MATH 2E: MULTIVARIABLE CALCULUS



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MATH 2E Multivariable Calculus

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TABLE OF CONTENTS

Licensing

Chapter 14: Multiple Integration

14.1: Iterated Integrals and Area

14.1E: Iterated Integrals and Area (Exercises)

14.2a: Double Integrals Over Rectangular Regions

14.2aE: Double Integrals Part 1 (Exercises)

14.2b: Double Integrals Over General Regions

14.2bE: Double Integrals Part 2 (Exercises)

14.3: Double Integration with Polar Coordinates

14.3b: Double Integrals in Polar Coordinates

14.3E: Double Integrals in Polar Coordinates (Exercises)

14.4: Triple Integrals

14.4E: Triple Integrals (Exercises)

14.4E: Triple Integrals (Exercises 2)

14.5: Triple Integrals in Cylindrical and Spherical Coordinates

14.6: Calculating Centers of Mass and Moments of Inertia

14.7: Change of Variables in Multiple Integrals (Jacobians)

Chapter 14 Review Exercises

Chapter 15: Vector Fields, Line Integrals, and Vector Theorems

15.0: Introduction to Vector Field Chapter

15.1: Vector Fields

15.1E: Vector Fields (Exercises)

15.2: Line Integrals

15.2: Line Integrals - Part 2

15.2E: Line Integrals (Exercises)

15.3: Conservative Vector Fields

15.3E: Conservative Vector Fields (Exercises)

15.4: Green's Theorem

15.4E: Green's Theorem (Exercises)

15.5: Divergence and Curl

15.5E: Divergence and Curl (Exercises)

15.6: Surface Integrals

15.7: Stokes' Theorem

15.8: The Divergence Theorem

Chapter 15 Review Exercises

Index
Glossary
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CHAPTER 14: MULTIPLE INTEGRATION

14.1: Iterated Integrals and Area

14.1E: Iterated Integrals and Area (Exercises)

14.2a: Double Integrals Over Rectangular Regions

14.2aE: Double Integrals Part 1 (Exercises)

14.2b: Double Integrals Over General Regions

14.2bE: Double Integrals Part 2 (Exercises)

14.3: Double Integration with Polar Coordinates

14.3b: Double Integrals in Polar Coordinates

14.3E: Double Integrals in Polar Coordinates (Exercises)

14.4: Triple Integrals

14.4E: Triple Integrals (Exercises)

14.4E: Triple Integrals (Exercises 2)

14.5: Triple Integrals in Cylindrical and Spherical Coordinates

14.6: Calculating Centers of Mass and Moments of Inertia

14.7: Change of Variables in Multiple Integrals (Jacobians)

Chapter 14 Review Exercises

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14.1: ITERATED INTEGRALS AND AREA

In the previous chapter we found that we could differentiate functions of several variables with respect to one variable, while treating all the other variables as constants or coefficients. We can integrate functions of several variables in a similar way. For instance, if we are told that $f_x(x,y) = 2xy$, we can treat y as staying constant and integrate to obtain $f(x,y)$:

$$\begin{aligned} f(x,y) &= \int f_x(x,y)\,dx \\ &= \int 2xy\,dx \\ &= x^2y + C. \end{aligned}$$

Make a careful note about the constant of integration, C. This "constant" is something with a derivative of 0 with respect to x, so it could be any expression that contains only constants and functions of y. For instance, if $f(x,y) = x^2y + \sin y + y^3 + 17$, then $f_x(x,y) = 2xy$. To signify that C is actually a function of y, we write:

$$f(x,y) = \int f_x(x,y)\,dx = x^2y + C(y). \tag{14.1.1}$$

Using this process we can even evaluate definite integrals.

Example 14.1.1: Integrating functions of more than one variable

Evaluate the integral $\int_1^{2y} 2xy\ dx$.

Solution

We find the indefinite integral as before, then apply the Fundamental Theorem of Calculus to evaluate the definite integral:

$$\begin{aligned} \int_1^{2y} 2xy\,dx &= x^2y\Big|_1^{2y} \\ &= (2y)^2y - (1)^2y \\ &= 4y^3 - y. \end{aligned}$$

We can also integrate with respect to y. In general,

$$\int_{h_1(y)}^{h_2(y)} f_x(x,y)\,dx = f(x,y)\Big|_{h_1(y)}^{h_2(y)} = f\big(h_2(y),y\big) - f\big(h_1(y),y\big), \tag{14.1.2}$$

and

$$\int_{g_1(x)}^{g_2(x)} f_y(x,y)\,dy = f(x,y)\Big|_{g_1(x)}^{g_2(x)} = f\big(x,g_2(x)\big) - f\big(x,g_1(x)\big). \tag{14.1.3}$$

Note that when integrating with respect to x, the bounds are functions of y (of the form $x = h_1(y)$ and $x = h_2(y)$) and the final result is also a function of y. When integrating with respect to y, the bounds are functions of x (of the form $y = g_1(x)$ and $y = g_2(x)$) and the final result is a function of x. Another example will help us understand this.

Example 14.1.2: Integrating functions of more than one variable

Evaluate $\int_1^x \left(5x^3y^{-3} + 6y^2\right)\, dy$.

Solution

We consider x as staying constant and integrate with respect to y:

$$\begin{aligned}\int_1^x \left(5x^3y^{-3} + 6y^2\right) dy &= \left(\frac{5x^3y^{-2}}{-2} + \frac{6y^3}{3}\right)\Big|_1^x \\ &= \left(-\frac{5}{2}x^3x^{-2} + 2x^3\right) - \left(-\frac{5}{2}x^3 + 2\right) \\ &= \frac{9}{2}x^3 - \frac{5}{2}x - 2.\end{aligned}$$

Note how the bounds of the integral are from $y = 1$ to $y = x$ and that the final answer is a function of x.

In the previous example, we integrated a function with respect to y and ended up with a function of x. We can integrate this as well. This process is known as **iterated integration**, or **multiple integration**.

Example 14.1.3: Integrating an integral

Evaluate $\int_1^2 \left(\int_1^x \left(5x^3y^{-3} + 6y^2\right) dy\right)\, dx$.

Solution

We follow a standard "order of operations" and perform the operations inside parentheses first (which is the integral evaluated in Example 14.1.2.)

$$\begin{aligned}\int_1^2 \left(\int_1^x \left(5x^3y^{-3} + 6y^2\right) dy\right)\, dx &= \int_1^2 \left(\left[\frac{5x^3y^{-2}}{-2} + \frac{6y^3}{3}\right]\Big|_1^x\right)\, dx \\ &= \int_1^2 \left(\frac{9}{2}x^3 - \frac{5}{2}x - 2\right)\, dx \\ &= \left(\frac{9}{8}x^4 - \frac{5}{4}x^2 - 2x\right)\Big|_1^2 \\ &= \frac{89}{8}.\end{aligned}$$

Note how the bounds of x were $x = 1$ to $x = 2$ and the final result was a number.

The previous example showed how we could perform something called an iterated integral; we do not yet know why we would be interested in doing so nor what the result, such as the number $89/8$, *means*. Before we investigate these questions, we offer some definitions.

Definition: Iterated Integration

Iterated integration is the process of repeatedly integrating the results of previous integrations. Integrating one integral is denoted as follows.

Let a, b, c and d be numbers and let $g_1(x)$, $g_2(x)$, $h_1(y)$ and $h_2(y)$ be functions of x and y, respectively. Then:

1. $\displaystyle\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\ dx\ dy = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \right)\ dy.$
2. $\displaystyle\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\ dy\ dx = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right)\ dx.$

Again make note of the bounds of these iterated integrals.

With $\displaystyle\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\ dx\ dy$, x varies from $h_1(y)$ to $h_2(y)$, whereas y varies from c to d. That is, the bounds of x are *curves*, the curves $x = h_1(y)$ and $x = h_2(y)$, whereas the bounds of y are *constants*, $y = c$ and $y = d$. It is useful to remember that when setting up and evaluating such iterated integrals, we integrate "from curve to curve, then from point to point."

We now begin to investigate *why* we are interested in iterated integrals and *what* they mean.

Area of a plane region

Consider the plane region R bounded by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$, shown in Figure 14.1.1. We learned in Section 7.1 (in Calculus I) that the area of R is given by

$$\int_a^b \big(g_2(x) - g_1(x)\big)\,dx. \tag{14.1.4}$$

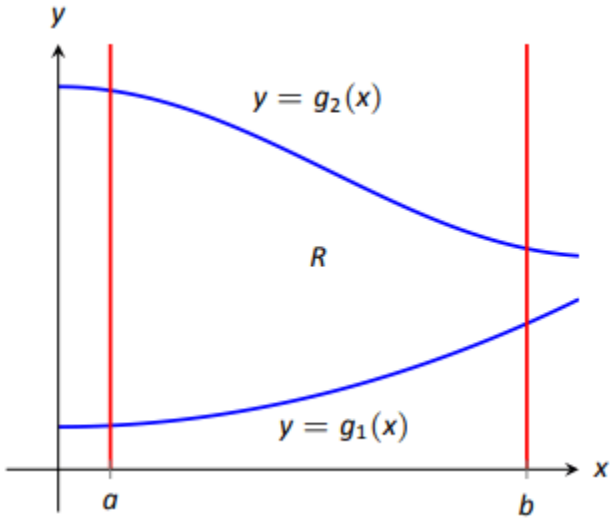


Figure 14.1.1: Calculating the area of a plane region R with an iterated integral.

We can view the expression $\big(g_2(x) - g_1(x)\big)$ as

$$\big(g_2(x) - g_1(x)\big) = \int_{g_1(x)}^{g_2(x)} 1\,dy = \int_{g_1(x)}^{g_2(x)} dy,$$

meaning we can express the area of R as an iterated integral:

$$\text{area of } R = \int_a^b \big(g_2(x) - g_1(x)\big)\,dx = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} dy \right) dx = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\,dx. \tag{14.1.5}$$

In short: a certain iterated integral can be viewed as giving the area of a plane region.

A region R could also be defined by $c \le y \le d$ and $h_1(y) \le x \le h_2(y)$, as shown in Figure 14.1.2. Using a process similar to that above, we have

$$\text{the area of } R = \int_c^d \int_{h_1(y)}^{h_2(y)} dx\, dy. \tag{14.1.6}$$

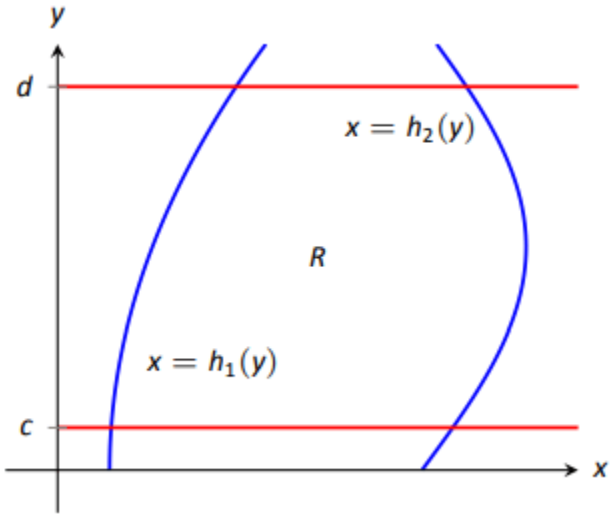


Figure 14.1.2: Calculating the area of a plane region R with an iterated integral.

We state this formally in a theorem.

THEOREM 14.1.1: Area of a plane region

1. Let R be a plane region bounded by $a \le x \le b$ and $g_1(x) \le y \le g_2(x)$, where g_1 and g_2 are continuous functions on $[a, b]$. The area A of R is
$$A = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\, dx. \tag{14.1.7}$$
2. Let R be a plane region bounded by $c \le y \le d$ and $h_1(y) \le x \le h_2(y)$, where h_1 and h_2 are continuous functions on $[c, d]$. The area A of R is
$$A = \int_c^d \int_{h_1(y)}^{h_2(y)} dx\, dy. \tag{14.1.8}$$

The following examples should help us understand this theorem.

Example 14.1.4: Area of a rectangle

Find the area A of the rectangle with corners $(-1, 1)$ and $(3, 3)$, as shown in Figure 14.1.3.

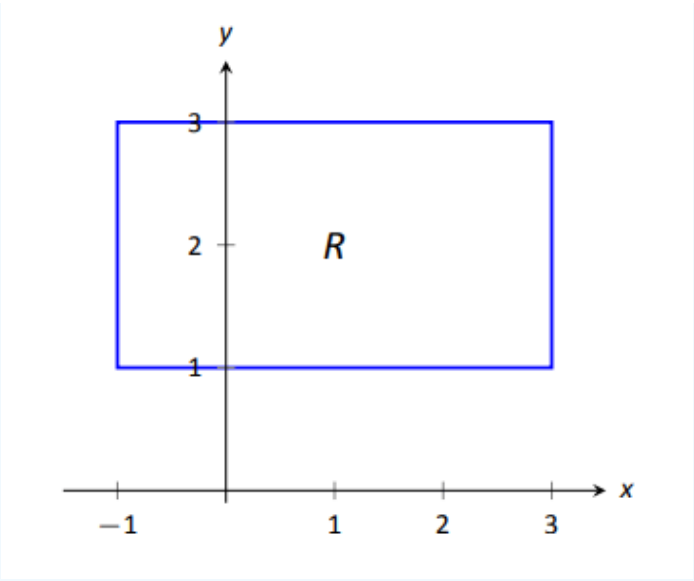


Figure 14.1.3: Calculating the area of a rectangle with an iterated integral in Example 14.1.4.

Solution

Multiple integration is obviously overkill in this situation, but we proceed to establish its use.

The region R is bounded by $x = -1$, $x = 3$, $y = 1$ and $y = 3$. Choosing to integrate with respect to y first, we have

$$A = \int_{-1}^{3}\int_{1}^{3} 1\,dy\,dx = \int_{-1}^{3}\left(y\Big|_{1}^{3}\right)dx = \int_{-1}^{3} 2\,dx = 2x\Big|_{-1}^{3} = 8.$$

We could also integrate with respect to x first, giving:

$$A = \int_{1}^{3}\int_{-1}^{3} 1\,dx\,dy = \int_{1}^{3}\left(x\Big|_{-1}^{3}\right)dy = \int_{1}^{3} 4\,dy = 4y\Big|_{1}^{3} = 8.$$

Clearly there are simpler ways to find this area, but it is interesting to note that this method works.

Example 14.1.5: Area of a triangle

Find the area A of the triangle with vertices at $(1, 1)$, $(3, 1)$ and $(5, 5)$, as shown in Figure 14.1.4.

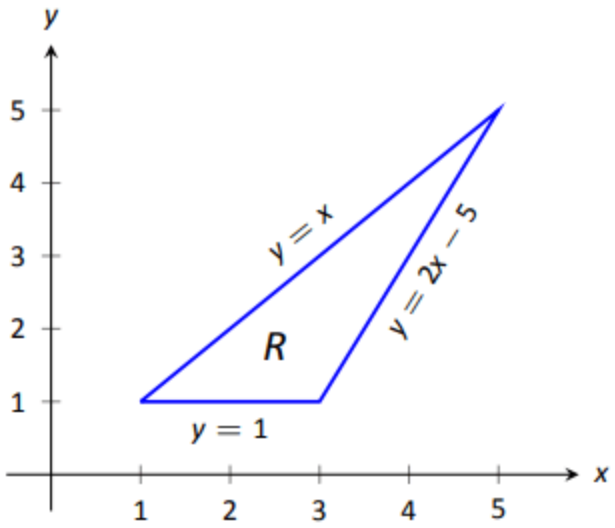


Figure 14.1.4: Calculating the area of a triangle with iterated integrals in Example 14.1.5.

Solution

The triangle is bounded by the lines as shown in the figure. Choosing to integrate with respect to x first gives that x is bounded by $x = y$ to $x = \frac{y+5}{2}$, while y is bounded by $y = 1$ to $y = 5$. (Recall that since x-values increase from left to right, the leftmost curve, $x = y$, is the lower bound and the rightmost curve, $x = (y + 5)/2$, is the upper bound.) The area is

$$\begin{aligned} A &= \int_1^5 \int_y^{\frac{y+5}{2}} dx\, dy \\ &= \int_1^5 \left(x \Big|_y^{\frac{y+5}{2}} \right) dy \\ &= \int_1^5 \left(-\frac{1}{2}y + \frac{5}{2} \right) dy \\ &= \left(-\frac{1}{4}y^2 + \frac{5}{2}y \right) \Big|_1^5 \\ &= 4. \end{aligned}$$

We can also find the area by integrating with respect to y first. In this situation, though, we have two functions that act as the lower bound for the region R, $y = 1$ and $y = 2x - 5$. This requires us to use two iterated integrals. Note how the x-bounds are different for each integral:

$$\begin{aligned} A &= \int_1^3 \int_1^x 1\, dy\, dx \quad + \quad \int_3^5 \int_{2x-5}^x 1\, dy\, dx \\ &= \int_1^3 \big(y\big)\Big|_1^x dx \quad + \quad \int_3^5 \big(y\big)\Big|_{2x-5}^x dx \\ &= \int_1^3 \big(x - 1\big)\, dx \quad + \quad \int_3^5 \big(-x + 5\big)\, dx \\ &= 2 \quad + \quad 2 \\ &= 4. \end{aligned}$$

As expected, we get the same answer both ways.

Example 14.1.6: Area of a plane region

Find the area of the region enclosed by $y = 2x$ and $y = x^2$, as shown in Figure 14.1.5.

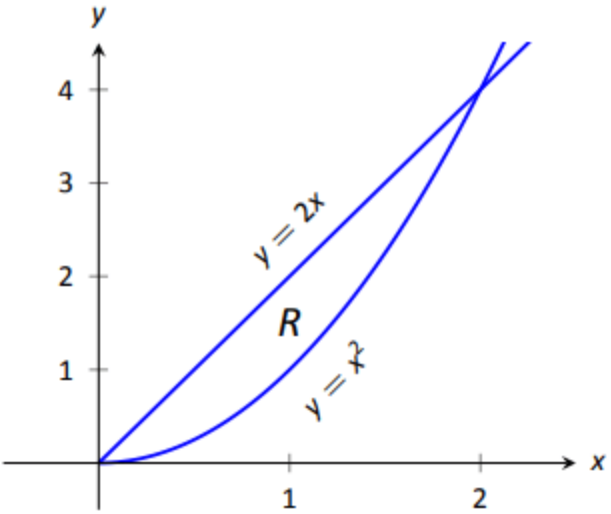


Figure 14.1.5: Calculating the area of a plane region with iterated integrals in Example 14.1.6.

Solution

Once again we'll find the area of the region using both orders of integration.

Using $dy\ dx$:

$$\int_0^2 \int_{x^2}^{2x} 1\, dy\, dx = \int_0^2 (2x - x^2)\, dx = \left(x^2 - \frac{1}{3}x^3\right)\Big|_0^2 = \frac{4}{3}. \tag{14.1.9}$$

Using $dx\ dy$:

$$\int_0^4 \int_{y/2}^{\sqrt{y}} 1\, dx\, dy = \int_0^4 (\sqrt{y} - y/2)\, dy = \left(\frac{2}{3}y^{3/2} - \frac{1}{4}y^2\right)\Big|_0^4 = \frac{4}{3}. \tag{14.1.10}$$

Changing Order of Integration

In each of the previous examples, we have been given a region R and found the bounds needed to find the area of R using both orders of integration. We integrated using both orders of integration to demonstrate their equality.

We now approach the skill of describing a region using both orders of integration from a different perspective. Instead of starting with a region and creating iterated integrals, we will start with an iterated integral and rewrite it in the other integration order. To do so, we'll need to understand the region over which we are integrating.

The simplest of all cases is when both integrals are bound by constants. The region described by these bounds is a rectangle (see Example 14.1.4), and so:

$$\int_a^b \int_c^d 1\, dy\, dx = \int_c^d \int_a^b 1\, dx\, dy. \tag{14.1.11}$$

When the inner integral's bounds are not constants, it is generally very useful to sketch the bounds to determine what the region we are integrating over looks like. From the sketch we can then rewrite the integral with the other order of integration.

Examples will help us develop this skill.

Example 14.1.7: Changing the order of integration

Rewrite the iterated integral $\int_0^6 \int_0^{x/3} 1 \; dy \; dx$ with the order of integration $dx \; dy$.

Solution

We need to use the bounds of integration to determine the region we are integrating over.

The bounds tell us that y is bounded by 0 and $x/3$; x is bounded by 0 and 6. We plot these four curves: $y = 0$, $y = x/3$, $x = 0$ and $x = 6$ to find the region described by the bounds. Figure 14.1.6 shows these curves, indicating that R is a triangle.

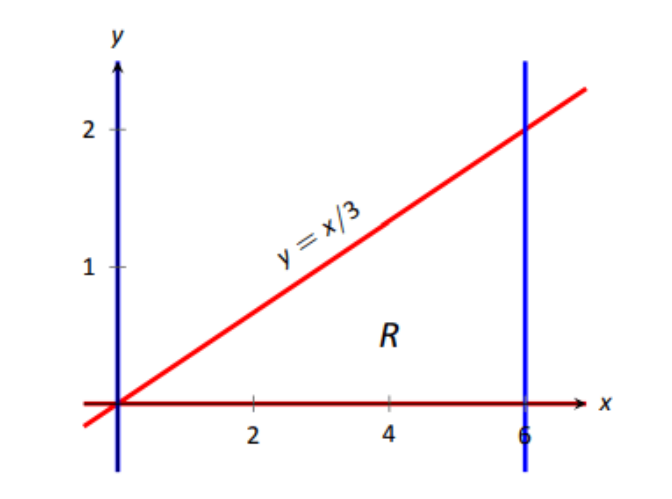


Figure 14.1.6: Sketching the region R described by the iterated integral in Example 14.1.7.

To change the order of integration, we need to consider the curves that bound the x-values. We see that the lower bound is $x = 3y$ and the upper bound is $x = 6$. The bounds on y are 0 to 2. Thus we can rewrite the integral as $\int_0^2 \int_{3y}^6 1 \; dx \; dy$.

Example 14.1.8: Changing the order of integration

Change the order of integration of $\int_0^4 \int_{y^2/4}^{(y+4)/2} 1 \; dx \; dy$.

Solution

We sketch the region described by the bounds to help us change the integration order. x is bounded below and above (i.e., to the left and right) by $x = y^2/4$ and $x = (y + 4)/2$ respectively, and y is bounded between 0 and 4. Graphing the previous curves, we find the region R to be that shown in Figure 14.1.7.

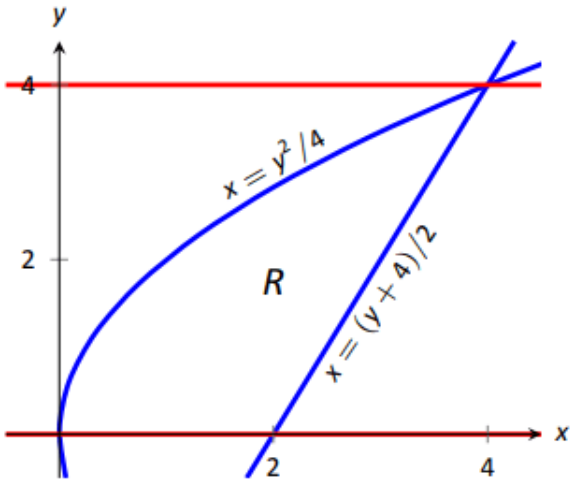


Figure 14.1.7: Drawing the region determined by the bounds of integration in Example 14.1.8.

To change the order of integration, we need to establish curves that bound y. The figure makes it clear that there are two lower bounds for y: $y = 0$ on $0 \le x \le 2$, and $y = 2x - 4$ on $2 \le x \le 4$. Thus we need two double integrals. The upper bound for each is $y = 2\sqrt{x}$. Thus we have

$$\int_0^4 \int_{y^2/4}^{(y+4)/2} 1\,dx\,dy = \int_0^2 \int_0^{2\sqrt{x}} 1\,dy\,dx + \int_2^4 \int_{2x-4}^{2\sqrt{x}} 1\,dy\,dx.$$

This section has introduced a new concept, the iterated integral. We developed one application for iterated integration: area between curves. However, this is not new, for we already know how to find areas bounded by curves.

In the next section we apply iterated integration to solve problems we currently do not know how to handle. The "real" goal of this section was not to learn a new way of computing area. Rather, our goal was to learn how to define a region in the plane using the bounds of an iterated integral. That skill is very important in the following sections.

14.1E: Iterated Integrals and Area (Exercises)

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14.1E: Iterated Integrals and Area (Exercises)

Terms and Concepts

1. When integrating $f_x(x,y)$ with respect to x, the constant of integration C is really which: $C(x)$ or $C(y)$? What does this mean?

Answer:

The constant of integration will be $C(y)$ since y is considered a constant in the integration and thus, the original function f could have terms that are functions of y that were lost when taking the partial derivative with respect to the variable x.

2. Integrating an iterated integral is called _____ _____.

Answer:

iterated integration or **multiple integration**

3. When evaluating an iterated integral of the form $\int_a^b \int_{g_1(x)}^{g_2(x)} \, dy \, dx$, we integrate from _____ to _____, then from _____ to _____.

Answer:

We integrate from **bottom** to **top**, then from **left** to **right**. Here, from $y = g_1(x)$ to $y = g_2(x)$, then from $x = a$ to $x = b$.
More generally, we integrate from **curve** to **curve** and then from **point** to **point**.

4. One understanding of an iterated integral is that $\int_a^b \int_{g_1(x)}^{g_2(x)} \, dy \, dx$ gives the _____ of a plane region.

Answer:

area

Problems

In Exercises 5-10, evaluate the integral and subsequent iterated integral.

5. (a) $\int_2^5 (6x^2 + 4xy - 3y^2) \, dy$
(b) $\int_{-3}^2 \int_2^5 (6x^2 + 4xy - 3y^2) \, dy \, dx$

Answer:

(a) $\int_2^5 (6x^2 + 4xy - 3y^2) \, dy = (6x^2y + 2xy^2 - y^3)\Big|_2^5 = 30x^2 + 50x - 125 - (12x^2 + 8x - 8) = 18x^2 + 42x - 117$
(b) $\int_{-3}^2 \int_2^5 (6x^2 + 4xy - 3y^2) \, dy \, dx = \int_{-3}^2 (18x^2 + 42x - 117) \, dx = (6x^3 + 21x^2 - 117x)\Big|_{-3}^2 = 48 + 84 - 234 - (-162 + 189 + 351) = -102 - 378 = -480$

6. (a) $\int_0^\pi (2x\cos y + \sin x) \, dx$
(b) $\int_0^{\pi/2} \int_0^\pi (2x\cos y + \sin x) \, dx \, dy$

7. (a) $\int_1^x (x^2y - y + 2) \, dy$

(b) $\int_0^2 \int_1^x (x^2y - y + 2)\ dy\ dx$

Answer:

(a) $\int_1^x (x^2y - y + 2)\ dy = \left(\frac{x^2y^2}{2} - \frac{y^2}{2} + 2y\right)\Big|_1^x = \frac{x^4}{2} - \frac{x^2}{2} + 2x - (\frac{x^2}{2} - \frac{1}{2} + 2) = \frac{x^4}{2} - x^2 + 2x - \frac{3}{2}$

(b) $\int_0^2 \int_1^x (x^2y - y + 2)\ dy\ dx = \int_0^2 \left(\frac{x^4}{2} - x^2 + 2x - \frac{3}{2}\right)\ dx = \left(\frac{x^5}{10} - \frac{x^3}{3} + x^2 - \frac{3}{2}x\right)\Big|_0^2 = \frac{32}{10} - \frac{8}{3} + 4 - 3 - 0 = \frac{96}{30} - \frac{80}{30} + \frac{30}{30} = \frac{23}{15}$

8. (a) $\int_y^{y^2} (x - y)\ dx$

(b) $\int_{-1}^1 \int_y^{y^2} (x - y)\ dx\ dy$

9. (a) $\int_0^y (\cos x \sin y)\ dx$

(b) $\int_0^\pi \int_0^y (\cos x \sin y)\ dx\ dy$

Answer:

(a) $\int_0^y (\cos x \sin y)\ dx = (\sin x \sin y)\Big|_0^y = \sin^2 y$

(b) $\int_0^\pi \int_0^y (\cos x \sin y)\ dx\ dy = \int_0^\pi \sin^2 y\ dy = \int_0^\pi \frac{1 - \cos 2y}{2}\ dy = \frac{y - \frac{1}{2}\sin 2y}{2}\Big|_0^\pi = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

10. (a) $\int_0^x \left(\frac{1}{1 + x^2}\right)\ dy$

(b) $\int_1^2 \int_0^x \left(\frac{1}{1 + x^2}\right)\ dy\ dx$

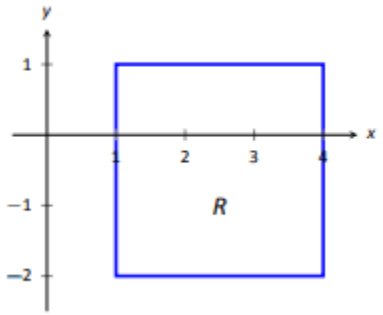
Answer:

(a) $\int_0^x \left(\frac{1}{1 + x^2}\right)\ dy = \frac{y}{1 + x^2}\Big|_0^x = \frac{x}{1 + x^2}$

(b) $\int_1^2 \int_0^x \left(\frac{1}{1 + x^2}\right)\ dy\ dx = \int_1^2 \left(\frac{x}{1 + x^2}\right)\ dx = \frac{1}{2}\ln(1 + x^2)\Big|_1^2 = \frac{1}{2}\ln(5) - \frac{1}{2}\ln(2) = \frac{1}{2}\ln(\frac{5}{2})$

In Exercises 11-16, a graph of a planar region R is given. Give the iterated integrals, with both orders of integration $dy\ dx$ and $dx\ dy$, that give the area of R. Evaluate one of the iterated integrals to find the area.

11.

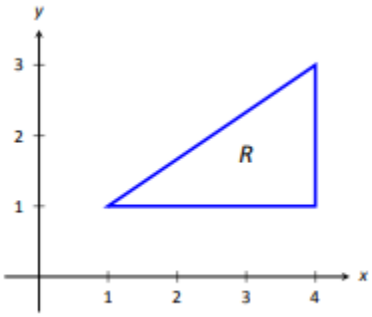


Answer:

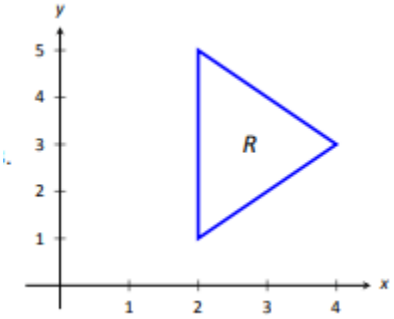
$\text{Area} = \int_1^4 \int_{-2}^1 1\ dy\ dx = \int_{-2}^1 \int_1^4 1\ dx\ dy$

$\int_{-2}^1 \int_1^4 1\ dx\ dy = \int_{-2}^1 x\Big|_1^4\ dy = \int_{-2}^1 3\ dy = 3y\Big|_{-2}^1 = 3(3) = 9\ \text{units}^2$

12.



13.



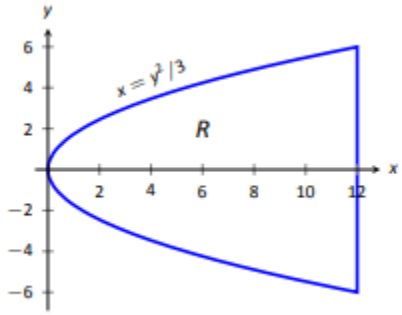
Answer:

$$\text{Area} = \int_2^4 \int_{x-1}^{7-x} 1 \; dy \; dx \quad = \quad \int_1^3 \int_2^{y+1} 1 \; dx \; dy \quad + \quad \int_3^5 \int_2^{7-y} 1 \; dx \; dy$$

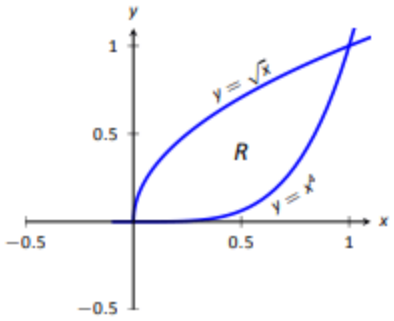
$$\int_2^4 \int_{x-1}^{7-x} 1 \; dy \; dx \quad = \quad \int_2^4 y\Big|_{x-1}^{7-x} \; dx \quad = \quad \int_2^4 (8-2x) \; dx \quad = \quad (8x - x^2)\Big|_2^4 = 32 - 16 - 16 + 4 \quad =$$

$$4 \text{ units}^2$$

14.



15.



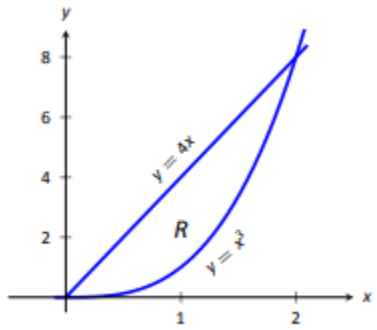
Answer:

$$\text{Area} = \int_0^1 \int_{x^4}^{\sqrt{x}} 1 \; dy \; dx \quad = \quad \int_0^1 \int_{y^2}^{\sqrt[4]{y}} 1 \; dx \; dy$$

$$\int_0^1 \int_{x^4}^{\sqrt{x}} dy \; dx \quad = \quad \int_0^1 y\Big|_{x^4}^{\sqrt{x}} dx \quad = \quad \int_0^1 (\sqrt{x} - x^4) \; dx \quad = \quad (\tfrac{2}{3}x^{3/2} - \tfrac{x^5}{5})\Big|_0^1 \quad = \quad \tfrac{2}{3} - \tfrac{1}{5} \quad =$$

$$\tfrac{10}{15} - \tfrac{3}{15} \; = \; \frac{7}{15} \text{ units}^2$$

16.

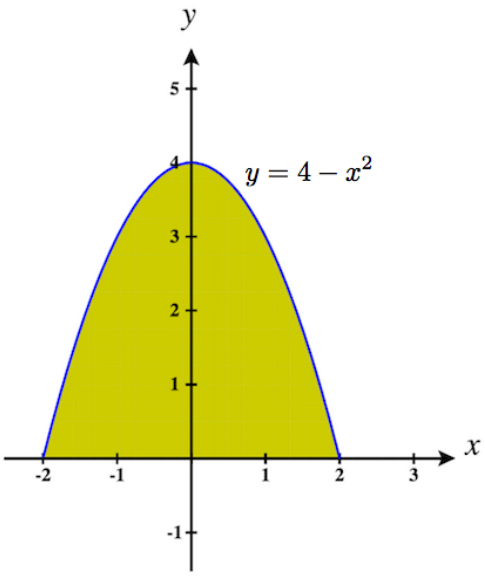


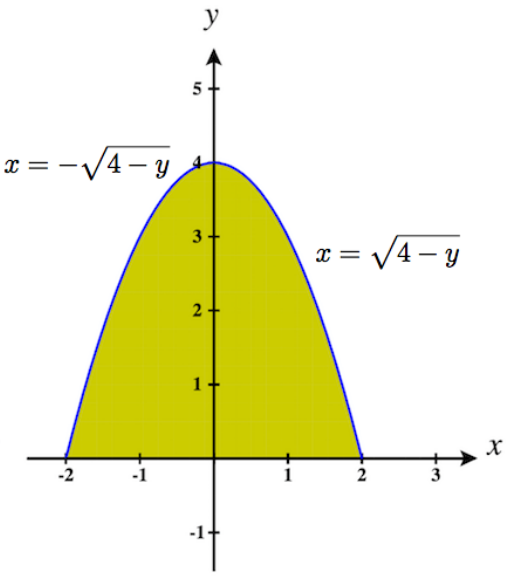
In Exercises 17-22, iterated integrals are given that compute the area of a region R in the xy-plane. Sketch the region R, and give the iterated integral(s) that give the area of R with the opposite order of integration.

17. $\int_{-2}^{2} \int_{0}^{4-x^2} dy \; dx$

Answer:

$$\int_{-2}^{2} \int_{0}^{4-x^2} dy \; dx \quad = \quad \int_0^4 \int_{-\sqrt{4-y}}^{\sqrt{4-y}} dx \; dy$$



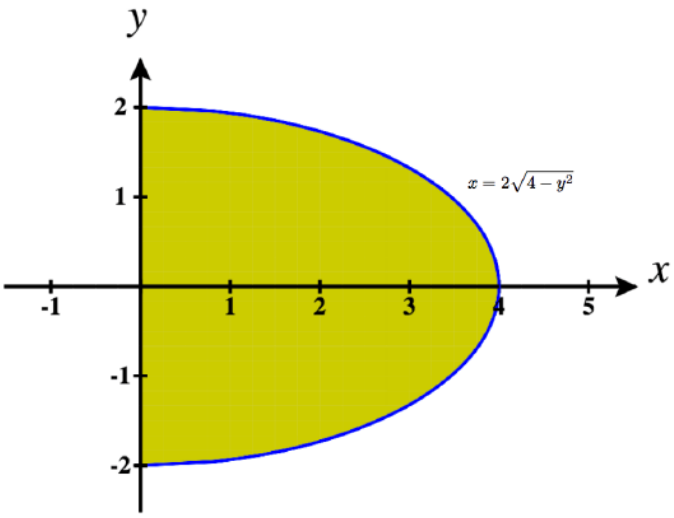


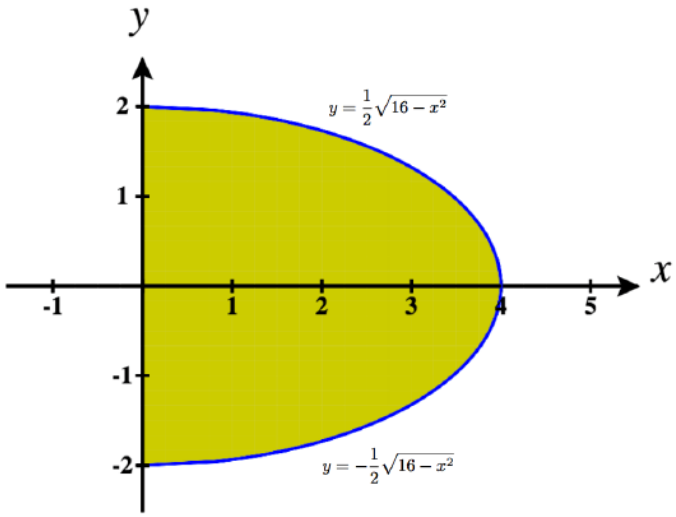
18. $\int_0^1 \int_{5-5x}^{5-5x^2} dy\ dx$

19. $\int_{-2}^{2} \int_0^{2\sqrt{4-y^2}} dx\ dy$

Answer:

$$\int_{-2}^{2} \int_0^{2\sqrt{4-y^2}} dx\ dy \quad = \quad \int_0^4 \int_{-\frac{1}{2}\sqrt{16-x^2}}^{\frac{1}{2}\sqrt{16-x^2}} dy\ dx$$



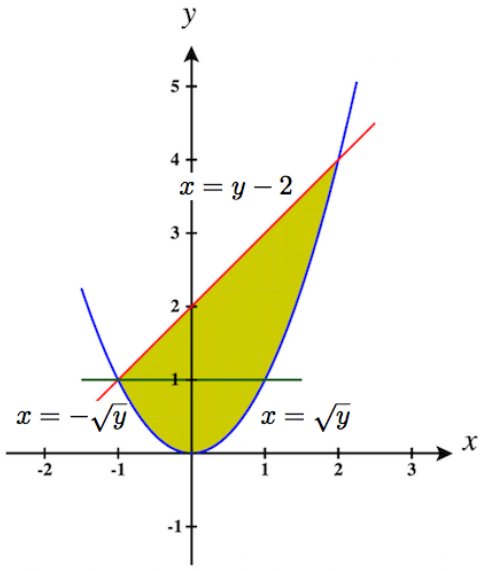


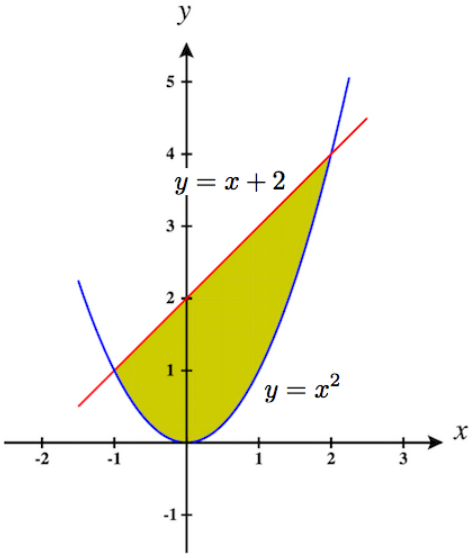
20. $\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \, dy \, dx$

21. $\int_{0}^{1} \int_{-\sqrt{y}}^{\sqrt{y}} \, dx \, dy + \int_{1}^{4} \int_{y-2}^{\sqrt{y}} \, dx \, dy$

Answer:

$$\int_{0}^{1} \int_{-\sqrt{y}}^{\sqrt{y}} \, dx \, dy + \int_{1}^{4} \int_{y-2}^{\sqrt{y}} \, dx \, dy \quad = \quad \int_{-1}^{2} \int_{x^2}^{x+2} \, dy \, dx$$





22. $\int_{-1}^{1} \int_{(x-1)/2}^{(1-x)/2} \, dy \, dx$

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14.2A: DOUBLE INTEGRALS OVER RECTANGULAR REGIONS

Learning Objectives

- Recognize when a function of two variables is integrable over a rectangular region.
- Recognize and use some of the properties of double integrals.
- Evaluate a double integral over a rectangular region by writing it as an iterated integral.
- Use a double integral to calculate the area of a region, volume under a surface, or average value of a function over a plane region.

In this section we investigate double integrals and show how we can use them to find the volume of a solid over a rectangular region in the xy-plane. Many of the properties of double integrals are similar to those we have already discussed for single integrals.

Volumes and Double Integrals

We begin by considering the space above a rectangular region R. Consider a continuous function $f(x,y) \geq 0$ of two variables defined on the closed rectangle R:

$$R = [a,b] \times [c,d] = \{(x,y) \in \mathbb{R}^2 | \, a \leq x \leq b, \, c \leq y \leq d\}$$

Here $[a,b] \times [c,d]$ denotes the Cartesian product of the two closed intervals $[a,b]$ and $[c,d]$. It consists of rectangular pairs (x,y) such that $a \leq x \leq b$ and $c \leq y \leq d$. The graph of f represents a surface above the xy-plane with equation $z = f(x,y)$ where z is the height of the surface at the point (x,y). Let S be the solid that lies above R and under the graph of f (Figure $14.2a.1$). The base of the solid is the rectangle R in the xy-plane. We want to find the volume V of the solid S.

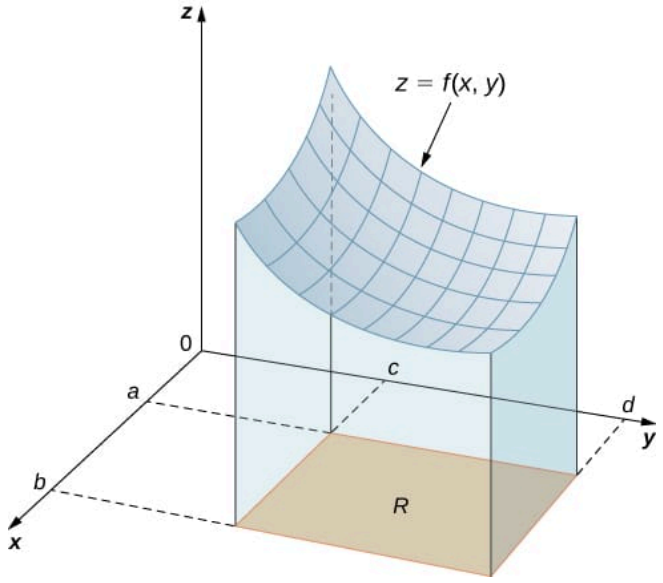


Figure $14.2a.1$: The graph of $f(x,y)$ *over the rectangle R in the* xy*-plane is a curved surface.*

We divide the region R into small rectangles R_{ij}, each with area ΔA and with sides Δx and Δy (Figure $14.2a.2$). We do this by dividing the interval $[a,b]$ into m subintervals and dividing the interval $[c,d]$ into n subintervals. Hence $\Delta x = \frac{b-a}{m}$, $\Delta y = \frac{d-c}{n}$, and $\Delta A = \Delta x \Delta y$.

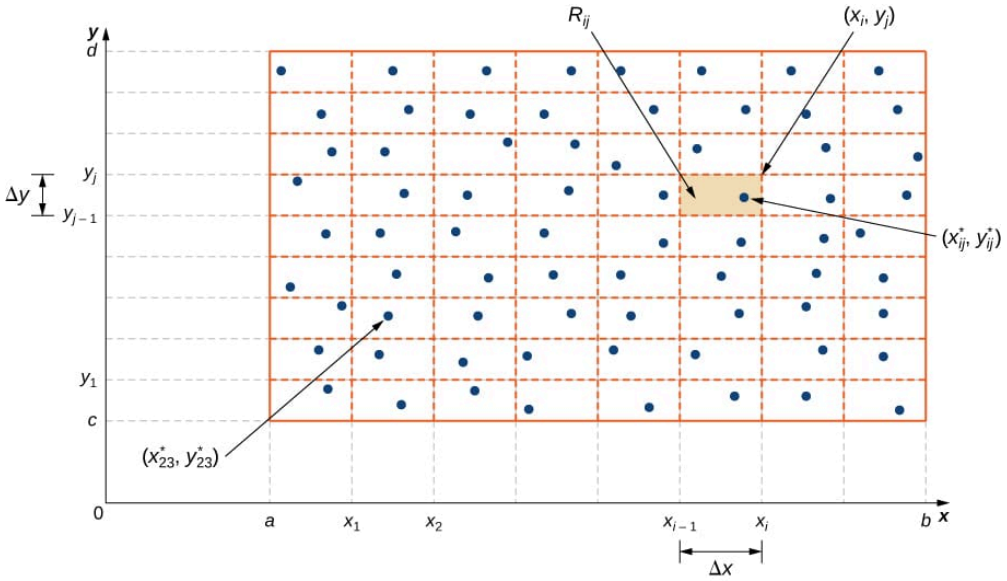


Figure $14.2a.2$: Rectangle R is divided into small rectangles R_{ij} each with area ΔA.

The volume of a thin rectangular box above R_{ij} is $f(x^*_{ij},\ y^*_{ij})\ \Delta A$, where $(x^*_{ij},\ y^*_{ij})$ is an arbitrary sample point in each R_{ij} as shown in the following figure, $f(x^*_{ij},\ y^*_{ij})$ is the height of the corresponding thin rectangular box, and ΔA is the area of each rectangle R_{ij}.

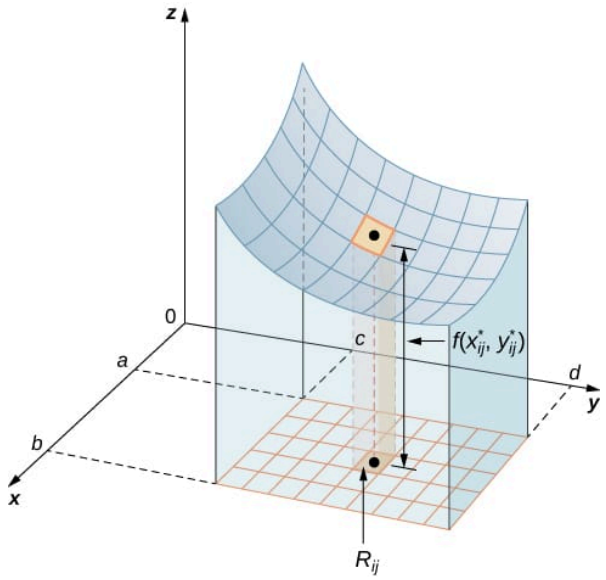


Figure $14.2a.3$: A thin rectangular box above R_{ij} with height $f(x^*_{ij},\ y^*_{ij})$.

Using the same idea for all the subrectangles, we obtain an approximate volume of the solid S as

$$V \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x^*_{ij},\ y^*_{ij})\Delta A.$$

This sum is known as a **double Riemann sum** and can be used to approximate the value of the volume of the solid. Here the double sum means that for each subrectangle we evaluate the function at the chosen point, multiply by the area of each rectangle, and then add all the results.

As we have seen in the single-variable case, we obtain a better approximation to the actual volume if m and n become larger.

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

or

$$V = \lim_{\Delta x,\, \Delta y\to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A.$$

Note that the sum approaches a limit in either case and the limit is the volume of the solid with the base R. Now we are ready to define the double integral.

Definition: Double Integral over a Rectangular Region R

The double integral of the function $f(x,\ y)$ over the rectangular region R in the xy-plane is defined as

$$\iint_R f(x,\, y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A.$$

If $f(x,y) \geq 0$, then the volume V of the solid S, which lies above R in the xy-plane and under the graph of f, is the double integral of the function $f(x,y)$ over the rectangle R. If the function is ever negative, then the double integral can be considered a "signed" volume in a manner similar to the way we defined net signed area in The Definite Integral.

Example $14.2a.1$: Setting up a Double Integral and Approximating It by Double Sums

Consider the function $z = f(x,\ y) = 3x^2 - y$ over the rectangular region $R = [0,2] \times [0,2]$ (Figure $14.2a.4$).

a. Set up a double integral for finding the value of the signed volume of the solid S that lies above R and "under" the graph of f.
b. Divide R into four squares with $m = n = 2$, and choose the sample point as the upper right corner point of each square (1,1),(2,1),(1,2), and (2,2) (Figure $14.2a.4$) to approximate the signed volume of the solid S that lies above R and "under" the graph of f.
c. Divide R into four squares with $m = n = 2$, and choose the sample point as the midpoint of each square: (1/2, 1/2), (3/2, 1/2), (1/2,3/2), and (3/2, 3/2) to approximate the signed volume.

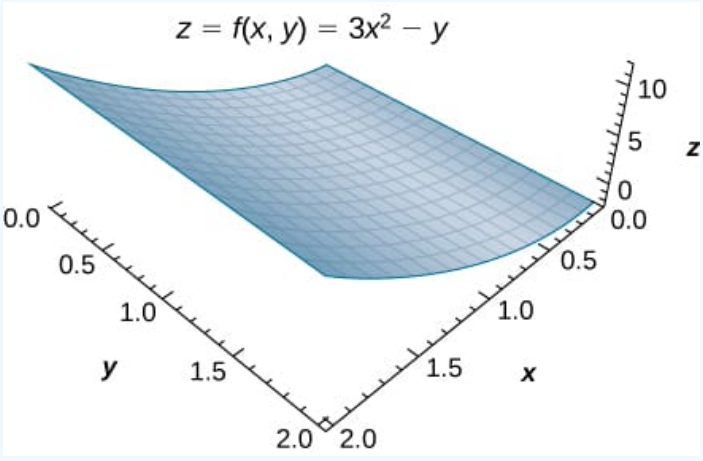


Figure $14.2a.4$: The function $z = f(x,y)$ graphed over the rectangular region $R = [0,2] \times [0,2]$.

Solution

a. As we can see, the function $z = f(x,y) = 3x^2 - y$ is above the plane. To find the signed volume of S, we need to divide the region R into small rectangles R_{ij}, each with area ΔA and with sides Δx and Δy, and choose (x_{ij}^*, y_{ij}^*) as sample points in each R_{ij}. Hence, a double integral is set up as

$$V = \iint_R (3x^2 - y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} [3(x_{ij}^*)^2 - y_{ij}^*]\Delta A.$$

b. Approximating the signed volume using a Riemann sum with $m = n = 2$ we have $\Delta A = \Delta x \Delta y = 1 \times 1 = 1$. Also, the sample points are (1, 1), (2, 1), (1, 2), and (2, 2) as shown in the following figure.

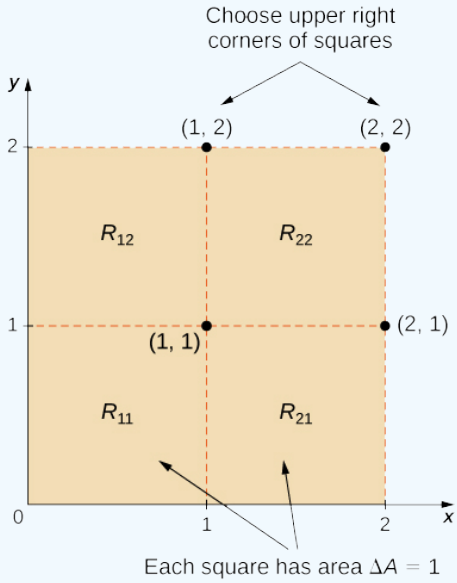


Figure $14.2a.5$: Subrectangles for the rectangular region $R = [0, 2] \times [0, 2]$.

Hence,

$$\begin{aligned}
V &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\Delta A \\
&= \sum_{i=1}^{2} (f(x_{i1}^*, y_{i1}^*) + f(x_{i2}^*, y_{i2}^*))\Delta A \\
&= f(x_{11}^*, y_{11}^*)\Delta A + f(x_{21}^*, y_{21}^*)\Delta A + f(x_{12}^*, y_{12}^*)\Delta A + f(x_{22}^*, y_{22}^*)\Delta A \\
&= f(1, 1)(1) + f(2, 1)(1) + f(1, 2)(1) + f(2, 2)(1) \\
&= (3 - 1)(1) + (12 - 1)(1) + (3 - 2)(1) + (12 - 2)(1) \\
&= 2 + 11 + 1 + 10 = 24.
\end{aligned}$$

c. Approximating the signed volume using a Riemann sum with $m = n = 2$ we have$\Delta A = \Delta x \Delta y = 1 \times 1 = 1$. In this case the sample points are (1/2, 1/2), (3/2, 1/2), (1/2, 3/2), and (3/2, 3/2).
Hence,

$$\begin{aligned}
V &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\Delta A \\
&= f(x_{11}^*, y_{11}^*)\Delta A + f(x_{21}^*, y_{21}^*)\Delta A + f(x_{12}^*, y_{12}^*)\Delta A + f(x_{22}^*, y_{22}^*)\Delta A \\
&= f(1/2, 1/2)(1) + f(3/2, 1/2)(1) + f(1/2, 3/2)(1) + f(3/2, 3/2)(1) \\
&= \left(\frac{3}{4} - \frac{1}{4}\right)(1) + \left(\frac{27}{4} - \frac{1}{2}\right)(1) + \left(\frac{3}{4} - \frac{3}{2}\right)(1) + \left(\frac{27}{4} - \frac{3}{2}\right)(1) \\
&= \frac{2}{4} + \frac{25}{4} + \left(-\frac{3}{4}\right) + \frac{21}{4} = \frac{45}{4} = 11.
\end{aligned}$$

Analysis

Notice that the approximate answers differ due to the choices of the sample points. In either case, we are introducing some error because we are using only a few sample points. Thus, we need to investigate how we can achieve an accurate answer.

Exercise $14.2a.1$

Use the same function$z = f(x, y) = 3x^2 - y$ over the rectangular region $R = [0, 2] \times [0, 2]$.

Divide R into the same four squares with $m = n = 2$, and choose the sample points as the upper left corner point of each square (0,1), (1,1), (0,2), and (1,2) (Figure $14.2a.5$) to approximate the signed volume of the solid S that lies above R and "under" the graph of f.

Hint available at the end of this section.

Answer available at the end of this section.

Note that we developed the concept of double integral using a rectangular region R. This concept can be extended to any general region. However, when a region is not rectangular, the subrectangles may not all fit perfectly into R, particularly if the base area is curved. We examine this situation in more detail in the next section, where we study regions that are not always rectangular and subrectangles may not fit perfectly in the region R. Also, the heights may not be exact if the surface $z = f(x, y)$ is curved. However, the errors on the sides and the height where the pieces may not fit perfectly within the solid S approach 0 as m and n approach infinity. Also, the double integral of the function $z = f(x, y)$ exists provided that the function f is not too discontinuous. If the function is bounded and continuous over R except on a finite number of smooth curves, then the double integral exists and we say that f is integrable over R.

Since $\Delta A = \Delta x \Delta y = \Delta y \Delta x$, we can express dA as $dx\ dy$ or $dy\ dx$. This means that, when we are using rectangular coordinates, the double integral over a region R denoted by

$$\iint_R f(x, y)\, dA$$

can be written as

$$\iint_R f(x, y)\, dx\, dy$$

or

$$\iint_R f(x, y)\, dy\, dx.$$

Now let's list some of the properties that can be helpful to compute double integrals.

Properties of Double Integrals

The properties of double integrals are very helpful when computing them or otherwise working with them. We list here six properties of double integrals. Properties 1 and 2 are referred to as the linearity of the integral, property 3 is the additivity of the integral, property 4 is the monotonicity of the integral, and property 5 is used to find the bounds of the integral. Property 6 is used if $f(x, y)$ is a product of two functions $g(x)$ and $h(y)$.

Theorem: Properties of Double Integrals

Assume that the functions $f(x, y)$ and $g(x, y)$ are integrable over the rectangular region R; S and T are subregions of R; and assume that m and M are real numbers.

i. The sum $f(x, y) + g(x, y)$ is integrable and

$$\iint_R [f(x,y)+g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA.$$

ii. If c is a constant, then $cf(x,y)$ is integrable and

$$\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA.$$

iii. If $R = S \cup T$ and $S \cap T = \emptyset$ except an overlap on the boundaries, then

$$\iint_R f(x,y)\,dA = \iint_S f(x,y)\,dA + \iint_T f(x,y)\,dA.$$

iv. If $f(x,y) \geq g(x,y)$ for (x,y) in R, then

$$\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA.$$

v. If $m \leq f(x,y) \leq M$ and $A(R) = \text{ the area of } R$, then

$$m \cdot A(R) \leq \iint_R f(x,y)\,dA \leq M \cdot A(R).$$

vi. In the case where $f(x,y)$ can be factored as a product of a function $g(x)$ of x only and a function $h(y)$ of y only, then over the region $R = \{(x,y) \mid a \leq x \leq b,\ c \leq y \leq d\}$, the double integral can be written as

$$\iint_R f(x,y)\,dA = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right).$$

These properties are used in the evaluation of double integrals, as we will see later. We will become skilled in using these properties once we become familiar with the computational tools of double integrals. So let's get to that now.

Iterated Integrals

So far, we have seen how to set up a double integral and how to obtain an approximate value for it. We can also imagine that evaluating double integrals by using the definition can be a very lengthy process if we choose larger values for m and n.Therefore, we need a practical and convenient technique for computing double integrals. In other words, we need to learn how to compute double integrals without employing the definition that uses limits and double sums.

The basic idea is that the evaluation becomes easier if we can break a double integral into single integrals by integrating first with respect to one variable and then with respect to the other. The key tool we need is called an iterated integral.

Definitions: Iterated Integrals

Assume a, b, c, and d are real numbers. We define an *iterated integral* for a function $f(x,y)$ over the rectangular region $R = [a,b] \times [c,d]$ as

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$$

or

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy.$$

The notation $\int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$ means that we integrate $f(x,y)$ with respect to y while holding x constant. Similarly, the notation $\int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$ means that we integrate $f(x,y)$ with respect to x while holding y constant. The fact that double integrals can

be split into iterated integrals is expressed in Fubini's theorem. Think of this theorem as an essential tool for evaluating double integrals.

Theorem: Fubini's Theorem

Suppose that $f(x,y)$ is a function of two variables that is continuous over a rectangular region $R = \{(x,y) \in \mathbb{R}^2 |\ a \le x \le b,\ c \le y \le d\}$. Then we see from Figure $14.2a.6$ that the double integral of f over the region equals an iterated integral,

$$\iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

More generally, Fubini's theorem is true if f is bounded on R and f is discontinuous only on a finite number of continuous curves. In other words, f has to be integrable over R.

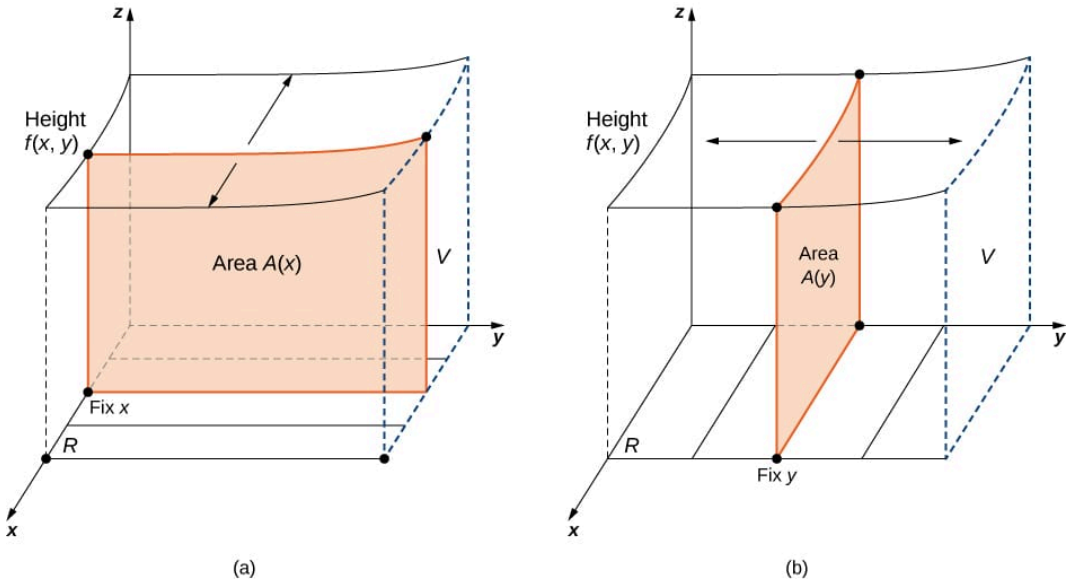


Figure $14.2a.6$: (a) Integrating first with respect to y and then with respect to x to find the area $A(x)$ and then the volume V; (b) integrating first with respect to x and then with respect to y to find the area $A(y)$ and then the volume V.

Example $14.2a.2$: Using Fubini's Theorem

Use Fubini's theorem to compute the double integral $\iint_R f(x,y)\ dA$ where $f(x,y) = x$ and $R = [0,2] \times [0,1]$.

Solution

Fubini's theorem offers an easier way to evaluate the double integral by the use of an iterated integral. Note how the boundary values of the region R become the upper and lower limits of integration.

$$\begin{aligned}
\iint_R f(x,y)\,dA &= \iint_R f(x,y)\,dx\,dy \\
&= \int_{y=0}^{y=1} \int_{x=0}^{x=2} x\,dx\,dy \\
&= \int_{y=0}^{y=1} \left[\frac{x^2}{2}\Big|_{x=0}^{x=2}\right] dy \\
&= \int_{y=0}^{y=1} 2\,dy = 2y\Big|_{y=0}^{y=1} = 2
\end{aligned}$$

The double integration in this example is simple enough to use Fubini's theorem directly, allowing us to convert a double integral into an iterated integral. Consequently, we are now ready to convert all double integrals to iterated integrals and demonstrate how the properties listed earlier can help us evaluate double integrals when the function $f(x, y)$ is more complex. Note that the order of integration can be changed (see Example 7).

Example $14.2a.3$: Illustrating Properties i and ii

Evaluate the double integral

$$\iint_R (xy - 3xy^2)\, dA, \text{ where } R = \{(x, y) \mid 0 \le x \le 2,\ 1 \le y \le 2\}.$$

Solution

This function has two pieces: one piece is xy and the other is $3xy^2$. Also, the second piece has a constant 3. Notice how we use properties i and ii to help evaluate the double integral.

$$\begin{aligned}
\iint_R (xy - 3xy^2)\, dA &= \iint_R xy\, dA + \iint_R (-3xy^2)\, dA && \text{Property i: Integral of a sum is the sum of the integrals.}\\
&= \int_{y=1}^{y=2} \int_{x=0}^{x=2} xy\, dx\, dy - \int_{y=1}^{y=2} \int_{x=0}^{x=2} 3xy^2\, dx\, dy && \text{Convert double integrals to iterated integrals.}\\
&= \int_{y=1}^{y=2} \left(\frac{x^2}{2} y\right)\Big|_{x=0}^{x=2} dy - 3\int_{y=1}^{y=2} \left(\frac{x^2}{2} y^2\right)\Big|_{x=0}^{x=2} dy && \text{Integrate with respect to } x, \text{ holding } y \text{ constant.}\\
&= \int_{y=1}^{y=2} 2y\, dy - \int_{y=1}^{y=2} 6y^2 dy && \text{Property ii: Placing the constant before the integral.}\\
&= 2\int_1^2 y\, dy - 6\int_1^2 y^2\, dy && \text{Integrate with respect to y.}\\
&= 2\frac{y^2}{2}\Big|_1^2 - 6\frac{y^3}{3}\Big|_1^2\\
&= y^2\Big|_1^2 - 2y^3\Big|_1^2\\
&= (4 - 1) - 2(8 - 1) = 3 - 2(7) = 3 - 14 = -11.
\end{aligned}$$

Example $14.2a.4$: Illustrating Property v.

Over the region $R = \{(x, y) \mid 1 \le x \le 3,\ 1 \le y \le 2\}$, we have $2 \le x^2 + y^2 \le 13$. Find a lower and an upper bound for the integral $\iint_R (x^2 + y^2)\ dA$.

Solution

For a lower bound, integrate the constant function 2 over the region R. For an upper bound, integrate the constant function 13 over the region R.

$$\int_1^2 \int_1^3 2\, dx\, dy = \int_1^2 [2x\Big|_1^3]\, dy = \int_1^2 2(2) dy = 4y\Big|_1^2 = 4(2 - 1) = 4$$

$$\int_1^2 \int_1^3 13 dx\, dy = \int_1^2 [13x\Big|_1^3]\, dy = \int_1^2 13(2)\, dy = 26y\Big|_1^2 = 26(2 - 1) = 26.$$

Hence, we obtain $4 \le \iint_R (x^2 + y^2)\ dA \le 26$.

Example 14.2a.5: Illustrating Property vi

Evaluate the integral $\iint_R e^y \cos x \; dA$ over the region $R = \{(x, y) \mid 0 \le x \le \frac{\pi}{2}, \; 0 \le y \le 1\}$.

Solution

This is a great example for property vi because the function $f(x, y)$ is clearly the product of two single-variable functions e^y and $\cos x$. Thus we can split the integral into two parts and then integrate each one as a single-variable integration problem.

$$\begin{aligned}\iint_R e^y \cos x \, dA &= \int_0^1 \int_0^{\pi/2} e^y \cos x \, dx \, dy \\ &= \left(\int_0^1 e^y dy\right)\left(\int_0^{\pi/2} \cos x \, dx\right) \\ &= (e^y\Big|_0^1)(\sin x\Big|_0^{\pi/2}) \\ &= e - 1.\end{aligned}$$

Exercise 14.2a.2

a. Use the properties of the double integral and Fubini's theorem to evaluate the integral

$$\int_0^1 \int_{-1}^3 (3 - x + 4y) \, dy \, dx.$$

b. Show that $0 \le \iint_R \sin \pi x \;\; \cos \pi y \; dA \le \dfrac{1}{32}$ where $R = \left(0, \frac{1}{4}\right)\left(\frac{1}{4}, \frac{1}{2}\right)$.

Hint available at the end of this section.

Answer available at the end of this section.

As we mentioned before, when we are using rectangular coordinates, the double integral over a region R denoted by $\iint_R f(x, y) \; dA$ can be written as $\iint_R \; f(x, y) \; dx \; dy$ or $\iint_R \; f(x, y) \; dy \; dx$. The next example shows that the results are the same regardless of which order of integration we choose.

Example 14.2a.6: Evaluating an Iterated Integral in Two Ways

Let's return to the function $f(x, y) = 3x^2 - y$ from Example 1, this time over the rectangular region $R = [0, 2] \times [0, 3]$. Use Fubini's theorem to evaluate $\iint_R f(x, y) \; dA$ in two different ways:

a. First integrate with respect to y and then with respect to x;
b. First integrate with respect to x and then with respect to y.

Solution

Figure 14.2a.6 shows how the calculation works in two different ways.

a. First integrate with respect to y and then integrate with respect to x:

$$\iint_R f(x,y)\,dA = \int_{x=0}^{x=2}\int_{y=0}^{y=3}(3x^2-y)\,dy\,dx$$
$$= \int_{x=0}^{x=2}\left(\int_{y=0}^{y=3}(3x^2-y)\,dy\right)dx = \int_{x=0}^{x=2}\left[3x^2y-\frac{y^2}{2}\Big|_{y=0}^{y=3}\right]dx$$
$$= \int_{x=0}^{x=2}\left(9x^2-\frac{9}{2}\right)dx = 3x^3-\frac{9}{2}x\Big|_{x=0}^{x=2} = 15.$$

b. First integrate with respect to x and then integrate with respect to y:

$$\iint_R f(x,y)\,dA = \int_{y=0}^{y=3}\int_{x=0}^{x=2}(3x^2-y)\,dx\,dy$$
$$= \int_{y=0}^{y=3}\left(\int_{x=0}^{x=2}(3x^2-y)\,dx\right)dy$$
$$= \int_{y=0}^{y=3}\left[x^3-xy\Big|_{x=0}^{x=2}\right]dy$$
$$= \int_{y=0}^{y=3}(8-2y)\,dy = 8y-y^2\Big|_{y=0}^{y=3} = 15.$$

Analysis

With either order of integration, the double integral gives us an answer of 15. We might wish to interpret this answer as a volume in cubic units of the solid S below the function $f(x,y)=3x^2-y$ over the region $R=[0,2]\times[0,3]$. However, remember that the interpretation of a double integral as a (non-signed) volume works only when the integrand f is a nonnegative function over the base region R.

Exercise $14.2a.3$

Evaluate

$$\int_{y=-3}^{y=2}\int_{x=3}^{x=5}(2-3x^2+y^2)\,dx\,dy.$$

Hint available at the end of this section.

Answer available at the end of this section.

In the next example we see that it can actually be beneficial to switch the order of integration to make the computation easier. We will come back to this idea several times in this chapter.

Example $14.2a.7$: Switching the Order of Integration

Consider the double integral $\iint_R x\ \sin(xy)\ dA$ over the region $R=\{(x,y)\mid 0\le x\le\pi,\ 1\le y\le 2\}$ (Figure $14.2a.7$).

a. Express the double integral in two different ways.
b. Analyze whether evaluating the double integral in one way is easier than the other and why.
c. Evaluate the integral.

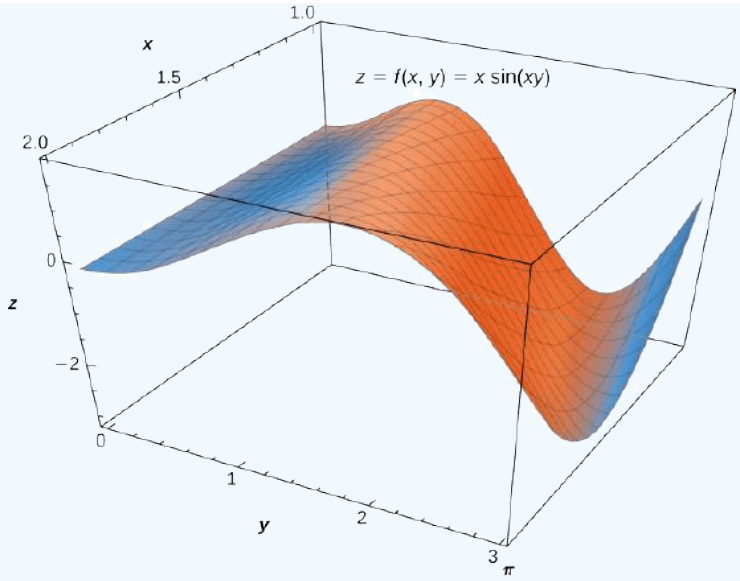


Figure 14.2a.7: The function $z = f(x, y) = x\ \sin(xy)$ over the rectangular region $R = [0, \pi] \times [1, 2]$.

a. We can express $\iint_R x\ \sin(xy)\ dA$ in the following two ways: first by integrating with respect to y and then with respect to x; second by integrating with respect to x and then with respect to y.

$$\iint_R x\ \sin(xy)\, dA = \int_{x=0}^{x=\pi} \int_{y=1}^{y=2} x\ \sin(xy)\, dy\, dx$$

Integrate first with respect to y.

$$= \int_{y=1}^{y=2} \int_{x=0}^{x=\pi} x\ \sin(xy)\, dx\, dy$$

Integrate first with respect to x.

b. If we want to integrate with respect to *y* first and then integrate with respect to x, we see that we can use the substitution $u = xy$, which gives $du = x\ dy$. Hence the inner integral is simply $\int \sin u\ du$ and we can change the limits to be functions of x,

$$\iint_R x\ \sin(xy)\, dA = \int_{x=0}^{x=\pi} \int_{y=1}^{y=2} x\ \sin(xy)\, dy\, dx = \int_{x=0}^{x=\pi} \left[\int_{u=x}^{u=2x} \sin(u)\, du \right] dx.$$

However, integrating with respect to x first and then integrating with respect to y requires integration by parts for the inner integral, with $u = x$ and $dv = \sin(xy)dx$

Then $du = dx$ and $v = -\frac{\cos(xy)}{y}$, so

$$\iint_R x \sin(xy)\, dA = \int_{y=1}^{y=2} \int_{x=0}^{x=\pi} x \sin(xy)\, dx\, dy = \int_{y=1}^{y=2} \left[-\frac{x\ \cos(xy)}{y} \Big|_{x=0}^{x=\pi} + \frac{1}{y} \int_{x=0}^{x=\pi} \cos(xy)\, dx \right] dy.$$

Since the evaluation is getting complicated, we will only do the computation that is easier to do, which is clearly the first method.

c. Evaluate the double integral using the easier way.

$$\begin{aligned}\iint_R x\,\sin(xy)\,dA &= \int_{x=0}^{x=\pi}\int_{y=1}^{y=2} x\,\sin(xy)\,dy\,dx \\ &= \int_{x=0}^{x=\pi}\left[\int_{u=x}^{u=2x}\sin(u)\,du\right]dx = \int_{x=0}^{x=\pi}\left[-\cos u\Big|_{u=x}^{u=2x}\right]dx \\ &= \int_{x=0}^{x=\pi}(-\cos 2x+\cos x)\,dx \\ &= \left(-\frac{1}{2}\sin 2x+\sin x\right)\Big|_{x=0}^{x=\pi} = 0.\end{aligned}$$

Exercise 14.2a.4

Evaluate the integral $\iint_R xe^{xy}\,dA$ where $R=[0,1]\times[0,\ln 5]$.

Hint available at the end of this section.

Answer available at the end of this section.

Applications of Double Integrals

Double integrals are very useful for finding the area of a region bounded by curves of functions. We describe this situation in more detail in the next section. However, if the region is a rectangular shape, we can find its area by integrating the constant function $f(x,y)=1$ over the region R.

Definition: Area of a Region R

The area of the region R is given by

$$A(R)=\iint_R 1\,dA.$$

This definition makes sense because using $f(x,y)=1$ and evaluating the integral make it a product of length and width. Let's check this formula with an example and see how this works.

Example 14.2a.8: Finding Area Using a Double Integral

Find the area of the region $R=\{\ (x,y)\ |\ 0\le x\le 3,\ \ 0\le y\le 2\}$ by using a double integral, that is, by integrating 1 over the region R.

Solution

The region is rectangular with length 3 and width 2, so we know that the area is 6. We get the same answer when we use a double integral:

$$A(R)=\int_0^2\int_0^3 1\,dx\,dy=\int_0^2\left[x\Big|_0^3\right]dy=\int_0^2 3dy=3\int_0^2 dy=3y\Big|_0^2=3(2)=6\text{ units}^2.$$

We have already seen how double integrals can be used to find the volume of a solid bounded above by a function $f(x,y)\ge 0$ over a region R provided $f(x,y)\ge 0$ for all (x,y) in R. Here is another example to illustrate this concept.

Example $14.2a.9$: Volume of an Elliptic Paraboloid

Find the volume V of the solid S that is bounded by the elliptic paraboloid $2x^2 + y^2 + z = 27$, the planes $x = 3$ and $y = 3$, and the three coordinate planes.

Solution

First notice the graph of the surface $z = 27 - 2x^2 - y^2$ in Figure $14.2a.8$(a) and above the square region $R_1 = [-3, 3] \times [-3, 3]$. However, we need the volume of the solid bounded by the elliptic paraboloid $2x^2 + y^2 + z = 27$, the planes $x = 3$ and $y = 3$, and the three coordinate planes.

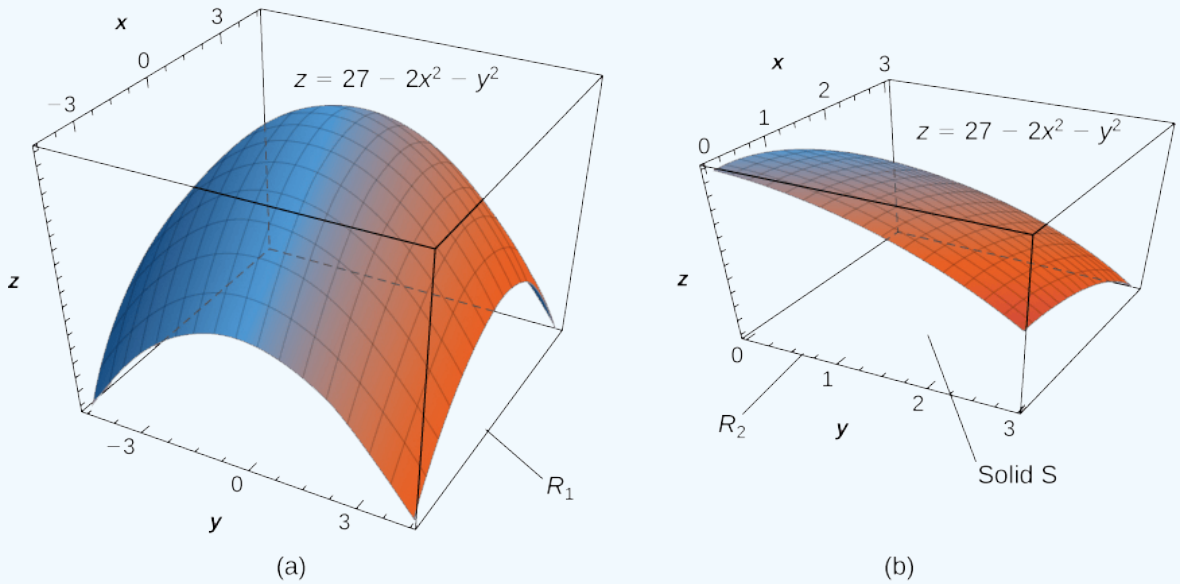


Figure $14.2a.8$: (a) The surface $z = 27 - 2x^2 - y^2$ above the square region $R1 = [-3, 3] \times [-3, 3]$. (b) The solid S lies under the surface $z = 27 - 2x^2 - y^2$ above the square region $R1 = [0, 3] \times [0, 3]$.

