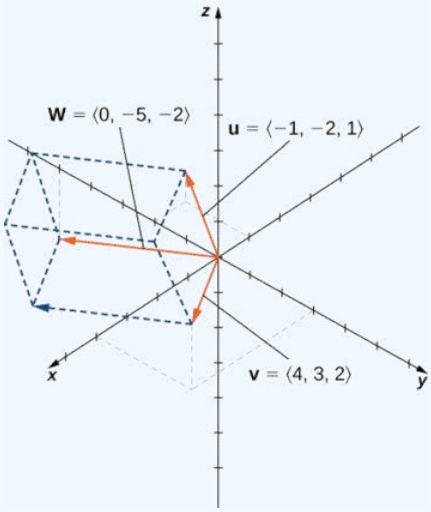


Figure 12.4.8

Solution

We have

$$\begin{aligned} \vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) &= \begin{vmatrix} -1 & -2 & 1 \\ 4 & 3 & 2 \\ 0 & -5 & -2 \end{vmatrix} \\ &= (-1) \begin{vmatrix} 3 & 2 \\ -5 & -2 \end{vmatrix} + 2 \begin{vmatrix} 4 & 2 \\ 0 & -2 \end{vmatrix} + \begin{vmatrix} 4 & 3 \\ 0 & -5 \end{vmatrix} \\ &= (-1)(-6+10) + 2(-8-0) + (-20-0) \\ &= -4 - 16 - 20 \\ &= -40. \end{aligned}$$

Thus, the volume of the parallelepiped is $|-40| = 40$ units^3

Exercise 12.4.11

Find the volume of the parallelepiped formed by the vectors $\vec{\mathbf{a}} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - \hat{\mathbf{k}}$, $\vec{\mathbf{b}} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}$, and $\vec{\mathbf{c}} = 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$.

Hint

Calculate the triple scalar product by finding a determinant.

Answer

8 units^3

Applications of the Cross Product

The cross product appears in many practical applications in mathematics, physics, and engineering. Let's examine some of these applications here, including the idea of torque, with which we began this section. Other applications show up in later chapters, particularly in our study of vector fields such as gravitational and electromagnetic fields (Introduction to Vector Calculus).

Example 12.4.12: Using the Triple Scalar Product

Use the triple scalar product to show that vectors $\vec{\mathbf{u}} = \langle 2, 0, 5\rangle$, $\vec{\mathbf{v}} = \langle 2, 2, 4\rangle$, and $\vec{\mathbf{w}} = \langle 1, -1, 3\rangle$ are coplanar—that is, show that these vectors lie in the same plane.

Solution

Start by calculating the triple scalar product to find the volume of the parallelepiped defined by $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$:

$$\begin{aligned}\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) &= \begin{vmatrix} 2 & 0 & 5 \\ 2 & 2 & 4 \\ 1 & -1 & 3 \end{vmatrix} \\ &= [2(2)(3) + (0)(4)(1) + 5(2)(-1)] - [5(2)(1) + (2)(4)(-1) + (0)(2)(3)] \\ &= 2 - 2 = 0.\end{aligned}$$

The volume of the parallelepiped is 0 units^3, so one of the dimensions must be zero. Therefore, the three vectors all lie in the same plane.

Exercise 12.4.12

Are the vectors $\vec{\mathbf{a}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$, $\vec{\mathbf{b}} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$, and $\vec{\mathbf{c}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ coplanar?

Hint

Calculate the triple scalar product.

Answer

No, the triple scalar product is $-4 \neq 0$, so the three vectors form the adjacent edges of a parallelepiped. They are not coplanar.

Example 12.4.13: Finding an Orthogonal Vector

Only a single plane can pass through any set of three noncolinear points. Find a vector orthogonal to the plane containing points $P = (9, -3, -2)$, $Q = (1, 3, 0)$, and $R = (-2, 5, 0)$.

Solution

The plane must contain vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$:

$$\overrightarrow{PQ} = \langle 1 - 9, 3 - (-3), 0 - (-2)\rangle = \langle -8, 6, 2\rangle$$

$$\overrightarrow{QR} = \langle -2 - 1, 5 - 3, 0 - 0\rangle = \langle -3, 2, 0\rangle.$$

The cross product $\overrightarrow{PQ} \times \overrightarrow{QR}$ produces a vector orthogonal to both $\overrightarrow{PQ}$ and $\overrightarrow{QR}$. Therefore, the cross product is orthogonal to the plane that contains these two vectors:

$$\overrightarrow{PQ} \times \overrightarrow{QR} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -8 & 6 & 2 \\ -3 & 2 & 0 \end{vmatrix}$$
$$= 0\hat{\mathbf{i}} - 6\hat{\mathbf{j}} - 16\hat{\mathbf{k}} - (-18\hat{\mathbf{k}} + 4\hat{\mathbf{i}} + 0\hat{\mathbf{j}})$$
$$= -4\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 2\hat{\mathbf{k}}.$$

We have seen how to use the triple scalar product and how to find a vector orthogonal to a plane. Now we apply the cross product to real-world situations.

Sometimes a force causes an object to rotate. For example, turning a screwdriver or a wrench creates this kind of rotational effect, called torque.

Definition: Torque

Torque, $\vec{\tau}$ (the Greek letter tau), measures the tendency of a force to produce rotation about an axis of rotation. Let $\vec{\mathbf{r}}$ be a vector with an initial point located on the axis of rotation and with a terminal point located at the point where the force is applied, and let vector $\vec{\mathbf{F}}$ represent the force. Then torque is equal to the cross product of $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$:

$$\vec{\tau} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}.$$

See Figure 12.4.9

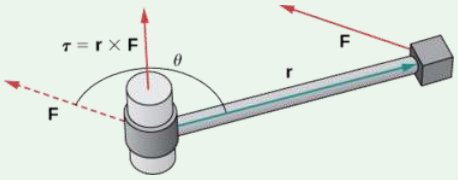


Figure 12.4.9: Torque measures how a force causes an object to rotate.

Think about using a wrench to tighten a bolt. The torque τ applied to the bolt depends on how hard we push the wrench (force) and how far up the handle we apply the force (distance). The torque increases with a greater force on the wrench at a greater distance from the bolt. Common units of torque are the newton-meter or foot-pound. Although torque is dimensionally equivalent to work (it has the same units), the two concepts are distinct. Torque is used specifically in the context of rotation, whereas work typically involves motion along a line.

Example 12.4.14: Evaluating Torque

A bolt is tightened by applying a force of 6 N to a 0.15-m wrench (Figure 12.4.10). The angle between the wrench and the force vector is 40°. Find the magnitude of the torque about the center of the bolt. Round the answer to two decimal places.

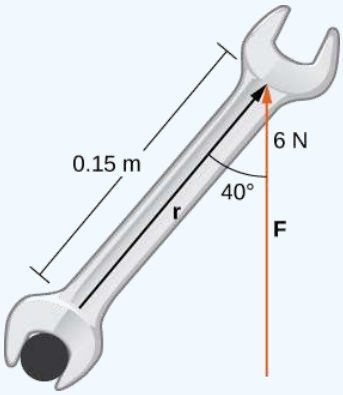


Figure 12.4.10: Torque describes the twisting action of the wrench.

Solution:

Substitute the given information into the equation defining torque:

$$\begin{aligned} \|\vec{\boldsymbol{\tau}}\| &= \|\vec{\mathbf{r}} \times \vec{\mathbf{F}}\| \\ &= \|\vec{\mathbf{r}}\| \, \|\vec{\mathbf{F}}\| \sin\theta \\ &= (0.15\,\text{m})(6\,\text{N}) \sin 40^\circ \\ &\approx 0.58\,\text{N·m}. \end{aligned}$$

Exercise 12.4.14

Calculate the force required to produce 15 N·m torque at an angle of 30° from a 150-cm rod.

Hint

$\|\vec{\boldsymbol{\tau}}\| = 15$ N·m and $\|\vec{\mathbf{r}}\| = 1.5$ m

Answer

20 N

Key Concepts

- The cross product $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$ of two vectors $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ is a vector orthogonal to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$. Its length is given by $\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\| = \|\vec{\mathbf{u}}\| \cdot \|\vec{\mathbf{v}}\| \cdot \sin\theta$, where θ is the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$. Its direction is given by the right-hand rule.
- The algebraic formula for calculating the cross product of two vectors,

  $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$, is

  $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (u_2 v_3 - u_3 v_2)\hat{\mathbf{i}} - (u_1 v_3 - u_3 v_1)\hat{\mathbf{j}} + (u_1 v_2 - u_2 v_1)\hat{\mathbf{k}}$.
- The cross product satisfies the following properties for vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$, and scalar c:

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = -(\vec{\mathbf{v}} \times \vec{\mathbf{u}})$$

$$\vec{\mathbf{u}} \times (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \times \vec{\mathbf{v}} + \vec{\mathbf{u}} \times \vec{\mathbf{w}}$$

$$c(\vec{\mathbf{u}} \times \vec{\mathbf{v}}) = (c\vec{\mathbf{u}}) \times \vec{\mathbf{v}} = \vec{\mathbf{u}} \times (c\vec{\mathbf{v}})$$

$$\vec{\mathbf{u}} \times \vec{\mathbf{0}} = \vec{\mathbf{0}} \times \vec{\mathbf{u}} = \vec{\mathbf{0}}$$

$$\vec{\mathbf{v}} \times \vec{\mathbf{v}} = \vec{\mathbf{0}}$$

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) \cdot \vec{\mathbf{w}}$$

- The cross product of vectors $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ is the determinant $\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
- If vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ form adjacent sides of a parallelogram, then the area of the parallelogram is given by $\|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\|$.
- The triple scalar product of vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ is $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})$.
- The volume of a parallelepiped with adjacent edges given by vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ is $V = |\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})|$.
- If the triple scalar product of vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$ is zero, then the vectors are coplanar. The converse is also true: If the vectors are coplanar, then their triple scalar product is zero.
- The cross product can be used to identify a vector orthogonal to two given vectors or to a plane.
- Torque $\vec{\boldsymbol{\tau}}$ measures the tendency of a force to produce rotation about an axis of rotation. If force $\vec{\mathbf{F}}$ is acting at a distance (displacement) $\vec{\mathbf{r}}$ from the axis, then torque is equal to the cross product of $\vec{\mathbf{r}}$ and $\vec{\mathbf{F}}$: $\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$.

Key Equations

- **The cross product of two vectors in terms of the unit vectors**

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (u_2 v_3 - u_3 v_2)\hat{\mathbf{i}} - (u_1 v_3 - u_3 v_1)\hat{\mathbf{j}} + (u_1 v_2 - u_2 v_1)\hat{\mathbf{k}}$$

Glossary

cross product

$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (u_2v_3 - u_3v_2)\hat{\mathbf{i}} - (u_1v_3 - u_3v_1)\hat{\mathbf{j}} + (u_1v_2 - u_2v_1)\hat{\mathbf{k}}$, where $\vec{\mathbf{u}} = \langle u_1, u_2, u_3 \rangle$ and $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$

determinant

a real number associated with a square matrix

parallelepiped

a three-dimensional prism with six faces that are parallelograms

torque

the effect of a force that causes an object to rotate

triple scalar product

the dot product of a vector with the cross product of two other vectors: $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})$

vector product

the cross product of two vectors

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12.5: Equations of Lines and Planes

Learning Objectives

- Write the vector, parametric, and symmetric equations of a line through a given point in a given direction, and a line through two given points.
- Find the distance from a point to a given line.
- Write the vector and scalar equations of a plane through a given point with a given normal.
- Find the distance from a point to a given plane.
- Find the angle between two planes.

By now, we are familiar with writing equations that describe a line in two dimensions. To write an equation for a line, we must know two points on the line, or we must know the direction of the line and at least one point through which the line passes. In two dimensions, we use the concept of slope to describe the orientation, or direction, of a line. In three dimensions, we describe the direction of a line using a vector parallel to the line. In this section, we examine how to use equations to describe lines and planes in space.

Equations for a Line in Space

Let's first explore what it means for two vectors to be parallel. Recall that parallel vectors must have the same or opposite directions. If two nonzero vectors, $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$, are parallel, we claim there must be a scalar, k, such that $\vec{\mathbf{u}} = k\vec{\mathbf{v}}$. If $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ have the same direction, simply choose

$$k = \frac{\|\vec{\mathbf{u}}\|}{\|\vec{\mathbf{v}}\|}.$$

If $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ have opposite directions, choose

$$k = -\frac{\|\vec{\mathbf{u}}\|}{\|\vec{\mathbf{v}}\|}.$$

Note that the converse holds as well. If $\vec{\mathbf{u}} = k\vec{\mathbf{v}}$ for some scalar k, then either $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ have the same direction $(k > 0)$ or opposite directions $(k < 0)$, so $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are parallel. Therefore, two nonzero vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are parallel if and only if $\vec{\mathbf{u}} = k\vec{\mathbf{v}}$ for some scalar k. By convention, the zero vector $\vec{\mathbf{0}}$ is considered to be parallel to all vectors.

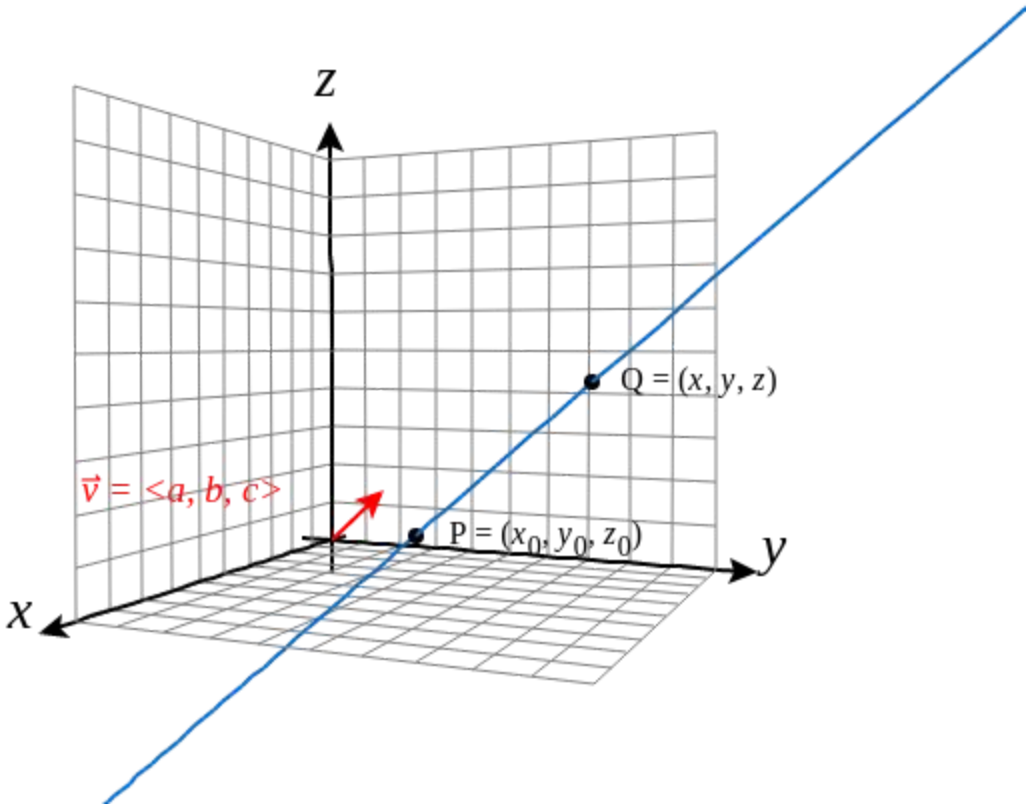


Figure 12.5.1: Vector $\vec{\mathbf{v}}$ is the direction vector for $\overrightarrow{PQ}$.

As in two dimensions, we can describe a line in space using a point on the line and the direction of the line, or a parallel vector, which we call the **direction vector** (Figure 12.5.1). Let L be a line in space passing through point $P(x_0, y_0, z_0)$. Let $\vec{\mathbf{v}} = \langle a, b, c \rangle$ be a vector parallel to L. Then, for any point on line $Q(x, y, z)$, we know that $\overrightarrow{PQ}$ is parallel to $\vec{\mathbf{v}}$. Thus, as we just discussed, there is a scalar, t, such that $\overrightarrow{PQ} = t\vec{\mathbf{v}}$, which gives

$$\begin{aligned} \overrightarrow{PQ} &= t\vec{\mathbf{v}} \\ \langle x - x_0, y - y_0, z - z_0 \rangle &= t\langle a, b, c \rangle \\ \langle x - x_0, y - y_0, z - z_0 \rangle &= \langle ta, tb, tc \rangle. \end{aligned} \tag{12.5.1}$$

Using vector operations, we can rewrite Equation 12.5.1

$$\begin{aligned} \langle x - x_0, y - y_0, z - z_0 \rangle &= \langle ta, tb, tc \rangle \\ \langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle &= t\langle a, b, c \rangle \\ \underbrace{\langle x, y, z \rangle}_{\vec{\mathbf{r}}} &= \underbrace{\langle x_0, y_0, z_0 \rangle}_{\vec{\mathbf{r}}_0} + t\underbrace{\langle a, b, c \rangle}_{\vec{\mathbf{v}}}. \end{aligned}$$

Setting $\vec{\mathbf{r}} = \langle x, y, z \rangle$ and $\vec{\mathbf{r}}_0 = \langle x_0, y_0, z_0 \rangle$, we now have the **vector equation of a line:**

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_0 + t\vec{\mathbf{v}}. \tag{12.5.2}$$

Equating components, Equation 12.5.2 shows that the following equations are simultaneously true: $x - x_0 = ta, y - y_0 = tb,$ and $z - z_0 = tc$. If we solve each of these equations for the component variables $x, y,$ and z, we get a set of equations in which each

variable is defined in terms of the parameter t and that, together, describe the line. This set of three equations forms a set of **parametric equations of a line:**

$$x = x_0 + ta$$
$$y = y_0 + tb$$
$$z = z_0 + tc.$$

If we solve each of the equations for t assuming a, b, and c are nonzero, we get a different description of the same line:

$$\frac{x - x_0}{a} = t$$
$$\frac{y - y_0}{b} = t$$
$$\frac{z - z_0}{c} = t.$$

Because each expression equals t, they all have the same value. We can set them equal to each other to create **symmetric equations of a line:**

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

We summarize the results in the following theorem.

Theorem: Parametric and Symmetric Equations of a Line

A line L parallel to vector $\vec{\mathbf{v}} = \langle a, b, c \rangle$ and passing through point $P(x_0, y_0, z_0)$ can be described by the following parametric equations:

$$x = x_0 + ta, y = y_0 + tb,$$

and

$$z = z_0 + tc.$$

If the constants a, b, and c are all nonzero, then L can be described by the symmetric equation of the line:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

The parametric equations of a line are not unique. Using a different parallel vector or a different point on the line leads to a different, equivalent representation. Each set of parametric equations leads to a related set of symmetric equations, so it follows that a symmetric equation of a line is not unique either.

Example 12.5.1: Equations of a Line in Space

Find parametric and symmetric equations of the line passing through points $(1, 4, -2)$ and $(-3, 5, 0)$.

Solution

First, identify a vector parallel to the line:

$$\vec{\mathbf{v}} = \langle -3 - 1, 5 - 4, 0 - (-2) \rangle = \langle -4, 1, 2 \rangle.$$

Use either of the given points on the line to complete the parametric equations:

$$x = 1 - 4t$$
$$y = 4 + t,$$

and

$$z = -2 + 2t.$$

Solve each equation for t to create the symmetric equation of the line:

$$\frac{x-1}{-4} = y-4 = \frac{z+2}{2}.$$

Exercise 12.5.1

Find parametric and symmetric equations of the line passing through points $(1,-3,2)$ and $(5,-2,8)$.

Hint:

Start by finding a vector parallel to the line.

Answer

Possible set of parametric equations: $x = 1+4t,\ y = -3+t,\ z = 2+6t;$ related set of symmetric equations:

$$\frac{x-1}{4} = y+3 = \frac{z-2}{6}$$

Sometimes we don't want the equation of a whole line, just a line segment. In this case, we limit the values of our parameter t. For example, let $P(x_0,y_0,z_0)$ and $Q(x_1,y_1,z_1)$ be points on a line, and let $\vec{\mathbf{p}} = \langle x_0,y_0,z_0\rangle$ and $\vec{\mathbf{q}} = \langle x_1,y_1,z_1\rangle$ be the associated position vectors. In addition, let $\vec{\mathbf{r}} = \langle x,y,z\rangle$. We want to find a vector equation for the line segment between P and Q. Using P as our known point on the line, and $\overrightarrow{PQ} = \langle x_1-x_0, y_1-y_0, z_1-z_0\rangle$ as the direction vector equation, Equation 12.5.2 gives

$$\vec{\mathbf{r}} = \vec{\mathbf{p}} + t(\overrightarrow{PQ}). \tag{12.5.3}$$

Equation 12.5.3 can be expanded using properties of vectors:

$$\begin{aligned}
\vec{\mathbf{r}} &= \vec{\mathbf{p}} + t(\overrightarrow{PQ}) \\
&= \langle x_0,y_0,z_0\rangle + t\langle x_1-x_0, y_1-y_0, z_1-z_0\rangle \\
&= \langle x_0,y_0,z_0\rangle + t(\langle x_1,y_1,z_1\rangle - \langle x_0,y_0,z_0\rangle) \\
&= \langle x_0,y_0,z_0\rangle + t\langle x_1,y_1,z_1\rangle - t\langle x_0,y_0,z_0\rangle \\
&= (1-t)\langle x_0,y_0,z_0\rangle + t\langle x_1,y_1,z_1\rangle \\
&= (1-t)\vec{\mathbf{p}} + t\vec{\mathbf{q}}.
\end{aligned}$$

Thus, the vector equation of the line passing through P and Q is

$$\vec{\mathbf{r}} = (1-t)\vec{\mathbf{p}} + t\vec{\mathbf{q}}.$$

Remember that we did not want the equation of the whole line, just the line segment between P and Q. Notice that when $t=0$, we have $\vec{\mathbf{r}} = \vec{\mathbf{p}}$, and when $t=1$, we have $\vec{\mathbf{r}} = \vec{\mathbf{q}}$. Therefore, the vector equation of the line segment between P and Q is

$$\vec{\mathbf{r}} = (1-t)\vec{\mathbf{p}} + t\vec{\mathbf{q}}, 0 \le t \le 1.$$

Going back to Equation 12.5.2, we can also find parametric equations for this line segment. We have

$$\begin{aligned}
\vec{\mathbf{r}} &= \vec{\mathbf{p}} + t(\overrightarrow{PQ}) \\
\langle x,y,z\rangle &= \langle x_0,y_0,z_0\rangle + t\langle x_1-x_0, y_1-y_0, z_1-z_0\rangle \\
&= \langle x_0+t(x_1-x_0), y_0+t(y_1-y_0), z_0+t(z_1-z_0)\rangle.
\end{aligned}$$

Then, the parametric equations are

$$\begin{aligned}
x &= x_0 + t(x_1-x_0) \\
y &= y_0 + t(y_1-y_0) \\
z &= z_0 + t(z_1-z_0), \quad 0 \le t \le 1.
\end{aligned} \tag{12.5.4}$$

Example 12.5.2: Parametric Equations of a Line Segment

Find parametric equations of the line segment between the points $P(2,1,4)$ and $Q(3,-1,3)$.

Solution

Start with the parametric equations for a line (Equations 12.5.4) and work with each component separately:

$$\begin{aligned} x &= x_0 + t(x_1 - x_0) \\ &= 2 + t(3-2) \\ &= 2 + t, \\ y &= y_0 + t(y_1 - y_0) \\ &= 1 + t(-1-1) \\ &= 1 - 2t, \end{aligned}$$

and

$$\begin{aligned} z &= z_0 + t(z_1 - z_0) \\ &= 4 + t(3-4) \\ &= 4 - t. \end{aligned}$$

Therefore, the parametric equations for the line segment are

$$\begin{aligned} x &= 2 + t \\ y &= 1 - 2t \\ z &= 4 - t, \quad 0 \le t \le 1. \end{aligned}$$

Exercise 12.5.2

Find parametric equations of the line segment between points $P(-1,3,6)$ and $Q(-8,2,4)$.

Answer

$x = -1 - 7t,\ y = 3 - t,\ z = 6 - 2t, \quad 0 \le t \le 1$

Distance between a Point and a Line

We already know how to calculate the distance between two points in space. We now expand this definition to describe the distance between a point and a line in space. Several real-world contexts exist when it is important to be able to calculate these distances. When building a home, for example, builders must consider "setback" requirements, when structures or fixtures have to be a certain distance from the property line. Air travel offers another example. Airlines are concerned about the distances between populated areas and proposed flight paths.

Let L be a line in the plane and let M be any point not on the line. Then, we define distance d from M to L as the length of line segment $\overline{MP}$, where P is a point on L such that $\overline{MP}$ is perpendicular to L (Figure 12.5.2).

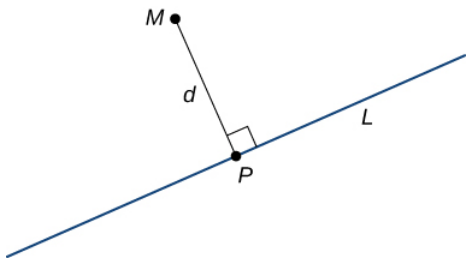


Figure 12.5.2: The distance from point M to line L is the length of $\overline{MP}$.

When we're looking for the distance between a line and a point in space, Figure 12.5.2 still applies. We still define the distance as the length of the perpendicular line segment connecting the point to the line. In space, however, there is no clear way to know which point on the line creates such a perpendicular line segment, so we select an arbitrary point on the line and use properties of vectors to calculate the distance. Therefore, let P be an arbitrary point on line L and let $\vec{\mathbf{v}}$ be a direction vector for L (Figure 12.5.3).

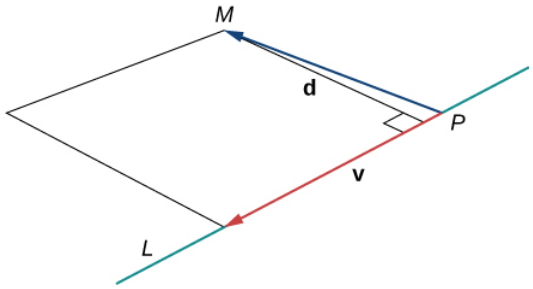


Figure 12.5.3: Vectors $\overrightarrow{PM}$ and $\vec{\mathbf{v}}$ form two sides of a parallelogram with base $\|\vec{\mathbf{v}}\|$ and height d, which is the distance between a line and a point in space.

Vectors $\overrightarrow{PM}$ and $\vec{\mathbf{v}}$ form two sides of a parallelogram with area $\|\overrightarrow{PM} \times \vec{\mathbf{v}}\|$. Using a formula from geometry, the area of this parallelogram can also be calculated as the product of its base and height:

$$\|\overrightarrow{PM} \times \vec{\mathbf{v}}\| = \|\vec{\mathbf{v}}\| d.$$

We can use this formula to find a general formula for the distance between a line in space and any point not on the line.

Distance from a Point to a Line

Let L be a line in space passing through point P with direction vector $\vec{\mathbf{v}}$. If M is any point not on L, then the distance from M to L is

$$d = \frac{\|\overrightarrow{PM} \times \vec{\mathbf{v}}\|}{\|\vec{\mathbf{v}}\|}.$$

Example 12.5.3: Calculating the Distance from a Point to a Line

Find the distance between the point $M = (1, 1, 3)$ and line $\dfrac{x-3}{4} = \dfrac{y+1}{2} = z - 3$.

Solution:

From the symmetric equations of the line, we know that vector $\vec{\mathbf{v}} = \langle 4, 2, 1 \rangle$ is a direction vector for the line. Setting the symmetric equations of the line equal to zero, we see that point $P(3, -1, 3)$ lies on the line. Then,

$$\begin{aligned} \overrightarrow{PM} &= \langle 1-3, 1-(-1), 3-3 \rangle \\ &= \langle -2, 2, 0 \rangle. \end{aligned}$$

To calculate the distance, we need to find $\overrightarrow{PM} \times \vec{\mathbf{v}}$:

$$\begin{aligned} \overrightarrow{PM} \times \vec{\mathbf{v}} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -2 & 2 & 0 \\ 4 & 2 & 1 \end{vmatrix} \\ &= (2-0)\hat{\mathbf{i}} - (-2-0)\hat{\mathbf{j}} + (-4-8)\hat{\mathbf{k}} \\ &= 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 12\hat{\mathbf{k}}. \end{aligned}$$

Therefore, the distance between the point and the line is (Figure 12.5.4)

$$
\begin{aligned}
d &= \frac{\|\overrightarrow{PM} \times \vec{\mathbf{v}}\|}{\|\vec{\mathbf{v}}\|} \\
&= \frac{\sqrt{2^2+2^2+12^2}}{\sqrt{4^2+2^2+1^2}} \\
&= \frac{2\sqrt{38}}{\sqrt{21}} \\
&= \frac{2\sqrt{798}}{21} \text{ units}
\end{aligned}
$$

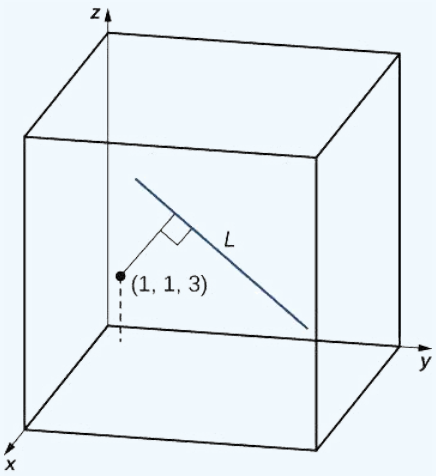


Figure 12.5.4: Point $(1,1,3)$ is approximately 2.7 units from the line with symmetric equations $\frac{x-3}{4} = \frac{y+1}{2} = z-3$.

? Exercise 12.5.3

Find the distance between point $(0,3,6)$ and the line with parametric equations $x = 1-t,\ y = 1+2t,\ z = 5+3t$.

Hint

Find a vector with initial point $(0,3,6)$ and a terminal point on the line, and then find a direction vector for the line.

Answer

$\sqrt{\frac{10}{7}} = \frac{\sqrt{70}}{7}$ units

Relationships between Lines

Given two lines in the two-dimensional plane, the lines are equal, they are parallel but not equal, or they intersect in a single point. In three dimensions, a fourth case is possible. If two lines in space are not parallel, but do not intersect, then the lines are said to be **skew lines** (Figure 12.5.5).

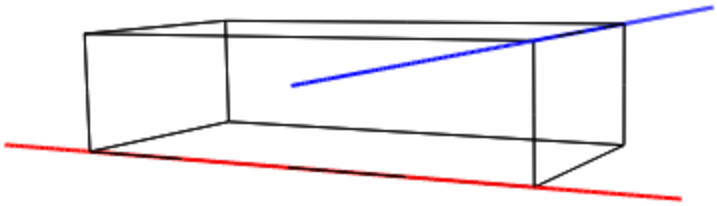


Figure 12.5.5: In three dimensions, it is possible that two lines do not cross, even when they have different directions.

To classify lines as parallel but not equal, equal, intersecting, or skew, we need to know two things: whether the direction vectors are parallel and whether the lines share a point (Figure 12.5.6).

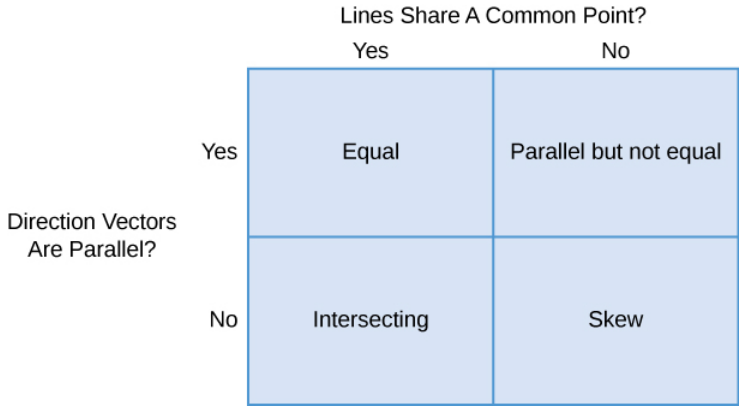


Figure 12.5.6: Determine the relationship between two lines based on whether their direction vectors are parallel and whether they share a point.

Example 12.5.4: Classifying Lines in Space

For each pair of lines, determine whether the lines are equal, parallel but not equal, skew, or intersecting.

a.

- $L_1: \ x = 2s - 1, \ y = s - 1, \ z = s - 4$

- $L_2:\ x=t-3,\ y=3t+8,\ z=5-2t$

b.

- $L_1:\ x=-y=z$
- $L_2:\ \dfrac{x-3}{2}=y=z-2$

c.

- $L_1:\ x=6s-1,\ y=-2s,\ z=3s+1$
- $L_2:\ \dfrac{x-4}{6}=\dfrac{y+3}{-2}=\dfrac{z-1}{3}$

Solution

a. Line L_1 has direction vector $\vec{\mathbf{v}}_1=\langle 2,1,1\rangle$; line L_2 has direction vector $\vec{\mathbf{v}}_2=\langle 1,3,-2\rangle$. Because the direction vectors are not parallel vectors, the lines are either intersecting or skew. To determine whether the lines intersect, we see if there is a point, (x,y,z), that lies on both lines. To find this point, we use the parametric equations to create a system of equalities:

$$2s-1=t-3;$$

$$s-1=3t+8;$$

$$s-4=5-2t.$$

By the first equation, $t=2s+2$. Substituting into the second equation yields

$$s-1=3(2s+2)+8$$

$$s-1=6s+6+8$$

$$5s=-15$$

$$s=-3.$$

Substitution into the third equation, however, yields a contradiction:

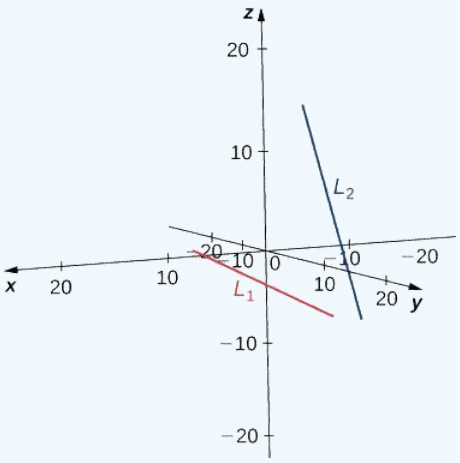
$$s-4=5-2(2s+2)$$

$$s-4=5-4s-4$$

$$5s=5$$

$$s=1.$$

There is no single point that satisfies the parametric equations for L_1 and L_2 simultaneously. These lines do not intersect, so they are skew (see the following figure).

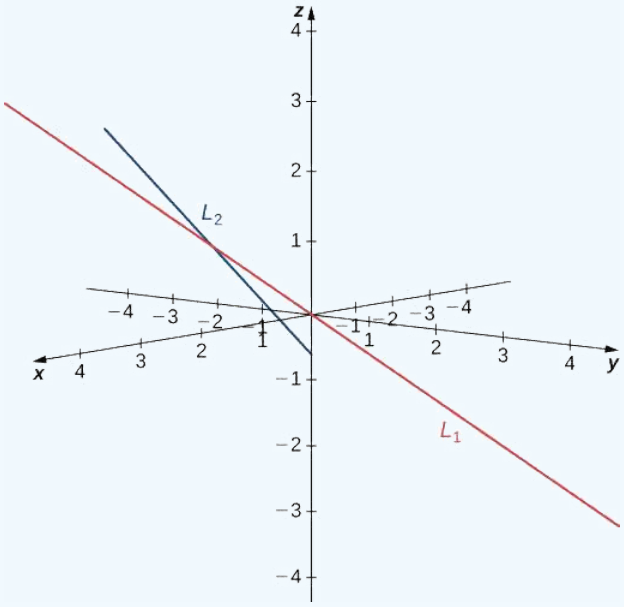


b. Line L_1 has direction vector $\vec{\mathbf{v}}_1=\langle 1,-1,1\rangle$ and passes through the origin, $(0,0,0)$. Line L_2 has a different direction vector, $\vec{\mathbf{v}}_2=\langle 2,1,1\rangle$, so these lines are not parallel or equal. Let r represent the parameter for line L_1 and let s represent the

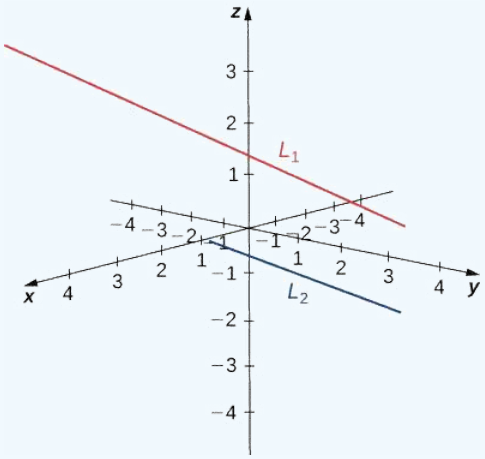
parameter for L_2:

$$\begin{array}{ll} \text{Line } L_1: & \text{Line } L_2: \\ x = r & x = 2s + 3 \\ y = -r & y = s \\ z = r & z = s + 2 \end{array}$$

Solve the system of equations to find $r = 1$ and $s = -1$. If we need to find the point of intersection, we can substitute these parameters into the original equations to get $(1, -1, 1)$ (see the following figure).



c. Lines L_1 and L_2 have equivalent direction vectors: $\vec{\mathbf{v}} = \langle 6, -2, 3 \rangle$. These two lines are parallel (see the following figure).



Exercise 12.5.4

Describe the relationship between the lines with the following parametric equations:

$$x = 1 - 4t,\ y = 3 + t,\ z = 8 - 6t$$

$$x = 2 + 3s,\ y = 2s,\ z = -1 - 3s.$$

Hint

Start by identifying direction vectors for each line. Is one a multiple of the other?

Answer

These lines are skew because their direction vectors are not parallel and there is no point (x, y, z) that lies on both lines.

Equations for a Plane

We know that a line is determined by two points. In other words, for any two distinct points, there is exactly one line that passes through those points, whether in two dimensions or three. Similarly, given any three points that do not all lie on the same line, there is a unique plane that passes through these points. Just as a line is determined by two points, a plane is determined by three.

This may be the simplest way to characterize a plane, but we can use other descriptions as well. For example, given two distinct, intersecting lines, there is exactly one plane containing both lines. A plane is also determined by a line and any point that does not lie on the line. These characterizations arise naturally from the idea that a plane is determined by three points. Perhaps the most surprising characterization of a plane is actually the most useful.

Imagine a pair of orthogonal vectors that share an initial point. Visualize grabbing one of the vectors and twisting it. As you twist, the other vector spins around and sweeps out a plane. Here, we describe that concept mathematically. Let $\vec{\mathbf{n}} = \langle a, b, c \rangle$ be a vector and $P = (x_0, y_0, z_0)$ be a point. Then the set of all points $Q = (x, y, z)$ such that $\overrightarrow{PQ}$ is orthogonal to $\vec{\mathbf{n}}$ forms a plane (Figure 12.5.7). We say that $\vec{\mathbf{n}}$ is a **normal vector**, or perpendicular to the plane. Remember, the dot product of orthogonal vectors is zero. This fact generates the **vector equation of a plane**:

$$\vec{\mathbf{n}} \cdot \overrightarrow{PQ} = 0.$$

Rewriting this equation provides additional ways to describe the plane:

$$\begin{aligned} \vec{\mathbf{n}} \cdot \overrightarrow{PQ} &= 0 \\ \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle &= 0 \\ a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0. \end{aligned}$$

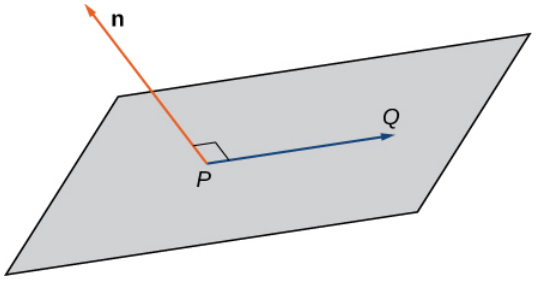


Figure 12.5.7: Given a point P and vector $\vec{\mathbf{n}}$, the set of all points Q with $\overrightarrow{PQ}$ orthogonal to $\vec{\mathbf{n}}$ forms a plane.

Definition: Scalar Equation of a Plane

Given a point P and vector $\vec{\mathbf{n}}$, the set of all points Q satisfying the equation $\vec{\mathbf{n}} \cdot \overrightarrow{PQ} = 0$ forms a plane. The equation

$$\vec{\mathbf{n}} \cdot \overrightarrow{PQ} = 0$$

is known as the *vector equation of a plane*.

The *scalar equation of a plane* (sometimes also called the *standard equation of a plane*) containing point $P = (x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c \rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

This equation can be expressed as $ax + by + cz + d = 0$, where $d = -ax_0 - by_0 - cz_0$. This form of the equation is sometimes called the *general form of the equation of a plane.*

As described earlier in this section, any three points that do not all lie on the same line determine a plane. Given three such points, we can find an equation for the plane containing these points.

Example 12.5.5: Writing an Equation of a Plane Given Three Points in the Plane

Write an equation for the plane containing points $P=(1,1,-2), Q=(0,2,1)$, and $R=(-1,-1,0)$ in both standard and general forms.

Solution

To write an equation for a plane, we must find a normal vector for the plane. We start by identifying two vectors in the plane:

$$\begin{aligned}\overrightarrow{PQ} &= \langle 0-1, 2-1, 1-(-2)\rangle \\ &= \langle -1, 1, 3\rangle \\ \overrightarrow{QR} &= \langle -1-0, -1-2, 0-1\rangle \\ &= \langle -1, -3, -1\rangle.\end{aligned}$$

The cross product $\overrightarrow{PQ}\times\overrightarrow{QR}$ is orthogonal to both $\overrightarrow{PQ}$ and $\overrightarrow{QR}$, so it is normal to the plane that contains these two vectors:

$$\begin{aligned}\vec{\mathbf{n}} &= \overrightarrow{PQ}\times\overrightarrow{QR} \\ &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 1 & 3 \\ -1 & -3 & -1 \end{vmatrix} \\ &= (-1+9)\hat{\mathbf{i}} - (1+3)\hat{\mathbf{j}} + (3+1)\hat{\mathbf{k}} \\ &= 8\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 4\hat{\mathbf{k}}.\end{aligned}$$

Thus, $\vec{\mathbf{n}} = \langle 8, -4, 4\rangle$, and we can choose any of the three given points to write an equation of the plane:

$$\begin{aligned}8(x-1) - 4(y-1) + 4(z+2) &= 0 \\ 8x - 4y + 4z + 4 &= 0.\end{aligned}$$

The scalar equations of a plane vary depending on the normal vector and point chosen.

Example 12.5.6: Writing an Equation for a Plane Given a Point and a Line

Find an equation of the plane that passes through point $(1,4,3)$ and contains the line given by $x=\dfrac{y-1}{2}=z+1$.

Solution

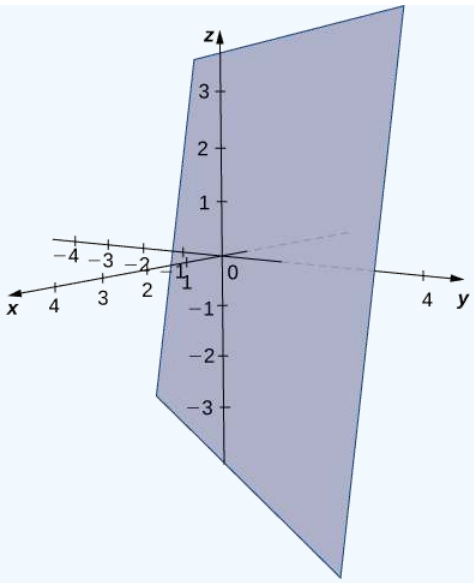
Symmetric equations describe the line that passes through point $(0,1,-1)$ parallel to vector $\vec{\mathbf{v}}_1=\langle 1,2,1\rangle$ (see the following figure). Use this point and the given point, $(1,4,3)$, to identify a second vector parallel to the plane:

$$\vec{\mathbf{v}}_2 = \langle 1-0, 4-1, 3-(-1)\rangle = \langle 1,3,4\rangle.$$

Use the cross product of these vectors to identify a normal vector for the plane:

$$\begin{aligned}\vec{\mathbf{n}} &= \vec{\mathbf{v}}_1\times\vec{\mathbf{v}}_2 \\ &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & 1 \\ 1 & 3 & 4 \end{vmatrix} \\ &= (8-3)\hat{\mathbf{i}} - (4-1)\hat{\mathbf{j}} + (3-2)\hat{\mathbf{k}} \\ &= 5\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}.\end{aligned}$$

The scalar equations for the plane are $5x-3(y-1)+(z+1)=0$ and $5x-3y+z+4=0$.



Exercise 12.5.6

Find an equation of the plane containing the lines L_1 and L_2:

$$L_1 : \ x = -y = z$$

$$L_2 : \ \frac{x-3}{2} = y = z-2.$$

Hint

Hint: The cross product of the lines' direction vectors gives a normal vector for the plane.

Answer

$$-2(x-1)+(y+1)+3(z-1)=0$$

or

$$-2x+y+3z=0$$

Now that we can write an equation for a plane, we can use the equation to find the distance d between a point P and the plane. It is defined as the shortest possible distance from P to a point on the plane.

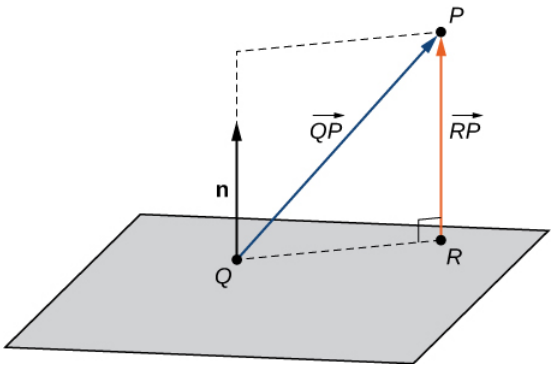


Figure 12.5.8: We want to find the shortest distance from point P to the plane. Let point R be the point in the plane such that, for any other point in the plane Q, $\|\overrightarrow{RP}\| < \|\overrightarrow{QP}\|$.

Just as we find the two-dimensional distance between a point and a line by calculating the length of a line segment perpendicular to the line, we find the three-dimensional distance between a point and a plane by calculating the length of a line segment

perpendicular to the plane. Let R be the point in the plane such that $\overrightarrow{RP}$ is orthogonal to the plane, and let Q be an arbitrary point in the plane. Then the projection of vector $\overrightarrow{QP}$ onto the normal vector describes vector $\overrightarrow{RP}$, as shown in Figure 12.5.8.

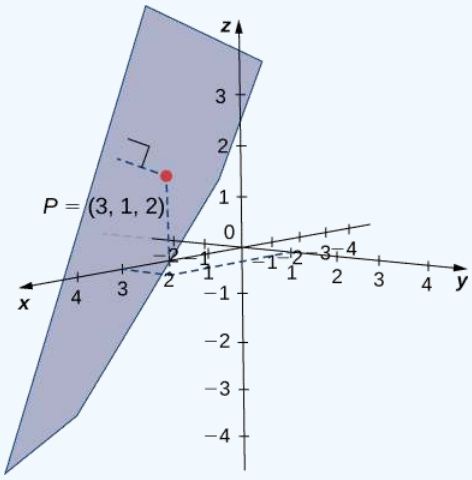
The Distance between a Plane and a Point

Suppose a plane with normal vector $\vec{\mathbf{n}}$ passes through point Q. The distance d from the plane to a point P not in the plane is given by

$$d = \|\text{proj}_{\vec{\mathbf{n}}}\, \overrightarrow{QP}\| = |\,\text{comp}_{\vec{\mathbf{n}}}\, \overrightarrow{QP}\,| = \frac{\left|\overrightarrow{QP} \cdot \vec{\mathbf{n}}\right|}{\|\vec{\mathbf{n}}\|}. \tag{12.5.5}$$

Example 12.5.7: Distance between a Point and a Plane

Find the distance between point $P = (3, 1, 2)$ and the plane given by $x - 2y + z = 5$ (see the following figure).



Solution

The coefficients of the plane's equation provide a normal vector for the plane: $\vec{\mathbf{n}} = \langle 1, -2, 1\rangle$. To find vector $\overrightarrow{QP}$, we need a point in the plane. Any point will work, so set $y = z = 0$ to see that point $Q = (5, 0, 0)$ lies in the plane. Find the component form of the vector from Q to P:

$$\overrightarrow{QP} = \langle 3 - 5, 1 - 0, 2 - 0\rangle = \langle -2, 1, 2\rangle.$$

Apply the distance formula from Equation 12.5.5:

$$\begin{aligned} d &= \frac{\left|\overrightarrow{QP} \cdot \vec{\mathbf{n}}\right|}{\|\vec{\mathbf{n}}\|} \\ &= \frac{|\langle -2, 1, 2\rangle \cdot \langle 1, -2, 1\rangle|}{\sqrt{1^2 + (-2)^2 + 1^2}} \\ &= \frac{|-2 - 2 + 2|}{\sqrt{6}} \\ &= \frac{2}{\sqrt{6}} = \frac{\sqrt{6}}{3} \text{ units.} \end{aligned}$$

Exercise 12.5.7

Find the distance between point $P = (5, -1, 0)$ and the plane given by $4x + 2y - z = 3$.

Hint

Point $(0, 0, -3)$ lies on the plane.

Answer

$$\frac{15}{\sqrt{21}} = \frac{5\sqrt{21}}{7} \text{ units}$$

Parallel and Intersecting Planes

We have discussed the various possible relationships between two lines in two dimensions and three dimensions. When we describe the relationship between two planes in space, we have only two possibilities: the two distinct planes are parallel or they intersect. When two planes are parallel, their normal vectors are parallel. When two planes intersect, the intersection is a line (Figure 12.5.9).

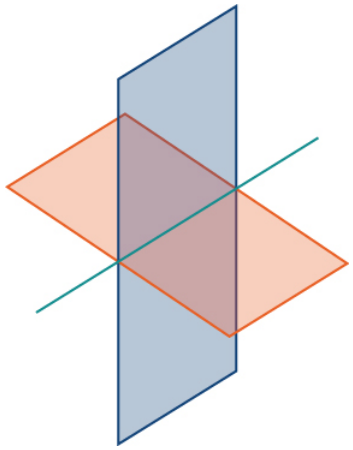
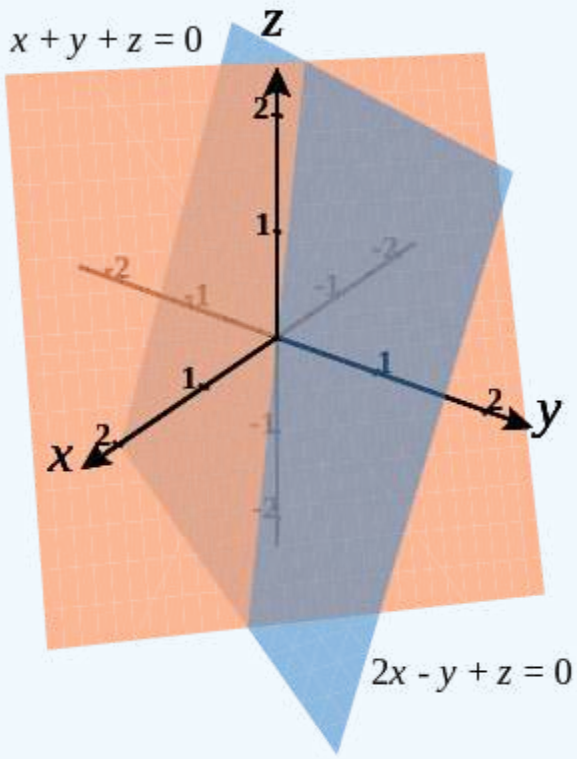


Figure 12.5.9: The intersection of two nonparallel planes is always a line.

We can use the equations of the two planes to find parametric equations for the line of intersection.

Example 12.5.8: Finding the Line of Intersection for Two Planes

Find parametric and symmetric equations for the line formed by the intersection of the planes given by $x + y + z = 0$ and $2x - y + z = 0$ (see the following figure).



Solution

Note that the two planes have nonparallel normals, so the planes intersect. Further, the origin satisfies each equation, so we know the line of intersection passes through the origin. Add the plane equations so we can eliminate one of the variables, in this case, y:

$$\begin{array}{r} x+y+z=0 \\ 2x-y+z=0 \\ \hline 3x+2z=0\,. \end{array}$$

This gives us $x=-\frac{2}{3}z$. We substitute this value into the first equation to express y in terms of z:

$$\begin{array}{r} x+y+z=0 \\ -\frac{2}{3}z+y+z=0 \\ y+\frac{1}{3}z=0 \\ y=-\frac{1}{3}z \end{array}.$$

We now have the first two variables, x and y, in terms of the third variable, z. Now we define z in terms of t. To eliminate the need for fractions, we choose to define the parameter t as $t=-\frac{1}{3}z$. Then, $z=-3t$. Substituting the parametric

representation of z back into the other two equations, we see that the parametric equations for the line of intersection are $x = 2t,\ y = t,\ z = -3t$. The symmetric equations for the line are $\dfrac{x}{2} = y = \dfrac{z}{-3}$.

? Exercise 12.5.8

Find parametric equations for the line formed by the intersection of planes $x + y - z = 3$ and $3x - y + 3z = 5$.

Hint

Add the two equations, then express z in terms of x. Then, express y in terms of x.

Answer

$x = t,\ y = 7 - 3t,\ z = 4 - 2t$

In addition to finding the equation of the line of intersection between two planes, we may need to find the angle formed by the intersection of two planes. For example, builders constructing a house need to know the angle where different sections of the roof meet to know whether the roof will look good and drain properly. We can use normal vectors to calculate the angle between the two planes. We can do this because the angle between the normal vectors is the same as the angle between the planes. Figure 12.5.10 shows why this is true.

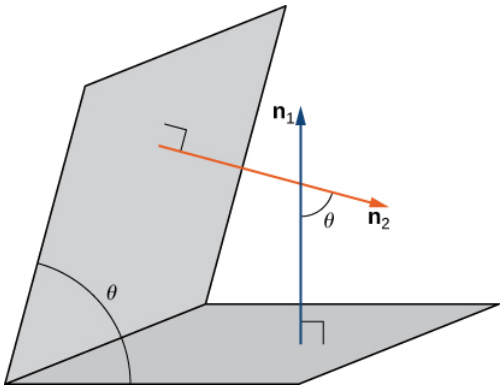


Figure 12.5.10: The angle between two planes has the same measure as the angle between the normal vectors for the planes.

We can find the measure of the angle θ between two intersecting planes by first finding the cosine of the angle, using the following equation:

$$\cos\theta = \frac{|\vec{\mathbf{n}}_1 \cdot \vec{\mathbf{n}}_2|}{\|\vec{\mathbf{n}}_1\|\|\vec{\mathbf{n}}_2\|}.$$

We can then use the angle to determine whether two planes are parallel or orthogonal or if they intersect at some other angle.

✓ Example 12.5.9: Finding the Angle between Two Planes

Determine whether each pair of planes is parallel, orthogonal, or neither. If the planes are intersecting, but not orthogonal, find the measure of the angle between them. Give the answer in radians and round to two decimal places.

a. $x + 2y - z = 8$ and $2x + 4y - 2z = 10$
b. $2x - 3y + 2z = 3$ and $6x + 2y - 3z = 1$
c. $x + y + z = 4$ and $x - 3y + 5z = 1$

Solution:

a. The normal vectors for these planes are $\vec{\mathbf{n}}_1 = \langle 1, 2, -1\rangle$ and $\vec{\mathbf{n}}_2 = \langle 2, 4, -2\rangle$. These two vectors are scalar multiples of each other. The normal vectors are parallel, so the planes are parallel.
b. The normal vectors for these planes are $\vec{\mathbf{n}}_1 = \langle 2, -3, 2\rangle$ and $\vec{\mathbf{n}}_2 = \langle 6, 2, -3\rangle$. Taking the dot product of these vectors, we have

$$\begin{aligned}\vec{\mathbf{n}}_1 \cdot \vec{\mathbf{n}}_2 &= \langle 2,-3,2\rangle \cdot \langle 6,2,-3\rangle \\ &= 2(6)-3(2)+2(-3)=0.\end{aligned}$$

The normal vectors are orthogonal, so the corresponding planes are orthogonal as well.

c. The normal vectors for these planes are $\vec{\mathbf{n}}_1 = \langle 1,1,1\rangle$ and $\vec{\mathbf{n}}_2 = \langle 1,-3,5\rangle$:

$$\begin{aligned}\cos\theta &= \frac{|\vec{\mathbf{n}}_1 \cdot \vec{\mathbf{n}}_2|}{\|\vec{\mathbf{n}}_1\|\|\vec{\mathbf{n}}_2\|} \\ &= \frac{|\langle 1,1,1\rangle \cdot \langle 1,-3,5\rangle|}{\sqrt{1^2+1^2+1^2}\sqrt{1^2+(-3)^2+5^2}} \\ &= \frac{3}{\sqrt{105}}\end{aligned}$$

Then $\theta = \arccos\frac{3}{\sqrt{105}} \approx 1.27$ rad.

Thus the angle between the two planes is about 1.27 rad, or approximately 73°.

Exercise 12.5.9

Find the measure of the angle between planes $x+y-z=3$ and $3x-y+3z=5$. Give the answer in radians and round to two decimal places.

Hint

Use the coefficients of the variables in each equation to find a normal vector for each plane.

Answer

1.44 rad

When we find that two planes are parallel, we may need to find the distance between them. To find this distance, we simply select a point in one of the planes. The distance from this point to the other plane is the distance between the planes.

Previously, we introduced the formula for calculating this distance in Equation 12.5.5:

$$d = \frac{\overrightarrow{QP}\cdot\vec{\mathbf{n}}}{\|\vec{\mathbf{n}}\|},$$

where Q is a point on the plane, P is a point not on the plane, and $\vec{n}$ is the normal vector that passes through point Q. Consider the distance from point (x_0,y_0,z_0) to plane $ax+by+cz+k=0$. Let (x_1,y_1,z_1) be any point in the plane. Substituting into the formula yields

$$\begin{aligned}d &= \frac{|a(x_0-x_1)+b(y_0-y_1)+c(z_0-z_1)|}{\sqrt{a^2+b^2+c^2}} \\ &= \frac{|ax_0+by_0+cz_0+k|}{\sqrt{a^2+b^2+c^2}}.\end{aligned}$$

We state this result formally in the following theorem.

Distance from a Point to a Plane

Let $P(x_0,y_0,z_0)$ be a point. The distance from P to plane $ax+by+cz+k=0$ is given by

$$d = \frac{|ax_0+by_0+cz_0+k|}{\sqrt{a^2+b^2+c^2}}.$$

Example 12.5.10: Finding the Distance between Parallel Planes

Find the distance between the two parallel planes given by $2x+y-z=2$ and $2x+y-z=8$.

Solution

Point $(1,0,0)$ lies in the first plane. The desired distance, then, is

$$\begin{aligned} d &= \frac{|ax_0+by_0+cz_0+k|}{\sqrt{a^2+b^2+c^2}} \\ &= \frac{|2(1)+1(0)+(-1)(0)+(-8)|}{\sqrt{2^2+1^2+(-1)^2}} \\ &= \frac{6}{\sqrt{6}} = \sqrt{6}\,\text{units} \end{aligned}$$

Exercise 12.5.10:

Find the distance between parallel planes $5x-2y+z=6$ and $5x-2y+z=-3$.

Hint

Set $x=y=0$ to find a point on the first plane.

Answer

$$\frac{9}{\sqrt{30}} = \frac{3\sqrt{30}}{10}\,\text{units}$$

Distance between Two Skew Lines

Finding the distance from a point to a line or from a line to a plane seems like a pretty abstract procedure. But, if the lines represent pipes in a chemical plant or tubes in an oil refinery or roads at an intersection of highways, confirming that the distance between them meets specifications can be both important and awkward to measure. One way is to model the two pipes as lines, using the techniques in this chapter, and then calculate the distance between them. The calculation involves forming vectors along the directions of the lines and using both the cross product and the dot product.

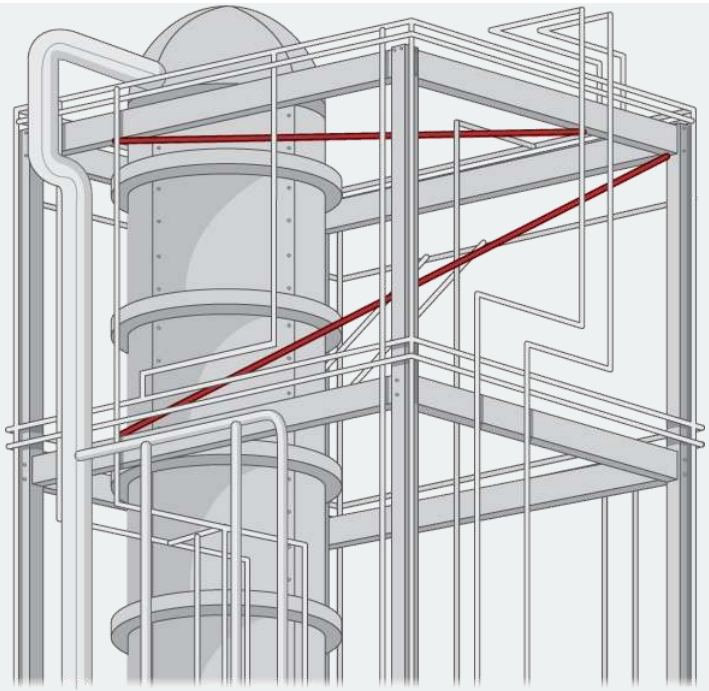


Figure 12.5.11: Industrial pipe installations often feature pipes running in different directions. How can we find the distance between two skew pipes?

The symmetric forms of two lines, L_1 and L_2, are

$$L_1 : \frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$$

$$L_2 : \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}.$$

You are to develop a formula for the distance d between these two lines, in terms of the values $a_1, b_1, c_1; a_2, b_2, c_2; x_1, y_1, z_1;$ and x_2, y_2, z_2. The distance between two lines is usually taken to mean the minimum distance, so this is the length of a line segment or the length of a vector that is perpendicular to both lines and intersects both lines.

1. First, write down two vectors, $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$, that lie along L_1 and L_2, respectively.

2. Find the cross product of these two vectors and call it $\vec{\mathbf{N}}$. This vector is perpendicular to $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$, and hence is perpendicular to both lines.

3. From vector $\vec{\mathbf{N}}$, form a unit vector $\vec{\mathbf{n}}$ in the same direction.

4. Use symmetric equations to find a convenient vector $\vec{\mathbf{v}}_{12}$ that lies between any two points, one on each line. Again, this can be done directly from the symmetric equations.

5. The dot product of two vectors is the magnitude of the projection of one vector onto the other—that is, $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = \|\vec{\mathbf{A}}\| \|\vec{\mathbf{B}}\| \cos\theta$, where θ is the angle between the vectors. Using the dot product, find the projection of vector $\vec{\mathbf{v}}_{12}$ found in step 4 onto unit vector $\vec{\mathbf{n}}$ found in step 3. This projection is perpendicular to both lines, and hence its length must be the perpendicular distance d between them. Note that the value of d may be negative, depending on your choice of vector $\vec{\mathbf{v}}_{12}$ or the order of the cross product, so use absolute value signs around the numerator.

6. Check that your formula gives the correct distance of $|-25|/\sqrt{198} \approx 1.78$ between the following two lines:

$$L_1 : \frac{x-5}{2} = \frac{y-3}{4} = \frac{z-1}{3}$$

$$L_2 : \frac{x-6}{3} = \frac{y-1}{5} = \frac{z}{7}.$$

7. Is your general expression valid when the lines are parallel? If not, why not? (Hint: What do you know about the value of the cross product of two parallel vectors? Where would that result show up in your expression for d?)

8. Demonstrate that your expression for the distance is zero when the lines intersect. Recall that two lines intersect if they are not parallel and they are in the same plane. Hence, consider the direction of $\vec{\mathbf{n}}$ and $\vec{\mathbf{v}}_{12}$. What is the result of their dot product?

9. Consider the following application. Engineers at a refinery have determined they need to install support struts between many of the gas pipes to reduce damaging vibrations. To minimize cost, they plan to install these struts at the closest points between adjacent skewed pipes. Because they have detailed schematics of the structure, they are able to determine the correct lengths of the struts needed, and hence manufacture and distribute them to the installation crews without spending valuable time making measurements.

The rectangular frame structure has the dimensions $4.0 \times 15.0 \times 10.0\,\text{m}$ (height, width, and depth). One sector has a pipe entering the lower corner of the standard frame unit and exiting at the diametrically opposed corner (the one farthest away at the top); call this L_1. A second pipe enters and exits at the two different opposite lower corners; call this L_2 (Figure 12.5.12).

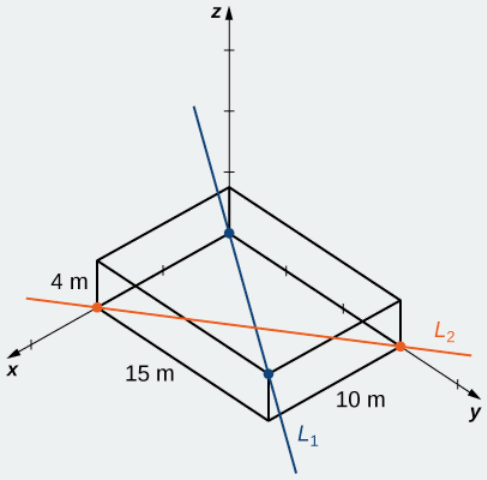


Figure 12.5.12: Two pipes cross through a standard frame unit.

Write down the vectors along the lines representing those pipes, find the cross product between them from which to create the unit vector $\vec{\mathbf{n}}$, define a vector that spans two points on each line, and finally determine the minimum distance between the lines. (Take the origin to be at the lower corner of the first pipe.) Similarly, you may also develop the symmetric equations for each line and substitute directly into your formula.

Key Concepts

- In three dimensions, the direction of a line is described by a direction vector. The vector equation of a line with direction vector $\vec{\mathbf{v}} = \langle a, b, c \rangle$ passing through point $P = (x_0, y_0, z_0)$ is $\vec{\mathbf{r}} = \vec{\mathbf{r}}_0 + t\vec{\mathbf{v}}$, where $\vec{\mathbf{r}}_0 = \langle x_0, y_0, z_0 \rangle$ is the position vector of point P. This equation can be rewritten to form the parametric equations of the line: $x = x_0 + ta, y = y_0 + tb$, and $z = z_0 + tc$. The line can also be described with the symmetric equations $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$.
- Let L be a line in space passing through point P with direction vector $\vec{\mathbf{v}}$. If Q is any point not on L, then the distance from Q to L is $d = \dfrac{\|\overrightarrow{PQ} \times \vec{\mathbf{v}}\|}{\|\vec{\mathbf{v}}\|}$.
- In three dimensions, two lines may be parallel but not equal, equal, intersecting, or skew.
- Given a point P and vector $\vec{\mathbf{n}}$, the set of all points Q satisfying equation $\vec{\mathbf{n}} \cdot \overrightarrow{PQ} = 0$ forms a plane. Equation $\vec{\mathbf{n}} \cdot \overrightarrow{PQ} = 0$ is known as the *vector equation of a plane.*
- The scalar equation of a plane containing point $P = (x_0, y_0, z_0)$ with normal vector $\vec{\mathbf{n}} = \langle a, b, c \rangle$ is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$. This equation can be expressed as $ax + by + cz + d = 0$, where $d = -ax_0 - by_0 - cz_0$. This form of the equation is sometimes called the general *form of the equation of a plane.*

- Suppose a plane with normal vector n passes through point Q. The distance D from the plane to point P not in the plane is given by

$$D = \|\text{proj}_{\vec{\mathbf{n}}}\overrightarrow{QP}\| = |\text{ comp}_{\vec{\mathbf{n}}}\overrightarrow{QP}| = \frac{\left|\overrightarrow{QP}\cdot\vec{\mathbf{n}}\right|}{\|\vec{\mathbf{n}}\|.}$$

- The normal vectors of parallel planes are parallel. When two planes intersect, they form a line.
- The measure of the angle θ between two intersecting planes can be found using the equation: $\cos\theta = \frac{|\vec{\mathbf{n}}_1 \cdot \vec{\mathbf{n}}_2|}{\|\vec{\mathbf{n}}_1\|\|\vec{\mathbf{n}}_2\|}$, where $\vec{\mathbf{n}}_1$ and $\vec{\mathbf{n}}_2$ are normal vectors to the planes.
- The distance D from point (x_0, y_0, z_0) to plane $ax + by + cz + d = 0$ is given by

$$D = \frac{|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

.

Key Equations

- **Vector Equation of a Line**

  $\vec{\mathbf{r}} = \vec{\mathbf{r}}_0 + t\vec{\mathbf{v}}$

- **Parametric Equations of a Line**

  $x = x_0 + ta,\ y = y_0 + tb,$ and $z = z_0 + tc$

- **Symmetric Equations of a Line**

  $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$

- **Vector Equation of a Plane**

  $\vec{\mathbf{n}} \cdot \overrightarrow{PQ} = 0$

- **Scalar Equation of a Plane**

  $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

- **Distance between a Plane and a Point**

  $d = \|\text{proj}_{\vec{\mathbf{n}}}\overrightarrow{QP}\| = |\text{ comp}_{\vec{\mathbf{n}}}\overrightarrow{QP}| = \frac{\left|\overrightarrow{QP}\cdot\vec{\mathbf{n}}\right|}{\|\vec{\mathbf{n}}\|}$

Glossary

direction vector

a vector parallel to a line that is used to describe the direction, or orientation, of the line in space

general form of the equation of a plane

an equation in the form $ax + by + cz + d = 0,$ where $\vec{\mathbf{n}} = \langle a, b, c\rangle$ is a normal vector of the plane, $P = (x_0, y_0, z_0)$ is a point on the plane, and $d = -ax_0 - by_0 - cz_0$

normal vector

a vector perpendicular to a plane

parametric equations of a line

the set of equations $x = x_0 + ta, y = y_0 + tb,$ and $z = z_0 + tc$ describing the line with direction vector $v = \langle a, b, c\rangle$ passing through point (x_0, y_0, z_0)

scalar equation of a plane

the equation $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ used to describe a plane containing point $P=(x_0,y_0,z_0)$ with normal vector $n=\langle a,b,c\rangle$ or its alternate form $ax+by+cz+d=0$, where $d=-ax_0-by_0-cz_0$

skew lines

two lines that are not parallel but do not intersect

symmetric equations of a line

the equations $\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$ describing the line with direction vector $v=\langle a,b,c\rangle$ passing through point (x_0,y_0,z_0)

vector equation of a line

the equation $\vec{\mathbf{r}}=\vec{\mathbf{r}}_0+t\vec{\mathbf{v}}$ used to describe a line with direction vector $\vec{\mathbf{v}}=\langle a,b,c\rangle$ passing through point $P=(x_0,y_0,z_0)$, where $\vec{\mathbf{r}}_0=\langle x_0,y_0,z_0\rangle$, is the position vector of point P

vector equation of a plane

the equation $\vec{\mathbf{n}}\cdot\overrightarrow{PQ}=0,$ where P is a given point in the plane, Q is any point in the plane, and $\vec{\mathbf{n}}$ is a normal vector of the plane

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12.6: Cylinders and Quadric Surfaces

Learning Objectives

- Convert from cylindrical to rectangular coordinates.
- Convert from rectangular to cylindrical coordinates.
- Convert from spherical to rectangular coordinates.
- Convert from rectangular to spherical coordinates.

The Cartesian coordinate system provides a straightforward way to describe the location of points in space. Some surfaces, however, can be difficult to model with equations based on the Cartesian system. This is a familiar problem; recall that in two dimensions, polar coordinates often provide a useful alternative system for describing the location of a point in the plane, particularly in cases involving circles. In this section, we look at two different ways of describing the location of points in space, both of them based on extensions of polar coordinates. As the name suggests, cylindrical coordinates are useful for dealing with problems involving cylinders, such as calculating the volume of a round water tank or the amount of oil flowing through a pipe. Similarly, spherical coordinates are useful for dealing with problems involving spheres, such as finding the volume of domed structures.

Cylindrical Coordinates

When we expanded the traditional Cartesian coordinate system from two dimensions to three, we simply added a new axis to model the third dimension. Starting with polar coordinates, we can follow this same process to create a new three-dimensional coordinate system, called the cylindrical coordinate system. In this way, cylindrical coordinates provide a natural extension of polar coordinates to three dimensions.

Definition: The Cylindrical Coordinate System

In the cylindrical coordinate system**,** a point in space (Figure $12.6.1$) is represented by the ordered triple (r, θ, z), where

- (r, θ) are the polar coordinates of the point's projection in the xy-plane
- z is the usual z-**coordinate** in the Cartesian coordinate system

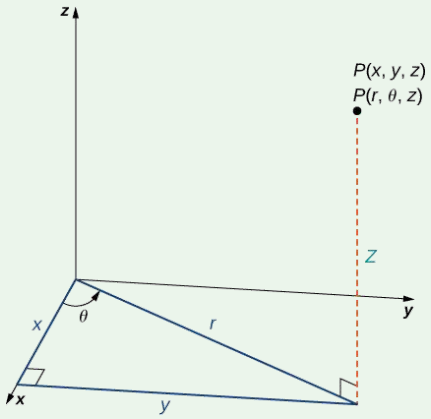


Figure 12.6.1: The right triangle lies in the xy-plane. The length of the hypotenuse is r and θ is the measure of the angle formed by the positive x-axis and the hypotenuse. The z-coordinate describes the location of the point above or below the xy-plane.

In the xy-plane, the right triangle shown in Figure $12.6.1$ provides the key to transformation between cylindrical and Cartesian, or rectangular, coordinates.

Conversion between Cylindrical and Cartesian Coordinates

The rectangular coordinates (x, y, z) and the cylindrical coordinates (r, θ, z) of a point are related as follows:

These equations are used to convert from cylindrical coordinates to rectangular coordinates.

- $x = r\cos\theta$
- $y = r\sin\theta$
- $z = z$

These equations are used to convert from rectangular coordinates to cylindrical coordinates

1. $r^2 = x^2 + y^2$
2. $\tan\theta = \dfrac{y}{x}$
3. $z = z$

As when we discussed conversion from rectangular coordinates to polar coordinates in two dimensions, it should be noted that the equation $\tan\theta = \dfrac{y}{x}$ has an infinite number of solutions. However, if we restrict θ to values between 0 and 2π, then we can find a unique solution based on the quadrant of the xy-plane in which original point (x, y, z) is located. Note that if $x = 0$, then the value of θ is either $\dfrac{\pi}{2}, \dfrac{3\pi}{2}$, or 0, depending on the value of y.

Notice that these equations are derived from properties of right triangles. To make this easy to see, consider point P in the xy-plane with rectangular coordinates $(x, y, 0)$ and with cylindrical coordinates $(r, \theta, 0)$, as shown in Figure 12.6.2

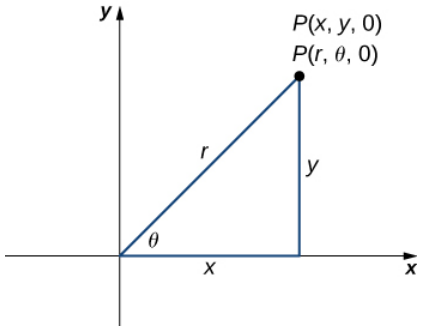


Figure 12.6.2: The Pythagorean theorem provides equation $r^2 = x^2 + y^2$. Right-triangle relationships tell us that $x = r\cos\theta, y = r\sin\theta,$ and $\tan\theta = y/x$.

Let's consider the differences between rectangular and cylindrical coordinates by looking at the surfaces generated when each of the coordinates is held constant. If c is a constant, then in rectangular coordinates, surfaces of the form $x = c, y = c,$ or $z = c$ are all planes. Planes of these forms are parallel to the yz-plane, the xz-plane, and the xy-plane, respectively. When we convert to cylindrical coordinates, the z-coordinate does not change. Therefore, in cylindrical coordinates, surfaces of the form $z = c$ are planes parallel to the xy-plane. Now, let's think about surfaces of the form $r = c$. The points on these surfaces are at a fixed distance from the z-axis. In other words, these surfaces are vertical circular cylinders. Last, what about $\theta = c$? The points on a surface of the form $\theta = c$ are at a fixed angle from the x-axis, which gives us a half-plane that starts at the z-axis (Figures 12.6.3 and 12.6.4).

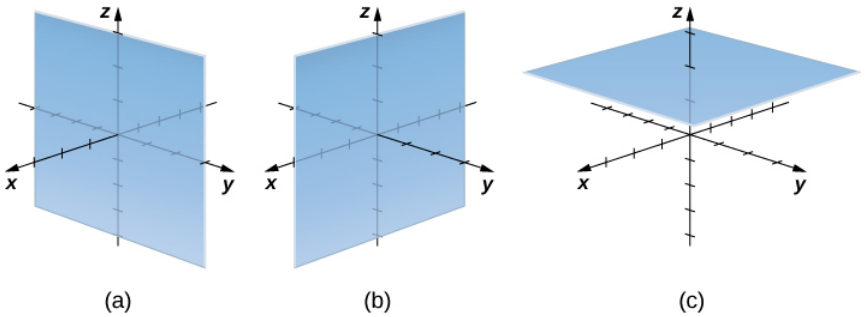


Figure 12.6.3: In rectangular coordinates, (a) surfaces of the form $x = c$ are planes parallel to the yz-plane, (b) surfaces of the form $y = c$ are planes parallel to the xz-plane, and (c) surfaces of the form $z = c$ are planes parallel to the xy-plane.

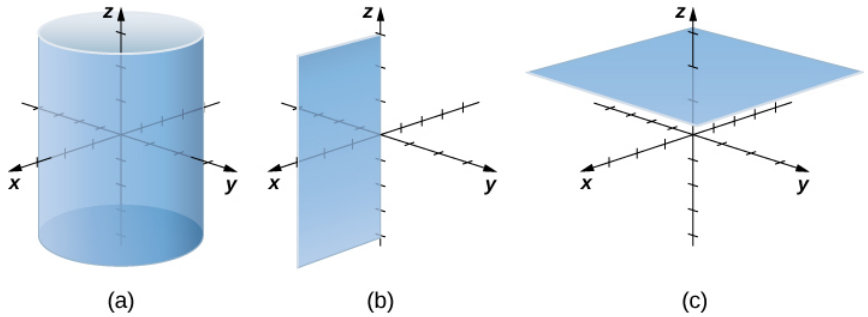


Figure 12.6.4: In cylindrical coordinates, (a) surfaces of the form $r = c$ are vertical cylinders of radius r, (b) surfaces of the form $\theta = c$ are half-planes at angle θ from the x-axis, and (c) surfaces of the form $z = c$ are planes parallel to the xy-plane.

Example 12.6.1: Converting from Cylindrical to Rectangular Coordinates

Plot the point with cylindrical coordinates $(4, \frac{2\pi}{3}, -2)$ and express its location in rectangular coordinates.

Solution

Conversion from cylindrical to rectangular coordinates requires a simple application of the equations listed in Conversion between Cylindrical and Cartesian Coordinates:

$$\begin{aligned} x &= r\cos\theta = 4\cos\frac{2\pi}{3} = -2 \\ y &= r\sin\theta = 4\sin\frac{2\pi}{3} = 2\sqrt{3} \\ z &= -2 \end{aligned}.$$

The point with cylindrical coordinates $(4, \frac{2\pi}{3}, -2)$ has rectangular coordinates $(-2, 2\sqrt{3}, -2)$ (Figure 12.6.5).

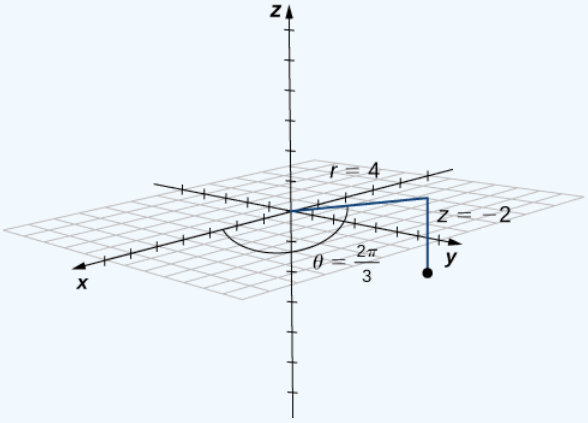


Figure 12.6.5: The projection of the point in the xy-plane is 4 units from the origin. The line from the origin to the point's projection forms an angle of $\frac{2\pi}{3}$ with the positive x-axis. The point lies 2 units below the xy-plane.

Exercise 12.6.1

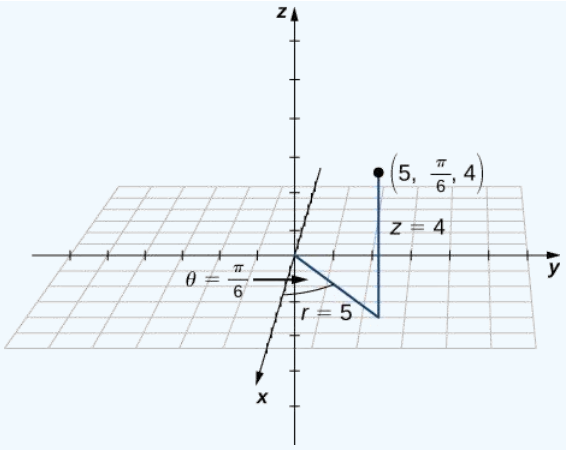
Point R has cylindrical coordinates $(5, \frac{\pi}{6}, 4)$. Plot R and describe its location in space using rectangular, or Cartesian, coordinates.

Hint

The first two components match the polar coordinates of the point in the xy-plane.

Answer

The rectangular coordinates of the point are $(\frac{5\sqrt{3}}{2}, \frac{5}{2}, 4)$.



If this process seems familiar, it is with good reason. This is exactly the same process that we followed in Introduction to Parametric Equations and Polar Coordinates to convert from polar coordinates to two-dimensional rectangular coordinates.

Example 12.6.2: Converting from Rectangular to Cylindrical Coordinates

Convert the rectangular coordinates $(1, -3, 5)$ to cylindrical coordinates.

Solution

Use the second set of equations from Conversion between Cylindrical and Cartesian Coordinates to translate from rectangular to cylindrical coordinates:

$$\begin{aligned} r^2 &= x^2 + y^2 \\ r &= \pm\sqrt{1^2 + (-3)^2} \\ &= \pm\sqrt{10}. \end{aligned}$$

We choose the positive square root, so $r = \sqrt{10}$.Now, we apply the formula to find θ. In this case, y is negative and x is positive, which means we must select the value of θ between $\dfrac{3\pi}{2}$ and 2π:

$$\begin{aligned} \tan\theta &= \frac{y}{x} & &= \frac{-3}{1} \\ \theta &= \arctan(-3) & &\approx 5.03 \text{ rad}. \end{aligned}$$

In this case, the z-coordinates are the same in both rectangular and cylindrical coordinates:

$$z = 5.$$

The point with rectangular coordinates $(1, -3, 5)$ has cylindrical coordinates approximately equal to $(\sqrt{10}, 5.03, 5)$.

Exercise 12.6.2

Convert point $(-8, 8, -7)$ from Cartesian coordinates to cylindrical coordinates.

Hint

$r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$

Answer

$(8\sqrt{2}, \frac{3\pi}{4}, -7)$

The use of cylindrical coordinates is common in fields such as physics. Physicists studying electrical charges and the capacitors used to store these charges have discovered that these systems sometimes have a cylindrical symmetry. These systems have complicated modeling equations in the Cartesian coordinate system, which make them difficult to describe and analyze. The equations can often be expressed in more simple terms using cylindrical coordinates. For example, the cylinder described by equation $x^2+y^2=25$ in the Cartesian system can be represented by cylindrical equation $r=5$.

Example 12.6.3: Identifying Surfaces in the Cylindrical Coordinate System

Describe the surfaces with the given cylindrical equations.

a. $\theta=\dfrac{\pi}{4}$
b. $r^2+z^2=9$
c. $z=r$

Solution

a. When the angle θ is held constant while r and z are allowed to vary, the result is a half-plane (Figure 12.6.6).

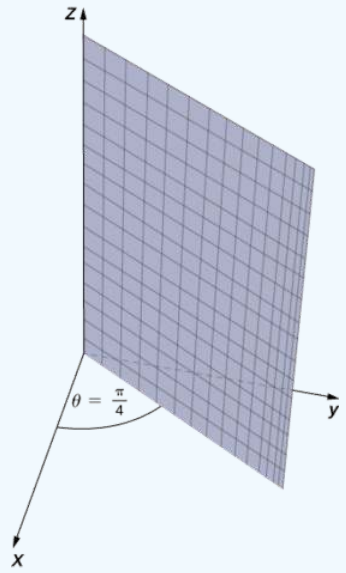


Figure 12.6.6: In polar coordinates, the equation $\theta=\pi/4$ describes the ray extending diagonally through the first quadrant. In three dimensions, this same equation describes a half-plane.

b. Substitute $r^2=x^2+y^2$ into equation $r^2+z^2=9$ to express the rectangular form of the equation: $x^2+y^2+z^2=9$. This equation describes a sphere centered at the origin with radius 3 (Figure 12.6.7).

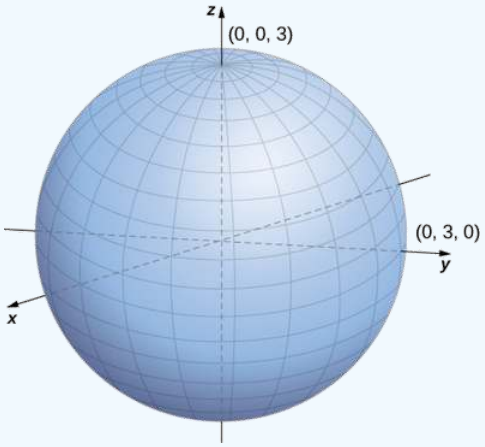


Figure 12.6.7: The sphere centered at the origin with radius 3 can be described by the cylindrical equation $r^2+z^2=9$.

c. To describe the surface defined by equation $z = r$, is it useful to examine traces parallel to the xy-plane. For example, the trace in plane $z = 1$ is circle $r = 1$, the trace in plane $z = 3$ is circle $r = 3$, and so on. Each trace is a circle. As the value of z increases, the radius of the circle also increases. The resulting surface is a cone (Figure 12.6.8).

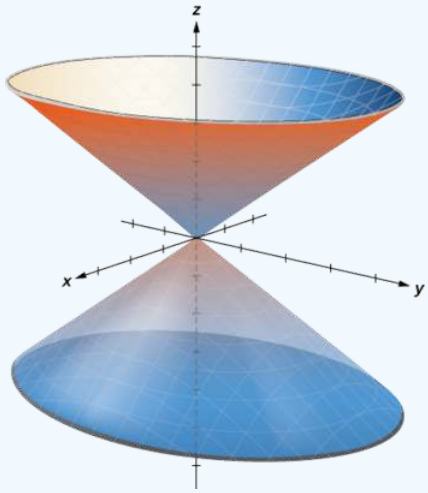


Figure 12.6.8: The traces in planes parallel to the xy-plane are circles. The radius of the circles increases as z increases.

? Exercise 12.6.3

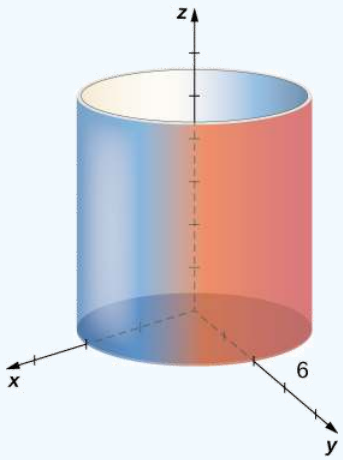
Describe the surface with cylindrical equation $r = 6$.

Hint

The θ and z components of points on the surface can take any value.

Answer

This surface is a cylinder with radius 6.



Spherical Coordinates

In the Cartesian coordinate system, the location of a point in space is described using an ordered triple in which each coordinate represents a distance. In the cylindrical coordinate system, the location of a point in space is described using two distances (r and z) and an angle measure (θ). In the spherical coordinate system, we again use an ordered triple to describe the location of a point in space. In this case, the triple describes one distance and two angles. Spherical coordinates make it simple to describe a sphere, just as cylindrical coordinates make it easy to describe a cylinder. Grid lines for spherical coordinates are based on angle measures, like those for polar coordinates.

Definition: spherical coordinate system

In the *spherical coordinate system*, a point P in space (Figure 12.6.9) is represented by the ordered triple (ρ, θ, φ) where

- ρ (the Greek letter rho) is the distance between P and the origin $(\rho \neq 0)$;
- θ is the same angle used to describe the location in cylindrical coordinates;
- φ (the Greek letter phi) is the angle formed by the positive z-axis and line segment $\overline{OP}$, where O is the origin and $0 \leq \varphi \leq \pi$.

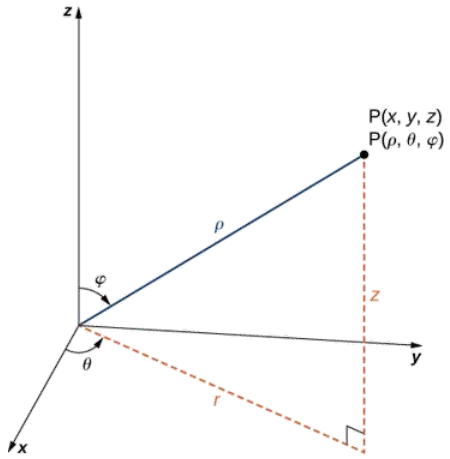


Figure 12.6.9: The relationship among spherical, rectangular, and cylindrical coordinates.

By convention, the origin is represented as $(0, 0, 0)$ in spherical coordinates.

HOW TO: Converting among Spherical, Cylindrical, and Rectangular Coordinates

Rectangular coordinates (x, y, z), cylindrical coordinates (r, θ, z), and spherical coordinates (ρ, θ, φ) of a point are related as follows:

Convert from spherical coordinates to rectangular coordinates

These equations are used to convert from spherical coordinates to rectangular coordinates.

- $x = \rho \sin\varphi \cos\theta$
- $y = \rho \sin\varphi \sin\theta$
- $z = \rho \cos\varphi$

Convert from rectangular coordinates to spherical coordinates

These equations are used to convert from rectangular coordinates to spherical coordinates.

- $\rho^2 = x^2 + y^2 + z^2$
- $\tan\theta = \dfrac{y}{x}$
- $\varphi = \arccos\left(\dfrac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$.

Convert from spherical coordinates to cylindrical coordinates

These equations are used to convert from spherical coordinates to cylindrical coordinates.

- $r = \rho \sin\varphi$
- $\theta = \theta$
- $z = \rho \cos\varphi$

Convert from cylindrical coordinates to spherical coordinates

These equations are used to convert from cylindrical coordinates to spherical coordinates.

- $\rho = \sqrt{r^2 + z^2}$
- $\theta = \theta$
- $\varphi = \arccos\left(\dfrac{z}{\sqrt{r^2 + z^2}}\right)$

The formulas to convert from spherical coordinates to rectangular coordinates may seem complex, but they are straightforward applications of trigonometry. Looking at Figure 12.6.10, it is easy to see that $r = \rho \sin\varphi$. Then, looking at the triangle in the xy-plane with r as its hypotenuse, we have $x = r\cos\theta = \rho\sin\varphi\cos\theta$. The derivation of the formula for y is similar. Figure 12.6.10 also shows that $\rho^2 = r^2 + z^2 = x^2 + y^2 + z^2$ and $z = \rho\cos\varphi$. Solving this last equation for φ and then substituting $\rho = \sqrt{r^2 + z^2}$ (from the first equation) yields $\varphi = \arccos\left(\dfrac{z}{\sqrt{r^2 + z^2}}\right)$. Also, note that, as before, we must be careful when using the formula $\tan\theta = \dfrac{y}{x}$ to choose the correct value of θ.

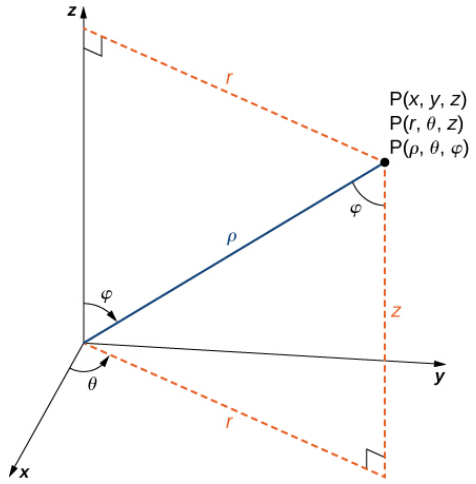


Figure 12.6.10: The equations that convert from one system to another are derived from right-triangle relationships.

As we did with cylindrical coordinates, let's consider the surfaces that are generated when each of the coordinates is held constant. Let c be a constant, and consider surfaces of the form $\rho = c$. Points on these surfaces are at a fixed distance from the origin and form a sphere. The coordinate θ in the spherical coordinate system is the same as in the cylindrical coordinate system, so surfaces of the form $\theta = c$ are half-planes, as before. Last, consider surfaces of the form $\varphi = c$. The points on these surfaces are at a fixed angle from the z-axis and form a half-cone (Figure 12.6.11).

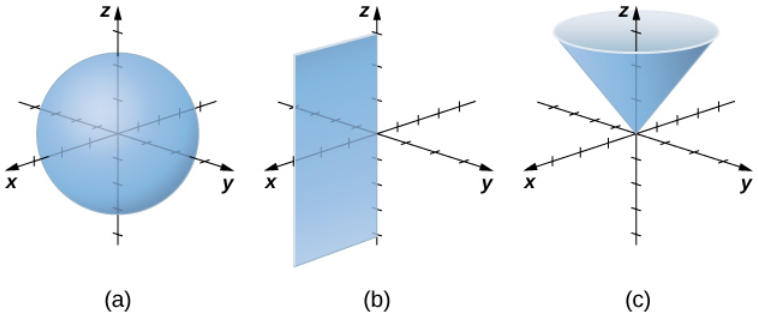


Figure 12.6.11: In spherical coordinates, surfaces of the form $\rho = c$ are spheres of radius ρ (a), surfaces of the form $\theta = c$ are half-planes at an angle θ from the x-axis (b), and surfaces of the form $\phi = c$ are half-cones at an angle ϕ from the z-axis (c).

Example 12.6.4: Converting from Spherical Coordinates

Plot the point with spherical coordinates $(8, \dfrac{\pi}{3}, \dfrac{\pi}{6})$ and express its location in both rectangular and cylindrical coordinates.

Solution

Use the equations in Converting among Spherical, Cylindrical, and Rectangular Coordinates to translate between spherical and cylindrical coordinates (Figure 12.6.12):

$$
\begin{aligned}
x &= \rho \sin\varphi \cos\theta \\
&= 8 \sin\left(\frac{\pi}{6}\right) \cos\left(\frac{\pi}{3}\right) \\
&= 8 \left(\frac{1}{2}\right) \frac{1}{2} \\
&= 2 \\
y &= \rho \sin\varphi \sin\theta \\
&= 8 \sin\left(\frac{\pi}{6}\right) \sin\left(\frac{\pi}{3}\right) \\
&= 8 \left(\frac{1}{2}\right) \frac{\sqrt{3}}{2} \\
&= 2\sqrt{3} \\
z &= \rho \cos\varphi \\
&= 8 \cos\left(\frac{\pi}{6}\right) \\
&= 8 \left(\frac{\sqrt{3}}{2}\right) \\
&= 4\sqrt{3}
\end{aligned}
$$

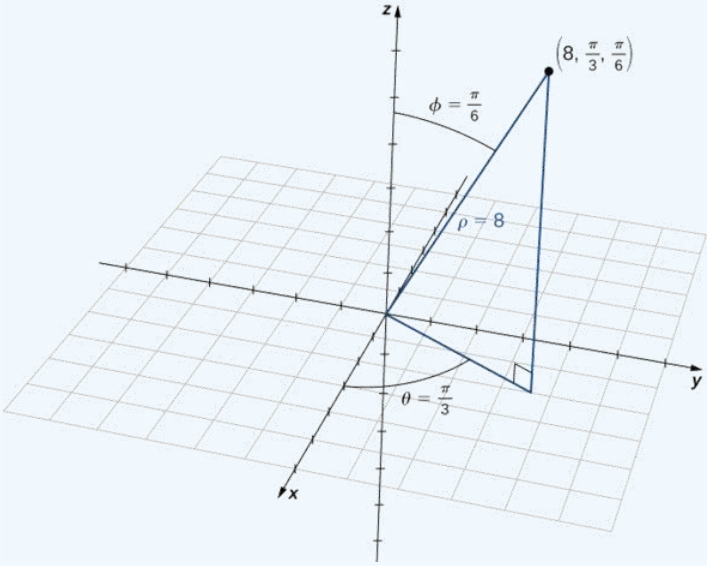


Figure 12.6.12: The projection of the point in the xy-plane is 4 units from the origin. The line from the origin to the point's projection forms an angle of $\pi/3$ with the positive x-axis. The point lies $4\sqrt{3}$ units above the xy-plane.

The point with spherical coordinates $(8, \frac{\pi}{3}, \frac{\pi}{6})$ has rectangular coordinates $(2, 2\sqrt{3}, 4\sqrt{3})$.

Finding the values in cylindrical coordinates is equally straightforward:

$$\begin{aligned} r &= \rho \sin \varphi \\ &= 8 \sin \frac{\pi}{6} \\ &= 4 \\ \theta &= \theta \\ z &= \rho \cos \varphi \\ &= 8 \cos \frac{\pi}{6} \\ &= 4\sqrt{3}. \end{aligned}$$

Thus, cylindrical coordinates for the point are $(4, \frac{\pi}{3}, 4\sqrt{3})$.

? Exercise 12.6.4

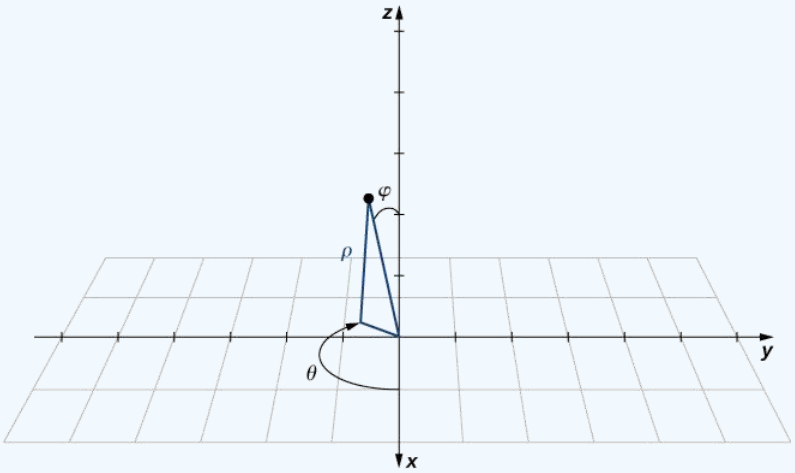
Plot the point with spherical coordinates $(2, -\frac{5\pi}{6}, \frac{\pi}{6})$ and describe its location in both rectangular and cylindrical coordinates.

Hint

Converting the coordinates first may help to find the location of the point in space more easily.

Answer

Cartesian: $(-\frac{\sqrt{3}}{2}, -\frac{1}{2}, \sqrt{3})$, cylindrical: $(1, -\frac{5\pi}{6}, \sqrt{3})$



✓ Example 12.6.5: Converting from Rectangular Coordinates

Convert the rectangular coordinates $(-1, 1, \sqrt{6})$ to both spherical and cylindrical coordinates.

Solution

Start by converting from rectangular to spherical coordinates:

$$\rho^2 = x^2 + y^2 + z^2 = (-1)^2 + 1^2 + (\sqrt{6})^2 = 8$$

$$\tan \theta = \frac{1}{-1}$$

$$\rho = 2\sqrt{2} \text{ and } \theta = \arctan(-1) = \frac{3\pi}{4}.$$

Because $(x, y) = (-1, 1)$, then the correct choice for θ is $\frac{3\pi}{4}$.

There are actually two ways to identify φ. We can use the equation $\varphi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$. A more simple approach, however, is to use equation $z = \rho\cos\varphi$. We know that $z = \sqrt{6}$ and $\rho = 2\sqrt{2}$, so

$$\sqrt{6} = 2\sqrt{2}\cos\varphi, \text{ so } \cos\varphi = \frac{\sqrt{6}}{2\sqrt{2}} = \frac{\sqrt{3}}{2}$$

and therefore $\varphi = \frac{\pi}{6}$. The spherical coordinates of the point are $(2\sqrt{2}, \frac{3\pi}{4}, \frac{\pi}{6})$.

To find the cylindrical coordinates for the point, we need only find r:

$$r = \rho\sin\varphi = 2\sqrt{2}\sin(\frac{\pi}{6}) = \sqrt{2}.$$

The cylindrical coordinates for the point are $(\sqrt{2}, \frac{3\pi}{4}, \sqrt{6})$.

Example 12.6.6: Identifying Surfaces in the Spherical Coordinate System

Describe the surfaces with the given spherical equations.

a. $\theta = \frac{\pi}{3}$
b. $\varphi = \frac{5\pi}{6}$
c. $\rho = 6$
d. $\rho = \sin\theta\sin\varphi$

Solution

a. The variable θ represents the measure of the same angle in both the cylindrical and spherical coordinate systems. Points with coordinates $(\rho, \frac{\pi}{3}, \varphi)$ lie on the plane that forms angle $\theta = \frac{\pi}{3}$ with the positive x-axis. Because $\rho > 0$, the surface described by equation $\theta = \frac{\pi}{3}$ is the half-plane shown in Figure 12.6.13

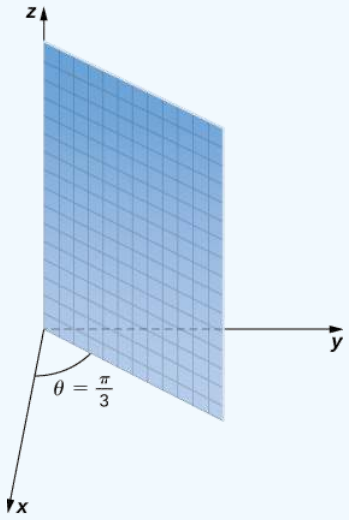


Figure 12.6.13: The surface described by equation $\theta = \frac{\pi}{3}$ is a half-plane.

b. Equation $\varphi = \frac{5\pi}{6}$ describes all points in the spherical coordinate system that lie on a line from the origin forming an angle measuring $\frac{5\pi}{6}$ rad with the positive z-axis. These points form a half-cone (Figure 12.6.14). Because there is only one value for φ that is measured from the positive z-axis, we do not get the full cone (with two pieces).

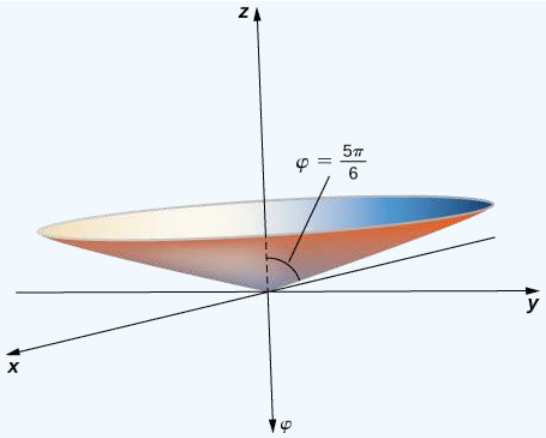


Figure 12.6.14: The equation $\varphi = \dfrac{5\pi}{6}$ describes a cone.

To find the equation in rectangular coordinates, use equation $\varphi = \arccos\left(\dfrac{z}{\sqrt{x^2+y^2+z^2}}\right)$.

$$\begin{aligned} \frac{5\pi}{6} &= \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \\ \cos\frac{5\pi}{6} &= \frac{z}{\sqrt{x^2+y^2+z^2}} \\ -\frac{\sqrt{3}}{2} &= \frac{z}{\sqrt{x^2+y^2+z^2}} \\ \frac{3}{4} &= \frac{z^2}{x^2+y^2+z^2} \\ \frac{3x^2}{4}+\frac{3y^2}{4}+\frac{3z^2}{4} &= z^2 \\ \frac{3x^2}{4}+\frac{3y^2}{4}-\frac{z^2}{4} &= 0. \end{aligned}$$

This is the equation of a cone centered on the z-axis.

c. Equation $\rho = 6$ describes the set of all points 6 units away from the origin—a sphere with radius 6 (Figure 12.6.15).

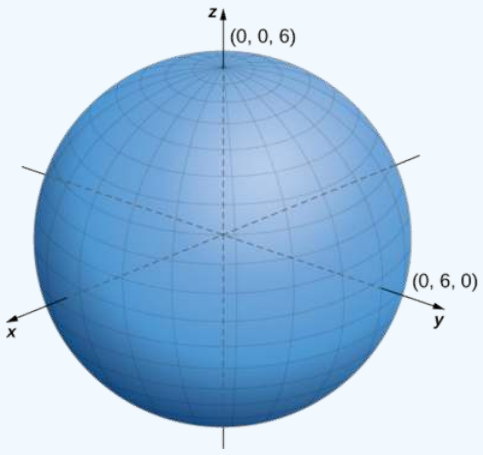


Figure 12.6.15: Equation $\rho = 6$ describes a sphere with radius 6.

d. To identify this surface, convert the equation from spherical to rectangular coordinates, using equations $y = \rho\sin\varphi\sin\theta$ and $\rho^2 = x^2+y^2+z^2$:

$\rho = \sin\theta\sin\varphi$

$\rho^2 = \rho\sin\theta\sin\varphi$ Multiply both sides of the equation by ρ.

$x^2 + y^2 + z^2 = y$ Substitute rectangular variables using the equations above.

$x^2 + y^2 - y + z^2 = 0$ Subtract y from both sides of the equation.

$x^2 + y^2 - y + \frac{1}{4} + z^2 = \frac{1}{4}$ Complete the square.

$x^2 + \left(y - \frac{1}{2}\right)^2 + z^2 = \frac{1}{4}$. Rewrite the middle terms as a perfect square.

The equation describes a sphere centered at point $(0, \frac{1}{2}, 0)$ with radius $\frac{1}{2}$.

? Exercise 12.6.5

Describe the surfaces defined by the following equations.

a. $\rho = 13$
b. $\theta = \frac{2\pi}{3}$
c. $\varphi = \frac{\pi}{4}$

Hint

Think about what each component represents and what it means to hold that component constant.

Answer a

This is the set of all points 13 units from the origin. This set forms a sphere with radius 13.

Answer b

This set of points forms a half plane. The angle between the half plane and the positive x-axis is $\theta = \frac{2\pi}{3}$.

Answer c

Let P be a point on this surface. The position vector of this point forms an angle of $\varphi = \frac{\pi}{4}$ with the positive z-axis, which means that points closer to the origin are closer to the axis. These points form a half-cone.

Spherical coordinates are useful in analyzing systems that have some degree of symmetry about a point, such as the volume of the space inside a domed stadium or wind speeds in a planet's atmosphere. A sphere that has Cartesian equation $x^2 + y^2 + z^2 = c^2$ has the simple equation $\rho = c$ in spherical coordinates.

In geography, latitude and longitude are used to describe locations on Earth's surface, as shown in Figure $12.6.16$ Although the shape of Earth is not a perfect sphere, we use spherical coordinates to communicate the locations of points on Earth. Let's assume Earth has the shape of a sphere with radius 4000 mi. We express angle measures in degrees rather than radians because latitude and longitude are measured in degrees.

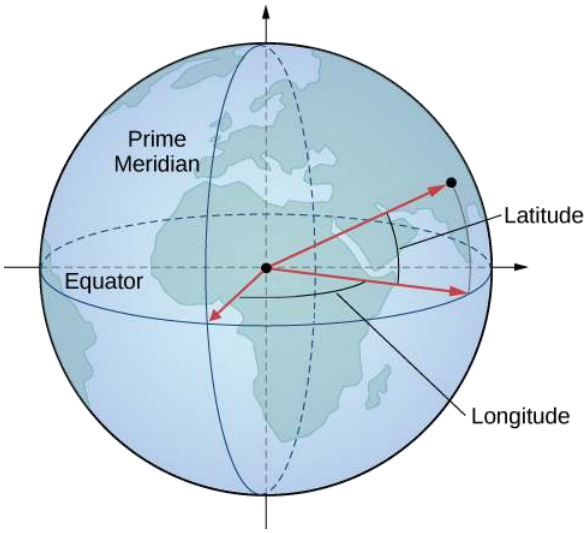


Figure 12.6.16: In the latitude–longitude system, angles describe the location of a point on Earth relative to the equator and the prime meridian.

Let the center of Earth be the center of the sphere, with the ray from the center through the North Pole representing the positive z-axis. The prime meridian represents the trace of the surface as it intersects the xz-plane. The equator is the trace of the sphere intersecting the xy-plane.

Example 12.6.7: Converting Latitude and Longitude to Spherical Coordinates

The latitude of Columbus, Ohio, is 40° N and the longitude is 83° W, which means that Columbus is 40° north of the equator. Imagine a ray from the center of Earth through Columbus and a ray from the center of Earth through the equator directly south of Columbus. The measure of the angle formed by the rays is 40°. In the same way, measuring from the prime meridian, Columbus lies 83° to the west. Express the location of Columbus in spherical coordinates.

Solution

The radius of Earth is 4000mi, so $\rho = 4000$. The intersection of the prime meridian and the equator lies on the positive x-axis. Movement to the west is then described with negative angle measures, which shows that $\theta = -83^\circ$, Because Columbus lies 40° north of the equator, it lies 50° south of the North Pole, so $\varphi = 50^\circ$. In spherical coordinates, Columbus lies at point $(4000, -83^\circ, 50^\circ)$.

Exercise 12.6.6

Sydney, Australia is at $34^\circ S$ and $151^\circ E$. Express Sydney's location in spherical coordinates.

Hint

Because Sydney lies south of the equator, we need to add 90° to find the angle measured from the positive z-axis.

Answer

$(4000, 151^\circ, 124^\circ)$

Cylindrical and spherical coordinates give us the flexibility to select a coordinate system appropriate to the problem at hand. A thoughtful choice of coordinate system can make a problem much easier to solve, whereas a poor choice can lead to unnecessarily complex calculations. In the following example, we examine several different problems and discuss how to select the best coordinate system for each one.

Example 12.6.8: Choosing the Best Coordinate System

In each of the following situations, we determine which coordinate system is most appropriate and describe how we would orient the coordinate axes. There could be more than one right answer for how the axes should be oriented, but we select an orientation that makes sense in the context of the problem. Note: There is not enough information to set up or solve these problems; we simply select the coordinate system (Figure 12.6.17).

a. Find the center of gravity of a bowling ball.
b. Determine the velocity of a submarine subjected to an ocean current.
c. Calculate the pressure in a conical water tank.
d. Find the volume of oil flowing through a pipeline.
e. Determine the amount of leather required to make a football.



(a) (b) (c)



(d) (e)

Figure 12.6.17: (credit: (a) modification of work by scl hua, Wikimedia, (b) modification of work by DVIDSHUB, Flickr, (c) modification of work by Michael Malak, Wikimedia, (d) modification of work by Sean Mack, Wikimedia, (e) modification of work by Elvert Barnes, Flickr)

Solution

a. Clearly, a bowling ball is a sphere, so spherical coordinates would probably work best here. The origin should be located at the physical center of the ball. There is no obvious choice for how the x-, y- and z-axes should be oriented. Bowling balls normally have a weight block in the center. One possible choice is to align the z-axis with the axis of symmetry of the weight block.
b. A submarine generally moves in a straight line. There is no rotational or spherical symmetry that applies in this situation, so rectangular coordinates are a good choice. The z-axis should probably point upward. The x- and y-axes could be aligned to point east and north, respectively. The origin should be some convenient physical location, such as the starting position of the submarine or the location of a particular port.

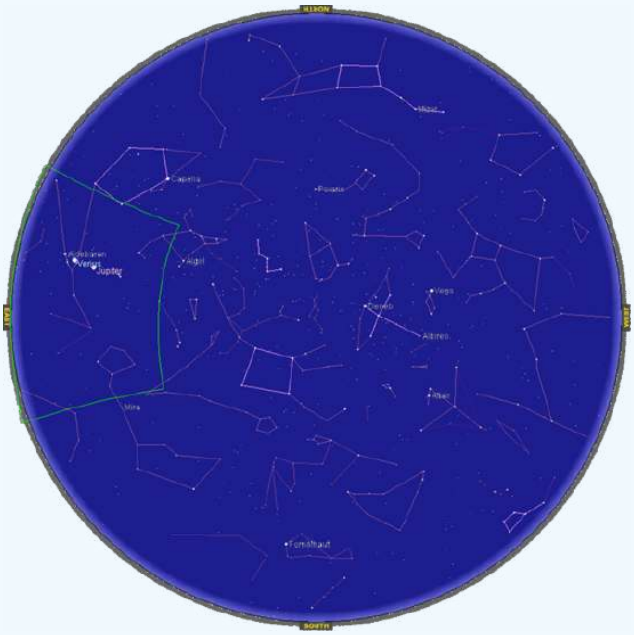
c. A cone has several kinds of symmetry. In cylindrical coordinates, a cone can be represented by equation $z = kr$, where k is a constant. In spherical coordinates, we have seen that surfaces of the form $\varphi = c$ are half-cones. Last, in rectangular coordinates, elliptic cones are quadric surfaces and can be represented by equations of the form $z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$. In this case, we could choose any of the three. However, the equation for the surface is more complicated in rectangular coordinates than in the other two systems, so we might want to avoid that choice. In addition, we are talking about a water tank, and the depth of the water might come into play at some point in our calculations, so it might be nice to have a component that represents height and depth directly. Based on this reasoning, cylindrical coordinates might be the best choice. Choose the z-axis to align with the axis of the cone. The orientation of the other two axes is arbitrary. The origin should be the bottom point of the cone.

d. A pipeline is a cylinder, so cylindrical coordinates would be best the best choice. In this case, however, we would likely choose to orient our z-axis with the center axis of the pipeline. The x-axis could be chosen to point straight downward or to some other logical direction. The origin should be chosen based on the problem statement. Note that this puts the z-axis in a horizontal orientation, which is a little different from what we usually do. It may make sense to choose an unusual orientation for the axes if it makes sense for the problem.

e. A football has rotational symmetry about a central axis, so cylindrical coordinates would work best. The z-axis should align with the axis of the ball. The origin could be the center of the ball or perhaps one of the ends. The position of the x-axis is arbitrary.

? Exercise 12.6.7

Which coordinate system is most appropriate for creating a star map, as viewed from Earth (see the following figure)?



How should we orient the coordinate axes?

Hint

What kinds of symmetry are present in this situation?

Answer

Spherical coordinates with the origin located at the center of the earth, the z-axis aligned with the North Pole, and the x-axis aligned with the prime meridian

Key Concepts

- In the cylindrical coordinate system, a point in space is represented by the ordered triple (r, θ, z), where (r, θ) represents the polar coordinates of the point's projection in the xy-plane and z represents the point's projection onto the z-axis.
- To convert a point from cylindrical coordinates to Cartesian coordinates, use equations $x = r\cos\theta, y = r\sin\theta,$ and $z = z$.
- To convert a point from Cartesian coordinates to cylindrical coordinates, use equations $r^2 = x^2 + y^2, \tan\theta = \dfrac{y}{x},$ and $z = z$.
- In the spherical coordinate system, a point P in space is represented by the ordered triple (ρ, θ, φ), where ρ is the distance between P and the origin $(\rho \neq 0), \theta$ is the same angle used to describe the location in cylindrical coordinates, and φ is the angle formed by the positive z-axis and line segment $\overline{OP}$, where O is the origin and $0 \leq \varphi \leq \pi$.
- To convert a point from spherical coordinates to Cartesian coordinates, use equations $x = \rho\sin\varphi\cos\theta, y = \rho\sin\varphi\sin\theta,$ and $z = \rho\cos\varphi$.
- To convert a point from Cartesian coordinates to spherical coordinates, use equations $\rho^2 = x^2 + y^2 + z^2, \tan\theta = \dfrac{y}{x},$ and $\varphi = \arccos\left(\dfrac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$.
- To convert a point from spherical coordinates to cylindrical coordinates, use equations $r = \rho\sin\varphi, \theta = \theta,$ and $z = \rho\cos\varphi$.
- To convert a point from cylindrical coordinates to spherical coordinates, use equations $\rho = \sqrt{r^2 + z^2}, \theta = \theta,$ and $\varphi = \arccos\left(\dfrac{z}{\sqrt{r^2 + z^2}}\right)$.

Glossary

cylindrical coordinate system

a way to describe a location in space with an ordered triple (r, θ, z), where (r, θ) represents the polar coordinates of the point's projection in the xy-plane, and z represents the point's projection onto the z-axis

spherical coordinate system

a way to describe a location in space with an ordered triple (ρ, θ, φ), where ρ is the distance between P and the origin $(\rho \neq 0), \theta$ is the same angle used to describe the location in cylindrical coordinates, and φ is the angle formed by the positive z-axis and line segment $\bar{OP}$, where O is the origin and $0 \leq \varphi \leq \pi$

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CHAPTER OVERVIEW

13: Vector Functions

A general Calculus Textmap organized around the textbook

Calculus: Early Transcendentals

by James Stewart

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Topic hierarchy

13.1: Vector Functions and Space Curves
13.2: Derivatives and Integrals of Vector Functions
13.3: Arc Length and Curvature
13.4: Motion in Space- Velocity and Acceleration

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13.1: Vector Functions and Space Curves

Learning Objectives

- Write the general equation of a vector-valued function in component form and unit-vector form.
- Recognize parametric equations for a space curve.
- Describe the shape of a helix and write its equation.
- Define the limit of a vector-valued function.

Our study of vector-valued functions combines ideas from our earlier examination of single-variable calculus with our description of vectors in three dimensions from the preceding chapter. In this section, we extend concepts from earlier chapters and also examine new ideas concerning curves in three-dimensional space. These definitions and theorems support the presentation of material in the rest of this chapter and also in the remaining chapters of the text.

Definition of a Vector-Valued Function

Our first step in studying the calculus of vector-valued functions is to define what exactly a vector-valued function is. We can then look at graphs of vector-valued functions and see how they define curves in both two and three dimensions.

Definition: Vector-valued Functions

A vector-valued function is a function of the form

$$\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} \quad \text{or} \quad \vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}},$$

where the component functions f, g, and h, are real-valued functions of the parameter t. Vector-valued functions are also written in the form

$$\vec{\mathbf{r}}(t) = \langle f(t),\, g(t)\rangle \quad \text{or} \quad \vec{\mathbf{r}}(t) = \langle f(t),\, g(t),\, h(t)\rangle.$$

In both cases, the first form of the function defines a two-dimensional vector-valued function; the second form describes a three-dimensional vector-valued function.

The parameter t can lie between two real numbers: $a \le t \le b$. Another possibility is that the value of t might take on all real numbers. Last, the component functions themselves may have domain restrictions that enforce restrictions on the value of t. We often use t as a parameter because t can represent time.

Example 13.1.1: Evaluating Vector-Valued Functions and Determining Domains

For each of the following vector-valued functions, evaluate $\vec{\mathbf{r}}(0)$, $\vec{\mathbf{r}}\left(\frac{\pi}{2}\right)$, and $\vec{\mathbf{r}}\left(\frac{2\pi}{3}\right)$. Do any of these functions have domain restrictions?

a. $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}}$
b. $\vec{\mathbf{r}}(t) = 3\tan t\,\hat{\mathbf{i}} + 4\sec t\,\hat{\mathbf{j}} + 5t\,\hat{\mathbf{k}}$

Solution

a. To calculate each of the function values, substitute the appropriate value of t into the function:

$$\begin{aligned}\vec{\mathbf{r}}(0) &= 4\cos(0)\hat{\mathbf{i}} + 3\sin(0)\hat{\mathbf{j}} \\ &= 4\hat{\mathbf{i}} + 0\hat{\mathbf{j}} = 4\hat{\mathbf{i}} \\ \vec{\mathbf{r}}\left(\frac{\pi}{2}\right) &= 4\cos\left(\frac{\pi}{2}\right)\hat{\mathbf{i}} + 3\sin\left(\frac{\pi}{2}\right)\hat{\mathbf{j}} \\ &= 0\hat{\mathbf{i}} + 3\hat{\mathbf{j}} = 3\hat{\mathbf{j}} \\ \vec{\mathbf{r}}\left(\frac{2\pi}{3}\right) &= 4\cos\left(\frac{2\pi}{3}\right)\hat{\mathbf{i}} + 3\sin\left(\frac{2\pi}{3}\right)\hat{\mathbf{j}} \\ &= 4\left(-\tfrac{1}{2}\right)\hat{\mathbf{i}} + 3\left(\tfrac{\sqrt{3}}{2}\right)\hat{\mathbf{j}} = -2\hat{\mathbf{i}} + \tfrac{3\sqrt{3}}{2}\hat{\mathbf{j}}\end{aligned}$$

To determine whether this function has any domain restrictions, consider the component functions separately. The first component function is $f(t) = 4\cos t$ and the second component function is $g(t) = 3\sin t$. Neither of these functions has a domain restriction, so the domain of $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}}$ is all real numbers.

b. To calculate each of the function values, substitute the appropriate value of t into the function:

$$\begin{aligned}\vec{\mathbf{r}}(0) &= 3\tan(0)\hat{\mathbf{i}} + 4\sec(0)\hat{\mathbf{j}} + 5(0)\hat{\mathbf{k}} \\ &= 0\hat{\mathbf{i}} + 4j + 0\hat{\mathbf{k}} = 4\hat{\mathbf{j}} \\ \vec{\mathbf{r}}\left(\frac{\pi}{2}\right) &= 3\tan\left(\frac{\pi}{2}\right)\hat{\mathbf{i}} + 4\sec\left(\frac{\pi}{2}\right)\hat{\mathbf{j}} + 5\left(\frac{\pi}{2}\right)\hat{\mathbf{k}}, \text{ which does not exist} \\ \vec{\mathbf{r}}\left(\frac{2\pi}{3}\right) &= 3\tan\left(\frac{2\pi}{3}\right)\hat{\mathbf{i}} + 4\sec\left(\frac{2\pi}{3}\right)\hat{\mathbf{j}} + 5\left(\frac{2\pi}{3}\right)\hat{\mathbf{k}} \\ &= 3(-\sqrt{3})\hat{\mathbf{i}} + 4(-2)\hat{\mathbf{j}} + \frac{10\pi}{3}\hat{\mathbf{k}} \\ &= (-3\sqrt{3})\hat{\mathbf{i}} - 8\hat{\mathbf{j}} + \frac{10\pi}{3}\hat{\mathbf{k}}\end{aligned}$$

To determine whether this function has any domain restrictions, consider the component functions separately. The first component function is $f(t) = 3\tan t$, the second component function is $g(t) = 4\sec t$, and the third component function is $h(t) = 5t$. The first two functions are not defined for odd multiples of $\frac{\pi}{2}$, so the function is not defined for odd multiples of $\frac{\pi}{2}$. Therefore,

$$\mathrm{D}_{\vec{\mathbf{r}}} = \left\{t \,|\, t \neq \frac{(2n+1)\pi}{2}\right\},$$

where n is any integer.

? Exercise 13.1.1

For the vector-valued function $\vec{\mathbf{r}}(t) = (t^2 - 3t)\,\hat{\mathbf{i}} + (4t+1)\,\hat{\mathbf{j}}$, evaluate $\vec{\mathbf{r}}(0)$, $\vec{\mathbf{r}}(1)$, and $\vec{\mathbf{r}}(-4)$. Does this function have any domain restrictions?

Hint

Substitute the appropriate values of t into the function.

Answer

$\vec{\mathbf{r}}(0) = \hat{\mathbf{j}}$, $\vec{\mathbf{r}}(1) = -2\hat{\mathbf{i}} + 5\hat{\mathbf{j}}$, $\vec{\mathbf{r}}(-4) = 28\hat{\mathbf{i}} - 15\hat{\mathbf{j}}$

The domain of $\vec{\mathbf{r}}(t) = (t^2 - 3t)\hat{\mathbf{i}} + (4t+1)\hat{\mathbf{j}}$ is all real numbers.

Example 13.1.1 illustrates an important concept. The domain of a vector-valued function consists of real numbers. The domain can be all real numbers or a subset of the real numbers. The range of a vector-valued function consists of vectors. Each real number in the domain of a vector-valued function is mapped to either a two- or a three-dimensional vector.

Graphing Vector-Valued Functions

Recall that a plane vector consists of two quantities: direction and magnitude. Given any point in the plane (the *initial point*), if we move in a specific direction for a specific distance, we arrive at a second point. This represents the *terminal point* of the vector. We calculate the components of the vector by subtracting the coordinates of the initial point from the coordinates of the terminal point.

A vector is considered to be in *standard position* if the initial point is located at the origin. When graphing a vector-valued function, we typically graph the vectors in the domain of the function in standard position, because doing so guarantees the uniqueness of the graph. This convention applies to the graphs of three-dimensional vector-valued functions as well. The graph of a vector-valued function of the form

$$\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}$$

consists of the set of all points $(f(t),\ g(t))$, and the path it traces is called a **plane curve**. The graph of a vector-valued function of the form

$$\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}$$

consists of the set of all points $(f(t),\ g(t),\ h(t))$, and the path it traces is called a **space curve**. Any representation of a plane curve or space curve using a vector-valued function is called a **vector parameterization** of the curve.

Each plane curve and space curve has an **orientation**, indicated by arrows drawn in on the curve, that shows the direction of motion along the curve as the value of the parameter t increases.

Example 13.1.2 : Graphing a Vector-Valued Function

Create a graph of each of the following vector-valued functions:

a. The plane curve represented by $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}}$, $0 \le t \le 2\pi$
b. The plane curve represented by $\vec{\mathbf{r}}(t) = 4\cos(t^3)\,\hat{\mathbf{i}} + 3\sin(t^3)\,\hat{\mathbf{j}}$, $0 \le t \le \sqrt[3]{2\pi}$
c. The space curve represented by $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 4\sin t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$, $0 \le t \le 4\pi$

Solution

a. As with any graph, we start with a table of values. We then graph each of the vectors in the second column of the table in standard position and connect the terminal points of each vector to form a curve (Figure 13.1.1). This curve turns out to be an ellipse centered at the origin.

Table 13.1.1: Table of Values for $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}}$, $0 \le t \le 2\pi$

t	$\vec{\mathbf{r}}(t)$	t	$\vec{\mathbf{r}}(t)$
0	$4\hat{\mathbf{i}}$	π	$-4\hat{\mathbf{i}}$
$\frac{\pi}{4}$	$2\sqrt{2}\hat{\mathbf{i}} + \frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$	$\frac{5\pi}{4}$	$-2\sqrt{2}\hat{\mathbf{i}} - \frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$
$\frac{\pi}{2}$	$3\hat{\mathbf{j}}$	$\frac{3\pi}{2}$	$-3\hat{\mathbf{j}}$
$\frac{3\pi}{4}$	$-2\sqrt{2}\hat{\mathbf{i}} + \frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$	$\frac{7\pi}{4}$	$2\sqrt{2}\hat{\mathbf{i}} - \frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$
2π	$4\hat{\mathbf{i}}$		

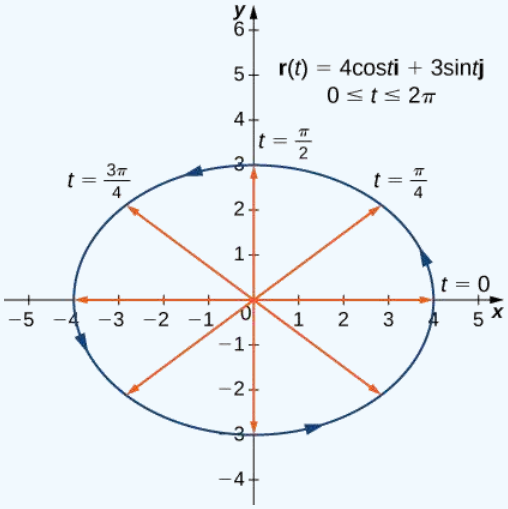


Figure 13.1.1: The graph of the first vector-valued function is an ellipse.

b. The table of values for $\vec{\mathbf{r}}(t) = 4\cos(t^3)\,\hat{\mathbf{i}} + 3\sin(t^3)\,\hat{\mathbf{j}}$, $0 \le t \le \sqrt[3]{2\pi}$ is as follows:

Table of Values for $\vec{\mathbf{r}}(t) = 4\cos(t^3)\,\hat{\mathbf{i}} + 3\sin(t^3)\,\hat{\mathbf{j}}$, $0 \le t \le \sqrt[3]{2\pi}$

t	$\vec{\mathbf{r}}(t)$	t	$\vec{\mathbf{r}}(t)$
0	$4\hat{\mathbf{i}}$	$\sqrt[3]{\pi}$	$-4\hat{\mathbf{i}}$
$\sqrt[3]{\frac{\pi}{4}}$	$2\sqrt{2}\hat{\mathbf{i}} + \frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$	$\sqrt[3]{\frac{5\pi}{4}}$	$-2\sqrt{2}\hat{\mathbf{i}} - \frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$
$\sqrt[3]{\frac{\pi}{2}}$	$3\hat{\mathbf{j}}$	$\sqrt[3]{\frac{3\pi}{2}}$	$-3\hat{\mathbf{j}}$
$\sqrt[3]{\frac{3\pi}{4}}$	$-2\sqrt{2}\hat{\mathbf{i}} + \frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$	$\sqrt[3]{\frac{7\pi}{4}}$	$2\sqrt{2}\hat{\mathbf{i}} - \frac{3\sqrt{2}}{2}\hat{\mathbf{j}}$
$\sqrt[3]{2\pi}$	$4\hat{\mathbf{i}}$		

The graph of this curve is also an ellipse centered at the origin.

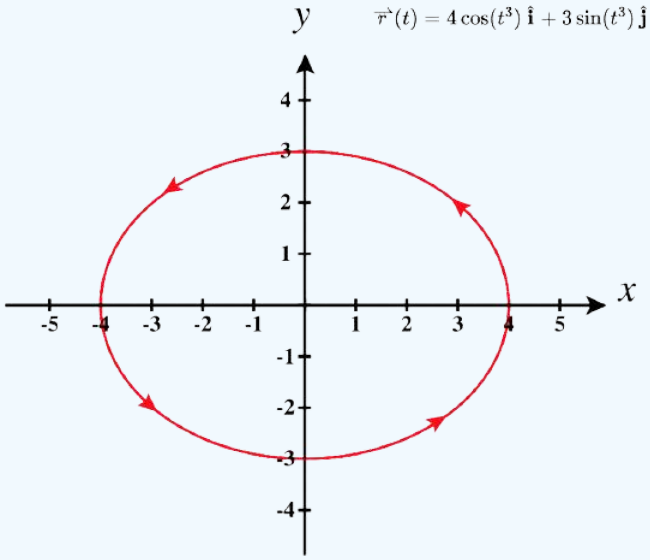


Figure 13.1.2: The graph of the second vector-valued function is also an ellipse.

c. We go through the same procedure for a three-dimensional vector function.

Table of Values for $\mathrm{r}(\mathrm{t}) = 4\cos \mathrm{t}\,\hat{\mathbf{i}} + 4\sin \mathrm{t}\,\hat{\mathbf{j}} + \mathrm{t}\hat{\mathbf{k}},\ 0 \le \mathrm{t} \le 4\pi$

t	$\vec{\mathbf{r}}(t)$	t	$\vec{\mathbf{r}}(t)$
0	$4\hat{\mathbf{i}}$	π	$-4\hat{\mathbf{i}} + \pi\hat{\mathbf{k}}$
$\frac{\pi}{4}$	$2\sqrt{2}\hat{\mathbf{i}} + 2\sqrt{2}\hat{\mathbf{j}} + \frac{\pi}{4}\hat{\mathbf{k}}$	$\frac{5\pi}{4}$	$-2\sqrt{2}\hat{\mathbf{i}} - 2\sqrt{2}\hat{\mathbf{j}} + \frac{5\pi}{4}\hat{\mathbf{k}}$
$\frac{\pi}{2}$	$4\hat{\mathbf{j}} + \frac{\pi}{2}\hat{\mathbf{k}}$	$\frac{3\pi}{2}$	$-4\hat{\mathbf{j}} + \frac{3\pi}{2}\hat{\mathbf{k}}$
$\frac{3\pi}{4}$	$-2\sqrt{2}\hat{\mathbf{i}} + 2\sqrt{2}\hat{\mathbf{j}} + \frac{3\pi}{4}\hat{\mathbf{k}}$	$\frac{7\pi}{4}$	$2\sqrt{2}\hat{\mathbf{i}} - 2\sqrt{2}\hat{\mathbf{j}} + \frac{7\pi}{4}\hat{\mathbf{k}}$
2π	$4\hat{\mathbf{j}} + 2\pi\hat{\mathbf{k}}$		

The values then repeat themselves, except for the fact that the coefficient of $\hat{\mathbf{k}}$ is always increasing (13.1.3). This curve is called a helix. Notice that if the $\hat{\mathbf{k}}$ component is eliminated, then the function becomes $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 4\sin t\,\hat{\mathbf{j}}$, which is a circle of radius 4 centered at the origin.

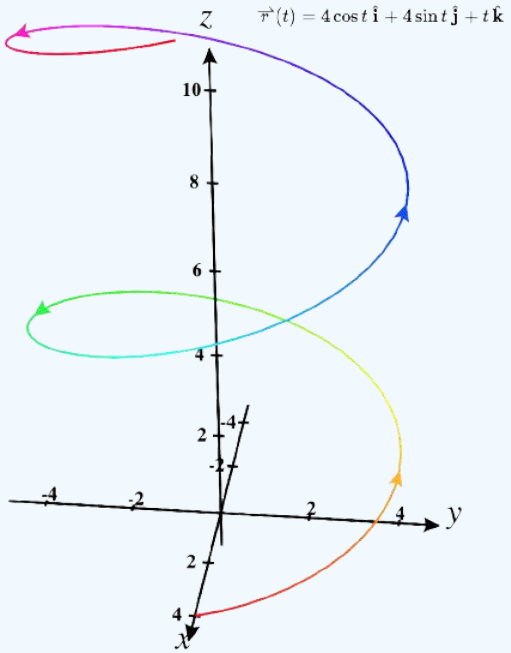


Figure 13.1.3: The graph of the third vector-valued function is a helix.

You may notice that the graphs in parts a. and b. are identical. This happens because the function describing the curve in part b. is a so-called reparameterization of the function describing the curve in part a. In fact, any curve has an infinite number of reparameterizations; for example, we can replace t with $2t$ in any of the three previous curves without changing the shape of the curve. The interval over which t is defined may change, but that is all. We return to this idea later in this chapter when we study arc-length parameterization. As mentioned, the name of the shape of the curve of the graph in 13.1.3 is a *helix*. The curve resembles a spring, with a circular cross-section looking down along the z-axis. It is possible for a helix to be elliptical in cross-section as well. For example, the vector-valued function $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}$ describes an elliptical helix. The projection of this helix into the xy-plane is an ellipse. Last, the arrows in the graph of this helix indicate the orientation of the curve as t progresses from 0 to 4π.

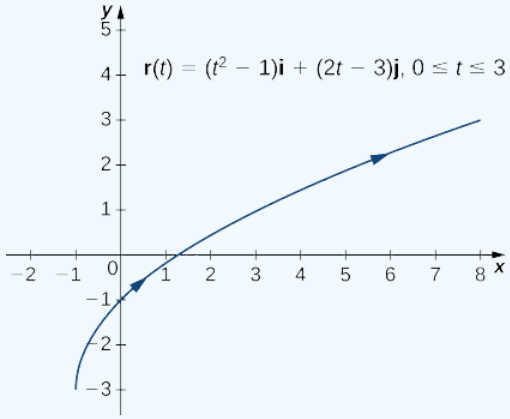
? Exercise 13.1.2

Create a graph of the vector-valued function $\vec{\mathbf{r}}(t) = (t^2 - 1)\hat{\mathbf{i}} + (2t - 3)\hat{\mathbf{j}}$, $0 \le t \le 3$.

Hint

Start by making a table of values, then graph the vectors for each value of t.

Answer



At this point, you may notice a similarity between vector-valued functions and parameterized curves. Indeed, given a vector-valued function $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}$ we can define $x = f(t)$ and $y = g(t)$. If a restriction exists on the values of t (for example, t is restricted to the interval $[a, b]$ for some constants $a < b$, then this restriction is enforced on the parameter. The graph of the parameterized function would then agree with the graph of the vector-valued function, except that the vector-valued graph would represent vectors rather than points. Since we can parameterize a curve defined by a function $y = f(x)$, it is also possible to represent an arbitrary plane curve by a vector-valued function.

Limits and Continuity of a Vector-Valued Function

We now take a look at the limit of a vector-valued function. This is important to understand to study the calculus of vector-valued functions.

Definition: limit of a vector-valued function

A vector-valued function $\vec{\mathbf{r}}$ approaches the limit $\vec{\mathbf{L}}$ as t approaches a, written

$$\lim_{t \to a} \vec{\mathbf{r}}(t) = \vec{\mathbf{L}},$$

provided

$$\lim_{t \to a} \left\| \vec{\mathbf{r}}(t) - \vec{\mathbf{L}} \right\| = 0.$$

This is a rigorous definition of the limit of a vector-valued function. In practice, we use the following theorem:

Theorem: Limit of a vector-valued function

Let f, g, and h be functions of t. Then the limit of the vector-valued function $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}$ as t approaches a is given by

$$\lim_{t \to a} \vec{\mathbf{r}}(t) = \left[\lim_{t \to a} f(t)\right] \hat{\mathbf{i}} + \left[\lim_{t \to a} g(t)\right] \hat{\mathbf{j}}, \tag{13.1.1}$$

provided the limits $\lim_{t \to a} f(t)$ and $\lim_{t \to a} g(t)$ exist.

Similarly, the limit of the vector-valued function $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$ as t approaches a is given by

$$\lim_{t \to a} \vec{\mathbf{r}}(t) = \left[\lim_{t \to a} f(t)\right] \hat{\mathbf{i}} + \left[\lim_{t \to a} g(t)\right] \hat{\mathbf{j}} + \left[\lim_{t \to a} h(t)\right] \hat{\mathbf{k}}, \tag{13.1.2}$$

provided the limits $\lim_{t \to a} f(t)$, $\lim_{t \to a} g(t)$ and $\lim_{t \to a} h(t)$ exist.

In the following example, we show how to calculate the limit of a vector-valued function.

Example 13.1.3: Evaluating the Limit of a Vector-Valued Function

For each of the following vector-valued functions, calculate $\lim\limits_{t\to 3}\vec{\mathbf{r}}(t)$ for

a. $\vec{\mathbf{r}}(t)=(t^2-3t+4)\hat{\mathbf{i}}+(4t+3)\hat{\mathbf{j}}$
b. $\vec{\mathbf{r}}(t)=\frac{2t-4}{t+1}\hat{\mathbf{i}}+\frac{t}{t^2+1}\hat{\mathbf{j}}+(4t-3)\hat{\mathbf{k}}$

Solution

a. Use Equation 13.1.1 and substitute the value $t=3$ into the two component expressions:

$$\begin{aligned}\lim_{t\to 3}\vec{\mathbf{r}}(t) &= \lim_{t\to 3}\left[(t^2-3t+4)\hat{\mathbf{i}}+(4t+3)\hat{\mathbf{j}}\right]\\ &= \left[\lim_{t\to 3}(t^2-3t+4)\right]\hat{\mathbf{i}}+\left[\lim_{t\to 3}(4t+3)\right]\hat{\mathbf{j}}\\ &= 4\hat{\mathbf{i}}+15\hat{\mathbf{j}}\end{aligned}$$

b. Use Equation 13.1.2 and substitute the value $t=3$ into the three component expressions:

$$\begin{aligned}\lim_{t\to 3}\vec{\mathbf{r}}(t) &= \lim_{t\to 3}\left(\frac{2t-4}{t+1}\hat{\mathbf{i}}+\frac{t}{t^2+1}\hat{\mathbf{j}}+(4t-3)\hat{\mathbf{k}}\right)\\ &= \left[\lim_{t\to 3}\left(\frac{2t-4}{t+1}\right)\right]\hat{\mathbf{i}}+\left[\lim_{t\to 3}\left(\frac{t}{t^2+1}\right)\right]\hat{\mathbf{j}}+\left[\lim_{t\to 3}(4t-3)\right]\hat{\mathbf{k}}\\ &= \tfrac{1}{2}\hat{\mathbf{i}}+\tfrac{3}{10}\hat{\mathbf{j}}+9\hat{\mathbf{k}}\end{aligned}$$

Exercise 13.1.3

Calculate $\lim\limits_{t\to 2}\vec{\mathbf{r}}(t)$ for the function $\vec{\mathbf{r}}(t)=\sqrt{t^2+3t-1}\ \hat{\mathbf{i}}-(4t-3)\hat{\mathbf{j}}-\sin\frac{(t+1)\pi}{2}\hat{\mathbf{k}}$

Hint

Use Equation 13.1.2 from the preceding theorem.

Answer

$$\lim_{t\to 2}\vec{\mathbf{r}}(t)=3\hat{\mathbf{i}}-5\hat{\mathbf{j}}+\hat{\mathbf{k}}$$

Now that we know how to calculate the limit of a vector-valued function, we can define **continuity at a point** for such a function.

Definitions

Let f, g, and h be functions of t. Then, the vector-valued function $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}$ is **continuous at point $t=a$** if the following three conditions hold:

1. $\vec{\mathbf{r}}(a)$ exists
2. $\lim\limits_{t\to a}\vec{\mathbf{r}}(t)$ exists
3. $\lim\limits_{t\to a}\vec{\mathbf{r}}(t)=\vec{\mathbf{r}}(a)$

Similarly, the vector-valued function $\vec{\mathbf{r}}(t)=f(t)\hat{\mathbf{i}}+g(t)\hat{\mathbf{j}}+h(t)\hat{\mathbf{k}}$ is **continuous at point $t=a$** if the following three conditions hold:

1. $\vec{\mathbf{r}}(a)$ exists
2. $\lim\limits_{t\to a}\vec{\mathbf{r}}(t)$ exists
3. $\lim\limits_{t\to a}\vec{\mathbf{r}}(t)=\vec{\mathbf{r}}(a)$

Summary

- A vector-valued function is a function of the form $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}}$ or $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$, where the component functions f, g, and h are real-valued functions of the parameter t.
- The graph of a vector-valued function of the form $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}}$ is called a *plane curve*. The graph of a vector-valued function of the form $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$ is called a *space curve*.
- It is possible to represent an arbitrary plane curve by a vector-valued function.
- To calculate the limit of a vector-valued function, calculate the limits of the component functions separately.

Key Equations

- **Vector-valued function**
  $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}}$ or $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$,or $\vec{\mathbf{r}}(t) = \langle f(t), g(t)\rangle$ or $\vec{\mathbf{r}}(t) = \langle f(t), g(t), h(t)\rangle$
- **Limit of a vector-valued function**
  $\lim\limits_{t\to a}\vec{\mathbf{r}}(t) = [\lim\limits_{t\to a} f(t)]\,\hat{\mathbf{i}} + [\lim\limits_{t\to a} g(t)]\,\hat{\mathbf{j}}$ or $\lim\limits_{t\to a}\vec{\mathbf{r}}(t) = [\lim\limits_{t\to a} f(t)]\,\hat{\mathbf{i}} + [\lim\limits_{t\to a} g(t)]\,\hat{\mathbf{j}} + [\lim\limits_{t\to a} h(t)]\hat{\mathbf{k}}$

Glossary

component functions

the component functions of the vector-valued function $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}}$ are $f(t)$ and $g(t)$, and the component functions of the vector-valued function $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$ are $f(t)$, $g(t)$ and $h(t)$

helix

a three-dimensional curve in the shape of a spiral

limit of a vector-valued function

a vector-valued function $\vec{\mathbf{r}}(t)$ has a limit $\vec{\mathbf{L}}$ as t approaches a if $\lim t \to a \left|\vec{\mathbf{r}}(t) - \vec{\mathbf{L}}\right| = 0$

plane curve

the set of ordered pairs $(f(t), g(t))$ together with their defining parametric equations $x = f(t)$ and $y = g(t)$

reparameterization

an alternative parameterization of a given vector-valued function

space curve

the set of ordered triples $(f(t), g(t), h(t))$ together with their defining parametric equations $x = f(t)$, $y = g(t)$ and $z = h(t)$

vector parameterization

any representation of a plane or space curve using a vector-valued function

vector-valued function

a function of the form $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}}$ or $\vec{\mathbf{r}}(t) = f(t)\hat{\mathbf{i}} + g(t)\hat{\mathbf{j}} + h(t)\hat{\mathbf{k}}$,where the component functions f, g, and h are real-valued functions of the parameter t.

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13.2: Derivatives and Integrals of Vector Functions

Learning Objectives

- Write an expression for the derivative of a vector-valued function.
- Find the tangent vector at a point for a given position vector.
- Find the unit tangent vector at a point for a given position vector and explain its significance.
- Calculate the definite integral of a vector-valued function.

To study the calculus of vector-valued functions, we follow a similar path to the one we took in studying real-valued functions. First, we define the derivative, then we examine applications of the derivative, then we move on to defining integrals. However, we will find some interesting new ideas along the way as a result of the vector nature of these functions and the properties of space curves.