Now let's look at the graph of the surface in Figure $14.2a.8$(b). We determine the volume V by evaluating the double integral over R_2:

$$
\begin{aligned}
V &= \iint_R z\,dA = \iint_R (27 - 2x^2 - y^2)\,dA \\
&= \int_{y=0}^{y=3} \int_{x=0}^{x=3} (27 - 2x^2 - y^2)\,dx\,dy && \text{Convert to literal integral.} \\
&= \int_{y=0}^{y=3} \left[27x - \frac{2}{3}x^3 - y^2 x\right]\Big|_{x=0}^{x=3} dy && \text{Integrate with respect to } x. \\
&= \int_{y=0}^{y=3} (63 - 3y^2)dy = 63y - y^3\Big|_{y=0}^{y=3} = 162.
\end{aligned}
$$

Exercise $14.2a.5$

Find the volume of the solid bounded above by the graph of $f(x, y) = xy\sin(x^2y)$ and below by the xy-plane on the rectangular region $R = [0, 1] \times [0, \pi]$.

Hint available at the end of this section.

Answer available at the end of this section.

Recall that we defined the average value of a function of one variable on an interval $[a, b]$ as

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Similarly, we can define the average value of a function of two variables over a region R. The main difference is that we divide by an area instead of the width of an interval.

Definition: Average Value of a Function

The **average value of a function** of two variables over a region R is

$$F_{ave} = \frac{1}{\text{Area of } R}\iint_R f(x,y)\,dx\,dy.$$

In the next example we find the average value of a function over a rectangular region. This is a good example of obtaining useful information for an integration by making individual measurements over a grid, instead of trying to find an algebraic expression for a function.

Example $14.2a.10$: Calculating Average Storm Rainfall

The weather map in Figure $14.2a.9$ shows an unusually moist storm system associated with the remnants of Hurricane Karl, which dumped 4–8 inches (100–200 mm) of rain in some parts of the Midwest on September 22–23, 2010. The area of rainfall measured 300 miles east to west and 250 miles north to south. Estimate the average rainfall over the entire area in those two days.

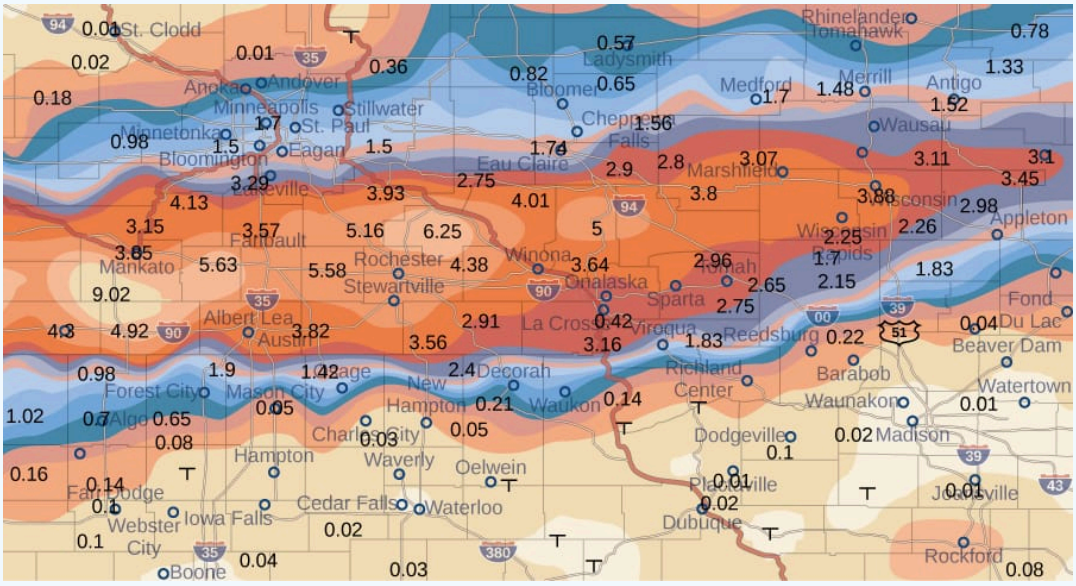


Figure $14.2a.9$: Effects of Hurricane Karl, which dumped 4–8 inches (100–200 mm) of rain in some parts of southwest Wisconsin, southern Minnesota, and southeast South Dakota over a span of 300 miles east to west and 250 miles north to south.

Solution

Place the origin at the southwest corner of the map so that all the values can be considered as being in the first quadrant and hence all are positive. Now divide the entire map into six rectangles ($m = 2$ and $n = 3$), as shown in Figure $14.2a.9$. Assume $f(x,y)$ denotes the storm rainfall in inches at a point approximately x miles to the east of the origin and y miles to the north of the origin. Let R represent the entire area of $250 \times 300 = 75000$ square miles. Then the area of each subrectangle is

$$\Delta A = \frac{1}{6}(75000) = 12500.$$

Assume $(x_{ij} *, y_{ij} *)$ are approximately the midpoints of each subrectangle R_{ij}. Note the color-coded region at each of these points, and estimate the rainfall. The rainfall at each of these points can be estimated as:

- At (x_{11}, y_{11}), the rainfall is 0.08.
- At (x_{12}, y_{12}), the rainfall is 0.08.
- At (x_{13}, y_{13}), the rainfall is 0.01.
- At (x_{21}, y_{21}), the rainfall is 1.70.
- At (x_{22}, y_{22}), the rainfall is 1.74.
- At (x_{23}, y_{23}), the rainfall is 3.00.

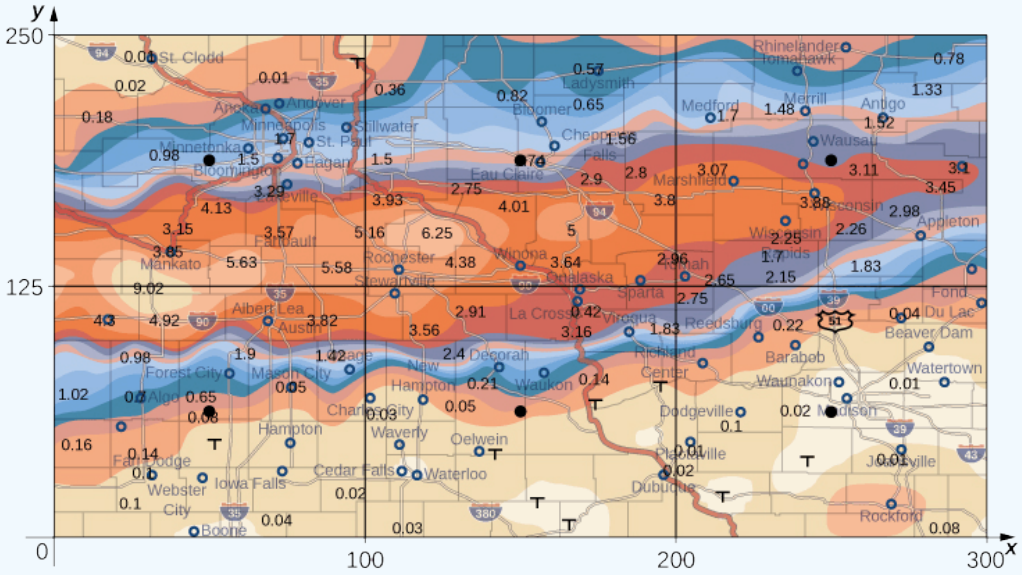


Figure $14.2a.10$: Storm rainfall with rectangular axes and showing the midpoints of each subrectangle.

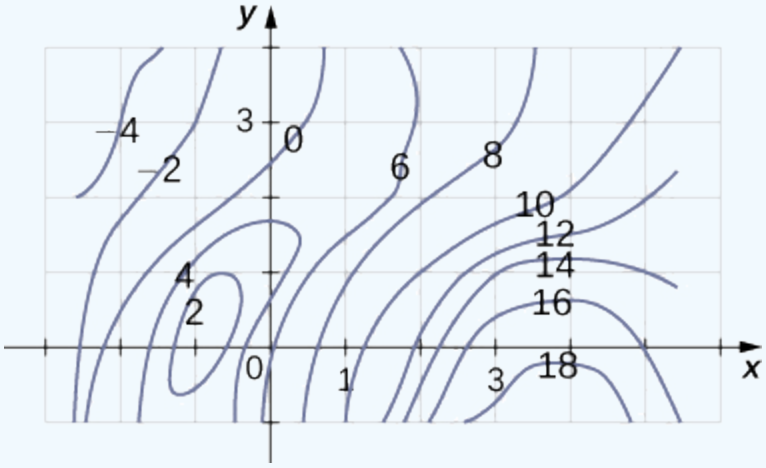
According to our definition, the average storm rainfall in the entire area during those two days was

$$
\begin{aligned}
f_{ave} = \frac{1}{Area\, R} \iint_R f(x,y)\, dx\, dy &= \frac{1}{75000} \iint_R f(x,y)\, dx\, dy \\
&\approx \frac{1}{75000} \sum_{i=1}^{3} \sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*) \Delta A \\
&= \frac{1}{75000} \left[f(x_{11}^*, y_{11}^*)\Delta A + f(x_{12}^*, y_{12}^*)\Delta A + f(x_{13}^*, y_{13}^*)\Delta A + f(x_{21}^*, y_{21}^*)\Delta A + f(x_{22}^*, y_{22}^*)\Delta A + f(x_{23}^*, y_{23}^*)\Delta A \right] \\
&\approx \frac{1}{75000} \Big[0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00 \Big] \Delta A \\
&= \frac{1}{75000} \Big[0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00 \Big] 12500 \\
&= \frac{1}{6} \Big[0.08 + 0.08 + 0.01 + 1.70 + 1.74 + 3.00 \Big] \\
&\approx 1.10 \text{ in.}
\end{aligned}
$$

During September 22–23, 2010 this area had an average storm rainfall of approximately 1.10 inches.

Exercise $14.2a.6$

A contour map is shown for a function $f(x,y)$ on the rectangle $R = [-3, 6] \times [-1, 4]$.



a. Use the midpoint rule with $m = 3$ and $n = 2$ to estimate the value of $\iint_R f(x, y)\ dA$.

b. Estimate the average value of the function $f(x, y)$.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- We can use a double Riemann sum to approximate the volume of a solid bounded above by a function of two variables over a rectangular region. By taking the limit, this becomes a double integral representing the volume of the solid.
- Properties of double integral are useful to simplify computation and find bounds on their values.
- We can use Fubini's theorem to write and evaluate a double integral as an iterated integral.
- Double integrals are used to calculate the area of a region, the volume under a surface, and the average value of a function of two variables over a rectangular region.

Key Equations

- $$\iint_R f(x, y)\, dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i j*, y_i j*)\, \Delta A$$
- $$\int_a^b \int_c^d f(x, y)\, dx\, dy = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$

  or

  $$\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy$$
- $$f_{ave} = \frac{1}{\text{Area of } R} \iint_R f(x, y)\, dx\, dy$$

Glossary

double integral
of the function $f(x, y)$ over the region R in the xy-plane is defined as the limit of a double Riemann sum,

$$\iint_R f(x, y)\, dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A.$$

double Riemann sum
of the function $f(x,y)$ over a rectangular region R is

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A,$$

where R is divided into smaller subrectangles R_{ij} and (x_{ij}^*, y_{ij}^*) is an arbitrary point in R_{ij}

Fubini's theorem
if $f(x,y)$ is a function of two variables that is continuous over a rectangular region $R = \{(x,y)^{\text{int}}\mathbb{R}^2 \mid a \le x \le b,\ c \le y \le d\}$, then the double integral of f over the region equals an iterated integral,

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dx\,dy = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

iterated integral
for a function $f(x,y)$ over the region R is

a. $\int_a^b \int_c^d f(x,y)\ dx\ dy = \int_a^b \left[\int_c^d f(x,y)\,dy\right]\ dx,$

b. $\int_c^d \int_a^b f(x,y)\ dx\ dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right]\ dy,$

where a, b, c, and d are any real numbers and $R = [a,b] \times [c,d]$

Answers

Exercise $14.2a.1$
Follow the steps of the previous example.

Exercise $14.2a.1$

$$V \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_{ij}^*, y_{ij}^*)\,\Delta A = 0$$

Exercise $14.2a.2$
Use properties i. and ii. and evaluate the iterated integral, and then use property v.

Exercise $14.2a.2$
a. 26
b. Answers may vary.

Exercise $14.2a.3$
Use Fubini's theorem.

Exercise $14.2a.3$
$-\frac{1340}{3}$

Exercise $14.2a.4$
Integrate with respect to y first.

Exercise $14.2a.4$
$\frac{4-\ln 5}{\ln 5}$

Exercise $14.2a.5$
Graph the function, set up the integral, and use an iterated integral.

Exercise $14.2a.5$
$\frac{\pi}{2}$

Exercise $14.2a.6$
Divide the region into six rectangles, and use the contour lines to estimate the values for $f(x,y)$.

Exercise $14.2a.6$
Answers to both parts a. and b. may vary.

14.2aE: Double Integrals Part 1 (Exercises)

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14.2aE: Double Integrals Part 1 (Exercises)

In exercises 1 and 2, use the midpoint rule with $m = 4$ and $n = 2$ to estimate the volume of the solid bounded by the surface $z = f(x, y)$, the vertical planes $x = 1$, $x = 2$, $y = 1$, and $y = 2$, and the horizontal plane $z = 0$.

1) $f(x, y) = 4x + 2y + 8xy$

Answer

27

2) $f(x, y) = 16x^2 + \frac{y}{2}$

In exercises 3 and 4, estimate the volume of the solid under the surface $z = f(x, y)$ and above the rectangular region R by using a Riemann sum with $m = n = 2$ and the sample points to be the lower left corners of the sub-rectangles of the partition.

3) $f(x, y) = \sin x - \cos y$, $R = [0, \pi] \times [0, \pi]$

Answer

0

4) $f(x, y) = \cos x + \cos y$, $R = [0, \pi] \times [0, \frac{\pi}{2}]$

5) Use the midpoint rule with $m = n = 2$ to estimate $\iint_R f(x, y)\ dA$, where the values of the function *f* on $R = [8, 10] \times [9, 11]$ are given in the following table.

	y				
x	9	9.5	10	10.5	11
8	9.8	5	6.7	5	5.6
8.5	9.4	4.5	8	5.4	3.4
9	8.7	4.6	6	5.5	3.4
9.5	6.7	6	4.5	5.4	6.7
10	6.8	6.4	5.5	5.7	6.8

Answer

21.3

6) The values of the function f on the rectangle $R = [0, 2] \times [7, 9]$ are given in the following table. Estimate the double integral $\iint_R f(x, y)\ dA$ by using a Riemann sum with $m = n = 2$. Select the sample points to be the upper right corners of the sub-squares of *R*.

	$y_0 = 7$	$y_1 = 8$	$y_2 = 9$
$x_0 = 0$	10.22	10.21	9.85
$x_1 = 1$	6.73	9.75	9.63
$x_2 = 2$	5.62	7.83	8.21

7) The depth of a children's 4-ft by 4-ft swimming pool, measured at 1-ft intervals, is given in the following table.

a. Estimate the volume of water in the swimming pool by using a Riemann sum with $m = n = 2$. Select the sample points using the midpoint rule on $R = [0, 4] \times [0, 4]$.
b. Find the average depth of the swimming pool.

	y				
x	0	1	2	3	4
0	1	1.5	2	2.5	3
1	1	1.5	2	2.5	3
2	1	1.5	1.5	2.5	3
3	1	1	1.5	2	2.5
4	1	1	1	1.5	2

Answer

a. 28 ft^3
b. 1.75 ft.

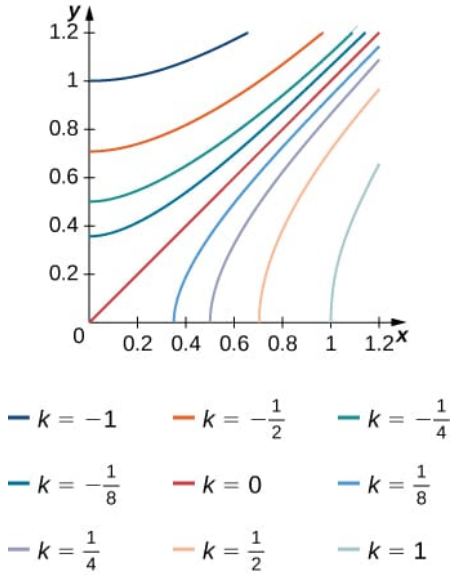
8) The depth of a 3-ft by 3-ft hole in the ground, measured at 1-ft intervals, is given in the following table.

a. Estimate the volume of the hole by using a Riemann sum with $m = n = 3$ and the sample points to be the upper left corners of the sub-squares of R.
b. Find the average depth of the hole.

	y			
x	0	1	2	3
0	6	6.5	6.4	6
1	6.5	7	7.5	6.5
2	6.5	6.7	6.5	6
3	6	6.5	5	5.6

9) The level curves $f(x, y) = k$ of the function f are given in the following graph, where k is a constant.

a. Apply the midpoint rule with $m = n = 2$ to estimate the double integral $\iint_R f(x, y)\ dA$, where $R = [0.2, 1] \times [0, 0.8]$.
b. Estimate the average value of the function f on R.



Answer

a. Answers will vary somewhat, but should be close to these: $f(0.4, 0.2) \approx 0.125$, $f(0.4, 0.6) \approx -0.2$, $f(0.8, 0.2)$

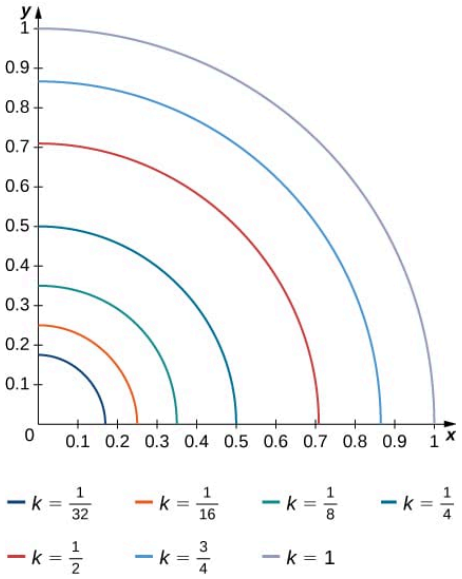
≈ 0.6, and $f(0.8, 0.6) \approx 0.3$.

Since the area of each rectangle is $\Delta A_i = (0.4)(0.4) = 0.16$ units^2, $\iint_R f(x,y)\ dA = \int_{0.2}^{1}\int_{0}^{0.8} f(x,y)\ dy\ dx$ $\approx f(0.4, 0.2)(0.16) + f(0.4, 0.6)(0.16) + f(0.8, 0.2)(0.16) + f(0.8, 0.6)(0.16) = 0.132$

b. $f_{ave} = \dfrac{\int_{0.2}^{1}\int_{0}^{0.8} f(x,y)\,dy\,dx}{\text{area of } R} \approx \dfrac{\int_{0.2}^{1}\int_{0}^{0.8} f(x,y)\,dy\,dx}{0.64} \approx 0.206$

10) The level curves $f(x, y) = k$ of the function f are given in the following graph, where k is a constant.

a. Apply the midpoint rule with $m = n = 2$ to estimate the double integral $\iint_R f(x,y)\ dA$, where $R = [0.1, 0.5] \times [0.1, 0.5]$.
b. Estimate the average value of the function *f* on R.

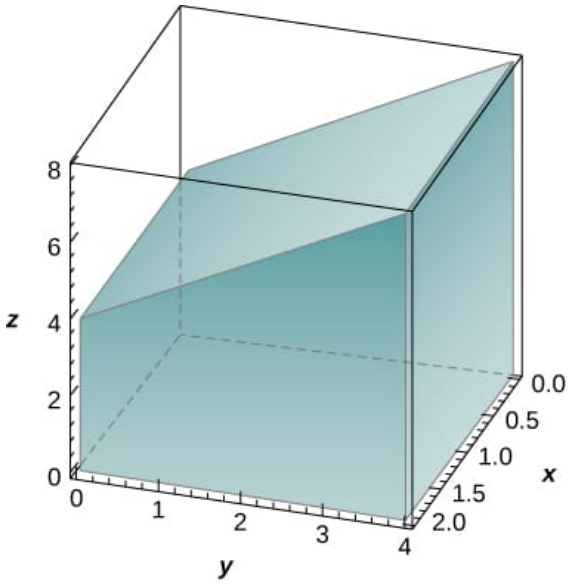


11) The solid lying under the surface $z = \sqrt{4 - y^2}$ and above the rectangular region$R = [0, 2] \times [0, 2]$ is illustrated in the following graph. Evaluate the double integral $\iint_R f(x, y)$, where $f(x, y) = \sqrt{4 - y^2}$ by finding the volume of the corresponding solid.

Answer

2π

12) The solid lying under the plane $z = y + 4$ and above the rectangular region $R = [0, 2] \times [0, 4]$ is illustrated in the following graph. Evaluate the double integral $\iint_R f(x,y)\ dA$, where $f(x, y) = y + 4$, by finding the volume of the corresponding solid.



In the exercises 13 - 20, calculate the integrals by reversing the order of integration.

13) $\int_{-1}^{1}\left(\int_{-2}^{2}(2x+3y+5)\,dx\right)\,dy$

Answer

40

14) $\int_{0}^{2}\left(\int_{0}^{1}(x+2e^{y}+3)\,dx\right)\,dy$

15) $\int_{1}^{27}\left(\int_{1}^{2}(\sqrt[3]{x}+\sqrt[3]{y})\,dy\right)\,dx$

Answer

$\frac{81}{2}+39\sqrt[3]{2}$

16) $\int_{1}^{16}\left(\int_{1}^{8}(\sqrt[4]{x}+2\sqrt[3]{y})\,dy\right)\,dx$

17) $\int_{\ln 2}^{\ln 3}\left(\int_{0}^{1}e^{x+y}\,dy\right)\,dx$

Answer

$e-1$

18) $\int_{0}^{2}\left(\int_{0}^{1}3^{x+y}\,dy\right)\,dx$

19) $\int_{1}^{6}\left(\int_{2}^{9}\frac{\sqrt{y}}{x^{2}}\,dy\right)\,dx$

Answer

$15-\frac{10\sqrt{2}}{9}$

20) $\int_{1}^{9}\left(\int_{4}^{2}\frac{\sqrt{x}}{y^{2}}\,dy\right)\,dx$

In exercises 21 - 34, evaluate the iterated integrals by choosing the order of integration.

21) $\int_0^{\pi}\int_0^{\pi/2} \sin(2x)\cos(3y)\ dx\ dy$

Answer

0

22) $\int_{\pi/12}^{\pi/8}\int_{\pi/4}^{\pi/3} [\cot x + \tan(2y)]\ dx\ dy$

23) $\int_1^e\int_1^e \left[\frac{1}{x}\sin(\ln x) + \frac{1}{y}\cos(\ln y)\right]\ dx\ dy$

Answer

$(e-1)(1+\sin 1-\cos 1)$

24) $\int_1^e\int_1^e \frac{\sin(\ln x)\cos(\ln y)}{xy}\ dx\ dy$

25) $\int_1^2\int_1^2 \left(\frac{\ln y}{x} + \frac{x}{2y+1}\right)\ dy\ dx$

Answer

$\frac{3}{4}\ln\left(\frac{5}{3}\right) + 2(\ln 2)^2 - \ln 2$

26) $\int_1^e\int_1^2 x^2\ln(x)\ dy\ dx$

27) $\int_1^{\sqrt{3}}\int_1^2 y\ \arctan\left(\frac{1}{x}\right)\ dy\ dx$

Answer

$\frac{1}{8}[(2\sqrt{3}-3)\pi + 6\ \ln 2]$

28) $\int_0^1\int_0^{1/2} (\arcsin x + \arcsin y)\ dy\ dx$

29) $\int_0^1\int_1^2 xe^{x+4y}\ dy\ dx$

Answer

$\frac{1}{4}e^4(e^4-1)$

30) $\int_1^2\int_0^1 xe^{x-y}\ dy\ dx$

31) $\int_1^e\int_1^e \left(\frac{\ln y}{\sqrt{y}} + \frac{\ln x}{\sqrt{x}}\right)\ dy\ dx$

Answer

$4(e-1)(2-\sqrt{e})$

32) $\int_1^e\int_1^e \left(\frac{x\ \ln y}{\sqrt{y}} + \frac{y\ \ln x}{\sqrt{x}}\right)\ dy\ dx$

33) $\int_0^1\int_1^2 \left(\frac{x}{x^2+y^2}\right)\ dy\ dx$

Answer

$-\frac{\pi}{4} + \ln\left(\frac{5}{4}\right) - \frac{1}{2}\ln 2 + \arctan 2$

34) $\int_0^1 \int_1^2 \frac{y}{x+y^2}\ dy\ dx$

In exercises 35 - 38, find the average value of the function over the given rectangles.

35)$f(x,y) = -x + 2y$, $R = [0,1] \times [0,1]$

Answer

$\frac{1}{2}$

36) $f(x,y) = x^4 + 2y^3$, $R = [1,2] \times [2,3]$

37) $f(x,y) = \sinh x + \sinh y$, $R = [0,1] \times [0,2]$

Answer

$\frac{1}{2}(2\ \cosh 1 + \cosh 2 - 3)$.

38) $f(x,y) = \arctan(xy)$, $R = [0,1] \times [0,1]$

39) Let f and g be two continuous functions such that $0 \le m_1 \le f(x) \le M_1$ for any $x \in [a,b]$ and $0 \le m_2 \le g(y) \le M_2$ for any$y \in [c,d]$. Show that the following inequality is true:

$$m_1 m_2 (b-a)(c-d) \le \int_a^b \int_c^d f(x)g(y)\,dydx \le M_1 M_2 (b-a)(c-d).$$

In exercises 40 - 43, use property v. of double integrals and the answer from the preceding exercise to show that the following inequalities are true.

40) $\frac{1}{e^2} \le \iint_R e^{-x^2-y^2}\ dA \le 1$, where $R = [0,1] \times [0,1]$

41) $\frac{\pi^2}{144} \le \iint_R \sin x \cos y\ dA \le \frac{\pi^2}{48}$, where $R = [\frac{\pi}{6}, \frac{\pi}{3}] \times [\frac{\pi}{6}, \frac{\pi}{3}]$

42) $0 \le \iint_R e^{-y}\ \cos x\ dA \le \frac{\pi}{2}$, where $R = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$

43) $0 \le \iint_R (\ln x)(\ln y)\ dA \le (e-1)^2$, where $R = [1,e] \times [1,e]$

44) Let f and g be two continuous functions such that $0 \le m_1 \le f(x) \le M_1$ for any $x \in [a,b]$ and $0 \le m_2 \le g(y) \le M_2$ for any $y \in [c,d]$. Show that the following inequality is true:

$$(m_1+m_2)(b-a)(c-d) \le \int_a^b \int_c^d |f(x)+g(y)|\ dy\ dx \le (M_1+M_2)(b-a)(c-d)$$

In exercises 45 - 48, use property v. of double integrals and the answer from the preceding exercise to show that the following inequalities are true.

45) $\frac{2}{e} \le \iint_R (e^{-x^2} + e^{-y^2})\ dA \le 2$, where $R = [0,1] \times [0,1]$

46) $\frac{\pi^2}{36} \iint_R (\sin x + \cos y)\ dA \le \frac{\pi^2\sqrt{3}}{36}$, where $R = [\frac{\pi}{6}, \frac{\pi}{3}] \times [\frac{\pi}{6}, \frac{\pi}{3}]$

47) $\frac{\pi}{2} e^{-\pi/2} \le \iint_R (\cos x + e^{-y})\ dA \le \pi$, where $R = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$

48) $\frac{1}{e} \le \iint_R (e^{-y} - \ln x)\ dA \le 2$, where $R = [0,1] \times [0,1]$

In exercises 49 - 50, the function f is given in terms of double integrals.

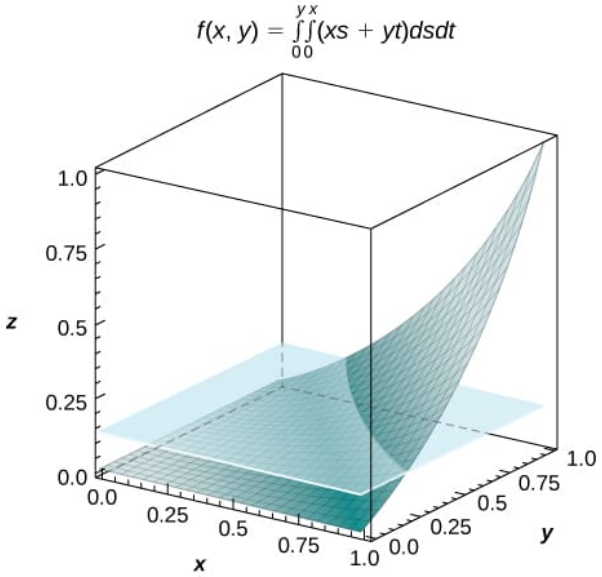
a. Determine the explicit form of the function f.
b. Find the volume of the solid under the surface $z = f(x,y)$ and above the region R.
c. Find the average value of the function f on R.
d. Use a computer algebra system (CAS) to plot $z = f(x,y)$ and $z = f_{ave}$ in the same system of coordinates.

49) [T] $f(x,y) = \int_0^y \int_0^x (xs + yt) ds\ dt$, where $(x,y)^{\text{int}} R = [0,1] \times [0,1]$

Answer

a. $f(x,y) = \frac{1}{2}xy(x^2 + y^2)$;
b. $V = \int_0^1 \int_0^1 f(x,y)\ dx\ dy = \frac{1}{8}$;
c. $f_{ave} = \frac{1}{8}$;

d.



50) [T] $f(x,y) = \int_0^x \int_0^y [\cos(s) + \cos(t)]\ dt\ ds$, where $(x,y)^{\text{int}} R = [0,3] \times [0,3]$

51) Show that if f and g are continuous on $[a,b]$ and $[c,d]$, respectively, then

$$\int_a^b \int_c^d |f(x) + g(y)| dy\ dx = (d-c) \int_a^b f(x)\ dx$$

$$+ \int_a^b \int_c^d g(y)\ dy\ dx = (b-a) \int_c^d g(y)\ dy + \int_c^d \int_a^b f(x)\ dx\ dy.$$

52) Show that $\int_a^b \int_c^d yf(x) + xg(y)\ dy\ dx = \frac{1}{2}(d^2 - c^2) \left(\int_a^b f(x)\, dx \right) + \frac{1}{2}(b^2 - a^2) \left(\int_c^d g(y)\, dy \right)$.

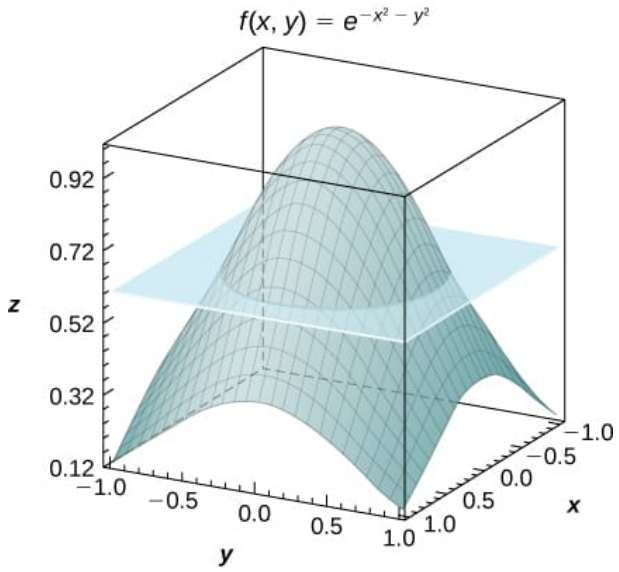
53) [T] Consider the function $f(x,y) = e^{-x^2-y^2}$, where $(x,y)^{\text{int}} R = [-1,1] \times [-1,1]$.

a. Use the midpoint rule with $m = n = 2, 4, \ldots, 10$ to estimate the double integral $I = \iint_R e^{-x^2-y^2} dA$. Round your answers to the nearest hundredths.
b. For $m = n = 2$, find the average value of *f* over the region *R*. Round your answer to the nearest hundredths.
c. Use a CAS to graph in the same coordinate system the solid whose volume is given by $\iint_R e^{-x^2-y^2} dA$ and the plane $z = f_{ave}$.

Answer

a. For $m = n = 2$, $I = 4e^{-0.5} \approx 2.43$
b. $f_{ave} = e^{-0.5} \simeq 0.61$;

c.



54) [T] Consider the function $f(x,y) = \sin(x^2)\ \cos(y^2)$, where $(x, y^{\text{int}} R = [-1,1] \times [-1,1]$.

a. Use the midpoint rule with $m = n = 2, 4, ..., 10$ to estimate the double integral $I = \iint_R \sin(x^2)\cos(y^2)\ dA$. Round your answers to the nearest hundredths.
b. For $m = n = 2$, find the average value of $f$over the region *R*. Round your answer to the nearest hundredths.
c. Use a CAS to graph in the same coordinate system the solid whose volume is given by $\iint_R \sin(x^2)\cos(y^2)\ dA$ and the plane $z = f_{ave}$.

In exercises 55 - 56, the functions f_n are given, where $n \geq 1$ is a natural number.

a. Find the volume of the solids S_n under the surfaces $z = f_n(x,y)$ and above the region R.
b. Determine the limit of the volumes of the solids S_n as n increases without bound.

55) $f(x,y) = x^n + y^n + xy,\ \ (x,y)^{\text{int}} R = [0,1] \times [0,1]$

Answer

a. $\frac{2}{n+1} + \frac{1}{4}$
b. $\frac{1}{4}$

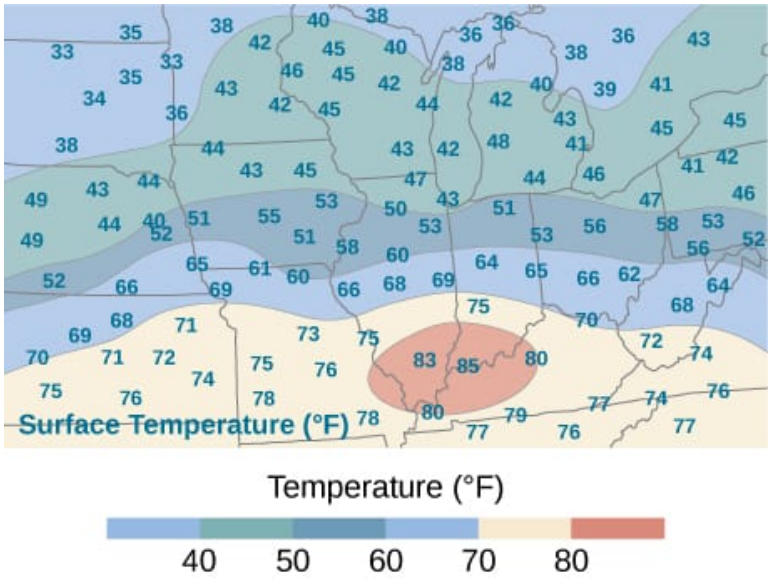
56) $f(x,y) = \frac{1}{x^n} + \frac{1}{y^n},\ \ (x,y)^{\text{int}} R = [1,2] \times [1,2]$

57) Show that the average value of a function f on a rectangular region $R = [a,b] \times [c,d]$ is $f_{ave} \approx \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)$,where (x_{ij}^*, y_{ij}^*) are the sample points of the partition of R, where $1 \leq i \leq m$ and $1 \leq j \leq n$.

58) Use the midpoint rule with $m = n$ to show that the average value of a function f on a rectangular region $R = [a,b] \times [c,d]$ is approximated by

$$f_{ave} \approx \frac{1}{n^2} \sum_{i,j=1}^{n} f\left(\frac{1}{2}(x_{i=1} + x_i),\ \frac{1}{2}(y_{j=1} + y_j)\right).$$

59) An isotherm map is a chart connecting points having the same temperature at a given time for a given period of time. Use the preceding exercise and apply the midpoint rule with $m = n = 2$ to find the average temperature over the region given in the following figure.



Answer

56.5° F; here $f(x_1^*, y_1^*) = 71,\ f(x_2^*, y_1^*) = 72,\ f(x_2^*, y_1^*) = 40,\ f(x_2^*, y_2^*) = 43$, where x_i^* and y_j^* are the midpoints of the subintervals of the partitions of $[a, b]$ and $[c, d]$, respectively.

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14.2B: DOUBLE INTEGRALS OVER GENERAL REGIONS

Learning Objectives

- Recognize when a function of two variables is integrable over a general region.
- Evaluate a double integral by computing an iterated integral over a region bounded by two vertical lines and two functions of x, or two horizontal lines and two functions of y.
- Simplify the calculation of an iterated integral by changing the order of integration.
- Use double integrals to calculate the volume of a region between two surfaces or the area of a plane region.
- Solve problems involving double improper integrals.

Previously, we studied the concept of double integrals and examined the tools needed to compute them. We learned techniques and properties to integrate functions of two variables over rectangular regions. We also discussed several applications, such as finding the volume bounded above by a function over a rectangular region, finding area by integration, and calculating the average value of a function of two variables.

In this section we consider double integrals of functions defined over a general bounded region D on the plane. Most of the previous results hold in this situation as well, but some techniques need to be extended to cover this more general case.

General Regions of Integration

An example of a general bounded region D on a plane is shown in Figure $14.2b.1$. Since D is bounded on the plane, there must exist a rectangular region R on the same plane that encloses the region D that is, a rectangular region R exists such that D is a subset of $R (D \subseteq R)$.

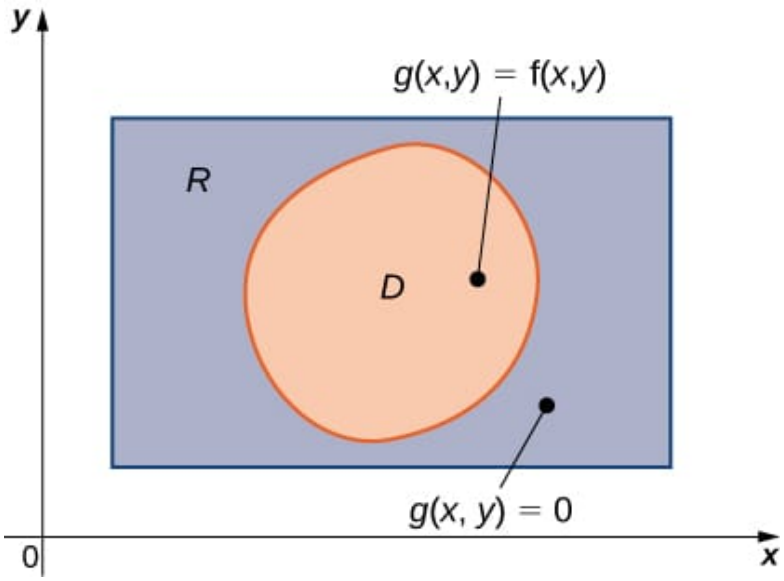


Figure $14.2b.1$: For a region D that is a subset of R, we can define a function $g(x, y)$ to equal $f(x, y)$ at every point in D and 0 at every point of R not in D.

Suppose $z = f(x, y)$ is defined on a general planar bounded region D as in Figure $14.2b.1$. In order to develop double integrals of f over D we extend the definition of the function to include all points on the rectangular region R and then use the concepts and tools from the preceding section. But how do we extend the definition of f to include all the points on R? We do this by defining a new function $g(x, y)$ on R as follows:

$$g(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \text{ is in } D \\ 0, & \text{if } (x,y) \text{ is in } R \text{ but not in } D \end{cases}$$

Note that we might have some technical difficulties if the boundary of D is complicated. So we assume the boundary to be a piecewise smooth and continuous simple closed curve. Also, since all the results developed in the section on Double Integrals over Rectangular Regions used an integrable function $f(x,y)$ we must be careful about $g(x,y)$ and verify that $g(x,y)$ is an integrable function over the rectangular region R. This happens as long as the region D is bounded by simple closed curves. For now we will concentrate on the descriptions of the regions rather than the function and extend our theory appropriately for integration.

We consider two types of planar bounded regions.

Definition: Type I and Type II regions

A region D in the (x,y)-plane is of Type I if it lies between two vertical lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$. That is (Figure $14.2b.2$),

$$D = \{(x,y) \,|\, a \le x \le b,\ g_1(x) \le y \le g_2(x)\}.$$

A region D in the xy-plane is of Type II if it lies between two horizontal lines and the graphs of two continuous functions $h_1(y)$ and $h_2(y)$. That is (Figure $14.2b.3$),

$$D = \{(x,y) \,|\, c \le y \le d,\ h_1(y) \le x \le h_2(y)\}.$$

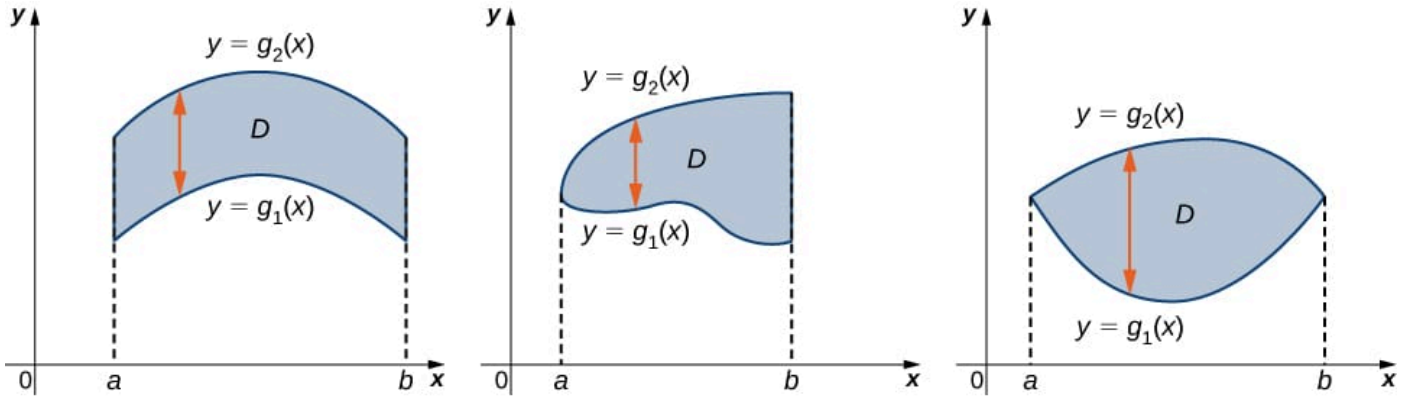


Figure $14.2b.2$. A Type I region lies between two vertical lines and the graphs of two functions of x.

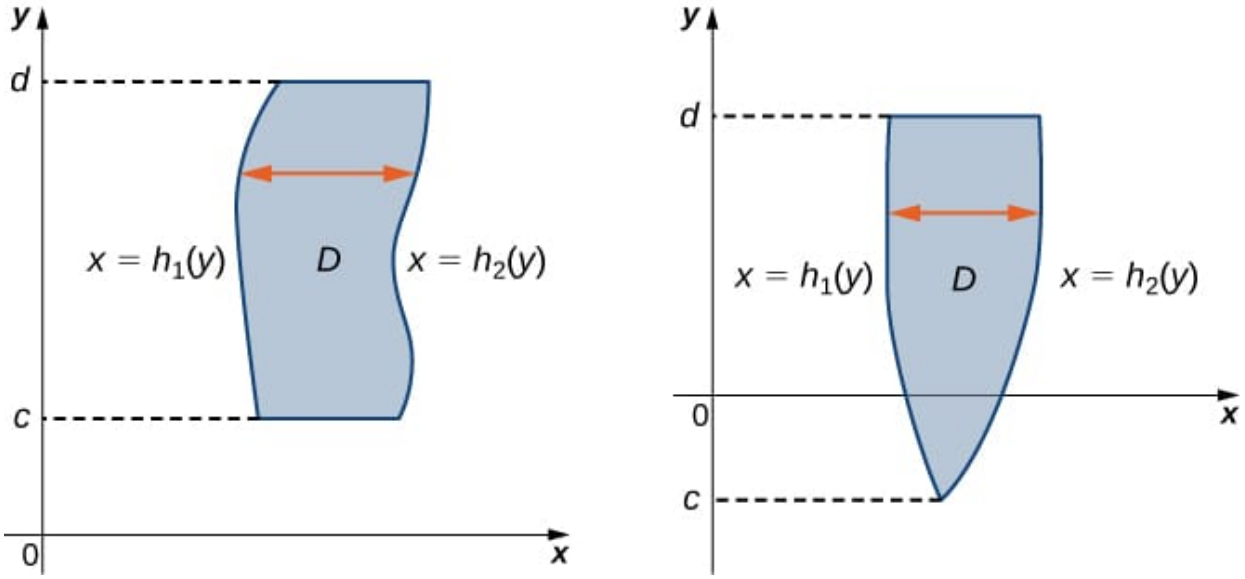


Figure $14.2b.3$: A Type II region lies between two horizontal lines and the graphs of two functions of y.

Example $14.2b.1$: Describing a Region as Type I and Also as Type II

Consider the region in the first quadrant between the functions $y = \sqrt{x}$ and $y = x^3$ (Figure $14.2b.4$). Describe the region first as Type I and then as Type II.

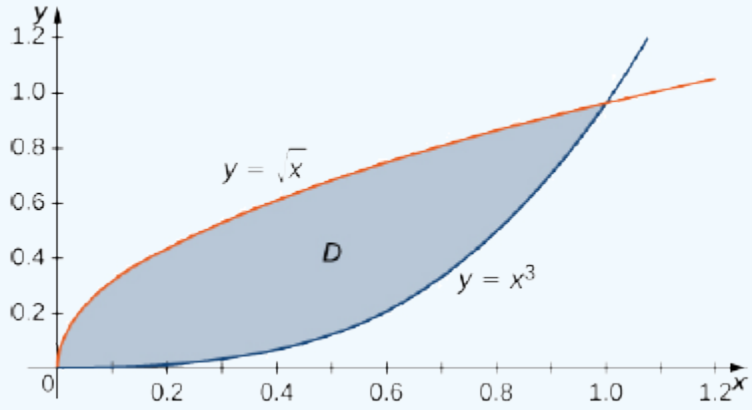


Figure $14.2b.4$: Region D can be described as Type I or as Type II.

When describing a region as Type I, we need to identify the function that lies above the region and the function that lies below the region. Here, region D is bounded above by $y = \sqrt{x}$ and below by $y = x^3$ in the interval for x in $[0, 1]$. Hence, as Type I, D is described as the set $\left\{(x, y) \mid 0 \le x \le 1,\ x^3 \le y \le \sqrt[3]{x}\right\}$.

However, when describing a region as Type II, we need to identify the function that lies on the left of the region and the function that lies on the right of the region. Here, the region D is bounded on the left by $x = y^2$ and on the right by $x = \sqrt[3]{y}$ in the interval for y in $[0, 1]$. Hence, as Type II, D is described as the set $\left\{(x, y) \mid 0 \le y \le 1,\ y^2 \le x \le \sqrt[3]{y}\right\}$.

Exercise $14.2b.1$

Consider the region in the first quadrant between the functions $y = 2x$ and $y = x^2$. Describe the region first as Type I and then as Type II.

Hint available at the end of this section.

Answer available at the end of this section.

Double Integrals over Non-rectangular Regions

To develop the concept and tools for evaluation of a double integral over a general, nonrectangular region, we need to first understand the region and be able to express it as Type I or Type II or a combination of both. Without understanding the regions, we will not be able to decide the limits of integrations in double integrals. As a first step, let us look at the following theorem.

Theorem: Double Integrals over Nonrectangular Regions

Suppose $g(x, y)$ is the extension to the rectangle R of the function $f(x, y)$ defined on the regions D and R as shown in Figure $14.2b.1$ inside R. Then $g(x, y)$ is integrable and we define the double integral of $f(x, y)$ over D by

$$\iint_D f(x, y)\, dA = \iint_R g(x, y)\, dA.$$

The right-hand side of this equation is what we have seen before, so this theorem is reasonable because R is a rectangle and

$\iint_R g(x,y)dA$ has been discussed in the preceding section. Also, the equality works because the values of $g(x,y)$ are 0 for any point (x,y) that lies outside D and hence these points do not add anything to the integral. However, it is important that the rectangle R contains the region D.

As a matter of fact, if the region D is bounded by smooth curves on a plane and we are able to describe it as Type I or Type II or a mix of both, then we can use the following theorem and not have to find a rectangle R containing the region.

Theorem: Fubini's Theorem (Strong Form)

For a function $f(x,y)$ that is continuous on a region D of Type I, we have

$$\iint_D f(x,y)\,dA = \iint_D f(x,y)\,dy\,dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right] dx.$$

Similarly, for a function $f(x,y)$ that is continuous on a region D of Type II, we have

$$\iint_D f(x,y)\,dA = \iint_D f(x,y)\,dx\,dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right] dy.$$

The integral in each of these expressions is an iterated integral, similar to those we have seen before. Notice that, in the inner integral in the first expression, we integrate $f(x,y)$ with x being held constant and the limits of integration being $g_1(x)$ and $g_2(x)$. In the inner integral in the second expression, we integrate $f(x,y)$ with y being held constant and the limits of integration are $h_1(x)$ and $h_2(x)$.

Example $14.2b.2$: Evaluating an Iterated Integral over a Type I Region

Evaluate the integral $\iint_D x^2e^{xy}\,dA$ where D is shown in Figure $14.2b.5$.

Solution

First construct the region as a Type I region (Figure $14.2b.5$). Here $D = \{(x,y) \mid 0 \le x \le 2,\ \frac{1}{2}x \le y \le 1\}$. Then we have

$$\iint_D x^2e^{xy}\,dA = \int_{x=0}^{x=2}\int_{y=1/2x}^{y=1} x^2e^{xy}\,dy\,dx.$$

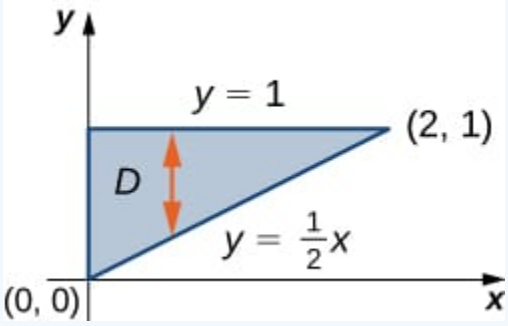


Figure $14.2b.5$: We can express region D as a Type I region and integrate from $y = \frac{1}{2}x$ to $y = 1$ between the lines $x = 0$ and $x = 2$.

Therefore, we have

$$\int_{x=0}^{x=2}\int_{y=\frac{1}{2}x}^{y=1} x^2e^{xy}\,dy\,dx = \int_{x=0}^{x=2}\left[\int_{y=\frac{1}{2}x}^{y=1} x^2e^{xy}\,dy\right]dx \qquad \text{Iterated integral for a Type I region.}$$

$$= \int_{x=0}^{x=2}\left[x^2\frac{e^{xy}}{x}\right]\Big|_{y=1/2x}^{y=1} dx \qquad \text{Integrate with respect to } y$$

$$= \int_{x=0}^{x=2}\left[xe^x - xe^{x^2/2}\right]dx \qquad \text{Integrate with respect to } x$$

$$= \left[xe^x - e^x - e^{\frac{1}{2}x^2}\right]\Big|_{x=0}^{x=2} = 2.$$

In Example 14.2b.2, we could have looked at the region in another way, such as $D = \{(x,y) \mid 0 \le y \le 1,\ 0 \le x \le 2y\}$ (Figure 14.2b.6).

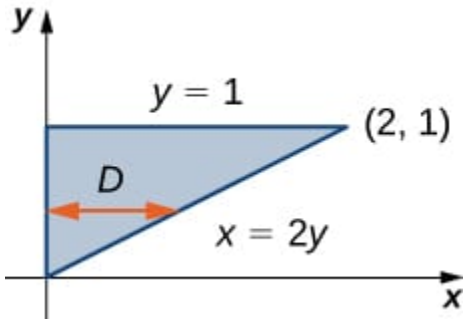


Figure 14.2b.6.

This is a Type II region and the integral would then look like

$$\iint_D x^2e^{xy}\,dA = \int_{y=0}^{y=1}\int_{x=0}^{x=2y} x^2e^{xy}\,dx\,dy.$$

However, if we integrate first with respect to x this integral is lengthy to compute because we have to use integration by parts twice.

Example 14.2b.3: Evaluating an Iterated Integral over a Type II Region

Evaluate the integral

$$\iint_D (3x^2 + y^2)\,dA$$

where $D = \{(x,y) \mid -2 \le y \le 3,\ y^2 - 3 \le x \le y + 3\}$.

Solution

Notice that D can be seen as either a Type I or a Type II region, as shown in Figure 14.2b.7. However, in this case describing D as Type I is more complicated than describing it as Type II. Therefore, we use D as a Type II region for the integration.

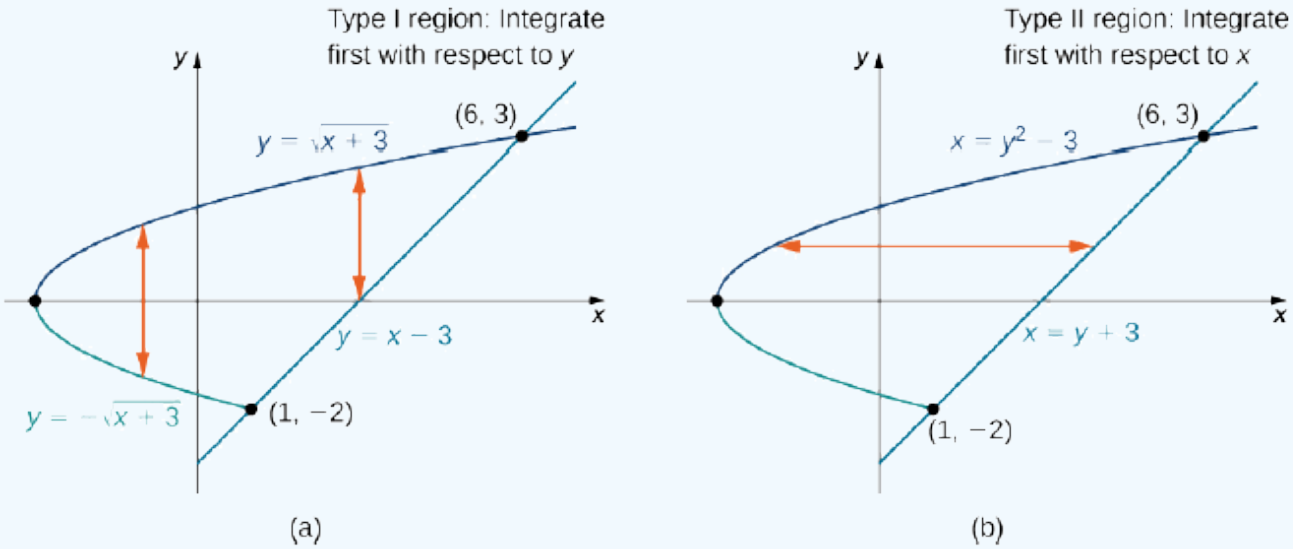


Figure 14.2b.7: The region D in this example can be either (a) Type I or (b) Type II.

Choosing this order of integration, we have

$$\iint_D (3x^2 + y^2)\, dA = \int_{y=-2}^{y=3} \int_{x=y^2-3}^{x=y+3} (3x^2 + y^2)\, dx\, dy$$

$$= \int_{y=-2}^{y=3} (x^3 + xy^2)\Big|_{y^2-3}^{y+3}\, dy \qquad \text{Iterated integral, Type II region}$$

$$= \int_{y=-2}^{y=3} \left((y+3)^3 + (y+3)y^2 - (y^2-3)^3 - (y^2-3)y^2\right)\, dy$$

$$= \int_{-2}^{3} (54 + 27y - 12y^2 + 2y^3 + 8y^4 - y^6)\, dy \qquad \text{Integrate with respect to } x.$$

$$= \left[54y + \frac{27y^2}{2} - 4y^3 + \frac{y^4}{2} + \frac{8y^5}{5} - \frac{y^7}{7}\right]_{-2}^{3}$$

$$= \frac{2375}{7}.$$

Exercise 14.2b.2

Sketch the region D and evaluate the iterated integral

$$\iint_D xy\, dy\, dx$$

where D is the region bounded by the curves $y = \cos\ x$ and $y = \sin\ x$ in the interval $[-3\pi/4,\ \pi/4]$.

Hint available at the end of this section.

Answer available at the end of this section.

Recall from Double Integrals over Rectangular Regions the properties of double integrals. As we have seen from the examples here, all these properties are also valid for a function defined on a non-rectangular bounded region on a plane. In particular, property 3 states:

If $R = S \cup T$ and $S \cap T = 0$ except at their boundaries, then

$$\iint_R f(x,y)\,dA = \iint_S f(x,y)\,dA + \iint_T f(x,y)\,dA.$$

Similarly, we have the following property of double integrals over a non-rectangular bounded region on a plane.

Theorem: Decomposing Regions into Smaller Regions

Suppose the region D can be expressed as $D = D_1 \cup D_2$ where D_1 and D_2 do not overlap except at their boundaries. Then

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA.$$

This theorem is particularly useful for non-rectangular regions because it allows us to split a region into a union of regions of Type I and Type II. Then we can compute the double integral on each piece in a convenient way, as in the next example.

Example $14.2b.4$: Decomposing Regions

Express the region D shown in Figure $14.2b.8$ as a union of regions of Type I or Type II, and evaluate the integral

$$\iint_D (2x+5y)\,dA.$$

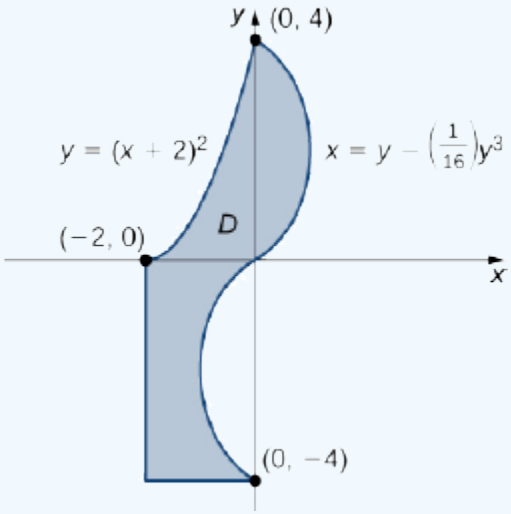


Figure $14.2b.8$: This region can be decomposed into a union of three regions of Type I or Type II.

Solution

The region D is not easy to decompose into any one type; it is actually a combination of different types. So we can write it as a union of three regions D_1, D_2, and D_3 where, $D_1 = \{(x,y) \mid -2 \le x \le 0,\ 0 \le y \le (x+2)^2\}$, $D_2 = \{(x,y) \mid 0 \le y \le 4,\ 0 \le x \le \left(y - \frac{1}{16}y^3\right)\}$, and $D_3 = \{(x,y) \mid -4 \le y \le 0,\ -2 \le x \le \left(y - \frac{1}{16}y^3\right)\}$. These regions are illustrated more clearly in Figure $14.2b.9$.

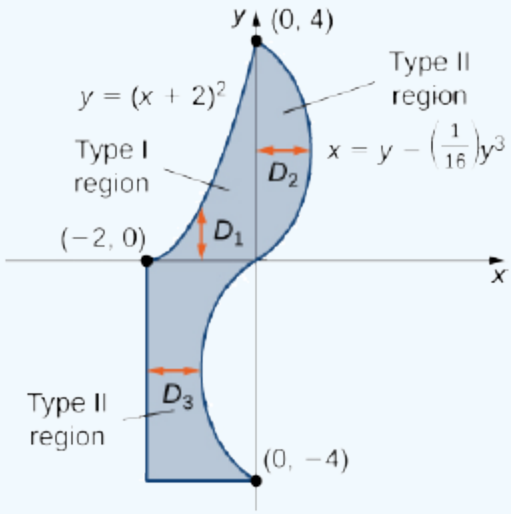


Figure 14.2b.9: Breaking the region into three subregions makes it easier to set up the integration.

Here D_1 is Type I and D_2 and D_3 are both of Type II. Hence,

$$\begin{aligned}\iint_D (2x+5y)\,dA &= \iint_{D_1} (2x+5y)\,dA + \iint_{D_2} (2x+5y)\,dA + \iint_{D_3} (2x+5y)\,dA \\ &= \int_{x=-2}^{x=0}\int_{y=0}^{y=(x+2)^2} (2x+5y)\,dy\,dx + \int_{y=0}^{y=4}\int_{x=0}^{x=y-(1/16)y^3} (2x+5y)\,dx\,dy + \int_{y=-4}^{y=0}\int_{x=-2}^{x=y-(1/16)y^3} (2x+5y)\,dx\,dy \\ &= \int_{x=-2}^{x=0}\left[\frac{1}{2}(2+x)^2(20+24x+5x^2)\right]dx + \int_{y=0}^{y=4}\left[\frac{1}{256}y^6 - \frac{7}{16}y^4 + 6y^2\right]dy + \int_{y=-4}^{y=0}\left[\frac{1}{256}y^6 - \frac{7}{16}y^4 + 6y^2 + 10y - 4\right]dy \\ &= \frac{40}{3} + \frac{1664}{35} - \frac{1696}{35} = \frac{1304}{105}.\end{aligned}$$

Now we could redo this example using a union of two Type II regions (see Exercise 14.2b.4 below).

Exercise 14.2b.3

Consider the region bounded by the curves $y = \ln x$ and $y = e^x$ in the interval $[1, 2]$. Decompose the region into smaller regions of Type II.

Hint available at the end of this section.

Answer available at the end of this section.

Exercise 14.2b.4

Redo Example 14.2b.4 using a union of two Type II regions.

Hint available at the end of this section.

Answer available at the end of this section.

Changing the Order of Integration

As we have already seen when we evaluate an iterated integral, sometimes one order of integration leads to a computation that is significantly simpler than the other order of integration. Sometimes the order of integration does not matter, but it is important to learn to recognize when a change in order will simplify our work.

Example 14.2b.5: Changing the Order of Integration

Reverse the order of integration in the iterated integral

$$\int_{x=0}^{x=\sqrt{2}} \int_{y=0}^{y=2-x^2} xe^{x^2}\, dy\, dx.$$

Then evaluate the new iterated integral.

Solution

The region as presented is of Type I. To reverse the order of integration, we must first express the region as Type II. Refer to Figure 14.2b.10.

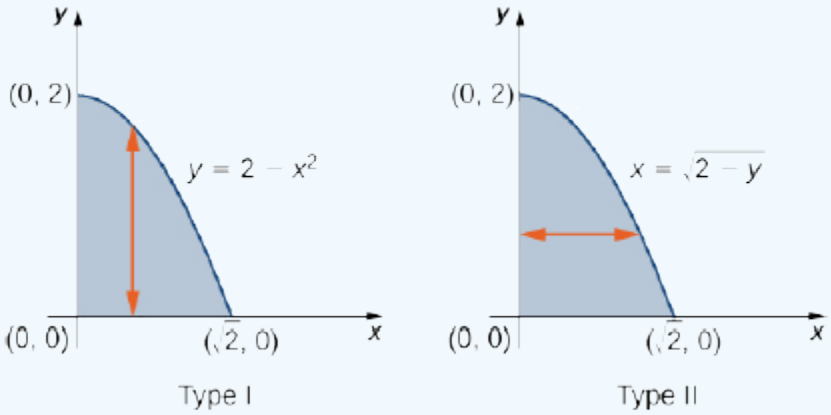


Figure 14.2b.10: Converting a region from Type I to Type II.

We can see from the limits of integration that the region is bounded above by $y = 2 - x^2$ and below by $y = 0$ where x is in the interval $[0, \sqrt{2}]$. By reversing the order, we have the region bounded on the left by $x = 0$ and on the right by $x = \sqrt{2-y}$ where y is in the interval $[0, 2]$. We solved $y = 2 - x^2$ in terms of x to obtain $x = \sqrt{2-y}$.

Hence

$$\int_0^{\sqrt{2}} \int_0^{2-x^2} xe^{x^2} dy\, dx = \int_0^2 \int_0^{\sqrt{2-y}} xe^{x^2}\, dx\, dy \qquad \text{Reverse the order of integration then use substitution.}$$

$$= \int_0^2 \left[\frac{1}{2} e^{x^2} \Big|_0^{\sqrt{2-y}} \right] dy = \int_0^2 \frac{1}{2}(e^{2-y} - 1)\, dy$$

$$= -\frac{1}{2}(e^{2-y} + y)\Big|_0^2 = \frac{1}{2}(e^2 - 3).$$

Example 14.2b.6: Evaluating an Iterated Integral by Reversing the Order of Integration

Consider the iterated integral

$$\iint_R f(x,y)\, dx\, dy$$

where $z = f(x,y) = x - 2y$ over a triangular region R that has sides on $x = 0,\ y = 0$, and the line $x + y = 1$. Sketch the region, and then evaluate the iterated integral by

a. integrating first with respect to y and then

b. integrating first with respect to x.

Solution

A sketch of the region appears in Figure $14.2b.11$.

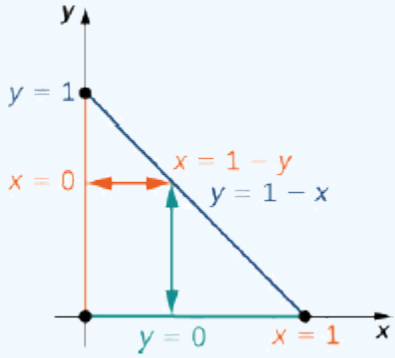


Figure $14.2b.11$: A triangular region R for integrating in two ways.

We can complete this integration in two different ways.

a. One way to look at it is by first integrating y from $y = 0$ to $y = 1 - x$ vertically and then integrating x from $x = 0$ to $x = 1$:

$$\iint_R f(x,y)\,dx\,dy = \int_{x=0}^{x=1}\int_{y=0}^{y=1-x}(x-2y)\,dy\,dx = \int_{x=0}^{x=1}\left(xy-y^2\right)\Big|_{y=0}^{y=1-x}dx$$

$$= \int_{x=0}^{x=1}\left[x(1-x)-(1-x)^2\right]\,dx = \int_{x=0}^{x=1}[-1+3x-2x^2]dx = \left[-x+\frac{3}{2}x^2-\frac{2}{3}x^3\right]\Big|_{x=0}^{x=1} = -\frac{1}{6}.$$

b. The other way to do this problem is by first integrating x from $x = 0$ to $x = 1 - y$ horizontally and then integrating y from $y = 0$ to $y = 1$:

$$\iint_R f(x,y)\,dx\,dy = \int_{y=0}^{y=1}\int_{x=0}^{x=1-y}(x-2y)\,dx\,dy = \int_{y=0}^{y=1}\left[\frac{1}{2}x^2-2xy\right]_{x=0}^{x=1-y}dy$$

$$= \int_{y=0}^{y=1}\left[\frac{1}{2}(1-y)^2-2y(1-y)\right]dy = \int_{y=0}^{y=1}\left[\frac{1}{2}-3y+\frac{5}{2}y^2\right]\,dy$$

$$= \left[\frac{1}{2}y-\frac{3}{2}y^2+\frac{5}{6}y^3\right]\Big|_{y=0}^{y=1} = -\frac{1}{6}.$$

Exercise $14.2b.5$

Evaluate the iterated integral $\iint_D (x^2+y^2)\,dA$ over the region D in the first quadrant between the functions $y = 2x$ and $y = x^2$. Evaluate the iterated integral by integrating first with respect to y and then integrating first with resect to x.

Hint available at the end of this section.

Answer available at the end of this section.

Calculating Volumes, Areas, and Average Values

We can use double integrals over general regions to compute volumes, areas, and average values. The methods are the same as those in Double Integrals over Rectangular Regions, but without the restriction to a rectangular region, we can now solve a wider variety of problems.

Example 14.2b.7: Finding the Volume of a Tetrahedron

Find the volume of the solid bounded by the planes $x = 0,\ y = 0,\ z = 0$, and $2x + 3y + z = 6$.

Solution

The solid is a tetrahedron with the base on the xy-plane and a height $z = 6 - 2x - 3y$. The base is the region D bounded by the lines, $x = 0$, $y = 0$ and $2x + 3y = 6$ where $z = 0$ (Figure 14.2b.12). Note that we can consider the region D as Type I or as Type II, and we can integrate in both ways.

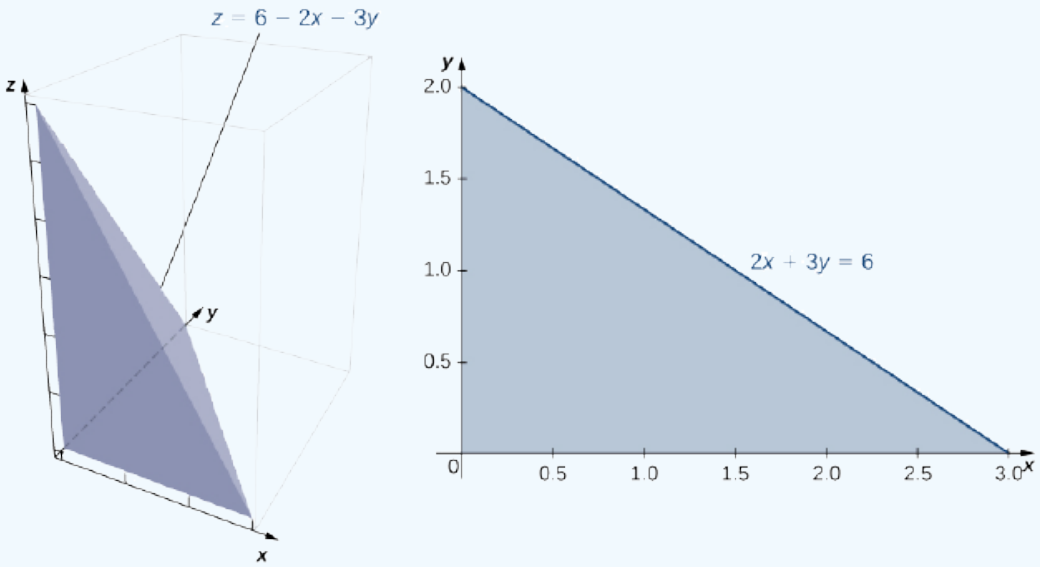


Figure 14.2b.12: A tetrahedron consisting of the three coordinate planes and the plane $z = 6 - 2x - 3y$, with the base bound by $x = 0,\ y = 0$, and $2x + 3y = 6$.

First, consider D as a Type I region, and hence $D = \{(x, y) \mid 0 \le x \le 3,\ 0 \le y \le 2 - \frac{2}{3}x\}$.

Therefore, the volume is

$$V = \int_{x=0}^{x=3} \int_{y=0}^{y=2-(2x/3)} (6 - 2x - 3y)\, dy\, dx = \int_{x=0}^{x=3} \left[\left(6y - 2xy - \frac{3}{2}y^2 \right) \Big|_{y=0}^{y=2-(2x/3)} \right] dx$$

$$= \int_{x=0}^{x=3} \left[\frac{2}{3}(x-3)^2 \right] dx = 6.$$

Now consider D as a Type II region, so $D = \{(x, y) \mid 0 \le y \le 2,\ 0 \le x \le 3 - \frac{3}{2}y\}$. In this calculation, the volume is

$$V = \int_{y=0}^{y=2} \int_{x=0}^{x=3-(3y/2)} (6 - 2x - 3y)\, dx\, dy = \int_{y=0}^{y=2} \left[(6x - x^2 - 3xy) \Big|_{x=0}^{x=3-(3y/2)} \right] dy$$

$$= \int_{y=0}^{y=2} \left[\frac{9}{4}(y-2)^2 \right] dy = 6.$$

Therefore, the volume is 6 cubic units.

Exercise 14.2b.6

Find the volume of the solid bounded above by $f(x, y) = 10 - 2x + y$ over the region enclosed by the curves $y = 0$ and $y = e^x$ where x is in the interval $[0, 1]$.

Hint available at the end of this section.

Finding the area of a rectangular region is easy, but finding the area of a non-rectangular region is not so easy. As we have seen, we can use double integrals to find a rectangular area. As a matter of fact, this comes in very handy for finding the area of a general non-rectangular region, as stated in the next definition.

Definition: Double Integrals

The area of a plane-bounded region D is defined as the double integral

$$\iint_D 1\, dA.$$

We have already seen how to find areas in terms of single integration. Here we are seeing another way of finding areas by using double integrals, which can be very useful, as we will see in the later sections of this chapter.

Example $14.2b.8$: Finding the Area of a Region

Find the area of the region bounded below by the curve $y = x^2$ and above by the line $y = 2x$ in the first quadrant (Figure $14.2b.13$).

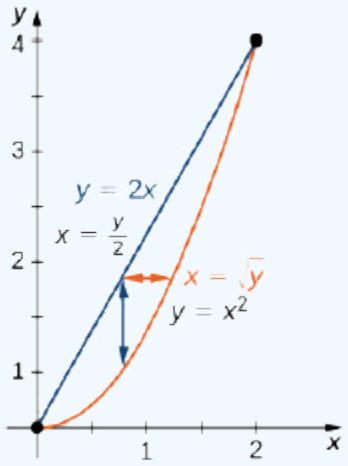


Figure $14.2b.13$: The region bounded by $y = x^2$ and $y = 2x$.

Solution

We just have to integrate the constant function $f(x, y) = 1$ over the region. Thus, the area A of the bounded region is $\int_{x=0}^{x=2} \int_{y=x^2}^{y=2x} dy\, dx$ or $\int_{y=0}^{y=4} \int_{x=y/2}^{x=\sqrt{y}} dx\, dy$:

$$
\begin{aligned}
A &= \iint_D 1\,dx\;dy \\
&= \int_{x=0}^{x=2} \int_{y=x^2}^{y=2x} 1\,dy\;dx \\
&= \int_{x=0}^{x=2} \left(y\Big|_{y=x^2}^{y=2x} \right) dx \\
&= \int_{x=0}^{x=2} (2x - x^2)\,dx \\
&= \left(x^2 - \frac{x^3}{3} \right) \Big|_0^2 = \frac{4}{3}.
\end{aligned}
$$

Exercise $14.2b.7$

Find the area of a region bounded above by the curve $y = x^3$ and below by $y = 0$ over the interval $[0, 3]$.

Answer available at the end of this section.

Answer available at the end of this section.

We can also use a double integral to find the average value of a function over a general region. The definition is a direct extension of the earlier formula.

Definition: The Average Value of a Function

If $f(x, y)$ is integrable over a plane-bounded region D with positive area $A(D)$, then the average value of the function is

$$f_{ave} = \frac{1}{A(D)} \iint_D f(x, y)\,dA.$$

Note that the area is $A(D) = \iint_D 1\;dA$.

Example $14.2b.9$: Finding an Average Value

Find the average value of the function $f(x, y) = 7xy^2$ on the region bounded by the line $x = y$ and the curve $x = \sqrt{y}$ (Figure $14.2b.14$).

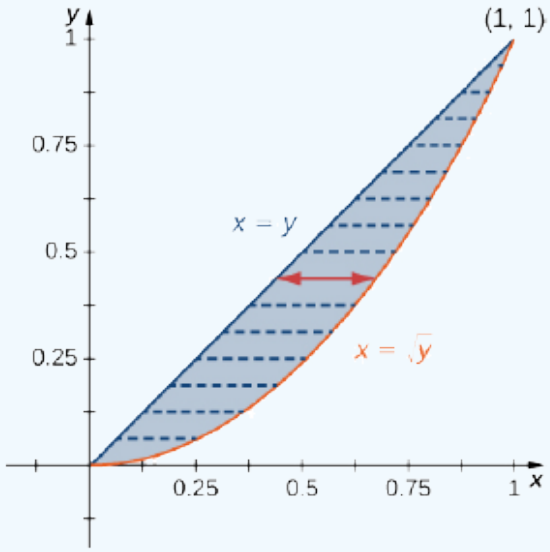


Figure $14.2b.14$: The region bounded by $x = y$ and $x = \sqrt{y}$.

Solution

First find the area $A(D)$ where the region D is given by the figure. We have

$$A(D) = \iint_D 1\,dA = \int_{y=0}^{y=1}\int_{x=y}^{x=\sqrt{y}} 1\,dx\;dy = \int_{y=0}^{y=1}\left[x\Big|_{x=y}^{x=\sqrt{y}}\right]dy = \int_{y=0}^{y=1}(\sqrt{y}-y)\,dy = \frac{2}{3}y^{2/3} - \frac{y^2}{2}\Big|_0^1 = \frac{1}{6}$$

Then the average value of the given function over this region is

$$f_{ave} = \frac{1}{A(D)}\iint_D f(x,y)\,dA = \frac{1}{A(D)}\int_{y=0}^{y=1}\int_{x=y}^{x=\sqrt{y}} 7xy^2\,dx\;dy = \frac{1}{1/6}\int_{y=0}^{y=1}\left[\frac{7}{2}x^2y^2\Big|_{x=y}^{x=\sqrt{y}}\right]dy$$
$$= 6\int_{y=0}^{y=1}\left[\frac{7}{2}y^2(y-y^2)\right]dy = 6\int_{y=0}^{y=1}\left[\frac{7}{2}(y^3-y^4)\right]dy = \frac{42}{2}\left(\frac{y^4}{4}-\frac{y^5}{5}\right)\Big|_0^1 = \frac{42}{40} = \frac{21}{20}.$$

Exercise $14.2b.8$

Find the average value of the function $f(x,y) = xy$ over the triangle with vertices $(0,0)$, $(1,0)$ and $(1,3)$.

Hint available at the end of this section.

Answer available at the end of this section.

Improper Double Integrals

An **improper double integral** is an integral $\iint_D f\;dA$ where either D is an unbounded region or f is an unbounded function. For example, $D = \{(x,y) \mid |x-y| \geq 2\}$ is an unbounded region, and the function $f(x,y) = 1/(1-x^2-2y^2)$ over the ellipse $x^2+3y^2 \geq 1$ is an unbounded function. Hence, both of the following integrals are improper integrals:

i. $$\iint_D xy\;dA \text{ where } D = \{(x,y)||\,x-y| \geq 2\};$$

ii. $$\iint_D \frac{1}{1-x^2-2y^2}\,dA \text{ where } D=\{(x,y)|\, x^2+3y^2\le 1\}.$$

In this section we would like to deal with improper integrals of functions over rectangles or simple regions such that f has only finitely many discontinuities. Not all such improper integrals can be evaluated; however, a form of Fubini's theorem does apply for some types of improper integrals.

Theorem: Fubini's Theorem for Improper Integrals

If D is a bounded rectangle or simple region in the plane defined by

$\{(x,y)\ : a\le x\le b,\ g(x)\le y\le h(x)\}$ and also by

$\{(x,y)\ : c\le y\le d,\ j(y)\le x\le k(y)\}$ and f is a nonnegative function on D with finitely many discontinuities in the interior of D then

$$\iint_D f\,dA=\int_{x=a}^{x=b}\int_{y=g(x)}^{y=h(x)} f(x,y)\,dy\,dx=\int_{y=c}^{y=d}\int_{x=j(y)}^{x=k(y)} f(x,y)\,dx\,dy$$

It is very important to note that we required that the function be nonnegative on D for the theorem to work. We consider only the case where the function has finitely many discontinuities inside D.

Example $14.2b.10$: Evaluating a Double Improper Integral

Consider the function $f(x,y)=\frac{e^y}{y}$ over the region $D=\{(x,y)\ : 0\le x\le 1,\ x\le y\le\sqrt{x}\}$.

Notice that the function is nonnegative and continuous at all points on D except $(0,0)$. Use Fubini's theorem to evaluate the improper integral.

Solution

First we plot the region D (Figure $14.2b.15$); then we express it in another way.