Derivatives of Vector-Valued Functions

Now that we have seen what a vector-valued function is and how to take its limit, the next step is to learn how to differentiate a vector-valued function. The definition of the derivative of a vector-valued function is nearly identical to the definition of a real-valued function of one variable. However, because the range of a vector-valued function consists of vectors, the same is true for the range of the derivative of a vector-valued function.

Definition: Derivative of Vector-Valued Functions

The derivative of a vector-valued function $\vec{\mathbf{r}}(t)$ is

$$\vec{\mathbf{r}}\,'(t) = \lim_{\Delta t \to 0} \frac{\vec{\mathbf{r}}(t+\Delta t) - \vec{\mathbf{r}}(t)}{\Delta t} \tag{13.2.1}$$

provided the limit exists. If $\vec{\mathbf{r}}\,'(t)$ exists, then $\vec{\mathbf{r}}(t)$ is differentiable at t. If $\vec{\mathbf{r}}\,'(t)$ exists for all t in an open interval (a, b) then $\vec{\mathbf{r}}(t)$ is differentiable over the interval (a, b). For the function to be differentiable over the closed interval $[a, b]$, the following two limits must exist as well:

$$\vec{\mathbf{r}}\,'(a) = \lim_{\Delta t \to 0^+} \frac{\vec{\mathbf{r}}(a+\Delta t) - \vec{\mathbf{r}}(a)}{\Delta t}$$

and

$$\vec{\mathbf{r}}\,'(b) = \lim_{\Delta t \to 0^-} \frac{\vec{\mathbf{r}}(b+\Delta t) - \vec{\mathbf{r}}(b)}{\Delta t}$$

Many of the rules for calculating derivatives of real-valued functions can be applied to calculating the derivatives of vector-valued functions as well. Recall that the derivative of a real-valued function can be interpreted as the slope of a tangent line or the instantaneous rate of change of the function. The derivative of a vector-valued function can be understood to be an instantaneous rate of change as well; for example, when the function represents the position of an object at a given point in time, the derivative represents its velocity at that same point in time.

We now demonstrate taking the derivative of a vector-valued function.

Example 13.2.1: Finding the Derivative of a Vector-Valued Function

Use the definition to calculate the derivative of the function

$$\vec{\mathbf{r}}(t) = (3t+4)\,\hat{\mathbf{i}} + (t^2 - 4t + 3)\,\hat{\mathbf{j}}.$$

Solution

Let's use Equation 13.2.1:

$$
\begin{aligned}
\vec{\mathbf{r}}\,'(t) &= \lim_{\Delta t \to 0} \frac{\vec{\mathbf{r}}(t+\Delta t) - \vec{\mathbf{r}}(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{[(3(t+\Delta t)+4)\,\hat{\mathbf{i}} + ((t+\Delta t)^2 - 4(t+\Delta t)+3)\,\hat{\mathbf{j}}] - [(3t+4)\,\hat{\mathbf{i}} + (t^2-4t+3)\,\hat{\mathbf{j}}]}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{(3t+3\Delta t+4)\,\hat{\mathbf{i}} - (3t+4)\,\hat{\mathbf{i}} + (t^2+2t\Delta t+(\Delta t)^2-4t-4\Delta t+3)\,\hat{\mathbf{j}} - (t^2-4t+3)\,\hat{\mathbf{j}}}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{(3\Delta t)\,\hat{\mathbf{i}} + (2t\Delta t + (\Delta t)^2 - 4\Delta t)\,\hat{\mathbf{j}}}{\Delta t} \\
&= \lim_{\Delta t \to 0} (3\,\hat{\mathbf{i}} + (2t+\Delta t-4)\,\hat{\mathbf{j}}) \\
&= 3\,\hat{\mathbf{i}} + (2t-4)\,\hat{\mathbf{j}}
\end{aligned}
$$

? Exercise 13.2.1

Use the definition to calculate the derivative of the function $\vec{\mathbf{r}}(t) = (2t^2+3)\,\hat{\mathbf{i}} + (5t-6)\,\hat{\mathbf{j}}$.

Hint

Use Equation 13.2.1.

Answer

$$\vec{\mathbf{r}}\,'(t) = 4t\,\hat{\mathbf{i}} + 5\,\hat{\mathbf{j}}$$

Notice that in the calculations in Example 13.2.1, we could also obtain the answer by first calculating the derivative of each component function, then putting these derivatives back into the vector-valued function. This is always true for calculating the derivative of a vector-valued function, whether it is in two or three dimensions. We state this in the following theorem. The proof of this theorem follows directly from the definitions of the limit of a vector-valued function and the derivative of a vector-valued function.

Theorem 13.2.1: Differentiation of Vector-Valued Functions

Let f, g, and h be differentiable functions of t.

1. If $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}$ then
$$\vec{\mathbf{r}}\,'(t) = f\,'(t)\,\hat{\mathbf{i}} + g'(t)\,\hat{\mathbf{j}}.$$
2. If $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}$ then
$$\vec{\mathbf{r}}\,'(t) = f\,'(t)\,\hat{\mathbf{i}} + g'(t)\,\hat{\mathbf{j}} + h\,'(t)\,\hat{\mathbf{k}}.$$

✓ Example 13.2.2: Calculating the Derivative of Vector-Valued Functions

Use Theorem 13.2.1 to calculate the derivative of each of the following functions.

a. $\vec{\mathbf{r}}(t) = (6t+8)\,\hat{\mathbf{i}} + (4t^2+2t-3)\,\hat{\mathbf{j}}$
b. $\vec{\mathbf{r}}(t) = 3\cos t\,\hat{\mathbf{i}} + 4\sin t\,\hat{\mathbf{j}}$
c. $\vec{\mathbf{r}}(t) = e^t \sin t\,\hat{\mathbf{i}} + e^t \cos t\,\hat{\mathbf{j}} - e^{2t}\,\hat{\mathbf{k}}$

Solution

We use Theorem 13.2.1 and what we know about differentiating functions of one variable.

a. The first component of
$$\vec{\mathbf{r}}(t) = (6t+8)\,\hat{\mathbf{i}} + (4t^2+2t-3)\,\hat{\mathbf{j}}$$

is $f(t)=6t+8$. The second component is $g(t)=4t^2+2t-3$. We have $f'(t)=6$ and $g'(t)=8t+2$, so the Theorem 13.2.1 gives $\vec{\mathbf{r}}\,'(t)=6\,\hat{\mathbf{i}}+(8t+2)\,\hat{\mathbf{j}}$.

b. The first component is $f(t)=3\cos t$ and the second component is $g(t)=4\sin t$. We have $f'(t)=-3\sin t$ and $g'(t)=4\cos t$, so we obtain $\vec{\mathbf{r}}\,'(t)=-3\sin t\,\hat{\mathbf{i}}+4\cos t\,\hat{\mathbf{j}}$.

c. The first component of $\vec{\mathbf{r}}(t)=e^t\sin t\,\hat{\mathbf{i}}+e^t\cos t\,\hat{\mathbf{j}}-e^{2t}\,\hat{\mathbf{k}}$ is $f(t)=e^t\sin t$, the second component is $g(t)=e^t\cos t$, and the third component is $h(t)=-e^{2t}$. We have $f'(t)=e^t(\sin t+\cos t)$, $g'(t)=e^t(\cos t-\sin t)$, and $h'(t)=-2e^{2t}$, so the theorem gives $\vec{\mathbf{r}}\,'(t)=e^t(\sin t+\cos t)\,\hat{\mathbf{i}}+e^t(\cos t-\sin t)\,\hat{\mathbf{j}}-2e^{2t}\,\hat{\mathbf{k}}$.

? Exercise 13.2.2

Calculate the derivative of the function

$$\vec{\mathbf{r}}(t)=(t\ln t)\,\hat{\mathbf{i}}+(5e^t)\,\hat{\mathbf{j}}+(\cos t-\sin t)\,\hat{\mathbf{k}}.$$

Hint

Identify the component functions and use Theorem 13.2.1

Answer

$$\vec{\mathbf{r}}\,'(t)=(1+\ln t)\,\hat{\mathbf{i}}+5e^t\,\hat{\mathbf{j}}-(\sin t+\cos t)\,\hat{\mathbf{k}}$$

We can extend to vector-valued functions the properties of the derivative that we presented previously. In particular, the constant multiple rule, the sum and difference rules, the product rule, and the chain rule all extend to vector-valued functions. However, in the case of the product rule, there are actually three extensions:

1. for a real-valued function multiplied by a vector-valued function,
2. for the dot product of two vector-valued functions, and
3. for the cross product of two vector-valued functions.

Theorem: Properties of the Derivative of Vector-Valued Functions

Let $\vec{\mathbf{r}}$ and $\vec{\mathbf{u}}$ be differentiable vector-valued functions of t, let f be a differentiable real-valued function of t, and let c be a scalar.

i.	$\frac{d}{dt}[c\vec{\mathbf{r}}(t)]$	$=$	$c\vec{\mathbf{r}}\,'(t)$	Scalar multiple
ii.	$\frac{d}{dt}[\vec{\mathbf{r}}(t)\pm\vec{\mathbf{u}}(t)]$	$=$	$\vec{\mathbf{r}}\,'(t)\pm\vec{\mathbf{u}}\,'(t)$	Sum and difference
iii.	$\frac{d}{dt}[f(t)\vec{\mathbf{u}}(t)]$	$=$	$f'(t)\vec{\mathbf{u}}(t)+f(t)\vec{\mathbf{u}}\,'(t)$	Scalar product
iv.	$\frac{d}{dt}[\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{u}}(t)]$	$=$	$\vec{\mathbf{r}}\,'(t)\cdot\vec{\mathbf{u}}(t)+\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{u}}\,'(t)$	Dot product
v.	$\frac{d}{dt}[\vec{\mathbf{r}}(t)\times\vec{\mathbf{u}}(t)]$	$=$	$\vec{\mathbf{r}}\,'(t)\times\vec{\mathbf{u}}(t)+\vec{\mathbf{r}}(t)\times\vec{\mathbf{u}}\,'(t)$	Cross product
vi.	$\frac{d}{dt}[\vec{\mathbf{r}}(f(t))]$	$=$	$\vec{\mathbf{r}}\,'(f(t))\cdot f'(t)$	Chain rule
vii.	If $\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}(t)$	$=$	c, then $\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}\,'(t)=0$.	

Proof

The proofs of the first two properties follow directly from the definition of the derivative of a vector-valued function. The third property can be derived from the first two properties, along with the product rule. Let $\vec{\mathbf{u}}(t)=g(t)\,\hat{\mathbf{i}}+h(t)\,\hat{\mathbf{j}}$. Then

$$\begin{aligned}\frac{d}{dt}[f(t)\vec{\mathbf{u}}(t)] &= \frac{d}{dt}[f(t)(g(t)\,\hat{\mathbf{i}} + h(t)\,\hat{\mathbf{j}})] \\ &= \frac{d}{dt}[f(t)g(t)\,\hat{\mathbf{i}} + f(t)h(t)\,\hat{\mathbf{j}}] \\ &= \frac{d}{dt}[f(t)g(t)]\,\hat{\mathbf{i}} + \frac{d}{dt}[f(t)h(t)]\,\hat{\mathbf{j}} \\ &= (f'(t)g(t) + f(t)g'(t))\,\hat{\mathbf{i}} + (f'(t)h(t) + f(t)h'(t))\,\hat{\mathbf{j}} \\ &= f'(t)\vec{\mathbf{u}}(t) + f(t)\vec{\mathbf{u}}'(t).\end{aligned}$$

To prove property iv. let $\vec{\mathbf{r}}(t) = f_1(t)\,\hat{\mathbf{i}} + g_1(t)\,\hat{\mathbf{j}}$ and $\vec{\mathbf{u}}(t) = f_2(t)\,\hat{\mathbf{i}} + g_2(t)\,\hat{\mathbf{j}}$. Then

$$\begin{aligned}\frac{d}{dt}[\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{u}}(t)] &= \frac{d}{dt}[f_1(t)f_2(t) + g_1(t)g_2(t)] \\ &= f_1'(t)f_2(t) + f_1(t)f_2'(t) + g_1'(t)g_2(t) + g_1(t)g_2'(t) = f_1'(t)f_2(t) + g_1'(t)g_2(t) + f_1(t)f_2'(t) + g_1(t)g_2'(t) \\ &= (f_1'\,\hat{\mathbf{i}} + g_1'\,\hat{\mathbf{j}})\cdot(f_2\,\hat{\mathbf{i}} + g_2\,\hat{\mathbf{j}}) + (f_1\,\hat{\mathbf{i}} + g_1\,\hat{\mathbf{j}})\cdot(f_2'\,\hat{\mathbf{i}} + g_2'\,\hat{\mathbf{j}}) \\ &= \vec{\mathbf{r}}'(t)\cdot\vec{\mathbf{u}}(t) + \vec{\mathbf{r}}(t)\cdot\vec{\mathbf{u}}'(t).\end{aligned}$$

The proof of property v. is similar to that of property iv. Property vi. can be proved using the chain rule. Last, property vii. follows from property iv:

$$\begin{aligned}\frac{d}{dt}[\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}(t)] &= \frac{d}{dt}[c] \\ \vec{\mathbf{r}}'(t)\cdot\vec{\mathbf{r}}(t) + \vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}'(t) &= 0 \\ 2\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}'(t) &= 0 \\ \vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}'(t) &= 0\end{aligned}$$

Now for some examples using these properties.

Example 13.2.3: Using the Properties of Derivatives of Vector-Valued Functions

Given the vector-valued functions

$$\vec{\mathbf{r}}(t) = (6t+8)\,\hat{\mathbf{i}} + (4t^2+2t-3)\,\hat{\mathbf{j}} + 5t\,\hat{\mathbf{k}}$$

and

$$\vec{\mathbf{u}}(t) = (t^2-3)\,\hat{\mathbf{i}} + (2t+4)\,\hat{\mathbf{j}} + (t^3-3t)\,\hat{\mathbf{k}},$$

calculate each of the following derivatives using the properties of the derivative of vector-valued functions.

a. $\dfrac{d}{dt}[\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{u}}(t)]$

b. $\dfrac{d}{dt}[\vec{\mathbf{u}}(t)\times\vec{\mathbf{u}}'(t)]$

Solution

We have $\vec{\mathbf{r}}'(t) = 6\,\hat{\mathbf{i}} + (8t+2)\,\hat{\mathbf{j}} + 5\,\hat{\mathbf{k}}$ and $\vec{\mathbf{u}}'(t) = 2t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + (3t^2-3)\,\hat{\mathbf{k}}$. Therefore, according to **property iv**:

1.

$$\begin{aligned}\frac{d}{dt}[\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{u}}(t)] &= \vec{\mathbf{r}}\,'(t)\cdot\vec{\mathbf{u}}(t)+\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{u}}\,'(t)\\ &= (6\,\hat{\mathbf{i}}+(8t+2)\,\hat{\mathbf{j}}+5\,\hat{\mathbf{k}})\cdot((t^2-3)\,\hat{\mathbf{i}}+(2t+4)\,\hat{\mathbf{j}}+(t^3-3t)\,\hat{\mathbf{k}})\\ &\quad+((6t+8)\,\hat{\mathbf{i}}+(4t^2+2t-3)\,\hat{\mathbf{j}}+5t\,\hat{\mathbf{k}})\cdot(2t\,\hat{\mathbf{i}}+2\,\hat{\mathbf{j}}+(3t^2-3)\,\hat{\mathbf{k}})\\ &= 6(t^2-3)+(8t+2)(2t+4)+5(t^3-3t)\\ &\quad+2t(6t+8)+2(4t^2+2t-3)+5t(3t^2-3)\\ &= 20t^3+42t^2+26t-16.\end{aligned}$$

2. First, we need to adapt **property v** for this problem:

$$\frac{d}{dt}[\vec{\mathbf{u}}(t)\times\vec{\mathbf{u}}\,'(t)] = \vec{\mathbf{u}}\,'(t)\times\vec{\mathbf{u}}\,'(t)+\vec{\mathbf{u}}(t)\times\vec{\mathbf{u}}\,''(t).$$

Recall that the cross product of any vector with itself is zero. Furthermore, $\vec{\mathbf{u}}\,''(t)$ represents the second derivative of $\vec{\mathbf{u}}(t)$:

$$\vec{\mathbf{u}}\,''(t) = \frac{d}{dt}[\vec{\mathbf{u}}\,'(t)] = \frac{d}{dt}[2t\,\hat{\mathbf{i}}+2\,\hat{\mathbf{j}}+(3t^2-3)\,\hat{\mathbf{k}}] = 2\,\hat{\mathbf{i}}+6t\,\hat{\mathbf{k}}.$$

Therefore,

$$\begin{aligned}\frac{d}{dt}[\vec{\mathbf{u}}(t)\times\vec{\mathbf{u}}\,'(t)] &= 0+((t^2-3)\,\hat{\mathbf{i}}+(2t+4)\,\hat{\mathbf{j}}+(t^3-3t)\,\hat{\mathbf{k}})\times(2\,\hat{\mathbf{i}}+6t\,\hat{\mathbf{k}})\\ &= \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ t^2-3 & 2t+4 & t^3-3t\\ 2 & 0 & 6t\end{vmatrix}\\ &= 6t(2t+4)\,\hat{\mathbf{i}}-(6t(t^2-3)-2(t^3-3t))\,\hat{\mathbf{j}}-2(2t+4)\,\hat{\mathbf{k}}\\ &= (12t^2+24t)\,\hat{\mathbf{i}}+(12t-4t^3)\,\hat{\mathbf{j}}-(4t+8)\,\hat{\mathbf{k}}.\end{aligned}$$

? Exercise 13.2.3

Calculate $\frac{d}{dt}[\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}\,'(t)]$ and $\frac{d}{dt}[\vec{\mathbf{u}}(t)\times\vec{\mathbf{r}}(t)]$ for the vector-valued functions:

- $\vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}}+\sin t\,\hat{\mathbf{j}}-e^{2t}\,\hat{\mathbf{k}}$
- $\vec{\mathbf{u}}(t) = t\,\hat{\mathbf{i}}+\sin t\,\hat{\mathbf{j}}+\cos t\,\hat{\mathbf{k}}$,

Hint

Follow the same steps as in Example 13.2.3

Answer

$$\frac{d}{dt}[\vec{\mathbf{r}}(t)\cdot\vec{\mathbf{r}}\,'(t)] = 8e^{4t}$$

$$\frac{d}{dt}[\vec{\mathbf{u}}(t)\times\vec{\mathbf{r}}(t)] = -(e^{2t}(\cos t+2\sin t)+\cos 2t)\,\hat{\mathbf{i}}+(e^{2t}(2t+1)-\sin 2t)\,\hat{\mathbf{j}}+(t\cos t+\sin t-\cos 2t)\,\hat{\mathbf{k}}$$

Tangent Vectors and Unit Tangent Vectors

Recall that the derivative at a point can be interpreted as the slope of the tangent line to the graph at that point. In the case of a vector-valued function, the derivative provides a tangent vector to the curve represented by the function. Consider the vector-valued function

$$\vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}}+\sin t\,\hat{\mathbf{j}} \tag{13.2.2}$$

The derivative of this function is

$$\vec{\mathbf{r}}\,'(t) = -\sin t\,\hat{\mathbf{i}}+\cos t\,\hat{\mathbf{j}}$$

If we substitute the value $t = \pi/6$ into both functions we get

$$\vec{\mathbf{r}}\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}\,\hat{\mathbf{i}} + \frac{1}{2}\,\hat{\mathbf{j}}$$

and

$$\vec{\mathbf{r}}\,'\left(\frac{\pi}{6}\right) = -\frac{1}{2}\,\hat{\mathbf{i}} + \frac{\sqrt{3}}{2}\,\hat{\mathbf{j}}.$$

The graph of this function appears in Figure 13.2.1, along with the vectors $\vec{\mathbf{r}}\left(\frac{\pi}{6}\right)$ and $\vec{\mathbf{r}}'\left(\frac{\pi}{6}\right)$.

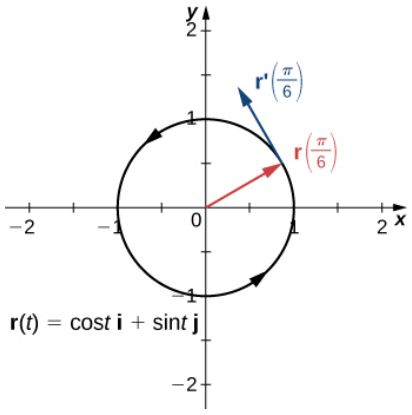


Figure 13.2.1: The tangent line at a point is calculated from the derivative of the vector-valued function $\vec{\mathbf{r}}(t)$.

Notice that the vector $\vec{\mathbf{r}}\,'\left(\frac{\pi}{6}\right)$ is tangent to the circle at the point corresponding to $t = \frac{\pi}{6}$. This is an example of a tangent vector to the plane curve defined by Equation 13.2.2.

Definition: principal unit tangent vector

Let C be a curve defined by a vector-valued function $\vec{\mathbf{r}}$, and assume that $\vec{\mathbf{r}}\,'(t)$ exists when $\mathrm{t} = \mathrm{t}_0$ A tangent vector $\vec{\mathbf{r}}$ at $t = t_0$ is any vector such that, when the tail of the vector is placed at point $\vec{\mathbf{r}}(t_0)$ on the graph, vector $\vec{\mathbf{r}}$ is tangent to curve C. Vector $\vec{\mathbf{r}}\,'(t_0)$ is an example of a tangent vector at point $t = t_0$. Furthermore, assume that $\vec{\mathbf{r}}\,'(t) \neq 0$. The principal unit tangent vector at t is defined to be

$$\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{r}}\,'(t)}{\|\vec{\mathbf{r}}\,'(t)\|},$$

provided $\|\vec{\mathbf{r}}\,'(t)\| \neq 0$.

The unit tangent vector is exactly what it sounds like: a unit vector that is tangent to the curve. To calculate a unit tangent vector, first find the derivative $\vec{\mathbf{r}}\,'(t)$. Second, calculate the magnitude of the derivative. The third step is to divide the derivative by its magnitude.

Example 13.2.4: Finding a Unit Tangent Vector

Find the unit tangent vector for each of the following vector-valued functions:

a. $\vec{\mathbf{r}}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}$
b. $\vec{\mathbf{u}}(t) = (3t^2 + 2t)\,\hat{\mathbf{i}} + (2 - 4t^3)\,\hat{\mathbf{j}} + (6t + 5)\,\hat{\mathbf{k}}$

Solution

a.

$$\begin{aligned}
\text{First step:} \quad \vec{\mathbf{r}}\,'(t) &= -\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}} \\
\text{Second step:} \quad \|\vec{\mathbf{r}}\,'(t)\| &= \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \\
\text{Third step:} \quad \vec{\mathbf{T}}(t) &= \frac{\vec{\mathbf{r}}\,'(t)}{\|\vec{\mathbf{r}}\,'(t)\|} = \frac{-\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}}}{1} = -\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}}
\end{aligned}$$

b.

$$\begin{aligned}
\text{First step:} \quad \vec{\mathbf{r}}\,'(t) &= (6t+2)\,\hat{\mathbf{i}} - 12t^2\,\hat{\mathbf{j}} + 6\,\hat{\mathbf{k}} \\
\text{Second step:} \quad \|\vec{\mathbf{r}}\,'(t)\| &= \sqrt{(6t+2)^2 + (-12t^2)^2 + 6^2} \\
&= \sqrt{144t^4 + 36t^2 + 24t + 40} \\
&= 2\sqrt{36t^4 + 9t^2 + 6t + 10} \\
\text{Third step:} \quad \vec{\mathbf{T}}(t) &= \frac{\vec{\mathbf{r}}\,'(t)}{\|\vec{\mathbf{r}}\,'(t)\|} = \frac{(6t+2)\,\hat{\mathbf{i}} - 12t^2\,\hat{\mathbf{j}} + 6\,\hat{\mathbf{k}}}{2\sqrt{36t^4 + 9t^2 + 6t + 10}} \\
&= \frac{3t+1}{\sqrt{36t^4 + 9t^2 + 6t + 10}}\,\hat{\mathbf{i}} - \frac{6t^2}{\sqrt{36t^4 + 9t^2 + 6t + 10}}\,\hat{\mathbf{j}} + \frac{3}{\sqrt{36t^4 + 9t^2 + 6t + 10}}\,\hat{\mathbf{k}}
\end{aligned}$$

Exercise 13.2.4

Find the unit tangent vector for the vector-valued function

$$\vec{\mathbf{r}}(t) = (t^2 - 3)\,\hat{\mathbf{i}} + (2t+1)\,\hat{\mathbf{j}} + (t-2)\,\hat{\mathbf{k}}.$$

Hint

Follow the same steps as in Example 13.2.4

Answer

$$\vec{\mathbf{T}}(t) = \frac{2t}{\sqrt{4t^2+5}}\,\hat{\mathbf{i}} + \frac{2}{\sqrt{4t^2+5}}\,\hat{\mathbf{j}} + \frac{1}{\sqrt{4t^2+5}}\,\hat{\mathbf{k}}$$

Integrals of Vector-Valued Functions

We introduced antiderivatives of real-valued functions in Antiderivatives and definite integrals of real-valued functions in The Definite Integral. Each of these concepts can be extended to vector-valued functions. Also, just as we can calculate the derivative of a vector-valued function by differentiating the component functions separately, we can calculate the antiderivative in the same manner. Furthermore, the Fundamental Theorem of Calculus applies to vector-valued functions as well.

The antiderivative of a vector-valued function appears in applications. For example, if a vector-valued function represents the velocity of an object at time t, then its antiderivative represents position. Or, if the function represents the acceleration of the object at a given time, then the antiderivative represents its velocity.

Definition: Definite and Indefinite Integrals of Vector-Valued Functions

Let f, g, and h be integrable real-valued functions over the closed interval $[a, b]$.

1. The indefinite integral of a vector-valued function $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}$ is

$$\int [f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}]\,dt = \left[\int f(t)\,dt\right]\,\hat{\mathbf{i}} + \left[\int g(t)\,dt\right]\,\hat{\mathbf{j}}.$$

The definite integral of a vector-valued function is

$$\int_a^b [f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}]\,dt = \left[\int_a^b f(t)\,dt\right]\,\hat{\mathbf{i}} + \left[\int_a^b g(t)\,dt\right]\,\hat{\mathbf{j}}.$$

2. The indefinite integral of a vector-valued function $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}$ is

$$\int [f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}]\,dt = \left[\int f(t)\,dt\right]\,\hat{\mathbf{i}} + \left[\int g(t)\,dt\right]\,\hat{\mathbf{j}} + \left[\int h(t)\,dt\right]\,\hat{\mathbf{k}}.$$

The definite integral of the vector-valued function is

$$\int_a^b [f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}]\,dt = \left[\int_a^b f(t)\,dt\right]\,\hat{\mathbf{i}} + \left[\int_a^b g(t)\,dt\right]\,\hat{\mathbf{j}} + \left[\int_a^b h(t)\,dt\right]\,\hat{\mathbf{k}}.$$

Since the indefinite integral of a vector-valued function involves indefinite integrals of the component functions, each of these component integrals contains an integration constant. They can all be different. For example, in the two-dimensional case, we can have

$$\int f(t)\,dt = F(t) + C_1 \text{ and } \int g(t)\,dt = G(t) + C_2,$$

where F and G are antiderivatives of f and g, respectively. Then

$$\begin{aligned}\int [f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}]\,dt &= \left[\int f(t)\,dt\right]\hat{\mathbf{i}} + \left[\int g(t)\,dt\right]\hat{\mathbf{j}} \\ &= (F(t)+C_1)\,\hat{\mathbf{i}} + (G(t)+C_2)\,\hat{\mathbf{j}} \\ &= F(t)\,\hat{\mathbf{i}} + G(t)\,\hat{\mathbf{j}} + C_1\,\hat{\mathbf{i}} + C_2\,\hat{\mathbf{j}} \\ &= F(t)\,\hat{\mathbf{i}} + G(t)\,\hat{\mathbf{j}} + \vec{\mathbf{C}}\end{aligned}$$

where $\vec{\mathbf{C}} = C_1\,\hat{\mathbf{i}} + C_2\,\hat{\mathbf{j}}$. Therefore, the *integration constants* becomes a *constant vector*.

Example 13.2.5: Integrating Vector-Valued Functions

Calculate each of the following integrals:

a. $\displaystyle\int [(3t^2+2t)\,\hat{\mathbf{i}} + (3t-6)\,\hat{\mathbf{j}} + (6t^3+5t^2-4)\,\hat{\mathbf{k}}]\,dt$

b. $\displaystyle\int [\langle t, t^2, t^3\rangle \times \langle t^3, t^2, t\rangle]\,dt$

c. $\displaystyle\int_0^{\frac{\pi}{3}} [\sin 2t\,\hat{\mathbf{i}} + \tan t\,\hat{\mathbf{j}} + e^{-2t}\,\hat{\mathbf{k}}]\,dt$

Solution

a. We use the first part of the definition of the integral of a space curve:

b. $$\begin{aligned}\int [(3t^2+2t)\,\hat{\mathbf{i}} + (3t-6)\,\hat{\mathbf{j}} + (6t^3+5t^2-4)\,\hat{\mathbf{k}}]\,dt &= \left[\int 3t^2+2t\,dt\right]\hat{\mathbf{i}} + \left[\int 3t-6\,dt\right]\hat{\mathbf{j}} + \left[\int 6t^3+5t^2-4\,dt\right]\hat{\mathbf{k}} \\ &= (t^3+t^2)\,\hat{\mathbf{i}} + \left(\frac{3}{2}t^2 - 6t\right)\hat{\mathbf{j}} + \left(\frac{3}{2}t^4 + \frac{5}{3}t^3 - 4t\right)\hat{\mathbf{k}} + \vec{\mathbf{C}}.\end{aligned}$$

c. First calculate $\langle t, t^2, t^3\rangle \times \langle t^3, t^2, t\rangle$:

$$\begin{aligned}\langle t, t^2, t^3\rangle \times \langle t^3, t^2, t\rangle &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ t & t^2 & t^3 \\ t^3 & t^2 & t \end{vmatrix} \\ &= (t^2(t) - t^3(t^2))\,\hat{\mathbf{i}} - (t^2 - t^3(t^3))\,\hat{\mathbf{j}} + (t(t^2) - t^2(t^3))\,\hat{\mathbf{k}} \\ &= (t^3 - t^5)\,\hat{\mathbf{i}} + (t^6 - t^2)\,\hat{\mathbf{j}} + (t^3 - t^5)\,\hat{\mathbf{k}}.\end{aligned}$$

Next, substitute this back into the integral and integrate:

$$\begin{aligned}\int [\langle t, t^2, t^3\rangle \times \langle t^3, t^2, t\rangle]\,dt &= \int (t^3 - t^5)\,\hat{\mathbf{i}} + (t^6 - t^2)\,\hat{\mathbf{j}} + (t^3 - t^5)\,\hat{\mathbf{k}}\,dt \\ &= \left(\frac{t^4}{4} - \frac{t^6}{6}\right)\hat{\mathbf{i}} + \left(\frac{t^7}{7} - \frac{t^3}{3}\right)\hat{\mathbf{j}} + \left(\frac{t^4}{4} - \frac{t^6}{6}\right)\hat{\mathbf{k}} + \vec{\mathbf{C}}.\end{aligned}$$

d. Use the second part of the definition of the integral of a space curve:

$$\begin{aligned}\int_0^{\frac{\pi}{3}} [\sin 2t\,\hat{\mathbf{i}} + \tan t\,\hat{\mathbf{j}} + e^{-2t}\,\hat{\mathbf{k}}]\,dt &= \left[\int_0^{\frac{\pi}{3}} \sin 2t\,dt\right]\hat{\mathbf{i}} + \left[\int_0^{\frac{\pi}{3}} \tan t\,dt\right]\hat{\mathbf{j}} + \left[\int_0^{\frac{\pi}{3}} e^{-2t}\,dt\right]\hat{\mathbf{k}} \\ &= \left(-\tfrac{1}{2}\cos 2t\right)\Big|_0^{\pi/3}\hat{\mathbf{i}} - \left(\ln|\cos t|\right)\Big|_0^{\pi/3}\hat{\mathbf{j}} - \left(\tfrac{1}{2}e^{-2t}\right)\Big|_0^{\pi/3}\hat{\mathbf{k}} \\ &= \left(-\tfrac{1}{2}\cos\tfrac{2\pi}{3} + \tfrac{1}{2}\cos 0\right)\hat{\mathbf{i}} - \left(\ln\left(\cos\tfrac{\pi}{3}\right) - \ln(\cos 0)\right)\hat{\mathbf{j}} - \left(\tfrac{1}{2}e^{-2\pi/3} - \tfrac{1}{2}e^{-2(0)}\right)\hat{\mathbf{k}} \\ &= \left(\tfrac{1}{4} + \tfrac{1}{2}\right)\hat{\mathbf{i}} - (-\ln 2)\,\hat{\mathbf{j}} - \left(\tfrac{1}{2}e^{-2\pi/3} - \tfrac{1}{2}\right)\hat{\mathbf{k}} \\ &= \tfrac{3}{4}\hat{\mathbf{i}} + (\ln 2)\,\hat{\mathbf{j}} + \left(\tfrac{1}{2} - \tfrac{1}{2}e^{-2\pi/3}\right)\hat{\mathbf{k}}.\end{aligned}$$

? Exercise 13.2.5

Calculate the following integral:

$$\int_1^3 [(2t+4)\,\hat{\mathbf{i}} + (3t^2 - 4t)\,\hat{\mathbf{j}}]\,dt$$

Hint

Use the definition of the definite integral of a plane curve.

Answer

$$\int_1^3 [(2t+4)\,\hat{\mathbf{i}} + (3t^2 - 4t)\,\hat{\mathbf{j}}]\,dt = 16\,\hat{\mathbf{i}} + 10\,\hat{\mathbf{j}}$$

Summary

- To calculate the derivative of a vector-valued function, calculate the derivatives of the component functions, then put them back into a new vector-valued function.
- Many of the properties of differentiation of scalar functions also apply to vector-valued functions.
- The derivative of a vector-valued function $\vec{\mathbf{r}}(t)$ is also a tangent vector to the curve. The unit tangent vector $\vec{\mathbf{T}}(t)$ is calculated by dividing the derivative of a vector-valued function by its magnitude.
- The antiderivative of a vector-valued function is found by finding the antiderivatives of the component functions, then putting them back together in a vector-valued function.
- The definite integral of a vector-valued function is found by finding the definite integrals of the component functions, then putting them back together in a vector-valued function.

Key Equations

- **Derivative of a vector-valued function**

$$\vec{\mathbf{r}}\,'(t) = \lim_{\Delta t \to 0} \frac{\vec{\mathbf{r}}(t+\Delta t) - \vec{\mathbf{r}}(t)}{\Delta t}$$

- **Principal unit tangent vector**

$$\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{r}}\,'(t)}{\|\vec{\mathbf{r}}\,'(t)\|}$$

- **Indefinite integral of a vector-valued function**

$$\int [f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}]\,dt = \left[\int f(t)\,dt\right]\hat{\mathbf{i}} + \left[\int g(t)\,dt\right]\hat{\mathbf{j}} + \left[\int h(t)\,dt\right]\hat{\mathbf{k}}$$

- **Definite integral of a vector-valued function**

$$\int_a^b [f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}]\,dt = \left[\int_a^b f(t)\,dt\right]\hat{\mathbf{i}} + \left[\int_a^b g(t)\,dt\right]\hat{\mathbf{j}} + \left[\int_a^b h(t)\,dt\right]\hat{\mathbf{k}}$$

Glossary

definite integral of a vector-valued function

the vector obtained by calculating the definite integral of each of the component functions of a given vector-valued function, then using the results as the components of the resulting function

derivative of a vector-valued function

the derivative of a vector-valued function $\vec{\mathbf{r}}(t)$ is $\vec{\mathbf{r}}\,'(t) = \lim_{\Delta t \to 0} \frac{\vec{\mathbf{r}}(t+\Delta t) - \vec{\mathbf{r}}(t)}{\Delta t}$, provided the limit exists

indefinite integral of a vector-valued function

a vector-valued function with a derivative that is equal to a given vector-valued function

principal unit tangent vector

a unit vector tangent to a curve C

tangent vector

to $\vec{\mathbf{r}}(t)$ at $t = t_0$ any vector $\vec{\mathbf{v}}$ such that, when the tail of the vector is placed at point $\vec{\mathbf{r}}(t_0)$ on the graph, vector $\vec{\mathbf{v}}$ is tangent to curve C

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13.3: Arc Length and Curvature

Learning Objectives

- Determine the length of a particle's path in space by using the arc-length function.
- Explain the meaning of the curvature of a curve in space and state its formula.
- Describe the meaning of the normal and binormal vectors of a curve in space.

In this section, we study formulas related to curves in both two and three dimensions, and see how they are related to various properties of the same curve. For example, suppose a vector-valued function describes the motion of a particle in space. We would like to determine how far the particle has traveled over a given time interval, which can be described by the arc length of the path it follows. Or, suppose that the vector-valued function describes a road we are building and we want to determine how sharply the road curves at a given point. This is described by the curvature of the function at that point. We explore each of these concepts in this section.

Arc Length for Vector Functions

We have seen how a vector-valued function describes a curve in either two or three dimensions. Recall that the formula for the arc length of a curve defined by the parametric functions $x = x(t), y = y(t), t_1 \leq t \leq t_2$ is given by

$$s = \int_{t_1}^{t_2} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

In a similar fashion, if we define a smooth curve using a vector-valued function $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}$, where $a \leq t \leq b$, the arc length is given by the formula

$$s = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt.$$

In three dimensions, if the vector-valued function is described by $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}$ over the same interval $a \leq t \leq b$, the arc length is given by

$$s = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, dt.$$

Theorem: Arc-Length Formulas for Plane and Space curves

Plane curve: Given a smooth curve C defined by the function $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}}$, where t lies within the interval $[a, b]$, the arc length of C over the interval is

$$s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt \tag{13.3.1}$$

$$= \int_a^b \|\vec{\mathbf{r}}\,'(t)\|\, dt. \tag{13.3.2}$$

Space curve: Given a smooth curve C defined by the function $\vec{\mathbf{r}}(t) = f(t)\,\hat{\mathbf{i}} + g(t)\,\hat{\mathbf{j}} + h(t)\,\hat{\mathbf{k}}$, where t lies within the interval $[a, b]$, the arc length of C over the interval is

$$s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt \tag{13.3.3}$$

$$= \int_a^b \|\vec{\mathbf{r}}\,'(t)\|\, dt. \tag{13.3.4}$$

The two formulas are very similar; they differ only in the fact that a space curve has three component functions instead of two. Note that the formulas are defined for smooth curves: curves where the vector-valued function $\vec{\mathbf{r}}(t)$ is differentiable with a non-

zero derivative. The smoothness condition guarantees that the curve has no cusps (or corners) that could make the formula problematic.

Example 13.3.1: Finding the Arc Length

Calculate the arc length for each of the following vector-valued functions:

a. $\vec{\mathbf{r}}(t) = (3t-2)\,\hat{\mathbf{i}} + (4t+5)\,\hat{\mathbf{j}}, \quad 1 \le t \le 5$
b. $\vec{\mathbf{r}}(t) = \langle t\cos t, t\sin t, 2t\rangle, 0 \le t \le 2\pi$

Solution

a. Using Equation 13.3.2, $\vec{\mathbf{r}}\,'(t) = 3\,\hat{\mathbf{i}} + 4\,\hat{\mathbf{j}}$, so

$$\begin{aligned} s &= \int_a^b \|\vec{\mathbf{r}}\,'(t)\|dt \\ &= \int_1^5 \sqrt{3^2+4^2}\,dt \\ &= \int_1^5 5dt = 5t\big|_1^5 = 20. \end{aligned}$$

b. Using Equation 13.3.4, $\vec{\mathbf{r}}\,'(t) = \langle \cos t - t\sin t, \sin t + t\cos t, 2\rangle$, so

$$\begin{aligned} s &= \int_a^b \| \vec{\mathbf{r}}\,'(t) \| \, dt \\ &= \int_0^{2\pi} \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 2^2}\; dt \\ &= \int_0^{2\pi} \sqrt{(\cos^2 t - 2t\sin t\cos t + t^2\sin^2 t) + (\sin^2 t + 2t\sin t\cos t + t^2\cos^2 t) + 4}\; dt \\ &= \int_0^{2\pi} \sqrt{\cos^2 t + \sin^2 t + t^2(\cos^2 t + \sin^2 t) + 4}\; dt \\ &= \int_0^{2\pi} \sqrt{t^2+5}\,dt \end{aligned}$$

Here we can use a table integration formula

$$\int \sqrt{u^2+a^2}\,du = \frac{u}{2}\sqrt{u^2+a^2} + \frac{a^2}{2}\ln\left|u+\sqrt{u^2+a^2}\right| + C,$$

so we obtain

$$\begin{aligned} \int_0^{2\pi} \sqrt{t^2+5}\,dt &= \frac{1}{2}\left(t\sqrt{t^2+5} + 5\ln\left|t+\sqrt{t^2+5}\right|\right)_0^{2\pi} \\ &= \frac{1}{2}\left(2\pi\sqrt{4\pi^2+5} + 5\ln\left(2\pi+\sqrt{4\pi^2+5}\right)\right) - \frac{5}{2}\ln\sqrt{5} \\ &\approx 25.343 \text{ units.} \end{aligned}$$

Exercise 13.3.1

Calculate the arc length of the parameterized curve

$$\vec{\mathbf{r}}(t) = \langle 2t^2+1, 2t^2-1, t^3\rangle, \quad 0 \le t \le 3.$$

Hint

Use Equation 13.3.4

Answer

$\vec{\mathbf{r}}\,'(t) = \langle 4t, 4t, 3t^2 \rangle$, so $s = \frac{1}{27}(113^{3/2} - 32^{3/2}) \approx 37.785$ units

We now return to the helix introduced earlier in this chapter. A vector-valued function that describes a helix can be written in the form

$$\vec{\mathbf{r}}(t) = R\cos\left(\frac{2\pi Nt}{h}\right)\hat{\mathbf{i}} + R\sin\left(\frac{2\pi Nt}{h}\right)\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}, 0 \le t \le h,$$

where R represents the radius of the helix, h represents the height (distance between two consecutive turns), and the helix completes N turns. Let's derive a formula for the arc length of this helix using Equation 13.3.4. First of all,

$$\vec{\mathbf{r}}\,'(t) = -\frac{2\pi NR}{h}\sin\left(\frac{2\pi Nt}{h}\right)\hat{\mathbf{i}} + \frac{2\pi NR}{h}\cos\left(\frac{2\pi Nt}{h}\right)\hat{\mathbf{j}} + \hat{\mathbf{k}}.$$

Therefore,

$$\begin{aligned}
s &= \int_a^b \|\vec{\mathbf{r}}\,'(t)\|dt \\
&= \int_0^h \sqrt{\left(-\frac{2\pi NR}{h}\sin\left(\frac{2\pi Nt}{h}\right)\right)^2 + \left(\frac{2\pi NR}{h}\cos\left(\frac{2\pi Nt}{h}\right)\right)^2 + 1^2}\,dt \\
&= \int_0^h \sqrt{\frac{4\pi^2N^2R^2}{h^2}\left(\sin^2\left(\frac{2\pi Nt}{h}\right) + \cos^2\left(\frac{2\pi Nt}{h}\right)\right) + 1}\,dt \\
&= \int_0^h \sqrt{\frac{4\pi^2N^2R^2}{h^2} + 1}\,dt \\
&= \left[t\sqrt{\frac{4\pi^2N^2R^2}{h^2} + 1}\right]_0^h \\
&= h\sqrt{\frac{4\pi^2N^2R^2 + h^2}{h^2}} \\
&= \sqrt{4\pi^2N^2R^2 + h^2}.
\end{aligned}$$

This gives a formula for the length of a wire needed to form a helix with N turns that has radius R and height h.

Arc-Length Parameterization

We now have a formula for the arc length of a curve defined by a vector-valued function. Let's take this one step further and examine what an **arc-length function** is.

If a vector-valued function represents the position of a particle in space as a function of time, then the arc-length function measures how far that particle travels as a function of time. The formula for the arc-length function follows directly from the formula for arc length:

$$s = \int_a^t \sqrt{(f'(u))^2 + (g'(u))^2 + (h'(u))^2}\,du. \tag{13.3.5}$$

If the curve is in two dimensions, then only two terms appear under the square root inside the integral. The reason for using the independent variable u is to distinguish between time and the variable of integration. Since $s(t)$ measures distance traveled as a function of time, $s'(t)$ measures the speed of the particle at any given time. Since we have a formula for $s(t)$ in Equation 13.3.5, we can differentiate both sides of the equation:

$$
\begin{aligned}
s'(t) &= \frac{d}{dt}\left[\int_a^t \sqrt{(f'(u))^2+(g'(u))^2+(h'(u))^2}\,du\right] \\
&= \frac{d}{dt}\left[\int_a^t \|\vec{\mathbf{r}}\,'(u)\|du\right] \\
&= \|\vec{\mathbf{r}}\,'(t)\|.
\end{aligned}
$$

If we assume that $\vec{\mathbf{r}}(t)$ defines a smooth curve, then the arc length is always increasing, so $s'(t) > 0$ for $t > a$. Last, if $\vec{\mathbf{r}}(t)$ is a curve on which $\|\vec{\mathbf{r}}\,'(t)\| = 1$ for all t, then

$$
s(t) = \int_a^t \|\vec{\mathbf{r}}\,'(u)\|\,du = \int_a^t 1\,du = t - a,
$$

which means that t represents the arc length as long as $a = 0$.

Theorem: Arc-Length Function

Let $\vec{\mathbf{r}}(t)$ describe a smooth curve for $t \geq a$. Then the arc-length function is given by

$$
s(t) = \int_a^t \|\vec{\mathbf{r}}\,'(u)\|\,du
$$

Furthermore,

$$
\frac{ds}{dt} = \|\vec{\mathbf{r}}\,'(t)\| > 0.
$$

If $\|\vec{\mathbf{r}}\,'(t)\| = 1$ for all $t \geq a$, then the parameter t represents the arc length from the starting point at $t = a$.

A useful application of this theorem is to find an alternative parameterization of a given curve, called an **arc-length parameterization**. Recall that any vector-valued function can be reparameterized via a change of variables. For example, if we have a function $\vec{\mathbf{r}}(t) = \langle 3\cos t, 3\sin t\rangle, 0 \leq t \leq 2\pi$ that parameterizes a circle of radius 3, we can change the parameter from t to $4t$, obtaining a new parameterization $\vec{\mathbf{r}}(t) = \langle 3\cos 4t, 3\sin 4t\rangle$. The new parameterization still defines a circle of radius 3, but now we need only use the values $0 \leq t \leq \pi/2$ to traverse the circle once.

Suppose that we find the arc-length function $s(t)$ and are able to solve this function for t as a function of s. We can then reparameterize the original function $\vec{\mathbf{r}}(t)$ by substituting the expression for t back into $\vec{\mathbf{r}}(t)$. The vector-valued function is now written in terms of the parameter s. Since the variable s represents the arc length, we call this an *arc-length parameterization* of the original function $\vec{\mathbf{r}}(t)$. One advantage of finding the arc-length parameterization is that the distance traveled along the curve starting from $s = 0$ is now equal to the parameter s. The arc-length parameterization also appears in the context of curvature (which we examine later in this section) and line integrals.

Example 13.3.2: Finding an Arc-Length Parameterization

Find the arc-length parameterization for each of the following curves:

a. $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 4\sin t\,\hat{\mathbf{j}}, \quad t \geq 0$
b. $\vec{\mathbf{r}}(t) = \langle t+3, 2t-4, 2t\rangle, \quad t \geq 3$

Solution

a. First we find the arc-length function using Equation 13.3.5:

$$\begin{aligned} s(t) &= \int_a^t \|\vec{\mathbf{r}}\,'(u)\|\,du \\ &= \int_0^t \|\langle -4\sin u, 4\cos u\rangle\|\,du \\ &= \int_0^t \sqrt{(-4\sin u)^2 + (4\cos u)^2}\,du \\ &= \int_0^t \sqrt{16\sin^2 u + 16\cos^2 u}\,du \\ &= \int_0^t 4\,du = 4t, \end{aligned}$$

which gives the relationship between the arc length s and the parameter t as $s = 4t$; so, $t = s/4$. Next we replace the variable t in the original function $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 4\sin t\,\hat{\mathbf{j}}$ with the expression $s/4$ to obtain

$$\vec{\mathbf{r}}(s) = 4\cos\left(\frac{s}{4}\right)\hat{\mathbf{i}} + 4\sin\left(\frac{s}{4}\right)\hat{\mathbf{j}}.$$

This is the arc-length parameterization of $\vec{\mathbf{r}}(t)$. Since the original restriction on t was given by $t \geq 0$, the restriction on s becomes $s/4 \geq 0$, or $s \geq 0$.

b. The arc-length function is given by Equation 13.3.5:

$$\begin{aligned} s(t) &= \int_a^t \|\vec{\mathbf{r}}\,'(u)\|\,du \\ &= \int_3^t \|\langle 1, 2, 2\rangle\|\,du \\ &= \int_3^t \sqrt{1^2 + 2^2 + 2^2}\,du \\ &= \int_3^t 3\,du \\ &= 3t - 9. \end{aligned}$$

Therefore, the relationship between the arc length s and the parameter t is $s = 3t - 9$, so $t = \frac{s}{3} + 3$. Substituting this into the original function $\vec{\mathbf{r}}(t) = \langle t + 3, 2t - 4, 2t\rangle$ yields

$$\vec{\mathbf{r}}(s) = \langle\left(\frac{s}{3} + 3\right) + 3,\ 2\left(\frac{s}{3} + 3\right) - 4,\ 2\left(\frac{s}{3} + 3\right)\rangle = \langle\frac{s}{3} + 6, \frac{2s}{3} + 2, \frac{2s}{3} + 6\rangle.$$

This is an arc-length parameterization of $\vec{\mathbf{r}}(t)$. The original restriction on the parameter t was $t \geq 3$, so the restriction on s is $(s/3) + 3 \geq 3$, or $s \geq 0$.

? Exercise 13.3.2

Find the arc-length function for the helix

$$\vec{\mathbf{r}}(t) = \langle 3\cos t, 3\sin t, 4t\rangle, \quad t \geq 0.$$

Then, use the relationship between the arc length and the parameter t to find an arc-length parameterization of $\vec{\mathbf{r}}(t)$.

Hint

Start by finding the arc-length function.

Answer

$s = 5t$, or $t = s/5$. Substituting this into $\vec{\mathbf{r}}(t) = \langle 3\cos t, 3\sin t, 4t\rangle$ gives

$$\vec{\mathbf{r}}(s) = \langle 3\cos\left(\frac{s}{5}\right), 3\sin\left(\frac{s}{5}\right), \frac{4s}{5}\rangle, \quad s \geq 0$$

Curvature

An important topic related to arc length is curvature. The concept of curvature provides a way to measure how sharply a smooth curve turns. A circle has constant curvature. The smaller the radius of the circle, the greater the curvature.

Think of driving down a road. Suppose the road lies on an arc of a large circle. In this case you would barely have to turn the wheel to stay on the road. Now suppose the radius is smaller. In this case you would need to turn more sharply to stay on the road. In the case of a curve other than a circle, it is often useful first to inscribe a circle to the curve at a given point so that it is tangent to the curve at that point and "hugs" the curve as closely as possible in a neighborhood of the point (Figure 13.3.1). The curvature of the graph at that point is then defined to be the same as the curvature of the inscribed circle.

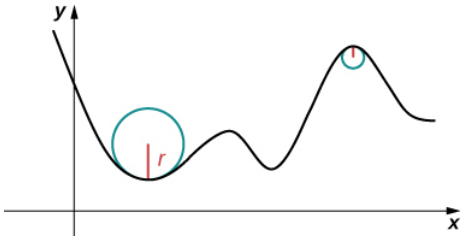


Figure 13.3.1: The graph represents the curvature of a function $y = f(x)$. The sharper the turn in the graph, the greater the curvature, and the smaller the radius of the inscribed circle.

Definition: Curvature

Let C be a smooth curve in the plane or in space given by $\vec{\mathbf{r}}(s)$, where s is the arc-length parameter. The curvature κ at s is

$$\kappa = \left\| \frac{d\vec{\mathbf{T}}}{ds} \right\| = \|\vec{\mathbf{T}}\,'(s)\|.$$

Visit this video for more information about the curvature of a space curve.

The formula in the definition of curvature is not very useful in terms of calculation. In particular, recall that $\vec{\mathbf{T}}(t)$ represents the unit tangent vector to a given vector-valued function $\vec{\mathbf{r}}(t)$, and the formula for $\vec{\mathbf{T}}(t)$ is

$$\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{r}}\,'(t)}{\|\vec{\mathbf{r}}\,'(t)\|}.$$

To use the formula for curvature, it is first necessary to express $\vec{\mathbf{r}}(t)$ in terms of the arc-length parameter s, then find the unit tangent vector $\vec{\mathbf{T}}(s)$ for the function $\vec{\mathbf{r}}(s)$, then take the derivative of $\vec{\mathbf{T}}(s)$ with respect to s. This is a tedious process. Fortunately, there are equivalent formulas for curvature.

Theorem: Alternate Formulas of Curvature

If C is a smooth curve given by $\vec{\mathbf{r}}(t)$, then the curvature κ of C at t is given by

$$\kappa = \frac{\|\vec{\mathbf{T}}\,'(t)\|}{\|\vec{\mathbf{r}}\,'(t)\|}. \tag{13.3.6}$$

If C is a three-dimensional curve, then the curvature can be given by the formula

$$\kappa = \frac{\|\vec{\mathbf{r}}\,'(t) \times \vec{\mathbf{r}}\,''(t)\|}{\|\vec{\mathbf{r}}\,'(t)\|^3}. \tag{13.3.7}$$

If C is the graph of a function $y = f(x)$ and both y' and y'' exist, then the curvature κ at point (x, y) is given by

$$\kappa = \frac{|y''|}{[1+(y')^2]^{3/2}}. \tag{13.3.8}$$

Proof

The first formula follows directly from the chain rule:

$$\frac{d\vec{\mathbf{T}}}{dt} = \frac{d\vec{\mathbf{T}}}{ds}\frac{ds}{dt},$$

where s is the arc length along the curve C. Dividing both sides by ds/dt, and taking the magnitude of both sides gives

$$\left\| \frac{d\vec{\mathbf{T}}}{ds} \right\| = \left\| \frac{\vec{\mathbf{T}}\,'(t)}{\frac{ds}{dt}} \right\|.$$

Since $ds/dt = \|\vec{\mathbf{r}}\,'(t)\|$, this gives the formula for the curvature κ of a curve C in terms of any parameterization of C:

$$\kappa = \frac{\|\vec{\mathbf{T}}\,'(t)\|}{\|\vec{\mathbf{r}}\,'(t)\|}.$$

In the case of a three-dimensional curve, we start with the formulas $\vec{\mathbf{T}}(t) = (\vec{\mathbf{r}}\,'(t))/\|\vec{\mathbf{r}}\,'(t)\|$ and $ds/dt = \|\vec{\mathbf{r}}\,'(t)\|$. Therefore, $\vec{\mathbf{r}}\,'(t) = (ds/dt)\vec{\mathbf{T}}(t)$. We can take the derivative of this function using the scalar product formula:

$$\vec{\mathbf{r}}\,''(t) = \frac{d^2s}{dt^2}\vec{\mathbf{T}}(t) + \frac{ds}{dt}\vec{\mathbf{T}}\,'(t).$$

Using these last two equations we get

$$\begin{aligned} \vec{\mathbf{r}}\,'(t) \times \vec{\mathbf{r}}\,''(t) &= \frac{ds}{dt}\vec{\mathbf{T}}(t) \times \left(\frac{d^2s}{dt^2}\vec{\mathbf{T}}(t) + \frac{ds}{dt}\vec{\mathbf{T}}\,'(t) \right) \\ &= \frac{ds}{dt}\frac{d^2s}{dt^2}\vec{\mathbf{T}}(t) \times \vec{\mathbf{T}}(t) + (\frac{ds}{dt})^2 \vec{\mathbf{T}}(t) \times \vec{\mathbf{T}}\,'(t). \end{aligned}$$

Since $\vec{\mathbf{T}}(t) \times \vec{\mathbf{T}}(t) = 0$, this reduces to

$$\vec{\mathbf{r}}\,'(t) \times \vec{\mathbf{r}}\,''(t) = \left(\frac{ds}{dt} \right)^2 \vec{\mathbf{T}}(t) \times \vec{\mathbf{T}}\,'(t).$$

Since $\vec{\mathbf{T}}\,'$ is parallel to $\vec{\mathbf{N}}$, and $\vec{\mathbf{T}}$ is orthogonal to $\vec{\mathbf{N}}$, it follows that $\vec{\mathbf{T}}$ and $\vec{\mathbf{T}}\,'$ are orthogonal. This means that $\|\vec{\mathbf{T}} \times \vec{\mathbf{T}}\,'\| = \|\vec{\mathbf{T}}\|\|\vec{\mathbf{T}}\,'\| \sin(\pi/2) = \|\vec{\mathbf{T}}\,'\|$, so

$$\|\vec{\mathbf{r}}\,'(t) \times \vec{\mathbf{r}}\,''(t)\| = \left(\frac{ds}{dt} \right)^2 \|\vec{\mathbf{T}}\,'(t)\|.$$

Now we solve this equation for $\|\vec{\mathbf{T}}\,'(t)\|$ and use the fact that $ds/dt = \|\vec{\mathbf{r}}\,'(t)\|$:

$$\|\vec{\mathbf{T}}\,'(t)\| = \frac{\|\vec{\mathbf{r}}\,'(t) \times \vec{\mathbf{r}}\,''(t)\|}{\|\vec{\mathbf{r}}\,'(t)\|^2}.$$

Then, we divide both sides by $\|\vec{\mathbf{r}}\,'(t)\|$. This gives

$$\kappa = \frac{\|\vec{\mathbf{T}}\,'(t)\|}{\|\vec{\mathbf{r}}\,'(t)\|} = \frac{\|\vec{\mathbf{r}}\,'(t) \times \vec{\mathbf{r}}\,''(t)\|}{\|\vec{\mathbf{r}}\,'(t)\|^3}.$$

This proves 13.3.7. To prove 13.3.8, we start with the assumption that curve C is defined by the function $y = f(x)$. Then, we can define $\vec{\mathbf{r}}(t) = x\,\hat{\mathbf{i}} + f(x)\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}$. Using the previous formula for curvature:

$$\vec{\mathbf{r}}\,'(t) = \hat{\mathbf{i}} + f'(x)\,\hat{\mathbf{j}}$$

$$\vec{\mathbf{r}}\,''(t) = f''(x)\,\hat{\mathbf{j}}$$

$$\vec{\mathbf{r}}\,'(t) \times \vec{\mathbf{r}}\,''(t) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & f'(x) & 0 \\ 0 & f''(x) & 0 \end{vmatrix} = f''(x)\,\hat{\mathbf{k}}.$$

Therefore,

$$\kappa = \frac{\|\vec{\mathbf{r}}\,'(t) \times \vec{\mathbf{r}}\,''(t)\|}{\|\vec{\mathbf{r}}\,'(t)\|^3} = \frac{|f''(x)|}{(1+[f'(x)]^2)^{3/2}}$$

Example 13.3.3: Finding Curvature

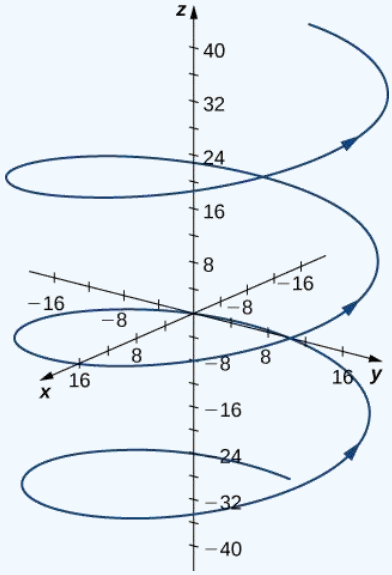
Find the curvature for each of the following curves at the given point:

a. $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} + 4\sin t\,\hat{\mathbf{j}} + 3t\,\hat{\mathbf{k}}, \quad t = \dfrac{4\pi}{3}$

b. $f(x) = \sqrt{4x - x^2}, x = 2$

Solution

a. This function describes a helix.



The curvature of the helix at $t = (4\pi)/3$ can be found by using 13.3.6. First, calculate $\vec{\mathbf{T}}(t)$:

$$\begin{aligned} \vec{\mathbf{T}}(t) &= \frac{\vec{\mathbf{r}}\,'(t)}{\|\vec{\mathbf{r}}\,'(t)\|} \\ &= \frac{\langle -4\sin t, 4\cos t, 3\rangle}{\sqrt{(-4\sin t)^2 + (4\cos t)^2 + 3^2}} \\ &= \langle -\frac{4}{5}\sin t, \frac{4}{5}\cos t, \frac{3}{5}\rangle. \end{aligned}$$

Next, calculate $\vec{\mathbf{T}}\,'(t)$:

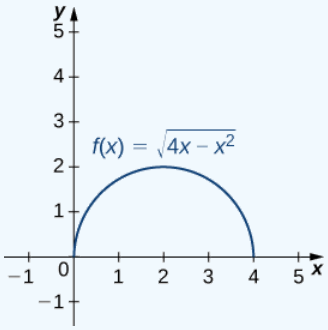
$$\vec{\mathbf{T}}\,'(t) = \langle -\frac{4}{5}\cos t, -\frac{4}{5}\sin t, 0\rangle.$$

Last, apply 13.3.6:

$$\kappa = \frac{\|\vec{\mathbf{T}}'(t)\|}{\|\vec{\mathbf{r}}'(t)\|} = \frac{\|\langle -\frac{4}{5}\cos t, -\frac{4}{5}\sin t, 0\rangle\|}{\|\langle -4\sin t, 4\cos t, 3\rangle\|}$$
$$= \frac{\sqrt{(-\frac{4}{5}\cos t)^2 + (-\frac{4}{5}\sin t)^2 + 0^2}}{\sqrt{(-4\sin t)^2 + (4\cos t)^2 + 3^2}}$$
$$= \frac{4/5}{5} = \frac{4}{25}.$$

The curvature of this helix is constant at all points on the helix.

2. This function describes a semicircle.



To find the curvature of this graph, we must use 13.3.8. First, we calculate y' and y'':

$$y = \sqrt{4x - x^2} = (4x - x^2)^{1/2}$$
$$y' = \frac{1}{2}(4x - x^2)^{-1/2}(4 - 2x) = (2 - x)(4x - x^2)^{-1/2}$$
$$y'' = -(4x - x^2)^{-1/2} + (2 - x)(-\frac{1}{2})(4x - x^2)^{-3/2}(4 - 2x)$$
$$= -\frac{4x - x^2}{(4x - x^2)^{3/2}} - \frac{(2 - x)^2}{(4x - x^2)^{3/2}}$$
$$= \frac{x^2 - 4x - (4 - 4x + x^2)}{(4x - x^2)^{3/2}}$$
$$= -\frac{4}{(4x - x^2)^{3/2}}.$$

Then, we apply 13.3.8:

$$\begin{aligned}\kappa &= \frac{|y''|}{[1+(y')^2]^{3/2}} \\ &= \frac{\left|-\frac{4}{(4x-x^2)^{3/2}}\right|}{\left[1+((2-x)(4x-x^2)^{-1/2})^2\right]^{3/2}} = \frac{\left|\frac{4}{(4x-x^2)^{3/2}}\right|}{\left[1+\frac{(2-x)^2}{4x-x^2}\right]^{3/2}} \\ &= \frac{\left|\frac{4}{(4x-x^2)^{3/2}}\right|}{\left[\frac{4x-x^2+x^2-4x+4}{4x-x^2}\right]^{3/2}} = \left|\frac{4}{(4x-x^2)^{3/2}}\right| \cdot \frac{(4x-x^2)^{3/2}}{8} \\ &= \frac{1}{2}.\end{aligned}$$

The curvature of this circle is equal to the reciprocal of its radius. There is a minor issue with the absolute value in 13.3.8; however, a closer look at the calculation reveals that the denominator is positive for any value of x.

? Exercise 13.3.3

Find the curvature of the curve defined by the function

$$y = 3x^2 - 2x + 4$$

at the point $x = 2$.

Hint

Use 13.3.8

Answer

$\kappa = \frac{6}{101^{3/2}} \approx 0.0059$

The Normal and Binormal Vectors

We have seen that the derivative $\vec{\mathbf{r}}\,'(t)$ of a vector-valued function is a tangent vector to the curve defined by $\vec{\mathbf{r}}(t)$, and the unit tangent vector $\vec{\mathbf{T}}(t)$ can be calculated by dividing $\vec{\mathbf{r}}\,'(t)$ by its magnitude. When studying motion in three dimensions, two other vectors are useful in describing the motion of a particle along a path in space: the principal unit normal vector and the **binormal vector**.

Definition: Binormal Vectors

Let C be a three-dimensional **smooth** curve represented by $\vec{\mathbf{r}}$ over an open interval I. If $\vec{\mathbf{T}}\,'(t) \neq \vec{\mathbf{0}}$, then the principal unit normal vector at t is defined to be

$$\vec{\mathbf{N}}(t) = \frac{\vec{\mathbf{T}}\,'(t)}{\|\vec{\mathbf{T}}\,'(t)\|}. \tag{13.3.9}$$

The binormal vector at t is defined as

$$\vec{\mathbf{B}}(t) = \vec{\mathbf{T}}(t) \times \vec{\mathbf{N}}(t), \tag{13.3.10}$$

where $\vec{\mathbf{T}}(t)$ is the unit tangent vector.

Note that, by definition, the binormal vector is orthogonal to both the unit tangent vector and the normal vector. Furthermore, $\vec{\mathbf{B}}(t)$ is always a unit vector. This can be shown using the formula for the magnitude of a cross product.

$$\|\vec{\mathbf{B}}(t)\| = \|\vec{\mathbf{T}}(t) \times \vec{\mathbf{N}}(t)\| = \|\vec{\mathbf{T}}(t)\|\|\vec{\mathbf{N}}(t)\| \sin\theta,$$

where θ is the angle between $\vec{\mathbf{T}}(t)$ and $\vec{\mathbf{N}}(t)$. Since $\vec{\mathbf{N}}(t)$ is the derivative of a unit vector, property (vii) of the derivative of a vector-valued function tells us that $\vec{\mathbf{T}}(t)$ and $\vec{\mathbf{N}}(t)$ are orthogonal to each other, so $\theta = \pi/2$. Furthermore, they are both unit vectors, so their magnitude is 1. Therefore, $\|\vec{\mathbf{T}}(t)\|\|\vec{\mathbf{N}}(t)\| \sin\theta = (1)(1)\sin(\pi/2) = 1$ and $\vec{\mathbf{B}}(t)$ is a unit vector.

The principal unit normal vector can be challenging to calculate because the unit tangent vector involves a quotient, and this quotient often has a square root in the denominator. In the three-dimensional case, finding the cross product of the unit tangent vector and the unit normal vector can be even more cumbersome. Fortunately, we have alternative formulas for finding these two vectors, and they are presented in Motion in Space.

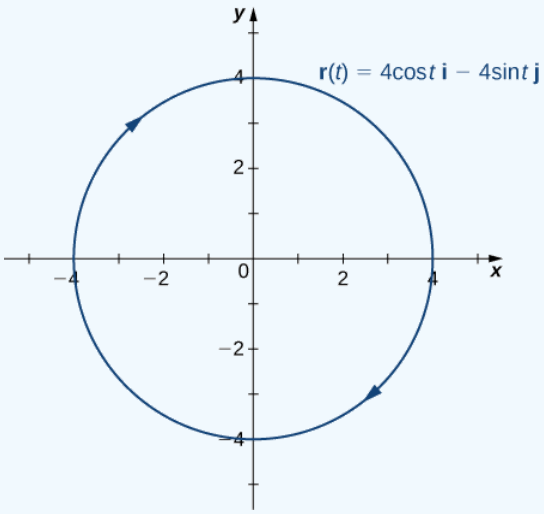
Example 13.3.4: Finding the Principal Unit Normal Vector and Binormal Vector

For each of the following vector-valued functions, find the principal unit normal vector. Then, if possible, find the binormal vector.

1. $\vec{\mathbf{r}}(t) = 4\cos t\,\hat{\mathbf{i}} - 4\sin t\,\hat{\mathbf{j}}$
2. $\vec{\mathbf{r}}(t) = (6t+2)\,\hat{\mathbf{i}} + 5t^2\,\hat{\mathbf{j}} - 8t\,\hat{\mathbf{k}}$

Solution

1. This function describes a circle.



To find the principal unit normal vector, we first must find the unit tangent vector $\vec{\mathbf{T}}(t)$:

$$\begin{aligned}
\vec{\mathbf{T}}(t) &= \frac{\vec{\mathbf{r}}\,'(t)}{\|\vec{\mathbf{r}}\,'(t)\|} \\
&= \frac{-4\sin t\,\hat{\mathbf{i}} - 4\cos t\,\hat{\mathbf{j}}}{\sqrt{(-4\sin t)^2 + (-4\cos t)^2}} \\
&= \frac{-4\sin t\,\hat{\mathbf{i}} - 4\cos t\,\hat{\mathbf{j}}}{\sqrt{16\sin^2 t + 16\cos^2 t}} \\
&= \frac{-4\sin t\,\hat{\mathbf{i}} - 4\cos t\,\hat{\mathbf{j}}}{\sqrt{16(\sin^2 t + \cos^2 t)}} \\
&= \frac{-4\sin t\,\hat{\mathbf{i}} - 4\cos t\,\hat{\mathbf{j}}}{4} \\
&= -\sin t\,\hat{\mathbf{i}} - \cos t\,\hat{\mathbf{j}}.
\end{aligned}$$

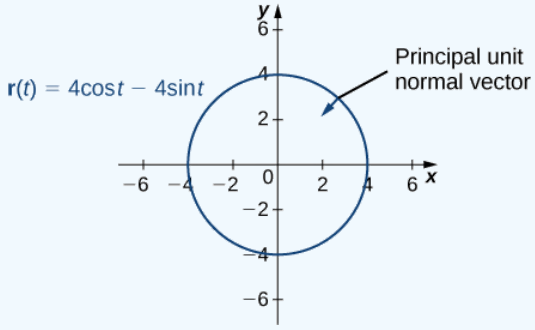
Next, we use 13.3.9:

$$\begin{aligned}\vec{\mathbf{N}}(t) &= \frac{\vec{\mathbf{T}}'(t)}{\|\vec{\mathbf{T}}'(t)\|} \\ &= \frac{-\cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}}{\sqrt{(-\cos t)^2 + (\sin t)^2}} \\ &= \frac{-\cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}}{\sqrt{\cos^2 t + \sin^2 t}} \\ &= -\cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}.\end{aligned}$$

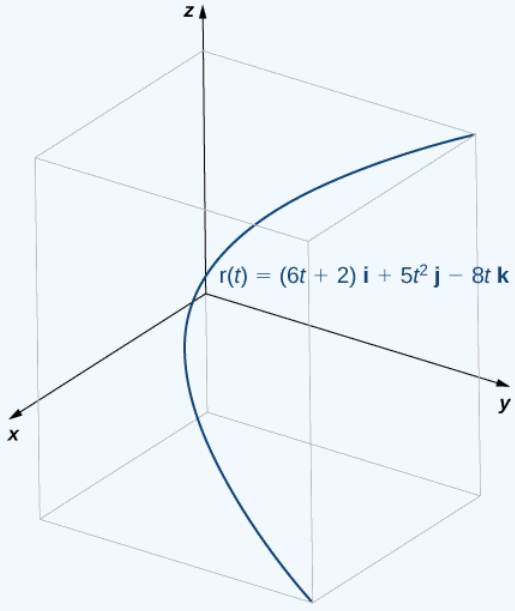
Notice that the unit tangent vector and the principal unit normal vector are orthogonal to each other for all values of t:

$$\begin{aligned}\vec{\mathbf{T}}(t) \cdot \vec{\mathbf{N}}(t) &= \langle -\sin t, -\cos t\rangle \cdot \langle -\cos t, \sin t\rangle \\ &= \sin t \cos t - \cos t \sin t \\ &= 0.\end{aligned}$$

Furthermore, the principal unit normal vector points toward the center of the circle from every point on the circle. Since $\vec{\mathbf{r}}(t)$ defines a curve in two dimensions, we cannot calculate the binormal vector.



2. This function looks like this:



To find the principal unit normal vector, we first find the unit tangent vector $\vec{\mathbf{T}}(t)$:

$$\begin{aligned}\vec{\mathbf{T}}(t) &= \frac{\vec{\mathbf{r}}\,'(t)}{\|\vec{\mathbf{r}}\,'(t)\|} \\ &= \frac{6\,\hat{\mathbf{i}} + 10t\,\hat{\mathbf{j}} - 8\,\hat{\mathbf{k}}}{\sqrt{6^2 + (10t)^2 + (-8)^2}} \\ &= \frac{6\,\hat{\mathbf{i}} + 10t\,\hat{\mathbf{j}} - 8\,\hat{\mathbf{k}}}{\sqrt{36 + 100t^2 + 64}} \\ &= \frac{6\,\hat{\mathbf{i}} + 10t\,\hat{\mathbf{j}} - 8\,\hat{\mathbf{k}}}{\sqrt{100(t^2+1)}} \\ &= \frac{3\,\hat{\mathbf{i}} - 5t\,\hat{\mathbf{j}} - 4\,\hat{\mathbf{k}}}{5\sqrt{t^2+1}} \\ &= \frac{3}{5}(t^2+1)^{-1/2}\,\hat{\mathbf{i}} - t(t^2+1)^{-1/2}\,\hat{\mathbf{j}} - \frac{4}{5}(t^2+1)^{-1/2}\,\hat{\mathbf{k}}.\end{aligned}$$

Next, we calculate $\vec{\mathbf{T}}\,'(t)$ and $\|\vec{\mathbf{T}}\,'(t)\|$:

$$\begin{aligned}\vec{\mathbf{T}}\,'(t) &= \frac{3}{5}(-\frac{1}{2})(t^2+1)^{-3/2}(2t)\,\hat{\mathbf{i}} - ((t^2+1)^{-1/2} - t(\frac{1}{2})(t^2+1)^{-3/2}(2t))\,\hat{\mathbf{j}} - \frac{4}{5}(-\frac{1}{2})(t^2+1)^{-3/2}(2t)\,\hat{\mathbf{k}} \\ &= -\frac{3t}{5(t^2+1)^{3/2}}\,\hat{\mathbf{i}} - \frac{1}{(t^2+1)^{3/2}}\,\hat{\mathbf{j}} + \frac{4t}{5(t^2+1)^{3/2}}\,\hat{\mathbf{k}} \\ \|\vec{\mathbf{T}}\,'(t)\| &= \sqrt{\left(-\frac{3t}{5(t^2+1)^{3/2}}\right)^2 + \left(-\frac{1}{(t^2+1)^{3/2}}\right)^2 + \left(\frac{4t}{5(t^2+1)^{3/2}}\right)^2} \\ &= \sqrt{\frac{9t^2}{25(t^2+1)^3} + \frac{1}{(t^2+1)^3} + \frac{16t^2}{25(t^2+1)^3}} \\ &= \sqrt{\frac{25t^2+25}{25(t^2+1)^3}} \\ &= \sqrt{\frac{1}{(t^2+1)^2}} \\ &= \frac{1}{t^2+1}.\end{aligned}$$

Therefore, according to 13.3.9:

$$\begin{aligned}\vec{\mathbf{N}}(t) &= \frac{\vec{\mathbf{T}}\,'(t)}{\|\vec{\mathbf{T}}\,'(t)\|} \\ &= \left(-\frac{3t}{5(t^2+1)^{3/2}}\,\hat{\mathbf{i}} - \frac{1}{(t^2+1)^{3/2}}\,\hat{\mathbf{j}} + \frac{4t}{5(t^2+1)^{3/2}}\,\hat{\mathbf{k}}\right)(t^2+1) \\ &= -\frac{3t}{5(t^2+1)^{1/2}}\,\hat{\mathbf{i}} - \frac{5}{5(t^2+1)^{1/2}}\,\hat{\mathbf{j}} + \frac{4t}{5(t^2+1)^{1/2}}\,\hat{\mathbf{k}} \\ &= -\frac{3t\,\hat{\mathbf{i}} + 5\,\hat{\mathbf{j}} - 4t\,\hat{\mathbf{k}}}{5\sqrt{t^2+1}}.\end{aligned}$$

Once again, the unit tangent vector and the principal unit normal vector are orthogonal to each other for all values of t:

$$\begin{aligned}\vec{\mathbf{T}}(t)\cdot\vec{\mathbf{N}}(t) &= \left(\frac{3\,\hat{\mathbf{i}}-5t\,\hat{\mathbf{j}}-4\,\hat{\mathbf{k}}}{5\sqrt{t^2+1}}\right)\cdot\left(-\frac{3t\,\hat{\mathbf{i}}+5\,\hat{\mathbf{j}}-4t\,\hat{\mathbf{k}}}{5\sqrt{t^2+1}}\right)\\ &= \frac{3(-3t)-5t(-5)-4(4t)}{25(t^2+1)}\\ &= \frac{-9t+25t-16t}{25(t^2+1)}\\ &= 0.\end{aligned}$$

Last, since $\vec{\mathbf{r}}(t)$ represents a three-dimensional curve, we can calculate the binormal vector using 13.3.10:

$$\begin{aligned}\vec{\mathbf{B}}(t) &= \vec{\mathbf{T}}(t)\times\vec{\mathbf{N}}(t)\\ &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{3}{5\sqrt{t^2+1}} & -\frac{5t}{5\sqrt{t^2+1}} & -\frac{4}{5\sqrt{t^2+1}} \\ -\frac{3t}{5\sqrt{t^2+1}} & -\frac{5}{5\sqrt{t^2+1}} & \frac{4t}{5\sqrt{t^2+1}} \end{vmatrix}\\ &= \left(\left(-\frac{5t}{5\sqrt{t^2+1}}\right)\left(\frac{4t}{5\sqrt{t^2+1}}\right)-\left(-\frac{4}{5\sqrt{t^2+1}}\right)\left(-\frac{5}{5\sqrt{t^2+1}}\right)\right)\hat{\mathbf{i}}\\ &\quad-\left(\left(\frac{3}{5\sqrt{t^2+1}}\right)\left(\frac{4t}{5\sqrt{t^2+1}}\right)-\left(-\frac{4}{5\sqrt{t^2+1}}\right)\left(-\frac{3t}{5\sqrt{t^2+1}}\right)\right)\hat{\mathbf{j}}\\ &\quad+\left(\left(\frac{3}{5\sqrt{t^2+1}}\right)\left(-\frac{5}{5\sqrt{t^2+1}}\right)-\left(-\frac{5t}{5\sqrt{t^2+1}}\right)\left(-\frac{3t}{5\sqrt{t^2+1}}\right)\right)\hat{\mathbf{k}}\\ &= \left(\frac{-20t^2-20}{25(t^2+1)}\right)\hat{\mathbf{i}}+\left(\frac{-15-15t^2}{25(t^2+1)}\right)\hat{\mathbf{k}}\\ &= -20\left(\frac{t^2+1}{25(t^2+1)}\right)\hat{\mathbf{i}}-15\left(\frac{t^2+1}{25(t^2+1)}\right)\hat{\mathbf{k}}\\ &= -\frac{4}{5}\,\hat{\mathbf{i}}-\frac{3}{5}\,\hat{\mathbf{k}}.\end{aligned}$$

? Exercise 13.3.4

Find the unit normal vector for the vector-valued function $\vec{\mathbf{r}}(t)=(t^2-3t)\,\hat{\mathbf{i}}+(4t+1)\,\hat{\mathbf{j}}$ and evaluate it at $t=2$.

Hint

First, find $\vec{\mathbf{T}}(t)$, then use 13.3.9

Answer

$\vec{\mathbf{N}}(2)=\frac{\sqrt{2}}{2}(\,\hat{\mathbf{i}}-\hat{\mathbf{j}})$

For any smooth curve in three dimensions that is defined by a vector-valued function, we now have formulas for the unit tangent vector $\vec{\mathbf{T}}$, the unit normal vector $\vec{\mathbf{N}}$, and the binormal vector $\vec{\mathbf{B}}$. The unit normal vector and the binormal vector form a plane that is perpendicular to the curve at any point on the curve, called the normal plane. In addition, these three vectors form a frame of reference in three-dimensional space called the *Frenet frame of reference* (also called the **TNB** frame) (Figure 13.3.2). Last, the plane determined by the vectors $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$ forms the osculating plane of C at any point P on the curve.

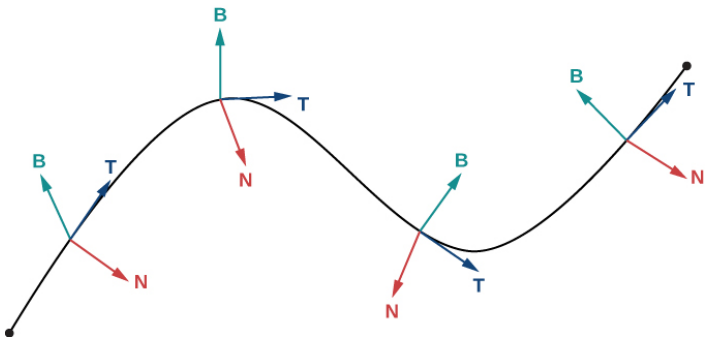


Figure 13.3.2: This figure depicts a Frenet frame of reference. At every point P *on a three-dimensional curve, the unit tangent, unit normal, and binormal vectors form a three-dimensional frame of reference.*

Suppose we form a circle in the osculating plane of C at point P on the curve. Assume that the circle has the same curvature as the curve does at point P and let the circle have radius r. Then, the curvature of the circle is given by $\frac{1}{r}$. We call r the radius of curvature of the curve, and it is equal to the reciprocal of the curvature. If this circle lies on the concave side of the curve and is tangent to the curve at point P, then this circle is called the *osculating circle* of C at P, as shown in Figure 13.3.3

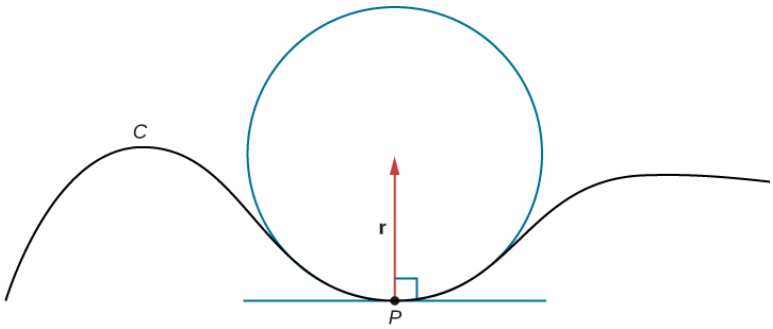


Figure 13.3.3: In this osculating circle, the circle is tangent to curve C at point P *and shares the same curvature.*

For more information on osculating circles, see this demonstration on curvature and torsion, this article on osculating circles, and this discussion of Serret formulas.

To find the equation of an osculating circle in two dimensions, we need find only the center and radius of the circle.

Example 13.3.5: Finding the Equation of an Osculating Circle

Find the equation of the osculating circle of the curve defined by the function $y = x^3 - 3x + 1$ at $x = 1$.

Solution

Figure 13.3.4shows the graph of $y = x^3 - 3x + 1$.

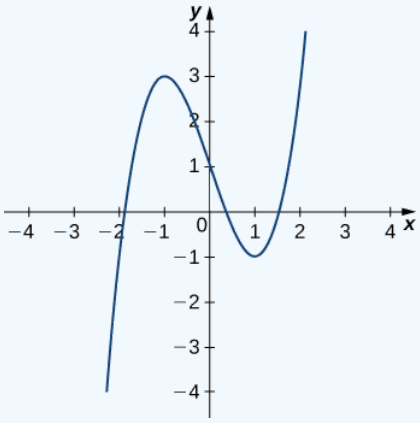


Figure 13.3.4: We want to find the osculating circle of this graph at the point where $x = 1$.

First, let's calculate the curvature at $x = 1$:

$$\kappa = \frac{|f''(x)|}{\left(1+[f'(x)]^2\right)^{3/2}} = \frac{|6x|}{(1+[3x^2-3]^2)^{3/2}}.$$

This gives $\kappa = 6$. Therefore, the radius of the osculating circle is given by $R = \frac{1}{\kappa} = \frac{1}{6}$. Next, we then calculate the coordinates of the center of the circle. When $x = 1$, the slope of the tangent line is zero. Therefore, the center of the osculating circle is directly above the point on the graph with coordinates $(1, -1)$. The center is located at $(1, -\frac{5}{6})$. The formula for a circle with radius r and center (h, k) is given by $(x-h)^2 + (y-k)^2 = r^2$. Therefore, the equation of the osculating circle is $(x-1)^2 + (y+\frac{5}{6})^2 = \frac{1}{36}$. The graph and its osculating circle appears in the following graph.

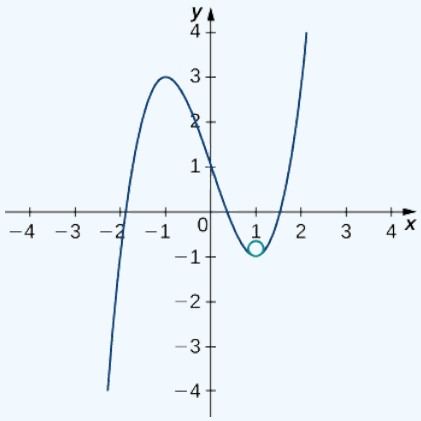


Figure 13.3.5: The osculating circle has radius $R = \frac{1}{6}$.

Exercise 13.3.5

Find the equation of the osculating circle of the curve defined by the vector-valued function $y = 2x^2 - 4x + 5$ at $x = 1$.

Hint

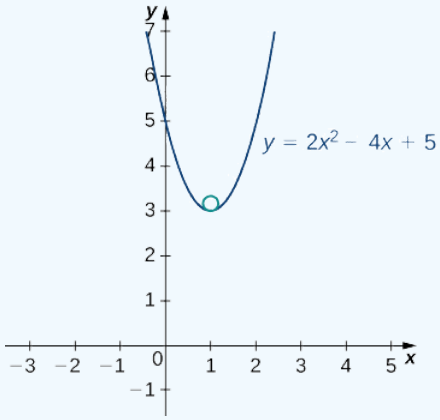
Use 13.3.8to find the curvature of the graph, then draw a graph of the function around $x = 1$ to help visualize the circle in relation to the graph.

Answer

$\kappa = \frac{4}{[1+(4x-4)^2]^{3/2}}$

At the point $x = 1$, the curvature is equal to 4. Therefore, the radius of the osculating circle is $\frac{1}{4}$.

A graph of this function appears next:



The vertex of this parabola is located at the point $(1,3)$. Furthermore, the center of the osculating circle is directly above the vertex. Therefore, the coordinates of the center are $(1, \frac{13}{4})$. The equation of the osculating circle is $(x-1)^2+(y-\frac{13}{4})^2=\frac{1}{16}$.

Key Concepts

- The arc-length function for a vector-valued function is calculated using the integral formula $s(t)=\int_a^b \|\vec{\mathbf{r}}\,'(t)\|\,dt$. This formula is valid in both two and three dimensions.
- The curvature of a curve at a point in either two or three dimensions is defined to be the curvature of the inscribed circle at that point. The arc-length parameterization is used in the definition of curvature.
- There are several different formulas for curvature. The curvature of a circle is equal to the reciprocal of its radius.
- The principal unit normal vector at t is defined to be

$$\vec{\mathbf{N}}(t)=\frac{\vec{\mathbf{T}}\,'(t)}{\|\vec{\mathbf{T}}\,'(t)\|}.$$

- The binormal vector at t is defined as $\vec{\mathbf{B}}(t)=\vec{\mathbf{T}}(t)\times\vec{\mathbf{N}}(t)$, where $\vec{\mathbf{T}}(t)$ is the unit tangent vector.
- The Frenet frame of reference is formed by the unit tangent vector, the principal unit normal vector, and the binormal vector.
- The osculating circle is tangent to a curve at a point and has the same curvature as the tangent curve at that point.

Key Equations

- **Arc length of space curve**
  $s=\int_a^b \sqrt{[f\,'(t)]^2+[g\,'(t)]^2+[h\,'(t)]^2}\,dt=\int_a^b \|\vec{\mathbf{r}}\,'(t)\|\,dt$
- **Arc-length function**
  $s(t)=\int_a^t \sqrt{f\,'(u))^2+(g\,'(u))^2+(h\,'(u))^2}\,du \text{ or } s(t)=\int_a^t \|\vec{\mathbf{r}}\,'(u)\|\,du$
- $\kappa=\frac{\|\vec{\mathbf{T}}\,'(t)\|}{\|\vec{\mathbf{r}}\,'(t)\|} \text{ or } \kappa=\frac{\|\vec{\mathbf{r}}\,'(t)\times\vec{\mathbf{r}}\,''(t)\|}{\|\vec{\mathbf{r}}\,'(t)\|^3} \text{ or } \kappa=\frac{|y''|}{[1+(y')^2]^{3/2}}$
- **Principal unit normal vector**
  $\vec{\mathbf{N}}(t)=\frac{\vec{\mathbf{T}}\,'(t)}{\|\vec{\mathbf{T}}\,'(t)\|}$
- **Binormal vector**
  $\vec{\mathbf{B}}(t)=\vec{\mathbf{T}}(t)\times\vec{\mathbf{N}}(t)$

Glossary

arc-length function

a function $s(t)$ that describes the arc length of curve C as a function of t

arc-length parameterization

a reparameterization of a vector-valued function in which the parameter is equal to the arc length

binormal vector

a unit vector orthogonal to the unit tangent vector and the unit normal vector

curvature

the derivative of the unit tangent vector with respect to the arc-length parameter

Frenet frame of reference

(TNB frame) a frame of reference in three-dimensional space formed by the unit tangent vector, the unit normal vector, and the binormal vector

normal plane

a plane that is perpendicular to a curve at any point on the curve

osculating circle

a circle that is tangent to a curve C at a point P and that shares the same curvature

osculating plane

the plane determined by the unit tangent and the unit normal vector

principal unit normal vector

a vector orthogonal to the unit tangent vector, given by the formula $\frac{\vec{\mathbf{T}}'(t)}{\|\vec{\mathbf{T}}'(t)\|}$

radius of curvature

the reciprocal of the curvature

smooth

curves where the vector-valued function $\vec{\mathbf{r}}(t)$ is differentiable with a non-zero derivative

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13.4: Motion in Space- Velocity and Acceleration

Learning Objectives

- Describe the velocity and acceleration vectors of a particle moving in space.
- Explain the tangential and normal components of acceleration.
- State Kepler's laws of planetary motion.

We have now seen how to describe curves in the plane and in space, and how to determine their properties, such as arc length and curvature. All of this leads to the main goal of this chapter, which is the description of motion along plane curves and space curves. We now have all the tools we need; in this section, we put these ideas together and look at how to use them.

Motion Vectors in the Plane and in Space

Our starting point is using vector-valued functions to represent the position of an object as a function of time. All of the following material can be applied either to curves in the plane or to space curves. For example, when we look at the orbit of the planets, the curves defining these orbits all lie in a plane because they are elliptical. However, a particle traveling along a helix moves on a curve in three dimensions.

Definition: Speed, Velocity, and Acceleration

Let $\vec{\mathbf{r}}(t)$ be a twice-differentiable vector-valued function of the parameter t that represents the position of an object as a function of time.

The velocity vector $\vec{\mathbf{v}}(t)$ of the object is given by

$$\text{Velocity } = \vec{\mathbf{v}}(t) = \vec{\mathbf{r}}\,'(t). \tag{13.4.1}$$

The acceleration vector $\vec{\mathbf{a}}(t)$ is defined to be

$$\text{Acceleration } = \vec{\mathbf{a}}(t) = \vec{\mathbf{v}}\,'(t) = \vec{\mathbf{r}}\,''(t). \tag{13.4.2}$$

The *speed* is defined to be

$$\text{Speed } = v(t) = \|\vec{\mathbf{v}}(t)\| = \|\vec{\mathbf{r}}\,'(t)\| = \frac{ds}{dt}. \tag{13.4.3}$$

Since $\vec{\mathbf{r}}(t)$ can be in either two or three dimensions, these vector-valued functions can have either two or three components. In two dimensions, we define $\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}$ and in three dimensions $\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$. Then the velocity, acceleration, and speed can be written as shown in the following table.

Table 13.4.1: Formulas for Position, Velocity, Acceleration, and Speed

Quantity	Two Dimensions	Three Dimensions
Position	$\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}$	$\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$
Velocity	$\vec{\mathbf{v}}(t) = x'(t)\hat{\mathbf{i}} + y'(t)\hat{\mathbf{j}}$	$\vec{\mathbf{v}}(t) = x'(t)\hat{\mathbf{i}} + y'(t)\hat{\mathbf{j}} + z'(t)\hat{\mathbf{k}}$
Acceleration	$\vec{\mathbf{a}}(t) = x''(t)\hat{\mathbf{i}} + y''(t)\hat{\mathbf{j}}$	$\vec{\mathbf{a}}(t) = x''(t)\hat{\mathbf{i}} + y''(t)\hat{\mathbf{j}} + z''(t)\hat{\mathbf{k}}$
Speed	$\|\vec{\mathbf{v}}(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2}$	$\|\vec{\mathbf{v}}(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$

Example 13.4.1: Studying Motion Along a Parabola

A particle moves in a parabolic path defined by the vector-valued function $\vec{\mathbf{r}}(t) = t^2\hat{\mathbf{i}} + \sqrt{5 - t^2}\,\hat{\mathbf{j}}$, where t measures time in seconds.

1. Find the velocity, acceleration, and speed as functions of time.
2. Sketch the curve along with the velocity vector at time $t = 1$.

Solution

1. We use Equations 13.4.1, 13.4.2, and 13.4.3:

$$\begin{aligned}
\vec{\mathbf{v}}(t) &= \vec{\mathbf{r}}\,'(t) = 2t\hat{\mathbf{i}} - \frac{t}{\sqrt{5-t^2}}\hat{\mathbf{j}} \\
\vec{\mathbf{a}}(t) &= \vec{\mathbf{v}}\,'(t) = 2\hat{\mathbf{i}} - 5(5-t^2)^{-\frac{3}{2}}\hat{\mathbf{j}} \\
||\vec{\mathbf{v}}(t)|| &= ||\vec{\mathbf{r}}\,'(t)|| \\
&= \sqrt{(2t)^2 + \left(-\frac{t}{\sqrt{5-t^2}}\right)^2} \\
&= \sqrt{4t^2 + \frac{t^2}{5-t^2}} \\
&= \sqrt{\frac{21t^2 - 4t^4}{5-t^2}}.
\end{aligned}$$

2. The graph of $\vec{\mathbf{r}}(t) = t^2\hat{\mathbf{i}} + \sqrt{5-t^2}\,\hat{\mathbf{j}}$ is a portion of a parabola (Figure 13.4.1).

When $t = 1$, $\vec{\mathbf{r}}(1) = (1)^2\hat{\mathbf{i}} + \sqrt{5-(1)^2}\,\hat{\mathbf{j}} = \hat{\mathbf{i}} + \sqrt{4}\hat{\mathbf{j}} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}}$.

Thus the particle would be located at the point $(1, 2)$ when $t = 1$.

The velocity vector at $t = 1$ is

$$\begin{aligned}
\vec{\mathbf{v}}(1) &= \vec{\mathbf{r}}\,'(1) = 2(1)\hat{\mathbf{i}} - \frac{1}{\sqrt{5-1^2}}\hat{\mathbf{j}} \\
&= 2\hat{\mathbf{i}} - \frac{1}{2}\hat{\mathbf{j}}
\end{aligned}$$

and the acceleration vector at $t = 1$ is

$$\vec{\mathbf{a}}(1) = \vec{\mathbf{v}}\,'(1) = 2\hat{\mathbf{i}} - 5(5-1^2)^{-3/2}\hat{\mathbf{j}} = 2\hat{\mathbf{i}} - \frac{5}{8}\hat{\mathbf{j}}.$$

Notice that the velocity vector is tangent to the path, as is always the case.

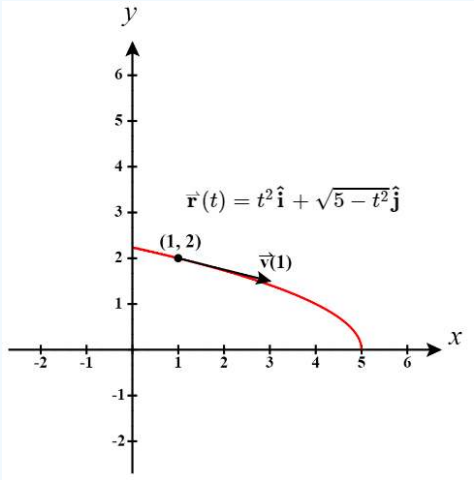


Figure 13.4.1: This graph depicts the velocity vector at time $t = 1$ for a particle moving in a parabolic path.

Exercise 13.4.1

A particle moves in a path defined by the vector-valued function $\vec{\mathbf{r}}(t) = (t^2 - 3t)\,\hat{\mathbf{i}} + (2t - 4)\,\hat{\mathbf{j}} + (t + 2)\,\hat{\mathbf{k}}$, where t measures time in seconds and where distance is measured in feet. Find the velocity, acceleration, and speed as functions of time.

Hint

Use Equations 13.4.1, 13.4.2, and 13.4.3

Answer

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{r}}\,'(t) = (2t - 3)\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

$$\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}\,'(t) = 2\,\hat{\mathbf{i}}$$

$$||\vec{\mathbf{r}}\,'(t)|| = \sqrt{(2t - 3)^2 + 2^2 + 1^2} = \sqrt{4t^2 - 12t + 14}$$

The units for velocity and speed are feet per second, and the units for acceleration are feet per second squared.

To gain a better understanding of the velocity and acceleration vectors, imagine you are driving along a curvy road. If you do not turn the steering wheel, you would continue in a straight line and run off the road. The speed at which you are traveling when you run off the road, coupled with the direction, gives a vector representing your velocity, as illustrated in Figure 13.4.2

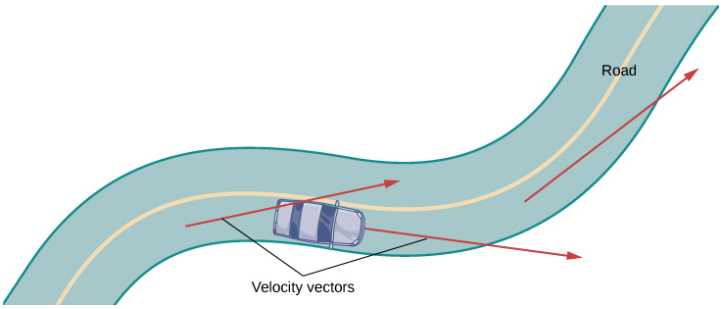


Figure 13.4.2: At each point along a road traveled by a car, the velocity vector of the car is tangent to the path traveled by the car.

However, the fact that you must turn the steering wheel to stay on the road indicates that your velocity is always changing (even if your speed is not) because your *direction* is constantly changing to keep you on the road. As you turn to the right, your acceleration vector also points to the right. As you turn to the left, your acceleration vector points to the left. This indicates that your velocity and acceleration vectors are constantly changing, regardless of whether your actual speed varies (Figure 13.4.3).

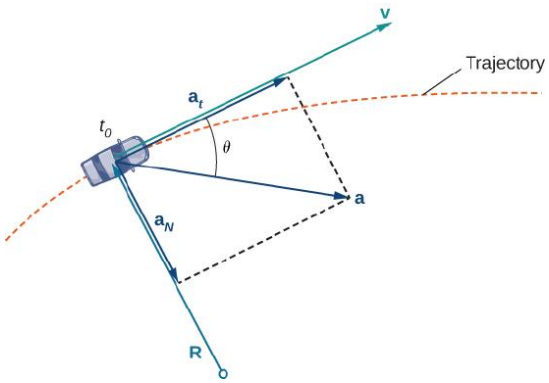


Figure 13.4.3: The dashed line represents the trajectory of an object (a car, for example). The acceleration vector points toward the inside of the turn at all times.

Components of the Acceleration Vector

We can combine some of the concepts discussed in Arc Length and Curvature with the acceleration vector to gain a deeper understanding of how this vector relates to motion in the plane and in space. Recall that the unit tangent vector $\vec{\mathbf{T}}$ and the unit

normal vector $\vec{\mathbf{N}}$ form an osculating plane at any point P on the curve defined by a vector-valued function $\vec{\mathbf{r}}(t)$. The following theorem shows that the acceleration vector $\vec{\mathbf{a}}(t)$ lies in the osculating plane and can be written as a linear combination of the unit tangent and the unit normal vectors.

Theorem 13.4.1: The Plane of the Acceleration Vector

The acceleration vector $\vec{\mathbf{a}}(t)$ of an object moving along a curve traced out by a twice-differentiable function $\vec{\mathbf{r}}(t)$ lies in the plane formed by the unit tangent vector $\vec{\mathbf{T}}(t)$ and the principal unit normal vector $\vec{\mathbf{N}}(t)$ to C. Furthermore,

$$\vec{\mathbf{a}}(t) = v'(t)\vec{\mathbf{T}}(t) + [v(t)]^2\kappa\vec{\mathbf{N}}(t)$$

Here, $v(t) = ||\vec{\mathbf{v}}(t)||$ is the speed of the object and κ is the curvature of C traced out by $\vec{\mathbf{r}}(t)$.

Proof

Because $\vec{\mathbf{v}}(t) = \vec{\mathbf{r}}\,'(t)$ and $\vec{\mathbf{T}}(t) = \dfrac{\vec{\mathbf{r}}\,'(t)}{||\vec{\mathbf{r}}\,'(t)||}$, we have $\vec{\mathbf{v}}(t) = ||\vec{\mathbf{r}}\,'(t)||\vec{\mathbf{T}}(t) = v(t)\vec{\mathbf{T}}(t)$.

Now we differentiate this equation:

$$\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}\,'(t) = \frac{d}{dt}\Big(v(t)\vec{\mathbf{T}}(t)\Big) = v'(t)\vec{\mathbf{T}}(t) + v(t)\vec{\mathbf{T}}\,'(t)$$

Since $\vec{\mathbf{N}}(t) = \dfrac{\vec{\mathbf{T}}\,'(t)}{||\vec{\mathbf{T}}\,'(t)||}$, we know $\vec{\mathbf{T}}\,'(t) = ||\vec{\mathbf{T}}\,'(t)||\vec{\mathbf{N}}(t)$, so

$$\vec{\mathbf{a}}(t) = v'(t)\vec{\mathbf{T}}(t) + v(t)||\vec{\mathbf{T}}\,'(t)||\vec{\mathbf{N}}(t).$$

A formula for curvature is $\kappa = \dfrac{||\vec{\mathbf{T}}\,'(t)||}{||\vec{\mathbf{r}}\,'(t)||}$, so $\vec{\mathbf{T}}\,'(t) = \kappa||\vec{\mathbf{r}}\,'(t)|| = \kappa v(t)$.

This gives $\vec{\mathbf{a}}(t) = v'(t)\vec{\mathbf{T}}(t) + \kappa(v(t))^2\vec{\mathbf{N}}(t)$.

□

The coefficients of $\vec{\mathbf{T}}(t)$ and $\vec{\mathbf{N}}(t)$ are referred to as the **tangential component of acceleration** and the **normal component of acceleration**, respectively. We write $a_{\vec{\mathbf{T}}}$ to denote the tangential component and $a_{\vec{\mathbf{N}}}$ to denote the normal component.

Theorem 13.4.2: Tangential and Normal Components of Acceleration

Let $\vec{\mathbf{r}}(t)$ be a vector-valued function that denotes the position of an object as a function of time. Then $\vec{\mathbf{a}}(t) = \vec{\mathbf{r}}\,''(t)$ is the acceleration vector. The tangential and normal components of acceleration $a_{\vec{\mathbf{T}}}$ and $a_{\vec{\mathbf{N}}}$ are given by the formulas

$$a_{\vec{\mathbf{T}}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{T}} = \frac{\vec{\mathbf{v}} \cdot \vec{\mathbf{a}}}{||\vec{\mathbf{v}}||} \tag{13.4.4}$$

and

$$a_{\vec{\mathbf{N}}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{N}} = \frac{||\vec{\mathbf{v}} \times \vec{\mathbf{a}}||}{||\vec{\mathbf{v}}||} = \sqrt{||\vec{\mathbf{a}}||^2 - \left(a_{\vec{\mathbf{T}}}\right)^2}. \tag{13.4.5}$$

These components are related by the formula

$$\vec{\mathbf{a}}(t) = a_{\vec{\mathbf{T}}}\vec{\mathbf{T}}(t) + a_{\vec{\mathbf{N}}}\vec{\mathbf{N}}(t). \tag{13.4.6}$$

Here $\vec{\mathbf{T}}(t)$ is the unit tangent vector to the curve defined by $\vec{\mathbf{r}}(t)$, and $\vec{\mathbf{N}}(t)$ is the unit normal vector to the curve defined by $\vec{\mathbf{r}}(t)$.

The normal component of acceleration is also called the *centripetal component of acceleration* or sometimes the *radial component of acceleration*. To understand centripetal acceleration, suppose you are traveling in a car on a circular track at a constant speed. Then, as we saw earlier, the acceleration vector points toward the center of the track at all times. As a rider in the car, you feel a pull toward the *outside* of the track because you are constantly turning. This sensation acts in the opposite direction of centripetal acceleration. The same holds true for non-circular paths. The reason is that your body tends to travel in a straight line and resists the force resulting from acceleration that push it toward the side. Note that at point B in Figure $13.4.4$ the acceleration vector is pointing backward. This is because the car is decelerating as it goes into the curve.

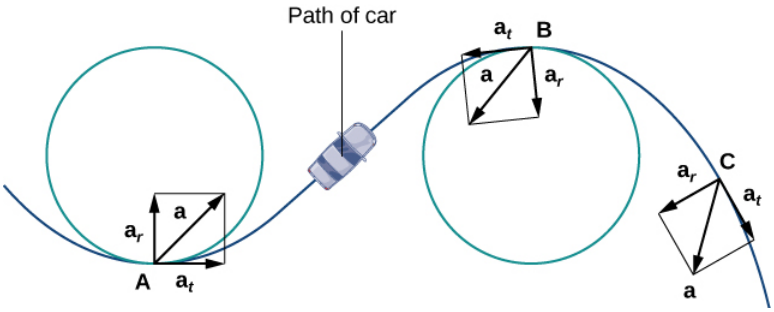


Figure 13.4.4: The tangential and normal components of acceleration can be used to describe the acceleration vector.

The tangential and normal unit vectors at any given point on the curve provide a frame of reference at that point. The tangential and normal components of acceleration are the projections of the acceleration vector onto $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$, respectively.

Example 13.4.2: Finding Components of Acceleration

A particle moves in a path defined by the vector-valued function $\vec{\mathbf{r}}(t) = t^2\,\hat{\mathbf{i}} + (2t-3)\,\hat{\mathbf{j}} + (3t^2-3t)\,\hat{\mathbf{k}}$, where t measures time in seconds and distance is measured in feet.

a. Find $a_{\vec{\mathbf{T}}}$ and $a_{\vec{\mathbf{N}}}$ as functions of t.
b. Find $a_{\vec{\mathbf{T}}}$ and $a_{\vec{\mathbf{N}}}$ at time $t = 2$.

Solution

a. Let's start deriving the velocity and acceleration functions:

$$\begin{aligned}\vec{\mathbf{v}}(t) &= \vec{\mathbf{r}}\,'(t)\\ &= 2t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + (6t-3)\,\hat{\mathbf{k}}\\ \vec{\mathbf{a}}(t) &= \vec{\mathbf{v}}\,'(t)\\ &= 2\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{k}}\end{aligned}$$

Now we apply Equation 13.4.4:

$$\begin{aligned}a_{\vec{\mathbf{T}}} &= \frac{\vec{\mathbf{v}}\cdot\vec{\mathbf{a}}}{||\vec{\mathbf{v}}||}\\ &= \frac{(2t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + (6t-3)\,\hat{\mathbf{k}})\cdot(2\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{k}})}{||2t\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + (6t-3)\,\hat{\mathbf{k}}||}\\ &= \frac{4t + 6(6t-3)}{\sqrt{(2t)^2 + 2^2 + (6t-3)^2}}\\ &= \frac{40t-18}{40t^2 - 36t + 13}\end{aligned}$$

Now we can apply Equation 13.4.5:

$$
\begin{aligned}
a_{\vec{\mathbf{N}}} &= \sqrt{||\vec{\mathbf{a}}||^2 - \left(a_{\vec{\mathbf{T}}}\right)^2} \\
&= \sqrt{||2\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{k}}||^2 - \left(\frac{40t-18}{\sqrt{40t^2-36t+13}}\right)^2} \\
&= \sqrt{4+36-\frac{(40t-18)^2}{40t^2-36t+13}} \\
&= \sqrt{\frac{40(40t^2-36t+13)-(1600t^2-1440t+324)}{40t^2-36t+13}} \\
&= \sqrt{\frac{196}{40t^2-36t+13}} \\
&= \frac{14}{\sqrt{40t^2-36t+13}}
\end{aligned}
$$

b. We must evaluate each of the answers from part a at $t=2$:

$$
\begin{aligned}
a_{\vec{\mathbf{T}}}(2) &= \frac{40(2)-18}{\sqrt{40(2)^2-36(2)+13}} \\
&= \frac{80-18}{\sqrt{160-72+13}} \\
&= \frac{62}{\sqrt{101}} \\
a_{\vec{\mathbf{N}}}(2) &= \frac{14}{\sqrt{40(2)^2-36(2)+13}} \\
&= \frac{14}{\sqrt{160-72+13}} = \frac{14}{\sqrt{101}}.
\end{aligned}
$$

The units of acceleration are feet per second squared, as are the units of the normal and tangential components of acceleration.

Exercise 13.4.2

An object moves in a path defined by the vector-valued function $\vec{\mathbf{r}}(t) = 4t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}}$, where t measures time in seconds.

a. Find $a_{\vec{\mathbf{T}}}$ and $a_{\vec{\mathbf{N}}}$ as functions of t.
b. Find $a_{\vec{\mathbf{T}}}$ and $a_{\vec{\mathbf{N}}}$ at time $t=-3$.

Hint

Use Equations 13.4.4and 13.4.5

Answer

a.

$$\begin{aligned} a_{\vec{\mathbf{T}}} &= \frac{\vec{\mathbf{v}}(t)\cdot\vec{\mathbf{a}}(t)}{||\vec{\mathbf{v}}(t)||} = \frac{\vec{\mathbf{r}}\,'(t)\cdot\vec{\mathbf{r}}\,''(t)}{||\vec{\mathbf{r}}\,'(t)||} \\ &= \frac{(4\,\hat{\mathbf{i}}+2t\,\hat{\mathbf{j}})\cdot(2\,\hat{\mathbf{j}})}{||4\,\hat{\mathbf{i}}+2t\,\hat{\mathbf{j}}||} \\ &= \frac{4t}{\sqrt{4^2+(2t)^2}} \\ &= \frac{2t}{\sqrt{2+t^2}} \end{aligned}$$

$$\begin{aligned} a_{\vec{\mathbf{N}}} &= \sqrt{||\vec{\mathbf{a}}||^2 - a_{\vec{\mathbf{T}}}^2} \\ &= \sqrt{||2\,\hat{\mathbf{j}}||^2 - \left(\frac{2t}{\sqrt{2+t^2}}\right)^2} \\ &= \sqrt{4-\frac{4t^2}{2+t^2}} \end{aligned}$$

b.

$$\begin{aligned} a_{\vec{\mathbf{T}}}(-3) &= \frac{2(-3)}{\sqrt{2+(-3)^2}} \\ &= \frac{-6}{\sqrt{11}} \end{aligned}$$

$$\begin{aligned} a_{\vec{\mathbf{N}}}(-3) &= \sqrt{4-\frac{4(-3)^2}{2+(-3)^2}} \\ &= \sqrt{4-\frac{36}{11}} \\ &= \sqrt{\frac{8}{11}} \\ &= \frac{2\sqrt{2}}{\sqrt{11}} \end{aligned}$$

Projectile Motion

Now let's look at an application of vector functions. In particular, let's consider the effect of gravity on the motion of an object as it travels through the air, and how it determines the resulting trajectory of that object. In the following, we ignore the effect of air resistance. This situation, with an object moving with an initial velocity but with no forces acting on it other than gravity, is known as projectile motion. It describes the motion of objects from golf balls to baseballs, and from arrows to cannonballs.

First we need to choose a coordinate system. If we are standing at the origin of this coordinate system, then we choose the positive y-axis to be up, the negative y-axis to be down, and the positive x-axis to be forward (i.e., away from the thrower of the object). The effect of gravity is in a downward direction, so Newton's second law tells us that the force on the object resulting from gravity is equal to the mass of the object times the acceleration resulting from gravity, or $\vec{\mathbf{F}}_g = m\vec{\mathbf{a}}$, where $\vec{\mathbf{F}}_g$ represents the force from gravity and $\vec{\mathbf{a}} = -g\,\hat{\mathbf{j}}$ represents the acceleration resulting from gravity at Earth's surface. The value of g in the English system of measurement is approximately 32 ft/sec^2 and it is approximately 9.8 m/sec^2 in the metric system. This is the only force acting on the object. Since gravity acts in a downward direction, we can write the force resulting from gravity in the form $\vec{\mathbf{F}}_g = -mg\,\hat{\mathbf{j}}$, as shown in Figure 13.4.5.

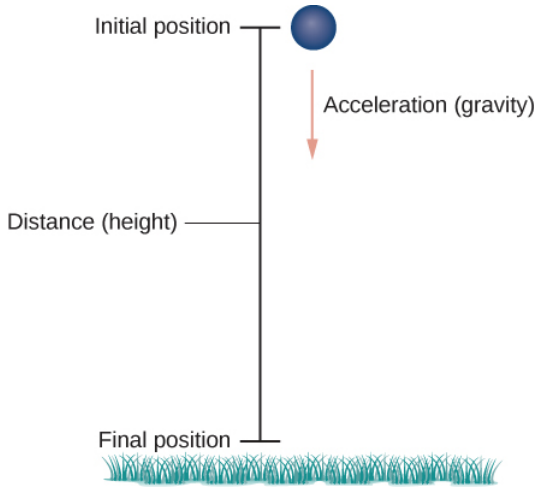


Figure 13.4.5: An object is falling under the influence of gravity.

Newton's second law also tells us that $F = m\vec{\mathbf{a}}$, where $\vec{\mathbf{a}}$ represents the acceleration vector of the object. This force must be equal to the force of gravity at all times, so we therefore know that

$$\begin{aligned} \vec{\mathbf{F}} &= \vec{\mathbf{F}}_g \\ m\vec{\mathbf{a}} &= -mg\,\hat{\mathbf{j}} \\ \vec{\mathbf{a}} &= -g\,\hat{\mathbf{j}}. \end{aligned}$$

Now we use the fact that the acceleration vector is the first derivative of the velocity vector. Therefore, we can rewrite the last equation in the form

$$\vec{\mathbf{v}}\,'(t) = -g\,\hat{\mathbf{j}}$$

By taking the antiderivative of each side of this equation we obtain

$$\vec{\mathbf{v}}(t) = \int -g\,\hat{\mathbf{j}}\; dt = -gt\,\hat{\mathbf{j}} + \vec{\mathbf{C}}_1$$

for some constant vector $\vec{\mathbf{C}}_1$. To determine the value of this vector, we can use the velocity of the object at a fixed time, say at time $t = 0$. We call this velocity the *initial velocity*: $\vec{\mathbf{v}}(0) = \vec{\mathbf{v}}_0$. Therefore, $\vec{\mathbf{v}}(0) = -g(0)\,\hat{\mathbf{j}} + \vec{\mathbf{C}}_1 = \vec{\mathbf{v}}_0$ and $\vec{\mathbf{C}}_1 = \vec{\mathbf{v}}_0$. This gives the velocity vector as $\vec{\mathbf{v}}(t) = -gt\,\hat{\mathbf{j}} + \vec{\mathbf{v}}_0$.

Next we use the fact that velocity $\vec{\mathbf{v}}(t)$ is the derivative of position $\vec{\mathbf{s}}(t)$. This gives the equation

$$\vec{\mathbf{s}}\,'(t) = -gt\,\hat{\mathbf{j}} + \vec{\mathbf{v}}_0.$$

Taking the antiderivative of both sides of this equation leads to

$$\begin{aligned} \vec{\mathbf{s}}(t) &= \int -gt\,\hat{\mathbf{j}} + \vec{\mathbf{v}}_0\; dt \\ &= -\frac{1}{2}gt^2\,\hat{\mathbf{j}} + \vec{\mathbf{v}}_0 t + \vec{\mathbf{C}}_2 \end{aligned}$$

with another unknown constant vector $\vec{\mathbf{C}}_2$. To determine the value of $\vec{\mathbf{C}}_2$, we can use the position of the object at a given time, say at time $t = 0$. We call this position the *initial position*: $\vec{\mathbf{s}}(0) = \vec{\mathbf{s}}_0$. Therefore, $\vec{\mathbf{s}}(0) = -(1/2)g(0)^2\,\hat{\mathbf{j}} + \vec{\mathbf{v}}_0(0) + \vec{\mathbf{C}}_2 = \vec{\mathbf{s}}_0$. This gives the position of the object at any time as

$$\vec{\mathbf{s}}(t) = -\frac{1}{2}gt^2\,\hat{\mathbf{j}} + \vec{\mathbf{v}}_0 t + \vec{\mathbf{s}}_0.$$

Let's take a closer look at the initial velocity and initial position. In particular, suppose the object is thrown upward from the origin at an angle θ to the horizontal, with initial speed $\vec{\mathbf{v}}_0$. How can we modify the previous result to reflect this scenario? First, we can

assume it is thrown from the origin. If not, then we can move the origin to the point from where it is thrown. Therefore, $\vec{\mathbf{s}}_0 = \vec{\mathbf{0}}$, as shown in Figure 13.4.6.

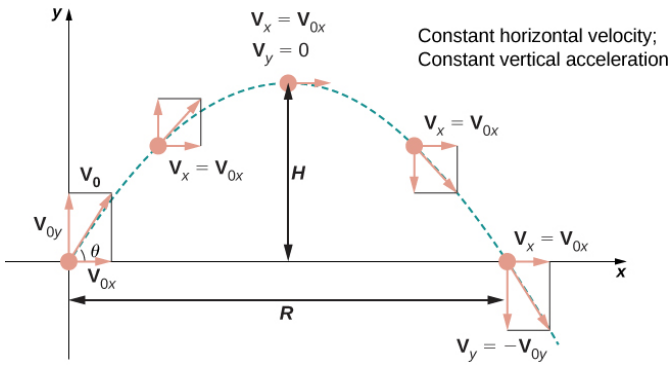


Figure 13.4.6: Projectile motion when the object is thrown upward at an angle θ. The horizontal motion is at constant velocity and the vertical motion is at constant acceleration.

We can rewrite the initial velocity vector in the form $\vec{\mathbf{v}}_0 = v_0 \cos\theta\,\hat{\mathbf{i}} + v_0 \sin\theta\,\hat{\mathbf{j}}$. Then the equation for the position function $\vec{\mathbf{s}}(t)$ becomes

$$\begin{aligned}\vec{\mathbf{s}}(t) &= -\frac{1}{2}gt^2\,\hat{\mathbf{j}} + v_0 t\cos\theta\,\hat{\mathbf{i}} + v_0 t\sin\theta\,\hat{\mathbf{j}} \\ &= v_0 t\cos\theta\,\hat{\mathbf{i}} + v_0 t\sin\theta\,\hat{\mathbf{j}} - \frac{1}{2}gt^2\,\hat{\mathbf{j}} \\ &= v_0 t\cos\theta\,\hat{\mathbf{i}} + \left(v_0 t\sin\theta - \frac{1}{2}gt^2\right)\,\hat{\mathbf{j}}.\end{aligned}$$

The coefficient of $\hat{\mathbf{i}}$ represents the horizontal component of $\vec{\mathbf{s}}(t)$ and is the horizontal distance of the object from the origin at time t. The maximum value of the horizontal distance (measured at the same initial and final altitude) is called the range R. The coefficient of $\hat{\mathbf{j}}$ represents the vertical component of $\vec{\mathbf{s}}(t)$ and is the altitude of the object at time t. The maximum value of the vertical distance is the height H.

Example 13.4.3: Motion of a Cannonball

During an Independence Day celebration, a cannonball is fired from a cannon on a cliff toward the water. The cannon is aimed at an angle of 30° above horizontal and the initial speed of the cannonball is 600 ft/sec. The cliff is 100 ft above the water (Figure 13.4.7).

a. Find the maximum height of the cannonball.
b. How long will it take for the cannonball to splash into the sea?
c. How far out to sea will the cannonball hit the water?

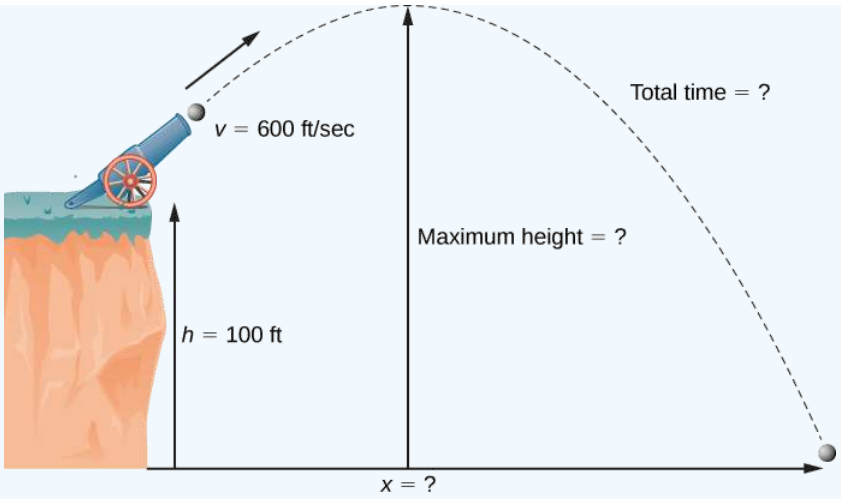


Figure 13.4.7: The flight of a cannonball (ignoring air resistance) is projectile motion.

Solution

We use the equation

$$\vec{\mathbf{s}}(t) = v_0 t\cos\theta\,\hat{\mathbf{i}} + \left(v_0 t\sin\theta - \frac{1}{2}gt^2\right)\,\hat{\mathbf{j}}$$

with $\theta = 30^\circ$, $g = 32\,\frac{\text{ft}}{\text{sec}^2}$, and $v_0 = 600\,\frac{\text{ft}}{\text{sec}}$. Then the position equation becomes

$$\begin{aligned}\vec{\mathbf{s}}(t) &= 600t(\cos 30^\circ)\,\hat{\mathbf{i}} + \left(600t\sin 30^\circ - \frac{1}{2}(32)t^2\right)\,\hat{\mathbf{j}}\\ &= 300t\sqrt{3}\,\hat{\mathbf{i}} + \left(300t - 16t^2\right)\,\hat{\mathbf{j}}\end{aligned}$$

a. The cannonball reaches its maximum height when the vertical component of its velocity is zero, because the cannonball is neither rising nor falling at that point. The velocity vector is

$$\begin{aligned}\vec{\mathbf{v}}(t) &= \vec{\mathbf{s}}\,'(t)\\ &= 300\sqrt{3}\,\hat{\mathbf{i}} + (300 - 32t)\,\hat{\mathbf{j}}\end{aligned}$$

Therefore, the vertical component of velocity is given by the expression $300 - 32t$. Setting this expression equal to zero and solving for t gives $t = 9.375$ sec. The height of the cannonball at this time is given by the vertical component of the position vector, evaluated at $t = 9.375$.

$$\begin{aligned}\vec{\mathbf{s}}(9.375) &= 300(9.375)\sqrt{3}\,\hat{\mathbf{i}} + (300(9.375) - 16(9.375)^2)\,\hat{\mathbf{j}}\\ &= 4871.39\,\hat{\mathbf{i}} + 1406.25\,\hat{\mathbf{j}}\end{aligned}$$

Therefore, the maximum height of the cannonball is 1406.39 ft above the cannon, or 1506.39 ft above sea level.

b. When the cannonball lands in the water, it is 100 ft below the cannon. Therefore, the vertical component of the position vector is equal to -100. Setting the vertical component of $\vec{\mathbf{s}}(t)$ equal to -100 and solving, we obtain

$$\begin{aligned} 300t - 16t^2 &= -100 \\ 16t^2 - 300t - 100 &= 0 \\ 4t^2 - 75 - 25 &= 0 \end{aligned}$$

$$\begin{aligned} t &= \frac{75 \pm \sqrt{(-75)^2 - 4(4)(-25)}}{2(4)} \\ &= \frac{75 \pm \sqrt{6025}}{8} \\ &= \frac{75 \pm 5\sqrt{241}}{8} \end{aligned}$$

The positive value of t that solves this equation is approximately 19.08. Therefore, the cannonball hits the water after approximately 19.08 sec.

c. To find the distance out to sea, we simply substitute the answer from part (b) into $\vec{\mathbf{s}}(t)$:

$$\begin{aligned} \vec{\mathbf{s}}(19.08) &= 300(19.08)\sqrt{3}\,\hat{\mathbf{i}} + \left(300(19.08) - 16(19.08)^2\right)\,\hat{\mathbf{j}} \\ &= 9914.26\,\hat{\mathbf{i}} - 100.7424\,\hat{\mathbf{j}} \end{aligned}$$

Therefore, the ball hits the water about 9914.26 ft away from the base of the cliff. Notice that the vertical component of the position vector is very close to −100, which tells us that the ball just hit the water. Note that 9914.26 feet is not the true range of the cannon since the cannonball lands in the ocean at a location below the cannon. The range of the cannon would be determined by finding how far out the cannonball is when its height is 100 ft above the water (the same as the altitude of the cannon).

? Exercise 13.4.3

An archer fires an arrow at an angle of 40° above the horizontal with an initial speed of 98 m/sec. The height of the archer is 171.5 cm. Find the horizontal distance the arrow travels before it hits the ground.

Hint

The equation for the position vector needs to account for the height of the archer in meters.

Answer

967.15 m

One final question remains: In general, what is the maximum distance a projectile can travel, given its initial speed? To determine this distance, we assume the projectile is fired from ground level and we wish it to return to ground level. In other words, we want to determine an equation for the range. In this case, the equation of projectile motion is

$$\vec{\mathbf{s}} = v_0 t\cos\theta\,\hat{\mathbf{i}} + \left(v_0 t\sin\theta - \frac{1}{2}gt^2\right)\,\hat{\mathbf{j}}.$$

Setting the second component equal to zero and solving for t yields

$$\begin{aligned} v_0 t\sin\theta - \frac{1}{2}gt^2 &= 0 \\ t\left(v_0\sin\theta - \frac{1}{2}gt\right) &= 0 \end{aligned}$$

Therefore, either $t = 0$ or $t = \dfrac{2v_0\sin\theta}{g}$. We are interested in the second value of t, so we substitute this into $\vec{\mathbf{s}}(t)$, which gives

$$\vec{\mathbf{s}}\left(\frac{2v_0\sin\theta}{g}\right) = v_0\left(\frac{2v_0\sin\theta}{g}\right)\cos\theta\,\hat{\mathbf{i}} + \left(v_0\left(\frac{2v_0\sin\theta}{g}\right)\sin\theta - \frac{1}{2}g\left(\frac{2v_0\sin\theta}{g}\right)^2\right)\hat{\mathbf{j}}$$
$$= \left(\frac{2v_0^2\sin\theta\cos\theta}{g}\right)\hat{\mathbf{i}}$$
$$= \frac{v_0^2\sin 2\theta}{g}\,\hat{\mathbf{i}}.$$

Thus, the expression for the range of a projectile fired at an angle θ is

$$R = \frac{v_0^2\sin 2\theta}{g}\,\hat{\mathbf{i}}.$$

The only variable in this expression is θ. To maximize the distance traveled, take the derivative of the coefficient of $\hat{\mathbf{i}}$ with respect to θ and set it equal to zero:

$$\frac{d}{d\theta}\left(\frac{v_0^2\sin 2\theta}{g}\right) = 0$$
$$\frac{2v_0^2\cos 2\theta}{g} = 0$$
$$\theta = 45^\circ$$

This value of θ) is the smallest positive value that makes the derivative equal to zero. Therefore, in the absence of air resistance, the best angle to fire a projectile (to maximize the range) is at a 45° angle. The distance it travels is given by

$$\vec{\mathbf{s}}\left(\frac{2v_0\sin 45^\circ}{g}\right) = \frac{v_0^2\sin 90^\circ}{g}\,\hat{\mathbf{i}} = \frac{v_0^2}{g}\,\hat{\mathbf{i}}$$

Therefore, the range for an angle of 45° is $\frac{v_0^2}{g}$ units.

Kepler's Laws

During the early 1600s, Johannes Kepler was able to use the amazingly accurate data from his mentor Tycho Brahe to formulate his three laws of planetary motion, now known as Kepler's laws of planetary motion. These laws also apply to other objects in the solar system in orbit around the Sun, such as comets (e.g., Halley's comet) and asteroids. Variations of these laws apply to satellites in orbit around Earth.

Theorem 13.4.2: Kepler's Laws of Planetary Motion

1. The path of any planet about the Sun is elliptical in shape, with the center of the Sun located at one focus of the ellipse (the law of ellipses).
2. A line drawn from the center of the Sun to the center of a planet sweeps out equal areas in equal time intervals (the law of equal areas) (Figure 13.4.8).
3. The ratio of the squares of the periods of any two planets is equal to the ratio of the cubes of the lengths of their semimajor orbital axes (the Law of Harmonies).

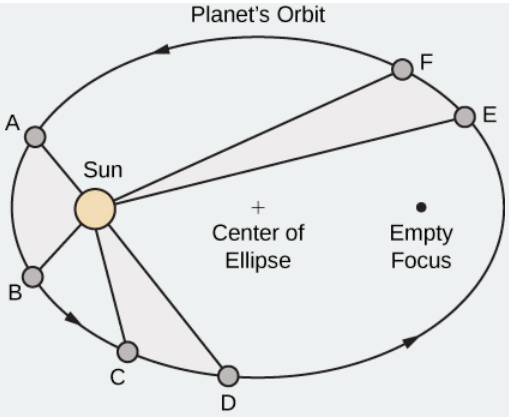


Figure 13.4.8: Kepler's first and second laws are pictured here. The Sun is located at a focus of the elliptical orbit of any planet. Furthermore, the shaded areas are all equal, assuming that the amount of time measured as the planet moves is the same for each region.

Kepler's third law is especially useful when using appropriate units. In particular, *1 astronomical unit* is defined to be the average distance from Earth to the Sun, and is now recognized to be 149,597,870,700 m or, approximately 93,000,000 mi. We therefore write 1 A.U. = 93,000,000 mi. Since the time it takes for Earth to orbit the Sun is 1 year, we use Earth years for units of time. Then, substituting 1 year for the period of Earth and 1 A.U. for the average distance to the Sun, Kepler's third law can be written as

$$T_p^2 = D_p^3$$

for any planet in the solar system, where T_P is the period of that planet measured in Earth years and D_P is the average distance from that planet to the Sun measured in astronomical units. Therefore, if we know the average distance from a planet to the Sun (in astronomical units), we can then calculate the length of its year (in Earth years), and vice versa.

Kepler's laws were formulated based on observations from Brahe; however, they were not proved formally until Sir Isaac Newton was able to apply calculus. Furthermore, Newton was able to generalize Kepler's third law to other orbital systems, such as a moon orbiting around a planet. Kepler's original third law only applies to objects orbiting the Sun.

Proof

Let's now prove Kepler's first law using the calculus of vector-valued functions. First we need a coordinate system. Let's place the Sun at the origin of the coordinate system and let the vector-valued function $\vec{\mathbf{r}}(t)$ represent the location of a planet as a function of time. Newton proved Kepler's law using his second law of motion and his law of universal gravitation. Newton's second law of motion can be written as $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$, where $\vec{\mathbf{F}}$ represents the net force acting on the planet. His law of universal gravitation can be written in the form $\vec{\mathbf{F}} = -\dfrac{GmM}{||\vec{\mathbf{r}}||^2} \cdot \dfrac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||}$, which indicates that the force resulting from the gravitational attraction of the Sun points back toward the Sun, and has magnitude $\dfrac{GmM}{||\vec{\mathbf{r}}||^2}$ (Figure 13.4.9).

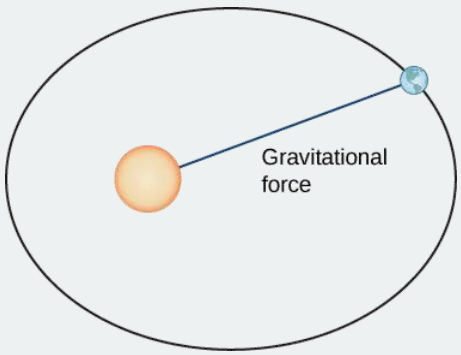


Figure 13.4.9: The gravitational force between Earth and the Sun is equal to the mass of the earth times its acceleration.

Setting these two forces equal to each other, and using the fact that $\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}\,'(t)$, we obtain

$$m\vec{\mathbf{v}}\,'(t) = -\frac{GmM}{\|\vec{\mathbf{r}}\|^2} \cdot \frac{\vec{\mathbf{r}}}{\|\vec{\mathbf{r}}\|},$$

which can be rewritten as

$$\frac{d\vec{\mathbf{v}}}{dt} = -\frac{GM}{||\vec{\mathbf{r}}||^3}\vec{\mathbf{r}}.$$

This equation shows that the vectors $d\vec{\mathbf{v}}/dt$ and $\vec{\mathbf{r}}$ are parallel to each other, so $d\vec{\mathbf{v}}/dt \times \vec{\mathbf{r}} = \vec{\mathbf{0}}$. Next, let's differentiate $\vec{\mathbf{r}} \times \vec{\mathbf{v}}$ with respect to time:

$$\frac{d}{dt}(\vec{\mathbf{r}} \times \vec{\mathbf{v}}) = \frac{d\vec{\mathbf{r}}}{dt} \times \vec{\mathbf{v}} + \vec{\mathbf{r}} \times \frac{d\vec{\mathbf{v}}}{dt} = \vec{\mathbf{v}} \times \vec{\mathbf{v}} + \vec{\mathbf{0}} = \vec{\mathbf{0}}. \tag{13.4.7}$$

This proves that $\vec{\mathbf{r}} \times \vec{\mathbf{v}}$ is a constant vector, which we call $\vec{\mathbf{C}}$. Since $\vec{\mathbf{r}}$ and $\vec{\mathbf{v}}$ are both perpendicular to $\vec{\mathbf{C}}$ for all values of t, they must lie in a plane perpendicular to $\vec{\mathbf{C}}$. Therefore, the motion of the planet lies in a plane.

Next we calculate the expression $d\vec{\mathbf{v}}/dt \times \vec{\mathbf{C}}$:

$$\frac{d\vec{\mathbf{v}}}{dt} \times \vec{\mathbf{C}} = -\frac{GM}{||\vec{\mathbf{r}}||^3}\vec{\mathbf{r}} \times (\vec{\mathbf{r}} \times \vec{\mathbf{v}}) = -\frac{GM}{||\vec{\mathbf{r}}||^3}[(\vec{\mathbf{r}} \cdot \vec{\mathbf{v}})\vec{\mathbf{r}} - (\vec{\mathbf{r}} \cdot \vec{\mathbf{r}})\vec{\mathbf{v}}]. \tag{13.4.8}$$

The last equality in Equation 13.4.8 is from the triple cross product formula (see the cross product section in Introduction to Vectors in Space). We need an expression for $\vec{\mathbf{r}} \cdot \vec{\mathbf{v}}$. To calculate this, we differentiate $\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}$ with respect to time:

$$\frac{d}{dt}(\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}) = \frac{d\vec{\mathbf{r}}}{dt} \cdot \vec{\mathbf{r}} + \vec{\mathbf{r}} \cdot \frac{d\vec{\mathbf{r}}}{dt} = 2\vec{\mathbf{r}} \cdot \frac{d\vec{\mathbf{r}}}{dt} = 2\vec{\mathbf{r}} \cdot \vec{\mathbf{v}}. \tag{13.4.9}$$

Since $\vec{\mathbf{r}} \cdot \vec{\mathbf{r}} = ||\vec{\mathbf{r}}||^2$, we also have

$$\frac{d}{dt}(\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}) = \frac{d}{dt}||\vec{\mathbf{r}}||^2 = 2||\vec{\mathbf{r}}||\frac{d}{dt}||\vec{\mathbf{r}}||. \tag{13.4.10}$$

Combining Equation 13.4.9 and Equation 13.4.10, we get

$$2\vec{\mathbf{r}} \cdot \vec{\mathbf{v}} = 2||\vec{\mathbf{r}}||\frac{d}{dt}||\vec{\mathbf{r}}||$$
$$\vec{\mathbf{r}} \cdot \vec{\mathbf{v}} = ||\vec{\mathbf{r}}||\frac{d}{dt}||\vec{\mathbf{r}}||.$$

Substituting this into Equation 13.4.8 gives us

$$\begin{aligned}\frac{d\vec{\mathbf{v}}}{dt} \times \vec{\mathbf{C}} &= -\frac{GM}{||\vec{\mathbf{r}}||^3}[(\vec{\mathbf{r}} \cdot \vec{\mathbf{v}})\vec{\mathbf{r}} - (\vec{\mathbf{r}} \cdot \vec{\mathbf{r}})\vec{\mathbf{v}}] \\ &= -\frac{GM}{||\vec{\mathbf{r}}||^3}\left[||\vec{\mathbf{r}}||\left(\frac{d}{dt}||\vec{\mathbf{r}}||\right)\vec{\mathbf{r}} - ||\vec{\mathbf{r}}||^2\vec{\mathbf{v}}\right] \\ &= -GM\left[\frac{1}{||\vec{\mathbf{r}}||^2}\left(\frac{d}{dt}||\vec{\mathbf{r}}||\right)\vec{\mathbf{r}} - \frac{1}{||\vec{\mathbf{r}}||}\vec{\mathbf{v}}\right] \\ &= GM\left[\frac{\vec{\mathbf{v}}}{||\vec{\mathbf{r}}||} - \frac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||^2}\left(\frac{d}{dt}||\vec{\mathbf{r}}||\right)\right].\end{aligned} \tag{13.4.11}$$

However,

$$\begin{aligned}\frac{d}{dt}\frac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||} &= \frac{\frac{d}{dt}(\vec{\mathbf{r}})||\vec{\mathbf{r}}|| - \vec{\mathbf{r}}\frac{d}{dt}||\vec{\mathbf{r}}||}{||\vec{\mathbf{r}}||^2} \\ &= \frac{\frac{d\vec{\mathbf{r}}}{dt}}{||\vec{\mathbf{r}}||} - \frac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||^2}\frac{d}{dt}||\vec{\mathbf{r}}|| \\ &= \frac{\vec{\mathbf{v}}}{||\vec{\mathbf{r}}||} - \frac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||^2}\frac{d}{dt}||\vec{\mathbf{r}}||.\end{aligned}$$

Therefore, Equation 13.4.11becomes

$$\frac{d\vec{\mathbf{v}}}{dt} \times \vec{\mathbf{C}} = GM\left(\frac{d}{dt}\frac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||}\right).$$

Since $\vec{\mathbf{C}}$ is a constant vector, we can integrate both sides and obtain

$$\vec{\mathbf{v}} \times \vec{\mathbf{C}} = GM\frac{\vec{\mathbf{r}}}{||\vec{\mathbf{r}}||} + \vec{\mathbf{D}},$$

where $\vec{\mathbf{D}}$ is a constant vector. Our goal is to solve for $||\vec{\mathbf{r}}||$. Let's start by calculating $\vec{\mathbf{r}} \cdot \left(\vec{\mathbf{v}} \times \vec{\mathbf{C}}\right)$:

$$\vec{\mathbf{r}} \cdot \left(\vec{\mathbf{v}} \times \vec{\mathbf{C}}\right) = GM\frac{||\vec{\mathbf{r}}||^2}{||\vec{\mathbf{r}}||} + \vec{\mathbf{r}} \cdot \vec{\mathbf{D}} = GM||\vec{\mathbf{r}}|| + \vec{\mathbf{r}} \cdot \vec{\mathbf{D}}.$$

However, $\vec{\mathbf{r}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{C}}) = (\vec{\mathbf{r}} \times \vec{\mathbf{v}}) \cdot \vec{\mathbf{C}}$, so

$$(\vec{\mathbf{r}} \times \vec{\mathbf{v}}) \cdot \vec{\mathbf{C}} = GM||\vec{\mathbf{r}}|| + \vec{\mathbf{r}} \cdot \vec{\mathbf{D}}.$$

Since $\vec{\mathbf{r}} \times \vec{\mathbf{v}} = \vec{\mathbf{C}}$, we have

$$||\vec{\mathbf{C}}||^2 = GM||\vec{\mathbf{r}}|| + \vec{\mathbf{r}} \cdot \vec{\mathbf{D}}.$$

Note that $\vec{\mathbf{r}} \cdot \vec{\mathbf{D}} = ||\vec{\mathbf{r}}||||\vec{\mathbf{D}}|| \cos\theta$, where θ is the angle between $\vec{\mathbf{r}}$ and $\vec{\mathbf{D}}$. Therefore,

$$||\vec{\mathbf{C}}||^2 = GM||\vec{\mathbf{r}}|| + ||\vec{\mathbf{r}}||||\vec{\mathbf{D}}|| \cos\theta$$

Solving for $||\vec{\mathbf{r}}||$,

$$||\vec{\mathbf{r}}|| = \frac{||\vec{\mathbf{C}}||^2}{GM + ||\vec{\mathbf{D}}|| \cos\theta} = \frac{||\vec{\mathbf{C}}||^2}{GM}\left(\frac{1}{1 + e\cos\theta}\right).$$

where $e = ||\vec{\mathbf{D}}||/GM$. This is the polar equation of a conic with a focus at the origin, which we set up to be the Sun. It is a hyperbola if $e > 1$, a parabola if $e = 1$, or an ellipse if $e < 1$. Since planets have closed orbits, the only possibility is an ellipse. However, at this point it should be mentioned that hyperbolic comets do exist. These are objects that are merely passing through the solar system at speeds too great to be trapped into orbit around the Sun. As they pass close enough to the Sun, the gravitational field of the Sun deflects the trajectory enough so the path becomes hyperbolic.

□

Kepler's third law of planetary motion can be modified to the case of one object in orbit around an object other than the Sun, such as the Moon around the Earth. In this case, Kepler's third law becomes

$$P^2 = \frac{4\pi^2 a^3}{G(m + M)}, \tag{13.4.12}$$

where *m* is the mass of the Moon and *M* is the mass of Earth, *a* represents the length of the major axis of the elliptical orbit, and *P* represents the period.

Example 13.4.4: Using Kepler's Third Law for Nonheliocentric Orbits

Given that the mass of the Moon is 7.35×10^{22} kg, the mass of Earth is 5.97×10^{24} kg, $G = 6.67 \times 10^{-11} \text{m/kg} \cdot \text{sec}^2$, and the period of the moon is 27.3 days, let's find the length of the major axis of the orbit of the Moon around Earth.

Solution

It is important to be consistent with units. Since the universal gravitational constant contains seconds in the units, we need to use seconds for the period of the Moon as well:

$$27.3 \text{ days} \times \frac{24 \text{ hr}}{1 \text{ day}} \times \frac{3600 \text{ esc}}{1 \text{ hour}} = 2,358,720 \text{ sec}$$

Substitute all the data into Equation 13.4.12and solve for a:

$$\begin{aligned} (2,358,720 sec)^2 &= \frac{4\pi^2 a^3}{\left(6.67 \times 10^{-11} \frac{m}{\text{kg} \times \text{sec}^2}\right)(7.35 \times 10^{22}\text{kg} + 5.97 \times 10^{24}\text{kg})} \\ 5.563 \times 10^{12} &= \frac{4\pi^2 a^3}{(6.67 \times 10^{-11}\text{m}^3)(6.04 \times 10^{24})} \\ (5.563 \times 10^{12})(6.67 \times 10^{-11}\text{m}^3)(6.04 \times 10^{24}) &= 4\pi^2 a^3 \\ a^3 &= \frac{2.241 \times 10^{27}}{4\pi^2}\text{m}^3 \\ a &= 3.84 \times 10^8 \text{m} \\ &\approx 384,000 \text{ km}. \end{aligned}$$

Analysis

According to solarsystem.nasa.gov, the actual average distance from the Moon to Earth is 384,400 km. This is calculated using reflectors left on the Moon by Apollo astronauts back in the 1960s.

Exercise 13.4.4

Titan is the largest moon of Saturn. The mass of Titan is approximately $1.35 \times 10^{23} kg$. The mass of Saturn is approximately 5.68×10^{26} kg. Titan takes approximately 16 days to orbit Saturn. Use this information, along with the universal gravitation constant $G = 6.67 \times 10^{-11} \text{m/kg} \cdot \text{sec}^2$ to estimate the distance from Titan to Saturn.

Hint

Make sure your units agree, then use Equation 13.4.12

Answer

$$a \approx 1.224 \times 10^9 \text{m} = 1,224,000 \text{km}$$

Example 13.4.5: Halley's Comet

We now return to the chapter opener, which discusses the motion of Halley's comet around the Sun. Kepler's first law states that Halley's comet follows an elliptical path around the Sun, with the Sun as one focus of the ellipse. The period of Halley's comet is approximately 76.1 years, depending on how closely it passes by Jupiter and Saturn as it passes through the outer solar system. Let's use $T = 76.1$ years. What is the average distance of Halley's comet from the Sun?



Solution

Using the equation $T^2 = D^3$ with $T = 76.1$, we obtain $D^3 = 5791.21$, so $D \approx 17.96$ A.U. This comes out to approximately 1.67×10^9 mi.

A natural question to ask is: What are the maximum (aphelion) and minimum (perihelion) distances from Halley's Comet to the Sun? The eccentricity of the orbit of Halley's Comet is 0.967 (Source: http://nssdc.gsfc.nasa.gov/planetary...cometfact.html). Recall that the formula for the eccentricity of an ellipse is $e = c/a$, where a is the length of the semimajor axis and c is the distance from the center to either focus. Therefore, $0.967 = c/17.96$ and $c \approx 17.37$ A.U. Subtracting this from a gives the perihelion distance $p = a - c = 17.96 - 17.37 = 0.59$ A.U. According to the National Space Science Data Center (Source: http://nssdc.gsfc.nasa.gov/planetary...cometfact.html), the perihelion distance for Halley's comet is 0.587 A.U. To calculate the aphelion distance, we add

$$P = a + c = 17.96 + 17.37 = 35.33 \text{ A.U.}$$

This is approximately 3.3×10^9 mi. The average distance from Pluto to the Sun is 39.5 A.U. (Source: http://www.oarval.org/furthest.htm), so it would appear that Halley's Comet stays just within the orbit of Pluto.

NAVIGATING A BANKED TURN

How fast can a racecar travel through a circular turn without skidding and hitting the wall? The answer could depend on several factors:

- The weight of the car;
- The friction between the tires and the road;
- The radius of the circle;
- The "steepness" of the turn.

In this project we investigate this question for NASCAR racecars at the Bristol Motor Speedway in Tennessee. Before considering this track in particular, we use vector functions to develop the mathematics and physics necessary for answering questions such as this.

A car of mass m moves with constant angular speed ω around a circular curve of radius R (Figure 13.4.9). The curve is banked at an angle θ. If the height of the car off the ground is h, then the position of the car at time t is given by the function $\vec{\mathbf{r}}(t) = < R\cos(\omega t), R\sin(\omega t), h >$.

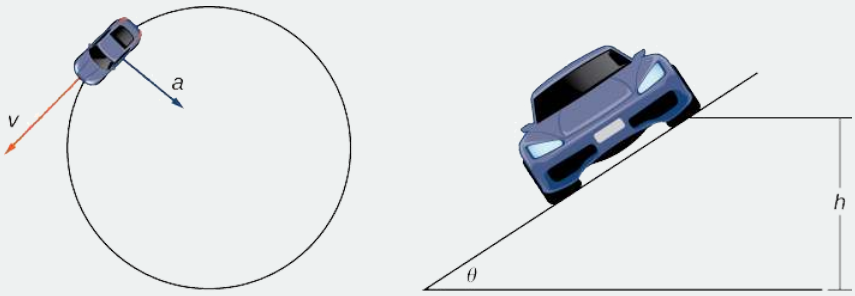


Figure 13.4.9: Views of a race car moving around a track.

1. Find the velocity function $\vec{\mathbf{v}}(t)$ of the car. Show that $\vec{\mathbf{v}}$ is tangent to the circular curve. This means that, without a force to keep the car on the curve, the car will shoot off of it.
2. Show that the speed of the car is ωR. Use this to show that $(2\pi 4)/\|\vec{\mathbf{v}}\| = (2\pi)/\omega$
3. Find the acceleration $\vec{\mathbf{a}}$. Show that this vector points toward the center of the circle and that $\|\vec{\mathbf{a}}\| = R\omega^2$.
4. The force required to produce this circular motion is called the *centripetal force*, and it is denoted $\vec{\mathbf{F}}_{cent}$. This force points toward the center of the circle (not toward the ground). Show that $\|\vec{\mathbf{F}}_{cent}\| = \left(m|\vec{\mathbf{v}}|^2\right)/R$.

As the car moves around the curve, three forces act on it: gravity, the force exerted by the road (this force is perpendicular to the ground), and the friction force (Figure 13.4.10). Because describing the frictional force generated by the tires and the road is complex, we use a standard approximation for the frictional force. Assume that $\vec{\mathbf{f}} = \mu\vec{\mathbf{N}}$ for some positive constant μ. The constant μ is called the *coefficient of friction*.

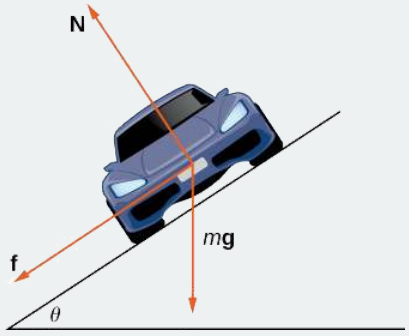


Figure 13.4.10: The car has three forces acting on it: gravity (denoted by $m\vec{\mathbf{g}}$), *the friction force* $\vec{\mathbf{f}}$, *and the force exerted by the road* $\vec{\mathbf{N}}$.

Let v_{max} denote the maximum speed the car can attain through the curve without skidding. In other words, v_{max} is the fastest speed at which the car can navigate the turn. When the car is traveling at this speed, the magnitude of the centripetal force is

$$\|\vec{\mathbf{F}}_{cent}\| = \frac{m(v_{max})^2}{R}.$$

The next three questions deal with developing a formula that relates the speed v_{max} to the banking angle θ.

5. Show that $\vec{\mathbf{N}}\cos\theta = m\vec{\mathbf{g}} + \vec{\mathbf{f}}\sin\theta$. Conclude that $\vec{\mathbf{N}} = (m\vec{\mathbf{g}})/(\cos\theta - \mu\sin\theta)$.
6. The centripetal force is the sum of the forces in the horizontal direction, since the centripetal force points toward the center of the circular curve. Show that

   $$\vec{\mathbf{F}}_{cent} = \vec{\mathbf{N}}\sin\theta + \vec{\mathbf{f}}\cos\theta.$$

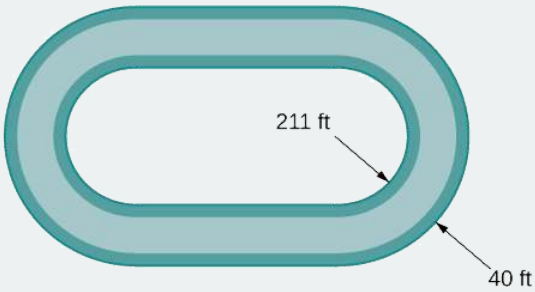
   Conclude that

   $$\vec{\mathbf{F}}_{cent} = \frac{\sin\theta + \mu\cos\theta}{cos\theta - \mu\sin\theta} m\vec{\mathbf{g}}.$$

7. Show that $(v_{\max})^2 = ((\sin\theta + \mu\, cos\theta)/(\cos\theta - \mu\sin\theta))gR$. Conclude that the maximum speed does not actually depend on the mass of the car.
   Now that we have a formula relating the maximum speed of the car and the banking angle, we are in a position to answer the questions like the one posed at the beginning of the project.
   The Bristol Motor Speedway is a NASCAR short track in Bristol, Tennessee. The track has the approximate shape shown in Figure 13.4.11. Each end of the track is approximately semicircular, so when cars make turns they are traveling along an approximately circular curve. If a car takes the inside track and speeds along the bottom of turn 1, the car travels along a semicircle of radius approximately 211 ft with a banking angle of 24°. If the car decides to take the outside track and speeds along the top of turn 1, then the car travels along a semicircle with a banking angle of 28°. (The track has variable angle banking.)



(a)



(b)

Figure 13.4.11: At the Bristol Motor Speedway, Bristol, Tennessee (a), the turns have an inner radius of about 211 ft and a width of 40 ft (b). (credit: part (a) photo by Raniel Diaz, Flickr)

The coefficient of friction for a normal tire in dry conditions is approximately 0.7. Therefore, we assume the coefficient for a NASCAR tire in dry conditions is approximately 0.98.

Before answering the following questions, note that it is easier to do computations in terms of feet and seconds, and then convert the answers to miles per hour as a final step.

8. In dry conditions, how fast can the car travel through the bottom of the turn without skidding?
9. In dry conditions, how fast can the car travel through the top of the turn without skidding?
10. In wet conditions, the coefficient of friction can become as low as 0.1. If this is the case, how fast can the car travel through the bottom of the turn without skidding?
11. Suppose the measured speed of a car going along the outside edge of the turn is 105 mph. Estimate the coefficient of friction for the car's tires.

Key Concepts

- If $\vec{\mathbf{r}}(t)$ represents the position of an object at time t, then $\vec{\mathbf{r}}\,'(t)$ represents the velocity and $\vec{\mathbf{r}}\,''(t)$ represents the acceleration of the object at time t. The magnitude of the velocity vector is speed.
- The acceleration vector always points toward the concave side of the curve defined by $\vec{\mathbf{r}}(t)$. The tangential and normal components of acceleration $a_{\vec{\mathbf{T}}}$ and $a_{\vec{\mathbf{N}}}$ are the projections of the acceleration vector onto the unit tangent and unit normal vectors to the curve.
- Kepler's three laws of planetary motion describe the motion of objects in orbit around the Sun. His third law can be modified to describe motion of objects in orbit around other celestial objects as well.
- Newton was able to use his law of universal gravitation in conjunction with his second law of motion and calculus to prove Kepler's three laws.

Key Equations

- **Velocity**

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{r}}\,'(t)$$

- **Acceleration**

$$\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}\,'(t) = \vec{\mathbf{r}}\,''(t)$$

- **Speed**

$$v(t) = ||\vec{\mathbf{v}}(t)|| = ||\vec{\mathbf{r}}\,'(t)|| = \frac{ds}{dt}$$

- **Tangential component of acceleration**

$$a_{\vec{\mathbf{T}}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{T}} = \frac{\vec{\mathbf{v}} \cdot \vec{\mathbf{a}}}{||\vec{\mathbf{v}}||}$$

- **Normal component of acceleration**

$$a_{\vec{\mathbf{N}}} = \vec{\mathbf{a}} \cdot \vec{\mathbf{N}} = \frac{||\vec{\mathbf{v}} \times \vec{\mathbf{a}}||}{||\vec{\mathbf{v}}||} = \sqrt{||\vec{\mathbf{a}}||^2 - \left(a_{\vec{\mathbf{T}}}\right)^2}$$

Glossary

acceleration vector
the second derivative of the position vector

Kepler's laws of planetary motion
three laws governing the motion of planets, asteroids, and comets in orbit around the Sun

normal component of acceleration
the coefficient of the unit normal vector $\vec{\mathbf{N}}$ when the acceleration vector is written as a linear combination of $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$

projectile motion
motion of an object with an initial velocity but no force acting on it other than gravity

tangential component of acceleration
the coefficient of the unit tangent vector $\vec{\mathbf{T}}$ when the acceleration vector is written as a linear combination of $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$

velocity vector
the derivative of the position vector

- Gilbert Strang (MIT) and Edwin "Jed" Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.
- Edited by Paul Seeburger
  Paul Seeburger added finding point $(1, 2)$ when $t = 1$ in Example 13.4.1.
  He also created Figure 13.4.1.

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CHAPTER OVERVIEW

14: Partial Derivatives

A general Calculus Textmap organized around the textbook

Calculus: Early Transcendentals

by James Stewart

I II III IV V VI VII VIII IX X XI XII XIII XIV XV XVI XVII

This Textmap is currently under construction... please be patient with us.

Template:HideTOC

Topic hierarchy

14.1: Functions of Several Variables
14.2: Limits and Continuity
14.3: Partial Derivatives
14.4: Tangent Planes and Linear Approximations
14.5: The Chain Rule
14.6: Directional Derivatives and the Gradient Vector
14.7: Maximum and Minimum Values
14.8: Lagrange Multipliers

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14.1: Functions of Several Variables

Learning Objectives

- Recognize a function of two variables and identify its domain and range.
- Sketch a graph of a function of two variables.
- Sketch several traces or level curves of a function of two variables.
- Recognize a function of three or more variables and identify its level surfaces.

Our first step is to explain what a function of more than one variable is, starting with functions of two independent variables. This step includes identifying the domain and range of such functions and learning how to graph them. We also examine ways to relate the graphs of functions in three dimensions to graphs of more familiar planar functions.

Functions of Two Variables

The definition of a function of two variables is very similar to the definition for a function of one variable. The main difference is that, instead of mapping values of one variable to values of another variable, we map ordered pairs of variables to another variable.

Definition: Function of Two Variables

A *function of two variables* $z = f(x, y)$ maps each ordered pair (x, y) in a subset D of the real plane R^2 to a unique real number z. The set D is called the domain of the function. The range of f is the set of all real numbers z that has at least one ordered pair $(x, y) \in D$ such that $f(x, y) = z$ as shown in Figure 14.1.1.

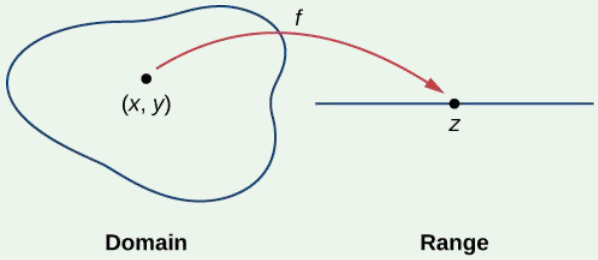


Figure 14.1.1: The domain of a function of two variables consists of ordered pairs (x, y).

Determining the domain of a function of two variables involves taking into account any domain restrictions that may exist. Let's take a look.

Example 14.1.1: Domains and Ranges for Functions of Two Variables

Find the domain and range of each of the following functions:

a. $f(x, y) = 3x + 5y + 2$
b. $g(x, y) = \sqrt{9 - x^2 - y^2}$

Solution

a. This is an example of a linear function in two variables. There are no values or combinations of x and y that cause $f(x, y)$ to be undefined, so the domain of f is R^2. To determine the range, first pick a value for z. We need to find a solution to the equation $f(x, y) = z$, or $3x - 5y + 2 = z$. One such solution can be obtained by first setting $y = 0$, which yields the equation $3x + 2 = z$. The solution to this equation is $x = \dfrac{z-2}{3}$, which gives the ordered pair $\left(\dfrac{z-2}{3}, 0\right)$ as a solution to the equation $f(x, y) = z$ for any value of z. Therefore, the range of the function is all real numbers, or R.

b. For the function $g(x, y)$ to have a real value, the quantity under the square root must be nonnegative:

$$9 - x^2 - y^2 \geq 0.$$

This inequality can be written in the form

$$x^2+y^2 \le 9.$$

Therefore, the domain of $g(x,y)$ is $\{(x,y) \in R^2 \mid x^2+y^2 \le 9\}$. The graph of this set of points can be described as a disk of radius 3 centered at the origin. The domain includes the boundary circle as shown in the following graph.

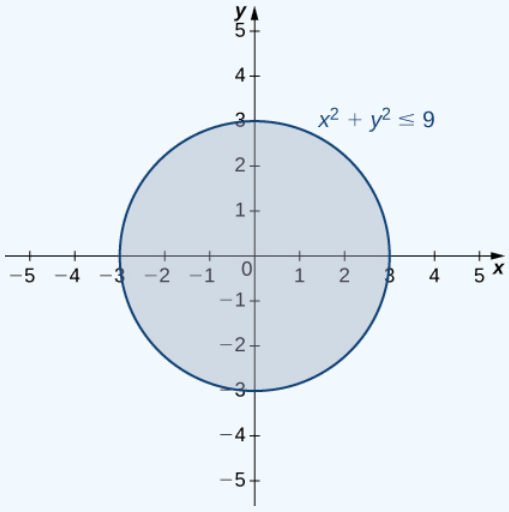


Figure 14.1.2: The domain of the function $g(x,y) = \sqrt{9-x^2-y^2}$ is a closed disk of radius 3.

To determine the range of $g(x,y) = \sqrt{9-x^2-y^2}$ we start with a point (x_0, y_0) on the boundary of the domain, which is defined by the relation $x^2+y^2 = 9$. It follows that $x_0^2+y_0^2 = 9$ and

$$\begin{aligned} g(x_0,y_0) &= \sqrt{9-x_0^2-y_0^2} \\ &= \sqrt{9-(x_0^2+y_0^2)} \\ &= \sqrt{9-9} \\ &= 0. \end{aligned}$$

If $x_0^2+y_0^2 = 0$ (in other words, $x_0 = y_0 = 0$) , then

$$\begin{aligned} g(x_0,y_0) &= \sqrt{9-x_0^2-y_0^2} \\ &= \sqrt{9-(x_0^2+y_0^2)} \\ &= \sqrt{9-0} = 3. \end{aligned}$$

This is the maximum value of the function. Given any value c between 0 and 3, we can find an entire set of points inside the domain of g such that $g(x,y) = c$:

$$\begin{aligned} \sqrt{9-x^2-y^2} &= c \\ 9-x^2-y^2 &= c^2 \\ x^2+y^2 &= 9-c^2. \end{aligned}$$

Since $9-c^2 > 0$, this describes a circle of radius $\sqrt{9-c^2}$ centered at the origin. Any point on this circle satisfies the equation $g(x,y) = c$. Therefore, the range of this function can be written in interval notation as $[0,3]$.

? Exercise 14.1.1

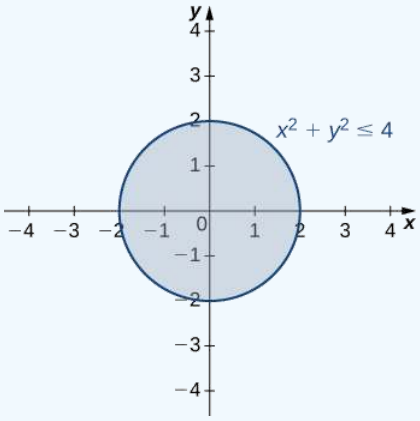
Find the domain and range of the function $f(x,y) = \sqrt{36-9x^2-9y^2}$.

Hint

Determine the set of ordered pairs that do not make the radicand negative.

Solution

The domain is $\{(x,y)|x^2+y^2\leq 4\}$ the shaded circle defined by the inequality $x^2+y^2\leq 4$, which has a circle of radius 2 as its boundary. The range is $[0,6]$.



Graphing Functions of Two Variables

Suppose we wish to graph the function $z=f(x,y)$. This function has two independent variables (x and y) and one dependent variable (z). When graphing a function $y=f(x)$ of one variable, we use the Cartesian plane. We are able to graph any ordered pair (x,y) in the plane, and every point in the plane has an ordered pair (x,y) associated with it. With a function of two variables, each ordered pair (x,y) in the domain of the function is mapped to a real number z. Therefore, the graph of the function f consists of ordered triples (x,y,z). The graph of a function $z=f(x,y)$ of two variables is called a surface.

To understand more completely the concept of plotting a set of ordered triples to obtain a surface in three-dimensional space, imagine the (x,y) coordinate system laying flat. Then, every point in the domain of the function f has a unique z-value associated with it. If z is positive, then the graphed point is located above the xy-plane, if z is negative, then the graphed point is located below the xy-plane. The set of all the graphed points becomes the two-dimensional surface that is the graph of the function f.

Example 14.1.2: Graphing Functions of Two Variables

Create a graph of each of the following functions:

a. $g(x,y)=\sqrt{9-x^2-y^2}$
b. $f(x,y)=x^2+y^2$

Solution

a. In Example $14.1.2$, we determined that the domain of $g(x,y)=\sqrt{9-x^2-y^2}$ is $\{(x,y)\in R^2\mid x^2+y^2\leq 9\}$ and the range is $\{z\in R^2\mid 0\leq z\leq 3\}$. When $x^2+y^2=9$ we have $g(x,y)=0$. Therefore any point on the circle of radius 3 centered at the origin in the xy-plane maps to $z=0$ in R^3. If $x^2+y^2=8$, then $g(x,y)=1$, so any point on the circle of radius $2\sqrt{2}$ centered at the origin in the xy-plane maps to $z=1$ in R^3. As x^2+y^2 gets closer to zero, the value of z approaches 3. When $x^2+y^2=0$, then $g(x,y)=3$. This is the origin in the xy-plane If x^2+y^2 is equal to any other value between 0 and 9, then $g(x,y)$ equals some other constant between 0 and 3. The surface described by this function is a hemisphere centered at the origin with radius 3 as shown in the following graph.

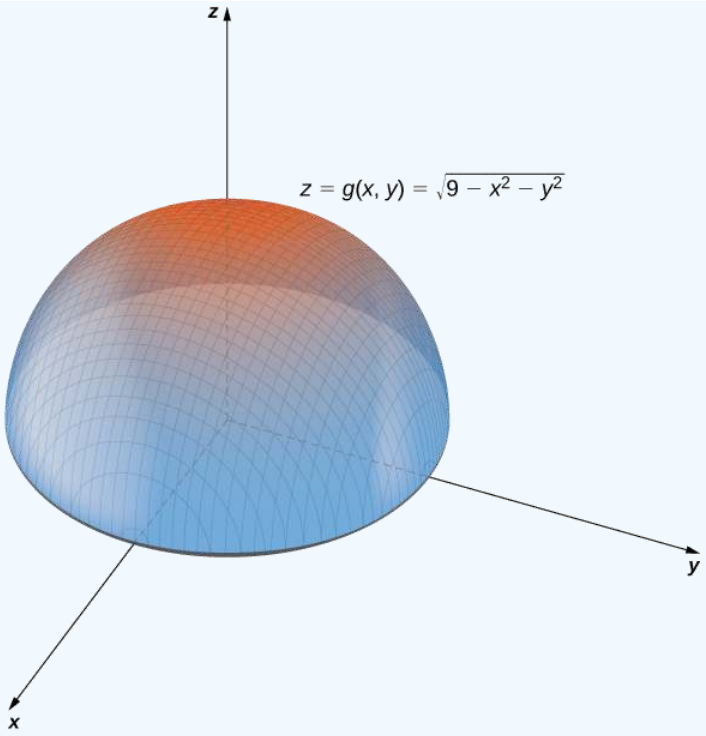


Figure 14.1.3: Graph of the hemisphere represented by the given function of two variables.

b. This function also contains the expression x^2+y^2 . Setting this expression equal to various values starting at zero, we obtain circles of increasing radius. The minimum value of $f(x,y)=x^2+y^2$ is zero (attained when $x=y=0$.. When $x=0$, the function becomes $z=y^2$, and when $y=0$, then the function becomes $z=x^2$. These are cross-sections of the graph, and are parabolas. Recall from Introduction to Vectors in Space that the name of the graph of $f(x,y)=x^2+y^2$ is a **paraboloid**. The graph of f appears in the following graph.

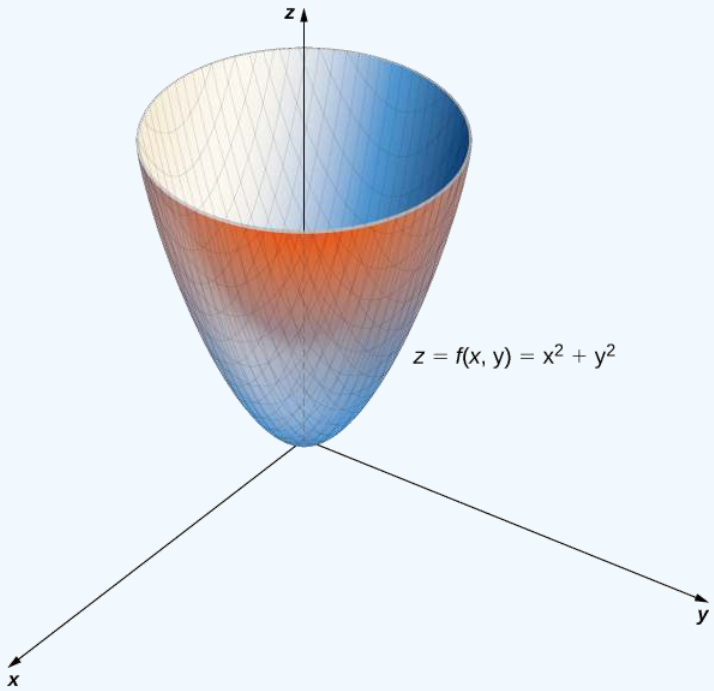


Figure 14.1.4: A paraboloid is the graph of the given function of two variables.

Example 14.1.3: Nuts and Bolts

A profit function for a hardware manufacturer is given by

$$f(x,y) = 16 - (x-3)^2 - (y-2)^2,$$

where x is the number of nuts sold per month (measured in thousands) and y represents the number of bolts sold per month (measured in thousands). Profit is measured in thousands of dollars. Sketch a graph of this function.

Solution

This function is a polynomial function in two variables. The domain of f consists of (x,y) coordinate pairs that yield a nonnegative profit:

$$16 - (x-3)^2 - (y-2)^2 \geq 0$$
$$(x-3)^2 + (y-2)^2 \leq 16.$$

This is a disk of radius 4 centered at $(3,2)$. A further restriction is that both x and y must be nonnegative. When $x = 3$ and $y = 2$, $f(x,y) = 16$. Note that it is possible for either value to be a noninteger; for example, it is possible to sell 2.5 thousand nuts in a month. The domain, therefore, contains thousands of points, so we can consider all points within the disk. For any $z < 16$, we can solve the equation $f(x,y) = 16$:

$$16 - (x-3)^2 - (y-2)^2 = z$$
$$(x-3)^2 + (y-2)^2 = 16 - z.$$

Since $z < 16,$ we know that $16 - z > 0,$ so the previous equation describes a circle with radius $\sqrt{16-z}$ centered at the point $(3,2)$. Therefore. the range of $f(x,y)$ is $\{z \in \mathbb{R} | z \leq 16\}$. The graph of $f(x,y)$ is also a paraboloid, and this paraboloid points downward as shown.

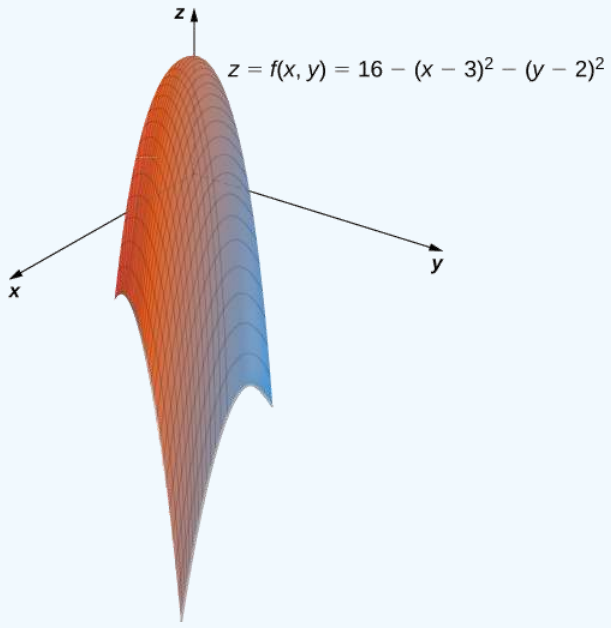
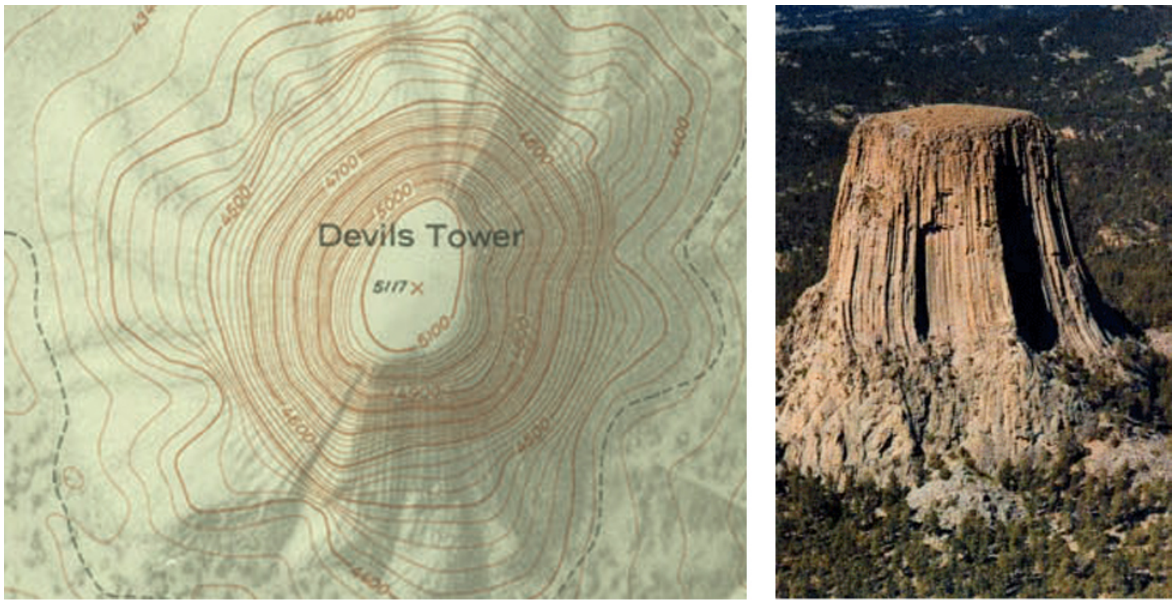


Figure 14.1.5: The graph of the given function of two variables is also a paraboloid.

Level Curves

If hikers walk along rugged trails, they might use a topographical map that shows how steeply the trails change. A *topographical map* contains curved lines called contour lines. Each contour line corresponds to the points on the map that have equal elevation (Figure 14.1.6). A level curve of a function of two variables $f(x,y)$ is completely analogous to a contour line on a topographical map.



(a) (b)

Figure 14.1.6: (a) A topographical map of Devil's Tower, Wyoming. Lines that are close together indicate very steep terrain. (b) A perspective photo of Devil's Tower shows just how steep its sides are. Notice the top of the tower has the same shape as the center of the topographical map.

Definition: Level Curves

Given a function $f(x,y)$ and a number c in the range of f, a level curve of a function of two variables for the value c is defined to be the set of points satisfying the equation $f(x,y)=c$.

Returning to the function $g(x,y)=\sqrt{9-x^2-y^2}$, we can determine the level curves of this function. The range of g is the closed interval $[0,3]$. First, we choose any number in this closed interval—say, $c=2$. The level curve corresponding to $c=2$ is described by the equation

$$\sqrt{9-x^2-y^2}=2.$$

To simplify, square both sides of this equation:

$$9-x^2-y^2=4.$$

Now, multiply both sides of the equation by -1 and add 9 to each side:

$$x^2+y^2=5.$$

This equation describes a circle centered at the origin with radius $\sqrt{5}$. Using values of c between 0 and 3 yields other circles also centered at the origin. If $c=3$, then the circle has radius 0, so it consists solely of the origin. Figure $14.1.7$ is a graph of the level curves of this function corresponding to $c=0,1,2$, and 3. Note that in the previous derivation it may be possible that we introduced extra solutions by squaring both sides. This is not the case here because the range of the square root function is nonnegative.

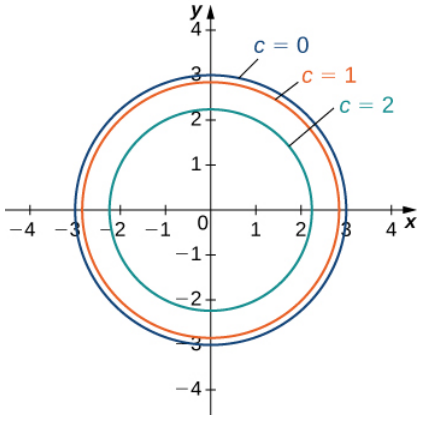


Figure 14.1.7: Level curves of the function $g(x,y)=\sqrt{9-x^2-y^2}$, using $c=0,1,2,$ and $3(c=3$ corresponds to the origin).

A graph of the various level curves of a function is called a *contour map*.

Example 14.1.4: Making a Contour Map

Given the function $f(x,y)=\sqrt{8+8x-4y-4x^2-y^2}$, find the level curve corresponding to $c=0$. Then create a contour map for this function. What are the domain and range of f?

Solution

To find the level curve for $c=0$, we set $f(x,y)=0$ and solve. This gives

$$0=\sqrt{8+8x-4y-4x^2-y^2}\ .$$

We then square both sides and multiply both sides of the equation by -1:

$$4x^2+y^2-8x+4y-8=0.$$

Now, we rearrange the terms, putting the x terms together and the y terms together, and add 8 to each side:

$$4x^2-8x+y^2+4y=8.$$

Next, we group the pairs of terms containing the same variable in parentheses, and factor 4 from the first pair:

$$4(x^2-2x)+(y^2+4y)=8.$$

Then we complete the square in each pair of parentheses and add the correct value to the right-hand side:

$$4(x^2-2x+1)+(y^2+4y+4)=8+4(1)+4.$$

Next, we factor the left-hand side and simplify the right-hand side:

$$4(x-1)^2+(y+2)^2=16.$$

Last, we divide both sides by 16 :

$$\frac{(x-1)^2}{4}+\frac{(y+2)^2}{16}=1. \tag{14.1.1}$$

This equation describes an ellipse centered at $(1,-2)$. The graph of this ellipse appears in the following graph.

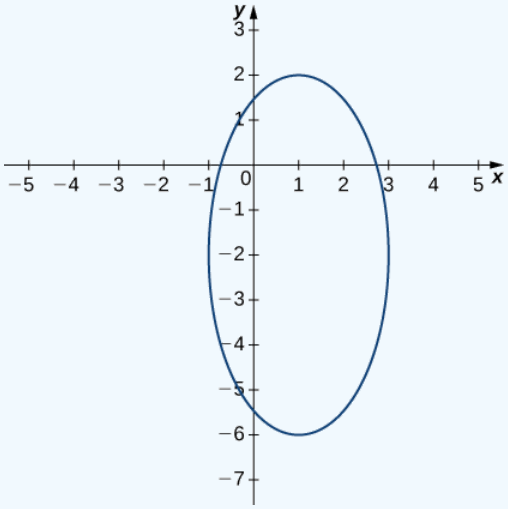


Figure 14.1.8: Level curve of the function $f(x,y)=\sqrt{8+8x-4y-4x^2-y^2}$ corresponding to $c=0$

We can repeat the same derivation for values of c less than 4. Then, Equation 14.1.1 becomes

$$\frac{4(x-1)^2}{16-c^2}+\frac{(y+2)^2}{16-c^2}=1$$

for an arbitrary value of c. Figure 14.1.9 shows a contour map for $f(x,y)$ using the values $c=0,1,2$, and 3. When $c=4$, the level curve is the point $(-1,2)$.

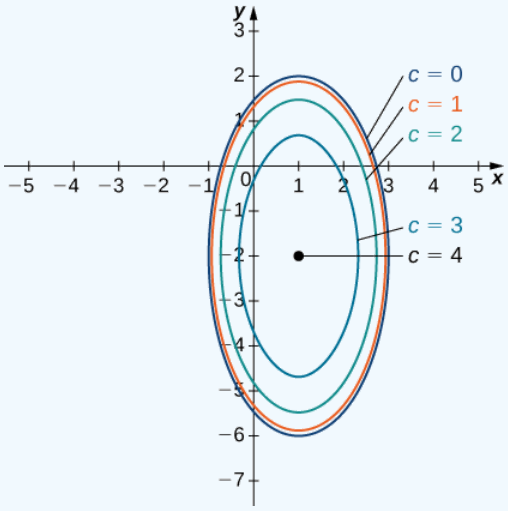


Figure 14.1.9: Contour map for the function $f(x,y)=\sqrt{8+8x-4y-4x^2-y^2}$ using the values $c=0,1,2,3,$ and 4.

Finding the Domain & Range

Since this is a square root function, the radicand must not be negative. So we have

$$8+8x-4y-4x^2-y^2\geq 0$$

Recognizing that the boundary of the domain is an ellipse, we repeat the steps we showed above to obtain

$$\frac{(x-1)^2}{4}+\frac{(y+2)^2}{16}\leq 1$$

So the domain of f can be written: $\left\{(x,y)\,|\,\frac{(x-1)^2}{4}+\frac{(y+2)^2}{16}\leq 1\right\}$.

To find the range of f, we need to consider the possible outputs of this square root function. We know the output cannot be negative, so we need to next check if its output is ever 0. From the work we completed above to find the level curve for $c=0$,

we know the value of f is 0 for any point on that level curve (on the ellipse, $\frac{(x-1)^2}{4}+\frac{(y+2)^2}{16}=1$). So we know the lower bound of the range of this function is 0.

To determine the upper bound for the range of the function in this problem, it's easier if we first complete the square under the radical.

$$\begin{aligned} f(x,y) &= \sqrt{8+8x-4y-4x^2-y^2} \\ &= \sqrt{8-4(x^2-2x\quad)-(y^2+4y\quad)} \\ &= \sqrt{8-4(x^2-2x+1-1)-(y^2+4y+4-4)} \\ &= \sqrt{8-4(x^2-2x+1)+4-(y^2+4y+4)+4} \\ &= \sqrt{16-4(x-1)^2-(y+2)^2} \end{aligned}$$

Now that we have f in this form, we can see how large the radicand can be. Since we are subtracting two perfect squares from 16, we know that the value of the radicand cannot be greater than 16. At the point $(1,-2)$, we can see the radicand will be 16 (since we will be subtracting 0 from 16 at this point. This gives us the maximum value of f, that is $f(1,-2)=\sqrt{16}=4$.

So the range of this function is $[0,4]$.

Exercise 14.1.2

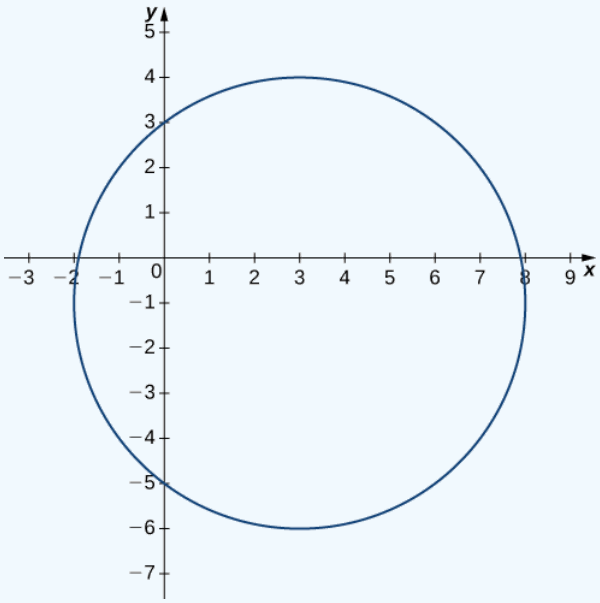
Find and graph the level curve of the function $g(x,y)=x^2+y^2-6x+2y$ corresponding to $c=15$.

Hint

First, set $g(x,y)=15$ and then complete the square.

Solution

The equation of the level curve can be written as $(x-3)^2+(y+1)^2=25,$ which is a circle with radius 5 centered at $(3,-1)$.



Another useful tool for understanding the **graph of a function of two variables** is called a vertical trace. Level curves are always graphed in the xy-plane, but as their name implies, vertical traces are graphed in the xz- or yz-planes.

Definition: Vertical Traces

Consider a function $z = f(x, y)$ with domain $D \subseteq \mathbb{R}^2$. A *vertical trace* of the function can be either the set of points that solves the equation $f(a, y) = z$ for a given constant $x = a$ or $f(x, b) = z$ for a given constant $y = b$.

Example 14.1.5: Finding Vertical Traces

Find vertical traces for the function $f(x, y) = \sin x \cos y$ corresponding to $x = -\frac{\pi}{4}, 0$, and $\frac{\pi}{4}$, and $y = -\frac{\pi}{4}, 0$, and $\frac{\pi}{4}$.

Solution

First set $x = -\frac{\pi}{4}$ in the equation $z = \sin x \cos y$:

$$z = \sin(-\frac{\pi}{4}) \cos y = -\frac{\sqrt{2} \cos y}{2} \approx -0.7071 \cos y.$$

This describes a cosine graph in the plane $x = -\frac{\pi}{4}$. The other values of z appear in the following table.

Vertical Traces Parallel to the $xz - Plane$ for the Function $f(x, y) = \sin x \cos y$

c	Vertical Trace for $x = c$
$-\frac{\pi}{4}$	$z = -\frac{\sqrt{2} \cos y}{2}$
0	$z = 0$
$\frac{\pi}{4}$	$z = \frac{\sqrt{2} \cos y}{2}$

In a similar fashion, we can substitute the $y - values$ in the equation $f(x, y)$ to obtain the traces in the $yz - plane$, as listed in the following table.

Vertical Traces Parallel to the $yz - Plane$ for the Function $f(x, y) = \sin x \cos y$

d	Vertical Trace for $y = d$
$\frac{\pi}{4}$	$z = \frac{\sqrt{2} \sin x}{2}$
0	$z = \sin x$
$-\frac{\pi}{4}$	$z = \frac{\sqrt{2} \sin x}{2}$

The three traces in the $xz - plane$ are cosine functions; the three traces in the $yz - plane$ are sine functions. These curves appear in the intersections of the surface with the planes $x = -\frac{\pi}{4}, x = 0, x = \frac{\pi}{4}$ and $y = -\frac{\pi}{4}, y = 0, y = \frac{\pi}{4}$ as shown in the following figure.

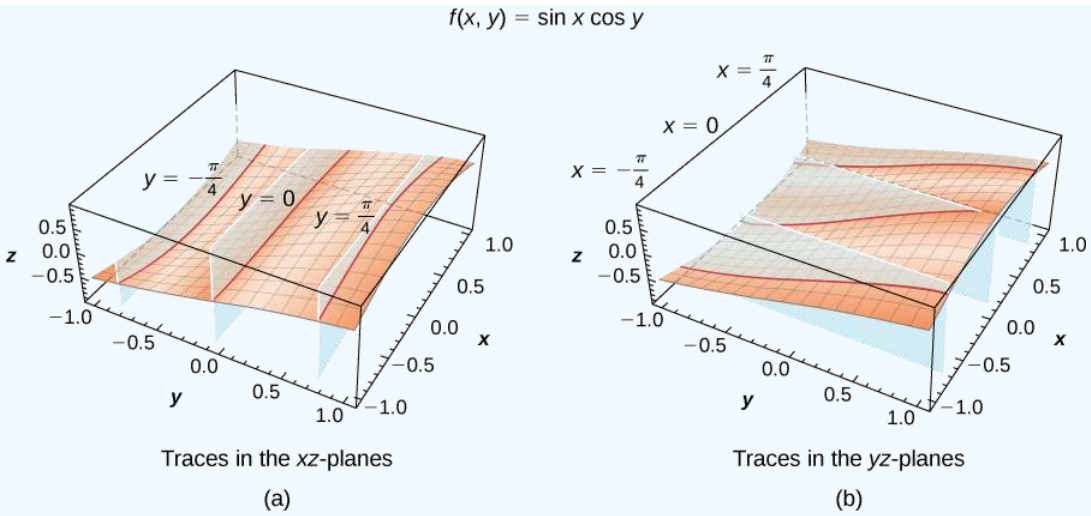


Figure 14.1.10: Vertical traces of the function $f(x,y)$ are cosine curves in the $xz-planes$ (a) and sine curves in the $yz-planes$ (b).

? Exercise 14.1.3

Determine the equation of the vertical trace of the function $g(x,y)=-x^2-y^2+2x+4y-1$ corresponding to $y=3$, and describe its graph.

Hint

Set $y=3$ in the equation $z=-x^2-y^2+2x+4y-1$ and complete the square.

Solution

$z=3-(x-1)^2$. This function describes a parabola opening downward in the plane $y=3$.

Functions of two variables can produce some striking-looking surfaces. Figure 14.1.11shows two examples.

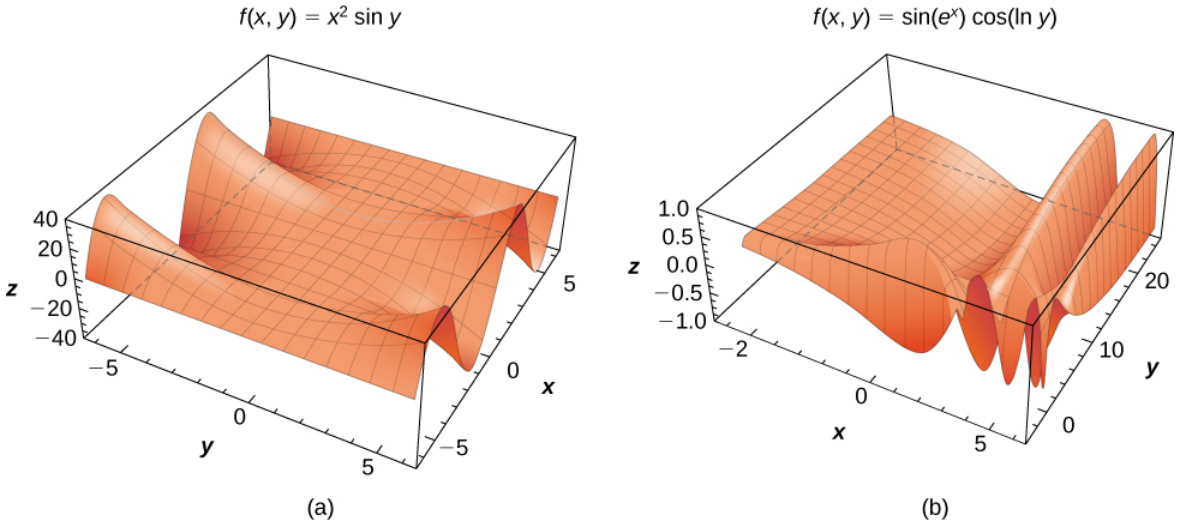


Figure 14.1.11: Examples of surfaces representing functions of two variables: (a) a combination of a power function and a sine function and (b) a combination of trigonometric, exponential, and logarithmic functions.

Functions of More Than Two Variables

So far, we have examined only functions of two variables. However, it is useful to take a brief look at functions of more than two variables. Two such examples are

$$\underbrace{f(x,y,z)=x^2-2xy+y^2+3yz-z^2+4x-2y+3x-6}_{\text{a polynomial in three variables}}$$

and

$$g(x,y,t)=(x^2-4xy+y^2)\sin t-(3x+5y)\cos t.$$

In the first function, (x,y,z) represents a point in space, and the function f maps each point in space to a fourth quantity, such as temperature or wind speed. In the second function, (x,y) can represent a point in the plane, and t can represent time. The function might map a point in the plane to a third quantity (for example, pressure) at a given time t. The method for finding the domain of a function of more than two variables is analogous to the method for functions of one or two variables.

Example 14.1.6: Domains for Functions of Three Variables

Find the domain of each of the following functions:

a. $f(x,y,z)=\dfrac{3x-4y+2z}{\sqrt{9-x^2-y^2-z^2}}$

b. $g(x,y,t)=\dfrac{\sqrt{2t-4}}{x^2-y^2}$

Solution:

a. For the function $f(x,y,z)=\dfrac{3x-4y+2z}{\sqrt{9-x^2-y^2-z^2}}$ to be defined (and be a real value), two conditions must hold:

1. The denominator **cannot** be zero.
2. The radicand **cannot** be negative.

Combining these conditions leads to the inequality

$$9-x^2-y^2-z^2>0.$$

Moving the variables to the other side and reversing the inequality gives the domain as

$$domain(f)=\{(x,y,z)\in R^3 \mid x^2+y^2+z^2<9\},$$

which describes a ball of radius 3 centered at the origin. (Note: The surface of the ball is not included in this domain.)

b. For the function $g(x,y,t)=\dfrac{\sqrt{2t-4}}{x^2-y^2}$ to be defined (and be a real value), two conditions must hold:

1. The radicand cannot be negative.
2. The denominator cannot be zero.

Since the radicand cannot be negative, this implies $2t-4\geq 0$, and therefore that $t\geq 2$. Since the denominator cannot be zero, $x^2-y^2\neq 0$, or $x^2\neq y^2$, Which can be rewritten as $y=\pm x$, which are the equations of two lines passing through the origin. Therefore, the domain of g is

$$domain(g)=\{(x,y,t)|y\neq\pm x, t\geq 2\}.$$

Exercise 14.1.4

Find the domain of the function $h(x,y,t)=(3t-6)\sqrt{y-4x^2+4}$.

Hint

Check for values that make radicands negative or denominators equal to zero.

Solution

$$domain(h)=\{(x,y,t)\in\mathbb{R}^3 \mid y\geq 4x^2-4\}$$

Functions of two variables have level curves, which are shown as curves in the $xy-plane$. However, when the function has three variables, the curves become surfaces, so we can define level surfaces for functions of three variables.

Definition: Level Surface of a Function of Three Variables

Given a function $f(x,y,z)$ and a number c in the range of f, a *level surface of a function of three variables* is defined to be the set of points satisfying the equation $f(x,y,z)=c$.

Example 14.1.7: Finding a Level Surface

Find the level surface for the function $f(x,y,z)=4x^2+9y^2-z^2$ corresponding to $c=1$.

Solution

The level surface is defined by the equation $4x^2+9y^2-z^2=1$. This equation describes a hyperboloid of one sheet as shown in Figure 14.1.12

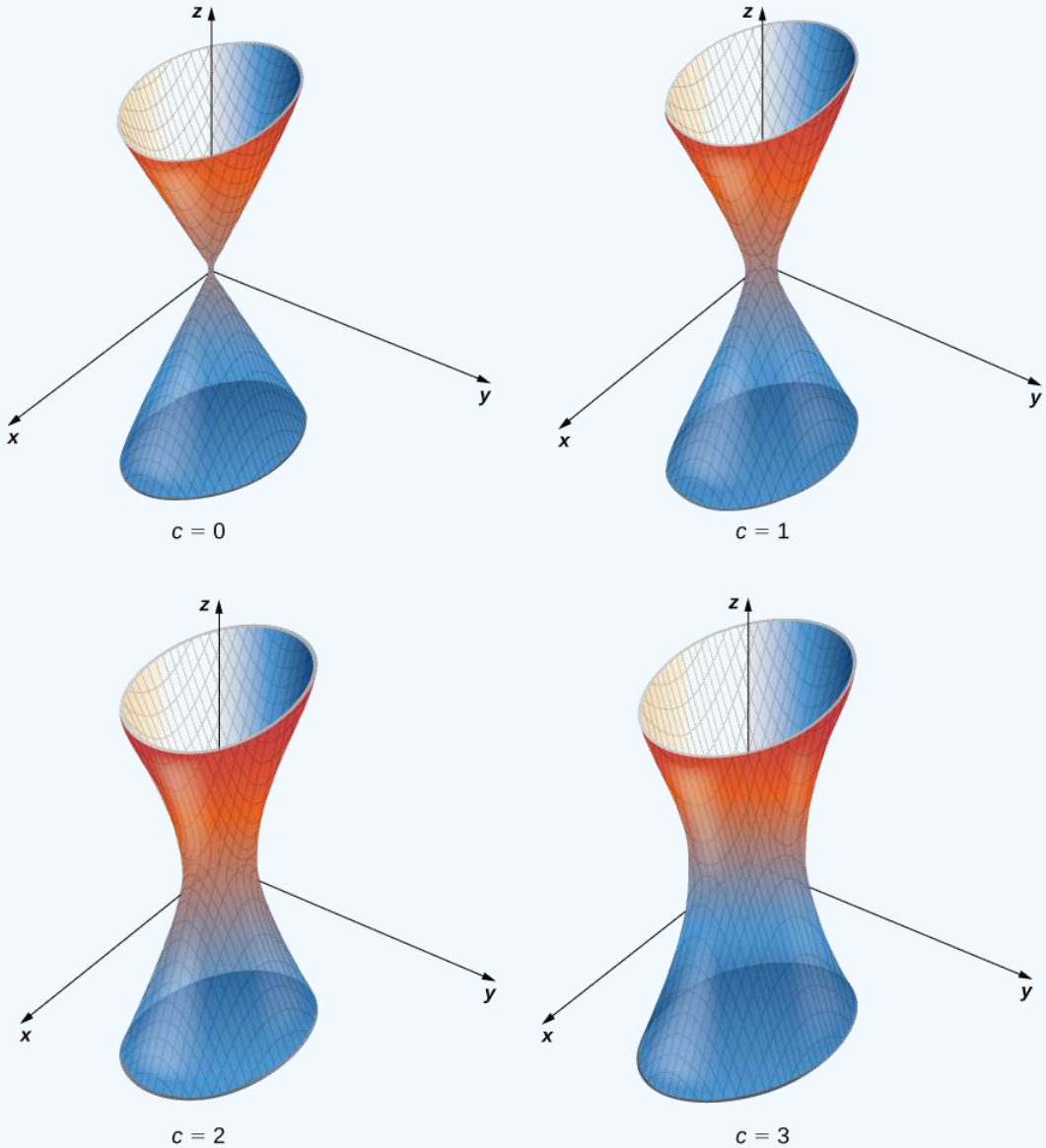


Figure 14.1.12: A hyperboloid of one sheet with some of its level surfaces.

? Exercise 14.1.5

Find the equation of the level surface of the function

$$g(x,y,z)=x^2+y^2+z^2-2x+4y-6z$$

corresponding to $c=2$, and describe the surface, if possible.

Hint

Set $g(x,y,z)=c$ and complete the square.

Solution

$(x-1)^2+(y+2)^2+(z-3)^2=16$ describes a sphere of radius 4 centered at the point $(1,-2,3)$.

Summary

- The graph of a function of two variables is a surface in $\mathbb{R}^3$ and can be studied using level curves and vertical traces.
- A set of level curves is called a contour map.

Key Equations

- **Vertical trace**

  $f(a,y)=z$ for $x=a$ or $f(x,b)=z$ for $y=b$

- **Level surface of a function of three variables**

  $f(x,y,z)=c$

Glossary

contour map
a plot of the various level curves of a given function $f(x,y)$

function of two variables
a function $z=f(x,y)$ that maps each ordered pair (x,y) in a subset D of R^2 to a unique real number z

graph of a function of two variables
a set of ordered triples (x,y,z) that satisfies the equation $z=f(x,y)$ plotted in three-dimensional Cartesian space

level curve of a function of two variables
the set of points satisfying the equation $f(x,y)=c$ for some real number c in the range of f

level surface of a function of three variables
the set of points satisfying the equation $f(x,y,z)=c$ for some real number c in the range of f

surface
the graph of a function of two variables, $z=f(x,y)$

vertical trace
the set of ordered triples (c,y,z) that solves the equation $f(c,y)=z$ for a given constant $x=c$ or the set of ordered triples (x,d,z) that solves the equation $f(x,d)=z$ for a given constant $y=d$

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14.2: Limits and Continuity

Learning Objectives

- Calculate the limit of a function of two variables.
- Learn how a function of two variables can approach different values at a boundary point, depending on the path of approach.
- State the conditions for continuity of a function of two variables.
- Verify the continuity of a function of two variables at a point.
- Calculate the limit of a function of three or more variables and verify the continuity of the function at a point.

We have now examined functions of more than one variable and seen how to graph them. In this section, we see how to take the limit of a function of more than one variable, and what it means for a function of more than one variable to be continuous at a point in its domain. It turns out these concepts have aspects that just don't occur with functions of one variable.

Limit of a Function of Two Variables

Recall from Section 2.5 that the definition of a limit of a function of one variable:

Let $f(x)$ be defined for all $x \neq a$ in an open interval containing a. Let L be a real number. Then

$$\lim_{x \to a} f(x) = L$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$, such that if $0 < |x - a| < \delta$ for all x in the domain of f, then

$$|f(x) - L| < \varepsilon.$$

Before we can adapt this definition to define a limit of a function of two variables, we first need to see how to extend the idea of an open interval in one variable to an open interval in two variables.

Definition: δ Disks

Consider a point $(a, b) \in \mathbb{R}^2$. A δ **disk** centered at point (a, b) is defined to be an open disk of radius δ centered at point (a, b) —that is,

$$\{(x, y) \in \mathbb{R}^2 \mid (x - a)^2 + (y - b)^2 < \delta^2\}$$

as shown in Figure 14.2.1.

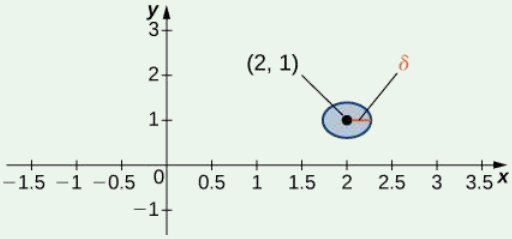


Figure 14.2.1: A δ disk centered around the point $(2, 1)$.

The idea of a δ disk appears in the definition of the limit of a function of two variables. If δ is small, then all the points (x, y) in the δ disk are close to (a, b). This is completely analogous to x being close to a in the definition of a limit of a function of one variable. In one dimension, we express this restriction as

$$a - \delta < x < a + \delta.$$

In more than one dimension, we use a δ disk.

Definition: limit of a function of two variables

Let f be a function of two variables, x and y. The limit of $f(x,y)$ as (x,y) approaches (a,b) is L, written

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for each $\varepsilon > 0$ there exists a small enough $\delta > 0$ such that for all points (x,y) in a δ disk around (a,b), except possibly for (a,b) itself, the value of $f(x,y)$ is no more than ε away from L (Figure 14.2.2).

Using symbols, we write the following: For any $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x,y) - L| < \varepsilon$$

whenever

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta.$$

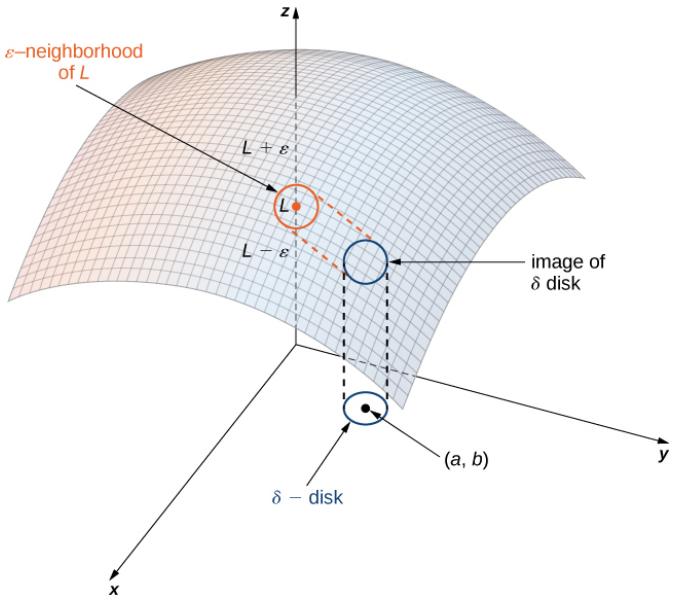


Figure 14.2.2: The limit of a function involving two variables requires that $f(x,y)$ be within ε of L whenever (x,y) is within δ of (a,b). The smaller the value of ε, the smaller the value of δ.

Proving that a limit exists using the definition of a **limit of a function of two variables** can be challenging. Instead, we use the following theorem, which gives us shortcuts to finding limits. The formulas in this theorem are an extension of the formulas in the limit laws theorem in the section titled The Limit Laws in single-variable calculus.

Theorem 14.2.1: Limit Laws for Functions of Two Variables

Let $f(x,y)$ and $g(x,y)$ be defined for all $(x,y) \neq (a,b)$ in a neighborhood around (a,b), and assume the neighborhood is contained completely inside the domain of f. Assume that L and M are real numbers such that

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

and

$$\lim_{(x,y)\to(a,b)} g(x,y) = M,$$

and let c be a constant. Then each of the following statements holds:

Constant Law:

$$\lim_{(x,y)\to(a,b)} c = c$$

Identity Laws:

$$\lim_{(x,y)\to(a,b)} x = a$$

$$\lim_{(x,y)\to(a,b)} y = b$$

Sum Law:

$$\lim_{(x,y)\to(a,b)} (f(x,y)+g(x,y)) = L+M$$

Difference Law:

$$\lim_{(x,y)\to(a,b)} (f(x,y)-g(x,y)) = L-M$$

Constant Multiple Law:

$$\lim_{(x,y)\to(a,b)} (cf(x,y)) = cL$$

Product Law:

$$\lim_{(x,y)\to(a,b)} (f(x,y)g(x,y)) = LM$$

Quotient Law:

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M} \text{ for } M \neq 0$$

Power Law:

$$\lim_{(x,y)\to(a,b)} (f(x,y))^n = L^n$$

for any positive integer n.

Root Law:

$$\lim_{(x,y)\to(a,b)} \sqrt[n]{f(x,y)} = \sqrt[n]{L}$$

for all L if n is odd and positive, and for $L \geq 0$ if n is even and positive.

The proofs of these properties are similar to those for the limits of functions of one variable. We can apply these laws to finding limits of various functions.

Example 14.2.1: Finding the Limit of a Function of Two Variables

Find each of the following limits:

a. $\lim_{(x,y)\to(2,-1)} (x^2-2xy+3y^2-4x+3y-6)$

b. $\lim_{(x,y)\to(2,-1)} \frac{2x+3y}{4x-3y}$

Solution

a. First use the sum and difference laws to separate the terms:

$$\lim_{(x,y)\to(2,-1)}(x^2-2xy+3y^2-4x+3y-6)$$
$$=\left(\lim_{(x,y)\to(2,-1)}x^2\right)-\left(\lim_{(x,y)\to(2,-1)}2xy\right)+\left(\lim_{(x,y)\to(2,-1)}3y^2\right)-\left(\lim_{(x,y)\to(2,-1)}4x\right)$$
$$+\left(\lim_{(x,y)\to(2,-1)}3y\right)-\left(\lim_{(x,y)\to(2,-1)}6\right).$$

Next, use the constant multiple law on the second, third, fourth, and fifth limits:

$$=(\lim_{(x,y)\to(2,-1)}x^2)-2(\lim_{(x,y)\to(2,-1)}xy)+3(\lim_{(x,y)\to(2,-1)}y^2)-4(\lim_{(x,y)\to(2,-1)}x)$$
$$+3(\lim_{(x,y)\to(2,-1)}y)-\lim_{(x,y)\to(2,-1)}6.$$

Now, use the power law on the first and third limits, and the product law on the second limit:

$$\left(\lim_{(x,y)\to(2,-1)}x\right)^2-2\left(\lim_{(x,y)\to(2,-1)}x\right)\left(\lim_{(x,y)\to(2,-1)}y\right)+3\left(\lim_{(x,y)\to(2,-1)}y\right)^2$$
$$-4\left(\lim_{(x,y)\to(2,-1)}x\right)+3\left(\lim_{(x,y)\to(2,-1)}y\right)-\lim_{(x,y)\to(2,-1)}6.$$

Last, use the identity laws on the first six limits and the constant law on the last limit:

$$\lim_{(x,y)\to(2,-1)}(x^2-2xy+3y^2-4x+3y-6)=(2)^2-2(2)(-1)+3(-1)^2-4(2)+3(-1)-6$$
$$=-6.$$

b. Before applying the quotient law, we need to verify that the limit of the denominator is nonzero. Using the difference law, constant multiple law, and identity law,

$$\lim_{(x,y)\to(2,-1)}(4x-3y)=\lim_{(x,y)\to(2,-1)}4x-\lim_{(x,y)\to(2,-1)}3y$$
$$=4(\lim_{(x,y)\to(2,-1)}x)-3(\lim_{(x,y)\to(2,-1)}y)$$
$$=4(2)-3(-1)=11.$$

Since the limit of the denominator is nonzero, the quotient law applies. We now calculate the limit of the numerator using the difference law, constant multiple law, and identity law:

$$\lim_{(x,y)\to(2,-1)}(2x+3y)=\lim_{(x,y)\to(2,-1)}2x+\lim_{(x,y)\to(2,-1)}3y$$
$$=2(\lim_{(x,y)\to(2,-1)}x)+3(\lim_{(x,y)\to(2,-1)}y)$$
$$=2(2)+3(-1)=1.$$

Therefore, according to the quotient law we have

$$\lim_{(x,y)\to(2,-1)}\frac{2x+3y}{4x-3y}=\frac{\lim_{(x,y)\to(2,-1)}(2x+3y)}{\lim_{(x,y)\to(2,-1)}(4x-3y)}$$
$$=\frac{1}{11}.$$

? Exercise 14.2.1:

Evaluate the following limit:

$$\lim_{(x,y)\to(5,-2)}\sqrt[3]{\frac{x^2-y}{y^2+x-1}}.$$

Hint

Use the limit laws.

Answer

$$\lim_{(x,y)\to(5,-2)} \sqrt[3]{\frac{x^2-y}{y^2+x-1}} = \frac{3}{2}$$

Since we are taking the limit of a function of two variables, the point (a,b) is in $\mathbb{R}^2$, and it is possible to approach this point from an infinite number of directions. Sometimes when calculating a limit, the answer varies depending on the path taken toward (a,b). If this is the case, then the limit fails to exist. In other words, the limit must be unique, regardless of path taken.

Example 14.2.2: Limits That Fail to Exist

Show that neither of the following limits exist:

a. $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy}{3x^2+y^2}$

b. $\lim\limits_{(x,y)\to(0,0)} \dfrac{4xy^2}{x^2+3y^4}$

Solution

a. The domain of the function $f(x,y) = \dfrac{2xy}{3x^2+y^2}$ consists of all points in the xy-plane except for the point $(0,0)$ (Figure 14.2.3). To show that the limit does not exist as (x,y) approaches $(0,0)$, we note that it is impossible to satisfy the definition of a limit of a function of two variables because of the fact that the function takes different values along different lines passing through point $(0,0)$. First, consider the line $y=0$ in the xy-plane. Substituting $y=0$ into $f(x,y)$ gives

$$f(x,0) = \frac{2x(0)}{3x^2+0^2} = 0$$

for any value of x. Therefore the value of f remains constant for any point on the x-axis, and as y approaches zero, the function remains fixed at zero.

Next, consider the line $y=x$. Substituting $y=x$ into $f(x,y)$ gives

$$f(x,x) = \frac{2x(x)}{3x^2+x^2} = \frac{2x^2}{4x^2} = \tfrac{1}{2}.$$

This is true for any point on the line $y=x$. If we let x approach zero while staying on this line, the value of the function remains fixed at $\frac{1}{2}$, regardless of how small x is.

Choose a value for ε that is less than $1/2$—say, $1/4$. Then, no matter how small a δ disk we draw around $(0,0)$, the values of $f(x,y)$ for points inside that δ disk will include both 0 and $\frac{1}{2}$. Therefore, the definition of limit at a point is never satisfied and the limit fails to exist.

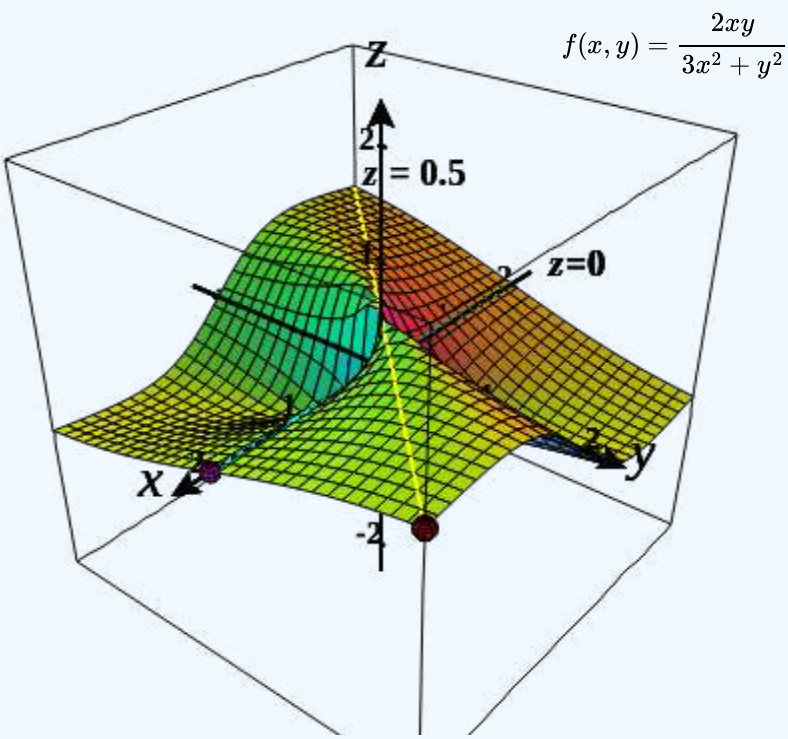


Figure 14.2.3: Graph of the function $f(x,y)=\dfrac{2xy}{3x^2+y^2}$. Along the line $y=0$, the function is equal to zero; along the line $y=x$, the function is equal to $\frac{1}{2}$.

b. In a similar fashion to a., we can approach the origin along any straight line passing through the origin. If we try the x-axis (i.e., $y=0$), then the function remains fixed at zero. The same is true for the y-axis. Suppose we approach the origin along a straight line of slope k. The equation of this line is $y=kx$. Then the limit becomes

$$\begin{aligned}\lim_{(x,y)\to(0,0)}\frac{4xy^2}{x^2+3y^4} &= \lim_{(x,y)\to(0,0)}\frac{4x(kx)^2}{x^2+3(kx)^4}\\ &= \lim_{(x,y)\to(0,0)}\frac{4k^2x^3}{x^2+3k^4x^4}\\ &= \lim_{(x,y)\to(0,0)}\frac{4k^2x}{1+3k^4x^2}\\ &= \frac{\lim\limits_{(x,y)\to(0,0)}(4k^2x)}{\lim\limits_{(x,y)\to(0,0)}(1+3k^4x^2)}\\ &= 0.\end{aligned}$$

regardless of the value of k. It would seem that the limit is equal to zero. What if we chose a curve passing through the origin instead? For example, we can consider the parabola given by the equation $x=y^2$. Substituting y^2 in place of x in $f(x,y)$ gives

$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+3y^4} = \lim_{(x,y)\to(0,0)} \frac{4(y^2)y^2}{(y^2)^2+3y^4}$$
$$= \lim_{(x,y)\to(0,0)} \frac{4y^4}{y^4+3y^4}$$
$$= \lim_{(x,y)\to(0,0)} 1$$
$$= 1.$$

By the same logic in part a, it is impossible to find a δ disk around the origin that satisfies the definition of the limit for any value of $\varepsilon < 1$. Therefore,

$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+3y^4}$$

does **not** exist.

? Exercise 14.2.2:

Show that

$$\lim_{(x,y)\to(2,1)} \frac{(x-2)(y-1)}{(x-2)^2+(y-1)^2}$$

does not exist.

Hint

Pick a line with slope k passing through point $(2,1)$.

Answer

If $y = k(x-2)+1$, then $\lim_{(x,y)\to(2,1)} \frac{(x-2)(y-1)}{(x-2)^2+(y-1)^2} = \frac{k}{1+k^2}$. Since the answer depends on k, the limit fails to exist.

Interior Points and Boundary Points

To study continuity and differentiability of a function of two or more variables, we first need to learn some new terminology.

Definition: interior and boundary points

Let S be a subset of $\mathbb{R}^2$ (Figure 14.2.4).

1. A point P_0 is called an *interior point* of S if there is a δ disk centered around P_0 contained completely in S.
2. A point P_0 is called a *boundary point* of S if every δ disk centered around P_0 contains points both inside and outside S.

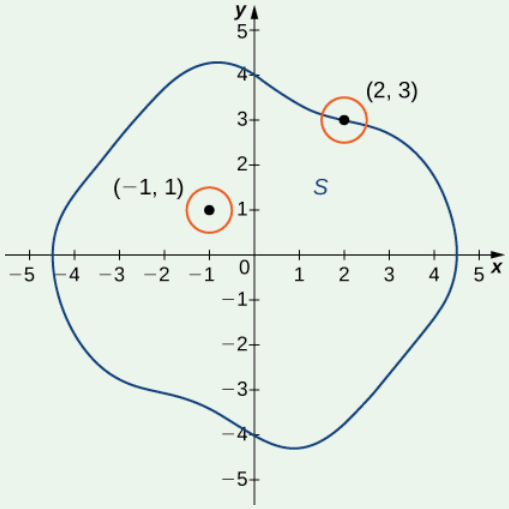


Figure 14.2.4: In the set S shown, $(-1,1)$ is an interior point and $(2,3)$ is a boundary point.

Definition: Open and closed sets

Let S be a subset of $\mathbb{R}^2$ (Figure 14.2.4).

1. S is called an *open set* if every point of S is an interior point.
2. S is called a *closed set* if it contains all its boundary points.

An example of an open set is a δ disk. If we include the boundary of the disk, then it becomes a closed set. A set that contains some, but not all, of its boundary points is neither open nor closed. For example if we include half the boundary of a δ disk but not the other half, then the set is neither open nor closed.

Definition: connected sets and Regions

Let S be a subset of $\mathbb{R}^2$ (Figure 14.2.4).

1. An open set S is a *connected set* if it cannot be represented as the union of two or more disjoint, nonempty open subsets.
2. A set S is a *region* if it is open, connected, and nonempty.

The definition of a limit of a function of two variables requires the δ disk to be contained inside the domain of the function. However, if we wish to find the limit of a function at a boundary point of the domain, the δ **disk** is not contained inside the domain. By definition, some of the points of the δ **disk** are inside the domain and some are outside. Therefore, we need only consider points that are inside both the δ disk and the domain of the function. This leads to the definition of the limit of a function at a boundary point.

Definition

Let f be a function of two variables, x and y, and suppose (a,b) is on the boundary of the domain of f. Then, the limit of $f(x,y)$ as (x,y) approaches (a,b) is L, written

$$\lim_{(x,y)\to(a,b)} f(x,y) = L,$$

if for any $\varepsilon > 0,$ there exists a number $\delta > 0$ such that for any point (x,y) inside the domain of f and within a suitably small distance positive δ of (a,b), the value of $f(x,y)$ is no more than ε away from L (Figure 14.2.2). Using symbols, we can write: For any $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x,y) - L| < \varepsilon \text{ whenever } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta.$$

Example 14.2.3: Limit of a Function at a Boundary Point

Prove

$$\lim_{(x,y)\to(4,3)} \sqrt{25-x^2-y^2} = 0.$$

Solution

The domain of the function $f(x,y)=\sqrt{25-x^2-y^2}$ is $\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 25\}$, which is a circle of radius 5 centered at the origin, along with its interior as shown in Figure 14.2.5.

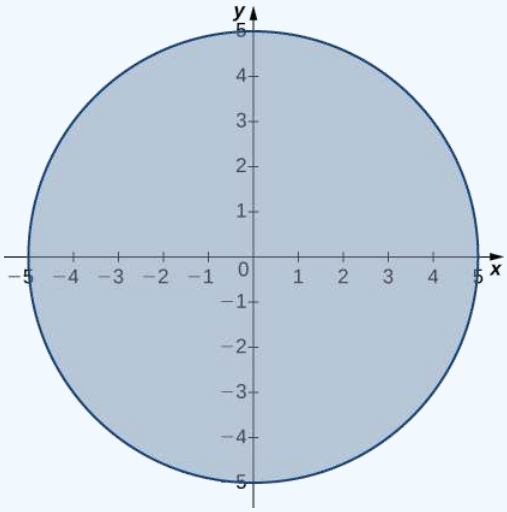


Figure 14.2.5: Domain of the function $f(x,y)=\sqrt{25-x^2-y^2}$.

We can use the limit laws, which apply to limits at the boundary of domains as well as interior points:

$$\begin{aligned}\lim_{(x,y)\to(4,3)} \sqrt{25-x^2-y^2} &= \sqrt{\lim_{(x,y)\to(4,3)}(25-x^2-y^2)}\\ &= \sqrt{\lim_{(x,y)\to(4,3)} 25 - \lim_{(x,y)\to(4,3)} x^2 - \lim_{(x,y)\to(4,3)} y^2}\\ &= \sqrt{25-4^2-3^2}\\ &= 0\end{aligned}$$

See the following graph.

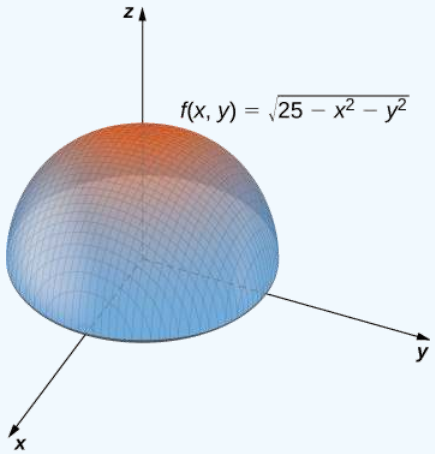


Figure 14.2.6: Graph of the function $f(x,y)=\sqrt{25-x^2-y^2}$.

Exercise 14.2.3

Evaluate the following limit:

$$\lim_{(x,y)\to(5,-2)} \sqrt{29-x^2-y^2}\,.$$

Hint

Determine the domain of $f(x,y)=\sqrt{29-x^2-y^2}$.

Answer

$$\lim_{(x,y)\to(5,-2)} \sqrt{29-x^2-y^2} = \sqrt{29-5^2-(-2)^2} = \sqrt{29-25-4} = \sqrt{0} = 0$$

Continuity of Functions of Two Variables

In Continuity, we defined the continuity of a function of one variable and saw how it relied on the limit of a function of one variable. In particular, three conditions are necessary for $f(x)$ to be continuous at point $x=a$

1. $f(a)$ exists.
2. $\lim\limits_{x\to a} f(x)$ exists.
3. $\lim\limits_{x\to a} f(x) = f(a)$.

These three conditions are necessary for continuity of a function of two variables as well.

Definition: continuous Functions

A function $f(x,y)$ is continuous at a point (a,b) in its domain if the following conditions are satisfied:

1. $f(a,b)$ exists.
2. $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists.
3. $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

Example 14.2.4: Demonstrating Continuity for a Function of Two Variables

Show that the function

$$f(x,y)=\frac{3x+2y}{x+y+1}$$

is continuous at point $(5,-3)$.

Solution

There are three conditions to be satisfied, per the definition of continuity. In this example, $a=5$ and $b=-3$.

1. $f(a,b)$ exists. This is true because the domain of the function f consists of those ordered pairs for which the denominator is nonzero (i.e., $x+y+1\neq 0$). Point $(5,-3)$ satisfies this condition. Furthermore,

$$f(a,b)=f(5,-3)=\frac{3(5)+2(-3)}{5+(-3)+1}=\frac{15-6}{2+1}=3.$$

2. $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists. This is also true:

$$\begin{aligned}\lim_{(x,y)\to(a,b)} f(x,y) &= \lim_{(x,y)\to(5,-3)} \frac{3x+2y}{x+y+1} \\ &= \frac{\lim\limits_{(x,y)\to(5,-3)}(3x+2y)}{\lim\limits_{(x,y)\to(5,-3)}(x+y+1)} \\ &= \frac{15-6}{5-3+1} \\ &= 3.\end{aligned}$$

3. $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. This is true because we have just shown that both sides of this equation equal three.

? Exercise 14.2.4

Show that the function

$$f(x,y) = \sqrt{26-2x^2-y^2}$$

is continuous at point $(2,-3)$.

Hint

Use the three-part definition of continuity.

Answer

1. The domain of f contains the ordered pair $(2,-3)$ because $f(a,b) = f(2,-3) = \sqrt{16-2(2)^2-(-3)^2} = 3$
2. $\lim\limits_{(x,y)\to(a,b)} f(x,y) = 3$
3. $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b) = 3$

Continuity of a function of any number of variables can also be defined in terms of delta and epsilon. A function of two variables is continuous at a point (x_0,y_0) in its domain if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, whenever $\sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$ it is true, $|f(x,y)-f(a,b)| < \varepsilon$. This definition can be combined with the formal definition (that is, the *epsilon–delta definition*) of continuity of a function of one variable to prove the following theorems:

Theorem 14.2.2: The Sum of Continuous Functions is Continuous

If $f(x,y)$ is continuous at (x_0,y_0), and $g(x,y)$ is continuous at (x_0,y_0), then $f(x,y)+g(x,y)$ is continuous at (x_0,y_0).

Theorem 14.2.3: The Product of Continuous Functions is Continuous

If $g(x)$ is continuous at x_0 and $h(y)$ is continuous at y_0, then $f(x,y) = g(x)h(y)$ is continuous at (x_0,y_0).

Theorem 14.2.4: The Composition of Continuous Functions is Continuous

Let g be a function of two variables from a domain $D \subseteq \mathbb{R}^2$ to a range $R \subseteq R$. Suppose g is continuous at some point $(x_0,y_0) \in D$ and define $z_0 = g(x_0,y_0)$. Let f be a function that maps R to R such that z_0 is in the domain of f. Last, assume f is continuous at z_0. Then $f \circ g$ is continuous at (x_0,y_0) as shown in Figure 14.2.7.

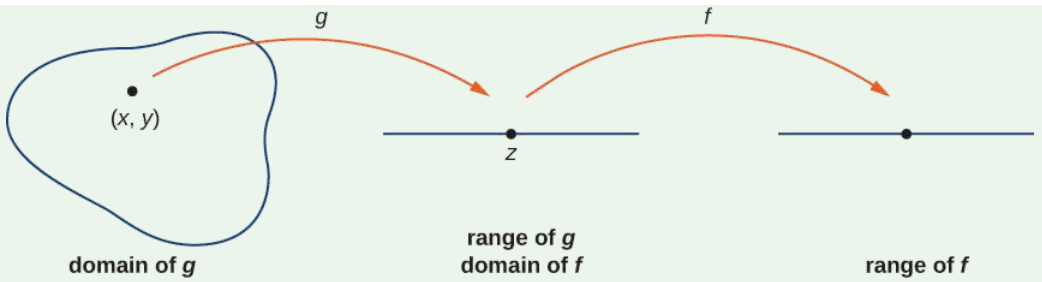


Figure 14.2.7: The composition of two continuous functions is continuous.

Let's now use the previous theorems to show continuity of functions in the following examples.

Example 14.2.5: More Examples of Continuity of a Function of Two Variables

Show that the functions $f(x,y) = 4x^3y^2$ and $g(x,y) = \cos(4x^3y^2)$ are continuous everywhere.

Solution

The polynomials $g(x) = 4x^3$ and $h(y) = y^2$ are continuous at every real number, and therefore by the product of continuous functions theorem, $f(x,y) = 4x^3y^2$ is continuous at every point (x,y) in the xy-plane. Since $f(x,y) = 4x^3y^2$ is continuous at every point (x,y) in the xy-plane and $g(x) = \cos x$ is continuous at every real number x, the continuity of the composition of functions tells us that $g(x,y) = \cos(4x^3y^2)$ is continuous at every point (x,y) in the xy-plane.

Exercise 14.2.5

Show that the functions $f(x,y) = 2x^2y^3 + 3$ and $g(x,y) = (2x^2y^3 + 3)^4$ are continuous everywhere.

Hint

Use the continuity of the sum, product, and composition of two functions.

Answer

The polynomials $g(x) = 2x^2$ and $h(y) = y^3$ are continuous at every real number; therefore, by the product of continuous functions theorem, $f(x,y) = 2x^2y^3$ is continuous at every point (x,y) in the xy-plane. Furthermore, any constant function is continuous everywhere, so $g(x,y) = 3$ is continuous at every point (x,y) in the xy-plane. Therefore, $f(x,y) = 2x^2y^3 + 3$ is continuous at every point (x,y) in the xy-plane. Last, $h(x) = x^4$ is continuous at every real number x, so by the continuity of composite functions theorem $g(x,y) = (2x^2y^3 + 3)^4$ is continuous at every point (x,y) in the xy-plane.

Functions of Three or More Variables

The limit of a function of three or more variables occurs readily in applications. For example, suppose we have a function $f(x,y,z)$ that gives the temperature at a physical location (x,y,z) in three dimensions. Or perhaps a function $g(x,y,z,t)$ can indicate air pressure at a location (x,y,z) at time t. How can we take a limit at a point in $\mathbb{R}^3$? What does it mean to be continuous at a point in four dimensions?

The answers to these questions rely on extending the concept of a δ disk into more than two dimensions. Then, the ideas of the limit of a function of three or more variables and the continuity of a function of three or more variables are very similar to the definitions given earlier for a function of two variables.

Definition: δ-balls

Let (x_0, y_0, z_0) be a point in $\mathbb{R}^3$. Then, a δ-ball in three dimensions consists of all points in $\mathbb{R}^3$ lying at a distance of less than δ from (x_0, y_0, z_0) —that is,

$$\{(x,y,z) \in \mathbb{R}^3 \mid \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} < \delta\}.$$

To define a δ-ball in higher dimensions, add additional terms under the radical to correspond to each additional dimension. For example, given a point $P=(w_0,x_0,y_0,z_0)$ in $\mathbb{R}^4$, a δ ball around P can be described by

$$\{(w,x,y,z)\in\mathbb{R}^4 \mid \sqrt{(w-w_0)^2+(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} < \delta\}.$$

To show that a limit of a function of three variables exists at a point (x_0,y_0,z_0), it suffices to show that for any point in a δ ball centered at (x_0,y_0,z_0), the value of the function at that point is arbitrarily close to a fixed value (the limit value). All the limit laws for functions of two variables hold for functions of more than two variables as well.

Example 14.2.6: Finding the Limit of a Function of Three Variables

Find

$$\lim_{(x,y,z)\to(4,1,-3)} \frac{x^2y-3z}{2x+5y-z}.$$

Solution

Before we can apply the quotient law, we need to verify that the limit of the denominator is nonzero. Using the difference law, the identity law, and the constant law,

$$\begin{aligned}\lim_{(x,y,z)\to(4,1,-3)}(2x+5y-z) &= 2\left(\lim_{(x,y,z)\to(4,1,-3)}x\right)+5\left(\lim_{(x,y,z)\to(4,1,-3)}y\right)-\left(\lim_{(x,y,z)\to(4,1,-3)}z\right)\\ &= 2(4)+5(1)-(-3)\\ &= 16.\end{aligned}$$

Since this is nonzero, we next find the limit of the numerator. Using the product law, power law, difference law, constant multiple law, and identity law,

$$\begin{aligned}\lim_{(x,y,z)\to(4,1,-3)}(x^2y-3z) &= \left(\lim_{(x,y,z)\to(4,1,-3)}x\right)^2\left(\lim_{(x,y,z)\to(4,1,-3)}y\right)-3\lim_{(x,y,z)\to(4,1,-3)}z\\ &= (4^2)(1)-3(-3)\\ &= 16+9\\ &= 25\end{aligned}$$

Last, applying the quotient law:

$$\lim_{(x,y,z)\to(4,1,-3)}\frac{x^2y-3z}{2x+5y-z} = \frac{\lim\limits_{(x,y,z)\to(4,1,-3)}(x^2y-3z)}{\lim\limits_{(x,y,z)\to(4,1,-3)}(2x+5y-z)} = \frac{25}{16}$$

Exercise 14.2.6

Find

$$\lim_{(x,y,z)\to(4,-1,3)}\sqrt{13-x^2-2y^2+z^2}$$

Hint

Use the limit laws and the continuity of the composition of functions.

Answer

$$\lim_{(x,y,z)\to(4,-1,3)}\sqrt{13-x^2-2y^2+z^2} = 2$$

Key Concepts

- To study limits and continuity for functions of two variables, we use a δ disk centered around a given point.
- A function of several variables has a limit if for any point in a δ ball centered at a point P, the value of the function at that point is arbitrarily close to a fixed value (the limit value).
- The limit laws established for a function of one variable have natural extensions to functions of more than one variable.
- A function of two variables is continuous at a point if the limit exists at that point, the function exists at that point, and the limit and function are equal at that point.

Glossary

boundary point
a point P_0 of R is a boundary point if every δ disk centered around P_0 contains points both inside and outside R

closed set
a set S that contains all its boundary points

connected set
an open set S that cannot be represented as the union of two or more disjoint, nonempty open subsets

δ disk
an open disk of radius δ centered at point (a, b)

δ ball
all points in $\mathbb{R}^3$ lying at a distance of less than δ from (x_0, y_0, z_0)

interior point
a point P_0 of $\mathbb{R}$ is a boundary point if there is a δ disk centered around P_0 contained completely in $\mathbb{R}$

open set
a set S that contains none of its boundary points

region
an open, connected, nonempty subset of $\mathbb{R}^2$

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14.3: Partial Derivatives

Learning Objectives

- Calculate the partial derivatives of a function of two variables.
- Calculate the partial derivatives of a function of more than two variables.
- Determine the higher-order derivatives of a function of two variables.
- Explain the meaning of a partial differential equation and give an example.

Now that we have examined limits and continuity of functions of two variables, we can proceed to study derivatives. Finding derivatives of functions of two variables is the key concept in this chapter, with as many applications in mathematics, science, and engineering as differentiation of single-variable functions. However, we have already seen that limits and continuity of multivariable functions have new issues and require new terminology and ideas to deal with them. This carries over into differentiation as well.

Derivatives of a Function of Two Variables

When studying derivatives of functions of one variable, we found that one interpretation of the derivative is an instantaneous rate of change of y as a function of x. Leibniz notation for the derivative is dy/dx, which implies that y is the dependent variable and x is the independent variable. For a function $z=f(x,y)$ of two variables, x and y are the independent variables and z is the dependent variable. This raises two questions right away: How do we adapt Leibniz notation for functions of two variables? Also, what is an interpretation of the derivative? The answer lies in partial derivatives.

Definition: Partial Derivatives

Let $f(x,y)$ be a function of two variables. Then the *partial derivative* of f with respect to x, written as $\partial f/\partial x$, or f_x, is defined as

$$\frac{\partial f}{\partial x}=f_x(x,y)=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h} \tag{14.3.1}$$

The partial derivative of f with respect to y, written as $\partial f/\partial y$, or f_y, is defined as

$$\frac{\partial f}{\partial y}=f_y(x,y)=\lim_{k\to 0}\frac{f(x,y+k)-f(x,y)}{k}. \tag{14.3.2}$$

This definition shows two differences already. First, the notation changes, in the sense that we still use a version of Leibniz notation, but the d in the original notation is replaced with the symbol ∂. (This rounded "d" is usually called "partial," so $\partial f/\partial x$ is spoken as the "partial of f with respect to x.") This is the first hint that we are dealing with partial derivatives. Second, we now have two different derivatives we can take, since there are two different independent variables. Depending on which variable we choose, we can come up with different partial derivatives altogether, and often do.

Example 14.3.1: Calculating Partial Derivatives from the Definition

Use the definition of the partial derivative as a limit to calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the function

$$f(x,y)=x^2-3xy+2y^2-4x+5y-12.$$

Solution

First, calculate $f(x+h,y)$.

$$\begin{aligned} f(x+h,y) &= (x+h)^2-3(x+h)y+2y^2-4(x+h)+5y-12 \\ &= x^2+2xh+h^2-3xy-3hy+2y^2-4x-4h+5y-12. \end{aligned}$$

Next, substitute this into Equation 14.3.1 and simplify:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}\\ &= \lim_{h\to 0}\frac{(x^2+2xh+h^2-3xy-3hy+2y^2-4x-4h+5y-12)-(x^2-3xy+2y^2-4x+5y-12)}{h}\\ &= \lim_{h\to 0}\frac{x^2+2xh+h^2-3xy-3hy+2y^2-4x-4h+5y-12-x^2+3xy-2y^2+4x-5y+12}{h}\\ &= \lim_{h\to 0}\frac{2xh+h^2-3hy-4h}{h}\\ &= \lim_{h\to 0}\frac{h(2x+h-3y-4)}{h}\\ &= \lim_{h\to 0}(2x+h-3y-4)\\ &= 2x-3y-4.\end{aligned}$$

To calculate $\frac{\partial f}{\partial y}$, first calculate $f(x,y+h)$:

$$\begin{aligned}f(x+h,y) &= x^2-3x(y+h)+2(y+h)^2-4x+5(y+h)-12\\ &= x^2-3xy-3xh+2y^2+4yh+2h^2-4x+5y+5h-12.\end{aligned}$$

Next, substitute this into Equation 14.3.2 and simplify:

$$\begin{aligned}\frac{\partial f}{\partial y} &= \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}\\ &= \lim_{h\to 0}\frac{(x^2-3xy-3xh+2y^2+4yh+2h^2-4x+5y+5h-12)-(x^2-3xy+2y^2-4x+5y-12)}{h}\\ &= \lim_{h\to 0}\frac{x^2-3xy-3xh+2y^2+4yh+2h^2-4x+5y+5h-12-x^2+3xy-2y^2+4x-5y+12}{h}\\ &= \lim_{h\to 0}\frac{-3xh+4yh+2h^2+5h}{h}\\ &= \lim_{h\to 0}\frac{h(-3x+4y+2h+5)}{h}\\ &= \lim_{h\to 0}(-3x+4y+2h+5)\\ &= -3x+4y+5\end{aligned}$$

? Exercise 14.3.1

Use the definition of the partial derivative as a limit to calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the function

$$f(x,y)=4x^2+2xy-y^2+3x-2y+5.$$

Hint

Use Equations 14.3.1 and 14.3.2 from the definition of partial derivatives.

Answer

$$\frac{\partial f}{\partial x}=8x+2y+3$$

$$\frac{\partial f}{\partial y}=2x-2y-2$$

The idea to keep in mind when calculating partial derivatives is to treat all independent variables, other than the variable with respect to which we are differentiating, as constants. Then proceed to differentiate as with a function of a single variable. To see why this is true, first fix y and define $g(x)=f(x,y)$ as a function of x. Then

$$\begin{aligned} g'(x) &= \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} \\ &= \frac{\partial f}{\partial x}. \end{aligned}$$

The same is true for calculating the partial derivative of f with respect to y. This time, fix x and define $h(y) = f(x, y)$ as a function of y. Then

$$\begin{aligned} h'(x) &= \lim_{k \to 0} \frac{h(x+k) - h(x)}{k} \\ &= \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k} \\ &= \frac{\partial f}{\partial y}. \end{aligned}$$

All differentiation rules apply.

Example 14.3.2: Calculating Partial Derivatives

Calculate $\partial f / \partial x$ and $\partial f / \partial y$ for the following functions by holding the opposite variable constant then differentiating:

a. $f(x, y) = x^2 - 3xy + 2y^2 - 4x + 5y - 12$
b. $g(x, y) = \sin(x^2 y - 2x + 4)$

Solution:

a. To calculate $\partial f / \partial x$, treat the variable y as a constant. Then differentiate $f(x, y)$ with respect to x using the sum, difference, and power rules:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left[x^2 - 3xy + 2y^2 - 4x + 5y - 12\right] \\ &= \frac{\partial}{\partial x}[x^2] - \frac{\partial}{\partial x}[3xy] + \frac{\partial}{\partial x}[2y^2] - \frac{\partial}{\partial x}[4x] + \frac{\partial}{\partial x}[5y] - \frac{\partial}{\partial x}[12] \\ &= 2x - 3y + 0 - 4 + 0 - 0 \\ &= 2x - 3y - 4. \end{aligned}$$

The derivatives of the third, fifth, and sixth terms are all zero because they do not contain the variable x, so they are treated as constant terms. The derivative of the second term is equal to the coefficient of x, which is $-3y$. Calculating $\partial f / \partial y$:

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left[x^2 - 3xy + 2y^2 - 4x + 5y - 12\right] \\ &= \frac{\partial}{\partial y}[x^2] - \frac{\partial}{\partial y}[3xy] + \frac{\partial}{\partial y}[2y^2] - \frac{\partial}{\partial y}[4x] + \frac{\partial}{\partial y}[5y] - \frac{\partial}{\partial y}[12] \\ &= 0 - 3x + 4y - 0 + 5 - 0 \\ &= -3x + 4y + 5. \end{aligned}$$

These are the same answers obtained in Example 14.3.1.

b. To calculate $\partial g / \partial x$, treat the variable y as a constant. Then differentiate $g(x, y)$ with respect to x using the chain rule and power rule:

$$\begin{aligned}\frac{\partial g}{\partial x} &= \frac{\partial}{\partial x}\left[\sin(x^2y-2x+4)\right] \\ &= \cos(x^2y-2x+4)\frac{\partial}{\partial x}[x^2y-2x+4] \\ &= (2xy-2)\cos(x^2y-2x+4).\end{aligned}$$

To calculate $\partial g/\partial y$, treat the variable x as a constant. Then differentiate $g(x,y)$ with respect to y using the chain rule and power rule:

$$\begin{aligned}\frac{\partial g}{\partial y} &= \frac{\partial}{\partial y}\left[\sin(x^2y-2x+4)\right] \\ &= \cos(x^2y-2x+4)\frac{\partial}{\partial y}[x^2y-2x+4] \\ &= x^2\cos(x^2y-2x+4).\end{aligned}$$

? Exercise 14.3.2

Calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the function

$$f(x,y)=\tan(x^3-3x^2y^2+2y^4)$$

by holding the opposite variable constant, then differentiating.

Hint

Use Equations 14.3.1and 14.3.1from the definition of partial derivatives.

Answer

$$\frac{\partial f}{\partial x}=(3x^2-6xy^2)\sec^2(x^3-3x^2y^2+2y^4)$$
$$\frac{\partial f}{\partial y}=(-6x^2y+8y^3)\sec^2(x^3-3x^2y^2+2y^4)$$

How can we interpret these partial derivatives? Recall that the graph of a function of two variables is a surface in R^3. If we remove the limit from the definition of the partial derivative with respect to x, the difference quotient remains:

$$\frac{f(x+h,y)-f(x,y)}{h}.$$

This resembles the difference quotient for the derivative of a function of one variable, except for the presence of the y variable. Figure 14.3.1illustrates a surface described by an arbitrary function $z=f(x,y)$.

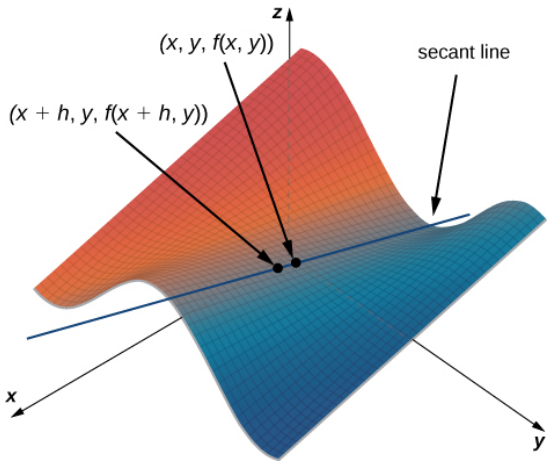


Figure 14.3.1: Secant line passing through the points $(x,y,f(x,y))$ and $(x+h,y,f(x+h,y))$.

In Figure 14.3.1, the value of h is positive. If we graph $f(x,y)$ and $f(x+h,y)$ for an arbitrary point (x,y), then the slope of the secant line passing through these two points is given by

$$\frac{f(x+h,y)-f(x,y)}{h}.$$

This line is parallel to the x-axis. Therefore, the slope of the secant line represents an average rate of change of the function f as we travel parallel to the x-axis. As h approaches zero, the slope of the secant line approaches the slope of the tangent line.

If we choose to change y instead of x by the same incremental value h, then the secant line is parallel to the y-axis and so is the tangent line. Therefore, $\partial f/\partial x$ represents the slope of the tangent line passing through the point $(x,y,f(x,y))$parallel to the x-axis and $\partial f/\partial y$ represents the slope of the tangent line passing through the point $(x,y,f(x,y))$parallel to the y-axis. If we wish to find the slope of a tangent line passing through the same point in any other direction, then we need what are called directional derivatives.

We now return to the idea of contour maps, which we introduced in Functions of Several Variables. We can use a **contour map** to estimate partial derivatives of a function $g(x,y)$.

Example 14.3.3: Partial Derivatives from a Contour Map

Use a contour map to estimate $\partial g/\partial x$ at the point $(\sqrt{5},0)$ for the function

$$g(x,y)=\sqrt{9-x^2-y^2}.$$

Solution

Figure 14.3.2represents a contour map for the function $g(x,y)$.

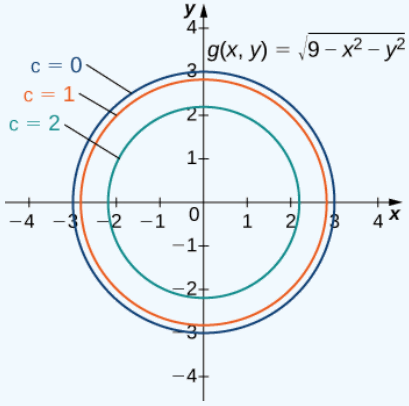


Figure 14.3.2: Contour map for the function $g(x,y)=\sqrt{9-x^2-y^2}$, using $c=0,1,2,$ and $3(c=3$ corresponds to the origin).

The inner circle on the contour map corresponds to $c=2$ and the next circle out corresponds to $c=1$. The first circle is given by the equation $2=\sqrt{9-x^2-y^2}$; the second circle is given by the equation $1=\sqrt{9-x^2-y^2}$. The first equation simplifies to $x^2+y^2=5$ and the second equation simplifies to $x^2+y^2=8$. The x-intercept of the first circle is $(\sqrt{5},0)$ and the x-intercept of the second circle is $(2\sqrt{2},0)$. We can estimate the value of $\partial g/\partial x$ evaluated at the point $(\sqrt{5},0)$ using the slope formula:

$$\begin{aligned}\left.\frac{\partial g}{\partial x}\right|_{(x,y)=(\sqrt{5},0)} &\approx \frac{g(\sqrt{5},0)-g(2\sqrt{2},0)}{\sqrt{5}-2\sqrt{2}}\\ &=\frac{2-1}{\sqrt{5}-2\sqrt{2}}\\ &=\frac{1}{\sqrt{5}-2\sqrt{2}}\approx -1.688.\end{aligned}$$

To calculate the exact value of $\partial g/\partial x$ evaluated at the point $(\sqrt{5},0)$, we start by finding $\partial g/\partial x$ using the chain rule. First, we rewrite the function as

$$g(x,y)=\sqrt{9-x^2-y^2}=(9-x^2-y^2)^{1/2}$$

and then differentiate with respect to x while holding y constant:

$$\frac{\partial g}{\partial x} = \frac{1}{2}(9-x^2-y^2)^{-1/2}(-2x)$$
$$= -\frac{x}{\sqrt{9-x^2-y^2}}.$$

Next, we evaluate this expression using $x=\sqrt{5}$ and $y=0$:

$$\left.\frac{\partial g}{\partial x}\right|_{(x,y)=(\sqrt{5},0)} = -\frac{\sqrt{5}}{\sqrt{9-(\sqrt{5})^2-(0)^2}}$$
$$= -\frac{\sqrt{5}}{\sqrt{4}}$$
$$= -\frac{\sqrt{5}}{2} \approx -1.118.$$

The estimate for the partial derivative corresponds to the slope of the secant line passing through the points $(\sqrt{5},0,g(\sqrt{5},0))$ and $(2\sqrt{2},0,g(2\sqrt{2},0))$. It represents an approximation to the slope of the tangent line to the surface through the point $(\sqrt{5},0,g(\sqrt{5},0))$, which is parallel to the x-axis.

? Exercise 14.3.3

Use a contour map to estimate $\partial f/\partial y$ at point $(0,\sqrt{2})$ for the function

$$f(x,y)=x^2-y^2.$$

Compare this with the exact answer.

Hint

Create a contour map for f using values of c from -3 to 3. Which of these curves passes through point $(0,\sqrt{2})$?

Answer

Using the curves corresponding to $c=-2$ and $c=-3$, we obtain

$$\left.\frac{\partial f}{\partial y}\right|_{(x,y)=(0,\sqrt{2})} \approx \frac{f(0,\sqrt{3})-f(0,\sqrt{2})}{\sqrt{3}-\sqrt{2}}$$
$$= \frac{-3-(-2)}{\sqrt{3}-\sqrt{2}}\cdot\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}$$
$$= -\sqrt{3}-\sqrt{2} \approx -3.146.$$

The exact answer is

$$\left.\frac{\partial f}{\partial y}\right|_{(x,y)=(0,\sqrt{2})} = (-2y|_{(x,y)=(0,\sqrt{2})} = -2\sqrt{2} \approx -2.828.$$

Functions of More Than Two Variables

Suppose we have a function of three variables, such as $w=f(x,y,z)$. We can calculate partial derivatives of w with respect to any of the independent variables, simply as extensions of the definitions for partial derivatives of functions of two variables.

Definition: Partial Derivatives

Let $f(x,y,z)$ be a function of three variables. Then, the partial derivative of f with respect to x, written as $\partial f/\partial x$, or f_x, is defined to be

$$\frac{\partial f}{\partial x} = f_x(x,y,z) = \lim_{h\to 0}\frac{f(x+h,y,z)-f(x,y,z)}{h}. \tag{14.3.3}$$

The *partial derivative* of f with *respect* to y, written as $\partial f/\partial y$, or f_y, is defined to be

$$\frac{\partial f}{\partial y} = f_y(x,y,z) = \lim_{k\to 0} \frac{f(x,y+k,z) - f(x,y,z)}{k}. \tag{14.3.4}$$

The *partial derivative* of f with respect to z, written as $\partial f/\partial z$, or f_z, is defined to be

$$\frac{\partial f}{\partial z} = f_z(x,y,z) = \lim_{m\to 0} \frac{f(x,y,z+m) - f(x,y,z)}{m}. \tag{14.3.5}$$

We can calculate a partial derivative of a function of three variables using the same idea we used for a function of two variables. For example, if we have a function f of x, y, and z, and we wish to calculate $\partial f/\partial x$, then we treat the other two independent variables as if they are constants, then differentiate with respect to x.

Example 14.3.4: Calculating Partial Derivatives for a Function of Three Variables

Use the limit definition of partial derivatives to calculate $\partial f/\partial x$ for the function

$$f(x,y,z) = x^2 - 3xy + 2y^2 - 4xz + 5yz^2 - 12x + 4y - 3z.$$

Then, find $\partial f/\partial y$ and $\partial f/\partial z$ by setting the other two variables constant and differentiating accordingly.

Solution:

We first calculate $\partial f/\partial x$ using Equation 14.3.3, then we calculate the other two partial derivatives by holding the remaining variables constant. To use the equation to find $\partial f/\partial x$, we first need to calculate $f(x+h,y,z)$:

$$\begin{aligned} f(x+h,y,z) &= (x+h)^2 - 3(x+h)y + 2y^2 - 4(x+h)z + 5yz^2 - 12(x+h) + 4y - 3z \\ &= x^2 + 2xh + h^2 - 3xy - 3xh + 2y^2 - 4xz - 4hz + 5yz^2 - 12x - 12h + 4y - 3z \end{aligned}$$

and recall that $f(x,y,z) = x^2 - 3xy + 2y^2 - 4zx + 5yz^2 - 12x + 4y - 3z$. Next, we substitute these two expressions into the equation:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \lim_{h\to 0}\left[\frac{\begin{array}{l} x^2 + 2xh + h^2 - 3xy - 3hy + 2y^2 - 4xz - 4hz + 5yz^2 - 12x - 12h + 4y - 3zh - x^2 - 3xy + 2y^2 - 4xz + 5yz^2 \\ - 12x + 4y - 3z \end{array}}{h}\right] \\ &= \lim_{h\to 0}\left[\frac{2xh + h^2 - 3hy - 4hz - 12h}{h}\right] \\ &= \lim_{h\to 0}\left[\frac{h(2x + h - 3y - 4z - 12)}{h}\right] \\ &= \lim_{h\to 0}(2x + h - 3y - 4z - 12) \\ &= 2x - 3y - 4z - 12. \end{aligned}$$

Then we find $\partial f/\partial y$ by holding x and z constant. Therefore, any term that does not include the variable y is constant, and its derivative is zero. We can apply the sum, difference, and power rules for functions of one variable:

$$\begin{aligned} &\frac{\partial}{\partial y}\left[x^2 - 3xy + 2y^2 - 4xz + 5yz^2 - 12x + 4y - 3z\right] \\ &= \frac{\partial}{\partial y}[x^2] - \frac{\partial}{\partial y}[3xy] + \frac{\partial}{\partial y}[2y^2] - \frac{\partial}{\partial y}[4xz] + \frac{\partial}{\partial y}[5yz^2] - \frac{\partial}{\partial y}[12x] + \frac{\partial}{\partial y}[4y] - \frac{\partial}{\partial z}[3z] \\ &= 0 - 3x + 4y - 0 + 5z^2 - 0 + 4 - 0 \\ &= -3x + 4y + 5z^2 + 4. \end{aligned}$$

To calculate $\partial f/\partial z$, we hold x and y constant and apply the sum, difference, and power rules for functions of one variable:

$$\begin{aligned}
&\frac{\partial}{\partial z}[x^2-3xy+2y^2-4xz+5yz^2-12x+4y-3z] \\
&=\frac{\partial}{\partial z}[x^2]-\frac{\partial}{\partial z}[3xy]+\frac{\partial}{\partial z}[2y^2]-\frac{\partial}{\partial z}[4xz]+\frac{\partial}{\partial z}[5yz^2]-\frac{\partial}{\partial z}[12x]+\frac{\partial}{\partial z}[4y]-\frac{\partial}{\partial z}[3z] \\
&=0-0+0-4x+10yz-0+0-3 \\
&=-4x+10yz-3
\end{aligned}$$

Exercise 14.3.4

Use the limit definition of partial derivatives to calculate $\partial f/\partial x$ for the function

$$f(x,y,z)=2x^2-4x^2y+2y^2+5xz^2-6x+3z-8.$$

Then find $\partial f/\partial y$ and $\partial f/\partial z$ by setting the other two variables constant and differentiating accordingly.

Hint

Use the strategy in the preceding example.

Answer

$$\frac{\partial f}{\partial x}=4x-8xy+5z^2-6,\ \frac{\partial f}{\partial y}=-4x^2+4y,\ \frac{\partial f}{\partial z}=10xz+3$$

Example 14.3.5: Calculating Partial Derivatives for a Function of Three Variables

Calculate the three partial derivatives of the following functions.

a. $f(x,y,z)=\dfrac{x^2y-4xz+y^2}{x-3yz}$

b. $g(x,y,z)=\sin(x^2y-z)+\cos(x^2-yz)$

Solution

In each case, treat all variables as constants except the one whose partial derivative you are calculating.

a.

$$\begin{aligned}
\frac{\partial f}{\partial x}&=\frac{\partial}{\partial x}\left[\frac{x^2y-4xz+y^2}{x-3yz}\right] \\
&=\frac{\frac{\partial}{\partial x}(x^2y-4xz+y^2)(x-3yz)-(x^2y-4xz+y^2)\frac{\partial}{\partial x}(x-3yz)}{(x-3yz)^2} \\
&=\frac{(2xy-4z)(x-3yz)-(x^2y-4xz+y^2)(1)}{(x-3yz)^2} \\
&=\frac{2x^2y-6xy^2z-4xz+12yz^2-x^2y+4xz-y^2}{(x-3yz)^2} \\
&=\frac{x^2y-6xy^2z-4xz+12yz^2+4xz-y^2}{(x-3yz)^2}
\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left[\frac{x^2y-4xz+y^2}{x-3yz}\right] \\ &= \frac{\frac{\partial}{\partial y}(x^2y-4xz+y^2)(x-3yz)-(x^2y-4xz+y^2)\frac{\partial}{\partial y}(x-3yz)}{(x-3yz)^2} \\ &= \frac{(x^2+2y)(x-3yz)-(x^2y-4xz+y^2)(-3z)}{(x-3yz)^2} \\ &= \frac{x^3-3x^2yz+2xy-6y^2z+3x^2yz-12xz^2+3y^2z}{(x-3yz)^2} \\ &= \frac{x^3+2xy-3y^2z-12xz^2}{(x-3yz)^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial z} &= \frac{\partial}{\partial z}\left[\frac{x^2y-4xz+y^2}{x-3yz}\right] \\ &= \frac{\frac{\partial}{\partial z}(x^2y-4xz+y^2)(x-3yz)-(x^2y-4xz+y^2)\frac{\partial}{\partial z}(x-3yz)}{(x-3yz)^2} \\ &= \frac{(-4x)(x-3yz)-(x^2y-4xz+y^2)(-3y)}{(x-3yz)^2} \\ &= \frac{-4x^2+12xyz+3x^2y^2-12xyz+3y^3}{(x-3yz)^2} \\ &= \frac{-4x^2+3x^2y^2+3y^3}{(x-3yz)^2}\end{aligned}$$

b.

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left[\sin(x^2y-z)+\cos(x^2-yz)\right] \\ &= (\cos(x^2y-z))\frac{\partial}{\partial x}(x^2y-z)-(\sin(x^2-yz))\frac{\partial}{\partial x}(x^2-yz) \\ &= 2xy\cos(x^2y-z)-2x\sin(x^2-yz) \\ \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}[\sin(x^2y-z)+\cos(x^2-yz)] \\ &= (\cos(x^2y-z))\frac{\partial}{\partial y}(x^2y-z)-(\sin(x^2-yz))\frac{\partial}{\partial y}(x^2-yz) \\ &= x^2\cos(x^2y-z)+z\sin(x^2-yz) \\ \frac{\partial f}{\partial z} &= \frac{\partial}{\partial z}[\sin(x^2y-z)+\cos(x^2-yz)] \\ &= (\cos(x^2y-z))\frac{\partial}{\partial z}(x^2y-z)-(\sin(x^2-yz))\frac{\partial}{\partial z}(x^2-yz) \\ &= -\cos(x^2y-z)+y\sin(x^2-yz)\end{aligned}$$

? Exercise 14.3.5

Calculate $\partial f/\partial x, \partial f/\partial y,$ and $\partial f/\partial z$ for the function

$$f(x,y,z)=\sec(x^2y)-\tan(x^3yz^2).$$

Hint

Use the strategy in the preceding example.

Answer

$$\frac{\partial f}{\partial x} = 2xy\sec(x^2y)\tan(x^2y) - 3x^2yz^2\sec^2(x^3yz^2)$$
$$\frac{\partial f}{\partial y} = x^2\sec(x^2y)\tan(x^2y) - x^3z^2\sec^2(x^3yz^2)$$
$$\frac{\partial f}{\partial z} = -2x^3yz\sec^2(x^3yz^2)$$

Higher-Order Partial Derivatives

Consider the function

$$f(x,y) = 2x^3 - 4xy^2 + 5y^3 - 6xy + 5x - 4y + 12.$$

Its partial derivatives are

$$\frac{\partial f}{\partial x} = 6x^2 - 4y^2 - 6y + 5$$

and

$$\frac{\partial f}{\partial y} = -8xy + 15y^2 - 6x - 4.$$

Each of these partial derivatives is a function of two variables, so we can calculate partial derivatives of these functions. Just as with derivatives of single-variable functions, we can call these *second-order derivatives*, third-order derivatives, and so on. In general, they are referred to as **higher-order partial derivatives.** There are four second-order partial derivatives for any function (provided they all exist):

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right]$$
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right]$$
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right].$$

An alternative notation for each is f_{xx}, f_{xy}, f_{yx}, and f_{yy}, respectively. Higher-order partial derivatives calculated with respect to different variables, such as f_{xy} and f_{yx}, are commonly called **mixed partial derivatives.**

Example 14.3.6: Calculating Second Partial Derivatives

Calculate all four second partial derivatives for the function

$$f(x,y) = xe^{-3y} + \sin(2x - 5y). \tag{14.3.6}$$

Solution:

To calculate $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y \partial x}$, we first calculate $\partial f/\partial x$:

$$\frac{\partial f}{\partial x} = e^{-3y} + 2\cos(2x - 5y). \tag{14.3.7}$$

To calculate $\frac{\partial^2 f}{\partial x^2}$, differentiate $\partial f/\partial x$ (Equation 14.3.7) with respect to x:

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] \\ &= \frac{\partial}{\partial x}[e^{-3y} + 2\cos(2x-5y)] \\ &= -4\sin(2x-5y).\end{aligned}$$

To calculate $\dfrac{\partial^2 f}{\partial y \partial x}$, differentiate $\partial f/\partial x$ (Equation 14.3.7) with respect to y:

$$\begin{aligned}\frac{\partial^2 f}{\partial y\,\partial x} &= \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] \\ &= \frac{\partial}{\partial y}[e^{-3y} + 2\cos(2x-5y)] \\ &= -3e^{-3y} + 10\sin(2x-5y).\end{aligned}$$

To calculate $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y^2}$, first calculate $\partial f/\partial y$:

$$\frac{\partial f}{\partial y} = -3xe^{-3y} - 5\cos(2x-5y). \tag{14.3.8}$$

To calculate $\dfrac{\partial^2 f}{\partial x \partial y}$, differentiate $\partial f/\partial y$ (Equation 14.3.8) with respect to x:

$$\begin{aligned}\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] \\ &= \frac{\partial}{\partial x}[-3xe^{-3y} - 5\cos(2x-5y)] \\ &= -3e^{-3y} + 10\sin(2x-5y).\end{aligned}$$

To calculate $\dfrac{\partial^2 f}{\partial y^2}$, differentiate $\partial f/\partial y$ (Equation 14.3.8) with respect to y:

$$\begin{aligned}\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] \\ &= \frac{\partial}{\partial y}[-3xe^{-3y} - 5\cos(2x-5y)] \\ &= 9xe^{-3y} - 25\sin(2x-5y).\end{aligned}$$

? Exercise 14.3.6

Calculate all four second partial derivatives for the function

$$f(x,y) = \sin(3x-2y) + \cos(x+4y).$$

Hint

Follow the same steps as in the previous example.

Answer

$$\frac{\partial^2 f}{\partial x^2} = -9\sin(3x-2y) - \cos(x+4y)$$

$$\frac{\partial^2 f}{\partial y \partial x} = 6\sin(3x-2y) - 4\cos(x+4y)$$

$$\frac{\partial^2 f}{\partial x \partial y} = 6\sin(3x-2y) - 4\cos(x+4y)$$

$$\frac{\partial^2 f}{\partial y^2} = -4\sin(3x-2y) - 16\cos(x+4y)$$

At this point we should notice that, in both Example 14.3.6 and the checkpoint, it was true that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$. Under certain conditions, this is always true. In fact, it is a direct consequence of the following theorem.

Theorem 14.3.1: Equality of Mixed Partial Derivatives (Clairaut's Theorem)

Suppose that $f(x,y)$ is defined on an open disk D that contains the point (a,b). If the functions f_{xy} and f_{yx} are continuous on D, then $f_{xy}(a,b) = f_{yx}(a,b)$.

Clairaut's theorem guarantees that as long as mixed second-order derivatives are continuous, the order in which we choose to differentiate the functions (i.e., which variable goes first, then second, and so on) does not matter. It can be extended to higher-order derivatives as well. The proof of Clairaut's theorem can be found in most advanced calculus books.

Two other second-order partial derivatives can be calculated for any function $f(x,y)$. The partial derivative f_{xx} is equal to the partial derivative of f_x with respect to x, and f_{yy} is equal to the partial derivative of f_y with respect to y.

Partial Differential Equations

Previously, we studied differential equations in which the unknown function had one independent variable. A partial differential equation is an equation that involves an unknown function of more than one independent variable and one or more of its partial derivatives. Examples of partial differential equations are

$$u_t = c^2(u_{xx} + u_{yy})$$
heat equation in two dimensions

$$u_{tt} = c^2(u_{xx} + u_{yy})$$
wave equation in two dimensions

$$u_{xx} + u_{yy} = 0$$
Laplace's equation in two dimensions

In the heat and wave equations, the unknown function u has three independent variables: t, x, and y with c is an arbitrary constant. The independent variables x and y are considered to be spatial variables, and the variable t represents time. In Laplace's equation, the unknown function u has two independent variables x and y.

Example 14.3.7: A Solution to the Wave Equation

Verify that

$$u(x,y,t) = 5\sin(3\pi x)\sin(4\pi y)\cos(10\pi t)$$

is a solution to the wave equation

$$u_{tt} = 4(u_{xx} + u_{yy}). \tag{14.3.9}$$

Solution

First, we calculate u_{tt}, u_{xx}, and u_{yy}:

$$\begin{aligned} u_{tt}(x,y,t) &= \frac{\partial}{\partial t}\left[\frac{\partial u}{\partial t}\right] \\ &= \frac{\partial}{\partial t}[5\sin(3\pi x)\sin(4\pi y)(-10\pi\sin(10\pi t))] \\ &= \frac{\partial}{\partial t}\left[-50\pi\sin(3\pi x)\sin(4\pi y)\sin(10\pi t)\right] \\ &= -500\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) \end{aligned}$$

$$\begin{aligned} u_{xx}(x,y,t) &= \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right] \\ &= \frac{\partial}{\partial x}\left[15\pi\cos(3\pi x)\sin(4\pi y)\cos(10\pi t)\right] \\ &= -45\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) \\ u_{yy}(x,y,t) &= \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial y}\right] \\ &= \frac{\partial}{\partial y}\left[5\sin(3\pi x)(4\pi\cos(4\pi y))\cos(10\pi t)\right] \\ &= \frac{\partial}{\partial y}\left[20\pi\sin(3\pi x)\cos(4\pi y)\cos(10\pi t)\right] \\ &= -80\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t). \end{aligned}$$

Next, we substitute each of these into the right-hand side of Equation 14.3.9 and simplify:

$$\begin{aligned} 4(u_{xx}+u_{yy}) &= 4(-45\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) + -80\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t)) \\ &= 4(-125\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t)) \\ &= -500\pi^2\sin(3\pi x)\sin(4\pi y)\cos(10\pi t) \\ &= u_{tt}. \end{aligned}$$

This verifies the solution.

? Exercise 14.3.7: A Solution to the Heat Equation

Verify that

$$u(x,y,t) = 2\sin\left(\frac{x}{3}\right)\sin\left(\frac{y}{4}\right)e^{-25t/16}$$

is a solution to the heat equation

$$u_t = 9(u_{xx}+u_{yy}).$$

Hint

Calculate the partial derivatives and substitute into the right-hand side.

Answer

TBA

Since the solution to the two-dimensional heat equation is a function of three variables, it is not easy to create a visual representation of the solution. We can graph the solution for fixed values of t, which amounts to snapshots of the heat distributions at fixed times. These snapshots show how the heat is distributed over a two-dimensional surface as time progresses. The graph of the preceding solution at time $t = 0$ appears in Figure 14.3.3. As time progresses, the extremes level out, approaching zero as t approaches infinity.

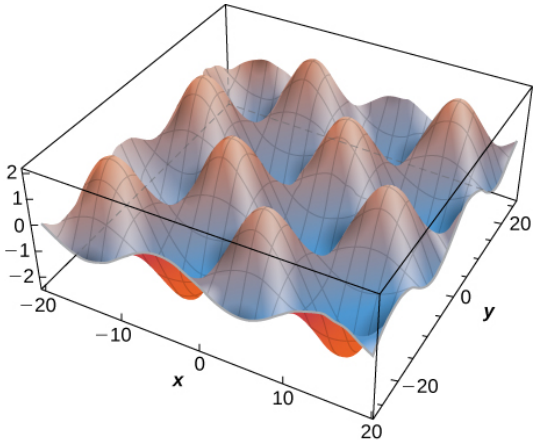


Figure 14.3.3

If we consider the heat equation in one dimension, then it is possible to graph the solution over time. The heat equation in one dimension becomes

$$u_t = c^2 u_{xx},$$

where c^2 represents the thermal diffusivity of the material in question. A solution of this differential equation can be written in the form

$$u_m(x,t) = e^{-\pi^2 m^2 c^2 t} \sin(m\pi x)$$

where m is any positive integer. A graph of this solution using $m = 1$ appears in Figure 14.3.4, where the initial temperature distribution over a wire of length 1 is given by $u(x,0) = \sin \pi x$. Notice that as time progresses, the wire cools off. This is seen because, from left to right, the highest temperature (which occurs in the middle of the wire) decreases and changes color from red to blue.

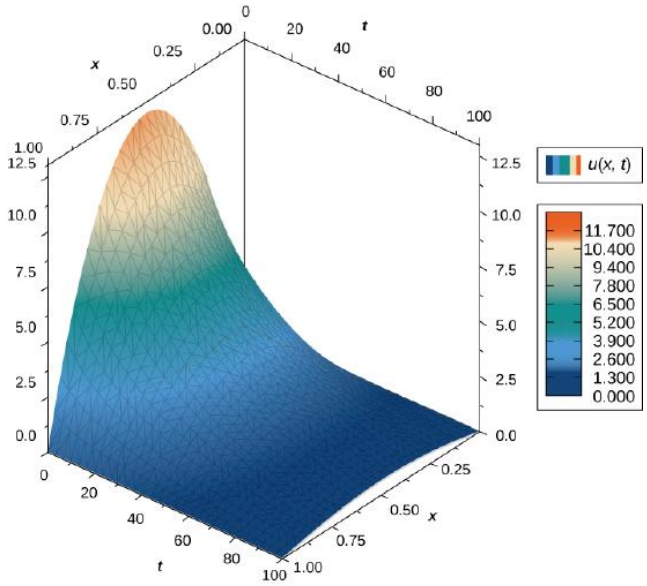
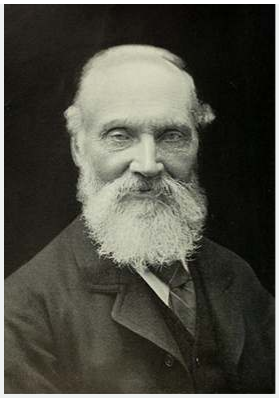


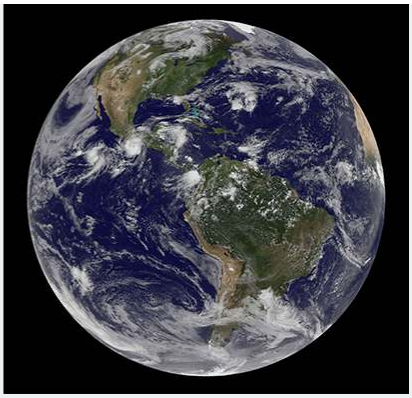
Figure 14.3.4: Graph of a solution of the heat equation in one dimension over time.

Lord Kelvin and the Age of Earth

During the late 1800s, the scientists of the new field of geology were coming to the conclusion that Earth must be “millions and millions” of years old. At about the same time, Charles Darwin had published his treatise on evolution. Darwin’s view was that evolution needed many millions of years to take place, and he made a bold claim that the Weald chalk fields, where important fossils were found, were the result of 300 million years of erosion.



(a)



(b)

Figure 14.3.5: (a) William Thomson (Lord Kelvin), 1824-1907, was a British physicist and electrical engineer; (b) Kelvin used the heat diffusion equation to estimate the age of Earth (credit: modification of work by NASA).

At that time, eminent physicist William Thomson (Lord Kelvin) used an important partial differential equation, known as the heat diffusion equation, to estimate the age of Earth by determining how long it would take Earth to cool from molten rock to what we had at that time. His conclusion was a range of **20** to **400** million years, but most likely about **50** million years. For many decades, the proclamations of this irrefutable icon of science did not sit well with geologists or with Darwin.

- Read Kelvin's paper on estimating the age of the Earth.

Kelvin made reasonable assumptions based on what was known in his time, but he also made several assumptions that turned out to be wrong. One incorrect assumption was that Earth is solid and that the cooling was therefore via conduction only, hence justifying the use of the diffusion equation. But the most serious error was a forgivable one—omission of the fact that Earth contains radioactive elements that continually supply heat beneath Earth's mantle. The discovery of radioactivity came near the end of Kelvin's life and he acknowledged that his calculation would have to be modified.

Kelvin used the simple one-dimensional model applied only to Earth's outer shell, and derived the age from graphs and the roughly known temperature gradient near Earth's surface. Let's take a look at a more appropriate version of the diffusion equation in radial coordinates, which has the form

$$\frac{\partial T}{\partial t} = K\left[\frac{\partial^2 T}{\partial^2 r} + \frac{2}{r}\frac{\partial T}{\partial r}\right] \tag{14.3.10}$$

.

Here, $T(r,t)$ is temperature as a function of r (measured from the center of Earth) and time t. K is the heat conductivity—for molten rock, in this case. The standard method of solving such a partial differential equation is by separation of variables, where we express the solution as the product of functions containing each variable separately. In this case, we would write the temperature as

$$T(r,t) = R(r)f(t).$$

1. Substitute this form into Equation 14.3.10and, noting that $f(t)$ is constant with respect to distance (r) and $R(r)$ is constant with respect to time (t), show that

$$\frac{1}{f}\frac{\partial f}{\partial t} = \frac{K}{R}\left[\frac{\partial^2 R}{\partial r^2} + \frac{2}{r}\frac{\partial R}{\partial r}\right].$$

2. This equation represents the separation of variables we want. The left-hand side is only a function of t and the right-hand side is only a function of r, and they must be equal for all values of r and t. Therefore, they both must be equal to a constant. Let's call that constant $-\lambda^2$. (The convenience of this choice is seen on substitution.) So, we have

$$\frac{1}{f}\frac{\partial f}{\partial t} = -\lambda^2 \text{and} \frac{K}{R}\left[\frac{\partial^2 R}{\partial r^2} + \frac{2}{r}\frac{\partial R}{\partial r}\right] = -\lambda^2.$$

3. Now, we can verify through direct substitution for each equation that the solutions are $f(t) = Ae^{-\lambda^2 t}$ and $R(r) = B\left(\frac{\sin \alpha r}{r}\right) + C\left(\frac{\cos \alpha r}{r}\right)$, where $\alpha = \lambda/\sqrt{K}$. Note that $f(t) = Ae^{+\lambda n^2 t}$ is also a valid solution, so we could have chosen $+\lambda^2$ for our constant. Can you see why it would not be valid for this case as time increases?
4. Let's now apply boundary conditions.
   a. The temperature must be finite at the center of Earth, $r = 0$. Which of the two constants, B or C, must therefore be zero to keep R finite at $r = 0$? (Recall that $\sin(\alpha r)/r \to \alpha =$ as $r \to 0$, but $\cos(\alpha r)/r$ behaves very differently.)
   b. Kelvin argued that when magma reaches Earth's surface, it cools very rapidly. A person can often touch the surface within weeks of the flow. Therefore, the surface reached a moderate temperature very early and remained nearly constant at a surface temperature T_s. For simplicity, let's set $T = 0$ at $r = R_E$ and find α such that this is the temperature there for all time t. (Kelvin took the value to be $300K \approx 80^\circ F$. We can add this $300K$ constant to our solution later.) For this to be true, the sine argument must be zero at $r = R_E$. Note that α has an infinite series of values that satisfies this condition. Each value of α represents a valid solution (each with its own value for A). The total or general solution is the sum of all these solutions.
   c. At $t = 0$, we assume that all of Earth was at an initial hot temperature T_0 (Kelvin took this to be about $7000K$.) The application of this boundary condition involves the more advanced application of Fourier coefficients. As noted in part b. each value of α_n represents a valid solution, and the general solution is a sum of all these solutions. This results in a series solution:

$$T(r,t) = \left(\frac{T_0 R_E}{\pi}\right) \sum_n \frac{(-1)^{n-1}}{n} e^{-\lambda n^2 t} \frac{\sin(\alpha_n r)}{r}$$

   where $\alpha_n = n\pi/R_E$.

Note how the values of α_n come from the boundary condition applied in part b. The term $\frac{-1^{n-1}}{n}$ is the constant A_n for each term in the series, determined from applying the Fourier method. Letting $\beta = \frac{\pi}{R_E}$, examine the first few terms of this solution shown here and note how λ^2 in the exponential causes the higher terms to decrease quickly as time progresses:

$$T(r,t)$$
$$= \frac{T_0 R_E}{\pi r}\left(e^{-K\beta^2 t}(\sin \beta r) - \frac{1}{2}e^{-4K\beta^2 t}(\sin 2\beta r) + \frac{1}{3}e^{-9K\beta^2 t}(\sin 3\beta r) - \frac{1}{4}e^{-16K\beta^2 t}(\sin 4\beta r) + \frac{1}{5}e^{-25K\beta^2 t}(\sin 5\beta r)\ldots\right).$$

Near time $t = 0$, many terms of the solution are needed for accuracy. Inserting values for the conductivity K and $\beta = \pi/R_E$ for time approaching merely thousands of years, only the first few terms make a significant contribution. Kelvin only needed to look at the solution near Earth's surface (Figure 14.3.6) and, after a long time, determine what time best yielded the estimated temperature gradient known during his era ($1^\circ F$ increase per $50ft$). He simply chose a range of times with a gradient close to this value. In Figure 14.3.6, the solutions are plotted and scaled, with the $300 - K$ surface temperature added. Note that the center of Earth would be relatively cool. At the time, it was thought Earth must be solid.

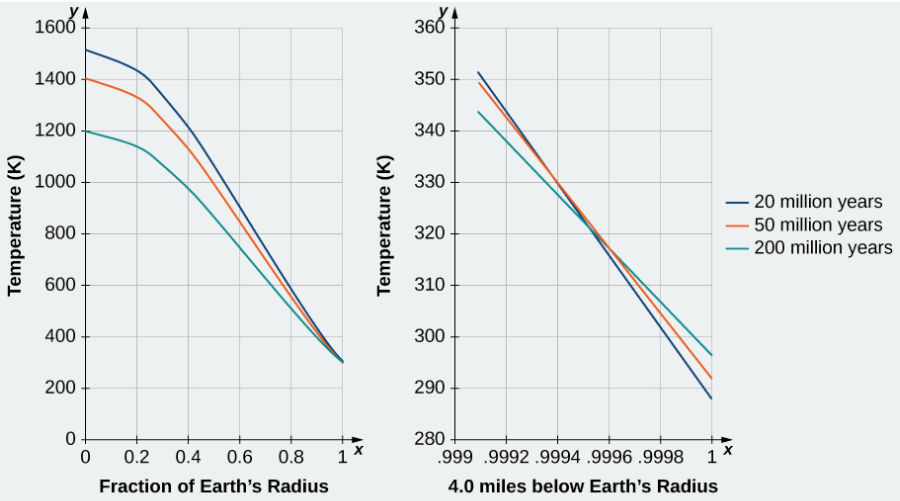


Figure 14.3.6: Temperature versus radial distance from the center of Earth. (a) Kelvin's results, plotted to scale. (b) A close-up of the results at a depth of 4.0 miles below Earth's surface.

Epilog

On May **20, 1904,** physicist **Ernest Rutherford** spoke at the Royal Institution to announce a revised calculation that included the contribution of radioactivity as a source of Earth's heat. In Rutherford's own words:

> "I came into the room, which was half-dark, and presently spotted Lord Kelvin in the audience, and realized that I was in for trouble at the last part of my speech dealing with the age of the Earth, where my views conflicted with his. To my relief, Kelvin fell fast asleep, but as I came to the important point, I saw the old bird sit up, open an eye and cock a baleful glance at me.
>
> Then a sudden inspiration came, and I said Lord Kelvin had limited the age of the Earth, provided no new source [of heat] was discovered. That prophetic utterance referred to what we are now considering tonight, radium! Behold! The old boy beamed upon me."

Rutherford calculated an age for Earth of about **500** million years. Today's accepted value of Earth's age is about **4.6** billion years.

Key Concepts

- A partial derivative is a derivative involving a function of more than one independent variable.
- To calculate a partial derivative with respect to a given variable, treat all the other variables as constants and use the usual differentiation rules.
- Higher-order partial derivatives can be calculated in the same way as higher-order derivatives.

Key Equations

Partial derivative of f with respect to x

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

Partial derivative of f with respect to y

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

Glossary

higher-order partial derivatives

second-order or higher partial derivatives, regardless of whether they are mixed partial derivatives

mixed partial derivatives

second-order or higher partial derivatives, in which at least two of the differentiations are with respect to different variables

partial derivative

a derivative of a function of more than one independent variable in which all the variables but one are held constant

partial differential equation

an equation that involves an unknown function of more than one independent variable and one or more of its partial derivatives

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14.4: Tangent Planes and Linear Approximations

Learning Objectives

- Determine the equation of a plane tangent to a given surface at a point.
- Use the tangent plane to approximate a function of two variables at a point.
- Explain when a function of two variables is differentiable.
- Use the total differential to approximate the change in a function of two variables.

In this section, we consider the problem of finding the tangent plane to a surface, which is analogous to finding the equation of a tangent line to a curve when the curve is defined by the graph of a function of one variable, $y = f(x)$. The slope of the tangent line at the point $x = a$ is given by $m = f'(a)$; what is the slope of a tangent plane? We learned about the equation of a plane in Equations of Lines and Planes in Space; in this section, we see how it can be applied to the problem at hand.

Tangent Planes

Intuitively, it seems clear that, in a plane, only one line can be tangent to a curve at a point. However, in three-dimensional space, many lines can be tangent to a given point. If these lines lie in the same plane, they determine the tangent plane at that point. A more intuitive way to think of a tangent plane is to assume the surface is smooth at that point (no corners). Then, a tangent line to the surface at that point in any direction does not have any abrupt changes in slope because the direction changes smoothly. Therefore, in a small-enough neighborhood around the point, a tangent plane touches the surface at that point only.

Definition: Tangent Lines

Let $P_0 = (x_0, y_0, z_0)$ be a point on a surface S, and let C be any curve passing through P_0 and lying entirely in S. If the tangent lines to all such curves C at P_0 lie in the same plane, then this plane is called the *tangent plane* to S at P_0 (Figure 14.4.1).

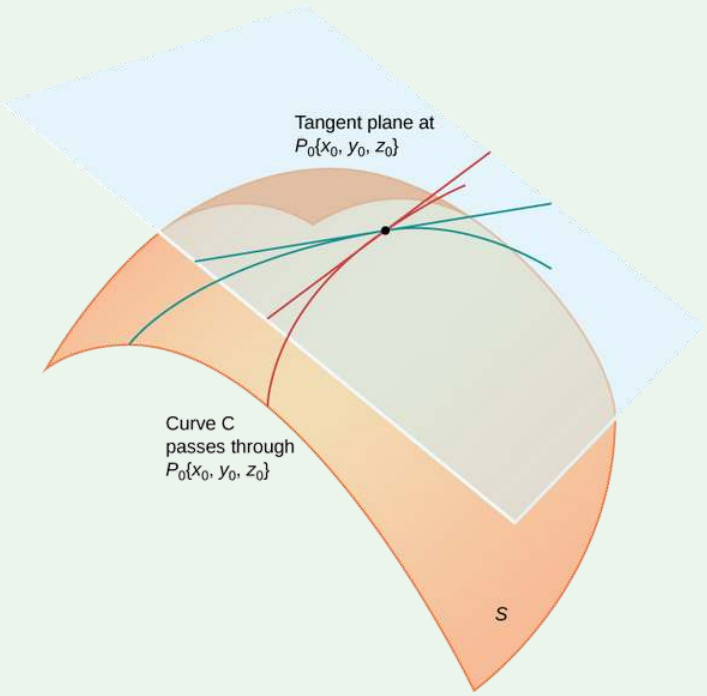


Figure 14.4.1: The tangent plane to a surface S at a point P_0 contains all the tangent lines to curves in S *that pass through* P_0.

For a tangent plane to a surface to exist at a point on that surface, it is sufficient for the function that defines the surface to be differentiable at that point. We define the term tangent plane here and then explore the idea intuitively.

Definition: Tangent Planes

Let S be a surface defined by a differentiable function $z = f(x, y)$, and let $P_0 = (x_0, y_0)$ be a point in the domain of f. Then, the equation of the tangent plane to S at P_0 is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \tag{14.4.1}$$

To see why this formula is correct, let's first find two tangent lines to the surface S. The equation of the tangent line to the curve that is represented by the intersection of S with the vertical trace given by $x = x_0$ is $z = f(x_0, y_0) + f_y(x_0, y_0)(y - y_0)$. Similarly, the equation of the tangent line to the curve that is represented by the intersection of S with the vertical trace given by $y = y_0$ is $z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0)$. A parallel vector to the first tangent line is $\vec{\mathbf{a}} = \hat{\mathbf{j}} + f_y(x_0, y_0)\,\hat{\mathbf{k}}$; a parallel vector to the second tangent line is $\vec{\mathbf{b}} = \hat{\mathbf{i}} + f_x(x_0, y_0)\,\hat{\mathbf{k}}$. We can take the cross product of these two vectors:

$$\begin{aligned}
\vec{\mathbf{a}} \times \vec{\mathbf{b}} &= (\hat{\mathbf{j}} + f_y(x_0, y_0)\,\hat{\mathbf{k}}) \times (\hat{\mathbf{i}} + f_x(x_0, y_0)\,\hat{\mathbf{k}}) \\
&= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 1 & f_y(x_0, y_0) \\ 1 & 0 & f_x(x_0, y_0) \end{vmatrix} \\
&= f_x(x_0, y_0)\,\hat{\mathbf{i}} + f_y(x_0, y_0)\,\hat{\mathbf{j}} - \hat{\mathbf{k}}.
\end{aligned}$$

This vector is perpendicular to both lines and is therefore perpendicular to the tangent plane. We can use this vector as a normal vector to the tangent plane, along with the point $P_0 = (x_0, y_0, f(x_0, y_0))$ in the equation for a plane:

$$\begin{aligned}
\vec{\mathbf{n}} \cdot ((x - x_0)\,\hat{\mathbf{i}} + (y - y_0)\,\hat{\mathbf{j}} + (z - f(x_0, y_0))\,\hat{\mathbf{k}}) &= 0 \\
(f_x(x_0, y_0)\,\hat{\mathbf{i}} + f_y(x_0, y_0)\,\hat{\mathbf{j}} - \hat{\mathbf{k}}) \cdot ((x - x_0)\,\hat{\mathbf{i}} + (y - y_0)\,\hat{\mathbf{j}} + (z - f(x_0, y_0))\,\hat{\mathbf{k}}) &= 0 \\
f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - f(x_0, y_0)) &= 0.
\end{aligned}$$

Solving this equation for z gives Equation 14.4.1.

Example 14.4.1: Finding a Tangent Plane

Find the equation of the tangent plane to the surface defined by the function $f(x, y) = 2x^2 - 3xy + 8y^2 + 2x - 4y + 4$ at point $(2, -1)$.

Solution

First, we must calculate $f_x(x, y)$ and $f_y(x, y)$, then use Equation with $x_0 = 2$ and $y_0 = -1$:

$$\begin{aligned}
f_x(x, y) &= 4x - 3y + 2 \\
f_y(x, y) &= -3x + 16y - 4 \\
f(2, -1) &= 2(2)^2 - 3(2)(-1) + 8(-1)^2 + 2(2) - 4(-1) + 4 = 34 \\
f_x(2, -1) &= 4(2) - 3(-1) + 2 = 13 \\
f_y(2, -1) &= -3(2) + 16(-1) - 4 = -26.
\end{aligned}$$

Then Equation 14.4.1 becomes

$$\begin{aligned}
z &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\
z &= 34 + 13(x - 2) - 26(y - (-1)) \\
z &= 34 + 13x - 26 - 26y - 26 \\
z &= 13x - 26y - 18.
\end{aligned}$$

(See the following figure).

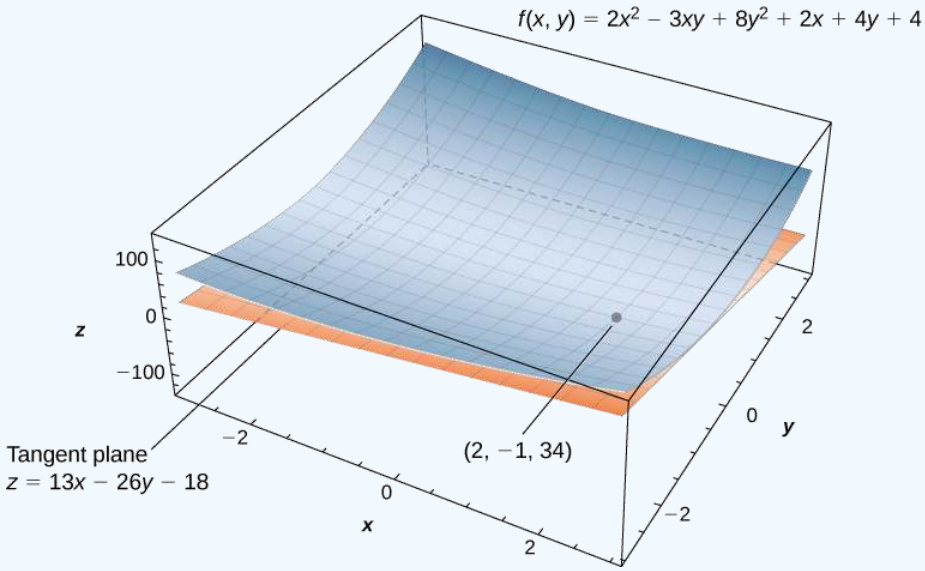


Figure 14.4.2: Calculating the equation of a tangent plane to a given surface at a given point.

Exercise 14.4.1

Find the equation of the tangent plane to the surface defined by the function $f(x,y) = x^3 - x^2y + y^2 - 2x + 3y - 2$ at point $(-1, 3)$.

Hint

First, calculate $f_x(x,y)$ and $f_y(x,y)$, then use Equation 14.4.1.

Answer

$z = 7x + 8y - 3$

Example 14.4.2: Finding Another Tangent Plane

Find the equation of the tangent plane to the surface defined by the function $f(x,y) = \sin(2x)\cos(3y)$ at the point $(\pi/3, \pi/4)$.