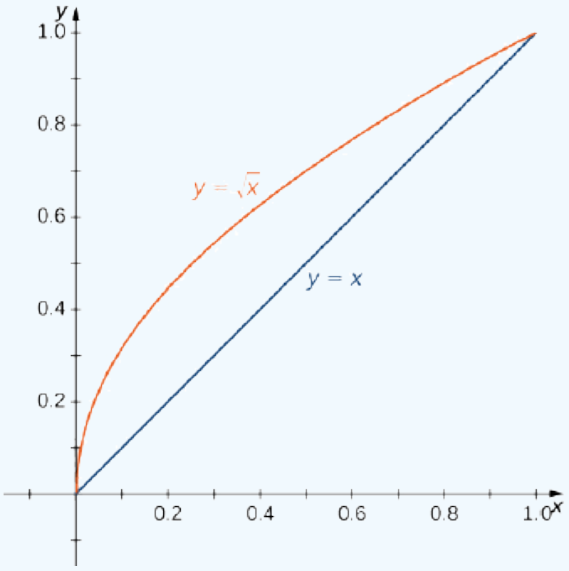


Figure $14.2b.15$: The function f is continuous at all points of the region D except $(0,0)$.

The other way to express the same region D is

$$D = \{(x, y) \,:\, 0 \le y \le 1,\ y^2 \le x \le y\}.$$

Thus we can use Fubini's theorem for improper integrals and evaluate the integral as

$$\int_{y=0}^{y=1} \int_{x=y^2}^{x=y} \frac{e^y}{y}\, dx\ dy.$$

Therefore, we have

$$\int_{y=0}^{y=1} \int_{x=y^2}^{x=y} \frac{e^y}{y}\, dx\ dy = \int_{y=0}^{y=1} \frac{e^y}{y} x \Big|_{x=y^2}^{x=y} dy = \int_{y=0}^{y=1} \frac{e^y}{y}(y - y^2)\, dy = \int_0^1 (e^y - ye^y)\, dy = e - 2.$$

As mentioned before, we also have an improper integral if the region of integration is unbounded. Suppose now that the function f is continuous in an unbounded rectangle R.

Theorem: Improper Integrals on an Unbounded Region

If R is an unbounded rectangle such as $R = \{(x, y) \ :\ a \le x \le \infty,\ c \le y \le \infty\}$, then when the limit exists, we have

$$\iint_R f(x, y)\, dA = \lim_{(b,d)\to(\infty,\infty)} \int_a^b \left(\int_c^d f(x, y)\, dy \right) dx = \lim_{(b,d)\to(\infty,\infty)} \int_c^d \left(\int_a^b f(x, y)\, dx \right) dy.$$

The following example shows how this theorem can be used in certain cases of improper integrals.

Example 14.2b.11

Evaluate the integral $\iint_R xye^{-x^2-y^2}\ dA$ where R is the first quadrant of the plane.

Solution

The region R is the first quadrant of the plane, which is unbounded. So

$$\begin{aligned}
\iint_R xye^{-x^2-y^2}\, dA &= \lim_{(b,d)\to(\infty,\infty)} \int_{x=0}^{x=b} \left(\int_{y=0}^{y=d} xye^{-x^2-y^2} dy \right) dx \\
&= \lim_{(b,d)\to(\infty,\infty)} \int_{x=0}^{x=b} \left(-\frac{1}{2} x \int_{y=0}^{y=d} -2ye^{-x^2-y^2} dy \right) dx \\
&= \lim_{(b,d)\to(\infty,\infty)} \int_{x=0}^{x=b} -\frac{1}{2} x \left(e^{-x^2-y^2} \right) \Big|_{y=0}^{y=d} dx \\
&= \lim_{(b,d)\to(\infty,\infty)} \int_{x=0}^{x=b} -\frac{1}{2} x \left(e^{-x^2-d^2} - e^{-x^2} \right) dx \\
&= \lim_{(b,d)\to(\infty,\infty)} \int_{x=0}^{x=b} \frac{1}{2} xe^{-x^2} \left(1 - e^{-d^2} \right) dx \\
&= \lim_{(b,d)\to(\infty,\infty)} \left(-\frac{1}{4} \right) \left(1 - e^{-d^2} \right) \int_{x=0}^{x=b} -2xe^{-x^2} dx \\
&= \lim_{(b,d)\to(\infty,\infty)} \left(-\frac{1}{4} \right) \left(1 - e^{-d^2} \right) e^{-x^2} \Big|_{x=0}^{x=b} dx \\
&= \lim_{(b,d)\to(\infty,\infty)} \frac{1}{4} \left(1 - e^{-b^2} \right) \left(1 - e^{-d^2} \right) \\
&= \lim_{(b,d)\to(\infty,\infty)} \frac{1}{4} \left(1 - \cancelto{0}{e^{-b^2}} \right) \left(1 - \cancelto{0}{e^{-d^2}} \right) \\
&= \frac{1}{4}
\end{aligned}$$

Thus,

$$\iint_R xye^{-x^2-y^2}\,dA$$

is convergent and the value is $\frac{1}{4}$.

Exercise $14.2b.9$

$$\iint_D \frac{y}{\sqrt{1-x^2-y^2}}dA$$

where $D = \{(x,y) \ : \ x \geq 0,\ y \geq 0,\ x^2+y^2 \leq 1\}$.

Hint available at the end of this section.

Answer available at the end of this section.

In some situations in probability theory, we can gain insight into a problem when we are able to use double integrals over general regions. Before we go over an example with a double integral, we need to set a few definitions and become familiar with some important properties.

Definition: Joint Density Function

Consider a pair of continuous random variables X and Y such as the birthdays of two people or the number of sunny and rainy days in a month. The **joint density function** f of X and Y satisfies the probability that (X,Y) lies in a certain region D :

$$P((X,Y)^{\text{int}}D) = \iint_D f(x,y)\,dA.$$

Since the probabilities can never be negative and must lie between 0 and 1 the joint density function satisfies the following inequality and equation:

$$f(x,y) \geq 0 \text{ and } \iint_R f(x,y)\,dA = 1.$$

Definition: Independent Random Variables

The variables X and Y are said to be **independent random variables** if their joint density function is the product of their individual density functions:

$$f(x,y) = f_1(x)f_2(y).$$

Example $14.2b.12$: Application to Probability

At Sydney's Restaurant, customers must wait an average of 15 minutes for a table. From the time they are seated until they have finished their meal requires an additional 40 minutes, on average. What is the probability that a customer spends less than an hour and a half at the diner, assuming that waiting for a table and completing the meal are independent events?

Solution

Waiting times are mathematically modeled by exponential density functions, with m being the average waiting time, as

$$f(t) = \begin{cases} 0, & \text{if } t < 0 \\ \dfrac{1}{m} e^{-t/m}, & \text{if } t \geq 0. \end{cases}$$

if X and Y are random variables for 'waiting for a table' and 'completing the meal,' then the probability density functions are, respectively,

$$f_1(x) = \begin{cases} 0, & \text{if } x < 0. \\ \dfrac{1}{15} e^{-x/15}, & \text{if } x \geq 0. \end{cases} \quad \text{and} \quad f_2(y) = \begin{cases} 0, & \text{if } y < 0 \\ \dfrac{1}{40} e^{-y/40}, & \text{if } y \geq 0. \end{cases}$$

Clearly, the events are independent and hence the joint density function is the product of the individual functions

$$f(x,y) = f_1(x) f_2(y) = \begin{cases} 0, & \text{if } x < 0 \text{ or } y < 0, \\ \dfrac{1}{600} e^{-x/15} e^{-y/40}, & \text{if } x, y \geq 0 \end{cases}$$

We want to find the probability that the combined time $X + Y$ is less than 90 minutes. In terms of geometry, it means that the region D is in the first quadrant bounded by the line $x + y = 90$ (Figure $14.2b.16$).

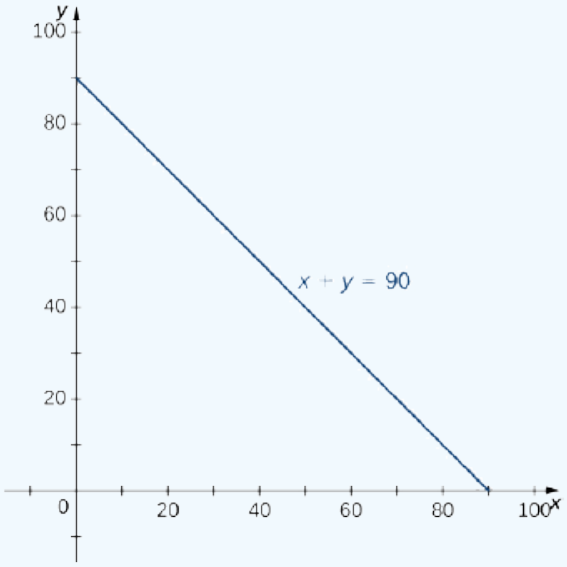


Figure $14.2b.16$: The region of integration for a joint probability density function.

Hence, the probability that (X, Y) is in the region D is

$$P(X + Y \leq 90) = P((X,Y)^{\text{int}} D) = \iint_D f(x,y)\, dA = \iint_D \frac{1}{600} e^{-x/15} e^{-y/40}\, dA.$$

Since $x + y = 90$ is the same as $y = 90 - x$, we have a region of Type I, so

$$D = \{(x,y)\,|\,0 \le x \le 90,\ 0 \le y \le 90 - x\},$$

$$\begin{aligned} P(X+Y \le 90) &= \frac{1}{600}\int_{x=0}^{x=90}\int_{y=0}^{y=90-x} e^{-x/15}e^{-y/40}dy\,dx \\ &= -\frac{1}{15}\int_{x=0}^{x=90} e^{-x/15}\int_{y=0}^{y=90-x} -\frac{1}{40}e^{-y/40}dy\,dx \\ &= -\frac{1}{15}\int_{x=0}^{x=90} e^{-x/15}e^{-y/40}\Big|_{y=0}^{y=90-x}\,dx \\ &= \quad \ldots \\ &\approx 0.8328 \end{aligned}$$

Thus, there is an 83.3% chance that a customer spends less than an hour and a half at the restaurant.

Another important application in probability that can involve improper double integrals is the calculation of expected values. First we define this concept and then show an example of a calculation.

Definition: Expected Values

In probability theory, we denote the **expected values** $E(X)$ and $E(Y)$ respectively, as the most likely outcomes of the events. The expected values $E(X)$ and $E(Y)$ are given by

$$E(X) = \iint_S x\,f(x,y)\,dA \text{ and } E(Y) = \iint_S y\,f(x,y)\,dA,$$

where S is the sample space of the random variables X and Y.

Example $14.2b.13$: Finding Expected Value

Find the expected time for the events 'waiting for a table' and 'completing the meal' in Example $14.2b.12$.

Solution

Using the first quadrant of the rectangular coordinate plane as the sample space, we have improper integrals for $E(X)$ and $E(Y)$. The expected time for a table is

$$\begin{aligned} E(X) &= \iint_S x\frac{1}{600}e^{-x/15}e^{-y/40}\,dA \\ &= \frac{1}{600}\int_{x=0}^{x=\infty}\int_{y=0}^{y=\infty} xe^{-x/15}e^{-y/40}dA \\ &= \frac{1}{600}\lim_{(a,b)\to(\infty,\infty)}\int_{x=0}^{x=a}\int_{y=0}^{y=b} xe^{-x/15}e^{-y/40}dx\,dy \\ &= \frac{1}{600}\left(\lim_{a\to\infty}\int_{x=0}^{x=a} xe^{-x/15}dx\right)\left(\lim_{b\to\infty}\int_{y=0}^{y=b} e^{-y/40}dy\right) \\ &= \frac{1}{600}\left((\lim_{a\to\infty}(-15e^{-x/15}(x+15)))\Big|_{x=0}^{x=a}\right)\left((\lim_{b\to\infty}(-40e^{-y/40}))\Big|_{y=0}^{y=b}\right) \\ &= \frac{1}{600}\left(\lim_{a\to\infty}(-15e^{-a/15}(a+15)+225)\right)\left(\lim_{b\to\infty}(-40e^{-b/40}+40)\right) \\ &= \frac{1}{600}(225)(40) = 15. \end{aligned}$$

A similar calculation shows that $E(Y) = 40$. This means that the expected values of the two random events are the average waiting time and the average dining time, respectively.

Exercise $14.2b.10$

The joint density function for two random variables X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{16250}(x^2+y^2), & \text{if } 0 \le x \le 15,\ 0 \le y \le 10 \\ 0, & \text{otherwise} \end{cases}$$

Find the probability that X is at most 10 and Y is at least 5.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- A general bounded region D on the plane is a region that can be enclosed inside a rectangular region. We can use this idea to define a double integral over a general bounded region.
- To evaluate an iterated integral of a function over a general nonrectangular region, we sketch the region and express it as a Type I or as a Type II region or as a union of several Type I or Type II regions that overlap only on their boundaries.
- We can use double integrals to find volumes, areas, and average values of a function over general regions, similarly to calculations over rectangular regions.
- We can use Fubini's theorem for improper integrals to evaluate some types of improper integrals.

Key Equations

- **Iterated integral over a Type I region**

$$\iint_D f(x,y)\,dA = \iint_D f(x,y)\,dy\,dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right] dx$$

- **Iterated integral over a Type II region**

$$\iint_D f(x,y)\,dA = \iint_D (x,y)\,dx\,dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right] dy$$

Glossary

improper double integral
: a double integral over an unbounded region or of an unbounded function

Type I
: a region D in the xy- plane is Type I if it lies between two vertical lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$

Type II
: a region D in the xy-plane is Type II if it lies between two horizontal lines and the graphs of two continuous functions $h_1(y)$ and $h_2(h)$

Answers

Exercise $14.2b.1$

Graph the functions, and draw vertical and horizontal lines.

Exercise $14.2b.1$

Type I and Type II are expressed as $\{(x,y) \mid 0 \le x \le 2,\ x^2 \le y \le 2x\}$ and $\{(x,y) \mid 0 \le y \le 4,\ \frac{1}{2}y \le x \le \sqrt{y}\}$, respectively.

Exercise $14.2b.2$

Express D as a Type I region, and integrate with respect to y first.

Exercise $14.2b.2$

$\frac{\pi}{4}$

Exercise $14.2b.3$

Sketch the region, and split it into three regions to set it up.

Exercise $14.2b.3$

$$\{(x,y) \mid 0 \le y \le 1,\ 1 \le x \le e^y\} \cup \{(x,y) \mid 1 \le y \le e,\ 1 \le x \le 2\} \cup \{(x,y) \mid e \le y \le e^2,\ \ln y \le x \le 2\}$$

Exercise $14.2b.4$

$$\{(x,y) \mid 0 \le y \le 4,\ \sqrt{y} - 2 \le x \le \left(y - \frac{1}{16}y^3\right)\} \cup \{(x,y) \mid -4 \le y \le 0,\ -2 \le x \le \left(y - \frac{1}{16}y^3\right)\}$$

Exercise $14.2b.4$

Same as in the example shown.

Exercise $14.2b.5$

Sketch the region and follow Example $14.2b.6$.

Exercise $14.2b.5$

$\frac{216}{35}$

Exercise $14.2b.6$

Sketch the region, and describe it as Type I.

$\frac{e^2}{4} + 10e - \frac{49}{4}$ cubic units

Exercise $14.2b.7$

Hint

Sketch the region.

Exercise $14.2b.7$

$\frac{81}{4}$ square units

Exercise $14.2b.8$

Express the line joining $(0,0)$ and $(1,3)$ as a function $y = g(x)$.

Exercise $14.2b.8$

$\frac{3}{4}$

Exercise $14.2b.9$

Notice that the integral is nonnegative and discontinuous on $x^2 + y^2 = 1$. Express the region D as $D = \{(x,y) \ : \ 0 \le x \le 1,\ 0 \le y \le \sqrt{1-x^2}\}$ and integrate using the method of substitution.

Exercise $14.2b.9$

$\frac{\pi}{4}$

Exercise $14.2b.10$

Compute the probability

$$P(X \le 10,\ Y \ge 5) = \int_{x=0}^{x=10} \int_{y=5}^{y=10} \frac{1}{16250}(x^2 + y^2) dy\ dx.$$

Exercise $14.2b.10$

$P(X \le 10,\ Y \ge 5) = \frac{11}{39} \approx 0.28205$

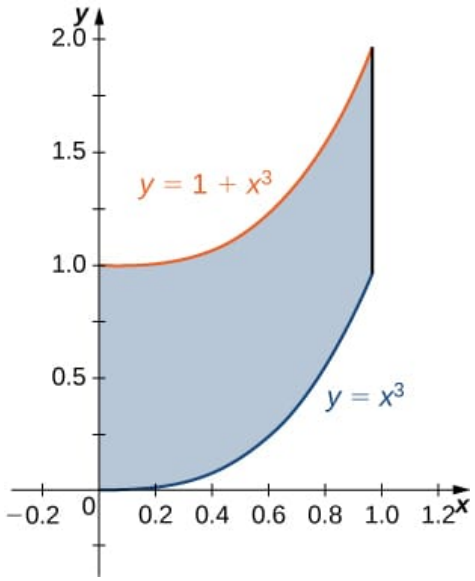
14.2bE: Double Integrals Part 2 (Exercises)

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14.2bE: Double Integrals Part 2 (Exercises)

1) The region D bounded by $y = x^3,\ y = x^3 + 1,\ x = 0$, and $x = 1$ as given in the following figure.



a. Classify this region as vertically simple (Type I) or horizontally simple (Type II).

Type:

Type I but not Type II

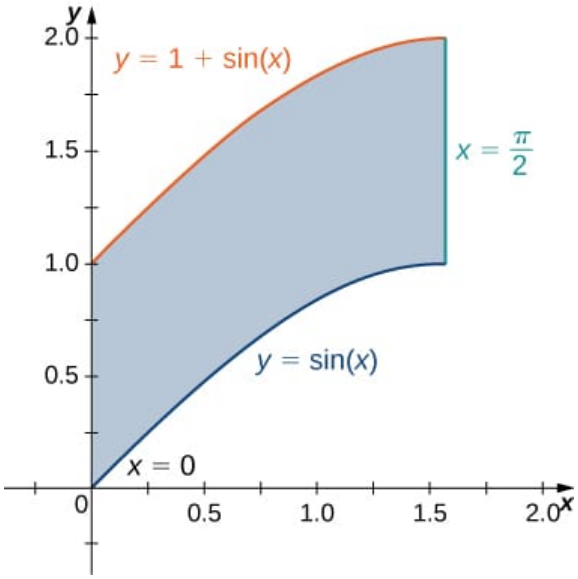
b. Find the area of the region D.

c. Find the average value of the function $f(x, y) = 3xy$ on the region graphed in the previous exercise.

Answer

$\frac{27}{20}$

2) The region D bounded by $y = \sin x,\ y = 1 + \sin x,\ x = 0$, and $x = \frac{\pi}{2}$ as given in the following figure.



a. Classify this region as vertically simple (Type I) or horizontally simple (Type II).

Type:

Type I but not Type II

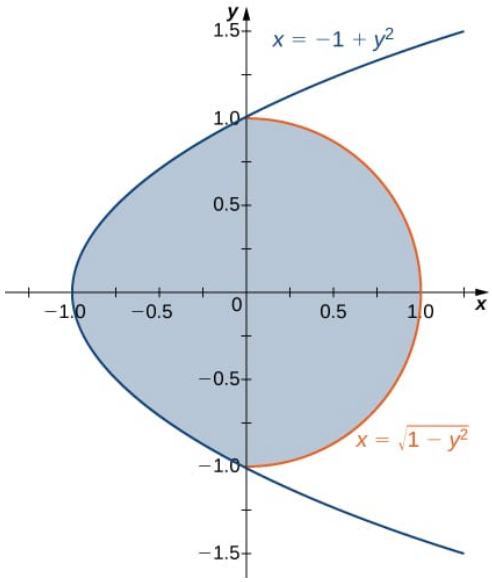
b. Find the area of the region D.

Answer

$\frac{\pi}{2}$ units^2

c. Find the average value of the function $f(x, y) = \cos x$ on the region D.

3) The region D bounded by $x = y^2 - 1$ and $x = \sqrt{1 - y^2}$ as given in the following figure.



a. Classify this region as vertically simple (Type I) or horizontally simple (Type II).

Type:

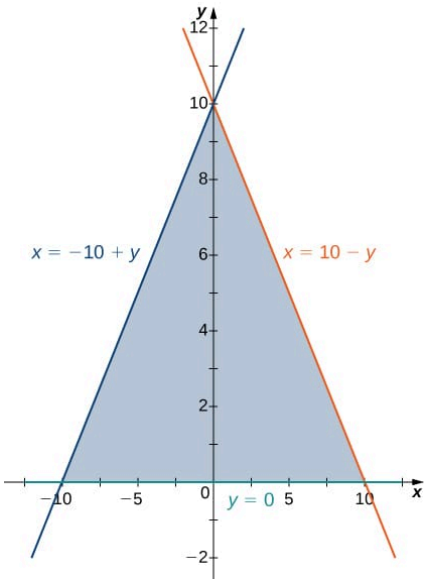
Type II but not Type I

b. Find the volume of the solid under the graph of the function $f(x,y) = xy + 1$ and above the region D.

Answer

$\frac{1}{6}(8 + 3\pi)$ units^3

4) The region D bounded by $y = 0,\ x = -10 + y$, and $x = 10 - y$ as given in the following figure.



a. Classify this region as vertically simple (Type I) or horizontally simple (Type II).

Type:

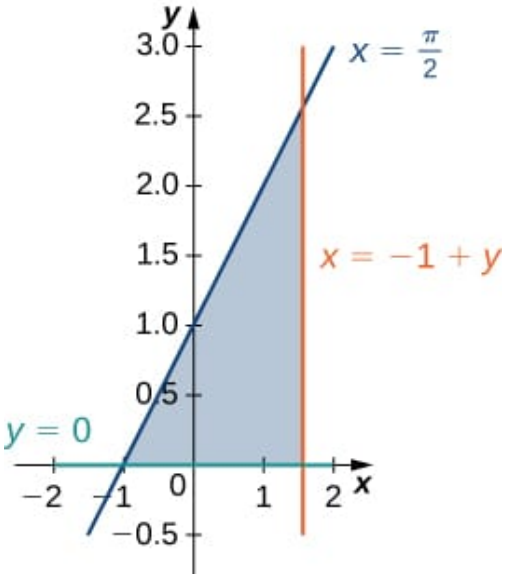
Type II but not Type I

b. Find the volume of the solid under the graph of the function $f(x,y) = x + y$ and above the region in the figure from the previous exercise.

Answer

$\frac{1000}{3}$ units^3

5) The region D bounded by $y = 0,\ x = y - 1,\ x = \frac{\pi}{2}$ as given in the following figure.

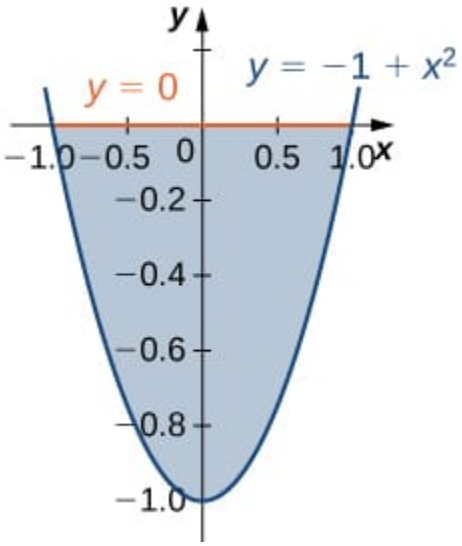


Classify this region as vertically simple (Type I) or horizontally simple (Type II).

Type:

Type I and Type II

6) The region D bounded by $y = 0$ and $y = x^2 - 1$ as given in the following figure.



Classify this region as vertically simple (Type I) or horizontally simple (Type II).

Type:

Type I and Type II

7) Let D be the region bounded by the curves of equations $y = \cos x$ and $y = 4 - x^2$ and the x-axis. Explain why D is neither of Type I nor II.

Answer

The region D is not of Type I: it does not lie between two vertical lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$. The region is not of Type II: it does not lie between two horizontal lines and the graphs of two continuous functions $h_1(y)$ and $h_2(y)$.

8) Let D be the region bounded by the curves of equations $y = x,\ y = -x$ and $y = 2 - x^2$. Explain why D is neither of Type I nor II.

In exercises 9 - 14, evaluate the double integral $\iint_D f(x,y)\ dA$ over the region D.

9) $f(x,y) = 1$ and

$D = \{(x,y)|\ 0 \le x \le \frac{\pi}{2},\ \sin x \le y \le 1 + \sin x\}$

Answer

$\frac{\pi}{2}$

10) $f(x,y) = 2$ and

$D = \{(x,y)|\ 0 \le y \le 1,\ y - 1 \le x \le \arccos y\}$

11) $f(x,y) = xy$ and

$D = \{(x,y)|\ -1 \le y \le 1,\ y^2 - 1 \le x \le \sqrt{1 - y^2}\}$

Answer

0

12) $f(x,y)=\sin y$ and D is the triangular region with vertices $(0,0),\ (0,3)$, and $(3,0)$

13) $f(x,y)=-x+1$ and D is the triangular region with vertices $(0,0),\ (0,2)$, and $(2,2)$

Answer

$\frac{2}{3}$

14) $f(x,y)=2x+4y$ and

$D=\left\{(x,y)|\ 0\le x\le 1,\ x^3\le y\le x^3+1\right\}$

In exercises 15 - 20, evaluate the iterated integrals.

15) $\int_0^1\int_{2\sqrt{x}}^{2\sqrt{x}+1}(xy+1)\ dy\ dx$

Answer

$\frac{41}{20}$

16) $\int_0^3\int_{2x}^{3x}(x+y^2)\ dy\ dx$

17) $\int_1^2\int_{-u^2-1}^{-u}(8uv)\ dv\ du$

Answer

-63

18) $\int_e^{e^2}\int_{\ln u}^{2}(v+\ln u)\ dv\ du$

19) $\int_0^{1/2}\int_{-\sqrt{1-4y^2}}^{\sqrt{1-4y^2}}4\ dx\ dy$

Answer

π

20) $\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}(2x+4y^3)\ dx\ dy$

21) Let D be the region bounded by $y=1-x^2,\ y=4-x^2$, and the x- and y-axes.

a. Show that $\iint_D x\ dA=\int_0^1\int_{1-x^2}^{4-x^2}x\ dy\ dx+\int_1^2\int_0^{4-x^2}x\ dy\ dx$ by dividing the region D into two regions of Type I.

b. Evaluate the integral $\iint_D x\ dA$.

22) Let D be the region bounded by $y=1,\ y=x,\ y=\ln x$, and the x-axis.

a. Show that $\iint_D y^2\ dA=\int_{-1}^0\int_{-x}^{2-x^2}y^2dy\ dx+\int_0^1\int_x^{2-x^2}y^2dy\ dx$ by dividing the region D into two regions of Type I, where $D=\left\{(x,y)\ |\ y\ge x, y\ge -x,\ y\le 2-x^2\right\}$.

b. Evaluate the integral $\iint_D y^2\ dA$.

23) Let D be the region bounded by $y = x^2$, $y = x + 2$, and $y = -x$.

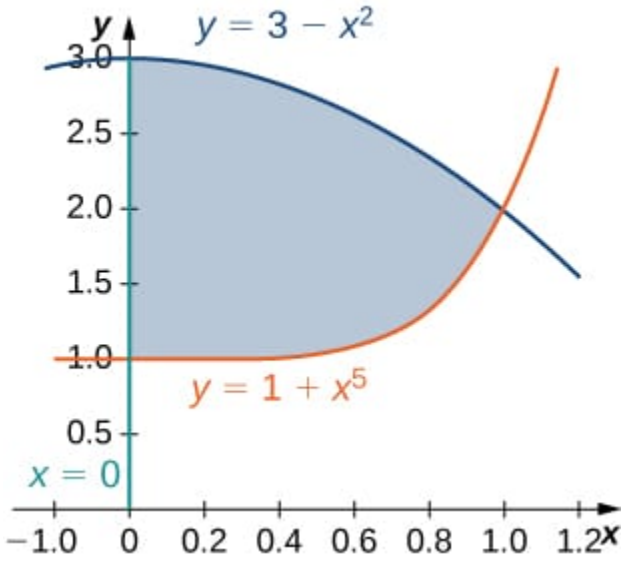
a. Show that $\iint_D x \; dA = \int_0^1 \int_{-y}^{\sqrt{y}} x \; dx \; dy + \int_1^4 \int_{y-2}^{\sqrt{y}} x \; dx \; dy$ by dividing the region D into two regions of Type II, where $D = \{(x, y) \mid y \geq x^2, \; y \geq -x, \; y \leq x + 2\}$.

b. Evaluate the integral $\iint_D x \; dA$.

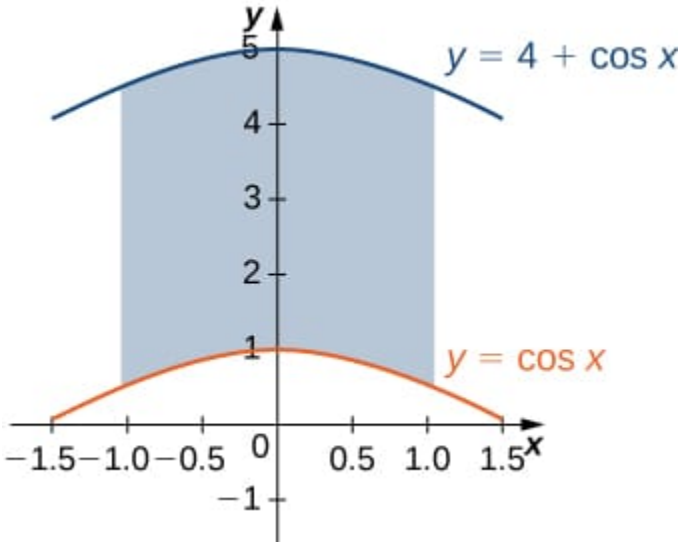
Answer

a. Answers may vary;
b. $\frac{7}{3}$

24) The region D bounded by $x = 0, y = x^5 + 1$, and $y = 3 - x^2$ is shown in the following figure. Find the area $A(D)$ of the region D.



25) The region D bounded by $y = \cos x, \; y = 4 + \cos x$, and $x = \pm\frac{\pi}{3}$ is shown in the following figure. Find the area $A(D)$ of the region D.



Answer

$\frac{8\pi}{3}$

26) Find the area $A(D)$ of the region $D = \{(x,y)|\ y \geq 1 - x^2, y \leq 4 - x^2,\ y \geq 0,\ x \geq 0\}$.

27) Let D be the region bounded by $y = 1,\ y = x,\ y = \ln x$, and the x-axis. Find the area $A(D)$ of the region D.

Answer

$\left(e - \frac{3}{2}\right)\ \text{units}^2$

28) Find the average value of the function $f(x,y) = \sin y$ on the triangular region with vertices $(0,0),\ (0,3)$, and $(3,0)$.

29) Find the average value of the function $f(x,y) = -x + 1$ on the triangular region with vertices $(0,0),\ (0,2)$, and $(2,2)$.

Answer

The average value of f on this triangular region is $\frac{1}{3}$.

In exercises 30 - 33, change the order of integration and evaluate the integral.

30) $\int_{-1}^{\pi/2} \int_{0}^{x+1} \sin x\ dy\ dx$

31) $\int_{0}^{1} \int_{x-1}^{1-x} x\ dy\ dx$

Answer

$$\int_0^1 \int_{x-1}^{1-x} x\ dy\ dx = \int_{-1}^0 \int_0^{y+1} x\ dx\ dy + \int_0^1 \int_0^{1-y} x\ dx\ dy = \frac{1}{3}$$

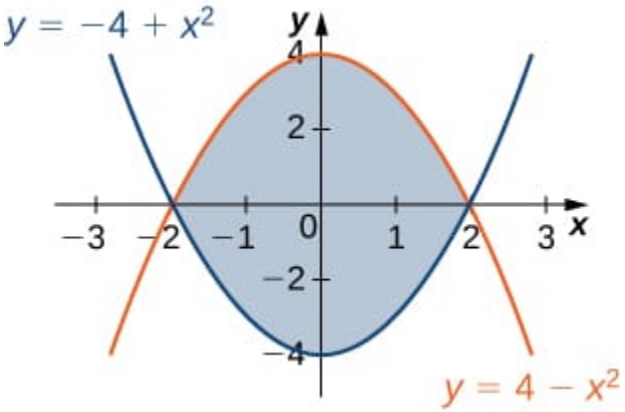
32) $\int_{-1}^{0} \int_{-\sqrt{y+1}}^{\sqrt{y+1}} y^2 dx\ dy$

33) $\int_{-1/2}^{1/2} \int_{-\sqrt{y^2+1}}^{\sqrt{y^2+1}} y\ dx\ dy$

Answer

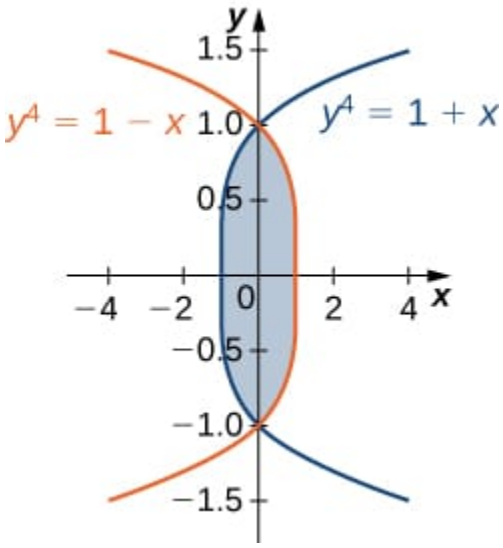
$$\int_{-1/2}^{1/2} \int_{-\sqrt{y^2+1}}^{\sqrt{y^2+1}} y\ dx\ dy = \int_1^2 \int_{-\sqrt{x^2-1}}^{\sqrt{x^2-1}} y\ dy\ dx = 0$$

34) The region D is shown in the following figure. Evaluate the double integral $\iint_D (x^2 + y)\ dA$ by using the easier order of integration.



35) The region D is shown in the following figure. Evaluate the double integral $\iint_D (x^2 - y^2)\ dA$ by using the easier order of

integration.



Answer

$$\iint_D (x^2 - y^2)dA = \int_{-1}^{1} \int_{y^4-1}^{1-y^4} (x^2 - y^2)dx\ dy = \frac{464}{4095}$$

36) Find the volume of the solid under the surface $z = 2x + y^2$ and above the region bounded by $y = x^5$ and $y = x$.

37) Find the volume of the solid under the plane $z = 3x + y$ and above the region determined by $y = x^7$ and $y = x$.

Answer

$\frac{4}{5}$ units^3

38) Find the volume of the solid under the plane $z = 3x + y$ and above the region bounded by $x = \tan y,\ x = -\tan y$, and $x = 1$.

39) Find the volume of the solid under the surface $z = x^3$ and above the plane region bounded by $x = \sin y,\ x = -\sin y$, and $x = 1$.

Answer

$\frac{5\pi}{32}$ units^3

40) Let g be a positive, increasing, and differentiable function on the interval $[a, b]$. Show that the volume of the solid under the surface $z = g'(x)$ and above the region bounded by $y = 0,\ y = g(x),\ x = a$, and $x = b$ is given by $\frac{1}{2}(g^2(b) - g^2(a))$.

41) Let g be a positive, increasing, and differentiable function on the interval $[a, b]$ and let k be a positive real number. Show that the volume of the solid under the surface $z = g'(x)$ and above the region bounded by $y = g(x),\ y = g(x) + k,\ x = a$, and $x = b$ is given by $k(g(b) - g(a))$.

42) Find the volume of the solid situated in the first octant and determined by the planes $z = 2,\ z = 0,\ x + y = 1,\ x = 0$, and $y = 0$.

43) Find the volume of the solid situated in the first octant and bounded by the planes $x + 2y = 1,\ x = 0,\ z = 4$, and $z = 0$.

Answer

1 units^3

44) Find the volume of the solid bounded by the planes $x + y = 1,\ x - y = 1,\ x = 0,\ z = 0$, and $z = 10$.

45) Find the volume of the solid bounded by the planes $x + y = 1,\ x - y = 1,\ x + y = -1\ x - y = -1,\ z = 1$, and $z = 0$

Answer

2 units^3

46) Let S_1 and S_2 be the solids situated in the first octant under the planes $x + y + z = 1$ and $x + y + 2z = 1$ respectively, and let S be the solid situated between $S_1,\ S_2,\ x = 0$, and $y = 0$.

a. Find the volume of the solid S_1.
b. Find the volume of the solid S_2.
c. Find the volume of the solid S by subtracting the volumes of the solids S_1 and S_2.

47) Let S_1 and S_2 be the solids situated in the first octant under the planes $2x + 2y + z = 2$ and $x + y + z = 1$ respectively, and let S be the solid situated between $S_1,\ S_2,\ x = 0$, and $y = 0$.

a. Find the volume of the solid S_1.
b. Find the volume of the solid S_2.
c. Find the volume of the solid S by subtracting the volumes of the solids S_1 and S_2.

Answer

a. $\frac{1}{3}\ \text{units}^3$
b. $\frac{1}{6}\ \text{units}^3$
c. $\frac{1}{6}\ \text{units}^3$

48) Let S_1 and S_2 be the solids situated in the first octant under the plane $x + y + z = 2$ and under the sphere $x^2 + y^2 + z^2 = 4$, respectively. If the volume of the solid S_2 is $\frac{4\pi}{3}$ determine the volume of the solid S situated between S_1 and S_2 by subtracting the volumes of these solids.

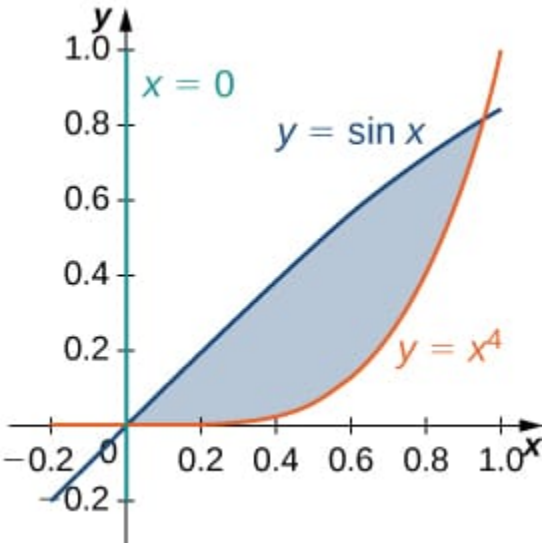
49) Let S_1 and S_2 be the solids situated in the first octant under the plane $x + y + z = 2$ and bounded by the cylinder $x^2 + y^2 = 4$, respectively.

a. Find the volume of the solid S_1.
b. Find the volume of the solid S_2.
c. Find the volume of the solid S situated between S_1 and S_2 by subtracting the volumes of the solids S_1 and S_2.

Answer

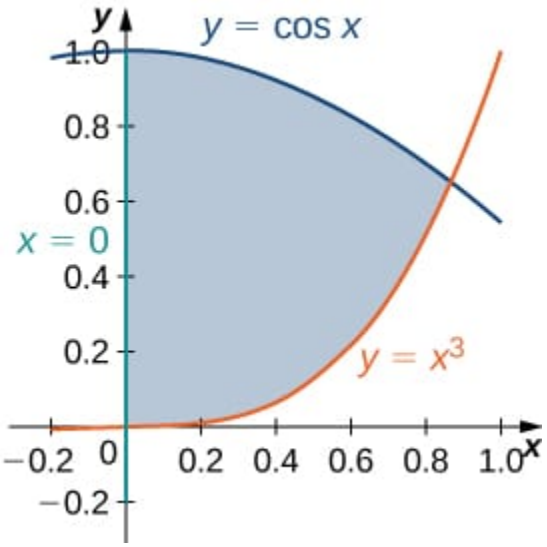
a. $\frac{4}{3}\ \text{units}^3$
b. $2\pi\ \text{units}^3$
c. $\frac{6\pi-4}{3}\ \text{units}^3$

50) [T] The following figure shows the region D bounded by the curves $y = \sin x,\ x = 0$, and $y = x^4$. Use a graphing calculator or CAS to find the x-coordinates of the intersection points of the curves and to determine the area of the region D. Round your answers to six decimal places.



51) [T] The region D bounded by the curves $y = \cos x,\ x = 0$, and $y = x^3$ is shown in the following figure. Use a graphing calculator

or CAS to find the x-coordinates of the intersection points of the curves and to determine the area of the region D. Round your answers to six decimal places.



Answer

0 and 0.865474; $A(D) = 0.621135 \text{ units}^3$

52) Suppose that (X, Y) is the outcome of an experiment that must occur in a particular region S in the xy-plane. In this context, the region S is called the sample space of the experiment and X and Y are random variables. If D is a region included in S, then the probability of (X, Y) being in D is defined as $P[(X, Y)^{\text{int}} D] = \iint_D p(x, y)dx\ dy$, where $p(x, y)$ is the joint probability density of the experiment. Here, $p(x, y)$ is a nonnegative function for which $\iint_S p(x, y)dx\ dy = 1$. Assume that a point (X, Y) is chosen arbitrarily in the square $[0, 3] \times [0, 3]$ with the probability density

$$p(x, y) = \frac{1}{9}(x, y)^{\text{int}}[0, 3] \times [0, 3],$$

$$p(x, y) = 0 \text{ otherwise}$$

Find the probability that the point (X, Y) is inside the unit square and interpret the result.

53) Consider X and Y two random variables of probability densities $p_1(x)$ and $p_2(x)$, respectively. The random variables X and Y are said to be independent if their joint density function is given by $p_(x, y) = p_1(x)p_2(y)$. At a drive-thru restaurant, customers spend, on average, 3 minutes placing their orders and an additional 5 minutes paying for and picking up their meals. Assume that placing the order and paying for/picking up the meal are two independent events X and Y. If the waiting times are modeled by the exponential probability densities

$$p_1(x) = \frac{1}{3}e^{-x/3}\ x \geq 0,$$

$$p_1(x) = 0 \text{ otherwise}$$

$$p_2(y) = \frac{1}{5}e^{-y/5}\ y \geq 0$$

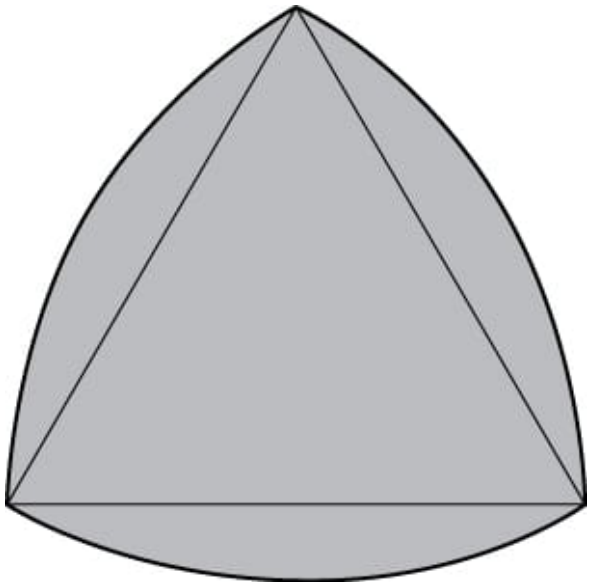
$$p_2(y) = 0 \text{ otherwise}$$

respectively, the probability that a customer will spend less than 6 minutes in the drive-thru line is given by $P[X + Y \leq 6] = \iint_D p(x, y)dx\ dy$, where $D = \{(x, y)|x \geq 0,\ y \geq 0,\ x + y \leq 6\}$. Find $P[X + Y \leq 6]$ and interpret the result.

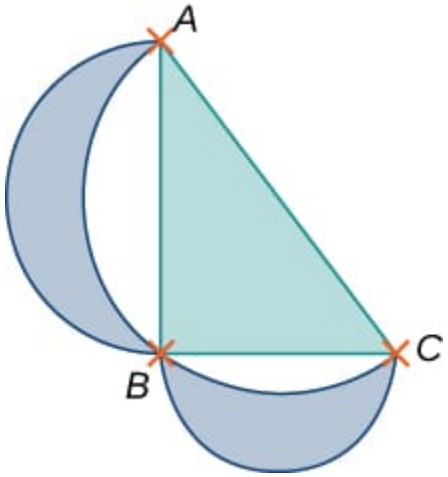
Answer

$P[X + Y \leq 6] = 1 + \frac{3}{2e^2} - \frac{5}{e^{6/5}} \approx 0.45$; there is a 45% chance that a customer will spend 6 minutes in the drive-thru line.

54) [T] The Reuleaux triangle consists of an equilateral triangle and three regions, each of them bounded by a side of the triangle and an arc of a circle of radius s centered at the opposite vertex of the triangle. Show that the area of the Reuleaux triangle in the following figure of side length s is $\frac{s^2}{2}(\pi - \sqrt{3})$.



55) [T] Show that the area of the lunes of Alhazen, the two blue lunes in the following figure, is the same as the area of the right triangle ABC. The outer boundaries of the lunes are semicircles of diameters AB and AC respectively, and the inner boundaries are formed by the circumcircle of the triangle ABC.



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14.3: DOUBLE INTEGRATION WITH POLAR COORDINATES

We have used iterated integrals to evaluate double integrals, which give the signed volume under a surface, $z = f(x, y)$, over a region R of the xy-plane. The integrand is simply $f(x, y)$, and the bounds of the integrals are determined by the region R.

Some regions R are easy to describe using rectangular coordinates -- that is, with equations of the form $y = f(x)$, $x = a$, etc. However, some regions are easier to handle if we represent their boundaries with polar equations of the form $r = f(\theta)$, $\theta = \alpha$, etc.

The basic form of the double integral is $\iint_R f(x, y)\, dA$. We interpret this integral as follows: over the region R, sum up lots of products of heights (given by $f(x_i, y_i)$) and areas (given by ΔA_i). That is, dA represents "a little bit of area." In rectangular coordinates, we can describe a small rectangle as having area $dx\ dy$ or $dy\ dx$ -- the area of a rectangle is simply length$\times$width -- a small change in x times a small change in y. Thus we replace dA in the double integral with $dx\ dy$ or $dy\ dx$.

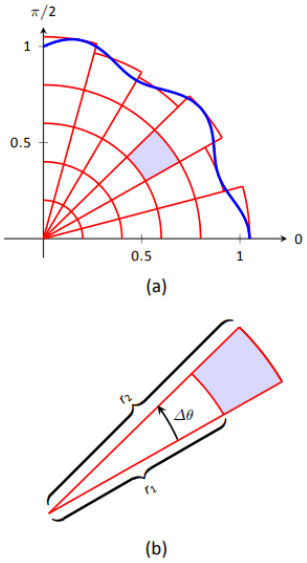


FIGURE 14.3.1

Now consider representing a region R with polar coordinates. Consider Figure $14.3.1a$. Let R be the region in the first quadrant bounded by the curve. We can approximate this region using the natural shape of polar coordinates: portions of sectors of circles. In the figure, one such region is shaded, shown again in part (b) of the figure.

As the area of a sector of a circle with radius r, subtended by an angle θ, is $A = \frac{1}{2}r^2\theta$, we can find the area of the shaded region. The whole sector has area $\frac{1}{2}r_2^2\Delta\theta$, whereas the smaller, unshaded sector has area $\frac{1}{2}r_1^2\Delta\theta$. The area of the shaded region is the difference of these areas:

$$\Delta A_i = \frac{1}{2}r_2^2\Delta\theta - \frac{1}{2}r_1^2\Delta\theta = \frac{1}{2}\left(r_2^2 - r_1^2\right)\left(\Delta\theta\right) = \frac{r_2 + r_1}{2}\left(r_2 - r_1\right)\Delta\theta. \tag{14.3.1}$$

Note that $(r_2 + r_1)/2$ is just the average of the two radii.

To approximate the region R, we use many such subregions; doing so shrinks the difference $r_2 - r_1$ between radii to 0 and shrinks the change in angle $\Delta\theta$ also to 0. We represent these infinitesimal changes in radius and angle as dr and $d\theta$, respectively. Finally, as dr is small, $r_2 \approx r_1$, and so $(r_2 + r_1)/2 \approx r_1$. Thus, when dr and $d\theta$ are small,

$$\Delta A_i \approx r_i\, dr\, d\theta. \tag{14.3.2}$$

Taking a limit, where the number of subregions goes to infinity and both $r_2 - r_1$ and $\Delta\theta$ go to 0, we get

$$dA = r\, dr\, d\theta. \tag{14.3.3}$$

So to evaluate $\iint_R f(x,y)\ dA$, replace dA with $r\ dr\ d\theta$. Convert the function $z = f(x,y)$ to a function with polar coordinates with the substitutions $x = r\cos\theta$, $y = r\sin\theta$. Finally, find bounds $g_1(\theta) \le r \le g_2(\theta)$ and $\alpha \le \theta \le \beta$ that describe R. This is the key principle of this section, so we restate it here as a Key Idea.

Key Idea: Evaluating Double Integrals with Polar Coordinates

Let R be a plane region bounded by the polar equations $\alpha \le \theta \le \beta$ and $g_1(\theta) \le r \le g_2(\theta)$. Then

$$\iint_R f(x,y)\ dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f\big(r\cos\theta, r\sin\theta\big)\ r\,dr\,d\theta. \tag{14.3.4}$$

Examples will help us understand this Key Idea.

Example 14.3.1: Evaluating a double integral with polar coordinates

Find the signed volume under the plane $z = 4 - x - 2y$ over the circle with equation $x^2 + y^2 = 1$.

Solution

The bounds of the integral are determined solely by the region R over which we are integrating. In this case, it is a circle with equation $x^2 + y^2 = 1$. We need to find polar bounds for this region. It may help to review polar coordinates earlier in this text; bounds for this circle are $0 \le r \le 1$ and $0 \le \theta \le 2\pi$.

We replace $f(x,y)$ with $f(r\cos\theta, r\sin\theta)$. That means we make the following substitutions:

$$4 - x - 2y \quad \Rightarrow \quad 4 - r\cos\theta - 2r\sin\theta.$$

Finally, we replace dA in the double integral with $r\ dr\ d\theta$. This gives the final iterated integral, which we evaluate:

$$\begin{aligned}
\iint_R f(x,y)\ dA &= \int_0^{2\pi}\int_0^1 \big(4 - r\cos\theta - 2r\sin\theta\big) r\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 \big(4r - r^2(\cos\theta - 2\sin\theta)\big) dr\,d\theta \\
&= \int_0^{2\pi} \left(2r^2 - \frac{1}{3}r^3(\cos\theta - 2\sin\theta)\right)\Big|_0^1 d\theta \\
&= \int_0^{2\pi} \left(2 - \frac{1}{3}\big(\cos\theta - 2\sin\theta\big)\right) d\theta \\
&= \left(2\theta - \frac{1}{3}\big(\sin\theta + 2\cos\theta\big)\right)\Big|_0^{2\pi} \\
&= 4\pi \approx 12.566.
\end{aligned}$$

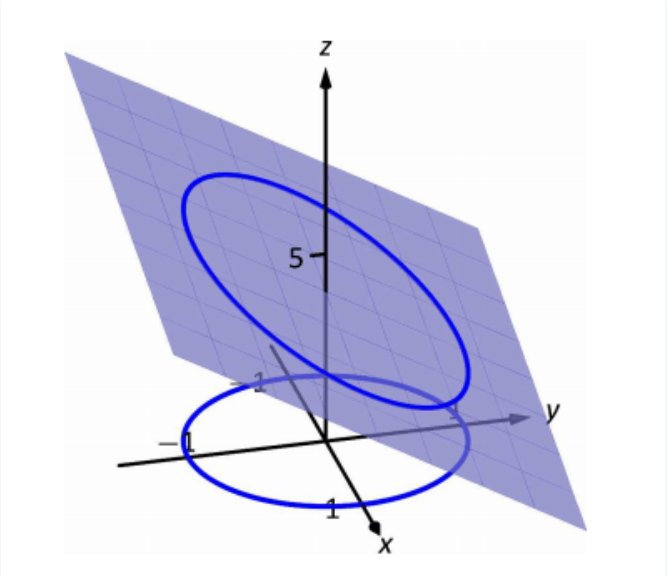


FIGURE 14.3.2

The surface and region R are shown in Figure 14.3.2.

Example 14.3.2: Evaluating a double integral with polar coordinates

Find the volume under the paraboloid $z = 4 - (x-2)^2 - y^2$ over the region bounded by the circles $(x-1)^2 + y^2 = 1$ and $(x-2)^2 + y^2 = 4$.

Solution

At first glance, this seems like a very hard volume to compute as the region R (shown in Figure 14.3.3a) has a hole in it, cutting out a strange portion of the surface, as shown in part (b) of the figure. However, by describing R in terms of polar equations, the volume is not very difficult to compute.

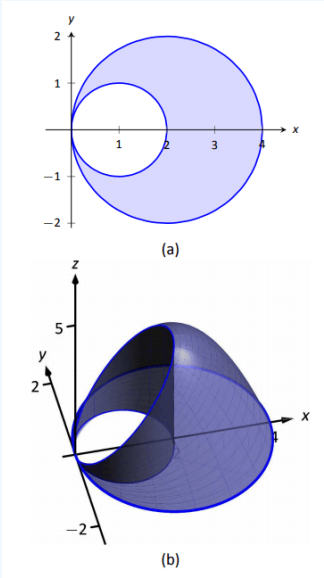


FIGURE 14.3.3

It is straightforward to show that the circle $(x-1)^2+y^2=1$ has polar equation $r=2\cos\theta$, and that the circle $(x-2)^2+y^2=4$ has polar equation $r=4\cos\theta$. Each of these circles is traced out on the interval $0\le\theta\le\pi$. The bounds on r are $2\cos\theta\le r\le 4\cos\theta$.

Replacing x with $r\cos\theta$ in the integrand, along with replacing y with $r\sin\theta$, prepares us to evaluate the double integral $\iint_R f(x,y)\,dA$:

$$\begin{aligned}\iint_R f(x,y)\,dA &= \int_0^{\pi}\int_{2\cos\theta}^{4\cos\theta}\Big(4-\big(r\cos\theta-2\big)^2-\big(r\sin\theta\big)^2\Big)r\,dr\,d\theta\\ &= \int_0^{\pi}\int_{2\cos\theta}^{4\cos\theta}\big(-r^3+4r^2\cos\theta\big)dr\,d\theta\\ &= \int_0^{\pi}\left(-\frac{1}{4}r^4+\frac{4}{3}r^3\cos\theta\right)\Big|_{2\cos\theta}^{4\cos\theta}d\theta\\ &= \int_0^{\pi}\left(\left[-\frac{1}{4}(256\cos^4\theta)+\frac{4}{3}(64\cos^4\theta)\right]-\right.\\ &\quad\left.\left[-\frac{1}{4}(16\cos^4\theta)+\frac{4}{3}(8\cos^4\theta)\right]\right)d\theta\\ &= \int_0^{\pi}\frac{44}{3}\cos^4\theta\,d\theta.\end{aligned}$$

To integrate $\cos^4\theta$, rewrite it as $\cos^2\theta\cos^2\theta$ and employ the power-reducing formula twice:

$$\begin{aligned}\cos^4\theta &= \cos^2\theta\cos^2\theta\\ &= \frac{1}{2}\big(1+\cos(2\theta)\big)\frac{1}{2}\big(1+\cos(2\theta)\big)\\ &= \frac{1}{4}\big(1+2\cos(2\theta)+\cos^2(2\theta)\big)\\ &= \frac{1}{4}\Big(1+2\cos(2\theta)+\frac{1}{2}\big(1+\cos(4\theta)\big)\Big)\\ &= \frac{3}{8}+\frac{1}{2}\cos(2\theta)+\frac{1}{8}\cos(4\theta).\end{aligned}$$

Picking up from where we left off above, we have

$$\begin{aligned}&= \int_0^{\pi}\frac{44}{3}\cos^4\theta\,d\theta\\ &= \int_0^{\pi}\frac{44}{3}\left(\frac{3}{8}+\frac{1}{2}\cos(2\theta)+\frac{1}{8}\cos(4\theta)\right)d\theta\\ &= \frac{44}{3}\left(\frac{3}{8}\theta+\frac{1}{4}\sin(2\theta)+\frac{1}{32}\sin(4\theta)\right)\Big|_0^{\pi}\\ &= \frac{11}{2}\pi\approx 17.279.\end{aligned}$$

While this example was not trivial, the double integral would have been *much* harder to evaluate had we used rectangular coordinates.

Example 14.3.3: Evaluating a double integral with polar coordinates

Find the volume under the surface $f(x,y)=\dfrac{1}{x^2+y^2+1}$ over the sector of the circle with radius a centered at the origin in the first quadrant, as shown in Figure 14.3.4.

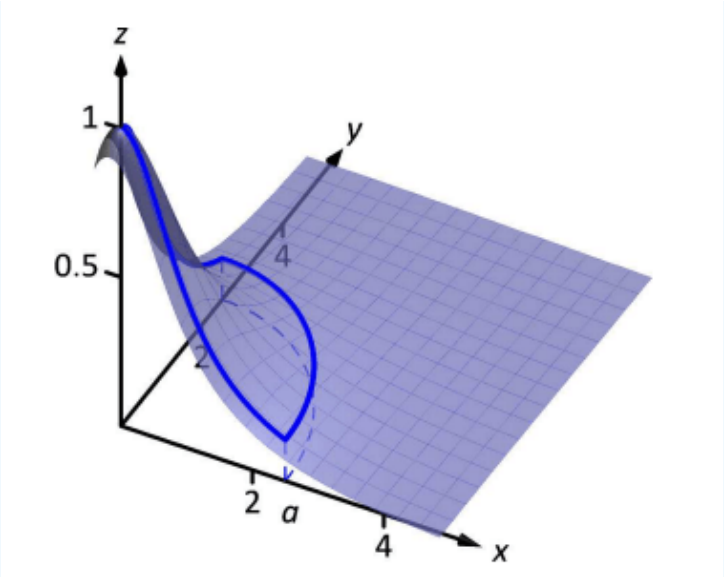


FIGURE 14.3.4

Solution

The region R we are integrating over is a circle with radius a, restricted to the first quadrant. Thus, in polar, the bounds on R are $0 \le r \le a$, $0 \le \theta \le \pi/2$. The integrand is rewritten in polar as

$$\frac{1}{x^2+y^2+1} \Rightarrow \frac{1}{r^2\cos^2\theta + r^2\sin^2\theta + 1} = \frac{1}{r^2+1}. \tag{14.3.5}$$

We find the volume as follows:

$$\begin{aligned}
\iint_R f(x,y)\,dA &= \int_0^{\pi/2}\int_0^a \frac{r}{r^2+1}\,dr\,d\theta \\
&= \int_0^{\pi/2} \frac{1}{2}\Big(\ln|r^2+1|\Big)\Big|_0^a\,d\theta \\
&= \int_0^{\pi/2} \frac{1}{2}\ln(a^2+1)\,d\theta \\
&= \left(\frac{1}{2}\ln(a^2+1)\theta\right)\Big|_0^{\pi/2} \\
&= \frac{\pi}{4}\ln(a^2+1).
\end{aligned}$$

Figure 14.3.4 shows that f shrinks to near 0 very quickly. Regardless, as a grows, so does the volume, without bound.

Note: Previous work has shown that there is finite *area* under $\frac{1}{x^2+1}$ over the entire x-axis. However, Example 14.3.3 shows that there is infinite *volume* under $\dfrac{1}{x^2+y^2+1}$ over the entire xy-plane.

Example 14.3.4: Finding the volume of a sphere

Find the volume of a sphere with radius a.

Solution

The sphere of radius a, centered at the origin, has equation $x^2+y^2+z^2=a^2$; solving for z, we have $z=\sqrt{a^2-x^2-y^2}$. This gives the upper half of a sphere. We wish to find the volume under this top half, then double it to find the total volume.

The region we need to integrate over is the circle of radius a, centered at the origin. Polar bounds for this equation are $0 \leq r \leq a$, $0 \leq \theta \leq 2\pi$.

All together, the volume of a sphere with radius a is:

$$\begin{aligned} 2\iint_R \sqrt{a^2-x^2-y^2}\, dA &= 2\int_0^{2\pi}\int_0^a \sqrt{a^2-(r\cos\theta)^2-(r\sin\theta)^2}\, r\, dr\, d\theta \\ &= 2\int_0^{2\pi}\int_0^a r\sqrt{a^2-r^2}\, dr\, d\theta. \end{aligned} \tag{14.3.6}$$

We can evaluate this inner integral with substitution. With $u = a^2 - r^2$, $du = -2r\ dr$. The new bounds of integration are $u(0) = a^2$ to $u(a) = 0$. Thus we have:

$$\begin{aligned} &= \int_0^{2\pi}\int_{a^2}^0 \left(-u^{1/2}\right) du\, d\theta \\ &= \int_0^{2\pi} \left(-\frac{2}{3}u^{3/2}\right)\Big|_{a^2}^0 d\theta \\ &= \int_0^{2\pi} \left(\frac{2}{3}a^3\right) d\theta \\ &= \left(\frac{2}{3}a^3\theta\right)\Big|_0^{2\pi} \\ &= \frac{4}{3}\pi a^3. \end{aligned}$$

Generally, the formula for the volume of a sphere with radius r is given as $4/3\pi r^3$; we have justified this formula with our calculation.

Example 14.3.5: Finding the volume of a solid

A sculptor wants to make a solid bronze cast of the solid shown in Figure 14.3.5, where the base of the solid has boundary, in polar coordinates, $r = \cos(3\theta)$, and the top is defined by the plane $z = 1 - x + 0.1y$. Find the volume of the solid.

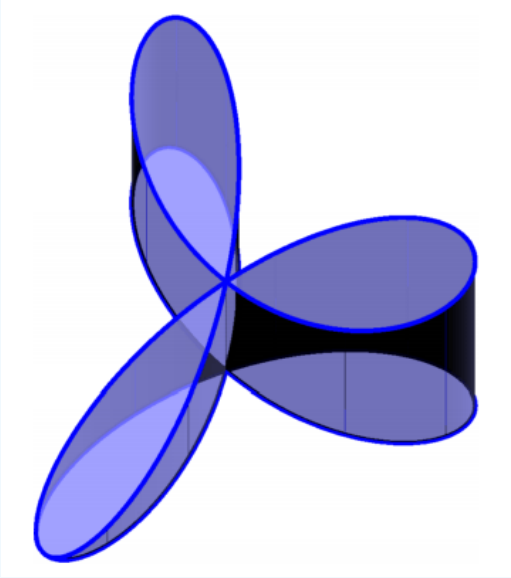


FIGURE 14.3.5

Solution

From the outset, we should recognize that knowing *how to set up* this problem is probably more important than knowing *how to compute the integrals*. The iterated integral to come is not "hard" to evaluate, though it is long, requiring lots of algebra. Once the proper iterated integral is determined, one can use readily--available technology to help compute the final answer.

The region R that we are integrating over is bound by $0 \le r \le \cos(3\theta)$, for $0 \le \theta \le \pi$ (note that this rose curve is traced out on the interval $[0, \pi]$, not $[0, 2\pi]$). This gives us our bounds of integration. The integrand is $z = 1 - x + 0.1y$; converting to polar, we have that the volume V is:

$$V = \iint_R f(x,y)\ dA = \int_0^{\pi} \int_0^{\cos(3\theta)} \Big(1 - r\cos\theta + 0.1r\sin\theta\Big) r\, dr\, d\theta.$$

Distributing the r, the inner integral is easy to evaluate, leading to

$$\int_0^{\pi} \left(\frac{1}{2}\cos^2(3\theta) - \frac{1}{3}\cos^3(3\theta)\cos\theta + \frac{0.1}{3}\cos^3(3\theta)\sin\theta\right)\ d\theta.$$

This integral takes time to compute by hand; it is rather long and cumbersome. The powers of cosine need to be reduced, and products like $\cos(3\theta)\cos\theta$ need to be turned to sums using the Product To Sum formulas in the back cover of this text.

We rewrite $\frac{1}{2}\cos^2(3\theta)$ as $\frac{1}{4}(1 + \cos(6\theta))$. We can also rewrite $\frac{1}{3}\cos^3(3\theta)\cos\theta$ as:

$$\frac{1}{3}\cos^3(3\theta)\cos\theta = \frac{1}{3}\cos^2(3\theta)\cos(3\theta)\cos\theta = \frac{1}{3}\frac{1+\cos(6\theta)}{2}\Big(\cos(4\theta) + \cos(2\theta)\Big).$$

This last expression still needs simplification, but eventually all terms can be reduced to the form $a\cos(m\theta)$ or $a\sin(m\theta)$ for various values of a and m.

We forgo the algebra and recommend the reader employ technology, such as WolframAlpha, to compute the numeric answer. Such technology gives:

$$\int_0^{\pi} \int_0^{\cos(3\theta)} \Big(1 - r\cos\theta + 0.1r\sin\theta\Big) r\, dr\, d\theta = \frac{\pi}{4} \approx 0.785u^3.$$

Since the units were not specified, we leave the result as almost 0.8 cubic units (meters, feet, etc.) Should the artist want to scale the piece uniformly, so that each rose petal had a length other than 1, she should keep in mind that scaling by a factor of k scales the volume by a factor of k^3.

We have used iterated integrals to find areas of plane regions and volumes under surfaces. Just as a single integral can be used to compute much more than "area under the curve," iterated integrals can be used to compute much more than we have thus far seen. The next two sections show two, among many, applications of iterated integrals.

14.3b: Double Integrals in Polar Coordinates

14.3E: Double Integrals in Polar Coordinates (Exercises)

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14.3b: Double Integrals in Polar Coordinates

Learning Objectives

- Recognize the format of a double integral over a polar rectangular region.
- Evaluate a double integral in polar coordinates by using an iterated integral.
- Recognize the format of a double integral over a general polar region.
- Use double integrals in polar coordinates to calculate areas and volumes.

Double integrals are sometimes much easier to evaluate if we change rectangular coordinates to polar coordinates. However, before we describe how to make this change, we need to establish the concept of a double integral in a polar rectangular region.

Polar Rectangular Regions of Integration

When we defined the double integral for a continuous function in rectangular coordinates—say, g over a region R in the xy -plane—we divided R into subrectangles with sides parallel to the coordinate axes. These sides have either constant x-values and/or constant y-values. In polar coordinates, the shape we work with is a polar rectangle, whose sides have constant r-values and/or constant θ-values. This means we can describe a polar rectangle as in Figure $14.3b.1a$, with $R = \{(r, \theta) \mid a \leq r \leq b, \ \alpha \leq \theta \leq \beta\}$.

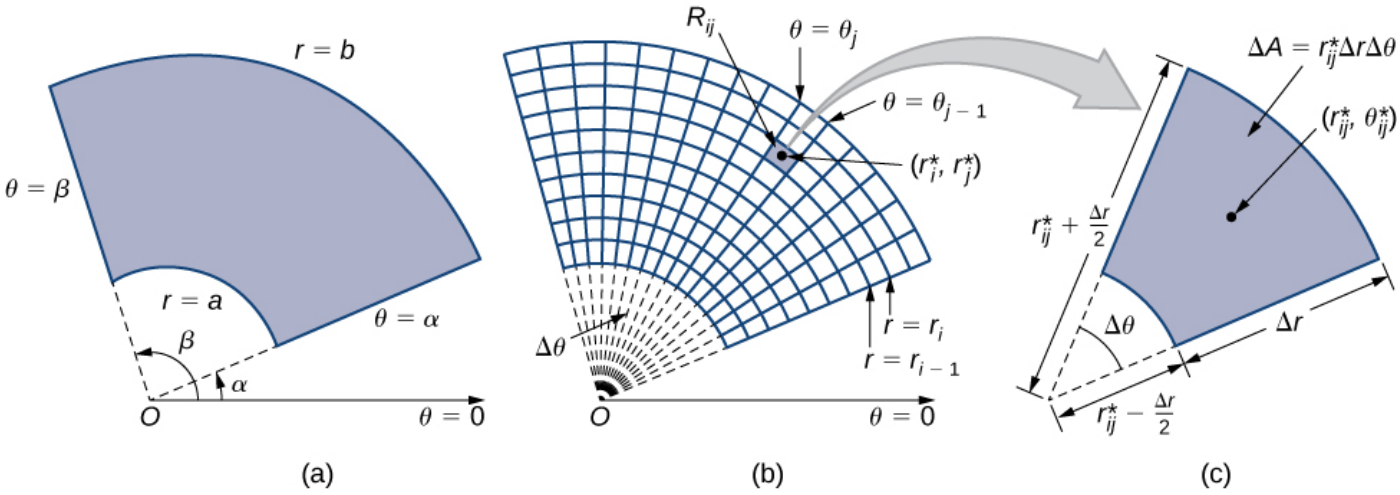


Figure $14.3b.1$: (a) A polar rectangle R (b) divided into subrectangles R_{ij} (c) Close-up of a subrectangle.

In this section, we are looking to integrate over polar rectangles. Consider a function $f(r, \theta)$ over a polar rectangle R. We divide the interval $[a, b]$ into m subintervals $[r_{i-1}, r_i]$ of length $\Delta r = (b - a)/m$ and divide the interval $[\alpha, \beta]$ into n subintervals $[\theta_{i-1}, \theta_i]$ of width $\Delta\theta = (\beta - \alpha)/n$. This means that the circles $r = r_i$ and rays $\theta = \theta_i$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ divide the polar rectangle R into smaller polar subrectangles R_{ij} (Figure $14.3b.1b$).

As before, we need to find the area ΔA of the polar subrectangle R_{ij} and the "polar" volume of the thin box above R_{ij}. Recall that, in a circle of radius r the length s of an arc subtended by a central angle of θ radians is $s = r\theta$. Notice that the polar rectangle R_{ij} looks a lot like a trapezoid with parallel sides $r_{i-1}\Delta\theta$ and $r_i\Delta\theta$ and with a width Δr. Hence the area of the polar subrectangle R_{ij} is

$$\Delta A = \frac{1}{2}\Delta r(r_{i-1}\Delta\theta + r_i\Delta\theta).$$

Simplifying and letting

$$r^*_{ij} = \frac{1}{2}(r_{i-1} + r_i)$$

we have $\Delta A = r_{ij}^* \Delta r \Delta\theta$.

Therefore, the polar volume of the thin box above R_{ij} (Figure $14.3b.2$) is

$$f(r_{ij}^*, \theta_{ij}^*)\Delta A = f(r_{ij}^*, \theta_{ij}^*) r_{ij}^* \Delta r \Delta\theta.$$

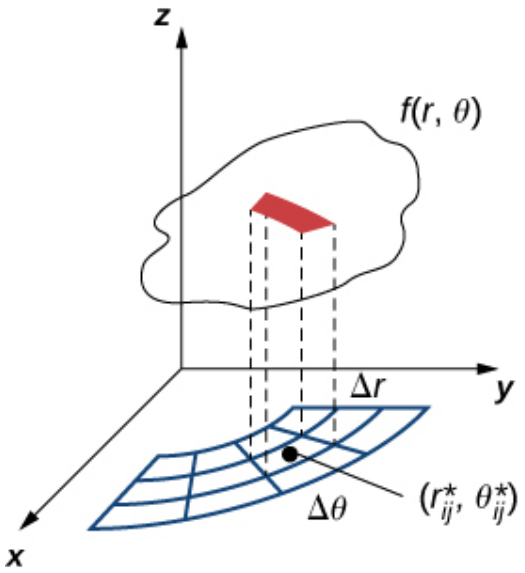


Figure $14.3b.2$: Finding the volume of the thin box above polar rectangle R_{ij}.

Using the same idea for all the subrectangles and summing the volumes of the rectangular boxes, we obtain a double Riemann sum as

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*) r_{ij}^* \Delta r \Delta\theta.$$

As we have seen before, we obtain a better approximation to the polar volume of the solid above the region R when we let m and n become larger. Hence, we define the polar volume as the limit of the double Riemann sum,

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*) r_{ij}^* \Delta r \Delta\theta.$$

This becomes the expression for the double integral.

Definition: The double integral in polar coordinates

The double integral of the function $f(r, \theta)$ over the polar rectangular region R in the $r\theta$-plane is defined as

$$\iint_R f(r,\theta)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*)\Delta A \tag{14.3b.1}$$

$$= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*) r_{ij}^* \Delta r \Delta\theta. \tag{14.3b.2}$$

Again, just as in section on Double Integrals over Rectangular Regions, the double integral over a polar rectangular region can be expressed as an iterated integral in polar coordinates. Hence,

$$\iint_R f(r,\theta)\, dA = \iint_R f(r,\theta)\, r\, dr\, d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=a}^{r=b} f(r,\theta)\, r\, dr\, d\theta.$$

Notice that the expression for dA is replaced by $r\ dr\ d\theta$ when working in polar coordinates. Another way to look at the polar double integral is to change the double integral in rectangular coordinates by substitution. When the function f is given in terms of x and y using $x = r\ \cos\ \theta,\ y = r\ \sin\ \theta$, and $dA = r\ dr\ d\theta$ changes it to

$$\iint_R f(x,y)\,dA = \iint_R f(r\ \cos\ \theta,\ r\ \sin\ \theta)\,r\,dr\,d\theta.$$

Note that all the properties listed in section on Double Integrals over Rectangular Regions for the double integral in rectangular coordinates hold true for the double integral in polar coordinates as well, so we can use them without hesitation.

Example $14.3b.1A$: Sketching a Polar Rectangular Region

Sketch the polar rectangular region

$$R = \{(r,\theta)\,|\,1 \le r \le 3, 0 \le \theta \le \pi\}.$$

Solution

As we can see from Figure $14.3b.3$, $r = 1$ and $r = 3$ are circles of radius 1 and 3 and $0 \le \theta \le \pi$ covers the entire top half of the plane. Hence the region R looks like a semicircular band.

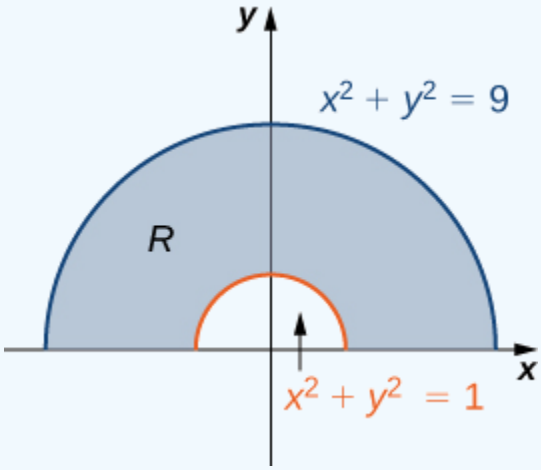


Figure $14.3b.3$: The polar region R lies between two semicircles.

Now that we have sketched a polar rectangular region, let us demonstrate how to evaluate a double integral over this region by using polar coordinates.

Example $14.3b.1B$: Evaluating a Double Integral over a Polar Rectangular Region

Evaluate the integral $\iint_R 3x\ dA$ over the region $R = \{(r,\theta)\ |\ 1 \le r \le 2,\ 0 \le \theta \le \pi\}$.

Solution

First we sketch a figure similar to Figure $14.3b.3$, but with outer radius $r = 2$. From the figure we can see that we have

$$\begin{aligned}\iint_R 3x\,dA &= \int_{\theta=0}^{\theta=\pi}\int_{r=1}^{r=2} 3r\cos\theta\, r\,dr\,d\theta \quad \text{Use an integral with correct limits of integration.}\\ &= \int_{\theta=0}^{\theta=\pi}\cos\theta\left[r^3\Big|_{r=1}^{r=2}\right]d\theta \quad \text{Integrate first with respect to } r.\\ &= \int_{\theta=0}^{\theta=\pi} 7\cos\theta\,d\theta\\ &= 7\sin\theta\Big|_{\theta=0}^{\theta=\pi} = 0.\end{aligned}$$

Exercise 14.3b.1

Sketch the region $D=\{(r,\theta)|1\le r\le 2,\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\}$, and evaluate $\iint_R x\ dA$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 14.3b.2A: Evaluating a Double Integral by Converting from Rectangular Coordinates

Evaluate the integral

$$\iint_R (1-x^2-y^2)\,dA$$

where R is the unit circle on the xy-plane.

Solution

The region R is a unit circle, so we can describe it as $R=\{(r,\theta)\mid 0\le r\le 1,\ 0\le\theta\le 2\pi\}$.

Using the conversion $x=r\ \cos\ \theta,\ y=r\ \sin\ \theta$, and $dA=r\ dr\ d\theta$, we have

$$\begin{aligned}\iint_R (1-x^2-y^2)\,dA &= \int_0^{2\pi}\int_0^1 (1-r^2)\,r\,dr\,d\theta\\ &= \int_0^{2\pi}\int_0^1 (r-r^3)\,dr\,d\theta\\ &= \int_0^{2\pi}\left[\frac{r^2}{2}-\frac{r^4}{4}\right]_0^1 d\theta\\ &= \int_0^{2\pi}\frac{1}{4}\,d\theta = \frac{\pi}{2}.\end{aligned}$$

Example 14.3b.2B: Evaluating a Double Integral by Converting from Rectangular Coordinates

Evaluate the integral

$$\iint_R (x+y)\,dA$$

where $R=\{(x,y)\mid 1\le x^2+y^2\le 4,\ x\le 0\}$.

Solution

We can see that R is an annular region that can be converted to polar coordinates and described as $R=\{(r,\theta)\,|\,1\le r\le 2,\ \frac{\pi}{2}\le\theta\le\frac{3\pi}{2}\}$ (see the following graph).

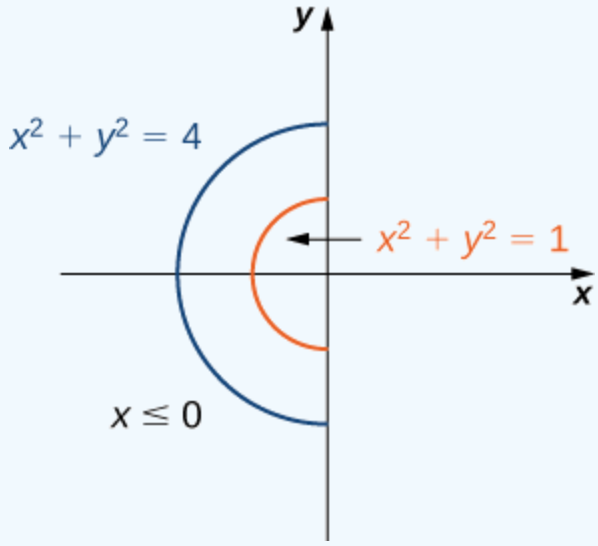


Figure 14.3b.4: The annular region of integration R.

Hence, using the conversion $x = r\ \cos\ \theta,\ y = r\ \sin\ \theta$, and $dA = r\ dr\ d\theta$, we have

$$\begin{aligned}\iint_R (x+y)\,dA &= \int_{\theta=\pi/2}^{\theta=3\pi/2}\int_{r=1}^{r=2}(r\ \cos\ \theta + r\ \sin\ \theta) r\,dr\,d\theta \\ &= \left(\int_{r=1}^{r=2} r^2\,dr\right)\left(\int_{\pi/2}^{3\pi/2}(\cos\ \theta + \sin\ \theta)\,d\theta\right) \\ &= \left[\frac{r^3}{3}\right]_1^2 [\sin\ \theta - \cos\ \theta]\Big|_{\pi/2}^{3\pi/2} \\ &= -\frac{14}{3}.\end{aligned}$$

Exercise 14.3b.2

Evaluate the integral

$$\iint_R (4 - x^2 - y^2)\,dA$$

where R is the circle of radius 2 on the xy-plane.