Solution

First, calculate $f_x(x,y)$ and $f_y(x,y)$, then use Equation 14.4.1 with $x_0 = \pi/3$ and $y_0 = \pi/4$:

$$f_x(x,y) = 2\cos(2x)\cos(3y)$$

$$f_y(x,y) = -3\sin(2x)\sin(3y)$$

$$f\left(\frac{\pi}{3}, \frac{\pi}{4}\right) = \sin\left(2\left(\frac{\pi}{3}\right)\right)\cos\left(3\left(\frac{\pi}{4}\right)\right) = \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{6}}{4}$$

$$f_x\left(\frac{\pi}{3}, \frac{\pi}{4}\right) = 2\cos\left(2\left(\frac{\pi}{3}\right)\right)\cos\left(3\left(\frac{\pi}{4}\right)\right) = 2\left(-\frac{1}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}$$

$$f_y\left(\frac{\pi}{3}, \frac{\pi}{4}\right) = -3\sin\left(2\left(\frac{\pi}{3}\right)\right)\sin\left(3\left(\frac{\pi}{4}\right)\right) = -3\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = -\frac{3\sqrt{6}}{4}.$$

Then Equation 14.4.1 becomes

$$\begin{aligned} z &= f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0) \\ &= -\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{2}\left(x-\frac{\pi}{3}\right)-\frac{3\sqrt{6}}{4}\left(y-\frac{\pi}{4}\right) \\ &= \frac{\sqrt{2}}{2}x-\frac{3\sqrt{6}}{4}y-\frac{\sqrt{6}}{4}-\frac{\pi\sqrt{2}}{6}+\frac{3\pi\sqrt{6}}{16} \end{aligned}$$

A tangent plane to a surface does not always exist at every point on the surface. Consider the piecewise function

$$f(x,y)=\begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}. \tag{14.4.2}$$

The graph of this function follows.

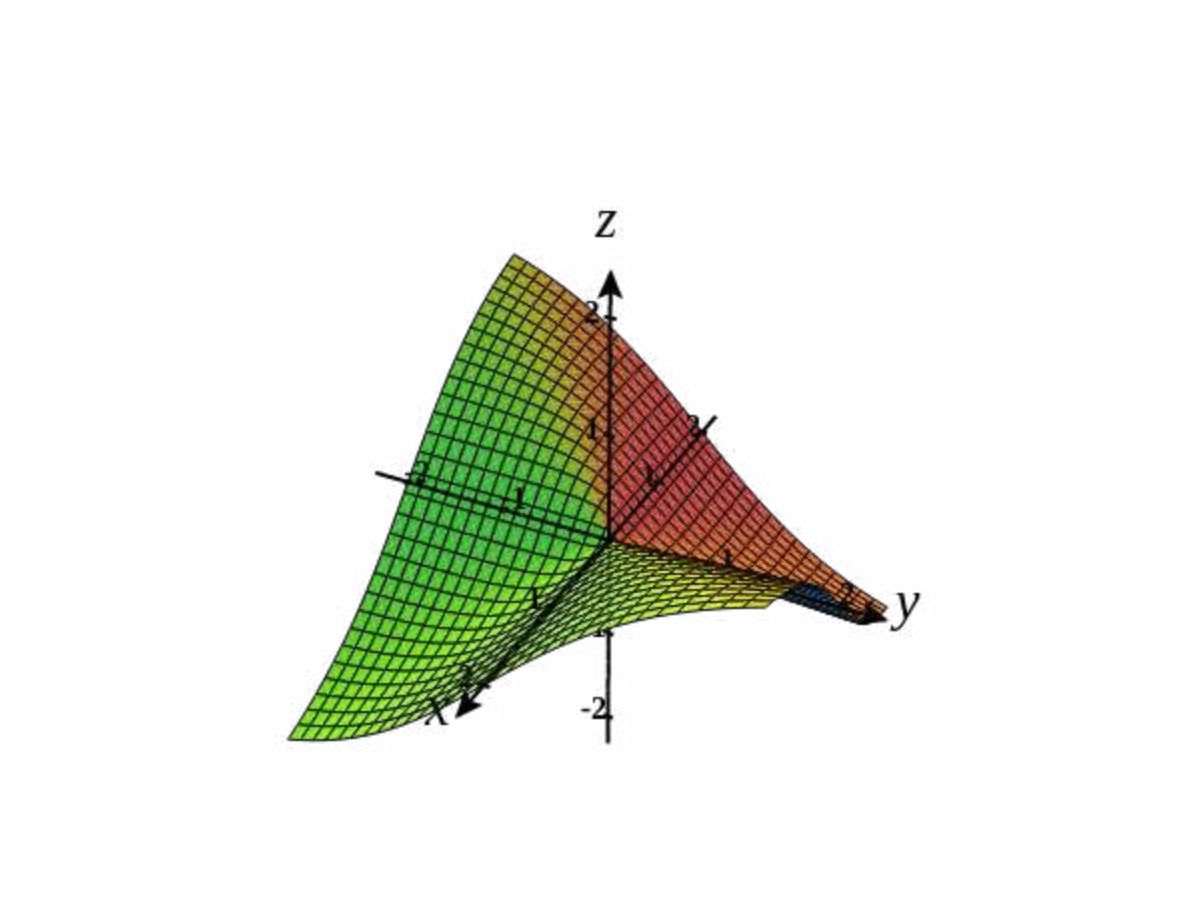


Figure* 14.4.3*: *Graph of a function that does not have a tangent plane at the origin. Dynamic figure powered by CalcPlot3D.*

If either $x=0$ or $y=0$, then $f(x,y)=0$, so the value of the function does not change on either the x- or y-axis. Therefore, $f_x(x,0)=f_y(0,y)=0$, so as either x or y approach zero, these partial derivatives stay equal to zero. Substituting them into Equation gives $z=0$ as the equation of the tangent line. However, if we approach the origin from a different direction, we get a different story. For example, suppose we approach the origin along the line $y=x$. If we put $y=x$ into the original function, it becomes

$$f(x,x)=\frac{x(x)}{\sqrt{x^2+(x)^2}}=\frac{x^2}{\sqrt{2x^2}}=\frac{|x|}{\sqrt{2}}.$$

When $x > 0$, the slope of this curve is equal to $\sqrt{2}/2$; when $x < 0$, the slope of this curve is equal to $-(\sqrt{2}/2)$. This presents a problem. In the definition of tangent plane, we presumed that all tangent lines through point P (in this case, the origin) lay in the same plane. This is clearly not the case here. When we study differentiable functions, we will see that this function is not differentiable at the origin.

Linear Approximations

Recall from Linear Approximations and Differentials that the formula for the linear approximation of a function $f(x)$ at the point $x = a$ is given by

$$y \approx f(a) + f'(a)(x - a).$$

The diagram for the linear approximation of a function of one variable appears in the following graph.

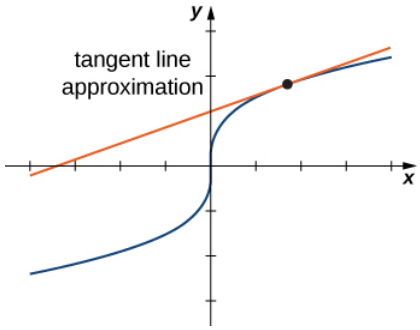


Figure 14.4.4: Linear approximation of a function in one variable.

The tangent line can be used as an approximation to the function $f(x)$ for values of x reasonably close to $x = a$. When working with a function of two variables, the tangent line is replaced by a tangent plane, but the approximation idea is much the same.

Definition: Linear Approximation

Given a function $z = f(x, y)$ with continuous partial derivatives that exist at the point (x_0, y_0), the linear approximation of f at the point (x_0, y_0) is given by the equation

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \tag{14.4.3}$$

Notice that this equation also represents the tangent plane to the surface defined by $z = f(x, y)$ at the point (x_0, y_0). The idea behind using a linear approximation is that, if there is a point (x_0, y_0) at which the precise value of $f(x, y)$ is known, then for values of (x, y) reasonably close to (x_0, y_0), the linear approximation (i.e., tangent plane) yields a value that is also reasonably close to the exact value of $f(x, y)$ (Figure 14.4.5). Furthermore the plane that is used to find the linear approximation is also the tangent plane to the surface at the point (x_0, y_0).

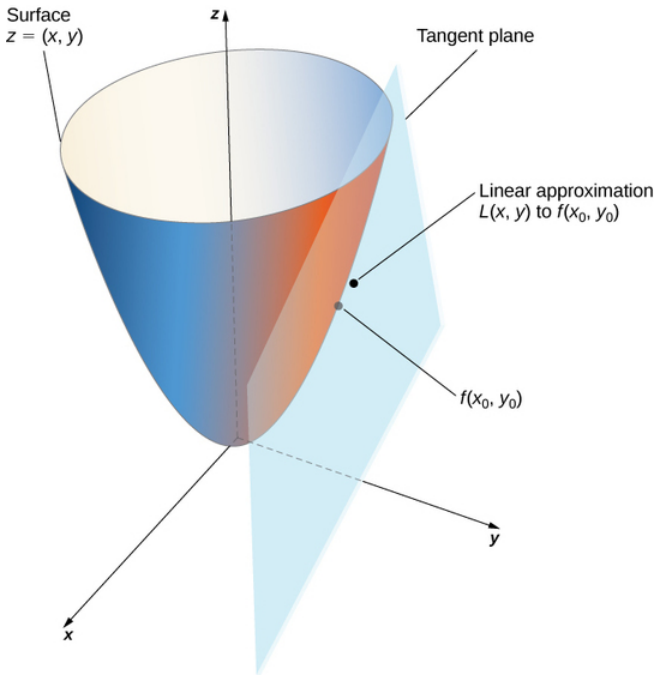


Figure 14.4.5: Using a tangent plane for linear approximation at a point.

Example 14.4.3: Using a Tangent Plane Approximation

Given the function $f(x,y)=\sqrt{41-4x^2-y^2}$, approximate $f(2.1,2.9)$ using point $(2,3)$ for (x_0,y_0). What is the approximate value of $f(2.1,2.9)$to four decimal places?

Solution

To apply Equation 14.4.3, we first must calculate $f(x_0,y_0)$, $f_x(x_0,y_0)$, and $f_y(x_0,y_0)$ using $x_0=2$ and $y_0=3$:

$$\begin{aligned} f(x_0,y_0) &= f(2,3)=\sqrt{41-4(2)^2-(3)^2}=\sqrt{41-16-9}=\sqrt{16}=4 \\ f_x(x,y) &= -\frac{4x}{\sqrt{41-4x^2-y^2}} \text{ so } f_x(x_0,y_0)=-\frac{4(2)}{\sqrt{41-4(2)^2-(3)^2}}=-2 \\ f_y(x,y) &= -\frac{y}{\sqrt{41-4x^2-y^2}} \text{ so } f_y(x_0,y_0)=-\frac{3}{\sqrt{41-4(2)^2-(3)^2}}=-\frac{3}{4}. \end{aligned}$$

Now we substitute these values into Equation 14.4.3:

$$\begin{aligned} L(x,y) &= f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0) \\ &= 4-2(x-2)-\frac{3}{4}(y-3) \\ &= \frac{41}{4}-2x-\frac{3}{4}y. \end{aligned}$$

Last, we substitute $x=2.1$ and $y=2.9$ into $L(x,y)$:

$$L(2.1,2.9)=\frac{41}{4}-2(2.1)-\frac{3}{4}(2.9)=10.25-4.2-2.175=3.875.$$

The approximate value of $f(2.1,2.9)$to four decimal places is

$$f(2.1,2.9)=\sqrt{41-4(2.1)^2-(2.9)^2}=\sqrt{14.95}\approx 3.8665,$$

which corresponds to a 0.2 error in approximation.

Exercise 14.4.2

Given the function $f(x,y)=e^{5-2x+3y}$, approximate $f(4.1,0.9)$using point $(4,1)$ for (x_0,y_0). What is the approximate value of $f(4.1,0.9)$to four decimal places?

Hint

First calculate $f(x_0,y_0)$, $f_x(x_0,y_0)$, and $f_y(x_0,y_0)$ using $x_0=4$ and $y_0=1$, then use Equation 14.4.3

Answer

$L(x,y)=6-2x+3y,$ so $L(4.1,0.9)=6-2(4.1)+3(0.9)=0.5\, f(4.1,0.9)=e^{5-2(4.1)+3(0.9)}=e^{-0.5}\approx 0.6065.$

Differentiability

When working with a function $y=f(x)$ of one variable, the function is said to be differentiable at a point $x=a$ if $f'(a)$ exists. Furthermore, if a function of one variable is differentiable at a point, the graph is "smooth" at that point (i.e., no corners exist) and a tangent line is well-defined at that point.

The idea behind differentiability of a function of two variables is connected to the idea of smoothness at that point. In this case, a surface is considered to be smooth at point P if a tangent plane to the surface exists at that point. If a function is differentiable at a point, then a tangent plane to the surface exists at that point. Recall the formula (Equation 14.4.1) for a tangent plane at a point (x_0,y_0) is given by

$$z=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

For a tangent plane to exist at the point (x_0,y_0), the partial derivatives must therefore exist at that point. However, this is not a sufficient condition for smoothness, as was illustrated in Figure 14.4.3. In that case, the partial derivatives existed at the origin, but the function also had a corner on the graph at the origin.

Definition: Differentiable Functions

A function $f(x,y)$is *differentiable* at a point $P(x_0,y_0)$ if, for all points (x,y) in a δ disk around P, we can write

$$f(x,y)=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)+E(x,y), \tag{14.4.4}$$

where the error term E satisfies

$$\lim_{(x,y)\to(x_0,y_0)}\frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}=0. \tag{14.4.5}$$

The last term in Equation 14.4.4 is referred to as the **error term** and it represents how closely the tangent plane comes to the surface in a small neighborhood (δ disk) of point P. For the function f to be differentiable at P, the function must be smooth—that is, the graph of f must be close to the tangent plane for points near P.

Example 14.4.4: Demonstrating Differentiability

Show that the function $f(x,y)=2x^2-4y$ is differentiable at point $(2,-3)$.

Solution

First, we calculate $f(x_0,y_0)$, $f_x(x_0,y_0)$, and $f_y(x_0,y_0)$ using $x_0=2$ and $y_0=-3,$ then we use Equation 14.4.4:

$$\begin{aligned} f(2,-3) &= 2(2)^2-4(-3)=8+12=20\\ f_x(2,-3) &= 4(2)=8\\ f_y(2,-3) &= -4. \end{aligned}$$

Therefore $m_1=8$ and $m_2=-4,$ and Equation 14.4.4becomes

$$
\begin{aligned}
f(x,y) &= f(2,-3)+f_x(2,-3)(x-2)+f_y(2,-3)(y+3)+E(x,y)\\
2x^2-4y &= 20+8(x-2)-4(y+3)+E(x,y)\\
2x^2-4y &= 20+8x-16-4y-12+E(x,y)\\
2x^2-4y &= 8x-4y-8+E(x,y)\\
E(x,y) &= 2x^2-8x+8.
\end{aligned}
$$

Next, we calculate the limit in Equation 14.4.5:

$$
\begin{aligned}
\lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} &= \lim_{(x,y)\to(2,-3)} \frac{2x^2-8x+8}{\sqrt{(x-2)^2+(y+3)^2}}\\
&= \lim_{(x,y)\to(2,-3)} \frac{2(x^2-4x+4)}{\sqrt{(x-2)^2+(y+3)^2}}\\
&= \lim_{(x,y)\to(2,-3)} \frac{2(x-2)^2}{\sqrt{(x-2)^2+(y+3)^2}}\\
&\le \lim_{(x,y)\to(2,-3)} \frac{2((x-2)^2+(y+3)^2)}{\sqrt{(x-2)^2+(y+3)^2}}\\
&= \lim_{(x,y)\to(2,-3)} 2\sqrt{(x-2)^2+(y+3)^2}\\
&= 0.
\end{aligned}
$$

Since $E(x,y)\ge 0$ for any value of x or y, the original limit must be equal to zero. Therefore, $f(x,y)=2x^2-4y$ is differentiable at point $(2,-3)$.

? Exercise 14.4.3

Show that the function $f(x,y)=3x-4y^2$ is differentiable at point $(-1,2)$.

Hint

First, calculate $f(x_0,y_0)$, $f_x(x_0,y_0)$, and $f_y(x_0,y_0)$ using $x_0=-1$ and $y_0=2$, then use Equation 14.4.5to find $E(x,y)$. Last, calculate the limit.

Answer

$$
f(-1,2) = -19, \quad f_x(-1,2)=3, \quad f_y(-1,2)=-16, \quad E(x,y)=-4(y-2)^2.
$$

$$
\begin{aligned}
\lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} &= \lim_{(x,y)\to(-1,2)} \frac{-4(y-2)^2}{\sqrt{(x+1)^2+(y-2)^2}}\\
&\le \lim_{(x,y)\to(-1,2)} \frac{-4((x+1)^2+(y-2)^2)}{\sqrt{(x+1)^2+(y-2)^2}}\\
&= \lim_{(x,y)\to(2,-3)} -4\sqrt{(x+1)^2+(y-2)^2}\\
&= 0.
\end{aligned}
$$

This function from (Equation 14.4.2)

$$
f(x,y)=\begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}
$$

is not differentiable at the origin (Figure 14.4.3). We can see this by calculating the partial derivatives. This function appeared earlier in the section, where we showed that $f_x(0,0) = f_y(0,0) = 0$. Substituting this information into Equations 14.4.4 and 14.4.5 using $x_0 = 0$ and $y_0 = 0$, we get

$$\begin{aligned} f(x,y) &= f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) + E(x,y) \\ E(x,y) &= \frac{xy}{\sqrt{x^2+y^2}}. \end{aligned}$$

Calculating

$$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}$$

gives

$$\begin{aligned} \lim_{(x,y)\to(x_0,y_0)} \frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} &= \lim_{(x,y)\to(0,0)} \frac{\dfrac{xy}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} \\ &= \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}. \end{aligned}$$

Depending on the path taken toward the origin, this limit takes different values. Therefore, the limit does not exist and the function f is not differentiable at the origin as shown in the following figure.

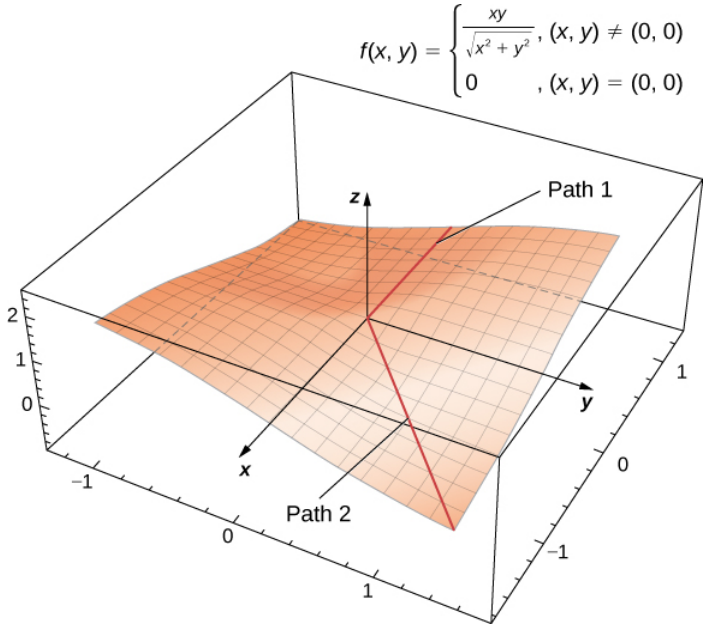


Figure 14.4.6: This function $f(x,y)$ (Equation 14.4.2) is not differentiable at the origin.

Differentiability and continuity for functions of two or more variables are connected, the same as for functions of one variable. In fact, with some adjustments of notation, the basic theorem is the same.

Theorem 14.4.1: Differentiability Implies Continuity

Let $z = f(x,y)$ be a function of two variables with (x_0,y_0) in the domain of f. If $f(x,y)$ is differentiable at (x_0,y_0), then $f(x,y)$ is continuous at (x_0,y_0).

Theorem 14.4.1 shows that if a function is differentiable at a point, then it is continuous there. However, if a function is continuous at a point, then it is not necessarily differentiable at that point. For example, the function discussed above (Equation 14.4.2)

$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

is *continuous* at the origin, but it is *not differentiable* at the origin. This observation is also similar to the situation in single-variable calculus.

We can further explore the connection between continuity and differentiability at a point. This next theorem says that if the function and its partial derivatives are continuous at a point, the function is differentiable.

Theorem 14.4.2: Continuity of First Partials Implies Differentiability

Let $z=f(x,y)$ be a function of two variables with (x_0,y_0) in the domain of f. If $f(x,y)$, $f_x(x,y)$, and $f_y(x,y)$ all exist in a neighborhood of (x_0,y_0) and are continuous at (x_0,y_0), then $f(x,y)$ is differentiable there.

Recall that earlier we showed that the function in Equation 14.4.2 was not differentiable at the origin. Let's calculate the partial derivatives f_x and f_y:

$$\frac{\partial f}{\partial x}=\frac{y^3}{(x^2+y^2)^{3/2}}$$

and

$$\frac{\partial f}{\partial y}=\frac{x^3}{(x^2+y^2)^{3/2}}.$$

The contrapositive of the preceding theorem states that if a function is not differentiable, then at least one of the hypotheses must be false. Let's explore the condition that $f_x(0,0)$ must be continuous. For this to be true, it must be true that

$$\lim_{(x,y)\to(0,0)} f_x(x,y)=f_x(0,0)$$

therefor

$$\lim_{(x,y)\to(0,0)} f_x(x,y)=\lim_{(x,y)\to(0,0)}\frac{y^3}{(x^2+y^2)^{3/2}}.$$

Let $x=ky$. Then

$$\begin{aligned}\lim_{(x,y)\to(0,0)}\frac{y^3}{(x^2+y^2)^{3/2}} &= \lim_{y\to0}\frac{y^3}{((ky)^2+y^2)^{3/2}}\\ &= \lim_{y\to0}\frac{y^3}{(k^2y^2+y^2)^{3/2}}\\ &= \lim_{y\to0}\frac{y^3}{|y|^3(k^2+1)^{3/2}}\\ &= \frac{1}{(k^2+1)^{3/2}}\lim_{y\to0}\frac{|y|}{y}.\end{aligned}$$

If $y>0$, then this expression equals $1/(k^2+1)^{3/2}$; if $y<0$, then it equals $-(1/(k^2+1)^{3/2})$. In either case, the value depends on k, so the limit fails to exist.

Differentials

In Linear Approximations and Differentials we first studied the concept of differentials. The differential of y, written dy, is defined as $f'(x)dx$. The differential is used to approximate $\Delta y=f(x+\Delta x)-f(x)$, where $\Delta x=dx$. Extending this idea to the linear approximation of a function of two variables at the point (x_0,y_0) yields the formula for the total differential for a function of two variables.

Definition: Total Differential

Let $z = f(x, y)$ be a function of two variables with (x_0, y_0) in the domain of f, and let Δx and Δy be chosen so that $(x_0 + \Delta x, y_0 + \Delta y)$ is also in the domain of f. If f is differentiable at the point (x_0, y_0), then the differentials dx and dy are defined as

$$dx = \Delta x$$

and

$$dy = \Delta y.$$

The differential dz, also called the ***total differential*** of $z = f(x, y)$ at (x_0, y_0), is defined as

$$dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy. \tag{14.4.6}$$

Notice that the symbol ∂ is not used to denote the total differential; rather, d appears in front of z. Now, let's define $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$. We use dz to approximate Δz, so

$$\Delta z \approx dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy.$$

Therefore, the differential is used to approximate the change in the function $z = f(x_0, y_0)$ at the point (x_0, y_0) for given values of Δx and Δy. Since $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$, this can be used further to approximate $f(x + \Delta x, y + \Delta y)$:

$$f(x + \Delta x, y + \Delta y) = f(x, y) + \Delta z \approx f(x, y) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y.$$

See the following figure.

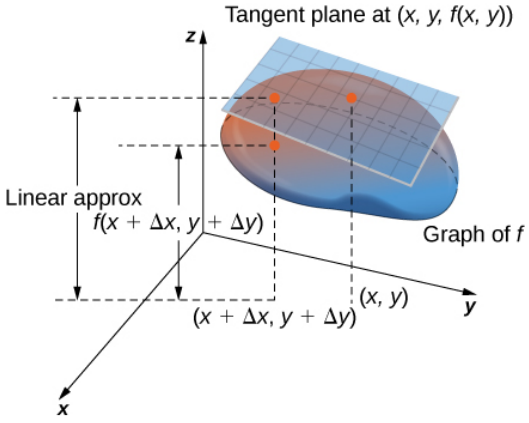


Figure 14.4.7: The linear approximation is calculated via the formula $f(x + \Delta x, y + \Delta y) \approx f(x, y) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$.

One such application of this idea is to determine error propagation. For example, if we are manufacturing a gadget and are off by a certain amount in measuring a given quantity, the differential can be used to estimate the error in the total volume of the gadget.

Example 14.4.5: Approximation by Differentials

Find the differential dz of the function $f(x, y) = 3x^2 - 2xy + y^2$ and use it to approximate Δz at point $(2, -3)$. Use $\Delta x = 0.1$ and $\Delta y = -0.05$. What is the exact value of Δz?

Solution

First, we must calculate $f(x_0, y_0)$, $f_x(x_0, y_0)$, and $f_y(x_0, y_0)$ using $x_0 = 2$ and $y_0 = -3$:

$$f(x_0,y_0) = f(2,-3) = 3(2)^2 - 2(2)(-3) + (-3)^2 = 12+12+9 = 33$$
$$f_x(x,y) = 6x - 2y$$
$$f_y(x,y) = -2x + 2y$$
$$f_x(x_0,y_0) = fx(2,-3) = 6(2) - 2(-3) = 12+6 = 18$$
$$f_y(x_0,y_0) = f_y(2,-3) = -2(2) + 2(-3) = -4-6 = -10.$$

Then, we substitute these quantities into Equation 14.4.6:

$$dz = f_x(x_0,y_0)dx + f_y(x_0,y_0)dy$$
$$dz = 18(0.1) - 10(-0.05) = 1.8 + 0.5 = 2.3.$$

This is the approximation to $\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)$. The exact value of Δz is given by

$$\begin{aligned}\Delta z &= f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)\\ &= f(2+0.1,-3-0.05) - f(2,-3)\\ &= f(2.1,-3.05) - f(2,-3)\\ &= 2.3425.\end{aligned}$$

? Exercise 14.4.4

Find the differential dz of the function $f(x,y) = 4y^2 + x^2y - 2xy$ and use it to approximate Δz at point $(1,-1)$. Use $\Delta x = 0.03$ and $\Delta y = -0.02$. What is the exact value of Δz?

Hint

First, calculate $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ using $x_0 = 1$ and $y_0 = -1$, then use Equation 14.4.6

Answer

$dz = 0.18$

$\Delta z = f(1.03,-1.02) - f(1,-1) = 0.180682$

Differentiability of a Function of Three Variables

All of the preceding results for differentiability of functions of two variables can be generalized to functions of three variables. First, the definition:

Definition: Differentiability at a Point

A function $f(x,y,z)$ is differentiable at a point $P(x_0,y_0,z_0)$ if for all points (x,y,z) in a δ disk around P we can write

$$f(x,y) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) + E(x,y,z),$$

where the error term E satisfies

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} \frac{E(x,y,z)}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} = 0.$$

If a function of three variables is differentiable at a point (x_0,y_0,z_0), then it is continuous there. Furthermore, continuity of first partial derivatives at that point guarantees differentiability.

Key Concepts

- The analog of a tangent line to a curve is a tangent plane to a surface for functions of two variables.
- Tangent planes can be used to approximate values of functions near known values.
- A function is differentiable at a point if it is ”smooth” at that point (i.e., no corners or discontinuities exist at that point).
- The total differential can be used to approximate the change in a function $z = f(x_0, y_0)$ at the point (x_0, y_0) for given values of Δx and Δy.

Key Equations

- **Tangent plane**

  $$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

- **Linear approximation**

  $$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

- **Total differential**

  $$dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy\ .$$

- **Differentiability (two variables)**

  $$f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + E(x, y),$$

  where the error term E satisfies

  $$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0\,.$$

- **Differentiability (three variables)**

  $$f(x, y) = f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) + E(x, y, z),$$

  where the error term E satisfies

  $$\lim_{(x,y,z)\to(x_0,y_0,z_0)} \frac{E(x, y, z)}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}} = 0\,.$$

Glossary

differentiable

a function $f(x, y)$ is differentiable at (x_0, y_0) if $f(x, y)$ can be expressed in the form $f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + E(x, y)$,

where the error term $E(x, y)$ satisfies $\lim_{(x,y)\to(x_0,y_0)} \dfrac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0$

linear approximation

given a function $f(x, y)$ and a tangent plane to the function at a point (x_0, y_0), we can approximate $f(x, y)$ for points near (x_0, y_0) using the tangent plane formula

tangent plane

given a function $f(x, y)$ that is differentiable at a point (x_0, y_0), the equation of the tangent plane to the surface $z = f(x, y)$ is given by $z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

total differential

the total differential of the function $f(x, y)$ at (x_0, y_0) is given by the formula $dz = f_x(x_0, y_0)dx + fy(x_0, y_0)dy$

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14.5: The Chain Rule

Learning Objectives

- State the chain rules for one or two independent variables.
- Use tree diagrams as an aid to understanding the chain rule for several independent and intermediate variables.
- Perform implicit differentiation of a function of two or more variables.

In single-variable calculus, we found that one of the most useful differentiation rules is the chain rule, which allows us to find the derivative of the composition of two functions. The same thing is true for multivariable calculus, but this time we have to deal with more than one form of the chain rule. In this section, we study extensions of the chain rule and learn how to take derivatives of compositions of functions of more than one variable.

Chain Rules for One or Two Independent Variables

Recall that the chain rule for the derivative of a composite of two functions can be written in the form

$$\frac{d}{dx}\Big(f(g(x))\Big) = f'\big(g(x)\big)g'(x).$$

In this equation, both $f(x)$ and $g(x)$ are functions of one variable. Now suppose that f is a function of two variables and g is a function of one variable. Or perhaps they are both functions of two variables, or even more. How would we calculate the derivative in these cases? The following theorem gives us the answer for the case of one independent variable.

Theorem 14.5.1: Chain Rule for One Independent Variable

Suppose that $x = g(t)$ and $y = h(t)$ are differentiable functions of t and $z = f(x, y)$ is a differentiable function of x and y. Then $z = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}, \tag{14.5.1}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at (x, y).

Proof

The proof of this theorem uses the definition of differentiability of a function of two variables. Suppose that f is differentiable at the point $P(x_0, y_0)$, where $x_0 = g(t_0)$ and $y_0 = h(t_0)$ for a fixed value of t_0. We wish to prove that $z = f\big(x(t), y(t)\big)$ is differentiable at $t = t_0$ and that Equation 14.5.1 holds at that point as well.

Since f is differentiable at P, we know that

$$z(t) = f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + E(x, y),$$

where

$$\lim_{(x,y)\to(x_0,y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0.$$

We then subtract $z_0 = f(x_0, y_0)$ from both sides of this equation:

$$\begin{aligned} z(t) - z(t_0) &= f(x(t), y(t)) - f(x(t_0), y(t_0)) \\ &= f_x(x_0, y_0)(x(t) - x(t_0)) + f_y(x_0, y_0)(y(t) - y(t_0)) + E(x(t), y(t)). \end{aligned}$$

Next, we divide both sides by $t - t_0$:

$$\frac{z(t) - z(t_0)}{t - t_0} = f_x(x_0, y_0)\frac{x(t) - x(t_0)}{t - t_0} + f_y(x_0, y_0)\frac{y(t) - y(t_0)}{t - t_0} + \frac{E(x(t), y(t))}{t - t_0}.$$

Then we take the limit as t approaches t_0:

$$\lim_{t\to t_0}\frac{z(t)-z(t_0)}{t-t_0} = f_x(x_0,y_0)\lim_{t\to t_0}\left(\frac{x(t)-x(t_0)}{t-t_0}\right) + f_y(x_0,y_0)\lim_{t\to t_0}\left(\frac{y(t)-y(t_0)}{t-t_0}\right) + \lim_{t\to t_0}\frac{E(x(t),y(t))}{t-t_0}.$$

The left-hand side of this equation is equal to dz/dt, which leads to

$$\frac{dz}{dt} = f_x(x_0,y_0)\frac{dx}{dt} + f_y(x_0,y_0)\frac{dy}{dt} + \lim_{t\to t_0}\frac{E(x(t),y(t))}{t-t_0}.$$

The last term can be rewritten as

$$\lim_{t\to t_0}\frac{E(x(t),y(t))}{t-t_0} = \lim_{t\to t_0}\frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\frac{\sqrt{(x-x_0)^2+(y-y_0)^2}}{t-t_0})$$
$$= \lim_{t\to t_0}\left(\frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\right)\lim_{t\to t_0}\left(\frac{\sqrt{(x-x_0)^2+(y-y_0)^2}}{t-t_0}\right).$$

As t approaches t_0, $\big(x(t),y(t)\big)$ approaches $\big(x(t_0),y(t_0)\big)$, so we can rewrite the last product as

$$\lim_{(x,y)\to(x_0,y_0)}\frac{E(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\lim_{(x,y)\to(x_0,y_0)}\left(\frac{\sqrt{(x-x_0)^2+(y-y_0)^2}}{t-t_0}\right).$$

Since the first limit is equal to zero, we need only show that the second limit is finite:

$$\lim_{(x,y)\to(x_0,y_0)}\frac{\sqrt{(x-x_0)^2+(y-y_0)^2}}{t-t_0} = \lim_{(x,y)\to(x_0,y_0)}\sqrt{\frac{(x-x_0)^2+(y-y_0)^2}{(t-t_0)^2}}$$
$$= \lim_{(x,y)\to(x_0,y_0)}\sqrt{\left(\frac{x-x_0}{t-t_0}\right)^2+\left(\frac{y-y_0}{t-t_0}\right)^2}$$
$$= \sqrt{\left[\lim_{(x,y)\to(x_0,y_0)}\left(\frac{x-x_0}{t-t_0}\right)\right]^2+\left[\lim_{(x,y)\to(x_0,y_0)}\left(\frac{y-y_0}{t-t_0}\right)\right]^2}.$$

Since $x(t)$ and $y(t)$ are both differentiable functions of t, both limits inside the last radical exist. Therefore, this value is finite. This proves the chain rule at $t=t_0$; the rest of the theorem follows from the assumption that all functions are differentiable over their entire domains.

□

Closer examination of Equation 14.5.1 reveals an interesting pattern. The first term in the equation is $\frac{\partial f}{\partial x}\cdot\frac{dx}{dt}$ and the second term is $\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$. Recall that when multiplying fractions, cancelation can be used. If we treat these derivatives as fractions, then each product "simplifies" to something resembling $\partial f/dt$. The variables x and y that disappear in this simplification are often called **intermediate variables**: they are independent variables for the function f, but are dependent variables for the variable t. Two terms appear on the right-hand side of the formula, and f is a function of two variables. This pattern works with functions of more than two variables as well, as we see later in this section.

Example 14.5.1: Using the Chain Rule

Calculate dz/dt for each of the following functions:

a. $z=f(x,y)=4x^2+3y^2, \quad x=x(t)=\sin t, \quad y=y(t)=\cos t$
b. $z=f(x,y)=\sqrt{x^2-y^2}, \quad x=x(t)=e^{2t}, \quad y=y(t)=e^{-t}$

Solution

a. To use the chain rule, we need four quantities—$\partial z/\partial x$, $\partial z/\partial y$, dx/dt, and dy/dt:

- $\dfrac{\partial z}{\partial x} = 8x$
- $\dfrac{dx}{dt} = \cos t$
- $\dfrac{\partial z}{\partial y} = 6y$
- $\dfrac{dy}{dt} = -\sin t$

Now, we substitute each of these into Equation 14.5.1:

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} \\ &= (8x)(\cos t) + (6y)(-\sin t) \\ &= 8x\cos t - 6y\sin t.\end{aligned}$$

This answer has three variables in it. To reduce it to one variable, use the fact that $x(t) = \sin t$ and $y(t) = \cos t$. We obtain

$$\begin{aligned}\frac{dz}{dt} &= 8x\cos t - 6y\sin t \\ &= 8(\sin t)\cos t - 6(\cos t)\sin t \\ &= 2\sin t\cos t.\end{aligned}$$

This derivative can also be calculated by first substituting $x(t)$ and $y(t)$ into $f(x,y)$, then differentiating with respect to t:

$$\begin{aligned}z = f(x,y) &= f\big(x(t), y(t)\big) \\ &= 4(x(t))^2 + 3(y(t))^2 \\ &= 4\sin^2 t + 3\cos^2 t.\end{aligned}$$

Then

$$\begin{aligned}\frac{dz}{dt} &= 2(4\sin t)(\cos t) + 2(3\cos t)(-\sin t) \\ &= 8\sin t\cos t - 6\sin t\cos t \\ &= 2\sin t\cos t,\end{aligned}$$

which is the same solution. However, it may not always be this easy to differentiate in this form.

b. To use the chain rule, we again need four quantities—$\partial z/\partial x, \partial z/dy, dx/dt$, and dy/dt :

- $\dfrac{\partial z}{\partial x} = \dfrac{x}{\sqrt{x^2 - y^2}}$
- $\dfrac{dx}{dt} = 2e^{2t}$
- $\dfrac{\partial z}{\partial y} = \dfrac{-y}{\sqrt{x^2 - y^2}}$
- $\dfrac{dy}{dt} = -e^{-t}$.

We substitute each of these into Equation 14.5.1:

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}\\ &= \left(\frac{x}{\sqrt{x^2-y^2}}\right)(2e^{2t})+\left(\frac{-y}{\sqrt{x^2-y^2}}\right)(-e^{-t})\\ &= \frac{2xe^{2t}-ye^{-t}}{\sqrt{x^2-y^2}}.\end{aligned}$$

To reduce this to one variable, we use the fact that $x(t)=e^{2t}$ and $y(t)=e^{-t}$. Therefore,

$$\begin{aligned}\frac{dz}{dt} &= \frac{2xe^2t+ye^{-t}}{\sqrt{x^2-y^2}}\\ &= \frac{2(e^{2t})e^{2t}+(e^{-t})e^{-t}}{\sqrt{e^{4t}-e^{-2t}}}\\ &= \frac{2e^{4t}+e^{-2t}}{\sqrt{e^{4t}-e^{-2t}}}.\end{aligned}$$

To eliminate negative exponents, we multiply the top by e^{2t} and the bottom by $\sqrt{e^{4t}}$:

$$\begin{aligned}\frac{dz}{dt} &= \frac{2e^{4t}+e^{-2t}}{\sqrt{e^{4t}-e^{-2t}}}\cdot\frac{e^{2t}}{\sqrt{e^{4t}}}\\ &= \frac{2e^{6t}+1}{\sqrt{e^{8t}-e^{2t}}}\\ &= \frac{2e^{6t}+1}{\sqrt{e^{2t}(e^{6t}-1)}}\\ &= \frac{2e^{6t}+1}{e^t\sqrt{e^{6t}-1}}.\end{aligned}$$

Again, this derivative can also be calculated by first substituting $x(t)$ and $y(t)$ into $f(x,y)$, then differentiating with respect to t:

$$\begin{aligned}z &= f(x,y)\\ &= f(x(t),y(t))\\ &= \sqrt{(x(t))^2-(y(t))^2}\\ &= \sqrt{e^{4t}-e^{-2t}}\\ &= (e^{4t}-e^{-2t})^{1/2}.\end{aligned}$$

Then

$$\begin{aligned}\frac{dz}{dt} &= \frac{1}{2}(e^{4t}-e^{-2t})^{-1/2}\left(4e^{4t}+2e^{-2t}\right)\\ &= \frac{2e^{4t}+e^{-2t}}{\sqrt{e^{4t}-e^{-2t}}}.\end{aligned}$$

This is the same solution.

Exercise 14.5.1

Calculate dz/dt given the following functions. Express the final answer in terms of t.

$$\begin{aligned} z = f(x,y) &= x^2 - 3xy + 2y^2, \\ x = x(t) &= 3\sin 2t, \\ y = y(t) &= 4\cos 2t \end{aligned}$$

Hint

Calculate $\partial z/\partial x$, $\partial z/dy$, dx/dt,and dy/dt, then use Equation 14.5.1.

Answer

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= (2x-3y)(6\cos 2t) + (-3x+4y)(-8\sin 2t) \\ &= -92\sin 2t\cos 2t - 72(\cos^2 2t - \sin^2 2t) \\ &= -46\sin 4t - 72\cos 4t. \end{aligned}$$

It is often useful to create a visual representation of Equation 14.5.1 for the chain rule. This is called a **tree diagram** for the chain rule for functions of one variable and it provides a way to remember the formula (Figure 14.5.1). This diagram can be expanded for functions of more than one variable, as we shall see very shortly.

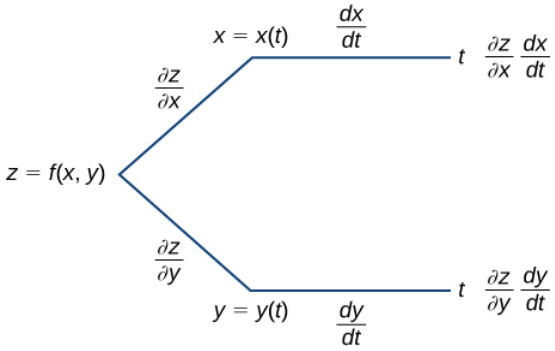


Figure 14.5.1: Tree diagram for the case $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\cdot\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\cdot\dfrac{dy}{dt}$.

In this diagram, the leftmost corner corresponds to $z = f(x,y)$. Since f has two **independent variables**, there are two lines coming from this corner. The upper branch corresponds to the variable x and the lower branch corresponds to the variable y. Since each of these variables is then dependent on one variable t, one branch then comes from x and one branch comes from y. Last, each of the branches on the far right has a label that represents the path traveled to reach that branch. The top branch is reached by following the x branch, then the t branch; therefore, it is labeled $(\partial z/\partial x)\times(dx/dt)$. The bottom branch is similar: first the y branch, then the t branch. This branch is labeled $(\partial z/\partial y)\times(dy/dt)$. To get the formula for dz/dt, add all the terms that appear on the rightmost side of the diagram. This gives us Equation.

In Chain Rule for Two Independent Variables, $z = f(x,y)$ is a function of x and y, and both $x = g(u,v)$ and $y = h(u,v)$ are functions of the independent variables u and v.

Theorem 14.5.2: Chain Rule for Two Independent Variables

Suppose $x = g(u,v)$ and $y = h(u,v)$ are differentiable functions of u and v, and $z = f(x,y)$ is a differentiable function of x and y. Then, $z = f(g(u,v),h(u,v))$ is a differentiable function of u and v, and

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \tag{14.5.2}$$

and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}. \tag{14.5.3}$$

We can draw a tree diagram for each of these formulas as well as follows.

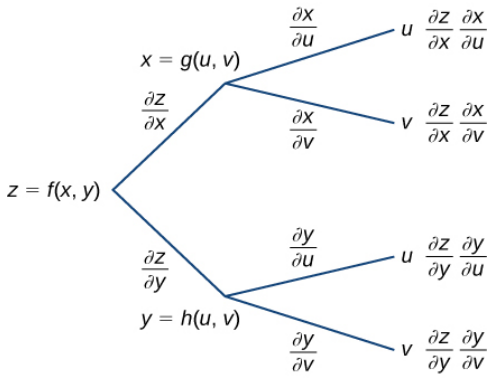


Figure 14.5.2: Tree diagram for $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$ and $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$.

To derive the formula for $\partial z/\partial u$, start from the left side of the diagram, then follow only the branches that end with u and add the terms that appear at the end of those branches. For the formula for $\partial z/\partial v$, follow only the branches that end with v and add the terms that appear at the end of those branches.

There is an important difference between these two chain rule theorems. In Chain Rule for One Independent Variable, the left-hand side of the formula for the derivative is not a partial derivative, but in Chain Rule for Two Independent Variables it is. The reason is that, in Chain Rule for One Independent Variable, z is ultimately a function of t alone, whereas in Chain Rule for Two Independent Variables, z is a function of both u and v.

Example 14.5.2: Using the Chain Rule for Two Variables

Calculate $\partial z/\partial u$ and $\partial z/\partial v$ using the following functions:

$$z = f(x,y) = 3x^2 - 2xy + y^2,\ x = x(u,v) = 3u + 2v,\ y = y(u,v) = 4u - v.$$

Solution

To implement the chain rule for two variables, we need six partial derivatives—$\partial z/\partial x,\ \partial z/\partial y,\ \partial x/\partial u,\ \partial x/\partial v,\ \partial y/\partial u$,and $\partial y/\partial v$:

$$\frac{\partial z}{\partial x} = 6x - 2y \qquad \frac{\partial z}{\partial y} = -2x + 2y$$
$$\frac{\partial x}{\partial u} = 3 \qquad \frac{\partial x}{\partial v} = 2$$
$$\frac{\partial y}{\partial u} = 4 \qquad \frac{\partial y}{\partial v} = -1.$$

To find $\partial z/\partial u$, we use Equation 14.5.2:

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}\\ &= 3(6x - 2y) + 4(-2x + 2y)\\ &= 10x + 2y.\end{aligned}$$

Next, we substitute $x(u,v) = 3u + 2v$ and $y(u,v) = 4u - v$:

$$\begin{aligned}\frac{\partial z}{\partial u} &= 10x + 2y \\ &= 10(3u + 2v) + 2(4u - v) \\ &= 38u + 18v.\end{aligned}$$

To find $\partial z/\partial v$, we use Equation 14.5.3:

$$\begin{aligned}\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\ &= 2(6x - 2y) + (-1)(-2x + 2y) \\ &= 14x - 6y.\end{aligned}$$

Then we substitute $x(u, v) = 3u + 2v$ and $y(u, v) = 4u - v$:

$$\begin{aligned}\frac{\partial z}{\partial v} &= 14x - 6y \\ &= 14(3u + 2v) - 6(4u - v) \\ &= 18u + 34v\end{aligned}$$

? Exercise 14.5.2

Calculate $\partial z/\partial u$ and $\partial z/\partial v$ given the following functions:

$$z = f(x, y) = \frac{2x - y}{x + 3y}, \quad x(u, v) = e^{2u}\cos 3v, \quad y(u, v) = e^{2u}\sin 3v.$$

Hint

Calculate $\partial z/\partial x$, $\partial z/\partial y$, $\partial x/\partial u$, $\partial x/\partial v$, $\partial y/\partial u$, and $\partial y/\partial v$, then use Equation 14.5.2 and Equation 14.5.3

Answer

$$\frac{\partial z}{\partial u} = 0, \quad \frac{\partial z}{\partial v} = \frac{-21}{(3\sin 3v + \cos 3v)^2}$$

The Generalized Chain Rule

Now that we've see how to extend the original chain rule to functions of two variables, it is natural to ask: Can we extend the rule to more than two variables? The answer is yes, as the *generalized chain rule* states.

Theorem 14.5.3: Generalized Chain Rule

Let $w = f(x_1, x_2, \ldots, x_m)$ be a differentiable function of m independent variables, and for each $i \in 1, \ldots, m$, let $x_i = x_i(t_1, t_2, \ldots, t_n)$ be a differentiable function of n independent variables. Then

$$\frac{\partial w}{\partial t_j} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial w}{\partial x_m}\frac{\partial x_m}{\partial t_j}$$

for any $j \in 1, 2, \ldots, n$.

In the next example we calculate the derivative of a function of three independent variables in which each of the three variables is dependent on two other variables.

✓ Example 14.5.3: Using the Generalized Chain Rule

Calculate $\partial w/\partial u$ and $\partial w/\partial v$ using the following functions:

$$\begin{aligned} w &= f(x,y,z) = 3x^2 - 2xy + 4z^2 \\ x &= x(u,v) = e^u \sin v \\ y &= y(u,v) = e^u \cos v \\ z &= z(u,v) = e^u. \end{aligned}$$

Solution

The formulas for $\partial w/\partial u$ and $\partial w/\partial v$ are

$$\begin{aligned} \frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u} \\ \frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v}. \end{aligned}$$

Therefore, there are nine different partial derivatives that need to be calculated and substituted. We need to calculate each of them:

$$\begin{aligned} \frac{\partial w}{\partial x} &= 6x - 2y \quad \frac{\partial w}{\partial y} = -2x \quad \frac{\partial w}{\partial z} = 8z \\ \frac{\partial x}{\partial u} &= e^u \sin v \quad \frac{\partial y}{\partial u} = e^u \cos v \quad \frac{\partial z}{\partial u} = e^u \\ \frac{\partial x}{\partial v} &= e^u \cos v \quad \frac{\partial y}{\partial v} = -e^u \sin v \quad \frac{\partial z}{\partial v} = 0. \end{aligned}$$

Now, we substitute each of them into the first formula to calculate $\partial w/\partial u$:

$$\begin{aligned} \frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial u} \\ &= (6x - 2y)e^u \sin v - 2xe^u \cos v + 8ze^u, \end{aligned}$$

then substitute $x(u,v) = e^u \sin v,\ y(u,v) = e^u \cos v,$ and $z(u,v) = e^u$ into this equation:

$$\begin{aligned} \frac{\partial w}{\partial u} &= (6x - 2y)e^u \sin v - 2xe^u \cos v + 8ze^u \\ &= (6e^u \sin v - 2eu \cos v)e^u \sin v - 2(e^u \sin v)e^u \cos v + 8e^{2u} \\ &= 6e^{2u}\sin^2 v - 4e^{2u}\sin v \cos v + 8e^{2u} \\ &= 2e^{2u}(3\sin^2 v - 2\sin v \cos v + 4). \end{aligned}$$

Next, we calculate $\partial w/\partial v$:

$$\begin{aligned} \frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial v} \\ &= (6x - 2y)e^u \cos v - 2x(-e^u \sin v) + 8z(0), \end{aligned}$$

then we substitute $x(u,v) = e^u \sin v,\ y(u,v) = e^u \cos v,$ and $z(u,v) = e^u$ into this equation:

$$\begin{aligned} \frac{\partial w}{\partial v} &= (6x - 2y)e^u \cos v - 2x(-e^u \sin v) \\ &= (6e^u \sin v - 2e^u \cos v)e^u \cos v + 2(e^u \sin v)(e^u \sin v) \\ &= 2e^{2u}\sin^2 v + 6e^{2u}\sin v \cos v - 2e^{2u}\cos^2 v \\ &= 2e^{2u}(\sin^2 v + \sin v \cos v - \cos^2 v). \end{aligned}$$

Exercise 14.5.3

Calculate $\partial w/\partial u$ and $\partial w/\partial v$ given the following functions:

$$w = f(x,y,z) = \frac{x+2y-4z}{2x-y+3z}$$
$$x = x(u,v) = e^{2u}\cos 3v$$
$$y = y(u,v) = e^{2u}\sin 3v$$
$$z = z(u,v) = e^{2u}.$$

Hint

Calculate nine partial derivatives, then use the same formulas from Example 14.5.3

Answer

$$\frac{\partial w}{\partial u} = 0$$
$$\frac{\partial w}{\partial v} = \frac{15-33\sin 3v+6\cos 3v}{(3+2\cos 3v-\sin 3v)^2}$$

Example 14.5.4: Drawing a Tree Diagram

Create a tree diagram for the case when

$$w = f(x,y,z), \quad x = x(t,u,v), \quad y = y(t,u,v), \quad z = z(t,u,v)$$

and write out the formulas for the three partial derivatives of w.

Solution

Starting from the left, the function f has three independent variables: x, y, and z. Therefore, three branches must be emanating from the first node. Each of these three branches also has three branches, for each of the variables t, u, and v.

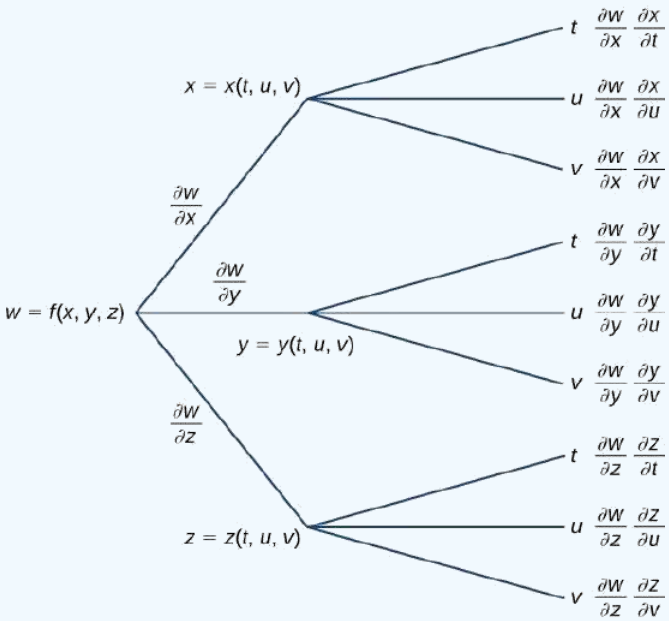


Figure 14.5.3: Tree diagram for a function of three variables, each of which is a function of three independent variables.

The three formulas are

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$
$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$
$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}.$$

? Exercise 14.5.4

Create a tree diagram for the case when

$$w = f(x, y), \quad x = x(t, u, v), \quad y = y(t, u, v)$$

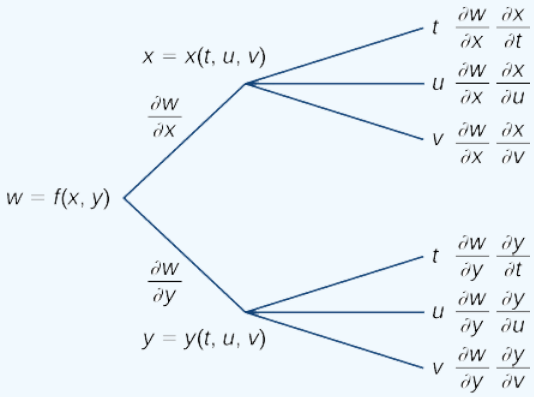
and write out the formulas for the three partial derivatives of w.

Hint

Determine the number of branches that emanate from each node in the tree.

Answer

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$$
$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u}$$
$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v}$$



Implicit Differentiation

Recall from implicit differentiation provides a method for finding dy/dx when y is defined implicitly as a function of x. The method involves differentiating both sides of the equation defining the function with respect to x, then solving for dy/dx. Partial derivatives provide an alternative to this method.

Consider the ellipse defined by the equation $x^2 + 3y^2 + 4y - 4 = 0$ as follows.

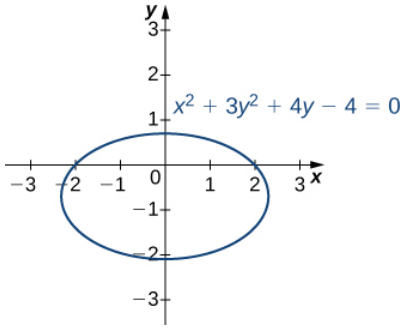


Figure 14.5.4: Graph of the ellipse defined by $x^2+3y^2+4y-4=0$.

This equation implicitly defines y as a function of x. As such, we can find the derivative dy/dx using the method of implicit differentiation:

$$\begin{aligned}\frac{d}{dx}(x^2+3y^2+4y-4) &= \frac{d}{dx}(0)\\ 2x+6y\frac{dy}{dx}+4\frac{dy}{dx} &= 0\\ (6y+4)\frac{dy}{dx} &= -2x\\ \frac{dy}{dx} &= -\frac{x}{3y+2}\end{aligned}$$

We can also define a function $z=f(x,y)$ by using the left-hand side of the equation defining the ellipse. Then $f(x,y)=x^2+3y^2+4y-4$. The ellipse $x^2+3y^2+4y-4=0$ can then be described by the equation $f(x,y)=0$. Using this function and the following theorem gives us an alternative approach to calculating dy/dx.

Theorem 14.5.4: Implicit Differentiation of a Function of Two or More Variables

Suppose the function $z=f(x,y)$ defines y implicitly as a function $y=g(x)$ of x via the equation $f(x,y)=0$. Then

$$\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y} \tag{14.5.4}$$

provided $f_y(x,y)\neq 0$.

If the equation $f(x,y,z)=0$ defines z implicitly as a differentiable function of x and y, then

$$\frac{dz}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial z} \quad \text{and} \quad \frac{dz}{dy}=-\frac{\partial f/\partial y}{\partial f/\partial z} \tag{14.5.5}$$

as long as $f_z(x,y,z)\neq 0$.

Equation 14.5.4 is a direct consequence of Equation 14.5.2. In particular, if we assume that y is defined implicitly as a function of x via the equation $f(x,y)=0$, we can apply the chain rule to find dy/dx :

$$\begin{aligned}\frac{d}{dx}f(x,y) &= \frac{d}{dx}(0)\\ \frac{\partial f}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx} &= 0\\ \frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dx} &= 0.\end{aligned}$$

Solving this equation for dy/dx gives Equation 14.5.4. Equation 14.5.4 can be derived in a similar fashion.

Let's now return to the problem that we started before the previous theorem. Using Theorem 14.5.4 and the function $f(x,y)=x^2+3y^2+4y-4$, we obtain

$$\frac{\partial f}{\partial x} = 2x$$
$$\frac{\partial f}{\partial y} = 6y + 4.$$

Then Equation 14.5.4 gives

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{2x}{6y+4} = -\frac{x}{3y+2},$$

which is the same result obtained by the earlier use of implicit differentiation.

Example 14.5.5: Implicit Differentiation by Partial Derivatives

a. Calculate dy/dx if y is defined implicitly as a function of x via the equation $3x^2 - 2xy + y^2 + 4x - 6y - 11 = 0$. What is the equation of the tangent line to the graph of this curve at point $(2, 1)$?
b. Calculate $\partial z/\partial x$ and $\partial z/\partial y$, given $x^2 e^y - yze^x = 0$.

Solution

a. Set $f(x, y) = 3x^2 - 2xy + y^2 + 4x - 6y - 11 = 0$, then calculate f_x and $f_y : f_x(x, y) = 6x - 2y + 4$ and $f_y(x, y) = -2x + 2y - 6$.

The derivative is given by

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = \frac{6x - 2y + 4}{-2x + 2y - 6} = \frac{3x - y + 2}{x - y + 3}.$$

The slope of the tangent line at point $(2, 1)$ is given by

$$\left.\frac{dy}{dx}\right|_{(x,y)=(2,1)} = \frac{3(2) - 1 + 2}{2 - 1 + 3} = \frac{7}{4}$$

To find the equation of the tangent line, we use the point-slope form (Figure 14.5.5):

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 1 &= \frac{7}{4}(x - 2) \\ y &= \frac{7}{4}x - \frac{7}{2} + 1 \\ y &= \frac{7}{4}x - \frac{5}{2}. \end{aligned}$$

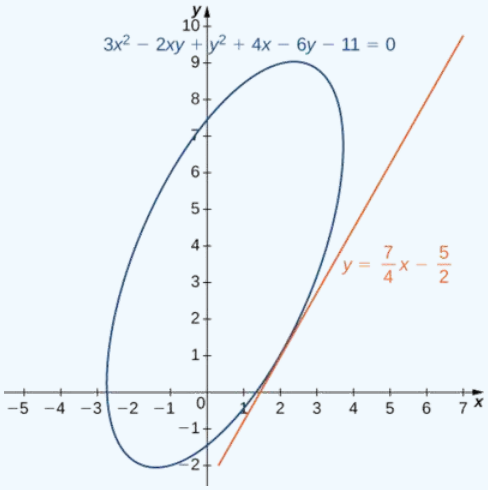


Figure 14.5.5: Graph of the rotated ellipse defined by $3x^2 - 2xy + y^2 + 4x - 6y - 11 = 0$.

b. We have $f(x,y,z)=x^2e^y-yze^x$. Therefore,

$$\frac{\partial f}{\partial x}=2xe^y-yze^x$$
$$\frac{\partial f}{\partial y}=x^2e^y-ze^x$$
$$\frac{\partial f}{\partial z}=-ye^x$$

Using Equation 14.5.5,

$$\frac{\partial z}{\partial x}=-\frac{\partial f/\partial x}{\partial f/\partial y} \qquad \text{and} \qquad \frac{\partial z}{\partial y}=-\frac{\partial f/\partial y}{\partial f/\partial z}$$
$$=-\frac{2xe^y-yze^x}{-ye^x} \qquad =-\frac{x^2e^y-ze^x}{-ye^x}$$
$$=\frac{2xe^y-yze^x}{ye^x} \qquad =\frac{x^2e^y-ze^x}{ye^x}$$

? Exercise 14.5.5

Find dy/dx if y is defined implicitly as a function of x by the equation $x^2+xy-y^2+7x-3y-26=0$. What is the equation of the tangent line to the graph of this curve at point $(3,-2)$?

Hint

Calculate $\partial f/dx$ and $\partial f/dy$, then use Equation 14.5.4

Solution

$$\frac{dy}{dx}=\frac{2x+y+7}{2y-x+3}\Big|_{(3,-2)}=\frac{2(3)+(-2)+7}{2(-2)-(3)+3}=-\frac{11}{4}$$

Equation of the tangent line: $y=-\frac{11}{4}x+\frac{25}{4}$

Key Concepts

- The chain rule for functions of more than one variable involves the partial derivatives with respect to all the independent variables.
- Tree diagrams are useful for deriving formulas for the chain rule for functions of more than one variable, where each independent variable also depends on other variables.

Key Equations

- **Chain rule, one independent variable**

  $$\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

- **Chain rule, two independent variables**

  $$\frac{dz}{du}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}\quad\frac{dz}{dv}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

- **Generalized chain rule**

  $$\frac{\partial w}{\partial t_j}=\frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_j}+\frac{\partial w}{\partial x_2}\frac{\partial x_1}{\partial t_j}+\cdots+\frac{\partial w}{\partial x_m}\frac{\partial x_m}{\partial t_j}$$

Glossary

generalized chain rule

the chain rule extended to functions of more than one independent variable, in which each independent variable may depend on one or more other variables

intermediate variable

given a composition of functions (e.g., $f(x(t), y(t))$), the intermediate variables are the variables that are independent in the outer function but dependent on other variables as well; in the function $f(x(t), y(t))$, the variables x and y are examples of intermediate variables

tree diagram

illustrates and derives formulas for the generalized chain rule, in which each independent variable is accounted for

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14.6: Directional Derivatives and the Gradient Vector

Learning Objectives

- Determine the directional derivative in a given direction for a function of two variables.
- Determine the gradient vector of a given real-valued function.
- Explain the significance of the gradient vector with regard to direction of change along a surface.
- Use the gradient to find the tangent to a level curve of a given function.
- Calculate directional derivatives and gradients in three dimensions.

A function $z = f(x, y)$ has two partial derivatives: $\partial z/\partial x$ and $\partial z/\partial y$. These derivatives correspond to each of the independent variables and can be interpreted as instantaneous rates of change (that is, as slopes of a tangent line). For example, $\partial z/\partial x$ represents the slope of a tangent line passing through a given point on the surface defined by $z = f(x, y)$, assuming the tangent line is parallel to the x**-axis**. Similarly, $\partial z/\partial y$ represents the slope of the tangent line parallel to the y**-axis.** Now we consider the possibility of a tangent line parallel to neither axis.

Directional Derivatives

We start with the graph of a surface defined by the equation $z = f(x, y)$. Given a point (a, b) in the domain of f, we choose a direction to travel from that point. We measure the direction using an angle θ, which is measured counterclockwise in the xy-plane, starting at zero from the positive x-axis (Figure 14.6.1). The distance we travel is h and the direction we travel is given by the unit vector $\vec{\mathbf{u}} = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$. Therefore, the z-coordinate of the second point on the graph is given by $z = f(a + h\cos\theta, b + h\sin\theta)$.

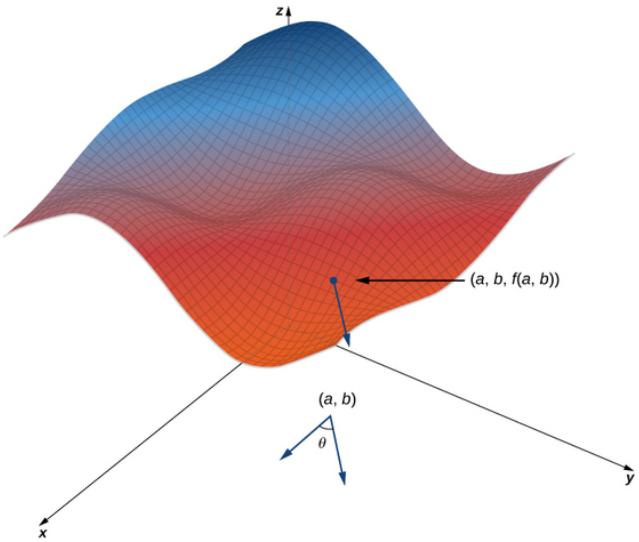


Figure 14.6.1: Finding the directional derivative at a point on the graph of $z = f(x, y)$. The slope of the blue arrow on the graph indicates the value of the directional derivative at that point.

We can calculate the slope of the secant line by dividing the difference in z-values by the length of the line segment connecting the two points in the domain. The length of the line segment is h. Therefore, the slope of the secant line is

$$m_{sec} = \frac{f(a + h\cos\theta, b + h\sin\theta) - f(a, b)}{h}$$

To find the slope of the tangent line in the same direction, we take the limit as h approaches zero.

Definition: Directional Derivatives

Suppose $z = f(x, y)$ is a function of two variables with a domain of D. Let $(a, b) \in D$ and define $\vec{\mathbf{u}} = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$. Then the directional derivative of f in the direction of $\vec{\mathbf{u}}$ is given by

$$D_{\vec{\mathbf{u}}} f(a, b) = \lim_{h \to 0} \frac{f(a + h\cos\theta, b + h\sin\theta) - f(a, b)}{h} \tag{14.6.1}$$

provided the limit exists.

Equation 14.6.1 provides a formal definition of the directional derivative that can be used in many cases to calculate a directional derivative.

Note that since the point (a,b) is chosen randomly from the domain D of the function f, we can use this definition to find the directional derivative as a function of x and y.

That is,

$$D_{\vec{\mathbf{u}}} f(x,y) = \lim_{h\to 0} \frac{f(x+h\cos\theta, y+h\sin\theta) - f(x,y)}{h} \tag{14.6.2}$$

Example 14.6.1: Finding a Directional Derivative from the Definition

Let $\theta = \arccos(3/5)$. Find the directional derivative $D_{\vec{\mathbf{u}}} f(x,y)$ of $f(x,y) = x^2 - xy + 3y^2$ in the direction of $\vec{\mathbf{u}} = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$.

Then determine $D_{\vec{\mathbf{u}}} f(-1,2)$.

Solution

First of all, since $\cos\theta = 3/5$ and θ is acute, this implies

$$\sin\theta = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25}} = \frac{4}{5}.$$

Using $f(x,y) = x^2 - xy + 3y^2$, we first calculate $f(x+h\cos\theta, y+h\sin\theta)$:

$$\begin{aligned} f(x+h\cos\theta, y+h\sin\theta) &= (x+h\cos\theta)^2 - (x+h\cos\theta)(y+h\sin\theta) + 3(y+h\sin\theta)^2 \\ &= x^2 + 2xh\cos\theta + h^2\cos^2\theta - xy - xh\sin\theta - yh\cos\theta - h^2\sin\theta\cos\theta + 3y^2 + 6yh\sin\theta + 3h^2\sin^2\theta \\ &= x^2 + 2xh(\frac{3}{5}) + \frac{9h^2}{25} - xy - \frac{4xh}{5} - \frac{3yh}{5} - \frac{12h^2}{25} + 3y^2 + 6yh(\frac{4}{5}) + 3h^2(\frac{16}{25}) \\ &= x^2 - xy + 3y^2 + \frac{2xh}{5} + \frac{9h^2}{5} + \frac{21yh}{5}. \end{aligned}$$

We substitute this expression into Equation 14.6.1 with $a = x$ and $b = y$:

$$\begin{aligned} D_{\vec{\mathbf{u}}} f(x,y) &= \lim_{h\to 0} \frac{f(x+h\cos\theta, y+h\sin\theta) - f(x,y)}{h} \\ &= \lim_{h\to 0} \frac{(x^2 - xy + 3y^2 + \frac{2xh}{5} + \frac{9h^2}{5} + \frac{21yh}{5}) - (x^2 - xy + 3y^2)}{h} \\ &= \lim_{h\to 0} \frac{\frac{2xh}{5} + \frac{9h^2}{5} + \frac{21yh}{5}}{h} \\ &= \lim_{h\to 0} \frac{2x}{5} + \frac{9h}{5} + \frac{21y}{5} \\ &= \frac{2x+21y}{5}. \end{aligned}$$

To calculate $D_{\vec{\mathbf{u}}} f(-1,2)$, we substitute $x = -1$ and $y = 2$ into this answer (Figure 14.6.2):

$$D_{\vec{\mathbf{u}}} f(-1,2) = \frac{2(-1)+21(2)}{5} = \frac{-2+42}{5} = 8.$$

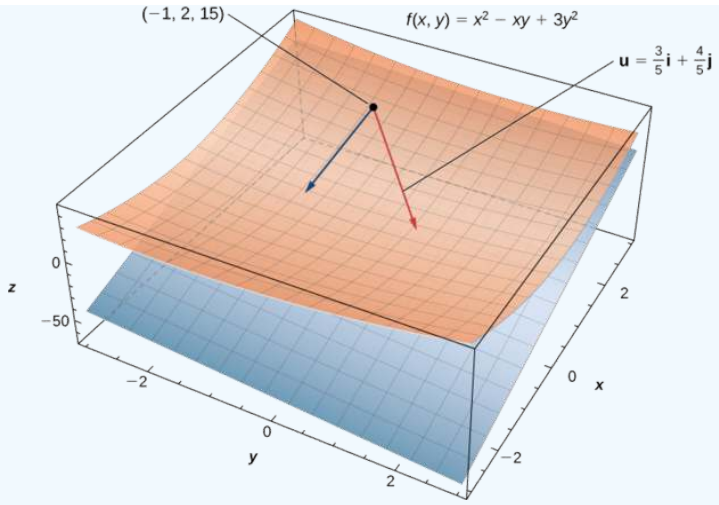


Figure 14.6.2: Finding the directional derivative in a given direction $\vec{\mathbf{u}}$ at a given point on a surface. The plane is tangent to the surface at the given point $(-1, 2, 15)$.

An easier approach to calculating directional derivatives that involves partial derivatives is outlined in the following theorem.

Theorem 14.6.1: Directional Derivative of a Function of Two Variables

Let $z = f(x, y)$ be a function of two variables x and y, and assume that f_x and f_y exist. Then the directional derivative of f in the direction of $\vec{\mathbf{u}} = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$ is given by

$$D_{\vec{\mathbf{u}}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta. \tag{14.6.3}$$

Proof

Applying the definition of a directional derivative stated above in Equation 14.6.1, the directional derivative of f in the direction of $\vec{\mathbf{u}} = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$ at a point (x_0, y_0) in the domain of f can be written

$$D_{\vec{\mathbf{u}}} f((x_0, y_0)) = \lim_{t\to 0} \frac{f(x_0 + t\cos\theta, y_0 + t\sin\theta) - f(x_0, y_0)}{t}.$$

Let $x = x_0 + t\cos\theta$ and $y = y_0 + t\sin\theta$, and define $g(t) = f(x, y)$. Since f_x and f_y both exist, we can use the chain rule for functions of two variables to calculate $g'(t)$:

$$g'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta.$$

If $t = 0$, then $x = x_0$ and $y = y_0$, so

$$g'(0) = f_x(x_0, y_0)\cos\theta + f_y(x_0, y_0)\sin\theta$$

By the definition of $g'(t)$, it is also true that

$$g'(0) = \lim_{t\to 0} \frac{g(t) - g(0)}{t} = \lim_{t\to 0} \frac{f(x_0 + t\cos\theta, y_0 + t\sin\theta) - f(x_0, y_0)}{t}.$$

Therefore, $D_{\vec{\mathbf{u}}} f(x_0, y_0) = f_x(x_0, y_0)\cos\theta + f_y(x_0, y_0)\sin\theta$.

Since the point (x_0, y_0) is an arbitrary point from the domain of f, this result holds for all points in the domain of f for which the partials f_x and f_y exist.

Therefore,

$$D_{\vec{\mathbf{u}}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta.$$

□

Example 14.6.2: Finding a Directional Derivative: Alternative Method

Let $\theta = \arccos(3/5)$. Find the directional derivative $D_{\vec{\mathbf{u}}} f(x,y)$ of $f(x,y) = x^2 - xy + 3y^2$ in the direction of $\vec{\mathbf{u}} = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$.

Then determine $D_{\vec{\mathbf{u}}} f(-1,2)$.

Solution

First, we must calculate the partial derivatives of f:

$$\begin{aligned} f_x(x,y) &= 2x - y \\ f_y(x,y) &= -x + 6y, \end{aligned}$$

Then we use Equation 14.6.3 with $\theta = \arccos(3/5)$:

$$\begin{aligned} D_{\vec{\mathbf{u}}} f(x,y) &= f_x(x,y)\cos\theta + f_y(x,y)\sin\theta \\ &= (2x-y)\frac{3}{5} + (-x+6y)\frac{4}{5} \\ &= \frac{6x}{5} - \frac{3y}{5} - \frac{4x}{5} + \frac{24y}{5} \\ &= \frac{2x+21y}{5}. \end{aligned}$$

To calculate $D_{\vec{\mathbf{u}}} f(-1,2)$, let $x = -1$ and $y = 2$:

$$D_{\vec{\mathbf{u}}} f(-1,2) = \frac{2(-1)+21(2)}{5} = \frac{-2+42}{5} = 8.$$

This is the same answer obtained in Example 14.6.1.

Exercise 14.6.1:

Find the directional derivative $D_{\vec{\mathbf{u}}} f(x,y)$ of $f(x,y) = 3x^2y - 4xy^3 + 3y^2 - 4x$ in the direction of $\vec{\mathbf{u}} = (\cos\frac{\pi}{3})\,\hat{\mathbf{i}} + (\sin\frac{\pi}{3})\,\hat{\mathbf{j}}$ using Equation 14.6.3.

What is $D_{\vec{\mathbf{u}}} f(3,4)$?

Hint

Calculate the partial derivatives and determine the value of θ.

Answer

$$D_{\vec{\mathbf{u}}} f(x,y) = \frac{(6xy - 4y^3 - 4)(1)}{2} + \frac{(3x^2 - 12xy^2 + 6y)\sqrt{3}}{2}$$

$$D_{\vec{\mathbf{u}}} f(3,4) = \frac{72 - 256 - 4}{2} + \frac{(27 - 576 + 24)\sqrt{3}}{2} = -94 - \frac{525\sqrt{3}}{2}$$

If the vector that is given for the direction of the derivative is not a unit vector, then it is only necessary to divide by the norm of the vector. For example, if we wished to find the directional derivative of the function in Example 14.6.2 in the direction of the vector $\langle -5, 12\rangle$, we would first divide by its magnitude to get $\vec{\mathbf{u}}$. This gives us $\vec{\mathbf{u}} = \langle -\frac{5}{13}, \frac{12}{13}\rangle$.

Then

$$\begin{aligned} D_{\vec{\mathbf{u}}} f(x,y) &= f_x(x,y)\cos\theta + f_y(x,y)\sin\theta \\ &= -\frac{5}{13}(2x-y) + \frac{12}{13}(-x+6y) \\ &= -\frac{22}{13}x + \frac{17}{13}y \end{aligned}$$

Gradient

The right-hand side of Equation 14.6.3 is equal to $f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$, which can be written as the dot product of two vectors. Define the first vector as $\vec{\boldsymbol{\nabla}} f(x,y) = f_x(x,y)\,\hat{\mathbf{i}} + f_y(x,y)\,\hat{\mathbf{j}}$ and the second vector as $\vec{\mathbf{u}} = (\cos\theta)\,\hat{\mathbf{i}} + (\sin\theta)\,\hat{\mathbf{j}}$. Then the right-hand

side of the equation can be written as the dot product of these two vectors:

$$D_{\vec{\mathbf{u}}} f(x,y) = \vec{\nabla} f(x,y) \cdot \vec{\mathbf{u}}. \tag{14.6.4}$$

The first vector in Equation 14.6.4 has a special name: the gradient of the function f. The symbol ∇ is called **nabla** and the vector $\vec{\nabla} f$ is read "**del** f."

Definition: The Gradient

Let $z = f(x,y)$ be a function of x and y such that f_x and f_y exist. The vector $\vec{\nabla} f(x,y)$ is called the **gradient** of f and is defined as

$$\vec{\nabla} f(x,y) = f_x(x,y)\,\hat{\mathbf{i}} + f_y(x,y)\,\hat{\mathbf{j}}. \tag{14.6.5}$$

The vector $\vec{\nabla} f(x,y)$ is also written as "**grad** f."

Example 14.6.3: Finding Gradients

Find the gradient $\vec{\nabla} f(x,y)$ of each of the following functions:

a. $f(x,y) = x^2 - xy + 3y^2$
b. $f(x,y) = \sin 3x \cos 3y$

Solution

For both parts a. and b., we first calculate the partial derivatives f_x and f_y, then use Equation 14.6.5.

a. $f_x(x,y) = 2x - y$ and $f_y(x,y) = -x + 6y$, so

$$\begin{aligned} \vec{\nabla} f(x,y) &= f_x(x,y)\,\hat{\mathbf{i}} + f_y(x,y)\,\hat{\mathbf{j}} \\ &= (2x - y)\,\hat{\mathbf{i}} + (-x + 6y)\,\hat{\mathbf{j}}. \end{aligned}$$

b. $f_x(x,y) = 3\cos 3x \cos 3y$ and $f_y(x,y) = -3\sin 3x \sin 3y$, so

$$\begin{aligned} \vec{\nabla} f(x,y) &= f_x(x,y)\,\hat{\mathbf{i}} + f_y(x,y)\,\hat{\mathbf{j}} \\ &= (3\cos 3x \cos 3y)\,\hat{\mathbf{i}} - (3\sin 3x \sin 3y)\,\hat{\mathbf{j}}. \end{aligned}$$

Exercise 14.6.2

Find the gradient $\vec{\nabla} f(x,y)$ of $f(x,y) = \dfrac{x^2 - 3y^2}{2x + y}$.

Hint

Calculate the partial derivatives, then use Equation 14.6.5.

Answer

$$\vec{\nabla} f(x,y) = \frac{2x^2 + 2xy + 6y^2}{(2x+y)^2}\,\hat{\mathbf{i}} - \frac{x^2 + 12xy + 3y^2}{(2x+y)^2}\,\hat{\mathbf{j}}$$

The gradient has some important properties. We have already seen one formula that uses the gradient: the formula for the directional derivative. Recall from The Dot Product that if the angle between two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is φ, then $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \|\vec{\mathbf{a}}\|\|\vec{\mathbf{b}}\| \cos\varphi$. Therefore, if the angle between $\vec{\nabla} f(x_0, y_0)$ and $\vec{\mathbf{u}} = (cos\theta)\,\hat{\mathbf{i}} + (sin\theta)\,\hat{\mathbf{j}}$ is φ, we have

$$D_{\vec{\mathbf{u}}} f(x_0,y_0) = \vec{\nabla} f(x_0,y_0) \cdot \vec{\mathbf{u}} = \|\vec{\nabla} f(x_0,y_0)\|\|\vec{\mathbf{u}}\| \cos\varphi = \|\vec{\nabla} f(x_0,y_0)\| \cos\varphi.$$

The $\|\vec{\mathbf{u}}\|$ disappears because $\vec{\mathbf{u}}$ is a unit vector. Therefore, the directional derivative is equal to the magnitude of the gradient evaluated at (x_0, y_0) multiplied by $\cos\varphi$. Recall that $\cos\varphi$ ranges from -1 to 1.

If $\varphi = 0$, then $\cos\varphi = 1$ and $\vec{\nabla} f(x_0, y_0)$ and $\vec{\mathbf{u}}$ both point in the same direction.

If $\varphi = \pi$, then $\cos\varphi = -1$ and $\vec{\nabla} f(x_0, y_0)$ and $\vec{\mathbf{u}}$ point in opposite directions.

In the first case, the value of $D_{\vec{\mathbf{u}}} f(x_0, y_0)$ is maximized; in the second case, the value of $D_{\vec{\mathbf{u}}} f(x_0, y_0)$ is minimized.

We can also see that if $\vec{\nabla} f(x_0, y_0) = \vec{0}$, then

$$D_{\vec{u}} f(x_0, y_0) = \vec{\nabla} f(x_0, y_0) \cdot \vec{u} = 0$$

for any vector $\vec{u}$. These three cases are outlined in the following theorem.

Theorem 14.6.2: Properties of the Gradient

Suppose the function $z = f(x, y)$ is differentiable at (x_0, y_0) (Figure 14.6.3).

i. If $\vec{\nabla} f(x_0, y_0) = \vec{0}$, then $D_{\vec{u}} f(x_0, y_0) = 0$ for any unit vector $\vec{u}$.
ii. If $\vec{\nabla} f(x_0, y_0) \neq \vec{0}$, then $D_{\vec{u}} f(x_0, y_0)$ is **maximized** when $\vec{u}$ points in the same direction as $\vec{\nabla} f(x_0, y_0)$. The maximum value of $D_{\vec{u}} f(x_0, y_0)$ is $\|\vec{\nabla} f(x_0, y_0)\|$.
iii. If $\vec{\nabla} f(x_0, y_0) \neq \vec{0}$, then $D_{\vec{u}} f(x_0, y_0)$ is **minimized** when $\vec{u}$ points in the opposite direction from $\vec{\nabla} f(x_0, y_0)$. The minimum value of $D_{\vec{u}} f(x_0, y_0)$ is $-\|\vec{\nabla} f(x_0, y_0)\|$.

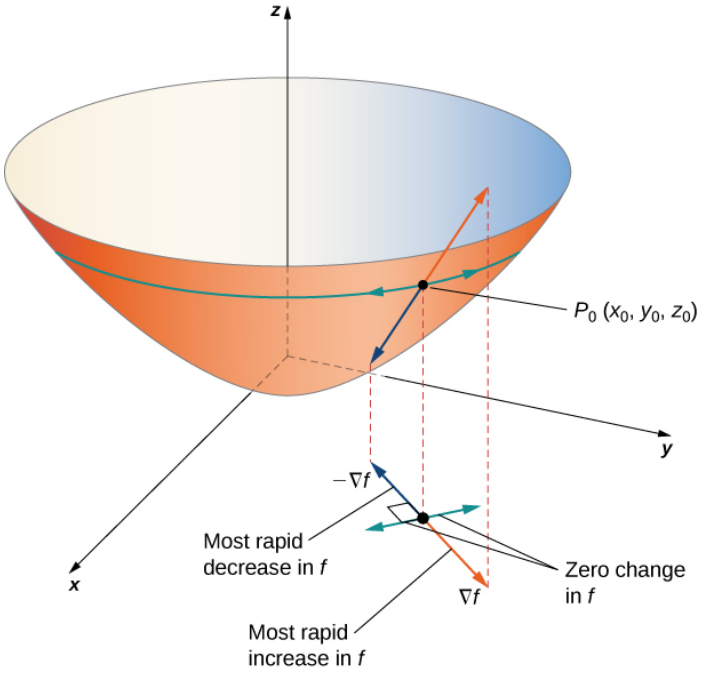


Figure 14.6.3: The gradient indicates the maximum and minimum values of the directional derivative at a point.

Example 14.6.4: Finding a Maximum Directional Derivative

Find the direction for which the directional derivative of $f(x, y) = 3x^2 - 4xy + 2y^2$ at $(-2, 3)$ is a maximum. What is the maximum value?

Solution

The maximum value of the directional derivative occurs when $\vec{\nabla} f$ and the unit vector point in the same direction. Therefore, we start by calculating $\vec{\nabla} f(x, y)$:

$$f_x(x, y) = 6x - 4y \text{ and } f_y(x, y) = -4x + 4y$$

so

$$\vec{\nabla} f(x, y) = f_x(x, y)\,\hat{\mathbf{i}} + f_y(x, y)\,\hat{\mathbf{j}} = (6x - 4y)\,\hat{\mathbf{i}} + (-4x + 4y)\,\hat{\mathbf{j}}.$$

Next, we evaluate the gradient at $(-2, 3)$:

$$\vec{\nabla} f(-2, 3) = (6(-2) - 4(3))\,\hat{\mathbf{i}} + (-4(-2) + 4(3))\,\hat{\mathbf{j}} = -24\,\hat{\mathbf{i}} + 20\,\hat{\mathbf{j}}.$$

We need to find a unit vector that points in the same direction as $\vec{\nabla} f(-2,3)$, so the next step is to divide $\vec{\nabla} f(-2,3)$ by its magnitude, which is $\sqrt{(-24)^2+(20)^2} = \sqrt{976} = 4\sqrt{61}$. Therefore,

$$\frac{\vec{\nabla} f(-2,3)}{\|\vec{\nabla} f(-2,3)\|} = \frac{-24}{4\sqrt{61}} i + \frac{20}{4\sqrt{61}} j = -\frac{6\sqrt{61}}{61}\,\hat{\mathbf{i}} + \frac{5\sqrt{61}}{61}\,\hat{\mathbf{j}}.$$

This is the unit vector that points in the same direction as $\vec{\nabla} f(-2,3)$. To find the angle corresponding to this unit vector, we solve the equations

$$\cos\theta = \frac{-6\sqrt{61}}{61} \text{ and } \sin\theta = \frac{5\sqrt{61}}{61}$$

for θ. Since cosine is negative and sine is positive, the angle must be in the second quadrant. Therefore, $\theta = \pi - \arcsin((5\sqrt{61})/61) \approx 2.45$ rad.

The maximum value of the directional derivative at $(-2,3)$ is $\|\vec{\nabla} f(-2,3)\| = 4\sqrt{61}$ (Figure 14.6.4).

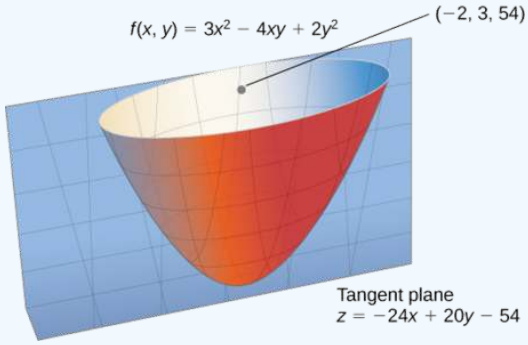


Figure 14.6.4: The maximum value of the directional derivative at $(-2,3)$ is in the direction of the gradient.

? Exercise 14.6.3

Find the direction for which the directional derivative of $g(x,y) = 4x - xy + 2y^2$ at $(-2,3)$ is a maximum. What is the maximum value?

Hint

Evaluate the gradient of g at point $(-2,3)$.

Answer

The gradient of g at $(-2,3)$ is $\vec{\nabla} g(-2,3) = \hat{\mathbf{i}} + 14\,\hat{\mathbf{j}}$. The unit vector that points in the same direction as $\vec{\nabla} g(-2,3)$ is

$$\frac{\vec{\nabla} g(-2,3)}{\|\vec{\nabla} g(-2,3)\|} = \frac{1}{\sqrt{197}}\,\hat{\mathbf{i}} + \frac{14}{\sqrt{197}}\,\hat{\mathbf{j}} = \frac{\sqrt{197}}{197}\,\hat{\mathbf{i}} + \frac{14\sqrt{197}}{197}\,\hat{\mathbf{j}},$$

which gives an angle of $\theta = \arcsin((14\sqrt{197})/197) \approx 1.499$ rad.

The maximum value of the directional derivative is $\|\vec{\nabla} g(-2,3)\| = \sqrt{197}$.

Figure 14.6.5 shows a portion of the graph of the function $f(x,y) = 3 + \sin x \sin y$. Given a point (a,b) in the domain of f, the maximum value of the directional derivative at that point is given by $\|\vec{\nabla} f(a,b)\|$. This would equal the rate of greatest ascent if the surface represented a topographical map. If we went in the opposite direction, it would be the rate of greatest descent.

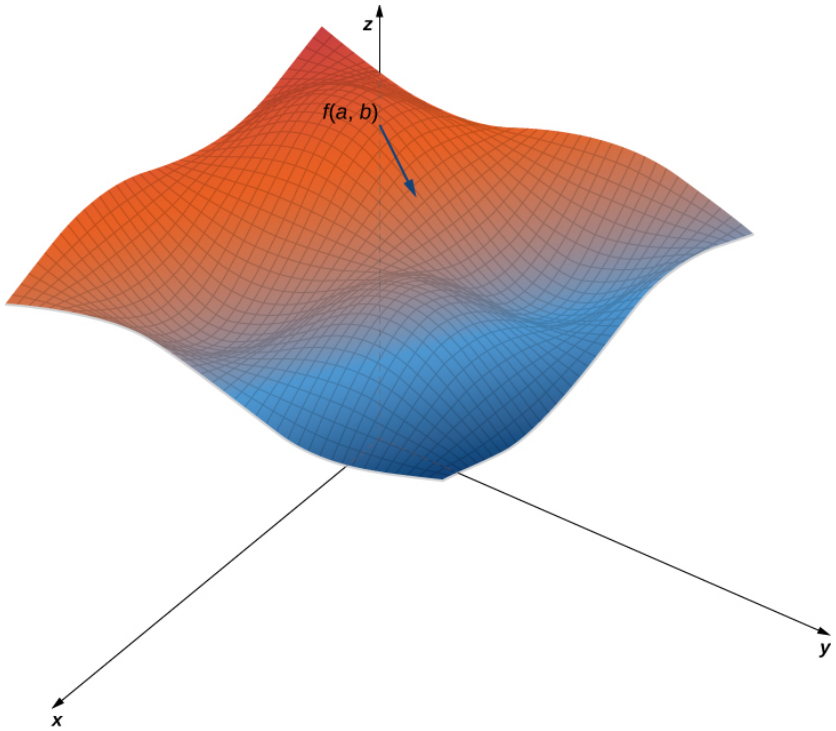


Figure 14.6.5: A typical surface in $\mathbb{R}^3$. Given a point on the surface, the directional derivative can be calculated using the gradient.

When using a topographical map, the steepest slope is always in the direction where the contour lines are closest together (Figure 14.6.6). This is analogous to the contour map of a function, assuming the level curves are obtained for equally spaced values throughout the range of that function.

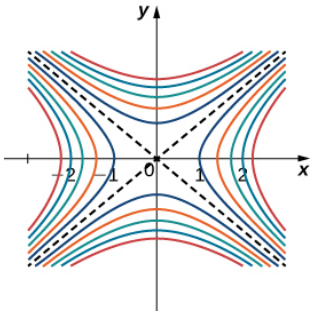


Figure 14.6.6: Contour map for the function $f(x,y) = x^2 - y^2$ using level values between -5 and 5.

Gradients and Level Curves

Recall that if a curve is defined parametrically by the function pair $(x(t), y(t))$, then the vector $x'(t)\,\hat{\mathbf{i}} + y'(t)\,\hat{\mathbf{j}}$ is tangent to the curve for every value of t in the domain. Now let's assume $z = f(x,y)$ is a differentiable function of x and y, and (x_0, y_0) is in its domain. Let's suppose further that $x_0 = x(t_0)$ and $y_0 = y(t_0)$ for some value of t, and consider the level curve $f(x,y) = k$. Define $g(t) = f(x(t), y(t))$ and calculate $g'(t)$ on the level curve. By the chain Rule,

$$g'(t) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t).$$

But $g'(t) = 0$ because $g(t) = k$ for all t. Therefore, on the one hand,

$$f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t) = 0;$$

on the other hand,

$$f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t) = \vec{\nabla} f(x,y) \cdot \langle x'(t), y'(t) \rangle.$$

Therefore,

$$\vec{\nabla} f(x,y) \cdot \langle x'(t), y'(t) \rangle = 0.$$

Thus, the dot product of these vectors is equal to zero, which implies they are orthogonal. However, the second vector is tangent to the level curve, which implies the gradient must be normal to the level curve, which gives rise to the following theorem.

Theorem 14.6.3: Gradient Is Normal to the Level Curve

Suppose the function $z = f(x, y)$ has continuous first-order partial derivatives in an open disk centered at a point (x_0, y_0). If $\vec{\nabla} f(x_0, y_0) \neq 0$, then $\vec{\nabla} f(x_0, y_0)$ is normal to the level curve of f at (x_0, y_0).

We can use this theorem to find tangent and normal vectors to level curves of a function.

Example 14.6.5: Finding Tangents to Level Curves

For the function $f(x, y) = 2x^2 - 3xy + 8y^2 + 2x - 4y + 4$, find a tangent vector to the level curve at point $(-2, 1)$. Graph the level curve corresponding to $f(x, y) = 18$ and draw in $\vec{\nabla} f(-2, 1)$ and a tangent vector.

Solution

First, we must calculate $\vec{\nabla} f(x, y)$:

$$f_x(x, y) = 4x - 3y + 2 \text{ and } f_y = -3x + 16y - 4 \text{ so } \vec{\nabla} f(x, y) = (4x - 3y + 2)\,\hat{\mathbf{i}} + (-3x + 16y - 4)\,\hat{\mathbf{j}}.$$

Next, we evaluate $\vec{\nabla} f(x, y)$ at $(-2, 1)$:

$$\vec{\nabla} f(-2, 1) = (4(-2) - 3(1) + 2)\,\hat{\mathbf{i}} + (-3(-2) + 16(1) - 4)\,\hat{\mathbf{j}} = -9\,\hat{\mathbf{i}} + 18\,\hat{\mathbf{j}}.$$

This vector is orthogonal to the curve at point $(-2, 1)$. We can obtain a tangent vector by reversing the components and multiplying either one by -1. Thus, for example, $-18\,\hat{\mathbf{i}} - 9\,\hat{\mathbf{j}}$ is a tangent vector (Figure 14.6.7).

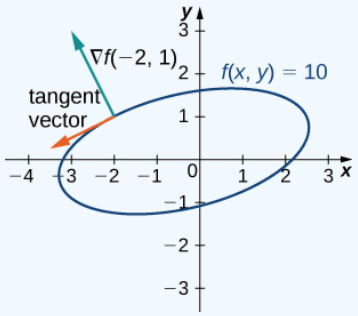


Figure 14.6.7: Tangent and normal vectors to $2x^2 - 3xy + 8y^2 + 2x - 4y + 4 = 18$ at point $(-2, 1)$.

Exercise 14.6.4

For the function $f(x, y) = x^2 - 2xy + 5y^2 + 3x - 2y + 3$, find the tangent to the level curve at point $(1, 1)$. Draw the graph of the level curve corresponding to $f(x, y) = 8$ and draw $\vec{\nabla} f(1, 1)$ and a tangent vector.

Hint

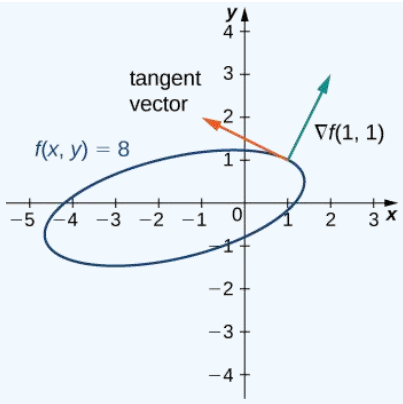
Calculate the gradient at point $(1, 1)$.

Answer

$\vec{\nabla} f(x, y) = (2x - 2y + 3)\,\hat{\mathbf{i}} + (-2x + 10y - 2)\,\hat{\mathbf{j}}$

$\vec{\nabla} f(1, 1) = 3\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{j}}$

Tangent vector: $6\,\hat{\mathbf{i}} - 3\,\hat{\mathbf{j}}$ or $-6\,\hat{\mathbf{i}} + 3\,\hat{\mathbf{j}}$



Three-Dimensional Gradients and Directional Derivatives

The definition of a gradient can be extended to functions of more than two variables.

Definition: Gradients in 3D

Let $w = f(x, y, z)$ be a function of three variables such that f_x, f_y, and f_z exist. The vector $\vec{\nabla} f(x, y, z)$ is called the gradient of f and is defined as

$$\vec{\nabla} f(x, y, z) = f_x(x, y, z)\,\hat{\mathbf{i}} + f_y(x, y, z)\,\hat{\mathbf{j}} + f_z(x, y, z)\,\hat{\mathbf{k}}. \tag{14.6.6}$$

$\vec{\nabla} f(x, y, z)$ can also be written as **grad** $f(x, y, z)$.

Calculating the gradient of a function in three variables is very similar to calculating the gradient of a function in two variables. First, we calculate the partial derivatives f_x, f_y, and f_z, and then we use Equation 14.6.6.

Example 14.6.6: Finding Gradients in Three Dimensions

Find the gradient $\vec{\nabla} f(x, y, z)$ of each of the following functions:

a. $f(x, y, z) = 5x^2 - 2xy + y^2 - 4yz + z^2 + 3xz$
b. $f(x, y, z) = e^{-2z} \sin 2x \cos 2y$

Solution

For both parts a. and b., we first calculate the partial derivatives f_x, f_y, and f_z, then use Equation 14.6.6.

a. $f_x(x, y, z) = 10x - 2y + 3z$, $f_y(x, y, z) = -2x + 2y - 4z$, and $f_z(x, y, z) = 3x - 4y + 2z$, so

$$\begin{aligned}\vec{\nabla} f(x, y, z) &= f_x(x, y, z)\,\hat{\mathbf{i}} + f_y(x, y, z)\,\hat{\mathbf{j}} + f_z(x, y, z)\,\hat{\mathbf{k}} \\ &= (10x - 2y + 3z)\,\hat{\mathbf{i}} + (-2x + 2y - 4z)\,\hat{\mathbf{j}} + (3x - 4y + 2z)\,\hat{\mathbf{k}}.\end{aligned}$$

b. $f_x(x, y, z) = 2e^{-2z} \cos 2x \cos 2y$, $f_y(x, y, z) = -2e^{-2z} \sin 2x \sin 2y$, and $f_z(x, y, z) = -2e^{-2z} \sin 2x \cos 2y$, so

$$\begin{aligned}\vec{\nabla} f(x, y, z) &= f_x(x, y, z)\,\hat{\mathbf{i}} + f_y(x, y, z)\,\hat{\mathbf{j}} + f_z(x, y, z)\,\hat{\mathbf{k}} \\ &= (2e^{-2z} \cos 2x \cos 2y)\,\hat{\mathbf{i}} + (-2e^{-2z} \sin 2x \sin 2y)\,\hat{\mathbf{j}} + (-2e^{-2z} \sin 2x \cos 2y)\,\hat{\mathbf{k}} \\ &= 2e^{-2z}(\cos 2x \cos 2y\,\hat{\mathbf{i}} - \sin 2x \sin 2y\,\hat{\mathbf{j}} - \sin 2x \cos 2y\,\hat{\mathbf{k}}).\end{aligned}$$

Exercise 14.6.5:

Find the gradient $\vec{\nabla} f(x, y, z)$ of $f(x, y, z) = \dfrac{x^2 - 3y^2 + z^2}{2x + y - 4z}$.

Answer

$$\vec{\nabla} f(x, y, z) = \frac{2x^2 + 2xy + 6y^2 - 8xz - 2z^2}{(2x + y - 4z)^2}\,\hat{\mathbf{i}} - \frac{x^2 + 12xy + 3y^2 - 24yz + z^2}{(2x + y - 4z)^2}\,\hat{\mathbf{j}} + \frac{4x^2 - 12y^2 - 4z^2 + 4xz + 2yz}{(2x + y - 4z)^2}\,\hat{\mathbf{k}}$$

The directional derivative can also be generalized to functions of three variables. To determine a direction in three dimensions, a vector with three components is needed. This vector is a unit vector, and the components of the unit vector are called *directional cosines***.** Given a three-dimensional unit vector $\vec{\mathbf{u}}$ in standard form (i.e., the initial point is at the origin), this vector forms three different angles with the positive x-, y-, and z-axes. Let's call these angles α, β, and γ. Then the directional cosines are given by $\cos\alpha, \cos\beta$, and $\cos\gamma$. These are the components of the unit vector $\vec{\mathbf{u}}$; since $\vec{\mathbf{u}}$ is a unit vector, it is true that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Definition: Directional Derivative of a Function of Three variables

Suppose $w = f(x, y, z)$ is a function of three variables with a domain of D. Let $(x_0, y_0, z_0) \in D$ and let $\vec{\mathbf{u}} = \cos\alpha\,\hat{\mathbf{i}} + \cos\beta\,\hat{\mathbf{j}} + \cos\gamma\,\hat{\mathbf{k}}$ be a unit vector. Then, the directional derivative of f in the direction of u is given by

$$D_{\vec{\mathbf{u}}} f(x_0, y_0, z_0) = \lim_{t\to 0} \frac{f(x_0 + t\cos\alpha, y_0 + t\cos\beta, z_0 + t\cos\gamma) - f(x_0, y_0, z_0)}{t}$$

provided the limit exists.

We can calculate the directional derivative of a function of three variables by using the gradient, leading to a formula that is analogous to Equation 14.6.3.

Theorem 14.6.4: Directional Derivative of a Function of Three Variables

Let $f(x, y, z)$ be a differentiable function of three variables and let $\vec{\mathbf{u}} = \cos\alpha\,\hat{\mathbf{i}} + \cos\beta\,\hat{\mathbf{j}} + \cos\gamma\,\hat{\mathbf{k}}$ be a unit vector. Then, the directional derivative of f in the direction of $\vec{\mathbf{u}}$ is given by

$$D_{\vec{\mathbf{u}}} f(x, y, z) = \vec{\nabla} f(x, y, z) \cdot \vec{\mathbf{u}} = f_x(x, y, z)\cos\alpha + f_y(x, y, z)\cos\beta + f_z(x, y, z)\cos\gamma. \tag{14.6.7}$$

The three angles α, β, and γ determine the unit vector $\vec{\mathbf{u}}$. In practice, we can use an arbitrary (nonunit) vector, then divide by its magnitude to obtain a unit vector in the desired direction.

Example 14.6.7: Finding a Directional Derivative in Three Dimensions

Calculate $D_{\vec{\mathbf{v}}} f(1, -2, 3)$ in the direction of $\vec{\mathbf{v}} = -\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}}$ for the function

$$f(x, y, z) = 5x^2 - 2xy + y^2 - 4yz + z^2 + 3xz.$$

Solution:

First, we find the magnitude of v:

$$\|\vec{\mathbf{v}}\| = \sqrt{(-1)^2 + (2)^2 + (2)^2} = \sqrt{9} = 3.$$

Therefore, $\dfrac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|} = \dfrac{-\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}}}{3} = -\dfrac{1}{3}\,\hat{\mathbf{i}} + \dfrac{2}{3}\,\hat{\mathbf{j}} + \dfrac{2}{3}\,\hat{\mathbf{k}}$ is a unit vector in the direction of $\vec{\mathbf{v}}$, so $\cos\alpha = -\dfrac{1}{3}, \cos\beta = \dfrac{2}{3}$, and $\cos\gamma = \dfrac{2}{3}$. Next, we calculate the partial derivatives of f:

$$\begin{aligned} f_x(x, y, z) &= 10x - 2y + 3z \\ f_y(x, y, z) &= -2x + 2y - 4z \\ f_z(x, y, z) &= -4y + 2z + 3x, \end{aligned}$$

then substitute them into Equation 14.6.7:

$$\begin{aligned} D_{\vec{\mathbf{v}}} f(x, y, z) &= f_x(x, y, z)\cos\alpha + f_y(x, y, z)\cos\beta + f_z(x, y, z)\cos\gamma \\ &= (10x - 2y + 3z)\left(-\tfrac{1}{3}\right) + (-2x + 2y - 4z)\left(\tfrac{2}{3}\right) + (-4y + 2z + 3x)\left(\tfrac{2}{3}\right) \\ &= -\frac{10x}{3} + \frac{2y}{3} - \frac{3z}{3} - \frac{4x}{3} + \frac{4y}{3} - \frac{8z}{3} - \frac{8y}{3} + \frac{4z}{3} + \frac{6x}{3} \\ &= -\frac{8x}{3} - \frac{2y}{3} - \frac{7z}{3}. \end{aligned}$$

Last, to find $D_{\vec{\mathbf{v}}} f(1, -2, 3)$, we substitute $x = 1$, $y = -2$, and $z = 3$:

$$\begin{aligned} D_{\vec{\mathbf{v}}} f(1,-2,3) &= -\frac{8(1)}{3} - \frac{2(-2)}{3} - \frac{7(3)}{3} \\ &= -\frac{8}{3} + \frac{4}{3} - \frac{21}{3} \\ &= -\frac{25}{3}. \end{aligned}$$

? Exercise 14.6.6:

Calculate $D_{\vec{\mathbf{v}}} f(x,y,z)$ and $D_{\vec{\mathbf{v}}} f(0,-2,5)$ in the direction of $\vec{\mathbf{v}} = -3\,\hat{\mathbf{i}} + 12\,\hat{\mathbf{j}} - 4\,\hat{\mathbf{k}}$ for the function

$$f(x,y,z) = 3x^2 + xy - 2y^2 + 4yz - z^2 + 2xz.$$

Hint

First, divide $\vec{\mathbf{v}}$ by its magnitude, calculate the partial derivatives of f, then use Equation 14.6.7.

Answer

$$D_{\vec{\mathbf{v}}} f(x,y,z) = -\frac{3}{13}(6x + y + 2z) + \frac{12}{13}(x - 4y + 4z) - \frac{4}{13}(2x + 4y - 2z)$$

$$D_{\vec{\mathbf{v}}} f(0,-2,5) = \frac{384}{13}$$

Summary

- A directional derivative represents a rate of change of a function in any given direction.
- The gradient can be used in a formula to calculate the directional derivative.
- The gradient indicates the direction of greatest change of a function of more than one variable.

Key Equations

- **directional derivative (two dimensions)**

$$D_{\vec{\mathbf{u}}} f(a,b) = \lim_{h \to 0} \frac{f(a + h\cos\theta, b + h\sin\theta) - f(a,b)}{h}$$

or

$$D_{\vec{\mathbf{u}}} f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta$$

- **gradient (two dimensions)**

$$\vec{\nabla} f(x,y) = f_x(x,y)\,\hat{\mathbf{i}} + f_y(x,y)\,\hat{\mathbf{j}}$$

- **gradient (three dimensions)**

$$\vec{\nabla} f(x,y,z) = f_x(x,y,z)\,\hat{\mathbf{i}} + f_y(x,y,z)\,\hat{\mathbf{j}} + f_z(x,y,z)\,\hat{\mathbf{k}}$$

- **directional derivative (three dimensions)**

$$D_{\vec{\mathbf{u}}} f(x,y,z) = \vec{\nabla} f(x,y,z) \cdot \vec{\mathbf{u}} = f_x(x,y,z)\cos\alpha + f_y(x,y,z)\cos\beta + f_x(x,y,z)\cos\gamma$$

Glossary

directional derivative

the derivative of a function in the direction of a given unit vector

gradient

the gradient of the function $f(x,y)$ is defined to be $\vec{\nabla} f(x,y) = (\partial f/\partial x)\,\hat{\mathbf{i}} + (\partial f/\partial y)\,\hat{\mathbf{j}}$, which can be generalized to a function of any number of independent variables

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14.7: Maximum and Minimum Values

Learning Objectives

- Use partial derivatives to locate critical points for a function of two variables.
- Apply a second derivative test to identify a critical point as a local maximum, local minimum, or saddle point for a function of two variables.
- Examine critical points and boundary points to find absolute maximum and minimum values for a function of two variables.

One of the most useful applications for derivatives of a function of one variable is the determination of maximum and/or minimum values. This application is also important for functions of two or more variables, but as we have seen in earlier sections of this chapter, the introduction of more independent variables leads to more possible outcomes for the calculations. The main ideas of finding critical points and using derivative tests are still valid, but new wrinkles appear when assessing the results.

Critical Points

For functions of a single variable, we defined critical points as the values of the variable at which the function's derivative equals zero or does not exist. For functions of two or more variables, the concept is essentially the same, except for the fact that we are now working with partial derivatives.

Definition: Critical Points

Let $z = f(x, y)$ be a function of two variables that is differentiable on an open set containing the point (x_0, y_0). The point (x_0, y_0) is called a *critical point* of a function of two variables f if one of the two following conditions holds:

1. $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$
2. Either $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

Example 14.7.1: Finding Critical Points

Find the critical points of each of the following functions:

a. $f(x, y) = \sqrt{4y^2 - 9x^2 + 24y + 36x + 36}$
b. $g(x, y) = x^2 + 2xy - 4y^2 + 4x - 6y + 4$

Solution

a. First, we calculate $f_x(x, y)$ and $f_y(x, y)$:

$$\begin{aligned} f_x(x, y) &= \frac{1}{2}(-18x + 36)(4y^2 - 9x^2 + 24y + 36x + 36)^{-1/2} \\ &= \frac{-9x + 18}{\sqrt{4y^2 - 9x^2 + 24y + 36x + 36}} \end{aligned}$$

$$\begin{aligned} f_y(x, y) &= \frac{1}{2}(8y + 24)(4y^2 - 9x^2 + 24y + 36x + 36)^{-1/2} \\ &= \frac{4y + 12}{\sqrt{4y^2 - 9x^2 + 24y + 36x + 36}} \end{aligned}.$$

Next, we set each of these expressions equal to zero:

$$\frac{-9x + 18}{\sqrt{4y^2 - 9x^2 + 24y + 36x + 36}} = 0$$

$$\frac{4y + 12}{\sqrt{4y^2 - 9x^2 + 24y + 36x + 36}} = 0.$$

Then, multiply each equation by its common denominator:

$$-9x+18 = 0$$
$$4y+12 = 0.$$

Therefore, $x = 2$ and $y = -3$, so $(2,-3)$ is a critical point of f.

We must also check for the possibility that the denominator of each partial derivative can equal zero, thus causing the partial derivative not to exist. Since the denominator is the same in each partial derivative, we need only do this once:

$$4y^2 - 9x^2 + 24y + 36x + 36 = 0. \tag{14.7.1}$$

Equation 14.7.1 represents a hyperbola. We should also note that the domain of f consists of points satisfying the inequality

$$4y^2 - 9x^2 + 24y + 36x + 36 \geq 0.$$

Therefore, any points on the hyperbola are not only critical points, they are also on the boundary of the domain. To put the hyperbola in standard form, we use the method of completing the square:

$$\begin{aligned} 4y^2 - 9x^2 + 24y + 36x + 36 &= 0 \\ 4y^2 - 9x^2 + 24y + 36x &= -36 \\ 4y^2 + 24y - 9x^2 + 36x &= -36 \\ 4(y^2 + 6y) - 9(x^2 - 4x) &= -36 \\ 4(y^2 + 6y + 9) - 9(x^2 - 4x + 4) &= -36 - 36 + 36 \\ 4(y+3)^2 - 9(x-2)^2 &= -36. \end{aligned}$$

Dividing both sides by -36 puts the equation in standard form:

$$\begin{aligned} \frac{4(y+3)^2}{-36} - \frac{9(x-2)^2}{-36} &= 1 \\ \frac{(x-2)^2}{4} - \frac{(y+3)^2}{9} &= 1. \end{aligned}$$

Notice that point $(2,-3)$ is the center of the hyperbola.

Thus, the critical points of the function f are $(2,-3)$ and all points on the hyperbola, $\frac{(x-2)^2}{4} - \frac{(y+3)^2}{9} = 1$.

b. First, we calculate $g_x(x,y)$ and $g_y(x,y)$:

$$\begin{aligned} g_x(x,y) &= 2x + 2y + 4 \\ g_y(x,y) &= 2x - 8y - 6. \end{aligned}$$

Next, we set each of these expressions equal to zero, which gives a system of equations in x and y:

$$\begin{aligned} 2x + 2y + 4 &= 0 \\ 2x - 8y - 6 &= 0. \end{aligned}$$

Subtracting the second equation from the first gives $10y + 10 = 0$, so $y = -1$. Substituting this into the first equation gives $2x + 2(-1) + 4 = 0$, so $x = -1$.

Therefore $(-1,-1)$ is a critical point of g. There are no points in $\mathbb{R}^2$ that make either partial derivative not exist.

Figure 14.7.1 shows the behavior of the surface at the critical point.

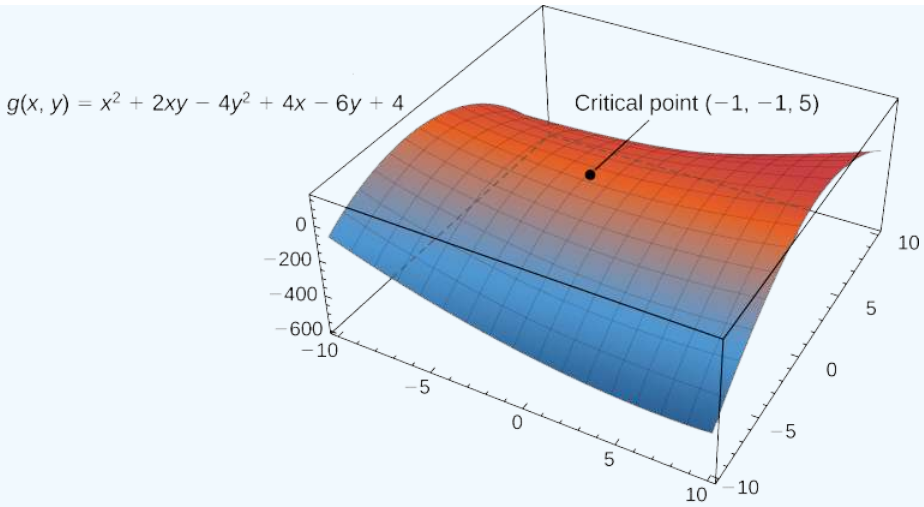


Figure 14.7.1: The function $g(x,y)$ has a critical point at $(-1,-1,5)$.

Exercise 14.7.1

Find the critical point of the function $f(x,y)=x^3+2xy-2x-4y$.

Hint

Calculate $f_x(x,y)$ and $f_y(x,y)$, then set them equal to zero.

Answer

The only critical point of f is $(2,-5)$.

The main purpose for determining critical points is to locate relative maxima and minima, as in single-variable calculus. When working with a function of one variable, the definition of a local extremum involves finding an interval around the critical point such that the function value is either greater than or less than all the other function values in that interval. When working with a function of two or more variables, we work with an open disk around the point.

Definition: Global and Local Extrema

Let $z=f(x,y)$ be a function of two variables that is defined and continuous on an open set containing the point (x_0,y_0). Then f has a **local maximum** at (x_0,y_0) if

$$f(x_0,y_0)\geq f(x,y)$$

for all points (x,y) within some disk centered at (x_0,y_0). The number $f(x_0,y_0)$ is called a local maximum value. If the preceding inequality holds for every point (x,y) in the domain of f, then f has a **global maximum** (also called an absolute maximum) at (x_0,y_0).

The function f has a local minimum at (x_0,y_0) if

$$f(x_0,y_0)\leq f(x,y)$$

for all points (x,y) within some disk centered at (x_0,y_0). The number $f(x_0,y_0)$ is called a local minimum value. If the preceding inequality holds for every point (x,y) in the domain of f, then f has a **global minimum** (also called an absolute minimum) at (x_0,y_0).

If $f(x_0,y_0)$ is either a local maximum or local minimum value, then it is called a **local extremum** (see the following figure).

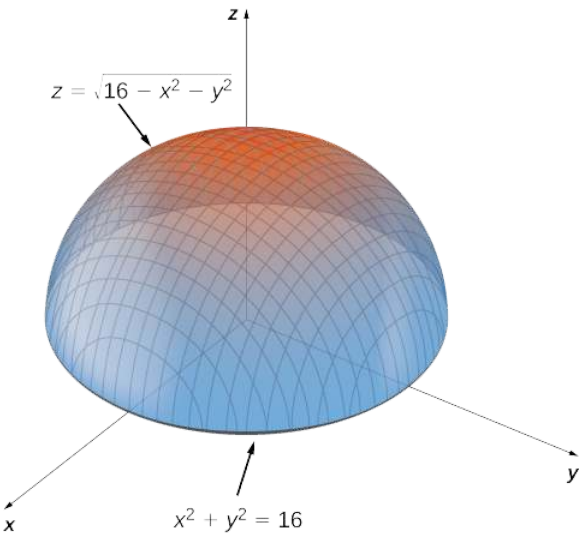


Figure 14.7.2: The graph of $z = \sqrt{16 - x^2 - y^2}$ has a maximum value when $(x, y) = (0, 0)$. It attains its minimum value at the boundary of its domain, which is the circle $x^2 + y^2 = 16$.

In Calculus 1, we showed that extrema of functions of one variable occur at critical points. The same is true for functions of more than one variable, as stated in the following theorem.

Fermat's Theorem for Functions of Two Variables

Let $z = f(x, y)$ be a function of two variables that is defined and continuous on an open set containing the point (x_0, y_0). Suppose f_x and f_y each exist at (x_0, y_0). If f has a local extremum at (x_0, y_0), then (x_0, y_0) is a critical point of f.

Second Derivative Test

Consider the function $f(x) = x^3$. This function has a critical point at $x = 0$, since $f'(0) = 3(0)^2 = 0$. However, f does not have an extreme value at $x = 0$. Therefore, the existence of a critical value at $x = x_0$ does not guarantee a local extremum at $x = x_0$. The same is true for a function of two or more variables. One way this can happen is at a **saddle point**. An example of a saddle point appears in the following figure.

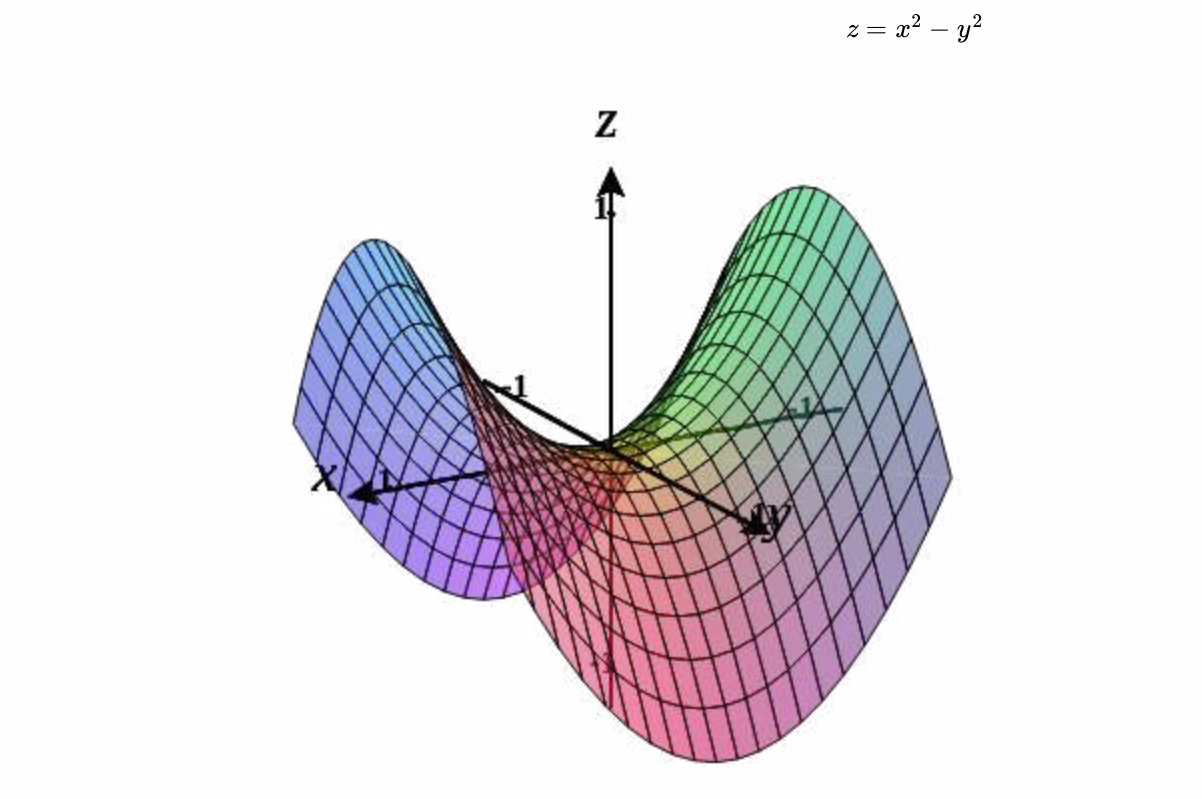


Figure 14.7.3: Graph of the function $z = x^2 - y^2$. This graph has a saddle point at the origin.

In this graph, the origin is a saddle point. This is because the first partial derivatives of f$(x, y) = x^2 - y^2$ are both equal to zero at this point, but it is neither a maximum nor a minimum for the function. Furthermore the vertical trace corresponding to $y = 0$ is $z = x^2$ (a parabola opening upward), but the vertical trace corresponding to $x = 0$ is $z = -y^2$ (a parabola opening downward). Therefore, it is both a global maximum for one trace and a global minimum for another.

Definition: Saddle Point

Given the function $z = f(x, y)$, the point $\big(x_0, y_0, f(x_0, y_0)\big)$ is a saddle point if both $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, but f does not have a local extremum at (x_0, y_0).

The second derivative test for a function of one variable provides a method for determining whether an extremum occurs at a critical point of a function. When extending this result to a function of two variables, an issue arises related to the fact that there are, in fact, four different second-order partial derivatives, although equality of mixed partials reduces this to three. The second derivative test for a function of two variables, stated in the following theorem, uses a **discriminant** D that replaces $f''(x_0)$ in the second derivative test for a function of one variable.

Second Derivative Test

Let $z = f(x, y)$ be a function of two variables for which the first- and second-order partial derivatives are continuous on some disk containing the point (x_0, y_0). Suppose $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$. Define the quantity

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - \big(f_{xy}(x_0, y_0)\big)^2.$$

Then:

i. If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a local minimum at (x_0, y_0).
ii. If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a local maximum at (x_0, y_0).
iii. If $D < 0$, then f has a saddle point at (x_0, y_0).
iv. If $D = 0$, then the test is inconclusive.

See Figure 14.7.4

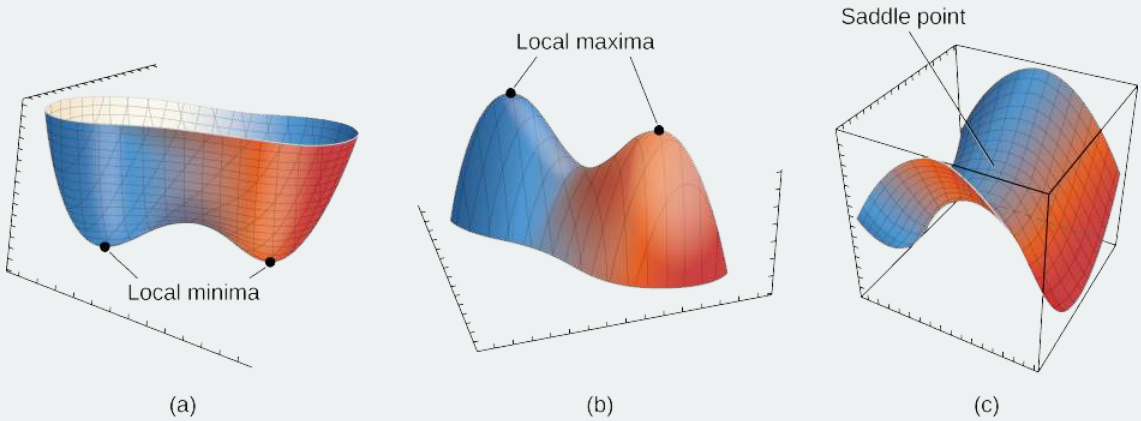


Figure 14.7.4: The second derivative test can often determine whether a function of two variables has local minima (a), local maxima (b), or a saddle point (c).

To apply the second derivative test, it is necessary that we first find the critical points of the function. There are several steps involved in the entire procedure, which are outlined in a problem-solving strategy.

Problem-Solving Strategy: Using the Second Derivative Test for Functions of Two Variables

Let $z = f(x, y)$ be a function of two variables for which the first- and second-order partial derivatives are continuous on some disk containing the point (x_0, y_0). To apply the second derivative test to find local extrema, use the following steps:

1. Determine the critical points (x_0, y_0) of the function f where $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. Discard any points where at least one of the partial derivatives does not exist.
2. Calculate the discriminant $D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - \big(f_{xy}(x_0, y_0)\big)^2$ for each critical point of f.
3. Apply the four cases of the test to determine whether each critical point is a local maximum, local minimum, or saddle point, or whether the theorem is inconclusive.

Example 14.7.2: Using the Second Derivative Test

Find the critical points for each of the following functions, and use the second derivative test to find the local extrema:

a. $f(x, y) = 4x^2 + 9y^2 + 8x - 36y + 24$

b. $g(x, y) = \dfrac{1}{3}x^3 + y^2 + 2xy - 6x - 3y + 4$

Solution

a. **Step 1** of the problem-solving strategy involves finding the critical points of f. To do this, we first calculate $f_x(x, y)$ and $f_y(x, y)$, then set each of them equal to zero:

$$\begin{aligned} f_x(x, y) &= 8x + 8 \\ f_y(x, y) &= 18y - 36. \end{aligned}$$

Setting them equal to zero yields the system of equations

$$\begin{aligned} 8x + 8 &= 0 \\ 18y - 36 &= 0. \end{aligned}$$

The solution to this system is $x = -1$ and $y = 2$. Therefore $(-1, 2)$ is a critical point of f.

Step 2 of the problem-solving strategy involves calculating D. To do this, we first calculate the second partial derivatives of f:

$$f_{xx}(x,y) = 8$$
$$f_{xy}(x,y) = 0$$
$$f_{yy}(x,y) = 18.$$

Therefore, $D = f_{xx}(-1,2)f_{yy}(-1,2) - \left(f_{xy}(-1,2)\right)^2 = (8)(18) - (0)^2 = 144.$

Step 3 states to apply the four cases of the test to classify the function's behavior at this critical point.

Since $D > 0$ and $f_{xx}(-1,2) > 0$, this corresponds to case 1. Therefore, f has a local minimum at $(-1,2)$ as shown in the following figure.

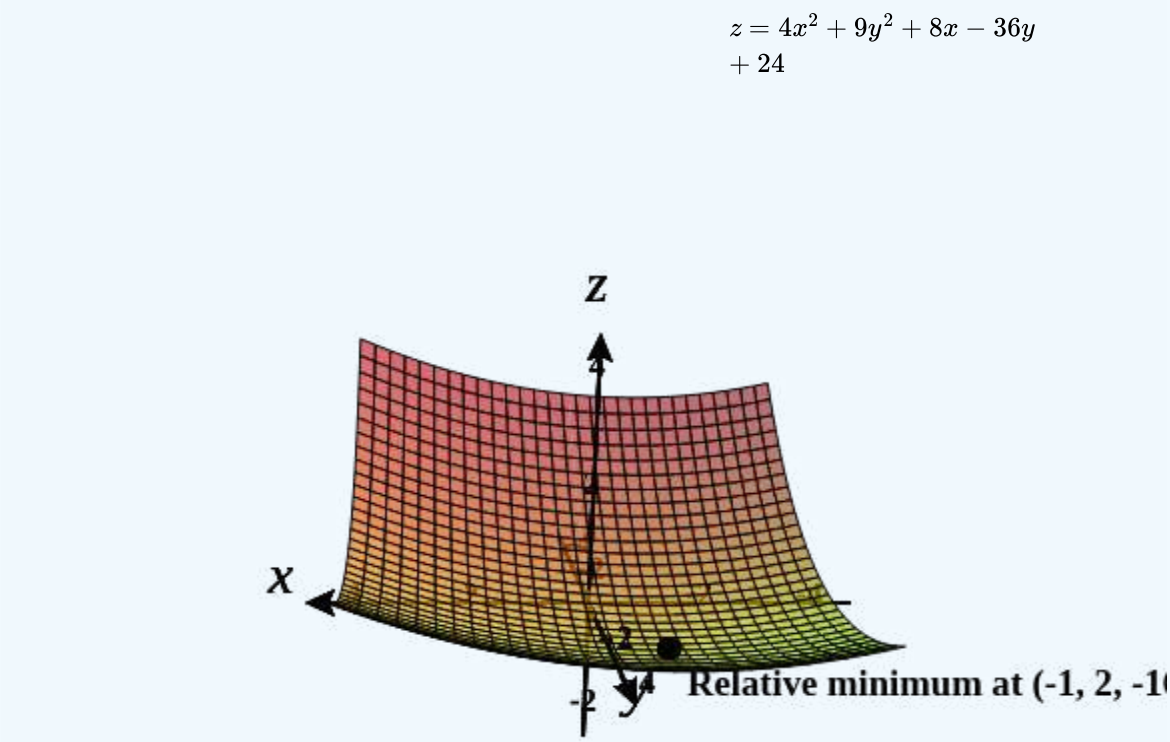


Figure 14.7.5 : The function $f(x,y)$ has a local minimum at $(-1,2,-16)$. Note the scale on the y-axis in this plot is in thousands.

b. For **step 1**, we first calculate $g_x(x,y)$ and $g_y(x,y)$, then set each of them equal to zero:

$$g_x(x,y) = x^2 + 2y - 6$$
$$g_y(x,y) = 2y + 2x - 3.$$

Setting them equal to zero yields the system of equations

$$x^2 + 2y - 6 = 0$$
$$2y + 2x - 3 = 0.$$

To solve this system, first solve the second equation for y. This gives $y = \dfrac{3-2x}{2}$. Substituting this into the first equation gives

$$x^2 + 3 - 2x - 6 = 0$$
$$x^2 - 2x - 3 = 0$$
$$(x-3)(x+1) = 0.$$

Therefore, $x=-1$ or $x=3$. Substituting these values into the equation $y=\dfrac{3-2x}{2}$ yields the critical points $\left(-1,\frac{5}{2}\right)$ and $\left(3,-\frac{3}{2}\right)$.

Step 2 involves calculating the second partial derivatives of g:

$$\begin{aligned} g_{xx}(x,y) &= 2x \\ g_{xy}(x,y) &= 2 \\ g_{yy}(x,y) &= 2. \end{aligned}$$

Then, we find a general formula for D:

$$\begin{aligned} D(x_0,y_0) &= g_{xx}(x_0,y_0)g_{yy}(x_0,y_0) - \left(g_{xy}(x_0,y_0)\right)^2 \\ &= (2x_0)(2) - 2^2 \\ &= 4x_0 - 4. \end{aligned}$$

Next, we substitute each critical point into this formula:

$$\begin{aligned} D\left(-1,\tfrac{5}{2}\right) &= (2(-1))(2) - (2)^2 = -4-4 = -8 \\ D\left(3,-\tfrac{3}{2}\right) &= (2(3))(2) - (2)^2 = 12-4 = 8. \end{aligned}$$

In step 3, we note that, applying Note to point $\left(-1,\frac{5}{2}\right)$ leads to case 3, which means that $\left(-1,\frac{5}{2}\right)$ is a saddle point. Applying the theorem to point $\left(3,-\frac{3}{2}\right)$ leads to case 1, which means that $\left(3,-\frac{3}{2}\right)$ corresponds to a local minimum as shown in the following figure.

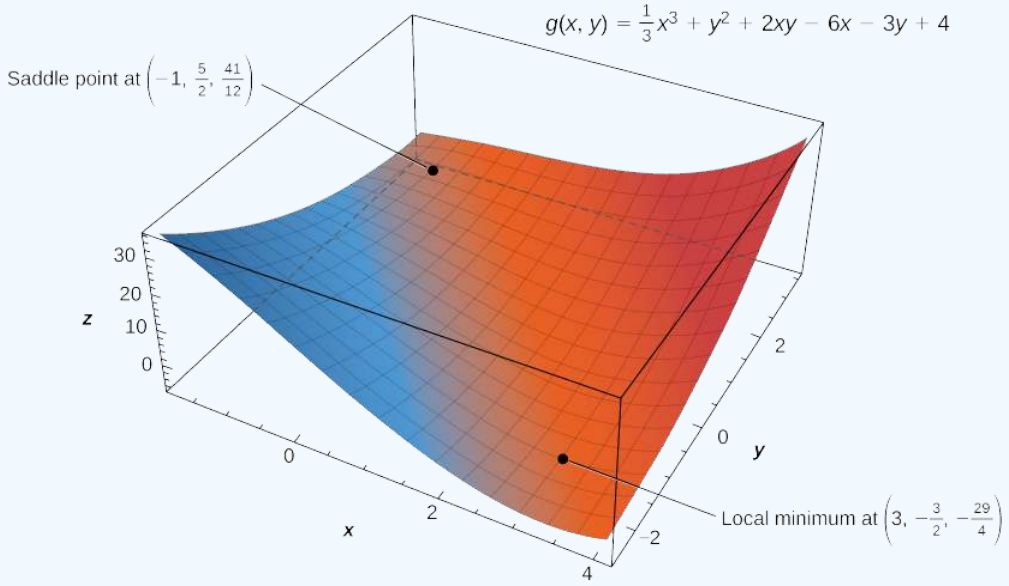


Figure 14.7.6: The function $g(x,y)$ has a local minimum and a saddle point.

? Exercise 14.7.2

Use the second derivative test to find the local extrema of the function

$$f(x,y) = x^3 + 2xy - 6x - 4y^2.$$

Hint

Follow the problem-solving strategy for applying the second derivative test.

Answer

$\left(\frac{4}{3}, \frac{1}{3}\right)$ is a saddle point, $\left(-\frac{3}{2}, -\frac{3}{8}\right)$ is a local maximum.

Absolute Maxima and Minima

When finding global extrema of functions of one variable on a closed interval, we start by checking the critical values over that interval and then evaluate the function at the endpoints of the interval. When working with a function of two variables, the closed interval is replaced by a closed, bounded set. A set is bounded if all the points in that set can be contained within a ball (or disk) of finite radius. First, we need to find the critical points inside the set and calculate the corresponding critical values. Then, it is necessary to find the maximum and minimum value of the function on the boundary of the set. When we have all these values, the largest function value corresponds to the global maximum and the smallest function value corresponds to the absolute minimum. First, however, we need to be assured that such values exist. The following theorem does this.

Extreme Value Theorem

A continuous function $f(x, y)$ on a closed and bounded set D in the plane attains an absolute maximum value at some point of D and an absolute minimum value at some point of D.

Now that we know any continuous function f defined on a closed, bounded set attains its extreme values, we need to know how to find them.

Finding Extreme Values of a Function of Two Variables

Assume $z = f(x, y)$ is a differentiable function of two variables defined on a closed, bounded set D. Then f will attain the absolute maximum value and the absolute minimum value, which are, respectively, the largest and smallest values found among the following:

1. The values of f at the critical points of f in D.
2. The values of f on the boundary of D.

The proof of this theorem is a direct consequence of the extreme value theorem and Fermat's theorem. In particular, if either extremum is not located on the boundary of D, then it is located at an interior point of D. But an interior point (x_0, y_0) of D that's an absolute extremum is also a local extremum; hence, (x_0, y_0) is a critical point of f by Fermat's theorem. Therefore the only possible values for the global extrema of f on D are the extreme values of f on the interior or boundary of D.

Problem-Solving Strategy: Finding Absolute Maximum and Minimum Values

Let $z = f(x, y)$ be a continuous function of two variables defined on a closed, bounded set D, and assume f is differentiable on D. To find the absolute maximum and minimum values of f on D, do the following:

1. Determine the critical points of f in D.
2. Calculate f at each of these critical points.
3. Determine the maximum and minimum values of f on the boundary of its domain.
4. The maximum and minimum values of f will occur at one of the values obtained in steps 2 and 3.

Finding the maximum and minimum values of f on the boundary of D can be challenging. If the boundary is a rectangle or set of straight lines, then it is possible to parameterize the line segments and determine the maxima on each of these segments, as seen in Example 14.7.3. The same approach can be used for other shapes such as circles and ellipses.

If the boundary of the set D is a more complicated curve defined by a function $g(x, y) = c$ for some constant c, and the first-order partial derivatives of g exist, then the method of Lagrange multipliers can prove useful for determining the extrema of f on the boundary which is introduced in Lagrange Multipliers.

Example 14.7.3: Finding Absolute Extrema

Use the problem-solving strategy for finding absolute extrema of a function to determine the absolute extrema of each of the following functions:

a. $f(x,y) = x^2 - 2xy + 4y^2 - 4x - 2y + 24$ on the domain defined by $0 \le x \le 4$ and $0 \le y \le 2$
b. $g(x,y) = x^2 + y^2 + 4x - 6y$ on the domain defined by $x^2 + y^2 \le 16$

Solution

a. Using the problem-solving strategy, step 1 involves finding the critical points of f on its domain. Therefore, we first calculate $f_x(x,y)$ and $f_y(x,y)$, then set them each equal to zero:

$$\begin{aligned} f_x(x,y) &= 2x - 2y - 4 \\ f_y(x,y) &= -2x + 8y - 2. \end{aligned}$$

Setting them equal to zero yields the system of equations

$$\begin{aligned} 2x - 2y - 4 &= 0 \\ -2x + 8y - 2 &= 0. \end{aligned}$$

The solution to this system is $x = 3$ and $y = 1$. Therefore $(3,1)$ is a critical point of f. Calculating $f(3,1)$ gives $f(3,1) = 17$.

The next step involves finding the extrema of f on the boundary of its domain. The boundary of its domain consists of four line segments as shown in the following graph:

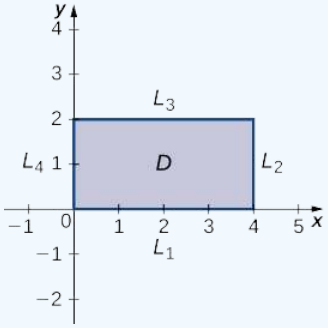


Figure 14.7.7: Graph of the domain of the function $f(x,y) = x^2 - 2xy + 4y^2 - 4x - 2y + 24$.

L_1 is the line segment connecting $(0,0)$ and $(4,0)$, and it can be parameterized by the equations $x(t) = t, y(t) = 0$ for $0 \le t \le 4$. Define $g(t) = f\big(x(t), y(t)\big)$. This gives $g(t) = t^2 - 4t + 24$. Differentiating g leads to $g'(t) = 2t - 4$. Therefore, g has a critical value at $t = 2$, which corresponds to the point $(2,0)$. Calculating $f(2,0)$ gives the z-value 20.

L_2 is the line segment connecting $(4,0)$ and $(4,2)$, and it can be parameterized by the equations $x(t) = 4, y(t) = t$ for $0 \le t \le 2$. Again, define $g(t) = f\big(x(t), y(t)\big)$. This gives $g(t) = 4t^2 - 10t + 24$. Then, $g'(t) = 8t - 10$. g has a critical value at $t = \frac{5}{4}$, which corresponds to the point $\left(4, \frac{5}{4}\right)$. Calculating $f\left(4, \frac{5}{4}\right)$ gives the z-value 17.75.

L_3 is the line segment connecting $(0,2)$ and $(4,2)$, and it can be parameterized by the equations $x(t) = t, y(t) = 2$ for $0 \le t \le 4$. Again, define $g(t) = f\big(x(t), y(t)\big)$. This gives $g(t) = t^2 - 8t + 36$. The critical value corresponds to the point $(4,2)$. So, calculating $f(4,2)$ gives the z-value 20.

L_4 is the line segment connecting $(0,0)$ and $(0,2)$, and it can be parameterized by the equations $x(t) = 0, y(t) = t$ for $0 \le t \le 2$. This time, $g(t) = 4t^2 - 2t + 24$ and the critical value $t = \frac{1}{4}$ correspond to the point $\left(0, \frac{1}{4}\right)$. Calculating $f\left(0, \frac{1}{4}\right)$ gives the z-value 23.75.

We also need to find the values of $f(x,y)$ at the corners of its domain. These corners are located at $(0,0), (4,0), (4,2)$ and $(0,2)$:

$$\begin{aligned} f(0,0) &= (0)^2 - 2(0)(0) + 4(0)^2 - 4(0) - 2(0) + 24 = 24 \\ f(4,0) &= (4)^2 - 2(4)(0) + 4(0)^2 - 4(4) - 2(0) + 24 = 24 \\ f(4,2) &= (4)^2 - 2(4)(2) + 4(2)^2 - 4(4) - 2(2) + 24 = 20 \\ f(0,2) &= (0)^2 - 2(0)(2) + 4(2)^2 - 4(0) - 2(2) + 24 = 36. \end{aligned}$$

The absolute maximum value is 36, which occurs at $(0,2)$, and the global minimum value is 20, which occurs at both $(4,2)$ and $(2,0)$ as shown in the following figure.

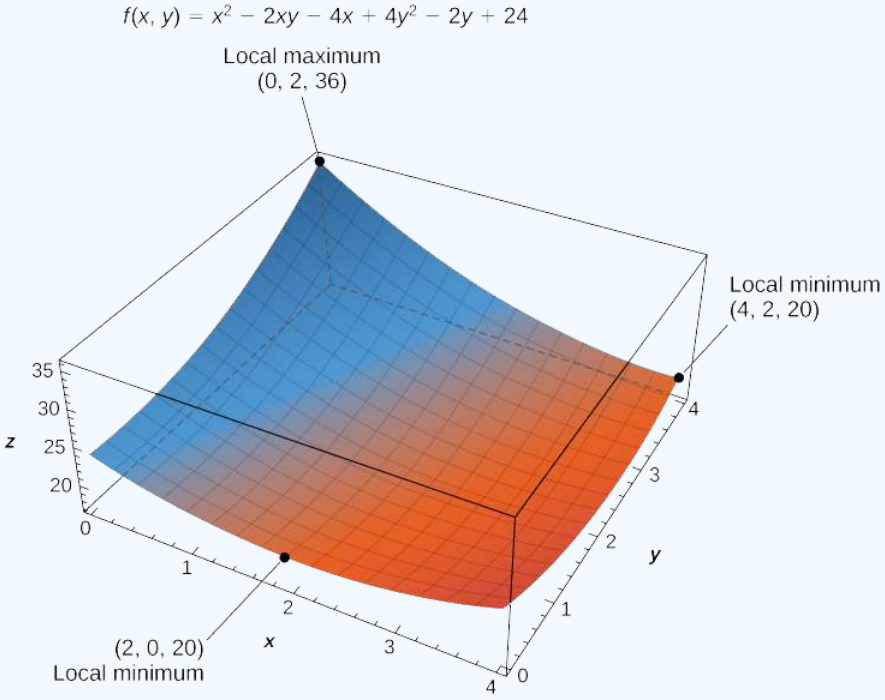


Figure 14.7.8: The function $f(x,y)$ has two global minima and one global maximum over its domain.

b. Using the problem-solving strategy, step 1 involves finding the critical points of g on its domain. Therefore, we first calculate $g_x(x,y)$ and $g_y(x,y)$, then set them each equal to zero:

$$\begin{aligned} g_x(x,y) &= 2x + 4 \\ g_y(x,y) &= 2y - 6. \end{aligned}$$

Setting them equal to zero yields the system of equations

$$\begin{aligned} 2x + 4 &= 0 \\ 2y - 6 &= 0. \end{aligned}$$

The solution to this system is $x = -2$ and $y = 3$. Therefore, $(-2,3)$ is a critical point of g. Calculating $g(-2,3)$, we get

$$g(-2,3) = (-2)^2 + 3^2 + 4(-2) - 6(3) = 4 + 9 - 8 - 18 = -13.$$

The next step involves finding the extrema of g on the boundary of its domain. The boundary of its domain consists of a circle of radius 4 centered at the origin as shown in the following graph.

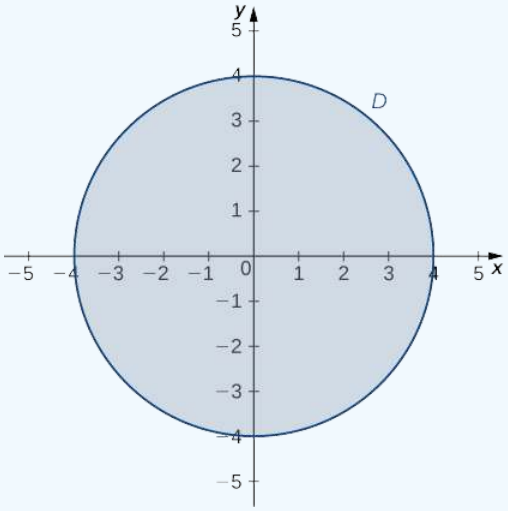


Figure 14.7.9: Graph of the restricted domain of the function $g(x,y)=x^2+y^2+4x-6y$.

The boundary of the domain of g can be parameterized using the functions $x(t)=4\cos t$, $y(t)=4\sin t$ for $0\le t\le 2\pi$. Define $h(t)=g\big(x(t),y(t)\big)$:

$$\begin{aligned} h(t) &= g\big(x(t),y(t)\big) \\ &= (4\cos t)^2+(4\sin t)^2+4(4\cos t)-6(4\sin t) \\ &= 16\cos^2 t+16\sin^2 t+16\cos t-24\sin t \\ &= 16+16\cos t-24\sin t. \end{aligned}$$

Setting $h'(t)=0$ leads to

$$\begin{aligned} -16\sin t-24\cos t &= 0 \\ -16\sin t &= 24\cos t \\ \frac{-16\sin t}{-16\cos t} &= \frac{24\cos t}{-16\cos t} \\ \tan t &= -\frac{3}{2}. \end{aligned}$$

This equation has two solutions over the interval $0\le t\le 2\pi$. One is $t=\pi-\arctan(\frac{3}{2})$ and the other is $t=2\pi-\arctan(\frac{3}{2})$. For the first angle,

$$\begin{aligned} \sin t &= \sin(\pi-\arctan(\tfrac{3}{2}))=\sin(\arctan(\tfrac{3}{2}))=\frac{3\sqrt{13}}{13} \\ \cos t &= \cos(\pi-\arctan(\tfrac{3}{2}))=-\cos(\arctan(\tfrac{3}{2}))=-\frac{2\sqrt{13}}{13}. \end{aligned}$$

Therefore, $x(t)=4\cos t=-\frac{8\sqrt{13}}{13}$ and $y(t)=4\sin t=\frac{12\sqrt{13}}{13}$, so $\left(-\frac{8\sqrt{13}}{13},\frac{12\sqrt{13}}{13}\right)$ is a critical point on the boundary and

$$\begin{aligned} g\left(-\tfrac{8\sqrt{13}}{13},\tfrac{12\sqrt{13}}{13}\right) &= \left(-\tfrac{8\sqrt{13}}{13}\right)^2+\left(\tfrac{12\sqrt{13}}{13}\right)^2+4\left(-\tfrac{8\sqrt{13}}{13}\right)-6\left(\tfrac{12\sqrt{13}}{13}\right) \\ &= \frac{144}{13}+\frac{64}{13}-\frac{32\sqrt{13}}{13}-\frac{72\sqrt{13}}{13} \\ &= \frac{208-104\sqrt{13}}{13}\approx -12.844. \end{aligned}$$

For the second angle,

$$\sin t = \sin(2\pi - \arctan(\tfrac{3}{2})) = -\sin(\arctan(\tfrac{3}{2})) = -\frac{3\sqrt{13}}{13}$$

$$\cos t = \cos(2\pi - \arctan(\tfrac{3}{2})) = \cos(\arctan(\tfrac{3}{2})) = \frac{2\sqrt{13}}{13}.$$

Therefore, $x(t) = 4\cos t = \frac{8\sqrt{13}}{13}$ and $y(t) = 4\sin t = -\frac{12\sqrt{13}}{13}$, so $\left(\frac{8\sqrt{13}}{13}, -\frac{12\sqrt{13}}{13}\right)$ is a critical point on the boundary and

$$\begin{aligned} g\left(\tfrac{8\sqrt{13}}{13}, -\tfrac{12\sqrt{13}}{13}\right) &= \left(\tfrac{8\sqrt{13}}{13}\right)^2 + \left(-\tfrac{12\sqrt{13}}{13}\right)^2 + 4\left(\tfrac{8\sqrt{13}}{13}\right) - 6\left(-\tfrac{12\sqrt{13}}{13}\right) \\ &= \frac{144}{13} + \frac{64}{13} + \frac{32\sqrt{13}}{13} + \frac{72\sqrt{13}}{13} \\ &= \frac{208 + 104\sqrt{13}}{13} \approx 44.844. \end{aligned}$$

The absolute minimum of g is -13, which is attained at the point $(-2, 3)$, which is an interior point of D. The absolute maximum of g is approximately equal to 44.844, which is attained at the boundary point $\left(\frac{8\sqrt{13}}{13}, -\frac{12\sqrt{13}}{13}\right)$. These are the absolute extrema of g on D as shown in the following figure.

$$f(x, y) = x^2 + 4x + y^2 - 6y$$

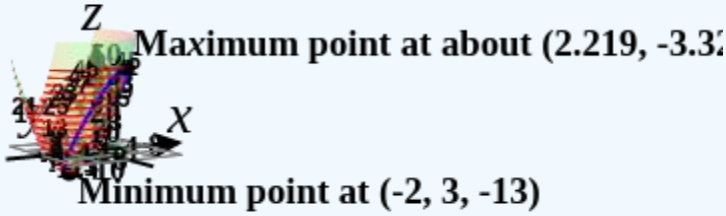


Figure 14.7.10: The function $f(x, y)$ has a local minimum and a local maximum.

Exercise 14.7.3:

Use the problem-solving strategy for finding absolute extrema of a function to find the absolute extrema of the function

$$f(x, y) = 4x^2 - 2xy + 6y^2 - 8x + 2y + 3$$

on the domain defined by $0 \le x \le 2$ and $-1 \le y \le 3$.

Hint

Calculate $f_x(x,y)$ and $f_y(x,y)$, and set them equal to zero. Then, calculate f for each critical point and find the extrema of f on the boundary of D.

Answer

The absolute minimum occurs at $(1,0) : f(1,0) = -1$.

The absolute maximum occurs at $(0,3) : f(0,3) = 63$.

Example 14.7.4: Profitable Golf Balls

Pro-T company has developed a profit model that depends on the number x of golf balls sold per month (measured in thousands), and the number of hours per month of advertising y, according to the function

$$z = f(x,y) = 48x + 96y - x^2 - 2xy - 9y^2,$$

where z is measured in thousands of dollars. The maximum number of golf balls that can be produced and sold is $50,000$, and the maximum number of hours of advertising that can be purchased is 25. Find the values of x and y that maximize profit, and find the maximum profit.



Figure 14.7.11: (credit: modification of work by oatsy40, Flickr)

Solution

Using the problem-solving strategy, step 1 involves finding the critical points of f on its domain. Therefore, we first calculate $f_x(x,y)$ and $f_y(x,y)$, then set them each equal to zero:

$$\begin{aligned} f_x(x,y) &= 48 - 2x - 2y \\ f_y(x,y) &= 96 - 2x - 18y. \end{aligned}$$

Setting them equal to zero yields the system of equations

$$\begin{aligned} 48 - 2x - 2y &= 0 \\ 96 - 2x - 18y &= 0. \end{aligned}$$

The solution to this system is $x = 21$ and $y = 3$. Therefore $(21,3)$ is a critical point of f. Calculating $f(21,3)$ gives $f(21,3) = 48(21) + 96(3) - 21^2 - 2(21)(3) - 9(3)^2 = 648$.

The domain of this function is $0 \le x \le 50$ and $0 \le y \le 25$ as shown in the following graph.

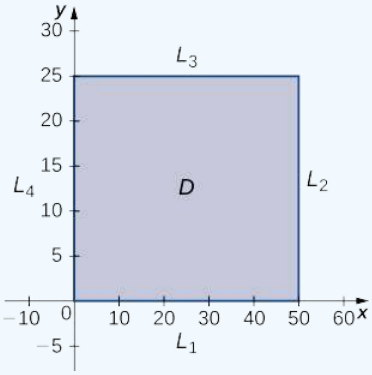


Figure 14.7.12: Graph of the domain of the function $f(x,y) = 48x + 96y - x^2 - 2xy - 9y^2$.

L_1 is the line segment connecting $(0,0)$ and $(50,0)$, and it can be parameterized by the equations $x(t) = t, y(t) = 0$ for $0 \le t \le 50$. We then define $g(t) = f(x(t), y(t))$:

$$\begin{aligned} g(t) &= f(x(t), y(t)) \\ &= f(t, 0) \\ &= 48t + 96(0) - y^2 - 2(t)(0) - 9(0)^2 \\ &= 48t - t^2. \end{aligned}$$

Setting $g'(t) = 0$ yields the critical point $t = 24$, which corresponds to the point $(24,0)$ in the domain of f. Calculating $f(24,0)$ gives 576.

L_2 is the line segment connecting $(50,0)$ and $(50,25)$, and it can be parameterized by the equations $x(t) = 50, y(t) = t$ for $0 \le t \le 25$. Once again, we define $g(t) = f\big(x(t), y(t)\big)$:

$$\begin{aligned} g(t) &= f\big(x(t), y(t)\big) \\ &= f(50, t) \\ &= 48(50) + 96t - 50^2 - 2(50)t - 9t^2 \\ &= -9t^2 - 4t - 100. \end{aligned}$$

This function has a critical point at $t = -\frac{2}{9}$, which corresponds to the point $(50, -29)$. This point is not in the domain of f.

L_3 is the line segment connecting $(0,25)$ and $(50,25)$, and it can be parameterized by the equations $x(t) = t, y(t) = 25$ for $0 \le t \le 50$. We define $g(t) = f\big(x(t), y(t)\big)$:

$$\begin{aligned} g(t) &= f\big(x(t), y(t)\big) \\ &= f(t, 25) \\ &= 48t + 96(25) - t^2 - 2t(25) - 9(25^2) \\ &= -t^2 - 2t - 3225. \end{aligned}$$

This function has a critical point at $t = -1$, which corresponds to the point $(-1, 25)$, which is not in the domain.

L_4 is the line segment connecting $(0,0)$ to $(0,25)$, and it can be parameterized by the equations $x(t) = 0, y(t) = t$ for $0 \le t \le 25$. We define $g(t) = f\big(x(t), y(t)\big)$:

$$\begin{aligned} g(t) &= f\big(x(t), y(t)\big) \\ &= f(0, t) \\ &= 48(0) + 96t - (0)^2 - 2(0)t - 9t^2 \\ &= 96t - 9t^2. \end{aligned}$$

This function has a critical point at $t = \frac{16}{3}$, which corresponds to the point $\left(0, \frac{16}{3}\right)$, which is on the boundary of the domain. Calculating $f\left(0, \frac{16}{3}\right)$ gives 256.

We also need to find the values of $f(x, y)$ at the corners of its domain. These corners are located at $(0, 0)$, $(50, 0)$, $(50, 25)$ and $(0, 25)$:

$$\begin{aligned} f(0,0) &= 48(0) + 96(0) - (0)^2 - 2(0)(0) - 9(0)^2 = 0 \\ f(50,0) &= 48(50) + 96(0) - (50)^2 - 2(50)(0) - 9(0)^2 = -100 \\ f(50,25) &= 48(50) + 96(25) - (50)^2 - 2(50)(25) - 9(25)^2 = -5825 \\ f(0,25) &= 48(0) + 96(25) - (0)^2 - 2(0)(25) - 9(25)^2 = -3225. \end{aligned}$$

The maximum value is 648, which occurs at $(21, 3)$. Therefore, a maximum profit of $\$648,000$ is realized when $21,000$ golf balls are sold and 3 hours of advertising are purchased per month as shown in the following figure.

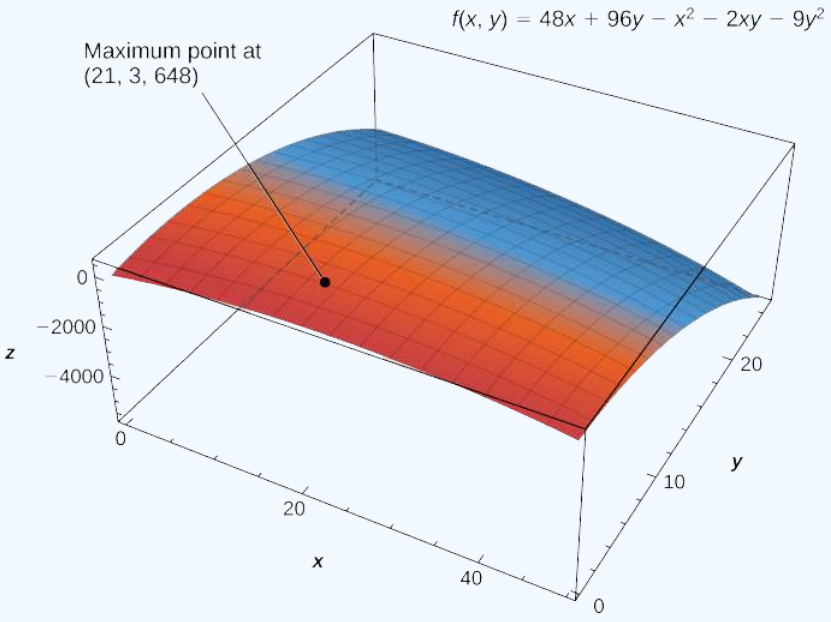


Figure 14.7.13: The profit function $f(x, y)$ has a maximum at $(21, 3, 648)$.

Key Concepts

- A critical point of the function $f(x, y)$ is any point (x_0, y_0) where either $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$, or at least one of $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ do not exist.
- A saddle point is a point (x_0, y_0) where $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$, but $f(x_0, y_0)$ is neither a maximum nor a minimum at that point.
- To find extrema of functions of two variables, first find the critical points, then calculate the discriminant and apply the second derivative test.

Key Equations

- **Discriminant**

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$$

Glossary

critical point of a function of two variables

the point (x_0, y_0) is called a critical point of $f(x, y)$ if one of the two following conditions holds:

1. $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$
2. At least one of $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ do not exist

discriminant

the discriminant of the function $f(x,y)$ is given by the formula $D = f_{xx}(x_0,y_0)f_{yy}(x_0,y_0) - (f_{xy}(x_0,y_0))^2$

saddle point

given the function $z = f(x,y)$, the point $(x_0, y_0, f(x_0,y_0))$ is a saddle point if both $f_x(x_0,y_0) = 0$ and $f_y(x_0,y_0) = 0$, but f does not have a local extremum at (x_0,y_0)

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14.8: Lagrange Multipliers

Learning Objectives

- Use the method of Lagrange multipliers to solve optimization problems with one constraint.
- Use the method of Lagrange multipliers to solve optimization problems with two constraints.

Solving optimization problems for functions of two or more variables can be similar to solving such problems in single-variable calculus. However, techniques for dealing with multiple variables allow us to solve more varied optimization problems for which we need to deal with additional conditions or constraints. In this section, we examine one of the more common and useful methods for solving optimization problems with constraints.

Lagrange Multipliers

In the previous section, an applied situation was explored involving maximizing a profit function, subject to certain **constraints.** In that example, the constraints involved a maximum number of golf balls that could be produced and sold in 1 month (x), and a maximum number of advertising hours that could be purchased per month (y). Suppose these were combined into a single budgetary constraint, such as $20x+4y \leq 216$, that took into account both the cost of producing the golf balls and the number of advertising hours purchased per month. The goal is still to maximize profit, but now there is a different type of constraint on the values of x and y. This constraint and the corresponding profit function

$$f(x,y) = 48x + 96y - x^2 - 2xy - 9y^2$$

is an example of an **optimization problem**, and the function $f(x,y)$ is called the **objective function**. A graph of various level curves of the function $f(x,y)$ follows.

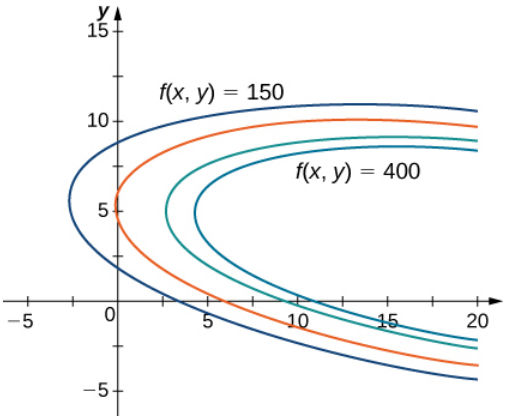


Figure 14.8.1: Graph showing level curves of the function $f(x,y) = 48x + 96y - x^2 - 2xy - 9y^2$ corresponding to $c = 150, 250, 350$, and 400.

In Figure 14.8.1, the value c represents different profit levels (i.e., values of the function f). As the value of c increases, the curve shifts to the right. Since our goal is to maximize profit, we want to choose a curve as far to the right as possible. If there were no restrictions on the number of golf balls the company could produce or the number of units of advertising available, then we could produce as many golf balls as we want, and advertise as much as we want, and there would be not be a maximum profit for the company. Unfortunately, we have a budgetary constraint that is modeled by the inequality $20x+4y \leq 216$. To see how this constraint interacts with the profit function, Figure 14.8.2 shows the graph of the line $20x+4y = 216$ superimposed on the previous graph.

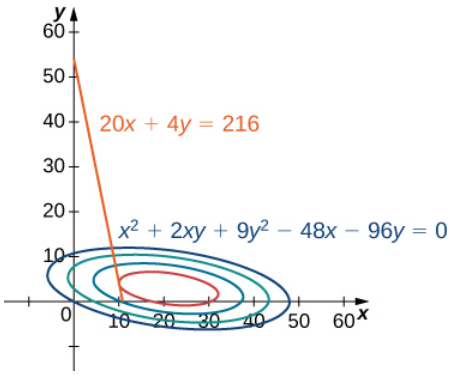


Figure 14.8.2: Graph of level curves of the function $f(x,y) = 48x + 96y - x^2 - 2xy - 9y^2$ corresponding to $c = 150, 250, 350,$ and 395. The red graph is the constraint function.

As mentioned previously, the maximum profit occurs when the level curve is as far to the right as possible. However, the level of production corresponding to this maximum profit must also satisfy the budgetary constraint, so the point at which this profit occurs must also lie on (or to the left of) the red line in Figure 14.8.2 Inspection of this graph reveals that this point exists where the line is tangent to the level curve of f. Trial and error reveals that this profit level seems to be around 395, when x and y are both just less than 5. We return to the solution of this problem later in this section. From a theoretical standpoint, at the point where the profit curve is tangent to the constraint line, the gradient of both of the functions evaluated at that point must point in the same (or opposite) direction. Recall that the gradient of a function of more than one variable is a vector. If two vectors point in the same (or opposite) directions, then one must be a constant multiple of the other. This idea is the basis of the **method of Lagrange multipliers.**

Theorem 14.8.1: Method of Lagrange Multipliers with One Constraint

Let f and g be functions of two variables with continuous partial derivatives at every point of some open set containing the smooth curve $g(x,y) = 0$. Suppose that f, when restricted to points on the curve $g(x,y) = 0$, has a local extremum at the point (x_0, y_0) and that $\vec{\nabla} g(x_0, y_0) \neq 0$. Then there is a number λ called a **Lagrange multiplier,** for which

$$\vec{\nabla} f(x_0, y_0) = \lambda \vec{\nabla} g(x_0, y_0).$$

Proof

Assume that a constrained extremum occurs at the point (x_0, y_0). Furthermore, we assume that the equation $g(x,y) = 0$ can be smoothly parameterized as

$$x = x(s) \text{ and } y = y(s)$$

where s is an arc length parameter with reference point (x_0, y_0) at $s = 0$. Therefore, the quantity $z = f(x(s), y(s))$ has a relative maximum or relative minimum at $s = 0$, and this implies that $\dfrac{dz}{ds} = 0$ at that point. From the chain rule,

$$\begin{aligned} \frac{dz}{ds} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} \\ &= \left(\frac{\partial f}{\partial x} \hat{\mathbf{i}} + \frac{\partial f}{\partial y} \hat{\mathbf{j}} \right) \cdot \left(\frac{\partial x}{\partial s} \hat{\mathbf{i}} + \frac{\partial y}{\partial s} \hat{\mathbf{j}} \right) \\ &= 0, \end{aligned}$$

where the derivatives are all evaluated at $s = 0$. However, the first factor in the dot product is the gradient of f, and the second factor is the unit tangent vector $\vec{\mathbf{T}}(0)$ to the constraint curve. Since the point (x_0, y_0) corresponds to $s = 0$, it follows from this equation that

$$\vec{\nabla} f(x_0, y_0) \cdot \vec{\mathbf{T}}(0) = 0,$$

which implies that the gradient is either the zero vector $\vec{\mathbf{0}}$ or it is normal to the constraint curve at a constrained relative extremum. However, the constraint curve $g(x,y)=0$ is a level curve for the function $g(x,y)$ so that if $\vec{\nabla} g(x_0,y_0)\neq 0$ then $\vec{\nabla} g(x_0,y_0)$ is normal to this curve at (x_0,y_0) It follows, then, that there is some scalar λ such that

$$\vec{\nabla} f(x_0,y_0)=\lambda\vec{\nabla} g(x_0,y_0)$$

□

To apply Theorem 14.8.1 to an optimization problem similar to that for the golf ball manufacturer, we need a problem-solving strategy.

Problem-Solving Strategy: Steps for Using Lagrange Multipliers

1. Determine the objective function $f(x,y)$ and the constraint function $g(x,y)$. Does the optimization problem involve maximizing or minimizing the objective function?
2. Set up a system of equations using the following template:

$$\vec{\nabla} f(x_0,y_0)=\lambda\vec{\nabla} g(x_0,y_0) \quad (14.8.1)$$
$$g(x_0,y_0)=0 \quad (14.8.2)$$

3. Solve for x_0 and y_0.
4. The largest of the values of f at the solutions found in step 3 maximizes f; the smallest of those values minimizes f.

Example 14.8.1: Using Lagrange Multipliers

Use the method of Lagrange multipliers to find the minimum value of $f(x,y)=x^2+4y^2-2x+8y$ subject to the constraint $x+2y=7$.

Solution

Let's follow the problem-solving strategy:

1. The objective function is $f(x,y)=x^2+4y^2-2x+8y$. To determine the constraint function, we must first subtract 7 from both sides of the constraint. This gives $x+2y-7=0$. The constraint function is equal to the left-hand side, so $g(x,y)=x+2y-7$. The problem asks us to solve for the minimum value of f, subject to the constraint (Figure 14.8.3).

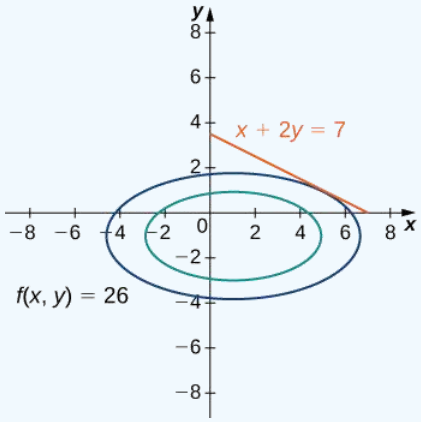


Figure 14.8.3: Graph of level curves of the function $f(x,y)=x^2+4y^2-2x+8y$ corresponding to $c=10$ and 26. The red graph is the constraint function.

2. We then must calculate the gradients of both f and g:

$$\vec{\nabla} f(x,y) = (2x-2)\,\hat{\mathbf{i}}+(8y+8)\,\hat{\mathbf{j}}$$
$$\vec{\nabla} g(x,y) = \hat{\mathbf{i}}+2\hat{\mathbf{j}}.$$

The equation $\vec{\nabla} f(x_0,y_0)=\lambda\vec{\nabla} g(x_0,y_0)$ becomes

$$(2x_0-2)\,\hat{\mathbf{i}}+(8y_0+8)\,\hat{\mathbf{j}}=\lambda\left(\hat{\mathbf{i}}+2\hat{\mathbf{j}}\right),$$

which can be rewritten as

$$(2x_0-2)\,\hat{\mathbf{i}}+(8y_0+8)\,\hat{\mathbf{j}}=\lambda\hat{\mathbf{i}}+2\lambda\hat{\mathbf{j}}.$$

Next, we set the coefficients of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ equal to each other:

$$\begin{aligned}2x_0-2&=\lambda\\8y_0+8&=2\lambda.\end{aligned}$$

The equation $g(x_0,y_0)=0$ becomes $x_0+2y_0-7=0$. Therefore, the system of equations that needs to be solved is

$$\begin{aligned}2x_0-2&=\lambda\\8y_0+8&=2\lambda\\x_0+2y_0-7&=0.\end{aligned}$$

3. This is a linear system of three equations in three variables. We start by solving the second equation for λ and substituting it into the first equation. This gives $\lambda=4y_0+4$, so substituting this into the first equation gives

$$2x_0-2=4y_0+4.$$

Solving this equation for x_0 gives $x_0=2y_0+3$. We then substitute this into the third equation:

$$\begin{aligned}(2y_0+3)+2y_0-7&=0\\4y_0-4&=0\\y_0&=1.\end{aligned}$$

Since $x_0=2y_0+3,$ this gives $x_0=5.$

4. Next, we evaluate $f(x,y)=x^2+4y^2-2x+8y$ at the point $(5,1)$,

$$f(5,1)=5^2+4(1)^2-2(5)+8(1)=27.$$

To ensure this corresponds to a minimum value on the constraint function, let's try some other points on the constraint from either side of the point $(5,1)$, such as the intercepts of $g(x,y)=0$, Which are $(7,0)$ and $(0,3.5)$.

We get $f(7,0)=35>27$ and $f(0,3.5)=77>27.$

So it appears that f has a relative minimum of 27 at $(5,1)$, subject to the given constraint.

? Exercise 14.8.1

Use the method of Lagrange multipliers to find the maximum value of

$$f(x,y)=9x^2+36xy-4y^2-18x-8y$$

subject to the constraint $3x+4y=32.$

Hint

Use the problem-solving strategy for the method of Lagrange multipliers.

Answer

Subject to the given constraint, f has a maximum value of 976 at the point $(8,2)$.

Let's now return to the problem posed at the beginning of the section.

Example 14.8.2: Golf Balls and Lagrange Multipliers

The golf ball manufacturer, Pro-T, has developed a profit model that depends on the number x of golf balls sold per month (measured in thousands), and the number of hours per month of advertising y, according to the function

$$z = f(x,y) = 48x + 96y - x^2 - 2xy - 9y^2,$$

where z is measured in thousands of dollars. The budgetary constraint function relating the cost of the production of thousands golf balls and advertising units is given by $20x + 4y = 216$. Find the values of x and y that maximize profit, and find the maximum profit.

Solution:

Again, we follow the problem-solving strategy:

1. The objective function is $f(x,y) = 48x + 96y - x^2 - 2xy - 9y^2$. To determine the constraint function, we first subtract 216 from both sides of the constraint, then divide both sides by 4, which gives $5x + y - 54 = 0$. The constraint function is equal to the left-hand side, so $g(x,y) = 5x + y - 54$. The problem asks us to solve for the maximum value of f, subject to this constraint.
2. So, we calculate the gradients of both f and g:

$$\begin{aligned} \vec{\nabla} f(x,y) &= (48 - 2x - 2y)\hat{\mathbf{i}} + (96 - 2x - 18y)\hat{\mathbf{j}} \\ \vec{\nabla} g(x,y) &= 5\hat{\mathbf{i}} + \hat{\mathbf{j}}. \end{aligned}$$

   The equation $\vec{\nabla} f(x_0,y_0) = \lambda \vec{\nabla} g(x_0,y_0)$ becomes

$$(48 - 2x_0 - 2y_0)\hat{\mathbf{i}} + (96 - 2x_0 - 18y_0)\hat{\mathbf{j}} = \lambda(5\hat{\mathbf{i}} + \hat{\mathbf{j}}),$$

   which can be rewritten as

$$(48 - 2x_0 - 2y_0)\hat{\mathbf{i}} + (96 - 2x_0 - 18y_0)\hat{\mathbf{j}} = \lambda 5\hat{\mathbf{i}} + \lambda\hat{\mathbf{j}}.$$

   We then set the coefficients of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ equal to each other:

$$\begin{aligned} 48 - 2x_0 - 2y_0 &= 5\lambda \\ 96 - 2x_0 - 18y_0 &= \lambda. \end{aligned}$$

   The equation $g(x_0,y_0) = 0$ becomes $5x_0 + y_0 - 54 = 0$. Therefore, the system of equations that needs to be solved is

$$\begin{aligned} 48 - 2x_0 - 2y_0 &= 5\lambda \\ 96 - 2x_0 - 18y_0 &= \lambda \\ 5x_0 + y_0 - 54 &= 0. \end{aligned}$$

3. We use the left-hand side of the second equation to replace λ in the first equation:

$$\begin{aligned} 48 - 2x_0 - 2y_0 &= 5(96 - 2x_0 - 18y_0) \\ 48 - 2x_0 - 2y_0 &= 480 - 10x_0 - 90y_0 \\ 8x_0 &= 432 - 88y_0 \\ x_0 &= 54 - 11y_0. \end{aligned}$$

   Then we substitute this into the third equation:

$$\begin{aligned} 5(54 - 11y_0) + y_0 - 54 &= 0 \\ 270 - 55y_0 + y_0 - 54 &= 0 \\ 216 - 54y_0 &= 0 \\ y_0 &= 4. \end{aligned}$$

   Since $x_0 = 54 - 11y_0$, this gives $x_0 = 10$.

4. We then substitute $(10,4)$ into $f(x,y)=48x+96y-x^2-2xy-9y^2,$ which gives

$$\begin{aligned} f(10,4) &= 48(10)+96(4)-(10)^2-2(10)(4)-9(4)^2 \\ &= 480+384-100-80-144 \\ &= 540. \end{aligned}$$

Therefore the maximum profit that can be attained, subject to budgetary constraints, is $\$540,000$ with a production level of $10,000$ golf balls and 4 hours of advertising bought per month. Let's check to make sure this truly is a maximum. The endpoints of the line that defines the constraint are $(10.8,0)$ and $(0,54)$ Let's evaluate f at both of these points:

$$\begin{aligned} f(10.8,0) &= 48(10.8)+96(0)-10.8^2-2(10.8)(0)-9(0^2) \\ &= 401.76 \\ f(0,54) &= 48(0)+96(54)-0^2-2(0)(54)-9(54^2) \\ &= -21,060. \end{aligned}$$

The second value represents a loss, since no golf balls are produced. Neither of these values exceed 540, so it seems that our extremum is a maximum value of f, subject to the given constraint.

? Exercise 14.8.2: Optimizing the Cobb-Douglas function

A company has determined that its production level is given by the Cobb-Douglas function $f(x,y)=2.5x^{0.45}y^{0.55}$ where x represents the total number of labor hours in 1 year and y represents the total capital input for the company. Suppose 1 unit of labor costs $\$40$ and 1 unit of capital costs $\$50$. Use the method of Lagrange multipliers to find the maximum value of $f(x,y)=2.5x^{0.45}y^{0.55}$ subject to a budgetary constraint of $\$500,000$ per year.

Hint

Use the problem-solving strategy for the method of Lagrange multipliers.

Answer

Subject to the given constraint, a maximum production level of 13890 occurs with 5625 labor hours and $\$5500$ of total capital input.

In the case of an objective function with three variables and a single constraint function, it is possible to use the method of Lagrange multipliers to solve an optimization problem as well. An example of an objective function with three variables could be the **Cobb-Douglas function** in Exercise 14.8.2 $f(x,y,z)=x^{0.2}y^{0.4}z^{0.4}$, where x represents the cost of labor, y represents capital input, and z represents the cost of advertising. The method is the same as for the method with a function of two variables; the equations to be solved are

$$\begin{aligned} \vec{\nabla} f(x,y,z) &= \lambda \vec{\nabla} g(x,y,z) \\ g(x,y,z) &= 0. \end{aligned}$$

✓ Example 14.8.3: Lagrange Multipliers with a Three-Variable objective function

Maximize the function $f(x,y,z)=x^2+y^2+z^2$ subject to the constraint $x+y+z=1$.

Solution

1. The objective function is $f(x,y,z)=x^2+y^2+z^2$. To determine the constraint function, we subtract 1 from each side of the constraint: $x+y+z-1=0$ which gives the constraint function as $g(x,y,z)=x+y+z-1$.

2. Next, we calculate $\vec{\nabla} f(x,y,z)$ and $\vec{\nabla} g(x,y,z)$:

$$\begin{aligned} \vec{\nabla} f(x,y,z) &= \langle 2x,2y,2z\rangle \\ \vec{\nabla} g(x,y,z) &= \langle 1,1,1\rangle. \end{aligned}$$

This leads to the equations

$$\begin{aligned}\langle 2x_0, 2y_0, 2z_0\rangle &= \lambda\langle 1,1,1\rangle \\ x_0 + y_0 + z_0 - 1 &= 0\end{aligned}$$

which can be rewritten in the following form:

$$\begin{aligned}2x_0 &= \lambda \\ 2y_0 &= \lambda \\ 2z_0 &= \lambda \\ x_0 + y_0 + z_0 - 1 &= 0.\end{aligned}$$

3. Since each of the first three equations has λ on the right-hand side, we know that $2x_0 = 2y_0 = 2z_0$ and all three variables are equal to each other. Substituting $y_0 = x_0$ and $z_0 = x_0$ into the last equation yields $3x_0 - 1 = 0$, so $x_0 = \frac{1}{3}$ and $y_0 = \frac{1}{3}$ and $z_0 = \frac{1}{3}$ which corresponds to a critical point on the constraint curve.

4. Then, we evaluate f at the point $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$:

$$f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{3}{9} = \frac{1}{3}$$

Therefore, a possible extremum of the function is $\frac{1}{3}$. To verify it is a minimum, choose other points that satisfy the constraint from either side of the point we obtained above and calculate f at those points. For example,

$$\begin{aligned}f(1,0,0) &= 1^2 + 0^2 + 0^2 = 1 \\ f(0,-2,3) &= 0^2 + (-2)^2 + 3^2 = 13.\end{aligned}$$

Both of these values are greater than $\frac{1}{3}$, leading us to believe the extremum is a minimum, subject to the given constraint.

? Exercise 14.8.3

Use the method of Lagrange multipliers to find the minimum value of the function

$$f(x,y,z) = x + y + z$$

subject to the constraint $x^2 + y^2 + z^2 = 1$.

Hint

Use the problem-solving strategy for the method of Lagrange multipliers with an objective function of three variables.

Answer

Evaluating f at both points we obtained, gives us,

$$\begin{aligned}f\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right) &= \frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{3} = \sqrt{3} \\ f\left(-\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right) &= -\frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{3} = -\sqrt{3}\end{aligned}$$

Since the constraint is continuous, we compare these values and conclude that f has a relative minimum of $-\sqrt{3}$ at the point $\left(-\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right)$, subject to the given constraint.

Problems with Two Constraints

The method of Lagrange multipliers can be applied to problems with more than one constraint. In this case the objective function, w is a function of three variables:

$$w = f(x, y, z)$$

and it is subject to two constraints:

$$g(x, y, z) = 0 \text{ and } h(x, y, z) = 0.$$

There are two Lagrange multipliers, λ_1 and λ_2, and the system of equations becomes

$$\begin{aligned} \vec{\nabla} f(x_0, y_0, z_0) &= \lambda_1 \vec{\nabla} g(x_0, y_0, z_0) + \lambda_2 \vec{\nabla} h(x_0, y_0, z_0) \\ g(x_0, y_0, z_0) &= 0 \\ h(x_0, y_0, z_0) &= 0 \end{aligned}$$

Example 14.8.4: Lagrange Multipliers with Two Constraints

Find the maximum and minimum values of the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

subject to the constraints $z^2 = x^2 + y^2$ and $x + y - z + 1 = 0$.

Solution

Let's follow the problem-solving strategy:

1. The objective function is $f(x, y, z) = x^2 + y^2 + z^2$. To determine the constraint functions, we first subtract z^2 from both sides of the first constraint, which gives $x^2 + y^2 - z^2 = 0$, so $g(x, y, z) = x^2 + y^2 - z^2$. The second constraint function is $h(x, y, z) = x + y - z + 1$.
2. We then calculate the gradients of f, g, and h:

$$\begin{aligned} \vec{\nabla} f(x, y, z) &= 2x\,\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} + 2z\hat{\mathbf{k}} \\ \vec{\nabla} g(x, y, z) &= 2x\,\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} - 2z\hat{\mathbf{k}} \\ \vec{\nabla} h(x, y, z) &= \hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}. \end{aligned}$$

The equation $\vec{\nabla} f(x_0, y_0, z_0) = \lambda_1 \vec{\nabla} g(x_0, y_0, z_0) + \lambda_2 \vec{\nabla} h(x_0, y_0, z_0)$ becomes

$$2x_0\,\hat{\mathbf{i}} + 2y_0\hat{\mathbf{j}} + 2z_0\hat{\mathbf{k}} = \lambda_1(2x_0\,\hat{\mathbf{i}} + 2y_0\hat{\mathbf{j}} - 2z_0\hat{\mathbf{k}}) + \lambda_2(\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}),$$

which can be rewritten as

$$2x_0\,\hat{\mathbf{i}} + 2y_0\hat{\mathbf{j}} + 2z_0\hat{\mathbf{k}} = (2\lambda_1 x_0 + \lambda_2)\hat{\mathbf{i}} + (2\lambda_1 y_0 + \lambda_2)\hat{\mathbf{j}} - (2\lambda_1 z_0 + \lambda_2)\hat{\mathbf{k}}.$$

Next, we set the coefficients of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ equal to each other:

$$\begin{aligned} 2x_0 &= 2\lambda_1 x_0 + \lambda_2 \\ 2y_0 &= 2\lambda_1 y_0 + \lambda_2 \\ 2z_0 &= -2\lambda_1 z_0 - \lambda_2. \end{aligned}$$

The two equations that arise from the constraints are $z_0^2 = x_0^2 + y_0^2$ and $x_0 + y_0 - z_0 + 1 = 0$. Combining these equations with the previous three equations gives

$$\begin{aligned} 2x_0 &= 2\lambda_1 x_0 + \lambda_2 \\ 2y_0 &= 2\lambda_1 y_0 + \lambda_2 \\ 2z_0 &= -2\lambda_1 z_0 - \lambda_2 \\ z_0^2 &= x_0^2 + y_0^2 \\ x_0 + y_0 - z_0 + 1 &= 0. \end{aligned}$$

3. The first three equations contain the variable λ_2. Solving the third equation for λ_2 and replacing into the first and second equations reduces the number of equations to four:

$$\begin{aligned} 2x_0 &= 2\lambda_1 x_0 - 2\lambda_1 z_0 - 2z_0 \\ 2y_0 &= 2\lambda_1 y_0 - 2\lambda_1 z_0 - 2z_0 \\ z_0^2 &= x_0^2 + y_0^2 \\ x_0 + y_0 - z_0 + 1 &= 0. \end{aligned}$$

Next, we solve the first and second equation for λ_1. The first equation gives $\lambda_1 = \dfrac{x_0 + z_0}{x_0 - z_0}$, the second equation gives $\lambda_1 = \dfrac{y_0 + z_0}{y_0 - z_0}$. We set the right-hand side of each equation equal to each other and cross-multiply:

$$\begin{aligned} \frac{x_0 + z_0}{x_0 - z_0} &= \frac{y_0 + z_0}{y_0 - z_0} \\ (x_0 + z_0)(y_0 - z_0) &= (x_0 - z_0)(y_0 + z_0) \\ x_0 y_0 - x_0 z_0 + y_0 z_0 - z_0^2 &= x_0 y_0 + x_0 z_0 - y_0 z_0 - z_0^2 \\ 2y_0 z_0 - 2x_0 z_0 &= 0 \\ 2z_0(y_0 - x_0) &= 0. \end{aligned}$$

Therefore, either $z_0 = 0$ or $y_0 = x_0$. If $z_0 = 0$, then the first constraint becomes $0 = x_0^2 + y_0^2$. The only real solution to this equation is $x_0 = 0$ and $y_0 = 0$, which gives the ordered triple $(0, 0, 0)$. This point does not satisfy the second constraint, so it is not a solution. Next, we consider $y_0 = x_0$, which reduces the number of equations to three:

$$\begin{aligned} y_0 &= x_0 \\ z_0^2 &= x_0^2 + y_0^2 \\ x_0 + y_0 - z_0 + 1 &= 0. \end{aligned}$$

We substitute the first equation into the second and third equations:

$$\begin{aligned} z_0^2 &= x_0^2 + x_0^2 \\ &= x_0 + x_0 - z_0 + 1 \qquad = 0. \end{aligned}$$

Then, we solve the second equation for z_0, which gives $z_0 = 2x_0 + 1$. We then substitute this into the first equation,

$$\begin{aligned} z_0^2 &= 2x_0^2 \\ (2x_0 + 1)^2 &= 2x_0^2 \\ 4x_0^2 + 4x_0 + 1 &= 2x_0^2 \\ 2x_0^2 + 4x_0 + 1 &= 0, \end{aligned}$$

and use the quadratic formula to solve for x_0:

$$x_0 = \frac{-4 \pm \sqrt{4^2 - 4(2)(1)}}{2(2)} = \frac{-4 \pm \sqrt{8}}{4} = \frac{-4 \pm 2\sqrt{2}}{4} = -1 \pm \frac{\sqrt{2}}{2}.$$

Recall $y_0 = x_0$, so this solves for y_0 as well. Then, $z_0 = 2x_0 + 1$, so

$$z_0 = 2x_0 + 1 = 2\left(-1 \pm \frac{\sqrt{2}}{2}\right) + 1 = -2 + 1 \pm \sqrt{2} = -1 \pm \sqrt{2}.$$

Therefore, there are two ordered triplet solutions:

$$\left(-1 + \frac{\sqrt{2}}{2}, -1 + \frac{\sqrt{2}}{2}, -1 + \sqrt{2}\right) \text{ and } \left(-1 - \frac{\sqrt{2}}{2}, -1 - \frac{\sqrt{2}}{2}, -1 - \sqrt{2}\right).$$

4. We substitute $\left(-1 + \dfrac{\sqrt{2}}{2}, -1 + \dfrac{\sqrt{2}}{2}, -1 + \sqrt{2}\right)$ into $f(x, y, z) = x^2 + y^2 + z^2$, which gives

$$f\left(-1+\frac{\sqrt{2}}{2},-1+\frac{\sqrt{2}}{2},-1+\sqrt{2}\right) = \left(-1+\frac{\sqrt{2}}{2}\right)^2+\left(-1+\frac{\sqrt{2}}{2}\right)^2+(-1+\sqrt{2})^2$$
$$= \left(1-\sqrt{2}+\frac{1}{2}\right)+\left(1-\sqrt{2}+\frac{1}{2}\right)+(1-2\sqrt{2}+2)$$
$$= 6-4\sqrt{2}.$$

Then, we substitute $\left(-1-\frac{\sqrt{2}}{2},-1+\frac{\sqrt{2}}{2},-1+\sqrt{2}\right)$ into $f(x,y,z)=x^2+y^2+z^2$, which gives

$$f\left(-1-\frac{\sqrt{2}}{2},-1+\frac{\sqrt{2}}{2},-1+\sqrt{2}\right) = \left(-1-\frac{\sqrt{2}}{2}\right)^2+\left(-1-\frac{\sqrt{2}}{2}\right)^2+(-1-\sqrt{2})^2$$
$$= \left(1+\sqrt{2}+\frac{1}{2}\right)+\left(1+\sqrt{2}+\frac{1}{2}\right)+(1+2\sqrt{2}+2)$$
$$= 6+4\sqrt{2}.$$

$6+4\sqrt{2}$ is the maximum value and $6-4\sqrt{2}$ is the minimum value of $f(x,y,z)$, subject to the given constraints.

? Exercise 14.8.4

Use the method of Lagrange multipliers to find the minimum value of the function

$$f(x,y,z)=x^2+y^2+z^2$$

subject to the constraints $2x+y+2z=9$ and $5x+5y+7z=29$.

Hint

Use the problem-solving strategy for the method of Lagrange multipliers with two constraints.

Answer

$f(2,1,2)=9$ is a minimum value of f, subject to the given constraints.

Key Concepts

- An objective function combined with one or more constraints is an example of an optimization problem.
- To solve optimization problems, we apply the method of Lagrange multipliers using a four-step problem-solving strategy.

Key Equations

- **Method of Lagrange multipliers, one constraint**

  $\vec{\nabla} f(x_0,y_0)=\lambda\vec{\nabla} g(x_0,y_0)$

  $g(x_0,y_0)=0$

- **Method of Lagrange multipliers, two constraints**

  $\vec{\nabla} f(x_0,y_0,z_0)=\lambda_1\vec{\nabla} g(x_0,y_0,z_0)+\lambda_2\vec{\nabla} h(x_0,y_0,z_0)$

  $g(x_0,y_0,z_0)=0$

  $h(x_0,y_0,z_0)=0$

Glossary

constraint

an inequality or equation involving one or more variables that is used in an optimization problem; the constraint enforces a limit on the possible solutions for the problem

Lagrange multiplier

the constant (or constants) used in the method of Lagrange multipliers; in the case of one constant, it is represented by the variable λ

method of Lagrange multipliers

a method of solving an optimization problem subject to one or more constraints

objective function

the function that is to be maximized or minimized in an optimization problem

optimization problem

calculation of a maximum or minimum value of a function of several variables, often using Lagrange multipliers

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CHAPTER OVERVIEW

15: Multiple Integrals

A general Calculus Textmap organized around the textbook

Calculus: Early Transcendentals

by James Stewart

I II III IV V VI VII VIII IX X XI XII XIII XIV XV XVI XVII

This Textmap is currently under construction... please be patient with us.

Template:HideTOC

Topic hierarchy

15.1: Double Integrals over Rectangles
15.2: Double Integrals over General Regions
15.3: Double Integrals in Polar Coordinates
15.4: Applications of Double Integrals
15.5: Surface Area
15.6: Triple Integrals
15.7: Triple Integrals in Cylindrical Coordinates
15.8: Triple Integrals in Spherical Coordinates
15.9: Change of Variables in Multiple Integrals

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15.1: Double Integrals over Rectangles

Learning Objectives

- Recognize when a function of two variables is integrable over a rectangular region.
- Recognize and use some of the properties of double integrals.
- Evaluate a double integral over a rectangular region by writing it as an iterated integral.
- Use a double integral to calculate the area of a region, volume under a surface, or average value of a function over a plane region.

In this section we investigate double integrals and show how we can use them to find the volume of a solid over a rectangular region in the xy-plane. Many of the properties of double integrals are similar to those we have already discussed for single integrals.

Volumes and Double Integrals

We begin by considering the space above a rectangular region R. Consider a continuous function $f(x, y) \geq 0$ of two variables defined on the closed rectangle R:

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \,|\, a \leq x \leq b,\ c \leq y \leq d\}$$

Here $[a, b] \times [c, d]$ denotes the Cartesian product of the two closed intervals $[a, b]$ and $[c, d]$. It consists of rectangular pairs (x, y) such that $a \leq x \leq b$ and $c \leq y \leq d$. The graph of f represents a surface above the xy-plane with equation $z = f(x, y)$ where z is the height of the surface at the point (x, y). Let S be the solid that lies above R and under the graph of f (Figure 15.1.1). The base of the solid is the rectangle R in the xy-plane. We want to find the volume V of the solid S.

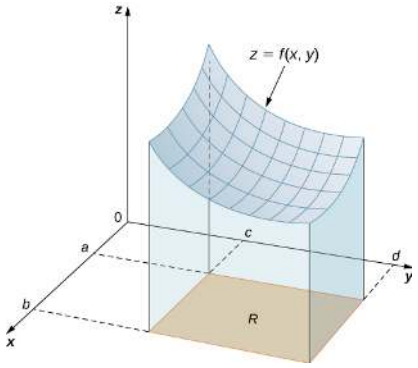


Figure 15.1.1: The graph of $f(x, y)$ over the rectangle R in the xy-plane is a curved surface.

We divide the region R into small rectangles R_{ij}, each with area ΔA and with sides Δx and Δy (Figure 15.1.2). We do this by dividing the interval $[a, b]$ into m subintervals and dividing the interval $[c, d]$ into n subintervals. Hence $\Delta x = \frac{b-a}{m}$, $\Delta y = \frac{d-c}{n}$, and $\Delta A = \Delta x \Delta y$.

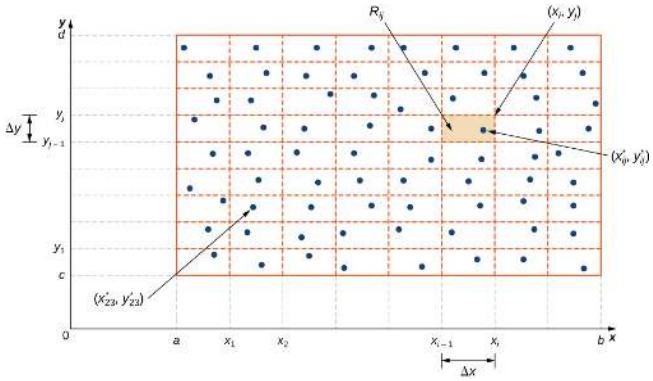


Figure 15.1.2: Rectangle R is divided into small rectangles R_{ij} each with area ΔA.

The volume of a thin rectangular box above R_{ij} is $f(x_{ij}^*, y_{ij}^*)\,\Delta A$, where (x_{ij}^*, y_{ij}^*) is an arbitrary sample point in each R_{ij} as shown in the following figure, $f(x_{ij}^*, y_{ij}^*)$ is the height of the corresponding thin rectangular box, and ΔA is the area of each rectangle R_{ij}.

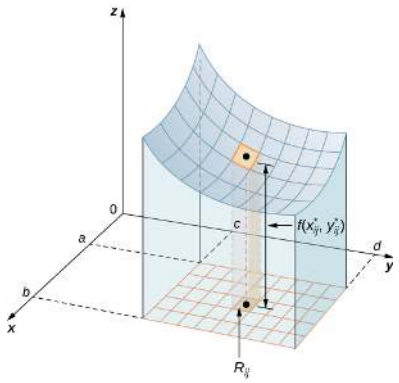


Figure 15.1.3: A thin rectangular box above R_{ij} with height $f(x_{ij}^*, y_{ij}^*)$.

Using the same idea for all the subrectangles, we obtain an approximate volume of the solid S as

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A.$$

This sum is known as a **double Riemann sum** and can be used to approximate the value of the volume of the solid. Here the double sum means that for each subrectangle we evaluate the function at the chosen point, multiply by the area of each rectangle, and then add all the results.

As we have seen in the single-variable case, we obtain a better approximation to the actual volume if m and n become larger.

$$V = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A$$

or

$$V = \lim_{\Delta x,\, \Delta y \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A.$$

Note that the sum approaches a limit in either case and the limit is the volume of the solid with the base R. Now we are ready to define the double integral.

Definition: Double Integral over a Rectangular Region R

The double integral of the function $f(x, y)$ over the rectangular region R in the xy-plane is defined as

$$\iint_R f(x, y) dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A.$$

If $f(x, y) \geq 0$, then the volume V of the solid S, which lies above R in the xy-plane and under the graph of f, is the double integral of the function $f(x, y)$ over the rectangle R. If the function is ever negative, then the double integral can be considered a "signed" volume in a manner similar to the way we defined net signed area in The Definite Integral.

Example 15.1.1: Setting up a Double Integral and Approximating It by Double Sums

Consider the function $z = f(x, y) = 3x^2 - y$ over the rectangular region $R = [0, 2] \times [0, 2]$ (Figure 15.1.4).

a. Set up a double integral for finding the value of the signed volume of the solid S that lies above R and "under" the graph of f.
b. Divide R into four squares with $m = n = 2$, and choose the sample point as the upper right corner point of each square (1,1),(2,1),(1,2), and (2,2) (Figure 15.1.4) to approximate the signed volume of the solid S that lies above R and "under" the graph of f.
c. Divide R into four squares with $m = n = 2$, and choose the sample point as the midpoint of each square: (1/2, 1/2), (3/2, 1/2), (1/2,3/2), and (3/2, 3/2) to approximate the signed volume.

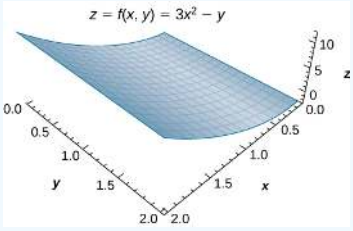


Figure 15.1.4: The function $z = f(x, y)$ graphed over the rectangular region $R = [0, 2] \times [0, 2]$.

Solution

a. As we can see, the function $z = f(x, y) = 3x^2 - y$ is above the plane. To find the signed volume of S, we need to divide the region R into small rectangles R_{ij}, each with area ΔA and with sides Δx and Δy, and choose (x_{ij}^*, y_{ij}^*) as sample points in each R_{ij}. Hence, a double integral is set up as

$$V = \iint_R (3x^2 - y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} [3(x_{ij}^*)^2 - y_{ij}^*]\Delta A.$$

b. Approximating the signed volume using a Riemann sum with $m = n = 2$ we have $\Delta A = \Delta x \Delta y = 1 \times 1 = 1$. Also, the sample points are (1, 1), (2, 1), (1, 2), and (2, 2) as shown in the following figure.

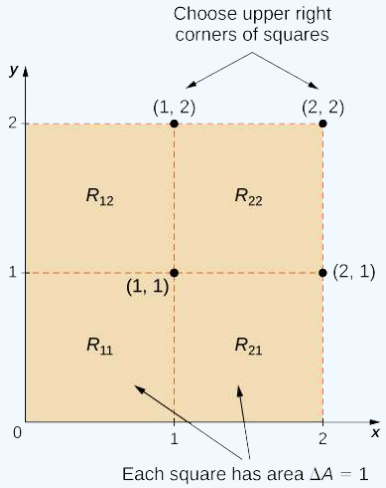


Figure 15.1.5: Subrectangles for the rectangular region $R = [0,2] \times [0,2]$.

Hence,

$$\begin{aligned} V &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\Delta A \\ &= \sum_{i=1}^{2} (f(x_{i1}^*, y_{i1}^*) + f(x_{i2}^*, y_{i2}^*))\Delta A \\ &= f(x_{11}^*, y_{11}^*)\Delta A + f(x_{21}^*, y_{21}^*)\Delta A + f(x_{12}^*, y_{12}^*)\Delta A + f(x_{22}^*, y_{22}^*)\Delta A \\ &= f(1,1)(1) + f(2,1)(1) + f(1,2)(1) + f(2,2)(1) \\ &= (3-1)(1) + (12-1)(1) + (3-2)(1) + (12-2)(1) \\ &= 2 + 11 + 1 + 10 = 24. \end{aligned}$$

c. Approximating the signed volume using a Riemann sum with $m = n = 2$ we have$\Delta A = \Delta x \Delta y = 1 \times 1 = 1$. In this case the sample points are (1/2, 1/2), (3/2, 1/2), (1/2, 3/2), and (3/2, 3/2).
Hence,

$$\begin{aligned} V &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\Delta A \\ &= f(x_{11}^*, y_{11}^*)\Delta A + f(x_{21}^*, y_{21}^*)\Delta A + f(x_{12}^*, y_{12}^*)\Delta A + f(x_{22}^*, y_{22}^*)\Delta A \\ &= f(1/2, 1/2)(1) + f(3/2, 1/2)(1) + f(1/2, 3/2)(1) + f(3/2, 3/2)(1) \\ &= \left(\frac{3}{4} - \frac{1}{4}\right)(1) + \left(\frac{27}{4} - \frac{1}{2}\right)(1) + \left(\frac{3}{4} - \frac{3}{2}\right)(1) + \left(\frac{27}{4} - \frac{3}{2}\right)(1) \\ &= \frac{2}{4} + \frac{25}{4} + \left(-\frac{3}{4}\right) + \frac{21}{4} = \frac{45}{4} = 11. \end{aligned}$$

Analysis

Notice that the approximate answers differ due to the choices of the sample points. In either case, we are introducing some error because we are using only a few sample points. Thus, we need to investigate how we can achieve an accurate answer.

? Exercise 15.1.1

Use the same function$z = f(x,y) = 3x^2 - y$ over the rectangular region $R = [0,2] \times [0,2]$.

Divide R into the same four squares with $m = n = 2$, and choose the sample points as the upper left corner point of each square (0,1), (1,1), (0,2), and (1,2) (Figure 15.1.5) to approximate the signed volume of the solid S that lies above R and "under" the graph of f.

Hint

Follow the steps of the previous example.

Answer

$$V \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\, \Delta A = 0$$

Note that we developed the concept of double integral using a rectangular region R. This concept can be extended to any general region. However, when a region is not rectangular, the subrectangles may not all fit perfectly into R, particularly if the base area is curved. We examine this situation in more detail in the next section, where we study regions that are not always rectangular and subrectangles may not fit perfectly in the region R. Also, the heights may not be exact if the surface $z = f(x, y)$ is curved. However, the errors on the sides and the height where the pieces may not fit perfectly within the solid S approach 0 as m and n approach infinity. Also, the double integral of the function $z = f(x, y)$ exists provided that the function f is not too discontinuous. If the function is bounded and continuous over R except on a finite number of smooth curves, then the double integral exists and we say that ff is integrable over R.

Since $\Delta A = \Delta x \Delta y = \Delta y \Delta x$, we can express dA as $dx\, dy$ or $dy\, dx$. This means that, when we are using rectangular coordinates, the double integral over a region R denoted by

$$\iint_R f(x, y)\, dA$$

can be written as

$$\iint_R f(x, y)\, dx\, dy$$

or

$$\iint_R f(x, y)\, dy\, dx.$$

Now let's list some of the properties that can be helpful to compute double integrals.

Properties of Double Integrals

The properties of double integrals are very helpful when computing them or otherwise working with them. We list here six properties of double integrals. Properties 1 and 2 are referred to as the linearity of the integral, property 3 is the additivity of the integral, property 4 is the monotonicity of the integral, and property 5 is used to find the bounds of the integral. Property 6 is used if $f(x, y)$ is a product of two functions $g(x)$ and $h(y)$.

Theorem: Properties of Double Integrals

Assume that the functions $f(x, y)$ and $g(x, y)$ are integrable over the rectangular region R; S and T are subregions of R; and assume that m and M are real numbers.

i. The sum $f(x, y) + g(x, y)$ is integrable and

$$\iint_R [f(x, y) + g(x, y)]\, dA = \iint_R f(x, y)\, dA + \iint_R g(x, y)\, dA.$$

ii. If c is a constant, then $cf(x, y)$ is integrable and

$$\iint_R cf(x, y)\, dA = c \iint_R f(x, y)\, dA.$$

iii. If $R = S \cup T$ and $S \cap T = \emptyset$ except an overlap on the boundaries, then

$$\iint_R f(x, y)\, dA = \iint_S f(x, y)\, dA + \iint_T f(x, y)\, dA.$$

iv. If $f(x, y) \geq g(x, y)$ for (x, y) in R, then

$$\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA.$$

v. If $m \leq f(x, y) \leq M$ and $A(R) =$ the area of R, then

$$m \cdot A(R) \leq \iint_R f(x, y)\, dA \leq M \cdot A(R).$$

vi. In the case where $f(x, y)$ can be factored as a product of a function $g(x)$ of x only and a function $h(y)$ of y only, then over the region $R = \{(x, y) \mid a \leq x \leq b,\ c \leq y \leq d\}$, the double integral can be written as

$$\iint_R f(x, y)\, dA = \left(\int_a^b g(x)\, dx \right) \left(\int_c^d h(y)\, dy \right).$$

These properties are used in the evaluation of double integrals, as we will see later. We will become skilled in using these properties once we become familiar with the computational tools of double integrals. So let's get to that now.

Iterated Integrals

So far, we have seen how to set up a double integral and how to obtain an approximate value for it. We can also imagine that evaluating double integrals by using the definition can be a very lengthy process if we choose larger values for m and n.Therefore, we need a practical and convenient technique for computing double integrals. In other words, we need to learn how to compute double integrals without employing the definition that uses limits and double sums.

The basic idea is that the evaluation becomes easier if we can break a double integral into single integrals by integrating first with respect to one variable and then with respect to the other. The key tool we need is called an iterated integral.

Definitions: Iterated Integrals

Assume a, b, c, and d are real numbers. We define an *iterated integral* for a function $f(x,y)$ over the rectangular region $R=[a,b]\times[c,d]$ as

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$$

or

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy.$$

The notation $\int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$ means that we integrate $f(x,y)$ with respect to y while holding x constant. Similarly, the notation $\int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$ means that we integrate $f(x,y)$ with respect to x while holding y constant. The fact that double integrals can be split into iterated integrals is expressed in Fubini's theorem. Think of this theorem as an essential tool for evaluating double integrals.

Theorem: Fubini's Theorem

Suppose that $f(x,y)$ is a function of two variables that is continuous over a rectangular region $R=\{(x,y)\in\mathbb{R}^2 \mid a\le x\le b,\ c\le y\le d\}$. Then we see from Figure 15.1.6 that the double integral of f over the region equals an iterated integral,

$$\iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

More generally, Fubini's theorem is true if f is bounded on R and f is discontinuous only on a finite number of continuous curves. In other words, f has to be integrable over R.

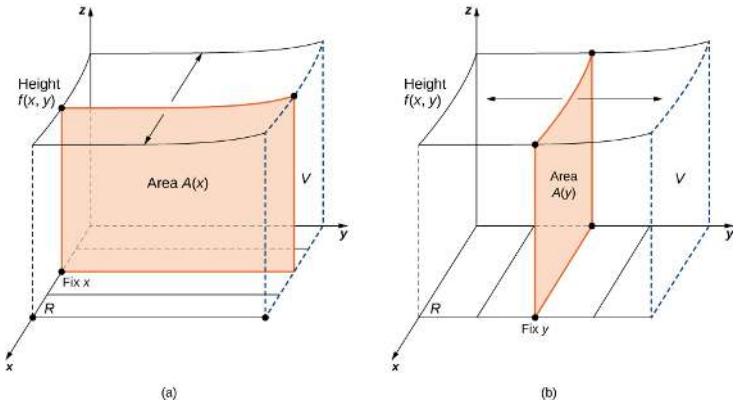


Figure 15.1.6: (a) Integrating first with respect to y and then with respect to x to find the area $A(x)$ and then the volume V; (b) integrating first with respect to x and then with respect to y to find the area $A(y)$ and then the volume V.

Example 15.1.2: Using Fubini's Theorem

Use Fubini's theorem to compute the double integral $\iint_R f(x,y)\,dA$ where $f(x,y)=x$ and $R=[0,2]\times[0,1]$.

Solution

Fubini's theorem offers an easier way to evaluate the double integral by the use of an iterated integral. Note how the boundary values of the region R become the upper and lower limits of integration.

$$\begin{aligned}\iint_R f(x,y)\,dA &= \iint_R f(x,y)\,dx\,dy \\ &= \int_{y=0}^{y=1}\int_{x=0}^{x=2} x\,dx\,dy \\ &= \int_{y=0}^{y=1}\left[\frac{x^2}{2}\Big|_{x=0}^{x=2}\right] dy \\ &= \int_{y=0}^{y=1} 2\,dy = 2y\Big|_{y=0}^{y=1} = 2\end{aligned}$$

The double integration in this example is simple enough to use Fubini's theorem directly, allowing us to convert a double integral into an iterated integral. Consequently, we are now ready to convert all double integrals to iterated integrals and demonstrate how the properties listed earlier can help us evaluate double integrals when the function $f(x,y)$ is more complex. Note that the order of integration can be changed (see Example 7).

Example 15.1.3: Illustrating Properties i and ii

Evaluate the double integral

$$\iint_R (xy - 3xy^2)\,dA, \text{ where } R = \{(x,y) \mid 0 \le x \le 2,\ 1 \le y \le 2\}.$$

Solution

This function has two pieces: one piece is xy and the other is $3xy^2$. Also, the second piece has a constant 3. Notice how we use properties i and ii to help evaluate the double integral.

$$\begin{aligned}\iint_R (xy-3xy^2)\,dA &= \iint_R xy\,dA + \iint_R (-3xy^2)\,dA && \text{Property i: Integral of a sum is the sum of the integrals.} \\ &= \int_{y=1}^{y=2}\int_{x=0}^{x=2} xy\,dx\,dy - \int_{y=1}^{y=2}\int_{x=0}^{x=2} 3xy^2\,dx\,dy && \text{Convert double integrals to iterated integrals.} \\ &= \int_{y=1}^{y=2}\left(\frac{x^2}{2}y\right)\Big|_{x=0}^{x=2} dy - 3\int_{y=1}^{y=2}\left(\frac{x^2}{2}y^2\right)\Big|_{x=0}^{x=2} dy && \text{Integrate with respect to } x\text{, holding } y \text{ constant.} \\ &= \int_{y=1}^{y=2} 2y\,dy - \int_{y=1}^{y=2} 6y^2 dy && \text{Property ii: Placing the constant before the integral.} \\ &= 2\int_1^2 y\,dy - 6\int_1^2 y^2\,dy && \text{Integrate with respect to y.} \\ &= 2\frac{y^2}{2}\Big|_1^2 - 6\frac{y^3}{3}\Big|_1^2 \\ &= y^2\Big|_1^2 - 2y^3\Big|_1^2 \\ &= (4-1) - 2(8-1) = 3 - 2(7) = 3 - 14 = -11.\end{aligned}$$

Example 15.1.4: Illustrating Property v.

Over the region $R = \{(x,y) \mid 1 \le x \le 3,\ 1 \le y \le 2\}$, we have $2 \le x^2 + y^2 \le 13$. Find a lower and an upper bound for the integral $\iint_R (x^2+y^2)\,dA$.

Solution

For a lower bound, integrate the constant function 2 over the region R. For an upper bound, integrate the constant function 13 over the region R.

$$\begin{aligned}\int_1^2\int_1^3 2\,dx\,dy &= \int_1^2 [2x\Big|_1^3]\,dy = \int_1^2 2(2)dy = 4y\Big|_1^2 = 4(2-1) = 4 \\ \int_1^2\int_1^3 13dx\,dy &= \int_1^2 [13x\Big|_1^3]\,dy = \int_1^2 13(2)\,dy = 26y\Big|_1^2 = 26(2-1) = 26.\end{aligned}$$

Hence, we obtain $4 \le \iint_R (x^2+y^2)\,dA \le 26$.

Example 15.1.5: Illustrating Property vi

Evaluate the integral $\iint_R e^y \cos x\,dA$ over the region $R = \{(x,y) \mid 0 \le x \le \frac{\pi}{2},\ 0 \le y \le 1\}$.

Solution

This is a great example for property vi because the function $f(x,y)$ is clearly the product of two single-variable functions e^y and $\cos x$. Thus we can split the integral into two parts and then integrate each one as a single-variable integration problem.

$$\begin{aligned}\iint_R e^y \cos x \, dA &= \int_0^1 \int_0^{\pi/2} e^y \cos x \, dx \, dy \\ &= \left(\int_0^1 e^y dy\right)\left(\int_0^{\pi/2} \cos x \, dx\right) \\ &= (e^y\Big|_0^1)(\sin x\Big|_0^{\pi/2}) \\ &= e-1.\end{aligned}$$

? Exercise 15.1.2

a. Use the properties of the double integral and Fubini's theorem to evaluate the integral

$$\int_0^1 \int_{-1}^3 (3-x+4y)\, dy\, dx.$$

b. Show that $0 \le \iint_R \sin \pi x \, \cos \pi y \, dA \le \frac{1}{32}$ where $R = \left(0, \frac{1}{4}\right)\left(\frac{1}{4}, \frac{1}{2}\right)$.

Hint

Use properties i. and ii. and evaluate the iterated integral, and then use property v.

Answer

a. 26

b. Answers may vary.

As we mentioned before, when we are using rectangular coordinates, the double integral over a region R denoted by $\iint_R f(x,y)\, dA$ can be written as $\iint_R f(x,y)\, dx\, dy$ or $\iint_R f(x,y)\, dy\, dx$. The next example shows that the results are the same regardless of which order of integration we choose.

✓ Example 15.1.6: Evaluating an Iterated Integral in Two Ways

Let's return to the function $f(x,y) = 3x^2 - y$ from Example 1, this time over the rectangular region $R = [0,2] \times [0,3]$. Use Fubini's theorem to evaluate $\iint_R f(x,y)\, dA$ in two different ways:

a. First integrate with respect to y and then with respect to x;
b. First integrate with respect to x and then with respect to y.

Solution

Figure 15.1.6 shows how the calculation works in two different ways.

a. First integrate with respect to y and then integrate with respect to x:

$$\begin{aligned}\iint_R f(x,y)\, dA &= \int_{x=0}^{x=2} \int_{y=0}^{y=3} (3x^2 - y)\, dy\, dx \\ &= \int_{x=0}^{x=2} \left(\int_{y=0}^{y=3} (3x^2 - y)\, dy\right) dx = \int_{x=0}^{x=2} \left[3x^2 y - \frac{y^2}{2}\Big|_{y=0}^{y=3}\right] dx \\ &= \int_{x=0}^{x=2} \left(9x^2 - \frac{9}{2}\right) dx = 3x^3 - \frac{9}{2}x\Big|_{x=0}^{x=2} = 15.\end{aligned}$$

b. First integrate with respect to x and then integrate with respect to y:

$$\begin{aligned}\iint_R f(x,y)\, dA &= \int_{y=0}^{y=3} \int_{x=0}^{x=2} (3x^2 - y)\, dx\, dy \\ &= \int_{y=0}^{y=3} \left(\int_{x=0}^{x=2} (3x^2 - y)\, dx\right) dy \\ &= \int_{y=0}^{y=3} \left[x^3 - xy\Big|_{x=0}^{x=2}\right] dy \\ &= \int_{y=0}^{y=3} (8-2y)\, dy = 8y - y^2\Big|_{y=0}^{y=3} = 15.\end{aligned}$$

Analysis

With either order of integration, the double integral gives us an answer of 15. We might wish to interpret this answer as a volume in cubic units of the solid S below the function $f(x,y)=3x^2-y$ over the region $R=[0,2]\times[0,3]$. However, remember that the interpretation of a double integral as a (non-signed) volume works only when the integrand f is a nonnegative function over the base region R.

? Exercise 15.1.3

Evaluate

$$\int_{y=-3}^{y=2}\int_{x=3}^{x=5}(2-3x^2+y^2)\,dx\,dy.$$

Hint

Use Fubini's theorem.

Answer

$-\frac{1340}{3}$

In the next example we see that it can actually be beneficial to switch the order of integration to make the computation easier. We will come back to this idea several times in this chapter.

✓ Example 15.1.7: Switching the Order of Integration

Consider the double integral $\iint_R x\,\sin(xy)\,dA$ over the region $R=\{(x,y)\,|\,0\le x\le\pi,\ 1\le y\le 2\}$ (Figure 15.1.7).

a. Express the double integral in two different ways.
b. Analyze whether evaluating the double integral in one way is easier than the other and why.
c. Evaluate the integral.

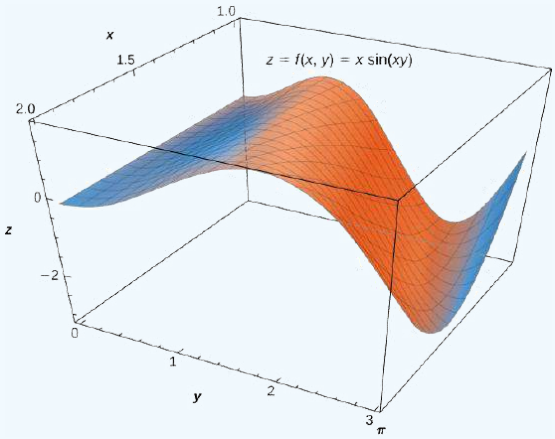


Figure 15.1.7: The function $z=f(x,y)=x\,\sin(xy)$ over the rectangular region $R=[0,\pi]\times[1,2]$.

a. We can express $\iint_R x\,\sin(xy)\,dA$ in the following two ways: first by integrating with respect to y and then with respect to x; second by integrating with respect to x and then with respect to y.

$$\iint_R x\,\sin(xy)\,dA=\int_{x=0}^{x=\pi}\int_{y=1}^{y=2}x\,\sin(xy)\,dy\,dx$$

Integrate first with respect to y.

$$=\int_{y=1}^{y=2}\int_{x=0}^{x=\pi}x\,\sin(xy)\,dx\,dy$$

Integrate first with respect to x.

b. If we want to integrate with respect to y first and then integrate with respect to x, we see that we can use the substitution $u=xy$, which gives $du=x\,dy$. Hence the inner integral is simply $\int\sin u\,du$ and we can change the limits to be functions of x,

$$\iint_R x\,\sin(xy)\,dA=\int_{x=0}^{x=\pi}\int_{y=1}^{y=2}x\,\sin(xy)\,dy\,dx=\int_{x=0}^{x=\pi}\left[\int_{u=x}^{u=2x}\sin(u)\,du\right]dx.$$

However, integrating with respect to x first and then integrating with respect to y requires integration by parts for the inner integral, with $u=x$ and $dv=\sin(xy)dx$

Then $du = dx$ and $v = -\frac{\cos(xy)}{y}$, so

$$\iint_R x\sin(xy)\,dA = \int_{y=1}^{y=2}\int_{x=0}^{x=\pi} x\sin(xy)\,dx\,dy = \int_{y=1}^{y=2}\left[-\frac{x\,\cos(xy)}{y}\Big|_{x=0}^{x=\pi} + \frac{1}{y}\int_{x=0}^{x=\pi}\cos(xy)\,dx\right]dy.$$

Since the evaluation is getting complicated, we will only do the computation that is easier to do, which is clearly the first method.

c. Evaluate the double integral using the easier way.

$$\begin{aligned}\iint_R x\,\sin(xy)\,dA &= \int_{x=0}^{x=\pi}\int_{y=1}^{y=2} x\,\sin(xy)\,dy\,dx \\ &= \int_{x=0}^{x=\pi}\left[\int_{u=x}^{u=2x}\sin(u)\,du\right]dx = \int_{x=0}^{x=\pi}\left[-\cos u\Big|_{u=x}^{u=2x}\right]dx \\ &= \int_{x=0}^{x=\pi}(-\cos 2x + \cos x)\,dx \\ &= \left(-\frac{1}{2}\sin 2x + \sin x\right)\Big|_{x=0}^{x=\pi} = 0.\end{aligned}$$

? Exercise 15.1.4

Evaluate the integral $\iint_R xe^{xy}\,dA$ where $R = [0,1]\times[0,\ln 5]$.

Hint

Integrate with respect to y first.

Answer

$\frac{4-\ln 5}{\ln 5}$

Applications of Double Integrals

Double integrals are very useful for finding the area of a region bounded by curves of functions. We describe this situation in more detail in the next section. However, if the region is a rectangular shape, we can find its area by integrating the constant function $f(x,y) = 1$ over the region R.

Definition: Area of a Region R

The area of the region R is given by

$$A(R) = \iint_R 1\,dA.$$

This definition makes sense because using $f(x,y) = 1$ and evaluating the integral make it a product of length and width. Let's check this formula with an example and see how this works.

Example 15.1.8: Finding Area Using a Double Integral

Find the area of the region $R = \{(x,y)\,|\,0 \le x \le 3,\ 0 \le y \le 2\}$ by using a double integral, that is, by integrating 1 over the region R.

Solution

The region is rectangular with length 3 and width 2, so we know that the area is 6. We get the same answer when we use a double integral:

$$A(R) = \int_0^2\int_0^3 1\,dx\,dy = \int_0^2 [x]_0^3\,dy = \int_0^2 3dy = 3\int_0^2 dy = 3y\Big|_0^2 = 3(2) = 6\text{ units}^2.$$

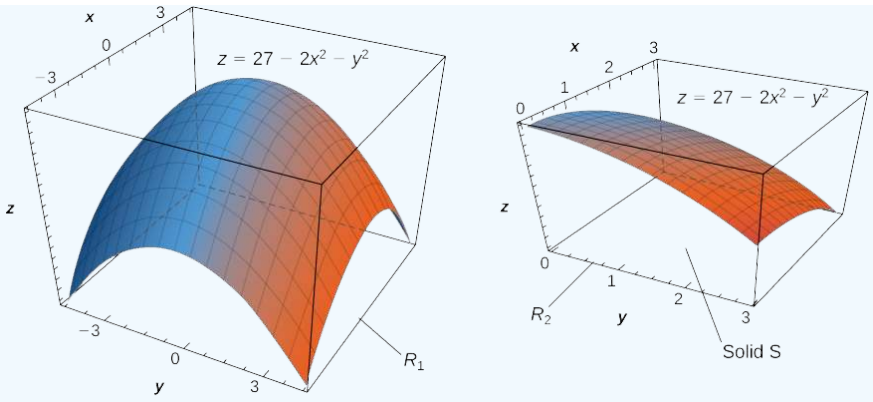
We have already seen how double integrals can be used to find the volume of a solid bounded above by a function $f(x,y) \ge 0$ over a region R provided $f(x,y) \ge 0$ for all (x,y) in R. Here is another example to illustrate this concept.

Example 15.1.9: Volume of an Elliptic Paraboloid

Find the volume V of the solid S that is bounded by the elliptic paraboloid $2x^2 + y^2 + z = 27$, the planes $x = 3$ and $y = 3$, and the three coordinate planes.

Solution

First notice the graph of the surface $z = 27 - 2x^2 - y^2$ in Figure 15.1.8(a) and above the square region $R_1 = [-3,3]\times[-3,3]$. However, we need the volume of the solid bounded by the elliptic paraboloid $2x^2 + y^2 + z = 27$, the planes $x = 3$ and $y = 3$, and the three coordinate planes.



(a) (b)

Figure 15.1.8: (a) The surface $z = 27 - 2x^2 - y^2$ above the square region $R1 = [-3,3] \times [-3,3]$. (b) The solid S lies under the surface $z = 27 - 2x^2 - y^2$ above the square region $R1 = [0,3] \times [0,3]$.

Now let's look at the graph of the surface in Figure 15.1.8(b). We determine the volume V by evaluating the double integral over R_2:

$$\begin{aligned} V &= \iint_R z\,dA = \iint_R (27 - 2x^2 - y^2)\,dA \\ &= \int_{y=0}^{y=3} \int_{x=0}^{x=3} (27 - 2x^2 - y^2)\,dx\,dy && \text{Convert to literal integral.} \\ &= \int_{y=0}^{y=3} \left[27x - \frac{2}{3}x^3 - y^2 x\right]\Big|_{x=0}^{x=3} dy && \text{Integrate with respect to } x. \\ &= \int_{y=0}^{y=3} (63 - 3y^2)dy = 63y - y^3\Big|_{y=0}^{y=3} = 162. \end{aligned}$$

? Exercise 15.1.5

Find the volume of the solid bounded above by the graph of $f(x,y) = xy\sin(x^2 y)$ and below by the xy-plane on the rectangular region $R = [0,1] \times [0,\pi]$.

Hint

Graph the function, set up the integral, and use an iterated integral.

Answer

$\frac{\pi}{2}$

Recall that we defined the average value of a function of one variable on an interval $[a,b]$ as

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Similarly, we can define the average value of a function of two variables over a region R. The main difference is that we divide by an area instead of the width of an interval.

Definition: Average Value of a Function

The **average value of a function** of two variables over a region R is

$$F_{ave} = \frac{1}{\text{Area of } R}\iint_R f(x,y)\,dx\,dy.$$

In the next example we find the average value of a function over a rectangular region. This is a good example of obtaining useful information for an integration by making individual measurements over a grid, instead of trying to find an algebraic expression for a function.

Example 15.1.10: Calculating Average Storm Rainfall

The weather map in Figure 15.1.9 shows an unusually moist storm system associated with the remnants of Hurricane Karl, which dumped 4–8 inches (100–200 mm) of rain in some parts of the Midwest on September 22–23, 2010. The area of rainfall measured 300 miles east to west and 250 miles north to south. Estimate the average rainfall over the entire area in those two days.

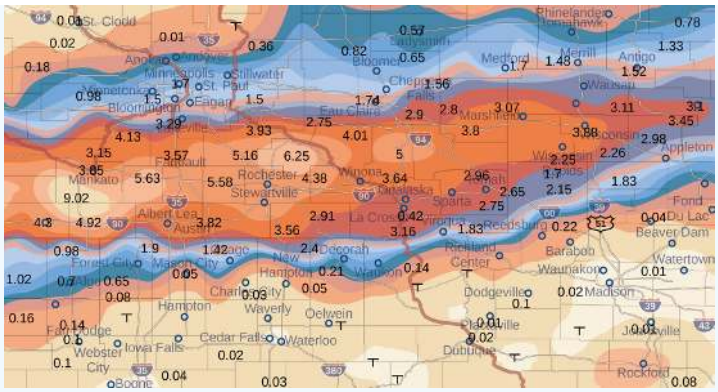


Figure 15.1.9: Effects of Hurricane Karl, which dumped 4–8 inches (100–200 mm) of rain in some parts of southwest Wisconsin, southern Minnesota, and southeast South Dakota over a span of 300 miles east to west and 250 miles north to south.

Solution

Place the origin at the southwest corner of the map so that all the values can be considered as being in the first quadrant and hence all are positive. Now divide the entire map into six rectangles ($m = 2$ and $n = 3$), as shown in Figure 15.1.9 Assume $f(x, y)$ denotes the storm rainfall in inches at a point approximately x miles to the east of the origin and y miles to the north of the origin. Let R represent the entire area of $250 \times 300 = 75000$ square miles. Then the area of each subrectangle is

$$\Delta A = \frac{1}{6}(75000) = 12500.$$

Assume $(x_{ij}*, y_{ij}*)$ are approximately the midpoints of each subrectangle R_{ij}. Note the color-coded region at each of these points, and estimate the rainfall. The rainfall at each of these points can be estimated as:

- At (x_{11}, y_{11}), the rainfall is 0.08.
- At (x_{12}, y_{12}), the rainfall is 0.08.
- At (x_{13}, y_{13}), the rainfall is 0.01.
- At (x_{21}, y_{21}), the rainfall is 1.70.
- At (x_{22}, y_{22}), the rainfall is 1.74.
- At (x_{23}, y_{23}), the rainfall is 3.00.

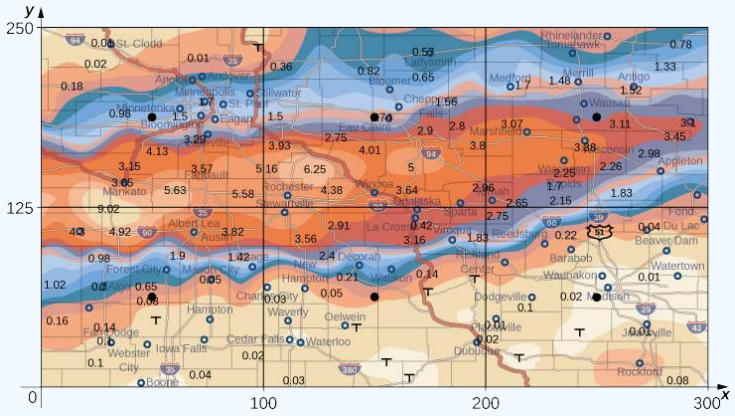


Figure 15.1.10: Storm rainfall with rectangular axes and showing the midpoints of each subrectangle.

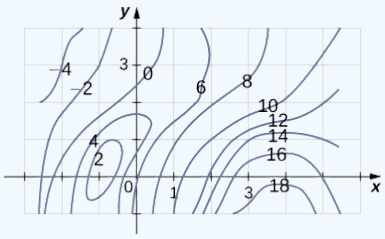
According to our definition, the average storm rainfall in the entire area during those two days was

$$
\begin{aligned}
f_{ave} = \frac{1}{Area\ R}\iint_R f(x,y)\,dx\,dy &= \frac{1}{75000}\iint_R f(x,y)\,dx\,dy \\
&\approx \frac{1}{75000}\sum_{i=1}^{3}\sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\Delta A \\
&= \frac{1}{75000}\Big[f(x_{11}^*, y_{11}^*)\Delta A + f(x_{12}^*, y_{12}^*)\Delta A + f(x_{13}^*, y_{13}^*)\Delta A + f(x_{21}^*, y_{21}^*)\Delta A + f(x_{22}^*, y_{22}^*)\Delta A + f(x_{23}^*, y_{23}^*)\Delta A\Big] \\
&\approx \frac{1}{75000}\Big[0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00\Big]\Delta A \\
&= \frac{1}{75000}\Big[0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00\Big]12500 \\
&= \frac{1}{6}\Big[0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00\Big] \\
&\approx 1.10 \text{ in.}
\end{aligned}
$$

During September 22–23, 2010 this area had an average storm rainfall of approximately 1.10 inches.

? Exercise 15.1.6

A contour map is shown for a function $f(x,y)$ on the rectangle $R = [-3,6]\times[-1,4]$.



a. Use the midpoint rule with $m = 3$ and $n = 2$ to estimate the value of $\iint_R f(x,y)\,dA$.

b. Estimate the average value of the function $f(x,y)$.

Hint

Divide the region into six rectangles, and use the contour lines to estimate the values for $f(x,y)$.

Answer

Answers to both parts a. and b. may vary.

Key Concepts

- We can use a double Riemann sum to approximate the volume of a solid bounded above by a function of two variables over a rectangular region. By taking the limit, this becomes a double integral representing the volume of the solid.
- Properties of double integral are useful to simplify computation and find bounds on their values.
- We can use Fubini's theorem to write and evaluate a double integral as an iterated integral.
- Double integrals are used to calculate the area of a region, the volume under a surface, and the average value of a function of two variables over a rectangular region.

Key Equations

- $$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i j*, y_i j*)\,\Delta A$$
- $$\int_a^b\int_c^d f(x,y)\,dx\,dy = \int_a^b\left[\int_c^d f(x,y)\,dy\right]dx$$

  or

  $$\int_c^d\int_a^b f(x,y)\,dx\,dy = \int_c^d\left[\int_a^b f(x,y)\,dx\right]dy$$
- $$f_{ave} = \frac{1}{\text{Area of } R}\iint_R f(x,y)\,dx\,dy$$

Glossary

double integral

of the function $f(x,y)$ over the region R in the xy-plane is defined as the limit of a double Riemann sum,

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A.$$

double Riemann sum

of the function $f(x,y)$ over a rectangular region R is

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A,$$

where R is divided into smaller subrectangles R_{ij} and (x_{ij}^*, y_{ij}^*) is an arbitrary point in R_{ij}

Fubini's theorem

if $f(x,y)$ is a function of two variables that is continuous over a rectangular region $R = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ c \le y \le d\}$, then the double integral of f over the region equals an iterated integral,

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dx\,dy = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

iterated integral

for a function $f(x,y)$ over the region R is

a. $\displaystyle\int_a^b \int_c^d f(x,y)\,dx\,dy = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx,$

b. $\displaystyle\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy,$

where a, b, c, and d are any real numbers and $R = [a,b] \times [c,d]$

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15.2: Double Integrals over General Regions

Learning Objectives

- Recognize when a function of two variables is integrable over a general region.
- Evaluate a double integral by computing an iterated integral over a region bounded by two vertical lines and two functions of x, or two horizontal lines and two functions of y.
- Simplify the calculation of an iterated integral by changing the order of integration.
- Use double integrals to calculate the volume of a region between two surfaces or the area of a plane region.
- Solve problems involving double improper integrals.

Previously, we studied the concept of double integrals and examined the tools needed to compute them. We learned techniques and properties to integrate functions of two variables over rectangular regions. We also discussed several applications, such as finding the volume bounded above by a function over a rectangular region, finding area by integration, and calculating the average value of a function of two variables.

In this section we consider double integrals of functions defined over a general bounded region D on the plane. Most of the previous results hold in this situation as well, but some techniques need to be extended to cover this more general case.

General Regions of Integration

An example of a general bounded region D on a plane is shown in Figure 15.2.1. Since D is bounded on the plane, there must exist a rectangular region R on the same plane that encloses the region D that is, a rectangular region R exists such that D is a subset of $R (D \subseteq R)$.

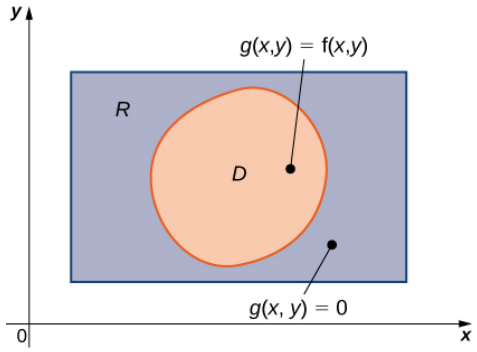


Figure 15.2.1: For a region D that is a subset of R, we can define a function $g(x, y)$ to equal $f(x, y)$ at every point in D and 0 at every point of R not in D.

Suppose $z = f(x, y)$ is defined on a general planar bounded region D as in Figure 15.2.1. In order to develop double integrals of f over D we extend the definition of the function to include all points on the rectangular region R and then use the concepts and tools from the preceding section. But how do we extend the definition of f to include all the points on R? We do this by defining a new function $g(x, y)$ on R as follows:

$$g(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \text{ is in } D \\ 0, & \text{if } (x,y) \text{ is in } R \text{ but not in } D \end{cases}$$

Note that we might have some technical difficulties if the boundary of D is complicated. So we assume the boundary to be a piecewise smooth and continuous simple closed curve. Also, since all the results developed in the section on Double Integrals over Rectangular Regions used an integrable function $f(x, y)$ we must be careful about $g(x, y)$ and verify that $g(x, y)$ is an integrable function over the rectangular region R. This happens as long as the region D is bounded by simple closed curves. For now we will concentrate on the descriptions of the regions rather than the function and extend our theory appropriately for integration.

We consider two types of planar bounded regions.

Definition: Type I and Type II regions

A region D in the (x, y)-plane is of Type I if it lies between two vertical lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$. That is (Figure 15.2.2),

$$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}.$$

A region D in the xy-plane is of Type II if it lies between two horizontal lines and the graphs of two continuous functions $h_1(y)$ and $h_2(y)$. That is (Figure 15.2.3),

$$D = \{(x, y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}.$$

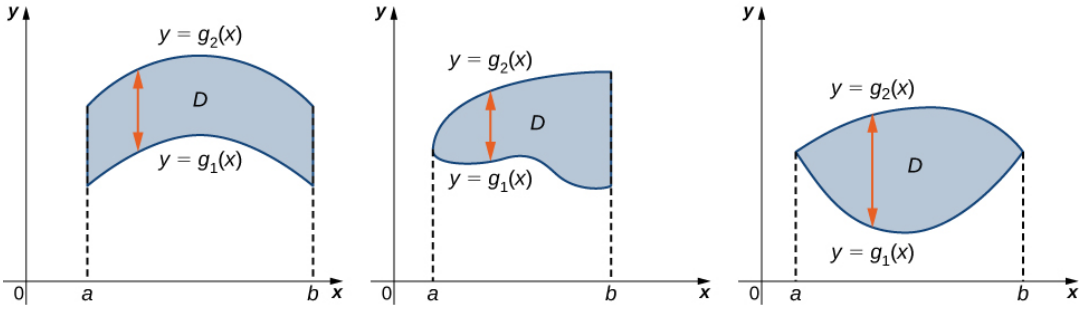


Figure 15.2.2. A Type I region lies between two vertical lines and the graphs of two functions of x.

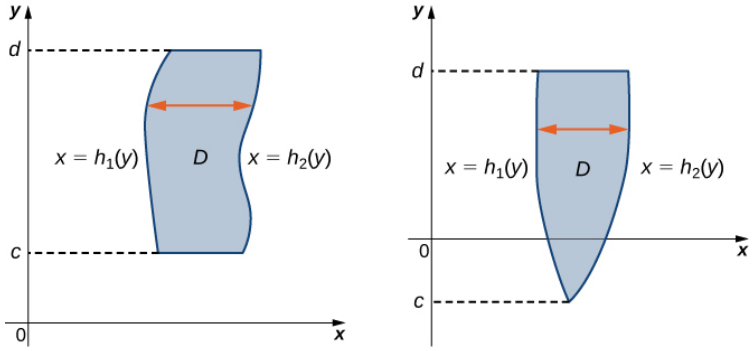


Figure 15.2.3: A Type II region lies between two horizontal lines and the graphs of two functions of y.

Example 15.2.1: Describing a Region as Type I and Also as Type II

Consider the region in the first quadrant between the functions $y = \sqrt{x}$ and $y = x^3$ (Figure 15.2.4). Describe the region first as Type I and then as Type II.

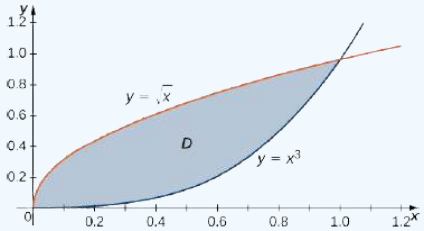


Figure 15.2.4: Region D can be described as Type I or as Type II.

When describing a region as Type I, we need to identify the function that lies above the region and the function that lies below the region. Here, region D is bounded above by $y = \sqrt{x}$ and below by $y = x^3$ in the interval for x in $[0, 1]$. Hence, as Type I, D is described as the set $\{(x, y) \mid 0 \le x \le 1,\ x^3 \le y \le \sqrt[3]{x}\}$.

However, when describing a region as Type II, we need to identify the function that lies on the left of the region and the function that lies on the right of the region. Here, the region D is bounded on the left by $x = y^2$ and on the right by $x = \sqrt[3]{y}$ in the interval for y in $[0, 1]$. Hence, as Type II, D is described as the set $\{(x, y) \mid 0 \le y \le 1,\ y^2 \le x \le \sqrt[3]{y}\}$.

Exercise 15.2.1

Consider the region in the first quadrant between the functions $y = 2x$ and $y = x^2$. Describe the region first as Type I and then as Type II.

Hint

Graph the functions, and draw vertical and horizontal lines.

Answer

Type I and Type II are expressed as $\{(x,y)\,|\,0\le x\le 2,\ x^2\le y\le 2x\}$ and $\{(x,y)|\,0\le y\le 4,\ \frac{1}{2}y\le x\le\sqrt{y}\}$, respectively.

Double Integrals over Non-rectangular Regions

To develop the concept and tools for evaluation of a double integral over a general, nonrectangular region, we need to first understand the region and be able to express it as Type I or Type II or a combination of both. Without understanding the regions, we will not be able to decide the limits of integrations in double integrals. As a first step, let us look at the following theorem.

Theorem: Double Integrals over Nonrectangular Regions

Suppose $g(x,y)$ is the extension to the rectangle R of the function $f(x,y)$ defined on the regions D and R as shown in Figure 15.2.1 inside R. Then $g(x,y)$ is integrable and we define the double integral of $f(x,y)$ over D by

$$\iint_D f(x,y)\,dA=\iint_R g(x,y)\,dA.$$

The right-hand side of this equation is what we have seen before, so this theorem is reasonable because R is a rectangle and $\iint_R g(x,y)dA$ has been discussed in the preceding section. Also, the equality works because the values of $g(x,y)$ are 0 for any point (x,y) that lies outside D and hence these points do not add anything to the integral. However, it is important that the rectangle R contains the region D.

As a matter of fact, if the region D is bounded by smooth curves on a plane and we are able to describe it as Type I or Type II or a mix of both, then we can use the following theorem and not have to find a rectangle R containing the region.

Theorem: Fubini's Theorem (Strong Form)

For a function $f(x,y)$ that is continuous on a region D of Type I, we have

$$\iint_D f(x,y)\,dA=\iint_D f(x,y)\,dy\,dx=\int_a^b\left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right]dx.$$

Similarly, for a function $f(x,y)$ that is continuous on a region D of Type II, we have

$$\iint_D f(x,y)\,dA=\iint_D f(x,y)\,dx\,dy=\int_c^d\left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right]dy.$$

The integral in each of these expressions is an iterated integral, similar to those we have seen before. Notice that, in the inner integral in the first expression, we integrate $f(x,y)$ with x being held constant and the limits of integration being $g_1(x)$ and $g_2(x)$. In the inner integral in the second expression, we integrate $f(x,y)$ with y being held constant and the limits of integration are $h_1(x)$ and $h_2(x)$.

Example 15.2.2: Evaluating an Iterated Integral over a Type I Region

Evaluate the integral $\iint_D x^2e^{xy}\,dA$ where D is shown in Figure 15.2.5.

Solution

First construct the region as a Type I region (Figure 15.2.5). Here $D=\{(x,y)\,|\,0\le x\le 2,\ \frac{1}{2}x\le y\le 1\}$. Then we have

$$\iint_D x^2e^{xy}\,dA=\int_{x=0}^{x=2}\int_{y=1/2x}^{y=1} x^2e^{xy}\,dy\,dx.$$

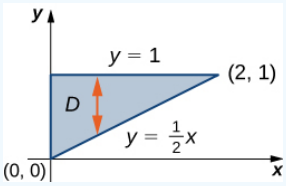


Figure 15.2.5: We can express region D as a Type I region and integrate from $y=\frac{1}{2}x$ to $y=1$ between the lines $x=0$ and $x=2$.

Therefore, we have

$$\int_{x=0}^{x=2}\int_{y=\frac{1}{2}x}^{y=1} x^2e^{xy}\,dy\,dx = \int_{x=0}^{x=2}\left[\int_{y=\frac{1}{2}x}^{y=1} x^2e^{xy}\,dy\right]dx \qquad \text{Iterated integral for a Type I region.}$$

$$= \int_{x=0}^{x=2}\left[x^2\frac{e^{xy}}{x}\right]\Big|_{y=1/2x}^{y=1}dx \qquad \text{Integrate with respect to } y$$

$$= \int_{x=0}^{x=2}\left[xe^x - xe^{x^2/2}\right]dx \qquad \text{Integrate with respect to } x$$

$$= \left[xe^x - e^x - e^{\frac{1}{2}x^2}\right]\Big|_{x=0}^{x=2} = 2.$$

In Example 15.2.2 we could have looked at the region in another way, such as $D = \{(x,y)\,|\,0 \le y \le 1,\ 0 \le x \le 2y\}$ (Figure 15.2.6).

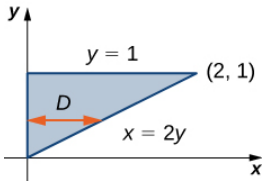


Figure 15.2.6.

This is a Type II region and the integral would then look like

$$\iint_D x^2e^{xy}\,dA = \int_{y=0}^{y=1}\int_{x=0}^{x=2y} x^2e^{xy}\,dx\,dy.$$

However, if we integrate first with respect to x this integral is lengthy to compute because we have to use integration by parts twice.

Example 15.2.3: Evaluating an Iterated Integral over a Type II Region

Evaluate the integral

$$\iint_D (3x^2 + y^2)\,dA$$

where $D = \{(x,y)\,|\,-2 \le y \le 3,\ y^2 - 3 \le x \le y + 3\}$.

Solution

Notice that D can be seen as either a Type I or a Type II region, as shown in Figure 15.2.7. However, in this case describing D as Type I is more complicated than describing it as Type II. Therefore, we use D as a Type II region for the integration.

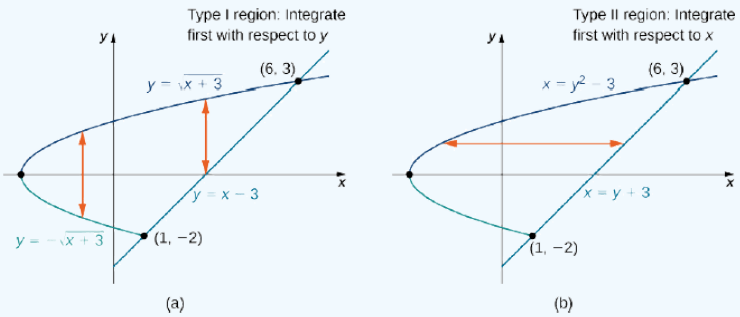


Figure 15.2.7: The region D in this example can be either (a) Type I or (b) Type II.

Choosing this order of integration, we have

$$\iint_D (3x^2+y^2)\,dA = \int_{y=-2}^{y=3}\int_{x=y^2-3}^{x=y+3}(3x^2+y^2)\,dx\,dy$$

$$= \int_{y=-2}^{y=3}(x^3+xy^2)\Big|_{y^2-3}^{y+3}\,dy \qquad \text{Iterated integral, Type II region}$$

$$= \int_{y=-2}^{y=3}\left((y+3)^3+(y+3)y^2-(y^2-3)^3-(y^2-3)y^2\right)\,dy$$

$$= \int_{-2}^{3}(54+27y-12y^2+2y^3+8y^4-y^6)\,dy \qquad \text{Integrate with respect to } x.$$

$$= \left[54y+\frac{27y^2}{2}-4y^3+\frac{y^4}{2}+\frac{8y^5}{5}-\frac{y^7}{7}\right]_{-2}^{3}$$

$$= \frac{2375}{7}.$$

? Exercise 15.2.2

Sketch the region D and evaluate the iterated integral

$$\iint_D xy\,dy\,dx$$

where D is the region bounded by the curves $y=\cos\,x$ and $y=\sin\,x$ in the interval $[-3\pi/4,\ \pi/4]$

Hint

Express D as a Type I region, and integrate with respect to y first.

Answer

$\frac{\pi}{4}$

Recall from Double Integrals over Rectangular Regions the properties of double integrals. As we have seen from the examples here, all these properties are also valid for a function defined on a non-rectangular bounded region on a plane. In particular, property 3 states:

If $R=S\cup T$ and $S\cap T=0$ except at their boundaries, then

$$\iint_R f(x,y)\,dA = \iint_S f(x,y)\,dA + \iint_T f(x,y)\,dA.$$

Similarly, we have the following property of double integrals over a non-rectangular bounded region on a plane.

Theorem: Decomposing Regions into Smaller Regions

Suppose the region D can be expressed as $D=D_1\cup D_2$ where D_1 and D_2 do not overlap except at their boundaries. Then

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA.$$

This theorem is particularly useful for non-rectangular regions because it allows us to split a region into a union of regions of Type I and Type II. Then we can compute the double integral on each piece in a convenient way, as in the next example.

Example 15.2.4: Decomposing Regions

Express the region D shown in Figure 15.2.8as a union of regions of Type I or Type II, and evaluate the integral

$$\iint_D (2x+5y)\,dA.$$

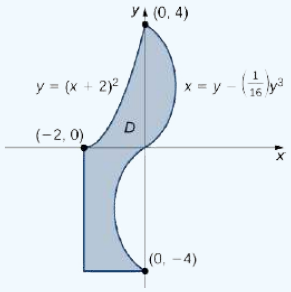


Figure 15.2.8: This region can be decomposed into a union of three regions of Type I or Type II.

Solution

The region D is not easy to decompose into any one type; it is actually a combination of different types. So we can write it as a union of three regions D_1, D_2, and D_3 where, $D_1 = \{(x,y) \mid -2 \le x \le 0,\ 0 \le y \le (x+2)^2\}$, $D_2 = \{(x,y) \mid 0 \le y \le 4,\ 0 \le x \le (y - \frac{1}{16}y^3)\}$, and $D_3 = \{(x,y) \mid -4 \le y \le 0,\ -2 \le x \le (y - \frac{1}{16}y^3)\}$. These regions are illustrated more clearly in Figure 15.2.9

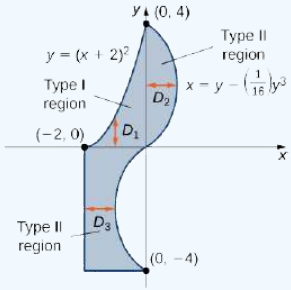


Figure 15.2.9: Breaking the region into three subregions makes it easier to set up the integration.

Here D_1 is Type I and D_2 and D_3 are both of Type II. Hence,

$$\begin{aligned}
\iint_D (2x+5y)\,dA &= \iint_{D_1} (2x+5y)\,dA + \iint_{D_2} (2x+5y)\,dA + \iint_{D_3} (2x+5y)\,dA \\
&= \int_{x=-2}^{x=0} \int_{y=0}^{y=(x+2)^2} (2x+5y)\,dy\,dx + \int_{y=0}^{y=4} \int_{x=0}^{x=y-(1/16)y^3} (2x+5y)\,dx\,dy + \int_{y=-4}^{y=0} \int_{x=-2}^{x=y-(1/16)y^3} (2x+5y)\,dx\,dy \\
&= \int_{x=-2}^{x=0} \left[\frac{1}{2}(2+x)^2(20+24x+5x^2)\right] dx + \int_{y=0}^{y=4} \left[\frac{1}{256}y^6 - \frac{7}{16}y^4 + 6y^2\right] dy + \\
&\qquad \int_{y=-4}^{y=0} \left[\frac{1}{256}y^6 - \frac{7}{16}y^4 + 6y^2 + 10y - 4\right] dy \\
&= \frac{40}{3} + \frac{1664}{35} - \frac{1696}{35} = \frac{1304}{105}.
\end{aligned}$$

Now we could redo this example using a union of two Type II regions (see Exercise 15.2.4below).

? Exercise 15.2.3

Consider the region bounded by the curves $y = \ln x$ and $y = e^x$ in the interval $[1, 2]$. Decompose the region into smaller regions of Type II.

Hint

Sketch the region, and split it into three regions to set it up.

Answer

$$\{(x,y) \mid 0 \le y \le 1,\ 1 \le x \le e^y\} \cup \{(x,y) \mid 1 \le y \le e,\ 1 \le x \le 2\} \cup \{(x,y) \mid e \le y \le e^2,\ \ln y \le x \le 2\}$$

? Exercise 15.2.4

Redo Example 15.2.4using a union of two Type II regions.

Hint

$$\{(x,y)\,|\,0\le y\le 4,\ \sqrt{y}-2\le x\le \left(y-\frac{1}{16}y^3\right)\}\cup\{(x,y)\,|\,-4\le y\le 0,\ -2\le x\le \left(y-\frac{1}{16}y^3\right)\}$$

Answer

Same as in the example shown.

Changing the Order of Integration

As we have already seen when we evaluate an iterated integral, sometimes one order of integration leads to a computation that is significantly simpler than the other order of integration. Sometimes the order of integration does not matter, but it is important to learn to recognize when a change in order will simplify our work.

✓ Example 15.2.5: Changing the Order of Integration

Reverse the order of integration in the iterated integral

$$\int_{x=0}^{x=\sqrt{2}}\int_{y=0}^{y=2-x^2} xe^{x^2}\,dy\,dx.$$

Then evaluate the new iterated integral.

Solution

The region as presented is of Type I. To reverse the order of integration, we must first express the region as Type II. Refer to Figure 15.2.10

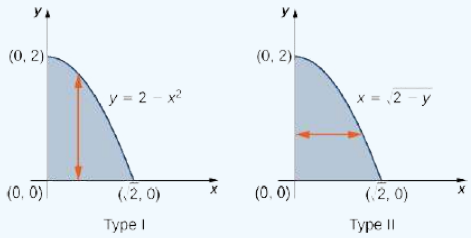


Figure 15.2.10: Converting a region from Type I to Type II.

We can see from the limits of integration that the region is bounded above by $y=2-x^2$ and below by $y=0$ where x is in the interval $[0,\sqrt{2}]$. By reversing the order, we have the region bounded on the left by $x=0$ and on the right by $x=\sqrt{2-y}$ where y is in the interval $[0,2]$. We solved $y=2-x^2$ in terms of x to obtain $x=\sqrt{2-y}$.

Hence

$$\int_0^{\sqrt{2}}\int_0^{2-x^2} xe^{x^2}\,dy\,dx = \int_0^2\int_0^{\sqrt{2-y}} xe^{x^2}\,dx\,dy \qquad \text{Reverse the order of integration then use substitution.}$$

$$= \int_0^2 \left[\frac{1}{2}e^{x^2}\Big|_0^{\sqrt{2-y}}\right]dy = \int_0^2 \frac{1}{2}(e^{2-y}-1)\,dy$$

$$= -\frac{1}{2}(e^{2-y}+y)\Big|_0^2 = \frac{1}{2}(e^2-3).$$

✓ Example 15.2.6: Evaluating an Iterated Integral by Reversing the Order of Integration

Consider the iterated integral

$$\iint_R f(x,y)\,dx\,dy$$

where $z = f(x,y) = x - 2y$ over a triangular region R that has sides on $x = 0,\ y = 0$, and the line $x + y = 1$. Sketch the region, and then evaluate the iterated integral by

a. integrating first with respect to y and then
b. integrating first with respect to x.

Solution

A sketch of the region appears in Figure 15.2.11

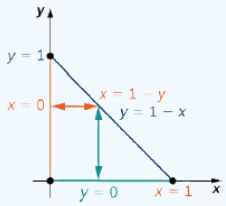


Figure 15.2.11: A triangular region R for integrating in two ways.

We can complete this integration in two different ways.

a. One way to look at it is by first integrating y from $y = 0$ to $y = 1 - x$ vertically and then integrating x from $x = 0$ to $x = 1$:

$$\iint_R f(x,y)\,dx\,dy = \int_{x=0}^{x=1}\int_{y=0}^{y=1-x}(x-2y)\,dy\,dx = \int_{x=0}^{x=1}\left(xy-y^2\right)\Big|_{y=0}^{y=1-x}dx$$

$$= \int_{x=0}^{x=1}\left[x(1-x)-(1-x)^2\right]\,dx = \int_{x=0}^{x=1}\left[-1+3x-2x^2\right]dx = \left[-x+\frac{3}{2}x^2-\frac{2}{3}x^3\right]\Big|_{x=0}^{x=1} = -\frac{1}{6}.$$

b. The other way to do this problem is by first integrating x from $x = 0$ to $x = 1 - y$ horizontally and then integrating y from $y = 0$ to $y = 1$:

$$\iint_R f(x,y)\,dx\,dy = \int_{y=0}^{y=1}\int_{x=0}^{x=1-y}(x-2y)\,dx\,dy = \int_{y=0}^{y=1}\left[\frac{1}{2}x^2-2xy\right]_{x=0}^{x=1-y}dy$$

$$= \int_{y=0}^{y=1}\left[\frac{1}{2}(1-y)^2-2y(1-y)\right]dy = \int_{y=0}^{y=1}\left[\frac{1}{2}-3y+\frac{5}{2}y^2\right]\,dy$$

$$= \left[\frac{1}{2}y-\frac{3}{2}y^2+\frac{5}{6}y^3\right]\Big|_{y=0}^{y=1} = -\frac{1}{6}.$$

? Exercise 15.2.5

Evaluate the iterated integral $\iint_D (x^2+y^2)\,dA$ over the region D in the first quadrant between the functions $y = 2x$ and $y = x^2$.

Evaluate the iterated integral by integrating first with respect to y and then integrating first with resect to x.

Hint

Sketch the region and follow Example 15.2.6

Answer

$\frac{216}{35}$

Calculating Volumes, Areas, and Average Values

We can use double integrals over general regions to compute volumes, areas, and average values. The methods are the same as those in Double Integrals over Rectangular Regions, but without the restriction to a rectangular region, we can now solve a wider variety of problems.

✓ Example 15.2.7: Finding the Volume of a Tetrahedron

Find the volume of the solid bounded by the planes $x = 0,\ y = 0,\ z = 0$, and $2x + 3y + z = 6$.

Solution

The solid is a tetrahedron with the base on the xy-plane and a height $z = 6 - 2x - 3y$. The base is the region D bounded by the lines, $x = 0$, $y = 0$ and $2x + 3y = 6$ where $z = 0$ (Figure 15.2.12). Note that we can consider the region D as Type I or as Type II, and we can integrate in both ways.

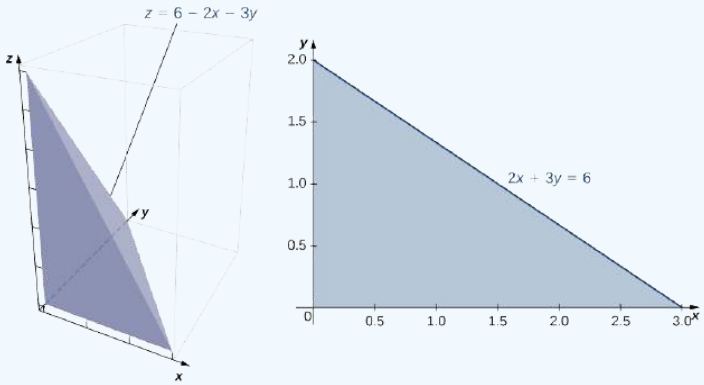


Figure 15.2.12: A tetrahedron consisting of the three coordinate planes and the plane $z = 6 - 2x - 3y$, with the base bound by $x = 0,\ y = 0$, and $2x + 3y = 6$.

First, consider D as a Type I region, and hence $D = \left\{(x, y) \mid 0 \le x \le 3,\ 0 \le y \le 2 - \frac{2}{3}x\right\}$.

Therefore, the volume is

$$V = \int_{x=0}^{x=3} \int_{y=0}^{y=2-(2x/3)} (6 - 2x - 3y)\, dy\, dx = \int_{x=0}^{x=3} \left[\left(6y - 2xy - \frac{3}{2}y^2\right)\Big|_{y=0}^{y=2-(2x/3)} \right] dx$$

$$= \int_{x=0}^{x=3} \left[\frac{2}{3}(x-3)^2\right] dx = 6.$$

Now consider D as a Type II region, so $D = \left\{(x, y) \mid 0 \le y \le 2,\ 0 \le x \le 3 - \frac{3}{2}y\right\}$. In this calculation, the volume is

$$V = \int_{y=0}^{y=2} \int_{x=0}^{x=3-(3y/2)} (6 - 2x - 3y)\, dx\, dy = \int_{y=0}^{y=2} \left[(6x - x^2 - 3xy)\Big|_{x=0}^{x=3-(3y/2)} \right] dy$$

$$= \int_{y=0}^{y=2} \left[\frac{9}{4}(y-2)^2\right] dy = 6.$$

Therefore, the volume is 6 cubic units.

? Exercise 15.2.6

Find the volume of the solid bounded above by $f(x, y) = 10 - 2x + y$ over the region enclosed by the curves $y = 0$ and $y = e^x$ where x is in the interval $[0, 1]$.

Hint

Sketch the region, and describe it as Type I.

Answer

$\frac{e^2}{4} + 10e - \frac{49}{4}$ cubic units

Finding the area of a rectangular region is easy, but finding the area of a non-rectangular region is not so easy. As we have seen, we can use double integrals to find a rectangular area. As a matter of fact, this comes in very handy for finding the area of a general non-rectangular region, as stated in the next definition.

Definition: Double Integrals

The area of a plane-bounded region D is defined as the double integral

$$\iint_D 1\, dA.$$

We have already seen how to find areas in terms of single integration. Here we are seeing another way of finding areas by using double integrals, which can be very useful, as we will see in the later sections of this chapter.

Example 15.2.8: Finding the Area of a Region

Find the area of the region bounded below by the curve $y = x^2$ and above by the line $y = 2x$ in the first quadrant (Figure 15.2.13).

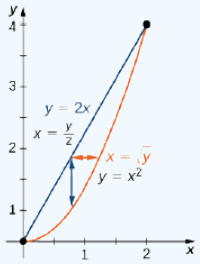


Figure 15.2.13: The region bounded by $y = x^2$ and $y = 2x$.

Solution

We just have to integrate the constant function $f(x, y) = 1$ over the region. Thus, the area A of the bounded region is $\int_{x=0}^{x=2} \int_{y=x^2}^{y=2x} dy\, dx$ or $\int_{y=0}^{y=4} \int_{x=y/2}^{x=\sqrt{y}} dx\, dy$:

$$\begin{aligned} A &= \iint_D 1\, dx\, dy \\ &= \int_{x=0}^{x=2} \int_{y=x^2}^{y=2x} 1\, dy\, dx \\ &= \int_{x=0}^{x=2} \left(y\Big|_{y=x^2}^{y=2x} \right) dx \\ &= \int_{x=0}^{x=2} (2x - x^2)\, dx \\ &= \left(x^2 - \frac{x^3}{3} \right) \Big|_0^2 = \frac{4}{3}. \end{aligned}$$

Exercise 15.2.7

Find the area of a region bounded above by the curve $y = x^3$ and below by $y = 0$ over the interval $[0, 3]$.

Hint

Sketch the region.

Answer

$\frac{81}{4}$ square units

We can also use a double integral to find the average value of a function over a general region. The definition is a direct extension of the earlier formula.

Definition: The Average Value of a Function

If $f(x, y)$ is integrable over a plane-bounded region D with positive area $A(D)$, then the average value of the function is

$$f_{ave} = \frac{1}{A(D)} \iint_D f(x, y)\, dA.$$

Note that the area is $A(D) = \iint_D 1\, dA$.

Example 15.2.9: Finding an Average Value

Find the average value of the function $f(x,y)=7xy^2$ on the region bounded by the line $x=y$ and the curve $x=\sqrt{y}$ (Figure 15.2.14).

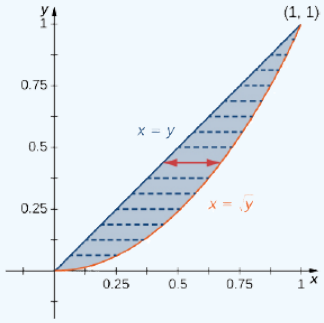


Figure 15.2.14: The region bounded by $x=y$ and $x=\sqrt{y}$.

Solution

First find the area $A(D)$ where the region D is given by the figure. We have

$$A(D)=\iint_D 1\,dA=\int_{y=0}^{y=1}\int_{x=y}^{x=\sqrt{y}} 1\,dx\,dy=\int_{y=0}^{y=1}\left[x\Big|_{x=y}^{x=\sqrt{y}}\right]dy=\int_{y=0}^{y=1}(\sqrt{y}-y)\,dy=\frac{2}{3}y^{2/3}-\frac{y^2}{2}\Big|_0^1=\frac{1}{6}$$

Then the average value of the given function over this region is

$$f_{ave}=\frac{1}{A(D)}\iint_D f(x,y)\,dA=\frac{1}{A(D)}\int_{y=0}^{y=1}\int_{x=y}^{x=\sqrt{y}}7xy^2\,dx\,dy=\frac{1}{1/6}\int_{y=0}^{y=1}\left[\frac{7}{2}x^2y^2\Big|_{x=y}^{x=\sqrt{y}}\right]dy$$

$$=6\int_{y=0}^{y=1}\left[\frac{7}{2}y^2(y-y^2)\right]dy=6\int_{y=0}^{y=1}\left[\frac{7}{2}(y^3-y^4)\right]dy=\frac{42}{2}\left(\frac{y^4}{4}-\frac{y^5}{5}\right)\Big|_0^1=\frac{42}{40}=\frac{21}{20}.$$

Exercise 15.2.8

Find the average value of the function $f(x,y)=xy$ over the triangle with vertices $(0,0),\ (1,0)$ and $(1,3)$.

Hint

Express the line joining $(0,0)$ and $(1,3)$ as a function $y=g(x)$.

Answer

$\frac{3}{4}$

Improper Double Integrals

An **improper double integral** is an integral $\iint_D f\,dA$ where either D is an unbounded region or f is an unbounded function. For example, $D=\{(x,y)\,|\,|x-y|\geq 2\}$ is an unbounded region, and the function $f(x,y)=1/(1-x^2-2y^2)$ over the ellipse $x^2+3y^2\geq 1$ is an unbounded function. Hence, both of the following integrals are improper integrals:

i. $$\iint_D xy\,dA \text{ where } D=\{(x,y)||\,x-y|\geq 2\};$$

ii. $$\iint_D \frac{1}{1-x^2-2y^2}\,dA \text{ where } D=\{(x,y)|\,x^2+3y^2\leq 1\}.$$

In this section we would like to deal with improper integrals of functions over rectangles or simple regions such that f has only finitely many discontinuities. Not all such improper integrals can be evaluated; however, a form of Fubini's theorem does apply for some types of improper integrals.

Theorem: Fubini's Theorem for Improper Integrals

If D is a bounded rectangle or simple region in the plane defined by

$\{(x,y) : a \le x \le b,\ g(x) \le y \le h(x)\}$ and also by

$\{(x,y) : c \le y \le d,\ j(y) \le x \le k(y)\}$ and f is a nonnegative function on D with finitely many discontinuities in the interior of D then

$$\iint_D f\, dA = \int_{x=a}^{x=b} \int_{y=g(x)}^{y=h(x)} f(x,y)\, dy\, dx = \int_{y=c}^{y=d} \int_{x=j(y)}^{x=k(y)} f(x,y)\, dx\, dy$$

It is very important to note that we required that the function be nonnegative on D for the theorem to work. We consider only the case where the function has finitely many discontinuities inside D.

Example 15.2.10: Evaluating a Double Improper Integral

Consider the function $f(x,y) = \frac{e^y}{y}$ over the region $D = \{(x,y) : 0 \le x \le 1,\ x \le y \le \sqrt{x}\}$.

Notice that the function is nonnegative and continuous at all points on D except $(0,0)$. Use Fubini's theorem to evaluate the improper integral.

Solution

First we plot the region D (Figure 15.2.15); then we express it in another way.

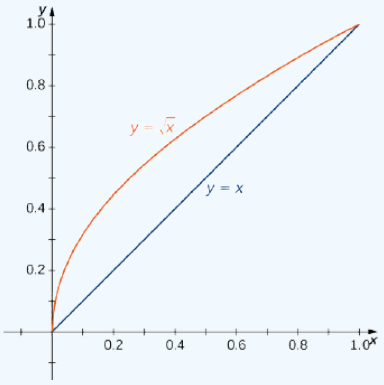


Figure 15.2.15: The function f is continuous at all points of the region D except $(0,0)$.

The other way to express the same region D is

$$D = \{(x,y) : 0 \le y \le 1,\ y^2 \le x \le y\}.$$

Thus we can use Fubini's theorem for improper integrals and evaluate the integral as

$$\int_{y=0}^{y=1} \int_{x=y^2}^{x=y} \frac{e^y}{y}\, dx\, dy.$$

Therefore, we have

$$\int_{y=0}^{y=1} \int_{x=y^2}^{x=y} \frac{e^y}{y}\, dx\, dy = \int_{y=0}^{y=1} \frac{e^y}{y} x \Big|_{x=y^2}^{x=y} dy = \int_{y=0}^{y=1} \frac{e^y}{y}(y - y^2)\, dy = \int_0^1 (e^y - ye^y)\, dy = e - 2.$$

As mentioned before, we also have an improper integral if the region of integration is unbounded. Suppose now that the function f is continuous in an unbounded rectangle R.

Theorem: Improper Integrals on an Unbounded Region

If R is an unbounded rectangle such as $R = \{(x,y) : a \le x \le \infty,\ c \le y \le \infty\}$, then when the limit exists, we have

$$\iint_R f(x,y)\, dA = \lim_{(b,d)\to(\infty,\infty)} \int_a^b \left(\int_c^d f(x,y)\, dy \right) dx = \lim_{(b,d)\to(\infty,\infty)} \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy.$$

The following example shows how this theorem can be used in certain cases of improper integrals.

Example 15.2.11

Evaluate the integral $\iint\limits_R xye^{-x^2-y^2}\,dA$ where R is the first quadrant of the plane.

Solution

The region R is the first quadrant of the plane, which is unbounded. So

$$\begin{aligned}\iint\limits_R xye^{-x^2-y^2}\,dA &= \lim_{(b,d)\to(\infty,\infty)}\int_{x=0}^{x=b}\left(\int_{y=0}^{y=d} xye^{-x^2-y^2}\,dy\right)dx \\ &= \lim_{(b,d)\to(\infty,\infty)}\int_{y=0}^{x=b} xye^{-x^2-y^2}\,dy \\ &= \lim_{(b,d)\to(\infty,\infty)}\frac{1}{4}\left(1-e^{-b^2}\right)\left(1-e^{-d^2}\right)=\frac{1}{4}\end{aligned}$$

Thus,

$$\iint\limits_R xye^{-x^2-y^2}\,dA$$

is convergent and the value is $\frac{1}{4}$.

Exercise 15.2.9

$$\iint\limits_D \frac{y}{\sqrt{1-x^2-y^2}}\,dA$$

where $D=\left\{(x,y)\,:\,x\geq 0,\ y\geq 0,\ x^2+y^2\leq 1\right\}$.

Hint

Notice that the integral is nonnegative and discontinuous on $x^2+y^2=1$. Express the region D as $D=\left\{(x,y)\,:\,0\leq x\leq 1,\ 0\leq y\leq\sqrt{1-x^2}\right\}$ and integrate using the method of substitution.

Answer

$\frac{\pi}{4}$

In some situations in probability theory, we can gain insight into a problem when we are able to use double integrals over general regions. Before we go over an example with a double integral, we need to set a few definitions and become familiar with some important properties.

Definition: Joint Density Function

Consider a pair of continuous random variables X and Y such as the birthdays of two people or the number of sunny and rainy days in a month. The **joint density function** f of X and Y satisfies the probability that (X,Y) lies in a certain region D:

$$P((X,Y)\in D)=\iint\limits_D f(x,y)\,dA.$$

Since the probabilities can never be negative and must lie between 0 and 1 the joint density function satisfies the following inequality and equation:

$$f(x,y)\geq 0 \text{ and } \iint\limits_R f(x,y)\,dA=1.$$

Definition: Independent Random Variables

The variables X and Y are said to be **independent random variables** if their joint density function is the product of their individual density functions:

$$f(x,y) = f_1(x)f_2(y).$$

Example 15.2.12: Application to Probability

At Sydney's Restaurant, customers must wait an average of 15 minutes for a table. From the time they are seated until they have finished their meal requires an additional 40 minutes, on average. What is the probability that a customer spends less than an hour and a half at the diner, assuming that waiting for a table and completing the meal are independent events?

Solution

Waiting times are mathematically modeled by exponential density functions, with m being the average waiting time, as

$$f(t) = \begin{cases} 0, & \text{if } t < 0 \\ \dfrac{1}{m}e^{-t/m}, & \text{if } t \geq 0. \end{cases}$$

if X and Y are random variables for 'waiting for a table' and 'completing the meal,' then the probability density functions are, respectively,

$$f_1(x) = \begin{cases} 0, & \text{if } x < 0. \\ \dfrac{1}{15}e^{-x/15}, & \text{if } x \geq 0. \end{cases} \quad \text{and} \quad f_2(y) = \begin{cases} 0, & \text{if } y < 0 \\ \dfrac{1}{40}e^{-y/40}, & \text{if } y \geq 0. \end{cases}$$

Clearly, the events are independent and hence the joint density function is the product of the individual functions

$$f(x,y) = f_1(x)f_2(y) = \begin{cases} 0, & \text{if } x < 0 \text{ or } y < 0, \\ \dfrac{1}{600}e^{-x/15}, & \text{if } x, y \geq 0 \end{cases}$$

We want to find the probability that the combined time $X+Y$ is less than 90 minutes. In terms of geometry, it means that the region D is in the first quadrant bounded by the line $x+y=90$ (Figure 15.2.16).

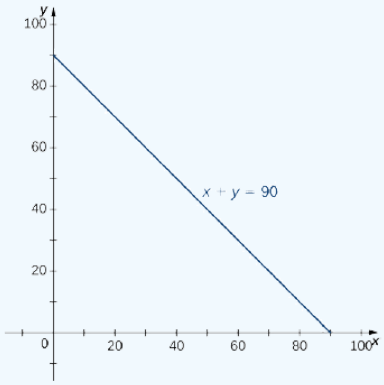


Figure 15.2.16: The region of integration for a joint probability density function.

Hence, the probability that (X,Y) is in the region D is

$$P(X+Y \leq 90) = P((X,Y) \in D) = \iint_D f(x,y)\,dA = \iint_D \frac{1}{600}e^{-x/15}e^{-y/40}\,dA.$$

Since $x+y=90$ is the same as $y = 90-x$, we have a region of Type I, so

$$D = \{(x,y) \mid 0 \leq x \leq 90,\ 0 \leq y \leq 90-x\},$$

$$P(X+Y \leq 90) = \frac{1}{600}\int_{x=0}^{x=90}\int_{y=0}^{y=90-x} e^{-(/15}e^{-y/40}dx\,dy = \frac{1}{600}\int_{x=0}^{x=90}\int_{y=0}^{y=90-x} e^{-x/15}e^{-y/40}dx\,dy$$

$$= \frac{1}{600}\int_{x=0}^{x=90}\int_{y=0}^{y=90-x} e^{-(x/15+y/40)}dx\,dy = 0.8328$$

Thus, there is an 83.2% chance that a customer spends less than an hour and a half at the restaurant.

Another important application in probability that can involve improper double integrals is the calculation of expected values. First we define this concept and then show an example of a calculation.

Definition: Expected Values

In probability theory, we denote the **expected values** $E(X)$ and $E(Y)$ respectively, as the most likely outcomes of the events. The expected values $E(X)$ and $E(Y)$ are given by

$$E(X) = \iint_S x\, f(x,y)\, dA \text{ and } E(Y) = \iint_S y\, f(x,y)\, dA,$$

where S is the sample space of the random variables X and Y.

Example 15.2.13: Finding Expected Value

Find the expected time for the events 'waiting for a table' and 'completing the meal' in Example 15.2.12

Solution

Using the first quadrant of the rectangular coordinate plane as the sample space, we have improper integrals for $E(X)$ and $E(Y)$. The expected time for a table is

$$\begin{aligned}
E(X) &= \iint_S x \frac{1}{600} e^{-x/15} e^{-y/40}\, dA \\
&= \frac{1}{600} \int_{x=0}^{x=\infty} \int_{y=0}^{y=\infty} x e^{-x/15} e^{-y/40} dA \\
&= \frac{1}{600} \lim_{(a,b)\to(\infty,\infty)} \int_{x=0}^{x=a} \int_{y=0}^{y=b} x e^{-x/15} e^{-y/40} dx\, dy \\
&= \frac{1}{600} \left(\lim_{a\to\infty} \int_{x=0}^{x=a} x e^{-x/15} dx \right) \left(\lim_{b\to\infty} \int_{y=0}^{y=b} e^{-y/40} dy \right) \\
&= \frac{1}{600} \left(\left(\lim_{a\to\infty} (-15e^{-x/15}(x+15)) \right) \Big|_{x=0}^{x=a} \right) \left(\left(\lim_{b\to\infty} (-40e^{-y/40}) \right) \Big|_{y=0}^{y=b} \right) \\
&= \frac{1}{600} \left(\lim_{a\to\infty} (-15e^{-a/15}(x+15) + 225) \right) \left(\lim_{b\to\infty} (-40e^{-b/40} + 40) \right) \\
&= \frac{1}{600} (225)(40) = 15.
\end{aligned}$$

A similar calculation shows that $E(Y) = 40$. This means that the expected values of the two random events are the average waiting time and the average dining time, respectively.

Exercise 15.2.10

The joint density function for two random variables X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{600}(x^2+y^2), & \text{if } \leq x \leq 15,\ 0 \leq y \leq 10 \\ 0, & \text{otherwise} \end{cases}$$

Find the probability that X is at most 10 and Y is at least 5.

Hint

Compute the probability

$$P(X \leq 10,\ Y \geq 5) = \int_{x=-\infty}^{10} \int_{y=5}^{y=10} \frac{1}{6000}(x^2+y^2) dy\, dx.$$

Answer

$\frac{55}{72} \approx 0.7638$

Key Concepts

- A general bounded region D on the plane is a region that can be enclosed inside a rectangular region. We can use this idea to define a double integral over a general bounded region.
- To evaluate an iterated integral of a function over a general nonrectangular region, we sketch the region and express it as a Type I or as a Type II region or as a union of several Type I or Type II regions that overlap only on their boundaries.
- We can use double integrals to find volumes, areas, and average values of a function over general regions, similarly to calculations over rectangular regions.
- We can use Fubini's theorem for improper integrals to evaluate some types of improper integrals.

Key Equations

- **Iterated integral over a Type I region**

$$\iint_D f(x,y)\,dA = \iint_D f(x,y)\,dy\,dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right] dx$$

- **Iterated integral over a Type II region**

$$\iint_D f(x,y)\,dA = \iint_D (x,y)\,dx\,dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right] dy$$

Glossary

improper double integral
a double integral over an unbounded region or of an unbounded function

Type I
a region D in the xy- plane is Type I if it lies between two vertical lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$

Type II
a region D in the xy-plane is Type II if it lies between two horizontal lines and the graphs of two continuous functions $h_1(y)$ and $h_2(h)$

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15.3: Double Integrals in Polar Coordinates

Learning Objectives

- Recognize the format of a double integral over a polar rectangular region.
- Evaluate a double integral in polar coordinates by using an iterated integral.
- Recognize the format of a double integral over a general polar region.
- Use double integrals in polar coordinates to calculate areas and volumes.

Double integrals are sometimes much easier to evaluate if we change rectangular coordinates to polar coordinates. However, before we describe how to make this change, we need to establish the concept of a double integral in a polar rectangular region.

Polar Rectangular Regions of Integration

When we defined the double integral for a continuous function in rectangular coordinates—say, g over a region R in the xy-plane—we divided R into subrectangles with sides parallel to the coordinate axes. These sides have either constant x-values and/or constant y-values. In polar coordinates, the shape we work with is a polar rectangle, whose sides have constant r-values and/or constant θ-values. This means we can describe a polar rectangle as in Figure 15.3.1a, with $R = \{(r, \theta) \mid a \le r \le b,\ \alpha \le \theta \le \beta\}$.

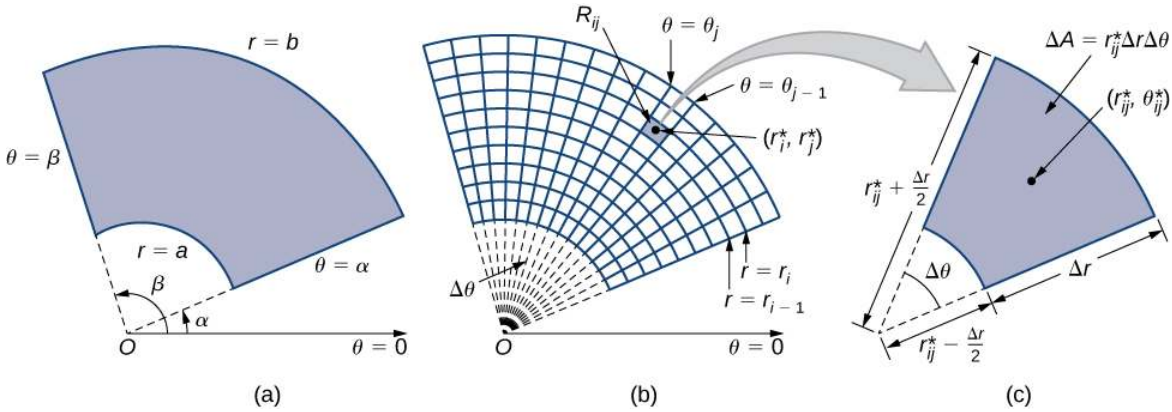


Figure 15.3.1: (a) A polar rectangle R (b) divided into subrectangles R_{ij} (c) Close-up of a subrectangle.

In this section, we are looking to integrate over polar rectangles. Consider a function $f(r, \theta)$ over a polar rectangle R. We divide the interval $[a, b]$ into m subintervals $[r_{i-1}, r_i]$ of length $\Delta r = (b-a)/m$ and divide the interval $[\alpha, \beta]$ into n subintervals $[\theta_{i-1}, \theta_i]$ of width $\Delta\theta = (\beta - \alpha)/n$. This means that the circles $r = r_i$ and rays $\theta = \theta_i$ for $1 \le i \le m$ and $1 \le j \le n$ divide the polar rectangle R into smaller polar subrectangles R_{ij} (Figure 15.3.1b).

As before, we need to find the area ΔA of the polar subrectangle R_{ij} and the "polar" volume of the thin box above R_{ij}. Recall that, in a circle of radius r the length s of an arc subtended by a central angle of θ radians is $s = r\theta$. Notice that the polar rectangle R_{ij} looks a lot like a trapezoid with parallel sides $r_{i-1}\Delta\theta$ and $r_i\Delta\theta$ and with a width Δr. Hence the area of the polar subrectangle R_{ij} is

$$\Delta A = \frac{1}{2}\Delta r(r_{i-1}\Delta\theta + r_i\Delta\theta).$$

Simplifying and letting

$$r^*_{ij} = \frac{1}{2}(r_{i-1} + r_i)$$

we have $\Delta A = r^*_{ij}\Delta r\Delta\theta$.

Therefore, the polar volume of the thin box above R_{ij} (Figure 15.3.2) is

$$f(r^*_{ij}, \theta^*_{ij})\Delta A = f(r^*_{ij}, \theta^*_{ij})r^*_{ij}\Delta r\Delta\theta.$$

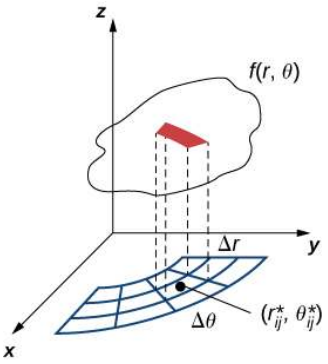


Figure 15.3.2: Finding the volume of the thin box above polar rectangle R_{ij}.

Using the same idea for all the subrectangles and summing the volumes of the rectangular boxes, we obtain a double Riemann sum as

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*) r_{ij}^* \Delta r \Delta\theta.$$

As we have seen before, we obtain a better approximation to the polar volume of the solid above the region R when we let m and n become larger. Hence, we define the polar volume as the limit of the double Riemann sum,

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*) r_{ij}^* \Delta r \Delta\theta.$$

This becomes the expression for the double integral.

Definition: The double integral in polar coordinates

The double integral of the function $f(r, \theta)$ over the polar rectangular region R in the $r\theta$-plane is defined as

$$\iint_R f(r,\theta) dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*) \Delta A \tag{15.3.1}$$

$$= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*) r_{ij}^* \Delta r \Delta\theta. \tag{15.3.2}$$

Again, just as in section on Double Integrals over Rectangular Regions, the double integral over a polar rectangular region can be expressed as an iterated integral in polar coordinates. Hence,

$$\iint_R f(r,\theta)\, dA = \iint_R f(r,\theta)\, r\, dr\, d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=a}^{r=b} f(r,\theta)\, r\, dr\, d\theta.$$

Notice that the expression for dA is replaced by $r\, dr\, d\theta$ when working in polar coordinates. Another way to look at the polar double integral is to change the double integral in rectangular coordinates by substitution. When the function f is given in terms of x and y using $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\, dr\, d\theta$ changes it to

$$\iint_R f(x,y)\, dA = \iint_R f(r\cos\theta,\, r\sin\theta)\, r\, dr\, d\theta.$$

Note that all the properties listed in section on Double Integrals over Rectangular Regions for the double integral in rectangular coordinates hold true for the double integral in polar coordinates as well, so we can use them without hesitation.

Example 15.3.1A: Sketching a Polar Rectangular Region

Sketch the polar rectangular region

$$R = \{(r,\theta) \mid 1 \le r \le 3, 0 \le \theta \le \pi\}.$$

Solution

As we can see from Figure 15.3.3, $r = 1$ and $r = 3$ are circles of radius 1 and 3 and $0 \le \theta \le \pi$ covers the entire top half of the plane. Hence the region R looks like a semicircular band.

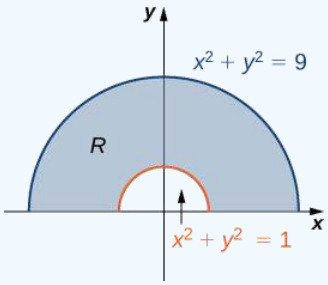


Figure 15.3.3: The polar region R lies between two semicircles.

Now that we have sketched a polar rectangular region, let us demonstrate how to evaluate a double integral over this region by using polar coordinates.

Example 15.3.1B: Evaluating a Double Integral over a Polar Rectangular Region

Evaluate the integral $\iint_R 3x\,dA$ over the region $R = \{(r, \theta) \mid 1 \le r \le 2,\ 0 \le \theta \le \pi\}$.

Solution

First we sketch a figure similar to Figure 15.3.3, but with outer radius $r = 2$. From the figure we can see that we have

$$\begin{aligned}
\iint_R 3x\,dA &= \int_{\theta=0}^{\theta=\pi} \int_{r=1}^{r=2} 3r\cos\theta\, r\,dr\,d\theta \quad \text{Use an integral with correct limits of integration.} \\
&= \int_{\theta=0}^{\theta=\pi} \cos\theta \left[r^3\Big|_{r=1}^{r=2}\right] d\theta \quad \text{Integrate first with respect to } r. \\
&= \int_{\theta=0}^{\theta=\pi} 7\cos\theta\,d\theta \\
&= 7\sin\theta\Big|_{\theta=0}^{\theta=\pi} = 0.
\end{aligned}$$

Exercise 15.3.1

Sketch the region $D = \{(r, \theta) | 1 \le r \le 2,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\}$, and evaluate $\iint_R x\,dA$.

Hint

Follow the steps in Example 15.3.1A

Answer

$\frac{14}{3}$

Example 15.3.2A: Evaluating a Double Integral by Converting from Rectangular Coordinates

Evaluate the integral

$$\iint_R (1 - x^2 - y^2)\,dA$$

where R is the unit circle on the xy-plane.

Solution

The region R is a unit circle, so we can describe it as $R = \{(r, \theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$.

Using the conversion $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\,dr\,d\theta$, we have

$$\begin{aligned}\iint_R (1 - x^2 - y^2)\,dA &= \int_0^{2\pi}\int_0^1 (1 - r^2)\,r\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^1 (r - r^3)\,dr\,d\theta \\ &= \int_0^{2\pi}\left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 d\theta \\ &= \int_0^{2\pi}\frac{1}{4}\,d\theta = \frac{\pi}{2}.\end{aligned}$$

Example 15.3.2B: Evaluating a Double Integral by Converting from Rectangular Coordinates

Evaluate the integral

$$\iint_R (x + y)\,dA$$

where $R = \{(x, y) \mid 1 \le x^2 + y^2 \le 4,\ x \le 0\}$.

Solution

We can see that R is an annular region that can be converted to polar coordinates and described as $R = \{(r, \theta) \mid 1 \le r \le 2,\ \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\}$ (see the following graph).

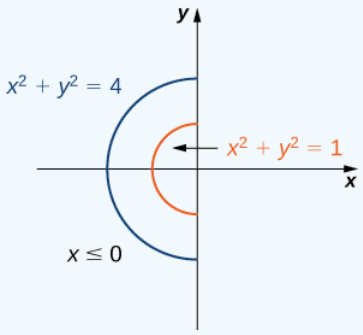


Figure 15.3.4: The annular region of integration R.

Hence, using the conversion $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\,dr\,d\theta$, we have

$$\begin{aligned}\iint_R (x + y)\,dA &= \int_{\theta=\pi/2}^{\theta=3\pi/2}\int_{r=1}^{r=2} (r\cos\theta + r\sin\theta)r\,dr\,d\theta \\ &= \left(\int_{r=1}^{r=2} r^2\,dr\right)\left(\int_{\pi/2}^{3\pi/2} (\cos\theta + \sin\theta)\,d\theta\right) \\ &= \left[\frac{r^3}{3}\right]_1^2 [\sin\theta - \cos\theta]\Big|_{\pi/2}^{3\pi/2} \\ &= -\frac{14}{3}.\end{aligned}$$

Exercise 15.3.2

Evaluate the integral

$$\iint_R (4 - x^2 - y^2)\,dA$$

where R is the circle of radius 2 on the xy-plane.

Hint

Follow the steps in the previous example.

Answer

8π

General Polar Regions of Integration

To evaluate the double integral of a continuous function by iterated integrals over general polar regions, we consider two types of regions, analogous to Type I and Type II as discussed for rectangular coordinates in section on Double Integrals over General Regions. It is more common to write polar equations as $r = f(\theta)$ than $\theta = f(r)$, so we describe a general polar region as $R = \{(r, \theta) \mid \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\}$ (Figure 15.3.5).

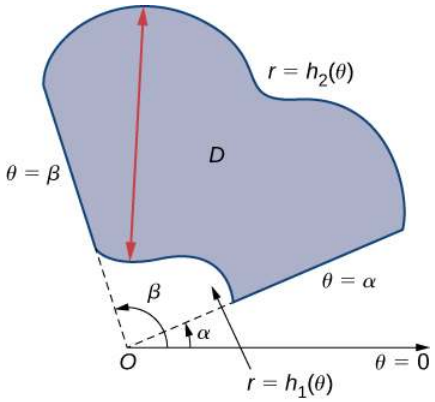


Figure 15.3.5: A general polar region between $\alpha \leq \theta \leq \beta$ and $h_1(\theta) \leq r \leq h_2(\theta)$.

Theorem: Double Integrals over General Polar Regions

If $f(r, \theta)$ is continuous on a general polar region D as described above, then

$$\iint_D f(r, \theta)\, r\, dr\, d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} f(r, \theta)\, r\, dr\, d\theta.$$

Example 15.3.3: Evaluating a Double Integral over a General Polar Region

Evaluate the integral

$$\iint_D r^2 \sin\theta\, r\, dr\, d\theta$$

where D is the region bounded by the polar axis and the upper half of the cardioid $r = 1 + \cos\theta$.

Solution

We can describe the region D as $\{(r, \theta) \mid 0 \leq \theta \leq \pi,\ 0 \leq r \leq 1 + \cos\theta\}$ as shown in Figure 15.3.6

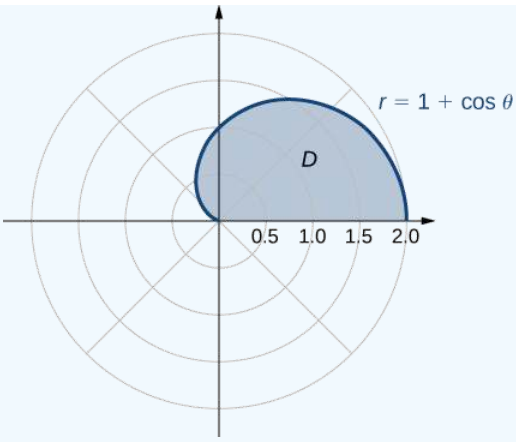


Figure 15.3.6: The region D is the top half of a cardioid.

Hence, we have

$$\begin{aligned}\iint_D r^2 \sin\theta\, r\, dr\, d\theta &= \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=1+\cos\theta} (r^2\sin\theta)\, r\, dr\, d\theta \\ &= \frac{1}{4}\int_{\theta=0}^{\theta=\pi} \left[r^4\right]\Big|_{r=0}^{r=1+\cos\ \theta} \sin\theta\, d\theta \\ &= \frac{1}{4}\int_{\theta=0}^{\theta=\pi} (1+\cos\theta)^4 \sin\theta\, d\theta \\ &= -\frac{1}{4}\left[\frac{(1+\cos\theta)^5}{5}\right]_0^{\pi} = \frac{8}{5}.\end{aligned}$$

? Exercise 15.3.3

Evaluate the integral

$$\iint_D r^2 \sin^2 2\theta\, r\, dr\, d\theta$$

where $D = \left\{(r,\theta)\,|\,0 \le \theta \le \pi,\ 0 \le r \le 2\sqrt{\cos 2\theta}\right\}$.

Hint

Graph the region and follow the steps in the previous example.

Answer

$\frac{\pi}{8}$

Polar Areas and Volumes

As in rectangular coordinates, if a solid S is bounded by the surface $z = f(r,\theta)$, as well as by the surfaces $r = a$, $r = b$, $\theta = \alpha$, and $\theta = \beta$, we can find the volume V of S by double integration, as

$$V = \iint_R f(r,\theta)\, r\, dr\, d\theta = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=a}^{r=b} f(r,\theta)\, r\, dr\, d\theta.$$

If the base of the solid can be described as $D = \{(r,\theta)|\alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$, then the double integral for the volume becomes

$$V = \iint_D f(r,\theta)\, r\, dr\, d\theta = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=h_1(\theta)}^{r=h_2(\theta)} f(r,\theta)\, r\, dr\, d\theta.$$

We illustrate this idea with some examples.