Hint available at the end of this section.

Answer available at the end of this section.

General Polar Regions of Integration

To evaluate the double integral of a continuous function by iterated integrals over general polar regions, we consider two types of regions, analogous to Type I and Type II as discussed for rectangular coordinates in section on Double Integrals over General Regions. It is more common to write polar equations as $r = f(\theta)$ than $\theta = f(r)$, so we describe a general polar region as $R = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$ (Figure 14.3b.5).

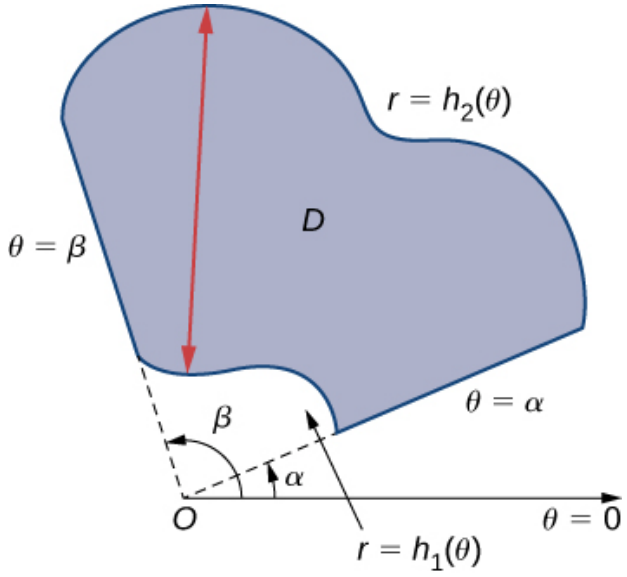


Figure $14.3b.5$: A general polar region between $\alpha \le \theta \le \beta$ and $h_1(\theta) \le r \le h_2(\theta)$.

Theorem: Double Integrals over General Polar Regions

If $f(r, \theta)$ is continuous on a general polar region D as described above, then

$$\iint_D f(r,\theta)\, r\, dr\, d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} f(r,\theta)\, r\, dr\, d\theta.$$

Example $14.3b.3$: Evaluating a Double Integral over a General Polar Region

Evaluate the integral

$$\iint_D r^2 \sin\theta\, r\, dr\, d\theta$$

where D is the region bounded by the polar axis and the upper half of the cardioid $r = 1 + \cos\ \theta$.

Solution

We can describe the region D as $\{(r, \theta) \mid 0 \le \theta \le \pi,\ 0 \le r \le 1 + \cos\ \theta\}$ as shown in Figure $14.3b.6$.

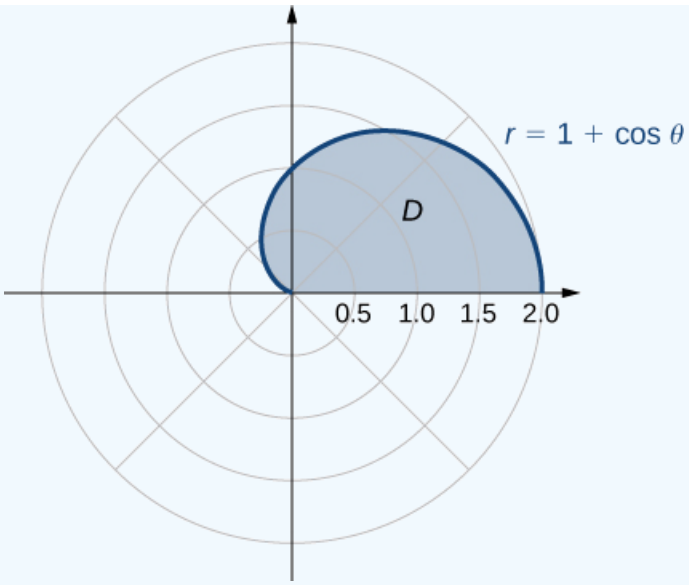


Figure $14.3b.6$: The region D is the top half of a cardioid.

Hence, we have

$$\begin{aligned}\iint_D r^2 \sin\,\theta\, r\, dr\, d\theta &= \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=1+\cos\theta} (r^2\sin\,\theta)\, r\, dr\, d\theta \\ &= \frac{1}{4}\int_{\theta=0}^{\theta=\pi} [r^4]\Big|_{r=0}^{r=1+\cos\,\theta} \sin\,\theta\, d\theta \\ &= \frac{1}{4}\int_{\theta=0}^{\theta=\pi} (1+\cos\,\theta)^4 \sin\,\theta\, d\theta \\ &= -\frac{1}{4}\left[\frac{(1+\cos\,\theta)^5}{5}\right]_0^{\pi} = \frac{8}{5}.\end{aligned}$$

Exercise $14.3b.3$

Evaluate the integral

$$\iint_D r^2 \sin^2 2\theta\, r\, dr\, d\theta$$

where $D = \left\{(r,\theta)\,|\,0 \le \theta \le \pi,\ 0 \le r \le 2\sqrt{\cos\,2\theta}\right\}$.

Hint available at the end of this section.

Answer available at the end of this section.

Polar Areas and Volumes

As in rectangular coordinates, if a solid S is bounded by the surface $z = f(r,\theta)$, as well as by the surfaces $r = a,\ \ r = b,\ \ \theta = \alpha$, and $\theta = \beta$, we can find the volume V of S by double integration, as

$$V = \iint_R f(r,\theta)\, r\, dr\, d\theta = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=a}^{r=b} f(r,\theta)\, r\, dr\, d\theta.$$

If the base of the solid can be described as $D = \{(r,\theta)|\alpha \le \theta \le \beta,\ \ h_1(\theta) \le r \le h_2(\theta)\}$, then the double integral for the volume becomes

$$V = \iint_D f(r,\theta)\, r\, dr\, d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} f(r,\theta)\, r\, dr\, d\theta.$$

We illustrate this idea with some examples.

Example $14.3b.4A$: Finding a Volume Using a Double Integral

Find the volume of the solid that lies under the paraboloid $z = 1 - x^2 - y^2$ and above the unit circle on the xy-plane (Figure $14.3b.7$).

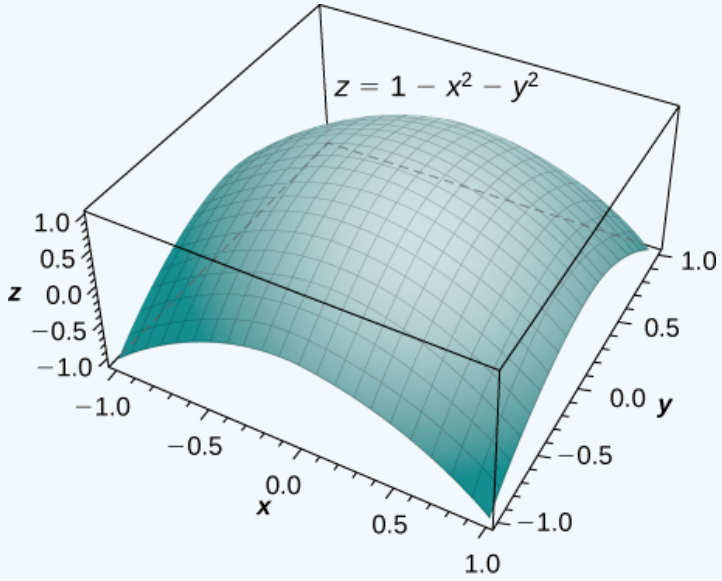


Figure $14.3b.7$: Finding the volume of a solid under a paraboloid and above the unit circle.

Solution

By the method of double integration, we can see that the volume is the iterated integral of the form

$$\iint_R (1 - x^2 - y^2)\, dA$$

where $R = \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$.

This integration was shown before in Example $14.3b.2A$, so the volume is $\frac{\pi}{2}$ cubic units.

Example $14.3b.4B$: Finding a Volume Using Double Integration

Find the volume of the solid that lies under the paraboloid $z = 4 - x^2 - y^2$ and above the disk $(x-1)^2 + y^2 = 1$ on the xy -plane. See the paraboloid in Figure $14.3b.8$ intersecting the cylinder $(x-1)^2 + y^2 = 1$ above the xy-plane.

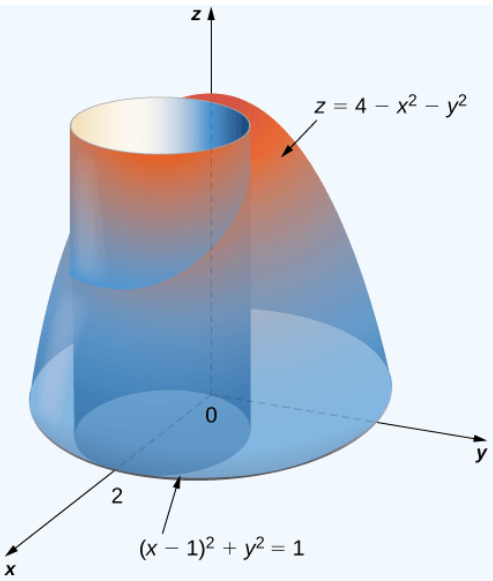


Figure $14.3b.8$: Finding the volume of a solid with a paraboloid cap and a circular base.

Solution

First change the disk $(x-1)^2+y^2=1$ to polar coordinates. Expanding the square term, we have $x^2-2x+1+y^2=1$. Then simplify to get $x^2+y^2=2x$, which in polar coordinates becomes $r^2=2r\ \cos\ \theta$ and then either $r=0$ or $r=2\ \cos\ \theta$. Similarly, the equation of the paraboloid changes to $z=4-r^2$. Therefore we can describe the disk $(x-1)^2+y^2=1$ on the xy -plane as the region

$$D=\{(r,\theta)\,|\,0\le\theta\le\pi,\ 0\le r\le 2\ \cos\theta\}.$$

Hence the volume of the solid bounded above by the paraboloid $z=4-x^2-y^2$ and below by $r=2\ \cos\theta$ is

$$\begin{aligned}V&=\iint_D f(r,\theta)\,r\,dr\,d\theta\\&=\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=2\ \cos\ \theta}(4-r^2)\,r\,dr\,d\theta\\&=\int_{\theta=0}^{\theta=\pi}\left[4\frac{r^2}{2}-\frac{r^4}{4}\Big|_0^{2\ \cos\ \theta}\right]d\theta\\&=\int_0^{\pi}[8\ \cos^2\theta-4\ \cos^4\theta]\,d\theta\\&=\left[\frac{5}{2}\theta+\frac{5}{2}\sin\ \theta\ \cos\ \theta-\sin\ \theta\cos^3\theta\right]_0^{\pi}=\frac{5}{2}\pi\ \text{units}^3.\end{aligned}$$

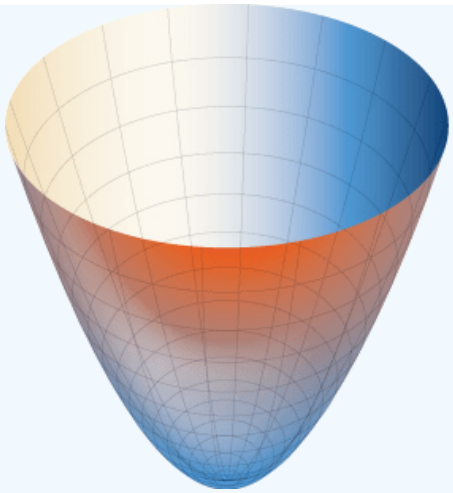
Notice in the next example that integration is not always easy with polar coordinates. Complexity of integration depends on the function and also on the region over which we need to perform the integration. If the region has a more natural expression in polar coordinates or if f has a simpler antiderivative in polar coordinates, then the change in polar coordinates is appropriate; otherwise, use rectangular coordinates.

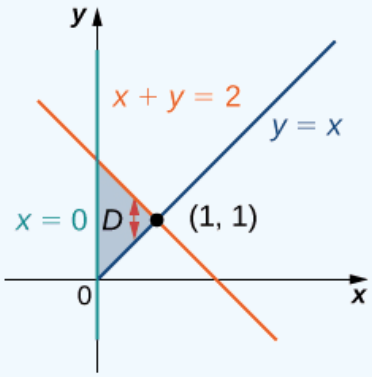
Example $14.3b.5A$: Finding a Volume Using a Double Integral

Find the volume of the region that lies under the paraboloid $z=x^2+y^2$ and above the triangle enclosed by the lines $y=x,\ x=0$, and $x+y=2$ in the xy-plane.

Solution

First examine the region over which we need to set up the double integral and the accompanying paraboloid.





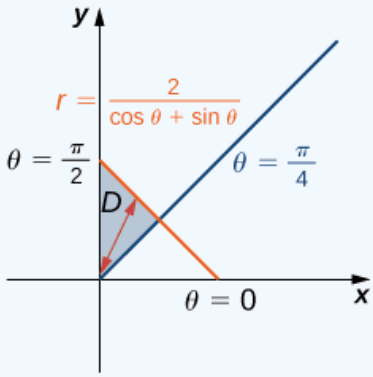


Figure 14.3b.9: Finding the volume of a solid under a paraboloid and above a given triangle.

The region D is $\{(x,y) \mid 0 \le x \le 1,\ x \le y \le 2-x\}$. Converting the lines $y = x,\ x = 0$, and $x + y = 2$ in the xy-plane to functions of r and θ we have $\theta = \pi/4,\ \theta = \pi/2$, and $r = 2/(\cos\ \theta + \sin\ \theta)$, respectively. Graphing the region on the xy- plane, we see that it looks like $D = \{(r,\theta) \mid \pi/4 \le \theta \le \pi/2,\ 0 \le r \le 2/(\cos\ \theta + \sin\ \theta)\}$.

Now converting the equation of the surface gives $z = x^2 + y^2 = r^2$. Therefore, the volume of the solid is given by the double integral

$$\begin{aligned} V &= \iint_D f(r,\theta)\, r\, dr\, d\theta \\ &= \int_{\theta=\pi/4}^{\theta=\pi/2} \int_{r=0}^{r=2/(\cos\ \theta+\sin\ \theta)} r^2 r\, dr d\theta \\ &= \int_{\pi/4}^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2/(\cos\ \theta+\sin\ \theta)} d\theta \\ &= \frac{1}{4}\int_{\pi/4}^{\pi/2} \left(\frac{2}{\cos\ \theta + \sin\ \theta}\right)^4 d\theta \\ &= \frac{16}{4}\int_{\pi/4}^{\pi/2} \left(\frac{1}{\cos\ \theta + \sin\ \theta}\right)^4 d\theta \\ &= 4\int_{\pi/4}^{\pi/2} \left(\frac{1}{\cos\ \theta + \sin\ \theta}\right)^4 d\theta. \end{aligned}$$

As you can see, this integral is very complicated. So, we can instead evaluate this double integral in rectangular coordinates as

$$V = \int_0^1 \int_x^{2-x} (x^2 + y^2)\, dy\, dx.$$

Evaluating gives

$$\begin{aligned} V &= \int_0^1 \int_x^{2-x} (x^2 + y^2)\, dy\, dx \\ &= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]\Big|_x^{2-x} dx \\ &= \int_0^1 \frac{8}{3} - 4x + 4x^2 - \frac{8x^3}{3}\, dx \\ &= \left[\frac{8x}{3} - 2x^2 + \frac{4x^3}{3} - \frac{2x^4}{3}\right]\Big|_0^1 \\ &= \frac{4}{3}\ \text{units}^3. \end{aligned}$$

To answer the question of how the formulas for the volumes of different standard solids such as a sphere, a cone, or a cylinder are found, we want to demonstrate an example and find the volume of an arbitrary cone.

Example $14.3b.5B$: Finding a Volume Using a Double Integral

Use polar coordinates to find the volume inside the cone $z = 2 - \sqrt{x^2 + y^2}$ and above the xy-plane.

Solution

The region D for the integration is the base of the cone, which appears to be a circle on the xy-plane (Figure $14.3b.10$).

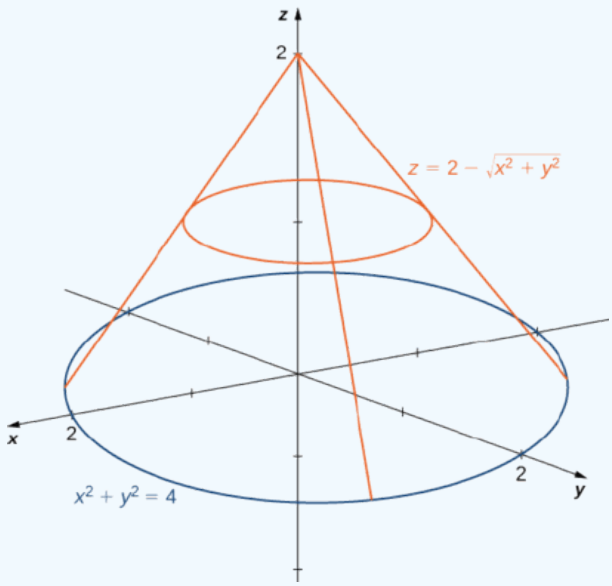


Figure $14.3b.10$: Finding the volume of a solid inside the cone and above the xy-plane.

We find the equation of the circle by setting $z = 0$:

$$\begin{aligned} 0 &= 2 - \sqrt{x^2 + y^2} \\ 2 &= \sqrt{x^2 + y^2} \\ x^2 + y^2 &= 4. \end{aligned}$$

This means the radius of the circle is 2 so for the integration we have $0 \le \theta \le 2\pi$ and $0 \le r \le 2$. Substituting $x = r\ \cos\theta$ and $y = r\ \sin\ \theta$ in the equation $z = 2 - \sqrt{x^2 + y^2}$ we have $z = 2 - r$. Therefore, the volume of the cone is

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} (2 - r)\, r\, dr\, d\theta = 2\pi \frac{4}{3} = \frac{8\pi}{3} \text{ cubic units.}$$

Analysis

Note that if we were to find the volume of an arbitrary cone with radius α units and height h units, then the equation of the cone would be $z = h - \frac{h}{a}\sqrt{x^2 + y^2}$.

We can still use Figure $14.3b.10$ and set up the integral as

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a} \left(h - \frac{h}{a} r\right) r\, dr\, d\theta.$$

Evaluating the integral, we get $\frac{1}{3}\pi a^2 h$.

Exercise $14.3b.5$

Use polar coordinates to find an iterated integral for finding the volume of the solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 16 - x^2 - y^2$.

Hint available at the end of this section.

Answer available at the end of this section.

As with rectangular coordinates, we can also use polar coordinates to find areas of certain regions using a double integral. As before, we need to understand the region whose area we want to compute. Sketching a graph and identifying the region can be helpful to realize the limits of integration. Generally, the area formula in double integration will look like

$$\text{Area of } A = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} 1 \, r \, dr \, d\theta.$$

Example $14.3b.6A$: Finding an Area Using a Double Integral in Polar Coordinates

Evaluate the area bounded by the curve $r = \cos \ 4\theta$.

Solution

Sketching the graph of the function $r = \cos \ 4\theta$ reveals that it is a polar rose with eight petals (see the following figure).

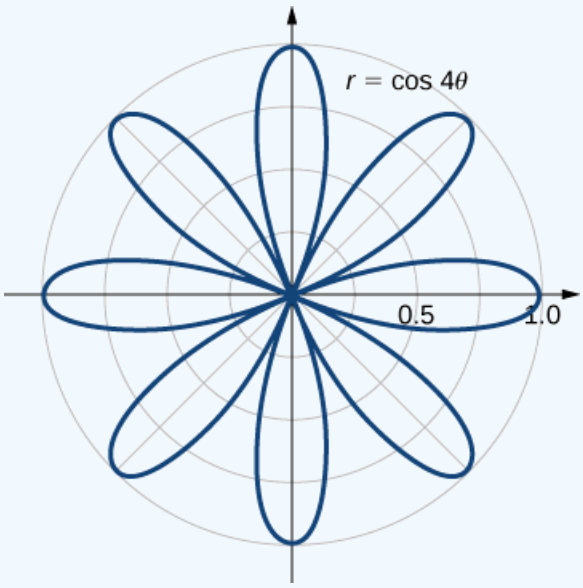


Figure $14.3b.11$: Finding the area of a polar rose with eight petals.

Using symmetry, we can see that we need to find the area of one petal and then multiply it by 8. Notice that the values of θ for which the graph passes through the origin are the zeros of the function $\cos \ 4\theta$, and these are odd multiples of $\pi/8$. Thus, one of the petals corresponds to the values of θ in the interval $[-\pi/8, \pi/8]$. Therefore, the area bounded by the curve $r = \cos \ 4\theta$ is

$$
\begin{aligned}
A &= 8\int_{\theta=-\pi/8}^{\theta=\pi/8}\int_{r=0}^{r=\cos\,4\theta} 1\,r\,dr\,d\theta \\
&= 8\int_{\theta=-\pi/8}^{\theta=\pi/8}\left[\frac{1}{2}r^2\Big|_0^{\cos\,4\theta}\right]d\theta \\
&= 8\int_{-\pi/8}^{\pi/8}\frac{1}{2}\cos^2 4\theta\,d\theta \\
&= 8\left[\frac{1}{4}\theta+\frac{1}{16}\sin\,4\theta\,\cos\,4\theta\Big|_{-\pi/8}^{\pi/8}\right] \\
&= 8\left[\frac{\pi}{16}\right]=\frac{\pi}{2}\text{ units}^2.
\end{aligned}
$$

Example $14.3b.6B$: Finding Area Between Two Polar Curves

Find the area enclosed by the circle $r = 3\ \cos\ \theta$ and the cardioid $r = 1 + \cos\ \theta$.

Solution

First and foremost, sketch the graphs of the region (Figure $14.3b.12$).

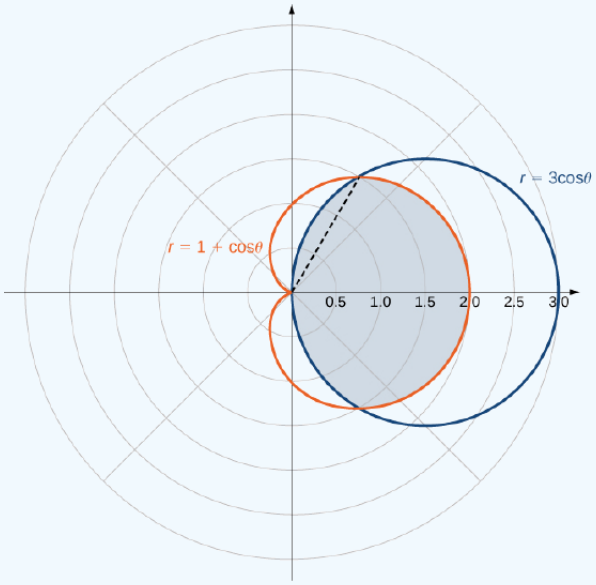


Figure $14.3b.12$: Finding the area enclosed by both a circle and a cardioid.

We can from see the symmetry of the graph that we need to find the points of intersection. Setting the two equations equal to each other gives

$$3\,\cos\,\theta = 1 + \cos\,\theta.$$

One of the points of intersection is $\theta = \pi/3$. The area above the polar axis consists of two parts, with one part defined by the cardioid from $\theta = 0$ to $\theta = \pi/3$ and the other part defined by the circle from $\theta = \pi/3$ to $\theta = \pi/2$. By symmetry, the total area is twice the area above the polar axis. Thus, we have

$$A = 2\left[\int_{\theta=0}^{\theta=\pi/3}\int_{r=0}^{r=1+\cos\,\theta} 1\,r\,dr\,d\theta + \int_{\theta=\pi/3}^{\theta=\pi/2}\int_{r=0}^{r=3\,\cos\,\theta} 1\,r\,dr\,d\theta\right].$$

Evaluating each piece separately, we find that the area is

$$A = 2\left(\frac{1}{4}\pi+\frac{9}{16}\sqrt{3}+\frac{3}{8}\pi-\frac{9}{16}\sqrt{3}\right)=2\left(\frac{5}{8}\pi\right)=\frac{5}{4}\pi\,\text{square units}.$$

Exercise 14.3b.6

Find the area enclosed inside the cardioid $r = 3 - 3\ \sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 14.3b.7: Evaluating an Improper Double Integral in Polar Coordinates

Evaluate the integral

$$\iint_{R^2} e^{-10(x^2+y^2)}\,dx\,dy.$$

Solution

This is an improper integral because we are integrating over an unbounded region R^2. In polar coordinates, the entire plane R^2 can be seen as $0 \le \theta \le 2\pi,\ 0 \le r \le \infty$.

Using the changes of variables from rectangular coordinates to polar coordinates, we have

$$\begin{aligned}
\iint_{R^2} e^{-10(x^2+y^2)}\,dx\,dy &= \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=\infty} e^{-10r^2}\,r\,dr\,d\theta = \int_{\theta=0}^{\theta=2\pi}\left(\lim_{a\to\infty}\int_{r=0}^{r=a} e^{-10r^2}r\,dr\right)d\theta \\
&= \left(\int_{\theta=0}^{\theta=2\pi}\right)d\theta\left(\lim_{a\to\infty}\int_{r=0}^{r=a} e^{-10r^2}r\,dr\right) \\
&= 2\pi\left(\lim_{a\to\infty}\int_{r=0}^{r=a} e^{-10r^2}r\,dr\right) \\
&= 2\pi\lim_{a\to\infty}\left(-\frac{1}{20}\right)\left(e^{-10r^2}\Big|_0^a\right) \\
&= 2\pi\left(-\frac{1}{20}\right)\lim_{a\to\infty}\left(e^{-10a^2}-1\right) \\
&= \frac{\pi}{10}.
\end{aligned}$$

Exercise 14.3b.7

Evaluate the integral

$$\iint_{R^2} e^{-4(x^2+y^2)}dx\,dy.$$

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- To apply a double integral to a situation with circular symmetry, it is often convenient to use a double integral in polar coordinates. We can apply these double integrals over a polar rectangular region or a general polar region, using an iterated integral similar to those used with rectangular double integrals.
- The area dA in polar coordinates becomes $r\ dr\ d\theta$.
- Use $x = r\ \cos\ \theta,\ y = r\ \sin\ \theta$, and $dA = r\ dr\ d\theta$ to convert an integral in rectangular coordinates to an integral in polar coordinates.

- Use $r^2 = x^2 + y^2$ and $\theta = tan^{-1}\left(\frac{y}{x}\right)$ to convert an integral in polar coordinates to an integral in rectangular coordinates, if needed.
- To find the volume in polar coordinates bounded above by a surface $z = f(r, \theta)$ over a region on the xy-plane, use a double integral in polar coordinates.

Key Equations

- **Double integral over a polar rectangular region** R

$$\iint_R f(r,\theta)dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*)\Delta A = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*)r_{ij}^*\Delta r\Delta\theta$$

- **Double integral over a general polar region**

$$\iint_D f(r,\theta)\, r\, dr\, d\theta = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=h_1(\theta)}^{r_2(\theta)} f(r,\theta)\, r\, dr\, d\theta$$

Glossary

polar rectangle
: the region enclosed between the circles $r = a$ and $r = b$ and the angles $\theta = \alpha$ and $\theta = \beta$; it is described as $R = \{(r, \theta) \mid a \le r \le b,\ \alpha \le \theta \le \beta\}$

Answers

Exercise $14.3b.1$

Follow the steps in Example $14.3b.1A$.

Exercise $14.3b.1$

$\frac{14}{3}$

Exercise $14.3b.2$

Follow the steps in the previous example.

Exercise $14.3b.2$

8π

Exercise $14.3b.3$

Graph the region and follow the steps in the previous example.

Exercise $14.3b.3$

$\frac{\pi}{8}$

Exercise $14.3b.5$

Sketching the graphs can help.

Exercise $14.3b.5$

$$V = \int_0^{2\pi}\int_0^{2\sqrt{2}} (16 - 2r^2)\, r\, dr\, d\theta = 64\pi \text{ cubic units.}$$

Exercise $14.3b.6$

Sketch the graph, and solve for the points of intersection.

Exercise $14.3b.6$

$$A = 2\int_{-\pi/2}^{\pi/6}\int_{1+\sin\theta}^{3-3\sin\theta} r\, dr\, d\theta = \left(8\pi + 9\sqrt{3}\right) \text{ units}^2$$

Exercise $14.3b.7$

Convert to the polar coordinate system.

Exercise $14.3b.7$

$\frac{\pi}{4}$

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14.3E: Double Integrals in Polar Coordinates (Exercises)

Terms and Concepts

1. When evaluating $\int\int_R f(x,y)\ dA$ using polar coordinates, $f(x,y)$ is replaced with _____ and dA is replaced with _____.

Answer

$f(x,y)$ is replaced with $f(r\cos\theta, r\sin\theta)$ and dA is replaced with $r\ dr\ d\theta$.

2. Why would one be interested in evaluating a double integral with polar coordinates?

Defining Polar Regions

In exercises 3 - 6, express the region R in polar coordinates.

3) R is the region of the disk of radius 2 centered at the origin that lies in the first quadrant.

Answer

$R = \{(r,\theta) \mid 0 \le r \le 2,\ 0 \le \theta \le \frac{\pi}{2}\}$

4) R is the region of the disk of radius 3 centered at the origin.

5) R is the region between the circles of radius 4 and radius 5 centered at the origin that lies in the second quadrant.

Answer

$R = \{(r,\theta) \mid 4 \le r \le 5,\ \frac{\pi}{2} \le \theta \le \pi\}$

6) R is the region bounded by the y-axis and $x = \sqrt{1-y^2}$.

7) R is the region bounded by the x-axis and $y = \sqrt{2-x^2}$.

Answer

$R = \{(r,\theta) \mid 0 \le r \le \sqrt{2},\ 0 \le \theta \le \pi\}$

8) $R = \{(x,y) \mid x^2 + y^2 \le 4x\}$

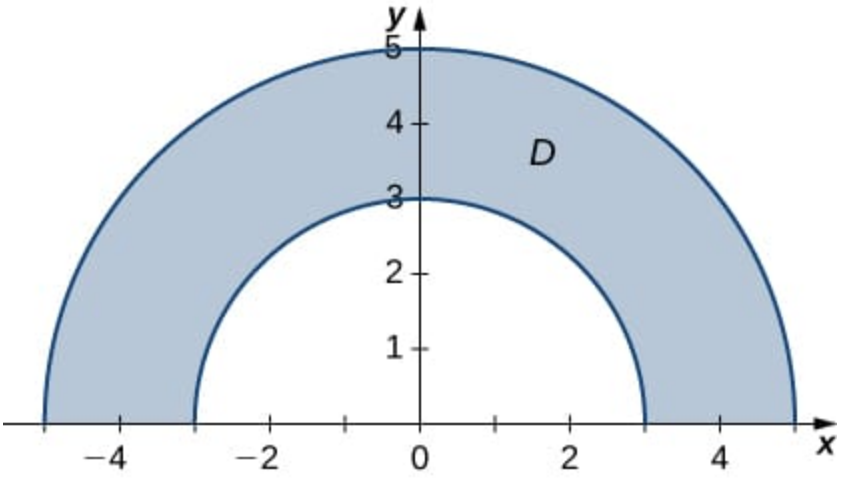
9) $R = \{(x,y) \mid x^2 + y^2 \le 4y\}$

Answer

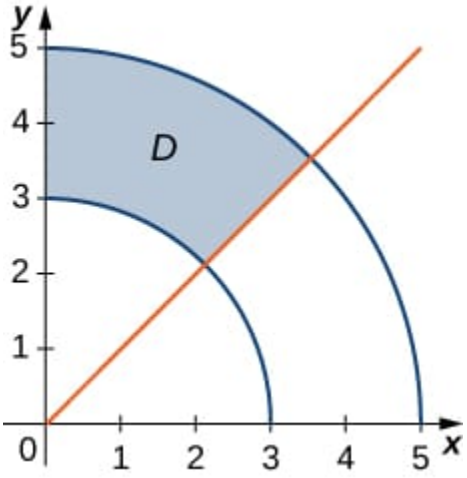
$R = \{(r,\theta) \mid 0 \le r \le 4\ \sin\theta,\ 0 \le \theta \le \pi\}$

In exercises 10 - 15, the graph of the polar rectangular region D is given. Express D in polar coordinates. In exercises 10 - 13 the regions are bounded by circles centered at the origin, the coordinate axes, and/or lines of the form $y = \pm x$ (each of which should be able to be identified by inspection).

10)



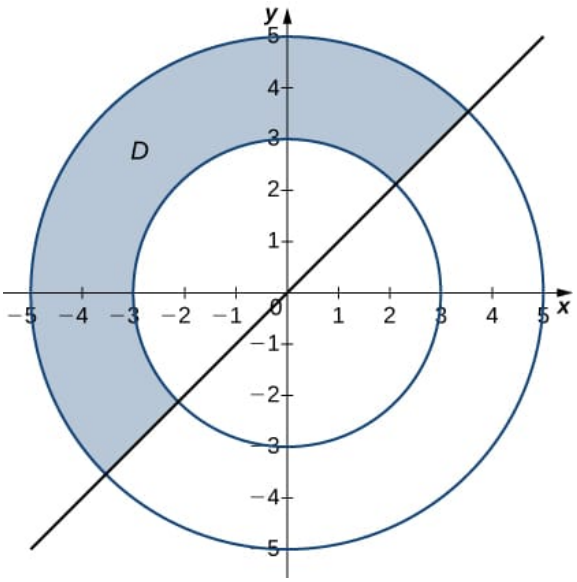
11)



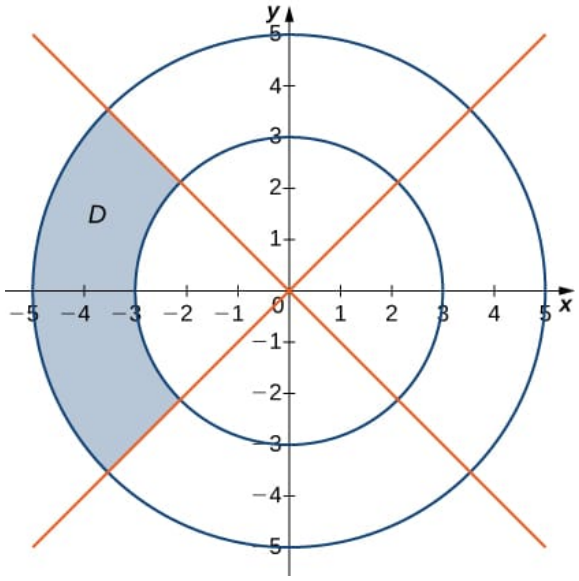
Answer

$$D = \{(r, \theta) \mid 3 \le r \le 5,\ \tfrac{\pi}{4} \le \theta \le \tfrac{\pi}{2}\}$$

12)



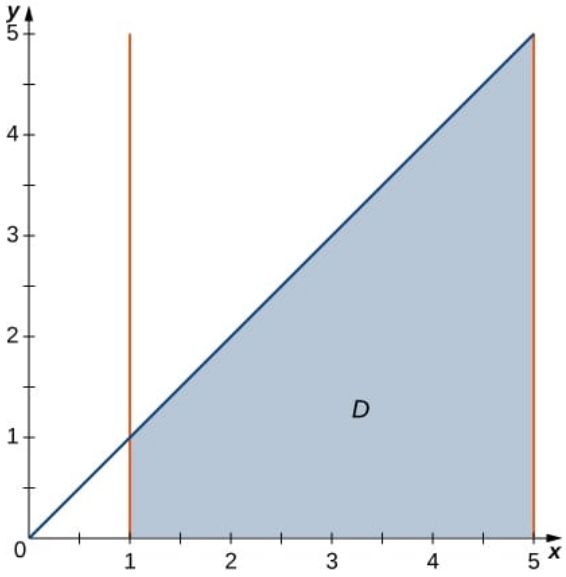
13)



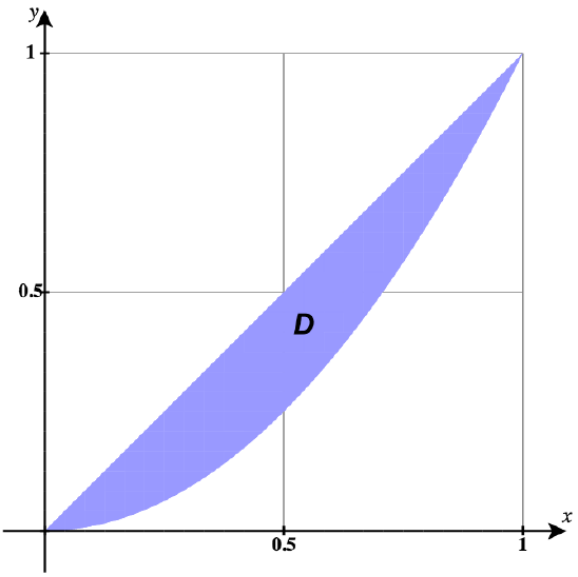
Answer

$$D = \left\{(r, \theta) \mid 3 \leq r \leq 5,\ \tfrac{3\pi}{4} \leq \theta \leq \tfrac{5\pi}{4}\right\}$$

14) In the following graph, the region D is situated below $y = x$ and is bounded by $x = 1,\ x = 5$, and $y = 0$.



15) In the following graph, the region D is bounded by $y = x$ and $y = x^2$.



Answer

$D = \{(r,\theta) \mid 0 \le r \le \tan\theta \ \sec\theta,\ 0 \le \theta \le \frac{\pi}{4}\}$

Evaluating Polar Double Integrals

In exercises 16 - 25, evaluate the double integral $\iint_R f(x,y)\ dA$ over the polar rectangular region R.

16) $f(x,y) = x^2 + y^2$, $R = \{(r,\theta) \mid 3 \le r \le 5,\ 0 \le \theta \le 2\pi\}$

17) $f(x,y) = x + y$, $R = \{(r,\theta) \mid 3 \le r \le 5,\ 0 \le \theta \le 2\pi\}$

Answer

0

18) $f(x,y) = x^2 + xy,\ R = \{(r,\theta) \mid 1 \le r \le 2,\ \pi \le \theta \le 2\pi\}$

19) $f(x,y) = x^4 + y^4,\ R = \{(r,\theta) \mid 1 \le r \le 2,\ \frac{3\pi}{2} \le \theta \le 2\pi\}$

Answer

$\frac{63\pi}{16}$

20) $f(x,y) = \sqrt[3]{x^2+y^2}$, where $R = \{(r,\theta) \mid 0 \le r \le 1,\ \frac{\pi}{2} \le \theta \le \pi\}$.

21) $f(x,y) = x^4 + 2x^2y^2 + y^4$, where $R = \{(r,\theta) \mid 3 \le r \le 4,\ \frac{\pi}{3} \le \theta \le \frac{2\pi}{3}\}$.

Answer

$\frac{3367\pi}{18}$

22) $f(x,y) = \sin(\arctan\frac{y}{x})$, where $R = \{(r,\theta) \mid 1 \le r \le 2,\ \frac{\pi}{6} \le \theta \le \frac{\pi}{3}\}$

23) $f(x,y) = \arctan(\frac{y}{x})$, where $R = \{(r,\theta) \mid 2 \le r \le 3,\ \frac{\pi}{4} \le \theta \le \frac{\pi}{3}\}$

Answer

$\frac{35\pi^2}{576}$

24) $\iint_R e^{x^2+y^2}\left[1 + 2\ \arctan\left(\frac{y}{x}\right)\right]\ dA,\ R = \{(r,\theta) \mid 1 \le r \le 2,\ \frac{\pi}{6} \le \theta\frac{\pi}{3}\}$

25) $\iint_R \left(e^{x^2+y^2}+x^4+2x^2y^2+y^4\right)\arctan\left(\frac{y}{x}\right)\ dA,\ R=\left\{(r,\theta)\mid 1\le r\le 2,\ \frac{\pi}{4}\le\theta\le\frac{\pi}{3}\right\}$

Answer

$\frac{7}{576}\pi^2(21-e+e^4)$

Converting Double Integrals to Polar Form

In exercises 26 - 29, the integrals have been converted to polar coordinates. Verify that the identities are true and choose the easiest way to evaluate the integrals, in rectangular or polar coordinates.

26) $\int_1^2\int_0^x (x^2+y^2)\ dy\ dx=\int_0^{\frac{\pi}{4}}\int_{\sec\theta}^{2\ \sec\theta} r^3\ dr\ d\theta$

27) $\int_2^3\int_0^x \frac{x}{\sqrt{x^2+y^2}}\ dy\ dx=\int_0^{\pi/4}\int_{2\sec\theta}^{3\sec\theta} r\ \cos\theta\ dr\ d\theta$

Answer

$\frac{5}{2}\ln(1+\sqrt{2})$

28) $\int_0^1\int_{x^2}^x \frac{1}{\sqrt{x^2+y^2}}\ dy\ dx=\int_0^{\pi/4}\int_0^{\tan\theta\ \sec\theta} dr\ d\theta$

29) $\int_0^1\int_{x^2}^x \frac{y}{\sqrt{x^2+y^2}}\ dy\ dx=\int_0^{\pi/4}\int_0^{\tan\theta\ \sec\theta} r\ \sin\theta\ dr\ d\theta$

Answer

$\frac{1}{6}(2-\sqrt{2})$

In exercises 30 - 37, draw the region of integration, R, labeling all limits of integration, convert the integrals to polar coordinates and evaluate them.

30) $\int_0^3\int_0^{\sqrt{9-y^2}} \left(x^2+y^2\right)\ dx\ dy$

31) $\int_0^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \left(x^2+y^2\right)^2\ dx\ dy$

Answer

$\int_0^{\pi}\int_0^2 r^5\ dr\ d\theta \quad = \quad \frac{32\pi}{3}$

32) $\int_0^1\int_0^{\sqrt{1-x^2}} (x+y)\ dy\ dx$

33) $\int_0^4\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \sin(x^2+y^2)\ dy\ dx$

Answer

$\int_{-\pi/2}^{\pi/2}\int_0^4 r\ \sin(r^2)\ dr\ d\theta \quad = \quad \pi\ \sin^2 8$

34) $\int_0^5\int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} \sqrt{x^2+y^2}\ dy\ dx$

35) $\int_{-4}^4\int_{-\sqrt{16-y^2}}^0 (2y-x)\ dx\ dy$

Answer

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^4 \left(2r\sin\theta - r\cos\theta\right)\ r\ dr\ d\theta \quad = \quad \frac{128}{3}$$

36) $\int_0^2 \int_y^{\sqrt{8-y^2}} (x+y)\ dx\ dy$

37) $\int_{-2}^{-1} \int_0^{\sqrt{4-x^2}} (x+5)\ dy\ dx + \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} (x+5)\ dy\ dx + \int_1^2 \int_0^{\sqrt{4-x^2}} (x+5)\ dy\ dx$

Answer

$$\int_0^{\pi} \int_1^2 \left(r\cos\theta + 5\right)\ r\ dr\ d\theta \quad = \quad \frac{15\pi}{2}$$

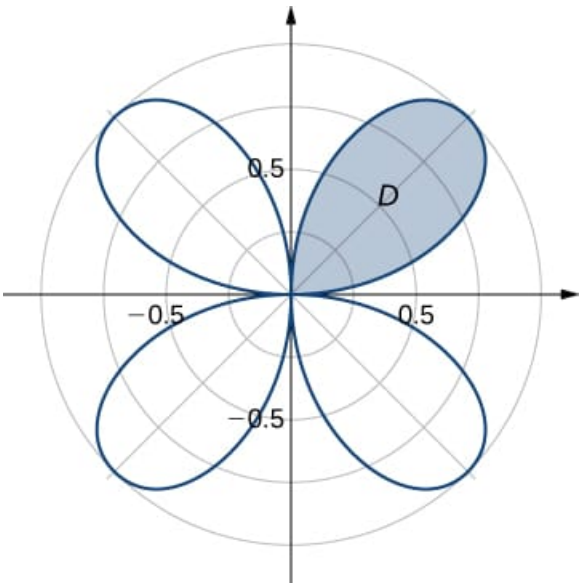
38) Evaluate the integral $\iint_D r\ dA$ where D is the region bounded by the polar axis and the upper half of the cardioid $r = 1 + \cos\theta$.

39) Find the area of the region D bounded by the polar axis and the upper half of the cardioid $r = 1 + \cos\theta$.

Answer

$\frac{3\pi}{4}$

40) Evaluate the integral $\iint_D r\ dA$, where D is the region bounded by the part of the four-leaved rose $r = \sin 2\theta$ situated in the first quadrant (see the following figure).



41) Find the total area of the region enclosed by the four-leaved rose $r = \sin 2\theta$ (see the figure in the previous exercise).

Answer

$\frac{\pi}{2}$

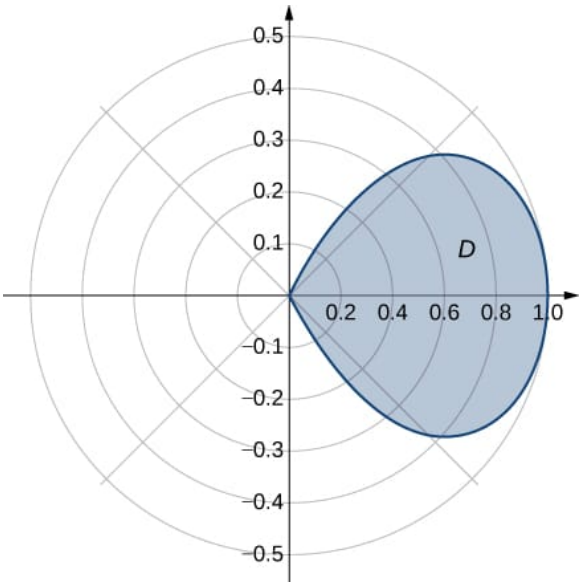
42) Find the area of the region D which is the region bounded by $y = \sqrt{4-x^2}$, $x = \sqrt{3}$, $x = 2$, and $y = 0$.

43) Find the area of the region D, which is the region inside the disk $x^2 + y^2 \leq 4$ and to the right of the line $x = 1$.

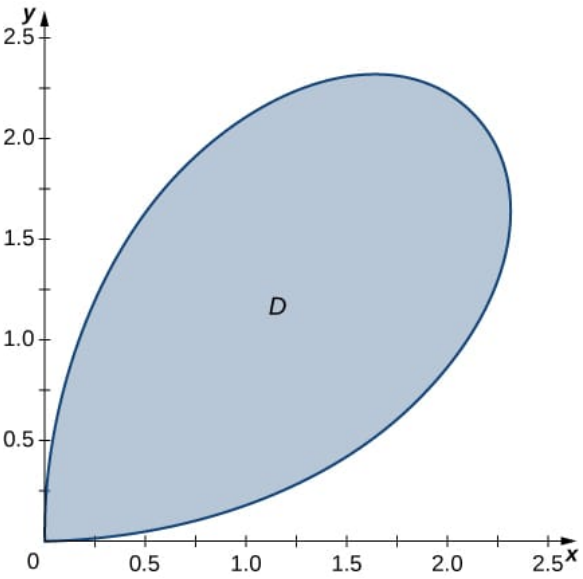
Answer

$\frac{1}{3}(4\pi - 3\sqrt{3})$

44) Determine the average value of the function $f(x,y) = x^2 + y^2$ over the region D bounded by the polar curve $r = \cos 2\theta$, where $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$ (see the following graph).



45) Determine the average value of the function $f(x,y) = \sqrt{x^2 + y^2}$ over the region D bounded by the polar curve $r = 3\sin 2\theta$, where $0 \leq \theta \leq \frac{\pi}{2}$ (see the following graph).



Answer

$\frac{16}{3\pi}$

46) Find the volume of the solid situated in the first octant and bounded by the paraboloid $z = 1 - 4x^2 - 4y^2$ and the planes $x = 0,\ y = 0$, and $z = 0$.

47) Find the volume of the solid bounded by the paraboloid $z = 2 - 9x^2 - 9y^2$ and the plane $z = 1$.

Answer

$\frac{\pi}{18}$

48)

1. Find the volume of the solid S_1 bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 1$.
2. Find the volume of the solid S_2 outside the double cone $z^2 = x^2 + y^2$ inside the cylinder $x^2 + y^2 = 1$, and above the plane $z = 0$.
3. Find the volume of the solid inside the cone $z^2 = x^2 + y^2$ and below the plane $z = 1$ by subtracting the volumes of the solids S_1 and S_2.

49)

1. Find the volume of the solid S_1 inside the unit sphere $x^2 + y^2 + z^2 = 1$ and above the plane $z = 0$.
2. Find the volume of the solid S_2 inside the double cone $(z - 1)^2 = x^2 + y^2$ and above the plane $z = 0$.
3. Find the volume of the solid outside the double cone $(z - 1)^2 = x^2 + y^2$ and inside the sphere $x^2 + y^2 + z^2 = 1$.

Answer

a. $\frac{2\pi}{3}$; b. $\frac{\pi}{2}$; c. $\frac{\pi}{6}$

In Exercises 50-51, special double integrals are presented that are especially well suited for evaluation in polar coordinates.

50) The surface of a right circular cone with height h and base radius a can be described by the equation $f(x, y) = h - h\sqrt{\frac{x^2}{a^2} + \frac{y^2}{a^2}}$, where the tip of the cone lies at $(0, 0, h)$ and the circular base lies in the xy-plane, centered at the origin.
Confirm that the volume of a right circular cone with height h and base radius a is $V = \frac{1}{3}\pi a^2 h$ by evaluating $\int\int_R f(x, y)\ dA$ in polar coordinates.

51) Consider $\int\int_R e^{-(x^2+y^2)}\ dA$.
(a) Why is this integral difficult to evaluate in rectangular coordinates, regardless of the region R?
(b) Let R be the region bounded by the circle of radius a centered at the origin. Evaluate the double integral using polar coordinates.
(c) Take the limit of your answer from (b), as $a \to \infty$. What does this imply about the volume under the surface of $e^{-(x^2+y^2)}$ over the entire xy-plane?

For the following two exercises, consider a spherical ring, which is a sphere with a cylindrical hole cut so that the axis of the cylinder passes through the center of the sphere (see the following figure).



52) If the sphere has radius 4 and the cylinder has radius 2 find the volume of the spherical ring.

53) A cylindrical hole of diameter 6 cm is bored through a sphere of radius 5 cm such that the axis of the cylinder passes through the center of the sphere. Find the volume of the resulting spherical ring.

Answer

$\frac{256\pi}{3}\ \text{cm}^3$

54) Find the volume of the solid that lies under the double cone $z^2 = 4x^2 + 4y^2$, inside the cylinder $x^2 + y^2 = x$, and above the plane $z = 0$.

55) Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, inside the cylinder $x^2 + y^2 = 1$ and above the plane $z = 0$.

Answer

$\frac{3\pi}{32}$

56) Find the volume of the solid that lies under the plane $x + y + z = 10$ and above the disk $x^2 + y^2 = 4x$.

57) Find the volume of the solid that lies under the plane $2x + y + 2z = 8$ and above the unit disk $x^2 + y^2 = 1$.

Answer

4π

58) A radial function f is a function whose value at each point depends only on the distance between that point and the origin of the system of coordinates; that is, $f(x, y) = g(r)$, where $r = \sqrt{x^2 + y^2}$. Show that if f is a continuous radial function, then

$$\iint_D f(x, y)dA = (\theta_2 - \theta_1)[G(R_2) - G(R_1)], \text{ where } G'(r) = rg(r)$$

and $(x, y)^{\text{int}}D = \{(r, \theta) \mid R_1 \le r \le R_2, \ 0 \le \theta \le 2\pi\}$, with $0 \le R_1 < R_2$ and $0 \le \theta_1 < \theta_2 \le 2\pi$.

59) Use the information from the preceding exercise to calculate the integral $\iint_D (x^2 + y^2)^3 dA$, where D is the unit disk.

Answer

$\frac{\pi}{4}$

60) Let $f(x, y) = \dfrac{F'(r)}{r}$ be a continuous radial function defined on the annular region $D = \{(r, \theta) \mid R_1 \le r \le R_2, \ 0 \le \theta \le 2\pi\}$, where $r = \sqrt{x^2 + y^2}$, $0 < R_1 < R_2$, and F is a differentiable function.

Show that $\iint_D f(x, y) \ dA = 2\pi[F(R_2) - F(R_1)]$.

61) Apply the preceding exercise to calculate the integral $\iint_D \dfrac{e^{\sqrt{x^2+y^2}}}{\sqrt{x^2 + y^2}} \ dx \ dy$ where D is the annular region between the circles of radii 1 and 2 situated in the third quadrant.

Answer

$\frac{1}{2}\pi e(e - 1)$

62) Let f be a continuous function that can be expressed in polar coordinates as a function of θ only; that is, $f(x, y) = h(\theta)$, where $(x, y)^{\text{int}}D = \{(r, \theta) \mid R_1 \le r \le R_2, \ \theta_1 \le \theta \le \theta_2\}$, with $0 \le R_1 < R_2$ and $0 \le \theta_1 < \theta_2 \le 2\pi$.

Show that $\iint_D f(x, y) \ dA = \dfrac{1}{2}(R_2^2 - R_1^2)[H(\theta_2) - H(\theta_1)]$, where H is an antiderivative of h.

63) Apply the preceding exercise to calculate the integral $\iint_D \dfrac{y^2}{x^2} \ dA$, where $D = \{(r, \theta) \mid 1 \le r \le 2, \ \frac{\pi}{6} \le \theta \le \frac{\pi}{3}\}$.

Answer

$\sqrt{3} - \frac{\pi}{4}$

64) Let f be a continuous function that can be expressed in polar coordinates as a function of θ only; that is $f(x, y) = g(r)h(\theta)$, where $(x, y)^{\text{int}}\{(r, \theta) \mid R_1 \le r \le R_2, \ \theta_1 \le \theta \le \theta_2\}$ with $0 \le R_1 < R_2$ and $0 \le \theta_1 < \theta_2 \le 2\pi$. Show that

$$\iint_D f(x, y) \, dA = [G(R_2) - G(R_1)] \ [H(\theta_2) - H(\theta_1)],$$

where G and H are antiderivatives of g and h, respectively.

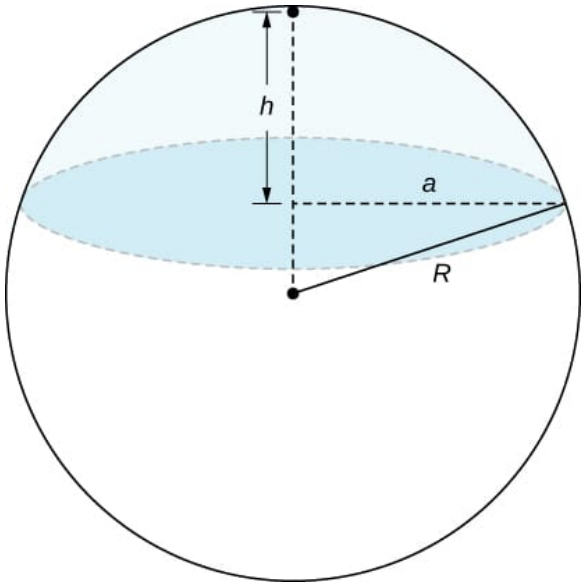
65) Evaluate $\iint_D \arctan\left(\frac{y}{x}\right)\sqrt{x^2+y^2}\,dA$, where $D=\{(r,\theta)\mid 2\le r\le 3,\ \frac{\pi}{4}\le\theta\le\frac{\pi}{3}\}$.

Answer

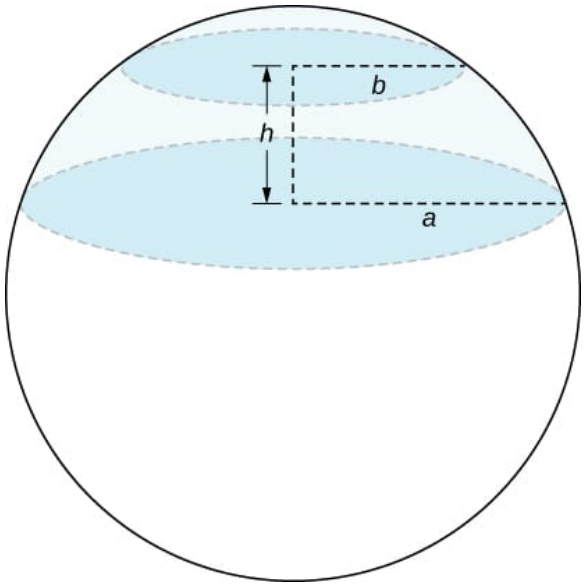
$\frac{133}{864}\pi^2$

66) A spherical cap is the region of a sphere that lies above or below a given plane.

a. Show that the volume of the spherical cap in the figure below is $\frac{1}{6}\pi h(3a^2+h^2)$.



b. A spherical segment is the solid defined by intersecting a sphere with two parallel planes. If the distance between the planes is h show that the volume of the spherical segment in the figure below is $\frac{1}{6}\pi h(3a^2+3b^2+h^2)$.



67) In statistics, the joint density for two independent, normally distributed events with a mean $\mu=0$ and a standard distribution σ is defined by $p(x,y)=\frac{1}{2\pi\sigma^2}e^{-\frac{x^2+y^2}{2\sigma^2}}$. Consider (X,Y), the Cartesian coordinates of a ball in the resting position after it was released from a position on the *z*-axis toward the xy-plane. Assume that the coordinates of the ball are independently normally distributed with a mean $\mu=0$ and a standard deviation of σ (in feet). The probability that the ball will stop no more than a feet from the origin is given by

$$P[X^2+Y^2 \le a^2] = \iint_D p(x,y)dy\ dx,$$

where D is the disk of radius a centered at the origin. Show that

$$P[X^2+Y^2 \le a^2] = 1 - e^{-a^2/2\sigma^2}.$$

68) The double improper integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2+y^2/2}\,dy\,dx$$

may be defined as the limit value of the double integrals $\iint_D e^{-x^2+y^2/2}\ dA$ over disks D_a of radii a centered at the origin, as a increases without bound; that is,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2+y^2/2}dy\ dx = \lim_{a\to\infty}\iint_{D_a} e^{-x^2+y^2/2}\,dA.$$

Use polar coordinates to show that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2+y^2/2}\ dy\ dx = 2\pi.$

69) Show that $\int_{-\infty}^{\infty} e^{-x^2/2}\ dx = \sqrt{2\pi}$ by using the relation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2+y^2/2}\,dy\,dx = \left(\int_{-\infty}^{\infty} e^{-x^2/2}dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2/2}dy\right).$$

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- Problems 1, 2, 34 - 37 and 50 - 51 are from Apex Calculus, Chapter 13.3
- Edited by Paul Seeburger (Monroe Community College)

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14.4: TRIPLE INTEGRALS

Learning Objectives

- Recognize when a function of three variables is integrable over a rectangular box.
- Evaluate a triple integral by expressing it as an iterated integral.
- Recognize when a function of three variables is integrable over a closed and bounded region.
- Simplify a calculation by changing the order of integration of a triple integral.
- Calculate the average value of a function of three variables.

Previously, we discussed the double integral of a function $f(x, y)$ of two variables over a rectangular region in the plane. In this section we define the triple integral of a function $f(x, y, z)$ of three variables over a rectangular solid box in space, $\mathbb{R}^3$. Later in this section we extend the definition to more general regions in $\mathbb{R}^3$.

Integrable Functions of Three Variables

We can define a rectangular box B in $\mathbb{R}^3$ as

$$B = \big\{(x, y, z) \,|\, a \le x \le b,\, c \le y \le d,\, e \le z \le f\big\}.$$

We follow a similar procedure to what we did in previously. We divide the interval $[a, b]$ into l subintervals $[x_{i-1}, x_i]$ of equal length Δx with

$$\Delta x = \frac{x_i - x_{i-1}}{l},$$

divide the interval $[c, d]$ into m subintervals $[y_{i-1}, y_i]$ of equal length Δy with

$$\Delta y = \frac{y_j - y_{j-1}}{m},$$

and divide the interval $[e, f]$ into n subintervals $[z_{i-1}, z_i]$ of equal length Δz with

$$\Delta z = \frac{z_k - z_{k-1}}{n}$$

Then the rectangular box B is subdivided into lmn subboxes:

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{i-1}, y_i] \times [z_{i-1}, z_i],$$

as shown in Figure $14.4.1$.

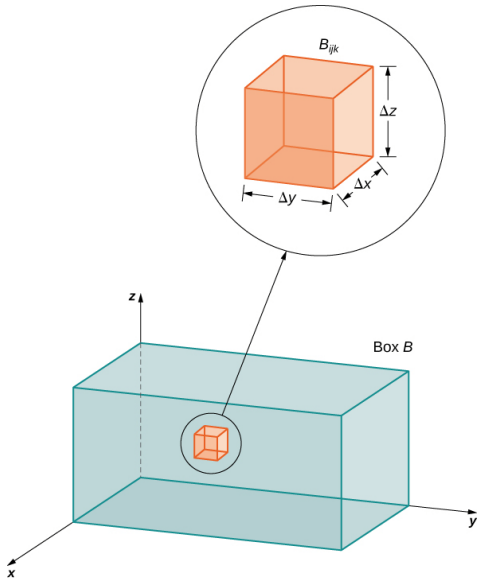


Figure 14.4.1: A rectangular box in $\mathbb{R}^3$ divided into subboxes by planes parallel to the coordinate planes.

For each i, j, and k, consider a sample point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ in each sub-box B_{ijk}. We see that its volume is $\Delta V = \Delta x \Delta y \Delta z$. Form the triple Riemann sum

$$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\, \Delta x \Delta y \Delta z.$$

We define the triple integral in terms of the limit of a triple Riemann sum, as we did for the double integral in terms of a double Riemann sum.

Definition: The triple integral

The triple integral of a function $f(x, y, z)$ over a rectangular box B is defined as

$$\lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\, \Delta x \Delta y \Delta z = \iiint_B f(x, y, z)\, dV$$

if this limit exists.

When the triple integral exists on B the function $f(x, y, z)$ is said to be integrable on B. Also, the triple integral exists if $f(x, y, z)$ is continuous on B. Therefore, we will use continuous functions for our examples. However, continuity is sufficient but not necessary; in other words, f is bounded on B and continuous except possibly on the boundary of B. The sample point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ can be any point in the rectangular sub-box B_{ijk} and all the properties of a double integral apply to a triple integral. Just as the double integral has many practical applications, the triple integral also has many applications, which we discuss in later sections.

Now that we have developed the concept of the triple integral, we need to know how to compute it. Just as in the case of the double integral, we can have an iterated triple integral, and consequently, a version of **Fubini's theorem** for triple integrals exists.

Fubini's Theorem for Triple Integrals

If $f(x, y, z)$ is continuous on a rectangular box $B = [a, b] \times [c, d] \times [e, f]$, then

$$\iiint_B f(x, y, z)\, dV = \int_e^f \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz.$$

This integral is also equal to any of the other five possible orderings for the iterated triple integral.

For a, b, c, d, e and f real numbers, the iterated triple integral can be expressed in six different orderings:

$$\int_e^f \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz = \int_e^f \left(\int_c^d \left(\int_a^b f(x,y,z)\,dx\right) dy\right) dz \tag{14.4.1}$$

$$= \int_c^d \left(\int_e^f \left(\int_a^b f(x,y,z)\,dx\right) dz\right) dy \tag{14.4.2}$$

$$= \int_a^b \left(\int_e^f \left(\int_c^d f(x,y,z)\,dy\right) dz\right) dx \tag{14.4.3}$$

$$= \int_e^f \left(\int_a^b \left(\int_c^d f(x,y,z)\,dy\right) dx\right) dz \tag{14.4.4}$$

$$= \int_c^d \left(\int_a^b \left(\int_e^f f(x,y,z)\,dz\right) dx\right) dy \tag{14.4.5}$$

$$= \int_a^b \left(\int_c^d \left(\int_e^f f(x,y,z)\,dz\right) dy\right) dx \tag{14.4.6}$$

For a rectangular box, the order of integration does not make any significant difference in the level of difficulty in computation. We compute triple integrals using Fubini's Theorem rather than using the Riemann sum definition. We follow the order of integration in the same way as we did for double integrals (that is, from inside to outside).

Example 14.4.1: Evaluating a Triple Integral

Evaluate the triple integral

$$\int_{z=0}^{z=1} \int_{y=2}^{y=4} \int_{x=-1}^{x=5} (x + yz^2)\,dx\,dy\,dz.$$

Solution

The order of integration is specified in the problem, so integrate with respect to x first, then y, and then z.

$$\begin{aligned}
&\int_{z=0}^{z=1} \int_{y=2}^{y=4} \int_{x=-1}^{x=5} (x + yz^2)\,dx\,dy\,dz \\
&= \int_{z=0}^{z=1} \int_{y=2}^{y=4} \left[\frac{x^2}{2} + xyz^2\Big|_{x=-1}^{x=5}\right] dy\,dz && \text{Integrate with respect to } x. \\
&= \int_{z=0}^{z=1} \int_{y=2}^{y=4} \left[12 + 6yz^2\right]\, dy\,dz && \text{Evaluate.} \\
&= \int_{z=0}^{z=1} \left[12y + 6\frac{y^2}{2}z^2\Big|_{y=2}^{y=4}\right] dz && \text{Integrate with respect to } y. \\
&= \int_{z=0}^{z=1} [24 + 36z^2]\,dz && \text{Evaluate.} \\
&= \left[24z + 36\frac{z^3}{3}\right]_{z=0}^{z=1} && \text{Integrate with respect to } z. \\
&= 36. && \text{Evaluate.}
\end{aligned}$$

Example 14.4.2: Evaluating a Triple Integral

Evaluate the triple integral

$$\iiint_B x^2yz\,dV$$

where $B = \{(x,y,z) \mid -2 \le x \le 1,\ 0 \le y \le 3,\ 1 \le z \le 5\}$ as shown in Figure 14.4.2.

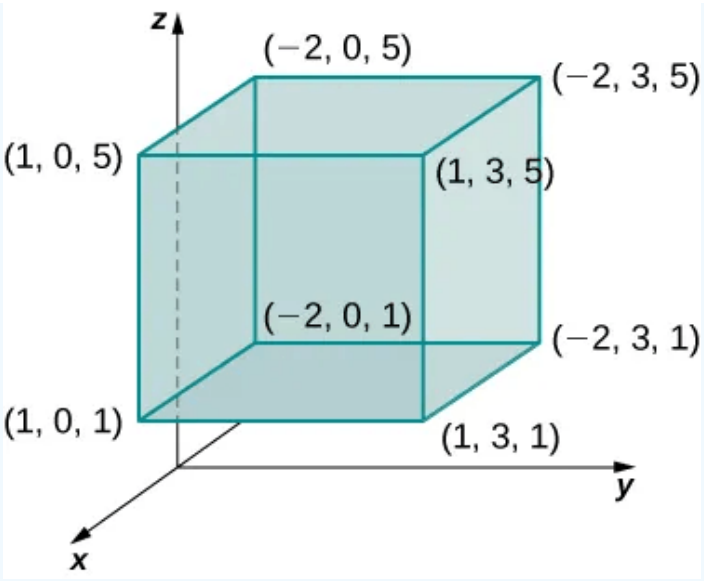


Figure 14.4.2: Evaluating a triple integral over a given rectangular box.

Solution

The order is not specified, but we can use the iterated integral in any order without changing the level of difficulty. Choose, say, to integrate y first, then x, and then z.

$$\begin{aligned}\iiint_B x^2yz\,dV &= \int_1^5\int_{-2}^1\int_0^3 [x^2yz]\,dy\,dx\,dz\\ &= \int_1^5\int_{-2}^1 \left[x^2\frac{y^2}{2}z\Big|_0^3\right]dx\,dz\\ &= \int_1^5\int_{-2}^1 \frac{9}{2}x^2z\,dx\,dz\\ &= \int_1^5 \left[\frac{9}{2}\frac{x^3}{3}z\Big|_{-2}^1\right]dz\\ &= \int_1^5 \frac{27}{2}z\,dz\\ &= \frac{27}{2}\frac{z^2}{2}\Big|_1^5 = \frac{27}{4}(25-1) = 162.\end{aligned}$$

Now try to integrate in a different order just to see that we get the same answer. Choose to integrate with respect to x first, then z, then y

$$\begin{aligned}\iiint_B x^2yz\,dV &= \int_0^3\int_1^5\int_{-2}^1 [x^2yz]\,dx\,dz\,dy\\ &= \int_0^3\int_1^5 \left[\frac{x^3}{3}yz\Big|_{-2}^1\right]dz\,dy\\ &= \int_0^3\int_1^5 3yz\,dz\,dy\\ &= \int_0^3 \left[3y\frac{z^2}{2}\Big|_1^5\right]dy\\ &= \int_0^3 36y\,dy\\ &= 36\frac{y^2}{2}\Big|_0^3 = 18(9-0) = 162.\end{aligned}$$

Exercise 14.4.1

Evaluate the triple integral

$$\iiint_B z \sin x \cos y \, dV$$

where $B = \{(x, y, z) \mid 0 \leq x \leq \pi, \ \frac{3\pi}{2} \leq y \leq 2\pi, \ 1 \leq z \leq 3\}$.

Hint available at the end of this section.

Triple Integrals over a General Bounded Region

We now expand the definition of the triple integral to compute a triple integral over a more **general bounded region** E in $\mathbb{R}^3$. The general bounded regions we will consider are of three types. First, let D be the bounded region that is a projection of E onto the xy-plane. Suppose the region E in $\mathbb{R}^3$ has the form

$$E = \{(x, y, z) \,|\, (x, y) \overset{\text{int}}{} D, u_1(x, y) \leq z \leq u_2(x, y)\}.$$

For two functions $z = u_1(x, y)$ and $u_2(x, y)$, such that $u_1(x, y) \leq u_2(x, y)$ for all (x, y) in D as shown in the following figure.

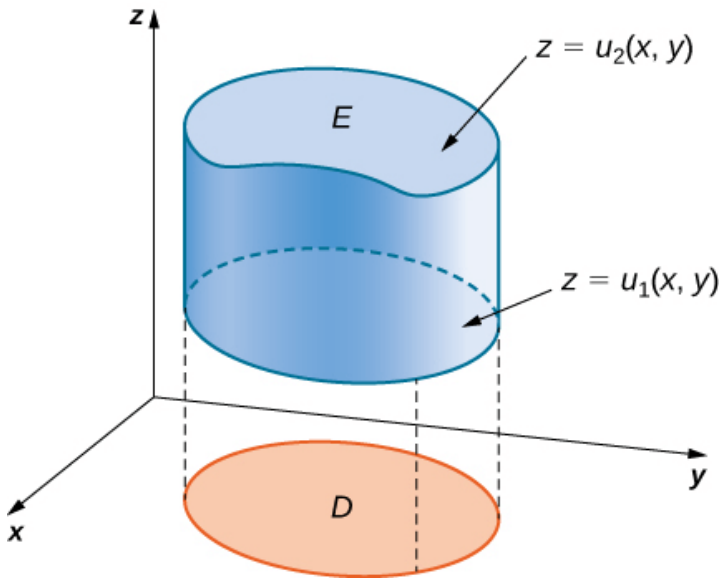


Figure 14.4.3: We can describe region E as the space between $u_1(x, y)$ and $u_2(x, y)$ above the projection D of E onto the xy-plane.

Triple Integral over a General Region

The triple integral of a continuous function $f(x, y, z)$ over a general three-dimensional region

$$E = \{(x, y, z) \,|\, (x, y) \overset{\text{int}}{} D, \ u_1(x, y) \leq z \leq u_2(x, y)\}$$

in $\mathbb{R}^3$, where D is the projection of E onto the xy-plane, is

$$\iiint_E f(x, y, z) \, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z) \, dz \right] dA.$$

Similarly, we can consider a general bounded region D in the xy-plane and two functions $y = u_1(x, z)$ and $y = u_2(x, z)$ such that $u_1(x, z) \leq u_2(x, z)$ for all (x, z) in D. Then we can describe the solid region E in $\mathbb{R}^3$ as

$$E = \{(x, y, z) \mid (x, z) \overset{\text{int}}{} D,\ u_1(x, z) \le z \le u_2(x, z)\}$$

where D is the projection of E onto the xy-plane and the triple integral is

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\, dy \right] dA.$$

Finally, if D is a general bounded region in the xy-plane and we have two functions $x = u_1(y, z)$ and $x = u_2(y, z)$ such that $u_1(y, z) \le u_2(y, z)$ for all (y, z) in D, then the solid region E in $\mathbb{R}^3$ can be described as

$$E = \{(x, y, z) \mid (y, z) \overset{\text{int}}{} D,\ u_1(y, z) \le z \le u_2(y, z)\}$$

where D is the projection of E onto the xy-plane and the triple integral is

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\, dx \right] dA.$$

Note that the region D in any of the planes may be of Type I or Type II as described in previously. If D in the xy-plane is of Type I (Figure 14.4.4), then

$$E = \{(x, y, z) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x),\ u_1(x, y) \le z \le u_2(x, y)\}.$$

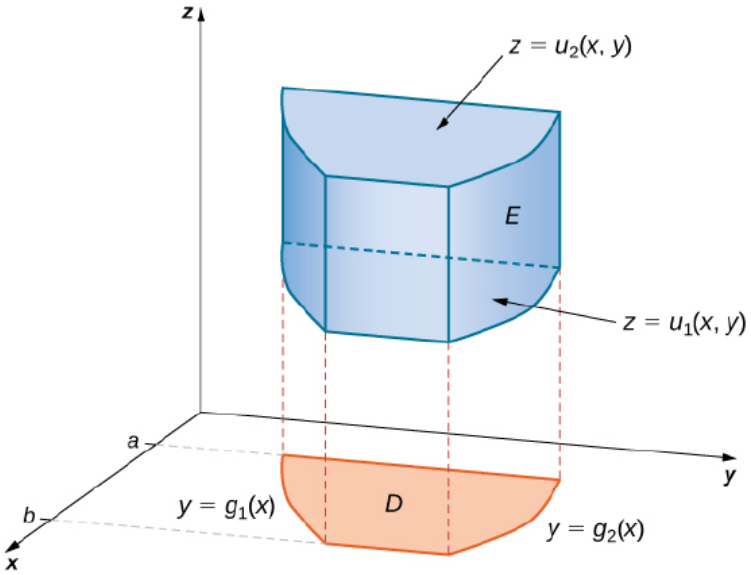


Figure 14.4.4: A box E where the projection D in the xy-plane is of Type I.

Then the triple integral becomes

$$\iiint_E f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\, dy\, dx.$$

If D in the xy-plane is of Type II (Figure 14.4.5), then

$$E = \{(x, y, z) \mid c \le x \le d, h_1(x) \le y \le h_2(x),\ u_1(x, y) \le z \le u_2(x, y)\}.$$

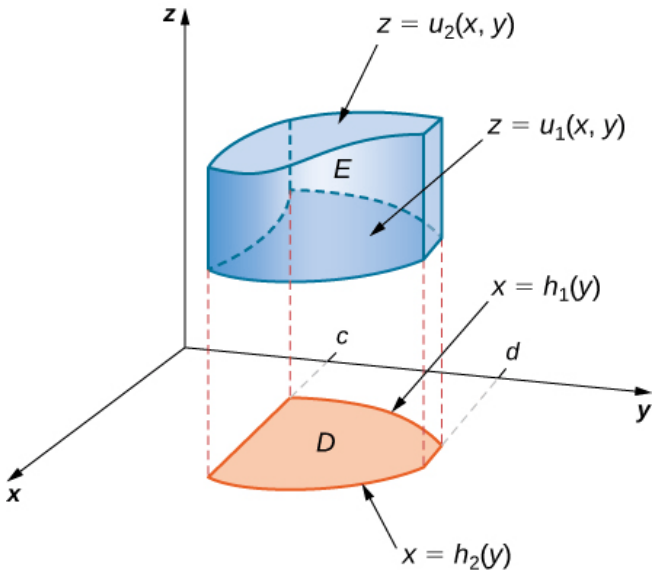


Figure 14.4.5: A box E where the projection D in the xy-plane is of Type II.

Then the triple integral becomes

$$\iiint_E f(x, y, z)\, dV = \int_{y=c}^{y=d} \int_{x=h_1(y)}^{x=h_2(y)} \int_{z=u_1(x,y)}^{z=u_2(x,y)} f(x, y, z)\, dz\, dx\, dy.$$

Example 14.4.3A: Evaluating a Triple Integral over a General Bounded Region

Evaluate the triple integral of the function $f(x, y, z) = 5x - 3y$ over the solid tetrahedron bounded by the planes $x = 0,\ y = 0,\ z = 0$, and $x + y + z = 1$.

Solution

Figure 14.4.6 shows the solid tetrahedron E and its projection D on the xy-plane.

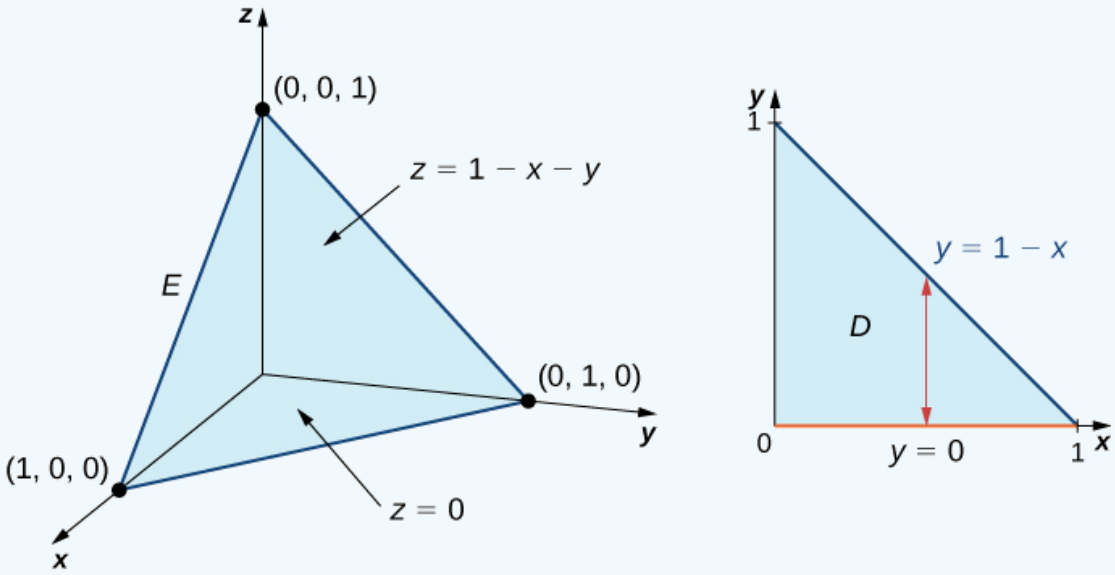


Figure 14.4.6: The solid E has a projection D on the xy-plane of Type I.

We can describe the solid region tetrahedron as

$$E = \{(x, y, z) \mid 0 \le x \le 1,\ 0 \le y \le 1 - x,\ 0 \le z \le 1 - x - y\}.$$

Hence, the triple integral is

$$\iiint_E f(x, y, z)\, dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} (5x - 3y)\, dz\, dy\, dx.$$

To simplify the calculation, first evaluate the integral $\int_{z=0}^{z=1-x-y} (5x - 3y)\ dz$. We have

$$\int_{z=0}^{z=1-x-y} (5x - 3y)\, dz = (5x - 3y)z \Big|_{z=0}^{z=1-x-y} = (5x - 3y)(1 - x - y).$$

Now evaluate the integral

$$\int_{y=0}^{y=1-x} (5x - 3y)(1 - x - y)\, dy,$$

obtaining

$$\int_{y=0}^{y=1-x} (5x - 3y)(1 - x - y)\, dy = \frac{1}{2}(x - 1)^2(6x - 1).$$

Finally evaluate

$$\int_{x=0}^{x=1} \frac{1}{2}(x - 1)^2(6x - 1)\, dx = \frac{1}{12}.$$

Putting it all together, we have

$$\iiint_E f(x, y, z)\, dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} (5x - 3y)\, dz\, dy\, dx = \frac{1}{12}.$$

Just as we used the double integral

$$\iint_D 1\, dA$$

to find the area of a general bounded region D we can use

$$\iiint_E 1\, dV$$

to find the volume of a general solid bounded region E. The next example illustrates the method.