Example 15.3.4A: Finding a Volume Using a Double Integral

Find the volume of the solid that lies under the paraboloid $z = 1 - x^2 - y^2$ and above the unit circle on the xy-plane (Figure 15.3.7).

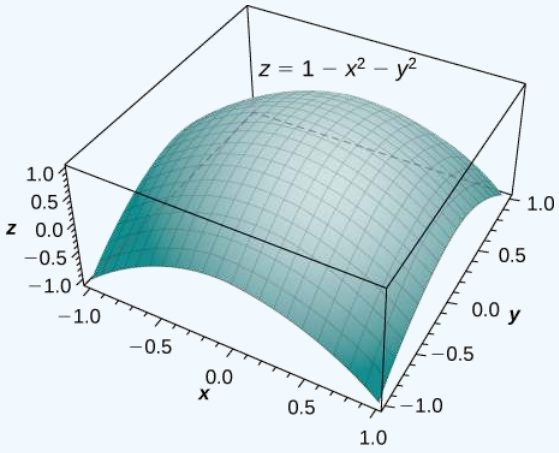


Figure 15.3.7: Finding the volume of a solid under a paraboloid and above the unit circle.

Solution

By the method of double integration, we can see that the volume is the iterated integral of the form

$$\iint_R (1 - x^2 - y^2)\, dA$$

where $R = \{(r, \theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$.

This integration was shown before in Example 15.3.2A, so the volume is $\frac{\pi}{2}$ cubic units.

Example 15.3.4B: Finding a Volume Using Double Integration

Find the volume of the solid that lies under the paraboloid $z = 4 - x^2 - y^2$ and above the disk $(x-1)^2 + y^2 = 1$ on the xy-plane. See the paraboloid in Figure 15.3.8intersecting the cylinder $(x-1)^2 + y^2 = 1$ above the xy-plane.

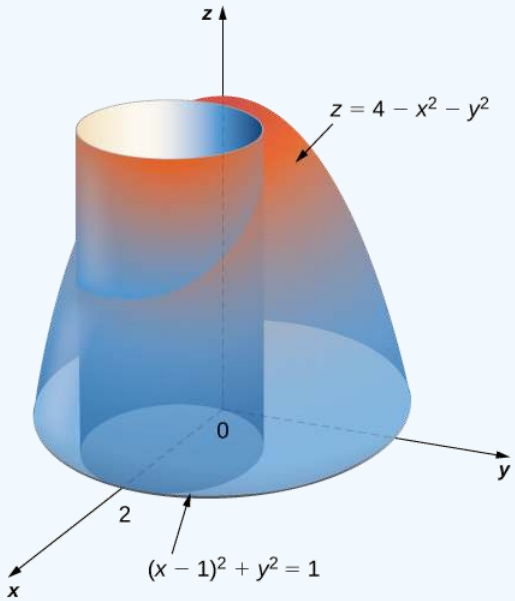


Figure 15.3.8: Finding the volume of a solid with a paraboloid cap and a circular base.

Solution

First change the disk $(x-1)^2+y^2=1$ to polar coordinates. Expanding the square term, we have $x^2-2x+1+y^2=1$. Then simplify to get $x^2+y^2=2x$, which in polar coordinates becomes $r^2=2r\cos\theta$ and then either $r=0$ or $r=2\cos\theta$. Similarly, the equation of the paraboloid changes to $z=4-r^2$. Therefore we can describe the disk $(x-1)^2+y^2=1$ on the xy-plane as the region

$$D=\{(r,\theta)\,|\,0\le\theta\le\pi,\ 0\le r\le 2\cos\theta\}.$$

Hence the volume of the solid bounded above by the paraboloid $z=4-x^2-y^2$ and below by $r=2\cos\theta$ is

$$\begin{aligned}V&=\iint_D f(r,\theta)\,r\,dr\,d\theta\\&=\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=2\cos\theta}(4-r^2)\,r\,dr\,d\theta\\&=\int_{\theta=0}^{\theta=\pi}\left[4\frac{r^2}{2}-\frac{r^4}{4}\Big|_0^{2\cos\theta}\right]d\theta\\&=\int_0^{\pi}[8\cos^2\theta-4\cos^4\theta]\,d\theta\\&=\left[\frac{5}{2}\theta+\frac{5}{2}\sin\theta\cos\theta-\sin\theta\cos^3\theta\right]_0^{\pi}=\frac{5}{2}\pi\text{ units}^3.\end{aligned}$$

Notice in the next example that integration is not always easy with polar coordinates. Complexity of integration depends on the function and also on the region over which we need to perform the integration. If the region has a more natural expression in polar coordinates or if f has a simpler antiderivative in polar coordinates, then the change in polar coordinates is appropriate; otherwise, use rectangular coordinates.

Example 15.3.5A: Finding a Volume Using a Double Integral

Find the volume of the region that lies under the paraboloid $z=x^2+y^2$ and above the triangle enclosed by the lines $y=x$, $x=0$, and $x+y=2$ in the xy-plane.

Solution

First examine the region over which we need to set up the double integral and the accompanying paraboloid.

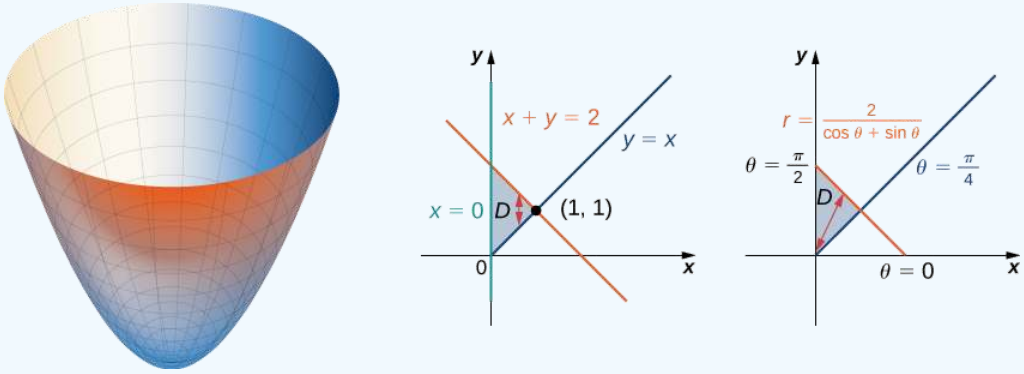


Figure 15.3.9: Finding the volume of a solid under a paraboloid and above a given triangle.

The region D is $\{(x,y)\,|\,0\le x\le 1,\ x\le y\le 2-x\}$. Converting the lines $y=x$, $x=0$, and $x+y=2$ in the xy-plane to functions of r and θ we have $\theta=\pi/4$, $\theta=\pi/2$, and $r=2/(\cos\theta+\sin\theta)$, respectively. Graphing the region on the xy-plane, we see that it looks like $D=\{(r,\theta)\,|\,\pi/4\le\theta\le\pi/2,\ 0\le r\le 2/(\cos\theta+\sin\theta)\}$.

Now converting the equation of the surface gives $z=x^2+y^2=r^2$. Therefore, the volume of the solid is given by the double integral

$$
\begin{aligned}
V &= \iint_D f(r,\theta)\, r\, dr\, d\theta \\
&= \int_{\theta=\pi/4}^{\theta=\pi/2} \int_{r=0}^{r=2/(\cos\ \theta+\sin\ \theta)} r^2 r\, dr d\theta \\
&= \int_{\pi/4}^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2/(\cos\ \theta+\sin\ \theta)} d\theta \\
&= \frac{1}{4}\int_{\pi/4}^{\pi/2} \left(\frac{2}{\cos\theta+\sin\theta}\right)^4 d\theta \\
&= \frac{16}{4}\int_{\pi/4}^{\pi/2} \left(\frac{1}{\cos\theta+\sin\theta}\right)^4 d\theta \\
&= 4\int_{\pi/4}^{\pi/2} \left(\frac{1}{\cos\theta+\sin\theta}\right)^4 d\theta.
\end{aligned}
$$

As you can see, this integral is very complicated. So, we can instead evaluate this double integral in rectangular coordinates as

$$V = \int_0^1 \int_x^{2-x} (x^2+y^2)\, dy\, dx.$$

Evaluating gives

$$
\begin{aligned}
V &= \int_0^1 \int_x^{2-x} (x^2+y^2)\, dy\, dx \\
&= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]\Big|_x^{2-x} dx \\
&= \int_0^1 \frac{8}{3} - 4x + 4x^2 - \frac{8x^3}{3}\, dx \\
&= \left[\frac{8x}{3} - 2x^2 + \frac{4x^3}{3} - \frac{2x^4}{3}\right]\Big|_0^1 \\
&= \frac{4}{3}\ \text{units}^3.
\end{aligned}
$$

To answer the question of how the formulas for the volumes of different standard solids such as a sphere, a cone, or a cylinder are found, we want to demonstrate an example and find the volume of an arbitrary cone.

Example $15.3.5B$: Finding a Volume Using a Double Integral

Use polar coordinates to find the volume inside the cone $z = 2 - \sqrt{x^2+y^2}$ and above the xy-plane.

Solution

The region D for the integration is the base of the cone, which appears to be a circle on the xy-plane (Figure $15.3.10$).

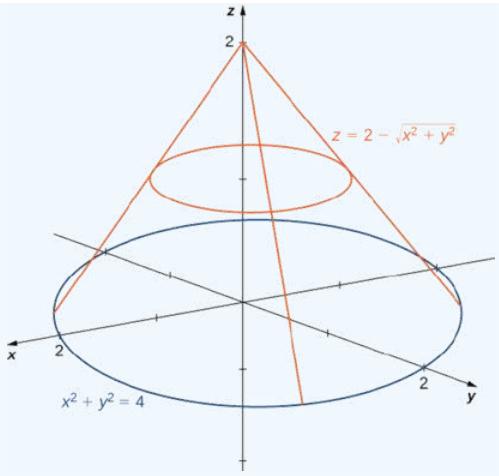


Figure 15.3.10: Finding the volume of a solid inside the cone and above the xy-plane.

We find the equation of the circle by setting $z = 0$:

$$\begin{aligned} 0 &= 2 - \sqrt{x^2 + y^2} \\ 2 &= \sqrt{x^2 + y^2} \\ x^2 + y^2 &= 4. \end{aligned}$$

This means the radius of the circle is 2 so for the integration we have $0 \le \theta \le 2\pi$ and $0 \le r \le 2$. Substituting $x = r\cos\theta$ and $y = r\sin\theta$ in the equation $z = 2 - \sqrt{x^2 + y^2}$ we have $z = 2 - r$. Therefore, the volume of the cone is

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} (2 - r)\, r\, dr\, d\theta = 2\pi \frac{4}{3} = \frac{8\pi}{3} \text{ cubic units.}$$

Analysis

Note that if we were to find the volume of an arbitrary cone with radius α units and height h units, then the equation of the cone would be $z = h - \frac{h}{a}\sqrt{x^2 + y^2}$.

We can still use Figure 15.3.10and set up the integral as

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a} \left(h - \frac{h}{a}r\right) r\, dr\, d\theta.$$

Evaluating the integral, we get $\frac{1}{3}\pi a^2 h$.

? Exercise 15.3.5

Use polar coordinates to find an iterated integral for finding the volume of the solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 16 - x^2 - y^2$.

Hint

Sketching the graphs can help.

Answer

$$V = \int_0^{2\pi} \int_0^{2\sqrt{2}} (16 - 2r^2)\, r\, dr\, d\theta = 64\pi \text{ cubic units.}$$

As with rectangular coordinates, we can also use polar coordinates to find areas of certain regions using a double integral. As before, we need to understand the region whose area we want to compute. Sketching a graph and identifying the region can be helpful to realize the limits of integration. Generally, the area formula in double integration will look like

$$\text{Area of } A = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} 1 \, r \, dr \, d\theta.$$

Example 15.3.6A: Finding an Area Using a Double Integral in Polar Coordinates

Evaluate the area bounded by the curve $r = \cos 4\theta$.

Solution

Sketching the graph of the function $r = \cos 4\theta$ reveals that it is a polar rose with eight petals (see the following figure).

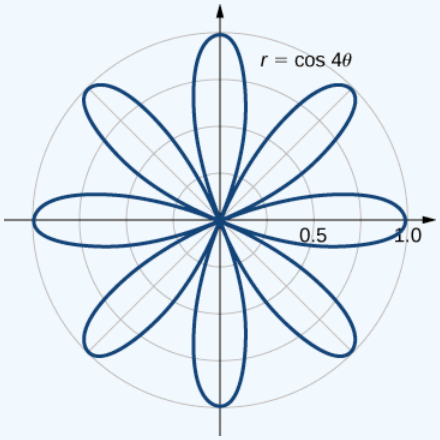


Figure 15.3.11: Finding the area of a polar rose with eight petals.

Using symmetry, we can see that we need to find the area of one petal and then multiply it by 8. Notice that the values of θ for which the graph passes through the origin are the zeros of the function $\cos 4\theta$, and these are odd multiples of $\pi/8$. Thus, one of the petals corresponds to the values of θ in the interval $[-\pi/8, \pi/8]$ Therefore, the area bounded by the curve $r = \cos 4\theta$ is

$$\begin{aligned} A &= 8 \int_{\theta=-\pi/8}^{\theta=\pi/8} \int_{r=0}^{r=\cos 4\theta} 1 \, r \, dr \, d\theta \\ &= 8 \int_{\theta=-\pi/8}^{\theta=\pi/8} \left[\frac{1}{2} r^2 \Big|_0^{\cos 4\theta} \right] d\theta \\ &= 8 \int_{-\pi/8}^{\pi/8} \frac{1}{2} \cos^2 4\theta \, d\theta \\ &= 8 \left[\frac{1}{4}\theta + \frac{1}{16} \sin 4\theta \cos 4\theta \Big|_{-\pi/8}^{\pi/8} \right] \\ &= 8 \left[\frac{\pi}{16} \right] = \frac{\pi}{2} \text{ units}^2. \end{aligned}$$

Example 15.3.6B: Finding Area Between Two Polar Curves

Find the area enclosed by the circle $r = 3 \cos \theta$ and the cardioid $r = 1 + \cos \theta$.

Solution

First and foremost, sketch the graphs of the region (Figure 15.3.12).

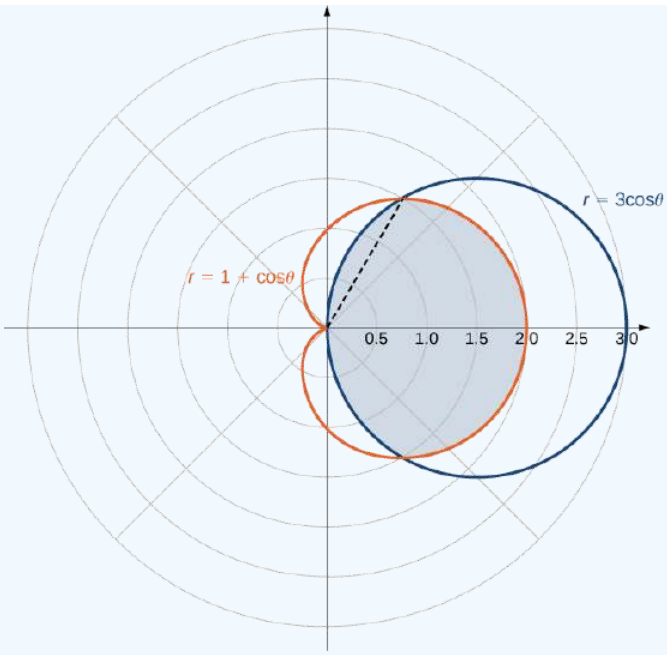


Figure 15.3.12: Finding the area enclosed by both a circle and a cardioid.

We can from see the symmetry of the graph that we need to find the points of intersection. Setting the two equations equal to each other gives

$$3\cos\theta = 1 + \cos\theta.$$

One of the points of intersection is $\theta = \pi/3$. The area above the polar axis consists of two parts, with one part defined by the cardioid from $\theta = 0$ to $\theta = \pi/3$ and the other part defined by the circle from $\theta = \pi/3$ to $\theta = \pi/2$. By symmetry, the total area is twice the area above the polar axis. Thus, we have

$$A = 2\left[\int_{\theta=0}^{\theta=\pi/3}\int_{r=0}^{r=1+\cos\theta} 1\, r\, dr\, d\theta + \int_{\theta=\pi/3}^{\theta=\pi/2}\int_{r=0}^{r=3\cos\theta} 1\, r\, dr\, d\theta\right].$$

Evaluating each piece separately, we find that the area is

$$A = 2\left(\frac{1}{4}\pi + \frac{9}{16}\sqrt{3} + \frac{3}{8}\pi - \frac{9}{16}\sqrt{3}\right) = 2\left(\frac{5}{8}\pi\right) = \frac{5}{4}\pi \text{ square units.}$$

? Exercise 15.3.6

Find the area enclosed inside the cardioid $r = 3 - 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

Hint

Sketch the graph, and solve for the points of intersection.

Answer

$$A = 2\int_{-\pi/2}^{\pi/6}\int_{1+\sin\theta}^{3-3\sin\theta} r\, dr\, d\theta = (8\pi + 9\sqrt{3}) \text{ units}^2$$

✓ Example 15.3.7: Evaluating an Improper Double Integral in Polar Coordinates

Evaluate the integral

$$\iint_{R^2} e^{-10(x^2+y^2)}\, dx\, dy.$$

Solution

This is an improper integral because we are integrating over an unbounded region R^2. In polar coordinates, the entire plane R^2 can be seen as $0 \le \theta \le 2\pi,\ 0 \le r \le \infty$.

Using the changes of variables from rectangular coordinates to polar coordinates, we have

$$\begin{aligned}\iint_{R^2} e^{-10(x^2+y^2)}\,dx\,dy &= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=\infty} e^{-10r^2}\,r\,dr\,d\theta = \int_{\theta=0}^{\theta=2\pi}\left(\lim_{a\to\infty}\int_{r=0}^{r=a} e^{-10r^2}\,r\,dr\right)d\theta \\ &= \left(\int_{\theta=0}^{\theta=2\pi}\right)d\theta\left(\lim_{a\to\infty}\int_{r=0}^{r=a} e^{-10r^2}\,r\,dr\right) \\ &= 2\pi\left(\lim_{a\to\infty}\int_{r=0}^{r=a} e^{-10r^2}\,r\,dr\right) \\ &= 2\pi\lim_{a\to\infty}\left(-\frac{1}{20}\right)\left(e^{-10r^2}\Big|_0^a\right) \\ &= 2\pi\left(-\frac{1}{20}\right)\lim_{a\to\infty}\left(e^{-10a^2}-1\right) \\ &= \frac{\pi}{10}.\end{aligned}$$

? Exercise 15.3.7

Evaluate the integral

$$\iint_{R^2} e^{-4(x^2+y^2)}dx\,dy.$$

Hint

Convert to the polar coordinate system.

Answer

$\frac{\pi}{4}$

Key Concepts

- To apply a double integral to a situation with circular symmetry, it is often convenient to use a double integral in polar coordinates. We can apply these double integrals over a polar rectangular region or a general polar region, using an iterated integral similar to those used with rectangular double integrals.
- The area dA in polar coordinates becomes $r\,dr\,d\theta$.
- Use $x = r\cos\theta,\ y = r\sin\theta$, and $dA = r\,dr\,d\theta$ to convert an integral in rectangular coordinates to an integral in polar coordinates.
- Use $r^2 = x^2 + y^2$ and $\theta = tan^{-1}\left(\frac{y}{x}\right)$ to convert an integral in polar coordinates to an integral in rectangular coordinates, if needed.
- To find the volume in polar coordinates bounded above by a surface $z = f(r,\theta)$ over a region on the xy-plane, use a double integral in polar coordinates.

Key Equations

- **Double integral over a polar rectangular region R**

$$\iint_R f(r,\theta)dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*,\theta_{ij}^*)\Delta A = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*,\theta_{ij}^*)r_{ij}^*\Delta r\Delta\theta$$

- **Double integral over a general polar region**

$$\iint_D f(r,\theta)\,r\,dr\,d\theta = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=h_1(\theta)}^{r_2(\theta)} f(r,\theta)\,r\,dr\,d\theta$$

Glossary

polar rectangle

the region enclosed between the circles $r = a$ and $r = b$ and the angles $\theta = \alpha$ and $\theta = \beta$; it is described as $R = \{(r, \theta) \mid a \leq r \leq b,\ \alpha \leq \theta \leq \beta\}$

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15.4: Applications of Double Integrals

Learning Objectives

- Recognize when a function of two variables is integrable over a rectangular region.
- Recognize and use some of the properties of double integrals.
- Evaluate a double integral over a rectangular region by writing it as an iterated integral.
- Use a double integral to calculate the area of a region, volume under a surface, or average value of a function over a plane region.

In this section we investigate double integrals and show how we can use them to find the volume of a solid over a rectangular region in the xy-plane. Many of the properties of double integrals are similar to those we have already discussed for single integrals.

Volumes and Double Integrals

We begin by considering the space above a rectangular region R. Consider a continuous function $f(x, y) \geq 0$ of two variables defined on the closed rectangle R:

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \,|\, a \leq x \leq b,\ c \leq y \leq d\}$$

Here $[a, b] \times [c, d]$ denotes the Cartesian product of the two closed intervals $[a, b]$ and $[c, d]$. It consists of rectangular pairs (x, y) such that $a \leq x \leq b$ and $c \leq y \leq d$. The graph of f represents a surface above the xy-plane with equation $z = f(x, y)$ where z is the height of the surface at the point (x, y). Let S be the solid that lies above R and under the graph of f (Figure 15.4.1). The base of the solid is the rectangle R in the xy-plane. We want to find the volume V of the solid S.

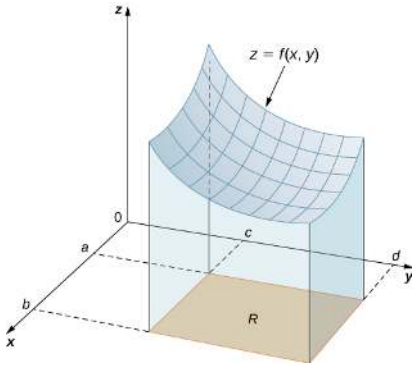


Figure 15.4.1: The graph of $f(x, y)$ over the rectangle R in the xy-plane is a curved surface.

We divide the region R into small rectangles R_{ij}, each with area ΔA and with sides Δx and Δy (Figure 15.4.2). We do this by dividing the interval $[a, b]$ into m subintervals and dividing the interval $[c, d]$ into n subintervals. Hence $\Delta x = \frac{b-a}{m}$, $\Delta y = \frac{d-c}{n}$, and $\Delta A = \Delta x \Delta y$.

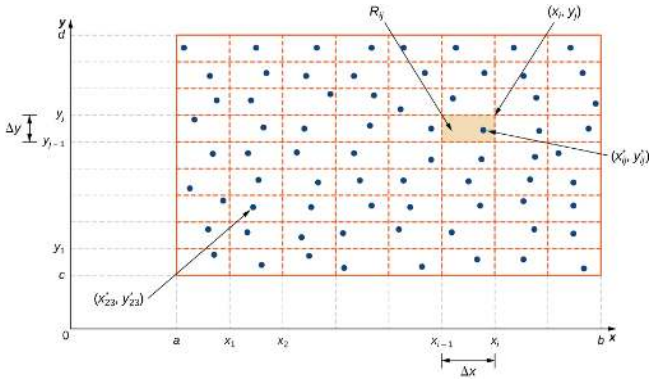


Figure 15.4.2: Rectangle R is divided into small rectangles R_{ij} each with area ΔA.

The volume of a thin rectangular box above R_{ij} is $f(x_{ij}^*, y_{ij}^*)\,\Delta A$, where (x_{ij}^*, y_{ij}^*) is an arbitrary sample point in each R_{ij} as shown in the following figure, $f(x_{ij}^*, y_{ij}^*)$ is the height of the corresponding thin rectangular box, and ΔA is the area of each rectangle R_{ij}.

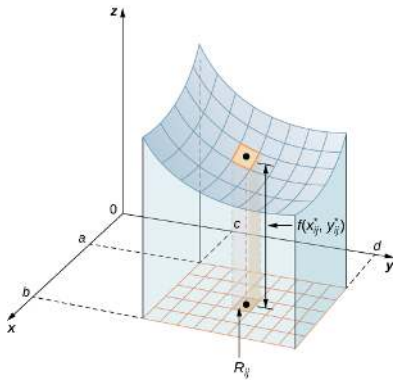


Figure 15.4.3: A thin rectangular box above R_{ij} with height $f(x_{ij}^*, y_{ij}^*)$.

Using the same idea for all the subrectangles, we obtain an approximate volume of the solid S as

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A.$$

This sum is known as a **double Riemann sum** and can be used to approximate the value of the volume of the solid. Here the double sum means that for each subrectangle we evaluate the function at the chosen point, multiply by the area of each rectangle, and then add all the results.

As we have seen in the single-variable case, we obtain a better approximation to the actual volume if m and n become larger.

$$V = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A$$

or

$$V = \lim_{\Delta x,\, \Delta y \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A.$$

Note that the sum approaches a limit in either case and the limit is the volume of the solid with the base R. Now we are ready to define the double integral.

Definition: Double Integral over a Rectangular Region R

The double integral of the function $f(x, y)$ over the rectangular region R in the xy-plane is defined as

$$\iint_R f(x, y) dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A.$$

If $f(x, y) \geq 0$, then the volume V of the solid S, which lies above R in the xy-plane and under the graph of f, is the double integral of the function $f(x, y)$ over the rectangle R. If the function is ever negative, then the double integral can be considered a "signed" volume in a manner similar to the way we defined net signed area in The Definite Integral.

Example 15.4.1: Setting up a Double Integral and Approximating It by Double Sums

Consider the function $z = f(x, y) = 3x^2 - y$ over the rectangular region $R = [0, 2] \times [0, 2]$ (Figure 15.4.4).

a. Set up a double integral for finding the value of the signed volume of the solid S that lies above R and "under" the graph of f.
b. Divide R into four squares with $m = n = 2$, and choose the sample point as the upper right corner point of each square (1,1),(2,1),(1,2), and (2,2) (Figure 15.4.4) to approximate the signed volume of the solid S that lies above R and "under" the graph of f.
c. Divide R into four squares with $m = n = 2$, and choose the sample point as the midpoint of each square: (1/2, 1/2), (3/2, 1/2), (1/2,3/2), and (3/2, 3/2) to approximate the signed volume.

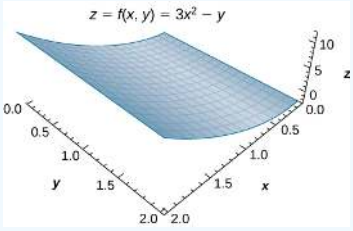


Figure 15.4.4: The function $z = f(x, y)$ graphed over the rectangular region $R = [0, 2] \times [0, 2]$.

Solution

a. As we can see, the function $z = f(x, y) = 3x^2 - y$ is above the plane. To find the signed volume of S, we need to divide the region R into small rectangles R_{ij}, each with area ΔA and with sides Δx and Δy, and choose (x_{ij}^*, y_{ij}^*) as sample points in each R_{ij}. Hence, a double integral is set up as

$$V = \iint_R (3x^2 - y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} [3(x_{ij}^*)^2 - y_{ij}^*]\Delta A.$$

b. Approximating the signed volume using a Riemann sum with $m = n = 2$ we have $\Delta A = \Delta x \Delta y = 1 \times 1 = 1$. Also, the sample points are (1, 1), (2, 1), (1, 2), and (2, 2) as shown in the following figure.

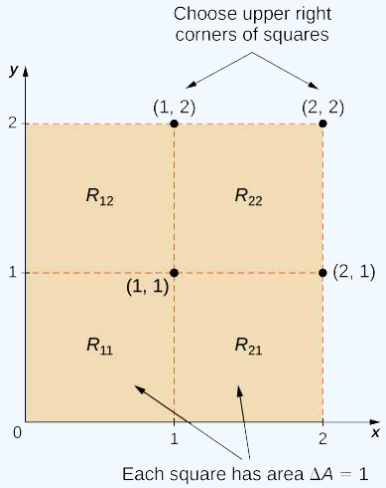


Figure 15.4.5: Subrectangles for the rectangular region $R = [0,2] \times [0,2]$.

Hence,

$$\begin{aligned} V &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\Delta A \\ &= \sum_{i=1}^{2} (f(x_{i1}^*, y_{i1}^*) + f(x_{i2}^*, y_{i2}^*))\Delta A \\ &= f(x_{11}^*, y_{11}^*)\Delta A + f(x_{21}^*, y_{21}^*)\Delta A + f(x_{12}^*, y_{12}^*)\Delta A + f(x_{22}^*, y_{22}^*)\Delta A \\ &= f(1,1)(1) + f(2,1)(1) + f(1,2)(1) + f(2,2)(1) \\ &= (3-1)(1) + (12-1)(1) + (3-2)(1) + (12-2)(1) \\ &= 2 + 11 + 1 + 10 = 24. \end{aligned}$$

c. Approximating the signed volume using a Riemann sum with $m = n = 2$ we have$\Delta A = \Delta x \Delta y = 1 \times 1 = 1$. In this case the sample points are (1/2, 1/2), (3/2, 1/2), (1/2, 3/2), and (3/2, 3/2).
Hence,

$$\begin{aligned} V &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\Delta A \\ &= f(x_{11}^*, y_{11}^*)\Delta A + f(x_{21}^*, y_{21}^*)\Delta A + f(x_{12}^*, y_{12}^*)\Delta A + f(x_{22}^*, y_{22}^*)\Delta A \\ &= f(1/2, 1/2)(1) + f(3/2, 1/2)(1) + f(1/2, 3/2)(1) + f(3/2, 3/2)(1) \\ &= \left(\frac{3}{4} - \frac{1}{4}\right)(1) + \left(\frac{27}{4} - \frac{1}{2}\right)(1) + \left(\frac{3}{4} - \frac{3}{2}\right)(1) + \left(\frac{27}{4} - \frac{3}{2}\right)(1) \\ &= \frac{2}{4} + \frac{25}{4} + \left(-\frac{3}{4}\right) + \frac{21}{4} = \frac{45}{4} = 11. \end{aligned}$$

Analysis

Notice that the approximate answers differ due to the choices of the sample points. In either case, we are introducing some error because we are using only a few sample points. Thus, we need to investigate how we can achieve an accurate answer.

? Exercise 15.4.1

Use the same function$z = f(x,y) = 3x^2 - y$ over the rectangular region $R = [0,2] \times [0,2]$.

Divide R into the same four squares with $m = n = 2$, and choose the sample points as the upper left corner point of each square (0,1), (1,1), (0,2), and (1,2) (Figure 15.4.5) to approximate the signed volume of the solid S that lies above R and "under" the graph of f.

Hint

Follow the steps of the previous example.

Answer

$$V \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\, \Delta A = 0$$

Note that we developed the concept of double integral using a rectangular region R. This concept can be extended to any general region. However, when a region is not rectangular, the subrectangles may not all fit perfectly into R, particularly if the base area is curved. We examine this situation in more detail in the next section, where we study regions that are not always rectangular and subrectangles may not fit perfectly in the region R. Also, the heights may not be exact if the surface $z = f(x, y)$ is curved. However, the errors on the sides and the height where the pieces may not fit perfectly within the solid S approach 0 as m and n approach infinity. Also, the double integral of the function $z = f(x, y)$ exists provided that the function f is not too discontinuous. If the function is bounded and continuous over R except on a finite number of smooth curves, then the double integral exists and we say that ff is integrable over R.

Since $\Delta A = \Delta x \Delta y = \Delta y \Delta x$, we can express dA as $dx\, dy$ or $dy\, dx$. This means that, when we are using rectangular coordinates, the double integral over a region R denoted by

$$\iint_R f(x, y)\, dA$$

can be written as

$$\iint_R f(x, y)\, dx\, dy$$

or

$$\iint_R f(x, y)\, dy\, dx.$$

Now let's list some of the properties that can be helpful to compute double integrals.

Properties of Double Integrals

The properties of double integrals are very helpful when computing them or otherwise working with them. We list here six properties of double integrals. Properties 1 and 2 are referred to as the linearity of the integral, property 3 is the additivity of the integral, property 4 is the monotonicity of the integral, and property 5 is used to find the bounds of the integral. Property 6 is used if $f(x, y)$ is a product of two functions $g(x)$ and $h(y)$.

Theorem: Properties of Double Integrals

Assume that the functions $f(x, y)$ and $g(x, y)$ are integrable over the rectangular region R; S and T are subregions of R; and assume that m and M are real numbers.

i. The sum $f(x, y) + g(x, y)$ is integrable and

$$\iint_R [f(x, y) + g(x, y)]\, dA = \iint_R f(x, y)\, dA + \iint_R g(x, y)\, dA.$$

ii. If c is a constant, then $cf(x, y)$ is integrable and

$$\iint_R cf(x, y)\, dA = c \iint_R f(x, y)\, dA.$$

iii. If $R = S \cup T$ and $S \cap T = \emptyset$ except an overlap on the boundaries, then

$$\iint_R f(x, y)\, dA = \iint_S f(x, y)\, dA + \iint_T f(x, y)\, dA.$$

iv. If $f(x, y) \geq g(x, y)$ for (x, y) in R, then

$$\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA.$$

v. If $m \leq f(x, y) \leq M$ and $A(R) =$ the area of R, then

$$m \cdot A(R) \leq \iint_R f(x, y)\, dA \leq M \cdot A(R).$$

vi. In the case where $f(x, y)$ can be factored as a product of a function $g(x)$ of x only and a function $h(y)$ of y only, then over the region $R = \{(x, y) \mid a \leq x \leq b,\ c \leq y \leq d\}$, the double integral can be written as

$$\iint_R f(x, y)\, dA = \left(\int_a^b g(x)\, dx \right) \left(\int_c^d h(y)\, dy \right).$$

These properties are used in the evaluation of double integrals, as we will see later. We will become skilled in using these properties once we become familiar with the computational tools of double integrals. So let's get to that now.

Iterated Integrals

So far, we have seen how to set up a double integral and how to obtain an approximate value for it. We can also imagine that evaluating double integrals by using the definition can be a very lengthy process if we choose larger values for m and n.Therefore, we need a practical and convenient technique for computing double integrals. In other words, we need to learn how to compute double integrals without employing the definition that uses limits and double sums.

The basic idea is that the evaluation becomes easier if we can break a double integral into single integrals by integrating first with respect to one variable and then with respect to the other. The key tool we need is called an iterated integral.

Definitions: Iterated Integrals

Assume a, b, c, and d are real numbers. We define an *iterated integral* for a function $f(x,y)$ over the rectangular region $R=[a,b]\times[c,d]$ as

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$$

or

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy.$$

The notation $\int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$ means that we integrate $f(x,y)$ with respect to y while holding x constant. Similarly, the notation $\int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$ means that we integrate $f(x,y)$ with respect to x while holding y constant. The fact that double integrals can be split into iterated integrals is expressed in Fubini's theorem. Think of this theorem as an essential tool for evaluating double integrals.

Theorem: Fubini's Theorem

Suppose that $f(x,y)$ is a function of two variables that is continuous over a rectangular region $R=\{(x,y)\in\mathbb{R}^2 \mid a\le x\le b,\ c\le y\le d\}$. Then we see from Figure 15.4.6 that the double integral of f over the region equals an iterated integral,

$$\iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

More generally, Fubini's theorem is true if f is bounded on R and f is discontinuous only on a finite number of continuous curves. In other words, f has to be integrable over R.

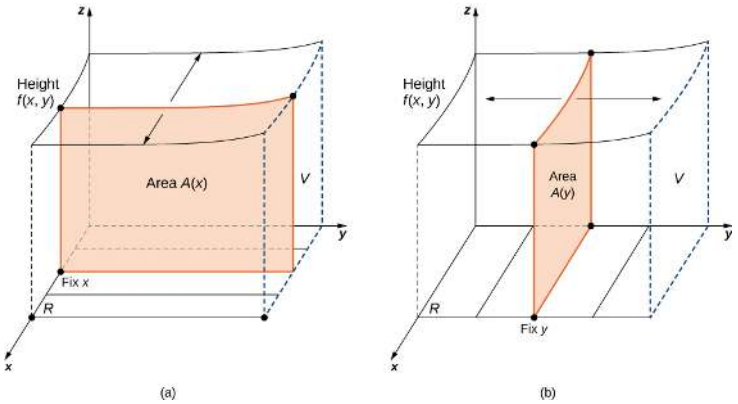


Figure 15.4.6: (a) Integrating first with respect to y and then with respect to x to find the area $A(x)$ and then the volume V; (b) integrating first with respect to x and then with respect to y to find the area $A(y)$ and then the volume V.

Example 15.4.2: Using Fubini's Theorem

Use Fubini's theorem to compute the double integral $\iint_R f(x,y)\,dA$ where $f(x,y)=x$ and $R=[0,2]\times[0,1]$.

Solution

Fubini's theorem offers an easier way to evaluate the double integral by the use of an iterated integral. Note how the boundary values of the region R become the upper and lower limits of integration.

$$\begin{aligned}\iint_R f(x,y)\,dA &= \iint_R f(x,y)\,dx\,dy \\ &= \int_{y=0}^{y=1}\int_{x=0}^{x=2} x\,dx\,dy \\ &= \int_{y=0}^{y=1}\left[\frac{x^2}{2}\Big|_{x=0}^{x=2}\right] dy \\ &= \int_{y=0}^{y=1} 2\,dy = 2y\Big|_{y=0}^{y=1} = 2\end{aligned}$$

The double integration in this example is simple enough to use Fubini's theorem directly, allowing us to convert a double integral into an iterated integral. Consequently, we are now ready to convert all double integrals to iterated integrals and demonstrate how the properties listed earlier can help us evaluate double integrals when the function $f(x,y)$ is more complex. Note that the order of integration can be changed (see Example 7).

Example 15.4.3: Illustrating Properties i and ii

Evaluate the double integral

$$\iint_R (xy - 3xy^2)\,dA, \text{ where } R = \{(x,y) \mid 0 \le x \le 2,\ 1 \le y \le 2\}.$$

Solution

This function has two pieces: one piece is xy and the other is $3xy^2$. Also, the second piece has a constant 3. Notice how we use properties i and ii to help evaluate the double integral.

$$\begin{aligned}\iint_R (xy-3xy^2)\,dA &= \iint_R xy\,dA + \iint_R (-3xy^2)\,dA && \text{Property i: Integral of a sum is the sum of the integrals.} \\ &= \int_{y=1}^{y=2}\int_{x=0}^{x=2} xy\,dx\,dy - \int_{y=1}^{y=2}\int_{x=0}^{x=2} 3xy^2\,dx\,dy && \text{Convert double integrals to iterated integrals.} \\ &= \int_{y=1}^{y=2}\left(\frac{x^2}{2}y\right)\Big|_{x=0}^{x=2} dy - 3\int_{y=1}^{y=2}\left(\frac{x^2}{2}y^2\right)\Big|_{x=0}^{x=2} dy && \text{Integrate with respect to } x\text{, holding } y \text{ constant.} \\ &= \int_{y=1}^{y=2} 2y\,dy - \int_{y=1}^{y=2} 6y^2 dy && \text{Property ii: Placing the constant before the integral.} \\ &= 2\int_1^2 y\,dy - 6\int_1^2 y^2\,dy && \text{Integrate with respect to y.} \\ &= 2\frac{y^2}{2}\Big|_1^2 - 6\frac{y^3}{3}\Big|_1^2 \\ &= y^2\Big|_1^2 - 2y^3\Big|_1^2 \\ &= (4-1) - 2(8-1) = 3 - 2(7) = 3 - 14 = -11.\end{aligned}$$

Example 15.4.4: Illustrating Property v.

Over the region $R = \{(x,y) \mid 1 \le x \le 3,\ 1 \le y \le 2\}$, we have $2 \le x^2 + y^2 \le 13$. Find a lower and an upper bound for the integral $\iint_R (x^2+y^2)\,dA$.

Solution

For a lower bound, integrate the constant function 2 over the region R. For an upper bound, integrate the constant function 13 over the region R.

$$\begin{aligned}\int_1^2\int_1^3 2\,dx\,dy &= \int_1^2 [2x\Big|_1^3]\,dy = \int_1^2 2(2)dy = 4y\Big|_1^2 = 4(2-1) = 4 \\ \int_1^2\int_1^3 13dx\,dy &= \int_1^2 [13x\Big|_1^3]\,dy = \int_1^2 13(2)\,dy = 26y\Big|_1^2 = 26(2-1) = 26.\end{aligned}$$

Hence, we obtain $4 \le \iint_R (x^2+y^2)\,dA \le 26$.

Example 15.4.5: Illustrating Property vi

Evaluate the integral $\iint_R e^y \cos x\,dA$ over the region $R = \{(x,y) \mid 0 \le x \le \frac{\pi}{2},\ 0 \le y \le 1\}$.

Solution

This is a great example for property vi because the function $f(x,y)$ is clearly the product of two single-variable functions e^y and $\cos x$. Thus we can split the integral into two parts and then integrate each one as a single-variable integration problem.

$$\begin{aligned}\iint_R e^y \cos x \, dA &= \int_0^1 \int_0^{\pi/2} e^y \cos x \, dx \, dy \\ &= \left(\int_0^1 e^y dy\right)\left(\int_0^{\pi/2} \cos x \, dx\right) \\ &= (e^y\Big|_0^1)(\sin x\Big|_0^{\pi/2}) \\ &= e-1.\end{aligned}$$

Exercise 15.4.2

a. Use the properties of the double integral and Fubini's theorem to evaluate the integral

$$\int_0^1 \int_{-1}^3 (3-x+4y)\, dy\, dx.$$

b. Show that $0 \le \iint_R \sin \pi x \, \cos \pi y \, dA \le \frac{1}{32}$ where $R = \left(0, \frac{1}{4}\right)\left(\frac{1}{4}, \frac{1}{2}\right)$.

Hint

Use properties i. and ii. and evaluate the iterated integral, and then use property v.

Answer

a. 26

b. Answers may vary.

As we mentioned before, when we are using rectangular coordinates, the double integral over a region R denoted by $\iint_R f(x,y)\, dA$ can be written as $\iint_R f(x,y)\, dx\, dy$ or $\iint_R f(x,y)\, dy\, dx$. The next example shows that the results are the same regardless of which order of integration we choose.

Example 15.4.6: Evaluating an Iterated Integral in Two Ways

Let's return to the function $f(x,y) = 3x^2 - y$ from Example 1, this time over the rectangular region $R = [0,2] \times [0,3]$. Use Fubini's theorem to evaluate $\iint_R f(x,y)\, dA$ in two different ways:

a. First integrate with respect to y and then with respect to x;
b. First integrate with respect to x and then with respect to y.

Solution

Figure 15.4.6 shows how the calculation works in two different ways.

a. First integrate with respect to y and then integrate with respect to x:

$$\begin{aligned}\iint_R f(x,y)\, dA &= \int_{x=0}^{x=2} \int_{y=0}^{y=3} (3x^2 - y)\, dy\, dx \\ &= \int_{x=0}^{x=2} \left(\int_{y=0}^{y=3} (3x^2 - y)\, dy\right) dx = \int_{x=0}^{x=2} \left[3x^2 y - \frac{y^2}{2}\Big|_{y=0}^{y=3}\right] dx \\ &= \int_{x=0}^{x=2} \left(9x^2 - \frac{9}{2}\right) dx = 3x^3 - \frac{9}{2}x\Big|_{x=0}^{x=2} = 15.\end{aligned}$$

b. First integrate with respect to x and then integrate with respect to y:

$$\begin{aligned}\iint_R f(x,y)\, dA &= \int_{y=0}^{y=3} \int_{x=0}^{x=2} (3x^2 - y)\, dx\, dy \\ &= \int_{y=0}^{y=3} \left(\int_{x=0}^{x=2} (3x^2 - y)\, dx\right) dy \\ &= \int_{y=0}^{y=3} \left[x^3 - xy\Big|_{x=0}^{x=2}\right] dy \\ &= \int_{y=0}^{y=3} (8-2y)\, dy = 8y - y^2\Big|_{y=0}^{y=3} = 15.\end{aligned}$$

Analysis

With either order of integration, the double integral gives us an answer of 15. We might wish to interpret this answer as a volume in cubic units of the solid S below the function $f(x,y)=3x^2-y$ over the region $R=[0,2]\times[0,3]$. However, remember that the interpretation of a double integral as a (non-signed) volume works only when the integrand f is a nonnegative function over the base region R.

? Exercise 15.4.3

Evaluate

$$\int_{y=-3}^{y=2}\int_{x=3}^{x=5}(2-3x^2+y^2)\,dx\,dy.$$

Hint

Use Fubini's theorem.

Answer

$-\frac{1340}{3}$

In the next example we see that it can actually be beneficial to switch the order of integration to make the computation easier. We will come back to this idea several times in this chapter.

✓ Example 15.4.7: Switching the Order of Integration

Consider the double integral $\iint_R x\,\sin(xy)\,dA$ over the region $R=\{(x,y)\mid 0\le x\le\pi,\ 1\le y\le 2\}$ (Figure 15.4.7).

a. Express the double integral in two different ways.
b. Analyze whether evaluating the double integral in one way is easier than the other and why.
c. Evaluate the integral.

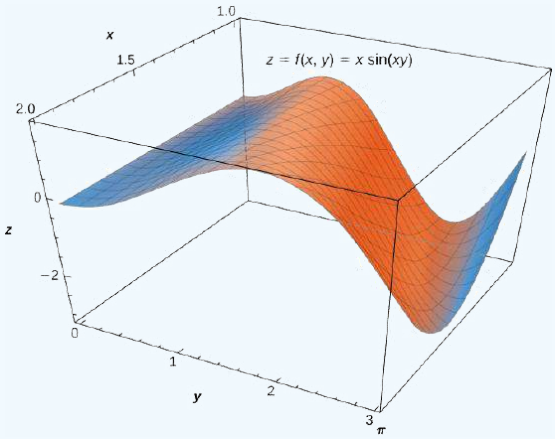


Figure 15.4.7: The function $z=f(x,y)=x\,\sin(xy)$ over the rectangular region $R=[0,\pi]\times[1,2]$.

a. We can express $\iint_R x\,\sin(xy)\,dA$ in the following two ways: first by integrating with respect to y and then with respect to x; second by integrating with respect to x and then with respect to y.

$$\iint_R x\,\sin(xy)\,dA=\int_{x=0}^{x=\pi}\int_{y=1}^{y=2}x\,\sin(xy)\,dy\,dx$$

Integrate first with respect to y.

$$=\int_{y=1}^{y=2}\int_{x=0}^{x=\pi}x\,\sin(xy)\,dx\,dy$$

Integrate first with respect to x.

b. If we want to integrate with respect to y first and then integrate with respect to x, we see that we can use the substitution $u=xy$, which gives $du=x\,dy$. Hence the inner integral is simply $\int\sin u\,du$ and we can change the limits to be functions of x,

$$\iint_R x\,\sin(xy)\,dA=\int_{x=0}^{x=\pi}\int_{y=1}^{y=2}x\,\sin(xy)\,dy\,dx=\int_{x=0}^{x=\pi}\left[\int_{u=x}^{u=2x}\sin(u)\,du\right]dx.$$

However, integrating with respect to x first and then integrating with respect to y requires integration by parts for the inner integral, with $u=x$ and $dv=\sin(xy)dx$

Then $du = dx$ and $v = -\frac{\cos(xy)}{y}$, so

$$\iint_R x\sin(xy)\,dA = \int_{y=1}^{y=2}\int_{x=0}^{x=\pi} x\sin(xy)\,dx\,dy = \int_{y=1}^{y=2}\left[-\frac{x\,\cos(xy)}{y}\Big|_{x=0}^{x=\pi} + \frac{1}{y}\int_{x=0}^{x=\pi}\cos(xy)\,dx\right]dy.$$

Since the evaluation is getting complicated, we will only do the computation that is easier to do, which is clearly the first method.

c. Evaluate the double integral using the easier way.

$$\begin{aligned}\iint_R x\,\sin(xy)\,dA &= \int_{x=0}^{x=\pi}\int_{y=1}^{y=2} x\,\sin(xy)\,dy\,dx \\ &= \int_{x=0}^{x=\pi}\left[\int_{u=x}^{u=2x}\sin(u)\,du\right]dx = \int_{x=0}^{x=\pi}\left[-\cos u\Big|_{u=x}^{u=2x}\right]dx \\ &= \int_{x=0}^{x=\pi}(-\cos 2x + \cos x)\,dx \\ &= \left(-\frac{1}{2}\sin 2x + \sin x\right)\Big|_{x=0}^{x=\pi} = 0.\end{aligned}$$

? Exercise 15.4.4

Evaluate the integral $\iint_R xe^{xy}\,dA$ where $R = [0,1]\times[0,\ln 5]$.

Hint

Integrate with respect to y first.

Answer

$\frac{4-\ln 5}{\ln 5}$

Applications of Double Integrals

Double integrals are very useful for finding the area of a region bounded by curves of functions. We describe this situation in more detail in the next section. However, if the region is a rectangular shape, we can find its area by integrating the constant function $f(x,y) = 1$ over the region R.

Definition: Area of a Region R

The area of the region R is given by

$$A(R) = \iint_R 1\,dA.$$

This definition makes sense because using $f(x,y) = 1$ and evaluating the integral make it a product of length and width. Let's check this formula with an example and see how this works.

Example 15.4.8: Finding Area Using a Double Integral

Find the area of the region $R = \{(x,y)\,|\,0 \le x \le 3,\ 0 \le y \le 2\}$ by using a double integral, that is, by integrating 1 over the region R.

Solution

The region is rectangular with length 3 and width 2, so we know that the area is 6. We get the same answer when we use a double integral:

$$A(R) = \int_0^2\int_0^3 1\,dx\,dy = \int_0^2 [x]_0^3\,dy = \int_0^2 3dy = 3\int_0^2 dy = 3y\Big|_0^2 = 3(2) = 6\text{ units}^2.$$

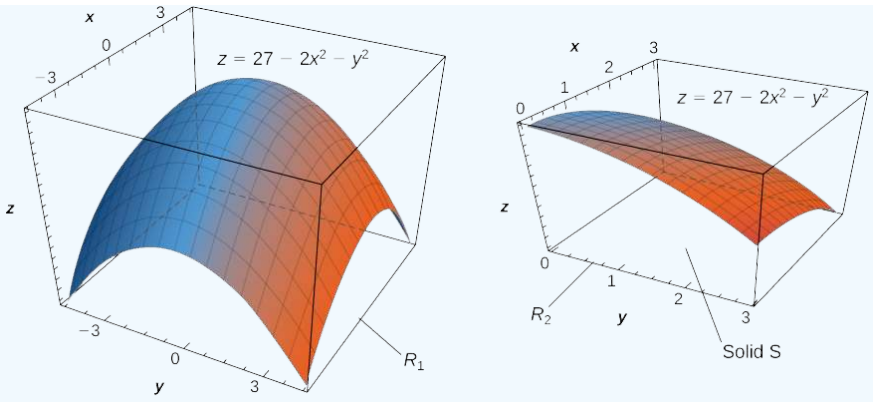
We have already seen how double integrals can be used to find the volume of a solid bounded above by a function $f(x,y) \ge 0$ over a region R provided $f(x,y) \ge 0$ for all (x,y) in R. Here is another example to illustrate this concept.

Example 15.4.9: Volume of an Elliptic Paraboloid

Find the volume V of the solid S that is bounded by the elliptic paraboloid $2x^2 + y^2 + z = 27$, the planes $x = 3$ and $y = 3$, and the three coordinate planes.

Solution

First notice the graph of the surface $z = 27 - 2x^2 - y^2$ in Figure 15.4.8(a) and above the square region $R_1 = [-3,3]\times[-3,3]$. However, we need the volume of the solid bounded by the elliptic paraboloid $2x^2 + y^2 + z = 27$, the planes $x = 3$ and $y = 3$, and the three coordinate planes.



(a) (b)

Figure 15.4.8: (a) The surface $z = 27 - 2x^2 - y^2$ above the square region $R1 = [-3,3] \times [-3,3]$. (b) The solid S lies under the surface $z = 27 - 2x^2 - y^2$ above the square region $R1 = [0,3] \times [0,3]$.

Now let's look at the graph of the surface in Figure 15.4.8(b). We determine the volume V by evaluating the double integral over R_2:

$$\begin{aligned} V &= \iint_R z\,dA = \iint_R (27 - 2x^2 - y^2)\,dA \\ &= \int_{y=0}^{y=3} \int_{x=0}^{x=3} (27 - 2x^2 - y^2)\,dx\,dy && \text{Convert to literal integral.} \\ &= \int_{y=0}^{y=3} \left[27x - \frac{2}{3}x^3 - y^2 x\right]\Big|_{x=0}^{x=3} dy && \text{Integrate with respect to } x. \\ &= \int_{y=0}^{y=3} (63 - 3y^2)dy = 63y - y^3\Big|_{y=0}^{y=3} = 162. \end{aligned}$$

? Exercise 15.4.5

Find the volume of the solid bounded above by the graph of $f(x,y) = xy\sin(x^2 y)$ and below by the xy-plane on the rectangular region $R = [0,1] \times [0,\pi]$.

Hint

Graph the function, set up the integral, and use an iterated integral.

Answer

$\frac{\pi}{2}$

Recall that we defined the average value of a function of one variable on an interval $[a,b]$ as

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Similarly, we can define the average value of a function of two variables over a region R. The main difference is that we divide by an area instead of the width of an interval.

Definition: Average Value of a Function

The **average value of a function** of two variables over a region R is

$$F_{ave} = \frac{1}{\text{Area of } R}\iint_R f(x,y)\,dx\,dy.$$

In the next example we find the average value of a function over a rectangular region. This is a good example of obtaining useful information for an integration by making individual measurements over a grid, instead of trying to find an algebraic expression for a function.

Example 15.4.10: Calculating Average Storm Rainfall

The weather map in Figure 15.4.9 shows an unusually moist storm system associated with the remnants of Hurricane Karl, which dumped 4–8 inches (100–200 mm) of rain in some parts of the Midwest on September 22–23, 2010. The area of rainfall measured 300 miles east to west and 250 miles north to south. Estimate the average rainfall over the entire area in those two days.

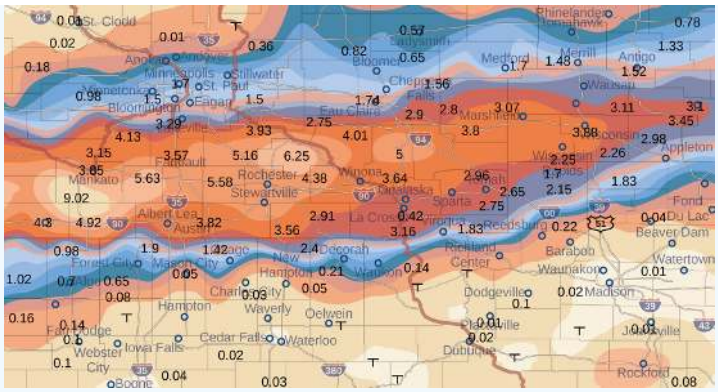


Figure 15.4.9: Effects of Hurricane Karl, which dumped 4–8 inches (100–200 mm) of rain in some parts of southwest Wisconsin, southern Minnesota, and southeast South Dakota over a span of 300 miles east to west and 250 miles north to south.

Solution

Place the origin at the southwest corner of the map so that all the values can be considered as being in the first quadrant and hence all are positive. Now divide the entire map into six rectangles ($m = 2$ and $n = 3$), as shown in Figure 15.4.9. Assume $f(x, y)$ denotes the storm rainfall in inches at a point approximately x miles to the east of the origin and y miles to the north of the origin. Let R represent the entire area of $250 \times 300 = 75000$ square miles. Then the area of each subrectangle is

$$\Delta A = \frac{1}{6}(75000) = 12500.$$

Assume $(x_{ij}*, y_{ij}*)$ are approximately the midpoints of each subrectangle R_{ij}. Note the color-coded region at each of these points, and estimate the rainfall. The rainfall at each of these points can be estimated as:

- At (x_{11}, y_{11}), the rainfall is 0.08.
- At (x_{12}, y_{12}), the rainfall is 0.08.
- At (x_{13}, y_{13}), the rainfall is 0.01.
- At (x_{21}, y_{21}), the rainfall is 1.70.
- At (x_{22}, y_{22}), the rainfall is 1.74.
- At (x_{23}, y_{23}), the rainfall is 3.00.

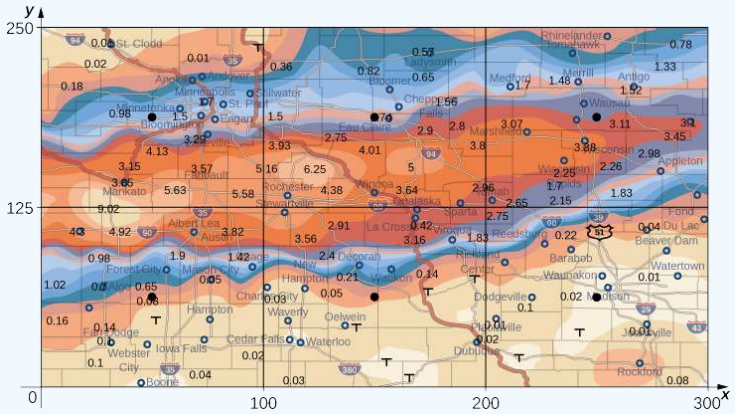


Figure 15.4.10: Storm rainfall with rectangular axes and showing the midpoints of each subrectangle.

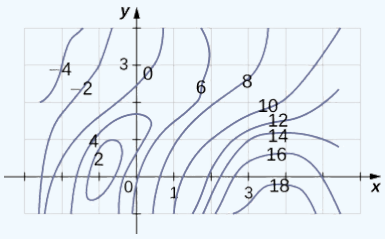
According to our definition, the average storm rainfall in the entire area during those two days was

$$
\begin{aligned}
f_{ave} = \frac{1}{Area\ R}\iint_R f(x,y)\,dx\,dy &= \frac{1}{75000}\iint_R f(x,y)\,dx\,dy \\
&\approx \frac{1}{75000}\sum_{i=1}^{3}\sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\Delta A \\
&= \frac{1}{75000}\left[f(x_{11}^*, y_{11}^*)\Delta A + f(x_{12}^*, y_{12}^*)\Delta A + f(x_{13}^*, y_{13}^*)\Delta A + f(x_{21}^*, y_{21}^*)\Delta A + f(x_{22}^*, y_{22}^*)\Delta A + f(x_{23}^*, y_{23}^*)\Delta A\right] \\
&\approx \frac{1}{75000}\Big[0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00\Big]\Delta A \\
&= \frac{1}{75000}\Big[0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00\Big]12500 \\
&= \frac{1}{6}\Big[0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00\Big] \\
&\approx 1.10 \text{ in.}
\end{aligned}
$$

During September 22–23, 2010 this area had an average storm rainfall of approximately 1.10 inches.

? Exercise 15.4.6

A contour map is shown for a function $f(x,y)$ on the rectangle $R = [-3,6]\times[-1,4]$.



a. Use the midpoint rule with $m = 3$ and $n = 2$ to estimate the value of $\iint_R f(x,y)\,dA$.

b. Estimate the average value of the function $f(x,y)$.

Hint

Divide the region into six rectangles, and use the contour lines to estimate the values for $f(x,y)$.

Answer

Answers to both parts a. and b. may vary.

Key Concepts

- We can use a double Riemann sum to approximate the volume of a solid bounded above by a function of two variables over a rectangular region. By taking the limit, this becomes a double integral representing the volume of the solid.
- Properties of double integral are useful to simplify computation and find bounds on their values.
- We can use Fubini's theorem to write and evaluate a double integral as an iterated integral.
- Double integrals are used to calculate the area of a region, the volume under a surface, and the average value of a function of two variables over a rectangular region.

Key Equations

- $$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i j*, y_i j*)\,\Delta A$$
- $$\int_a^b\int_c^d f(x,y)\,dx\,dy = \int_a^b\left[\int_c^d f(x,y)\,dy\right]dx$$

  or

  $$\int_c^d\int_a^b f(x,y)\,dx\,dy = \int_c^d\left[\int_a^b f(x,y)\,dx\right]dy$$
- $$f_{ave} = \frac{1}{\text{Area of } R}\iint_R f(x,y)\,dx\,dy$$

Glossary

double integral

of the function $f(x,y)$ over the region R in the xy-plane is defined as the limit of a double Riemann sum,

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A.$$

double Riemann sum

of the function $f(x,y)$ over a rectangular region R is

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A,$$

where R is divided into smaller subrectangles R_{ij} and (x_{ij}^*, y_{ij}^*) is an arbitrary point in R_{ij}

Fubini's theorem

if $f(x,y)$ is a function of two variables that is continuous over a rectangular region $R = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ c \le y \le d\}$, then the double integral of f over the region equals an iterated integral,

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dx\,dy = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

iterated integral

for a function $f(x,y)$ over the region R is

a. $\displaystyle\int_a^b \int_c^d f(x,y)\,dx\,dy = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx,$

b. $\displaystyle\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy,$

where a, b, c, and d are any real numbers and $R = [a,b] \times [c,d]$

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15.6: Triple Integrals

Learning Objectives

- Recognize when a function of three variables is integrable over a rectangular box.
- Evaluate a triple integral by expressing it as an iterated integral.
- Recognize when a function of three variables is integrable over a closed and bounded region.
- Simplify a calculation by changing the order of integration of a triple integral.
- Calculate the average value of a function of three variables.

Previously, we discussed the double integral of a function $f(x,y)$ of two variables over a rectangular region in the plane. In this section we define the triple integral of a function $f(x,y,z)$ of three variables over a rectangular solid box in space, $\mathbb{R}^3$. Later in this section we extend the definition to more general regions in $\mathbb{R}^3$.

Integrable Functions of Three Variables

We can define a rectangular box B in $\mathbb{R}^3$ as

$$B = \{(x,y,z) \mid a \le x \le b,\ c \le y \le d,\ e \le z \le f\}.$$

We follow a similar procedure to what we did in previously. We divide the interval $[a,b]$ into l subintervals $[x_{i-1}, x_i]$ of equal length Δx with

$$\Delta x = \frac{x_i - x_{i-1}}{l},$$

divide the interval $[c,d]$ into m subintervals $[y_{i-1}, y_i]$ of equal length Δy with

$$\Delta y = \frac{y_j - y_{j-1}}{m},$$

and divide the interval $[e,f]$ into n subintervals $[z_{i-1}, z_i]$ of equal length Δz with

$$\Delta z = \frac{z_k - z_{k-1}}{n}$$

Then the rectangular box B is subdivided into lmn subboxes:

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{i-1}, y_i] \times [z_{i-1}, z_i],$$

as shown in Figure 15.6.1.

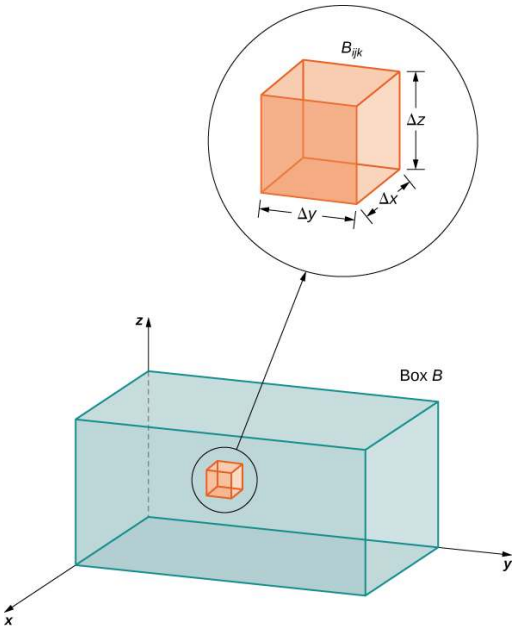


Figure 15.6.1: A rectangular box in $\mathbb{R}^3$ divided into subboxes by planes parallel to the coordinate planes.

For each i, j, and k, consider a sample point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ in each sub-box B_{ijk}. We see that its volume is $\Delta V = \Delta x \Delta y \Delta z$. Form the triple Riemann sum

$$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\, \Delta x \Delta y \Delta z.$$

We define the triple integral in terms of the limit of a triple Riemann sum, as we did for the double integral in terms of a double Riemann sum.

Definition: The triple integral

The triple integral of a function $f(x,y,z)$ over a rectangular box B is defined as

$$\lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\,\Delta x\Delta y\Delta z = \iiint_B f(x,y,z)\,dV$$

if this limit exists.

When the triple integral exists on B the function $f(x,y,z)$ is said to be integrable on B. Also, the triple integral exists if $f(x,y,z)$ is continuous on B. Therefore, we will use continuous functions for our examples. However, continuity is sufficient but not necessary; in other words, f is bounded on B and continuous except possibly on the boundary of B. The sample point $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ can be any point in the rectangular sub-box B_{ijk} and all the properties of a double integral apply to a triple integral. Just as the double integral has many practical applications, the triple integral also has many applications, which we discuss in later sections.

Now that we have developed the concept of the triple integral, we need to know how to compute it. Just as in the case of the double integral, we can have an iterated triple integral, and consequently, a version of **Fubini's theorem** for triple integrals exists.

Fubini's Theorem for Triple Integrals

If $f(x,y,z)$ is continuous on a rectangular box $B=[a,b]\times[c,d]\times[e,f]$, then

$$\iiint_B f(x,y,z)\,dV = \int_e^f\int_c^d\int_a^b f(x,y,z)\,dx\,dy\,dz.$$

This integral is also equal to any of the other five possible orderings for the iterated triple integral.

For a, b, c, d, e and f real numbers, the iterated triple integral can be expressed in six different orderings:

$$\int_e^f\int_c^d\int_a^b f(x,y,z)\,dx\,dy\,dz = \int_e^f\left(\int_c^d\left(\int_a^b f(x,y,z)\,dx\right)dy\right)dz \tag{15.6.1}$$

$$= \int_c^d\left(\int_e^f\left(\int_a^b f(x,y,z)\,dx\right)dz\right)dy \tag{15.6.2}$$

$$= \int_a^b\left(\int_e^f\left(\int_c^d f(x,y,z)\,dy\right)dz\right)dx \tag{15.6.3}$$

$$= \int_e^f\left(\int_a^b\left(\int_c^d f(x,y,z)\,dy\right)dx\right)dz \tag{15.6.4}$$

$$= \int_c^d\left(\int_a^b\left(\int_c^d f(x,y,z)\,dz\right)dx\right)dy \tag{15.6.5}$$

$$= \int_a^b\left(\int_c^d\left(\int_e^f f(x,y,z)\,dz\right)dy\right)dx \tag{15.6.6}$$

For a rectangular box, the order of integration does not make any significant difference in the level of difficulty in computation. We compute triple integrals using Fubini's Theorem rather than using the Riemann sum definition. We follow the order of integration in the same way as we did for double integrals (that is, from inside to outside).

Example 15.6.1: Evaluating a Triple Integral

Evaluate the triple integral

$$\int_{z=0}^{z=1}\int_{y=2}^{y=4}\int_{x=-1}^{x=5}(x+yz^2)\,dx\,dy\,dz.$$

Solution

The order of integration is specified in the problem, so integrate with respect to x first, then y, and then z.

$$
\begin{aligned}
&\int_{z=0}^{z=1}\int_{y=2}^{y=4}\int_{x=-1}^{x=5}(x+yz^2)\,dx\,dy\,dz \\
&=\int_{z=0}^{z=1}\int_{y=2}^{y=4}\left[\frac{x^2}{2}+xyz^2\Big|_{x=-1}^{x=5}\right]dy\,dz && \text{Integrate with respect to } x. \\
&=\int_{z=0}^{z=1}\int_{y=2}^{y=4}\left[12+6yz^2\right]dy\,dz && \text{Evaluate.} \\
&=\int_{z=0}^{z=1}\left[12y+6\frac{y^2}{2}z^2\Big|_{y=2}^{y=4}\right]dz && \text{Integrate with respect to } y. \\
&=\int_{z=0}^{z=1}\left[24+36z^2\right]dz && \text{Evaluate.} \\
&=\left[24z+36\frac{z^3}{3}\right]_{z=0}^{z=1} && \text{Integrate with respect to } z. \\
&=36. && \text{Evaluate.}
\end{aligned}
$$

Example 15.6.2: Evaluating a Triple Integral

Evaluate the triple integral

$$\iiint_B x^2yz\,dV$$

where $B=\{(x,y,z)\mid -2\le x\le 1,\ 0\le y\le 3,\ 1\le z\le 5\}$ as shown in Figure 15.6.2

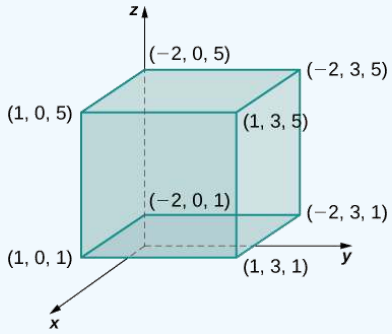


Figure 15.6.2: Evaluating a triple integral over a given rectangular box.

Solution

The order is not specified, but we can use the iterated integral in any order without changing the level of difficulty. Choose, say, to integrate y first, then x, and then z.

$$
\begin{aligned}
\iiint_B x^2yz\,dV &= \int_1^5\int_{-2}^1\int_0^3 [x^2yz]\,dy\,dx\,dz \\
&= \int_1^5\int_{-2}^1\left[x^2\frac{y^3}{3}z\Big|_0^3\right]dx\,dz \\
&= \int_1^5\int_{-2}^1 \frac{y}{2}x^2z\,dx\,dz \\
&= \int_1^5\left[\frac{9}{2}\frac{x^3}{3}z\Big|_{-2}^1\right]dz \\
&= \int_1^5 \frac{27}{2}z\,dz \\
&= \frac{27}{2}\frac{z^2}{2}\Big|_1^5 = 162.
\end{aligned}
$$

Now try to integrate in a different order just to see that we get the same answer. Choose to integrate with respect to x first, then z, then y

$$\begin{aligned}\iiint_B x^2yz\,dV &= \int_0^3\int_1^5\int_{-2}^1 [x^2yz]\,dx\,dz\,dy \\ &= \int_0^3\int_1^5 \left[\frac{x^3}{3}yz\Big|_{-2}^1\right] dz\,dy \\ &= \int_0^3\int_1^5 3yz\,dz\,dy \\ &= \int_0^3 \left[3y\frac{z^2}{2}\Big|_1^5\right] dy \\ &= \int_0^3 36y\,dy \\ &= 36\frac{y^2}{2}\Big|_0^3 = 18(9-0) = 162.\end{aligned}$$

? Exercise 15.6.1

Evaluate the triple integral

$$\iiint_B z\sin x\,\cos y\,dV$$

where $B = \left\{(x,y,z)\,|\,0\le x\le\pi,\ \frac{3\pi}{2}\le y\le 2\pi,\ 1\le z\le 3\right\}$.

Hint

Follow the steps in the previous example.

Answer

$$\iiint_B z\sin x\,\cos y\,dV = 8$$

Triple Integrals over a General Bounded Region

We now expand the definition of the triple integral to compute a triple integral over a more **general bounded region *E*** in $\mathbb{R}^3$. The general bounded regions we will consider are of three types. First, let D be the bounded region that is a projection of E onto the xy-plane. Suppose the region E in $\mathbb{R}^3$ has the form

$$E = \left\{(x,y,z)\,|\,(x,y)\in D, u_1(x,y)\le z\le u_2(x,y)\right\}.$$

For two functions $z = u_1(x,y)$ and $u_2(x,y)$, such that $u_1(x,y)\le u_2(x,y)$ for all (x,y) in D as shown in the following figure.

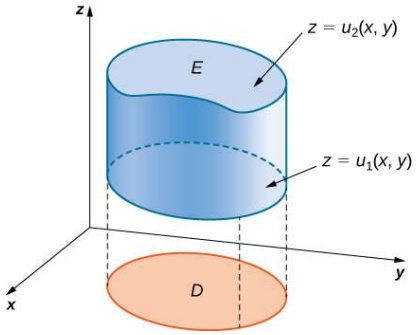


Figure 15.6.3: We can describe region E as the space between $u_1(x,y)$ and $u_2(x,y)$ above the projection D of E onto the xy-plane.

Triple Integral over a General Region

The triple integral of a continuous function $f(x,y,z)$ over a general three-dimensional region

$$E = \left\{(x,y,z)\,|\,(x,y)\in D,\ u_1(x,y)\le z\le u_2(x,y)\right\}$$

in $\mathbb{R}^3$, where D is the projection of E onto the xy-plane, is

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA.$$

Similarly, we can consider a general bounded region D in the xy-plane and two functions $y = u_1(x, z)$ and $y = u_2(x, z)$ such that $u_1(x, z) \le u_2(x, z)$ for all (x, z) in D. Then we can describe the solid region E in $\mathbb{R}^3$ as

$$E = \{(x, y, z) \mid (x, z) \in D,\ u_1(x, z) \le z \le u_2(x, z)\}$$

where D is the projection of E onto the xy-plane and the triple integral is

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\, dy \right] dA.$$

Finally, if D is a general bounded region in the xy-plane and we have two functions $x = u_1(y, z)$ and $x = u_2(y, z)$ such that $u_1(y, z) \le u_2(y, z)$ for all (y, z) in D, then the solid region E in $\mathbb{R}^3$ can be described as

$$E = \{(x, y, z) \mid (y, z) \in D,\ u_1(y, z) \le z \le u_2(y, z)\}$$

where D is the projection of E onto the xy-plane and the triple integral is

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\, dx \right] dA.$$

Note that the region D in any of the planes may be of Type I or Type II as described in previously. If D in the xy-plane is of Type I (Figure 15.6.4), then

$$E = \{(x, y, z) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x),\ u_1(x, y) \le z \le u_2(x, y)\}.$$

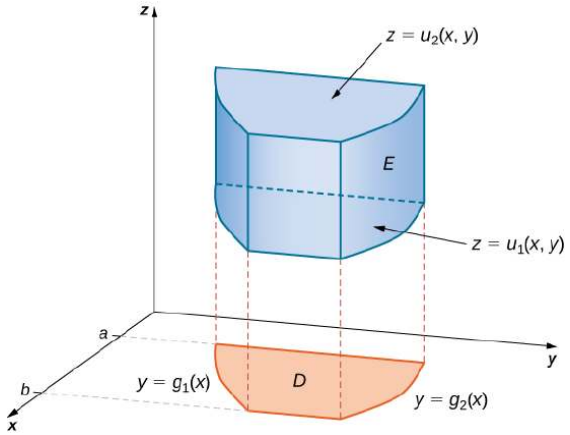


Figure 15.6.4: A box E where the projection D in the xy-plane is of Type I.

Then the triple integral becomes

$$\iiint_E f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\, dy\, dx.$$

If D in the xy-plane is of Type II (Figure 15.6.5), then

$$E = \{(x, y, z) \mid c \le x \le d, h_1(x) \le y \le h_2(x),\ u_1(x, y) \le z \le u_2(x, y)\}.$$

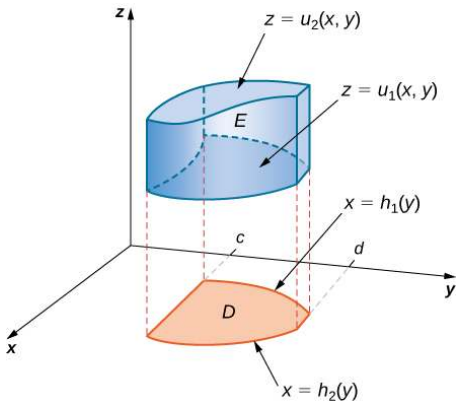


Figure 15.6.5: A box E where the projection D in the xy-plane is of Type II.

Then the triple integral becomes

$$\iiint_E f(x,y,z)\,dV = \int_{y=c}^{y=d}\int_{x=h_1(y)}^{x=h_2(y)}\int_{z=u_1(x,y)}^{z=u_2(x,y)} f(x,y,z)\,dz\,dx\,dy.$$

Example 15.6.3A: Evaluating a Triple Integral over a General Bounded Region

Evaluate the triple integral of the function $f(x,y,z) = 5x - 3y$ over the solid tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

Solution

Figure 15.6.6 shows the solid tetrahedron E and its projection D on the xy-plane.

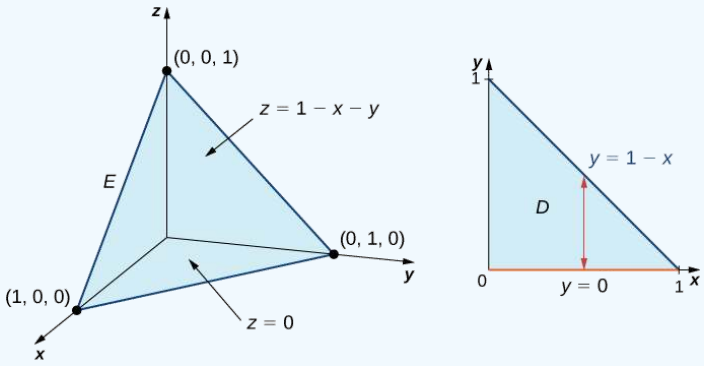


Figure 15.6.6: The solid E has a projection D on the xy-plane of Type I.

We can describe the solid region tetrahedron as

$$E = \{(x,y,z) \mid 0 \le x \le 1,\ 0 \le y \le 1 - x,\ 0 \le z \le 1 - x - y\}.$$

Hence, the triple integral is

$$\iiint_E f(x,y,z)\,dV = \int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\int_{z=0}^{z=1-x-y} (5x - 3y)\,dz\,dy\,dx.$$

To simplify the calculation, first evaluate the integral $\int_{z=0}^{z=1-x-y} (5x - 3y)\,dz$. We have

$$\int_{z=0}^{z=1-x-y} (5x - 3y)\,dz = (5x - 3y)z\Big|_{z=0}^{z=1-x-y} = (5x - 3y)(1 - x - y).$$

Now evaluate the integral

$$\int_{y=0}^{y=1-x} (5x - 3y)(1 - x - y)\,dy,$$

obtaining

$$\int_{y=0}^{y=1-x} (5x - 3y)(1 - x - y)\,dy = \frac{1}{2}(x-1)^2(6x-1).$$

Finally evaluate

$$\int_{x=0}^{x=1} \frac{1}{2}(x-1)^2(6x-1)\,dx = \frac{1}{12}.$$

Putting it all together, we have

$$\iiint_E f(x,y,z)\,dV = \int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\int_{z=0}^{z=1-x-y} (5x - 3y)\,dz\,dy\,dx = \frac{1}{12}.$$

Just as we used the double integral

$$\iint_D 1\,dA$$

to find the area of a general bounded region D we can use

$$\iiint_E 1\,dV$$

to find the volume of a general solid bounded region E. The next example illustrates the method.

Example 15.6.3B: Finding a Volume by Evaluating a Triple Integral

Find the volume of a right pyramid that has the square base in the xy-plane $[-1,1]\times[-1,1]$ and vertex at the point $(0,0,1)$ as shown in the following figure.

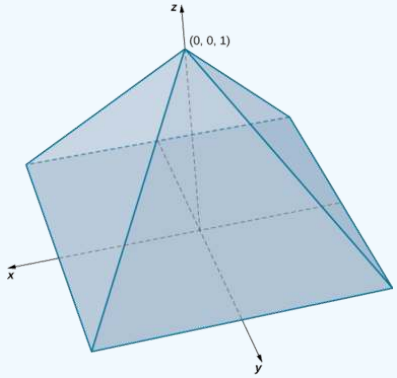


Figure 15.6.7: Finding the volume of a pyramid with a square base.

Solution

In this pyramid the value of z changes from 0 to 1 and at each height z the cross section of the pyramid for any value of z is the square

$$[-1+z,\,1-z]\times[-1+z,\,1-z].$$

Hence, the volume of the pyramid is

$$\iiint_E 1\,dV$$

where

$$E=\{(x,y,z)\,|\,0\le z\le 1,\ -1+z\le y\le 1-z,\ -1+z\le x\le 1-z\}.$$

Thus, we have

$$\begin{aligned}\iiint_E 1\,dV &= \int_{z=0}^{z=1}\int_{y=-1+z}^{y=1-z}\int_{x=-1+z}^{x=1-z} 1\,dx\,dy\,dz\\ &= \int_{z=0}^{z=1}\int_{y=-1+z}^{y=1-z}(2-2z)\,dy\,dz\\ &= \int_{z=0}^{z=1}(2-2z)^2\,dz=\frac{4}{3}.\end{aligned}$$

Hence, the volume of the pyramid is $\frac{4}{3}$ cubic units.

Exercise 15.6.3

Consider the solid sphere $E=\{(x,y,z)\,|\,x^2+y^2+z^2=9\}$. Write the triple integral

$$\iiint_E f(x,y,z)\,dV$$

for an arbitrary function f as an iterated integral. Then evaluate this triple integral with $f(x,y,z)=1$. Notice that this gives the volume of a sphere using a triple integral.

Hint

Follow the steps in the previous example. Use symmetry.

Answer

$$\begin{aligned}\iiint_E 1\,dV = 8\int_{x=-3}^{x=3}\int_{y=-\sqrt{9-z^2}}^{y=\sqrt{9-z^2}}\int_{z=-\sqrt{9-x^2-y^2}}^{z=\sqrt{9-x^2-y^2}} 1\,dz\,dy\,dx\\ = 36\pi \text{ cubic units.}\end{aligned}$$

Changing the Order of Integration

As we have already seen in double integrals over general bounded regions, changing the order of the integration is done quite often to simplify the computation. With a triple integral over a rectangular box, the order of integration does not change the level of difficulty of the calculation. However, with a triple integral over a general bounded region, choosing an appropriate order of integration can simplify the computation quite a bit. Sometimes making the change to polar coordinates can also be very helpful. We demonstrate two examples here.

Example 15.6.4: Changing the Order of Integration

Consider the iterated integral

$$\int_{x=0}^{x=1}\int_{y=0}^{y=x^2}\int_{z=0}^{z=y} f(x,y,z)\,dz\,dy\,dx.$$

The order of integration here is first with respect to z, then y, and then x. Express this integral by changing the order of integration to be first with respect to x, then z, and then y. Verify that the value of the integral is the same if we let $f(x,y,z)=xyz$.

Solution

The best way to do this is to sketch the region E and its projections onto each of the three coordinate planes. Thus, let

$$E=\{(x,y,z)\,|\,0\le x\le 1,\ 0\le y\le x^2,\ 0\le z\le y\}.$$

and

$$\int_{x=0}^{x=1}\int_{y=0}^{y=x^2}\int_{z=0}^{z=x^2} f(x,y,z)\,dz\,dy\,dx=\iiint_E f(x,y,z)\,dV.$$

We need to express this triple integral as

$$\int_{y=c}^{y=d}\int_{z=v_2(y)}^{z=v_2(y)}\int_{x=u_1(y,z)}^{x=u_2(y,z)} f(x,y,z)\,dx\,dz\,dy.$$

Knowing the region E we can draw the following projections (Figure 15.6.8):

on the xy-plane is $D_1=\{(x,y)\,|\,0\le x\le 1,\ 0\le y\le x^2\}=\{(x,y)\,|\,0\le y\le 1,\ \sqrt{y}\le x\le 1\}$,

on the yz-plane is $D_2=\{(y,z)\,|\,0\le y\le 1,\ 0\le z\le y^2\}$, and

on the xz-plane is $D_3=\{(x,z)\,|\,0\le x\le 1,\ 0\le z\le x^2\}$.

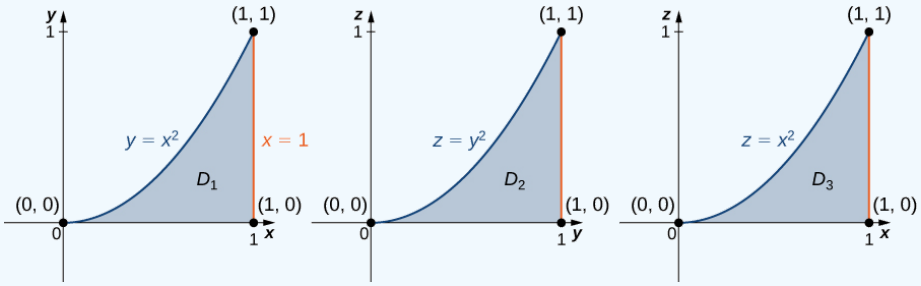


Figure 15.6.8. The three cross sections of E on the three coordinate planes.

Now we can describe the same region E as $\{(x,y,z)\,|\,0\le y\le 1,\ 0\le z\le y^2,\ \sqrt{y}\le x\le 1\}$, and consequently, the triple integral becomes

$$\int_{y=c}^{y=d}\int_{z=v_2(y)}^{z=v_2(y)}\int_{x=u_1(y,z)}^{x=u_2(y,z)} f(x,y,z)\,dx\,dz\,dy=\int_{y=0}^{y=1}\int_{z=0}^{z=x^2}\int_{x=\sqrt{y}}^{x=1} f(x,y,z)\,dx\,dz\,dy$$

Now assume that $f(x,y,z)=xyz$ in each of the integrals. Then we have

$$\int_{x=0}^{x=1}\int_{y=0}^{y=x^2}\int_{z=0}^{z=y^2} xyz\,dz\,dy\,dx = \int_{x=0}^{x=1}\int_{y=0}^{y=x^2}\left[xy\frac{z^2}{2}\Big|_{z=0}^{z=y^2}\right] dy\,dx$$
$$= \int_{x=0}^{x=1}\int_{y=0}^{y=x^2}\left(x\frac{y^5}{2}\right) dy\,dx$$
$$= \int_{x=0}^{x=1}\left[x\frac{y^6}{12}\Big|_{y=0}^{y=x^2}\right] dx$$
$$= \int_{x=0}^{x=1}\frac{x^{13}}{12}dx = \frac{x^{14}}{168}\Big|_{x=0}^{x=1}$$
$$= \frac{1}{168},$$
$$\int_{y=0}^{y=1}\int_{z=0}^{z=y^2}\int_{x=\sqrt{y}}^{x=1} xyz\,dx\,dz\,dy = \int_{y=0}^{y=1}\int_{z=0}^{z=y^2}\left[yz\frac{x^2}{2}\Big|_{\sqrt{y}}^{1}\right] dz\,dy$$
$$= \int_{y=0}^{y=1}\int_{z=0}^{z=y^2}\left(\frac{yz}{2}-\frac{y^2z}{2}\right) dz\,dy$$
$$= \int_{y=0}^{y=1}\left[\frac{yz^2}{4}-\frac{y^2z^2}{4}\Big|_{z=0}^{z=y^2}\right] dy$$
$$= \int_{y=0}^{y=1}\left(\frac{y^5}{4}-\frac{y^6}{4}\right) dy$$
$$= \left(\frac{y^6}{24}-\frac{y^7}{28}\right)\Big|_{y=0}^{y=1}$$
$$= \frac{1}{168}.$$

The answers match.

? Exercise 15.6.4

Write five different iterated integrals equal to the given integral

$$\int_{z=0}^{z=4}\int_{y=0}^{y=4-z}\int_{x=0}^{x=\sqrt{y}} f(x,y,z)\,dx\,dy\,dz.$$

Hint

Follow the steps in the previous example, using the region E as $\{(x,y,z)\,|\,0\le z\le 4,\ 0\le y\le 4-z,\ 0\le x\le\sqrt{y}\}$, and describe and sketch the projections onto each of the three planes, five different times.

Answer

$$(i)\int_{z=0}^{z=4}\int_{x=0}^{x=\sqrt{4-z}}\int_{y=x^2}^{y=4-z} f(x,y,z)\,dy\,dx\,dz,\ (ii)\int_{y=0}^{y=4}\int_{z=0}^{z=4-y}\int_{x=0}^{x=\sqrt{y}} f(x,y,z)\,dx\,dz\,dy,\ (iii)\int_{y=0}^{y=4}\int_{x=0}^{x=\sqrt{y}}\int_{z=0}^{Z=4-y} f(x,y,z)\,dz\,dx\,dy,$$
$$(iv)\int_{x=0}^{x=2}\int_{y=x^2}^{y=4}\int_{z=0}^{z=4-y} f(x,y,z)\,dz\,dy\,dx,\ (v)\int_{x=0}^{x=2}\int_{z=0}^{z=4-x^2}\int_{y=x^2}^{y=4-z} f(x,y,z)\,dy\,dz\,dx$$

✓ Example 15.6.5: Changing Integration Order and Coordinate Systems

Evaluate the triple integral

$$\iiint_E \sqrt{x^2+z^2}\,dV,$$

where E is the region bounded by the paraboloid $y=x^2+z^2$ (Figure 15.6.9) and the plane $y=4$.

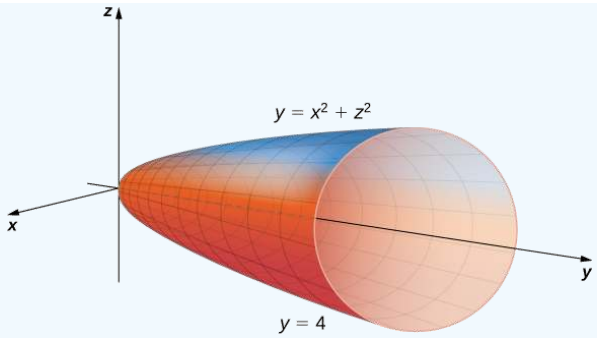


Figure 15.6.9. Integrating a triple integral over a paraboloid.

Solution

The projection of the solid region E onto the xy-plane is the region bounded above by $y = 4$ and below by the parabola $y = x^2$ as shown.

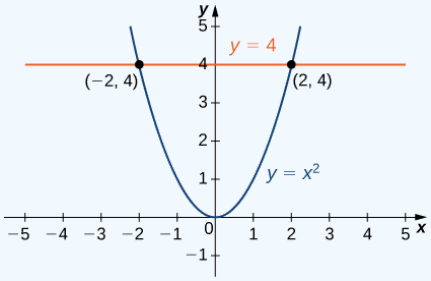


Figure 15.6.10. Cross section in the xy-plane of the paraboloid in Figure 15.6.9.

Thus, we have

$$E = \{(x, y, z) \mid -2 \le x \le 2,\ x^2 \le y \le 4,\ -\sqrt{y - x^2} \le z \le \sqrt{y - x^2}\}.$$

The triple integral becomes

$$\iiint_E \sqrt{x^2 + z^2}\, dV = \int_{x=-2}^{x=2} \int_{y=x^2}^{y=4} \int_{z=-\sqrt{y-x^2}}^{z=\sqrt{y-x^2}} \sqrt{x^2 + z^2}\, dz\, dy\, dx.$$

This expression is difficult to compute, so consider the projection of E onto the xz-plane. This is a circular disc $x^2 + z^2 \le 4$. So we obtain

$$\iiint_E \sqrt{x^2 + z^2}\, dV = \int_{x=-2}^{x=2} \int_{y=x^2}^{y=4} \int_{z=-\sqrt{y-x^2}}^{z=\sqrt{y-x^2}} \sqrt{x^2 + z^2}\, dz\, dy\, dx = \int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} \int_{y=x^2+z^2}^{y=4} \sqrt{x^2 + z^2}\, dy\, dz\, dx.$$

Here the order of integration changes from being first with respect to z then y and then x to being first with respect to y then to z and then to x. It will soon be clear how this change can be beneficial for computation. We have

$$\int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} \int_{y=x^2+z^2}^{y=4} \sqrt{x^2 + z^2}\, dy\, dz\, dx = \int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} (4 - x^2 - z^2)\sqrt{x^2 + z^2}\, dz\, dx.$$

Now use the polar substitution $x = r\cos\theta$, $z = r\sin\theta$, and $dz\, dx = r\, dr\, d\theta$ in the xz-plane. This is essentially the same thing as when we used polar coordinates in the xy-plane, except we are replacing y by z. Consequently the limits of integration change and we have, by using $r^2 = x^2 + z^2$,

$$\int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} (4 - x^2 - z^2)\sqrt{x^2 + z^2}\, dz\, dx = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} (4 - r^2) r r\, dr\, d\theta = \int_0^{2\pi} \left[\frac{4r^3}{3} - \frac{r^5}{5}\Big|_0^2\right] d\theta =$$

$$\int_0^{2\pi} \frac{64}{15}\, d\theta = \frac{128\pi}{15}$$

Average Value of a Function of Three Variables

Recall that we found the average value of a function of two variables by evaluating the double integral over a region on the plane and then dividing by the area of the region. Similarly, we can find the average value of a function in three variables by evaluating the triple integral over a solid region and then dividing by the volume of the solid.

Average Value of a Function of Three Variables

If $f(x, y, z)$ is integrable over a solid bounded region E with positive volume $V(E)$, then the average value of the function is

$$f_{ave} = \frac{1}{V(E)} \iiint_E f(x,y,z)\, dV.$$

Note that the volume is

$$V(E) = \iiint_E 1\, dV.$$

Example 15.6.6: Finding an Average Temperature

The temperature at a point (x,y,z) of a solid E bounded by the coordinate planes and the plane $x+y+z=1$ is $T(x,y,z) = (xy+8z+20)\,^\circ\mathrm{C}$. Find the average temperature over the solid.

Solution

Use the theorem given above and the triple integral to find the numerator and the denominator. Then do the division. Notice that the plane $x+y+z=1$ has intercepts $(1,0,0)$, $(0,1,0)$,and $(0,0,1)$. The region E looks like

$$E = \{(x,y,z) \mid 0 \le x \le 1,\ 0 \le y \le 1-x,\ 0 \le z \le 1-x-y\}.$$

Hence the triple integral of the temperature is

$$\iiint_E f(x,y,z)\, dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} (xy+8z+20)\, dz\, dy\, dx = \frac{147}{40}.$$

The volume evaluation is

$$V(E) = \iiint_E 1\, dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} 1\, dz\, dy\, dx = \frac{1}{6}.$$

Hence the average value is

$$T_{ave} = \frac{147/40}{1/6} = \frac{6(147)}{40} = \frac{441}{20}\,^\circ\mathrm{C}$$

.

Exercise 15.6.6

Find the average value of the function $f(x,y,z) = xyz$ over the cube with sides of length 4 units in the first octant with one vertex at the origin and edges parallel to the coordinate axes.

Hint

Follow the steps in the previous example.

Answer

$f_{ave} = 8$

Key Concepts

- To compute a triple integral we use Fubini's theorem, which states that if $f(x,y,z)$ is continuous on a rectangular box $B = [a,b] \times [c,d] \times [e,f]$, then

$$\iiint_B f(x,y,z)\, dV = \int_e^f \int_c^d \int_a^b f(x,y,z)\, dx\, dy\, dz$$

  and is also equal to any of the other five possible orderings for the iterated triple integral.
- To compute the volume of a general solid bounded region E we use the triple integral

$$V(E) = \iiint_E 1\, dV.$$

- Interchanging the order of the iterated integrals does not change the answer. As a matter of fact, interchanging the order of integration can help simplify the computation.
- To compute the average value of a function over a general three-dimensional region, we use

$$f_{ave} = \frac{1}{V(E)} \iiint_E f(x,y,z)\, dV.$$

Key Equations

- **Triple integral**

$$\lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta x \Delta y \Delta z = \iiint_B f(x, y, z)\, dV$$

Glossary

triple integral

the triple integral of a continuous function $f(x, y, z)$ over a rectangular solid box B is the limit of a Riemann sum for a function of three variables, if this limit exists

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15.7: Triple Integrals in Cylindrical Coordinates

Learning Objectives

- Evaluate a triple integral by changing to cylindrical coordinates.
- Evaluate a triple integral by changing to spherical coordinates.

Earlier in this chapter we showed how to convert a double integral in rectangular coordinates into a double integral in polar coordinates in order to deal more conveniently with problems involving circular symmetry. A similar situation occurs with triple integrals, but here we need to distinguish between cylindrical symmetry and spherical symmetry. In this section we convert triple integrals in rectangular coordinates into a triple integral in either cylindrical or spherical coordinates.

Also recall the chapter prelude, which showed the opera house l'Hemisphèric in Valencia, Spain. It has four sections with one of the sections being a theater in a five-story-high sphere (ball) under an oval roof as long as a football field. Inside is an IMAX screen that changes the sphere into a planetarium with a sky full of 9000 twinkling stars. Using triple integrals in spherical coordinates, we can find the volumes of different geometric shapes like these.

Review of Cylindrical Coordinates

As we have seen earlier, in two-dimensional space $\mathbb{R}^2$ a point with rectangular coordinates (x, y) can be identified with (r, θ) in polar coordinates and vice versa, where $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$ and $\tan\theta = \left(\frac{y}{x}\right)$ are the relationships between the variables.

In three-dimensional space $\mathbb{R}^3$ a point with rectangular coordinates (x, y, z) can be identified with cylindrical coordinates (r, θ, z) and vice versa. We can use these same conversion relationships, adding z as the vertical distance to the point from the $(xy$-plane as shown in 15.7.1.

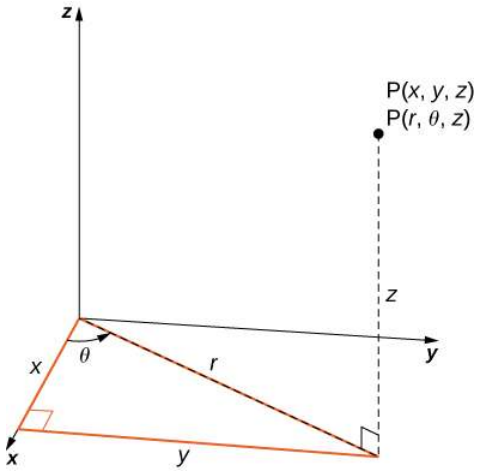


Figure 15.7.1: Cylindrical coordinates are similar to polar coordinates with a vertical z coordinate added.

To convert from rectangular to cylindrical coordinates, we use the conversion

- $x = r\cos\theta$
- $y = r\sin\theta$
- $z = z$

To convert from cylindrical to rectangular coordinates, we use

- $r^2 = x^2 + y^2$ and
- $\theta = \tan^{-1}\left(\frac{y}{x}\right)$
- $z = z$

Note that that z-coordinate remains the same in both cases.

In the two-dimensional plane with a rectangular coordinate system, when we say $x = k$ (constant) we mean an unbounded vertical line parallel to the y-axis and when $y = l$ (constant) we mean an unbounded horizontal line parallel to the x-axis. With the polar

coordinate system, when we say $r = c$ (constant), we mean a circle of radius c units and when $\theta = \alpha$ (constant) we mean an infinite ray making an angle α with the positive x-axis.

Similarly, in three-dimensional space with rectangular coordinates (x, y, z) the equations $x = k$, $y = l$ and $z = m$ where k, l and m are constants, represent unbounded planes parallel to the yz-plane, xz-plane and xy-plane, respectively. With cylindrical coordinates (r, θ, z), by $r = c$, $\theta = \alpha$, and $z = m$, where c, α, and m are constants, we mean an unbounded vertical cylinder with the z-axis as its radial axis; a plane making a constant angle α with the xy-plane; and an unbounded horizontal plane parallel to the xy-plane, respectively. This means that the circular cylinder $x^2 + y^2 = c^2$ in rectangular coordinates can be represented simply as $r = c$ in cylindrical coordinates. (Refer to Cylindrical and Spherical Coordinates for more review.)

Integration in Cylindrical Coordinates

Triple integrals can often be more readily evaluated by using cylindrical coordinates instead of rectangular coordinates. Some common equations of surfaces in rectangular coordinates along with corresponding equations in cylindrical coordinates are listed in Table 15.7.1. These equations will become handy as we proceed with solving problems using triple integrals.

Table 15.7.1: Equations of Some Common Shapes

	Circular cylinder	Circular cone	Sphere	Paraboloid
Rectangular	$x^2 + y^2 = c^2$	$z^2 = c^2(x^2 + y^2)$	$x^2 + y^2 + z^2 = c^2$	$z = c(x^2 + y^2)$
Cylindrical	$r = c$	$z = cr$	$r^2 + z^2 = c^2$	$z = cr^2$

As before, we start with the simplest bounded region B in $\mathbb{R}^3$ to describe in cylindrical coordinates, in the form of a cylindrical box, $B = \{(r, \theta, z) | a \le r \le b,\ \alpha \le \theta \le \beta,\ c \le z \le d\}$ (Figure 15.7.2). Suppose we divide each interval into l, m, and n subdivisions such that $\Delta r = \frac{b-a}{l}$, $\Delta\theta = \frac{\beta-\alpha}{m}$, and $\Delta z = \frac{d-c}{n}$. Then we can state the following definition for a triple integral in cylindrical coordinates.

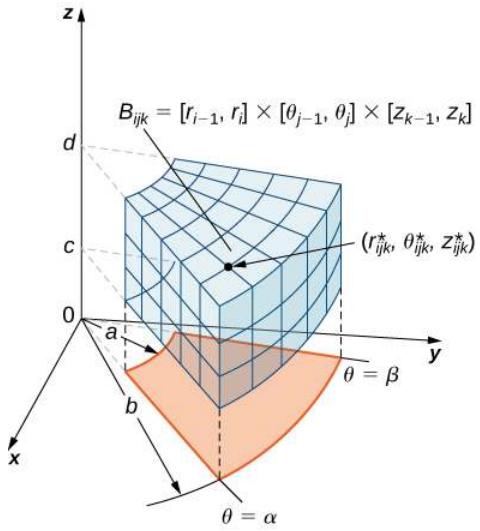


Figure 15.7.2: A cylindrical box B described by cylindrical coordinates.

DEFINITION: triple integral in cylindrical coordinates

Consider the cylindrical box (expressed in cylindrical coordinates)

$$B = \{(r, \theta, z) | a \le r \le b,\ \alpha \le \theta \le \beta,\ c \le z \le d\}.$$

If the function $f(r, \theta, z)$ is continuous on B and if $(r^*_{ijk}, \theta^*_{ijk}, z^*_{ijk})$ is any sample point in the cylindrical subbox $B_{ijk} = |r_{i-1}, r_i| \times |\theta_{j-1}, \theta_j| \times |z_{k-1}, k_i|$ (Figure 15.7.2), then we can define the *triple integral in cylindrical coordinates* as the limit of a triple Riemann sum, provided the following limit exists:

$$\lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(r^*_{ijk},\theta^*_{ijk},z^*_{ijk})\,\Delta r\Delta\theta\Delta z.$$

Note that if $g(x,y,z)$ is the function in rectangular coordinates and the box B is expressed in rectangular coordinates, then the triple integral

$$\iiint_B g(x,y,z)\,dV$$

is equal to the triple integral

$$\iiint_B g(r\,\cos\theta,\ r\,\sin\theta,\ z)r\,dr\,d\theta\,dz$$

and we have

$$\iiint_B g(x,y,z)\,dV=\iiint_B g(r\,\cos\theta,\ r\,\sin\theta,\ z)\,r\,dr\,d\theta\,dz=\iiint_B f(r,\theta\,z)\,r\,dr\,d\theta\,dz.$$

As mentioned in the preceding section, all the properties of a double integral work well in triple integrals, whether in rectangular coordinates or cylindrical coordinates. They also hold for iterated integrals. To reiterate, in cylindrical coordinates, Fubini's theorem takes the following form:

Theorem: Fubini's Theorem in Cylindrical Coordinates

Suppose that $g(x,y,z)$ is continuous on a rectangular box B which when described in cylindrical coordinates looks like $B=\{(r,\theta,z)|a\le r\le b,\ \alpha\le\theta\le\beta,\ c\le z\le d\}$.

Then $g(x,y,z)=g(r\,\cos\theta,r\,\sin\theta,z)=f(r,\theta,z)$ and

$$\iiint_B g(x,y,z)\,dV=\int_c^d\int_\beta^\alpha\int_a^b f(r,\theta,z)\,r\,dr\,d\theta\,dz.$$

The iterated integral may be replaced equivalently by any one of the other five iterated integrals obtained by integrating with respect to the three variables in other orders.

Cylindrical coordinate systems work well for solids that are symmetric around an axis, such as cylinders and cones. Let us look at some examples before we define the triple integral in cylindrical coordinates on general cylindrical regions.

Example 15.7.1: Evaluating a Triple Integral over a Cylindrical Box

Evaluate the triple integral

$$\iiint_B (zr\,\sin\theta)r\,dr\,d\theta\,dz$$

where the cylindrical box B is $B=\{(r,\theta,z)|0\le r\le 2,\ 0\le\theta\le\pi/2,\ 0,\le z\le 4\}$.

Solution

As stated in Fubini's theorem, we can write the triple integral as the iterated integral

$$\iiint_B (zr\,\sin\theta)r\,dr\,d\theta\,dz=\int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=2}\int_{z=0}^{z=4}(zr\,\sin\theta)\,r\,dz\,dr\,d\theta.$$

The evaluation of the iterated integral is straightforward. Each variable in the integral is independent of the others, so we can integrate each variable separately and multiply the results together. This makes the computation much easier:

$$\int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=2}\int_{z=0}^{z=4}(zr\,\sin\,\theta)\,r\,dz\,dr\,d\theta=\left(\int_0^{\pi/2}\sin\,\theta\,d\theta\right)\left(\int_0^2 r^2dr\right)\left(\int_0^4 z\,dz\right)$$
$$=\left(-\cos\theta|_0^{\pi/2}\right)\left(\frac{r^3}{3}\Big|_0^2\right)\left(\frac{z^2}{2}\Big|_0^4\right)=\frac{64}{3}.$$

? Exercise 15.7.1:

Evaluate the triple integral

$$\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=1}\int_{z=0}^{z=4}(rz\,\sin\,\theta)\,r\,dz\,dr\,d\theta.$$

Hint

Follow the same steps as in the previous example.

Answer

8

If the cylindrical region over which we have to integrate is a general solid, we look at the projections onto the coordinate planes. Hence the triple integral of a continuous function $f(r,\theta,z)$ over a general solid region $E=\{(r,\theta,z)|(r,\theta)\in D,u_1(r,\theta)\le z\le u_2(r,\theta)\}$ in $\mathbb{R}^3$ where D is the projection of E onto the $r\theta$-plane, is

$$\iiint_E f(r,\theta,z)\,r\,dr\,d\theta\,dz=\iint_D\left[\int_{u_1(r,\theta)}^{u_2(r,\theta)}f(r,\theta,z)dz\right]r\,dr\,d\theta.$$

In particular, if $D=\{(r,\theta)|G_1(\theta)\le r\le g_2(\theta),\alpha\le\theta\le\beta\}$, then we have

$$\iiint_E f(r,\theta,z)\,r\,dr\,d\theta=\int_{\theta=\alpha}^{\theta=\beta}\int_{r=g_1(\theta)}^{r=g_2(\theta)}\int_{z=u_1(r,\theta)}^{z=u_2(r,\theta)}f(r,\theta,z)\,r\,dz\,dr\,d\theta.$$

Similar formulas exist for projections onto the other coordinate planes. We can use polar coordinates in those planes if necessary.

✓ Example 15.7.2: Setting up a Triple Integral in Cylindrical Coordinates over a General Region

Consider the region E inside the right circular cylinder with equation $r=2\,\sin\,\theta$, bounded below by the $r\theta$-plane and bounded above by the sphere with radius 4 centered at the origin (Figure 15.5.3). Set up a triple integral over this region with a function $f(r,\theta,z)$ in cylindrical coordinates.

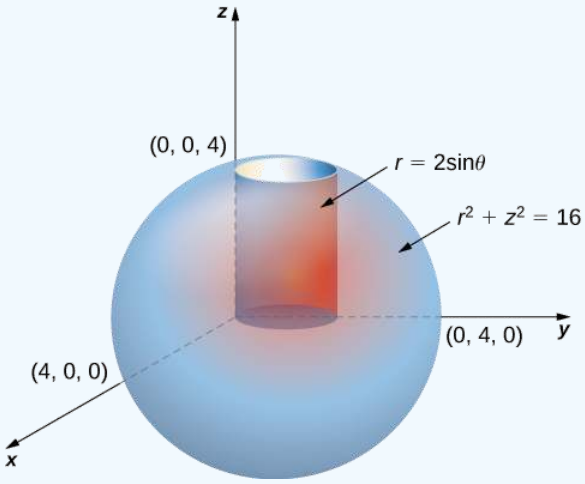


Figure 15.7.3: Setting up a triple integral in cylindrical coordinates over a cylindrical region.

Solution

First, identify that the equation for the sphere is $r^2+z^2=16$. We can see that the limits for z are from 0 to $z=\sqrt{16-r^2}$. Then the limits for r are from 0 to $r=2\sin\theta$. Finally, the limits for θ are from 0 to π. Hence the region is $E=\{(r,\theta,z)|0\le\theta\le\pi,\ 0\le r\le 2\sin\theta,\ 0\le z\le\sqrt{16-r^2}\}$. Therefore, the triple integral is

$$\iiint_E f(r,\theta,z)\,r\,dz\,dr\,d\theta=\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=2\sin\theta}\int_{z=0}^{z=\sqrt{16-r^2}} f(r,\theta,z)\,r\,dz\,dr\,d\theta.$$

? Exercise 15.7.2:

Consider the region inside the right circular cylinder with equation $r=2\sin\theta$ bounded below by the $r\theta$-plane and bounded above by $z=4-y$. Set up a triple integral with a function $f(r,\theta,z)$ in cylindrical coordinates.

Hint

Analyze the region, and draw a sketch.

Answer

$$\iiint_E f(r,\theta,z)r\,dz\,dr\,d\theta=\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=2\sin\theta}\int_{z=0}^{z=4-r\sin\theta} f(r,\theta,z)r\,dz\,dr\,d\theta.$$

✓ Example 15.7.3: Setting up a Triple Integral in Two Ways

Let E be the region bounded below by the cone $z=\sqrt{x^2+y^2}$ and above by the paraboloid $z=2-x^2-y^2$. (Figure 15.5.4). Set up a triple integral in cylindrical coordinates to find the volume of the region, using the following orders of integration:

a. $dz\,dr\,d\theta$

b. $dr\,dz\,d\theta$

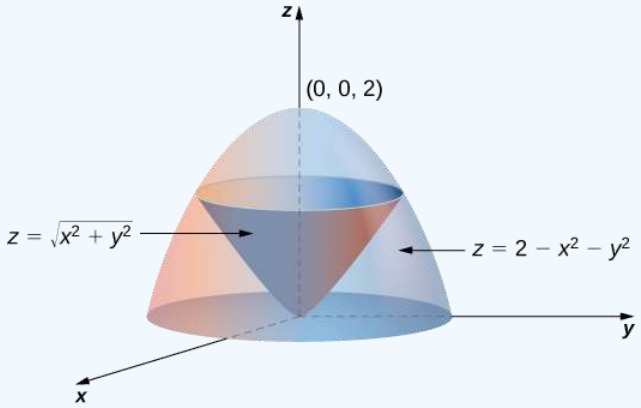


Figure 15.7.4: Setting up a triple integral in cylindrical coordinates over a conical region.

Solution

a. The cone is of radius 1 where it meets the paraboloid. Since $z=2-x^2-y^2=2-r^2$ and $z=\sqrt{x^2+y^2}=r$ (assuming r is nonnegative), we have $2-r^2=r$. Solving, we have $r^2+r-2=(r+2)(r-1)=0$. Since $r\ge 0$, we have $r=1$. Therefore $z=1$. So the intersection of these two surfaces is a circle of radius 1 in the plane $z=1$. The cone is the lower bound for z and the paraboloid is the upper bound. The projection of the region onto the xy-plane is the circle of radius 1 centered at the origin.

Thus, we can describe the region as $E=\{(r,\theta,z)|0\le\theta\le 2\pi,\ 0\le r\le 1,\ r\le z\le 2-r^2\}$.

Hence the integral for the volume is

$$V = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=r}^{z=2-r^2} r\, dz\, dr\, d\theta.$$

b. We can also write the cone surface as $r = z$ and the paraboloid as $r^2 = 2 - z$. The lower bound for r is zero, but the upper bound is sometimes the cone and the other times it is the paraboloid. The plane $z = 1$ divides the region into two regions. Then the region can be described as

$$E = \{(r, \theta, z) | 0 \leq \theta \leq 2\pi,\ 0 \leq z \leq 1,\ 0 \leq r \leq z\} \cup \{(r, \theta, z) | 0 \leq \theta \leq 2\pi,\ 1 \leq z \leq 2,\ 0 \leq r \leq \sqrt{2 - z}\}.$$

Now the integral for the volume becomes

$$V = \int_{\theta=0}^{\theta=2\pi} \int_{z=0}^{z=1} \int_{r=0}^{r=z} r\, dr\, dz\, d\theta + \int_{\theta=0}^{\theta=2\pi} \int_{z=1}^{z=2} \int_{r=0}^{r=\sqrt{2-z}} r\, dr\, dz\, d\theta.$$

Exercise 15.7.3:

Redo the previous example with the order of integration $d\theta\, dz\, dr$.

Hint

Note that θ is independent of r and z.

Answer

$E = \{(r, \theta, z) | 0 \leq \theta \leq 2\pi,\ 0 \leq z \leq 1,\ 0 \leq r \leq 2 - z^2\}$ and

$$V = \int_{r=0}^{r=1} \int_{z=0}^{z=2-r^2} \int_{\theta=0}^{\theta=2\pi} r\, d\theta\, dz\, dr.$$

Example 15.7.4: Finding a Volume with Triple Integrals in Two Ways

Let E be the region bounded below by the $r\theta$-plane, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$ (Figure 15.5.5). Set up a triple integral in cylindrical coordinates to find the volume of the region using the following orders of integration, and in each case find the volume and check that the answers are the same:

a. $dz\, dr\, d\theta$
b. $dr\, dz\, d\theta$.

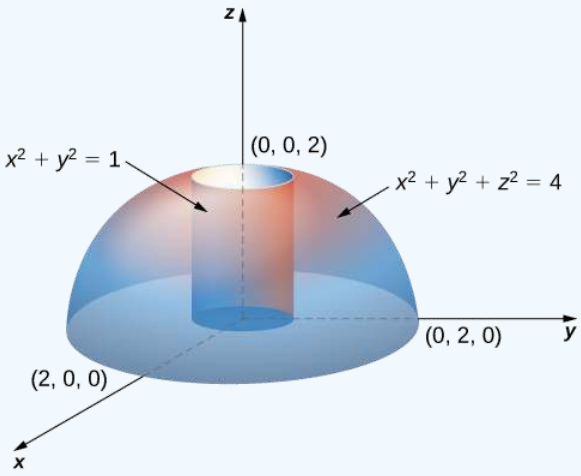


Figure 15.7.5: Finding a cylindrical volume with a triple integral in cylindrical coordinates.

Solution

a. Note that the equation for the sphere is

$$x^2 + y^2 + z^2 = 4 \text{ or } r^2 + z^2 = 4$$

and the equation for the cylinder is

$$x^2+y^2=1 \text{ or } r^2=1.$$

Thus, we have for the region E

$$E=\{(r,\theta,z)|0\le z\le\sqrt{4-r^2},\ 0\le r\le 1,\ 0\le\theta\le 2\pi\}$$

Hence the integral for the volume is

$$\begin{aligned} V(E) &= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}\int_{z=0}^{z=\sqrt{4-r^2}} r\,dz\,dr\,d\theta \\ &= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}\left[rz\Big|_{z=0}^{z=\sqrt{4-r^2}}\right]dr\,d\theta \\ &= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}\left(r\sqrt{4-r^2}\right)dr\,d\theta \\ &= \int_0^{2\pi}\left(\frac{8}{3}-\sqrt{3}\right)d\theta = 2\pi\left(\frac{8}{3}-\sqrt{3}\right) \text{ cubic units.} \end{aligned}$$

b. Since the sphere is $x^2+y^2+z^2=4$, which is $r^2+z^2=4$, and the cylinder is $x^2+y^2=1$, which is $r^2=1$, we have $1+z^2=4$, that is, $z^2=3$. Thus we have two regions, since the sphere and the cylinder intersect at $(1,\sqrt{3})$ in the rz-plane

$$E_1=\{(r,\theta,z)|0\le r\le\sqrt{4-r^2},\ \sqrt{3}\le z\le 2,\ 0\le\theta\le 2\pi\}$$

and

$$E_2=\{(r,\theta,z)|0\le r\le 1,\ 0\le z\le\sqrt{3},\ 0\le\theta\le 2\pi\}.$$

Hence the integral for the volume is

$$\begin{aligned} V(E) &= \int_{\theta=0}^{\theta=2\pi}\int_{z=\sqrt{3}}^{z=2}\int_{r=0}^{r=\sqrt{4-r^2}} r\,dr\,dz\,d\theta+\int_{\theta=0}^{\theta=2\pi}\int_{z=0}^{z=\sqrt{3}}\int_{r=0}^{r=1} r\,dr\,dz\,d\theta \\ &= \sqrt{3}\pi+\left(\frac{16}{3}-3\sqrt{3}\right)\pi \\ &= 2\pi\left(\frac{8}{3}-\sqrt{3}\right) \text{ cubic units.} \end{aligned}$$

? Exercise 15.7.4

Redo the previous example with the order of integration $d\theta\,dz\,dr$.

Hint

A figure can be helpful. Note that θ is independent of r and z.

Answer

$E_2=\{(r,\theta,z)|0\le\theta\le 2\pi,\ 0\le r\le 1,\ r\le z\le\sqrt{4-r^2}\}$ and

$$V=\int_{r=0}^{r=1}\int_{z=r}^{z=\sqrt{4-r^2}}\int_{\theta=0}^{\theta=2\pi} r\,d\theta\,dz\,dr.$$

Review of Spherical Coordinates

In three-dimensional space $\mathbb{R}^3$ in the spherical coordinate system, we specify a point P by its distance ρ from the origin, the polar angle θ from the positive x-axis (same as in the cylindrical coordinate system), and the angle φ from the positive z-axis and the line OP (Figure 15.7.6). Note that $\rho>0$ and $0\le\varphi\le\pi$. (Refer to Cylindrical and Spherical Coordinates for a review.) Spherical coordinates are useful for triple integrals over regions that are symmetric with respect to the origin.

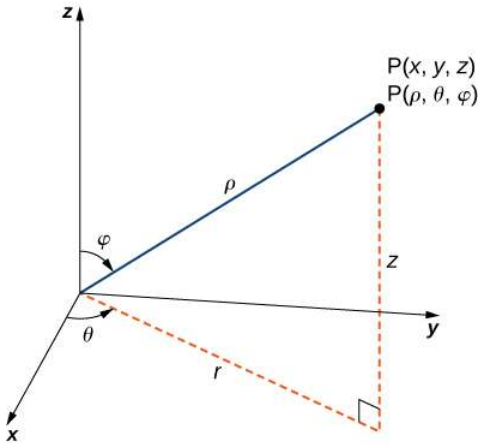


Figure 15.7.6: The spherical coordinate system locates points with two angles and a distance from the origin.

Recall the relationships that connect rectangular coordinates with spherical coordinates.

From spherical coordinates to rectangular coordinates:

$$x = \rho \sin \varphi \cos\theta,\ y = \rho \sin \varphi \sin \theta,\ and\ z = \rho \cos \varphi.$$

From rectangular coordinates to spherical coordinates:

$$\rho^2 = x^2 + y^2 + z^2,\ \tan \theta = \frac{y}{x},\ \varphi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right).$$

Other relationships that are important to know for conversions are

- $r = \rho \sin \varphi$
- $\theta = \theta$ These equations are used to convert from spherical coordinates to cylindrical coordinates.
- $z = \rho \cos \varphi$

and

- $\rho = \sqrt{r^2 + z^2}$
- $\theta = \theta$ These equations are used to convert from cylindrical coordinates to spherical coordinates.
- $\varphi = \arccos\left(\frac{z}{\sqrt{r^2+z^2}}\right)$

15.7.7 shows a few solid regions that are convenient to express in spherical coordinates.

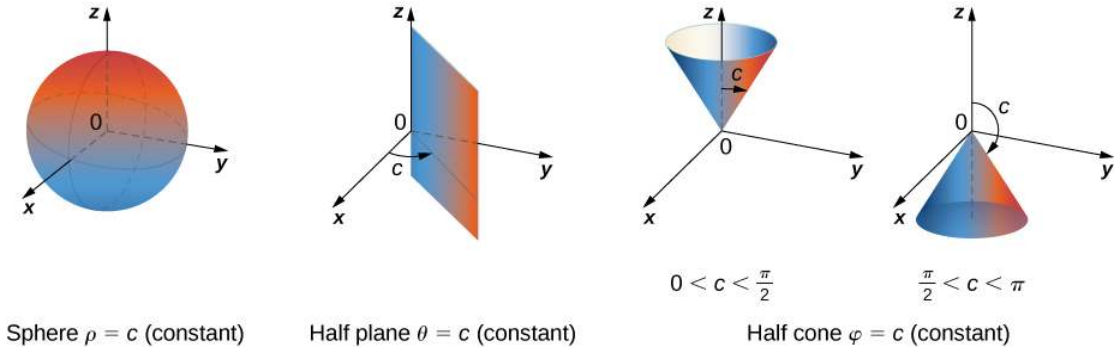


Figure 15.7.7: Spherical coordinates are especially convenient for working with solids bounded by these types of surfaces. (The letter c indicates a constant.)

Integration in Spherical Coordinates

We now establish a triple integral in the spherical coordinate system, as we did before in the cylindrical coordinate system. Let the function $f(\rho, \theta, \varphi)$ be continuous in a bounded spherical box, $B = \{(\rho, \theta, \varphi) | a \le \rho \le b,\ \alpha \le \theta \le \beta,\ \gamma \le \varphi \le \psi\}$. We then divide each interval into l, m, n and n subdivisions such that $\Delta\rho = \frac{b-a}{l}$, $\Delta\theta = \frac{\beta-\alpha}{m}$. $\Delta\varphi = \frac{\psi-\gamma}{n}$. Now we can illustrate the

following theorem for triple integrals in spherical coordinates with $(\rho^*_{ijk}, \theta^*_{ijk}, \varphi^*_{ijk})$ being any sample point in the spherical subbox B_{ijk}. For the volume element of the subbox ΔV in spherical coordinates, we have $\Delta V = (\Delta\rho)\,(\rho\Delta\varphi)\,(\rho\,\sin\,\varphi\,\Delta\theta)$, as shown in the following figure.

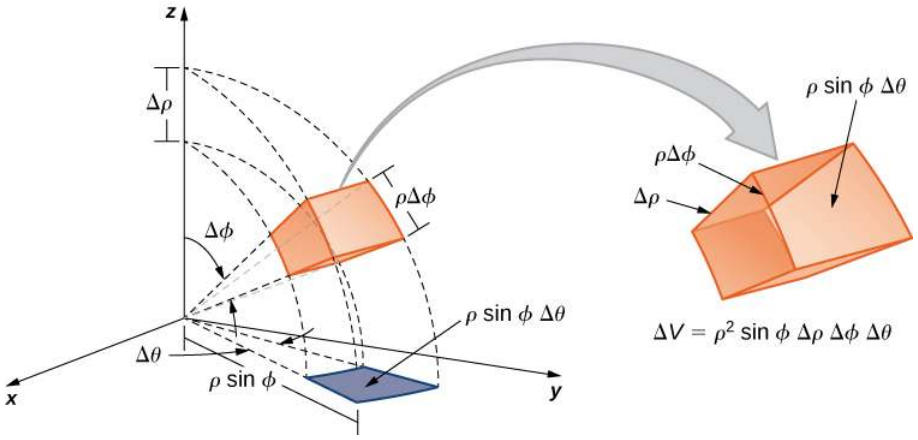


Figure 15.7.8: The volume element of a box in spherical coordinates.

Definition: triple integral in spherical coordinates

The **triple integral in spherical coordinates** is the limit of a triple Riemann sum,

$$\lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(\rho^*_{ijk}, \theta^*_{ijk}, \varphi^*_{ijk})(\rho^*_{ijk})^2\,\sin\,\varphi\Delta\rho\,\Delta\theta\Delta\varphi$$

provided the limit exists.

As with the other multiple integrals we have examined, all the properties work similarly for a triple integral in the spherical coordinate system, and so do the iterated integrals. Fubini's theorem takes the following form.

Theorem: Fubini's Theorem for Spherical Coordinates

If $f(\rho, \theta, \varphi)$ is continuous on a spherical solid box $B = [a, b] \times [\alpha, \beta] \times [\gamma, \psi]$, then

$$\iiint_B f(\rho, \theta, \varphi)\,\rho^2\,\sin\,\varphi\,d\rho\,d\varphi\,d\theta = \int_{\varphi=\gamma}^{\varphi=\psi}\int_{\theta=\alpha}^{\theta=\beta}\int_{\rho=a}^{\rho=b} f(\rho, \theta, \varphi)\,\rho^2\,\sin\,\varphi\,d\rho\,d\varphi\,d\theta.$$

This iterated integral may be replaced by other iterated integrals by integrating with respect to the three variables in other orders.

As stated before, spherical coordinate systems work well for solids that are symmetric around a point, such as spheres and cones. Let us look at some examples before we consider triple integrals in spherical coordinates on general spherical regions.

Example 15.7.5: Evaluating a Triple Integral in Spherical Coordinates

Evaluate the iterated triple integral

$$\int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\pi/2}\int_{\rho=0}^{\rho=1} \rho^2\,\sin\,\varphi\,d\rho\,d\varphi\,d\theta.$$

Solution

As before, in this case the variables in the iterated integral are actually independent of each other and hence we can integrate each piece and multiply:

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^2\,\sin\,\varphi\,d\rho\,d\varphi\,d\theta = \int_0^{2\pi} d\theta\int_0^{\pi/2}\sin\,\varphi\,d\varphi\int_0^1 \rho^2 d\rho = (2\pi)\,(1)\left(\frac{1}{3}\right) = \frac{2\pi}{3}$$

The concept of triple integration in spherical coordinates can be extended to integration over a general solid, using the projections onto the coordinate planes. Note that dV and dA mean the increments in volume and area, respectively. The variables V and A are used as the variables for integration to express the integrals.

The triple integral of a continuous function $f(\rho,\theta,\varphi)$ over a general solid region

$$E=\{(\rho,\theta,\varphi)|(\rho,\theta)\in D, u_1(\rho,\theta)\le\varphi\le u_2(\rho,\theta)\}$$

in $\mathbb{R}^3$, where D is the projection of E onto the $\rho\theta$-plane, is

$$\iiint_E f(\rho,\theta,\varphi)\,dV=\iint_D\left[\int_{u_1(\rho,\theta)}^{u_2(\rho,\theta)} f(\rho,\theta,\varphi)\,d\varphi\right]\,dA.$$

In particular, if $D=\{(\rho,\theta)|g_1(\theta)\le\rho\le g_2(\theta),\ \alpha\le\theta\le\beta\}$, the we have

$$\iiint_E f(\rho,\theta,\varphi)\,dV=\int_\alpha^\beta\int_{g_1(\theta)}^{g_2(\theta)}\int_{u_1(\rho,\theta)}^{u_2(\rho,\theta)} f(\rho,\theta,\varphi)\rho^2\sin\varphi\,d\varphi\,d\rho\,d\theta.$$

Similar formulas occur for projections onto the other coordinate planes.

Example 15.7.6: Setting up a Triple Integral in Spherical Coordinates

Set up an integral for the volume of the region bounded by the cone $z=\sqrt{3(x^2+y^2)}$ and the hemisphere $z=\sqrt{4-x^2-y^2}$ (see the figure below).

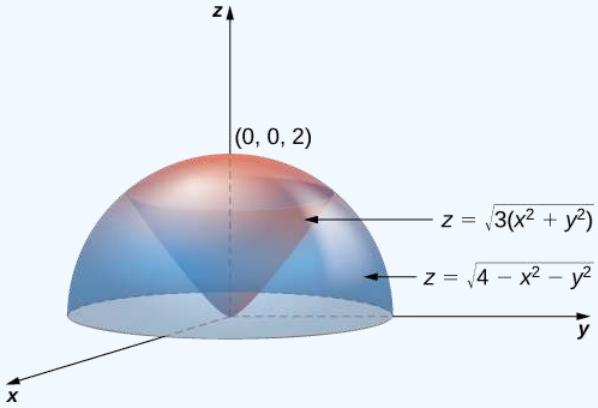


Figure 15.7.9: A region bounded below by a cone and above by a hemisphere.

Solution

Using the conversion formulas from rectangular coordinates to spherical coordinates, we have:

For the cone: $z=\sqrt{3(x^2+y^2)}$ or $\rho\cos\varphi=\sqrt{3}\rho\sin\varphi$ or $\tan\varphi=\frac{1}{\sqrt{3}}$ or $\varphi=\frac{\pi}{6}$.

For the sphere: $z=\sqrt{4-x^2-y^2}$ or $z^2+x^2+y^2=4$ or $\rho^2=4$ or $\rho=2$.

Thus, the triple integral for the volume is

$$V(E)=\int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\pi/6}\int_{\rho=0}^{\rho=2}\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta.$$

Exercise 15.7.5

Set up a triple integral for the volume of the solid region bounded above by the sphere $\rho=2$ and bounded below by the cone $\varphi=\pi/3$.

Hint

Follow the steps of the previous example.

Answer

$$V(E) = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi/3} \int_{\rho=0}^{\rho=2} \rho^2 \sin\,\varphi \, d\rho \, d\varphi \, d\theta$$

Example 15.7.7: Interchanging Order of Integration in Spherical Coordinates

Let E be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the sphere $z = x^2 + y^2 + z^2$ (Figure 15.5.10). Set up a triple integral in spherical coordinates and find the volume of the region using the following orders of integration:

a. $d\rho \, d\phi \, d\theta$
b. $d\varphi \, d\rho \, d\theta$

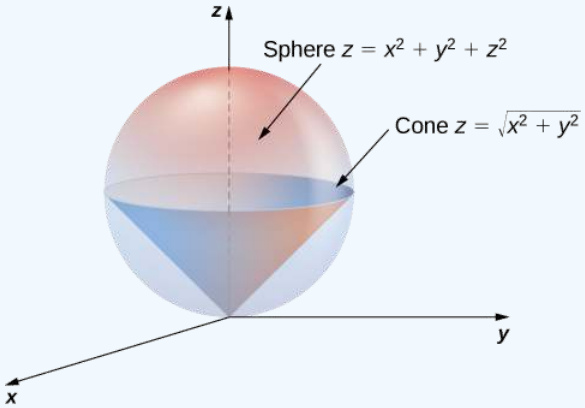


Figure 15.7.10:. A region bounded below by a cone and above by a sphere.

Solution

a. Use the conversion formulas to write the equations of the sphere and cone in spherical coordinates.

For the sphere:

$$\begin{aligned} x^2 + y^2 + z^2 &= z \\ \rho^2 &= \rho \cos\,\varphi \\ \rho &= \cos\,\varphi. \end{aligned}$$

For the cone:

$$\begin{aligned} z &= \sqrt{x^2 + y^2} \\ \rho \cos\,\varphi &= \sqrt{\rho^2 \sin^2\,\varphi \cos^2\theta + \rho^2 \sin^2\,\varphi \sin^2\theta} \\ \rho \cos\,\varphi &= \sqrt{\rho^2 \sin^2\varphi\,(\cos^2\theta + \sin^2\theta)} \\ \rho \cos\,\varphi &= \rho \sin\,\varphi \\ \cos\,\varphi &= \sin\,\varphi \\ \varphi &= \pi/4. \end{aligned}$$

Hence the integral for the volume of the solid region E becomes

$$V(E) = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi/4} \int_{\rho=0}^{\rho=\cos\,\varphi} \rho^2 \sin\,\varphi \, d\rho \, d\varphi \, d\theta.$$

b. Consider the $\varphi\rho$-plane. Note that the ranges for φ and ρ (from part a.) are

$$\begin{aligned} 0 \le \varphi &\le \pi/4 \\ 0 \le \rho &\le \cos\varphi \end{aligned}$$

The curve $\rho = \cos\,\varphi$ meets the line $\varphi = \pi/4$ at the point $(\pi/4, \sqrt{2}/2)$ Thus, to change the order of integration, we need to use two pieces:

$$0 \le \rho \le \sqrt{2}/2,\ 0 \le \varphi \le \pi/4$$

and

$$\sqrt{2}/2 \le \rho \le 1,\ 0 \le \varphi \le \cos^{-1}\,\rho.$$

Hence the integral for the volume of the solid region E becomes

$$V(E) = \int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=\sqrt{2}/2}\int_{\varphi=0}^{\varphi=\pi/4} \rho^2\,\sin\,\varphi\,d\varphi\,d\rho\,d\theta + \int_{\theta=0}^{\theta=2\pi}\int_{\rho=\sqrt{2}/2}^{\rho=1}\int_{\varphi=0}^{\varphi=\cos^{-1}\,\rho} \rho^2\,\sin\,\varphi\,d\varphi\,d\rho\,d\theta$$

In each case, the integration results in $V(E) = \frac{\pi}{8}$.

Before we end this section, we present a couple of examples that can illustrate the conversion from rectangular coordinates to cylindrical coordinates and from rectangular coordinates to spherical coordinates.

Example 15.7.8: Converting from Rectangular Coordinates to Cylindrical Coordinates

Convert the following integral into cylindrical coordinates:

$$\int_{y=-1}^{y=1}\int_{x=0}^{x=\sqrt{1-y^2}}\int_{z=x^2+y^2}^{z=\sqrt{x^2+y^2}} xyz\,dz\,dx\,dy.$$

Solution

The ranges of the variables are

$$\begin{aligned} -1 &\le y \le y \\ 0 &\le x \le \sqrt{1-y^2} \\ x^2+y^2 &\le z \le \sqrt{x^2+y^2}. \end{aligned}$$

The first two inequalities describe the right half of a circle of radius 1. Therefore, the ranges for θ and r are

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \text{ and } 0 \le r \le 1.$$

The limits of z are $r^2 \le z \le r$, hence

$$\int_{y=-1}^{y=1}\int_{x=0}^{x=\sqrt{1-y^2}}\int_{z=x^2+y^2}^{z=\sqrt{x^2+y^2}} xyz\,dz\,dx\,dy = \int_{\theta=-\pi/2}^{\theta=\pi/2}\int_{r=0}^{r=1}\int_{z=r^2}^{z=r} r(r\,\cos\theta)\,(r\,\sin\,\theta)\,z\,dz\,dr\,d\theta.$$

Example 15.7.9: Converting from Rectangular Coordinates to Spherical Coordinates

Convert the following integral into spherical coordinates:

$$\int_{y=0}^{y=3}\int_{x=0}^{x=\sqrt{9-y^2}}\int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)\,dz\,dx\,dy.$$

Solution

The ranges of the variables are

$$\begin{aligned} 0 &\le y \le 3 \\ 0 &\le x \le \sqrt{9-y^2} \\ \sqrt{x^2+y^2} &\le z \le \sqrt{18-x^2-y^2}. \end{aligned}$$

The first two ranges of variables describe a quarter disk in the first quadrant of the xy-plane. Hence the range for θ is $0 \le \theta \le \frac{\pi}{2}$.

The lower bound $z = \sqrt{x^2 + y^2}$ is the upper half of a cone and the upper bound $z = \sqrt{18 - x^2 - y^2}$ is the upper half of a sphere. Therefore, we have $0 \le \rho \le \sqrt{18}$, which is $0 \le \rho \le 3\sqrt{2}$.

For the ranges of φ we need to find where the cone and the sphere intersect, so solve the equation

$$\begin{aligned} r^2 + z^2 &= 18 \\ \left(\sqrt{x^2 + y^2}\right)^2 + z^2 &= 18 \\ z^2 + z^2 &= 18 \\ 2z^2 &= 18 \\ z^2 &= 9 \\ z &= 3. \end{aligned}$$

This gives

$$\begin{aligned} 3\sqrt{2}\cos\varphi &= 3 \\ \cos\varphi &= \frac{1}{\sqrt{2}} \\ \varphi &= \frac{\pi}{4}. \end{aligned}$$

Putting this together, we obtain

$$\int_{y=0}^{y=3} \int_{x=0}^{x=\sqrt{9-y^2}} \int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{18-x^2-y^2}} (x^2 + y^2 + z^2)\, dz\, dx\, dy = \int_{\varphi=0}^{\varphi=\pi/4} \int_{\theta=0}^{\theta=\pi/2} \int_{\rho=0}^{\rho=3\sqrt{2}} \rho^4 \sin\varphi\, d\rho\, d\theta\, d\varphi.$$

? Exercise 15.7.6:

Use rectangular, cylindrical, and spherical coordinates to set up triple integrals for finding the volume of the region inside the sphere $x^2 + y^2 + z^2 = 4$ but outside the cylinder $x^2 + y^2 = 1$.

Answer: Rectangular

$$\int_{x=-2}^{x=2} \int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} \int_{z=-\sqrt{4-x^2-y^2}}^{z=\sqrt{4-x^2-y^2}} dz\, dy\, dx - \int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} \int_{z=-\sqrt{4-x^2-y^2}}^{z=\sqrt{4-x^2-y^2}} dz\, dy\, dx.$$

Answer: Cylindrical

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=1}^{r=2} \int_{z=-\sqrt{4-r^2}}^{z=\sqrt{4-r^2}} r\, dz\, dr\, d\theta.$$

Answer: Spherical

$$\int_{\varphi=\pi/6}^{\varphi=5\pi/6} \int_{\theta=0}^{\theta=2\pi} \int_{\rho=\csc\varphi}^{\rho=2} \rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi.$$

Now that we are familiar with the spherical coordinate system, let's find the volume of some known geometric figures, such as spheres and ellipsoids.

✓ Example 15.7.10: Chapter Opener: Finding the Volume of l'Hemisphèric

Find the volume of the spherical planetarium in l'Hemisphèric in Valencia, Spain, which is five stories tall and has a radius of approximately 50 ft, using the equation $x^2 + y^2 + z^2 = r^2$.



Figure 15.7.11: (credit: modification of work by Javier Yaya Tur, Wikimedia Commons)

Solution

We calculate the volume of the ball in the first octant, where $x \leq 0$, $y \leq 0$, and $z \leq 0$, using spherical coordinates, and then multiply the result by 8 for symmetry. Since we consider the region D as the first octant in the integral, the ranges of the variables are

$$0 \leq \varphi \leq \frac{\pi}{2},\ 0 \leq \rho \leq r,\ 0 \leq \theta \leq \frac{\pi}{2}.$$

Therefore,

$$\begin{aligned} V = \iiint_D dx\, dy\, dz &= 8 \int_{\theta=0}^{\theta=\pi/2} \int_{\rho=0}^{\rho=\pi} \int_{\varphi=0}^{\varphi=\pi/2} \rho^2 \sin\theta\, d\varphi\, d\rho\, d\varphi \\ &= 8 \int_{\varphi=0}^{\varphi=\pi/2} d\varphi \int_{\rho=0}^{\rho=r} \rho^2 d\rho \int_{\theta=0}^{\theta=\pi/2} \sin\theta\, d\theta \\ &= 8 \left(\frac{\pi}{2}\right) \left(\frac{r^3}{3}\right) (1) \\ &= \frac{4}{3}\pi r^3. \end{aligned}$$

This exactly matches with what we knew. So for a sphere with a radius of approximately 50 ft, the volume is $\frac{4}{3}\pi(50)^3 \approx 523,600\ ft^3$.

For the next example we find the volume of an ellipsoid.

Example 15.7.11: Finding the Volume of an Ellipsoid

Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Solution

We again use symmetry and evaluate the volume of the ellipsoid using spherical coordinates. As before, we use the first octant $x \leq 0$, $y \leq 0$, and $z \leq 0$ and then multiply the result by 8.

In this case the ranges of the variables are

$$0 \leq \varphi \leq \frac{\pi}{2}\ 0 \leq \rho \leq 1, \text{ and } 0 \leq \theta \leq \frac{\pi}{2}.$$

Also, we need to change the rectangular to spherical coordinates in this way:

$$x = a\rho\ \cos\ \varphi\ \sin\ \theta,\ y = b\rho\ \sin\ \varphi\ \sin\ \theta,\ \text{and}\ z = cp\ \cos\theta.$$

Then the volume of the ellipsoid becomes

$$\begin{aligned} V &= \iiint_D dx\, dy\, dz \\ &= 8\int_{\theta=0}^{\theta=\pi/2}\int_{\rho=0}^{\rho=1}\int_{\varphi=0}^{\varphi=\pi/2} abc\,\rho^2\sin\theta\, d\varphi\, d\rho\, d\theta \\ &= 8abc\int_{\varphi=0}^{\varphi=\pi/2} d\varphi \int_{\rho=0}^{\rho=1}\rho^2 d\rho \int_{\theta=0}^{\theta=\pi/2}\sin\theta\, d\theta \\ &= 8abc\left(\frac{\pi}{2}\right)\left(\frac{1}{3}\right)(1) \\ &= \frac{4}{3}\pi abc. \end{aligned}$$

Example 15.7.12: Finding the Volume of the Space Inside an Ellipsoid and Outside a Sphere

Find the volume of the space inside the ellipsoid $\frac{x^2}{75^2}+\frac{y^2}{80^2}+\frac{z^2}{90^2}=1$ and outside the sphere $x^2+y^2+z^2=50^2$.

Solution

This problem is directly related to the l'Hemisphèric structure. The volume of space inside the ellipsoid and outside the sphere might be useful to find the expense of heating or cooling that space. We can use the preceding two examples for the volume of the sphere and ellipsoid and then substract.

First we find the volume of the ellipsoid using $a=75$ ft, $b=80$ ft, and $c=90$ ft in the result from Example. Hence the volume of the ellipsoid is

$$V_{ellipsoid}=\frac{4}{3}\pi(75)(80)(90)\approx 2,262,000\ ft^3.$$

From Example, the volume of the sphere is

$$V_{sphere}\approx 523,600\ ft^3.$$

Therefore, the volume of the space inside the ellipsoid $\frac{x^2}{75^2}+\frac{y^2}{80^2}+\frac{z^2}{90^2}=1$ and outside the sphere $x^2+y^2+z^2=50^2$ is approximately

$$V_{Hemispheric}=V_{ellipsoid}-V_{sphere}=1,738,400\ ft^3.$$

Student Project: Hot air balloons

Hot air ballooning is a relaxing, peaceful pastime that many people enjoy. Many balloonist gatherings take place around the world, such as the Albuquerque International Balloon Fiesta. The Albuquerque event is the largest hot air balloon festival in the world, with over 500 balloons participating each year.



Figure 15.7.12: Balloons lift off at the 2001 Albuquerque International Balloon Fiesta. (credit: David Herrera, Flickr)

As the name implies, hot air balloons use hot air to generate lift. (Hot air is less dense than cooler air, so the balloon floats as long as the hot air stays hot.) The heat is generated by a propane burner suspended below the opening of the basket. Once the balloon takes off, the pilot controls the altitude of the balloon, either by using the burner to heat the air and ascend or by using a vent near the top of the balloon to release heated air and descend. The pilot has very little control over where the balloon

goes, however—balloons are at the mercy of the winds. The uncertainty over where we will end up is one of the reasons balloonists are attracted to the sport.

In this project we use triple integrals to learn more about hot air balloons. We model the balloon in two pieces. The top of the balloon is modeled by a half sphere of radius 28

feet. The bottom of the balloon is modeled by a frustum of a cone (think of an ice cream cone with the pointy end cut off). The radius of the large end of the frustum is 28 feet and the radius of the small end of the frustum is 28 feet. A graph of our balloon model and a cross-sectional diagram showing the dimensions are shown in the following figure.

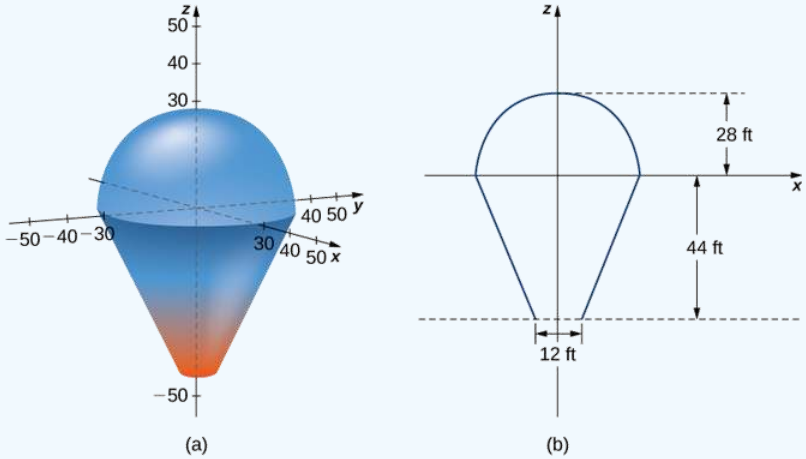


Figure 15.7.13: (a) Use a half sphere to model the top part of the balloon and a frustum of a cone to model the bottom part of the balloon. (b) A cross section of the balloon showing its dimensions.

We first want to find the volume of the balloon. If we look at the top part and the bottom part of the balloon separately, we see that they are geometric solids with known volume formulas. However, it is still worthwhile to set up and evaluate the integrals we would need to find the volume. If we calculate the volume using integration, we can use the known volume formulas to check our answers. This will help ensure that we have the integrals set up correctly for the later, more complicated stages of the project.

1. Find the volume of the balloon in two ways.

a. Use triple integrals to calculate the volume. Consider each part of the balloon separately. (Consider using spherical coordinates for the top part and cylindrical coordinates for the bottom part.)

b. Verify the answer using the formulas for the volume of a sphere, $V = \frac{4}{3}\pi r^3$, and for the volume of a cone, $V = \frac{1}{3}\pi r^2 h$.

In reality, calculating the temperature at a point inside the balloon is a tremendously complicated endeavor. In fact, an entire branch of physics (thermodynamics) is devoted to studying heat and temperature. For the purposes of this project, however, we are going to make some simplifying assumptions about how temperature varies from point to point within the balloon. Assume that just prior to liftoff, the temperature (in degrees Fahrenheit) of the air inside the balloon varies according to the function

$$T_0(r, \theta, z) = \frac{z - r}{10} + 210.$$

2. What is the average temperature of the air in the balloon just prior to liftoff? (Again, look at each part of the balloon separately, and do not forget to convert the function into spherical coordinates when looking at the top part of the balloon.)

Now the pilot activates the burner for 10 seconds. This action affects the temperature in a 12-foot-wide column 20 feet high, directly above the burner. A cross section of the balloon depicting this column in shown in the following figure

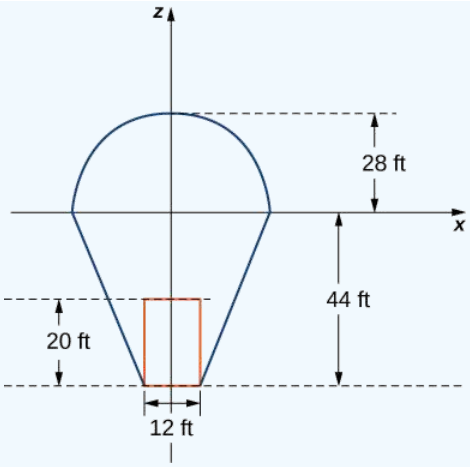


Figure 15.7.14: Activating the burner heats the air in a 20-foot-high, 12-foot-wide column directly above the burner.

Assume that after the pilot activates the burner for 10 seconds, the temperature of the air in the column described above *increases* according to the formula

$$H(r,\theta,z) = -2z - 48.$$

Then the temperature of the air in the column is given by

$$T_1(r,\theta,z) = \frac{z-r}{10} + 210 + (-2z - 48),$$

while the temperature in the remainder of the balloon is still given by

$$T_0(r,\theta,z) = \frac{z-r}{10} + 210.$$

3. Find the average temperature of the air in the balloon after the pilot has activated the burner for 10 seconds.

Key Concepts

- To evaluate a triple integral in cylindrical coordinates, use the iterated integral

$$\int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} \int_{z=u_1(r,\theta)}^{u_2(r,\theta)} f(r,\theta,z) r \, dz \, dr \, d\theta.$$

- To evaluate a triple integral in spherical coordinates, use the iterated integral

$$\int_{\theta=\alpha}^{\theta=\beta} \int_{\rho=g_1(\theta)}^{\rho=g_2(\theta)} \int_{\varphi=u_1(r,\theta)}^{u_2(r,\theta)} f(\rho,\theta,\varphi)\, \rho^2 \sin\varphi \, d\varphi \, d\rho \, d\theta.$$

Key Equations

- **Triple integral in cylindrical coordinates**

$$\iiint_B g(s,y,z)dV = \iiint_B g(r\cos\theta,\ r\sin\theta,\ z) r \, dr \, d\theta \, dz = \iiint_B f(r,\theta,z) r \, dr \, d\theta \, dz$$

- **Triple integral in spherical coordinates**

$$\iiint_B f(\rho,\theta,\varphi)\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \int_{\varphi=\gamma}^{\varphi=\psi} \int_{\theta=\alpha}^{\theta=\beta} \int_{\rho=a}^{\rho=b} f(\rho,\theta,\varphi)\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

Glossary

triple integral in cylindrical coordinates

the limit of a triple Riemann sum, provided the following limit exists:

$$lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}f(r^*_{ijk},\theta^*_{ijk},s^*_{ijk})r^*_{ijk}\Delta r\Delta\theta\Delta z$$

triple integral in spherical coordinates

the limit of a triple Riemann sum, provided the following limit exists:

$$lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}f(\rho^*_{ijk},\theta^*_{ijk},\varphi^*_{ijk})(\rho^*_{ijk})^2\sin\varphi\Delta\rho\Delta\theta\Delta\varphi$$

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15.8: Triple Integrals in Spherical Coordinates

Learning Objectives

- Evaluate a triple integral by changing to cylindrical coordinates.
- Evaluate a triple integral by changing to spherical coordinates.

Earlier in this chapter we showed how to convert a double integral in rectangular coordinates into a double integral in polar coordinates in order to deal more conveniently with problems involving circular symmetry. A similar situation occurs with triple integrals, but here we need to distinguish between cylindrical symmetry and spherical symmetry. In this section we convert triple integrals in rectangular coordinates into a triple integral in either cylindrical or spherical coordinates.

Also recall the chapter prelude, which showed the opera house l'Hemisphèric in Valencia, Spain. It has four sections with one of the sections being a theater in a five-story-high sphere (ball) under an oval roof as long as a football field. Inside is an IMAX screen that changes the sphere into a planetarium with a sky full of 9000 twinkling stars. Using triple integrals in spherical coordinates, we can find the volumes of different geometric shapes like these.

Review of Cylindrical Coordinates

As we have seen earlier, in two-dimensional space $\mathbb{R}^2$ a point with rectangular coordinates (x, y) can be identified with (r, θ) in polar coordinates and vice versa, where $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$ and $\tan\theta = \left(\frac{y}{x}\right)$ are the relationships between the variables.

In three-dimensional space $\mathbb{R}^3$ a point with rectangular coordinates (x, y, z) can be identified with cylindrical coordinates (r, θ, z) and vice versa. We can use these same conversion relationships, adding z as the vertical distance to the point from the $(xy$-plane as shown in 15.8.1.

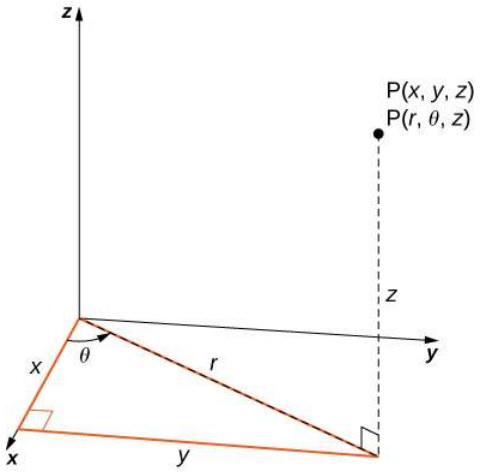


Figure 15.8.1: Cylindrical coordinates are similar to polar coordinates with a vertical z coordinate added.

To convert from rectangular to cylindrical coordinates, we use the conversion

- $x = r\cos\theta$
- $y = r\sin\theta$
- $z = z$

To convert from cylindrical to rectangular coordinates, we use

- $r^2 = x^2 + y^2$ and
- $\theta = \tan^{-1}\left(\frac{y}{x}\right)$
- $z = z$

Note that that z-coordinate remains the same in both cases.

In the two-dimensional plane with a rectangular coordinate system, when we say $x = k$ (constant) we mean an unbounded vertical line parallel to the y-axis and when $y = l$ (constant) we mean an unbounded horizontal line parallel to the x-axis. With the polar

coordinate system, when we say $r = c$ (constant), we mean a circle of radius c units and when $\theta = \alpha$ (constant) we mean an infinite ray making an angle α with the positive x-axis.

Similarly, in three-dimensional space with rectangular coordinates (x, y, z) the equations $x = k$, $y = l$ and $z = m$ where k, l and m are constants, represent unbounded planes parallel to the yz-plane, xz-plane and xy-plane, respectively. With cylindrical coordinates (r, θ, z), by $r = c$, $\theta = \alpha$, and $z = m$, where c, α, and m are constants, we mean an unbounded vertical cylinder with the z-axis as its radial axis; a plane making a constant angle α with the xy-plane; and an unbounded horizontal plane parallel to the xy-plane, respectively. This means that the circular cylinder $x^2 + y^2 = c^2$ in rectangular coordinates can be represented simply as $r = c$ in cylindrical coordinates. (Refer to Cylindrical and Spherical Coordinates for more review.)

Integration in Cylindrical Coordinates

Triple integrals can often be more readily evaluated by using cylindrical coordinates instead of rectangular coordinates. Some common equations of surfaces in rectangular coordinates along with corresponding equations in cylindrical coordinates are listed in Table 15.8.1. These equations will become handy as we proceed with solving problems using triple integrals.

Table 15.8.1: Equations of Some Common Shapes

	Circular cylinder	Circular cone	Sphere	Paraboloid
Rectangular	$x^2 + y^2 = c^2$	$z^2 = c^2(x^2 + y^2)$	$x^2 + y^2 + z^2 = c^2$	$z = c(x^2 + y^2)$
Cylindrical	$r = c$	$z = cr$	$r^2 + z^2 = c^2$	$z = cr^2$

As before, we start with the simplest bounded region B in $\mathbb{R}^3$ to describe in cylindrical coordinates, in the form of a cylindrical box, $B = \{(r, \theta, z) | a \le r \le b,\ \alpha \le \theta \le \beta,\ c \le z \le d\}$ (Figure 15.8.2). Suppose we divide each interval into l, m, and n subdivisions such that $\Delta r = \frac{b-a}{l}$, $\Delta\theta = \frac{\beta-\alpha}{m}$, and $\Delta z = \frac{d-c}{n}$. Then we can state the following definition for a triple integral in cylindrical coordinates.

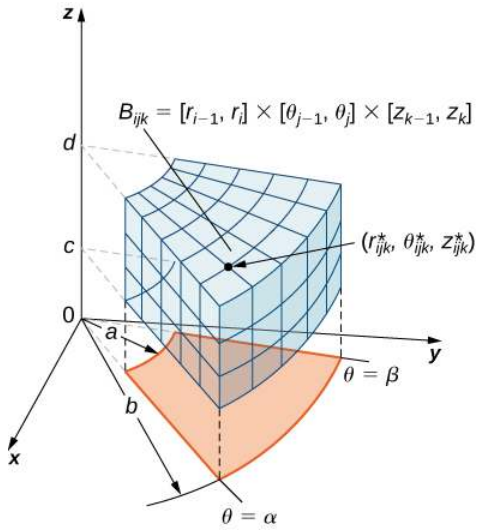


Figure 15.8.2: A cylindrical box B described by cylindrical coordinates.

DEFINITION: triple integral in cylindrical coordinates

Consider the cylindrical box (expressed in cylindrical coordinates)

$$B = \{(r, \theta, z) | a \le r \le b,\ \alpha \le \theta \le \beta,\ c \le z \le d\}.$$

If the function $f(r, \theta, z)$ is continuous on B and if $(r^*_{ijk}, \theta^*_{ijk}, z^*_{ijk})$ is any sample point in the cylindrical subbox $B_{ijk} = |r_{i-1}, r_i| \times |\theta_{j-1}, \theta_j| \times |z_{k-1}, k_i|$ (Figure 15.8.2), then we can define the *triple integral in cylindrical coordinates* as the limit of a triple Riemann sum, provided the following limit exists:

$$\lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(r^*_{ijk},\theta^*_{ijk},z^*_{ijk})\,\Delta r\Delta\theta\Delta z.$$

Note that if $g(x,y,z)$ is the function in rectangular coordinates and the box B is expressed in rectangular coordinates, then the triple integral

$$\iiint_B g(x,y,z)\,dV$$

is equal to the triple integral

$$\iiint_B g(r\,\cos\theta,\ r\,\sin\theta,\ z)r\,dr\,d\theta\,dz$$

and we have

$$\iiint_B g(x,y,z)\,dV=\iiint_B g(r\,\cos\theta,\ r\,\sin\theta,\ z)\,r\,dr\,d\theta\,dz=\iiint_B f(r,\theta\,z)\,r\,dr\,d\theta\,dz.$$

As mentioned in the preceding section, all the properties of a double integral work well in triple integrals, whether in rectangular coordinates or cylindrical coordinates. They also hold for iterated integrals. To reiterate, in cylindrical coordinates, Fubini's theorem takes the following form:

Theorem: Fubini's Theorem in Cylindrical Coordinates

Suppose that $g(x,y,z)$ is continuous on a rectangular box B which when described in cylindrical coordinates looks like $B=\{(r,\theta,z)|a\le r\le b,\ \alpha\le\theta\le\beta,\ c\le z\le d\}$.

Then $g(x,y,z)=g(r\,\cos\theta,r\,\sin\theta,z)=f(r,\theta,z)$ and

$$\iiint_B g(x,y,z)\,dV=\int_c^d\int_\beta^\alpha\int_a^b f(r,\theta,z)\,r\,dr\,d\theta\,dz.$$

The iterated integral may be replaced equivalently by any one of the other five iterated integrals obtained by integrating with respect to the three variables in other orders.

Cylindrical coordinate systems work well for solids that are symmetric around an axis, such as cylinders and cones. Let us look at some examples before we define the triple integral in cylindrical coordinates on general cylindrical regions.

Example 15.8.1: Evaluating a Triple Integral over a Cylindrical Box

Evaluate the triple integral

$$\iiint_B (zr\,\sin\theta)r\,dr\,d\theta\,dz$$

where the cylindrical box B is $B=\{(r,\theta,z)|0\le r\le 2,\ 0\le\theta\le\pi/2,\ 0,\le z\le 4\}$.

Solution

As stated in Fubini's theorem, we can write the triple integral as the iterated integral

$$\iiint_B (zr\,\sin\theta)r\,dr\,d\theta\,dz=\int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=2}\int_{z=0}^{z=4}(zr\,\sin\theta)\,r\,dz\,dr\,d\theta.$$

The evaluation of the iterated integral is straightforward. Each variable in the integral is independent of the others, so we can integrate each variable separately and multiply the results together. This makes the computation much easier:

$$\int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=2}\int_{z=0}^{z=4}(zr\,\sin\,\theta)\,r\,dz\,dr\,d\theta=\left(\int_0^{\pi/2}\sin\,\theta\,d\theta\right)\left(\int_0^2 r^2dr\right)\left(\int_0^4 z\,dz\right)$$
$$=\left(-\cos\theta|_0^{\pi/2}\right)\left(\frac{r^3}{3}\Big|_0^2\right)\left(\frac{z^2}{2}\Big|_0^4\right)=\frac{64}{3}.$$

? Exercise 15.8.1:

Evaluate the triple integral

$$\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=1}\int_{z=0}^{z=4}(rz\,\sin\,\theta)\,r\,dz\,dr\,d\theta.$$

Hint

Follow the same steps as in the previous example.

Answer

8

If the cylindrical region over which we have to integrate is a general solid, we look at the projections onto the coordinate planes. Hence the triple integral of a continuous function $f(r,\theta,z)$ over a general solid region $E=\{(r,\theta,z)|(r,\theta)\in D,u_1(r,\theta)\le z\le u_2(r,\theta)\}$ in $\mathbb{R}^3$ where D is the projection of E onto the $r\theta$-plane, is

$$\iiint_E f(r,\theta,z)\,r\,dr\,d\theta\,dz=\iint_D\left[\int_{u_1(r,\theta)}^{u_2(r,\theta)}f(r,\theta,z)dz\right]r\,dr\,d\theta.$$

In particular, if $D=\{(r,\theta)|G_1(\theta)\le r\le g_2(\theta),\alpha\le\theta\le\beta\}$, then we have

$$\iiint_E f(r,\theta,z)\,r\,dr\,d\theta=\int_{\theta=\alpha}^{\theta=\beta}\int_{r=g_1(\theta)}^{r=g_2(\theta)}\int_{z=u_1(r,\theta)}^{z=u_2(r,\theta)}f(r,\theta,z)\,r\,dz\,dr\,d\theta.$$

Similar formulas exist for projections onto the other coordinate planes. We can use polar coordinates in those planes if necessary.

✓ Example 15.8.2: Setting up a Triple Integral in Cylindrical Coordinates over a General Region

Consider the region E inside the right circular cylinder with equation $r=2\,\sin\,\theta$, bounded below by the $r\theta$-plane and bounded above by the sphere with radius 4 centered at the origin (Figure 15.5.3). Set up a triple integral over this region with a function $f(r,\theta,z)$ in cylindrical coordinates.

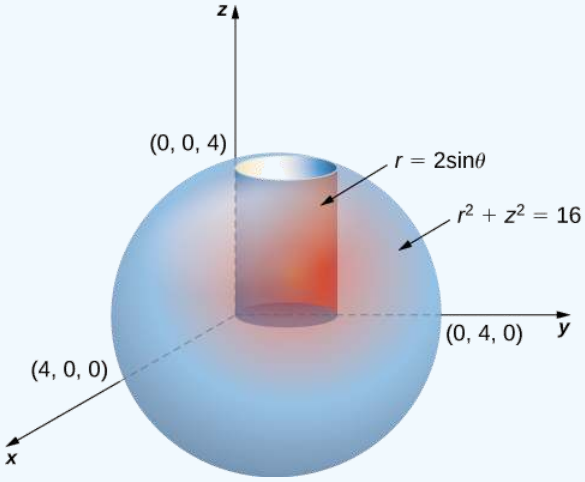


Figure 15.8.3: Setting up a triple integral in cylindrical coordinates over a cylindrical region.

Solution

First, identify that the equation for the sphere is $r^2+z^2=16$. We can see that the limits for z are from 0 to $z=\sqrt{16-r^2}$. Then the limits for r are from 0 to $r=2\sin\theta$. Finally, the limits for θ are from 0 to π. Hence the region is $E=\{(r,\theta,z)|0\le\theta\le\pi,\ 0\le r\le 2\sin\theta,\ 0\le z\le\sqrt{16-r^2}\}$. Therefore, the triple integral is

$$\iiint_E f(r,\theta,z)\,r\,dz\,dr\,d\theta=\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=2\sin\theta}\int_{z=0}^{z=\sqrt{16-r^2}} f(r,\theta,z)\,r\,dz\,dr\,d\theta.$$

? Exercise 15.8.2:

Consider the region inside the right circular cylinder with equation $r=2\sin\theta$ bounded below by the $r\theta$-plane and bounded above by $z=4-y$. Set up a triple integral with a function $f(r,\theta,z)$ in cylindrical coordinates.

Hint

Analyze the region, and draw a sketch.

Answer

$$\iiint_E f(r,\theta,z)r\,dz\,dr\,d\theta=\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=2\sin\theta}\int_{z=0}^{z=4-r\sin\theta} f(r,\theta,z)r\,dz\,dr\,d\theta.$$

✓ Example 15.8.3: Setting up a Triple Integral in Two Ways

Let E be the region bounded below by the cone $z=\sqrt{x^2+y^2}$ and above by the paraboloid $z=2-x^2-y^2$. (Figure 15.5.4). Set up a triple integral in cylindrical coordinates to find the volume of the region, using the following orders of integration:

a. $dz\,dr\,d\theta$

b. $dr\,dz\,d\theta$

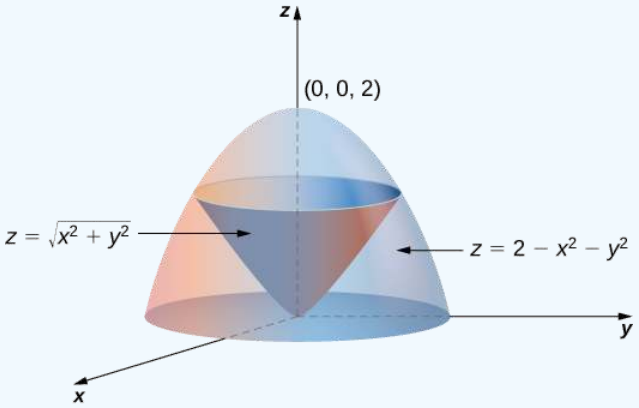


Figure 15.8.4: Setting up a triple integral in cylindrical coordinates over a conical region.

Solution

a. The cone is of radius 1 where it meets the paraboloid. Since $z=2-x^2-y^2=2-r^2$ and $z=\sqrt{x^2+y^2}=r$ (assuming r is nonnegative), we have $2-r^2=r$. Solving, we have $r^2+r-2=(r+2)(r-1)=0$. Since $r\ge 0$, we have $r=1$. Therefore $z=1$. So the intersection of these two surfaces is a circle of radius 1 in the plane $z=1$. The cone is the lower bound for z and the paraboloid is the upper bound. The projection of the region onto the xy-plane is the circle of radius 1 centered at the origin.

Thus, we can describe the region as $E=\{(r,\theta,z)|0\le\theta\le 2\pi,\ 0\le r\le 1,\ r\le z\le 2-r^2\}$.

Hence the integral for the volume is

$$V = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=r}^{z=2-r^2} r\, dz\, dr\, d\theta.$$

b. We can also write the cone surface as $r = z$ and the paraboloid as $r^2 = 2 - z$. The lower bound for r is zero, but the upper bound is sometimes the cone and the other times it is the paraboloid. The plane $z = 1$ divides the region into two regions. Then the region can be described as

$$E = \{(r, \theta, z) | 0 \leq \theta \leq 2\pi,\ 0 \leq z \leq 1,\ 0 \leq r \leq z\} \cup \{(r, \theta, z) | 0 \leq \theta \leq 2\pi,\ 1 \leq z \leq 2,\ 0 \leq r \leq \sqrt{2 - z}\}.$$

Now the integral for the volume becomes

$$V = \int_{\theta=0}^{\theta=2\pi} \int_{z=0}^{z=1} \int_{r=0}^{r=z} r\, dr\, dz\, d\theta + \int_{\theta=0}^{\theta=2\pi} \int_{z=1}^{z=2} \int_{r=0}^{r=\sqrt{2-z}} r\, dr\, dz\, d\theta.$$

Exercise 15.8.3:

Redo the previous example with the order of integration $d\theta\, dz\, dr$.

Hint

Note that θ is independent of r and z.

Answer

$E = \{(r, \theta, z) | 0 \leq \theta \leq 2\pi,\ 0 \leq z \leq 1,\ 0 \leq r \leq 2 - z^2\}$ and

$$V = \int_{r=0}^{r=1} \int_{z=0}^{z=2-r^2} \int_{\theta=0}^{\theta=2\pi} r\, d\theta\, dz\, dr.$$

Example 15.8.4: Finding a Volume with Triple Integrals in Two Ways

Let E be the region bounded below by the $r\theta$-plane, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$ (Figure 15.5.5). Set up a triple integral in cylindrical coordinates to find the volume of the region using the following orders of integration, and in each case find the volume and check that the answers are the same:

a. $dz\, dr\, d\theta$
b. $dr\, dz\, d\theta$.

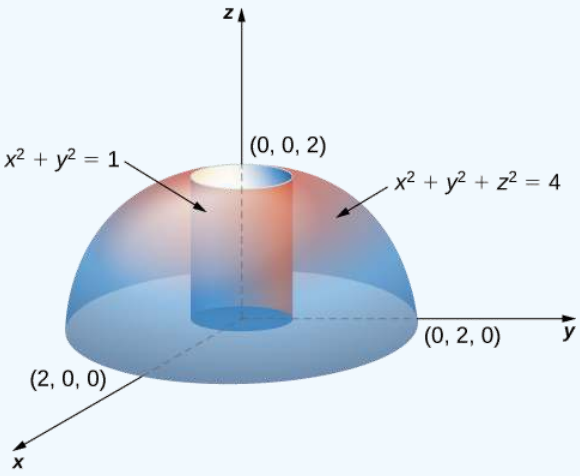


Figure 15.8.5: Finding a cylindrical volume with a triple integral in cylindrical coordinates.

Solution

a. Note that the equation for the sphere is

$$x^2 + y^2 + z^2 = 4 \text{ or } r^2 + z^2 = 4$$

and the equation for the cylinder is

$$x^2+y^2=1 \text{ or } r^2=1.$$

Thus, we have for the region E

$$E=\{(r,\theta,z)|0\le z\le\sqrt{4-r^2},\ 0\le r\le 1,\ 0\le\theta\le 2\pi\}$$

Hence the integral for the volume is

$$\begin{aligned} V(E) &= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}\int_{z=0}^{z=\sqrt{4-r^2}} r\,dz\,dr\,d\theta \\ &= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}\left[rz\Big|_{z=0}^{z=\sqrt{4-r^2}}\right]dr\,d\theta \\ &= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1}\left(r\sqrt{4-r^2}\right)dr\,d\theta \\ &= \int_0^{2\pi}\left(\frac{8}{3}-\sqrt{3}\right)d\theta = 2\pi\left(\frac{8}{3}-\sqrt{3}\right) \text{ cubic units.} \end{aligned}$$

b. Since the sphere is $x^2+y^2+z^2=4$, which is $r^2+z^2=4$, and the cylinder is $x^2+y^2=1$, which is $r^2=1$, we have $1+z^2=4$, that is, $z^2=3$. Thus we have two regions, since the sphere and the cylinder intersect at $(1,\sqrt{3})$ in the rz-plane

$$E_1=\{(r,\theta,z)|0\le r\le\sqrt{4-r^2},\ \sqrt{3}\le z\le 2,\ 0\le\theta\le 2\pi\}$$

and

$$E_2=\{(r,\theta,z)|0\le r\le 1,\ 0\le z\le\sqrt{3},\ 0\le\theta\le 2\pi\}.$$

Hence the integral for the volume is

$$\begin{aligned} V(E) &= \int_{\theta=0}^{\theta=2\pi}\int_{z=\sqrt{3}}^{z=2}\int_{r=0}^{r=\sqrt{4-r^2}} r\,dr\,dz\,d\theta+\int_{\theta=0}^{\theta=2\pi}\int_{z=0}^{z=\sqrt{3}}\int_{r=0}^{r=1} r\,dr\,dz\,d\theta \\ &= \sqrt{3}\pi+\left(\frac{16}{3}-3\sqrt{3}\right)\pi \\ &= 2\pi\left(\frac{8}{3}-\sqrt{3}\right) \text{ cubic units.} \end{aligned}$$

? Exercise 15.8.4

Redo the previous example with the order of integration $d\theta\,dz\,dr$.

Hint

A figure can be helpful. Note that θ is independent of r and z.

Answer

$E_2=\{(r,\theta,z)|0\le\theta\le 2\pi,\ 0\le r\le 1,\ r\le z\le\sqrt{4-r^2}\}$ and

$$V=\int_{r=0}^{r=1}\int_{z=r}^{z=\sqrt{4-r^2}}\int_{\theta=0}^{\theta=2\pi} r\,d\theta\,dz\,dr.$$

Review of Spherical Coordinates

In three-dimensional space $\mathbb{R}^3$ in the spherical coordinate system, we specify a point P by its distance ρ from the origin, the polar angle θ from the positive x-axis (same as in the cylindrical coordinate system), and the angle φ from the positive z-axis and the line OP (Figure 15.8.6). Note that $\rho>0$ and $0\le\varphi\le\pi$. (Refer to Cylindrical and Spherical Coordinates for a review.) Spherical coordinates are useful for triple integrals over regions that are symmetric with respect to the origin.

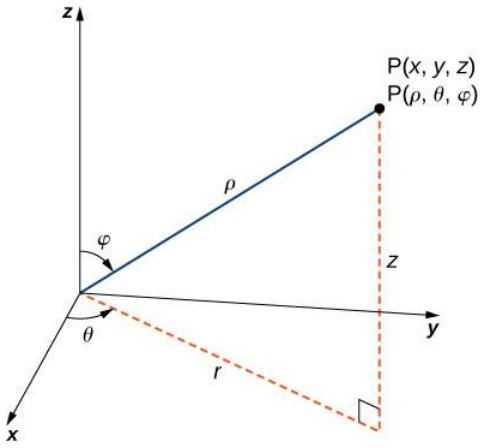


Figure 15.8.6: The spherical coordinate system locates points with two angles and a distance from the origin.

Recall the relationships that connect rectangular coordinates with spherical coordinates.

From spherical coordinates to rectangular coordinates:

$$x = \rho \sin \varphi \cos\theta,\ y = \rho \sin \varphi \sin \theta,\ and\ z = \rho \cos \varphi.$$

From rectangular coordinates to spherical coordinates:

$$\rho^2 = x^2 + y^2 + z^2,\ \tan \theta = \frac{y}{x},\ \varphi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right).$$

Other relationships that are important to know for conversions are

- $r = \rho \sin \varphi$
- $\theta = \theta$ These equations are used to convert from spherical coordinates to cylindrical coordinates.
- $z = \rho \cos \varphi$

and

- $\rho = \sqrt{r^2 + z^2}$
- $\theta = \theta$ These equations are used to convert from cylindrical coordinates to spherical coordinates.
- $\varphi = \arccos\left(\frac{z}{\sqrt{r^2+z^2}}\right)$

15.8.7 shows a few solid regions that are convenient to express in spherical coordinates.

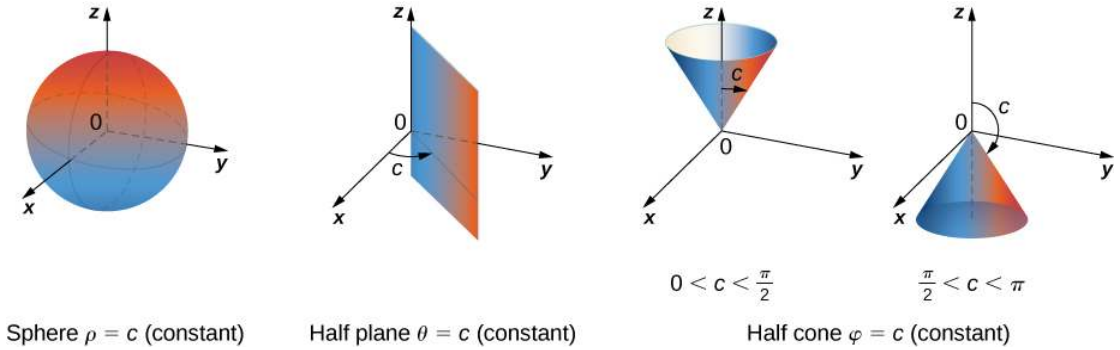


Figure 15.8.7: Spherical coordinates are especially convenient for working with solids bounded by these types of surfaces. (The letter c indicates a constant.)

Integration in Spherical Coordinates

We now establish a triple integral in the spherical coordinate system, as we did before in the cylindrical coordinate system. Let the function $f(\rho, \theta, \varphi)$ be continuous in a bounded spherical box, $B = \{(\rho, \theta, \varphi) | a \le \rho \le b,\ \alpha \le \theta \le \beta,\ \gamma \le \varphi \le \psi\}$. We then divide each interval into l, m, n and n subdivisions such that $\Delta\rho = \frac{b-a}{l}$, $\Delta\theta = \frac{\beta-\alpha}{m}$. $\Delta\varphi = \frac{\psi-\gamma}{n}$. Now we can illustrate the

following theorem for triple integrals in spherical coordinates with $(\rho^*_{ijk}, \theta^*_{ijk}, \varphi^*_{ijk})$ being any sample point in the spherical subbox B_{ijk}. For the volume element of the subbox ΔV in spherical coordinates, we have $\Delta V = (\Delta\rho)\,(\rho\Delta\varphi)\,(\rho\,\sin\,\varphi\,\Delta\theta)$, as shown in the following figure.

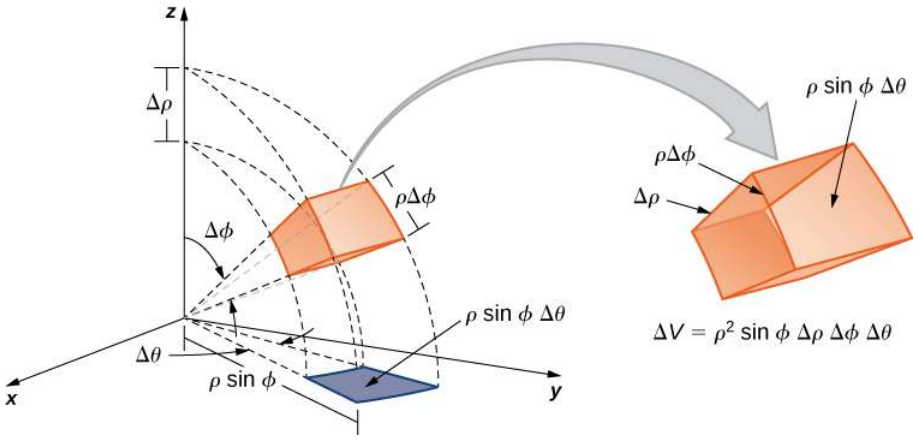


Figure 15.8.8: The volume element of a box in spherical coordinates.

Definition: triple integral in spherical coordinates

The **triple integral in spherical coordinates** is the limit of a triple Riemann sum,

$$\lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(\rho^*_{ijk}, \theta^*_{ijk}, \varphi^*_{ijk})(\rho^*_{ijk})^2\,\sin\,\varphi\Delta\rho\,\Delta\theta\Delta\varphi$$

provided the limit exists.

As with the other multiple integrals we have examined, all the properties work similarly for a triple integral in the spherical coordinate system, and so do the iterated integrals. Fubini's theorem takes the following form.

Theorem: Fubini's Theorem for Spherical Coordinates

If $f(\rho, \theta, \varphi)$ is continuous on a spherical solid box $B = [a, b] \times [\alpha, \beta] \times [\gamma, \psi]$, then

$$\iiint_B f(\rho, \theta, \varphi)\,\rho^2\,\sin\,\varphi\,d\rho\,d\varphi\,d\theta = \int_{\varphi=\gamma}^{\varphi=\psi}\int_{\theta=\alpha}^{\theta=\beta}\int_{\rho=a}^{\rho=b} f(\rho, \theta, \varphi)\,\rho^2\,\sin\,\varphi\,d\rho\,d\varphi\,d\theta.$$

This iterated integral may be replaced by other iterated integrals by integrating with respect to the three variables in other orders.

As stated before, spherical coordinate systems work well for solids that are symmetric around a point, such as spheres and cones. Let us look at some examples before we consider triple integrals in spherical coordinates on general spherical regions.

Example 15.8.5: Evaluating a Triple Integral in Spherical Coordinates

Evaluate the iterated triple integral

$$\int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\pi/2}\int_{\rho=0}^{\rho=1} \rho^2\,\sin\,\varphi\,d\rho\,d\varphi\,d\theta.$$

Solution

As before, in this case the variables in the iterated integral are actually independent of each other and hence we can integrate each piece and multiply:

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^2\,\sin\,\varphi\,d\rho\,d\varphi\,d\theta = \int_0^{2\pi} d\theta\int_0^{\pi/2}\sin\,\varphi\,d\varphi\int_0^1 \rho^2 d\rho = (2\pi)\,(1)\left(\frac{1}{3}\right) = \frac{2\pi}{3}$$

The concept of triple integration in spherical coordinates can be extended to integration over a general solid, using the projections onto the coordinate planes. Note that dV and dA mean the increments in volume and area, respectively. The variables V and A are used as the variables for integration to express the integrals.

The triple integral of a continuous function $f(\rho,\theta,\varphi)$ over a general solid region

$$E=\{(\rho,\theta,\varphi)|(\rho,\theta)\in D, u_1(\rho,\theta)\le\varphi\le u_2(\rho,\theta)\}$$

in $\mathbb{R}^3$, where D is the projection of E onto the $\rho\theta$-plane, is

$$\iiint_E f(\rho,\theta,\varphi)\,dV=\iint_D\left[\int_{u_1(\rho,\theta)}^{u_2(\rho,\theta)} f(\rho,\theta,\varphi)\,d\varphi\right]\,dA.$$

In particular, if $D=\{(\rho,\theta)|g_1(\theta)\le\rho\le g_2(\theta),\ \alpha\le\theta\le\beta\}$, the we have

$$\iiint_E f(\rho,\theta,\varphi)\,dV=\int_\alpha^\beta\int_{g_1(\theta)}^{g_2(\theta)}\int_{u_1(\rho,\theta)}^{u_2(\rho,\theta)} f(\rho,\theta,\varphi)\rho^2\sin\varphi\,d\varphi\,d\rho\,d\theta.$$

Similar formulas occur for projections onto the other coordinate planes.

Example 15.8.6: Setting up a Triple Integral in Spherical Coordinates

Set up an integral for the volume of the region bounded by the cone $z=\sqrt{3(x^2+y^2)}$ and the hemisphere $z=\sqrt{4-x^2-y^2}$ (see the figure below).

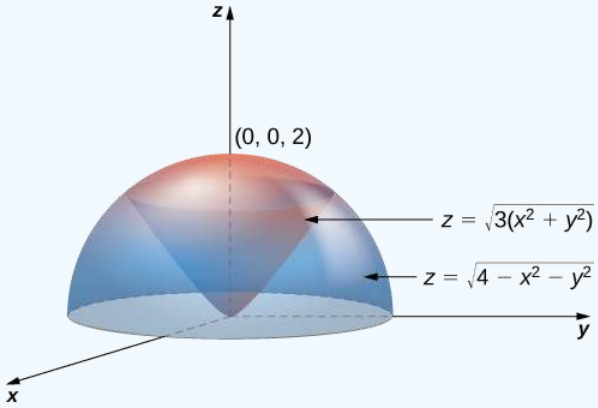


Figure 15.8.9: A region bounded below by a cone and above by a hemisphere.

Solution

Using the conversion formulas from rectangular coordinates to spherical coordinates, we have:

For the cone: $z=\sqrt{3(x^2+y^2)}$ or $\rho\cos\varphi=\sqrt{3}\rho\sin\varphi$ or $\tan\varphi=\frac{1}{\sqrt{3}}$ or $\varphi=\frac{\pi}{6}$.

For the sphere: $z=\sqrt{4-x^2-y^2}$ or $z^2+x^2+y^2=4$ or $\rho^2=4$ or $\rho=2$.

Thus, the triple integral for the volume is

$$V(E)=\int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\pi/6}\int_{\rho=0}^{\rho=2}\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta.$$

Exercise 15.8.5

Set up a triple integral for the volume of the solid region bounded above by the sphere $\rho=2$ and bounded below by the cone $\varphi=\pi/3$.

Hint

Follow the steps of the previous example.

Answer

$$V(E) = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi/3} \int_{\rho=0}^{\rho=2} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

Example 15.8.7: Interchanging Order of Integration in Spherical Coordinates

Let E be the region bounded below by the cone $z = \sqrt{x^2+y^2}$ and above by the sphere $z = x^2+y^2+z^2$ (Figure 15.5.10). Set up a triple integral in spherical coordinates and find the volume of the region using the following orders of integration:

a. $d\rho \, d\phi \, d\theta$
b. $d\varphi \, d\rho \, d\theta$

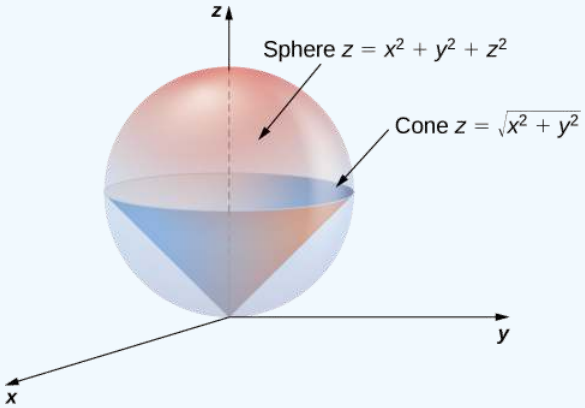


Figure 15.8.10:. A region bounded below by a cone and above by a sphere.

Solution

a. Use the conversion formulas to write the equations of the sphere and cone in spherical coordinates.

For the sphere:

$$\begin{aligned} x^2+y^2+z^2 &= z \\ \rho^2 &= \rho\cos\varphi \\ \rho &= \cos\varphi. \end{aligned}$$

For the cone:

$$\begin{aligned} z &= \sqrt{x^2+y^2} \\ \rho\cos\varphi &= \sqrt{\rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta} \\ \rho\cos\varphi &= \sqrt{\rho^2\sin^2\varphi\,(\cos^2\theta+\sin^2\theta)} \\ \rho\cos\varphi &= \rho\sin\varphi \\ \cos\varphi &= \sin\varphi \\ \varphi &= \pi/4. \end{aligned}$$

Hence the integral for the volume of the solid region E becomes

$$V(E) = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi/4} \int_{\rho=0}^{\rho=\cos\varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta.$$

b. Consider the $\varphi\rho$-plane. Note that the ranges for φ and ρ (from part a.) are

$$\begin{aligned} 0 \le \varphi &\le \pi/4 \\ 0 \le \rho &\le \cos\varphi \end{aligned}$$

The curve $\rho = \cos\,\varphi$ meets the line $\varphi = \pi/4$ at the point $(\pi/4, \sqrt{2}/2)$ Thus, to change the order of integration, we need to use two pieces:

$$0 \le \rho \le \sqrt{2}/2,\ 0 \le \varphi \le \pi/4$$

and

$$\sqrt{2}/2 \le \rho \le 1,\ 0 \le \varphi \le \cos^{-1}\,\rho.$$

Hence the integral for the volume of the solid region E becomes

$$V(E) = \int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=\sqrt{2}/2}\int_{\varphi=0}^{\varphi=\pi/4} \rho^2\,\sin\,\varphi\,d\varphi\,d\rho\,d\theta + \int_{\theta=0}^{\theta=2\pi}\int_{\rho=\sqrt{2}/2}^{\rho=1}\int_{\varphi=0}^{\varphi=\cos^{-1}\,\rho} \rho^2\,\sin\,\varphi\,d\varphi\,d\rho\,d\theta$$

In each case, the integration results in $V(E) = \frac{\pi}{8}$.

Before we end this section, we present a couple of examples that can illustrate the conversion from rectangular coordinates to cylindrical coordinates and from rectangular coordinates to spherical coordinates.

Example 15.8.8: Converting from Rectangular Coordinates to Cylindrical Coordinates

Convert the following integral into cylindrical coordinates:

$$\int_{y=-1}^{y=1}\int_{x=0}^{x=\sqrt{1-y^2}}\int_{z=x^2+y^2}^{z=\sqrt{x^2+y^2}} xyz\,dz\,dx\,dy.$$

Solution

The ranges of the variables are

$$\begin{aligned} -1 &\le y \le y \\ 0 &\le x \le \sqrt{1-y^2} \\ x^2+y^2 &\le z \le \sqrt{x^2+y^2}. \end{aligned}$$

The first two inequalities describe the right half of a circle of radius 1. Therefore, the ranges for θ and r are

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \text{ and } 0 \le r \le 1.$$

The limits of z are $r^2 \le z \le r$, hence

$$\int_{y=-1}^{y=1}\int_{x=0}^{x=\sqrt{1-y^2}}\int_{z=x^2+y^2}^{z=\sqrt{x^2+y^2}} xyz\,dz\,dx\,dy = \int_{\theta=-\pi/2}^{\theta=\pi/2}\int_{r=0}^{r=1}\int_{z=r^2}^{z=r} r(r\,\cos\theta)\,(r\,\sin\,\theta)\,z\,dz\,dr\,d\theta.$$

Example 15.8.9: Converting from Rectangular Coordinates to Spherical Coordinates

Convert the following integral into spherical coordinates:

$$\int_{y=0}^{y=3}\int_{x=0}^{x=\sqrt{9-y^2}}\int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)\,dz\,dx\,dy.$$

Solution

The ranges of the variables are

$$\begin{aligned} 0 &\le y \le 3 \\ 0 &\le x \le \sqrt{9-y^2} \\ \sqrt{x^2+y^2} &\le z \le \sqrt{18-x^2-y^2}. \end{aligned}$$

The first two ranges of variables describe a quarter disk in the first quadrant of the xy-plane. Hence the range for θ is $0 \le \theta \le \frac{\pi}{2}$.

The lower bound $z = \sqrt{x^2 + y^2}$ is the upper half of a cone and the upper bound $z = \sqrt{18 - x^2 - y^2}$ is the upper half of a sphere. Therefore, we have $0 \le \rho \le \sqrt{18}$, which is $0 \le \rho \le 3\sqrt{2}$.

For the ranges of φ we need to find where the cone and the sphere intersect, so solve the equation

$$\begin{aligned} r^2 + z^2 &= 18 \\ \left(\sqrt{x^2 + y^2}\right)^2 + z^2 &= 18 \\ z^2 + z^2 &= 18 \\ 2z^2 &= 18 \\ z^2 &= 9 \\ z &= 3. \end{aligned}$$

This gives

$$\begin{aligned} 3\sqrt{2}\cos\varphi &= 3 \\ \cos\varphi &= \frac{1}{\sqrt{2}} \\ \varphi &= \frac{\pi}{4}. \end{aligned}$$

Putting this together, we obtain

$$\int_{y=0}^{y=3} \int_{x=0}^{x=\sqrt{9-y^2}} \int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{18-x^2-y^2}} (x^2 + y^2 + z^2)\, dz\, dx\, dy = \int_{\varphi=0}^{\varphi=\pi/4} \int_{\theta=0}^{\theta=\pi/2} \int_{\rho=0}^{\rho=3\sqrt{2}} \rho^4 \sin\varphi\, d\rho\, d\theta\, d\varphi.$$

? Exercise 15.8.6:

Use rectangular, cylindrical, and spherical coordinates to set up triple integrals for finding the volume of the region inside the sphere $x^2 + y^2 + z^2 = 4$ but outside the cylinder $x^2 + y^2 = 1$.

Answer: Rectangular

$$\int_{x=-2}^{x=2} \int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} \int_{z=-\sqrt{4-x^2-y^2}}^{z=\sqrt{4-x^2-y^2}} dz\, dy\, dx - \int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} \int_{z=-\sqrt{4-x^2-y^2}}^{z=\sqrt{4-x^2-y^2}} dz\, dy\, dx.$$

Answer: Cylindrical

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=1}^{r=2} \int_{z=-\sqrt{4-r^2}}^{z=\sqrt{4-r^2}} r\, dz\, dr\, d\theta.$$

Answer: Spherical

$$\int_{\varphi=\pi/6}^{\varphi=5\pi/6} \int_{\theta=0}^{\theta=2\pi} \int_{\rho=\csc\varphi}^{\rho=2} \rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi.$$

Now that we are familiar with the spherical coordinate system, let's find the volume of some known geometric figures, such as spheres and ellipsoids.

✓ Example 15.8.10: Chapter Opener: Finding the Volume of l'Hemisphèric

Find the volume of the spherical planetarium in l'Hemisphèric in Valencia, Spain, which is five stories tall and has a radius of approximately 50 ft, using the equation $x^2 + y^2 + z^2 = r^2$.



Figure 15.8.11: (credit: modification of work by Javier Yaya Tur, Wikimedia Commons)

Solution

We calculate the volume of the ball in the first octant, where $x \leq 0$, $y \leq 0$, and $z \leq 0$, using spherical coordinates, and then multiply the result by 8 for symmetry. Since we consider the region D as the first octant in the integral, the ranges of the variables are

$$0 \leq \varphi \leq \frac{\pi}{2},\ 0 \leq \rho \leq r,\ 0 \leq \theta \leq \frac{\pi}{2}.$$

Therefore,

$$\begin{aligned} V = \iiint_D dx\, dy\, dz &= 8 \int_{\theta=0}^{\theta=\pi/2} \int_{\rho=0}^{\rho=\pi} \int_{\varphi=0}^{\varphi=\pi/2} \rho^2 \sin\theta \, d\varphi \, d\rho \, d\varphi \\ &= 8 \int_{\varphi=0}^{\varphi=\pi/2} d\varphi \int_{\rho=0}^{\rho=r} \rho^2 d\rho \int_{\theta=0}^{\theta=\pi/2} \sin\theta \, d\theta \\ &= 8 \left(\frac{\pi}{2}\right) \left(\frac{r^3}{3}\right) (1) \\ &= \frac{4}{3}\pi r^3. \end{aligned}$$

This exactly matches with what we knew. So for a sphere with a radius of approximately 50 ft, the volume is $\frac{4}{3}\pi(50)^3 \approx 523,600\ ft^3$.

For the next example we find the volume of an ellipsoid.

Example 15.8.11: Finding the Volume of an Ellipsoid

Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Solution

We again use symmetry and evaluate the volume of the ellipsoid using spherical coordinates. As before, we use the first octant $x \leq 0$, $y \leq 0$, and $z \leq 0$ and then multiply the result by 8.

In this case the ranges of the variables are

$$0 \leq \varphi \leq \frac{\pi}{2}\ 0 \leq \rho \leq 1, \text{ and } 0 \leq \theta \leq \frac{\pi}{2}.$$

Also, we need to change the rectangular to spherical coordinates in this way:

$$x = a\rho\cos\varphi\sin\theta,\ y = b\rho\sin\varphi\sin\theta, \text{ and } z = cp\cos\theta.$$

Then the volume of the ellipsoid becomes

$$
\begin{aligned}
V &= \iiint_D dx\, dy\, dz \\
&= 8 \int_{\theta=0}^{\theta=\pi/2} \int_{\rho=0}^{\rho=1} \int_{\varphi=0}^{\varphi=\pi/2} abc\, \rho^2 \sin\theta\, d\varphi\, d\rho\, d\theta \\
&= 8abc \int_{\varphi=0}^{\varphi=\pi/2} d\varphi \int_{\rho=0}^{\rho=1} \rho^2 d\rho \int_{\theta=0}^{\theta=\pi/2} \sin\theta\, d\theta \\
&= 8abc \left(\frac{\pi}{2}\right)\left(\frac{1}{3}\right)(1) \\
&= \frac{4}{3}\pi abc.
\end{aligned}
$$

Example 15.8.12: Finding the Volume of the Space Inside an Ellipsoid and Outside a Sphere

Find the volume of the space inside the ellipsoid $\frac{x^2}{75^2}+\frac{y^2}{80^2}+\frac{z^2}{90^2}=1$ and outside the sphere $x^2+y^2+z^2=50^2$.

Solution

This problem is directly related to the l'Hemisphèric structure. The volume of space inside the ellipsoid and outside the sphere might be useful to find the expense of heating or cooling that space. We can use the preceding two examples for the volume of the sphere and ellipsoid and then substract.

First we find the volume of the ellipsoid using $a=75$ ft, $b=80$ ft, and $c=90$ ft in the result from Example. Hence the volume of the ellipsoid is

$$V_{ellipsoid} = \frac{4}{3}\pi(75)(80)(90) \approx 2,262,000\ ft^3.$$

From Example, the volume of the sphere is

$$V_{sphere} \approx 523,600\ ft^3.$$

Therefore, the volume of the space inside the ellipsoid $\frac{x^2}{75^2}+\frac{y^2}{80^2}+\frac{z^2}{90^2}=1$ and outside the sphere $x^2+y^2+z^2=50^2$ is approximately

$$V_{Hemispheric} = V_{ellipsoid} - V_{sphere} = 1,738,400\ ft^3.$$

Student Project: Hot air balloons

Hot air ballooning is a relaxing, peaceful pastime that many people enjoy. Many balloonist gatherings take place around the world, such as the Albuquerque International Balloon Fiesta. The Albuquerque event is the largest hot air balloon festival in the world, with over 500 balloons participating each year.



Figure 15.8.12: Balloons lift off at the 2001 Albuquerque International Balloon Fiesta. (credit: David Herrera, Flickr)

As the name implies, hot air balloons use hot air to generate lift. (Hot air is less dense than cooler air, so the balloon floats as long as the hot air stays hot.) The heat is generated by a propane burner suspended below the opening of the basket. Once the balloon takes off, the pilot controls the altitude of the balloon, either by using the burner to heat the air and ascend or by using a vent near the top of the balloon to release heated air and descend. The pilot has very little control over where the balloon

goes, however—balloons are at the mercy of the winds. The uncertainty over where we will end up is one of the reasons balloonists are attracted to the sport.

In this project we use triple integrals to learn more about hot air balloons. We model the balloon in two pieces. The top of the balloon is modeled by a half sphere of radius 28

feet. The bottom of the balloon is modeled by a frustum of a cone (think of an ice cream cone with the pointy end cut off). The radius of the large end of the frustum is 28 feet and the radius of the small end of the frustum is 28 feet. A graph of our balloon model and a cross-sectional diagram showing the dimensions are shown in the following figure.

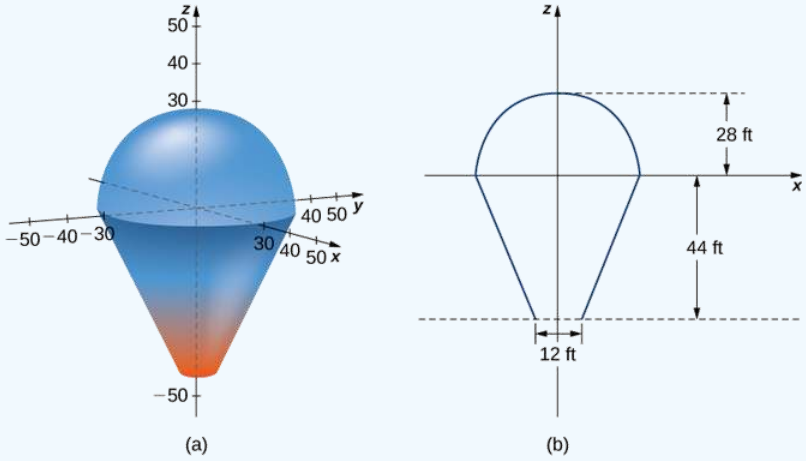


Figure 15.8.13: (a) Use a half sphere to model the top part of the balloon and a frustum of a cone to model the bottom part of the balloon. (b) A cross section of the balloon showing its dimensions.

We first want to find the volume of the balloon. If we look at the top part and the bottom part of the balloon separately, we see that they are geometric solids with known volume formulas. However, it is still worthwhile to set up and evaluate the integrals we would need to find the volume. If we calculate the volume using integration, we can use the known volume formulas to check our answers. This will help ensure that we have the integrals set up correctly for the later, more complicated stages of the project.

1. Find the volume of the balloon in two ways.

a. Use triple integrals to calculate the volume. Consider each part of the balloon separately. (Consider using spherical coordinates for the top part and cylindrical coordinates for the bottom part.)

b. Verify the answer using the formulas for the volume of a sphere, $V = \frac{4}{3}\pi r^3$, and for the volume of a cone, $V = \frac{1}{3}\pi r^2 h$.

In reality, calculating the temperature at a point inside the balloon is a tremendously complicated endeavor. In fact, an entire branch of physics (thermodynamics) is devoted to studying heat and temperature. For the purposes of this project, however, we are going to make some simplifying assumptions about how temperature varies from point to point within the balloon. Assume that just prior to liftoff, the temperature (in degrees Fahrenheit) of the air inside the balloon varies according to the function

$$T_0(r, \theta, z) = \frac{z - r}{10} + 210.$$

2. What is the average temperature of the air in the balloon just prior to liftoff? (Again, look at each part of the balloon separately, and do not forget to convert the function into spherical coordinates when looking at the top part of the balloon.)

Now the pilot activates the burner for 10 seconds. This action affects the temperature in a 12-foot-wide column 20 feet high, directly above the burner. A cross section of the balloon depicting this column in shown in the following figure

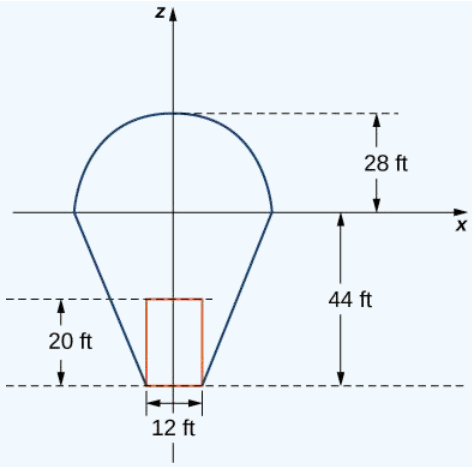


Figure 15.8.14: Activating the burner heats the air in a 20-foot-high, 12-foot-wide column directly above the burner.

Assume that after the pilot activates the burner for 10 seconds, the temperature of the air in the column described above *increases* according to the formula

$$H(r,\theta,z) = -2z - 48.$$

Then the temperature of the air in the column is given by

$$T_1(r,\theta,z) = \frac{z-r}{10} + 210 + (-2z - 48),$$

while the temperature in the remainder of the balloon is still given by

$$T_0(r,\theta,z) = \frac{z-r}{10} + 210.$$

3. Find the average temperature of the air in the balloon after the pilot has activated the burner for 10 seconds.

Key Concepts

- To evaluate a triple integral in cylindrical coordinates, use the iterated integral

$$\int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} \int_{z=u_1(r,\theta)}^{u_2(r,\theta)} f(r,\theta,z) r \, dz \, dr \, d\theta.$$

- To evaluate a triple integral in spherical coordinates, use the iterated integral

$$\int_{\theta=\alpha}^{\theta=\beta} \int_{\rho=g_1(\theta)}^{\rho=g_2(\theta)} \int_{\varphi=u_1(r,\theta)}^{u_2(r,\theta)} f(\rho,\theta,\varphi)\, \rho^2 \sin\varphi \, d\varphi \, d\rho \, d\theta.$$

Key Equations

- **Triple integral in cylindrical coordinates**

$$\iiint_B g(s,y,z)dV = \iiint_B g(r\cos\theta,\ r\sin\theta,\ z) r \, dr \, d\theta \, dz = \iiint_B f(r,\theta,z) r \, dr \, d\theta \, dz$$

- **Triple integral in spherical coordinates**

$$\iiint_B f(\rho,\theta,\varphi)\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \int_{\varphi=\gamma}^{\varphi=\psi} \int_{\theta=\alpha}^{\theta=\beta} \int_{\rho=a}^{\rho=b} f(\rho,\theta,\varphi)\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

Glossary

triple integral in cylindrical coordinates

the limit of a triple Riemann sum, provided the following limit exists:

$$lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}f(r^*_{ijk},\theta^*_{ijk},s^*_{ijk})r^*_{ijk}\Delta r\Delta\theta\Delta z$$

triple integral in spherical coordinates

the limit of a triple Riemann sum, provided the following limit exists:

$$lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}f(\rho^*_{ijk},\theta^*_{ijk},\varphi^*_{ijk})(\rho^*_{ijk})^2\sin\varphi\Delta\rho\Delta\theta\Delta\varphi$$

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15.9: Change of Variables in Multiple Integrals

Given the difficulty of evaluating multiple integrals, the reader may be wondering if it is possible to simplify those integrals using a suitable substitution for the variables. The answer is yes, though it is a bit more complicated than the substitution method which you learned in single-variable calculus.

Recall that if you are given, for example, the definite integral

$$\int_1^2 x^3\sqrt{x^2-1}\,dx \tag{15.9.1}$$

then you would make the substitution

$$\begin{aligned} u &= x^2-1 \Rightarrow x^2 = u+1 \\ du &= 2x\,dx \end{aligned} \tag{15.9.2}$$

which changes the limits of integration

$$\begin{aligned} x &= 1 \Rightarrow u = 0 \\ x &= 2 \Rightarrow u = 3 \end{aligned} \tag{15.9.3}$$

so that we get

$$\begin{aligned} \int_1^2 x^3\sqrt{x^2-1} &= \int_1^2 \frac{1}{2}x^2\cdot 2x\sqrt{x^2-1}\,dx \\ &= \int_0^3 \frac{1}{2}(u+1)\sqrt{u}\,du \\ &= \frac{1}{2}\int_0^3 \left(u^{3/2}+u^{1/2}\right)du, \text{ which can be easily integrated to give} \\ &= \frac{14\sqrt{3}}{5} \end{aligned}$$

Let us take a different look at what happened when we did that substitution, which will give some motivation for how substitution works in multiple integrals. First, we let $u = x^2-1$. On the interval of integration $[1,2]$, the function $x \mapsto x^2-1$ is strictly increasing (and maps $[1,2]$ onto $[0,3]$)and hence has an inverse function (defined on the interval $[0,3]$). That is, on $[0,3]$ we can define x as a function of u, namely

$$x = g(u) = \sqrt{u+1} \tag{15.9.4}$$

Then substituting that expression for x into the function $f(x) = x^3\sqrt{x^2-1}$ gives

$$f(x) = f(g(u)) = (u+1)^{3/2}\sqrt{u} \tag{15.9.5}$$

and we see that

$$\begin{aligned} \frac{dx}{du} = g'(u) \Rightarrow dx &= g'(u)\,du \\ dx &= \frac{1}{2}(u+1)^{-1/2}\,du \end{aligned} \tag{15.9.6}$$

so since

$$\begin{aligned} g(0) = 1 \Rightarrow 0 &= g^{-1}(1) \\ g(3) = 2 \Rightarrow 3 &= g^{-1}(2) \end{aligned} \tag{15.9.7}$$

then performing the substitution as we did earlier gives

$$\int_1^2 f(x)\,dx = \int_1^2 x^3\sqrt{x^2-1}\,dx \tag{15.9.8}$$

$$= \int_0^3 \frac{1}{2}(u+1)\sqrt{u}\,du, \text{ which can be written as}$$

$$= \int_0^3 (u+1)^{3/2}\sqrt{u}\cdot\frac{1}{2}(u+1)^{-1/2}\,du, \text{ which means}$$

$$\int_1^2 f(x)\,dx = \int_{g^{-1}(1)}^{g^{-1}(2)} f(g(u))g'(u)\,du$$

In general, if $x = g(u)$ is a one-to-one, differentiable function from an interval $[c,d]$ (which you can think of as being on the "u-axis") onto an interval $[a,b]$ (on the x-axis), which means that $g'(u) \neq 0$ on the interval (c,d), so that $a = g(c)$ and $b = g(d)$, then $c = g^{-1}(a)$ and $d = g^{-1}(b)$, and

$$\int_a^b f(x)\,dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))g'(u)\,du \tag{15.9.9}$$

This is called the *change of variable* formula for integrals of single-variable functions, and it is what you were implicitly using when doing integration by substitution. This formula turns out to be a special case of a more general formula which can be used to evaluate multiple integrals. We will state the formulas for double and triple integrals involving real-valued functions of two and three variables, respectively. We will assume that all the functions involved are continuously differentiable and that the regions and solids involved all have "reasonable" boundaries. The proof of the following theorem is beyond the scope of the text.

Theorem 15.9.1: Change of Variables Formula for Multiple Integrals

Let $x = x(u,v)$ and $y = y(u,v)$ define a one-to-one mapping of a region R' in the uv-plane onto a region R in the xy-plane such that the determinant

$$J(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \tag{15.9.10}$$

is never in R'. Then

$$\iint_R f(x,y)\,dA(x,y) = \iint_{R'} f(x(u,v),y(u,v))\,|J(u,v)|\,dA(u,v) \tag{15.9.11}$$

We use the notation $dA(x,y)$ and $dA(u,v)$ to denote the area element in the (x,y) and (u,v) coordinates, respectively.

Similarly, if $x = x(u,v,w)$, $y = y(u,v,w)$ and $z = z(u,v,w)$ define a one-to-one mapping of a solid S' in uvw-space onto a solid S in xyz-space such that the determinant

$$J(u,v,w) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} \tag{15.9.12}$$

is never 0 in S', then

$$\iiint_S f(x,y,z)dV(x,y,z) = \iiint_{S'} f(x(u,v,w),y(u,v,w),z(u,v,w))|J(u,v,w)|dV(u,v,w) \tag{15.9.13}$$

The determinant $J(u,v)$ in Equation 15.9.10is called the Jacobian of x and y with respect to u and v, and is sometimes written as

$$J(u,v)=\frac{\partial(x,y)}{\partial(u,v)} \tag{15.9.14}$$

Similarly, the Jacobian $J(u,v,w)$ of three variables is sometimes written as

$$J(u,v,w)=\frac{\partial(x,y,z)}{\partial(u,v,w)} \tag{15.9.15}$$

Notice that Equation 15.9.11is saying that $dA(x,y)=|J(u,v)|dA(u,v)$, which you can think of as a two-variable version of the relation $dx=g'(u)\,du$ in the single-variable case.

The following example shows how the change of variables formula is used.

Example 15.9.1

Evaluate

$$\iint_R e^{\frac{x-y}{x+y}}\,dA$$

where $R=(x,y):x\geq 0,y\geq 0,x+y\leq 1$.

Solution

First, note that evaluating this double integral *without* using substitution is probably impossible, at least in a closed form. By looking at the numerator and denominator of the exponent of e, we will try the substitution $u=x-y$ and $v=x+y$. To use the change of variables Formula 15.9.11, we need to write both x and y in terms of u and v. So solving for x and y gives $x=\frac{1}{2}(u+v)$ and $y=\frac{1}{2}(v-u)$. In Figure 15.9.1 below, we see how the mapping $x=x(u,v)=\frac{1}{2}(u+v),y=y(u,v)=\frac{1}{2}(v-u)$ maps the region R' onto R in a one-to-one manner.

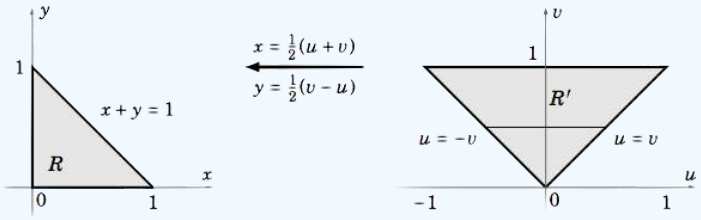


Figure 15.9.1: The regions R and R

Now we see that

$$J(u,v)=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\frac{1}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{vmatrix}=\frac{1}{2}\Rightarrow|J(u,v)|=\left|\frac{1}{2}\right|=\frac{1}{2}$$

so using horizontal slices in R', we have

$$\iint_R e^{\frac{x-y}{x+y}}\,dA = \iint_{R'} f(x(u,v),y(u,v))|J(u,v)|dA$$
$$= \int_0^1 \int_{-v}^{v} e^{u/v}\frac{1}{2}\,du\,dv$$
$$= \int_0^1 \left(\frac{v}{2}e^{u/v}\Big|_{u=-v}^{u=v}\right)dv$$
$$= \int_0^1 \frac{v}{2}(e-e^1)dv$$
$$= \frac{v^2}{4}(e-e^1)\Big|_0^1 = \frac{1}{4}\left(e-\frac{1}{e}\right) = \frac{e^2-1}{4e}$$

The change of variables formula can be used to evaluate double integrals in polar coordinates. Letting

$$x = x(r,\theta) = r\cos\theta \text{ and } y = y(r,\theta) = r\sin\theta, \tag{15.9.16}$$

we have

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r \Rightarrow |J(u,v)| = |r| = r \tag{15.9.17}$$

so we have the following formula:

Double Integral in Polar Coordinates

$$\iint_R f(x,y)\,dx\,dy = \iint_{R'} f(r\cos\theta, r\sin\theta)r\,dr\,d\theta \tag{15.9.18}$$

where the mapping $x = r\cos\theta, y = r\sin\theta$ maps the region R' in the $r\theta$-plane onto the region R in the xy-plane in a one-to-one manner.

Example 15.9.2: Volume of Paraboloid

Find the volume V inside the paraboloid $z = x^2 + y^2$ for $0 \le z \le 1$

Solution

Using vertical slices, we see that

$$V = \iint_R (1-z)dA = \iint_R (1-(x^2+y^2))dA$$

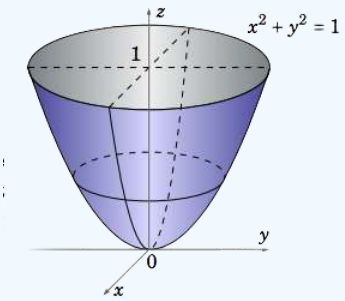


Figure 15.9.2: $z = x^2 + y^2$

where $R = (x, y) : x^2 + y^2 \leq 1$ is the unit disk in $\mathbb{R}^2$ (see Figure 15.9.2). In polar coordinates (r, θ) we know that $x^2 + y^2 = r^2$ and that the unit disk R is the set $R' = (r, \theta) : 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi$. Thus,

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^1 (1 - r^2) r \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^1 (r - r^3) dr \, d\theta \\ &= \int_o^{2\pi} \left(\frac{r^2}{2} - \frac{r^4}{4} \Big|_{r=0}^{r=1} \right) d\theta \\ &= \int_0^{2\pi} \frac{1}{4} \, d\theta \\ &= \frac{\pi}{2} \end{aligned}$$

Example 15.9.3: Volume of Cone

Find the volume V inside the cone $z = \sqrt{x^2 + y^2}$ for $0 \leq z \leq 1$.

Solution

Using vertical slices, we see that

$$V = \iint_R (1 - z) dA = \iint_R \left(1 - \sqrt{x^2 + y^2} \right) dA$$

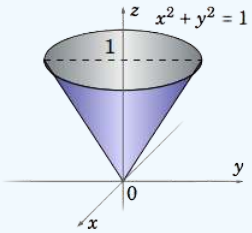


Figure 15.9.3: $z = \sqrt{x^2 + y^2}$

where $R = (x, y) : x^2 + y^2 \leq 1$ is the unit disk in $\mathbb{R}^2$ (see Figure 15.9.3). In polar coordinates (r, θ) we know that $\sqrt{x^2 + y^2} = r$ and that the unit disk R is the set $R' = (r, \theta) : 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi$. Thus,

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^1 (1 - r) r \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^1 (r - r^2) dr \, d\theta \\ &= \int_0^{2\pi} \left(\frac{r^2}{2} - \frac{r^3}{3} \Big|_{r=0}^{r=1} \right) d\theta \\ &= \int_0^{2\pi} \frac{1}{6} d\theta \\ &= \frac{\pi}{3} \end{aligned}$$

In a similar fashion, it can be shown (see Exercises 5-6) that triple integrals in cylindrical and spherical coordinates take the following forms:

Triple Integral in Cylindrical Coordinates

$$\iiint_S f(x,y,z)dx\,dy\,dz = \iiint_{S'} (r\cos\theta, r\sin\theta, z) r\,dr\,d\theta\,dz \tag{15.9.19}$$

where the mapping $x = r\cos\theta, y = r\sin\theta, z = z$ maps the solid S' in $r\theta z$-space onto the solid S in xyz-space in a one-to-one manner.

Triple Integral in Spherical Coordinates

$$\iiint_S f(x,y,z)dx\,dy\,dz = \iiint_{S'} f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\rho^2\sin\varphi d\rho\,d\varphi\,d\theta \tag{15.9.20}$$

where the mapping $x = \rho\sin\varphi\cos\theta, y = \rho\sin\varphi\sin\theta, z = \rho\cos\varphi$ maps the solid S' in $\rho\varphi\theta$- space onto the solid S in xyz-space in a one-to-one manner.

Example 15.9.4

For $a > 0$, find the volume V inside the sphere $S = x^2 + y^2 + z^2 = a^2$.

Solution

We see that S is the set $\rho = a$ in spherical coordinates, so

$$\begin{aligned} V &= \iiint_S 1dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} 1\rho^2\sin\varphi d\rho\,d\varphi\,d\theta \\ &= \int_0^{2\pi}\int_0^{\pi}\left(\frac{\rho^3}{3}\Big|_{\rho=0}^{\rho=a}\right)\sin\varphi d\varphi\,d\theta = \int_0^{2\pi}\int_0^{\pi}\frac{a^3}{3}\sin\varphi d\varphi\,d\theta \\ &= \int_0^{2\pi}\left(-\frac{a^3}{3}\cos\varphi\Big|_{\varphi=0}^{\varphi=\pi}\right)d\theta = \int_0^{2\pi}\frac{2a^3}{3}d\theta = \frac{4\pi a^3}{3} \end{aligned}$$

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CHAPTER OVERVIEW

16: Vector Calculus

A general Calculus Textmap organized around the textbook

Calculus: Early Transcendentals

by James Stewart

I II III IV V VI VII VIII IX X XI XII XIII XIV XV XVI XVII

This Textmap is currently under construction... please be patient with us.

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16.3: The Fundamental Theorem for Line Integrals
16.4: Green's Theorem
16.5: Curl and Divergence
16.6: Parametric Surfaces and Their Areas
16.7: Surface Integrals
16.8: Stokes' Theorem
16.9: The Divergence Theorem

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16.1: Vector Fields

We have already seen that a convenient way to describe a line in three dimensions is to provide a vector that "points to" every point on the line as a parameter t varies, like

$$\langle 1,2,3\rangle + t\langle 1,-2,2\rangle = \langle 1+t, 2-2t, 3+2t\rangle.$$

Except that this gives a particularly simple geometric object, there is nothing special about the individual functions of t that make up the coordinates of this vector---any vector with a parameter, like $\langle f(t), g(t), h(t)\rangle$, will describe some curve in three dimensions as t varies through all possible values.

Example 13.1.1

Describe the curves $\langle \cos t, \sin t, 0\rangle$, $\langle \cos t, \sin t, t\rangle$, and $\langle \cos t, \sin t, 2t\rangle$.

Solution

As t varies, the first two coordinates in all three functions trace out the points on the unit circle, starting with $(1,0)$ when $t=0$ and proceeding counter-clockwise around the circle as t increases. In the first case, the z coordinate is always 0, so this describes precisely the unit circle in the x-y plane. In the second case, the x and y coordinates still describe a circle, but now the z coordinate varies, so that the height of the curve matches the value of t. When $t=\pi$, for example, the resulting vector is $\langle -1, 0, \pi\rangle$. A bit of thought should convince you that the result is a helix. In the third vector, the z coordinate varies twice as fast as the parameter t, so we get a stretched out helix. Both are shown in figure 13.1.1. On the left is the first helix, shown for t between 0 and 4π; on the right is the second helix, shown for t between 0 and 2π. Both start and end at the same point, but the first helix takes two full "turns" to get there, because its z coordinate grows more slowly.

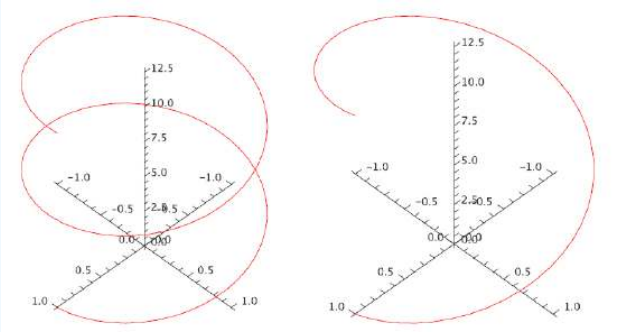


Figure 13.1.1. Two helixes.

A vector expression of the form $\langle f(t), g(t), h(t)\rangle$ is called a **vector function**; it is a function from the real numbers $\mathbb{R}$ to the set of all three-dimensional vectors.

We can alternately think of it as three separate functions, $x=f(t)$, $y=g(t)$, and $z=h(t)$, that describe points in space. In this case we usually refer to the set of equations as **parametric equations** for the curve, just as for a line. While the parameter t in a vector function might represent any one of a number of physical quantities, or be simply a "pure number", it is often convenient and useful to think of t as representing time. The vector function then tells you where in space
a particular object is at any time.

Vector functions can be difficult to understand, that is, difficult to picture. When available, computer software can be very helpful. When working by hand, one useful approach is to consider the "projections" of the curve onto the three standard coordinate planes. We have already done this in part: in example 13.1.1 we noted that all three curves project to a circle in the x-y plane, since $\langle \cos t, \sin t\rangle$ is a two dimensional vector function for the unit circle.

Example 13.1.2

Graph the projections of $\langle \cos t, \sin t, 2t\rangle$ onto the x-z plane and the y-z plane.

Solution

The two dimensional vector function for the projection onto the x-z plane is $\langle \cos t, 2t \rangle$, or in parametric form, $x = \cos t$, $z = 2t$. By eliminating t we get the equation $x = \cos(z/2)$, the familiar curve shown on the left in Figure ***13.1.2*** . For the projection onto the y-z plane, we start with the vector function $\langle \sin t, 2t \rangle$, which is the same as $y = \sin t$, $z = 2t$. Eliminating t gives $y = \sin(z/2)$, as shown on the right in figure 13.1.2.

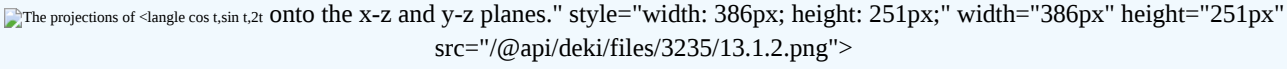


Figure 13.1.2. *The projections of $\langle \cos t, \sin t, 2t \rangle$ onto the x-z and y-z planes.*

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- David Guichard (Whitman College)
- Integrated by Justin Marshall.

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16.2: Line Integrals

In single-variable calculus you learned how to integrate a real-valued function $f(x)$ over an interval $[a,b]$ in $\mathbb{R}^1$. This integral (usually called a *Riemann integral*) can be thought of as an integral over a path in $\mathbb{R}^1$, since an interval (or collection of intervals) is really the only kind of "path" in $\mathbb{R}^1$. You may also recall that if $f(x)$ represented the force applied along the x-axis to an object at position x in $[a,b]$, then the *work* W done in moving that object from position $x=a$ to $x=b$ was defined as the integral:

$$W=\int_a^b f(x)dx \tag{16.2.1}$$

In this section, we will see how to define the integral of a function (either real-valued or vector-valued) of two variables over a general path (i.e. a curve) in $\mathbb{R}^2$. This definition will be motivated by the physical notion of work. We will begin with real-valued functions of two variables.

In physics, the intuitive idea of work is that

$$\text{Work}=\text{Force}\times\text{Distance} \tag{16.2.2}$$

Suppose that we want to find the total amount W of work done in moving an object along a curve C in $\mathbb{R}^2$ with a smooth parametrization $x=x(t), y=y(t), a\le t\le b$, with a force $f(x,y)$ which varies with the position (x,y) of the object and is applied in the direction of motion along C (see Figure 16.2.1below).

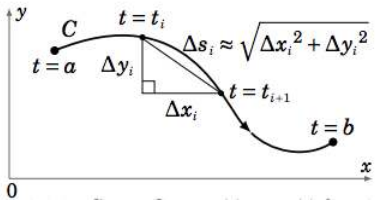


Figure 16.2.1 Curve $C: x=x(t)$, $y=y(t)$ for t in $[a,b]$

We will assume for now that the function $f(x,y)$ is continuous and real-valued, so we only consider the magnitude of the force. Partition the interval $[a,b]$ as follows:

$$a=t_0<t_1<t_2<\cdots<t_{n-1}<t_n=b, \text{ for some integer } n\ge 2 \tag{16.2.3}$$

As we can see from Figure 16.2.1, over a typical subinterval $[t_i,t_{i+1}]$ the distance Δs_i traveled along the curve is approximately $\sqrt{\Delta x_i^2+\Delta y_i^2}$, by the Pythagorean Theorem. Thus, if the subinterval is small enough then the work done in moving the object along that piece of the curve is approximately

$$\text{Force}\times\text{Distance}\approx f(x_{i*},y_{i*})\sqrt{\Delta x_i^2+\Delta y_i^2} \tag{16.2.4}$$

where $(x_{i*},y_{i*})=(x(t_{i*}),y(t_{i*}))$ for some t_{i*} in $[t_i,t_{i+1}]$, and so

$$W\approx\sum_{i=0}^{n-1} f(x_{i*},y_{i*})\sqrt{\Delta x_i^2+\Delta y_i^2} \tag{16.2.5}$$

is approximately the total amount of work done over the entire curve. But since

$$\sqrt{\Delta x_i^2+\Delta y_i^2}=\sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2+\left(\frac{\Delta y_i}{\Delta t_i}\right)^2}\,\Delta t_i \tag{16.2.6}$$

where $\Delta t_i=t_{i+1}-t_i$, then

$$W\approx\sum_{i=0}^{n-1} f(x_{i*},y_{i*})\sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2+\left(\frac{\Delta y_i}{\Delta t_i}\right)^2}\,\Delta t_i \tag{16.2.7}$$

Taking the limit of that sum as the length of the largest subinterval goes to 0, the sum over all subintervals becomes the integral from $t=a$ to $t=b$, $\Delta x_i\,\Delta t_i$ and $\Delta y_i\,\Delta t_i$ become $x'(t)$ and $y'(t)$, respectively, and $f(x_{i*},y_{i*})$ becomes $f(x(t),y(t))$, so

that

$$W = \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2}\, dt \tag{16.2.8}$$

The integral on the right side of the above equation gives us our idea of how to define, for any real-valued function $f(x, y)$, the integral of $f(x, y)$ along the curve C, called a *line integral*:

Definition 16.2.1: Line Integral of a scalar Field

For a real-valued function $f(x, y)$ and a curve C in $\mathbb{R}^2$, parametrized by $x = x(t), y = y(t), a \le t \le b$, the **line integral** of $f(x, y)$ **along** C **with respect to arc length** s is

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2}\, dt \tag{16.2.9}$$

The symbol ds is the differential of the arc length function

$$s = s(t) = \int_a^t \sqrt{x'(u)^2 + y'(u)^2}\, du \tag{16.2.10}$$

which you may recognize from Section 1.9 as the length of the curve C over the interval $[a, t]$, for all t in $[a, b]$. That is,

$$ds = s'(t)\, dt = \sqrt{x'(t)^2 + y'(t)^2}\, dt, \tag{16.2.11}$$

by the Fundamental Theorem of Calculus.

For a general real-valued function $f(x, y)$, what does the line integral $\int_C f(x, y)\, ds$ represent? The preceding discussion of ds gives us a clue. You can think of differentials as infinitesimal lengths. So if you think of $f(x, y)$ as the height of a picket fence along C, then $f(x, y)\, ds$ can be thought of as approximately the area of a section of that fence over some infinitesimally small section of the curve, and thus the line integral $\int_C f(x, y)\, ds$ is the total area of that picket fence (see Figure 16.2.2).

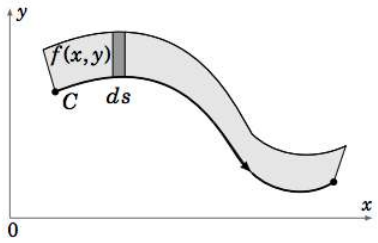


Figure 16.2.2: Area of shaded rectangle = height × width ≈ $f(x, y)\, ds$

Example 16.2.1

Use a line integral to show that the lateral surface area A of a right circular cylinder of radius r and height h is $2\pi rh$.

Solution

We will use the right circular cylinder with base circle C given by $x^2 + y^2 = r^2$ and with height h in the positive z direction (see Figure 16.2.3). Parametrize C as follows:

$$x = x(t) = r\cos t, y = y(t) = r\sin t, 0 \le t \le 2\pi \tag{16.2.12}$$

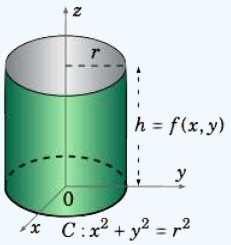


Figure 16.2.3

$$\begin{aligned} A &= \int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{x'(t)^2+y'(t)^2}\,dt \\ &= \int_0^{2\pi} h\sqrt{(-r\sin t)^2+(r\cos t)^2}\;dt \\ &= h\int_0^{2\pi} r\sqrt{\sin^2 t+\cos^2 t}\,dt \\ &= rh\int_0^{2\pi} 1\,dt = 2\pi rh \end{aligned}$$

Note in Example 16.2.1 that if we had traversed the circle C twice, i.e. let t vary from 0 to 4π then we would have gotten an area of $4\pi rh$, i.e. twice the desired area, even though the *curve* itself is still the same (namely, a circle of radius r). Also, notice that we traversed the circle in the counter-clockwise direction. If we had gone in the clockwise direction, using the parametrization

$$x = x(t) = r\cos(2\pi - t), y = y(t) = r\sin(2\pi - t), 0 \le t \le 2\pi, \tag{16.2.13}$$

then it is easy to verify (Exercise 12) that the value of the line integral is unchanged.

In general, it can be shown (Exercise 15) that reversing the direction in which a curve C is traversed leaves $\int_C f(x,y)\,ds$ unchanged, for any $f(x,y)$. If a curve C has a parametrization $x = x(t), y = y(t), a \le t \le b$, then denote by $-C$ the same curve as C but traversed in the opposite direction. Then $-C$ is parametrized by

$$x = x(a+b-t), y = y(a+b-t), a \le t \le b, \tag{16.2.14}$$

and we have

$$\int_C f(x,y)\,ds = \int_{-C} f(x,y)\,ds. \tag{16.2.15}$$

Notice that our definition of the line integral was with respect to the arc length parameter s. We can also define

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))x'(t)\,dt \tag{16.2.16}$$

as the *line integral* of $f(x,y)$ along C with respect to x, and

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))y'(t)\,dt \tag{16.2.17}$$

as the *line integral* of $f(x,y)$ *along* C *with respect to* y.

In the derivation of the formula for a line integral, we used the idea of work as force multiplied by distance. However, we know that force is actually a *vector*. So it would be helpful to develop a vector form for a line integral. For this, suppose that we have a function $f(x,y)$ defined on $\mathbb{R}^2$ by

$$\mathbf{f}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$$

for some continuous real-valued functions $P(x,y)$ and $Q(x,y)$ on $\mathbb{R}^2$. Such a function f is called a vector field on $\mathbb{R}^2$. It is defined at *points* in $\mathbb{R}^2$, and its values are *vectors* in $\mathbb{R}^2$. For a curve C with a smooth parametrization $x = x(t), y = y(t), a \le t \le b$, let

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$$

be the position vector for a point $(x(t), y(t))$ on C. Then $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j}$ and so

$$\begin{aligned}\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy &= \int_a^b P(x(t),y(t))x'(t)\,dt + \int_a^b Q(x(t),y(t))y'(t)\,dt \\ &= \int_a^b (P(x(t),y(t))x'(t) + Q(x(t),y(t))y'(t))\,dt \\ &= \int_a^b \mathbf{f}(x(t),y(t))\cdot\mathbf{r}'(t)dt\end{aligned}$$

by definition of $f(x,y)$. Notice that the function $f(x(t),y(t))\cdot r'(t)$ is a *real-valued* function on $[a,b]$, so the last integral on the right looks somewhat similar to our earlier definition of a line integral. This leads us to the following definition:

Definition 16.2.2: Line Integral of a vector Field

For a vector field $\mathbf{f}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ and a curve C with a smooth parametrization $x = x(t), y = y(t), a \le t \le b$, the **line integral of f along** C is

$$\int_C \mathbf{f}\cdot d\mathbf{r} = \int_C P(x,y)\,dx + \int_C Q(x,y)\,dy \tag{16.2.18}$$

$$= \int_a^b \mathbf{f}(x(t),y(t))\cdot\mathbf{r}'(t)\,dt \tag{16.2.19}$$

where $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ is the position vector for points on C.

We use the notation $d\mathbf{r} = \mathbf{r}'(t)\,dt = dx\mathbf{i} + dy\mathbf{j}$ to denote the differential of the vector-valued function $\mathbf{r}$. The line integral in Definition 16.2.2 is often called a *line integral of a vector field* to distinguish it from the line integral in Definition 16.2.1 which is called a *line integral* of a *scalar field*. For convenience we will often write

$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy = \int_C P(x,y)\,dx + Q(x,y)\,dy,$$

where it is understood that the line integral along C is being applied to both P and Q. The quantity $P(x,y)\,dx + Q(x,y)\,dy$ is known as a **differential form**. For a real-valued function $F(x,y)$, the differential of F is

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy. \tag{16.2.20}$$

A differential form $P(x,y)\,dx + Q(x,y)\,dy$ is called **exact** if it equals dF for some function $F(x,y)$.

Recall that if the points on a curve C have position vector $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, then $\mathbf{r}'(t)$ is a tangent vector to C at the point $(x(t), y(t))$ in the direction of increasing t (which we call the *direction* of C). Since C is a smooth curve, then $\mathbf{r}'(t) \neq \mathbf{0}$ on $[a,b]$ and hence

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

is the unit tangent vector to C at $(x(t), y(t))$. Putting Definitions 16.2.1 and 16.2.2 together we get the following theorem:

Theorem 16.2.1

For a vector field $\mathbf{f}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ and a curve C with a smooth parametrization $x = x(t), y = y(t), a \le t \le b$ and position vector $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$,

$$\int_C \mathbf{f}\cdot d\mathbf{r} = \int_C \mathbf{f}\cdot\mathbf{T}\,ds, \tag{16.2.21}$$

where $\mathbf{T}(t) = \dfrac{\mathbf{r}\,'(t)}{\|\mathbf{r}\,'(t)\|}$ is the unit tangent vector to C at $(x(t), y(t))$.

If the vector field $\mathbf{f}(x, y)$ represents the force moving an object along a curve C, then the work W done by this force is

$$W = \int_C \mathbf{f} \cdot \mathbf{T}\, ds = \int_C \mathbf{f} \cdot d\mathbf{r} \tag{16.2.22}$$

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16.3: The Fundamental Theorem for Line Integrals

One way to write the Fundamental Theorem of Calculus is:

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

That is, to compute the integral of a derivative f' we need only compute the values of f at the endpoints. Something similar is true for line integrals of a certain form.

Theorem: Fundamental Theorem of Line Integrals

Suppose a curve C is given by the vector function $\mathbf{r}(t)$, with $\mathbf{a} = \mathbf{r}(a)$ and $\mathbf{b} = \mathbf{r}(b)$. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{b}) - f(\mathbf{a}),$$

provided that $\mathbf{r}$ is sufficiently nice.

Proof

We write $\mathbf{r} = \langle x(t), y(t), z(t)\rangle$, so that $\mathbf{r}' = \langle x'(t), y'(t), z'(t)\rangle$. Also, we know that $\nabla f = \langle f_x, f_y, f_z\rangle$. Then

$$\int_C \nabla f \cdot d\mathbf{r} = \int_a^b \langle f_x, f_y, f_z\rangle \cdot \langle x'(t), y'(t), z'(t)\rangle\,dt = \int_a^b f_x x' + f_y y' + f_z z'\,dt.$$

By the chain rule (see section 14.4) $f_x x' + f_y y' + f_z z' = df/dt$, where f in this context means $f(x(t), y(t), z(t))$, a function of t. In other words, all we have is

$$\int_a^b f'(t)\,dt = f(b) - f(a).$$

In this context, $f(a) = f(x(a), y(a), z(a))$. Since $\mathbf{a} = \mathbf{r}(a) = \langle x(a), y(a), z(a)\rangle$, we can write $f(a) = f(\mathbf{a})$---this is a bit of a cheat, since we are simultaneously using f to mean $f(t)$ and $f(x, y, z)$, and since $f(x(a), y(a), z(a))$ is not technically the same as $f(\langle x(a), y(a), z(a)\rangle)$, but the concepts are clear and the different uses are compatible. Doing the same for b, we get

$$\int_C \nabla f \cdot d\mathbf{r} = \int_a^b f'(t)\,dt = f(b) - f(a) = f(\mathbf{b}) - f(\mathbf{a}).$$

□

This theorem, like the Fundamental Theorem of Calculus, says roughly that if we integrate a "derivative-like function" (f' or ∇f) the result depends only on the values of the original function (f) at the endpoints.

If a vector field $\mathbf{F}$ is the gradient of a function,

$$\mathbf{F} = \nabla f$$

then we say that $\mathbf{F}$ is a conservative vector field. If $\mathbf{F}$ is a conservative force field, then the integral for work, $\int_C \mathbf{F} \cdot d\mathbf{r}$, is in the form required by the Fundamental Theorem of Line Integrals. This means that in a conservative force field, the amount of work required to move an object from point $\mathbf{a}$ to point $\mathbf{b}$ depends only on those points, not on the path taken between them. In physics, forces that can ascribed to a conservative vector field are called conservative forces and are important for many applications.

Example 16.3.2:

An object moves in the force field

$$\mathbf{F} = \left\langle \frac{-x}{(x^2+y^2+z^2)^{3/2}}, \frac{-y}{(x^2+y^2+z^2)^{3/2}}, \frac{-z}{(x^2+y^2+z^2)^{3/2}} \right\rangle,$$

along the curve $\mathbf{r} = \langle 1+t, t^3, t\cos(\pi t)\rangle$ as t ranges from 0 to 1. Find the work done by the force on the object.

Solution

The straightforward way to do this involves substituting the components of $\mathbf{r}$ into $\mathbf{F}$, forming the dot product $\mathbf{F}\cdot\mathbf{r}'$, and then trying to compute the integral, but this integral is extraordinarily messy, perhaps impossible to compute. But since $\mathbf{F} = \nabla(1/\sqrt{x^2+y^2+z^2})$ we need only substitute:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \frac{1}{\sqrt{x^2+y^2+z^2}}\Bigg|_{(1,0,0)}^{(2,1,-1)} = \frac{1}{\sqrt{6}} - 1.$$

Another immediate consequence of the Fundamental Theorem involves **closed paths**. A path C is closed if it forms a loop, so that traveling over the C curve brings you back to the starting point. If C is a closed path, we can integrate around it starting at any point $\mathbf{a}$; since the starting and ending points are the same,

$$\int_C \nabla f\cdot d\mathbf{r} = f(\mathbf{a}) - f(\mathbf{a}) = 0.$$

For example, in a gravitational field (an inverse square law field) the amount of work required to move an object around a closed path is zero. Of course, it's only the *net* amount of work that is zero. It may well take a great deal of work to get from point $\mathbf{a}$ to point $\mathbf{b}$, but then the return trip will "produce" work. For example, it takes work to pump water from a lower to a higher elevation, but if you then let gravity pull the water back down, you can recover work by running a water wheel or generator. (In the real world you won't recover all the work because of various losses along the way.)

To make use of the Fundamental Theorem of Line Integrals, we need to be able to spot conservative vector fields $\mathbf{F}$ and to compute f so that $\mathbf{F} = \nabla f$. Suppose that $\mathbf{F} = \langle P, Q\rangle = \nabla f$. Then $P = f_x$ and $Q = f_y$, and provided that f is sufficiently nice, we know from Clairaut's Theorem that $P_y = f_{xy} = f_{yx} = Q_x$. If we compute P_y and Q_x and find that they are not equal, then $\mathbf{F}$ is not conservative. If $P_y = Q_x$, then, again provided that $\mathbf{F}$ is sufficiently nice, we can be assured that $\mathbf{F}$ is conservative. Ultimately, what's important is that we be able to find f; as this amounts to finding anti-derivatives, we may not always succeed.

Example 16.3.3

Find an f so that $\langle 3+2xy, x^2-3y^2\rangle = \nabla f$.

Solution

First, note that

$$\frac{\partial}{\partial y}(3+2xy) = 2x \qquad \text{and} \qquad \frac{\partial}{\partial x}(x^2-3y^2) = 2x,$$

so the desired f does exist. This means that $f_x = 3+2xy$, so that $f = 3x + x^2y + g(y)$; the first two terms are needed to get $3+2xy$, and the $g(y)$ could be any function of y, as it would disappear upon taking a derivative with respect to x. Likewise, since $f_y = x^2 - 3y^2$, $f = x^2y - y^3 + h(x)$. The question now becomes, is it possible to find $g(y)$ and $h(x)$ so that

$$3x + x^2y + g(y) = x^2y - y^3 + h(x),$$

and of course the answer is yes: $g(y) = -y^3$, $h(x) = 3x$. Thus, $f = 3x + x^2y - y^3$.

We can test a vector field $\mathbf{F} = \langle P, Q, R\rangle$ in a similar way. Suppose that $\langle P, Q, R\rangle = \langle f_x, f_y, f_z\rangle$. If we temporarily hold z constant, then $f(x,y,z)$ is a function of x and y, and by Clairaut's Theorem $P_y = f_{xy} = f_{yx} = Q_x$. Likewise, holding y constant implies $P_z = f_{xz} = f_{zx} = R_x$, and with x constant we get $Q_z = f_{yz} = f_{zy} = R_y$. Conversely, if we find that $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$ then $\mathbf{F}$ is conservative.

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16.4: Green's Theorem

We will now see a way of evaluating the line integral of a *smooth* vector field around a simple closed curve. A vector field $\mathbf{f}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is **smooth** if its component functions $P(x,y)$ and $Q(x,y)$ are smooth. We will use *Green's Theorem* (sometimes called *Green's Theorem in the plane*) to relate the *line* integral around a closed curve with a *double* integral over the region inside the curve:

Theorem 4.7: Green's Theorem

Let R be a region in $\mathbb{R}^2$ whose boundary is a simple closed curve C which is piecewise smooth. Let $\mathbf{f}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ be a smooth vector field defined on both R and C. Then

$$\oint_C \mathbf{f} \cdot d\mathbf{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA, \tag{16.4.1}$$

where C is traversed so that R is always on the left side of C.

Proof: We will prove the theorem in the case for a *simple* region R, that is, where the boundary curve C can be written as $C = C_1 \cup C_2$ in two distinct ways:

$$C_1 = \text{ the curve } y = y_1(x) \text{ from the point } X_1 \text{ to the point } X_2 \tag{16.4.2}$$

$$C_2 = \text{ the curve } y = y_2(x) \text{ from the point } X_2 \text{ to the point } X_1, \tag{16.4.3}$$

where X_1 and X_2 are the points on C farthest to the left and right, respectively; and

$$C_1 = \text{ the curve } x = x_1(y) \text{ from the point } Y_2 \text{ to the point } Y_1 \tag{16.4.4}$$

$$C_2 = \text{ the curve } x = x_2(y) \text{ from the point } Y_1 \text{ to the point } Y_2, \tag{16.4.5}$$

where Y_1 and Y_2 are the lowest and highest points, respectively, on C. See Figure 4.3.1.

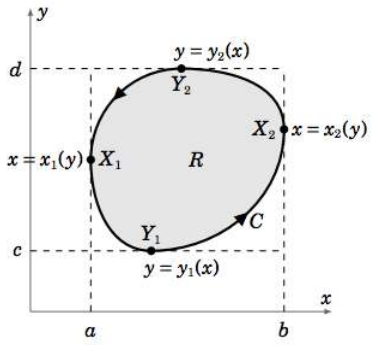


Figure 4.3.1

Integrate $P(x,y)$ around C using the representation $C = C_1 \cup C_2$ given by Equation 16.4.3 and Equation 16.4.4.

Since $y = y_1(x)$ along C_1 (as x goes from a to b) and $y = y_2(x)$ along C_2 (as x goes from b to a), as we see from Figure 4.3.1, then we have

$$\begin{aligned}
\oint_C P(x,y)\,dx &= \int_{C_1} P(x,y)\,dx + \int_{C_2} P(x,y)\,dx \\
&= \int_a^b P(x,y_1(x))\,dx + \int_b^a P(x,y_2(x))\,dx \\
&= \int_a^b P(x,y_1(x))\,dx - \int_a^b P(x,y_2(x))\,dx \\
&= -\int_a^b \left(P(x,y_2(x)) - P(x,y_1(x))\right)dx \\
&= -\int_a^b \left(P(x,y)\Big|_{y=y_1(x)}^{y=y_2(x)}\right)dx \\
&= -\int_a^b \int_{y_1(x)}^{y_2(x)} \frac{\partial P(x,y)}{\partial y}\,dy\,dx \text{ (by the Fundamental Theorem of Calculus)} \\
&= -\iint_R \frac{\partial P}{\partial y}\,dA. \qquad (16.4.6)
\end{aligned}$$

Likewise, integrate $Q(x,y)$ around C using the representation $C = C_1 \cup C_2$ given by Equation 16.4.5 and Equation ???. Since $x = x_1(y)$ along C_1 (as y goes from d to c) and $x = x_2(y)$ along C_2 (as y goes from c to d), as we see from Figure 4.3.1, then we have

$$\begin{aligned}
\oint_C Q(x,y)\,dy &= \int_{C_1} Q(x,y)\,dy + \int_{C_2} Q(x,y)\,dy \\
&= \int_d^c Q(x_1(y),y)\,dy + \int_c^d Q(x_2(y),y)\,dy \\
&= -\int_c^d Q(x_1(y),y)\,dy + \int_c^d Q(x_2(y),y)\,dy \\
&= \int_c^d \left(Q(x_2(y),y) - Q(x_1(y),y)\right)dy \\
&= \int_c^d \left(Q(x,y)\Big|_{x=x_1(y)}^{x=x_2(y)}\right)dy \\
&= \int_c^d \int_{x_1(y)}^{x_2(y)} \frac{\partial Q(x,y)}{\partial x}\,dx\,dy \text{ (by the Fundamental Theorem of Calculus)} \\
&= \iint_R \frac{\partial Q}{\partial x}\,dA, \text{ and so}
\end{aligned}$$

$$\begin{aligned}
\oint_C \mathbf{f}\cdot d\mathbf{r} &= \oint_C P(x,y)\,dx + \oint_C Q(x,y)\,dy \\
&= -\iint_R \frac{\partial P}{\partial y}\,dA + \iint_R \frac{\partial Q}{\partial x}\,dA \\
&= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA.
\end{aligned}$$

(QED)

Though we proved Green's Theorem only for a simple region R, the theorem can also be proved for more general regions (say, a union of simple regions).

Example 4.7

Evaluate $\oint_C (x^2+y^2)\,dx + 2xy\,dy$, where C is the boundary (traversed counterclockwise) of the region $R = (x,y) : 0 \le x \le 1, 2x^2 \le y \le 2x$.

R is the shaded region in Figure 4.3.2. By Green's Theorem, for $P(x,y) = x^2+y^2$ and $Q(x,y) = 2xy$, we have

$$\begin{aligned}\oint_C (x^2+y^2)\,dx + 2xy\,dy &= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \\ &= \iint_R (2y-2y)\,dA = \iint_R 0\,dA = 0.\end{aligned}$$

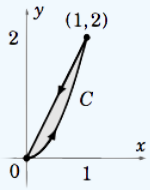


Figure 4.3.2

We actually already knew that the answer was zero. Recall from Example 4.5 in Section 4.2 that the vector field $\mathbf{f}(x,y) = (x^2+y^2)\mathbf{i} + 2xy\mathbf{j}$ has a potential function $F(x,y) = \frac{1}{3}x^3 + xy^2$, and so $\oint_C \mathbf{f}\cdot d\mathbf{r} = 0$ by Corollary 4.6.

Example 4.8

Let $\mathbf{f}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$, where

$$P(x,y) = \frac{-y}{x^2+y^2} \text{ and } Q(x,y) = \frac{x}{x^2+y^2},$$

and let $R = (x,y) : 0 < x^2+y^2 \le 1$. For the boundary curve $C : x^2+y^2 = 1$, traversed counterclockwise, it was shown in Exercise 9(b) in Section 4.2 that $\oint_C \mathbf{f}\cdot d\mathbf{r} = 2\pi$. But

$$\frac{\partial Q}{\partial x} = \frac{y^2+x^2}{(x^2+y^2)^2} = \frac{\partial P}{\partial y} \Rightarrow \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_R 0\,dA = 0$$

This would seem to contradict Green's Theorem. However, note that R is not the entire region enclosed by C, since the point $(0,0)$ is not contained in R. That is, R has a "hole" at the origin, so Green's Theorem does not apply.

If we modify the region R to be the annulus $R = (x,y) : 1/4 \le x^2+y^2 \le 1$ (see Figure 4.3.3), and take the "boundary" C of R to be $C = C_1 \cup C_2$, where C_1 is the unit circle $x^2+y^2 = 1$ traversed counterclockwise and C_2 is the circle $x^2+y^2 = 1/4$ traversed *clockwise*, then it can be shown (see Exercise 8) that

$$\oint_C \mathbf{f}\cdot d\mathbf{r} = 0$$

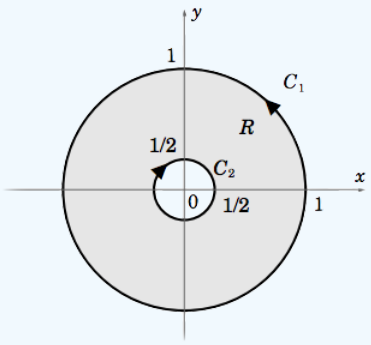


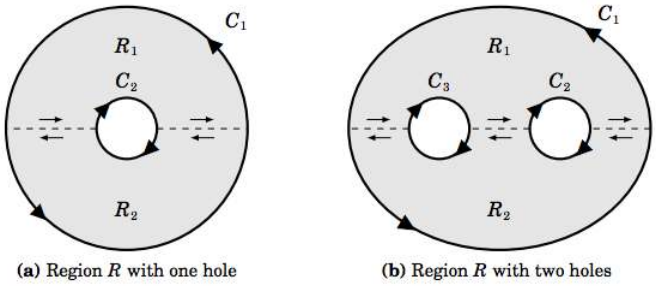
Figure 4.3.3 The annulus R

We would still have $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = 0$, so for this R we would have

$$\oint_C \mathbf{f} \cdot d\mathbf{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA,$$

which shows that Green's Theorem holds for the annular region R.

It turns out that Green's Theorem can be extended to *multiply connected regions*, that is, regions like the annulus in Example 4.8, which have one or more *regions* cut out from the interior, as opposed to discrete points being cut out. For such regions, the "outer" boundary and the "inner" boundaries are traversed so that R is always on the left side.



(a) Region R with one hole (b) Region R with two holes

Figure 4.3.4 Multiply connected regions

The intuitive idea for why Green's Theorem holds for multiply connected regions is shown in Figure 4.3.4 above. The idea is to cut "slits" between the boundaries of a multiply connected region R so that R is divided into subregions which do not have any "holes". For example, in Figure 4.3.4(a) the region R is the union of the regions R_1 and R_2, which are divided by the slits indicated by the dashed lines. Those slits are part of the boundary of both R_1 and R_2, and we traverse then in the manner indicated by the arrows. Notice that along each slit the boundary of R_1 is traversed in the opposite direction as that of R_2, which means that the line integrals of \textbf{f} along those slits cancel each other out. Since R_1 and R_2 do not have holes in them, then Green's Theorem holds in each subregion, so that

$$\oint_{bdy\ of\ R_1} \mathbf{f} \cdot d\mathbf{r} = \iint_{R_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \text{ and } \oint_{bdy\ of\ R_2} \mathbf{f} \cdot d\mathbf{r} = \iint R_2 \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

But since the line integrals along the slits cancel out, we have

$$\oint_{C_1 \cup C_2} \mathbf{f} \cdot d\mathbf{r} = \oint_{bdy\ of\ R_1} \mathbf{f} \cdot d\mathbf{r} + \oint_{bdy\ of\ R_2} \mathbf{f} \cdot d\mathbf{r},$$

and so

$$\oint_{C_1 \cup C_2} \mathbf{f} \cdot d\mathbf{r} = \iint_{R_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA + \iint_{R_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA,$$

which shows that Green's Theorem holds in the region R. A similar argument shows that the theorem holds in the region with two holes shown in Figure 4.3.4(b).

We know from Corollary 4.6 that when a smooth vector field $\mathbf{f}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ on a region R (whose boundary is a piecewise smooth, simple closed curve C) has a potential in R, then $\oint_C \mathbf{f} \cdot d\mathbf{r} = 0$. And if the potential $F(x, y)$ is smooth in R, then $\frac{\partial F}{\partial x} = P$ and $\frac{\partial F}{\partial y} = Q$, and so we know that

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y} \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \text{ in } R$$

Conversely, if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ in R then

$$\oint_C \mathbf{f} \cdot d\mathbf{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \iint_R 0 \, dA = 0$$

For a **simply connected** region R(i.e. a region with no holes), the following can be shown:

The following statements are equivalent for a simply connected region R in $\mathbb{R}^2$:

a. $\mathbf{f}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ has a smooth potential $F(x, y)$ in R
b. $\int_C \mathbf{f} \cdot d\mathbf{r}$ is independent of the path for any curve C in R
c. $\oint_C \mathbf{f} \cdot d\mathbf{r} = 0$ for every simple closed curve C in R
d. $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ in R (in this case, the differential form $Pdx + Qdy$ is exact)

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16.5: Curl and Divergence

In this final section we will establish some relationships between the gradient, divergence and curl, and we will also introduce a new quantity called the *Laplacian*. We will then show how to write these quantities in cylindrical and spherical coordinates.

Gradient

For a real-valued function $f(x,y,z)$ on $\mathbb{R}^3$, the gradient $\nabla f(x,y,z)$ is a vector-valued function on $\mathbb{R}^3$, that is, its value at a point (x,y,z) is the vector

$$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

in $\mathbb{R}^3$, where each of the partial derivatives is evaluated at the point (x,y,z). So in this way, you can think of the *symbol* ∇ as being "applied" to a real-valued function f to produce a vector ∇f.

It turns out that the divergence and curl can also be expressed in terms of the symbol ∇. This is done by thinking of ∇ as a *vector* in $\mathbb{R}^3$, namely

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}. \tag{16.5.1}$$

Here, the symbols $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ are to be thought of as "partial derivative operators" that will get "applied" to a real-valued function, say $f(x,y,z)$, to produce the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$. For instance, $\frac{\partial}{\partial x}$ "applied" to $f(x,y,z)$ produces $\frac{\partial f}{\partial x}$.

Is ∇ *really* a vector? Strictly speaking, no, since $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ are not actual numbers. But it helps to *think* of ∇ as a vector, especially with the divergence and curl, as we will soon see. The process of "applying" $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$ to a real-valued function $f(x,y,z)$ is normally thought of as *multiplying* the quantities:

$$\left(\frac{\partial}{\partial x}\right)(f) = \frac{\partial f}{\partial x}, \left(\frac{\partial}{\partial y}\right)(f) = \frac{\partial f}{\partial y}, \left(\frac{\partial}{\partial z}\right)(f) = \frac{\partial f}{\partial z}$$

For this reason, ∇ is often referred to as the "del operator", since it "operates" on functions.

Divergence

For example, it is often convenient to write the divergence div $\mathbf{f}$ as $\nabla \cdot \mathbf{f}$, since for a vector field $\mathbf{f}(x,y,z) = f_1(x,y,z)\mathbf{i} + f_2(x,y,z)\mathbf{j} + f_3(x,y,z)\mathbf{k}$, the dot product of $\mathbf{f}$ with ∇ (thought of as a vector) makes sense:

$$\begin{aligned}
\nabla \cdot \mathbf{f} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot (f_1(x,y,z)\mathbf{i} + f_2(x,y,z)\mathbf{j} + f_3(x,y,z)\mathbf{k}) \\
&= \left(\frac{\partial}{\partial x}\right)(f_1) + \left(\frac{\partial}{\partial y}\right)(f_2) + \left(\frac{\partial}{\partial z}\right)(f_3) \\
&= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \\
&= \operatorname{div} \mathbf{f}
\end{aligned}$$

We can also write curl $\mathbf{f}$ in terms of ∇, namely as $\nabla \times \mathbf{f}$, since for a vector field $\mathbf{f}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$, we have:

$$\nabla \times \mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x,y,z) & Q(x,y,z) & R(x,y,z) \end{vmatrix}$$
$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$
$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$
$$= \text{curl } \mathbf{f}$$

For a real-valued function $f(x,y,z)$, the gradient $\nabla f(x,y,z) = \dfrac{\partial f}{\partial x}\mathbf{i} + \dfrac{\partial f}{\partial y}\mathbf{j} + \dfrac{\partial f}{\partial z}\mathbf{k}$ is a vector field, so we can take its divergence:

$$\begin{aligned} \text{div } \nabla f &= \nabla \cdot \nabla f \\ &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) \\ &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \end{aligned}$$

Note that this is a real-valued function, to which we will give a special name:

Definition 4.7: Laplacian

For a real-valued function $f(x,y,z)$, the **Laplacian** of f, denoted by Δf, is given by

$$\Delta f(x,y,z) = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}. \tag{16.5.2}$$

Often the notation $\nabla^2 f$ is used for the Laplacian instead of Δf, using the convention $\nabla^2 = \nabla \cdot \nabla$.

Example 4.17

Let $\mathbf{r}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the position vector field on $\mathbb{R}^3$. Then $\|\mathbf{r}(x,y,z)\|^2 = \mathbf{r} \cdot \mathbf{r} = x^2 + y^2 + z^2$ is a real-valued function. Find

a. the gradient of $\|\mathbf{r}\|^2$
b. the divergence of $\mathbf{r}$
c. the curl of $\mathbf{r}$
d. the Laplacian of $\|\mathbf{r}\|^2$

Solution:

(a) $\nabla \|\mathbf{r}\|^2 = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = 2\mathbf{r}$

(b) $\nabla \cdot \mathbf{r} = \dfrac{\partial}{\partial x}(x) + \dfrac{\partial}{\partial y}(y) + \dfrac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$

(c)

$$\nabla \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = (0-0)\mathbf{i} - (0-0)\mathbf{j} + (0-0)\mathbf{k} = \mathbf{0}$$

(d) $\Delta\|\mathbf{r}\|^2 = \frac{\partial^2}{\partial x^2}(x^2+y^2+z^2) + \frac{\partial^2}{\partial y^2}(x^2+y^2+z^2) + \frac{\partial^2}{\partial z^2}(x^2+y^2+z^2) = 2+2+2 = 6$

Note that we could have calculated $\Delta\|\mathbf{r}\|^2$ another way, using the ∇ notation along with parts (a) and (b):

$$\Delta\|\mathbf{r}\|^2 = \nabla\cdot\nabla\|\mathbf{r}\|^2 = \nabla\cdot 2\mathbf{r} = 2\nabla\cdot\mathbf{r} = 2(3) = 6$$

Notice that in Example 4.17 if we take the curl of the gradient of $\|\mathbf{r}\|^2$ we get

$$\nabla\times(\nabla\|\mathbf{r}\|^2) = \nabla\times 2\mathbf{r} = 2\nabla\times\mathbf{r} = 2\mathbf{0} = \mathbf{0}.$$

The following theorem shows that this will be the case in general:

Theorem 4.15.

For any smooth real-valued function $f(x,y,z)$, $\nabla\times(\nabla f) = \mathbf{0}$.

Proof

We see by the smoothness of f that

$$\nabla\times(\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \tag{16.5.3}$$

$$= \left(\frac{\partial^2 f}{\partial y\partial z} - \frac{\partial^2 f}{\partial z\partial y}\right)\mathbf{i} - \left(\frac{\partial^2 f}{\partial x\partial z} - \frac{\partial^2 f}{\partial z\partial x}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x\partial y} - \frac{\partial^2 f}{\partial y\partial x}\right)\mathbf{k} = \mathbf{0}, \tag{16.5.4}$$

since the mixed partial derivatives in each component are equal.

□

Corollary 4.16

If a vector field $f(x,y,z)$ has a potential, then curl $\mathbf{f} = \mathbf{0}$.

Another way of stating Theorem 4.15 is that gradients are irrotational. Also, notice that in Example 4.17 if we take the divergence of the curl of $\mathbf{r}$ we trivially get

$$\nabla\cdot(\nabla\times\mathbf{r}) = \nabla\cdot\mathbf{0} = 0. \tag{16.5.5}$$

The following theorem shows that this will be the case in general:

Theorem 4.17.

For any smooth vector field $\mathbf{f}(x,y,z)$, $\nabla\cdot(\nabla\times\mathbf{f}) = 0$.

The proof is straightforward and left as an exercise for the reader.

Corollary 4.18

The flux of the curl of a smooth vector field $f(x,y,z)$ through any closed surface is zero.

Proof: Let Σ be a closed surface which bounds a solid S. The flux of $\nabla\times\mathbf{f}$ through Σ is

$$\iint_{\Sigma} (\nabla \times \mathbf{f}) \cdot d\sigma = \iiint_{S} \nabla \cdot (\nabla \times \mathbf{f})\, dV \text{ (by the Divergence Theorem)} \tag{16.5.6}$$

$$= \iiint_{S} 0\, dV \text{ (by Theorem 4.17)} \tag{16.5.7}$$

$$= 0 \tag{16.5.8}$$

(**QED**)

There is another method for proving Theorem 4.15 which can be useful, and is often used in physics. Namely, if the surface integral $\iint_{\Sigma} f(x,y,z)\, d\sigma = 0$ for *all* surfaces Σ in some solid region (usually all of $\mathbb{R}^3$), then we must have $f(x,y,z) = 0$ throughout that region. The proof is not trivial, and physicists do not usually bother to prove it. But the result is true, and can also be applied to double and triple integrals.

For instance, to prove Theorem 4.15, assume that $f(x,y,z)$ is a smooth real-valued function on $\mathbb{R}^3$. Let C be a simple closed curve in $\mathbb{R}^3$ and let Σ be any capping surface for C (i.e. Σ is orientable and its boundary is C). Since ∇f is a vector field, then

$$\iint_{\Sigma} (\nabla \times (\nabla f)) \cdot \mathbf{n}\, d\sigma = \oint_{C} \nabla f \cdot d\mathbf{r} \text{ by Stokes' Theorem, so}$$

$$= 0 \text{ by Corollary 4.13.}$$

Since the choice of Σ was arbitrary, then we must have $(\nabla \times (\nabla f)) \cdot \mathbf{n} = 0$ throughout $\mathbb{R}^3$, where $\mathbf{n}$ is any unit vector. Using $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ in place of $\mathbf{n}$, we see that we must have $\nabla \times (\nabla f) = \mathbf{0}$ in $\mathbb{R}^3$, which completes the proof.

Example 4.18

A system of electric charges has a *charge density* $\rho(x,y,z)$ and produces an electrostatic field $\mathbf{E}(x,y,z)$ at points (x,y,z) in space. *Gauss' Law* states that

$$\iint_{\Sigma} \mathbf{E} \cdot d\sigma = 4\pi \iiint_{S} \rho\, dV$$

for any closed surface Σ which encloses the charges, with S being the solid region enclosed by Σ. Show that $\nabla \cdot \mathbf{E} = 4\pi\rho$. This is one of *Maxwell's Equations*.

Solution

By the Divergence Theorem, we have

$$\iiint_{S} \nabla \cdot \mathbf{E} dV = \iint_{\Sigma} \mathbf{E} \cdot d\sigma$$

$$= 4\pi \iiint_{S} \rho\, dV \text{ by Gauss' Law, so combining the integrals gives}$$

$$\iiint_{S} (\nabla \cdot \mathbf{E} - 4\pi\rho)\, dV = 0 \text{ , so}$$

$$\nabla \cdot \mathbf{E} - 4\pi\rho = 0 \text{ since } \Sigma \text{ and hence } S \text{ was arbitrary, so}$$

$$\nabla \cdot \mathbf{E} = 4\pi\rho.$$

Often (especially in physics) it is convenient to use other coordinate systems when dealing with quantities such as the gradient, divergence, curl and Laplacian. We will present the formulas for these in cylindrical and spherical coordinates.

Recall from Section 1.7 that a point (x,y,z) can be represented in cylindrical coordinates (r,θ,z), where $x = r\cos\theta, y = r\sin\theta, z = z$. At each point

(r, θ, z), let $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$ be unit vectors in the direction of increasing r, θ, z, respectively (see Figure 4.6.1). Then $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$ form an orthonormal set of vectors. Note, by the right-hand rule, that $\mathbf{e}_z \times \mathbf{e}_r = \mathbf{e}_\theta$.

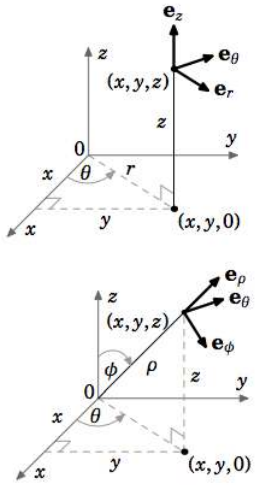


Figure 4.6.1 Orthonormal vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$ in cylindrical coordinates (left) and spherical coordinates (right).

Similarly, a point (x, y, z) can be represented in spherical coordinates (ρ, θ, φ), where $x = \rho \sin\varphi \cos\theta, y = \rho \sin\varphi \sin\theta, z = \rho \cos\varphi$. At each point (ρ, θ, φ), let $\mathbf{e}_\rho, \mathbf{e}_\theta, \mathbf{e}_\varphi$ be unit vectors in the direction of increasing ρ, θ, φ, respectively (see Figure 4.6.2). Then the vectors $\mathbf{e}_\rho, \mathbf{e}_\theta, \mathbf{e}_\varphi$ are orthonormal. By the right-hand rule, we see that $\mathbf{e}_\theta \times \mathbf{e}_\rho = \mathbf{e}_\varphi$.

We can now summarize the expressions for the gradient, divergence, curl and Laplacian in Cartesian, cylindrical and spherical coordinates in the following tables:

Cartesian

(x, y, z): Scalar function F; Vector field $\mathbf{f} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$

- gradient : $\nabla F = \dfrac{\partial F}{\partial x}\mathbf{i} + \dfrac{\partial F}{\partial y}\mathbf{j} + \dfrac{\partial F}{\partial z}\mathbf{k}$
- divergence : $\nabla \cdot \mathbf{f} = \dfrac{\partial f_1}{\partial x} + \dfrac{\partial f_2}{\partial y} + \dfrac{\partial f_3}{\partial z}$
- curl : $\nabla \times \mathbf{f} = \left(\dfrac{\partial f_3}{\partial y} - \dfrac{\partial f_2}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial f_1}{\partial z} - \dfrac{\partial f_3}{\partial x}\right)\mathbf{j} + \left(\dfrac{\partial f_2}{\partial x} - \dfrac{\partial f_1}{\partial y}\right)\mathbf{k}$
- Laplacian : $\Delta F = \dfrac{\partial^2 F}{\partial x^2} + \dfrac{\partial^2 F}{\partial y^2} + \dfrac{\partial^2 F}{\partial z^2}$

Cylindrical

(r, θ, z): Scalar function F; Vector field $\mathbf{f} = f_r\mathbf{e}_r + f_\theta\mathbf{e}_\theta + f_z\mathbf{e}_z$

- gradient : $\nabla F = \dfrac{\partial F}{\partial r}\mathbf{e}_r + \dfrac{1}{r}\dfrac{\partial F}{\partial \theta}\mathbf{e}_\theta + \dfrac{\partial F}{\partial z}\mathbf{e}_z$
- divergence : $\nabla \cdot \mathbf{f} = \dfrac{1}{r}\dfrac{\partial}{\partial r}(rf_r) + \dfrac{1}{r}\dfrac{\partial f_\theta}{\partial \theta} + \dfrac{\partial f_z}{\partial z}$
- curl : $\nabla \times \mathbf{f} = \left(\dfrac{1}{r}\dfrac{\partial f_z}{\partial \theta} - \dfrac{\partial f_\theta}{\partial z}\right)\mathbf{e}_r + \left(\dfrac{\partial f_r}{\partial z} - \dfrac{\partial f_z}{\partial r}\right)\mathbf{e}_\theta + \dfrac{1}{r}\left(\dfrac{\partial}{\partial r}(rf_\theta) - \dfrac{\partial f_r}{\partial \theta}\right)\mathbf{e}_z$
- Laplacian : $\Delta F = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial F}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 F}{\partial \theta^2} + \dfrac{\partial^2 F}{\partial z^2}$

Spherical

(ρ, θ, φ): Scalar function F; Vector field $\mathbf{f} = f_\rho \mathbf{e}_\rho + f_\theta \mathbf{e}_\theta + f_\varphi \mathbf{e}_\varphi$

- gradient : $\nabla F = \dfrac{\partial F}{\partial \rho}\mathbf{e}_\rho + \dfrac{1}{\rho \sin\varphi}\dfrac{\partial F}{\partial \theta}\mathbf{e}_\theta + \dfrac{1}{\rho}\dfrac{\partial F}{\partial \varphi}\mathbf{e}_\varphi$
- divergence : $\nabla \cdot \mathbf{f} = \dfrac{1}{\rho^2}\dfrac{\partial}{\partial \rho}(\rho^2 f_\rho) + \dfrac{1}{\rho}\sin\varphi\dfrac{\partial f_\theta}{\partial \theta} + \dfrac{1}{\rho \sin\varphi}\dfrac{\partial}{\partial \varphi}(\sin\varphi f_\theta)$
- curl : $\nabla \times \mathbf{f} = \dfrac{1}{\rho \sin\varphi}\left(\dfrac{\partial}{\partial \varphi}(\sin\varphi f_\theta) - \dfrac{\partial f_\varphi}{\partial \theta}\right)\mathbf{e}_\rho + \dfrac{1}{\rho}\left(\dfrac{\partial}{\partial \rho}(\rho f_\varphi) - \dfrac{\partial f_\rho}{\partial \varphi}\right)\mathbf{e}_\theta + \left(\dfrac{1}{\rho \sin\varphi}\dfrac{\partial f_\rho}{\partial \theta} - \dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}(\rho f_\theta)\right)\mathbf{e}_\varphi$
- Laplacian : $\Delta F = \dfrac{1}{\rho^2}\dfrac{\partial}{\partial \rho}\left(\rho^2 \dfrac{\partial F}{\partial \rho}\right) + \dfrac{1}{\rho^2 \sin^2\varphi}\dfrac{\partial^2 F}{\partial \theta^2} + \dfrac{1}{\rho^2 \sin\varphi}\dfrac{\partial}{\partial \varphi}\left(\sin\varphi \dfrac{\partial F}{\partial \varphi}\right)$

The derivation of the above formulas for cylindrical and spherical coordinates is straightforward but extremely tedious. The basic idea is to take the Cartesian equivalent of the quantity in question and to substitute into that formula using the appropriate coordinate transformation. As an example, we will derive the formula for the gradient in spherical coordinates.

Goal: Show that the gradient of a real-valued function $F(\rho, \theta, \varphi)$ in spherical coordinates is:

$$\nabla F = \frac{\partial F}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho \sin\varphi}\frac{\partial F}{\partial \theta}\mathbf{e}_\theta + \frac{1}{\rho}\frac{\partial F}{\partial \varphi}\mathbf{e}_\varphi$$

Idea: In the Cartesian gradient formula $\nabla F(x, y, z) = \dfrac{\partial F}{\partial x}\mathbf{i} + \dfrac{\partial F}{\partial y}\mathbf{j} + \dfrac{\partial F}{\partial z}\mathbf{k}$, put the Cartesian basis vectors **i**, **j**, **k** in terms of the spherical coordinate basis vectors $\mathbf{e}_\rho, \mathbf{e}_\theta, \mathbf{e}_\varphi$ and functions of ρ, θ and φ. Then put the partial derivatives $\dfrac{\partial F}{\partial x}, \dfrac{\partial F}{\partial y}, \dfrac{\partial F}{\partial z}$ in terms of $\dfrac{\partial F}{\partial \rho}, \dfrac{\partial F}{\partial \theta}, \dfrac{\partial F}{\partial \varphi}$ and functions of ρ, θ and φ.

Step 1: Get formulas for $\mathbf{e}_\rho, \mathbf{e}_\theta, \mathbf{e}_\varphi$ in terms of **i**, **j**, **k**.

We can see from Figure 4.6.2 that the unit vector $\mathbf{e}_\rho$ in the ρ direction at a general point (ρ, θ, φ) is $\mathbf{e}_\rho = \dfrac{\mathbf{r}}{\|\mathbf{r}\|}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the position vector of the point in Cartesian coordinates. Thus,

$$\mathbf{e}_\rho = \frac{\mathbf{r}}{\|\mathbf{r}\|} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}},$$

so using $x = \rho \sin\varphi \cos\theta, y = \rho \sin\varphi \sin\theta, z = \rho \cos\varphi,$ and $\rho = \sqrt{x^2 + y^2 + z^2}$, we get:

$$\boxed{\mathbf{e}_\rho = \sin\varphi \cos\theta \mathbf{i} + \sin\varphi \sin\theta \mathbf{j} + \cos\varphi \mathbf{k}}$$

Now, since the angle θ is measured in the xy-plane, then the unit vector $\mathbf{e}_\theta$ in the θ direction must be parallel to the xy-plane. That is, $\mathbf{e}_\theta$ is of the form $a\mathbf{i} + b\mathbf{j} + 0\mathbf{k}$. To figure out what a and b are, note that since $\mathbf{e}_\theta \perp \mathbf{e}_\rho$, then in particular $\mathbf{e}_\theta \perp \mathbf{e}_\rho$ when $\mathbf{e}_\rho$ is in the xy-plane. That occurs when the angle φ is $\pi/2$. Putting $\varphi = \pi/2$ into the formula for $\mathbf{e}_\rho$ gives $\mathbf{e}_\rho = \cos\theta \mathbf{i} + \sin\theta \mathbf{j} + 0\mathbf{k}$, and we see that a vector perpendicular to that is $-\sin\theta \mathbf{i} + \cos\theta \mathbf{j} + 0\mathbf{k}$. Since this vector is also a unit vector and points in the (positive) θ direction, it must be $\mathbf{e}_\theta$:

$$\boxed{\mathbf{e}_\theta = -\sin\theta \mathbf{i} + \cos\theta \mathbf{j} + 0\mathbf{k}}$$

Lastly, since $\mathbf{e}_\varphi = \mathbf{e}_\theta \times \mathbf{e}_\rho$, we get:

$$\boxed{\mathbf{e}_\varphi = \cos\varphi \cos\theta \mathbf{i} + \cos\varphi \sin\theta \mathbf{j} - \sin\varphi \mathbf{k}}$$

Step 2: Use the three formulas from Step 1 to solve for **i**, **j**, **k** in terms of $\mathbf{e}_\rho, \mathbf{e}_\theta, \mathbf{e}_\varphi$.

This comes down to solving a system of three equations in three unknowns. There are many ways of doing this, but we will do it by combining the formulas for $\mathbf{e}_\rho$ and $\mathbf{e}_\varphi$ to eliminate k, which will give us an equation involving just **i** and **j**. This, with the formula for $\mathbf{e}_\theta$, will then leave us with a system of two equations in two unknowns (**i** and **j**), which we will use to solve first for **j** then for **i**. Lastly, we will solve for **k**.

First, note that

$$\sin\varphi\mathbf{e}_\rho + \cos\varphi\mathbf{e}_\varphi = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$$

so that

$$\sin\theta(\sin\varphi\mathbf{e}_\rho + \cos\varphi\mathbf{e}_\varphi) + \cos\theta\mathbf{e}_\theta = (\sin^2\theta + \cos^2\theta)\mathbf{j} = \mathbf{j},$$

and so:

$$\boxed{\mathbf{j} = \sin\varphi\sin\theta\mathbf{e}_\rho + \cos\theta\mathbf{e}_\theta + \cos\varphi\sin\theta\mathbf{e}_\varphi}$$

Likewise, we see that

$$\cos\theta(\sin\varphi\mathbf{e}_\rho + \cos\varphi\mathbf{e}_\varphi) - \sin\theta\mathbf{e}_\theta = (\cos^2\theta + \sin^2\theta)\mathbf{i} = \mathbf{i},$$

and so:

$$\boxed{\mathbf{i} = \sin\varphi\cos\theta\mathbf{e}_\rho - \sin\theta\mathbf{e}_\theta + \cos\varphi\cos\theta\mathbf{e}_\varphi}$$

Lastly, we see that:

$$\boxed{\mathbf{k} = \cos\varphi\mathbf{e}_\rho - \sin\varphi\mathbf{e}_\varphi}$$

Step 3: Get formulas for $\dfrac{\partial F}{\partial\rho}, \dfrac{\partial F}{\partial\theta}, \dfrac{\partial F}{\partial\varphi}$ in terms of $\dfrac{\partial F}{\partial x}, \dfrac{\partial F}{\partial y}, \dfrac{\partial F}{\partial z}$.

By the Chain Rule, we have

$$\begin{aligned}
\frac{\partial F}{\partial\rho} &= \frac{\partial F}{\partial x}\frac{\partial x}{\partial\rho} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial\rho} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial\rho},\\
\frac{\partial F}{\partial\theta} &= \frac{\partial F}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial\theta} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial\theta},\\
\frac{\partial F}{\partial\varphi} &= \frac{\partial F}{\partial x}\frac{\partial x}{\partial\varphi} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial\varphi} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial\varphi},
\end{aligned}$$

which yields:

$$\boxed{\begin{aligned}
\frac{\partial F}{\partial\rho} &= \sin\varphi\cos\theta\frac{\partial F}{\partial x} + \sin\varphi\sin\theta\frac{\partial F}{\partial y} + \cos\varphi\frac{\partial F}{\partial z}\\
\frac{\partial F}{\partial\theta} &= -\rho\sin\varphi\sin\theta\frac{\partial F}{\partial x} + \rho\sin\varphi\cos\theta\frac{\partial F}{\partial y}\\
\frac{\partial F}{\partial\varphi} &= \rho\cos\varphi\cos\theta\frac{\partial F}{\partial x} + \rho\cos\varphi\sin\theta\frac{\partial F}{\partial y} - \rho\sin\varphi\frac{\partial F}{\partial z}
\end{aligned}} \tag{16.5.9}$$

Step 4: Use the three formulas from Step 3 to solve for $\dfrac{\partial F}{\partial x}, \dfrac{\partial F}{\partial y}, \dfrac{\partial F}{\partial z}$ in terms of $\dfrac{\partial F}{\partial\rho}, \dfrac{\partial F}{\partial\theta}, \dfrac{\partial F}{\partial\varphi}$.

Again, this involves solving a system of three equations in three unknowns. Using a similar process of elimination as in Step 2, we get:

$$\boxed{\begin{aligned}
\frac{\partial F}{\partial x} &= \frac{1}{\rho\sin\varphi}\left(\rho\sin^2\varphi\cos\theta\frac{\partial F}{\partial\rho} - \sin\theta\frac{\partial F}{\partial\theta} + \sin\varphi\cos\varphi\cos\theta\frac{\partial F}{\partial\varphi}\right)\\
\frac{\partial F}{\partial y} &= \frac{1}{\rho\sin\varphi}\left(\rho\sin^2\varphi\sin\theta\frac{\partial F}{\partial\rho} + \cos\theta\frac{\partial F}{\partial\theta} + \sin\varphi\cos\varphi\sin\theta\frac{\partial F}{\partial\varphi}\right)\\
\frac{\partial F}{\partial z} &= \frac{1}{\rho}\left(\rho\cos\varphi\frac{\partial F}{\partial\rho} - \sin\varphi\frac{\partial F}{\partial\varphi}\right)
\end{aligned}} \tag{16.5.10}$$

Step 5: Substitute the formulas for **i**, **j**, **k** from Step 2 and the formulas for $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$ from Step 4 into the Cartesian gradient formula $\nabla F(x,y,z) = \frac{\partial F}{\partial x}\mathbf{i} + \frac{\partial F}{\partial y}\mathbf{j} + \frac{\partial F}{\partial z}\mathbf{k}$.

Doing this last step is perhaps the most tedious, since it involves simplifying $3\times 3 + 3\times 3 + 2\times 2 = 22$ terms! Namely,

$$\begin{aligned}\nabla F &= \frac{1}{\rho\sin\varphi}\left(\rho\sin^2\varphi\cos\theta\frac{\partial F}{\partial \rho} - \sin\theta\frac{\partial F}{\partial \theta} + \sin\varphi\cos\varphi\cos\theta\frac{\partial F}{\partial \varphi}\right)(\sin\varphi\cos\theta\mathbf{e}_\rho - \sin\theta\mathbf{e}_\theta + \cos\varphi\cos\theta\mathbf{e}_\varphi)\\ &+ \frac{1}{\rho\sin\varphi}\left(\rho\sin^2\varphi\sin\theta\frac{\partial F}{\partial \rho} + \cos\theta\frac{\partial F}{\partial \theta} + \sin\varphi\cos\varphi\sin\theta\frac{\partial F}{\partial \varphi}\right)(\sin\varphi\sin\theta\mathbf{e}_\rho + \cos\theta\mathbf{e}_\theta + \cos\varphi\sin\theta\mathbf{e}_\varphi)\\ &+ \frac{1}{\rho}\left(\rho\cos\varphi\frac{\partial F}{\partial \rho} - \sin\varphi\frac{\partial F}{\partial \varphi}\right)(\cos\varphi\mathbf{e}_\rho - \sin\varphi\mathbf{e}_\varphi),\end{aligned}$$

which we see has 8 terms involving $\mathbf{e}_\rho$, 6 terms involving $\mathbf{e}_\theta$, and 8 terms involving $\mathbf{e}_\varphi$. But the algebra is straightforward and yields the desired result:

$$\nabla F = \frac{\partial F}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho\sin\varphi}\frac{\partial F}{\partial \theta}\mathbf{e}_\theta + \frac{1}{\rho}\frac{\partial F}{\partial \varphi}\mathbf{e}_\varphi \quad \checkmark \tag{16.5.11}$$

Example 4.19

In Example 4.17 we showed that $\nabla\|\mathbf{r}\|^2 = 2\mathbf{r}$ and $\Delta\|\mathbf{r}\|^2 = 6$, where $\mathbf{r}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in Cartesian coordinates. Verify that we get the same answers if we switch to spherical coordinates.

Solution

Since $\|\mathbf{r}\|^2 = x^2 + y^2 + z^2 = \rho^2$ in spherical coordinates, let $F(\rho,\theta,\varphi) = \rho^2$ (so that $F(\rho,\theta,\varphi) = \|\mathbf{r}\|^2$). The gradient of F in spherical coordinates is

$$\begin{aligned}\nabla F &= \frac{\partial F}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho\sin\varphi}\frac{\partial F}{\partial \theta}\mathbf{e}_\theta + \frac{1}{\rho}\frac{\partial F}{\partial \varphi}\mathbf{e}_\varphi\\ &= 2\rho\mathbf{e}_\rho + \frac{1}{\rho\sin\varphi}(0)\mathbf{e}_\theta + \frac{1}{\rho}(0)\mathbf{e}_\varphi\\ &= 2\rho\mathbf{e}_\rho = 2\rho\frac{\mathbf{r}}{\|\mathbf{r}\|}, \text{ as we showed earlier, so}\\ &= 2\rho\frac{\mathbf{r}}{\rho} = 2\mathbf{r}, \text{ as expected. And the Laplacian is}\\ \Delta F &= \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial F}{\partial \rho}\right) + \frac{1}{\rho^2\sin^2\varphi}\frac{\partial^2 F}{\partial \theta^2} + \frac{1}{\rho^2\sin\varphi}\frac{\partial}{\partial \varphi}\left(\sin\varphi\frac{\partial F}{\partial \varphi}\right)\\ &= \frac{1}{\rho^2}\frac{\partial}{\partial \rho}(\rho^2 2\rho) + \frac{1}{\rho^2\sin\varphi}(0) + \frac{1}{\rho^2\sin\varphi}\frac{\partial}{\partial \varphi}(\sin\varphi(0))\\ &= \frac{1}{\rho^2}\frac{\partial}{\partial \rho}(2\rho^3) + 0 + 0\\ &= \frac{1}{\rho^2}(6\rho^2) = 6, \text{ as expected.}\end{aligned}$$

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16.6: Parametric Surfaces and Their Areas

We have now seen many kinds of functions. When we talked about parametric curves, we defined them as functions from $\mathbb{R}$ to $\mathbb{R}^2$ (plane curves) or $\mathbb{R}$ to $\mathbb{R}^3$ (space curves). Because each of these has its domain $\mathbb{R}$, they are one dimensional (you can only go forward or backward). In this section, we investigate how to parameterize two dimensional surfaces. Below is the definition.

Definition: Parametric Surfaces

A *parametric surface* is a function with domain $\mathbb{R}^2$ and range $\mathbb{R}^3$.

We typically use the variables u and v for the domain and x, y, and z for the range. We often use vector notation to exhibit parametric surfaces.

Example 16.6.1

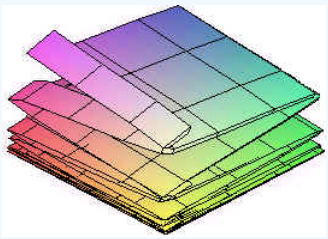
A sphere of radius 7 can be parameterized by

$$\mathbf{r}(u,v) = 7\cos u \sin v\hat{\mathbf{i}} + 7\sin u \sin v\hat{\mathbf{j}} + 7\cos v\hat{\mathbf{k}}$$

Notice that we have just used spherical coordinates with the radius held at 7.

We can use a computer to graph a parametric surface. Below is the graph of the surface

$$\mathbf{r}(u,v) = \sin u\,\hat{\mathbf{i}} + \cos v\hat{\mathbf{j}} + \exp(2u^{\frac{1}{3}} + 2v^{\frac{1}{3}})\hat{\mathbf{k}}.$$



Example 16.6.2

Represent the surface

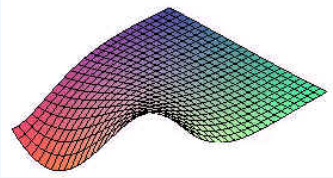
$$z = e^x \cos(x-y)$$

parametrically.

Solution

The idea is similar to parametric curves. We just let $x = u$ and $y = v$, to get

$$\mathbf{r}(u,v) = u\,\hat{\mathbf{i}} + v\hat{\mathbf{j}} + e^u \cos(u-v)\hat{\mathbf{k}}.$$



Example 16.6.3

A surface is created by revolving the curve

$$y = \cos x$$

about the x-axis. Find parametric equations for this surface.

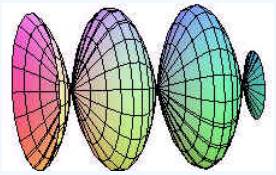
Solution

For a fixed value of x, we get a circle of radius $\cos x$. Now use polar coordinates (in the yz-plane) to get

$$\mathbf{r}(u,v) = u\hat{\mathbf{i}} + r\cos v\hat{\mathbf{j}} + r\sin v\hat{\mathbf{k}}.$$

Since $u = x$ and $\mathbf{r} = \cos x$, we can substitute $cosu$ for $\mathbf{r}$ in the above equation to get

$$\mathbf{r}(u,v) = u\hat{\mathbf{i}} + \cos u\cos v\hat{\mathbf{j}} + \cos u\sin v\hat{\mathbf{k}}.$$



Normal Vectors and Tangent Planes

We have already learned how to find a normal vector of a surface that is presented as a function of tow variables, namely find the gradient vector. To find the normal vector to a surface $\mathbf{r}(t)$ that is defined parametrically, we proceed as follows.

The partial derivatives

$$\mathbf{r}_u(u_0,v_0) \quad \text{and} \quad \mathbf{r}_v(u_0,v_0)$$

will lie on the tangent plane to the surface at the point (u_0,v_0). This is true, because fixing one variable constant and letting the other vary, produced a curve on the surface through (u_0,v_0). $\mathbf{r}_u(u_0,v_0)$ will be tangent to this curve. The tangent plane contains all vectors tangent to curves passing through the point.

To find a normal vector, we just cross the two tangent vectors.

Example 16.6.4

Find the equation of the tangent plane to the surface

$$\mathbf{r}(u,v) = (u^2 - v^2)\hat{\mathbf{i}} + (u+v)\hat{\mathbf{j}} + (uv)\hat{\mathbf{k}}$$

at the point $(1,2)$.

Solution

We have

$$\mathbf{r}_u(u,v) = (2u)\hat{\mathbf{i}} + \hat{\mathbf{j}} + v\hat{\mathbf{k}}$$

$$\mathbf{r}_v(u,v) = (-2v)\hat{\mathbf{i}} + \hat{\mathbf{j}} + u\hat{\mathbf{k}}$$

so that

$$\mathbf{r}_u(1,2) = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$$

$$\mathbf{r}_v(1,2) = -4\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$$

$$\mathbf{r}(1,2) = -3\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 3\hat{\mathbf{k}}.$$

Now cross these vectors together to get

$$r_u \times r_v = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 1 & 2 \\ -4 & 1 & 1 \end{vmatrix} \tag{16.6.1}$$

$$= -\hat{\mathbf{i}} - 10\hat{\mathbf{j}} + 6\hat{\mathbf{k}}. \tag{16.6.2}$$

We now have the normal vector and a point $(-3,3,2)$. We use the normal vector-point equation for a plane

$$-1(x+3)-10(y-3)+6(z-2)=0$$

$$-x-10y+6z=-15 \quad \text{or} \quad x+10y-6z=15.$$

Surface Area

To find the surface area of a parametrically defined surface, we proceed in a similar way as in the case as a surface defined by a function. Instead of projecting down to the region in the xy-plane, we project back to a region in the uv-plane. We cut the region into small rectangles which map approximately to small parallelograms with adjacent defining vectors $\mathbf{r}_u$ and $\mathbf{r}_v$. The area of these parallelograms will equal the magnitude of the cross product of $\mathbf{r}_u$ and $\mathbf{r}_v$. Finally add the areas up and take the limit as the rectangles get small. This will produce a double integral.

Definition: Area of a Parametric Surface

Let S be a smooth surface defined parametrically by

$$\mathbf{r}(u,v)=x(u,v)\hat{\mathbf{i}}+y(u,v)\hat{\mathbf{j}}+z(u,v)\hat{\mathbf{k}}$$

where u and v are contained in a region $\mathbb{R}$. Then the surface area of S is given by

$$SA=\iint_R ||r_u \times r_v||\, dudv.$$

Since the magnitude of a cross product involves a square root, the integral in the surface area formula is usually impossible or nearly impossible to evaluate without power series or by approximation techniques.

Example 16.6.5

Find the surface area of the surface given by

$$\mathbf{r}(u,v)=(v^2)\hat{\mathbf{i}}+(u-v)\hat{\mathbf{j}}+(u^2)\hat{\mathbf{k}} \quad 0\le u\le 2 \quad 1\le v\le 4.$$

Solution

We calculate

$$\mathbf{r}_u(u,v)=\hat{\mathbf{j}}+2u\hat{\mathbf{k}}$$

$$\mathbf{r}_v(u,v)=(2v)\hat{\mathbf{i}}+\hat{\mathbf{j}}.$$

The cross product is

$$||\mathbf{r}\times\mathbf{r}|| = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 1 & 2u \\ 2v & -1 & 0 \end{vmatrix} \tag{16.6.3}$$

$$= ||2u\hat{\mathbf{i}}+4uv\hat{\mathbf{j}}-2v\hat{\mathbf{k}}|| \tag{16.6.4}$$

$$= 2\sqrt{u^2+4u^2v^2+v^2}\,. \tag{16.6.5}$$

The surface area formula gives

$$SA=\int_0^2\int_1^4 2\sqrt{4u^2v^2+v^2}\; dvdu.$$

This integral is probably impossible to compute exactly. Instead, a calculator can be used to obtain a surface area of 70.9.

Larry Green (Lake Tahoe Community College)

- Integrated by Justin Marshall.

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16.7: Surface Integrals

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16.8: Stokes' Theorem

So far the only types of line integrals which we have discussed are those along curves in $\mathbb{R}^2$. But the definitions and properties which were covered in Sections 4.1 and 4.2 can easily be extended to include functions of three variables, so that we can now discuss line integrals along curves in $\mathbb{R}^3$.