Example $14.4.3B$: Finding a Volume by Evaluating a Triple Integral

Find the volume of a right pyramid that has the square base in the xy-plane $[-1, 1] \times [-1, 1]$ and vertex at the point $(0, 0, 1)$ as shown in the following figure.

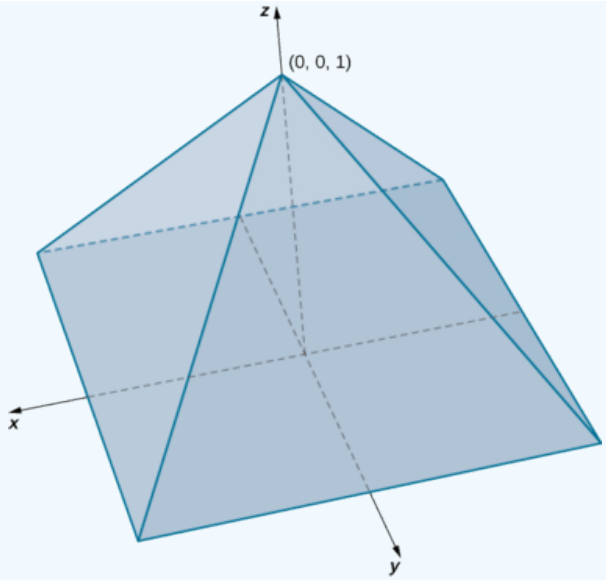


Figure 14.4.7: Finding the volume of a pyramid with a square base.

Solution

In this pyramid the value of z changes from 0 to 1 and at each height z the cross section of the pyramid for any value of z is the square

$$[-1+z,\, 1-z] \times [-1+z,\, 1-z].$$

Hence, the volume of the pyramid is

$$\iiint_E 1\, dV$$

where

$$E = \{(x, y, z) \,|\, 0 \le z \le 1,\, -1+z \le y \le 1-z,\, -1+z \le x \le 1-z\}.$$

Thus, we have

$$\begin{aligned} \iiint_E 1\, dV &= \int_{z=0}^{z=1} \int_{y=-1+z}^{y=1-z} \int_{x=-1+z}^{x=1-z} 1\, dx\, dy\, dz \\ &= \int_{z=0}^{z=1} \int_{y=-1+z}^{y=1-z} (2-2z)\, dy\, dz \\ &= \int_{z=0}^{z=1} (2-2z)^2\, dz = \frac{4}{3}. \end{aligned}$$

Hence, the volume of the pyramid is $\frac{4}{3}$ cubic units.

Exercise 14.4.3

Consider the solid sphere $E = \{(x, y, z) \mid x^2 + y^2 + z^2 = 9\}$. Write the triple integral

$$\iiint_E f(x, y, z)\, dV$$

for an arbitrary function f as an iterated integral. Then evaluate this triple integral with $f(x, y, z) = 1$. Notice that this gives the volume of a sphere using a triple integral.

Hint available at the end of this section.

Changing the Order of Integration

As we have already seen in double integrals over general bounded regions, changing the order of the integration is done quite often to simplify the computation. With a triple integral over a rectangular box, the order of integration does not change the level of difficulty of the calculation. However, with a triple integral over a general bounded region, choosing an appropriate order of integration can simplify the computation quite a bit. Sometimes making the change to polar coordinates can also be very helpful. We demonstrate two examples here.

Example 14.4.4: Changing the Order of Integration

Consider the iterated integral

$$\int_{x=0}^{x=1}\int_{y=0}^{y=x^2}\int_{z=0}^{z=y} f(x,y,z)\,dz\,dy\,dx.$$

The order of integration here is first with respect to *z*, then *y*, and then *x*. Express this integral by changing the order of integration to be first with respect to x, then z, and then y. Verify that the value of the integral is the same if we let $f(x,y,z) = xyz$.

Solution

The best way to do this is to sketch the region E and its projections onto each of the three coordinate planes. Thus, let

$$E = \{(x,y,z)\,|\,0 \le x \le 1,\ 0 \le y \le x^2,\ 0 \le z \le y\}.$$

and

$$\int_{x=0}^{x=1}\int_{y=0}^{y=x^2}\int_{z=0}^{z=x^2} f(x,y,z)\,dz\,dy\,dx = \iiint_E f(x,y,z)\,dV.$$

We need to express this triple integral as

$$\int_{y=c}^{y=d}\int_{z=v_1(y)}^{z=v_2(y)}\int_{x=u_1(y,z)}^{x=u_2(y,z)} f(x,y,z)\,dx\,dz\,dy.$$

Knowing the region E we can draw the following projections (Figure 14.4.8):

on the xy-plane is $D_1 = \{(x,y)\ |\ 0 \le x \le 1,\ \ 0 \le y \le x^2\} = \{(x,y)\ |\ 0 \le y \le 1,\ \ \sqrt{y} \le x \le 1\}$,

on the yz-plane is $D_2 = \{(y,z)\ |\ 0 \le y \le 1,\ \ 0 \le z \le y^2\}$, and

on the xz-plane is $D_3 = \{(x,z)\ |\ 0 \le x \le 1,\ \ 0 \le z \le x^2\}$.

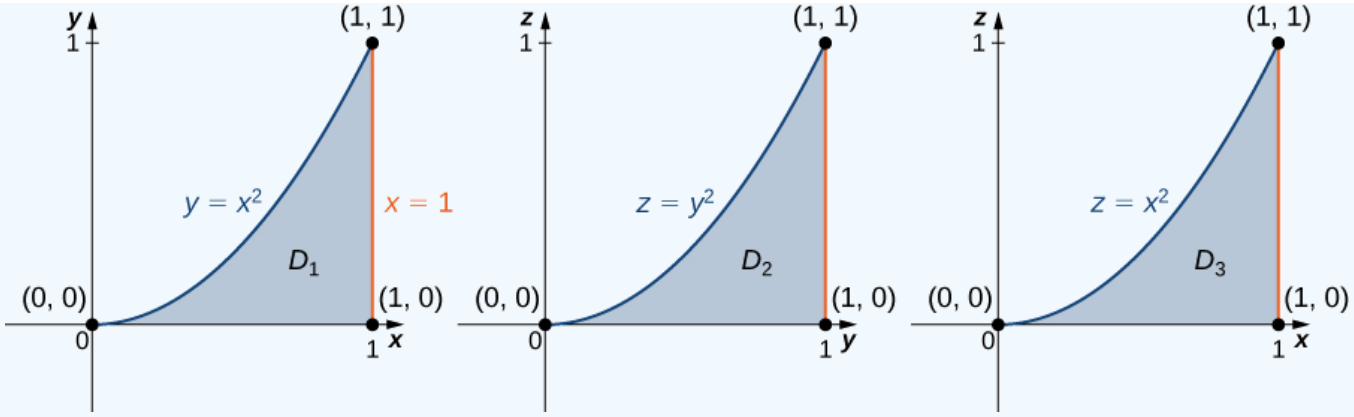


Figure 14.4.8. The three cross sections of E on the three coordinate planes.

Now we can describe the same region E as $\{(x,y,z) \mid 0 \le y \le 1,\ 0 \le z \le y^2,\ \sqrt{y} \le x \le 1\}$, and consequently, the triple integral becomes

$$\int_{y=c}^{y=d}\int_{z=v_1(y)}^{z=v_2(y)}\int_{x=u_1(y,z)}^{x=u_2(y,z)} f(x,y,z)\,dx\,dz\,dy = \int_{y=0}^{y=1}\int_{z=0}^{z=x^2}\int_{x=\sqrt{y}}^{x=1} f(x,y,z)\,dx\,dz\,dy$$

Now assume that $f(x,y,z) = xyz$ in each of the integrals. Then we have

$$\begin{aligned}
\int_{x=0}^{x=1}\int_{y=0}^{y=x^2}\int_{z=0}^{z=y^2} xyz\,dz\,dy\,dx &= \int_{x=0}^{x=1}\int_{y=0}^{y=x^2}\left[xy\frac{z^2}{2}\Big|_{z=0}^{z=y^2}\right]dy\,dx \\
&= \int_{x=0}^{x=1}\int_{y=0}^{y=x^2}\left(x\frac{y^5}{2}\right)dy\,dx \\
&= \int_{x=0}^{x=1}\left[x\frac{y^6}{12}\Big|_{y=0}^{y=x^2}\right]dx \\
&= \int_{x=0}^{x=1}\frac{x^{13}}{12}dx = \frac{x^{14}}{168}\Big|_{x=0}^{x=1} \\
&= \frac{1}{168},
\end{aligned}$$

$$\begin{aligned}
\int_{y=0}^{y=1}\int_{z=0}^{z=y^2}\int_{x=\sqrt{y}}^{x=1} xyz\,dx\,dz\,dy &= \int_{y=0}^{y=1}\int_{z=0}^{z=y^2}\left[yz\frac{x^2}{2}\Big|_{\sqrt{y}}^{1}\right]dz\,dy \\
&= \int_{y=0}^{y=1}\int_{z=0}^{z=y^2}\left(\frac{yz}{2}-\frac{y^2z}{2}\right)dz\,dy \\
&= \int_{y=0}^{y=1}\left[\frac{yz^2}{4}-\frac{y^2z^2}{4}\Big|_{z=0}^{z=y^2}\right]dy \\
&= \int_{y=0}^{y=1}\left(\frac{y^5}{4}-\frac{y^6}{4}\right)dy \\
&= \left(\frac{y^6}{24}-\frac{y^7}{28}\right)\Big|_{y=0}^{y=1} \\
&= \frac{1}{168}.
\end{aligned}$$

The answers match.

Exercise 14.4.4

Write five different iterated integrals equal to the given integral

$$\int_{z=0}^{z=4}\int_{y=0}^{y=4-z}\int_{x=0}^{x=\sqrt{y}} f(x,y,z)\,dx\,dy\,dz.$$

Hint available at the end of this section.

Example 14.4.5: Changing Integration Order and Coordinate Systems

Evaluate the triple integral

$$\iiint_E \sqrt{x^2+z^2}\,dV,$$

where E is the region bounded by the paraboloid $y=x^2+z^2$ (Figure 14.4.9) and the plane $y=4$.

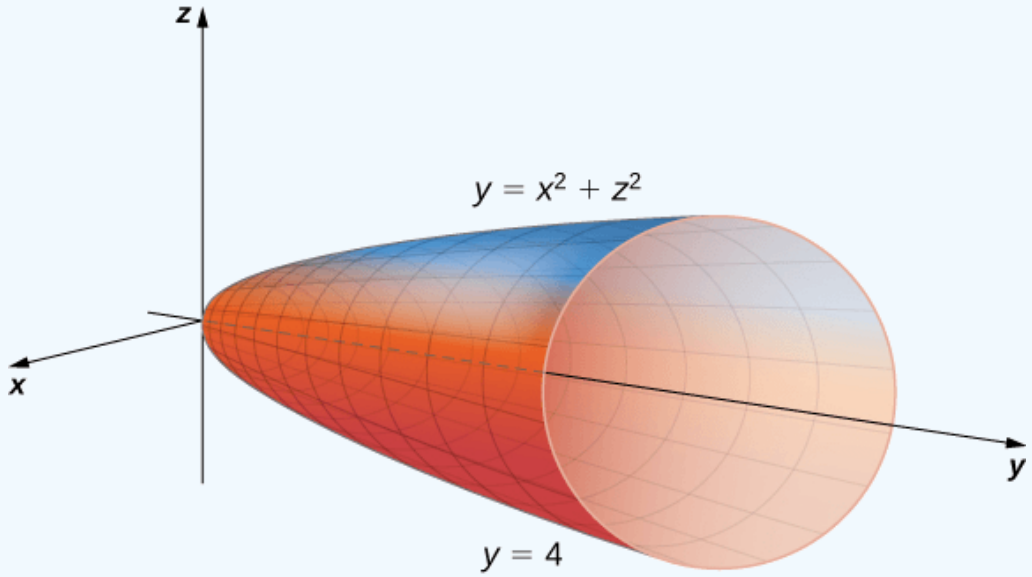


Figure 14.4.9. Integrating a triple integral over a paraboloid.

Solution

The projection of the solid region E onto the xy-plane is the region bounded above by $y=4$ and below by the parabola $y=x^2$ as shown.

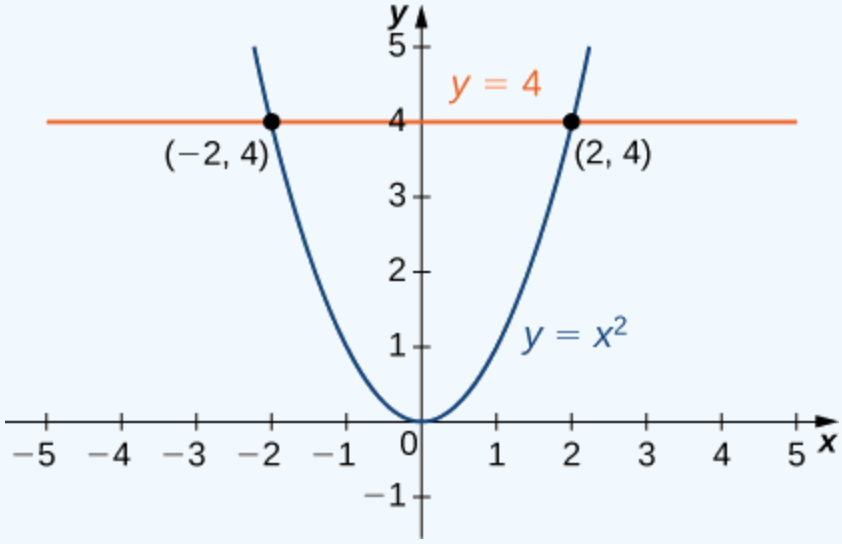


Figure 14.4.10. Cross section in the xy-plane of the paraboloid in Figure 14.4.9.

Thus, we have

$$E = \{(x, y, z) \mid -2 \le x \le 2,\ x^2 \le y \le 4,\ -\sqrt{y - x^2} \le z \le \sqrt{y - x^2}\}.$$

The triple integral becomes

$$\iiint_E \sqrt{x^2 + z^2}\, dV = \int_{x=-2}^{x=2} \int_{y=x^2}^{y=4} \int_{z=-\sqrt{y-x^2}}^{z=\sqrt{y-x^2}} \sqrt{x^2 + z^2}\, dz\, dy\, dx.$$

This expression is difficult to compute, so consider the projection of E onto the xz-plane. This is a circular disc $x^2 + z^2 \le 4$. So we obtain

$$\iiint_E \sqrt{x^2 + z^2}\, dV = \int_{x=-2}^{x=2} \int_{y=x^2}^{y=4} \int_{z=-\sqrt{y-x^2}}^{z=\sqrt{y-x^2}} \sqrt{x^2 + z^2}\, dz\, dy\, dx = \int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} \int_{y=x^2+z^2}^{y=4} \sqrt{x^2 + z^2}\, dy\, dz\, dx.$$

Here the order of integration changes from being first with respect to z then y and then x to being first with respect to y then to z and then to x. It will soon be clear how this change can be beneficial for computation. We have

$$\int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} \int_{y=x^2+z^2}^{y=4} \sqrt{x^2 + z^2}\, dy\, dz\, dx = \int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} (4 - x^2 - z^2)\sqrt{x^2 + z^2}\, dz\, dx.$$

Now use the polar substitution $x = r\ \cos\ \theta,\ z = r\ \sin\ \theta$, and $dz\ dx = r\ dr\ d\theta$ in the xz-plane. This is essentially the same thing as when we used polar coordinates in the xy-plane, except we are replacing y by z. Consequently the limits of integration change and we have, by using $r^2 = x^2 + z^2$,

$$\int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} (4 - x^2 - z^2)\sqrt{x^2 + z^2}\, dz\, dx = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} (4 - r^2) r r\, dr\, d\theta = \int_0^{2\pi} \left[\frac{4r^3}{3} - \frac{r^5}{5}\Big|_0^2\right] d\theta = \int_0^{2\pi} \frac{64}{15}\, d\theta = \frac{128\pi}{15}$$

Average Value of a Function of Three Variables

Recall that we found the average value of a function of two variables by evaluating the double integral over a region on the plane and then dividing by the area of the region. Similarly, we can find the average value of a function in three variables by evaluating the triple integral over a solid region and then dividing by the volume of the solid.

Average Value of a Function of Three Variables

If $f(x,y,z)$ is integrable over a solid bounded region E with positive volume $V\ (E)$, then the average value of the function is

$$f_{ave} = \frac{1}{V\,(E)} \iiint_E f(x,y,z)\,dV.$$

Note that the volume is

$$V\,(E) = \iiint_E 1\,dV.$$

Example 14.4.6: Finding an Average Temperature

The temperature at a point (x,y,z) of a solid E bounded by the coordinate planes and the plane $x+y+z=1$ is $T(x,y,z) = (xy+8z+20)$ °C. Find the average temperature over the solid.

Solution

Use the theorem given above and the triple integral to find the numerator and the denominator. Then do the division. Notice that the plane $x+y+z=1$ has intercepts $(1,0,0),\ (0,1,0)$, and $(0,0,1)$. The region E looks like

$$E = \{(x,y,z)\,|\,0 \le x \le 1,\ 0 \le y \le 1-x,\ 0 \le z \le 1-x-y\}.$$

Hence the triple integral of the temperature is

$$\iiint_E f(x,y,z)\,dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} (xy+8z+20)\,dz\,dy\,dx = \frac{147}{40}.$$

The volume evaluation is

$$V\,(E) = \iiint_E 1\,dV = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} \int_{z=0}^{z=1-x-y} 1\,dz\,dy\,dx = \frac{1}{6}.$$

Hence the average value is

$$T_{ave} = \frac{147/40}{1/6} = \frac{6(147)}{40} = \frac{441}{20}\ ^\circ\text{C}$$

.

Exercise 14.4.6

Find the average value of the function $f(x,y,z) = xyz$ over the cube with sides of length 4 units in the first octant with one vertex at the origin and edges parallel to the coordinate axes.

Hint available at the end of this section.

Key Concepts

- To compute a triple integral we use Fubini's theorem, which states that if $f(x,y,z)$ is continuous on a rectangular box $B = [a,b] \times [c,d] \times [e,f]$, then

  $$\iiint_B f(x,y,z)\,dV = \int_e^f \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

  and is also equal to any of the other five possible orderings for the iterated triple integral.

- To compute the volume of a general solid bounded region E we use the triple integral

$$V(E) = \iiint_E 1\, dV.$$

- Interchanging the order of the iterated integrals does not change the answer. As a matter of fact, interchanging the order of integration can help simplify the computation.
- To compute the average value of a function over a general three-dimensional region, we use

$$f_{ave} = \frac{1}{V(E)} \iiint_E f(x,y,z)\, dV.$$

Key Equations

- **Triple integral**

$$\lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\, \Delta x \Delta y \Delta z = \iiint_B f(x,y,z)\, dV$$

Glossary

triple integral
: the triple integral of a continuous function $f(x,y,z)$ over a rectangular solid box B is the limit of a Riemann sum for a function of three variables, if this limit exists

Answers

Exercise 14.4.1

Follow the steps in the previous example.

$$\iiint_B z \sin x \cos y\, dV = 8$$

Exercise 14.4.3

Follow the steps in the previous example. Use symmetry.

$$\iiint_E 1\, dV = 8 \int_{x=-3}^{x=3} \int_{y=-\sqrt{9-z^2}}^{y=\sqrt{9-z^2}} \int_{z=-\sqrt{9-x^2-y^2}}^{z=\sqrt{9-x^2-y^2}} 1\, dz\, dy\, dx$$
$$= 36\pi \text{ cubic units.}$$

Exercise 14.4.4

Follow the steps in the previous example, using the region E as $\{(x,y,z) \mid 0 \le z \le 4,\ 0 \le y \le 4-z,\ 0 \le x \le \sqrt{y}\}$, and describe and sketch the projections onto each of the three planes, five different times.

$$(i) \int_{z=0}^{z=4} \int_{x=0}^{x=\sqrt{4-z}} \int_{y=x^2}^{y=4-z} f(x,y,z)\, dy\, dx\, dz,\ (ii) \int_{y=0}^{y=4} \int_{z=0}^{z=4-y} \int_{x=0}^{x=\sqrt{y}} f(x,y,z)\, dx\, dz\, dy,\ (iii) \int_{y=0}^{y=4} \int_{x=0}^{x=\sqrt{y}} \int_{z=0}^{Z=4-y} f(x,y,z)\, dz\, dx\, dy,$$
$$(iv) \int_{x=0}^{x=2} \int_{y=x^2}^{y=4} \int_{z=0}^{z=4-y} f(x,y,z)\, dz\, dy\, dx,\ (v) \int_{x=0}^{x=2} \int_{z=0}^{z=4-x^2} \int_{y=x^2}^{y=4-z} f(x,y,z)\, dy\, dz\, dx$$

Exercise 14.4.6

Follow the steps in the previous example.

$f_{ave} = 8$

14.4E: Triple Integrals (Exercises)

14.4E: Triple Integrals (Exercises 2)

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14.4E: Triple Integrals (Exercises)

Terms and Concepts

1. The strategy for establishing bounds for triple integrals is "from _____ to _____, then from _____ to _____ and then from _____ to _____."

Answer:

We integrate from **surface** to **surface**, then from **curve** to **curve** and then from **point** to **point**.

2. Give an informal interpretation of what $\iiint_Q dV$ means.

Answer:

$\iiint_Q dV$ = Volume of the solid region Q

3. Give two uses of triple integration.

Answer:

To compute total mass or average density of a solid object, given a density function or to compute the average temperature in a solid region or object.

4. If an object has a constant density δ and a volume V, what is its mass?

Answer:

It's mass is δV.

Volume of Solid Regions

In Exercises 5-8, two surfaces $f_1(x,y)$ and $f_2(x,y)$ and a region R in the xy-plane are given. Set up and evaluate the triple integral that represents the volume between these surfaces over R.

5. $f_1(x,y) = 8 - x^2 - y^2,\ \ f_2(x,y) = 2x + y;$
R is the square with corners $(-1,-1)$ and $(1,1)$.

Answer:

$$\text{V} = \int_{-1}^{1}\int_{-1}^{1}\int_{2x+y}^{8-x^2-y^2} dz\ dy\ dx \quad = \quad \frac{88}{3}\ \text{units}^3$$

6. $f_1(x,y) = x^2 + y^2,\ \ f_2(x,y) = -x^2 - y^2;$
R is the square with corners $(0,0)$ and $(2,3)$.

7. $f_1(x,y) = \sin x \cos y,\ \ f_2(x,y) = \cos x \sin y + 2;$
R is the triangle with corners $(0,0),\ \ (\pi,0)$ and (π,π).

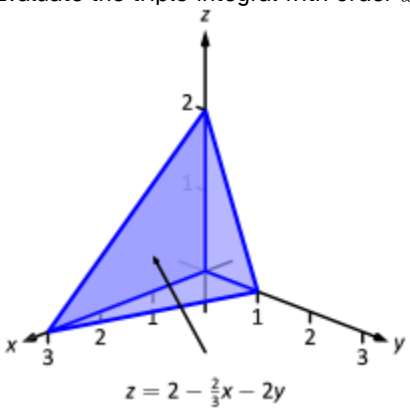
Answer:

$$\text{V} = \int_0^{\pi}\int_0^{x}\int_{\sin x \cos y}^{\cos x \sin y + 2} dz\ dy\ dx \quad = \quad (\pi^2 - \pi)\ \text{units}^3 \quad \approx 6.72801\ \text{units}^3$$

8. $f_1(x,y) = 2x^2 + 2y^2 + 3,\ \ f_2(x,y) = 6 - x^2 - y^2;$
R is the circle $x^2 + y^2 = 1$.

In Exercises 9-16, a domain D is described by its bounding surfaces, along with a graph. Set up the triple integral that gives the volume of D in the indicated order(s) of integration, and evaluate the triple integral to find this volume.

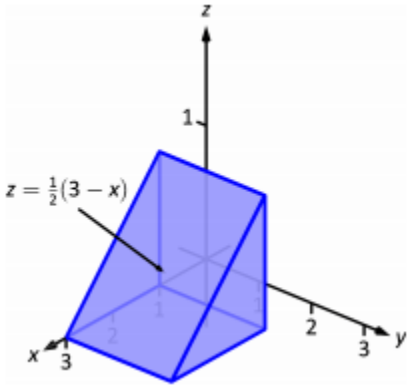
9. D is bounded by the coordinate planes and $z = 2 - \frac{2}{3}x - 2y$.
Evaluate the triple integral with order $dz\ dy\ dx$.



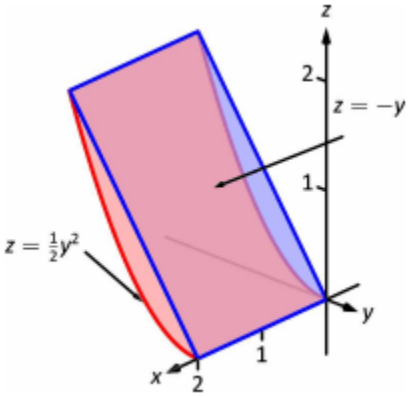
Answer:

$$\text{V} = \int_0^3 \int_0^{1-\frac{x}{3}} \int_0^{2-\frac{2}{3}x-2y} dz\ dy\ dx \quad = \quad 1\ \text{unit}^3$$

10. D is bounded by the planes $y = 0, y = 2, x = 1, z = 0$ and $z = (2 - x)/2$.
Evaluate the triple integral with order $dx\ dy\ dz$.



11. D is bounded by the planes $x = 0, x = 2, z = -y$ and by $z = y^2/2$.
Evaluate the triple integral with orders $dy\ dz\ dx$ and $dz\ dy\ dx$ to verify that you obtain the same volume either way.

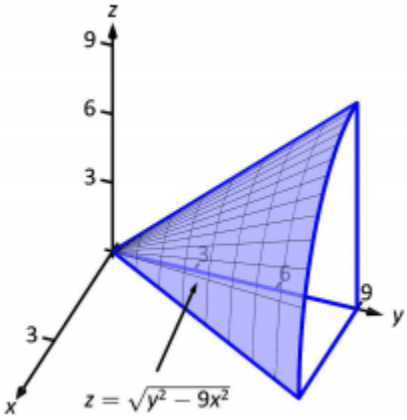


Answer:

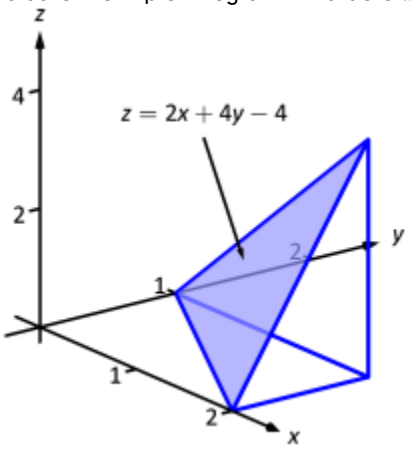
$$\text{V} = \int_0^2 \int_0^2 \int_{-\sqrt{2z}}^{-z} dy\ dz\ dx \quad = \quad \frac{4}{3}\ \text{unit}^3$$

$$V = \int_0^2 \int_{-2}^{0} \int_{\frac{y^2}{2}}^{-y} dz\ dy\ dx \quad = \quad \frac{4}{3}\ \text{unit}^3$$

12. D is bounded by the planes $z = 0, y = 9, x = 0$ and by $z = \sqrt{y^2 - 9x^2}$.
Do not evaluate any triple integral. Just set this one up in the orders: $dz\ dy\ dx$, $dx\ dy\ dz$, and $dy\ dz\ dx$.



13. D is bounded by the planes $x = 2, y = 1, z = 0$ and $z = 2x + 4y - 4$.
Evaluate the triple integral with orders $dz\ dy\ dx$ and $dx\ dy\ dz$ to verify that you obtain the same volume either way.

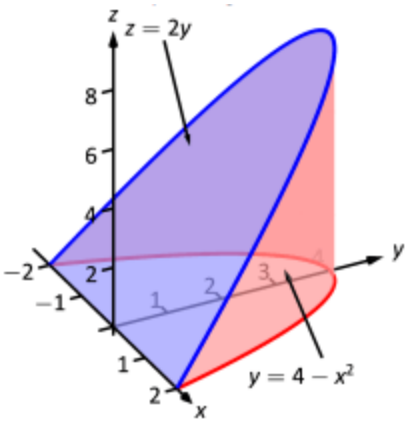


Answer:

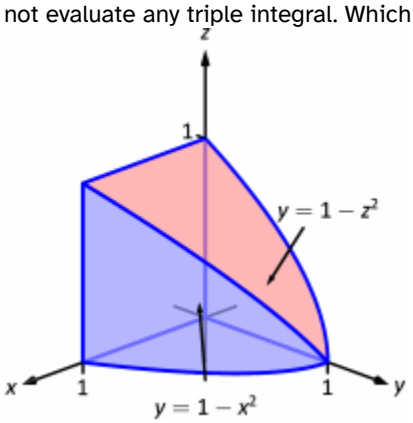
$$V = \int_0^2 \int_{1-\frac{x}{2}}^{1} \int_0^{2x+4y-4} dz\ dy\ dx \quad = \quad \frac{4}{3}\ \text{units}^3$$

$$V = \int_0^4 \int_{\frac{z}{4}}^{1} \int_{(z-4y+4)/2}^{2} dx\ dy\ dz \quad = \quad \frac{4}{3}\ \text{units}^3$$

14. D is bounded by the plane $z = 2y$ and by $y = 4 - x^2$.
Evaluate the triple integral with order $dz\ dy\ dx$.



15. D is bounded by the coordinate planes and $y = 1 - x^2$ and $y = 1 - z^2$.
Do not evaluate any triple integral. Which order would be easier to evaluate: $dz\ dy\ dx$ or $dy\ dz\ dx$? Explain why.



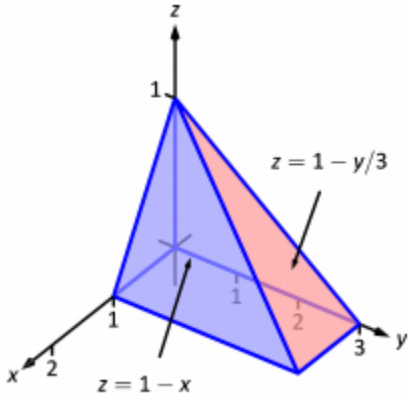
Answer:

$$V = \int_0^1 \int_0^{1-x^2} \int_0^{\sqrt{1-y}} dz\ dy\ dx$$

$$V = \int_0^1 \int_0^x \int_0^{1-x^2} dy\ dz\ dx + \int_0^1 \int_x^1 \int_0^{1-z^2} dy\ dz\ dx$$

The first one is easier since it only requires evaluation of a single integral, although both can be evaluated fairly easily.

16. D is bounded by the coordinate planes and by $z = 1 - y/3$ and $z = 1 - x$.
Evaluate the triple integral with order $dx\ dy\ dz$.



Evaluating General Triple Integrals

In exercises 17 - 20, evaluate the triple integrals over the rectangular solid box B.

17. $\iiint_B (2x + 3y^2 + 4z^3)\ dV$, where $B = \{(x, y, z) \mid 0 \le x \le 1,\ 0 \le y \le 2,\ 0 \le z \le 3\}$

Answer:

192

18. $\iiint_B (xy + yz + xz)\ dV$, where $B = \{(x, y, z) \mid 1 \le x \le 2,\ 0 \le y \le 2,\ 1 \le z \le 3\}$

19. $\iiint_B (x\ cos\ y + z)\ dV$, where $B = \{(x, y, z) \mid 0 \le x \le 1,\ 0 \le y \le \pi,\ -1 \le z \le 1\}$

Answer:

0

20. $\iiint_B (z\ sin\ x + y^2)\ dV$, where $B = \{(x, y, z) \mid 0 \le x \le \pi,\ 0 \le y \le 1,\ -1 \le z \le 2\}$

In Exercises 21 - 24, evaluate the triple integral.

21. $\int_{-\pi/2}^{\pi/2} \int_0^{\pi} \int_0^{\pi} (\cos x \sin y \sin z)\ dz\ dy\ dx$

Answer:

8

22. $\int_0^1 \int_0^x \int_0^{x+y} (x + y + z)\ dz\ dy\ dx$

23. $\int_0^{\pi} \int_0^1 \int_0^z (\sin(yz))\ dx\ dy\ dz$

Answer:

π

24. $\int_{\pi}^{\pi^2} \int_x^{x^3} \int_{-y^2}^{y^2} (\cos x \sin y \sin z)\ dz\ dy\ dx$

Average Value of a Function

25. Find the average value of the function $f(x, y, z) = x + y + z$ over the parallelepiped determined by $x = 0,\ x = 1,\ y = 0,\ y = 3,\ z = 0$, and $z = 5$.

Answer:

$\frac{9}{2}$

26. Find the average value of the function $f(x, y, z) = xyz$ over the solid $E = [0, 1] \times [0, 1] \times [0, 1]$ situated in the first octant.

Approximating Triple Integrals

27. The midpoint rule for the triple integral $\iiint_B f(x, y, z)\ dV$ over the rectangular solid box B is a generalization of the midpoint rule for double integrals. The region B is divided into subboxes of equal sizes and the integral is approximated by the triple Riemann sum

$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(\bar{x}_i, \bar{y}_j, \bar{z}_k)\Delta V,$$

where $(\bar{x}_i, \bar{y}_j, \bar{z}_k)$ is the center of the box B_{ijk} and ΔV is the volume of each subbox. Apply the midpoint rule to approximate

$$\iiint_B x^2\, dV$$

over the solid $B = \{(x, y, z) \mid 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$ by using a partition of eight cubes of equal size. Round your answer to three decimal places.

Answer:

$$\iiint_B f(x, y, z)\ dV \quad \approx \quad \tfrac{5}{16} \approx 0.313$$

28. [T] a. Apply the midpoint rule to approximate

$$\iiint_B e^{-x^2}\, dV$$

over the solid $B = \{(x, y, z) | 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$ by using a partition of eight cubes of equal size. Round your answer to three decimal places.

b. Use a CAS to improve the above integral approximation in the case of a partition of n^3 cubes of equal size, where $n = 3, 4, ..., 10$.

Applications

29. Suppose that the temperature in degrees Celsius at a point (x, y, z) of a solid E bounded by the coordinate planes and the plane $x + y + z = 5$ is given by:

$$T(x, y, z) = xz + 5z + 10$$

Find the average temperature over the solid.

Answer:

17.5° C

30. Suppose that the temperature in degrees Fahrenheit at a point (x, y, z) of a solid E bounded by the coordinate planes and the plane $x + y + z = 5$ is given by:

$$T(x, y, z) = x + y + xy$$

Find the average temperature over the solid.

31. If the charge density at an arbitrary point (x, y, z) of a solid E is given by the function $\rho(x, y, z)$, then the total charge inside the solid is defined as the triple integral $\iiint_E \rho(x, y, z)\ dV$. Assume that the charge density of the solid E enclosed by the paraboloids $x = 5 - y^2 - z^2$ and $x = y^2 + z^2 - 5$ is equal to the distance from an arbitrary point of E to the origin. Set up the integral that gives the total charge inside the solid E.

Answer:

$$\text{Total Charge inside the Solid } E \quad = \quad \int_{-\sqrt{5}}^{\sqrt{5}} \int_{-\sqrt{5-y^2}}^{\sqrt{5-y^2}} \int_{y^2+z^2-5}^{5-y^2-z^2} \sqrt{x^2 + y^2 + z^2}\ dx\ dz\ dy$$

32. Show that the volume of a regular right hexagonal pyramid of edge length a is $\dfrac{a^3\sqrt{3}}{2}$ by using triple integrals.

Contributors

- Problems 17 - 20 and 25 - 32 are from Section 15.4, OpenStax Calculus 3 by

  Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is

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- Problems 1 - 16 and 21 - 24 are from Apex Calculus, Section 13.6.
- Edited by Paul Seeburger (Monroe Community College)

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14.4E: Triple Integrals (Exercises 2)

Terms and Concepts

1. The strategy for establishing bounds for triple integrals is "from _____ to _____, then from _____ to _____ and then from _____ to _____."

Answer:

We integrate from **surface** to **surface**, then from **curve** to **curve** and then from **point** to **point**.

2. Give an informal interpretation of what $\int\int\int_Q \ dV$ means.

Answer:

$\int\int\int_Q \ dV$ = Volume of the solid region Q

3. Give two uses of triple integration.

Answer:

To compute total mass or average density of a solid object, given a density function or to compute the average temperature in a solid region or object.

4. If an object has a constant density δ and a volume V, what is its mass?

Answer:

It's mass is δV.

Volume of Solid Regions

In Exercises 5-8, two surfaces $f_1(x,y)$ and $f_2(x,y)$ and a region R in the xy-plane are given. Set up and evaluate the triple integral that represents the volume between these surfaces over R.

5. $f_1(x,y) = 8 - x^2 - y^2, \ \ f_2(x,y) = 2x + y;$
R is the square with corners $(-1,-1)$ and $(1,1)$.

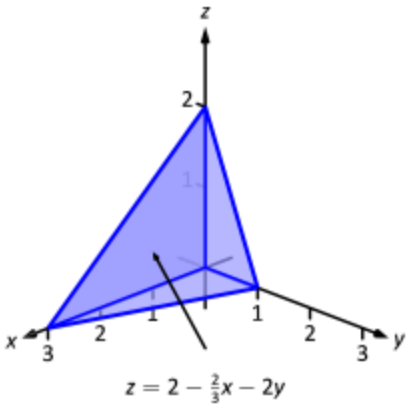
6. $f_1(x,y) = x^2 + y^2, \ \ f_2(x,y) = -x^2 - y^2;$
R is the square with corners $(0,0)$ and $(2,3)$.

7. $f_1(x,y) = \sin x \cos y, \ \ f_2(x,y) = \cos x \sin y + 2;$
R is the triangle with corners $(0,0), \ \ (\pi,0)$ and (π,π).

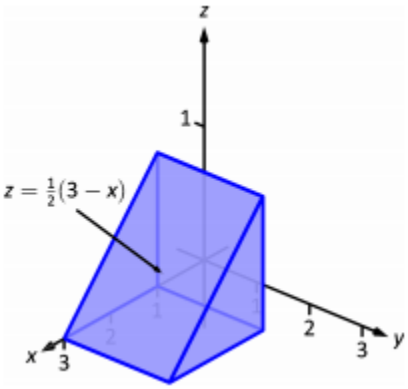
8. $f_1(x,y) = 2x^2 + 2y^2 + 3, \ \ f_2(x,y) = 6 - x^2 - y^2;$
R is the circle $x^2 + y^2 = 1$.

In Exercises 9-16, a domain D is described by its bounding surfaces, along with a graph. Set up the triple integral that gives the volume of D in the indicated order of integration, and evaluate the triple integral to find this volume.

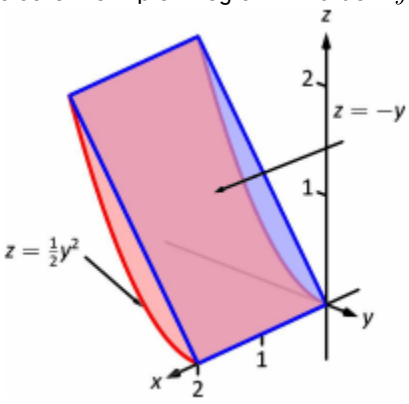
9. D is bounded by the coordinate planes and $z = 2 - \frac{2}{3}x - 2y$.
Evaluate the triple integral with order $dz \ dy \ dx$.



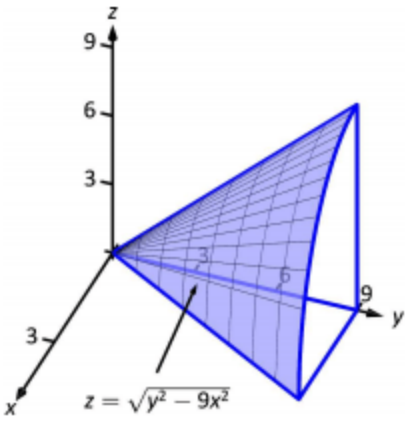
10. D is bounded by the planes $y = 0, y = 2, x = 1, z = 0$ and $z = (2 - x)/2$. Evaluate the triple integral with order $dx\ dy\ dz$.



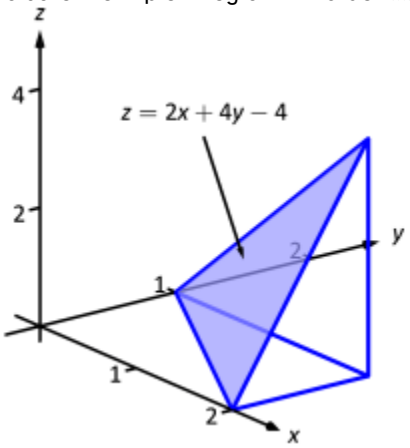
11. D is bounded by the planes $x = 0, x = 2, z = -y$ and by $z = y^2/2$. Evaluate the triple integral with order $dy\ dz\ dx$.



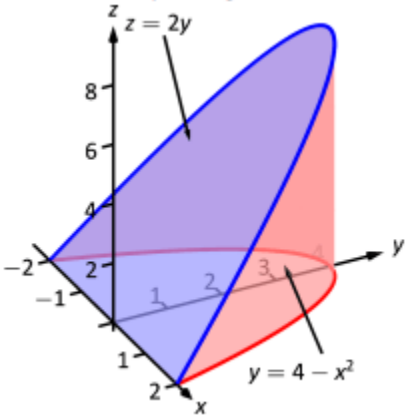
12. D is bounded by the planes $z = 0, y = 9, x = 0$ and by $z = \sqrt{y^2 - 9x^2}$. Do not evaluate any triple integral. Just set this one up.



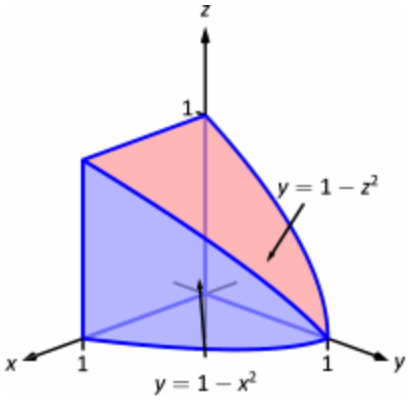
13. D is bounded by the planes $x = 2, y = 1, z = 0$ and $z = 2x + 4y - 4$.
Evaluate the triple integral with order $dx\ dy\ dz$.



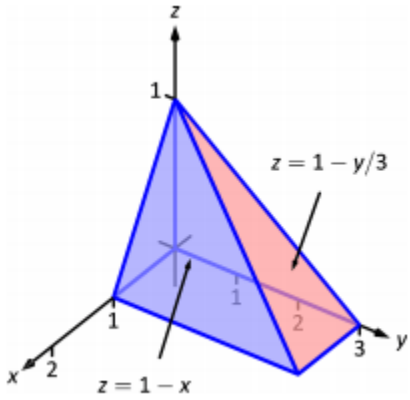
14. D is bounded by the plane $z = 2y$ and by $y = 4 - x^2$.
Evaluate the triple integral with order $dz\ dy\ dx$.



15. D is bounded by the coordinate planes and $y = 1 - x^2$ and $y = 1 - z^2$.
Do not evaluate any triple integral. Which order would be easier to evaluate: $dz\ dy\ dx$ or $dy\ dz\ dx$? Explain why.



16. D is bounded by the coordinate planes and by $z = 1 - y/3$ and $z = 1 - x$.
Evaluate the triple integral with order $dx\ dy\ dz$.



In Exercises 17-20, evaluate the triple integral.

17. $\int_{-\pi/2}^{\pi/2}\int_{0}^{\pi}\int_{0}^{\pi}(\cos x \sin y \sin z)\ dz\ dy\ dx$

18. $\int_{0}^{1}\int_{0}^{x}\int_{0}^{x+y}(x+y+z)\ dz\ dy\ dx$

19. $\int_{0}^{\pi}\int_{0}^{1}\int_{0}^{z}(\sin(yz))\ dx\ dy\ dz$

20. $\int_{\pi}^{\pi^2}\int_{x}^{x^3}\int_{-y^2}^{y^2}(\cos x \sin y \sin z)\ dz\ dy\ dx$

In the following exercises, evaluate the triple integrals over the rectangular solid box B.

$$\iiint_B (2x + 3y^2 + 4z^3)\ dV, \tag{14.4E.1}$$

where $B = \{(x, y, z) | 0 \le x \le 1,\ 0 \le y \le 2,\ 0 \le z \le 3\}$

[Hide Solution]

192

$$\iiint_B (xy + yz + xz)\ dV, \tag{14.4E.2}$$

where $B = \{(x, y, z) | 1 \leq x \leq 2,\ 0 \leq y \leq 2,\ 1 \leq z \leq 3\}$

$$\iiint_B (x\ cos\ y + z)\ dV, \tag{14.4E.3}$$

where $B = \{(x, y, z) | 0 \leq x \leq 1,\ 0 \leq y \leq \pi,\ -1 \leq z \leq 1\}$

[Hide solution]

0

$$\iiint_B (z\ sin\ x + y^2)\ dV, \tag{14.4E.4}$$

where $B = \{(x, y, z) | 0 \leq x \leq \pi,\ 0 \leq y \leq 1,\ -1 \leq z \leq 2\}$

In the following exercises, change the order of integration by integrating first with respect to z, then x, then y.

$$\int_0^1 \int_1^2 \int_2^3 (x^2 + ln\ y + z)\ dx\ dy\ dz \tag{14.4E.5}$$

[Hide Solution]

$$\int_0^1 \int_1^2 \int_2^3 (x^2 + ln\ y + z)\ dx\ dy\ dz = \frac{35}{6} + 2\ ln 2 \tag{14.4E.6}$$

$$\int_0^1 \int_{-1}^1 \int_0^3 (ze^x + 2y)\ dx\ dy\ dz \tag{14.4E.7}$$

$$\int_{-1}^2 \int_1^3 \int_0^4 \left(x^2 z + \frac{1}{y}\right)\ dx\ dy\ dz \tag{14.4E.8}$$

[Hide solution]

$$\int_{-1}^2 \int_1^3 \int_0^4 \left(x^2 z + \frac{1}{y}\right)\ dx\ dy\ dz = 64 + 12\ ln\ 3 \tag{14.4E.9}$$

$$\int_1^2 \int_{-2}^{-1} \int_0^1 \frac{x + y}{z}\ dx\ dy\ dz \tag{14.4E.10}$$

Let F, G, and H be continuous functions on $[a, b]$, $[c, d]$, and $[e, f]$, respectively, where $a,\ b,\ c,\ d,\ e$, and f are real numbers such that $a < b,\ c < d$, and $e < f$. Show that

$$\int_a^b \int_c^d \int_e^f F(x)\ G(y)\ H(z)\ dz\ dy\ dx = \left(\int_a^b F(x)\ dx\right)\left(\int_c^d G(y)\ dy\right)\left(\int_e^f H(z)\ dz\right). \tag{14.4E.11}$$

Let F, G, and H be differential functions on $[a, b]$, $[c, d]$, and $[e, f]$, respectively, where $a,\ b,\ c,\ d,\ e$, and f are real numbers such that $a < b,\ c < d$, and $e < f$. Show that

$$\int_a^b \int_c^d \int_e^f F'(x)\ G'(y)\ H'(z)\ dz\ dy\ dx = [F(b) - F(a)]\ [G(d) - G(c)]\ H(f) - H(e)]. \tag{14.4E.12}$$

In the following exercises, evaluate the triple integrals over the bounded region

$E = \{(x, y, z) | a \leq x \leq b,\ h_1(x) \leq y \leq h_2(x),\ e \leq z \leq f\}$.

$$\iiint_E (2x + 5y + 7z)\ dV, \tag{14.4E.13}$$

where $E = \{(x, y, z) | 0 \leq x \leq 1,\ 0 \leq y \leq -x + 1,\ 1 \leq z \leq 2\}$

[Hide solution]

$\frac{77}{12}$

$$\iiint_E (y\ ln\ x + z)\ dV, \tag{14.4E.14}$$

where $E = \{(x, y, z) | 1 \le x \le e,\ 0 \le yln\ x,\ 0 \le z \le 1\}$

$$\iiint_E (sin\ x + sin\ y) dV, \tag{14.4E.15}$$

where $E = \{(x, y, z) | 0 \le x \le \frac{\pi}{2},\ -cos\ x \le ycos\ x,\ -1 \le z \le 1\}$

[Hide Solution]

2

$$\iiint_E (xy + yz + xz) dV \tag{14.4E.16}$$

where $E = \{(x, y, z) | 0 \le x \le 1,\ -x^2 \le y \le x^2,\ 0 \le z \le 1\}$

In the following exercises, evaluate the triple integrals over the indicated bounded region E.

$$\iiint_E (x + 2yz)\ dV, \tag{14.4E.17}$$

where $E = \{(x, y, z) | 0 \le x \le 1,\ 0 \le y \le x,\ 0 \le z \le 5 - x - y\}$

[Hide Solution]

$\frac{430}{120}$

$$\iiint_E (x^3 + y^3 + z^3)\ dV, \tag{14.4E.18}$$

where $E = \{(x, y, z) | 0 \le x \le 2,\ 0 \le y \le 2x,\ 0 \le z \le 4 - x - y\}$

$$\iiint_E y\ dV, \tag{14.4E.19}$$

where $E = \{(x, y, z) | -1 \le x \le 1,\ -\sqrt{1 - x^2} \le y \le \sqrt{1 - x^2},\ 0 \le z \le 1 - x^2 - y^2\}$

[Hide Solution]

0

$$\iiint_E x\ dV, \tag{14.4E.20}$$

where $E = \{(x, y, z) | -2 \le x \le 2,\ -4\sqrt{1 - x^2} \le y \le \sqrt{4 - x^2},\ 0 \le z \le 4 - x^2 - y^2\}$

In the following exercises, evaluate the triple integrals over the bounded region E of the form

$E = \{(x, y, z) | g_1(y) \le x \le g_2(y),\ c \le y \le d,\ e \le z \le f\}$.

$$\iiint_E x^2\ dV, \tag{14.4E.21}$$

where $E = \{(x, y, z) | 1 - y^2 \le x \le y^2 - 1,\ -1 \le y \le 1,\ 1 \le z \le 2\}$

[Hide Solution]

$-\frac{64}{105}$

$$\iiint_E (sin\ x + y)\ dV, \tag{14.4E.22}$$

where $E = \{(x, y, z) | -y^4 \leq x \leq y^4,\ 0 \leq y \leq 2,\ 0 \leq z \leq 4\}$

$$\iiint_E (x - yz)\ dV, \tag{14.4E.23}$$

where $E = \{(x, y, z) | -y^6 \leq x \leq \sqrt{y},\ 0 \leq y \leq 1x,\ -1 \leq z \leq 1\}$

[Hide Solution]

$\frac{11}{26}$

$$\iiint_E z\ dV, \tag{14.4E.24}$$

where $E = \{(x, y, z) | 2 - 2y \leq x \leq 2 + \sqrt{y},\ 0 \leq y \leq 1x,\ 2 \leq z \leq 3\}$

In the following exercises, evaluate the triple integrals over the bounded region

$E = \{(x, y, z) | g_1(y) \leq x \leq g_2(y),\ c \leq y \leq d,\ u_1(x, y) \leq z \leq u_2(x, y)\}$

$$\iiint_E z\ dV, \tag{14.4E.25}$$

where $E = \{(x, y, z) | -y \leq x \leq y,\ 0 \leq y \leq 1,\ 0 \leq z \leq 1 - x^4 - y^4\}$

[Hide Solution]

$\frac{113}{450}$

$$\iiint_E (xz + 1)\ dV, \tag{14.4E.26}$$

where $E = \{(x, y, z) | 0 \leq x \leq \sqrt{y},\ 0 \leq y \leq 2,\ 0 \leq z \leq 1 - x^2 - y^2\}$

$$\iiint_E (x - z)\ dV, \tag{14.4E.27}$$

where $E = \{(x, y, z) | -\sqrt{1 - y^2} \leq x \leq y,\ 0 \leq y \leq \frac{1}{2}x,\ 0 \leq z \leq 1 - x^2 - y^2\}$

[Hide Solution]

$\frac{1}{160}(6\sqrt{3} - 41)$

$$\iiint_E (x + y)\ dV, \tag{14.4E.28}$$

where $E = \{(x, y, z) | 0 \leq x \leq \sqrt{1 - y^2},\ 0 \leq y \leq 1x,\ 0 \leq z \leq 1 - x\}$

In the following exercises, evaluate the triple integrals over the bounded region

$E = \{(x, y, z) | (x, y)^{\text{int}} D,\ u_1(x, y)x \leq z \leq u_2(x, y)\}$, where D is the projection of E onto the xy-plane

$$\iint_D \left(\int_1^2 (x + y)\ dz \right)\ dA, \tag{14.4E.29}$$

where $D = \{(x, y) | x^2 + y^2 \leq 1\}$

[Hide Solution]

$\frac{3\pi}{2}$

$$\iint_D \left(\int_1^3 x(z + 1)\ dz \right)\ dA, \tag{14.4E.30}$$

where $D = \{(x, y) | x^2 - y^2 \geq 1,\ x \leq \sqrt{5}\}$

$$\iint_D \left(\int_0^{10-x-y} (x+2z)\ dz \right)\ dA, \tag{14.4E.31}$$

where $D = \{(x,y) | y \geq 0,\ x \geq 0,\ x+y \leq 10\}$

[Hide Solution]

1250

$$\iint_D \left(\int_0^{4x^2+4y^2} y\ dz \right)\ dA, \tag{14.4E.32}$$

where $D = \{(x,y) | x^2+y^2 \leq 4,\ y \geq 1,\ x \geq 0\}$

The solid E bounded by $y^2+z^2=9,\ z=0$, and $x=5$ is shown in the following figure. Evaluate the integral

$$\iiint_E z\ dV \tag{14.4E.33}$$

by integrating first with respect to z, then y, and then x.

A solid arching shape that reaches its maximum along the y axis with z = 3. The shape reach zero at y = plus or minus 3, and the graph is truncated at x = 0 and 5.

[Hide Solution]

$$\int_0^5 \int_{-3}^3 \int_0^{\sqrt{9-y^2}} z\ dz\ dy\ dx = 90 \tag{14.4E.34}$$

The solid E bounded by $y=\sqrt{x},\ x=4,\ y=0$, and $z=1$ is given in the following figure. Evaluate the integral

$$\iiint_E xyz\ dV \tag{14.4E.35}$$

by integrating first with respect to x, then y, and then z.

A quarter section of an oval cylinder with z from negative 2 to positive 1. The solid is bounded by y = 0 and x = 4, and the top of the shape runs from (0, 0, 1) to (4, 2, 1) in a gentle arc.

[T] The volume of a solid E is given by the integral

$$\int_{-2}^0 \int_x^0 \int_0^{x^2+y^2} dz\ dy\ dx. \tag{14.4E.36}$$

Use a computer algebra system (CAS) to graph E and find its volume. Round your answer to two decimal places.

[Hide Solution]

$V = 5.33$

A complex shape that starts at the origin and reaches its maximum at (negative 2, negative 2, 8). The shape is truncated by the x = y plane, the x = 0 plane, the y = negative 2 plane, the z = 0 plane, and a complex triangular-like shape with curved edges and sides (negative 2, negative 2, 8), (0, 0, 0), and (0, negative 2, 4).

[T] The volume of a solid E is given by the integral

$$\int_{-1}^0 \int_{-x^3}^0 \int_0^{1+\sqrt{x^2+y^2}} dz\ dy\ dx. \tag{14.4E.37}$$

Use a CAS to graph E and find its volume V. Round your answer to two decimal places.

In the following exercises, use two circular permutations of the variables $x,\ y$, and z to write new integrals whose values equal the value of the original integral. A circular permutation of $x,\ y$, and z is the arrangement of the numbers in one of the following orders: $y,\ z$, and x or $z,\ x$, and y.

$$\int_0^1 \int_1^3 \int_2^4 (x^2z^2+1)dx\ dy\ dz \tag{14.4E.38}$$

[Hide Solution]

$$\int_0^1 \int_1^3 \int_2^4 (y^2z^2+1)dz\ dx\ dy; \tag{14.4E.39}$$

$$\int_0^1 \int_1^3 \int_2^4 (x^2z^2+1)dx\ dy\ dz \tag{14.4E.40}$$

$$\int_0^3 \int_0^1 \int_0^{-x+1} (2x+5y+7z)dy\ dx\ dz \tag{14.4E.41}$$

$$\int_0^1 \int_{-y}^{y} \int_0^{1-x^4-y^4} ln\ xdz\ dx\ dy \tag{14.4E.42}$$

$$\int_{-1}^1 \int_0^1 \int_{-y^6}^{\sqrt{y}} (x+yz)dx\ dy\ dz \tag{14.4E.43}$$

Set up the integral that gives the volume of the solid E bounded by $y^2 = x^2 + z^2$ and $y = a^2$, where $a > 0$.

[Hide Solution]

$$V = \int_{-a}^{a} \int_{-\sqrt{a^2-z^2}}^{\sqrt{a^2-z^2}} \int_{\sqrt{x^2+z^2}}^{a^2} dy\ dx\ dz \tag{14.4E.44}$$

Set up the integral that gives the volume of the solid E bounded by $x = y^2 + z^2$ and $x = a^2$, where $a > 0$.

Find the average value of the function $f(x, y, z) = x + y + z$ over the parallelepiped determined by $x + 0,\ x = 1,\ y = 0,\ y = 3,\ z = 0$, and $z = 5$.

[Hide Solution]

$\frac{9}{2}$

Find the average value of the function $f(x, y, z) = xyz$ over the solid $E = [0, 1] \times [0, 1] \times [0, 1]$ situated in the first octant.

Find the volume of the solid E that lies under the plane $x + y + z = 9$ and whose projection onto the xy-plane is bounded by $x = sin^{-1}y,\ y = 0$, and $x = \frac{\pi}{2}$.

Consider the pyramid with the base in the xy-plane of $[-2, 2] \times [-2, 2]$ and the vertex at the point $(0, 0, 8)$.

a. Show that the equations of the planes of the lateral faces of the pyramid are $4y + z = 8,\ 4y - z = -8,\ 4x + z = 8$, and $-4x + z = 8$.

b. Find the volume of the pyramid.

[Hide Solution]

a. Answers may vary; b. $\frac{128}{3}$

Consider the pyramid with the base in the xy-plane of $[-3, 3] \times [-3, 3]$ and the vertex at the point $(0, 0, 9)$.

a. Show that the equations of the planes of the side faces of the pyramid are $3y + z = 9,\ 3y + z = 9,\ y = 0$ and $x = 0$.

b. Find the volume of the pyramid.

The solid E bounded by the sphere of equation $x^2 + y^2 + z^2 = r^2$ with $r > 0$ and located in the first octant is represented in the following figure.

The eighth of a sphere of radius 2 with center at the origin for positive x, y, and z.

a. Write the triple integral that gives the volume of E by integrating first with respect to z, then with y, and then with x.

b. Rewrite the integral in part a. as an equivalent integral in five other orders.

[Hide Solution]

$$a.\ \int_0^4 \int_0^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2-y^2}} dz\ dy\ dx;\ b.\ \int_0^2 \int_0^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2-y^2}} dz\ dx\ dy, \tag{14.4E.45}$$

$$\int_0^r \int_0^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2-y^2}} dy\ dx\ dz,\ \int_0^r \int_0^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2-y^2}} dy\ dz\ dx, \tag{14.4E.46}$$

$$\int_0^r \int_0^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2-y^2}} dx\ dy\ dz,\ \int_0^r \int_0^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2-y^2}} dx\ dz\ dy, \tag{14.4E.47}$$

The solid E bounded by the sphere of equation $9x^2 + 4y^2 + z^2 = 1$ and located in the first octant is represented in the following figure.

In the first octant, a complex shape is shown that is roughly a solid ovoid with center the origin, height 1, width 0.5, and length 0.35.

a. Write the triple integral that gives the volume of E by integrating first with respect to z then with y and then with x.

b. Rewrite the integral in part a. as an equivalent integral in five other orders.

Find the volume of the prism with vertices $(0,0,0),\ (2,0,0),\ (2,3,0),\ (0,3,0),\ (0,0,1)$, and $(2,0,1)$.

[Hide Solution]

3

Find the volume of the prism with vertices $(0,0,0),\ (4,0,0),\ (4,6,0),\ (0,6,0),\ (0,0,1)$, and $(4,0,1)$.

The solid E bounded by $z = 10 - 2x - y$ and situated in the first octant is given in the following figure. Find the volume of the solid.

A tetrahedron bounded by the x y, y z, and x z planes and a triangle with vertices (0, 0, 10), (5, 0, 0), and (0, 10, 0).

[Hide Solution]

$\frac{250}{3}$

The solid E bounded by $z = 1 - x^2$ and situated in the first octant is given in the following figure. Find the volume of the solid.

A complex shape in the first octant with height 1, width 5, and length 1. The shape appears to be a slightly deformed quarter of a cylinder of radius 1 and width 5.

The midpoint rule for the triple integral

$$\iiint_B f(x,y,z)dV \tag{14.4E.48}$$

over the rectangular solid box B is a generalization of the midpoint rule for double integrals. The region B is divided into subboxes of equal sizes and the integral is approximated by the triple Riemann sum

$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(\bar{x}_i, \bar{y}_j, \bar{z}_k)\Delta V, \tag{14.4E.49}$$

where $(\bar{x}_i, \bar{y}_j, \bar{z}_k)$ is the center of the box B_{ijk} and ΔV is the volume of each subbox. Apply the midpoint rule to approximate

$$\iiint_B x^2 dV \tag{14.4E.50}$$

over the solid $B = \{(x, y, z) | 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ 0 \leq z \leq 1\}$ by using a partition of eight cubes of equal size. Round your answer to three decimal places.

[Hide Solution]

$\frac{5}{16} \approx 0.313$

[T]

a. Apply the midpoint rule to approximate

$$\iiint_B e^{-x^2} dV \tag{14.4E.51}$$

over the solid $B = \{(x, y, z) | 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ 0 \leq z \leq 1\}$ by using a partition of eight cubes of equal size. Round your answer to three decimal places.

b. Use a CAS to improve the above integral approximation in the case of a partition of n^3 cubes of equal size, where $n = 3, 4, ..., 10$.

Suppose that the temperature in degrees Celsius at a point (x, y, z) of a solid E bounded by the coordinate planes and $x + y + z = 5$ is $T(x, y, z) = xz + 5z + 10$. Find the average temperature over the solid.

[Hide Solution]

$\frac{35}{2}$

Suppose that the temperature in degrees Fahrenheit at a point (x, y, z) of a solid E bounded by the coordinate planes and $x + y + z = 5$ is $T(x, y, z) = x + y + xy$. Find the average temperature over the solid.

Show that the volume of a right square pyramid of height h and side length a is $v = \frac{ha^2}{3}$ by using triple integrals.

Show that the volume of a regular right hexagonal prism of edge length a is $\frac{3a^3\sqrt{3}}{2}$ by using triple integrals.

Show that the volume of a regular right hexagonal pyramid of edge length a is $\frac{a^3\sqrt{3}}{2}$ by using triple integrals.

If the charge density at an arbitrary point (x, y, z) of a solid E is given by the function $\rho(x, y, z)$, then the total charge inside the solid is defined as the triple integral

$$\iiint_E \rho(x, y, z) dV. \tag{14.4E.52}$$

Assume that the charge density of the solid E enclosed by the paraboloids $x = 5 - y^2 - z^2$ and $x = y^2 + z^2 - 5$ is equal to the distance from an arbitrary point of E to the origin. Set up the integral that gives the total charge inside the solid E.

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14.5: Triple Integrals in Cylindrical and Spherical Coordinates

Learning Objectives

- Evaluate a triple integral by changing to cylindrical coordinates.
- Evaluate a triple integral by changing to spherical coordinates.

Earlier in this chapter we showed how to convert a double integral in rectangular coordinates into a double integral in polar coordinates in order to deal more conveniently with problems involving circular symmetry. A similar situation occurs with triple integrals, but here we need to distinguish between cylindrical symmetry and spherical symmetry. In this section we convert triple integrals in rectangular coordinates into a triple integral in either cylindrical or spherical coordinates.

Also recall the chapter prelude, which showed the opera house l'Hemisphèric in Valencia, Spain. It has four sections with one of the sections being a theater in a five-story-high sphere (ball) under an oval roof as long as a football field. Inside is an IMAX screen that changes the sphere into a planetarium with a sky full of 9000 twinkling stars. Using triple integrals in spherical coordinates, we can find the volumes of different geometric shapes like these.

Review of Cylindrical Coordinates

As we have seen earlier, in two-dimensional space $\mathbb{R}^2$ a point with rectangular coordinates (x, y) can be identified with (r, θ) in polar coordinates and vice versa, where $x = r\ \cos\theta$, $y = r\ \sin\ \theta$, $r^2 = x^2 + y^2$ and $\tan\ \theta = \frac{y}{x}$ are the relationships between the variables.

In three-dimensional space $\mathbb{R}^3$ a point with rectangular coordinates (x, y, z) can be identified with cylindrical coordinates (r, θ, z) and vice versa. We can use these same conversion relationships, adding z as the vertical distance to the point from the $(xy$-plane as shown in $14.5.1$.

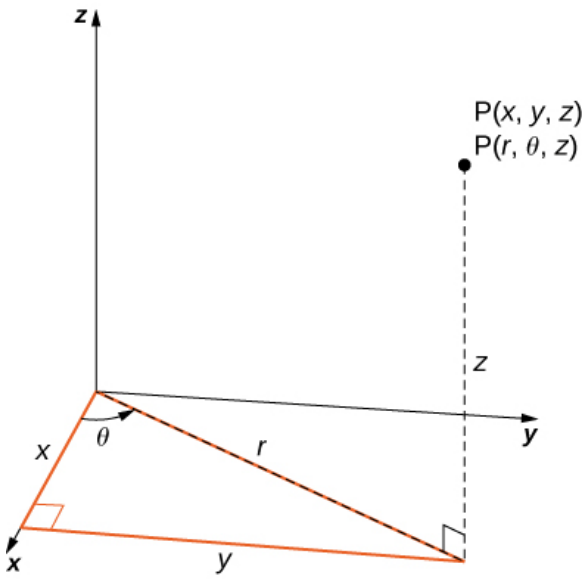


Figure $14.5.1$: Cylindrical coordinates are similar to polar coordinates with a vertical z coordinate added.

To convert from rectangular to cylindrical coordinates, we use the conversion

- $x = r\ \cos\theta$
- $y = r\ \sin\ \theta$

- $z = z$

To convert from cylindrical to rectangular coordinates, we use

- $r^2 = x^2 + y^2$ and
- $\theta = \tan^{-1}\left(\frac{y}{x}\right)$
- $z = z$

Note that that z-coordinate remains the same in both cases.

In the two-dimensional plane with a rectangular coordinate system, when we say $x = k$ (constant) we mean an unbounded vertical line parallel to the y-axis and when $y = l$ (constant) we mean an unbounded horizontal line parallel to the x-axis. With the polar coordinate system, when we say $r = c$ (constant), we mean a circle of radius c units and when $\theta = \alpha$ (constant) we mean an infinite ray making an angle α with the positive x-axis.

Similarly, in three-dimensional space with rectangular coordinates (x, y, z) the equations $x = k,\ \ y = l$ and $z = m$ where $k,\ \ l$ and m are constants, represent unbounded planes parallel to the yz-plane, xz-plane and xy-plane, respectively. With cylindrical coordinates (r, θ, z), by $r = c,\ \ \theta = \alpha$, and $z = m$, where c, α, and m are constants, we mean an unbounded vertical cylinder with the z-axis as its radial axis; a plane making a constant angle α with the xy-plane; and an unbounded horizontal plane parallel to the xy-plane, respectively. This means that the circular cylinder $x^2 + y^2 = c^2$ in rectangular coordinates can be represented simply as $r = c$ in cylindrical coordinates. (Refer to Cylindrical and Spherical Coordinates for more review.)

Integration in Cylindrical Coordinates

Triple integrals can often be more readily evaluated by using cylindrical coordinates instead of rectangular coordinates. Some common equations of surfaces in rectangular coordinates along with corresponding equations in cylindrical coordinates are listed in Table 14.5.1. These equations will become handy as we proceed with solving problems using triple integrals.

Table 14.5.1: Equations of Some Common Shapes

	Circular cylinder	Circular cone	Sphere	Paraboloid
Rectangular	$x^2 + y^2 = c^2$	$z^2 = c^2(x^2 + y^2)$	$x^2 + y^2 + z^2 = c^2$	$z = c(x^2 + y^2)$
Cylindrical	$r = c$	$z = cr$	$r^2 + z^2 = c^2$	$z = cr^2$

As before, we start with the simplest bounded region B in $\mathbb{R}^3$ to describe in cylindrical coordinates, in the form of a cylindrical box, $B = \{(r, \theta, z) | a \le r \le b,\ \ \alpha \le \theta \le \beta,\ \ c \le z \le d\}$ (Figure 14.5.2). Suppose we divide each interval into $l,\ \ m$, and n subdivisions such that $\Delta r = \frac{b \cdot a}{l},\ \ \Delta\theta = \frac{\beta \cdot \alpha}{m}$, and $\Delta z = \frac{d \cdot c}{n}$. Then we can state the following definition for a triple integral in cylindrical coordinates.

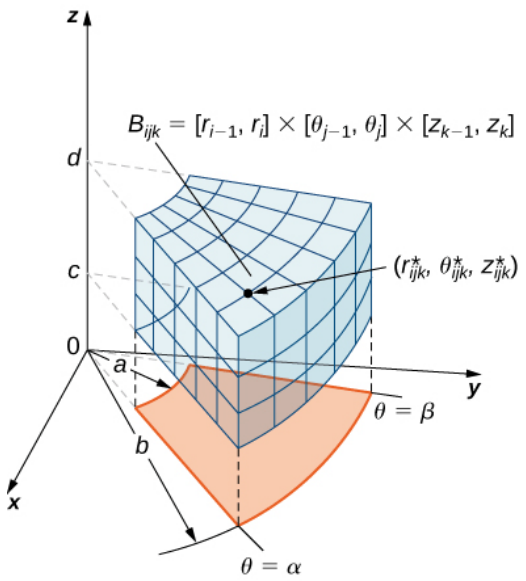


Figure 14.5.2: A cylindrical box B described by cylindrical coordinates.

DEFINITION: triple integral in cylindrical coordinates

Consider the cylindrical box (expressed in cylindrical coordinates)

$$B = \{(r, \theta, z) | a \le r \le b,\ \alpha \le \theta \le \beta,\ c \le z \le d\}.$$

If the function $f(r, \theta, z)$ is continuous on B and if $(r^*_{ijk}, \theta^*_{ijk}, z^*_{ijk})$ is any sample point in the cylindrical subbox $B_{ijk} = |r_{i-1}, r_i| \times |\theta_{j-1}, \theta_j| \times |z_{k-1}, k_i|$ (Figure 14.5.2), then we can define the *triple integral in cylindrical coordinates* as the limit of a triple Riemann sum, provided the following limit exists:

$$\lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(r^*_{ijk}, \theta^*_{ijk}, z^*_{ijk})\, \Delta r \Delta\theta \Delta z.$$

Note that if $g(x, y, z)$ is the function in rectangular coordinates and the box B is expressed in rectangular coordinates, then the triple integral

$$\iiint_B g(x, y, z)\, dV$$

is equal to the triple integral

$$\iiint_B g(r\, \cos\theta,\ r\, \sin\,\theta,\ z) r\, dr\, d\theta\, dz$$

and we have

$$\iiint_B g(x, y, z)\, dV = \iiint_B g(r\, \cos\theta,\ r\, \sin\,\theta,\ z)\, r\, dr\, d\theta\, dz = \iiint_B f(r, \theta\, z)\, r\, dr\, d\theta\, dz.$$

As mentioned in the preceding section, all the properties of a double integral work well in triple integrals, whether in rectangular coordinates or cylindrical coordinates. They also hold for iterated integrals. To reiterate, in cylindrical coordinates, Fubini's theorem takes the following form:

Theorem: Fubini's Theorem in Cylindrical Coordinates

Suppose that $g(x,y,z)$ is continuous on a rectangular box B which when described in cylindrical coordinates looks like $B = \{(r,\theta,z)|a \le r \le b,\ \alpha \le \theta \le \beta,\ c \le z \le d\}$.

Then $g(x,y,z) = g(r\ \cos\theta, r\ \sin\ \theta, z) = f(r,\theta,z)$ and

$$\iiint_B g(x,y,z)\,dV = \int_c^d \int_\beta^\alpha \int_a^b f(r,\theta,z)\,r\,dr\,d\theta\,dz.$$

The iterated integral may be replaced equivalently by any one of the other five iterated integrals obtained by integrating with respect to the three variables in other orders.

Cylindrical coordinate systems work well for solids that are symmetric around an axis, such as cylinders and cones. Let us look at some examples before we define the triple integral in cylindrical coordinates on general cylindrical regions.

Example 14.5.1: Evaluating a Triple Integral over a Cylindrical Box

Evaluate the triple integral

$$\iiint_B (zr\ \sin\ \theta) r\,dr\,d\theta\,dz$$

where the cylindrical box B is $B = \{(r,\theta,z)|0 \le r \le 2,\ 0 \le \theta \le \pi/2,\ 0, \le z \le 4\}$.

Solution

As stated in Fubini's theorem, we can write the triple integral as the iterated integral

$$\iiint_B (zr\ \sin\ \theta) r\,dr\,d\theta\,dz = \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=2} \int_{z=0}^{z=4} (zr\ \sin\ \theta)\,r\,dz\,dr\,d\theta.$$

The evaluation of the iterated integral is straightforward. Each variable in the integral is independent of the others, so we can integrate each variable separately and multiply the results together. This makes the computation much easier:

$$\int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=2} \int_{z=0}^{z=4} (zr\ \sin\ \theta)\,r\,dz\,dr\,d\theta = \left(\int_0^{\pi/2} \sin\,\theta\,d\theta\right)\left(\int_0^2 r^2 dr\right)\left(\int_0^4 z\,dz\right) = \left(-\cos\theta\Big|_0^{\pi/2}\right)\left(\frac{r^3}{3}\Big|_0^2\right)\left(\frac{z^2}{2}\Big|_0^4\right) = \frac{64}{3}.$$

Exercise 14.5.1:

Evaluate the triple integral

$$\int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=1} \int_{z=0}^{z=4} (rz\ \sin\ \theta)\,r\,dz\,dr\,d\theta.$$

Hint available at the end of this section.

If the cylindrical region over which we have to integrate is a general solid, we look at the projections onto the coordinate planes. Hence the triple integral of a continuous function $f(r,\theta,z)$ over a general solid region $E = \{(r,\theta,z)|(r,\theta)^{\text{int}} D, u_1(r,\theta) \le z \le u_2(r,\theta)\}$ in $\mathbb{R}^3$ where D is the projection of E onto the $r\theta$-plane, is

$$\iiint_E f(r,\theta,z)\,r\,dr\,d\theta\,dz = \iint_D \left[\int_{u_1(r,\theta)}^{u_2(r,\theta)} f(r,\theta,z)dz\right] r\,dr\,d\theta.$$

In particular, if $D = \{(r,\theta)|g_1(\theta) \le r \le g_2(\theta), \alpha \le \theta \le \beta\}$, then we have

$$\iiint_E f(r,\theta,z)\,r\,dr\,d\theta = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=g_1(\theta)}^{r=g_2(\theta)}\int_{z=u_1(r,\theta)}^{z=u_2(r,\theta)} f(r,\theta,z)\,r\,dz\,dr\,d\theta.$$

Similar formulas exist for projections onto the other coordinate planes. We can use polar coordinates in those planes if necessary.

Example 14.5.2: Setting up a Triple Integral in Cylindrical Coordinates over a General Region

Consider the region E inside the right circular cylinder with equation $r = 2\ \sin\ \theta$, bounded below by the $r\theta$-plane and bounded above by the sphere with radius 4 centered at the origin (Figure 15.5.3). Set up a triple integral over this region with a function $f(r,\theta,z)$ in cylindrical coordinates.

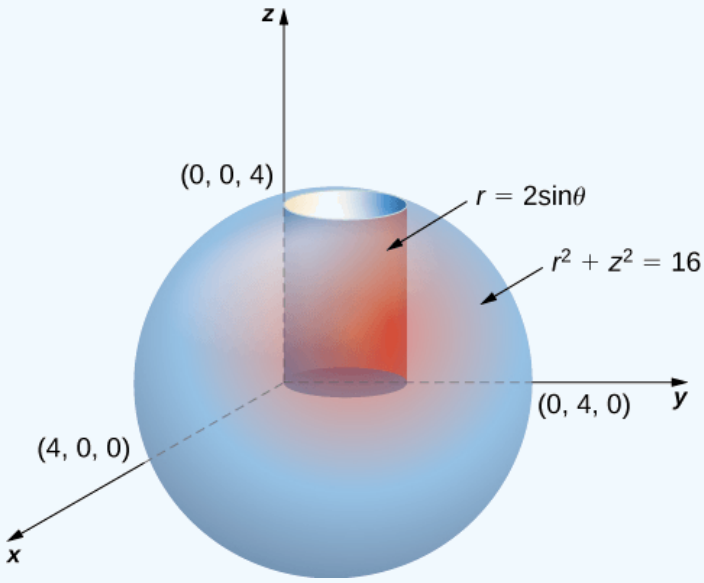


Figure 14.5.3: Setting up a triple integral in cylindrical coordinates over a cylindrical region.

Solution

First, identify that the equation for the sphere is $r^2 + z^2 = 16$. We can see that the limits for z are from 0 to $z = \sqrt{16 - r^2}$. Then the limits for r are from 0 to $r = 2\ \sin\ \theta$. Finally, the limits for θ are from 0 to π. Hence the region is $E = \{(r,\theta,z)|0 \le \theta \le \pi,\ 0 \le r \le 2\ \sin\ \theta,\ 0 \le z \le \sqrt{16 - r^2}\}$. Therefore, the triple integral is

$$\iiint_E f(r,\theta,z)\,r\,dz\,dr\,d\theta = \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=2\ \sin\ \theta}\int_{z=0}^{z=\sqrt{16-r^2}} f(r,\theta,z)\,r\,dz\,dr\,d\theta.$$

Exercise 14.5.2:

Consider the region inside the right circular cylinder with equation $r = 2\ \sin\ \theta$ bounded below by the $r\theta$-plane and bounded above by $z = 4 - y$. Set up a triple integral with a function $f(r,\theta,z)$ in cylindrical coordinates.

Hint available at the end of this section.

Example 14.5.3: Setting up a Triple Integral in Two Ways

Let E be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$. (Figure 15.5.4). Set up a triple integral in cylindrical coordinates to find the volume of the region, using the following orders of integration:

a. $dz\ dr\ d\theta$

b. $dr\ dz\ d\theta$

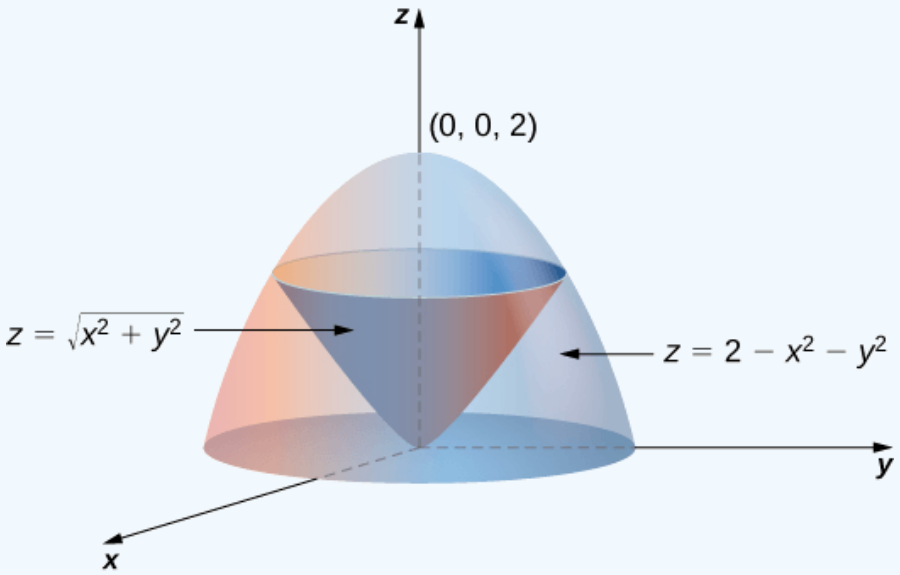


Figure 14.5.4: Setting up a triple integral in cylindrical coordinates over a conical region.

Solution

a. The cone is of radius 1 where it meets the paraboloid. Since $z = 2 - x^2 - y^2 = 2 - r^2$ and $z = \sqrt{x^2 + y^2} = r^2$ (assuming r is nonnegative), we have $2 - r^2 = r$. Solving, we have $r^2 + r - 2 = (r + 2)(r - 1) = 0$. Since $r \geq 0$, we have $r = 1$. Therefore $z = 1$. So the intersection of these two surfaces is a circle of radius 1 in the plane $z = 1$. The cone is the lower bound for z and the paraboloid is the upper bound. The projection of the region onto the xy-plane is the circle of radius 1 centered at the origin.

Thus, we can describe the region as $E = \{(r, \theta, z) | 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 1,\ r \leq z \leq 2 - r^2\}$.

Hence the integral for the volume is

$$V = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=r}^{z=2-r^2} r\, dz\, dr\, d\theta.$$

b. We can also write the cone surface as $r = z$ and the paraboloid as $r^2 = 2 - z$. The lower bound for r is zero, but the upper bound is sometimes the cone and the other times it is the paraboloid. The plane $z = 1$ divides the region into two regions. Then the region can be described as

$$E = \{(r, \theta, z) | 0 \leq \theta \leq 2\pi,\ 0 \leq z \leq 1,\ 0 \leq r \leq z\} \cup \{(r, \theta, z) | 0 \leq \theta \leq 2\pi,\ 1 \leq z \leq 2,\ 0 \leq r \leq \sqrt{2 - z}\}.$$

Now the integral for the volume becomes

$$V = \int_{\theta=0}^{\theta=2\pi} \int_{z=0}^{z=1} \int_{r=0}^{r=z} r\, dr\, dz\, d\theta + \int_{\theta=0}^{\theta=2\pi} \int_{z=1}^{z=2} \int_{r=0}^{r=\sqrt{2-z}} r\, dr\, dz\, d\theta.$$

Exercise 14.5.3:

Redo the previous example with the order of integration $d\theta\ dz\ dr$.

Hint available at the end of this section.

Example 14.5.4: Finding a Volume with Triple Integrals in Two Ways

Let *E* be the region bounded below by the $r\theta$-plane, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$ (Figure 15.5.5). Set up a triple integral in cylindrical coordinates to find the volume of the region using the following orders of integration, and in each case find the volume and check that the answers are the same:

a. $dz\ dr\ d\theta$
b. $dr\ dz\ d\theta$.

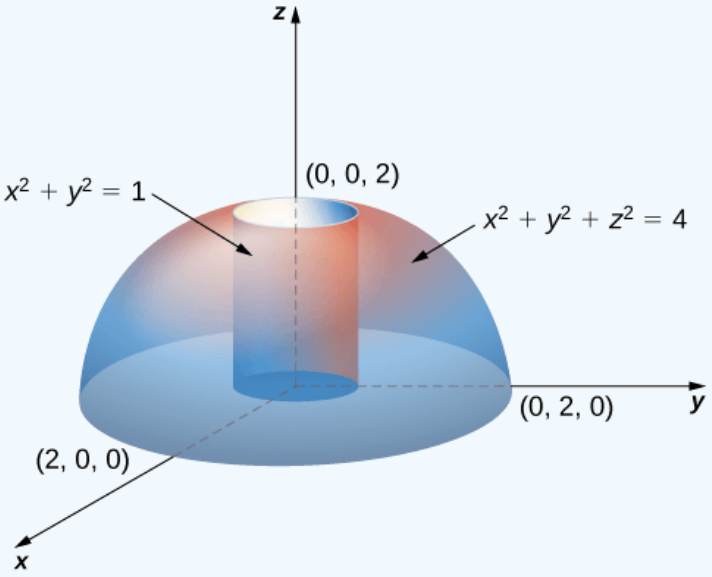


Figure 14.5.5: Finding a cylindrical volume with a triple integral in cylindrical coordinates.

Solution

a. Note that the equation for the sphere is

$$x^2 + y^2 + z^2 = 4 \text{ or } r^2 + z^2 = 4$$

and the equation for the cylinder is

$$x^2 + y^2 = 1 \text{ or } r^2 = 1.$$

Thus, we have for the region E

$$E = \{(r, \theta, z) | 0 \le z \le \sqrt{4 - r^2},\ 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$$

Hence the integral for the volume is

$$\begin{aligned} V(E) &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=0}^{z=\sqrt{4-r^2}} r\, dz\, dr\, d\theta \\ &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \left[rz \Big|_{z=0}^{z=\sqrt{4-r^2}} \right] dr\, d\theta \\ &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \left(r\sqrt{4 - r^2} \right) dr\, d\theta \\ &= \int_0^{2\pi} \left(\frac{8}{3} - \sqrt{3} \right) d\theta = 2\pi \left(\frac{8}{3} - \sqrt{3} \right) \text{ cubic units.} \end{aligned}$$

b. Since the sphere is $x^2 + y^2 + z^2 = 4$, which is $r^2 + z^2 = 4$, and the cylinder is $x^2 + y^2 = 1$, which is $r^2 = 1$, we have $1 + z^2 = 4$, that is, $z^2 = 3$. Thus we have two regions, since the sphere and the cylinder intersect at $(1, \sqrt{3})$ in the rz-plane

$$E_1 = \{(r, \theta, z) | 0 \leq r \leq \sqrt{4 - r^2},\ \sqrt{3} \leq z \leq 2,\ 0 \leq \theta \leq 2\pi\}$$

and

$$E_2 = \{(r, \theta, z) | 0 \leq r \leq 1,\ 0 \leq z \leq \sqrt{3},\ 0 \leq \theta \leq 2\pi\}.$$

Hence the integral for the volume is

$$\begin{aligned} V(E) &= \int_{\theta=0}^{\theta=2\pi} \int_{z=\sqrt{3}}^{z=2} \int_{r=0}^{r=\sqrt{4-r^2}} r\, dr\, dz\, d\theta + \int_{\theta=0}^{\theta=2\pi} \int_{z=0}^{z=\sqrt{3}} \int_{r=0}^{r=1} r\, dr\, dz\, d\theta \\ &= \sqrt{3}\pi + \left(\frac{16}{3} - 3\sqrt{3}\right)\pi \\ &= 2\pi\left(\frac{8}{3} - \sqrt{3}\right) \text{ cubic units.} \end{aligned}$$

Exercise 14.5.4

Redo the previous example with the order of integration $d\theta\ dz\ dr$.

Hint available at the end of this section.

Review of Spherical Coordinates

In three-dimensional space $\mathbb{R}^3$ in the spherical coordinate system, we specify a point P by its distance ρ from the origin, the polar angle θ from the positive x-axis (same as in the cylindrical coordinate system), and the angle φ from the positive z-axis and the line OP (Figure 14.5.6). Note that $\rho > 0$ and $0 \leq \varphi \leq \pi$. (Refer to Cylindrical and Spherical Coordinates for a review.) Spherical coordinates are useful for triple integrals over regions that are symmetric with respect to the origin.

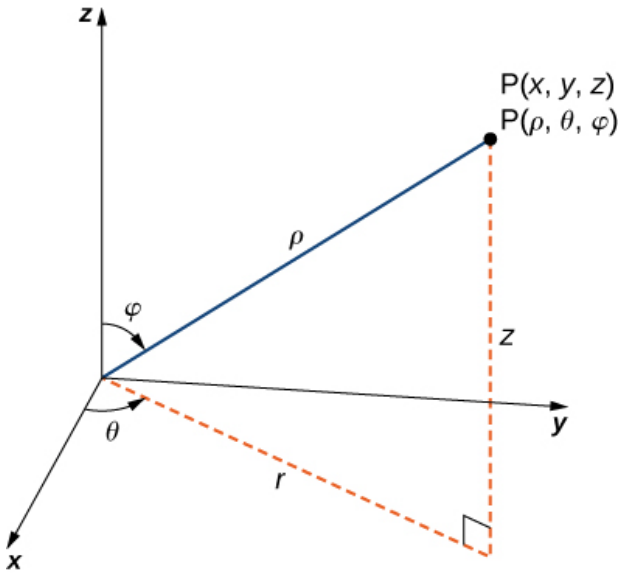


Figure 14.5.6: The spherical coordinate system locates points with two angles and a distance from the origin.

Recall the relationships that connect rectangular coordinates with spherical coordinates.

From spherical coordinates to rectangular coordinates:

$$x = \rho \sin \varphi \cos\theta,\ y = \rho \sin \varphi \sin \theta,\ and\, z = \rho \cos \varphi.$$

From rectangular coordinates to spherical coordinates:

$$\rho^2 = x^2 + y^2 + z^2,\ \tan\theta = \frac{y}{x},\ \varphi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right).$$

Other relationships that are important to know for conversions are

- $r = \rho\ \sin\ \varphi$
- $\theta = \theta$ These equations are used to convert from spherical coordinates to cylindrical coordinates.
- $z = \rho\ \cos\ \varphi$

and

- $\rho = \sqrt{r^2 + z^2}$
- $\theta = \theta$ These equations are used to convert from cylindrical coordinates to spherical coordinates.
- $\varphi = \arccos\left(\frac{z}{\sqrt{r^2+z^2}}\right)$

14.5.7 shows a few solid regions that are convenient to express in spherical coordinates.

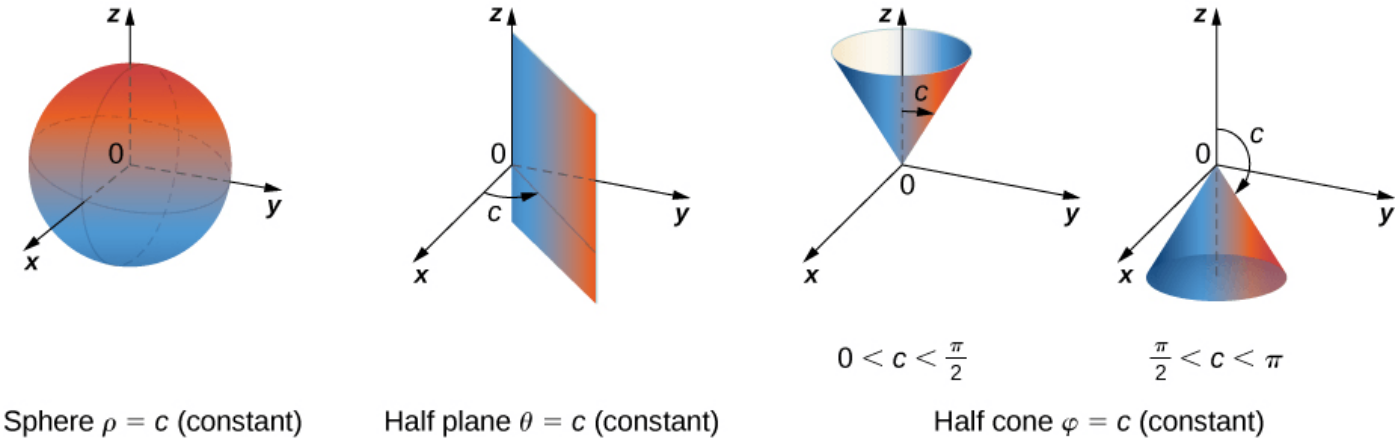


Figure 14.5.7: Spherical coordinates are especially convenient for working with solids bounded by these types of surfaces. (The letter c indicates a constant.)

Integration in Spherical Coordinates

We now establish a triple integral in the spherical coordinate system, as we did before in the cylindrical coordinate system. Let the function $f(\rho, \theta, \varphi)$ be continuous in a bounded spherical box, $B = \{(\rho, \theta, \varphi) | a \le \rho \le b,\ \alpha \le \theta \le \beta,\ \gamma \le \varphi \le \psi\}$. We then divide each interval into l, m, n and n subdivisions such that $\Delta\rho = \frac{b-a}{l},\ \Delta\theta = \frac{\beta-\alpha}{m}.\ \Delta\varphi = \frac{\psi-\gamma}{n}$. Now we can illustrate the following theorem for triple integrals in spherical coordinates with $(\rho^*_{ijk}, \theta^*_{ijk}, \varphi^*_{ijk})$ being any sample point in the spherical subbox B_{ijk}. For the volume element of the subbox ΔV in spherical coordinates, we have $\Delta V = (\Delta\rho)\ (\rho\Delta\varphi)\ (\rho\ \sin\ \varphi\ \Delta\theta)$, as shown in the following figure.

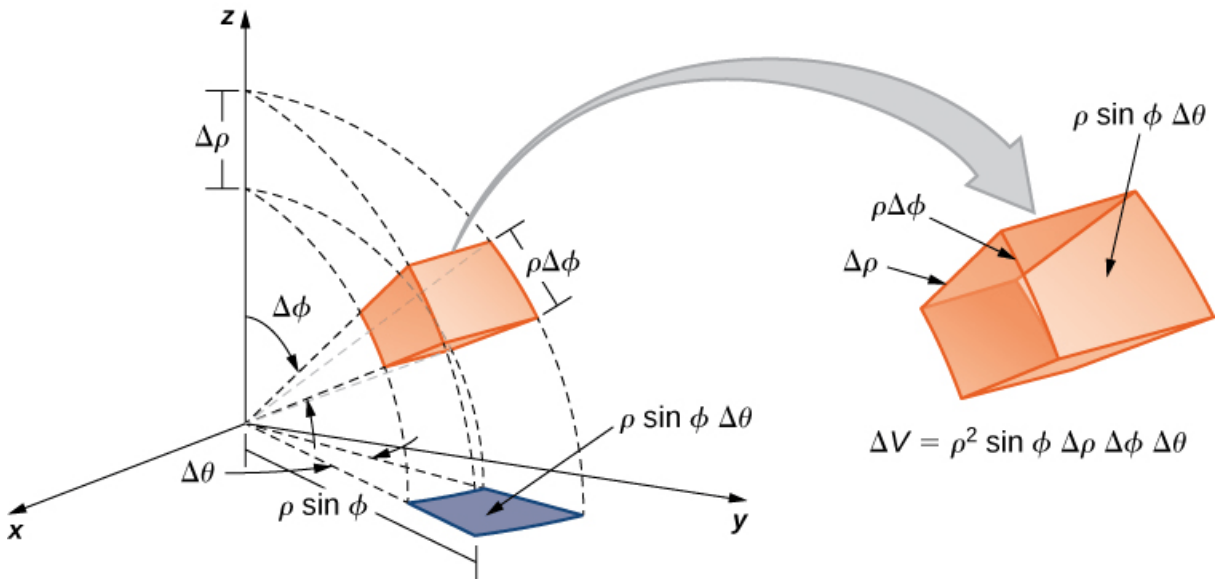


Figure 14.5.8: The volume element of a box in spherical coordinates.

Definition: triple integral in spherical coordinates

The **triple integral in spherical coordinates** is the limit of a triple Riemann sum,

$$\lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(\rho^*_{ijk},\theta^*_{ijk},\varphi^*_{ijk})(\rho^*_{ijk})^2 \sin\,\varphi\Delta\rho\,\Delta\theta\Delta\varphi$$

provided the limit exists.

As with the other multiple integrals we have examined, all the properties work similarly for a triple integral in the spherical coordinate system, and so do the iterated integrals. Fubini's theorem takes the following form.

Theorem: Fubini's Theorem for Spherical Coordinates

If $f(\rho,\theta,\varphi)$ is continuous on a spherical solid box $B=[a,b]\times[\alpha,\beta]\times[\gamma,\psi]$, then

$$\iiint_B f(\rho,\theta,\varphi)\,\rho^2\sin\,\varphi\,d\rho\,d\varphi\,d\theta = \int_{\varphi=\gamma}^{\varphi=\psi}\int_{\theta=\alpha}^{\theta=\beta}\int_{\rho=a}^{\rho=b} f(\rho,\theta,\varphi)\,\rho^2\sin\,\varphi\,d\rho\,d\varphi\,d\theta.$$

This iterated integral may be replaced by other iterated integrals by integrating with respect to the three variables in other orders.

As stated before, spherical coordinate systems work well for solids that are symmetric around a point, such as spheres and cones. Let us look at some examples before we consider triple integrals in spherical coordinates on general spherical regions.

Example 14.5.5: Evaluating a Triple Integral in Spherical Coordinates

Evaluate the iterated triple integral

$$\int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\pi/2}\int_{\rho=0}^{\rho=1}\rho^2\sin\,\varphi\,d\rho\,d\varphi\,d\theta.$$

Solution

As before, in this case the variables in the iterated integral are actually independent of each other and hence we can integrate each piece and multiply:

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^2 \sin\,\varphi\, d\rho\, d\varphi\, d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin\,\varphi\, d\varphi \int_0^1 \rho^2 d\rho = (2\pi)\,(1)\,\left(\frac{1}{3}\right) = \frac{2\pi}{3}$$

The concept of triple integration in spherical coordinates can be extended to integration over a general solid, using the projections onto the coordinate planes. Note that dV and dA mean the increments in volume and area, respectively. The variables V and A are used as the variables for integration to express the integrals.

The triple integral of a continuous function $f(\rho,\theta,\varphi)$ over a general solid region

$$E = \{(\rho,\theta,\varphi)|(\rho,\theta)^{\text{int}} D, u_1(\rho,\theta) \le \varphi \le u_2(\rho,\theta)\}$$

in $\mathbb{R}^3$, where D is the projection of E onto the $\rho\theta$-plane, is

$$\iiint_E f(\rho,\theta,\varphi)\, dV = \iint_D \left[\int_{u_1(\rho,\theta)}^{u_2(\rho,\theta)} f(\rho,\theta,\varphi)\, d\varphi\right] dA.$$

In particular, if $D = \{(\rho,\theta)|g_1(\theta) \le \rho \le g_2(\theta),\ \alpha \le \theta \le \beta\}$, the we have

$$\iiint_E f(\rho,\theta,\varphi)\, dV = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{u_1(\rho,\theta)}^{u_2(\rho,\theta)} f(\rho,\theta,\varphi)\rho^2 \sin\,\varphi\, d\varphi\, d\rho\, d\theta.$$

Similar formulas occur for projections onto the other coordinate planes.

Example 14.5.6: Setting up a Triple Integral in Spherical Coordinates

Set up an integral for the volume of the region bounded by the cone $z = \sqrt{3(x^2+y^2)}$ and the hemisphere $z = \sqrt{4-x^2-y^2}$ (see the figure below).

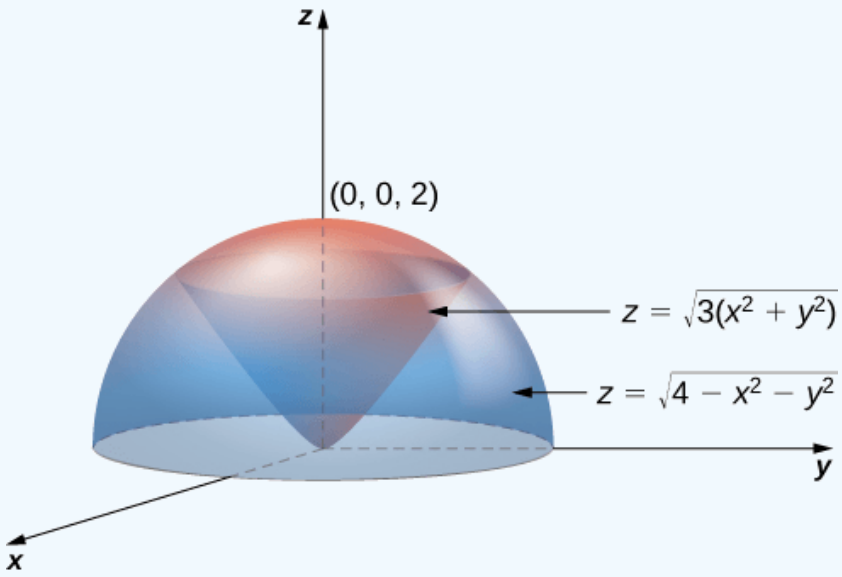


Figure 14.5.9: A region bounded below by a cone and above by a hemisphere.

Solution

Using the conversion formulas from rectangular coordinates to spherical coordinates, we have:

For the cone: $z = \sqrt{3(x^2+y^2)}$ or $\rho\ \cos\ \varphi = \sqrt{3}\rho\ \sin\ \varphi$ or $\tan\ \varphi = \frac{1}{\sqrt{3}}$ or $\varphi = \frac{\pi}{6}$.

For the sphere: $z = \sqrt{4 - x^2 - y^2}$ or $z^2 + x^2 + y^2 = 4$ or $\rho^2 = 4$ or $\rho = 2$.

Thus, the triple integral for the volume is

$$V(E) = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi/6} \int_{\rho=0}^{\rho=2} \rho^2 \sin\,\varphi \, d\rho \, d\varphi \, d\theta.$$

Exercise 14.5.5

Set up a triple integral for the volume of the solid region bounded above by the sphere $\rho = 2$ and bounded below by the cone $\varphi = \pi/3$.

Hint available at the end of this section.

Example 14.5.7: Interchanging Order of Integration in Spherical Coordinates

Let E be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the sphere $z = x^2 + y^2 + z^2$ (Figure 15.5.10). Set up a triple integral in spherical coordinates and find the volume of the region using the following orders of integration:

a. $d\rho \; d\phi \; d\theta$
b. $d\varphi \; d\rho \; d\theta$

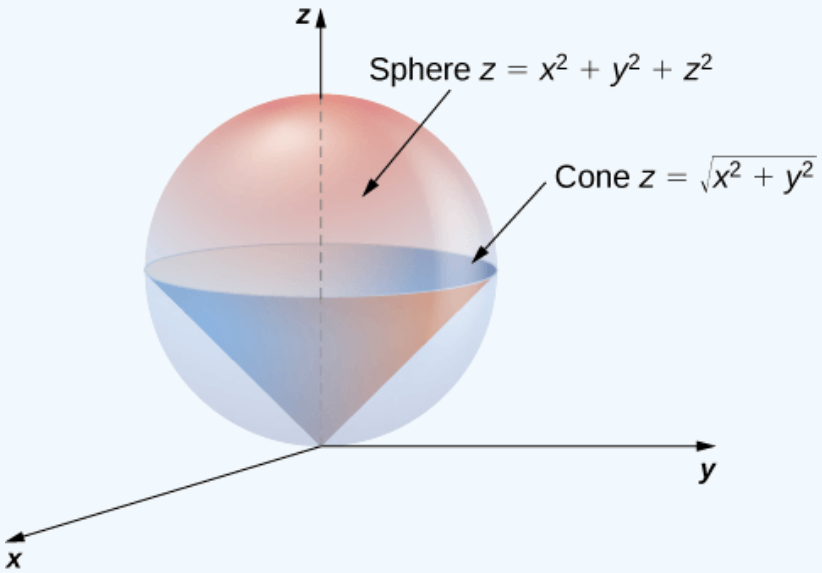


Figure 14.5.10: A region bounded below by a cone and above by a sphere.

Solution

a. Use the conversion formulas to write the equations of the sphere and cone in spherical coordinates.

For the sphere:

$$\begin{aligned} x^2 + y^2 + z^2 &= z \\ \rho^2 &= \rho \cos\,\varphi \\ \rho &= \cos\,\varphi. \end{aligned}$$

For the cone:

$$\begin{aligned} z &= \sqrt{x^2+y^2} \\ \rho\cos\varphi &= \sqrt{\rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta} \\ \rho\cos\varphi &= \sqrt{\rho^2\sin^2\varphi(\cos^2\theta+\sin^2\theta)} \\ \rho\cos\varphi &= \rho\sin\varphi \\ \cos\varphi &= \sin\varphi \\ \varphi &= \pi/4. \end{aligned}$$

Hence the integral for the volume of the solid region E becomes

$$V(E) = \int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\pi/4}\int_{\rho=0}^{\rho=\cos\varphi}\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta.$$

b. Consider the $\varphi\rho$-plane. Note that the ranges for φ and ρ (from part a.) are

$$\begin{aligned} 0 &\le \varphi \le \pi/4 \\ 0 &\le \rho \le \cos\varphi \end{aligned}$$

The curve $\rho = \cos\ \varphi$ meets the line $\varphi = \pi/4$ at the point $(\pi/4, \sqrt{2}/2)$. Thus, to change the order of integration, we need to use two pieces:

$$0 \le \rho \le \sqrt{2}/2,\ 0 \le \varphi \le \pi/4$$

and

$$\sqrt{2}/2 \le \rho \le 1,\ 0 \le \varphi \le \cos^{-1}\rho.$$

Hence the integral for the volume of the solid region E becomes

$$V(E) = \int_{\theta=0}^{\theta=2\pi}\int_{\rho=0}^{\rho=\sqrt{2}/2}\int_{\varphi=0}^{\varphi=\pi/4}\rho^2\sin\varphi\,d\varphi\,d\rho\,d\theta + \int_{\theta=0}^{\theta=2\pi}\int_{\rho=\sqrt{2}/2}^{\rho=1}\int_{\varphi=0}^{\varphi=\cos^{-1}\rho}\rho^2\sin\varphi\,d\varphi\,d\rho\,d\theta$$

In each case, the integration results in $V(E) = \frac{\pi}{8}$.

Before we end this section, we present a couple of examples that can illustrate the conversion from rectangular coordinates to cylindrical coordinates and from rectangular coordinates to spherical coordinates.

Example 14.5.8: Converting from Rectangular Coordinates to Cylindrical Coordinates

Convert the following integral into cylindrical coordinates:

$$\int_{y=-1}^{y=1}\int_{x=0}^{x=\sqrt{1-y^2}}\int_{z=x^2+y^2}^{z=\sqrt{x^2+y^2}} xyz\,dz\,dx\,dy.$$

Solution

The ranges of the variables are

$$\begin{aligned} -1 &\le y \le y \\ 0 &\le x \le \sqrt{1-y^2} \\ x^2+y^2 &\le z \le \sqrt{x^2+y^2}. \end{aligned}$$

The first two inequalities describe the right half of a circle of radius 1. Therefore, the ranges for θ and r are

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \text{ and } 0 \le r \le 1.$$

The limits of z are $r^2 \le z \le r$, hence

$$\int_{y=-1}^{y=1}\int_{x=0}^{x=\sqrt{1-y^2}}\int_{z=x^2+y^2}^{z=\sqrt{x^2+y^2}} xyz\,dz\,dx\,dy = \int_{\theta=-\pi/2}^{\theta=\pi/2}\int_{r=0}^{r=1}\int_{z=r^2}^{z=r} r(r\,\cos\theta)\,(r\,\sin\,\theta)\,z\,dz\,dr\,d\theta.$$

Example 14.5.9: Converting from Rectangular Coordinates to Spherical Coordinates

Convert the following integral into spherical coordinates:

$$\int_{y=0}^{y=3}\int_{x=0}^{x=\sqrt{9-y^2}}\int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)\,dz\,dx\,dy.$$

Solution

The ranges of the variables are

$$\begin{aligned} 0 \le y &\le 3 \\ 0 \le x &\le \sqrt{9-y^2} \\ \sqrt{x^2+y^2} \le z &\le \sqrt{18-x^2-y^2}. \end{aligned}$$

The first two ranges of variables describe a quarter disk in the first quadrant of the xy-plane. Hence the range for θ is $0 \le \theta \le \frac{\pi}{2}$.

The lower bound $z=\sqrt{x^2+y^2}$ is the upper half of a cone and the upper bound $z=\sqrt{18-x^2-y^2}$ is the upper half of a sphere. Therefore, we have $0 \le \rho \le \sqrt{18}$, which is $0 \le \rho \le 3\sqrt{2}$.

For the ranges of φ we need to find where the cone and the sphere intersect, so solve the equation

$$\begin{aligned} r^2+z^2 &= 18 \\ \left(\sqrt{x^2+y^2}\right)^2+z^2 &= 18 \\ z^2+z^2 &= 18 \\ 2z^2 &= 18 \\ z^2 &= 9 \\ z &= 3. \end{aligned}$$

This gives

$$\begin{aligned} 3\sqrt{2}\,\cos\,\varphi &= 3 \\ \cos\,\varphi &= \frac{1}{\sqrt{2}} \\ \varphi &= \frac{\pi}{4}. \end{aligned}$$

Putting this together, we obtain

$$\int_{y=0}^{y=3}\int_{x=0}^{x=\sqrt{9-y^2}}\int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)\,dz\,dx\,dy = \int_{\varphi=0}^{\varphi=\pi/4}\int_{\theta=0}^{\theta=\pi/2}\int_{\rho=0}^{\rho=3\sqrt{2}} \rho^4\,\sin\,\varphi\,d\rho\,d\theta\,d\varphi.$$

Exercise 14.5.6:

Use rectangular, cylindrical, and spherical coordinates to set up triple integrals for finding the volume of the region inside the sphere $x^2+y^2+z^2=4$ but outside the cylinder $x^2+y^2=1$.

Answer: Rectangular available at the end of this section.

Now that we are familiar with the spherical coordinate system, let's find the volume of some known geometric figures, such as

spheres and ellipsoids.

Example 14.5.10: Chapter Opener: Finding the Volume of l'Hemisphèric

Find the volume of the spherical planetarium in l'Hemisphèric in Valencia, Spain, which is five stories tall and has a radius of approximately 50 ft, using the equation $x^2 + y^2 + z^2 = r^2$.



Figure 14.5.11: (credit: modification of work by Javier Yaya Tur, Wikimedia Commons)

Solution

We calculate the volume of the ball in the first octant, where $x \leq 0,\ y \leq 0$, and $z \leq 0$, using spherical coordinates, and then multiply the result by 8 for symmetry. Since we consider the region D as the first octant in the integral, the ranges of the variables are

$$0 \leq \varphi \leq \frac{\pi}{2},\ 0 \leq \rho \leq r,\ 0 \leq \theta \leq \frac{\pi}{2}.$$

Therefore,

$$\begin{aligned} V = \iiint_D dx\, dy\, dz &= 8 \int_{\theta=0}^{\theta=\pi/2} \int_{\rho=0}^{\rho=\pi} \int_{\varphi=0}^{\varphi=\pi/2} \rho^2 \sin\,\theta\, d\varphi\, d\rho\, d\varphi \\ &= 8 \int_{\varphi=0}^{\varphi=\pi/2} d\varphi \int_{\rho=0}^{\rho=r} \rho^2 d\rho \int_{\theta=0}^{\theta=\pi/2} \sin\,\theta\, d\theta \\ &= 8\left(\frac{\pi}{2}\right)\left(\frac{r^3}{3}\right)(1) \\ &= \frac{4}{3}\pi r^3. \end{aligned}$$

This exactly matches with what we knew. So for a sphere with a radius of approximately 50 ft, the volume is $\frac{4}{3}\pi(50)^3 \approx 523,600\ ft^3$.

For the next example we find the volume of an ellipsoid.

Example 14.5.11: Finding the Volume of an Ellipsoid

Find the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Solution

We again use symmetry and evaluate the volume of the ellipsoid using spherical coordinates. As before, we use the first octant $x \leq 0,\ \ y \leq 0$, and $z \leq 0$ and then multiply the result by 8.

In this case the ranges of the variables are

$$0 \leq \varphi \leq \frac{\pi}{2}\, 0 \leq \rho \leq 1, \text{ and } 0 \leq \theta \leq \frac{\pi}{2}.$$

Also, we need to change the rectangular to spherical coordinates in this way:

$$x = a\rho \cos \varphi \sin \theta,\ y = b\rho \sin \varphi \sin \theta, \text{ and } z = cp \cos \theta.$$

Then the volume of the ellipsoid becomes

$$\begin{aligned} V &= \iiint_D dx\, dy\, dz \\ &= 8 \int_{\theta=0}^{\theta=\pi/2} \int_{\rho=0}^{\rho=1} \int_{\varphi=0}^{\varphi=\pi/2} abc\, \rho^2 \sin \theta\, d\varphi\, d\rho\, d\theta \\ &= 8abc \int_{\varphi=0}^{\varphi=\pi/2} d\varphi \int_{\rho=0}^{\rho=1} \rho^2 d\rho \int_{\theta=0}^{\theta=\pi/2} \sin \theta\, d\theta \\ &= 8abc \left(\frac{\pi}{2}\right) \left(\frac{1}{3}\right) (1) \\ &= \frac{4}{3}\pi abc. \end{aligned}$$

Example 14.5.12: Finding the Volume of the Space Inside an Ellipsoid and Outside a Sphere

Find the volume of the space inside the ellipsoid $\frac{x^2}{75^2} + \frac{y^2}{80^2} + \frac{z^2}{90^2} = 1$ and outside the sphere $x^2 + y^2 + z^2 = 50^2$.

Solution

This problem is directly related to the l'Hemisphèric structure. The volume of space inside the ellipsoid and outside the sphere might be useful to find the expense of heating or cooling that space. We can use the preceding two examples for the volume of the sphere and ellipsoid and then substract.

First we find the volume of the ellipsoid using $a = 75$ ft, $b = 80$ ft, and $c = 90$ ft in the result from Example. Hence the volume of the ellipsoid is

$$V_{ellipsoid} = \frac{4}{3}\pi(75)(80)(90) \approx 2,262,000\ ft^3.$$

From Example, the volume of the sphere is

$$V_{sphere} \approx 523,600\ ft^3.$$

Therefore, the volume of the space inside the ellipsoid $\frac{x^2}{75^2} + \frac{y^2}{80^2} + \frac{z^2}{90^2} = 1$ and outside the sphere $x^2 + y^2 + z^2 = 50^2$ is approximately

$$V_{Hemispheric} = V_{ellipsoid} - V_{sphere} = 1,738,400\ ft^3.$$

Student Project: Hot air balloons

Hot air ballooning is a relaxing, peaceful pastime that many people enjoy. Many balloonist gatherings take place around the world, such as the Albuquerque International Balloon Fiesta. The Albuquerque event is the largest hot air balloon festival in the world, with over 500 balloons participating each year.



Figure 14.5.12: Balloons lift off at the 2001 Albuquerque International Balloon Fiesta. (credit: David Herrera, Flickr)

As the name implies, hot air balloons use hot air to generate lift. (Hot air is less dense than cooler air, so the balloon floats as long as the hot air stays hot.) The heat is generated by a propane burner suspended below the opening of the basket. Once the balloon takes off, the pilot controls the altitude of the balloon, either by using the burner to heat the air and ascend or by using a vent near the top of the balloon to release heated air and descend. The pilot has very little control over where the balloon goes, however—balloons are at the mercy of the winds. The uncertainty over where we will end up is one of the reasons balloonists are attracted to the sport.

In this project we use triple integrals to learn more about hot air balloons. We model the balloon in two pieces. The top of the balloon is modeled by a half sphere of radius 28

feet. The bottom of the balloon is modeled by a frustum of a cone (think of an ice cream cone with the pointy end cut off). The radius of the large end of the frustum is 28 feet and the radius of the small end of the frustum is 28 feet. A graph of our balloon model and a cross-sectional diagram showing the dimensions are shown in the following figure.

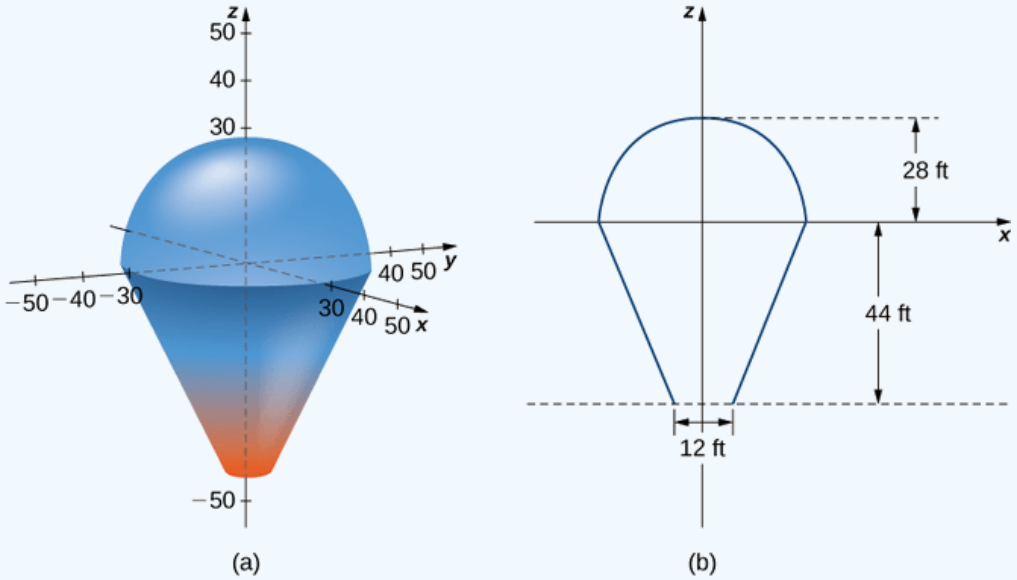


Figure 14.5.13: (a) Use a half sphere to model the top part of the balloon and a frustum of a cone to model the bottom part of the balloon. (b) A cross section of the balloon showing its dimensions.

We first want to find the volume of the balloon. If we look at the top part and the bottom part of the balloon separately, we see that they are geometric solids with known volume formulas. However, it is still worthwhile to set up and evaluate the integrals we would need to find the volume. If we calculate the volume using integration, we can use the known volume formulas to check our answers. This will help ensure that we have the integrals set up correctly for the later, more complicated stages of the project.

1. Find the volume of the balloon in two ways.

a. Use triple integrals to calculate the volume. Consider each part of the balloon separately. (Consider using spherical coordinates for the top part and cylindrical coordinates for the bottom part.)

b. Verify the answer using the formulas for the volume of a sphere, $V = \frac{4}{3}\pi r^3$, and for the volume of a cone, $V = \frac{1}{3}\pi r^2 h$.

In reality, calculating the temperature at a point inside the balloon is a tremendously complicated endeavor. In fact, an entire branch of physics (thermodynamics) is devoted to studying heat and temperature. For the purposes of this project, however, we are going to make some simplifying assumptions about how temperature varies from point to point within the balloon. Assume that just prior to liftoff, the temperature (in degrees Fahrenheit) of the air inside the balloon varies according to the function

$$T_0(r, \theta, z) = \frac{z - r}{10} + 210.$$

2. What is the average temperature of the air in the balloon just prior to liftoff? (Again, look at each part of the balloon separately, and do not forget to convert the function into spherical coordinates when looking at the top part of the balloon.)

Now the pilot activates the burner for 10 seconds. This action affects the temperature in a 12-foot-wide column 20 feet high, directly above the burner. A cross section of the balloon depicting this column in shown in the following figure

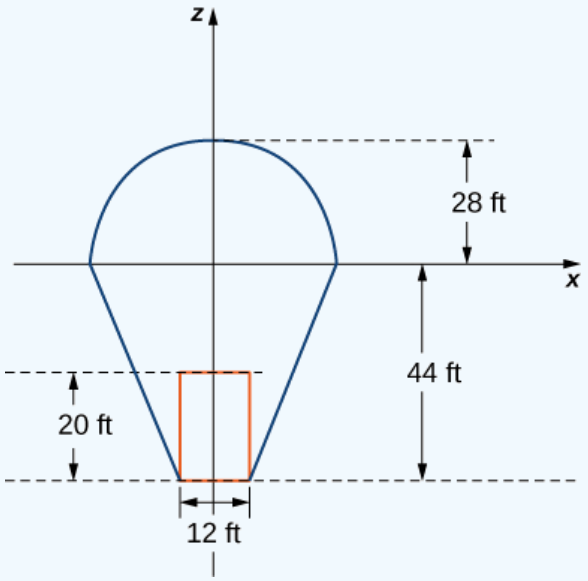


Figure $14.5.14$: Activating the burner heats the air in a 20-foot-high, 12-foot-wide column directly above the burner.

Assume that after the pilot activates the burner for 10 seconds, the temperature of the air in the column described above *increases* according to the formula

$$H(r, \theta, z) = -2z - 48.$$

Then the temperature of the air in the column is given by

$$T_1(r, \theta, z) = \frac{z - r}{10} + 210 + (-2z - 48),$$

while the temperature in the remainder of the balloon is still given by

$$T_0(r, \theta, z) = \frac{z - r}{10} + 210.$$

3. Find the average temperature of the air in the balloon after the pilot has activated the burner for 10 seconds.

Key Concepts

- To evaluate a triple integral in cylindrical coordinates, use the iterated integral

$$\int_{\theta=\alpha}^{\theta=\beta}\int_{r=g_1(\theta)}^{r=g_2(\theta)}\int_{z=u_1(r,\theta)}^{u_2(r,\theta)} f(r,\theta,z)r\,dz\,dr\,d\theta.$$

- To evaluate a triple integral in spherical coordinates, use the iterated integral

$$\int_{\theta=\alpha}^{\theta=\beta}\int_{\rho=g_1(\theta)}^{\rho=g_2(\theta)}\int_{\varphi=u_1(r,\theta)}^{u_2(r,\theta)} f(\rho,\theta,\varphi)\,\rho^2\sin\varphi\,d\varphi\,d\rho\,d\theta.$$

Key Equations

- **Triple integral in cylindrical coordinates**

$$\iiint_B g(s,y,z)dV = \iiint_B g(r\,\cos\theta,\ r\,\sin\,\theta,\ z)r\,dr\,d\theta\,dz = \iiint_B f(r,\theta,z)r\,dr\,d\theta\,dz$$

- **Triple integral in spherical coordinates**

$$\iiint_B f(\rho,\theta,\varphi)\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \int_{\varphi=\gamma}^{\varphi=\psi}\int_{\theta=\alpha}^{\theta=\beta}\int_{\rho=a}^{\rho=b} f(\rho,\theta,\varphi)\rho^2\sin\,\varphi\,d\rho\,d\varphi\,d\theta$$

Glossary

triple integral in cylindrical coordinates

the limit of a triple Riemann sum, provided the following limit exists:

$$lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(r^*_{ijk},\theta^*_{ijk},s^*_{ijk})r^*_{ijk}\Delta r\Delta\theta\Delta z$$

triple integral in spherical coordinates

the limit of a triple Riemann sum, provided the following limit exists:

$$lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(\rho^*_{ijk},\theta^*_{ijk},\varphi^*_{ijk})(\rho^*_{ijk})^2\sin\,\varphi\Delta\rho\Delta\theta\Delta\varphi$$

Answers

Exercise 14.5.1**:**

Follow the same steps as in the previous example.

8

Exercise 14.5.2**:**

Analyze the region, and draw a sketch.

$$\iiint_E f(r,\theta,z)r\,dz\,dr\,d\theta = \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=2\,\sin\,\theta}\int_{z=0}^{z=4-r\,\sin\,\theta} f(r,\theta,z)r\,dz\,dr\,d\theta.$$

Exercise 14.5.3**:**

Note that θ is independent of r and z.

$E=\{(r,\theta,z)|0\le\theta\le 2\pi,\ 0\le z\le 1,\ 0\le r\le 2-z^2\}$ and

$$V=\int_{r=0}^{r=1}\int_{z=0}^{z=2-r^2}\int_{\theta=0}^{\theta=2\pi} r\,d\theta\,dz\,dr.$$

Exercise 14.5.4

A figure can be helpful. Note that θ is independent of r and z.

$E_2=\{(r,\theta,z)|0\le\theta\le 2\pi,\ 0\le r\le 1,\ r\le z\le\sqrt{4-r^2}\}$ and

$$V=\int_{r=0}^{r=1}\int_{z=r}^{z=\sqrt{4-r^2}}\int_{\theta=0}^{\theta=2\pi} r\,d\theta\,dz\,dr.$$

Exercise 14.5.5

Follow the steps of the previous example.

$$V(E) = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi/3} \int_{\rho=0}^{\rho=2} \rho^2 \sin\,\varphi \, d\rho \, d\varphi \, d\theta$$

Exercise 14.5.6**:**

$$\int_{x=-2}^{x=2} \int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} \int_{z=-\sqrt{4-x^2-y^2}}^{z=\sqrt{4-x^2-y^2}} dz\, dy\, dx - \int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} \int_{z=-\sqrt{4-x^2-y^2}}^{z=\sqrt{4-x^2-y^2}} dz\, dy\, dx.$$

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=1}^{r=2} \int_{z=-\sqrt{4-r^2}}^{z=\sqrt{4-r^2}} r\, dz\, dr\, d\theta.$$

$$\int_{\varphi=\pi/6}^{\varphi=5\pi/6} \int_{\theta=0}^{\theta=2\pi} \int_{\rho=\csc\,\varphi}^{\rho=2} \rho^2 \sin\,\varphi \, d\rho\, d\theta\, d\varphi.$$

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14.6: Calculating Centers of Mass and Moments of Inertia

Learning Objectives

- Use double integrals to locate the center of mass of a two-dimensional object.
- Use double integrals to find the moment of inertia of a two-dimensional object.
- Use triple integrals to locate the center of mass of a three-dimensional object.

We have already discussed a few applications of multiple integrals, such as finding areas, volumes, and the average value of a function over a bounded region. In this section we develop computational techniques for finding the center of mass and moments of inertia of several types of physical objects, using double integrals for a lamina (flat plate) and triple integrals for a three-dimensional object with variable density. The density is usually considered to be a constant number when the lamina or the object is homogeneous; that is, the object has uniform density.

Center of Mass in Two Dimensions

The center of mass is also known as the center of gravity if the object is in a uniform gravitational field. If the object has uniform density, the center of mass is the geometric center of the object, which is called the centroid. Figure $14.6.1$ shows a point P as the center of mass of a lamina. The lamina is perfectly balanced about its center of mass.

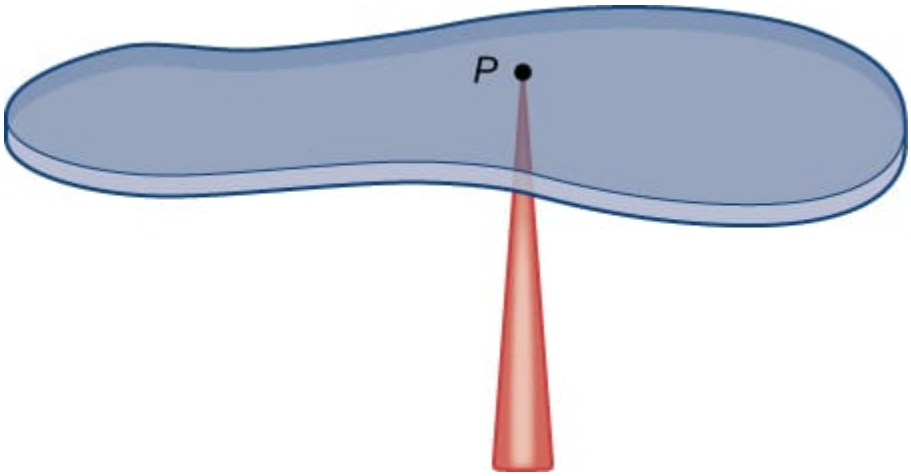


Figure $14.6.1$: A lamina is perfectly balanced on a spindle if the lamina's center of mass sits on the spindle.

To find the coordinates of the center of mass $P(\bar{x},\bar{y})$ of a lamina, we need to find the moment M_x of the lamina about the x-axis and the moment M_y about the y-axis. We also need to find the mass m of the lamina. Then

$$\bar{x}=\frac{M_y}{m}$$

and

$$\bar{y}=\frac{M_x}{m}.$$

Refer to Moments and Centers of Mass for the definitions and the methods of single integration to find the center of mass of a one-dimensional object (for example, a thin rod). We are going to use a similar idea here except that the object is a two-dimensional lamina and we use a double integral.

If we allow a constant density function, then $\bar{x}=\dfrac{M_y}{m}$ and $\bar{y}=\dfrac{M_x}{m}$ give the *centroid* of the lamina.

Suppose that the lamina occupies a region R in the xy-plane and let $\rho(x, y)$ be its density (in units of mass per unit area) at any point (x, y). Hence,

$$\rho(x, y) = \lim_{\Delta A \to 0} \frac{\Delta m}{\Delta A}$$

where Δm and ΔA are the mass and area of a small rectangle containing the point (x, y) and the limit is taken as the dimensions of the rectangle go to 0 (see the following figure).

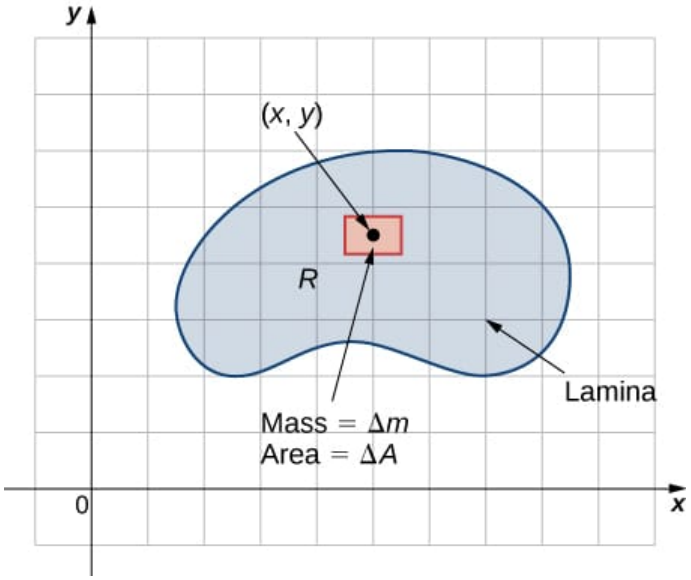


Figure 14.6.2: The density of a lamina at a point is the limit of its mass per area in a small rectangle about the point as the area goes to zero.

Just as before, we divide the region R into tiny rectangles R_{ij} with area ΔA and choose (x^*_{ij}, y^*_{ij}) as sample points. Then the mass m_{ij} of each R_{ij} is equal to $\rho(x^*_{ij}, y^*_{ij})\Delta A$ (Figure 14.6.2). Let k and l be the number of subintervals in x and y respectively. Also, note that the shape might not always be rectangular but the limit works anyway, as seen in previous sections.

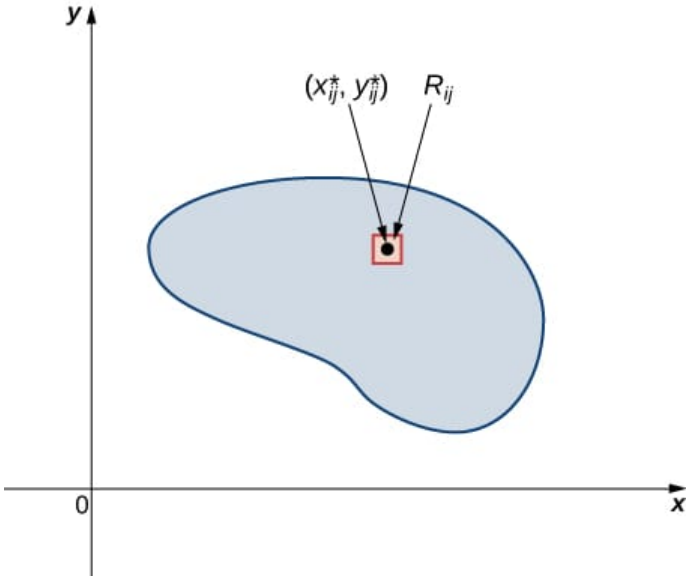


Figure 14.6.3: Subdividing the lamina into tiny rectangles R_{ij} each containing a sample point (x^*_{ij}, y^*_{ij}).

Hence, the mass of the lamina is

$$m = \lim_{k,l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} m_{ij} = \lim_{k,l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x^*_{ij}, y^*_{ij})\Delta A = \iint_R \rho(x, y) dA.$$

Let's see an example now of finding the total mass of a triangular lamina.

Example 14.6.1: Finding the Total Mass of a Lamina

Consider a triangular lamina R with vertices $(0,0),\ (0,3),\ (3,0)$ and with density $\rho(x,y) = xy\ kg/m^2$. Find the total mass.

Solution

A sketch of the region R is always helpful, as shown in the following figure.

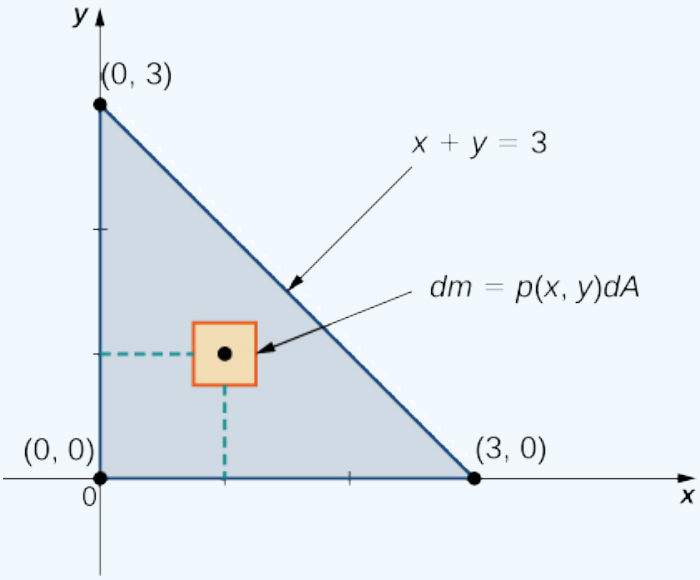


Figure 14.6.4: A lamina in the xy-plane with density $\rho(x,y) = xy$.

Using the expression developed for mass, we see that

$$m = \iint_R dm = \iint_R \rho(x,y)dA = \int_{x=0}^{x=3}\int_{y=0}^{y=3-x} xy\,dy\,dx = \int_{x=0}^{x=3}\left[x\frac{y^2}{2}\Big|_{y=0}^{y=3}\right]dx = \int_{x=0}^{x=3}\frac{1}{2}x(3-x)^2dx = \left[\frac{9x^2}{4} - x^3 + \frac{x^4}{8}\right]\Big|_{x=0}^{x=3} =$$

The computation is straightforward, giving the answer $m = \dfrac{27}{8}\ kg$.

Exercise 14.6.1

Consider the same region R as in the previous example, and use the density function $\rho(x,y) = \sqrt{xy}$. Find the total mass.

Answer available at the end of this section.

Now that we have established the expression for mass, we have the tools we need for calculating moments and centers of mass. The moment M_z about the x-axis for R is the limit of the sums of moments of the regions R_{ij} about the x-axis. Hence

$$M_x = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}(y_{ij}^*)m_{ij} = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}(y_{ij}^*)\rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_R y\rho(x,y)\,dA$$

Similarly, the moment M_y about the y-axis for R is the limit of the sums of moments of the regions R_{ij} about the y-axis. Hence

$$M_y = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}(x_{ij}^*)m_{ij} = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}(x_{ij}^*)\rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_R x\rho(x,y)\,dA$$

Example 14.6.2: Finding Moments

Consider the same triangular lamina R with vertices $(0,0),\ (0,3),\ (3,0)$ and with density $\rho(x,y)=xy$. Find the moments M_x and M_y.

Solution

Use double integrals for each moment and compute their values:

$$M_x=\iint_R y\rho(x,y)\,dA=\int_{x=0}^{x=3}\int_{y=0}^{y=3-x} xy^2\,dy\,dx=\frac{81}{20},$$

$$M_y=\iint_R x\rho(x,y)\,dA=\int_{x=0}^{x=3}\int_{y=0}^{y=3-x} x^2y\,dy\,dx=\frac{81}{20},$$

The computation is quite straightforward.

Exercise 14.6.2

Consider the same lamina R as above and use the density function $\rho(x,y)=\sqrt{xy}$. Find the moments M_x and M_y.

Answer available at the end of this section.

Finally we are ready to restate the expressions for the center of mass in terms of integrals. We denote the *x*-coordinate of the center of mass by $\bar{x}$ and the *y*-coordinate by $\bar{y}$. Specifically,

$$\bar{x}=\frac{M_y}{m}=\frac{\iint_R x\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA}$$

and

$$\bar{y}=\frac{M_x}{m}=\frac{\iint_R y\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA}$$

Example 14.6.3: center of mass

Again consider the same triangular region R with vertices $(0,0),\ (0,3),\ (3,0)$ and with density function $\rho(x,y)=xy$. Find the center of mass.

Solution

Using the formulas we developed, we have

$$\bar{x}=\frac{M_y}{m}=\frac{\iint_R x\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA}=\frac{81/20}{27/8}=\frac{6}{5},$$

$$\bar{y}=\frac{M_x}{m}=\frac{\iint_R y\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA}=\frac{81/20}{27/8}=\frac{6}{5}.$$

Therefore, the center of mass is the point $\left(\frac{6}{5},\frac{6}{5}\right)$.

Analysis

If we choose the density $\rho(x,y)$ instead to be uniform throughout the region (i.e., constant), such as the value 1 (any constant will do), then we can compute the centroid,

$$x_c = \frac{M_y}{m} = \frac{\iint_R x\,dA}{\iint_R dA} = \frac{9/2}{9/2} = 1,$$

$$y_c = \frac{M_x}{m} = \frac{\iint_R y\,dA}{\iint_R dA} = \frac{9/2}{9/2} = 1.$$

Notice that the center of mass $\left(\frac{6}{5}, \frac{6}{5}\right)$ is not exactly the same as the centroid $(1, 1)$ of the triangular region. This is due to the variable density of R If the density is constant, then we just use $\rho(x, y) = c$ (constant). This value cancels out from the formulas, so for a constant density, the center of mass coincides with the centroid of the lamina.

Exercise 14.6.3

Again use the same region R as above and use the density function $\rho(x, y) = \sqrt{xy}$. Find the center of mass.

Answer available at the end of this section.

Once again, based on the comments at the end of Example 14.6.3, we have expressions for the centroid of a region on the plane:

$$x_c = \frac{M_y}{m} = \frac{\iint_R x\,dA}{\iint_R dA} \text{ and } y_c = \frac{M_x}{m} = \frac{\iint_R y\,dA}{\iint_R dA}.$$

We should use these formulas and verify the centroid of the triangular region referred to in the last three examples.

Example 14.6.4: Finding Mass, Moments, and Center of Mass

Find the mass, moments, and the center of mass of the lamina of density $\rho(x, y) = x + y$ occupying the region R under the curve $y = x^2$ in the interval $0 \le x \le 2$ (see the following figure).

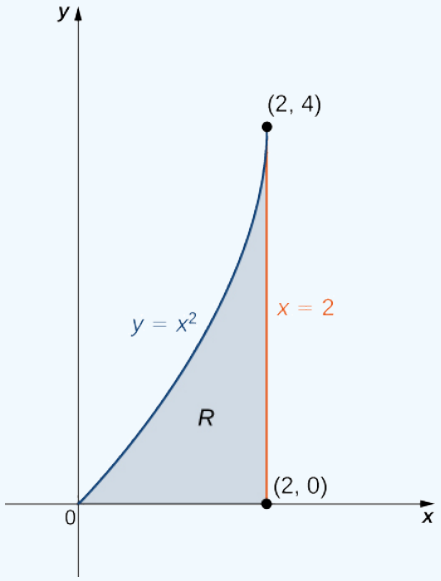


Figure 14.6.5: Locating the center of mass of a lamina R with density $\rho(x, y) = x + y$.

Solution

First we compute the mass m. We need to describe the region between the graph of $y = x^2$ and the vertical lines $x = 0$ and $x = 2$:

$$m = \iint_R dm = \iint_R \rho(x,y)\,dA = \int_{x=0} x = 2\int_{y=0}^{y=x^2} (x+y)dy\,dx = \int_{x=0}^{x=2} \left[xy + \frac{y^2}{2}\Big|_{y=0}^{y=x^2} \right] dx$$

$$= \int_{x=0}^{x=2} \left[x^3 + \frac{x^4}{2} \right] dx = \left[\frac{x^4}{4} + \frac{x^5}{10} \right]\Big|_{x=0}^{x=2} = \frac{36}{5}.$$

Now compute the moments M_x and M_y:

$$M_x = \iint_R y\rho(x,y)\,dA = \int_{x=0}^{x=2} \int_{y=0}^{y=x^2} y(x+y)\,dy\,dx = \frac{80}{7},$$

$$M_y = \iint_R x\rho(x,y)\,dA = \int_{x=0}^{x=2} \int_{y=0}^{y=x^2} x(x+y)\,dy\,dx = \frac{176}{15}.$$

Finally, evaluate the center of mass,

$$\bar{x} = \frac{M_y}{m} = \frac{\iint_R x\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA} = \frac{176/15}{36/5} = \frac{44}{27},$$

$$\bar{y} = \frac{M_x}{m} = \frac{\iint_R y\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA} = \frac{80/7}{36/5} = \frac{100}{63}.$$

Hence the center of mass is $(\bar{x},\bar{y}) = \left(\frac{44}{27}, \frac{100}{63}\right)$.

Exercise 14.6.4

Calculate the mass, moments, and the center of mass of the region between the curves $y = x$ and $y = x^2$ with the density function $\rho(x,y) = x$ in the interval $0 \le x \le 1$.

Answer available at the end of this section.

Example 14.6.5: Finding a Centroid

Find the centroid of the region under the curve $y = e^x$ over the interval $1 \le x \le 3$ (Figure 14.6.6).

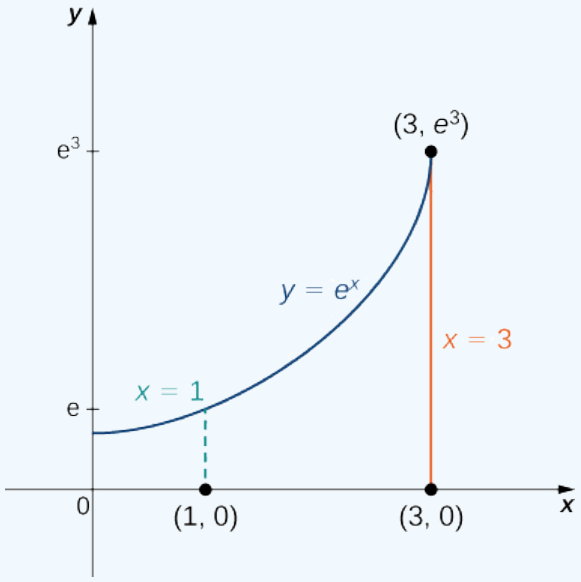


Figure 14.6.6: Finding a centroid of a region below the curve $y = e^x$.

Solution

To compute the centroid, we assume that the density function is constant and hence it cancels out:

$$x_c = \frac{M_y}{m} = \frac{\iint_R x\,dA}{\iint_R dA} \; and \; y_c = \frac{M_x}{m} = \frac{\iint_R y\,dA}{\iint_R dA},$$

$$x_c = \frac{M_y}{m} = \frac{\iint_R x\,dA}{\iint_R dA} = \frac{\int_{x=1}^{x=3}\int_{y=0}^{y=e^x} x\,dy\,dx}{\int_{x=1}^{x=3}\int_{y=0}^{y=e^x} dy\,dx} = \frac{\int_{x=1}^{x=3} xe^x dx}{\int_{x=1}^{x=3} e^x dx} = \frac{2e^3}{e^3-e} = \frac{2e^2}{e^2-1},$$

$$y_c = \frac{M_x}{m} = \frac{\iint_R y\,dA}{\iint_R dA} = \frac{\int_{x=1}^{x=3}\int_{y=0}^{y=e^x} y\,dy\,dx}{\int_{x=1}^{x=3}\int_{y=0}^{y=e^x} dy\,dx} = \frac{\int_{x=1}^{x=3} \frac{e^{2x}}{2} dx}{\int_{x=1}^{x=3} e^x dx} = \frac{\frac{1}{4}e^2(e^4-1)}{e(e^2-1)} = \frac{1}{4}e(e^2+1).$$

Thus the centroid of the region is

$$(x_c, y_c) = \left(\frac{2e^2}{e^2-1}, \frac{1}{4}e(e^2+1)\right).$$

Exercise 14.6.5

Calculate the centroid of the region between the curves $y = x$ and $y = \sqrt{x}$ with uniform density in the interval $0 \le x \le 1$.

Answer available at the end of this section.

Moments of Inertia

For a clear understanding of how to calculate moments of inertia using double integrals, we need to go back to the general definition in Section 6.6. The moment of inertia of a particle of mass m about an axis is mr^2 where r is the distance of the particle from the axis. We can see from Figure 14.6.3 that the moment of inertia of the subrectangle R_{ij} about the x-axis is $(y_{ij}^*)^2\rho(x_{ij}^*, y_{ij}^*)\Delta A$. Similarly, the moment of inertia of the subrectangle R_{ij} about the y-axis is $(x_{ij}^*)^2\rho(x_{ij}^*, y_{ij}^*)\Delta A$. The moment of inertia is related to the rotation of the mass; specifically, it measures the tendency of the mass to resist a change in rotational motion about an axis.

The moment of inertia I_x about the x-axis for the region R is the limit of the sum of moments of inertia of the regions R_{ij} about the x-axis. Hence

$$I_x = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}(y_{ij}^*)^2 m_{ij} = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}(y_{ij}^*)^2\rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_R y^2\rho(x,y)\,dA.$$

Similarly, the moment of inertia I_y about the y-axis for R is the limit of the sum of moments of inertia of the regions R_{ij} about the y-axis. Hence

$$I_y = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}(x_{ij}^*)^2 m_{ij} = \lim_{k,l\to\infty}\sum_{i=1}^{k}\sum_{j=1}^{l}(x_{ij}^*)^2\rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_R x^2\rho(x,y)\,dA.$$

Sometimes, we need to find the moment of inertia of an object about the origin, which is known as the polar moment of inertia. We denote this by I_0 and obtain it by adding the moments of inertia I_x and I_y. Hence

$$I_0 = I_x + I_y = \iint_R (x^2+y^2)\rho(x,y)\,dA.$$

All these expressions can be written in polar coordinates by substituting $x = r\ \cos\ \theta,\ y = r\ \sin\ \theta$, and $dA = r\ dr\ d\theta$. For example, $I_0 = \iint_R r^2\rho(r\ \cos\ \theta,\ r\ \sin\ \theta)\ dA$.

Example 14.6.6: Finding Moments of Inertia for a Triangular Lamina

Use the triangular region R with vertices $(0,0),\ (2,2)$, and $(2,0)$ and with density $\rho(x,y)=xy$ as in previous examples. Find the moments of inertia.

Solution

Using the expressions established above for the moments of inertia, we have

$$I_x=\iint_R y^2\rho(x,y)\,dA=\int_{x=0}^{x=2}\int_{y=0}^{y=x} xy^3\,dy\,dx=\frac{8}{3},$$

$$I_y=\iint_R x^2\rho(x,y)\,dA=\int_{x=0}^{x=2}\int_{y=0}^{y=x} x^3y\,dy\,dx=\frac{16}{3},$$

$$I_0=\iint_R (x^2+y^2)\rho(x,y)\,dA=\int_0^2\int_0^x (x^2+y^2)xy\,dy\,dx=I_x+I_y=8$$

Exercise 14.6.6

Again use the same region R as above and the density function $\rho(x,y)=\sqrt{xy}$. Find the moments of inertia.

Answer available at the end of this section.

As mentioned earlier, the moment of inertia of a particle of mass m about an axis is mr^2 where r is the distance of the particle from the axis, also known as the **radius of gyration**.

Hence the radii of gyration with respect to the x-axis, the y-axis and the origin are

$$R_x=\sqrt{\frac{I_x}{m}},\ R_y=\sqrt{\frac{I_y}{m}},\ and\ R_0=\sqrt{\frac{I_0}{m}},$$

respectively. In each case, the radius of gyration tells us how far (perpendicular distance) from the axis of rotation the entire mass of an object might be concentrated. The moments of an object are useful for finding information on the balance and torque of the object about an axis, but radii of gyration are used to describe the distribution of mass around its centroidal axis. There are many applications in engineering and physics. Sometimes it is necessary to find the radius of gyration, as in the next example.

Example 14.6.7: Finding the Radius of Gyration for a Triangular Lamina

Consider the same triangular lamina R with vertices $(0,0),\ (2,2)$, and $(2,0)$ and with density $\rho(x,y)=xy$ as in previous examples. Find the radii of gyration with respect to the x-axis the y-axis and the origin.

Solution

If we compute the mass of this region we find that $m=2$. We found the moments of inertia of this lamina in Example 14.6.4. From these data, the radii of gyration with respect to the x-axis, y-axis and the origin are, respectively,

$$R_x=\sqrt{\frac{I_x}{m}}=\sqrt{\frac{8/3}{2}}=\sqrt{\frac{8}{6}}=\frac{2\sqrt{3}}{3}, \tag{14.6.1}$$

$$R_y=\sqrt{\frac{I_y}{m}}=\sqrt{\frac{16/3}{2}}=\sqrt{\frac{8}{3}}=\frac{2\sqrt{6}}{3}, \tag{14.6.2}$$

$$R_0=\sqrt{\frac{I_0}{m}}=\sqrt{\frac{8}{2}}=\sqrt{4}=2. \tag{14.6.3}$$

Exercise 14.6.7

Use the same region R from Example 14.6.7 and the density function $\rho(x,y)=\sqrt{xy}$. Find the radii of gyration with respect to the x-axis, the y-axis, and the origin.

Hint available at the end of this section.

Answer available at the end of this section.

Center of Mass and Moments of Inertia in Three Dimensions

All the expressions of double integrals discussed so far can be modified to become triple integrals.

Definition

If we have a solid object Q with a density function $\rho(x,y,z)$ at any point (x,y,z) in space, then its mass is

$$m=\iiint_Q \rho(x,y,z)\,dV.$$

Its moments about the xy-plane the xz-plane and the yz-plane are

$$M_{xy}=\iiint_Q z\rho(x,y,z)\,dV,\ M_{xz}=\iiint_Q y\rho(x,y,z)\,dV,\ M_{yz}=\iiint_Q x\rho(x,y,z)\,dV.$$

If the center of mass of the object is the point $(\bar{x},\bar{y},\bar{z})$, then

$$\bar{x}=\frac{M_{yz}}{m},\ \bar{y}=\frac{M_{xz}}{m},\ \bar{z}=\frac{M_{xy}}{m}.$$

Also, if the solid object is homogeneous (with constant density), then the center of mass becomes the centroid of the solid. Finally, the moments of inertia about the yz-plane, xz-plane, and the xy-plane are

$$I_x=\iiint_Q (y^2+z^2)\,\rho(x,y,z)\,dV,$$

$$I_y=\iiint_Q (x^2+z^2)\,\rho(x,y,z)\,dV,$$

$$I_z=\iiint_Q (x^2+y^2)\,\rho(x,y,z)\,dV.$$

Example 14.6.8: Finding the Mass of a Solid

Suppose that Q is a solid region bounded by $x+2y+3z=6$ and the coordinate planes and has density $\rho(x,y,z)=x^2yz$. Find the total mass.

Solution

The region Q is a tetrahedron (Figure 14.6.7) meeting the axes at the points $(6,0,0)$, $(0,3,0)$, and $(0,0,2)$. To find the limits of integration, let $z=0$ in the slanted plane $z=\frac{1}{3}(6-x-2y)$. Then for x and y find the projection of Q onto the xy-plane, which is bounded by the axes and the line $x+2y=6$. Hence the mass is

$$m=\iiint_Q \rho(x,y,z)\,dV=\int_{x=0}^{x=6}\int_{y=0}^{y=1/2(6-x)}\int_{z=0}^{z=1/3(6-x-2y)} x^2yz\,dz\,dy\,dx=\frac{108}{35}$$

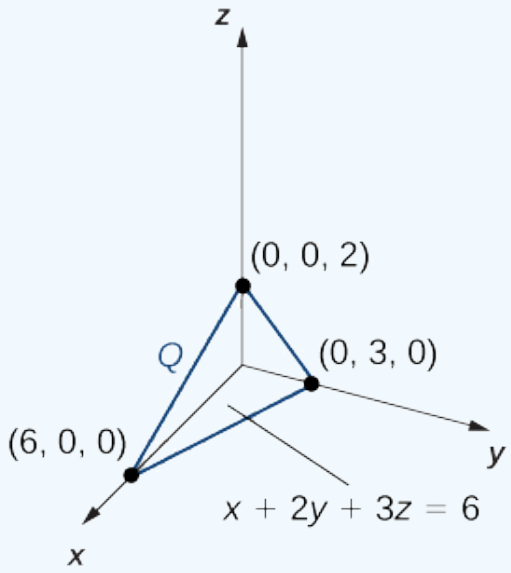


Figure 14.6.7: Finding the mass of a three-dimensional solid Q.

Exercise 14.6.8

Consider the same region Q (Figure 14.6.7), and use the density function $\rho(x, y, z) = xy^2z$. Find the mass.

Hint available at the end of this section.

Answer available at the end of this section.

Example 14.6.9: Finding the Center of Mass of a Solid

Suppose Q is a solid region bounded by the plane $x + 2y + 3z = 6$ and the coordinate planes with density $\rho(x, y, z) = x^2yz$ (see Figure 14.6.7). Find the center of mass using decimal approximation.

Solution

We have used this tetrahedron before and know the limits of integration, so we can proceed to the computations right away. First, we need to find the moments about the xy-plane, the xz-plane, and the yz-plane:

$$M_{xy} = \iiint_Q z\rho(x, y, z)\, dV = \int_{x=0}^{x=6} \int_{y=0}^{y=\frac{1}{2}(6-x)} \int_{z=0}^{z=\frac{1}{3}(6-x-2y)} x^2yz^2\, dz\, dy\, dx = \frac{54}{35} \approx 1.543,$$

$$M_{xz} = \iiint_Q y\rho(x, y, z)\, dV = \int_{x=0}^{x=6} \int_{y=0}^{y=\frac{1}{2}(6-x)} \int_{z=0}^{z=\frac{1}{3}(6-x-2y)} x^2y^2z\, dz\, dy\, dx = \frac{81}{35} \approx 2.314,$$

$$M_{yz} = \iiint_Q x\rho(x, y, z)\, dV = \int_{x=0}^{x=6} \int_{y=0}^{y=\frac{1}{2}(6-x)} \int_{z=0}^{z=\frac{1}{3}(6-x-2y)} x^3yz\, dz\, dy\, dx = \frac{243}{35} \approx 6.943.$$

Hence the center of mass is

$$\bar{x} = \frac{M_{yz}}{m},\ \bar{y} = \frac{M_{xz}}{m},\ \bar{z} = \frac{M_{xy}}{m},$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{243/35}{108/35} = \frac{243}{108} = 2.25,$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{81/35}{108/35} = \frac{81}{108} = 0.75,$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{54/35}{108/35} = \frac{54}{108} = 0.5$$

The center of mass for the tetrahedron Q is the point $(2.25, 0.75, 0.5)$.

Exercise 14.6.9

Consider the same region Q (Figure 14.6.7) and use the density function $\rho(x, y, z) = xy^2z$. Find the center of mass.

Hint available at the end of this section.

Answer available at the end of this section.

We conclude this section with an example of finding moments of inertia I_x, I_y, and I_z.

Example 14.6.10: Finding the Moments of Inertia of a Solid

Suppose that Q is a solid region and is bounded by $x + 2y + 3z = 6$ and the coordinate planes with density $\rho(x, y, z) = x^2yz$ (see Figure 14.6.7). Find the moments of inertia of the tetrahedron Q about the yz-plane, the xz-plane, and the xy-plane.

Solution

Once again, we can almost immediately write the limits of integration and hence we can quickly proceed to evaluating the moments of inertia. Using the formula stated before, the moments of inertia of the tetrahedron Q about the yz-plane, the xz -plane, and the xy-plane are

$$I_x = \iiint_Q (y^2 + z^2)\rho(x, y, z)\, dV,$$

$$I_y = \iiint_Q (x^2 + z^2)\rho(x, y, z)\, dV,$$

and

$$I_z = \iiint_Q (x^2 + y^2)\rho(x, y, z)\, dV \; with \; \rho(x, y, z) = x^2yz.$$

Proceeding with the computations, we have

$$I_x = \iiint_Q (y^2 + z^2)x^2\rho(x, y, z)\, dV$$

$$= \int_{x=0}^{x=6} \int_{y=0}^{y=\frac{1}{2}(6-x)} \int_{z=0}^{z=\frac{1}{3}(6-x-2y)} (y^2 + z^2)x^2yz\, dz\, dy\, dx = \frac{117}{35} \approx 3.343,$$

$$I_y = \iiint_Q (x^2 + z^2)x^2\rho(x, y, z)\, dV$$

$$= \int_{x=0}^{x=6} \int_{y=0}^{y=\frac{1}{2}(6-x)} \int_{z=0}^{z=\frac{1}{3}(6-x-2y)} (x^2 + z^2)x^2yz\, dz\, dy\, dx = \frac{684}{35} \approx 19.543,$$

$$I_z = \iiint_Q (x^2+y^2)x^2\rho(x,y,z)\,dV$$

$$= \int_{x=0}^{x=6}\int_{y=0}^{y=\frac{1}{2}(6-x)}\int_{z=0}^{z=\frac{1}{3}(6-x-2y)} (x^2+y^2)x^2yz\,dz\,dy\,dx = \frac{729}{35} \approx 20.829.$$

Thus, the moments of inertia of the tetrahedron Q about the yz-plane, the xz-plane, and the xy-plane are $117/35,\ 684/35$, and $729/35$, respectively.

Exercise 14.6.10

Consider the same region Q (Figure 14.6.7), and use the density function $\rho(x,y,z) = xy^2z$. Find the moments of inertia about the three coordinate planes.

Answer available at the end of this section.