Definition 16.8.1: Line Integrals

For a real-valued function $f(x,y,z)$ and a curve C in $\mathbb{R}^3$, parametrized by $x=x(t), y=y(t), z=z(t), a\le t\le b$, the **line integral** of $f(x,y,z)$ **along** C **with respect to arc length** s is

$$\int_C f(x,y,z)\,ds=\int_a^b f(x(t),y(t),z(t))\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}\,dt. \tag{16.8.1}$$

The **line integral of** $f(x,y,z)$ along C **with respect to** x is

$$\int_C f(x,y,z)\,dx=\int_a^b f(x(t),y(t),z(t))x'(t)\,dt. \tag{16.8.2}$$

The **line integral of** $f(x,y,z)$ along C **with respect to** y is

$$\int_C f(x,y,z)\,dy=\int_a^b f(x(t),y(t),z(t))y'(t)\,dt. \tag{16.8.3}$$

The **line integral of** $f(x,y,z)$ along C **with respect to** z is

$$\int_C f(x,y,z)\,dz=\int_a^b f(x(t),y(t),z(t))z'(t)\,dt. \tag{16.8.4}$$

Similar to the two-variable case, if $f(x,y,z)\ge 0$ then the line integral $\int_C f(x,y,z)\,ds$ can be thought of as the total area of the "picket fence" of height $f(x,y,z)$ at each point along the curve C in $\mathbb{R}^3$.

Vector fields in $\mathbb{R}^3$ are defined in a similar fashion to those in $\mathbb{R}^2$, which allows us to define the line integral of a vector field along a curve in $\mathbb{R}^3$.

Definition 16.8.2

For a vector field $\mathbf{f}(x,y,z)=P(x,y,z)\mathbf{i}+Q(x,y,z)\mathbf{j}+R(x,y,z)\mathbf{k}$ and a curve C in $\mathbb{R}^3$ with a smooth parametrization $x=x(t), y=y(t), z=z(t), a\le t\le b$, the **line integral of $\boldsymbol{f}$ along** C is

$$\int_C \mathbf{f}\cdot d\mathbf{r} = \int_C P(x,y,z)\,dx+\int_C Q(x,y,z)\,dy+\int_C R(x,y,z)\,dz \tag{16.8.5}$$

$$=\int_a^b \mathbf{f}(x(t),y(t),z(t))\cdot\mathbf{r}'(t)dt, \tag{16.8.6}$$

where $\mathbf{r}(t)=x(t)\mathbf{i}+y(t)\mathbf{j}+z(t)\mathbf{k}$ is the position vector for points on C.

Similar to the two-variable case, if $\mathbf{f}(x,y,z)$ represents the force applied to an object at a point (x,y,z) then the line integral $\int_C \mathbf{f}\cdot d\mathbf{r}$ represents the work done by that force in moving the object along the curve C in $\mathbb{R}^3$.

Some of the most important results we will need for line integrals in $\mathbb{R}^3$ are stated below without proof (the proofs are similar to their two-variable equivalents).

Theorem 16.8.1

For a vector field $\mathbf{f}(x,y,z)=P(x,y,z)\mathbf{i}+Q(x,y,z)\mathbf{j}+R(x,y,z)\mathbf{k}$ and a curve C with a smooth parametrization $x=x(t), y=y(t), z=z(t), a\le t\le b$ and position vector $\mathbf{r}(t)=x(t)\mathbf{i}+y(t)\mathbf{j}+z(t)\mathbf{k}$,

$$\int_C \mathbf{f}\cdot d\mathbf{r} = \int_C \mathbf{f}\cdot\mathbf{T}\,ds, \tag{16.8.7}$$

where $\mathbf{T}(t) = \dfrac{\mathbf{r}\,'(t)}{\|\mathbf{r}\,'(t)\|}$ is the unit tangent vector to C at $(x(t), y(t), z(t))$.

Theorem 16.8.2: Chain Rule

If $w = f(x,y,z)$ is a continuously differentiable function of $x, y,$ and $z,$ and $x = x(t), y = y(t)$ and $z = z(t)$ are differentiable functions of $t,$ then w is a differentiable function of t, and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}. \tag{16.8.8}$$

Also, if $x = x(t_1,t_2), y = y(t_1,t_2)$ and $z = z(t_1,t_2)$ are continuously differentiable function of (t_1,t_2), then

$$\frac{\partial w}{\partial t_1} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t_1} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t_1} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t_1} \tag{16.8.9}$$

and

$$\frac{\partial w}{\partial t_2} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t_2} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t_2} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t_2} \tag{16.8.10}$$

Theorem 16.8.3: Potential

Let $\mathbf{f}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$ be a vector field in some solid S, with P, Q and R continuously differentiable functions on S. Let C be a smooth curve in S parametrized by $x = x(t), y = y(t), z = z(t), a \le t \le b$. Suppose that there is a real-valued function $F(x,y,z)$ such that $\nabla F = \mathbf{f}$ on S. Then

$$\int_C \mathbf{f}\cdot d\mathbf{r} = F(B) - F(A), \tag{16.8.11}$$

where $A = (x(a), y(a), z(a))$ and $B = (x(b), y(b), z(b))$ are the endpoints of C.

Corollary

If a vector field \(\textbf{f}\) has a potential in a solid S, then $\oint_C \mathbf{f}\cdot d\mathbf{r} = 0$ for any closed curve C in S (i.e. $\oint_C \nabla F\cdot d\mathbf{r} = 0$ for any real-valued function $F(x,y,z)$).

Example 16.8.1

Let $f(x,y,z) = z$ and let C be the curve in $\mathbb{R}^3$ parametrized by

$$x = t\sin t, \quad y = t\cos t, \quad z = t, \quad 0 \le t \le 8\pi.$$

Evaluate $\int_C f(x,y,z)\,ds$. (Note: C is called a *conical helix*. See Figure 4.5.1).

Solution

Since $x\,'(t) = \sin t + t\cos t, y\,'(t) = \cos t - t\sin t,$ and $z'(t) = 1$, we have

$$\begin{aligned} x\,'(t)^2 + y\,'(t)^2 + z'(t)^2 &= (\sin^2 t + 2t\sin t\cos t + t^2\cos^2 t) + (\cos^2 t - 2t\sin t\cos t + t^2\sin^2 t) + 1 \\ &= t^2(\sin^2 t + \cos^2 t) + \sin^2 t + \cos^2 t + 1 \\ &= t^2 + 2, \end{aligned}$$

so since $f(x(t), y(t), z(t)) = z(t) = t$ along the curve C, then

$$\int_C f(x,y,z)\,ds = \int_0^{8\pi} f(x(t),y(t),z(t))\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}\,dt$$
$$= \int_0^{8\pi} t\sqrt{t^2+2}\,dt$$
$$= \left(\frac{1}{3}(t^2+2)^{3/2}\right)\Big|_0^{8\pi} = \frac{1}{3}\left((64\pi^2+2)^{3/2}-2\sqrt{2}\right).$$

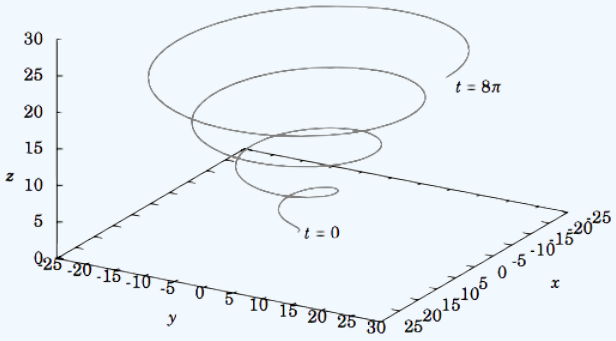


Figure 4.5.1 Conical helix C

Example 16.8.2

Let $\mathbf{f}(x,y,z) = x\mathbf{i}+y\mathbf{j}+2z\mathbf{k}$ be a vector field in $\mathbb{R}^3$. Using the same curve C from Example 4.12, evaluate $\int_C \mathbf{f}\cdot d\mathbf{r}$.

Solution:

It is easy to see that $F(x,y,z) = \frac{x^2}{2}+\frac{y^2}{2}+z^2$ is a potential for $\mathbf{f}(x,y,z)$ (i.e. $\nabla F = \mathbf{f}$).

So by Theorem 4.12 we know that

$$\int_C \mathbf{f}\cdot d\mathbf{r} = F(B)-F(A), \text{ where } A=(x(0),y(0),z(0)) \text{ and } B=(x(8\pi),y(8\pi),z(8\pi)), \text{ so}$$
$$= F(8\pi\sin 8\pi, 8\pi\cos 8\pi, 8\pi) - F(0\sin 0, 0\cos 0, 0)$$
$$= F(0,8\pi,8\pi) - F(0,0,0)$$
$$= 0+\frac{(8\pi)^2}{2}+(8\pi)^2-(0+0+0) = 96\pi^2.$$

We will now discuss a generalization of Green's Theorem in $\mathbb{R}^2$ to *orientable* surfaces in $\mathbb{R}^3$, called *Stokes' Theorem*. A surface Σ in $\mathbb{R}^3$ is **orientable** if there is a continuous vector field $\mathbf{N}$ in $\mathbb{R}^3$ such that $\mathbf{N}$ is nonzero and normal to Σ (i.e. perpendicular to the tangent plane) at each point of Σ. We say that such an $\mathbf{N}$ is a *normal vector field*.

For example, the unit sphere $x^2+y^2+z^2=1$ is orientable, since the continuous vector field $\mathbf{N}(x,y,z) = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ is nonzero and normal to the sphere at each point. In fact, $-\mathbf{N}(x,y,z)$ is another normal vector field (see Figure 4.5.2). We see in this case that $\mathbf{N}(x,y,z)$ is what we have called an outward normal vector, and $-\mathbf{N}(x,y,z)$ is an *inward* normal vector. These "outward" and "inward" normal vector fields on the sphere correspond to an "outer" and "inner" side, respectively, of the sphere. That is, we say that the sphere is a *two-sided* surface. Roughly, "two-sided" means "orientable". Other examples of two-sided, and hence orientable, surfaces are cylinders, paraboloids, ellipsoids, and planes.

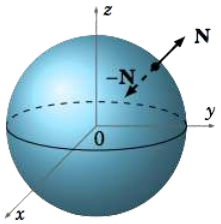


Figure 4.5.2

You may be wondering what kind of surface would *not* have two sides. An example is the **Möbius strip**, which is constructed by taking a thin rectangle and connecting its ends at the opposite corners, resulting in a "twisted" strip (see Figure 4.5.3).

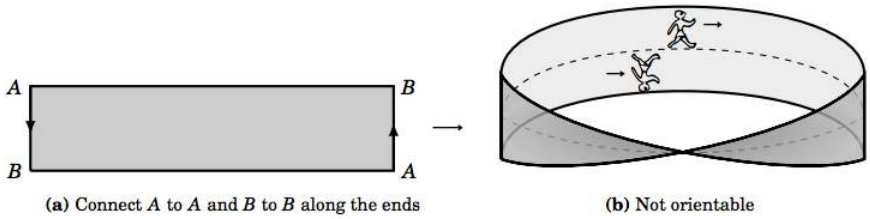


Figure 4.5.3: Möbius strip

If you imagine walking along a line down the center of the Möbius strip, as in Figure 4.5.3(b), then you arrive back at the same place from which you started but upside down! That is, your *orientation* changed even though your motion was continuous along that center line. Informally, thinking of your vertical direction as a normal vector field along the strip, there is a discontinuity at your starting point (and, in fact, at every point) since your vertical direction takes two different values there. The Möbius strip has only *one side*, and hence is nonorientable.

For an orientable surface Σ which has a boundary curve C, pick a unit normal vector $\mathbf{n}$ such that if you walked along C with your head pointing in the direction of $\mathbf{n}$, then the surface would be on your left. We say in this situation that $\mathbf{n}$ is a *positive unit normal vector* and that C is traversed $\mathbf{n}$-*positively*. We can now state Stokes' Theorem:

Theorem 16.8.4: Stoke's Theorem

Let Σ be an orientable surface in $\mathbb{R}^3$ whose boundary is a simple closed curve C, and let $\mathbf{f}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$ be a smooth vector field defined on some subset of $\mathbb{R}^3$ that contains Σ. Then

$$\oint_C \mathbf{f} \cdot d\mathbf{r} = \iint_\Sigma (\operatorname{curl} \mathbf{f}) \cdot \mathbf{n}\, d\sigma, \tag{16.8.12}$$

where

$$\operatorname{curl} \mathbf{f} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}, \tag{16.8.13}$$

$\mathbf{n}$ is a positive unit normal vector over Σ, and C is traversed $\mathbf{n}$-positively.

Proof: As the general case is beyond the scope of this text, we will prove the theorem only for the special case where Σ is the graph of $z = z(x,y)$ for some smooth real-valued function $z(x,y)$, with (x,y) varying over a region D in $\mathbb{R}^2$.

Projecting Σ onto the xy-plane, we see that the closed curve C (the boundary curve of Σ) projects onto a closed curve C_D which is the boundary curve of D (see Figure 4.5.4). Assuming that C has a smooth parametrization, its projection C_D in the xy-plane also has a smooth parametrization, say

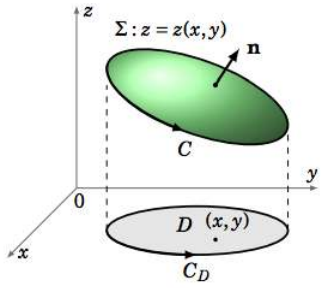


Figure 4.5.4

$$C_D : x = x(t), y = y(t), a \le t \le b,$$

and so C can be parametrized (in $\mathbb{R}^3$) as

$$C : x = x(t),\ y = y(t),\ z = z(x(t), y(t)),\ a \le t \le b,$$

since the curve C is part of the surface $z = z(x, y)$. Now, by the Chain Rule (Theorem 4.4 in Section 4.2), for $z = z(x(t), y(t))$ as a function of t , we know that

$$z'(t) = \frac{\partial z}{\partial x} x'(t) + \frac{\partial z}{\partial y} y'(t),$$

and so

$$\begin{aligned}
\oint_C \mathbf{f} \cdot d\mathbf{r} &= \int_C P(x, y, z)\, dx + Q(x, y, z)\, dy + R(x, y, z)\, dz \\
&= \int_a^b \left(Px'(t) + Qy'(t) + R\left(\frac{\partial z}{\partial x} x'(t) + \frac{\partial z}{\partial y} y'(t) \right) \right) dt \\
&= \int_a^b \left(\left(P + R\frac{\partial z}{\partial x} \right) x'(t) + \left(Q + R\frac{\partial z}{\partial y} \right) y'(t) \right) dt \\
&= \int_{C_D} \tilde{P}(x, y)\, dx + \tilde{Q}(x, y)\, dy,
\end{aligned}$$

where

$$\begin{aligned}
\tilde{P}(x, y) &= P(x, y, z(x, y)) + R(x, y, z(x, y)) \frac{\partial z}{\partial x}(x, y), \text{ and} \\
\tilde{Q}(x, y) &= Q(x, y, z(x, y)) + R(x, y, z(x, y)) \frac{\partial z}{\partial y}(x, y)
\end{aligned}$$

for (x, y) in D. Thus, by Green's Theorem applied to the region D, we have

$$\oint_C \mathbf{f} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial \tilde{Q}}{\partial x} - \frac{\partial \tilde{P}}{\partial y} \right) dA. \tag{16.8.14}$$

Thus,

$$\begin{aligned}
\frac{\partial \tilde{Q}}{\partial x} &= \frac{\partial}{\partial x} \left(Q(x, y, z(x, y)) + R(x, y, z(x, y)) \frac{\partial z}{\partial y}(x, y) \right), \text{ so by the Product Rule we get} \\
&= \frac{\partial}{\partial x} (Q(x, y, z(x, y))) + \left(\frac{\partial}{\partial x} R(x, y, z(x, y)) \right) \frac{\partial z}{\partial y}(x, y) + R(x, y, z(x, y)) \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y}(x, y) \right)
\end{aligned}$$

Now, by Equation 16.8.9 in Theorem 4.11, we have

$$\begin{aligned}\frac{\partial}{\partial x}(Q(x,y,z(x,y))) &= \frac{\partial Q}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial Q}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial Q}{\partial z}\frac{\partial z}{\partial x}\\ &= \frac{\partial Q}{\partial x}\cdot 1+\frac{\partial Q}{\partial y}\cdot 0+\frac{\partial Q}{\partial z}\frac{\partial z}{\partial x}\\ &= \frac{\partial Q}{\partial x}+\frac{\partial Q}{\partial z}\frac{\partial z}{\partial x}.\end{aligned}$$

Similarly,

$$\frac{\partial}{\partial x}(R(x,y,z(x,y))) = \frac{\partial R}{\partial x}+\frac{\partial R}{\partial z}\frac{\partial z}{\partial x}.$$

Thus,

$$\begin{aligned}\frac{\partial \tilde{Q}}{\partial x} &= \frac{\partial Q}{\partial x}+\frac{\partial Q}{\partial z}\frac{\partial z}{\partial x}+\left(\frac{\partial R}{\partial x}+\frac{\partial R}{\partial z}\frac{\partial z}{\partial x}\right)\frac{\partial z}{\partial y}+R(x,y,z(x,y))\frac{\partial^2 z}{\partial x\partial y}\\ &= \frac{\partial Q}{\partial x}+\frac{\partial Q}{\partial z}\frac{\partial z}{\partial x}+\frac{\partial R}{\partial x}\frac{\partial z}{\partial y}+\frac{\partial R}{\partial z}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}+R\frac{\partial^2 z}{\partial x\partial y}.\end{aligned}$$

In a similar fashion, we can calculate

$$\frac{\partial \tilde{P}}{\partial y} = \frac{\partial P}{\partial y}+\frac{\partial P}{\partial z}\frac{\partial z}{\partial y}+\frac{\partial R}{\partial y}\frac{\partial z}{\partial x}+\frac{\partial R}{\partial z}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x}+R\frac{\partial^2 z}{\partial y\partial x}.$$

So subtracting gives

$$\frac{\partial \tilde{Q}}{\partial x}-\frac{\partial \tilde{P}}{\partial y} = \left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)\frac{\partial z}{\partial x}+\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)\frac{\partial z}{\partial y}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right) \tag{16.8.15}$$

since $\dfrac{\partial^2 z}{\partial x\partial y} = \dfrac{\partial^2 z}{\partial y\partial x}$ by the smoothness of $z = z(x,y)$. Hence, by Equation 16.8.14,

$$\oint_C \mathbf{f}\cdot d\mathbf{r} = \iint_D \left(-\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\frac{\partial z}{\partial x}-\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\frac{\partial z}{\partial y}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\right)dA \tag{16.8.16}$$

after factoring out a -1 from the terms in the first two products in Equation 16.8.15.

Now, recall from Section 2.3 (see p.76) that the vector $\mathbf{N} = -\dfrac{\partial z}{\partial x}\,\mathbf{i}-\dfrac{\partial z}{\partial y}\mathbf{j}+\mathbf{k}$ is normal to the tangent plane to the surface $z = z(x,y)$ at each point of Σ. Thus,

$$\mathbf{n} = \frac{\mathbf{N}}{\|\mathbf{N}\|} = \frac{-\dfrac{\partial z}{\partial x}\,\mathbf{i}-\dfrac{\partial z}{\partial y}\mathbf{j}+\mathbf{k}}{\sqrt{1+\left(\dfrac{\partial z}{\partial x}\right)^2+\left(\dfrac{\partial z}{\partial y}\right)^2}}$$

is in fact a positive unit normal vector to Σ (see Figure 4.5.4). Hence, using the parametrization $\mathbf{r}(x,y) = x\mathbf{i}+y\mathbf{j}+z(x,y)\mathbf{k}$, for (x,y) in D, of the surface Σ, we have $\dfrac{\partial \mathbf{r}}{\partial x} = \mathbf{i}+\dfrac{\partial z}{\partial x}\mathbf{k}$ and $\dfrac{\partial \mathbf{r}}{\partial y} = \mathbf{j}+\dfrac{\partial z}{\partial y}\mathbf{k}$, and so $\left\|\dfrac{\partial \mathbf{r}}{\partial x}\times\dfrac{\partial \mathbf{r}}{\partial y}\right\| = \sqrt{1+\left(\dfrac{\partial z}{\partial x}\right)^2+\left(\dfrac{\partial z}{\partial y}\right)^2}$. So we see that using Equation 16.8.13 for curl $\mathbf{f}$, we have

$$\iint_{\Sigma} (\text{curl } \mathbf{f}) \cdot \mathbf{n}\, d\sigma = \iint_{D} (\text{curl } \mathbf{f}) \left\| \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} \right\| dA$$

$$= \iint_{D} \left(\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} \right) \cdot \left(-\frac{\partial z}{\partial x} \mathbf{i} - \frac{\partial z}{\partial y} \mathbf{j} + \mathbf{k} \right) dA$$

$$= \iint_{D} \left(- \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \frac{\partial z}{\partial x} - \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \frac{\partial z}{\partial y} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \right) dA,$$

which, upon comparing to Equation 16.8.16, proves the Theorem.

(QED)

Note: The condition in Stokes' Theorem that the surface Σ have a (continuously varying) positive unit normal vector $\mathbf{n}$ and a boundary curve C traversed $\mathbf{n}$-positively can be expressed more precisely as follows: if $\mathbf{r}(t)$ is the position vector for C and $\mathbf{T}(t) = \mathbf{r}'(t)/\|\mathbf{r}'(t)\|$ is the unit tangent vector to C, then the vectors $\mathbf{T}$, $\mathbf{n}$, $\mathbf{T} \times \mathbf{n}$ form a right-handed system.

Also, it should be noted that Stokes' Theorem holds even when the boundary curve C is piecewise smooth.

Example 16.8.3

Verify Stokes' Theorem for $\mathbf{f}(x, y, z) = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$ when Σ is the paraboloid $z = x^2 + y^2$ such that $z \leq 1$ (see Figure 4.5.5).

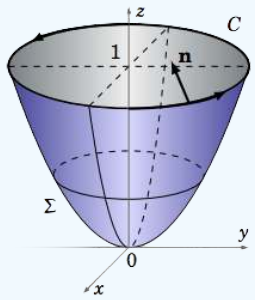


Figure 4.5.5 $z = x^2 + y^2$

Solution:

The positive unit normal vector to the surface $z = z(x, y) = x^2 + y^2$ is

$$\mathbf{n} = \frac{-\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}} = \frac{-2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}}{\sqrt{1 + 4x^2 + 4y^2}},$$

and curl $\mathbf{f}$ = (1−0)**i**+(1−0)**j**+(1−0)**k** = **i**+**j**+**k**, so

$$(\text{curl } \mathbf{f}) \cdot \mathbf{n} = (-2x - 2y + 1)/\sqrt{1 + 4x^2 + 4y^2}\,.$$

Since Σ can be parametrized as $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + (x^2 + y^2)\mathbf{k}$ for (x, y) in the region $D = (x, y) : x^2 + y^2 \leq 1$, then

$$\begin{aligned}
\iint_\Sigma (\operatorname{curl}\mathbf{f})\cdot\mathbf{n}\,d\sigma &= \iint_D (\operatorname{curl}\mathbf{f})\left\|\frac{\partial\mathbf{r}}{\partial x}\times\frac{\partial\mathbf{r}}{\partial y}\right\| dA \\
&= \iint_D \frac{-2x-2y+1}{\sqrt{1+4x^2+4y^2}}\sqrt{1+4x^2+4y^2}\;dA \\
&= \iint_D (-2x-2y+1)\,dA, \text{ so switching to polar coordinates gives} \\
&= \int_0^{2\pi}\int_0^1 (-2r\cos\theta-2r\sin\theta+1)\,r\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 (-2r^2\cos\theta-2r^2\sin\theta+r)\,dr\,d\theta \\
&= \int_0^{2\pi}\left(-\frac{2r^3}{3}\cos\theta-\frac{2r^3}{3}\sin\theta+\frac{r^2}{2}\Big|_{r=0}^{r=1}\right)d\theta \\
&= \int_0^{2\pi}\left(-\frac{2}{3}\cos\theta-\frac{2}{3}\sin\theta+\frac{1}{2}\right)d\theta \\
&= -\frac{2}{3}\sin\theta+\frac{2}{3}\cos\theta+\frac{1}{2}\theta\Big|_0^{2\pi} = \pi.
\end{aligned}$$

The boundary curve C is the unit circle $x^2+y^2=1$ laying in the plane $z=1$ (see Figure 4.5.5), which can be parametrized as $x=\cos t, y=\sin t, z=1$ for $0\le t\le 2\pi$. So

$$\begin{aligned}
\oint_C \mathbf{f}\cdot d\mathbf{r} &= \int_0^{2\pi}((1)(-\sin t)+(\cos t)(\cos t)+(\sin t)(0))\,dt \\
&= \int_0^{2\pi}\left(-\sin t+\frac{1+\cos 2t}{2}\right)dt \quad \left(\text{here we used } \cos^2 t=\frac{1+\cos 2t}{2}\right) \\
&= \cos t+\frac{t}{2}+\frac{\sin 2t}{4}\Big|_0^{2\pi} = \pi.
\end{aligned}$$

So we see that $\oint_C \mathbf{f}\cdot d\mathbf{r} = \iint_\Sigma (\operatorname{curl}\mathbf{f})\cdot\mathbf{n}d\sigma$, as predicted by Stokes' Theorem.

The line integral in the preceding example was far simpler to calculate than the surface integral, but this will not always be the case.

Example 16.8.4

Let Σ be the elliptic paraboloid $z=\frac{x^2}{4}+\frac{y^2}{9}$ for $z\le 1$, and let C be its boundary curve. Calculate $\oint_C \mathbf{f}\cdot d\mathbf{r}$ for $\mathbf{f}(x,y,z)=(9xz+2y)\mathbf{i}+(2x+y^2)\mathbf{j}+(-2y^2+2z)\mathbf{k}$, where C is traversed counterclockwise

Solution

The surface is similar to the one in Example 16.8.3, except now the boundary curve C is the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$ laying in the plane $z=1$. In this case, using Stokes' Theorem is easier than computing the line integral directly. As in Example 4.14, at each point $(x,y,z(x,y))$ on the surface $z=z(x,y)=\frac{x^2}{4}+\frac{y^2}{9}$ the vector

$$\mathbf{n}=\frac{-\frac{\partial z}{\partial x}\mathbf{i}-\frac{\partial z}{\partial y}\mathbf{j}+\mathbf{k}}{\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}}=\frac{-\frac{x}{2}\mathbf{i}-\frac{2y}{9}\mathbf{j}+\mathbf{k}}{\sqrt{1+\frac{x^2}{4}+\frac{4y^2}{9}}},$$

is a positive unit normal vector to Σ. And calculating the curl of $\mathbf{f}$ gives

$$\text{curl } \mathbf{f} = (-4y-0)\mathbf{i}+(9x-0)\mathbf{j}+(2-2)\mathbf{k} = -4y\mathbf{i}+9x\mathbf{j}+0\mathbf{k},$$

so

$$(\text{curl } \mathbf{f})\cdot\mathbf{n} = \frac{(-4y)(-\frac{x}{2})+(9x)(-\frac{2y}{9})+(0)(1)}{\sqrt{1+\frac{x^2}{4}+\frac{4y^2}{9}}} = \frac{2xy-2xy+0}{\sqrt{1+\frac{x^2}{4}+\frac{4y^2}{9}}} = 0,$$

and so by Stokes' Theorem

$$\oint_C \mathbf{f}\cdot d\mathbf{r} = \iint_\Sigma (\text{curl } \mathbf{f})\cdot\mathbf{n}\, d\sigma = \iint_\Sigma 0\, d\sigma = 0.$$

In physical applications, for a simple closed curve C the line integral $\oint_C \mathbf{f}\cdot d\mathbf{r}$ is often called the **circulation** of $\mathbf{f}$ around C. For example, if $\mathbf{E}$ represents the electrostatic field due to a point charge, then it turns out that curl $\mathbf{E}=\mathbf{0}$, which means that the circulation $\oint_C \mathbf{E}\cdot d\mathbf{r} = 0$ by Stokes' Theorem. Vector fields which have zero curl are often called *irrotational* fields.

In fact, the term curl was created by the 19th century Scottish physicist James Clerk Maxwell in his study of electromagnetism, where it is used extensively. In physics, the curl is interpreted as a measure of *circulation density*. This is best seen by using another definition of curl $\mathbf{f}$ which is equivalent to the definition given by Equation 16.8.13 Namely, for a point (x,y,z) in $\mathbb{R}^3$,

$$\mathbf{n}\cdot(\text{curl } \mathbf{f}(x,y,z) = \lim_{S\to 0}\frac{1}{S}\oint_C \mathbf{f}\cdot d\mathbf{r}, \tag{16.8.17}$$

where S is the surface area of a surface Σ containing the point (x,y,z) and with a simple closed boundary curve C and positive unit normal vector $\mathbf{n}$ at (x,y,z). In the limit, think of the curve C shrinking to the point (x,y,z), which causes Σ, the surface it bounds, to have smaller and smaller surface area. That ratio of circulation to surface area in the limit is what makes the curl a rough measure of circulation density (i.e. circulation per unit area).

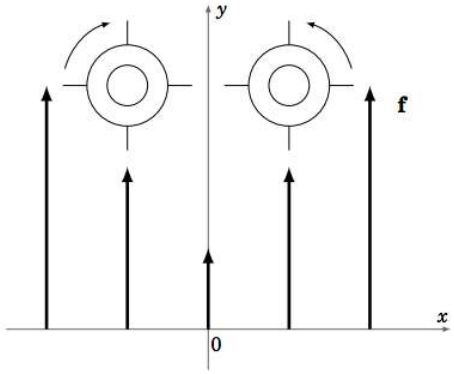


Figure 4.5.6 Curl and rotation

An idea of how the curl of a vector field is related to rotation is shown in Figure 4.5.6. Suppose we have a vector field $\mathbf{f}(x,y,z)$ which is always parallel to the xy-plane at each point (x,y,z) and that the vectors grow larger the further the point (x,y,z) is from the y-axis. For example, $\mathbf{f}(x,y,z) = (1+x^2)\mathbf{j}$. Think of the vector field as representing the flow of water, and imagine dropping two wheels with paddles into that water flow, as in Figure 4.5.6. Since the flow is stronger (i.e. the magnitude of $\mathbf{f}$ is larger) as you move away from the y-axis, then such a wheel would rotate counterclockwise if it were dropped to the right of the y-axis, and it would rotate clockwise if it were dropped to the left of the y-axis. In both cases the curl would be nonzero (curl $\mathbf{f}(x,y,z) = 2x\mathbf{k}$ in our example) and would obey the right-hand rule, that is, curl $\mathbf{f}(x,y,z)$ points in the direction of your thumb as you cup your right hand in the direction of the rotation of the wheel. So the curl points outward (in the positive z-direction) if $x>0$ and points inward (in the negative z-direction) if $x<0$. Notice that if all the vectors had the same direction *and* the same magnitude, then the wheels would not rotate and hence there would be no curl (which is why such fields are called irrotational, meaning no rotation).

Finally, by Stokes' Theorem, we know that if C is a simple closed curve in some solid region S in $\mathbb{R}^3$ and if $\mathbf{f}(x, y, z)$ is a smooth vector field such that curl $\mathbf{f} = 0$ in S, then

$$\oint_C \mathbf{f} \cdot d\mathbf{r} = \iint_\Sigma (\text{curl } \mathbf{f} \cdot \mathbf{n}\, d\sigma = \iint_\Sigma \mathbf{0} \cdot \mathbf{n}\, d\sigma = \iint_\Sigma 0\, d\sigma = 0,$$

where Σ is any orientable surface inside S whose boundary is C (such a surface is sometimes called a *capping surface* for C). So similar to the two-variable case, we have a threedimensional version of a result from Section 4.3, for solid regions in $\mathbb{R}^3$ which are **simply connected** (i.e. regions having no holes):

The following statements are equivalent for a simply connected solid region S in $\mathbb{R}^3$:

a. $\mathbf{f}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ has a smooth potential $F(x, y, z)$ in S
b. $\int_C \mathbf{f} \cdot d\mathbf{r}$ is independent of the path for any curve C in S
c. $\oint_C \mathbf{f} \cdot d\mathbf{r} = 0$ for every simple closed curve C in S
d. $\dfrac{\partial R}{\partial y} = \dfrac{\partial Q}{\partial z}, \dfrac{\partial P}{\partial z} = \dfrac{\partial R}{\partial x}$, and $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$ in S (i.e. curl $\mathbf{f} = \mathbf{0}$ in S)

Part (d) is also a way of saying that the differential form $P\,dx + Q\,dy + R\,dz$ is exact.

Example 16.8.5

Determine if the vector field $\mathbf{f}(x, y, z) = xyz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ has a potential in $\mathbb{R}^3$.

Solution

Since $\mathbb{R}^3$ is simply connected, we just need to check whether curl **f** = **0** throughout $\mathbb{R}^3$, that is,

$$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \text{and } \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

throughout $\mathbb{R}^3$, where $P(x, y, z) = xyz, Q(x, y, z) = xz,$ and $R(x, y, z) = xy$. But we see that

$$\frac{\partial P}{\partial z} = xy, \ \frac{\partial R}{\partial x} = y \quad \Rightarrow \quad \frac{\partial P}{\partial z} \neq \frac{\partial R}{\partial x} \text{ for some } (x, y, z) \text{ in } \mathbb{R}^3.$$

Thus, $f(x, y, z)$ does not have a potential in $\mathbb{R}^3$.

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16.9: The Divergence Theorem

In this final section we will establish some relationships between the gradient, divergence and curl, and we will also introduce a new quantity called the *Laplacian*. We will then show how to write these quantities in cylindrical and spherical coordinates.

Gradient

For a real-valued function $f(x,y,z)$ on $\mathbb{R}^3$, the gradient $\nabla f(x,y,z)$ is a vector-valued function on $\mathbb{R}^3$, that is, its value at a point (x,y,z) is the vector

$$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

in $\mathbb{R}^3$, where each of the partial derivatives is evaluated at the point (x,y,z). So in this way, you can think of the *symbol* ∇ as being "applied" to a real-valued function f to produce a vector ∇f.

It turns out that the divergence and curl can also be expressed in terms of the symbol ∇. This is done by thinking of ∇ as a *vector* in $\mathbb{R}^3$, namely

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}. \tag{16.9.1}$$

Here, the symbols $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ are to be thought of as "partial derivative operators" that will get "applied" to a real-valued function, say $f(x,y,z)$, to produce the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$. For instance, $\frac{\partial}{\partial x}$ "applied" to $f(x,y,z)$ produces $\frac{\partial f}{\partial x}$.

Is ∇ *really* a vector? Strictly speaking, no, since $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ are not actual numbers. But it helps to *think* of ∇ as a vector, especially with the divergence and curl, as we will soon see. The process of "applying" $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$ to a real-valued function $f(x,y,z)$ is normally thought of as *multiplying* the quantities:

$$\left(\frac{\partial}{\partial x}\right)(f) = \frac{\partial f}{\partial x}, \left(\frac{\partial}{\partial y}\right)(f) = \frac{\partial f}{\partial y}, \left(\frac{\partial}{\partial z}\right)(f) = \frac{\partial f}{\partial z}$$

For this reason, ∇ is often referred to as the "del operator", since it "operates" on functions.

Divergence

For example, it is often convenient to write the divergence div $\mathbf{f}$ as $\nabla \cdot \mathbf{f}$, since for a vector field $\mathbf{f}(x,y,z) = f_1(x,y,z)\mathbf{i} + f_2(x,y,z)\mathbf{j} + f_3(x,y,z)\mathbf{k}$, the dot product of $\mathbf{f}$ with ∇ (thought of as a vector) makes sense:

$$\begin{aligned}
\nabla \cdot \mathbf{f} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot (f_1(x,y,z)\mathbf{i} + f_2(x,y,z)\mathbf{j} + f_3(x,y,z)\mathbf{k}) \\
&= \left(\frac{\partial}{\partial x}\right)(f_1) + \left(\frac{\partial}{\partial y}\right)(f_2) + \left(\frac{\partial}{\partial z}\right)(f_3) \\
&= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \\
&= \operatorname{div} \mathbf{f}
\end{aligned}$$

We can also write curl $\mathbf{f}$ in terms of ∇, namely as $\nabla \times \mathbf{f}$, since for a vector field $\mathbf{f}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$, we have:

$$\nabla \times \mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x,y,z) & Q(x,y,z) & R(x,y,z) \end{vmatrix}$$
$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$
$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$
$$= \text{curl } \mathbf{f}$$

For a real-valued function $f(x,y,z)$, the gradient $\nabla f(x,y,z) = \dfrac{\partial f}{\partial x}\mathbf{i} + \dfrac{\partial f}{\partial y}\mathbf{j} + \dfrac{\partial f}{\partial z}\mathbf{k}$ is a vector field, so we can take its divergence:

$$\begin{aligned} \text{div } \nabla f &= \nabla \cdot \nabla f \\ &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) \\ &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \end{aligned}$$

Note that this is a real-valued function, to which we will give a special name:

Definition 4.7: Laplacian

For a real-valued function $f(x,y,z)$, the **Laplacian** of f, denoted by Δf, is given by

$$\Delta f(x,y,z) = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}. \tag{16.9.2}$$

Often the notation $\nabla^2 f$ is used for the Laplacian instead of Δf, using the convention $\nabla^2 = \nabla \cdot \nabla$.

Example 4.17

Let $\mathbf{r}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the position vector field on $\mathbb{R}^3$. Then $\|\mathbf{r}(x,y,z)\|^2 = \mathbf{r} \cdot \mathbf{r} = x^2 + y^2 + z^2$ is a real-valued function. Find

a. the gradient of $\|\mathbf{r}\|^2$
b. the divergence of $\mathbf{r}$
c. the curl of $\mathbf{r}$
d. the Laplacian of $\|\mathbf{r}\|^2$

Solution:

(a) $\nabla \|\mathbf{r}\|^2 = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = 2\mathbf{r}$

(b) $\nabla \cdot \mathbf{r} = \dfrac{\partial}{\partial x}(x) + \dfrac{\partial}{\partial y}(y) + \dfrac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$

(c)

$$\nabla \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = (0-0)\mathbf{i} - (0-0)\mathbf{j} + (0-0)\mathbf{k} = \mathbf{0}$$

(d) $\Delta\|\mathbf{r}\|^2 = \frac{\partial^2}{\partial x^2}(x^2+y^2+z^2) + \frac{\partial^2}{\partial y^2}(x^2+y^2+z^2) + \frac{\partial^2}{\partial z^2}(x^2+y^2+z^2) = 2+2+2 = 6$

Note that we could have calculated $\Delta\|\mathbf{r}\|^2$ another way, using the ∇ notation along with parts (a) and (b):

$$\Delta\|\mathbf{r}\|^2 = \nabla\cdot\nabla\|\mathbf{r}\|^2 = \nabla\cdot 2\mathbf{r} = 2\nabla\cdot\mathbf{r} = 2(3) = 6$$

Notice that in Example 4.17 if we take the curl of the gradient of $\|\mathbf{r}\|^2$ we get

$$\nabla\times(\nabla\|\mathbf{r}\|^2) = \nabla\times 2\mathbf{r} = 2\nabla\times\mathbf{r} = 2\mathbf{0} = \mathbf{0}.$$

The following theorem shows that this will be the case in general:

Theorem 4.15.

For any smooth real-valued function $f(x,y,z)$, $\nabla\times(\nabla f) = \mathbf{0}$.

Proof

We see by the smoothness of f that

$$\nabla\times(\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \tag{16.9.3}$$

$$= \left(\frac{\partial^2 f}{\partial y\partial z} - \frac{\partial^2 f}{\partial z\partial y}\right)\mathbf{i} - \left(\frac{\partial^2 f}{\partial x\partial z} - \frac{\partial^2 f}{\partial z\partial x}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x\partial y} - \frac{\partial^2 f}{\partial y\partial x}\right)\mathbf{k} = \mathbf{0}, \tag{16.9.4}$$

since the mixed partial derivatives in each component are equal.

□

Corollary 4.16

If a vector field $f(x,y,z)$ has a potential, then curl $\mathbf{f} = \mathbf{0}$.

Another way of stating Theorem 4.15 is that gradients are irrotational. Also, notice that in Example 4.17 if we take the divergence of the curl of $\mathbf{r}$ we trivially get

$$\nabla\cdot(\nabla\times\mathbf{r}) = \nabla\cdot\mathbf{0} = 0. \tag{16.9.5}$$

The following theorem shows that this will be the case in general:

Theorem 4.17.

For any smooth vector field $\mathbf{f}(x,y,z)$, $\nabla\cdot(\nabla\times\mathbf{f}) = 0$.

The proof is straightforward and left as an exercise for the reader.

Corollary 4.18

The flux of the curl of a smooth vector field $f(x,y,z)$ through any closed surface is zero.

Proof: Let Σ be a closed surface which bounds a solid S. The flux of $\nabla\times\mathbf{f}$ through Σ is

$$\iint_{\Sigma} (\nabla \times \mathbf{f}) \cdot d\sigma = \iiint_{S} \nabla \cdot (\nabla \times \mathbf{f})\, dV \text{ (by the Divergence Theorem)} \tag{16.9.6}$$

$$= \iiint_{S} 0\, dV \text{ (by Theorem 4.17)} \tag{16.9.7}$$

$$= 0 \tag{16.9.8}$$

(**QED**)

There is another method for proving Theorem 4.15 which can be useful, and is often used in physics. Namely, if the surface integral $\iint_{\Sigma} f(x,y,z)\, d\sigma = 0$ for *all* surfaces Σ in some solid region (usually all of $\mathbb{R}^3$), then we must have $f(x,y,z) = 0$ throughout that region. The proof is not trivial, and physicists do not usually bother to prove it. But the result is true, and can also be applied to double and triple integrals.

For instance, to prove Theorem 4.15, assume that $f(x,y,z)$ is a smooth real-valued function on $\mathbb{R}^3$. Let C be a simple closed curve in $\mathbb{R}^3$ and let Σ be any capping surface for C (i.e. Σ is orientable and its boundary is C). Since ∇f is a vector field, then

$$\iint_{\Sigma} (\nabla \times (\nabla f)) \cdot \mathbf{n}\, d\sigma = \oint_{C} \nabla f \cdot d\mathbf{r} \text{ by Stokes' Theorem, so}$$

$$= 0 \text{ by Corollary 4.13.}$$

Since the choice of Σ was arbitrary, then we must have $(\nabla \times (\nabla f)) \cdot \mathbf{n} = 0$ throughout $\mathbb{R}^3$, where $\mathbf{n}$ is any unit vector. Using $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ in place of $\mathbf{n}$, we see that we must have $\nabla \times (\nabla f) = \mathbf{0}$ in $\mathbb{R}^3$, which completes the proof.

Example 4.18

A system of electric charges has a *charge density* $\rho(x,y,z)$ and produces an electrostatic field $\mathbf{E}(x,y,z)$ at points (x,y,z) in space. *Gauss' Law* states that

$$\iint_{\Sigma} \mathbf{E} \cdot d\sigma = 4\pi \iiint_{S} \rho\, dV$$

for any closed surface Σ which encloses the charges, with S being the solid region enclosed by Σ. Show that $\nabla \cdot \mathbf{E} = 4\pi\rho$. This is one of *Maxwell's Equations*.

Solution

By the Divergence Theorem, we have

$$\iiint_{S} \nabla \cdot \mathbf{E} dV = \iint_{\Sigma} \mathbf{E} \cdot d\sigma$$

$$= 4\pi \iiint_{S} \rho\, dV \text{ by Gauss' Law, so combining the integrals gives}$$

$$\iiint_{S} (\nabla \cdot \mathbf{E} - 4\pi\rho)\, dV = 0 \text{ , so}$$

$$\nabla \cdot \mathbf{E} - 4\pi\rho = 0 \text{ since } \Sigma \text{ and hence } S \text{ was arbitrary, so}$$

$$\nabla \cdot \mathbf{E} = 4\pi\rho.$$

Often (especially in physics) it is convenient to use other coordinate systems when dealing with quantities such as the gradient, divergence, curl and Laplacian. We will present the formulas for these in cylindrical and spherical coordinates.

Recall from Section 1.7 that a point (x,y,z) can be represented in cylindrical coordinates (r,θ,z), where $x = r\cos\theta, y = r\sin\theta, z = z$. At each point

(r, θ, z), let $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$ be unit vectors in the direction of increasing r, θ, z, respectively (see Figure 4.6.1). Then $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$ form an orthonormal set of vectors. Note, by the right-hand rule, that $\mathbf{e}_z \times \mathbf{e}_r = \mathbf{e}_\theta$.

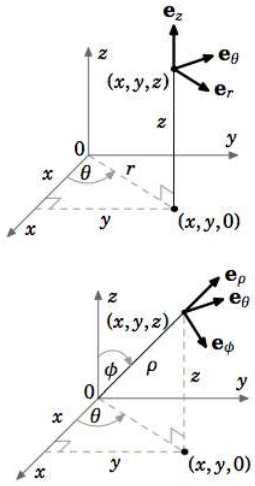


Figure 4.6.1 Orthonormal vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$ in cylindrical coordinates (left) and spherical coordinates (right).

Similarly, a point (x, y, z) can be represented in spherical coordinates (ρ, θ, φ), where $x = \rho \sin\varphi \cos\theta, y = \rho \sin\varphi \sin\theta, z = \rho \cos\varphi$. At each point (ρ, θ, φ), let $\mathbf{e}_\rho, \mathbf{e}_\theta, \mathbf{e}_\varphi$ be unit vectors in the direction of increasing ρ, θ, φ, respectively (see Figure 4.6.2). Then the vectors $\mathbf{e}_\rho, \mathbf{e}_\theta, \mathbf{e}_\varphi$ are orthonormal. By the right-hand rule, we see that $\mathbf{e}_\theta \times \mathbf{e}_\rho = \mathbf{e}_\varphi$.

We can now summarize the expressions for the gradient, divergence, curl and Laplacian in Cartesian, cylindrical and spherical coordinates in the following tables:

Cartesian

(x, y, z): Scalar function F; Vector field $\mathbf{f} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$

- gradient : $\nabla F = \dfrac{\partial F}{\partial x}\mathbf{i} + \dfrac{\partial F}{\partial y}\mathbf{j} + \dfrac{\partial F}{\partial z}\mathbf{k}$
- divergence : $\nabla \cdot \mathbf{f} = \dfrac{\partial f_1}{\partial x} + \dfrac{\partial f_2}{\partial y} + \dfrac{\partial f_3}{\partial z}$
- curl : $\nabla \times \mathbf{f} = \left(\dfrac{\partial f_3}{\partial y} - \dfrac{\partial f_2}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial f_1}{\partial z} - \dfrac{\partial f_3}{\partial x}\right)\mathbf{j} + \left(\dfrac{\partial f_2}{\partial x} - \dfrac{\partial f_1}{\partial y}\right)\mathbf{k}$
- Laplacian : $\Delta F = \dfrac{\partial^2 F}{\partial x^2} + \dfrac{\partial^2 F}{\partial y^2} + \dfrac{\partial^2 F}{\partial z^2}$

Cylindrical

(r, θ, z): Scalar function F; Vector field $\mathbf{f} = f_r\mathbf{e}_r + f_\theta\mathbf{e}_\theta + f_z\mathbf{e}_z$

- gradient : $\nabla F = \dfrac{\partial F}{\partial r}\mathbf{e}_r + \dfrac{1}{r}\dfrac{\partial F}{\partial \theta}\mathbf{e}_\theta + \dfrac{\partial F}{\partial z}\mathbf{e}_z$
- divergence : $\nabla \cdot \mathbf{f} = \dfrac{1}{r}\dfrac{\partial}{\partial r}(rf_r) + \dfrac{1}{r}\dfrac{\partial f_\theta}{\partial \theta} + \dfrac{\partial f_z}{\partial z}$
- curl : $\nabla \times \mathbf{f} = \left(\dfrac{1}{r}\dfrac{\partial f_z}{\partial \theta} - \dfrac{\partial f_\theta}{\partial z}\right)\mathbf{e}_r + \left(\dfrac{\partial f_r}{\partial z} - \dfrac{\partial f_z}{\partial r}\right)\mathbf{e}_\theta + \dfrac{1}{r}\left(\dfrac{\partial}{\partial r}(rf_\theta) - \dfrac{\partial f_r}{\partial \theta}\right)\mathbf{e}_z$
- Laplacian : $\Delta F = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial F}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 F}{\partial \theta^2} + \dfrac{\partial^2 F}{\partial z^2}$

Spherical

(ρ, θ, φ): Scalar function F; Vector field $\mathbf{f} = f_\rho \mathbf{e}_\rho + f_\theta \mathbf{e}_\theta + f_\varphi \mathbf{e}_\varphi$

- gradient : $\nabla F = \dfrac{\partial F}{\partial \rho}\mathbf{e}_\rho + \dfrac{1}{\rho \sin\varphi}\dfrac{\partial F}{\partial \theta}\mathbf{e}_\theta + \dfrac{1}{\rho}\dfrac{\partial F}{\partial \varphi}\mathbf{e}_\varphi$
- divergence : $\nabla \cdot \mathbf{f} = \dfrac{1}{\rho^2}\dfrac{\partial}{\partial \rho}(\rho^2 f_\rho) + \dfrac{1}{\rho}\sin\varphi\dfrac{\partial f_\theta}{\partial \theta} + \dfrac{1}{\rho \sin\varphi}\dfrac{\partial}{\partial \varphi}(\sin\varphi f_\theta)$
- curl : $\nabla \times \mathbf{f} = \dfrac{1}{\rho \sin\varphi}\left(\dfrac{\partial}{\partial \varphi}(\sin\varphi f_\theta) - \dfrac{\partial f_\varphi}{\partial \theta}\right)\mathbf{e}_\rho + \dfrac{1}{\rho}\left(\dfrac{\partial}{\partial \rho}(\rho f_\varphi) - \dfrac{\partial f_\rho}{\partial \varphi}\right)\mathbf{e}_\theta + \left(\dfrac{1}{\rho \sin\varphi}\dfrac{\partial f_\rho}{\partial \theta} - \dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}(\rho f_\theta)\right)\mathbf{e}_\varphi$
- Laplacian : $\Delta F = \dfrac{1}{\rho^2}\dfrac{\partial}{\partial \rho}\left(\rho^2 \dfrac{\partial F}{\partial \rho}\right) + \dfrac{1}{\rho^2 \sin^2\varphi}\dfrac{\partial^2 F}{\partial \theta^2} + \dfrac{1}{\rho^2 \sin\varphi}\dfrac{\partial}{\partial \varphi}\left(\sin\varphi \dfrac{\partial F}{\partial \varphi}\right)$

The derivation of the above formulas for cylindrical and spherical coordinates is straightforward but extremely tedious. The basic idea is to take the Cartesian equivalent of the quantity in question and to substitute into that formula using the appropriate coordinate transformation. As an example, we will derive the formula for the gradient in spherical coordinates.

Goal: Show that the gradient of a real-valued function $F(\rho, \theta, \varphi)$ in spherical coordinates is:

$$\nabla F = \frac{\partial F}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho \sin\varphi}\frac{\partial F}{\partial \theta}\mathbf{e}_\theta + \frac{1}{\rho}\frac{\partial F}{\partial \varphi}\mathbf{e}_\varphi$$

Idea: In the Cartesian gradient formula $\nabla F(x, y, z) = \dfrac{\partial F}{\partial x}\mathbf{i} + \dfrac{\partial F}{\partial y}\mathbf{j} + \dfrac{\partial F}{\partial z}\mathbf{k}$, put the Cartesian basis vectors **i**, **j**, **k** in terms of the spherical coordinate basis vectors $\mathbf{e}_\rho, \mathbf{e}_\theta, \mathbf{e}_\varphi$ and functions of ρ, θ and φ. Then put the partial derivatives $\dfrac{\partial F}{\partial x}, \dfrac{\partial F}{\partial y}, \dfrac{\partial F}{\partial z}$ in terms of $\dfrac{\partial F}{\partial \rho}, \dfrac{\partial F}{\partial \theta}, \dfrac{\partial F}{\partial \varphi}$ and functions of ρ, θ and φ.

Step 1: Get formulas for $\mathbf{e}_\rho, \mathbf{e}_\theta, \mathbf{e}_\varphi$ in terms of **i**, **j**, **k**.

We can see from Figure 4.6.2 that the unit vector $\mathbf{e}_\rho$ in the ρ direction at a general point (ρ, θ, φ) is $\mathbf{e}_\rho = \dfrac{\mathbf{r}}{\|\mathbf{r}\|}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the position vector of the point in Cartesian coordinates. Thus,

$$\mathbf{e}_\rho = \frac{\mathbf{r}}{\|\mathbf{r}\|} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}},$$

so using $x = \rho \sin\varphi \cos\theta, y = \rho \sin\varphi \sin\theta, z = \rho \cos\varphi,$ and $\rho = \sqrt{x^2 + y^2 + z^2}$, we get:

$$\boxed{\mathbf{e}_\rho = \sin\varphi \cos\theta \mathbf{i} + \sin\varphi \sin\theta \mathbf{j} + \cos\varphi \mathbf{k}}$$

Now, since the angle θ is measured in the xy-plane, then the unit vector $\mathbf{e}_\theta$ in the θ direction must be parallel to the xy-plane. That is, $\mathbf{e}_\theta$ is of the form $a\mathbf{i} + b\mathbf{j} + 0\mathbf{k}$. To figure out what a and b are, note that since $\mathbf{e}_\theta \perp \mathbf{e}_\rho$, then in particular $\mathbf{e}_\theta \perp \mathbf{e}_\rho$ when $\mathbf{e}_\rho$ is in the xy-plane. That occurs when the angle φ is $\pi/2$. Putting $\varphi = \pi/2$ into the formula for $\mathbf{e}_\rho$ gives $\mathbf{e}_\rho = \cos\theta \mathbf{i} + \sin\theta \mathbf{j} + 0\mathbf{k}$, and we see that a vector perpendicular to that is $-\sin\theta \mathbf{i} + \cos\theta \mathbf{j} + 0\mathbf{k}$. Since this vector is also a unit vector and points in the (positive) θ direction, it must be $\mathbf{e}_\theta$:

$$\boxed{\mathbf{e}_\theta = -\sin\theta \mathbf{i} + \cos\theta \mathbf{j} + 0\mathbf{k}}$$

Lastly, since $\mathbf{e}_\varphi = \mathbf{e}_\theta \times \mathbf{e}_\rho$, we get:

$$\boxed{\mathbf{e}_\varphi = \cos\varphi \cos\theta \mathbf{i} + \cos\varphi \sin\theta \mathbf{j} - \sin\varphi \mathbf{k}}$$

Step 2: Use the three formulas from Step 1 to solve for **i**, **j**, **k** in terms of $\mathbf{e}_\rho, \mathbf{e}_\theta, \mathbf{e}_\varphi$.

This comes down to solving a system of three equations in three unknowns. There are many ways of doing this, but we will do it by combining the formulas for $\mathbf{e}_\rho$ and $\mathbf{e}_\varphi$ to eliminate k, which will give us an equation involving just **i** and **j**. This, with the formula for $\mathbf{e}_\theta$, will then leave us with a system of two equations in two unknowns (**i** and **j**), which we will use to solve first for **j** then for **i**. Lastly, we will solve for **k**.

First, note that

$$\sin\varphi\mathbf{e}_\rho + \cos\varphi\mathbf{e}_\varphi = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$$

so that

$$\sin\theta(\sin\varphi\mathbf{e}_\rho + \cos\varphi\mathbf{e}_\varphi) + \cos\theta\mathbf{e}_\theta = (\sin^2\theta + \cos^2\theta)\mathbf{j} = \mathbf{j},$$

and so:

$$\boxed{\mathbf{j} = \sin\varphi\sin\theta\mathbf{e}_\rho + \cos\theta\mathbf{e}_\theta + \cos\varphi\sin\theta\mathbf{e}_\varphi}$$

Likewise, we see that

$$\cos\theta(\sin\varphi\mathbf{e}_\rho + \cos\varphi\mathbf{e}_\varphi) - \sin\theta\mathbf{e}_\theta = (\cos^2\theta + \sin^2\theta)\mathbf{i} = \mathbf{i},$$

and so:

$$\boxed{\mathbf{i} = \sin\varphi\cos\theta\mathbf{e}_\rho - \sin\theta\mathbf{e}_\theta + \cos\varphi\cos\theta\mathbf{e}_\varphi}$$

Lastly, we see that:

$$\boxed{\mathbf{k} = \cos\varphi\mathbf{e}_\rho - \sin\varphi\mathbf{e}_\varphi}$$

Step 3: Get formulas for $\frac{\partial F}{\partial \rho}, \frac{\partial F}{\partial \theta}, \frac{\partial F}{\partial \varphi}$ in terms of $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$.

By the Chain Rule, we have

$$\begin{aligned}
\frac{\partial F}{\partial \rho} &= \frac{\partial F}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \rho} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial \rho}, \\
\frac{\partial F}{\partial \theta} &= \frac{\partial F}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial \theta}, \\
\frac{\partial F}{\partial \varphi} &= \frac{\partial F}{\partial x}\frac{\partial x}{\partial \varphi} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \varphi} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial \varphi},
\end{aligned}$$

which yields:

$$\boxed{\begin{aligned}
\frac{\partial F}{\partial \rho} &= \sin\varphi\cos\theta\frac{\partial F}{\partial x} + \sin\varphi\sin\theta\frac{\partial F}{\partial y} + \cos\varphi\frac{\partial F}{\partial z} \\
\frac{\partial F}{\partial \theta} &= -\rho\sin\varphi\sin\theta\frac{\partial F}{\partial x} + \rho\sin\varphi\cos\theta\frac{\partial F}{\partial y} \\
\frac{\partial F}{\partial \varphi} &= \rho\cos\varphi\cos\theta\frac{\partial F}{\partial x} + \rho\cos\varphi\sin\theta\frac{\partial F}{\partial y} - \rho\sin\varphi\frac{\partial F}{\partial z}
\end{aligned}} \tag{16.9.9}$$

Step 4: Use the three formulas from Step 3 to solve for $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$ in terms of $\frac{\partial F}{\partial \rho}, \frac{\partial F}{\partial \theta}, \frac{\partial F}{\partial \varphi}$.

Again, this involves solving a system of three equations in three unknowns. Using a similar process of elimination as in Step 2, we get:

$$\boxed{\begin{aligned}
\frac{\partial F}{\partial x} &= \frac{1}{\rho\sin\varphi}\left(\rho\sin^2\varphi\cos\theta\frac{\partial F}{\partial \rho} - \sin\theta\frac{\partial F}{\partial \theta} + \sin\varphi\cos\varphi\cos\theta\frac{\partial F}{\partial \varphi}\right) \\
\frac{\partial F}{\partial y} &= \frac{1}{\rho\sin\varphi}\left(\rho\sin^2\varphi\sin\theta\frac{\partial F}{\partial \rho} + \cos\theta\frac{\partial F}{\partial \theta} + \sin\varphi\cos\varphi\sin\theta\frac{\partial F}{\partial \varphi}\right) \\
\frac{\partial F}{\partial z} &= \frac{1}{\rho}\left(\rho\cos\varphi\frac{\partial F}{\partial \rho} - \sin\varphi\frac{\partial F}{\partial \varphi}\right)
\end{aligned}} \tag{16.9.10}$$

Step 5: Substitute the formulas for **i**, **j**, **k** from Step 2 and the formulas for $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$ from Step 4 into the Cartesian gradient formula $\nabla F(x,y,z) = \frac{\partial F}{\partial x}\mathbf{i} + \frac{\partial F}{\partial y}\mathbf{j} + \frac{\partial F}{\partial z}\mathbf{k}$.

Doing this last step is perhaps the most tedious, since it involves simplifying $3\times 3 + 3\times 3 + 2\times 2 = 22$ terms! Namely,

$$\begin{aligned}\nabla F &= \frac{1}{\rho\sin\varphi}\left(\rho\sin^2\varphi\cos\theta\frac{\partial F}{\partial \rho} - \sin\theta\frac{\partial F}{\partial \theta} + \sin\varphi\cos\varphi\cos\theta\frac{\partial F}{\partial \varphi}\right)(\sin\varphi\cos\theta\mathbf{e}_\rho - \sin\theta\mathbf{e}_\theta + \cos\varphi\cos\theta\mathbf{e}_\varphi)\\ &+ \frac{1}{\rho\sin\varphi}\left(\rho\sin^2\varphi\sin\theta\frac{\partial F}{\partial \rho} + \cos\theta\frac{\partial F}{\partial \theta} + \sin\varphi\cos\varphi\sin\theta\frac{\partial F}{\partial \varphi}\right)(\sin\varphi\sin\theta\mathbf{e}_\rho + \cos\theta\mathbf{e}_\theta + \cos\varphi\sin\theta\mathbf{e}_\varphi)\\ &+ \frac{1}{\rho}\left(\rho\cos\varphi\frac{\partial F}{\partial \rho} - \sin\varphi\frac{\partial F}{\partial \varphi}\right)(\cos\varphi\mathbf{e}_\rho - \sin\varphi\mathbf{e}_\varphi),\end{aligned}$$

which we see has 8 terms involving $\mathbf{e}_\rho$, 6 terms involving $\mathbf{e}_\theta$, and 8 terms involving $\mathbf{e}_\varphi$. But the algebra is straightforward and yields the desired result:

$$\nabla F = \frac{\partial F}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho\sin\varphi}\frac{\partial F}{\partial \theta}\mathbf{e}_\theta + \frac{1}{\rho}\frac{\partial F}{\partial \varphi}\mathbf{e}_\varphi \quad \checkmark \tag{16.9.11}$$

Example 4.19

In Example 4.17 we showed that $\nabla\|\mathbf{r}\|^2 = 2\mathbf{r}$ and $\Delta\|\mathbf{r}\|^2 = 6,$ where $\mathbf{r}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in Cartesian coordinates. Verify that we get the same answers if we switch to spherical coordinates.

Solution

Since $\|\mathbf{r}\|^2 = x^2 + y^2 + z^2 = \rho^2$ in spherical coordinates, let $F(\rho,\theta,\varphi) = \rho^2$ (so that $F(\rho,\theta,\varphi) = \|\mathbf{r}\|^2$). The gradient of F in spherical coordinates is

$$\begin{aligned}\nabla F &= \frac{\partial F}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho\sin\varphi}\frac{\partial F}{\partial \theta}\mathbf{e}_\theta + \frac{1}{\rho}\frac{\partial F}{\partial \varphi}\mathbf{e}_\varphi\\ &= 2\rho\mathbf{e}_\rho + \frac{1}{\rho\sin\varphi}(0)\mathbf{e}_\theta + \frac{1}{\rho}(0)\mathbf{e}_\varphi\\ &= 2\rho\mathbf{e}_\rho = 2\rho\frac{\mathbf{r}}{\|\mathbf{r}\|},\ \text{as we showed earlier, so}\\ &= 2\rho\frac{\mathbf{r}}{\rho} = 2\mathbf{r},\ \text{as expected. And the Laplacian is}\\ \Delta F &= \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial F}{\partial \rho}\right) + \frac{1}{\rho^2\sin^2\varphi}\frac{\partial^2 F}{\partial \theta^2} + \frac{1}{\rho^2\sin\varphi}\frac{\partial}{\partial \varphi}\left(\sin\varphi\frac{\partial F}{\partial \varphi}\right)\\ &= \frac{1}{\rho^2}\frac{\partial}{\partial \rho}(\rho^2 2\rho) + \frac{1}{\rho^2\sin\varphi}(0) + \frac{1}{\rho^2\sin\varphi}\frac{\partial}{\partial \varphi}(\sin\varphi(0))\\ &= \frac{1}{\rho^2}\frac{\partial}{\partial \rho}(2\rho^3) + 0 + 0\\ &= \frac{1}{\rho^2}(6\rho^2) = 6,\ \text{as expected.}\end{aligned}$$

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CHAPTER OVERVIEW

17: Second-Order Differential Equations

A general Calculus Textmap organized around the textbook

Calculus: Early Transcendentals

by James Stewart

I II III IV V VI VII VIII IX X XI XII XIII XIV XV XVI XVII

This Textmap is currently under construction... please be patient with us.

Template:HideTOC

Topic hierarchy

17.1: Second-Order Linear Equations
17.2: Nonhomogeneous Linear Equations
17.3: Applications of Second-Order Differential Equations
17.4: Series Solutions of Differential Equations

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17.1: Second-Order Linear Equations

Learning Objectives

- Recognize homogeneous and nonhomogeneous linear differential equations.
- Determine the characteristic equation of a homogeneous linear equation.
- Use the roots of the characteristic equation to find the solution to a homogeneous linear equation.
- Solve initial-value and boundary-value problems involving linear differential equations.

When working with differential equations, usually the goal is to find a solution. In other words, we want to find a function (or functions) that satisfies the differential equation. The technique we use to find these solutions varies, depending on the form of the differential equation with which we are working. Second-order differential equations have several important characteristics that can help us determine which solution method to use. In this section, we examine some of these characteristics and the associated terminology.

Homogeneous Linear Equations

Consider the second-order differential equation

$$xy'' + 2x^2y' + 5x^3y = 0.$$

Notice that y and its derivatives appear in a relatively simple form. They are multiplied by functions of x, but are not raised to any powers themselves, nor are they multiplied together. As discussed in previously, first-order equations with similar characteristics are said to be linear. The same is true of second-order equations. Also note that all the terms in this differential equation involve either y or one of its derivatives. There are no terms involving only functions of x. Equations like this, in which every term contains y or one of its derivatives, are called **homogeneous**.

Not all differential equations are homogeneous. Consider the differential equation

$$xy'' + 2x^2y' + 5x^3y = x^2.$$

The x^2 term on the right side of the equal sign does not contain y or any of its derivatives. Therefore, this differential equation is *nonhomogeneous*.

Definition: Homogeneous and Nonhomogeneous Linear Equations

A second-order differential equation is linear if it can be written in the form

$$a_2(x)y'' + a)1(x)y' + a_0(x)y = r(x), \tag{17.1.1}$$

where $a_2(x), a_1(x), a_0(x)$, and $r(x)$ are real-valued functions and $a_2(x)$ is not identically zero. If $r(x) \equiv 0$—in other words, if $r(x) = 0$ for every value of x—the equation is said to be a *homogeneous linear equation*. If $r(x) \neq 0$ for some value of x, the equation is said to be a *nonhomogeneous linear equation*.

In linear differential equations, y and its derivatives can be raised only to the first power and they may not be multiplied by one another. Terms involving y^2 or $\sqrt{y'}$ make the equation nonlinear. Functions of y and its derivatives, such as $\sin y$ or $e^{y'}$, are similarly prohibited in linear differential equations.

Note that equations may not always be given in standard form (the form shown in the definition). It can be helpful to rewrite them in that form to decide whether they are linear, or whether a linear equation is homogeneous.

Example 17.1.1: Classifying Second-Order Equations

Classify each of the following equations as linear or nonlinear. If the equation is linear, determine further whether it is homogeneous or nonhomogeneous.

a. $y'' + 3x^4y' + x^2y^2 = x^3$
b. $(\sin x)y'' + (\cos x)y' + 3y = 0$
c. $4t^2x'' + 3txx' + 4x = 0$

d. $5y'' + y = 4x^5$
e. $(\cos x)y'' - \sin y' + (\sin x)y - \cos x = 0$
f. $8ty'' - 6t^2y' + 4ty - 3t^2 = 0$
g. $\sin(x^2)y'' - (\cos x)y' + x^2y = y' - 3$
h. $y'' + 5xy' - 3y = \cos y$

Solution

a. This equation is nonlinear because of the y^2 term.
b. This equation is linear. There is no term involving a power or function of y, and the coefficients are all functions of x.The equation is already written in standard form, and $r(x)$ is identically zero, so the equation is homogeneous.
c. This equation is nonlinear. Note that, in this case, x is the dependent variable and $t$is the independent variable. The second term involves the product of x and x', so the equation is nonlinear.
d. This equation is linear. Since $r(x) = 4x^5$, the equation is nonhomogeneous.
e. This equation is nonlinear, because of the $\sin y'$ term.
f. This equation is linear. Rewriting it in standard form gives

$$8t^2y'' - 6t^2y' + 4ty = 3t^2.$$

With the equation in standard form, we can see that $r(t) = 3t^2$, so the equation is nonhomogeneous.
g. This equation looks like it's linear, but we should rewrite it in standard form to be sure. We get

$$\sin(x^2)y'' - (\cos x + 1)y' + x^2y = -3.$$

This equation is, indeed, linear. With $r(x) = -3$, it is nonhomogeneous.
h. This equation is nonlinear because of the $\cos y$ term.

? Exercise 17.1.1

Classify each of the following equations as linear or nonlinear. If the equation is linear, determine further whether it is homogeneous or nonhomogeneous.

a. $(y'')2 - y' + 8x^3y = 0$
b. $(\sin t)y'' + \cos t - 3ty' = 0$

Hint

Write the equation in standard form (Equation 17.1.1) if necessary. Check for powers or functions of y and its derivatives.

Answer a

Nonlinear Linear

Answer b

nonhomogeneous

Later in this section, we will see some techniques for solving specific types of differential equations. Before we get to that, however, let's get a feel for how solutions to linear differential equations behave. In many cases, solving differential equations depends on making educated guesses about what the solution might look like. Knowing how various types of solutions behave will be helpful.

✓ Example 17.1.2: Verifying a Solution

Consider the linear, homogeneous differential equation

$$x^2y'' - xy' - 3y = 0.$$

Looking at this equation, notice that the coefficient functions are polynomials, with higher powers of x associated with higher-order derivatives of y. Show that $y = x^3$ is a solution to this differential equation.

Solution

Let $y = x^3$. Then $y' = 3x^2$ and $y'' = 6x$. Substituting into the differential equation, we see that

$$\begin{aligned} x^2y'' - xy' - 3y &= x^2(6x) - x(3x^2) - 3(x^3) \\ &= 6x^3 - 3x^3 - 3x^3 \\ &= 0. \end{aligned}$$

? Exercise 17.1.2

Show that $y = 2x^2$ is a solution to the differential equation

$$\frac{1}{2}x^2y'' - xy' + y = 0. \tag{17.1.2}$$

Hint

Calculate the derivatives and substitute them into the differential equation.

Answer

This requires calculating y' and y''.

$$y' = \frac{dy}{dx} = 4x$$

and

$$y'' = \frac{dy'}{dx} = 4$$

Inserting these derivatives along with $y = 2x^2$ into Equation 17.1.2

$$\begin{aligned} \frac{1}{2}x^2y'' - xy' + y &\stackrel{?}{=} 0 \\ \frac{1}{2}x^2(4) - x(4x) + 2x^2 &\stackrel{?}{=} 0 \\ 2x^2 - 4x^2 + 2x^2 &\stackrel{\checkmark}{=} 0 \end{aligned}$$

Yes, this is a solution to the differential equation in Equation 17.1.2

Although simply finding any solution to a differential equation is important, mathematicians and engineers often want to go beyond finding *one* solution to a differential equation to finding *all* solutions to a differential equation. In other words, we want to find a general solution**.** Just as with first-order differential equations, a general solution (or family of solutions) gives the entire set of solutions to a differential equation. An important difference between first-order and second-order equations is that, with second-order equations, we typically need to find two different solutions to the equation to find the general solution. If we find two solutions, then any linear combination of these solutions is also a solution. We state this fact as the following theorem.

Theorem: SUPERPOSITION PRINCIPLE

If $y_1(x)$ and $y_2(x)$ are solutions to a linear homogeneous differential equation, then the function

$$y(x) = c_1y_1(x) + c_2y_2(x), \tag{17.1.3}$$

where c_1 and c_2 are constants, is also a solution.

The proof of this superposition principle theorem is left as an exercise.

Example 17.1.3: Verifying the Superposition Principle

Consider the differential equation

$$y'' - 4y' - 5y = 0.$$

Given that e^{-x} and e^{5x} are solutions to this differential equation, show that $4e^{-x} + e^{5x}$ is a solution.

Solution

Although this can be done through a simple application of the Superposition principle (Equation 17.1.3), but we can also confirm it is a solution via an approach like in Example 17.1.2 We have

$$\begin{aligned} y(x) &= 4e^{-x} + e^{5x} \\ y'(x) &= -4e^{-x} + 5e^{5x} \\ y''(x) &= 4e^{-x} + 25e^{5x}. \end{aligned}$$

Then

$$\begin{aligned} y'' - 4y' - 5y &\stackrel{?}{=} (4e^{-x} + 25e^{5x}) - 4(-4e^{-x} + 5e^{5x}) - 5(4e^{-x} + e^{5x}) \\ &\stackrel{?}{=} 4e^{-x} + 25e^{5x} + 16e^{-x} - 20e^{5x} - 20e^{-x} - 5e^{5x} \\ &\stackrel{\checkmark}{=} 0. \end{aligned}$$

Thus, $y(x) = 4e^{-x} + e^{5x}$ is a solution.

Exercise 17.1.3

Consider the differential equation

$$y'' + 5y' + 6y = 0.$$

Given that e^{-2x} and e^{-3x} are solutions to this differential equation, show that $3e^{-2x} + 6e^{-3x}$ is a solution.

Hint

Differentiate the function and substitute into the differential equation.

Answer

Although this can be a simple application of the Superposition principle (Equation 17.1.3), we can also set through it like in Example 17.1.2 We have

$$\begin{aligned} y(x) &= 3e^{-2x} + 6e^{-3x} \\ y'(x) &= -6e^{-2x} - 18e^{-3x} \\ y''(x) &= 12e^{-2x} + 54e^{3x}. \end{aligned}$$

Then

$$\begin{aligned} y'' + 5y' + 6y &= (12e^{-2x} + 54e^{3x}) + 5(-6e^{-2x} - 18e^{-3x}) + 6(3e^{-2x} + 6e^{3x}) \\ &\stackrel{?}{=} \cancel{12e^{-2x}} + \cancel{54e^{3x}} - \cancel{30e^{-2x}} - \cancel{90e^{3x}} + \cancel{18e^{-2x}} + \cancel{36e^{3x}} \\ &\stackrel{\checkmark}{=} 0. \end{aligned}$$

Thus, $3e^{-2x} + 6e^{-3x}$ is a solution to the differential equation

Unfortunately, to find the general solution to a second-order differential equation, it is not enough to find any two solutions and then combine them. Consider the differential equation

$$x'' + 7x' + 12x = 0.$$

Both e^{-3t} and $2e^{-3t}$ are solutions (you can check this). However,

$$x(t) = c_1 e^{-3t} + c_2(2e^{-3t})$$

is *not* the general solution. This expression does not account for all solutions to the differential equation. In particular, it fails to account for the function e^{-4t}, which is also a solution to the differential equation. It turns out that to find the general solution to a second-order differential equation, we must find two *linearly independent solutions*. We define that terminology here.

Definition: Linearly Dependent functions

A set of functions $f_1(x),\ f_2(x), \ldots, f_n(x)$ is said to be *linearly dependent* if there are constants $c_1,\ c_2, \ldots c_n$, not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for all x over the interval of interest. A set of functions that is not linearly dependent is said to be *linearly independent*.

In this chapter, we usually test sets of only two functions for linear independence, which allows us to simplify this definition. From a practical perspective, we see that two functions are linearly dependent if either one of them is identically zero or if they are constant multiples of each other.

First we show that if the functions meet the conditions given previously, then they are linearly dependent. If one of the functions is identically zero—say, $f_2(x) \equiv 0$—then choose $c_1 = 0$ and $c_2 = 1$, and the condition for linear dependence is satisfied. If, on the other hand, neither $f_1(x)$ nor $f_2(x)$ is identically zero, but $f_1(x) = Cf_2(x)$ for some constant C, then choose $c_1 = C$ and $c_2 = -1$, and again, the condition is satisfied.

Next, we show that if two functions are linearly dependent, then either one is identically zero or they are constant multiples of one another. Assume $f_1(x)$ and $f_2(x)$ are linearly independent. Then, there are constants, c_1 and c_2, not both zero, such that

$$c_1 f_1(x) + c_2 f_2(x) = 0$$

for all x over the interval of interest. Then,

$$c_1 f_1(x) = -c_2 f_2(x).$$

Now, since we stated that c_1 and c_2 can't both be zero, assume $c_2 \neq 0$. Then, there are two cases: either $c_1 = 0$ or $c_1 \neq 0$. If $c_1 = 0$, then

$$0 = -c_2 f_2(x)$$
$$0 = f_2(x),$$

so one of the functions is identically zero. Now suppose $c_1 \neq 0$. Then,

$$f_1(x) = \left(-\frac{c_2}{c_1}\right) f_2(x)$$

and we see that the functions are constant multiples of one another.

Theorem: Linear Dependence of Two Functions

Two functions, $f_1(x)$ and $f_2(x)$, are said to be linearly dependent if either one of them is identically zero or if $f_1(x) = Cf_2(x)$ for some constant C and for all x over the interval of interest. Functions that are not linearly dependent are said to be *linearly independent*.

Example 17.1.4: Testing for Linear Dependence

Determine whether the following pairs of functions are linearly dependent or linearly independent.

a. $f_1(x) = x^2$ and $f_2(x) = 5x^2$
b. $f_1(x) = \sin x$ and $f_2(x) = \cos x$
c. $f_1(x) = e^{3x}$ and $f_2(x) = e^{-3x}$

d. $f_1(x) = 3x$ and $f_2(x) = 3x + 1$

Solution

a. $f_2(x) = 5f_1(x)$, so the functions are linearly dependent.
b. There is no constant C such that $f_1(x) = Cf_2(x)$, so the functions are linearly independent.
c. There is no constant C such that $f_1(x) = Cf_2(x)$, so the functions are linearly independent. Don't get confused by the fact that the exponents are constant multiples of each other. With two exponential functions, unless the exponents are equal, the functions are linearly independent.
d. There is no constant C such that $f_1(x) = Cf_2(x)$, so the functions are linearly independent.

Exercise 17.1.4

Determine whether the following pairs of functions are linearly dependent or linearly independent: $f_1(x) = e^x$ and $f_2(x) = 3e^{3x}$.

Hint

Are the functions constant multiples of one another?

Answer

Linearly independent

If we are able to find two linearly independent solutions to a second-order differential equation, then we can combine them to find the general solution. This result is formally stated in the following theorem.

Theorem: General Solution to a Homogeneous Equation

If $y_1(x)$ and $y_2(x)$ are linearly independent solutions to a second-order, linear, homogeneous differential equation, then the general solution is given by

$$y(x) = c_1y_1(x) + c_2y_2(x),$$

where c_1 and c_2 are constants.

When we say a family of functions is the general solution to a differential equation, we mean that

1. every expression of that form is a solution and
2. every solution to the differential equation can be written in that form, which makes this theorem extremely powerful.

If we can find two linearly independent solutions to a second order differential equation, we have, effectively, found *all* solutions to the second order differential equation—quite a remarkable statement. The proof of this theorem is beyond the scope of this text.

Example 17.1.5: Writing the General Solution

If $y_1(t) = e^{3t}$ and $y_2(t) = e^{-3t}$ are solutions to $y'' - 9y = 0$, what is the general solution?

Solution

Note that y_1 and y_2 are not constant multiples of one another, so they are linearly independent. Then, the general solution to the differential equation is

$y(t) = c_1e^{3t} + c_2e^{-3t}$.

Exercise 17.1.5

If $y_1(x) = e^{3x}$ and $y_2(x) = xe^{3x}$ are solutions to $y'' - 6y' + 9y = 0$, what is the general solution?

Hint

Check for linear independence first.

Answer

$y(x) = c_1 e^{3x} + c_2 x e^{3x}$

Second-Order Equations with Constant Coefficients

Now that we have a better feel for linear differential equations, we are going to concentrate on solving second-order equations of the form

$$ay'' + by' + cy = 0, \tag{17.2}$$

where a, b, and c are constants.

Since all the coefficients are constants, the solutions are probably going to be functions with derivatives that are constant multiples of themselves. We need all the terms to cancel out, and if taking a derivative introduces a term that is not a constant multiple of the original function, it is difficult to see how that term cancels out. Exponential functions have derivatives that are constant multiples of the original function, so let's see what happens when we try a solution of the form $y(x) = e^{\lambda x}$, where λ (the lowercase Greek letter lambda) is some constant.

If $y(x) = e^{\lambda x}$, then $y'(x) = \lambda e^{\lambda x}$ and $y'' = \lambda^2 e^{\lambda x}$. Substituting these expressions into Equation 17.1.1, we get

$$\begin{aligned} ay'' + by' + cy &= a(\lambda^2 e^{\lambda x}) + b(\lambda e^{\lambda x}) + ce^{\lambda x} \\ &= e^{\lambda x}(a\lambda^2 + b\lambda + c). \end{aligned}$$

Since $e^{\lambda x}$ is never zero, this expression can be equal to zero for all x only if

$$a\lambda^2 + b\lambda + c = 0.$$

We call this the *characteristic equation* of the differential equation.

Definition: characteristic equation

The *characteristic equation* of the second order differential equation $ay'' + by' + cy = 0$ is

$$a\lambda^2 + b\lambda + c = 0.$$

The characteristic equation is very important in finding solutions to differential equations of this form. We can solve the characteristic equation either by factoring or by using the quadratic formula

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This gives three cases. The characteristic equation has

1. distinct real roots;
2. a single, repeated real root; or
3. complex conjugate roots.

We consider each of these cases separately.

Case 1: Distinct Real Roots

If the characteristic equation has distinct real roots λ_1 and λ_2, then $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ are linearly independent solutions to Example 17.1.1, and the general solution is given by

$$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x},$$

where c_1 and c_2 are constants.

For example, the differential equation $y'' + 9y' + 14y = 0$ has the associated characteristic equation $\lambda^2 + 9\lambda + 14 = 0$. This factors into $(\lambda + 2)(\lambda + 7) = 0$, which has roots $\lambda_1 = -2$ and $\lambda_2 = -7$. Therefore, the general solution to this differential equation is

$$y(x) = c_1 e^{-2x} + c_2 e^{-7x}.$$

Case 2: Single Repeated Real Root

Things are a little more complicated if the characteristic equation has a repeated real root, λ. In this case, we know $e^{\lambda x}$ is a solution to Equation 17.1.1, but it is only one solution and we need two linearly independent solutions to determine the general solution. We might be tempted to try a function of the form $ke^{\lambda x}$, where k is some constant, but it would not be linearly independent of $e^{\lambda x}$. Therefore, let's try $xe^{\lambda x}$ as the second solution. First, note that by the quadratic formula,

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

But, λ is a repeated root, so the discriminate $(b^2 - 4ac)$ is zero and $\lambda = \frac{-b}{2a}$. Thus, if $y = xe^{\lambda x}$, we have

$$y' = e^{\lambda x} + \lambda x e^{\lambda x}$$
$$y'' = 2\lambda e^{\lambda x} + \lambda^2 x e^{\lambda x}.$$

Substituting both expressions into Equation 17.1.1, we see that

$$\begin{aligned} ay'' + by' + cy &= a(2\lambda e^{\lambda x} + \lambda^2 x e^{\lambda x}) + b(e^{\lambda x} + \lambda x e^{\lambda x}) + cxe^{\lambda x} \\ &= xe^{\lambda x}(a\lambda^2 + b\lambda + c) + e^{\lambda x}(2a\lambda + b) \\ &= xe^{\lambda x}(0) + e^{\lambda x}(2a(-b2a) + b) \\ &= 0 + e^{\lambda x}(0) \\ &\overset{\checkmark}{=} 0. \end{aligned}$$

This shows that $xe^{\lambda x}$ is a solution to Equation 17.1.1. Since $e^{\lambda x}$ and $xe^{\lambda x}$ are linearly independent, when the characteristic equation has a repeated root λ, the general solution to Equation 17.1.1 is given by

$$y(x) = c_1 e^{\lambda x} + c_2 x e^{\lambda x},$$

where c_1 and c_2 are constants.

For example, the differential equation $y'' + 12y' + 36y = 0$ has the associated characteristic equation

$$\lambda^2 + 12\lambda + 36 = 0.$$

This factors into $(\lambda + 6)^2 = 0,$ which has a repeated root $\lambda = -6$. Therefore, the general solution to this differential equation is

$$y(x) = c_1 e^{-6x} + c_2 x e^{-6x}.$$

Case 3: Complex Conjugate Roots

The third case we must consider is when $b^2 - 4ac < 0.$ In this case, when we apply the quadratic formula, we are taking the square root of a negative number. We must use the imaginary number $i = \sqrt{-1}$ to find the roots, which take the form $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i.$ The **complex number** $\alpha + \beta i$ is called the *conjugate* of $\alpha - \beta i$. Thus, we see that when the discriminate $b^2 - 4ac$ is negative, the roots of our characteristic equation are always **complex conjugates**.

This creates a little bit of a problem for us. If we follow the same process we used for distinct real roots—using the roots of the characteristic equation as the coefficients in the exponents of exponential functions—we get the functions $e^{(\alpha+\beta i)x}$ and $e^{(\alpha-\beta i)x}$ as our solutions. However, there are problems with this approach. First, these functions take on complex (imaginary) values, and a complete discussion of such functions is beyond the scope of this text. Second, even if we were comfortable with complex-value functions, in this course we do not address the idea of a derivative for such functions. So, if possible, we'd like to find two linearly independent *real-value* solutions to the differential equation. For purposes of this development, we are going to manipulate and differentiate the functions $e^{(\alpha+\beta i)x}$ and $e^{(\alpha-\beta i)x}$ as if they were real-value functions. For these particular functions, this approach is valid mathematically, but be aware that there are other instances when complex-value functions do not follow the same rules as real-value functions. Those of you interested in a more in-depth discussion of complex-value functions should consult a complex analysis text.

Based on the roots $\alpha \pm \beta i$ of the characteristic equation, the functions $e^{(\alpha+\beta i)x}$ and $e^{(\alpha-\beta i)x}$ are linearly independent solutions to the differential equation and the general solution is given by

$$y(x) = c_1 e^{(\alpha+\beta i)x} + c_2 e^{(\alpha-\beta i)x}.$$

Using some smart choices for c_1 and c_2, and a little bit of algebraic manipulation, we can find two linearly independent, real-value solutions to Equation 17.1.1 and express our general solution in those terms.

We encountered exponential functions with complex exponents earlier. One of the key tools we used to express these exponential functions in terms of sines and cosines was Euler's formula, which tells us that

$$\underbrace{e^{i\theta} = \cos\theta + i\sin\theta}_{\text{Euler's formula}} \tag{17.1.4}$$

for all real numbers θ.

Going back to the general solution, we have

$$\begin{aligned} y(x) &= c_1 e^{(\alpha+\beta i)x} + c_2 e^{(\alpha-\beta i)x} \\ &= c_1 e^{\alpha x} e^{\beta i x} + c_2 e^{\alpha x} e^{-\beta i x} \\ &= e^{\alpha x}(c_1 e^{\beta i x} + c_2 e^{-\beta i x}). \end{aligned}$$

Applying Euler's formula (Equation 17.1.4) together with the identities $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$, we get

$$\begin{aligned} y(x) &= e^{\alpha x}[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos(-\beta x) + i\sin(-\beta x))] \\ &= e^{\alpha x}[(c_1 + c_2)\cos\beta x + (c_1 - c_2)i\sin\beta x]. \end{aligned} \tag{17.1.5}$$

Now, if we choose $c_1 = c_2 = \frac{1}{2}$, the second term is zero and we get

$$y(x) = e^{\alpha x}\cos\beta x$$

as a real-value solution to Equation 17.1.1. Similarly, if we choose $c_1 = -\frac{i}{2}$ and $c_2 = \frac{i}{2}$, the first term of Equation 17.1.5 is zero and we get

$$y(x) = e^{\alpha x}\sin\beta x$$

as a second, linearly independent, real-value solution to Equation 17.1.1.

Based on this, we see that if the characteristic equation has complex conjugate roots $\alpha \pm \beta i$, then the general solution to Equation 17.1.1 is given by

$$\begin{aligned} y(x) &= c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x \\ &= e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x), \end{aligned}$$

where c_1 and c_2 are constants.

For example, the differential equation $y'' - 2y' + 5y = 0$ has the associated characteristic equation $\lambda^2 - 2\lambda + 5 = 0$. By the quadratic formula, the roots of the characteristic equation are $1 \pm 2i$. Therefore, the general solution to this differential equation is

$$y(x) = e^x(c_1\cos 2x + c_2\sin 2x).$$

Summary of Results

We can solve second-order, linear, homogeneous differential equations with constant coefficients by finding the roots of the associated characteristic equation. The form of the general solution varies, depending on whether the characteristic equation has distinct, real roots; a single, repeated real root; or complex conjugate roots. The three cases are summarized in Table 17.1.1.

Table 17.1.1: Summary of Characteristic Equation Cases

Characteristic Equation Roots	General Solution to the Differential Equation
Distinct real roots, λ_1 and λ_2	$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

Characteristic Equation Roots	General Solution to the Differential Equation
A repeated real root, λ	$y(x) = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$
Complex conjugate roots $\alpha \pm \beta i$	$y(x) = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

PROBLEM-SOLVING STRATEGY: USING THE CHARACTERISTIC EQUATION TO SOLVE SECOND-ORDER DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

1. Write the differential equation in the form $a'' + by' + cy = 0$.
2. Find the corresponding characteristic equation $a\lambda^2 + b\lambda + c = 0$.
3. Either factor the characteristic equation or use the quadratic formula to find the roots.
4. Determine the form of the general solution based on whether the characteristic equation has distinct, real roots; a single, repeated real root; or complex conjugate roots.

Example 17.1.6: Solving Second-Order Equations with Constant Coefficients

Find the general solution to the following differential equations. Give your answers as functions of x.

a. $y'' + 3y' - 4y = 0$
b. $y'' + 6y' + 13y = 0$
c. $y'' + 2y' + y = 0$
d. $y'' - 5y' = 0$
e. $y'' - 16y = 0$
f. $y'' + 16y = 0$

Solution

Note that all these equations are already given in standard form (step 1).

1. The characteristic equation is $\lambda^2 + 3\lambda - 4 = 0$ (step 2). This factors into $(\lambda + 4)(\lambda - 1) = 0$, so the roots of the characteristic equation are $\lambda_1 = -4$ and $\lambda_2 = 1$ (step 3). Then the general solution to the differential equation is
$$y(x) = c_1 e^{-4x} + c_2 e^{x}. \quad \text{(step 1)}$$
2. The characteristic equation is $\lambda^2 + 6\lambda + 13 = 0$ (step 2). Applying the quadratic formula, we see this equation has complex conjugate roots $-3 \pm 2i$ (step 3). Then the general solution to the differential equation is
$$y(t) = e^{-3t}(c_1 \cos 2t + c_2 \sin 2t). \quad \text{(step 2)}$$
3. The characteristic equation is $\lambda^2 + 2\lambda + 1 = 0$ (step 2). This factors into $(\lambda + 1)2 = 0,$ so the characteristic equation has a repeated real root $\lambda = -1$ (step 3). Then the general solution to the differential equation is
$$y(t) = c_1 e^{-t} + c_2 t e^{-t}. \quad \text{(step 3)}$$
4. The characteristic equation is $\lambda^2 - 5\lambda$ (step 2). This factors into $\lambda(\lambda - 5) = 0,$ so the roots of the characteristic equation are $\lambda_1 = 0$ and $\lambda_2 = 5$ (step 3). Note that $e^{0x} = e^0 = 1$, so our first solution is just a constant. Then the general solution to the differential equation is
$$y(x) = c_1 + c_2 e^{5x}. \quad \text{(step 4)}$$
5. The characteristic equation is $\lambda^2 - 16 = 0$ (step 2). This factors into $(\lambda + 4)(\lambda - 4) = 0,$ so the roots of the characteristic equation are $\lambda_1 = 4$ and $\lambda_2 = -4$ (step 3). Then the general solution to the differential equation is
$$y(x) = c_1 e^{4x} + c_2 e^{-4x}. \quad \text{(step 5)}$$
6. The characteristic equation is $\lambda^2 + 16 = 0$ (step 2). This has complex conjugate roots $\pm 4i$ (step 3). Note that $e^{0x} = e^0 = 1$, so the exponential term in our solution is just a constant. Then the general solution to the differential equation is
$$y(t) = c_1 \cos 4t + c_2 \sin 4t. \quad \text{(step 6)}$$

Exercise 17.1.6

Find the general solution to the following differential equations:

a. $y'' - 2y' + 10y = 0$
b. $y'' + 14y' + 49y = 0$

Hint

Find the roots of the characteristic equation.

Answer a

$y(x) = e^x(c_1 \cos 3x + c_2 \sin 3x)$

Answer b

$y(x) = c_1 e^{-7x} + c_2 x e^{-7x}$

Initial-Value Problems and Boundary-Value Problems

So far, we have been finding general solutions to differential equations. However, differential equations are often used to describe physical systems, and the person studying that physical system usually knows something about the state of that system at one or more points in time. For example, if a constant-coefficient differential equation is representing how far a motorcycle shock absorber is compressed, we might know that the rider is sitting still on his motorcycle at the start of a race, time $t = t_0$. This means the system is at equilibrium, so $y(t_0) = 0$, and the compression of the shock absorber is not changing, so $y'(t_0) = 0$. With these two initial conditions and the general solution to the differential equation, we can find the *specific* solution to the differential equation that satisfies both initial conditions. This process is known as *solving an initial-value problem*. (Recall that we discussed initial-value problems in Introduction to Differential Equations.) Note that second-order equations have two arbitrary constants in the general solution, and therefore we require two initial conditions to find the solution to the initial-value problem.

Sometimes we know the condition of the system at two different times. For example, we might know $y(t_0) = y_0$ and $y(t_1) = y_1$. These conditions are called boundary conditions, and finding the solution to the differential equation that satisfies the boundary conditions is called solving a boundary-value problem.

Mathematicians, scientists, and engineers are interested in understanding the conditions under which an initial-value problem or a boundary-value problem has a unique solution. Although a complete treatment of this topic is beyond the scope of this text, it is useful to know that, within the context of constant-coefficient, second-order equations, initial-value problems are guaranteed to have a unique solution as long as two initial conditions are provided. Boundary-value problems, however, are not as well behaved. Even when two boundary conditions are known, we may encounter boundary-value problems with unique solutions, many solutions, or no solution at all.

Example 17.1.7: Solving an Initial-Value Problem

Solve the following initial-value problem: $y'' + 3y' - 4y = 0,\ y(0) = 1,\ y'(0) = -9$.

Solution

We already solved this differential equation in Example 17.6a. and found the general solution to be

$$y(x) = c_1 e^{-4x} + c_2 e^x.$$

Then

$$y'(x) = -4c_1 e^{-4x} + c_2 e^x.$$

When $x = 0$, we have $y(0) = c_1 + c_2$ and $y'(0) = -4c_1 + c_2$. Applying the initial conditions, we have

$$\begin{aligned} c_1 + c_2 &= 1 \\ -4c_1 + c_2 &= -9. \end{aligned}$$

Then $c_1 = 1 - c_2$. Substituting this expression into the second equation, we see that

$$\begin{aligned} -4(1-c_2)+c_2 &= -9 \\ -4+4c_2+c_2 &= -9 \\ 5c_2 &= -5 \\ c_2 &= -1. \end{aligned}$$

So, $c_1 = 2$ and the solution to the initial-value problem is

$$y(x) = 2e^{-4x} - e^{x}.$$

Exercise 17.1.7

Solve the initial-value problem $y'' - 3y' - 10y = 0, \quad y(0) = 0,\ y'(0) = 7.$

Hint

Use the initial conditions to determine values for c_1 and c_2.

Answer

$$y(x) = -e^{-2x} + e^{5x}$$

Example 17.1.8: Solving an Initial-Value Problem and Graphing the Solution

Solve the following initial-value problem and graph the solution:

$$y'' + 6y' + 13y = 0, \quad y(0) = 0,\ y'(0) = 2$$

Solution

We already solved this differential equation in Example $17.1.6b$ and found the general solution to be

$$y(x) = e^{-3x}(c_1 \cos 2x + c_2 \sin 2x).$$

Then

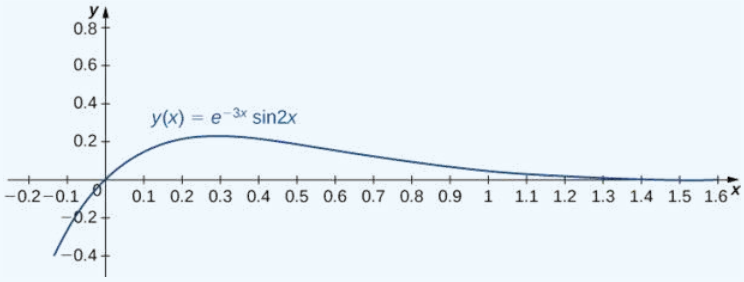
$$y'(x) = e^{-3x}(-2c_1 \sin 2x + 2c_2 \cos 2x) - 3e^{-3x}(c_1 \cos 2x + c_2 \sin 2x).$$

When $x = 0$, we have $y(0) = c_1$ and $y'(0) = 2c_2 - 3c_1$. Applying the initial conditions, we obtain

$$\begin{aligned} c_1 &= 0 \\ -3c_1 + 2c_2 &= 2. \end{aligned}$$

Therefore, $c_1 = 0,\ c_2 = 1,$ and the solution to the initial value problem is shown in the following graph.

$$y = e^{-3x} \sin 2x.$$



Exercise 17.1.8

Solve the following initial-value problem and graph the solution: $y'' - 2y' + 10y = 0, \quad y(0) = 2,\ y'(0) = -1$

Hint

Use the initial conditions to determine values for c_1 and c_2.

Answer

$$y(x) = e^x(2\cos 3x - \sin 3x)$$

Example 17.1.9: Initial-Value Problem Representing a Spring-Mass System

The following initial-value problem models the position of an object with mass attached to a spring. Spring-mass systems are examined in detail in Applications. The solution to the differential equation gives the position of the mass with respect to a neutral (equilibrium) position (in meters) at any given time. (Note that for spring-mass systems of this type, it is customary to define the downward direction as positive.)

$$y'' + 2y' + y = 0, \quad y(0) = 1,\ y'(0) = 0$$

Solve the initial-value problem and graph the solution. What is the position of the mass at time $t = 2$ sec? How fast is the mass moving at time $t = 1$ sec? In what direction?

Solution

In Example Example $17.1.6c$ we found the general solution to this differential equation to be

$$y(t) = c_1 e^{-t} + c_2 t e^{-t}.$$

Then

$$y'(t) = -c_1 e^{-t} + c_2(-te^{-t} + e^{-t}).$$

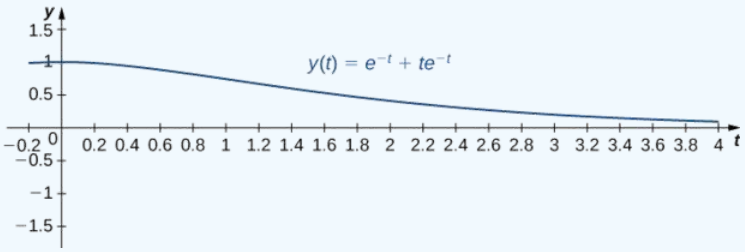
When $t = 0$, we have $y(0) = c_1$ and $y'(0) = c_1 + c_2$. Applying the initial conditions, we obtain

$$\begin{aligned} c_1 &= 1 \\ -c_1 + c_2 &= 0. \end{aligned}$$

Thus, $c_1 = 1, c_2 = 1,$ and the solution to the initial value problem is

$$y(t) = e^{-t} + te^{-t}.$$

This solution is represented in the following graph. At time $t = 2$, the mass is at position $y(2) = e^{-2} + 2e^{-2} = 3e^{-2} \approx 0.406$ m below equilibrium.



To calculate the velocity at time $t = 1$, we need to find the derivative. We have $y(t) = e^{-t} + te^{-t},$ so

$$y'(t) = -e^{-t} + e^{-t} - te^{-t} = -te^{-t}.$$

Then $y'(1) = -e^{-1} \approx -0.3679$. At time $t = 1$, the mass is moving upward at 0.3679m/sec.

Exercise 17.1.9

Suppose the following initial-value problem models the position (in feet) of a mass in a spring-mass system at any given time. Solve the initial-value problem and graph the solution. What is the position of the mass at time $t = 0.3$ sec? How fast is it moving at time $t = 0.1$ sec? In what direction?

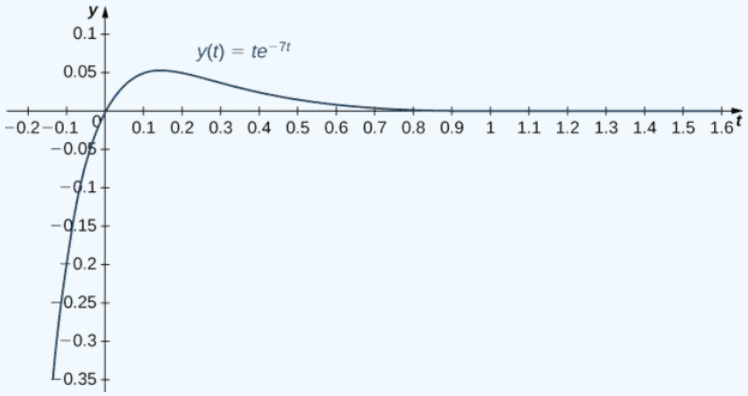
$$y'' + 14y' + 49y = 0, \quad y(0) = 0,\ y'(0) = 1$$

Hint

Use the initial conditions to determine values for c_1 and c_2.

Answer

$$y(t) = te^{-7t}$$



At time $t = 0.3,\ y(0.3) = 0.3e^{(-7*0.3)} = 0.3e^{-2.1} \approx 0.0367$. The mass is 0.0367ft below equilibrium. At time $t = 0.1,\ y'(0.1) = 0.3e^{-0.7} \approx 0.1490$.The mass is moving downward at a speed of 0.1490ft/sec.

Example 17.1.10: Solving a Boundary-Value Problem

In Example 17.6f. we solved the differential equation $y'' + 16y = 0$ and found the general solution to be $y(t) = c_1 \cos 4t + c_2 \sin 4t$. If possible, solve the boundary-value problem if the boundary conditions are the following:

a. $y(0) = 0, y(\frac{\pi}{4}) = 0$
b. $y(0) = 1, y(0) = 1, y(\frac{\pi}{8}) = 0$
c. $y(\frac{\pi}{8}) = 0, y(\frac{3\pi}{8}) = 2$

Solution

We have

$$y(x) = c_1 \cos 4t + c_2 \sin 4t.$$

1. Applying the first boundary condition given here, we get $y(0) = c_1 = 0$. So the solution is of the form $y(t) = c_2 \sin 4t$. When we apply the second boundary condition, though, we get $y(\frac{\pi}{4}) = c_2 \sin(4(\frac{\pi}{4})) = c_2 \sin \pi = 0$ for all values of c_2. The boundary conditions are not sufficient to determine a value for c_2, so this boundary-value problem has infinitely many solutions. Thus, $y(t) = c_2 \sin 4t$ is a solution for any value of c_2.
2. Applying the first boundary condition given here, we get $y(0) = c_1 = 1$. Applying the second boundary condition gives $y(\frac{\pi}{8}) = c_2 = 0,$ so $c_2 = 0$. In this case, we have a unique solution: $y(t) = \cos 4t$.
3. Applying the first boundary condition given here, we get $y(\frac{\pi}{8}) = c_2 = 0$. However, applying the second boundary condition gives $y(\frac{3\pi}{8}) = -c_2 = 2,$ so $c_2 = -2$. We cannot have $c_2 = 0 = -2,$ so this boundary value problem has no solution.

Key Concepts

- Second-order differential equations can be classified as linear or nonlinear, homogeneous or nonhomogeneous.
- To find a general solution for a homogeneous second-order differential equation, we must find two linearly independent solutions. If $y_1(x)$ and $y_2(x)$ are linearly independent solutions to a second-order, linear, homogeneous differential equation, then the general solution is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

- To solve homogeneous second-order differential equations with constant coefficients, find the roots of the characteristic equation. The form of the general solution varies depending on whether the characteristic equation has distinct, real roots; a single, repeated real root; or complex conjugate roots.
- Initial conditions or boundary conditions can then be used to find the specific solution to a differential equation that satisfies those conditions, except when there is no solution or infinitely many solutions.

Key Equations

- **Linear second-order differential equation**

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = r(x)$$

- **Second-order equation with constant coefficients**

$$ay'' + by' + cy = 0$$

Glossary

boundary conditions

the conditions that give the state of a system at different times, such as the position of a spring-mass system at two different times

boundary-value problem

a differential equation with associated boundary conditions

characteristic equation

the equation $a\lambda^2 + b\lambda + c = 0$ for the differential equation $ay'' + by' + cy = 0$

homogeneous linear equation

a second-order differential equation that can be written in the form $a_2(x)y'' + a_1(x)y' + a_0(x)y = r(x)$, but $r(x) = 0$ for every value of x

nonhomogeneous linear equation

a second-order differential equation that can be written in the form $a_2(x)y'' + a_1(x)y' + a_0(x)y = r(x)$, but $r(x) \neq 0$ for some value of x

linearly dependent

a set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ for whichthere are constants $c_1, c_2, \ldots, c_n$, not all zero, such that $c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$ for all x in the interval of interest

linearly independent

a set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ for which there are no constants $c_1, c_2, \ldots, c_n$, such that $c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$ for all x in the interval of interest

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17.2: Nonhomogeneous Linear Equations

Now we consider second order equations of the form $a\ddot{y}+b\dot{y}+cy=f(t)$, with a, b, and c constant. Of course, if $a=0$ this is really a first order equation, so we assume $a\neq 0$. Also, much as in exercise 20 of section 17.5, if $c=0$ we can solve the related first order equation $a\dot{h}+bh=f(t)$, and then solve $h=\dot{y}$ for y. So we will only examine examples in which $c\neq 0$.

Suppose that $y_1(t)$ and $y_2(t)$ are solutions to $a\ddot{y}+b\dot{y}+cy=f(t)$, and consider the function $h=y_1-y_2$. We substitute this function into the left hand side of the differential equation and simplify:

$$a(y_1-y_2)''+b(y_1-y_2)'+c(y_1-y_2)=ay_1''+by_1'+cy_1-(ay_2''+by_2'+cy_2)=f(t)-f(t)=0.$$

So h is a solution to the homogeneous equation $a\ddot{y}+b\dot{y}+cy=0$. Since we know how to find all such h, then with just one particular solution y_2 we can express all possible solutions y_1, namely, $y_1=h+y_2$, where now h is the general solution to the homogeneous equation. Of course, this is exactly how we approached the first order linear equation.

To make use of this observation we need a method to find a single solution y_2. This turns out to be somewhat more difficult than the first order case, but if $f(t)$ is of a certain simple form, we can find a solution using the **method of undetermined coefficients**, sometimes more whimsically called the **method of judicious guessing.**

Example 17.2.1:

Solve the differential equation $\ddot{y}-\dot{y}-6y=18t^2+5$.

Solution

The general solution of the homogeneous equation is $Ae^{3t}+Be^{-2t}$. We guess that a solution to the non-homogeneous equation might look like $f(t)$ itself, namely, a quadratic $y=at^2+bt+c$. Substituting this guess into the differential equation we get

$$\ddot{y}-\dot{y}-6y=2a-(2at+b)-6(at^2+bt+c)=-6at^2+(-2a-6b)t+(2a-b-6c).$$

We want this to equal $18t^2+5$, so we need

$$\begin{aligned}-6a&=18\\-2a-6b&=0\\2a-b-6c&=5\end{aligned}$$

This is a system of three equations in three unknowns and is not hard to solve: $a=-3$, $b=1$, $c=-2$. Thus the general solution to the differential equation is $Ae^{3t}+Be^{-2t}-3t^2+t-2$.

So the "judicious guess" is a function with the same form as $f(t)$ but with undetermined (or better, yet to be determined) coefficients. This works whenever $f(t)$ is a polynomial.

Example 17.2.2:

Consider the initial value problem $m\ddot{y}+ky=-mg$, $y(0)=2$, $\dot{y}(0)=50$. The left hand side represents a mass-spring system with no damping, i.e., $b=0$. Unlike the homogeneous case, we now consider the force due to gravity, $-mg$, assuming the spring is vertical at the surface of the earth, so that $g=980$. To be specific, let us take $m=1$ and $k=100$. The general solution to the homogeneous equation is $A\cos(10t)+B\sin(10t)$. For the solution to the non-homogeneous equation we guess simply a constant $y=a$, since $-mg=-980$ is a constant. Then $\ddot{y}+100y=100a$ so $a=-980/100=-9.8$. The desired general solution is then $A\cos(10t)+B\sin(10t)-9.8$. Substituting the initial conditions we get

$$\begin{aligned}2&=A-9.8\\50&=10B\end{aligned}$$

so $A=11.8$ and $B=5$ and the solution is $11.8\cos(10t)+5\sin(10t)-9.8$.

More generally, this method can be used when a function similar to $f(t)$ has derivatives that are also similar to $f(t)$; in the examples so far, since $f(t)$ was a polynomial, so were its derivatives. The method will work if $f(t)$ has the form

$p(t)e^{\alpha t}\cos(\beta t)+q(t)e^{\alpha t}\sin(\beta t)$, where $p(t)$ and $q(t)$ are polynomials; when $\alpha=\beta=0$ this is simply $p(t)$, a polynomial. In the most general form it is not simple to describe the appropriate judicious guess; we content ourselves with some examples to illustrate the process.

Example 17.2.3:

Find the general solution to $\ddot{y}+7\dot{y}+10y=e^{3t}$. The characteristic equation is $r^2+7r+10=(r+5)(r+2)$, so the solution to the homogeneous equation is $Ae^{-5t}+Be^{-2t}$. For a particular solution to the inhomogeneous equation we guess Ce^{3t}. Substituting we get

$$9Ce^{3t}+21Ce^{3t}+10Ce^{3t}=e^{3t}40C.$$

When $C=1/40$ this is equal to $f(t)=e^{3t}$, so the solution is $Ae^{-5t}+Be^{-2t}+(1/40)e^{3t}$.

Example 17.2.4:

Find the general solution to $\ddot{y}+7\dot{y}+10y=e^{-2t}$. Following the last example we might guess Ce^{-2t}, but since this is a solution to the homogeneous equation it cannot work. Instead we guess Cte^{-2t}. Then

$$(-2Ce^{-2t}-2Ce^{-2t}+4Cte^{-2t})+7(Ce^{-2t}-2Cte^{-2t})+10Cte^{-2t}=e^{-2t}(-3C).$$

Then $C=-1/3$ and the solution is $Ae^{-5t}+Be^{-2t}-(1/3)te^{-2t}$.

In general, if $f(t)=e^{kt}$ and k is one of the roots of the characteristic equation, then we guess Cte^{kt} instead of Ce^{kt}. If k is the only root of the characteristic equation, then Cte^{kt} will not work, and we must guess Ct^2e^{kt}.

Example 17.2.5:

Find the general solution to $\ddot{y}-6\dot{y}+9y=e^{3t}$. The characteristic equation is $r^2-6r+9=(r-3)^2$, so the general solution to the homogeneous equation is $Ae^{3t}+Bte^{3t}$. Guessing Ct^2e^{3t} for the particular solution, we get

$$(9Ct^2e^{3t}+6Cte^{3t}+6Cte^{3t}+2Ce^{3t})-6(3Ct^2e^{3t}+2Cte^{3t})+9Ct^2e^{3t}=e^{3t}2C.$$

The solution is thus $Ae^{3t}+Bte^{3t}+(1/2)t^2e^{3t}$.

It is common in various physical systems to encounter an $f(t)$ of the form $a\cos(\omega t)+b\sin(\omega t)$.

Example 17.2.6:

Find the general solution to $\ddot{y}+6\dot{y}+25y=\cos(4t)$. The roots of the characteristic equation are $-3\pm4i$, so the solution to the homogeneous equation is $e^{-3t}(A\cos(4t)+B\sin(4t))$. For a particular solution, we guess $C\cos(4t)+D\sin(4t)$. Substituting as usual:

$$\begin{aligned}(-16C\cos(4t) \ &+-16D\sin(4t))+6(-4C\sin(4t)+4D\cos(4t))+25(C\cos(4t)+D\sin(4t))\\ &=(24D+9C)\cos(4t)+(-24C+9D)\sin(4t).\end{aligned}$$

To make this equal to $\cos(4t)$ we need

$$\begin{aligned}24D+9C&=1\\ 9D-24C&=0\end{aligned}$$

which gives $C=1/73$ and $D=8/219$. The full solution is then

$$e^{-3t}(A\cos(4t)+B\sin(4t))+(1/73)\cos(4t)+(8/219)\sin(4t).$$

The function $e^{-3t}(A\cos(4t)+B\sin(4t))$ is a damped oscillation as in example 17.5.3, while $(1/73)\cos(4t)+(8/219)\sin(4t)$ is a simple undamped oscillation. As t increases, the sum $e^{-3t}(A\cos(4t)+B\sin(4t))$ approaches zero, so the solution \[e^{-3t}(A\cos(4t)+B\sin(4t))+(1/73)\cos(4t)+(8/219)\sin(4t)\[becomes more and more like the simple oscillation $(1/73)\cos(4t)+(8/219)\sin(4t)$---notice that the initial conditions don't matter to this long term

behavior. The damped portion is called the **transient part of solution**, and the simple oscillation is called the **steady state part of solution**. A physical example is a mass-spring system. If the only force on the mass is due to the spring, then the behavior of the system is a damped oscillation. If in addition an external force is applied to the mass, and if the force varies according to a function of the form $a\cos(\omega t)+b\sin(\omega t)$, then the long term behavior will be a simple oscillation determined by the steady state part of the general solution; the initial position of the mass will not matter.

As with the exponential form, such a simple guess may not work.

Example 17.2.4:

Find the general solution to $\ddot{y}+16y=-\sin(4t)$. The roots of the characteristic equation are $\pm 4i$, so the solution to the homogeneous equation is $A\cos(4t)+B\sin(4t)$. Since both $\cos(4t)$ and $\sin(4t)$ are solutions to the homogeneous equation, $C\cos(4t)+D\sin(4t)$ is also, so it cannot be a solution to the non-homogeneous equation. Instead, we guess $Ct\cos(4t)+Dt\sin(4t)$. Then substituting:

$$\begin{aligned}(-16Ct\cos(4t) \ &-16D\sin(4t)+8D\cos(4t)-8C\sin(4t)))+16(Ct\cos(4t)+Dt\sin(4t))\\ &=8D\cos(4t)-8C\sin(4t).\end{aligned}$$

Thus $C=1/8$, $D=0$, and the solution is $C\cos(4t)+D\sin(4t)+(1/8)t\cos(4t)$.

In general, if $f(t)=a\cos(\omega t)+b\sin(\omega t)$, and $\pm\omega i$ are the roots of the characteristic equation, then instead of $C\cos(\omega t)+D\sin(\omega t)$ we guess $Ct\cos(\omega t)+Dt\sin(\omega t)$.

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17.3: Applications of Second-Order Differential Equations

Let us consider to the example of a mass on a spring. We now examine the case of forced oscillations, which we did not yet handle. That is, we consider the equation

$$mx'' + cx' + kx = F(t)$$

for some nonzero $F(t)$. The setup is again: m is mass, c is friction, k is the spring constant, and $F(t)$ is an external force acting on the mass.

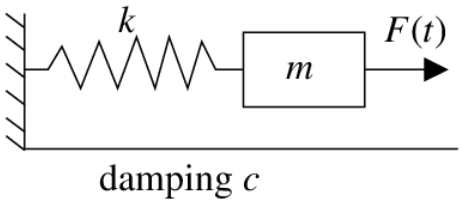


Figure 17.3.1

What we are interested in is periodic forcing, such as noncentered rotating parts, or perhaps loud sounds, or other sources of periodic force. Once we learn about Fourier series in Chapter 4, we will see that we cover all periodic functions by simply considering $F(t) = F_0 \cos(\omega t)$ (or sine instead of cosine, the calculations are essentially the same).

Undamped Forced Motion and Resonance

First let us consider undamped $c = 0$ motion for simplicity. We have the equation

$$mx'' + kx = F_0 \cos(\omega t)$$

This equation has the complementary solution (solution to the associated homogeneous equation)

$$x_c = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ is the *natural frequency* (angular), which is the frequency at which the system "wants to oscillate" without external interference.

Let us suppose that $\omega_0 \neq \omega$. We try the solution $x_p = A\cos(\omega t)$ and solve for A. Note that we need not have sine in our trial solution as on the left hand side we will only get cosines anyway. If you include a sine it is fine; you will find that its coefficient will be zero.

We solve using the method of undetermined coefficients. We find that

$$x_p = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

We leave it as an exercise to do the algebra required.

The general solution is

$$x = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

or written another way

$$x = C \cos(\omega_0 t - y) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

Hence it is a superposition of two cosine waves at different frequencies.

Example 17.3.1

Take

$$0.5x'' + 8x = 10\cos(\pi t), \quad x(0) = 0, \quad x'(0) = 0$$

Let us compute. First we read off the parameters: $\omega = \pi, \omega_0 = \sqrt{\frac{8}{0.5}} = 4, F_0 = 10, m = 0.5$. The general solution is

$$x = C_1\cos(4t) + C_2\sin(4t) + \frac{20}{16-\pi^2}\cos(\pi t)$$

Solve for C_1 and C_2 using the initial conditions. It is easy to see that $C_1 = \frac{-20}{16-\pi^2}$ and $C_2 = 0$. Hence

$$x = \frac{20}{16-\pi^2}(\cos(\pi t) - \cos(4t))$$

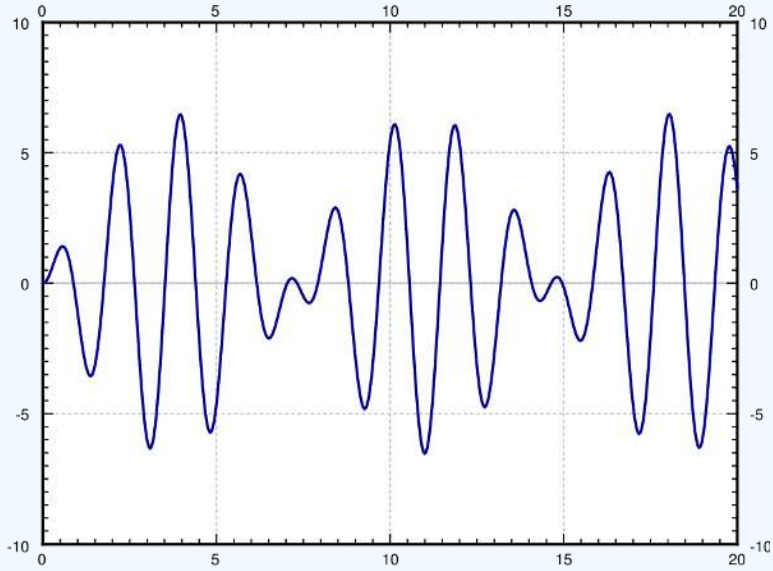


Figure 17.3.2: Graph of $\frac{20}{16-\pi^2}(\cos(\pi t) - \cos(4t))$.

Notice the "beating" behavior in Figure 17.3.2 First use the trigonometric identity

$$2\sin(\frac{A-B}{2})\sin(\frac{A+B}{2}) = \cos B - \cos A$$

to get that

$$x = \frac{20}{16-\pi^2}(2\sin(\frac{4-\pi}{2}t)\sin(\frac{4+\pi}{2}t))$$

Notice that x is a high frequency wave modulated by a low frequency wave.

Now suppose that $\omega_0 = \omega$. Obviously, we cannot try the solution $A\cos(\omega t)$ and then use the method of undetermined coefficients. We notice that $\cos(\omega t)$ solves the associated homogeneous equation. Therefore, we need to try $x_p = At\cos(\omega t) + Bt\sin(\omega t)$. This time we do need the sine term since the second derivative of $t\cos(\omega t)$ does contain sines. We write the equation

$$x'' + \omega^2 x = \frac{F_0}{m}\cos(\omega t)$$

Plugging x_p into the left hand side we get

$$2B\omega\cos(\omega t) - 2A\omega\sin(\omega t) = \frac{F_0}{m}\cos(\omega t)$$

Hence $A = 0$ and $B = \frac{F_0}{2m\omega}$. Our particular solution is $\frac{F_0}{2m\omega}t\sin(\omega t)$ and our general solution is

$$x = C_1 \cos(\omega t) + C_2 \sin(\omega t) + \frac{F_0}{2m\omega} t \sin(\omega t)$$

The important term is the last one (the particular solution we found). We can see that this term grows without bound as $t \to \infty$. In fact it oscillates between $\frac{F_0 t}{2m\omega}$ and $\frac{-F_0 t}{2m\omega}$. The first two terms only oscillate between $\pm\sqrt{C_1^2 + C_2^2}$, which becomes smaller and smaller in proportion to the oscillations of the last term as t gets larger. In Figure 17.3.3 we see the graph with $C_1 = C_2 = 0, F_0 = 2, m = 1, \omega = \pi$.

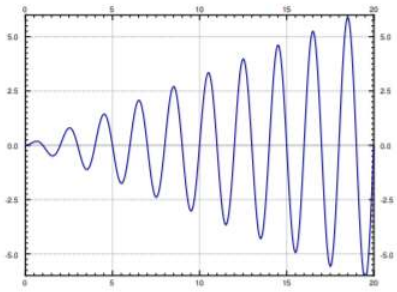


Figure 17.3.3: Graph of $\frac{1}{\pi} t \sin(\pi t)$.

By forcing the system in just the right frequency we produce very wild oscillations. This kind of behavior is called resonance or perhaps pure resonance. Sometimes resonance is desired. For example, remember when as a kid you could start swinging by just moving back and forth on the swing seat in the "correct frequency"? You were trying to achieve resonance. The force of each one of your moves was small, but after a while it produced large swings.

On the other hand resonance can be destructive. In an earthquake some buildings collapse while others may be relatively undamaged. This is due to different buildings having different resonance frequencies. So figuring out the resonance frequency can be very important.

A common (but wrong) example of destructive force of resonance is the Tacoma Narrows bridge failure. It turns out there was a different phenomenon at play.[1]

Damped Forced Motion and Practical Resonance

In real life things are not as simple as they were above. There is, of course, some damping. Our equation becomes

$$mx'' + cx' + kx = F_0 \cos(\omega t), \tag{17.3.1}$$

for some $c > 0$. We have solved the homogeneous problem before. We let

$$p = \frac{c}{2m} \quad \omega_0 = \sqrt{\frac{k}{m}}$$

We replace equation (17.3.1) with

$$x'' + 2px' + \omega_0^2 x = \frac{F_0}{m} \cos(\omega t)$$

The roots of the characteristic equation of the associated homogeneous problem are $r_1, r_2 = -p \pm \sqrt{p^2 - \omega_0^2}$. The form of the general solution of the associated homogeneous equation depends on the sign of $p^2 - \omega_0^2$, or equivalently on the sign of $c^2 - 4km$, as we have seen before. That is,

$$x_c = \begin{cases} C_1 e^{r_1 t} + C_2 e^{r_2 t}, & \text{if } c^2 > 4km, \\ C_1 e^{pt} + C_2 t e^{-pt}, & \text{if } c^2 = 4km, \\ e^{-pt}(C_1 \cos(\omega_1 t) + C_2 \sin(\omega_1 t)), & \text{if } c^2 < 4km, \end{cases}$$

where $\omega_1 = \sqrt{\omega_0^2 - p^2}$. In any case, we can see that $x_c(t) \to 0$ as $t \to \infty$. Furthermore, there can be no conflicts when trying to solve for the undetermined coefficients by trying $x_p = A\cos(\omega t) + B\sin(\omega t)$. Let us plug in and solve for A and B. We get (the tedious details are left to reader)

$$((\omega_0^2-\omega^2)B-2\omega pA)\sin(\omega t)+((\omega_0^2-\omega^2)A+2\omega pB)\cos(\omega t)=\frac{F_0}{m}\cos(\omega t)$$

We get that

$$A=\frac{(\omega_0^2-\omega^2)F_0}{m(2\omega p)^2+m\left(\omega_0^2-\omega^2\right)^2}$$

$$B=\frac{2\omega pF_0}{m(2\omega p)^2+m\left(\omega_0^2-\omega^2\right)^2}$$

We also compute $C=\sqrt{A^2+B^2}$ to be

$$C=\frac{F_0}{m\sqrt{(2\omega p)^2+\left(\omega_0^2-\omega^2\right)^2}}$$

Thus our particular solution is

$$x_P=\frac{(\omega_0^2-\omega^2)F_0}{m(2\omega p)^2+m\left(\omega_0^2-\omega^2\right)^2}\cos(\omega t)+\frac{2\omega pF_0}{m(2\omega p)^2+m\left(\omega_0^2-\omega^2\right)^2}\sin(\omega t)$$

Or in the alternative notation we have amplitude C and phase shift γ where (if $\omega\neq\omega_0$)

$$\tan\gamma=\frac{B}{A}=\frac{2\omega p}{\omega_0^2-\omega^2}$$

Hence we have

$$x_p=\frac{F_0}{m\sqrt{(2\omega p)^2+\left(\omega_0^2-\omega^2\right)^2}}\cos(\omega t-\gamma)$$

If $\omega=\omega_0$ we see that $A=0, B=C=\frac{F_0}{2m\omega p}$, and $\gamma=\frac{\pi}{2}$.

The exact formula is not as important as the idea. Do not memorize the above formula, you should instead remember the ideas involved. For different forcing function F, you will get a different formula for x_p. So there is no point in memorizing this specific formula. You can always recompute it later or look it up if you really need it.

For reasons we will explain in a moment, we call $x_c$the transient solution and denote it by x_{tr}. We call the x_p we found above the steady periodic solution and denote it by x_{sp}. The general solution to our problem is

$$x=x_c+x_p=x_{tr}+x_{sp}$$

We note that $x_c=x_{tr}$ goes to zero as $t\to\infty$, as all the terms involve an exponential with a negative exponent. Hence for large t, the effect of x_{tr} is negligible and we will essentially only see x_{sp}. Hence the name transient. Notice that x_{sp} involves no arbitrary constants, and the initial conditions will only affect x_{tr}. This means that the effect of the initial conditions will be negligible after some period of time. Because of this behavior, we might as well focus on the steady periodic solution and ignore the transient solution. See Figure 17.3.4for a graph of different initial conditions.

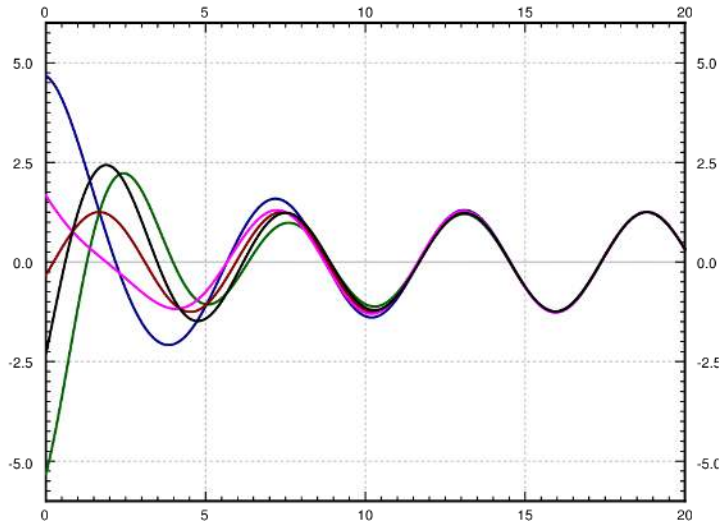


Figure 17.3.4: Solutions with different initial conditions for parameters $k = 1, m = 1, F_0 = 1, c = 0.7,$ and $\omega = 1.1$.

Notice that the speed at which x_{tr} goes to zero depends on P (and hence c). The bigger P is (the bigger c is), the "faster" x_{tr} becomes negligible. So the smaller the damping, the longer the "transient region." This agrees with the observation that when $c = 0$, the initial conditions affect the behavior for all time (i.e. an infinite "transient region").

Let us describe what we mean by resonance when damping is present. Since there were no conflicts when solving with undetermined coefficient, there is no term that goes to infinity. What we will look at however is the maximum value of the amplitude of the steady periodic solution. Let C be the amplitude of x_{sp}. If we plot C as a function of ω (with all other parameters fixed) we can find its maximum. We call the ω that achieves this maximum the practical resonance frequency. We call the maximal amplitude $C(\omega)$ the practical resonance amplitude. Thus when damping is present we talk of practical resonance rather than pure resonance. A sample plot for three different values of c is given in Figure 17.3.5 As you can see the practical resonance amplitude grows as damping gets smaller, and any practical resonance can disappear when damping is large.

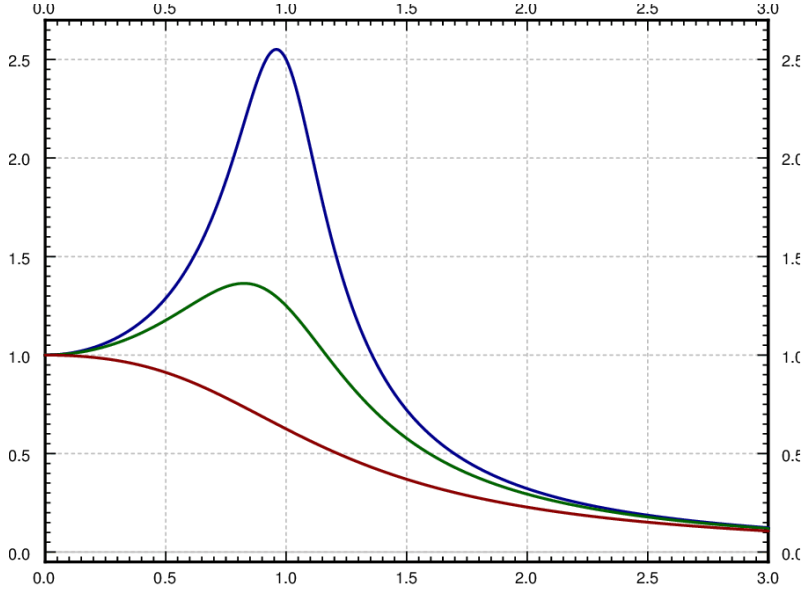


Figure 17.3.5: Graph of $C(\omega)$ showing practical resonance with parameters $k = 1, m = 1, F_0 = 1$. The top line is with $c = 0.4$, the middle line with $c = 0.8$, and the bottom line with $c = 1.6$.

To find the maximum we need to find the derivative $C'(\omega)$. Computation shows

$$C'(\omega) = \frac{-4\omega(2p^2 + \omega^2 - \omega_0^2)F_0}{m\left((2\omega p)^2 + (\omega_0^2 - \omega^2)\right)^{3/2}}$$

This is zero either when $\omega = 0$ or when $2p^2 + \omega^2 - \omega_0^2 = 0$. In other words, $C'(\omega) = 0$ when

$$\omega = \sqrt{\omega_0^2 - 2p^2} \text{ or } \omega = 0$$

It can be shown that if $\omega_0^2 - 2p^2$ is positive, then $\sqrt{\omega_0^2 - 2p^2}$ is the practical resonance frequency (that is the point where $C(\omega)$ is maximal, note that in this case $C'(\omega) > 0$ for small ω). If $\omega = 0$ is the maximum, then essentially there is no practical resonance since we assume that $\omega > 0$ in our system. In this case the amplitude gets larger as the forcing frequency gets smaller.

If practical resonance occurs, the frequency is smaller than ω_0. As the damping c (and hence P) becomes smaller, the practical resonance frequency goes to ω_0. So when damping is very small, ω_0 is a good estimate of the resonance frequency. This behavior agrees with the observation that when $c = 0$, then ω_0 is the resonance frequency.

Another interesting observation to make is that when $\omega \to \infty$, then $\omega \to 0$. This means that if the forcing frequency gets too high it does not manage to get the mass moving in the mass-spring system. This is quite reasonable intuitively. If we wiggle back and forth really fast while sitting on a swing, we will not get it moving at all, no matter how forceful. Fast vibrations just cancel each other out before the mass has any chance of responding by moving one way or the other.

The behavior is more complicated if the forcing function is not an exact cosine wave, but for example a square wave. A general periodic function will be the sum (superposition) of many cosine waves of different frequencies. The reader is encouraged to come back to this section once we have learned about the Fourier series.

Footnotes

[1]K. Billah and R. Scanlan, Resonance, Tacoma Narrows Bridge Failure, and Undergraduate Physics Textbooks, American Journal of Physics, 59(2), 1991, 118–124, http://www.ketchum.org/billah/Billah-Scanlan.pdf

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17.4: Series Solutions of Differential Equations

Learning Objectives

- Use power series to solve first-order and second-order differential equations.

Previously, we studied how functions can be represented as power series, $y(x) = \sum_{n=0}^{\infty} a_n x^n$. We also saw that we can find series representations of the derivatives of such functions by differentiating the power series term by term. This gives

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

and

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

In some cases, these power series representations can be used to find solutions to differential equations.

The examples and exercises in this section were chosen for which power solutions exist. However, it is not always the case that power solutions exist. Those of you interested in a more rigorous treatment of this topic should review the differential equations section of the LibreTexts.

Problem-Solving Strategy: Finding Power Series Solutions to Differential Equations

1. Assume the differential equation has a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

2. Differentiate the power series term by term to get

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

and

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

3. Substitute the power series expressions into the differential equation.
4. Re-index sums as necessary to combine terms and simplify the expression.
5. Equate coefficients of like powers of x to determine values for the coefficients a_n in the power series.
6. Substitute the coefficients back into the power series and write the solution.

Example 17.4.1: Series Solutions to Differential Equations

Find a power series solution for the following differential equations.

a. $y'' - y = 0$
b. $(x^2 - 1)y'' + 6xy' + 4y = -4$

Solution

Part a

Assume

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad \text{(step 1)}$$

Then,

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad \text{(step 2A)}$$

and

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \qquad \text{(step 2B)}$$

We want to find values for the coefficients a_n such that

$$\begin{aligned} y'' - y &= 0 \\ \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n &= 0. \end{aligned} \qquad \text{(step 3)}$$

We want the indices on our sums to match so that we can express them using a single summation. That is, we want to rewrite the first summation so that it starts with $n = 0$.

To re-index the first term, replace n with $n+2$ inside the sum, and change the lower summation limit to $n = 0$. We get

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n.$$

This gives

$$\begin{aligned} \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x_n &= 0 \\ \sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} - a_n] x^n &= 0. \end{aligned} \qquad \text{(step 4)}$$

Because power series expansions of functions are unique, this equation can be true only if the coefficients of each power of x are zero. So we have

$$(n+2)(n+1) a_{n+2} - a_n = 0 \text{ for } n = 0, 1, 2, \ldots.$$

This recurrence relationship allows us to express each coefficient a_n in terms of the coefficient two terms earlier. This yields one expression for even values of n and another expression for odd values of n. Looking first at the equations involving even values of n, we see that

$$\begin{aligned} a_2 &= \frac{a_0}{2} \\ a_4 &= \frac{a_2}{4 \cdot 3} = \frac{a_0}{4!} \\ a_6 &= \frac{a_4}{6 \cdot 5} = \frac{a_0}{6!} \\ &\vdots \end{aligned}$$

Thus, in general, when n is even,

$$a_n = \frac{a_0}{n!}. \qquad \text{(step 5)}$$

For the equations involving odd values of n, we see that

$$a_3 = \frac{a_1}{3 \cdot 2} = \frac{a_1}{3!}$$
$$a_5 = \frac{a_3}{5 \cdot 4} = \frac{a_1}{5!}$$
$$a_7 = \frac{a_5}{7 \cdot 6} = \frac{a_1}{7!}$$
$$\vdots$$

Therefore, in general, when n is odd,

$$a_n = \frac{a_1}{n!}. \qquad \text{(step 5)}$$

Putting this together, we have

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_1 x + \frac{a_0}{2} x^2 + \frac{a_1}{3!} x^3 + \frac{a_0}{4!} x^4 + \frac{a_1}{5!} x^5 + \cdots. \end{aligned}$$

Re-indexing the sums to account for the even and odd values of n separately, we obtain

$$y(x) = a_0 \sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^{2k+1}. \qquad \text{(step 6)}$$

Analysis for part a.

As expected for a second-order differential equation, this solution depends on two arbitrary constants. However, note that our differential equation is a constant-coefficient differential equation, yet the power series solution does not appear to have the familiar form (containing exponential functions) that we are used to seeing. Furthermore, since $y(x) = c_1 e^x + c_2 e^{-x}$ is the general solution to this equation, we must be able to write any solution in this form, and it is not clear whether the power series solution we just found can, in fact, be written in that form.

Fortunately, after writing the power series representations of e^x and e^{-x}, and doing some algebra, we find that if we choose

$$c_0 = \frac{(a_0 + a_1)}{2}, c_1 = \frac{(a_0 - a_1)}{2},$$

we then have $a_0 = c_0 + c_1$ and $a_1 = c_0 - c_1$, and

$$\begin{aligned} y(x) &= a_0 + a_1 x + \frac{a_0}{2} x^2 + \frac{a_1}{3!} x^3 + \frac{a_0}{4!} x^4 + \frac{a_1}{5!} x^5 + \cdots \\ &= (c_0 + c_1) + (c_0 - c_1)x + \frac{(c_0 + c_1)}{2} x^2 + \frac{(c_0 - c_1)}{3!} x^3 + \frac{(c_0 + c_1)}{4!} x^4 + \frac{(c_0 - c_1)}{5!} x^5 + \cdots \\ &= c_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} + c_1 \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \\ &= c_0 e^x + c_1 e^{-x}. \end{aligned}$$

So we have, in fact, found the same general solution. Note that this choice of c_1 and c_2 is not obvious. This is a case when we know what the answer should be, and have essentially "reverse-engineered" our choice of coefficients.

Part b

Assume

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad \text{(step 1)}$$

Then,

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad \text{(step 2)}$$

and

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \qquad \text{(step 2)}$$

We want to find values for the coefficients a_n such that

$$\begin{aligned}
(x^2-1)y''+6xy'+4y &= -4 \\
(x^2-1)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 6x\sum_{n=1}^{\infty} n a_n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n &= -4 \\
x^2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 6x\sum_{n=1}^{\infty} n a_n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^n &= -4.
\end{aligned}$$

Taking the external factors inside the summations, we get

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} 6n a_n x^n + \sum_{n=0}^{\infty} 4a_n x^n = -4. \qquad \text{(step 3)}$$

Now, in the first summation, we see that when $n = 0$ or $n = 1$, the term evaluates to zero, so we can add these terms back into our sum to get

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n = \sum_{n=0}^{\infty} n(n-1)a_n x^n.$$

Similarly, in the third term, we see that when $n = 0$, the expression evaluates to zero, so we can add that term back in as well. We have

$$\sum_{n=1}^{\infty} 6n a_n x^n = \sum_{n=0}^{\infty} 6n a_n x^n.$$

Then, we need only shift the indices in our second term. We get

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n.$$

Thus, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} 6n a_n x^n + \sum_{n=0}^{\infty} 4a_n x^n &= -4 \qquad \text{(step 4)} \\
\sum_{n=0}^{\infty} [n(n-1)a_n - (n+2)(n+1)a_{n+2} + 6n a_n + 4a_n] x^n &= -4 \\
\sum_{n=0}^{\infty} [(n^2-n)a_n + 6n a_n + 4a_n - (n+2)(n+1)a_{n+2}] x^n &= -4 \\
\sum_{n=0}^{\infty} [n^2 a_n + 5n a_n + 4a_n - (n+2)(n+1)a_{n+2}] x^n &= -4 \\
\sum_{n=0}^{\infty} [(n^2+5n+4)a_n - (n+2)(n+1)a_{n+2}] x^n &= -4 \\
\sum_{n=0}^{\infty} [(n+4)(n+1)a_n - (n+2)(n+1)a_{n+2}] x^n &= -4
\end{aligned}$$

Looking at the coefficients of each power of x, we see that the constant term must be equal to -4, and the coefficients of all other powers of x must be zero. Then, looking first at the constant term,

$$\begin{aligned} 4a_0 - 2a_2 &= -4 \\ a_2 &= 2a_0 + 2 \end{aligned} \qquad \text{(step 3)}$$

For $n \geq 1$, we have

$$\begin{aligned} (n+4)(n+1)a_n - (n+2)(n+1)a_{n+2} &= 0 \\ (n+1)[(n+4)a_n - (n+2)a_{n+2}] &= 0. \end{aligned}$$

Since $n \geq 1,\ n+1 \neq 0,$ we see that

$$(n+4)a_n - (n+2)a_{n+2} = 0$$

and thus

$$a_{n+2} = \frac{n+4}{n+2}a_n.$$

For even values of n, we have

$$\begin{aligned} a_4 &= \frac{6}{4}(2a_0+2) = 3a_0+3 \\ a_6 &= \frac{8}{6}(3a_0+3) = 4a_0+4 \\ &\vdots \end{aligned}$$

In general,

$$a_{2k} = (k+1)(a_0+1). \qquad \text{(step 5)}$$

For odd values of n, we have

$$\begin{aligned} a_3 &= \frac{5}{3}a_1 \\ a_5 &= \frac{7}{5}a_3 = \frac{7}{3}a_1 \\ a_7 &= \frac{9}{7}a_5 = \frac{9}{3}a_1 = 3a_1 \\ &\vdots \end{aligned}$$

In general,

$$a_{2k+1} = \frac{2k+3}{3}a_1. \qquad \text{(step 5 continued)}$$

Putting this together, we have

$$y(x) = \sum_{k=0}^{\infty}(k+1)(a_0+1)x^{2k} + \sum_{k=0}^{\infty}\left(\frac{2k+3}{3}\right)a_1x^{2k+1}. \qquad \text{(step 6)}$$

? Exercise 17.4.1

Find a power series solution for the following differential equations.

a. $y' + 2xy = 0$
b. $(x+1)y' = 3y$

Hint

Follow the problem-solving strategy.

Answer a

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n} = a_0 e^{-x^2}$$

Answer b

$$y(x) = a_0 (x+1)^3$$

Bessel functions

We close this section with a brief introduction to *Bessel functions*. Complete treatment of Bessel functions is well beyond the scope of this course, but we get a little taste of the topic here so we can see how series solutions to differential equations are used in real-world applications. The Bessel equation of order n is given by

$$x^2 y'' + xy' + (x^2 - n^2)y = 0.$$

This equation arises in many physical applications, particularly those involving cylindrical coordinates, such as the vibration of a circular drum head and transient heating or cooling of a cylinder. In the next example, we find a power series solution to the Bessel equation of order 0.

Example 17.4.2: Power Series Solution to the Bessel Equation

Find a power series solution to the Bessel equation of order 0 and graph the solution.

Solution

The Bessel equation of order 0 is given by

$$x^2 y'' + xy' + x^2 y = 0.$$

We assume a solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$. Then $y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and $y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$. Substituting this into the differential equation, we get

$$\begin{aligned}
x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} + x^2 \sum_{n=0}^{\infty} a_n x^n &= 0 && \text{Substitution.} \\
\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^{n+2} &= 0 && \text{Bring external factors within sums.} \\
\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n &= 0 && \text{Re-index third sum.} \\
\sum_{n=2}^{\infty} n(n-1) a_n x^n + a_1 x + \sum_{n=2}^{\infty} n a_n x^n + \sum_{n=2}^{\infty} a_{n-2} x^n &= 0 && \text{Separate } n=1 \text{ term from second sum.} \\
a_1 x + \sum_{n=2}^{\infty} [n(n-1) a_n + n a_n + a_{n-2}] x^n &= 0 && \text{Collect summation terms.} \\
a_1 x + \sum_{n=2}^{\infty} [(n^2 - n) a_n + n a_n + a_{n-2}] x^n &= 0 && \text{Multiply through in first term.} \\
a_1 x + \sum_{n=2}^{\infty} [n^2 a_n + a_{n-2}] x^n &= 0. && \text{Simplify.}
\end{aligned}$$

Then, $a_1 = 0$, and for $n \geq 2$,

$$n^2 a_n + a_{n-2} = 0$$
$$a_n = -\frac{1}{n^2} a_{n-2}.$$

Because $a_1 = 0$, all odd terms are zero. Then, for even values of n, we have

$$a_2 = -\frac{1}{2^2} a_0$$
$$a_4 = -\frac{1}{4^2} a_2 = \frac{1}{4^2 \cdot 2^2} a_0.$$
$$a_6 = -\frac{1}{6^2} a_4 = -\frac{1}{6^2 \cdot 4^2 \cdot 2^2} a_0$$

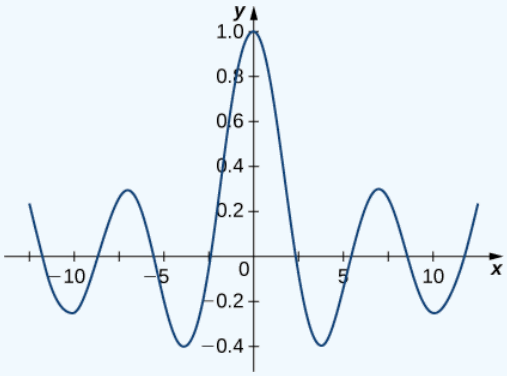
In general,

$$a_{2k} = \frac{(-1)^k}{(2)^{2k}(k!)^2} a_0.$$

Thus, we have

$$y(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2)^{2k}(k!)^2} x^{2k}.$$

The graph appears below.



? Exercise 17.4.2

Verify that the expression found in Example 17.4.2is a solution to the Bessel equation of order 0.

Hint

Differentiate the power series term by term and substitute it into the differential equation.

Key Concepts

- Power series representations of functions can sometimes be used to find solutions to differential equations.
- Differentiate the power series term by term and substitute into the differential equation to find relationships between the power series coefficients.

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