Key Concepts

Finding the mass, center of mass, moments, and moments of inertia in double integrals:

- For a lamina R with a density function $\rho(x,y)$ at any point (x,y) in the plane, the mass is
$$m = \iint_R \rho(x,y)\,dA.$$
- The moments about the x-axis and y-axis are
$$M_x = \iint_R y\rho(x,y)\,dA \text{ and } M_y = \iint_R x\rho(x,y)\,dA.$$
- The center of mass is given by $\bar{x} = \dfrac{M_y}{m},\ \bar{y} = \dfrac{M_x}{m}$.
- The center of mass becomes the centroid of the plane when the density is constant.
- The moments of inertia about the x-axis, y-axis, and the origin are
$$I_x = \iint_R y^2\rho(x,y)\,dA,\ I_y = \iint_R x^2\rho(x,y)\,dA, \text{ and } I_0 = I_x + I_y = \iint_R (x^2+y^2)\rho(x,y)\,dA.$$

Finding the mass, center of mass, moments, and moments of inertia in triple integrals:

- For a solid object Q with a density function $\rho(x,y,z)$ at any point (x,y,z) in space, the mass is
$$m = \iiint_Q \rho(x,y,z)\,dV.$$
- The moments about the xy-plane, the xz-plane, and the yz-plane are
$$M_{xy} = \iiint_Q z\rho(x,y,z)\,dV,\ M_{xz} = \iiint_Q y\rho(x,y,z)\,dV,\ M_{yz} = \iiint_Q x\rho(x,y,z)\,dV$$
- The center of mass is given by $\bar{x} = \dfrac{M_{yz}}{m},\ \bar{y} = \dfrac{M_{xz}}{m},\ \bar{z} = \dfrac{M_{xy}}{m}$.
- The center of mass becomes the centroid of the solid when the density is constant.
- The moments of inertia about the yz-plane, the xz-plane, and the xy-plane are
$$I_x = \iiint_Q (y^2+z^2)\,\rho(x,y,z)\,dV,\ I_y = \iiint_Q (x^2+z^2)\,\rho(x,y,z)\,dV,\ I_z = \iiint_Q (x^2+y^2)\,\rho(x,y,z)\,dV.$$

Key Equations

- **Mass of a lamina**

$$m = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} m_{ij} = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_R \rho(x,y)\,dA$$

- **Moment about the *x*-axis**

$$M_x = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} (y_{ij}^*) m_{ij} = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} (y_{ij}^*)\rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_R y\rho(x,y)\,dA$$

- **Moment about the *y*-axis**

$$M_y = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} (x_{ij}^*) m_{ij} = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} (x_{ij}^*)\rho(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_R x\rho(x,y)\,dA$$

- **Center of mass of a lamina**

$$\bar{x} = \frac{M_y}{m} = \frac{\iint_R x\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA} \; and \; \bar{y} = \frac{M_x}{m} = \frac{\iint_R y\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA}$$

Glossary

radius of gyration
: the distance from an object's center of mass to its axis of rotation

Answers

Exercise 14.6.1

$\frac{9\pi}{8}\ kg$

Exercise 14.6.2

$M_x = \frac{81\pi}{64}$ and $M_y = \frac{81\pi}{64}$

Exercise 14.6.3

$\bar{x} = \frac{M_y}{m} = \frac{81\pi/64}{9\pi/8} = \frac{9}{8}$ and $\bar{y} = \frac{M_x}{m} = \frac{81\pi}{9\pi/8} = \frac{0}{8}$.

Exercise 14.6.4

$\bar{x} = \frac{M_y}{m} = \frac{1/20}{1/12} = \frac{3}{5}$ and $\bar{y} = \frac{M_x}{m} = \frac{1/24}{1/12} = \frac{1}{2}$

Exercise 14.6.5

$x_c = \frac{M_y}{m} = \frac{1/15}{1/6} = \frac{2}{5}$ and $y_c = \frac{M_x}{m} = \frac{1/12}{1/6} = \frac{1}{2}$

Exercise 14.6.6

$$I_x = \int_{x=0}^{x=2}\int_{y=0}^{y=x} y^2\sqrt{xy}\,dy\,dx = \frac{64}{35}$$

and

$$I_y = \int_{x=0}^{x=2}\int_{y=0}^{y=x} x^2\sqrt{xy}\,dy\,dx = \frac{64}{35}.$$

Also,

$$I_0 = \int_{x=0}^{x=2}\int_{y=0}^{y=x} (x^2+y^2)\sqrt{xy}\,dy\,dx = \frac{128}{21}$$

Exercise 14.6.7

Follow the steps shown in the previous example.

Exercise 14.6.7

$R_x = \frac{6\sqrt{35}}{35}$, $R_y = \frac{6\sqrt{15}}{15}$, and $R_0 = \frac{4\sqrt{42}}{7}$.

Exercise 14.6.8

Follow the steps in the previous example.

Exercise 14.6.8

$\frac{54}{35} = 1.543$

Exercise 14.6.9

Check that $M_{xy} = \frac{27}{35}$, $M_{xz} = \frac{243}{140}$, and $M_{yz} = \frac{81}{35}$. Then use m from a previous checkpoint question.

Exercise 14.6.9

$\left(\frac{3}{2}, \frac{9}{8}, \frac{1}{2}\right)$

Exercise 14.6.10

The moments of inertia of the tetrahedron Q about the yz-plane, the xz-plane, and the xy-plane are $99/35,\ 36/7$ and $243/35$, respectively.

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14.7: Change of Variables in Multiple Integrals (Jacobians)

Learning Objectives

- Determine the image of a region under a given transformation of variables.
- Compute the Jacobian of a given transformation.
- Evaluate a double integral using a change of variables.
- Evaluate a triple integral using a change of variables.

Recall from Substitution Rule the method of integration by substitution. When evaluating an integral such as

$$\int_2^3 x(x^2-4)^5 dx,$$

we substitute $u = g(x) = x^2 - 4$. Then $du = 2x\ dx$ or $x\ dx = \frac{1}{2}du$ and the limits change to $u = g(2) = 2^2 - 4 = 0$ and $u = g(3) = 9 - 4 = 5$. Thus the integral becomes

$$\int_0^5 \frac{1}{2}u^5 du$$

and this integral is much simpler to evaluate. In other words, when solving integration problems, we make appropriate substitutions to obtain an integral that becomes much simpler than the original integral.

We also used this idea when we transformed double integrals in rectangular coordinates to polar coordinates and transformed triple integrals in rectangular coordinates to cylindrical or spherical coordinates to make the computations simpler. More generally,

$$\int_a^b f(x)dx = \int_c^d f(g(u))g'(u)du,$$

Where $x = g(u),\ dx = g'(u)du$, and $u = c$ and $u = d$ satisfy $c = g(a)$ and $d = g(b)$.

A similar result occurs in double integrals when we substitute

- $x = f(r,\theta) = r\ \cos\ \theta$
- $y = g(r,\theta) = r\ \sin\ \theta$, and
- $dA = dx\ dy = r\ dr\ d\theta$.

Then we get

$$\iint_R f(x,y)dA = \iint_S (r\cos\theta,\ r\sin\theta)r\,dr\,d\theta$$

where the domain R is replaced by the domain S in polar coordinates. Generally, the function that we use to change the variables to make the integration simpler is called a transformation or mapping.

Planar Transformations

A planar transformation T is a function that transforms a region G in one plane into a region R in another plane by a change of variables. Both G and R are subsets of R^2. For example, Figure $14.7.1$ shows a region G in the uv-plane transformed into a region R in the xy-plane by the change of variables $x = g(u,v)$ and $y = h(u,v)$, or sometimes we write $x = x(u,v)$ and $y = y(u,v)$. We shall typically assume that each of these functions has continuous first partial derivatives, which means $g_u,\ g_v,\ h_u$, and h_v exist and are also continuous. The need for this requirement will become clear soon.

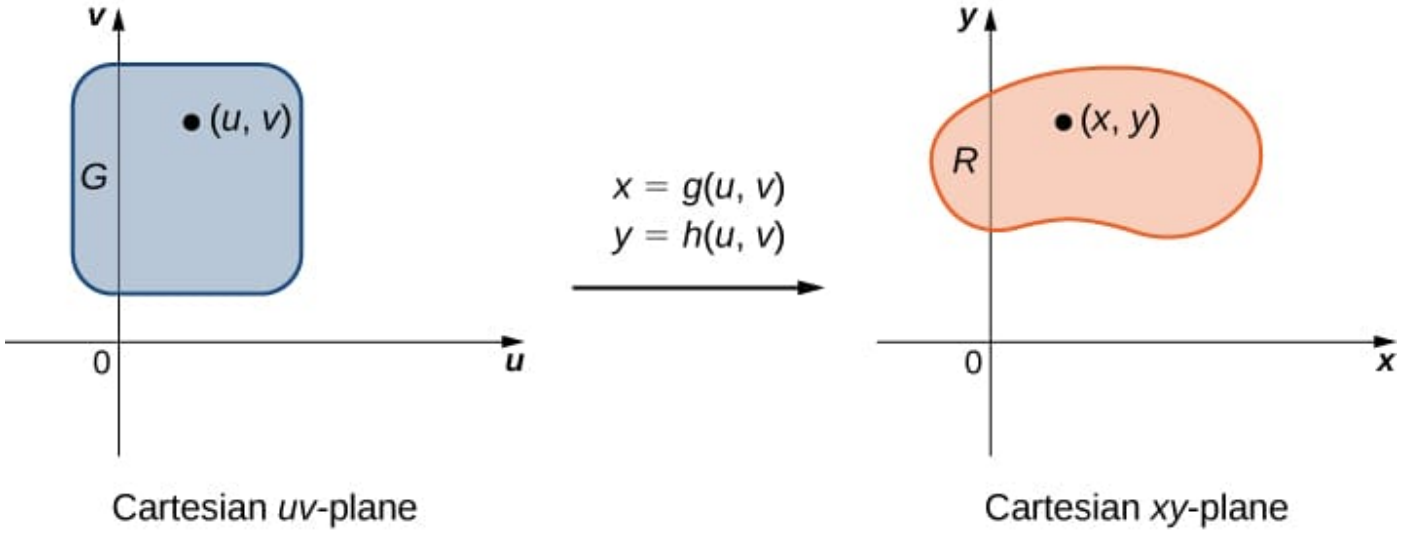


Figure 14.7.1: The transformation of a region G in the uv-plane into a region R in the xy-plane.

Definition: one-to-one transformation

A transformation $T: \ G \to R$, defined as $T(u, v) = (x, y)$, is said to be a one-to-one transformation if no two points map to the same image point.

To show that T s a one-to-one transformation, we assume $T(u_1, v_1) = T(u_2, v_2)$ and show that as a consequence we obtain$(u_1, v_1) = (u_2, v_2)$. If the transformation T is one-to-one in the domain G, then the inverse T^{-1} exists with the domain R such that $T^{-1} \circ T$ and $T \circ T^{-1}$ are identity functions.

Figure 14.7.2 shows the mapping $T(u, v) = (x, y)$ where x and y are related to u and v by the equations $x = g(u, v)$ and $y = h(u, v)$. The region G is the domain of T and the region R is the range of T, also known as the *image* of G under the transformation T.

Example 14.7.1A: Determining How the Transformation Works

Suppose a transformation T is defined as $T(r, \theta) = (x, y)$ where $x = r \ \cos \ \theta, \ y = r \ \sin \ \theta$. Find the image of the polar rectangle $G = \{(r, \theta) | 0 \leq r \leq 1, \ 0 \leq \theta \leq \pi/2\}$ in the $r\theta$-plane to a region R in the xy-plane. Show that T is a one-to-one transformation in G and find $T^{-1}(x, y)$.

Solution

Since r varies from 0 to 1 in the $r\theta$-plane, we have a circular disc of radius 0 to 1 in the xy-plane. Because θ varies from 0 to $\pi/2$ in the $r\theta$-plane, we end up getting a quarter circle of radius 1 in the first quadrant of the xy-plane (Figure 14.7.2). Hence R is a quarter circle bounded by $x^2 + y^2 = 1$ in the first quadrant.

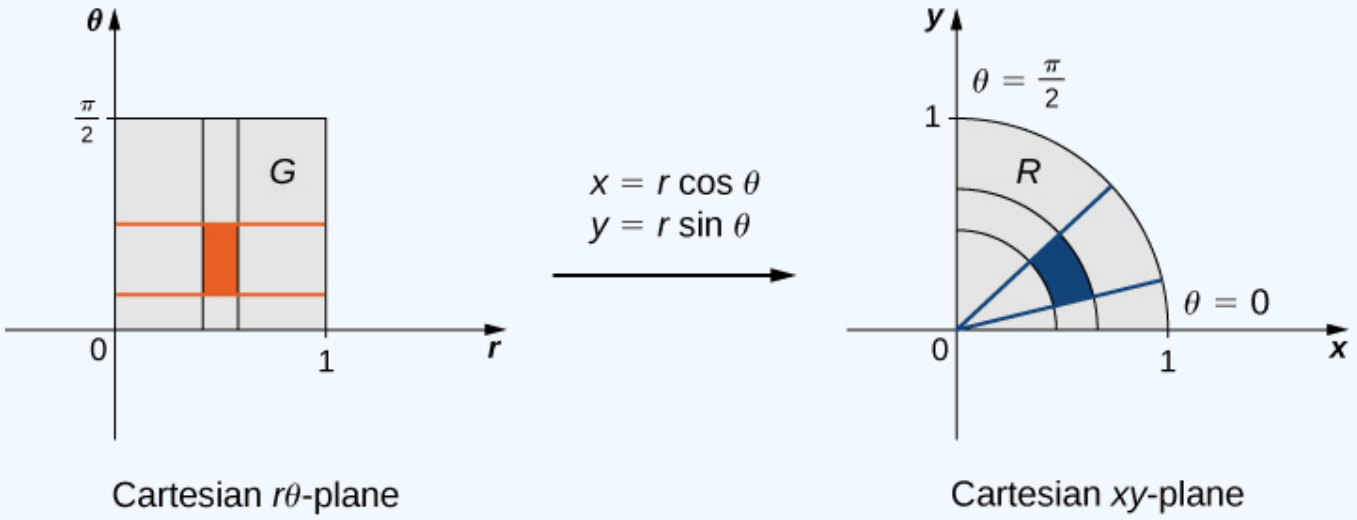


Figure 14.7.2: A rectangle in the $r\theta$-plane is mapped into a quarter circle in the xy-plane.

In order to show that T is a one-to-one transformation, assume $T(r_1, \theta_1) = T(r_2, \theta_2)$ and show as a consequence that $(r_1, \theta_1) = (r_2, \theta_2)$. In this case, we have

$$T(r_1, \theta_1) = T(r_2, \theta_2),$$

$$(x_1, y_1) = (x_1, y_1),$$

$$(r_1 \cos\ \theta_1, r_1 \sin\ \theta_1) = (r_2 \cos\ \theta_2, r_2 \sin\ \theta_2),$$

$$r_1 \cos\ \theta_1 = r_2 \cos\ \theta_2,\ r_1 \sin\ \theta_1 = r_2 \sin\ \theta_2.$$

Dividing, we obtain

$$\frac{r_1 \cos\ \theta_1}{r_1 \sin\ \theta_1} = \frac{r_2 \cos\ \theta_2}{r_2 \sin\ \theta_2}$$

$$\frac{\cos\ \theta_1}{\sin\ \theta_1} = \frac{\cos\ \theta_2}{\sin\ \theta_2}$$

$$\tan\ \theta_1 = \tan\ \theta_2$$

$$\theta_1 = \theta_2$$

since the tangent function is one-one function in the interval $0 \leq \theta \leq \pi/2$. Also, since $0 \leq r \leq 1$, we have $r_1 = r_2,\ \ \theta_1 = \theta_2$. Therefore, $(r_1, \theta_1) = (r_2, \theta_2)$ and T is a one-to-one transformation from G to R.

To find $T^{-1}(x, y)$ solve for r, θ in terms of x, y. We already know that $r^2 = x^2 + y^2$ and $\tan\ \ \theta = \frac{y}{x}$. Thus $T^{-1}(x, y) = (r, \theta)$ is defined as $r = \sqrt{x^2 + y^2}$ and $\tan^{-1}\left(\frac{y}{x}\right)$.

Example 14.7.1B: Finding the Image under T

Let the transformation T be defined by $T(u, v) = (x, y)$ where $x = u^2 - v^2$ and $y = uv$. Find the image of the triangle in the uv -plane with vertices $(0, 0),\ \ (0, 1)$, and $(1, 1)$.

Solution

The triangle and its image are shown in Figure 14.7.3. To understand how the sides of the triangle transform, call the side that joins $(0, 0)$ and $(0, 1)$ side A, the side that joins $(0, 0)$ and $(1, 1)$ side B, and the side that joins $(1, 1)$ and $(0, 1)$ side C.

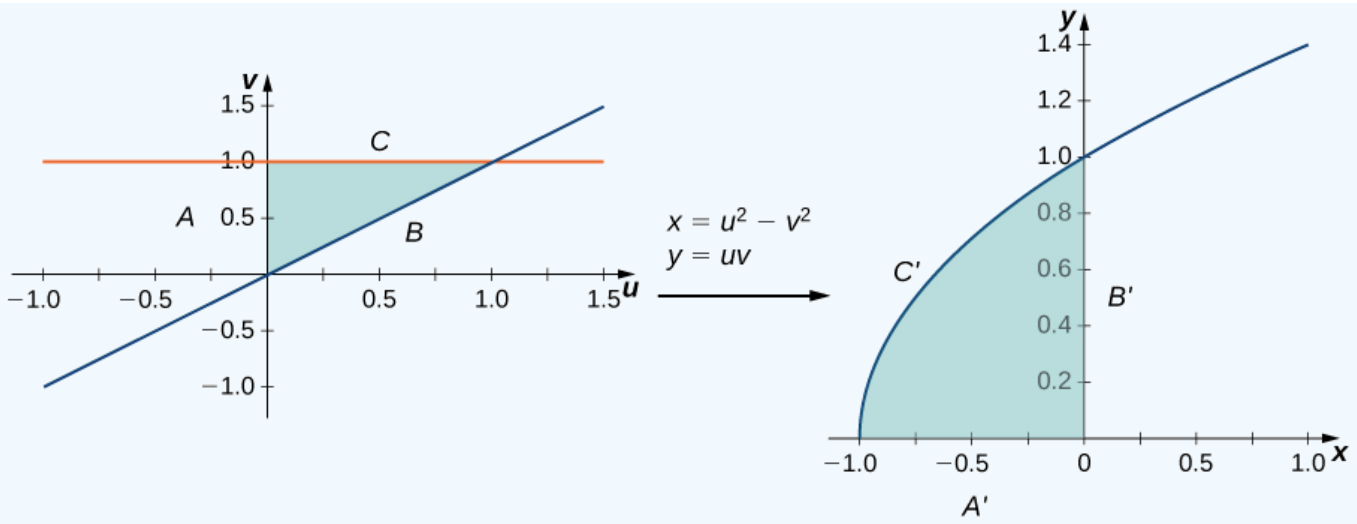


Figure 14.7.3: A triangular region in the uv-plane is transformed into an image in the xy-plane.

- For the side A : $u = 0,\ 0 \le v \le 1$ transforms to $x = -v^2,\ y = 0$ so this is the side A' that joins $(-1, 0)$ and $(0, 0)$.
- For the side B : $u = v,\ 0 \le u \le 1$ transforms to $x = 0,\ y = u^2$ so this is the side B' that joins $(0, 0)$ and $(0, 1)$.
- For the side C : $0 \le u \le 1,\ v = 1$ transforms to $x = u^2 - 1,\ y = u$ (hence $x = y^2 - 1$ so this is the side C' that makes the upper half of the parabolic arc joining $(-1, 0)$ and $(0, 1)$.

All the points in the entire region of the triangle in the uv-plane are mapped inside the parabolic region in the xy-plane.

Exercise 14.7.1

Let a transformation T be defined as $T(u, v) = (x, y)$ where $x = u + v,\ y = 3v$. Find the image of the rectangle $G = \{(u, v) :\ 0 \le u \le 1,\ 0 \le v \le 2\}$ from the uv-plane after the transformation into a region R in the xy-plane. Show that T is a one-to-one transformation and find $T^{-1}(x, y)$.

Hint available at the end of this section.

Answer available at the end of this section.

Jacobians

Recall that we mentioned near the beginning of this section that each of the component functions must have continuous first partial derivatives, which means that g_u, g_v, h_u and h_v exist and are also continuous. A transformation that has this property is called a C^1 transformation (here C denotes continuous). Let $T(u, v) = (g(u, v),\ h(u, v))$, where $x = g(u, v)$ and $y = h(u, v)$ be a one-to-one C^1 transformation. We want to see how it transforms a small rectangular region $S,\ \Delta u$ units by Δv units, in the uv-plane (Figure 14.7.4).

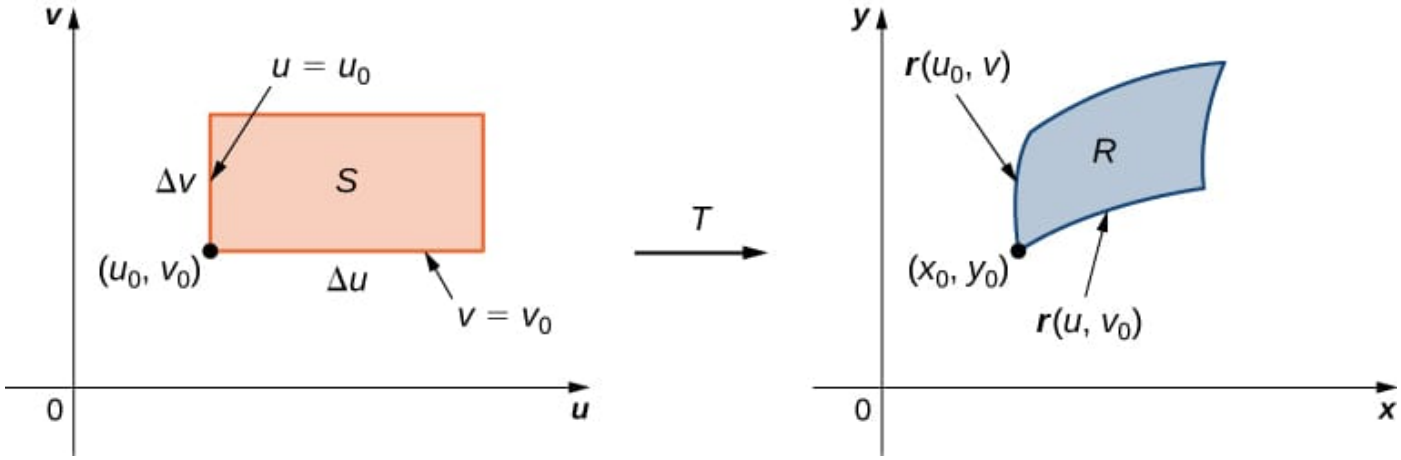


Figure 14.7.4: A small rectangle S in the uv-plane is transformed into a region R in the xy-plane.

Since $x = g(u, v)$ and $y = h(u, v)$, we have the position vector $r(u, v) = g(u, v)i + h(u, v)j$ of the image of the point (u, v). Suppose that $(u_0, v_0)$is the coordinate of the point at the lower left corner that mapped to $(x_0, y_0) = T(u_0, v_0)$ The line $v = v_0$ maps to the image curve with vector function $r(u, v_0)$, and the tangent vector at (x_0, y_0) to the image curve is

$$r_u = g_u(u_0, v_0)i + h_v(u_0, v_0)j = \frac{\partial x}{\partial u}i + \frac{\partial y}{\partial u}j.$$

Similarly, the line $u = u_0$ maps to the image curve with vector function $r(u_0, v)$, and the tangent vector at (x_0, y_0) to the image curve is

$$r_v = g_v(u_0, v_0)i + h_u(u_0, v_0)j = \frac{\partial x}{\partial v}i + \frac{\partial y}{\partial v}j.$$

Now, note that

$$r_u = \lim_{\Delta u \to 0} \frac{r(u_0 + \Delta u, v_0) - r(u_0, v_0)}{\Delta u} \, so \, r(u_0 + \Delta u, v_0) - r(u_0, v_0) \approx \Delta u r_u.$$

Similarly,

$$r_v = \lim_{\Delta v \to 0} \frac{r(u_0, v_0 + \Delta v) - r(u_0, v_0)}{\Delta v} \, so \, r(u_0, v_0 + \Delta v) - r(u_0, v_0) \approx \Delta v r_v.$$

This allows us to estimate the area ΔA of the image R by finding the area of the parallelogram formed by the sides $\Delta v r_v$ and $\Delta u r_u$. By using the cross product of these two vectors by adding the **k**th component as 0, the area ΔA of the image R (refer to The Cross Product) is approximately $|\Delta u r_u \times \Delta v r_v| = |r_u \times r_v| \Delta u \Delta v$. In determinant form, the cross product is

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} k = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right) k$$

Since $|k| = 1$, we have

$\Delta A \approx |r_u \times r_v| \Delta u \Delta v = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right) \Delta u \Delta v.$

Definition: Jacobian

The *Jacobian* of the C^1 transformation $T(u, v) = (g(u, v),\ h(u, v))$ is denoted by $J(u, v)$ and is defined by the 2×2 determinant

$$J(u,v)=\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u}\\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}\end{vmatrix}=\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right).$$

Using the definition, we have

$$\Delta A\approx J(u,v)\Delta u\Delta v=\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\Delta u\Delta v.$$

Note that the Jacobian is frequently denoted simply by

$$J(u,v)=\frac{\partial(x,y)}{\partial(u,v)}.$$

Note also that

$$\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u}\\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}\end{vmatrix}=\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}.$$

Hence the notation $J(u,v)=\frac{\partial(x,y)}{\partial(u,v)}$ suggests that we can write the Jacobian determinant with partials of x in the first row and partials of y in the second row.

Example $14.7.2A$: Finding the Jacobian

Find the Jacobian of the transformation given in Example $14.7.1A$.

Solution

The transformation in the example is $T(r,\theta)=(r\ \cos\ \theta,\ r\ \sin\ \theta)$ where $x=r\ \cos\ \theta$ and $y=r\ \sin\ \theta$. Thus the Jacobian is

$$J(r,\theta)=\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r\ \cos^2\theta+r\ \sin^2\theta=r(\cos^2\theta+\sin^2\theta)=r.$$

Example $14.7.2B$: Finding the Jacobian

Find the Jacobian of the transformation given in Example $14.7.1B$.

Solution

The transformation in the example is $T(u,v)=(u^2-v^2,uv)$ where $x=u^2-v^2$ and $y=uv$. Thus the Jacobian is

$$J(u,v)=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}2u & -2v\\ v & u\end{vmatrix}=2u^2+2v^2.$$

Exercise $14.7.2$

Find the Jacobian of the transformation given in the previous checkpoint: $T(u,v)=(u+v,2v)$.

Hint available at the end of this section.

Answer available at the end of this section.

Change of Variables for Double Integrals

We have already seen that, under the change of variables $T(u,v)=(x,y)$ where $x=g(u,v)$ and $y=h(u,v)$, a small region ΔA in the xy-plane is related to the area formed by the product $\Delta u \Delta v$ in the uv-plane by the approximation

$$\Delta A \approx J(u,v)\Delta u,\ \Delta v.$$

Now let's go back to the definition of double integral for a minute:

$$\iint_R f(x,y) fA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij},y_{ij})\Delta A.$$

Referring to Figure 14.7.5, observe that we divided the region S in the uv-plane into small subrectangles S_{ij} and we let the subrectangles R_{ij} in the xy-plane be the images of S_{ij} under the transformation $T(u,v)=(x,y)$.

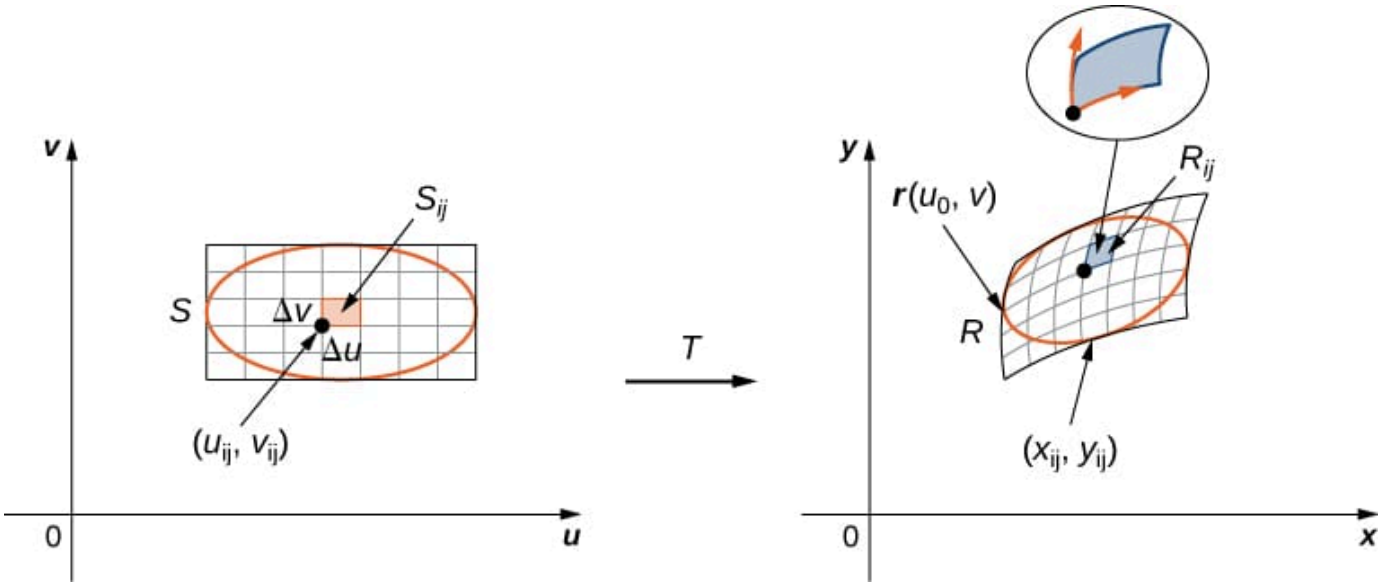


Figure 14.7.5: The subrectangles S_{ij} in the uv-plane transform into subrectangles R_{ij} in the xy-plane.

Then the double integral becomes

$$\iint_R = f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij},y_{ij})\Delta A = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(g(u_{ij},v_{ij}),\ h(u_{ij},v_{ij}))|J(u_{ij},v_{ij})|\Delta u \Delta v.$$

Notice this is exactly the double Riemann sum for the integral

$$\iint_S f(g(u,v),\ h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv.$$

Change of Variables for Double Integrals

Let $T(u,v)=(x,y)$ where $x=g(u,v)$ and $y=h(u,v)$ be a one-to-one C^1 transformation, with a nonzero Jacobian on the interior of the region S in the uv-plane it maps S into the region R in the xy-plane. If f is continuous on R, then

$$\iint_R f(x,y)dA = \iint_S f(g(u,v),\ h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv.$$

With this theorem for double integrals, we can change the variables from (x, y) to (u, v) in a double integral simply by replacing

$$dA = dx\, dy = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv$$

when we use the substitutions $x = g(u, v)$ and $y = h(u, v)$ and then change the limits of integration accordingly. This change of variables often makes any computations much simpler.

Example 14.7.3: Changing Variables from Rectangular to Polar Coordinates

Consider the integral

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2 + y^2} dy\, dx.$$

Use the change of variables $x = r\ \cos\ \theta$ and $y = r\ \sin\ \theta$, and find the resulting integral.

Solution

First we need to find the region of integration. This region is bounded below by $y = 0$ and above by $y = \sqrt{2x - x^2}$ (Figure 14.7.6).

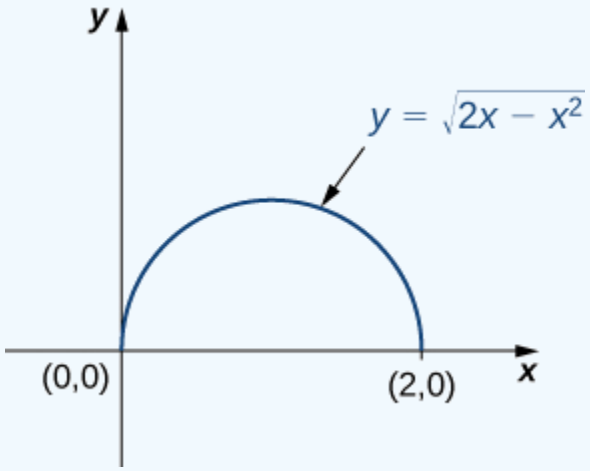


Figure 14.7.6: Changing a region from rectangular to polar coordinates.

Squaring and collecting terms, we find that the region is the upper half of the circle $x^2 + y^2 - 2x = 0$, that is $y^2 + (x - 1)^2 = 1$. In polar coordinates, the circle is $r = 2\ cos\ \theta$ so the region of integration in polar coordinates is bounded by $0 \le r \le \cos\ \theta$ and $0 \le \theta \le \frac{\pi}{2}$.

The Jacobian is $J(r, \theta) = r$, as shown in Example 14.7.2A. Since $r \ge 0$, we have $|J(r, \theta)| = r$.

The integrand $\sqrt{x^2 + y^2}$ changes to r in polar coordinates, so the double iterated integral is

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2 + y^2} dy\, dx = \int_0^{\pi/2} \int_0^{2\ cos\ \theta} r|j(r, \theta)| dr\, d\theta = \int_0^{\pi/2} \int_0^{2\ cos\ \theta} r^2 dr\, d\theta.$$

Exercise 14.7.3

Considering the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy\ dx$, use the change of variables $x = r\ cos\ \theta$ and $y = r\ sin\ \theta$ and find the resulting integral.

Hint available at the end of this section.

Answer available at the end of this section.

Notice in the next example that the region over which we are to integrate may suggest a suitable transformation for the integration. This is a common and important situation.

Example 14.7.4: Changing Variables

Consider the integral

$$\iint_R (x-y)dy\,dx,$$

where R is the parallelogram joining the points $(1,2),\ (3,4),\ (4,3)$, and $(6,5)$ (Figure 14.7.7). Make appropriate changes of variables, and write the resulting integral.

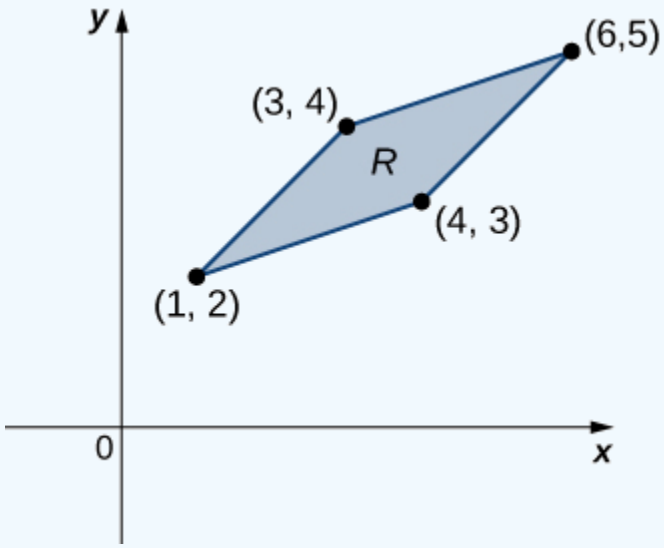


Figure 14.7.7: The region of integration for the given integral.

Solution

First, we need to understand the region over which we are to integrate. The sides of the parallelogram are $x-y+1,\ x-y-1=0,\ x-3y+5=0$ and $x-3y+9=0$ (Figure 14.7.8). Another way to look at them is $x-y=-1,\ x-y=1,\ x-3y=-5$, and $x-3y=9$.

Clearly the parallelogram is bounded by the lines $y=x+1,\ y=x-1,\ y=\frac{1}{3}(x+5)$, and $y=\frac{1}{3}(x+9)$.

Notice that if we were to make $u=x-y$ and $v=x-3y$, then the limits on the integral would be $-1\le u\le 1$ and $-9\le v\le -5$.

To solve for x and y, we multiply the first equation by 3 and subtract the second equation, $3u-v=(3x-3y)-(x-3y)=2x$. Then we have $x=\frac{3u-v}{2}$. Moreover, if we simply subtract the second equation from the first, we get $u-v=(x-y)-(x-3y)=2y$ and $y=\frac{u-v}{2}$.

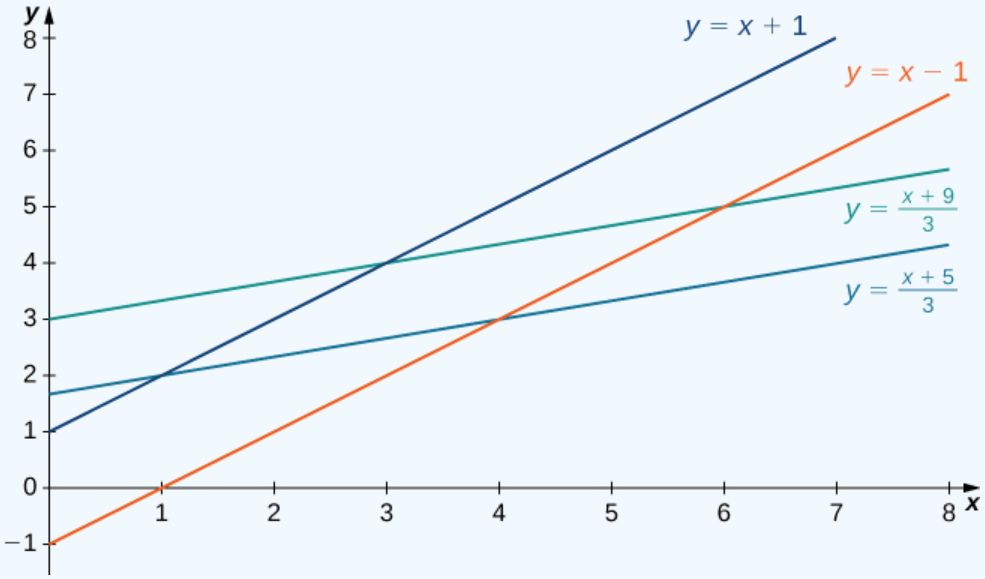


Figure 14.7.8: A parallelogram in the xy-plane that we want to transform by a change in variables.

Thus, we can choose the transformation

$$T(u,v) = \left(\frac{3u-v}{2}, \frac{u-v}{2}\right)$$

and compute the Jacobian $J(u,v)$. We have

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 3/2 & -1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{3}{4} + \frac{1}{4} = -\frac{1}{2}$$

Therefore, $|J(u,v)| = \frac{1}{2}$. Also, the original integrand becomes

$$x - y = \frac{1}{2}[3u - v - u + v] = \frac{1}{2}[3u - u] = \frac{1}{2}[2u] = u.$$

Therefore, by the use of the transformation T, the integral changes to

$$\iint_R (x-y)dy\,dx = \int_{-9}^{-5}\int_{-1}^{1} J(u,v)u\,du\,dv = \int_{-9}^{-5}\int_{-1}^{1}\left(\frac{1}{2}\right)u\,du\,dv,$$

which is much simpler to compute.

Exercise 14.7.4

Make appropriate changes of variables in the integral

$$\iint_R \frac{4}{(x-y)^2}dy\,dx,$$

where R is the trapezoid bounded by the lines $x - y = 2,\ x - y = 4,\ x = 0$, and $y = 0$. Write the resulting integral.

Hint available at the end of this section.

Answer available at the end of this section.

We are ready to give a problem-solving strategy for change of variables.

Problem-Solving Strategy: Change of Variables

1. Sketch the region given by the problem in the xy-plane and then write the equations of the curves that form the boundary.
2. Depending on the region or the integrand, choose the transformations $x = g(u, v)$ and $y = h(u, v)$.
3. Determine the new limits of integration in the uv-plane.
4. Find the Jacobian $J(u, v)$.
5. In the integrand, replace the variables to obtain the new integrand.
6. Replace $dy\ dx$ or $dx\ dy$, whichever occurs, by $J(u, v)du\ dv$.

In the next example, we find a substitution that makes the integrand much simpler to compute.

Example 14.7.5: Evaluating an Integral

Using the change of variables $u = x - y$ and $v = x + y$, evaluate the integral

$$\iint_R (x - y)e^{x^2-y^2} dA,$$

where R is the region bounded by the lines $x + y = 1$ and $x + y = 3$ and the curves $x^2 - y^2 = -1$ and $x^2 - y^2 = 1$ (see the first region in Figure 14.7.9).

Solution

As before, first find the region R and picture the transformation so it becomes easier to obtain the limits of integration after the transformations are made (Figure 14.7.9).

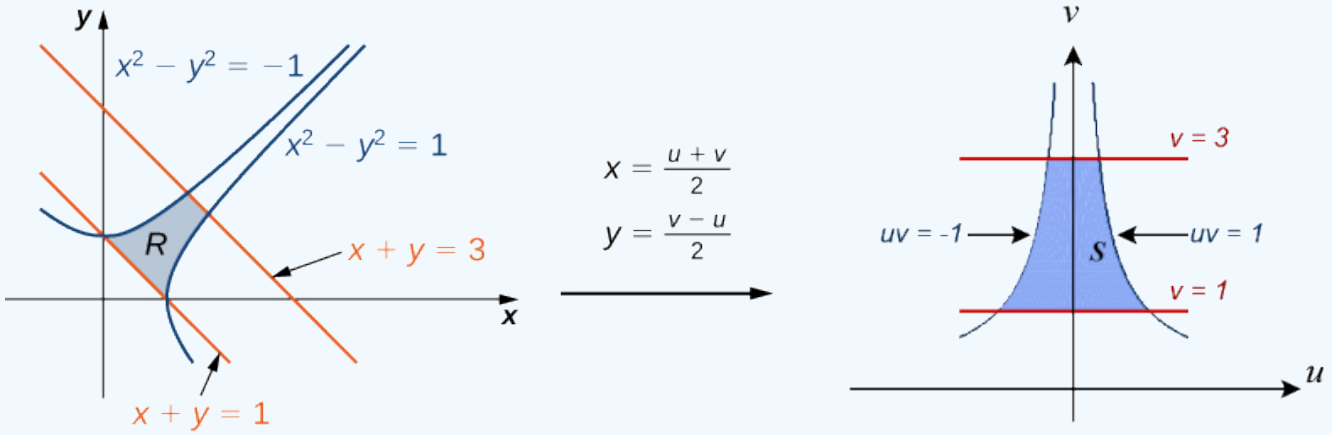


Figure 14.7.9: Transforming the region R into the region S to simplify the computation of an integral.

Given $u = x - y$ and $v = x + y$, we have $x = \frac{u+v}{2}$ and $y = \frac{v-u}{2}$ and hence the transformation to use is $T(u, v) = (\frac{u+v}{2}, \frac{v-u}{2})$. The lines $x + y = 1$ and $x + y = 3$ become $v = 1$ and $v = 3$, respectively. The curves $x^2 - y^2 = 1$ and $x^2 - y^2 = -1$ become $uv = 1$ and $uv = -1$, respectively.

Thus we can describe the region S (see the second region Figure 14.7.9) as

$$S = \left\{(u, v)|1 \le v \le 3,\ \frac{-1}{v} \le u \le \frac{1}{v}\right\}.$$

The Jacobian for this transformation is

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = \frac{1}{2}.$$

Therefore, by using the transformation T, the integral changes to

$$\iint_R (x-y)e^{x^2-y^2}\,dA = \frac{1}{2}\int_1^3 \int_{-1/v}^{1/v} ue^{uv}\,du\,dv.$$

Doing the evaluation, we have

$$\frac{1}{2}\int_1^3 \int_{-1/v}^{1/v} ue^{uv}\,du\,dv = \frac{2}{3e} \approx 0.245.$$

Exercise 14.7.5

Using the substitutions $x = v$ and $y = \sqrt{u+v}$, evaluate the integral $\iint_R y\ \sin(y^2 - x)\ dA$, where R is the region bounded by the lines $y = \sqrt{x},\ x = 2$ and $y = 0$.

Hint available at the end of this section.

Answer available at the end of this section.

Change of Variables for Triple Integrals

Changing variables in triple integrals works in exactly the same way. Cylindrical and spherical coordinate substitutions are special cases of this method, which we demonstrate here.

Suppose that G is a region in uvw-space and is mapped to D in xyz-space (Figure 14.7.10) by a one-to-one C^1 transformation $T(u,v,w) = (x,y,z)$ where $x = g(u,v,w),\ y = h(u,v,w)$, and $z = k(u,v,w)$.

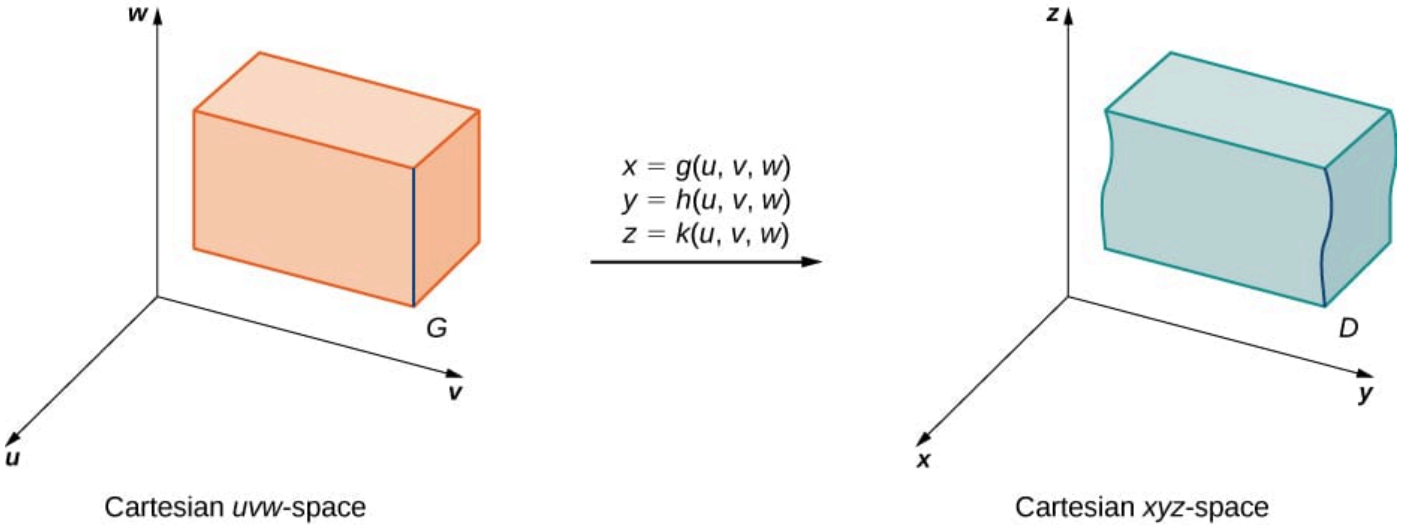


Figure 14.7.10: A region G in uvw-space mapped to a region D in xyz-space.

Then any function $F(x,y,z)$ defined on D can be thought of as another function $H(u,v,w)$ that is defined on G:

$$F(x,y,z) = F(g(u,v,w),\ h(u,v,w),\ k(u,v,w)) = H(u,v,w).$$

Now we need to define the Jacobian for three variables.

Definition: Jacobian determinant

The *Jacobian determinant* $J(u,v,w)$ in three variables is defined as follows:

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

This is also the same as

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

The Jacobian can also be simply denoted as $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.

With the transformations and the Jacobian for three variables, we are ready to establish the theorem that describes change of variables for triple integrals.

Change of Variables for Triple Integrals

Let $T(u,v,w) = (x,y,z)$ where $x = g(u,v,w),\ \ y = h(u,v,w)$, and $z = k(u,v,w)$, be a one-to-one C^1 transformation, with a nonzero Jacobian, that maps the region G in the uvw-space into the region D in the xyz-space. As in the two-dimensional case, if F is continuous on D, then

$$\iiint_D F(x,y,z)dV = \iiint_G f(g(u,v,w)\,h(u,v,w),\ k(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw \tag{14.7.1}$$

$$= \iiint_G H(u,v,w)|J(u,v,w)|du\,dv\,dw. \tag{14.7.2}$$

Let us now see how changes in triple integrals for cylindrical and spherical coordinates are affected by this theorem. We expect to obtain the same formulas as in Triple Integrals in Cylindrical and Spherical Coordinates.

Example 14.7.6A: Obtaining Formulas in Triple Integrals for Cylindrical and Spherical Coordinates

Derive the formula in triple integrals for

a. cylindrical and
b. spherical coordinates.

Solution

A.

For cylindrical coordinates, the transformation is $T(r,\theta,z) = (x,y,z)$ from the Cartesian $r\theta z$-space to the Cartesian xyz-space (Figure 14.7.11). Here $x = r\ \cos\ \theta,\ \ y = r\ \sin\theta$ and $z = z$. The Jacobian for the transformation is

$$J(r,\theta,z)=\frac{\partial(x,y,z)}{\partial(r,\theta,z)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z}\end{vmatrix}$$

$$\begin{vmatrix}\cos\theta & -r\sin\theta & 0\\ \sin\theta & r\cos\theta & 0\\ 0 & 0 & 1\end{vmatrix}=r\,\cos^2\theta+r\,\sin^2\theta=r.$$

We know that $r\geq 0$, so $|J(r,\theta,z)|=r$. Then the triple integral is

$$\iiint_D f(x,y,z)dV=\iiint_G f(r\,\cos\theta,\,r\,\sin\theta,\,z)r\,dr\,d\theta\,dz.$$

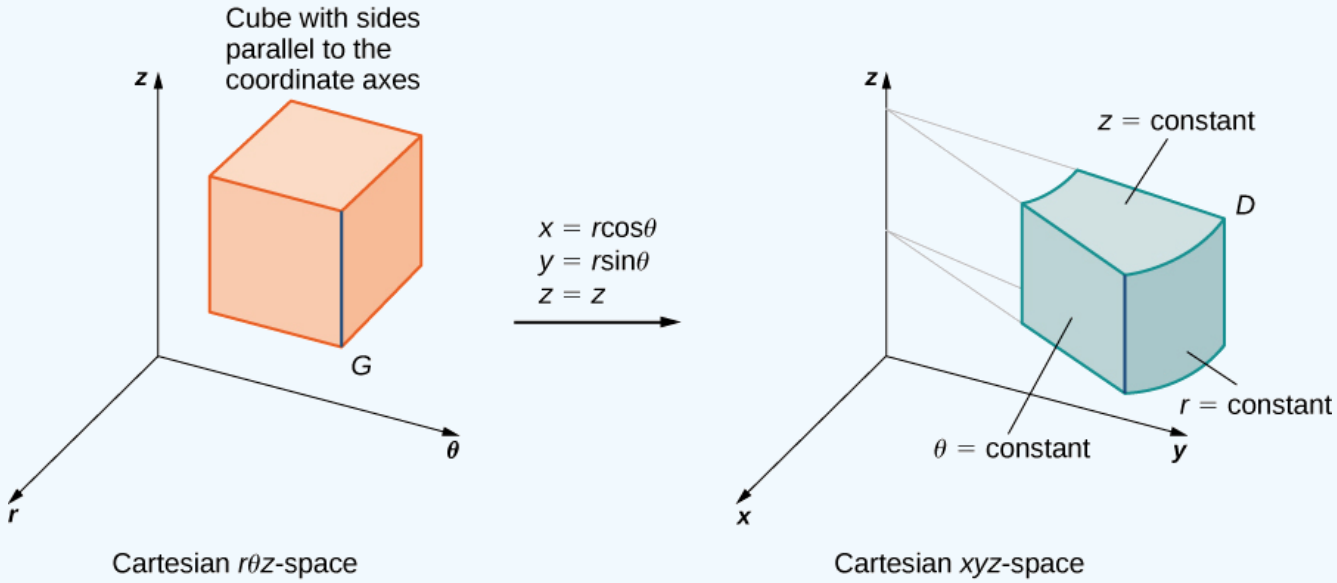


Figure 14.7.11: The transformation from rectangular coordinates to cylindrical coordinates can be treated as a change of variables from region G in $r\theta z$-space to region D in xyz-space.

B.

For spherical coordinates, the transformation is $T(\rho,\theta,\varphi)$ from the Cartesian $\rho\theta\varphi$-space to the Cartesian xyz-space (Figure 14.7.12). Here $x=\rho\ \sin\varphi\ \cos\theta,\ \ y=\rho\ \sin\varphi\ \sin\theta$, and $z=\rho\ \cos\varphi$. The Jacobian for the transformation is

$$J(\rho,\theta,\varphi)=\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}=\begin{vmatrix}\frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi}\\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi}\\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi}\end{vmatrix}=\begin{vmatrix}\sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta\\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta\\ \cos\varphi & 0 & -\rho\sin\varphi\end{vmatrix}.$$

Expanding the determinant with respect to the third row:

$$\begin{aligned}&=\cos\varphi\begin{vmatrix}-\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta\\ \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta\end{vmatrix}-\rho\sin\varphi\begin{vmatrix}\sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta\\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta\end{vmatrix}\\ &=\cos\varphi(-\rho^2\sin\varphi\,\cos\varphi\,\sin^2\theta-\rho^2\,\sin\varphi\,\cos\varphi\,\cos^2\theta)\\ &\quad-\rho\sin\varphi(\rho\sin^2\varphi\cos^2\theta+\rho\sin^2\varphi\sin^2\theta)\\ &=-\rho^2\sin\varphi\cos^2\varphi(\sin^2\theta+\cos^2\theta)-\rho^2\sin\varphi\sin^2\varphi(\sin^2\theta+\cos^2\theta)\\ &=-\rho^2\sin\varphi\cos^2\varphi-\rho^2\sin\varphi\sin^2\varphi\\ &=-\rho\sin\varphi(\cos^2\varphi+\sin^2\varphi)=-\rho^2\sin\varphi.\end{aligned}$$

Since $0\leq\varphi\leq\pi$, we must have $\sin\varphi\geq 0$. Thus $|J(\rho,\theta,\varphi)|=|-\rho^2\sin\varphi|=\rho^2\sin\varphi$.

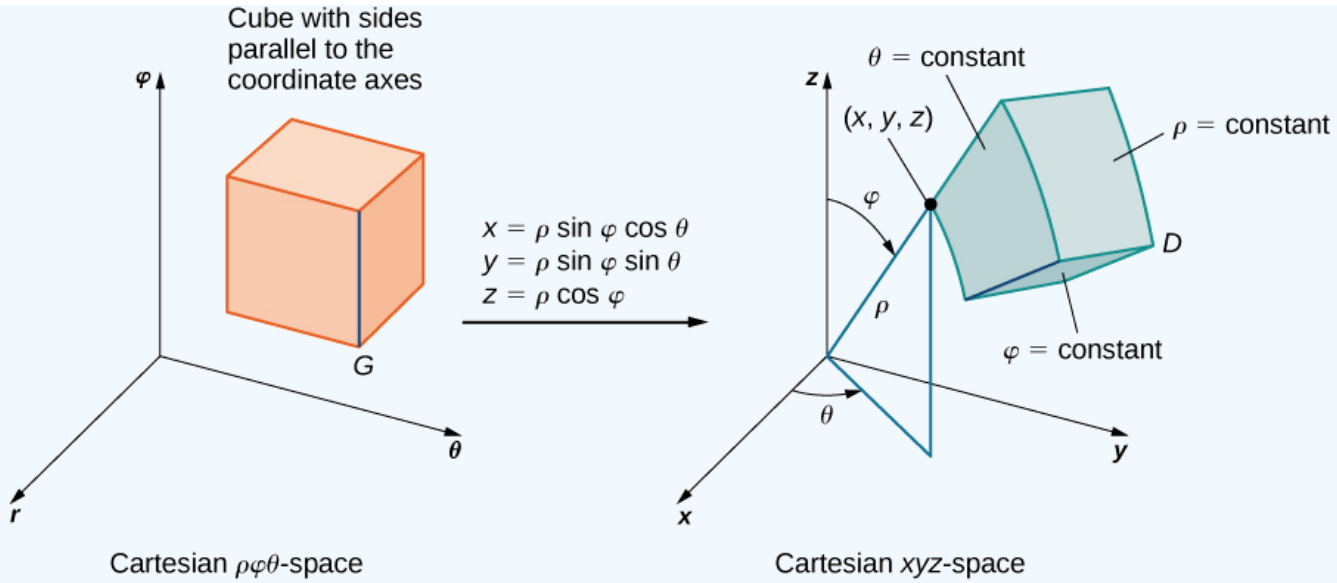


Figure 14.7.12: The transformation from rectangular coordinates to spherical coordinates can be treated as a change of variables from region G in $\rho\theta\varphi$-space to region D in xyz-space.

Then the triple integral becomes

$$\iiint_D f(x,y,z)dV = \iiint_G f(\rho\,\sin\varphi\,\cos\theta,\ \rho\,\sin\varphi\,\sin\theta, \rho\,\cos\varphi)\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta.$$

Let's try another example with a different substitution.

Example 14.7.6B: Evaluating a Triple Integral with a Change of Variables

Evaluate the triple integral

$$\int_0^3\int_0^4\int_{y/2}^{(y/2)+1}\left(x+\frac{z}{3}\right)dx\,dy\,dz$$

In xyz-space by using the transformation

$u=(2x-y)/2,\ \ v=y/2$, and $w=z/3$.

Then integrate over an appropriate region in uvw-space.

Solution

As before, some kind of sketch of the region G in xyz-space over which we have to perform the integration can help identify the region D in uvw-space (Figure 14.7.13). Clearly G in xyz-space is bounded by the planes $x=y/2,\ \ x=(y/2)+1,\ \ y=0,\ \ y=4,\ \ z=0$, and $z=4$. We also know that we have to use $u=(2x-y)/2,\ \ v=y/2$, and $w=z/3$ for the transformations. We need to solve for x, y and z. Here we find that $x=u+v,\ \ y=2v$, and $z=3w$.

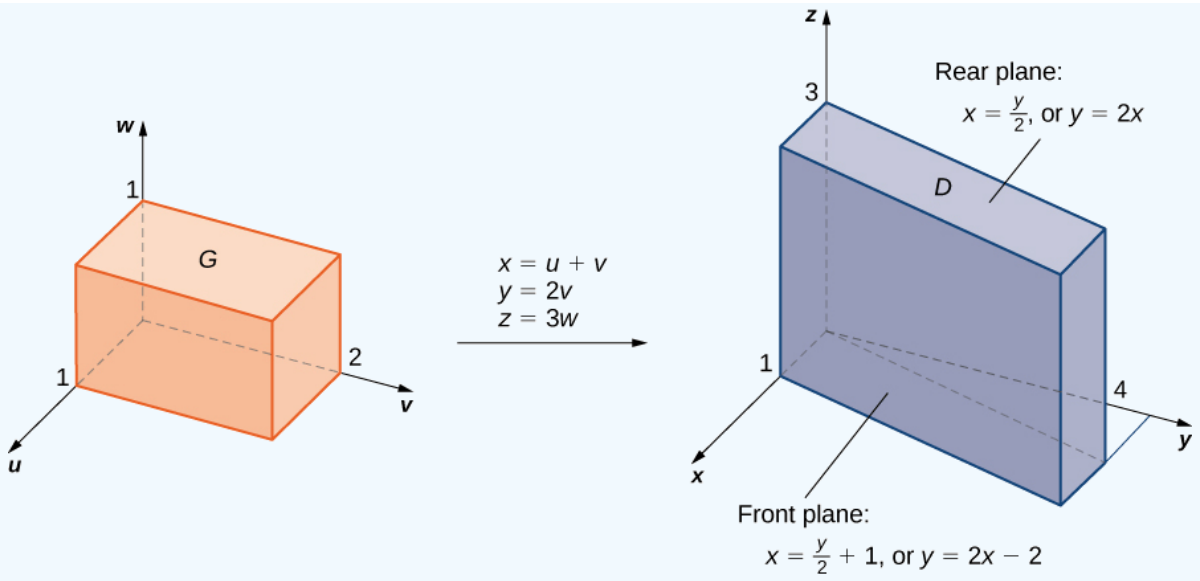


Figure 14.7.13: The region G in uvw-space is transformed to region D in xyz-space.

Using elementary algebra, we can find the corresponding surfaces for the region G and the limits of integration in uvw-space. It is convenient to list these equations in a table.

Equations in xyz for the region D	Corresponding equations in uvw for the region G	Limits for the integration in uvw
$x = y/2$	$u + v = 2v/2 = v$	$u = 0$
$x = y/2$	$u + v = (2v/2) + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$
$z = 0$	$3w = 0$	$w = 0$
$z = 3$	$3w = 3$	$w = 1$

Now we can calculate the Jacobian for the transformation:

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6.$$

The function to be integrated becomes

$$f(x,y,z) = x + \frac{z}{3} = u + v + \frac{3w}{3} = u + v + w.$$

We are now ready to put everything together and complete the problem.

$$\int_0^3 \int_0^4 \int_{y/2}^{(y/2)+1} \left(x + \frac{z}{3}\right) dx\, dy\, dz = \int_0^1 \int_0^2 \int_0^1 (u+v+w)|J(u,v,w)|du\, dv\, dw$$
$$= \int_0^1 \int_0^2 \int_0^1 (u+v+w)|6|du\, dv\, dw$$
$$= 6\int_0^1 \int_0^2 \int_0^1 (u+v+w)\, du\, dv\, dw$$
$$= 6\int_0^1 \int_0^2 \left[\frac{u^2}{2} + vu + wu\right]_0^1 dv\, dw$$
$$= 6\int_0^1 \int_0^2 \left(\frac{1}{2} + v + u\right) dv\, dw$$
$$= 6\int_0^1 \left[\frac{1}{2}v + \frac{v^2}{2} + wv\right]_0^2 dw$$
$$= 6\int_0^1 (3+2w)\, dw = 6\left[3w + w^2\right]_0^1 = 24.$$

Exercise 14.7.6

Let D be the region in xyz-space defined by $1 \le x \le 2,\ 0 \le xy \le 2$, and $0 \le z \le 1$.

Evaluate $\iiint_D (x^2y + 3xyz)\ dx\ dy\ dz$ by using the transformation $u = x,\ v = xy$, and $w = 3z$.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- A transformation T is a function that transforms a region G in one plane (space) into a region R. in another plane (space) by a change of variables.
- A transformation $T : G \to R$ defined as $T(u,v) = (x,y)$ (or $T(u,v,w) = (x,y,z)$)is said to be a one-to-one transformation if no two points map to the same image point.
- If f is continuous on R, then

$$\iint_R f(x,y)dA = \iint_S f(g(u,v),\ h(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv.$$

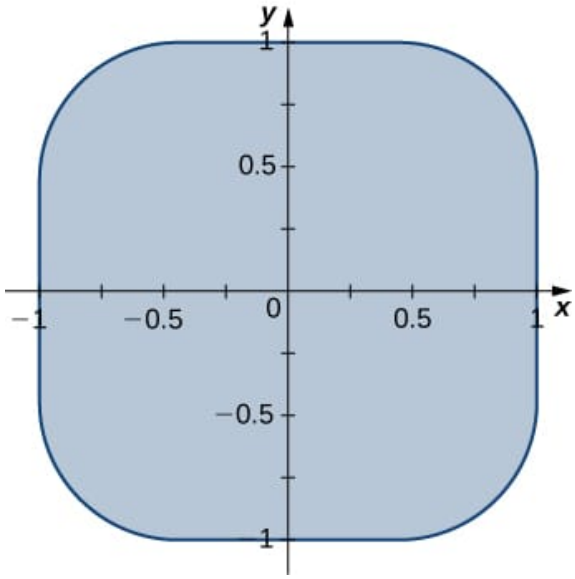
- If F is continuous on R, then

$$\iiint_R F(x,y,z)\, dV = \iiint_G F(g(u,v,w),\ h(u,v,w),\ k(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\, dv\, dw$$
$$= \iiint_G H(u,v,w)|J(u,v,w)|\, du\, dv\, dw.$$

[T] Lamé ovals (or superellipses) are plane curves of equations $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 1$, where *a*, *b*, and *n* are positive real numbers.

a. Use a CAS to graph the regions R bounded by Lamé ovals for $a = 1,\ b = 2,\ n = 4$ and $n = 6$ respectively.

b. Find the transformations that map the region R bounded by the Lamé oval $x^4 + y^4 = 1$ also called a squircle and graphed in the following figure, into the unit disk.



c. Use a CAS to find an approximation of the area $A(R)$ of the region R bounded by $x^4 + y^4 = 1$. Round your answer to two decimal places.

[T] Lamé ovals have been consistently used by designers and architects. For instance, Gerald Robinson, a Canadian architect, has designed a parking garage in a shopping center in Peterborough, Ontario, in the shape of a superellipse of the equation $(\frac{x}{a})^n + (\frac{y}{b})^n = 1$ with $\frac{a}{b} = \frac{9}{7}$ and $n = e$. Use a CAS to find an approximation of the area of the parking garage in the case $a = 900$ yards, $b = 700$ yards, and $n = 2.72$ yards.

[Hide Solution]

$A(R) \simeq 83,999.2$

Chapter Review Exercises

True or False? Justify your answer with a proof or a counterexample.

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dy\,dx$$

Fubini's theorem can be extended to three dimensions, as long as f is continuous in all variables.

[Hide solution]

True.

The integral

$$\int_0^{2\pi} \int_0^1 \int_0^1 dz\,dr\,d\theta$$

represents the volume of a right cone.

The Jacobian of the transformation for $x = u^2 - 2v,\ \ y = 3v - 2uv$ is given by $-4u^2 + 6u + 4v$.

[Hide Solution]

False.

Evaluate the following integrals.

$$\iint_R (5x^3y^2 - y^2)\, dA,\ R = \{(x,y)|0 \le x \le 2,\ 1 \le y \le 4\}$$

$$\iint_D \frac{y}{3x^2+1} dA,\ D = \{(x,y)|0 \le x \le 1,\ -x \le y \le x\}$$

[Hide Solution]

0

$$\iint_D \sin(x^2+y^2) dA$$

where D is a disk of radius 2 centered at the origin

$$\int_0^1 \int_0^1 xye^{x^2} dx\, dy$$

[Hide Solution]

$\frac{1}{4}$

$$\int_{-1}^1 \int_0^z \int_0^{x-z} 6dy\, dx\, dz$$

$$\iiint_R 3y\, dV,$$

where $R = \{(x,y,z)|0 \le x \le 1,\ 0 \le y \le x,\ 0 \le z \le \sqrt{9-y^2}\}$

[Hide Solution]

1.475

$$\int_0^2 \int_0^{2\pi} \int_r^1 r\, dz\, d\theta\, dr$$

$$\int_0^{2\pi} \int_0^{\pi/2} \int_1^3 \rho^2 \sin(\varphi) d\rho\, d\varphi,\ d\theta$$

[Hide Solution]

$\frac{52}{3}\pi$

$$\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\, dy\, sx$$

For the following problems, find the specified area or volume.

The area of region enclosed by one petal of $r = \cos(4\theta)$.

[Hide Solution]

$\frac{\pi}{16}$

The volume of the solid that lies between the paraboloid $z = 2x^2 + 2y^2$ and the plane $z = 8$.

The volume of the solid bounded by the cylinder $x^2 + y^2 = 16$ and from $z = 1$ to $z + x = 2$.

[Hide Solution]

93.291

The volume of the intersection between two spheres of radius 1, the top whose center is $(0, 0, 0.25)$ and the bottom, which is centered at $(0, 0, 0)$.

For the following problems, find the center of mass of the region.

$\rho(x, y) = xy$ on the circle with radius 1 in the first quadrant only.

[Hide Solution]

$(\frac{8}{15}, \frac{8}{15})$

$\rho(x, y) = (y + 1)\sqrt{x}$ in the region bounded by $y = e^x,\ \ y = 0$, and $x = 1$.

$\rho(x, y, z) = z$ on the inverted cone with radius 2 and height 2.

$(0, 0, \frac{8}{5})$

The volume an ice cream cone that is given by the solid above $z = \sqrt{(x^2 + y^2)}$ and below $z^2 + x^2 + y^2 = z$.

The following problems examine Mount Holly in the state of Michigan. Mount Holly is a landfill that was converted into a ski resort. The shape of Mount Holly can be approximated by a right circular cone of height 1100 ft and radius 6000 ft.

If the compacted trash used to build Mount Holly on average has a density $400\ \ lb/ft^3$, find the amount of work required to build the mountain.

[Hide Solution]

$1.452\pi \times 10^{15}$ ft-lb

In reality, it is very likely that the trash at the bottom of Mount Holly has become more compacted with all the weight of the above trash. Consider a density function with respect to height: the density at the top of the mountain is still density $400\ \ lb/ft^3$ and the density increases. Every 100 feet deeper, the density doubles. What is the total weight of Mount Holly?

The following problems consider the temperature and density of Earth's layers.

[T] The temperature of Earth's layers is exhibited in the table below. Use your calculator to fit a polynomial of degree 3 to the temperature along the radius of the Earth. Then find the average temperature of Earth. (*Hint*: begin at 0 in the inner core and increase outward toward the surface)

Layer	Depth from center (km)	Temperature ^oC
Rocky Crust	0 to 40	0
Upper Mantle	40 to 150	870
Mantle	400 to 650	870
Inner Mantel	650 to 2700	870
Molten Outer Core	2890 to 5150	4300
Inner Core	5150 to 6378	7200

Source: http://www.enchantedlearning.com/sub...h/Inside.shtml

[Hide Solution]

$y = -1.238 \times 10^{-7}x^3 + 0.001196x^2 - 3.666x + 7208$; average temperature approximately 2800^oC

[T] The density of Earth's layers is displayed in the table below. Using your calculator or a computer program, find the best-fit quadratic equation to the density. Using this equation, find the total mass of Earth.

Layer	Depth from center (km)	Density (g/cm^3)
Inner Core	0	12.95
Outer Core	1228	11.05
Mantle	3488	5.00
Upper Mantle	6338	3.90
Crust	6378	2.55

Source: http://hyperphysics.phy-astr.gsu.edu...rthstruct.html

The following problems concern the Theorem of Pappus (see Moments and Centers of Mass for a refresher), a method for calculating volume using centroids. Assuming a region R, when you revolve around the x-axis the volume is given by $V_x = 2\pi A\bar{y}$, and when you revolve around the y-axis the volume is given by $V_y = 2\pi A\bar{x}$, where A is the area of R. Consider the region bounded by $x^2 + y^2 = 1$ and above $y = x + 1$.

Find the volume when you revolve the region around the x-axis.

[Hide Solution]

$\frac{\pi}{3}$

Find the volume when you revolve the region around the y-axis.

Glossary

Jacobian

the Jacobian $J(u, v)$ in two variables is a 2×2 determinant:

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix};$$

the Jacobian $J(u, v, w)$ in three variables is a 3×3 determinant:

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}$$

one-to-one transformation
a transformation $T : G \to R$ defined as $T(u, v) = (x, y)$ is said to be one-to-one if no two points map to the same image point

planar transformation
a function T that transforms a region G in one plane into a region R in another plane by a change of variables

transformation
a function that transforms a region GG in one plane into a region RR in another plane by a change of variables

Answers

Exercise 14.7.1
Follow the steps of Example $14.7.1B$.

Exercise 14.7.1

$T^{-1}(x,y) = (u,v)$ where $u = \frac{3x-y}{3}$ and $v = \frac{y}{3}$

Exercise 14.7.2

Follow the steps in the previous two examples.

Exercise 14.7.2

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$$

Exercise 14.7.3

Follow the steps in the previous example.

Exercise 14.7.3

$$\int_0^{\pi/2} \int_0^1 r^3 dr\, d\theta$$

Exercise 14.7.4

Follow the steps in the previous example.

Exercise 14.7.4

$x = \frac{1}{2}(v+u)$ and $y = \frac{1}{2}(v-u)$

and

$$\int_2^4 \int_{-u}^{u} \left(\frac{1}{2}\right) \cdot \frac{4}{u^2}\, dv\, du.$$

Exercise 14.7.5

Sketch a picture and find the limits of integration.

Exercise 14.7.5

$\frac{1}{2}(\sin 2 - 2)$

Exercise 14.7.6

Make a table for each surface of the regions and decide on the limits, as shown in the example.

Exercise 14.7.6

$$\int_0^3 \int_0^2 \int_1^2 \left(\frac{v}{3} + \frac{vw}{3u}\right) du\, dv\, dw = 2 + \ln 8$$

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Chapter 14 Review Exercises

13.1: Iterated Integrals and Area

Terms and Concepts

1. When integrating $f_x(x, y)$ with respect to x, the constant of integration C is really which: $C(x)$ or $C(y)$? What does this mean?

2. Integrating an integral is called _____ _____.

3. When evaluating an iterated integral, we integrate from _____ to _____, then from _____ to _____.

4. One understanding of an iterated integral is that $\int_a^b \int_{g_1(x)}^{g_2(x)} dy\ dx$ gives the _____ of a plane region.

Problems

In Exercises 5-10, evaluate the integral and subsequent iterated integral.

5.
(a) $\int_2^5 (6x^2 + 4xy - 3y^2)\ dy$
(b) $\int_{-3}^2 \int_2^5 (6x^2 + 4xy - 3y^2)\ dy\ dx$

6.
(a) $\int_0^{\pi} (2x\cos y + \sin x)\ dx$
(b) $\int_0^{\pi/2} \int_0^{\pi} (2x\cos y + \sin x)\ dx\ dy$

7.
(a) $\int_1^x (x^2y - y + 2)\ dy$
(b) $\int_0^2 \int_1^x (x^2y - y + 2)\ dy\ dx$

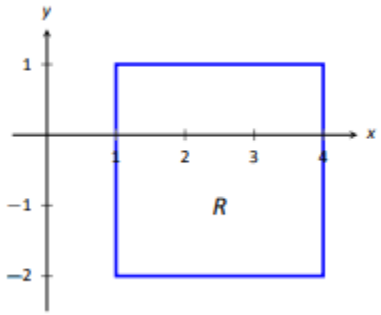
8.
(a) $\int_y^{y^2} (x - y)\ dx$
(b) $\int_{-1}^1 \int_y^{y^2} (x - y)\ dx\ dy$

9.
(a) $\int_0^y (\cos x \sin y)\ dx$
(b) $\int_0^{\pi} \int_0^y (\cos x \sin y)\ dx\ dy$

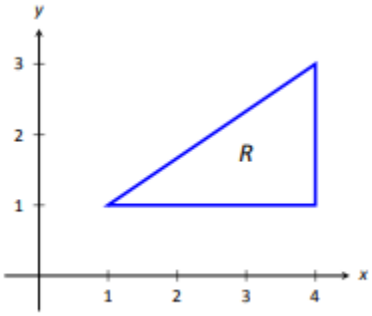
10.
(a) $\int_0^x \left(\frac{1}{1+x^2}\right)\ dy$
(b) $\int_1^2 \int_0^x \left(\frac{1}{1+x^2}\right)\ dy\ dx$

In Exercises 11-16, a graph of a planar region R is given. Give the iterated integrals, with both orders of integration $dy\ dx$ and $dx\ dy$, that give the area of R. Evaluate one of the iterated integrals to find the area.

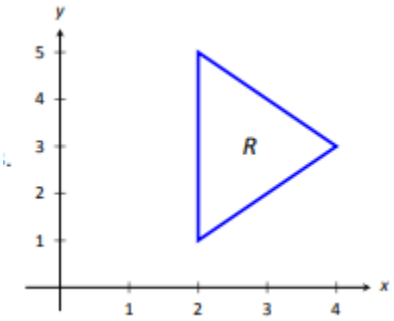
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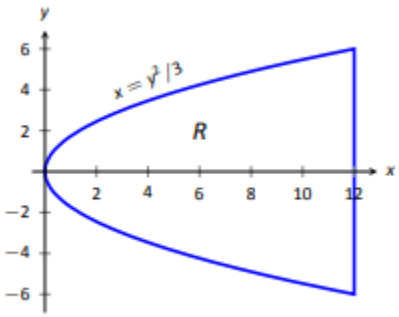
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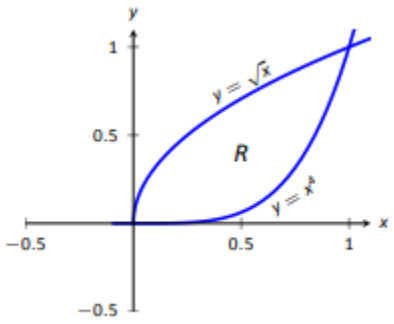
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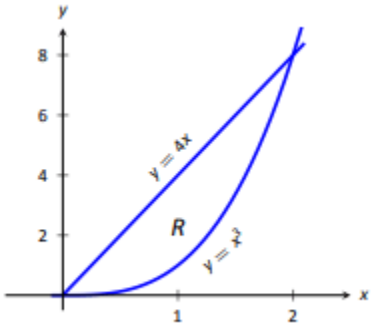
14.



15.



16.



In Exercises 17-22, iterated integrals are given that compute the area of a region R in the xy-plane. Sketch the region R, and give the iterated integral(s) that give the area of R with the opposite order of integration.

17. $\int_{-2}^{2}\int_{0}^{4-x^2} dy\ dx$

18. $\int_{0}^{1}\int_{5-5x}^{5-5x^2} dy\ dx$

19. $\int_{-2}^{2}\int_{0}^{2\sqrt{4-y^2}} dx\ dy$

20. $\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dy\ dx$

21. $\int_{0}^{1}\int_{-\sqrt{y}}^{\sqrt{y}} dx\ dy + \int_{1}^{4}\int_{y-2}^{\sqrt{y}} dx\ dy$

22. $\int_{-1}^{1}\int_{(x-1)/2}^{(1-x)/2} dy\ dx$

13.2: Double Integration and Volume

Terms and Concepts

1. An integral can be interpreted as giving the signed area over an interval; a double integral can be interpreted as giving the signed _____ over a region.

2. Explain why the following statement is false: "Fubini's Theorem states that $\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\ dy\ dx = \int_a^b \int_{g_1(y)}^{g_2(y)} f(x,y)\ dx\ dy$."

3. Explain why if $f(x,y) > 0$ over a region R, then $\iint_R f(x,y)\ dA > 0$.

4. If $\iint_R f(x,y)\ dA = \iint_R g(x,y)\ dA$, does this imply $f(x,y) = g(x,y)$?

Problems

In Exercises 5-10,
(a) Evaluate the given iterated integral, and
(b) rewrite the integral using the other order of integration.

5. $\int_1^2 \int_{-1}^1 \left(\frac{x}{y} + 3\right)\ dx\ dy$

6. $\int_{-\pi/2}^{\pi/2} \int_0^{\pi} (\sin x \cos y\,dy\,dx)$

7. $\int_0^4 \int_0^{-x/2+2} \left(3x^2 - y + 2\right)\ dy\ dx$

8. $\int_1^3 \int_y^3 \left(x^2 y - x y^2\right)\ dx\ dy$

9. $\int_0^2 1 \int_{-\sqrt{1-y}}^{\sqrt{1-y}} (x + y + 2)\ dx\ dy$

10. $\int_0^9 \int_{y/3}^{\sqrt{3}} \left(x y^2\right)\ dx\ dy$

In Exercises 11-18:
(a) Sketch the region R given by the problem.
(b) Set up the iterated integrals, in both orders, that evaluate the given double integral for the described region R.
(c) Evaluate one of the iterated integrals to find the signed volume under the surface $z = f(x,y)$ over the region R.

11. $\iint_R x^2 y\ dA$, where R is bounded by $y = \sqrt{x}$ and $y = x^2$.

12. $\iint_R x^2 y\ dA$, where R is bounded by $y = \sqrt[3]{x}$ and $y = x^3$.

13. $\iint_R x^2 - y^2\ dA$, where R is the rectangle with corners $(-1,-1), (1,-1), (1,1)$ and $(-1,1)$.

14. $\iint_R y e^x\ dA$, where R is bounded by $x = 0,\ x = y^2$ and $y = 1$.

15. $\iint_R (6 - 3x - 2y)\ dA$, where R is bounded by $x = 0, y = 0$ and $3x + 2y = 6$.

16. $\iint_R e^y\ dA$, where R is bounded by $y = \ln x$ and $y = \frac{1}{e-1}(x-1)$.

17. $\iint_R (x^3 y - x)\ dA$, where R is the half of the circle $x^2 + y^2 = 9$ in the first and second quadrants.

18. $\iint_R (4 - sy)\ dA$, where R is bounded by $y = 0, y = x/e$ and $y = \ln x$.

In Exercises 19-22, state why it is difficult/impossible to integrate the iterated integral in the given order of integration. Change the order of integration and evaluate the new iterated integral.

19. $\int_0^4 \int_{y/2}^2 e^{x^2}\ dx\ dy$

20. $\int_0^{\sqrt{\pi/2}} \int_x^{\sqrt{\pi/2}} \cos(y^2)\ dy\ dx$

21. $\int_0^1 \int_y^1 \frac{2y}{x^2+y^2}\ dx\ dy$

22. $\int_{-1}^1 \int_1^2 \frac{x\tan^2 y}{1+\ln y}\ dy\ dx$

In Exercises 23-26, find the average value of f over the region R. Notice how these functions and regions are related to the iterated integrals given in Exercises 5-8.

23. $f(x,y) = \frac{x}{y} + 3$; R is the rectangle with opposite corners $(-1,1)$ and $(1,2)$.

24. $f(x,y) = \sin x \cos y$; R is bounded by $x = 0, x = \pi, y = -\pi/2$ and $y = \pi/2$.

25. $f(x,y) = 3x^2 - y + 2$; R is bounded by the lines $y = 0, y = 2 - x/2$ and $x = 0$.

26. $f(x,y) = x^2y - xy^2$; R is bounded by $y = x, y = 1$ and $x = 3$.

13.3: Double Integration with Polar Coordinates

Terms and Concepts

1. When evaluating $\iint_R f(x,y)\ dA$ using polar coordinates, $f(x,y)$ is replaced with _____ and dA is replaced with _____.

2. Why would one be interested in evaluating a double integral with polar coordinates?

Problems

In Exercises 3-10, a function $f(x,y)$ is given and a region R of the xy-plane is described. Set up and evaluate $\iint_R f(x,y)\ dA$.

3. $f(x,y) = 3x - y + 4$; R is the region enclosed by the circle $x^2 + y^2 = 1$.

4. $f(x,y) = 4x + 4y$; R is the region enclosed by the circle $x^2 + y^2 = 4$.

5. $f(x,y) = 8 - y$; R is the region enclosed by the circles with polar equations $r = \cos\theta$ and $r = 3\cos\theta$.

6. $f(x,y) = 4$; R is the region enclosed by the petal of the rose curve $r = \sin(2\theta)$ in the first quadrant.

7. $f(x,y) = \ln(x^2 + y^2)$; R is the annulus enclosed by the circles \(x^2+y^2=1\text{ and }x^2+y^2=4.

8. $f(x,y) = 1 - x^2 - y^2$; R is the region enclosed by the circle $x^2 + y^2 = 1$.

9. $f(x,y) = x^2 - y^2$; R is the region enclosed by the circle $x^2 + y^2 = 36$ in the first and fourth quadrants.

10. $f(x,y) = (x-y)/(x+y)$; R is the region enclosed by the lines $y = x, y = 0$ and the circle $x^2 + y^2 = 1$ in the first quadrant.

In Exercises 11-14, an iterated integral in rectangular coordinates is given. Rewrite the integral using polar coordinates

and evaluate the new double integral.

11. $\int_0^5 \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} \sqrt{x^2+y^2} dy\ dx$

12. $\int_{-4}^4 \int_{-\sqrt{16-y^2}}^0 (2y-x) dx\ dy$

13. $\int_0^2 \int_y^{\sqrt{8-y^2}} (x+y)\ dx\ dy$

14. $\int_{-2}^{-1} \int_0^{\sqrt{4-x^2}} (x+5) dy\ dx + \int_{-1}^1 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} (x+5)\ dy\ dx + \int_1^2 \int_0^{\sqrt{4-x^2}} (x+5)\ dy\ dx$

In Exercises 15-16, special double integrals are presented that are especially well suited for evaluation in polar coordinates.

15. Consider $\iint_R e^{-(x^2+y^2)} dA$.
(a) Why is this integral difficult to evaluate in rectangular coordinates, regardless of the region R?
(b) Let R be the region bounded by the circle of radius *a* centered at the origin. Evaluate the double integral using polar coordinates.
(c) Take the limit of your answer from (b), as $a \to \infty$. What does this imply about the volume under the surface of $e^{-(x^2+y^2)}$ over the entire xy-plane?

16. The surface of a right circular cone with height h and base radius a can be described by the equation $f(x,y) = h - h\sqrt{\frac{x^2}{a^2} + \frac{y^2}{a^2}}$, where the tip of the cone lies at $(0,0,h)$ and the circular base lies in the xy-plane, centered at the origin.
Confirm that the volume of a right circular cone with height h and base radius a is $V = \frac{1}{3}\pi a^2 h$ by evaluating $\iint_R f(x,y)\ dA$ in polar coordinates.

13.4: Center of Mass

Terms and Concepts

1. Why is it easy to use "mass" and "weight" interchangeably, even though they are different measures?

2. Given a point (x,y), the value of *x* is a measure of distance from the _____-axis.

3. We can think of $\iint_R dm$ as meaning "sum up lots of _____."

4. What is a "discrete planar system?"

5. Why does M_x use $\iint_R y\delta(x,y)\ dA$ instead of $\iint_R x\delta(x,y)\ dA$; that is, why do we use "y" and not "x"?

6. Describe a situation where the center of mass of a lamina does not lie within the region of the lamina itself.

Problems

In Exercises 7-10, point masses are given along a line or in the plane. Find the center of mass $\overline{x}$ or $(\overline{x},\overline{y})$, as appropriate. (All masses are in grams and distances are in cm.)

7. $m_1 = 4$ at $x = 1$; $m_2 = 3$ at $x = 3$; $m_3 = 5$ at $x = 10$

8. $m_1 = 2$ at $x = -3$; $m_2 = 2$ at $x = -1$; $m_3 = 3$ at $x = 0$; $m_4 = 3$ at $x = 7$

9. $m_1 = 2$ at $(-2, 2)$; $m_2 = 2$ at $(2, -2)$; $m_3 = 20$ at $(0, 4)$

10. $m_1 = 1$ at $(-1, 1)$; $m_2 = 2$ at $(-1, 1)$; $m_3 = 2$ at $(1, 1)$; $m_4 = 1$ at $(1, -1)$

In Exercises 11-18, find the mass/weight of the lamina described by the region R in the plane and its density function $\delta(x, y)$.

11. R is the rectangle with corners $(1, -3), (1, 2), (7, 2)$ and $(7, -3)$; $\delta(x, y) = 5$gm/cm^2

12. R is the rectangle with corners $(1, -3), (1, 2), (7, 2)$ and $(7, -3)$; $\delta(x, y) = (x + y^2)$gm/cm^2

13. R is the triangle with corners $(-1, 0), (1, 0),$ and $(0, 1)$; $\delta(x, y) = 2$lb/in^2

14. R is the triangle with corners $(0, 0), (1, 0),$ and $(0, 1)$; $\delta(x, y) = (x^2 + y^2 + 1)$lb/in^2

15. R is the circle centered at the origin with radius 2; $\delta(x, y) = (x + y + 4)$kg/m^2

16. R is the circle sector bounded by $x^2 + y^2 = 25$ in the first quadrant; $\delta(x, y) = (\sqrt{x^2 + y^2} + 1)$kg/m^2

17. R is the annulus in the first and second quadrants bounded by $x^2 + y^2 = 9$ and $x^2 + y^2 = 36$; $\delta(x, y) = 4$lb/ft^2

18. R is the annulus in the first and second quadrants bounded by $x^2 + y^2 = 9$ and $x^+y^2 = 36$; $\delta(x, y) = \sqrt{x^2 + y^2}$lb/ft^2

In Exercises 19-26, find the center of mass of the lamina described by the region R in the plane and its density function $\delta(x, y)$.

Note: these are the same lamina as in Exercises 11-18.

19. R is the rectangle with corners $(1, -3), (1, 2), (7, 2)$ and $(7, -3)$; $\delta(x, y) = 5$gm/cm^2

20. R is the rectangle with corners $(1, -3), (1, 2), (7, 2)$ and $(7, -3)$; $\delta(x, y) = (x + y^2)$gm/cm^2

21. R is the triangle with corners $(-1, 0), (1, 0),$ and $(0, 1)$; $\delta(x, y) = 2$lb/in^2

22. R is the triangle with corners $(0, 0), (1, 0),$ and $(0, 1)$; $\delta(x, y) = (x^2 + y^2 + 1)$lb/in^2

23. R is the circle centered at the origin with radius 2; $\delta(x, y) = (x + y + 4)$kg/m^2

24. R is the circle sector bounded by $x^2 + y^2 = 25$ in the first quadrant; $\delta(x, y) = (\sqrt{x^2 + y^2} + 1)$kg/m^2

25. R is the annulus in the first and second quadrants bounded by $x^2 + y^2 = 9$ and $x^2 + y^2 = 36$; $\delta(x, y) = 4$lb/ft^2

26. R is the annulus in the first and second quadrants bounded by $x^2 + y^2 = 9$ and $x^+y^2 = 36$; $\delta(x, y) = \sqrt{x^2 + y^2}$lb/ft^2

The moment of inertia i is a measure of the tendency of lamina to resist rotating about an axis or continue to rotate about an axis. i_x is the moment of inertia about the x-axis, i_x is the moment of inertia about the x-axis, and i_o is the moment of inertia about the origin. These are computed as follows:

- $i_x = \iint_R y^2 \, dm$
- $i_y = \iint_R x^2 \, dm$
- $i_o = \iint_R (x^2 + y^2) \, dm$

In Exercises 27-30, a lamina corresponding to a planar region R is given with a mass of 16 units. For each, compute i_x, i_y and i_o.

27. R is the 4 x 4 square with corners $(-2, -2)$ and $(2, 2)$ with density $\delta(x, y) = 1$.

28. R is the 8 x 2 rectangle with corners $(-4, -1)$ and $(4, 1)$ with density $\delta(x, y) = 1$.

29. R is the 4 x 2 rectangle with corners $(-2,-1)$ and $(2,1)$ with density $\delta(x,y)=2$.

30. R is the circle with radius 2 centered at the origin with density $\delta(x,y)=4/\pi$.

13.5: Surface Area

Terms and Concepts

1. "Surface area" is analogous to what previously studied concept?

2. To approximate the area of a small portion of a surface, we computed the area of its _____ plane.

3. We interpret $\iint_R dS$ as "sum up lots of little _____ _____."

4. Why is it important to know how to set up a double integral to compute surface area, even if the resulting integral is hard to evaluate?

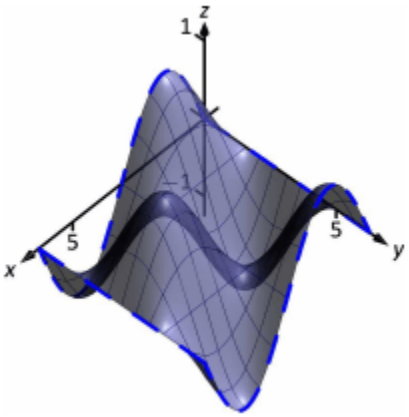
5. Why do $z=f(x,y)$ and $z=g(x,y)=f(x,y)+h$, for some real number h, have the same surface area over a region R?

6. Let $z=f(x,y)$ and $z=g(x,y)=2f(x,y)$. Why is the surface area of g over a region R not twice the surface area of f over R?

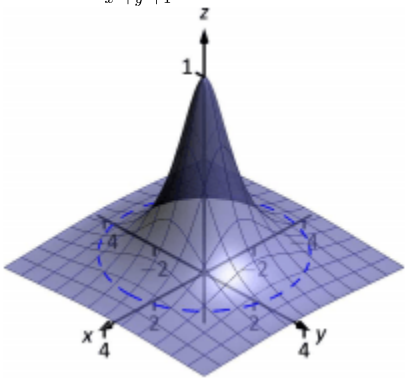
Problems

In Exercises 7-10, set up the iterated integral that computes the surfaces area of the given surface over the region R.

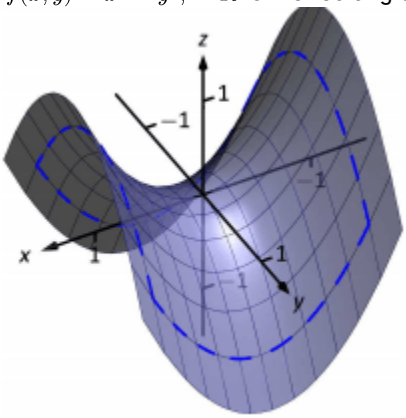
7. $f(x,y)=\sin x\cos y$; R is the rectangle with bounds $0\le x\le 2\pi$, $0\le y\le 2\pi$.



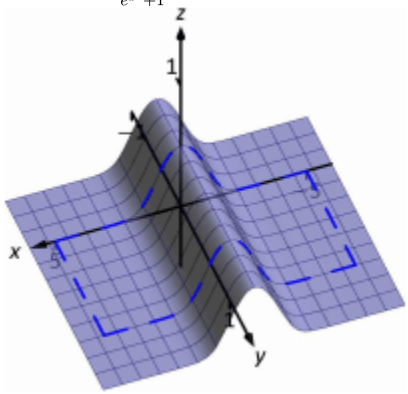
8. $f(x,y)=\frac{1}{x^2+y^2+1}$; R is the circle $x^2+y^2=9$.



9. $f(x,y) = x^2 - y^2$; R is the rectangle with opposite corners $(-1,-1)$ and $1,1)$.



10. $f(x,y) = \frac{1}{e^{x^2}+1}$; R is the rectangle bounded by $-5 \le x \le 5$ and $0 \le y \le 1$.



In Exercises 11-19, find the area of the given surface over the region R.

11. $f(x,y) = 3x - 7y + 2$; R is the rectangle with opposite corners $(-1,0)$ and $(1,3)$.

12. $f(x,y) = 2x + 2y + 2$; R is the triangle with corners $(0,0), (1,0)$ and $(0,1)$.

13. $f(x,y) = x^2 + y^2 + 10$; R is the circle $x^2 + y^2 = 16$.

14. $f(x,y) = -2x + 4y^2 + 7$ over R, the triangle bounded by $y = -x, y = x, 0 \le y \le 1$.

15. $f(x,y) = x^2 + y$ over R, the triangle bounded by $y = 2x, y = 0$ and $x = 2$.

16. $f(x,y) = \frac{2}{3}x^{3/2}$ over R, the rectangle with opposite corners $(0,0)$ and $(1,1)$.

17. $f(x,y) = 10 - 2\sqrt{x^2+y^2}$ over R, the circle $x^2 + y^2 = 25$. (This is the cone with height 10 and base radius 5; be sure to compare your result with the known formula.)

18. Find the surface area of the sphere with radius 5 by doubling the surface area of $f(x,y) = \sqrt{25 - x^2 - y^2}$ over R, the circle $x^2 + y^2 = 25$. (Be sure to compare your result with the known formula.)

19. Find the surface area of the ellipse formed by restricting the plane $f(x,y) = cx + dy + h$ to the region R, the circle $x^2 + y^2 = 1$, where c, d and h are some constants. Your answer should be given in terms of c and d; why does the value of h not matter?

13.6: Volume Between Surfaces and Triple Integration

Terms and Concepts

1. The strategy for establishing bounds for triple integrals is "_____ to _____, _____ and _____ to _____."

2. Give an informal interpretation of what "$\int\int\int_D \, dV$" means.

3. Give two uses of triple integration.

4. If an object has a constant density δ and a volume$V$, what is its mass?

Problems

In Exercises 5-8, two surfaces $f_1(x,y)$ and $f_2(x,y)$ and a region R in the xy-plane are given. Set up and evaluate the double integral that finds the volume between these surfaces over R.

5. $f_x(x,y) = 8 - x^2 - y^2, \;\; f_2(x,y) = 2x + y$;
R is the square with corners $(-1,-1)$ and $(1,1)$.

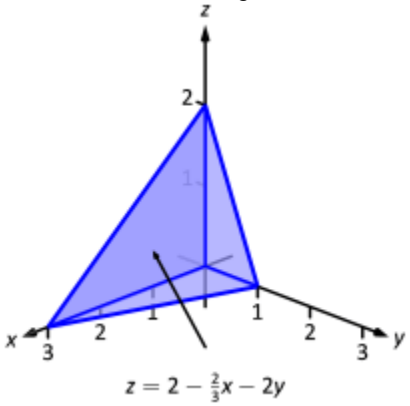
6. $f_x(x,y) = x^2 + y^2, \;\; f_2(x,y) = -x^2 - y^2$;
R is the square with corners $(0,0)$ and $(2,3)$.

7. $f_x(x,y) = \sin x \cos y, \;\; f_2(x,y) = \cos x \sin y + 2$;
R is the triangle with corners $(0,0), (\pi,0)$ and (π,π).

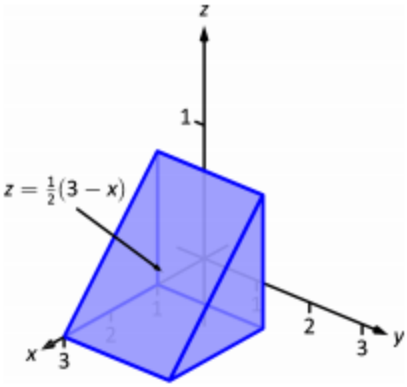
8. $f_x(x,y) = 2x^2 + 2y^2 + 3, \;\; f_2(x,y) = 6 - x^2 - y^2$;
R is the circle $x^2 + y^2 = 1$.

In Exercises 9-16, a domain D is described by its bounding surfaces, along with a graph. Set up the triple integrals that give the volume of D in all 6 orders of integration, and find the volume of D by evaluating the indicated triple integral.

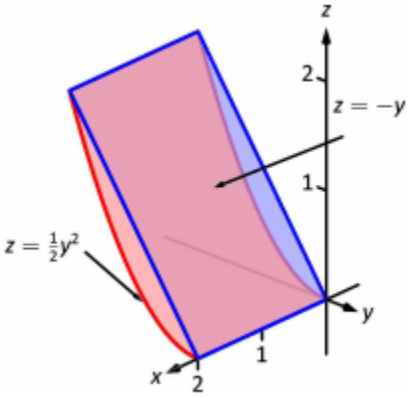
9. D is bounded by the coordinate planes and $z = 2 - 2x/3 - 2y$.
Evaluate the triple integral with order *dz dy dz*.



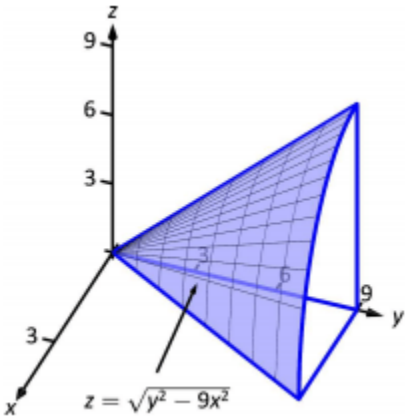
10. D is bounded by the planes $y = 0, y = 2, x = 1, z = 0$ and $z = (2 - x)/2$.
Evaluate the triple integral with order *dx dy dz*.



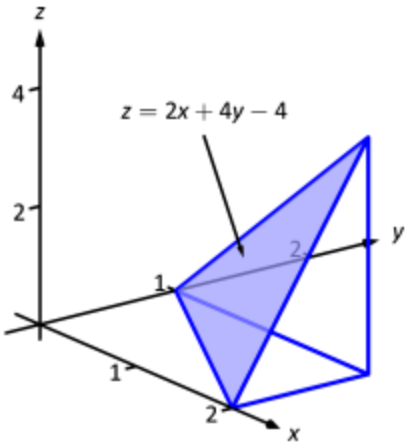
11. D is bounded by the planes $x = 0, x = 2, z = -y$ and by $z = y^2/2$.
Evaluate the triple integral with order *dy dz dx*.



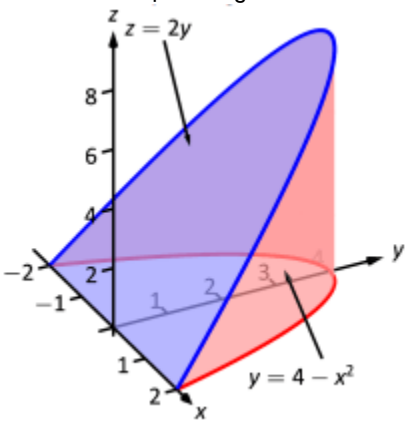
12. D is bounded by the planes \(z=0,y=9, x=0\text{ and by }\)z=\sqrt{y^2-9x^2}\).
Do not evaluate any triple integral.



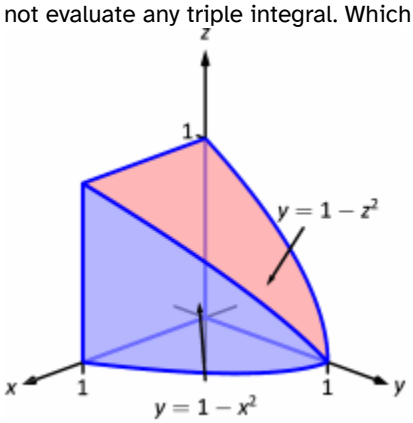
13. D is bounded by the planes $x = 2, y = 1, z = 0$ and $z = 2x + 4y - 4$.
Evaluate the triple integral with order *dx dy dz*.



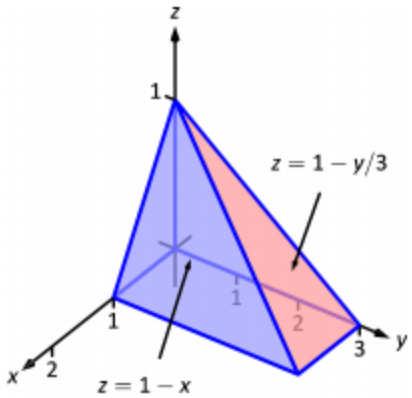
14. D is bounded by the plane $z = 2y$ and by $y = 4 - x^2$.
Evaluate the triple integral with order *dz dy dz*.



15. D is bounded by the coordinate planes and $y = 1 - x^2$ and $y = 1 - z^2$.
Do not evaluate any triple integral. Which order is easier to evaluate: *dz dy dx* or *dy dz dx*? Explain why.



16. D is bounded by the coordinate planes and by $z = 1 - y/3$ and $z = 1 - x$.
Evaluate the triple integral with order *dx dy dz*.



In Exercises 17-20, evaluate the triple integral.

17. $\int_{-\pi/2}^{\pi/2}\int_{0}^{\pi}\int_{0}^{\pi}(\cos x \sin y \sin z)dz\ dy\ dx$

18. $\int_{0}^{1}\int_{0}^{x}\int_{0}^{x+y}(x+y+z)dz\ dy\ dx$

19. $\int_{0}^{\pi}\int_{0}^{1}\int_{0}^{z}(\sin(yz))dx\ dy\ dz$

20. $\int_{\pi}^{\pi^2}\int_{x}^{x^3}\int_{-y^2}^{y^2}(\cos x \sin y \sin z)dz\ dy\ dx$

In Exercises 21-24, find the center of mass of the solid represented by the indicated space region D with density function $\delta(x,y,z)$.

21. D is bounded by the coordinate planes and $z = 2 - 2x/3 - 2y$; $\delta(x,y,z) = 10\text{g/cm}^3$.
(Note: this is the same region as used in Exercise 9.)

22. D is bounded by the planes $y = 0, y = 2, x = 1, z = 0$ and $z = (3-x)/2$; $\delta(x,y,z) = 2\text{g/cm}^3$.
(Note: this is the same region as used in Exercise 10.)

23. D is bounded by the planes $x = 2, y = 1, z = 0$ and $z = 2x + 4y - 4$; $\delta(x,y,z) = x^2\text{lb/in}^3$.
(Note: this is the same region as used in Exercise 13.)

24. D is bounded by the planes $z = 2y$ and by $y = 4 - x^2$. $\delta(x,y,z) = y^2\text{lb/in}^3$.
(Note: this is the same region as used in Exercise 14.)

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CHAPTER 15: VECTOR FIELDS, LINE INTEGRALS, AND VECTOR THEOREMS

15.0: Introduction to Vector Field Chapter

15.1: Vector Fields

15.1E: Vector Fields (Exercises)

15.2: Line Integrals

15.2: Line Integrals - Part 2

15.2E: Line Integrals (Exercises)

15.3: Conservative Vector Fields

15.3E: Conservative Vector Fields (Exercises)

15.4: Green's Theorem

15.4E: Green's Theorem (Exercises)

15.5: Divergence and Curl

15.5E: Divergence and Curl (Exercises)

15.6: Surface Integrals

15.7: Stokes' Theorem

15.8: The Divergence Theorem

Chapter 15 Review Exercises

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15.0: Introduction to Vector Field Chapter

Hurricanes are huge storms that can produce tremendous amounts of damage to life and property, especially when they reach land. Predicting where and when they will strike and how strong the winds will be is of great importance for preparing for protection or evacuation. Scientists rely on studies of rotational vector fields for their forecasts.



Figure 15.0.1: Hurricanes form from rotating winds driven by warm temperatures over the ocean. Meteorologists forecast the motion of hurricanes by studying the rotating vector fields of their wind velocity. Shown is Cyclone Catarina in the South Atlantic Ocean in 2004, as seen from the International Space Station. (credit: modification of work by NASA)

In this chapter, we learn to model new kinds of integrals over fields such as magnetic fields, gravitational fields, or velocity fields. We also learn how to calculate the work done on a charged particle traveling through a magnetic field, the work done on a particle with mass traveling through a gravitational field, and the volume per unit time of water flowing through a net dropped in a river.

All these applications are based on the concept of a vector field, which we explore in this chapter. Vector fields have many applications because they can be used to model real fields such as electromagnetic or gravitational fields. A deep understanding of physics or engineering is impossible without an understanding of vector fields. Furthermore, vector fields have mathematical properties that are worthy of study in their own right. In particular, vector fields can be used to develop several higher-dimensional versions of the Fundamental Theorem of Calculus.

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15.1: VECTOR FIELDS

Learning Objectives

- Recognize a vector field in a plane or in space.
- Sketch a vector field from a given equation.
- Identify a conservative field and its associated potential function.

Vector fields are an important tool for describing many physical concepts, such as gravitation and electromagnetism, which affect the behavior of objects over a large region of a plane or of space. They are also useful for dealing with large-scale behavior such as atmospheric storms or deep-sea ocean currents. In this section, we examine the basic definitions and graphs of vector fields so we can study them in more detail in the rest of this chapter.

Examples of Vector Fields

How can we model the gravitational force exerted by multiple astronomical objects? How can we model the velocity of water particles on the surface of a river? Figure $15.1.1$ gives visual representations of such phenomena.

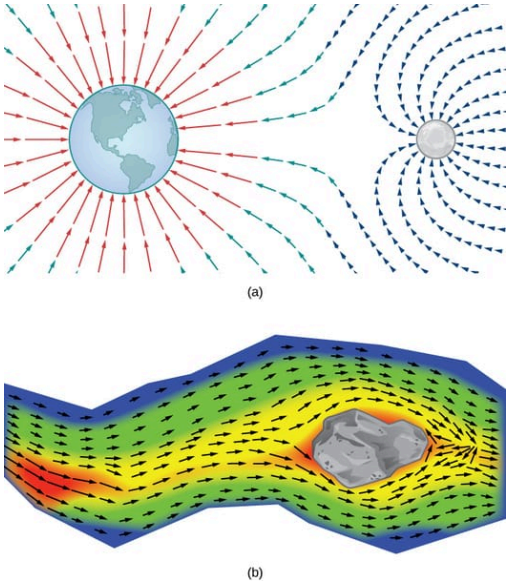


Figure $15.1.1$ (a) The gravitational field exerted by two astronomical bodies on a small object. (b) The vector velocity field of water on the surface of a river shows the varied speeds of water. Red indicates that the magnitude of the vector is greater, so the water flows more quickly; blue indicates a lesser magnitude and a slower speed of water flow.

Figure $15.1.1a$ shows a gravitational field exerted by two astronomical objects, such as a star and a planet or a planet and a moon. At any point in the figure, the vector associated with a point gives the net gravitational force exerted by the two objects on an object of unit mass. The vectors of largest magnitude in the figure are the vectors closest to the larger object. The larger object has greater mass, so it exerts a gravitational force of greater magnitude than the smaller object.

Figure $15.1.1b$ shows the velocity of a river at points on its surface. The vector associated with a given point on the river's surface gives the velocity of the water at that point. Since the vectors to the left of the figure are small in magnitude, the water is flowing slowly on that part of the surface. As the water moves from left to right, it encounters some rapids around a rock. The speed of the water increases, and a whirlpool occurs in part of the rapids.

Each figure illustrates an example of a vector field. Intuitively, a vector field is a map of vectors. In this section, we study vector fields in $\mathbb{R}^2$ and $\mathbb{R}^3$.

DEFINITION: Vector Field

- A vector field $\vecs{F}$ in $\mathbb{R}^2$ is an assignment of a two-dimensional vector $\vecs{F}(x, y)$ to each point (x, y) of a subset D of $\mathbb{R}^2$. The subset D is the domain of the vector field.
- A vector field $\vecs{F}$ in $\mathbb{R}^3$ is an assignment of a three-dimensional vector $\vecs{F}(x, y, z)$ to each point (x, y, z) of a subset D of $\mathbb{R}^3$. The subset D is the domain of the vector field.

Vector Fields in $\mathbb{R}^2$

A vector field in $\mathbb{R}^2$ can be represented in either of two equivalent ways. The first way is to use a vector with components that are two-variable functions:

$$\vecs{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$$

The second way is to use the standard unit vectors:

$$\vecs{F}(x, y) = P(x, y)\,\hat{\mathbf{i}} + Q(x, y)\,\hat{\mathbf{j}}.$$

A vector field is said to be *continuous* if its component functions are continuous.

Example 15.1.1: Finding a Vector Associated with a Given Point

Let $\vecs{F}(x, y) = (2y^2 + x - 4)\ \hat{\mathbf{i}} + \cos(x)\ \hat{\mathbf{j}}$ be a vector field in $\mathbb{R}^2$. Note that this is an example of a continuous vector field since both component functions are continuous. What vector is associated with point $(0, -1)$?

Solution

Substitute the point values for x and y:

$$\begin{aligned}\vecs{F}(0, -1) &= (2(-1)^2 + 0 - 4)\,\hat{\mathbf{i}} + \cos(0)\,\hat{\mathbf{j}} \\ &= -2\,\hat{\mathbf{i}} + \hat{\mathbf{j}}.\end{aligned}$$

Exercise 15.1.1

Let $\vecs{G}(x, y) = x^2y\ \hat{\mathbf{i}} - (x + y)\ \hat{\mathbf{j}}$ be a vector field in $\mathbb{R}^2$. What vector is associated with the point $(-2, 3)$?

Hint available at the end of this section.

Drawing a Vector Field

We can now represent a vector field in terms of its components of functions or unit vectors, but representing it visually by sketching it is more complex because the domain of a vector field is in $\mathbb{R}^2$, as is the range. Therefore the "graph" of a vector field in $\mathbb{R}^2$ lives in four-dimensional space. Since we cannot represent four-dimensional space visually, we instead draw vector fields in $\mathbb{R}^2$ in a plane itself. To do this, draw the vector associated with a given point at the point in a plane. For example, suppose the vector associated with point $(4, -1)$ is $\langle 3, 1\rangle$. Then, we would draw vector $\langle 3, 1\rangle$ at point $(4, -1)$.

We should plot enough vectors to see the general shape, but not so many that the sketch becomes a jumbled mess. If we were to plot the image vector at each point in the region, it would fill the region completely and is useless. Instead, we can choose points at the intersections of grid lines and plot a sample of several vectors from each quadrant of a rectangular coordinate system in $\mathbb{R}^2$.

There are two types of vector fields in $\mathbb{R}^2$ on which this chapter focuses: radial fields and rotational fields. Radial fields model certain gravitational fields and energy source fields, and rotational fields model the movement of a fluid in a vortex. In a radial

field, all vectors either point directly toward or directly away from the origin. Furthermore, the magnitude of any vector depends only on its distance from the origin. In a radial field, the vector located at point (x, y) is perpendicular to the circle centered at the origin that contains point (x, y), and all other vectors on this circle have the same magnitude.

Example 15.1.2: Drawing a Radial Vector Field

Sketch the vector field $\vecs{F}(x, y) = \frac{x}{2}\hat{\mathbf{i}} + \frac{y}{2}\hat{\mathbf{j}}$.

Solution

To sketch this vector field, choose a sample of points from each quadrant and compute the corresponding vector. The following table gives a representative sample of points in a plane and the corresponding vectors.

Table 15.1.1

(x, y)	$\vecs{F}(x, y)$	(x, y)	$\vecs{F}(x, y)$	(x, y)	$\vecs{F}(x, y)$
$(1, 0)$	$\langle \frac{1}{2}, 0 \rangle$	$(2, 0)$	$\langle 1, 0 \rangle$	$(1, 1)$	$\langle \frac{1}{2}, \frac{1}{2} \rangle$
$(0, 1)$	$\langle 0, \frac{1}{2} \rangle$	$(0, 2)$	$\langle 0, 1 \rangle$	$(-1, 1)$	$\langle -\frac{1}{2}, \frac{1}{2} \rangle$
$(-1, 0)$	$\langle -\frac{1}{2}, 0 \rangle$	$(-2, 0)$	$\langle -1, 0 \rangle$	$(-1, -1)$	$\langle -\frac{1}{2}, -\frac{1}{2} \rangle$
$(0, -1)$	$\langle 0, -\frac{1}{2} \rangle$	$(0, -2)$	$\langle 0, -1 \rangle$	$(1, -1)$	$\langle \frac{1}{2}, -\frac{1}{2} \rangle$

Figure $15.1.2a$ shows the vector field. To see that each vector is perpendicular to the corresponding circle, Figure $15.1.2b$ shows circles overlain on the vector field.

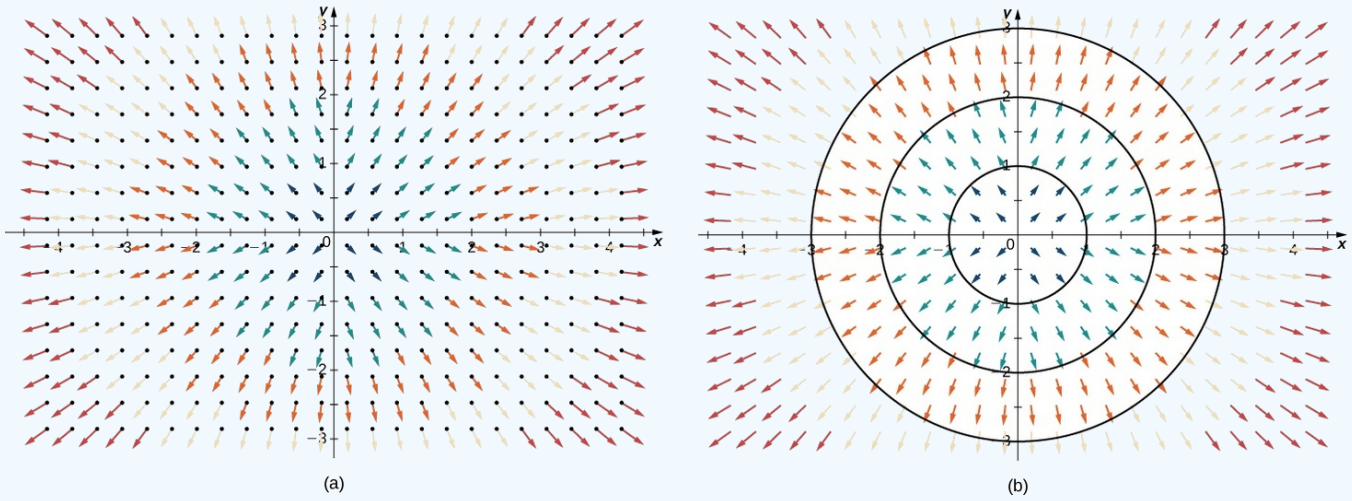


Figure 15.1.2: (a) A visual representation of the radial vector field $\vecs{F}(x, y) = \frac{x}{2}\hat{\mathbf{i}} + \frac{y}{2}\hat{\mathbf{j}}$. (b) The radial vector field $\vec{F}(x, y) = \frac{x}{2}\hat{\mathbf{i}} + \frac{y}{2}\hat{\mathbf{j}}$ with overlaid circles. Notice that each vector is perpendicular to the circle on which it is located.

Exercise 15.1.2

Draw the radial field $\vecs{F}(x, y) = -\frac{x}{3}\hat{\mathbf{i}} - \frac{y}{3}\hat{\mathbf{j}}$.

Hint available at the end of this section.

In contrast to radial fields, in a *rotational field*, the vector at point (x, y) is tangent (not perpendicular) to a circle with radius r

$= \sqrt{x^2 + y^2}$. In a standard rotational field, all vectors point either in a clockwise direction or in a counterclockwise direction, and the magnitude of a vector depends only on its distance from the origin. Both of the following examples are clockwise rotational fields, and we see from their visual representations that the vectors appear to rotate around the origin.

Example 15.1.3: Drawing a Rotational Vector Field

Sketch the vector field $\vecs F(x, y) = \langle y, \ -x \rangle$.

Solution

Create a table (see the one that follows) using a representative sample of points in a plane and their corresponding vectors. Figure 15.1.3 shows the resulting vector field.

Table 15.1.2

(x, y)	$\vecs F(x, y)$	(x, y)	$\vecs F(x, y)$	(x, y)	$\vecs F(x, y)$
$(1, 0)$	$\langle 0, -1 \rangle$	$(2, 0)$	$\langle 0, -2 \rangle$	$(1, 1)$	$\langle 1, -1 \rangle$
$(0, 1)$	$\langle 1, 0 \rangle$	$(0, 2)$	$\langle 2, 0 \rangle$	$(-1, 1)$	$\langle 1, 1 \rangle$
$(-1, 0)$	$\langle 0, 1 \rangle$	$(-2, 0)$	$\langle 0, 2 \rangle$	$(-1, -1)$	$\langle -1, 1 \rangle$
$(0, -1)$	$\langle -1, 0 \rangle$	$(0, -2)$	$\langle -2, 0 \rangle$	$(1, -1)$	$\langle -1, -1 \rangle$

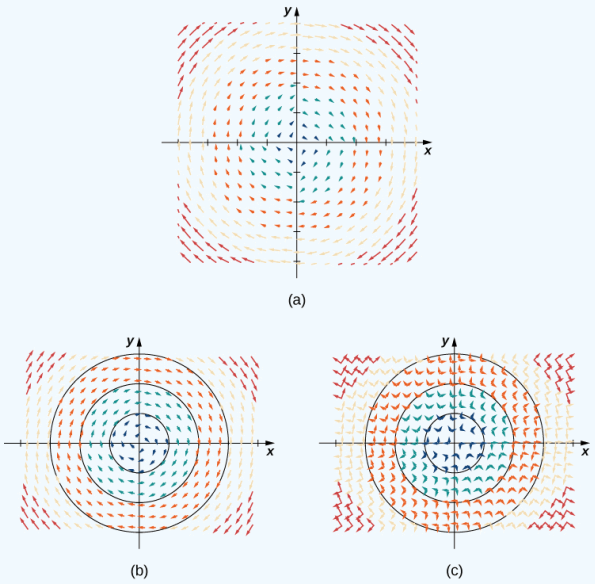


Figure 15.1.3: (a) A visual representation of vector field $\vecs F(x, y) = \langle y, -x \rangle$. (b) Vector field $\vecs F(x, y) = \langle y, -x \rangle$ with circles centered at the origin. (c) Vector $\vecs F(a, b)$ is perpendicular to radial vector $\langle a, b \rangle$ at point (a, b).

Analysis

Note that vector $\vecs F(a, b) = \langle b, -a \rangle$ points clockwise and is perpendicular to radial vector $\langle a, b \rangle$. (We can verify this assertion by computing the dot product of the two vectors: $\langle a, b \rangle \cdot \langle -b, a \rangle = -ab + ab = 0$.) Furthermore, vector $\langle b, -a \rangle$ has length $r = \sqrt{a^2 + b^2}$. Thus, we have a complete description of this rotational vector field: the vector associated with point (a, b) is the vector with length *r* tangent to the circle with radius *r*, and it points in the clockwise direction.

Sketches such as that in Figure 15.1.3 are often used to analyze major storm systems, including hurricanes and cyclones. In the northern hemisphere, storms rotate counterclockwise; in the southern hemisphere, storms rotate clockwise. (This is an effect caused by Earth's rotation about its axis and is called the Coriolis Effect.)



Figure 15.1.4: (credit: modification of work by NASA)

Example 15.1.4: Sketching a Vector Field

Sketch vector field $\vecs{F}(x,y) = \dfrac{y}{x^2+y^2}\hat{\mathbf{i}}, -\dfrac{x}{x^2+y^2}\hat{\mathbf{j}}$.

Solution

To visualize this vector field, first note that the dot product $\vecs{F}(a,b) \cdot (a\ \hat{\mathbf{i}} + b\ \hat{\mathbf{j}})$ is zero for any point (a,b). Therefore, each vector is tangent to the circle on which it is located. Also, as $(a,b) \to (0,0)$, the magnitude of $\vecs{F}(a,b)$ goes to infinity. To see this, note that

$$||\vecs{F}(a,b)|| = \sqrt{\frac{a^2+b^2}{(a^2+b^2)^2}} = \sqrt{\frac{1}{a^2+b^2}}.$$

Since $\dfrac{1}{a^2+b^2} \to \infty$ as $(a,b) \to (0,0)$, then $||\vecs{F}(a,b)|| \to \infty$ as $(a,b) \to (0,0)$. This vector field looks similar to the vector field in Example 15.1.3, but in this case the magnitudes of the vectors close to the origin are large. Table 15.1.3 shows a sample of points and the corresponding vectors, and Figure 15.1.5 shows the vector field. Note that this vector field models the whirlpool motion of the river in Figure 15.1.5(b). The domain of this vector field is all of $\mathbb{R}^2$ except for point $(0,0)$.

Table 15.1.3

(x,y)	$\vecs{F}(x,y)$	(x,y)	$\vecs{F}(x,y)$	(x,y)	$\vecs{F}(x,y)$
$(1,0)$	$\langle 0,-1\rangle$	$(2,0)$	$\langle 0,-\frac{1}{2}\rangle$	$(1,1)$	$\langle \frac{1}{2},-\frac{1}{2}\rangle$
$(0,1)$	$\langle 1,0\rangle$	$(0,2)$	$\langle \frac{1}{2},0\rangle$	$(-1,1)$	$\langle \frac{1}{2},\frac{1}{2}\rangle$
$(-1,0)$	$\langle 0,1\rangle$	$(-2,0)$	$\langle 0,\frac{1}{2}\rangle$	$(-1,-1)$	$\langle -\frac{1}{2},\frac{1}{2}\rangle$
$(0,-1)$	$\langle -1,0\rangle$	$(0,-2)$	$\langle -\frac{1}{2},0\rangle$	$(1,-1)$	$\langle -\frac{1}{2},-\frac{1}{2}\rangle$

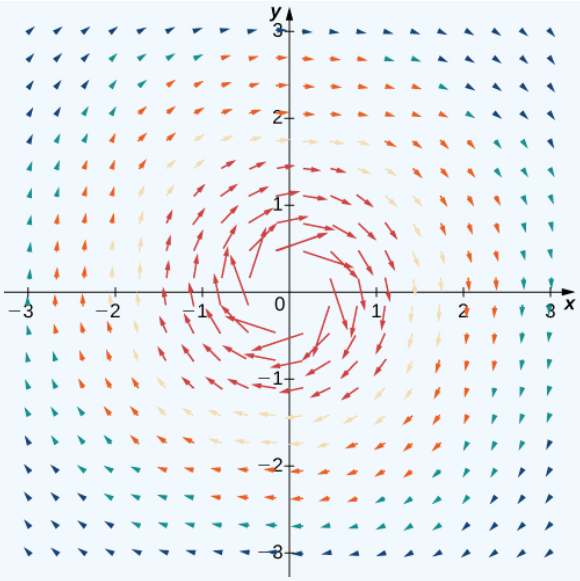


Figure 15.1.5: A visual representation of vector field $\vecs F(x,y) = \dfrac{y}{x^2+y^2}\hat{\mathbf{i}} - \dfrac{x}{x^2+y^2}\hat{\mathbf{j}}$. *This vector field could be used to model whirlpool motion of a fluid.*

Exercise 15.1.4

Sketch vector field $\vecs F(x,y) = \langle -2y,\ 2x\rangle$. Is the vector field radial, rotational, or neither?

Hint available at the end of this section.

Example 15.1.5: Velocity Field of a Fluid

Suppose that $\vecs v(x,y) = -\dfrac{2y}{x^2+y^2}\hat{\mathbf{i}} + \dfrac{2x}{x^2+y^2}\hat{\mathbf{j}}$ is the velocity field of a fluid. How fast is the fluid moving at point $(1,-1)$? (Assume the units of speed are meters per second.)

Solution

To find the velocity of the fluid at point $(1,-1)$, substitute the point into $\vecs v$:

$$\vecs v(1,-1) = \frac{-2(-1)}{1+1}\hat{\mathbf{i}} + \frac{2(1)}{1+1}\hat{\mathbf{j}} = \hat{\mathbf{i}} + \hat{\mathbf{j}}.$$

The speed of the fluid at $(1,-1)$ is the magnitude of this vector. Therefore, the speed is $||\hat{\mathbf{i}} + \hat{\mathbf{j}}|| = \sqrt{2}$ m/sec.

Exercise 15.1.5

Vector field $\vecs v(x,y) = \langle 4|x|,\ 1\rangle$ models the velocity of water on the surface of a river. What is the speed of the water at point $(2,3)$? Use meters per second as the units.

Hint available at the end of this section.

We have examined vector fields that contain vectors of various magnitudes, but just as we have unit vectors, we can also have a unit vector field. A vector field $\vecs F$ is a **unit vector field** if the magnitude of each vector in the field is 1. In a unit vector field, the only relevant information is the direction of each vector.

Example 15.1.6: A Unit Vector Field

Show that vector field $\vecs{F}(x,y) = \left\langle \frac{y}{\sqrt{x^2+y^2}}, -\frac{x}{\sqrt{x^2+y^2}} \right\rangle$ is a unit vector field.

Solution

To show that $\vecs{F}$ is a unit field, we must show that the magnitude of each vector is 1. Note that

$$\begin{aligned}\sqrt{\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 + \left(-\frac{x}{\sqrt{x^2+y^2}}\right)^2} &= \sqrt{\frac{y^2}{x^2+y^2} + \frac{x^2}{x^2+y^2}} \\ &= \sqrt{\frac{x^2+y^2}{x^2+y^2}} \\ &= 1\end{aligned}$$

Therefore, $\vecs{F}$ is a unit vector field.

Exercise 15.1.6

Is vector field $\vecs{F}(x,y) = \langle -y,\ x\rangle$ a unit vector field?

Hint available at the end of this section.

Why are unit vector fields important? Suppose we are studying the flow of a fluid, and we care only about the direction in which the fluid is flowing at a given point. In this case, the speed of the fluid (which is the magnitude of the corresponding velocity vector) is irrelevant, because all we care about is the direction of each vector. Therefore, the unit vector field associated with velocity is the field we would study.

If $\vecs{F} = \langle P, Q, R\rangle$ is a vector field, then the corresponding unit vector field is $\left\langle \frac{P}{||\vecs{F}||}, \frac{Q}{||\vecs{F}||}, \frac{R}{||\vecs{F}||} \right\rangle$. Notice that if $\vecs{F}(x,y) = \langle y,\ -x\rangle$ is the vector field from Example 15.1.6, then the magnitude of $\vecs{F}$ is $\sqrt{x^2+y^2}$, and therefore the corresponding unit vector field is the field $\vecs{G}$ from the previous example.

If $\vecs{F}$ is a vector field, then the process of dividing $\vecs{F}$ by its magnitude to form unit vector field $\vecs{F}/||\vecs{F}||$ is called normalizing the field $\vecs{F}$.

Vector Fields in $\mathbb{R}^3$

We have seen several examples of vector fields in $\mathbb{R}^2$; let's now turn our attention to vector fields in $\mathbb{R}^3$. These vector fields can be used to model gravitational or electromagnetic fields, and they can also be used to model fluid flow or heat flow in three dimensions. A two-dimensional vector field can really only model the movement of water on a two-dimensional slice of a river (such as the river's surface). Since a river flows through three spatial dimensions, to model the flow of the entire depth of the river, we need a vector field in three dimensions.

The extra dimension of a three-dimensional field can make vector fields in $\mathbb{R}^3$ more difficult to visualize, but the idea is the same. To visualize a vector field in $\mathbb{R}^3$, plot enough vectors to show the overall shape. We can use a similar method to visualizing a vector field in $\mathbb{R}^2$ by choosing points in each octant.

Just as with vector fields in $\mathbb{R}^2$, we can represent vector fields in $\mathbb{R}^3$ with component functions. We simply need an extra component function for the extra dimension. We write either

$$\vecs{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$$

or

$$\vecs F(x,y,z) = P(x,y,z)\hat{\mathbf{i}} + Q(x,y,z)\hat{\mathbf{j}} + R(x,y,z)\hat{\mathbf{k}}.$$

Example 15.1.7: Sketching a Vector Field in Three Dimensions

Describe vector field $\vecs F(x,y,z) = \langle 1,\ 1,\ z\rangle$.

Solution

For this vector field, the x- and y-components are constant, so every point in $\mathbb{R}^3$ has an associated vector with x- and y-components equal to one. To visualize $\vecs F$, we first consider what the field looks like in the xy-plane. In the xy-plane, $z = 0$. Hence, each point of the form $(a,b,0)$ has vector $\langle 1,1,0\rangle$ associated with it. For points not in the xy-plane but slightly above it, the associated vector has a small but positive z-component, and therefore the associated vector points slightly upward. For points that are far above the xy-plane, the z-component is large, so the vector is almost vertical. Figure 15.1.6 shows this vector field.

Figure 15.1.6: A visual representation of vector field $\vecs F(x,y,z) = \langle 1,1,z\rangle$.

Exercise 15.1.7

Sketch vector field $\vecs G(x,y,z) = \langle 2,\ \frac{z}{2},\ 1\rangle$.

Hint available at the end of this section.

Answer available at the end of this section.

In the next example, we explore one of the classic cases of a three-dimensional vector field: a gravitational field.

Example 15.1.8: Describing a Gravitational Vector Field

Newton's law of gravitation states that $\vecs F = -G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}}$, where G is the universal gravitational constant. It describes the gravitational field exerted by an object (object 1) of mass m_1 located at the origin on another object (object 2) of mass m_2 located at point (x,y,z). Field $\vecs F$ denotes the gravitational force that object 1 exerts on object 2, r is the distance between the two objects, and $\hat{\mathbf{r}}$ indicates the unit vector from the first object to the second. The minus sign shows that the gravitational force attracts toward the origin; that is, the force of object 1 is attractive. Sketch the vector field associated with this equation.

Solution

Since object 1 is located at the origin, the distance between the objects is given by $r = \sqrt{x^2+y^2+z^2}$. The unit vector from object 1 to object 2 is $\hat{\mathbf{r}} = \frac{\langle x,y,z\rangle}{||\langle x,y,z\rangle||}$, and hence $\hat{\mathbf{r}} = \langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r}\rangle$. Therefore, gravitational vector field $\vecs F$ exerted by object 1 on object 2 is

$$\vecs F = -Gm_1m_2\langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\rangle.$$

This is an example of a radial vector field in $\mathbb{R}^3$.

Figure 15.1.7 shows what this gravitational field looks like for a large mass at the origin. Note that the magnitudes of the vectors increase as the vectors get closer to the origin.

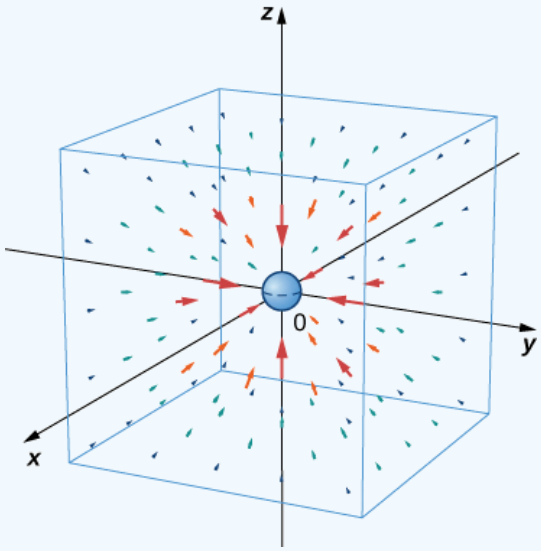


Figure 15.1.7: A visual representation of gravitational vector field $\vecs{F} = -Gm_1m_2\langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \rangle$ for a large mass at the origin.

Exercise 15.1.8

The mass of asteroid 1 is 750,000 kg and the mass of asteroid 2 is 130,000 kg. Assume asteroid 1 is located at the origin, and asteroid 2 is located at $(15, -5, 10)$, measured in units of 10 to the eighth power kilometers. Given that the universal gravitational constant is $G = 6.67384 \times 10^{-11} m^3 kg^{-1} s^{-2}$, find the gravitational force vector that asteroid 1 exerts on asteroid 2.

Hint available at the end of this section.

Gradient Fields (Conservative Fields)

In this section, we study a special kind of vector field called a gradient field or a **conservative field**. These vector fields are extremely important in physics because they can be used to model physical systems in which energy is conserved. Gravitational fields and electric fields associated with a static charge are examples of gradient fields.

Recall that if f is a (scalar) function of x and y, then the gradient of f is

$$\text{grad } f = \vecs{\nabla} f(x, y) = f_x(x, y)\hat{\mathbf{i}} + f_y(x, y)\hat{\mathbf{j}}.$$

We can see from the form in which the gradient is written that $\vecs{\nabla} f$ is a vector field in $\mathbb{R}^2$. Similarly, if f is a function of x, y, and z, then the gradient of f is

$$\text{grad } f = \vecs{\nabla} f(x, y, z) = f_x(x, y, z)\hat{\mathbf{i}} + f_y(x, y, z)\hat{\mathbf{j}} + f_z(x, y, z)\hat{\mathbf{k}}.$$

The gradient of a three-variable function is a vector field in $\mathbb{R}^3$. A gradient field is a vector field that can be written as the gradient of a function, and we have the following definition.

DEFINITION: Gradient Field

A vector field $\vecs{F}$ in $\mathbb{R}^2$ or in $\mathbb{R}^3$ is a gradient field if there exists a scalar function f such that $\vecs{\nabla} f = \vecs{F}$.

Example 15.1.9: Sketching a Gradient Vector Field

Use technology to plot the gradient vector field of $f(x,y) = x^2y^2$.

Solution

The gradient of f is $\vecs{\nabla} f(x,y) = \langle 2xy^2,\ 2x^2y\rangle$. To sketch the vector field, use a computer algebra system such as Mathematica. Figure 15.1.8 shows $\vecs{\nabla} f$.

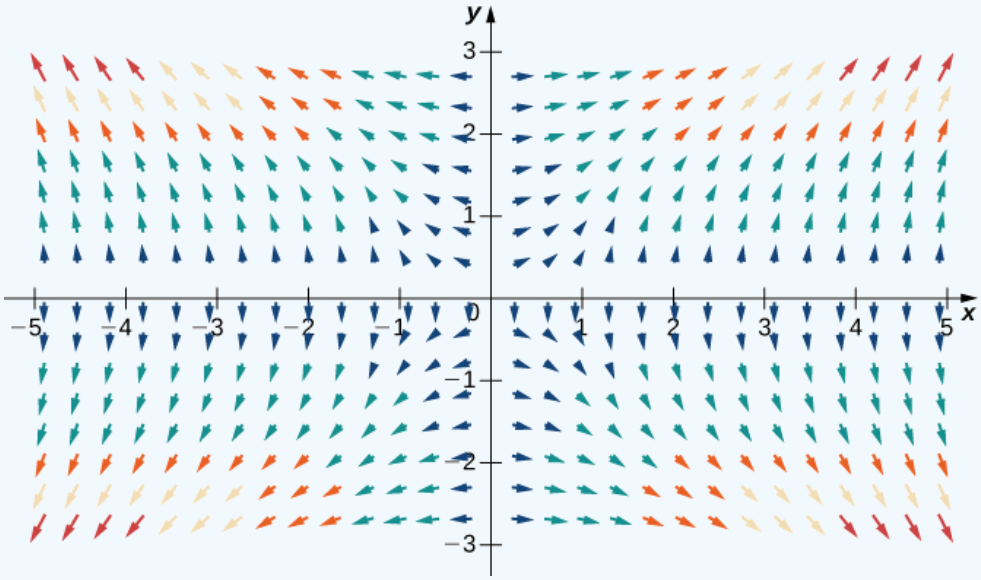


Figure 15.1.8: The gradient vector field is $\vecs{\nabla} f$, where $f(x,y) = x^2y^2$.

Exercise 15.1.9

Use technology to plot the gradient vector field of $f(x,y) = \sin x \cos y$.

Hint available at the end of this section.

Consider the function $f(x,y) = x^2y^2$ from Example 15.1.9. Figure 15.1.9 shows the level curves of this function overlaid on the function's gradient vector field. The gradient vectors are perpendicular to the level curves, and the magnitudes of the vectors get larger as the level curves get closer together, because closely grouped level curves indicate the graph is steep, and the magnitude of the gradient vector is the largest value of the directional derivative. Therefore, you can see the local steepness of a graph by investigating the corresponding function's gradient field.

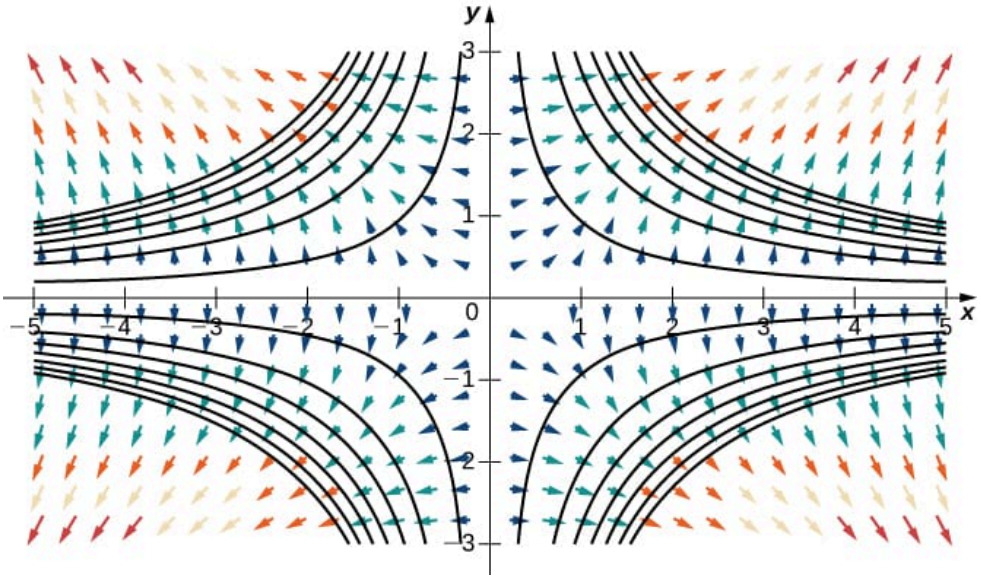


Figure 15.1.9: The gradient field of $f(x,y) = x^2y^2$ and several level curves of f. Notice that as the level curves get closer together, the magnitude of the gradient vectors increases.

As we learned earlier, a vector field $\vecs F$ is a conservative vector field, or a gradient field if there exists a scalar function f such that $\vecs\nabla f = \vecs F$. In this situation, f is called a **potential function** for $\vecs F$. Conservative vector fields arise in many applications, particularly in physics. The reason such fields are called *conservative* is that they model forces of physical systems in which energy is conserved. We study conservative vector fields in more detail later in this chapter.

You might notice that, in some applications, a potential function f for $\vecs F$ is defined instead as a function such that $-\vecs\nabla f = \vecs F$. This is the case for certain contexts in physics, for example.

Example 15.1.10: Verifying a Potential Function

Is $f(x,y,z) = x^2yz - \sin(xy)$ a potential function for vector field

$$\vecs F(x,y,z) = \langle 2xyz - y\cos(xy), x^2z - x\cos(xy), x^2y\rangle?$$

Solution

We need to confirm whether $\vecs\nabla f = \vecs F$. We have

$$\begin{aligned} f_x(x,y) &= 2xyz - y\cos(xy) \\ f_y(x,y) &= x^2z - x\cos(xy). \\ f_z(x,y) &= x^2y \end{aligned}$$

Therefore, $\vecs\nabla f = \vecs F$ and f is a potential function for $\vecs F$.

Exercise 15.1.10

Is $f(x,y,z) = x^2\cos(yz) + y^2z^2$ a potential function for $\vecs F(x,y,z) = \langle 2x\cos(yz), -x^2z\sin(yz) + 2yz^2, y^2\rangle$?

Hint available at the end of this section.

Example 15.1.11: Verifying a Potential Function

The velocity of a fluid is modeled by field $\vecs{v}(x,y) = \langle xy, \frac{x^2}{2} - y\rangle$. Verify that $f(x,y) = \dfrac{x^2y}{2} - \dfrac{y^2}{2}$ is a potential function for $\vecs{v}$.

Solution

To show that f is a potential function, we must show that $\vecs{\nabla} f = \vecs{v}$. Note that $f_x(x,y) = xy$ and $f_y(x,y) = \dfrac{x^2}{2} - y$. Therefore, $\vecs{\nabla} f(x,y) = \langle xy, \frac{x^2}{2} - y\rangle$ and f is a potential function for $\vecs{v}$ (Figure 15.1.10).

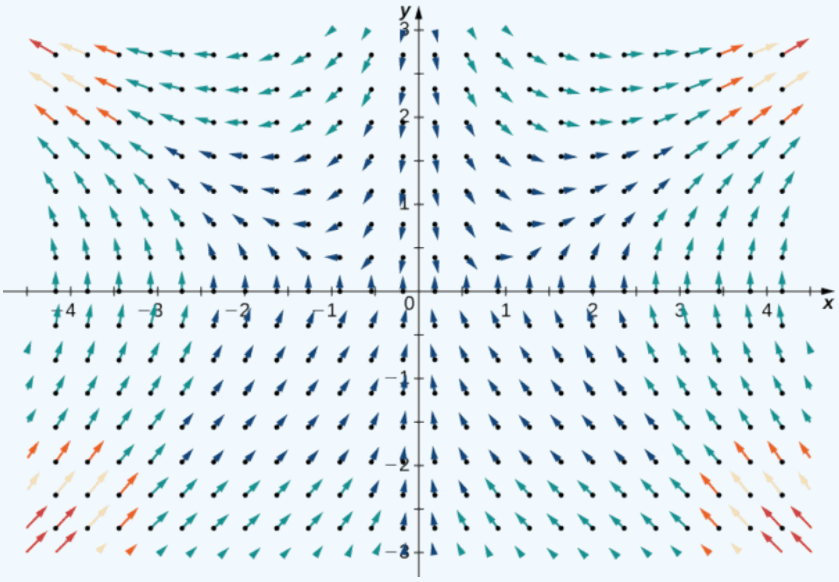


Figure 15.1.10: Velocity field $\vecs{v}(x,y)$ has a potential function and is a conservative field.

Exercise 15.1.11

Verify that $f(x,y) = x^3y^2 + x$ is a potential function for velocity field $\vecs{v}(x,y) = \langle 3x^2y^2 + 1, 2x^3y\rangle$.

Hint available at the end of this section.

If $\vecs{F}$ is a conservative vector field, then there is at least one potential function f such that $\vecs{\nabla} f = \vecs{F}$. But, could there be more than one potential function? If so, is there any relationship between two potential functions for the same vector field? Before answering these questions, let's recall some facts from single-variable calculus to guide our intuition. Recall that if $k(x)$ is an integrable function, then k has infinitely many antiderivatives. Furthermore, if $\vecs{F}$ and $\vecs{G}$ are both antiderivatives of k, then $\vecs{F}$ and $\vecs{G}$ **differ only by a constant**. That is, there is some number C such that $\vecs{F}(x) = \vecs{G}(x) + C$.

Now let $\vecs{F}$ be a conservative vector field and let f and g be potential functions for $\vecs{F}$. Since the gradient is like a derivative, $\vecs{F}$ being conservative means that $\vecs{F}$ is "integrable" with "antiderivatives" f and g. Therefore, if the analogy with single-variable calculus is valid, we expect there is some constant C such that $f(x) = g(x) + C$. The next theorem says that this is indeed the case.

To state the next theorem with precision, we need to assume the domain of the vector field is connected and open. To be connected means if P_1 and P_2 are any two points in the domain, then you can walk from P_1 to P_2 along a path that stays entirely inside the domain.

UNIQUENESS OF POTENTIAL FUNCTIONS

Let $\vecs F$ be a conservative vector field on an open and connected domain and let f and g be functions such that $\vecs\nabla f = \vecs F$ and $\vecs\nabla g = \vecs F$. Then, there is a constant C such that $f = g + C$.

Proof

Since f and g are both potential functions for $\vecs F$, then $\vecs\nabla(f-g) = \vecs\nabla f - \vecs\nabla g = \vecs F - \vecs F = \vecs 0$. Let $h = f - g$, then we have $\vecs\nabla h = \vecs 0$.We would like to show that h is a constant function.

Assume h is a function of x and y (the logic of this proof extends to any number of independent variables). Since $\vecs\nabla h = \vecs 0$, we have $h_x(x,y) = 0$ and $h_y(x,y) = 0$. The expression $h_x(x,y) = 0$ implies that h is a constant function with respect to x—that is, $h(x,y) = k_1(y)$ for some function k_1. Similarly, $h_y(x,y) = 0$ implies $h(x,y) = k_2(x)$ for some function k_2. Therefore, function h depends only on y and also depends only on x. Thus, $h(x,y) = C$ for some constant C on the connected domain of $\vecs F$. Note that we really do need connectedness at this point; if the domain of $\vecs F$ came in two separate pieces, then k could be a constant C_1 on one piece but could be a different constant C_2 on the other piece. Since $f - g = h = C$, we have that $f = g + C$, as desired.

□

Conservative vector fields also have a special property called the ***cross-partial property***. This property helps test whether a given vector field is conservative.

THE CROSS-PARTIAL PROPERTY OF CONSERVATIVE VECTOR FIELDS

Let $\vecs F$ be a vector field in two or three dimensions such that the component functions of $\vecs F$ have continuous second-order mixed-partial derivatives on the domain of $\vecs F$.

If $\vecs F(x,y) = \langle P(x,y), Q(x,y)\rangle$ is a conservative vector field in $\mathbb{R}^2$, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

If $\vecs F(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ is a conservative vector field in $\mathbb{R}^3$, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$
$$\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$
$$\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}.$$

Proof

Since $\vecs F$ is conservative, there is a function $f(x,y)$ such that $\vecs\nabla f = \vecs F$. Therefore, by the definition of the gradient, $f_x = P$ and $f_y = Q$. By Clairaut's theorem, $f_{xy} = f_{yx}$, But, $f_{xy} = P_y$ and $f_{yx} = Q_x$, and thus $P_y = Q_x$.

□

Clairaut's theorem gives a fast proof of the cross-partial property of conservative vector fields in $\mathbb{R}^3$, just as it did for vector fields in $\mathbb{R}^2$.

The Cross-Partial Property of Conservative Vector Fields shows that most vector fields are not conservative. The cross-partial

property is difficult to satisfy in general, so most vector fields won't have equal cross-partials.

Show that rotational vector field $\vecs F(x,y) = \langle y,\ -x\rangle$ is not conservative.

Solution

Let $P(x,y) = y$ and $Q(x,y) = -x$. If $\vecs F$ is conservative, then the cross-partials would be equal—that is, P_y would equal Q_x.Therefore, to show that $\vecs F$ is not conservative, check that $P_y \neq Q_x$. Since $P_y = 1$ and $Q_x = -1$, the vector field is not conservative.

Exercise 15.1.12

Show that vector field $\vecs F(x,y) = xy\ \hat{\mathbf{i}} - x^2y\ \hat{\mathbf{j}}$ is not conservative.

Hint available at the end of this section.

Example 15.1.13: Showing a Vector Field Is Not Conservative

Is vector field $\vecs F(x,y,z) = \langle 7, -2, x^3\rangle$ conservative?

Solution

Let $P(x,y,z) = 7$, $Q(x,y,z) = -2$, and $R(x,y,z) = x^3$. If $\vecs F$ is conservative, then all three cross-partial equations will be satisfied—that is, if $\vecs F$ is conservative, then P_y would equal Q_x, Q_z would equal R_y, and R_x would equal P_z. Note that

$$P_y = Q_x = R_y = Q_z = 0$$

so the first two necessary equalities hold. However, $R_x(x,y,z) = x^3$ and $P_z(x,y,z) = 0$ so $R_x \neq P_z$. Therefore, $\vecs F$ is not conservative.

Exercise 15.1.13

Is vector field $\vecs G(x,y,z) = \langle y,\ x,\ xyz\rangle$ conservative?

Hint available at the end of this section.

We conclude this section with a word of warning: The Cross-Partial Property of Conservative Vector Fields says that if $\vecs F$ is conservative, then $\vecs F$ has the cross-partial property. The theorem does *not* say that, if $\vecs F$ has the cross-partial property, then $\vecs F$ is conservative (the converse of an implication is not logically equivalent to the original implication). In other words, The Cross-Partial Property of Conservative Vector Fields can only help determine that a field is not conservative; it does not let you conclude that a vector field is conservative.

For example, consider vector field $\vecs F(x,y) = \langle x^2y, \dfrac{x^3}{3}\rangle$. This field has the cross-partial property, so it is natural to try to use The Cross-Partial Property of Conservative Vector Fields to conclude this vector field is conservative. However, this is a misapplication of the theorem. We learn later how to conclude that $\vecs F$ is conservative.

Key Concepts

- A vector field assigns a vector $\vecs F(x,y)$ to each point (x,y) in a subset D of $\mathbb{R}^2$ or $\mathbb{R}^3$. $\vecs F(x,y,z)$ to each point (x,y,z) in a subset D of $\mathbb{R}^3$.
- Vector fields can describe the distribution of vector quantities such as forces or velocities over a region of the plane or of space. They are in common use in such areas as physics, engineering, meteorology, oceanography.
- We can sketch a vector field by examining its defining equation to determine relative magnitudes in various locations and

then drawing enough vectors to determine a pattern.
- A vector field $\vecs F$ is called conservative if there exists a scalar function f such that $\vecs\nabla f = \vecs F$.

Key Equations

- **Vector field in $\mathbb{R}^2$**
  $\vecs F(x,y) = \langle P(x,y),\ Q(x,y)\rangle$
  or
  $\vecs F(x,y) = P(x,y)\ \hat{\mathbf{i}} + Q(x,y)\ \hat{\mathbf{j}}$
- **Vector field in $\mathbb{R}^3$**
  $\vecs F(x,y,z) = \langle P(x,y,z),\ Q(x,y,z),\ R(x,y,z)\rangle$
  or
  $\vecs F(x,y,z) = P(x,y,z)\ \hat{\mathbf{i}} + Q(x,y,z)\ \hat{\mathbf{j}} + R(x,y,z)\ \hat{\mathbf{k}}$

Glossary

conservative field
: a vector field for which there exists a scalar function f such that $\vecs\nabla f = \vecs F$

gradient field
: a vector field $\vecs F$ for which there exists a scalar function f such that $\vecs\nabla f = \vecs F$; in other words, a vector field that is the gradient of a function; such vector fields are also called *conservative*

potential function
: a scalar function f such that $\vecs\nabla f = \vecs F$

radial field
: a vector field in which all vectors either point directly toward or directly away from the origin; the magnitude of any vector depends only on its distance from the origin

rotational field
: a vector field in which the vector at point (x,y) is tangent to a circle with radius $r = \sqrt{x^2+y^2}$; in a rotational field, all vectors flow either clockwise or counterclockwise, and the magnitude of a vector depends only on its distance from the origin

unit vector field
: a vector field in which the magnitude of every vector is 1

vector field
: measured in $\mathbb{R}^2$, an assignment of a vector $\vecs F(x,y)$ to each point (x,y) of a subset D of $\mathbb{R}^2$; in $\mathbb{R}^3$, an assignment of a vector $\vecs F(x,y,z)$ to each point (x,y,z) of a subset D of $\mathbb{R}^3$

Answers

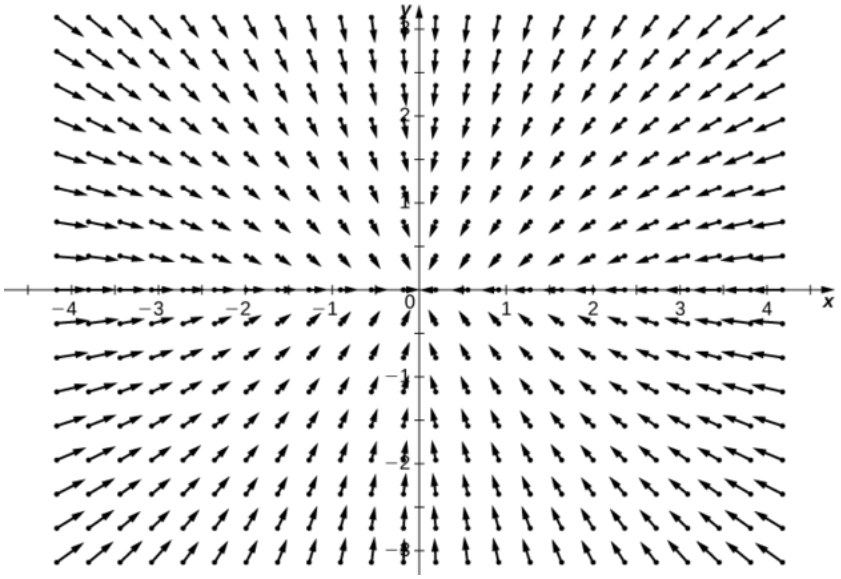
Exercise 15.1.1

Substitute the point values into the vector function.

$\vecs G(-2,3) = 12\hat{\mathbf{i}} - \hat{\mathbf{j}}$

Exercise 15.1.2

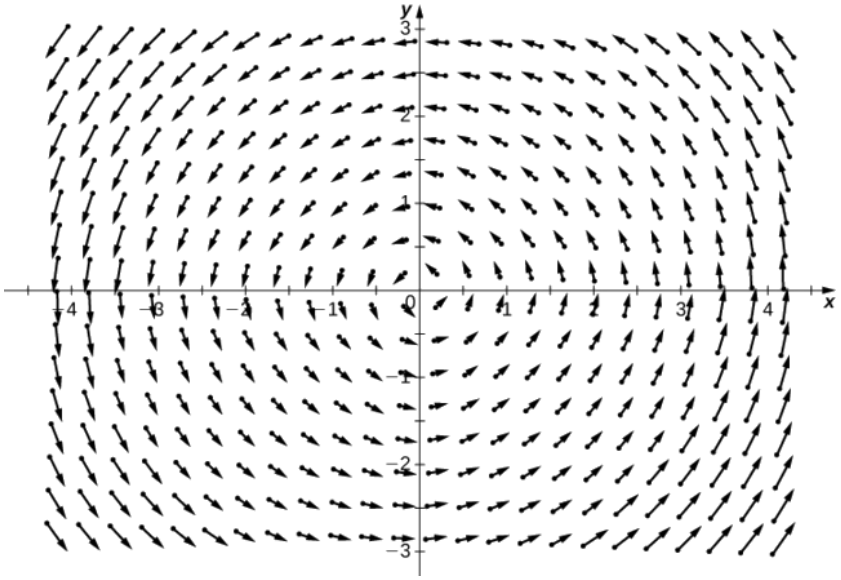
Sketch enough vectors to get an idea of the shape.



Exercise 15.1.4

Substitute enough points into $\vecs F$ to get an idea of the shape.

Rotational



Exercise 15.1.5

Remember, speed is the magnitude of velocity.

$\sqrt{65}$ m/sec

Exercise 15.1.6

Calculate the magnitude of $\vecs F$ at an arbitrary point (x, y).

No.

Exercise 15.1.7

Substitute enough points into the vector field to get an idea of the general shape.

Exercise 15.1.7

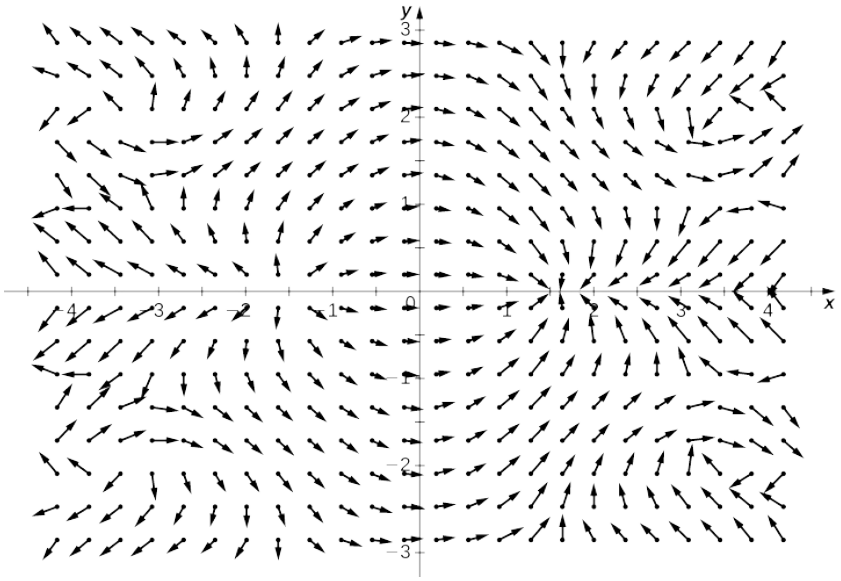
Exercise 15.1.8

Follow Example 15.1.8 and first compute the distance between the asteroids.

1.49063×10^{-18}, 4.96876×10^{-19}, 9.93752×10^{-19} N

Exercise 15.1.9

Find the gradient of f.



Exercise 15.1.10

Compute the gradient of f.

No

Exercise 15.1.11

Calculate the gradient.

$\vecs\nabla f(x,y) = \vecs v(x,y)$

Exercise 15.1.12

Check the cross-partials.

$P_y(x,y) = x$ and $Q_x(x,y) = -2xy$. Since $P_y(x,y) \neq Q_x(x,y)$, $\vecs F$ is not conservative.

Exercise 15.1.13

Check the cross-partials.

No

15.1E: Vector Fields (Exercises)

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15.1E: Vector Fields (Exercises)

1. The domain of vector field $\vecs F = \vecs F(x, y)$ is a set of points (x, y) in a plane, and the range of $\vecs F$ is a set of *what* in the plane?

Answer

Vectors

For exercises 2 - 4, determine whether the statement is *true or false*.

2. Vector field $\vecs F = \langle 3x^2, 1\rangle$ is a gradient field for both $\phi_1(x, y) = x^3 + y$ and $\phi_2(x, y) = y + x^3 + 100$.

3. Vector field $\vecs F = \dfrac{\langle y, x\rangle}{\sqrt{x^2 + y^2}}$ is constant in direction and magnitude on a unit circle.

Answer

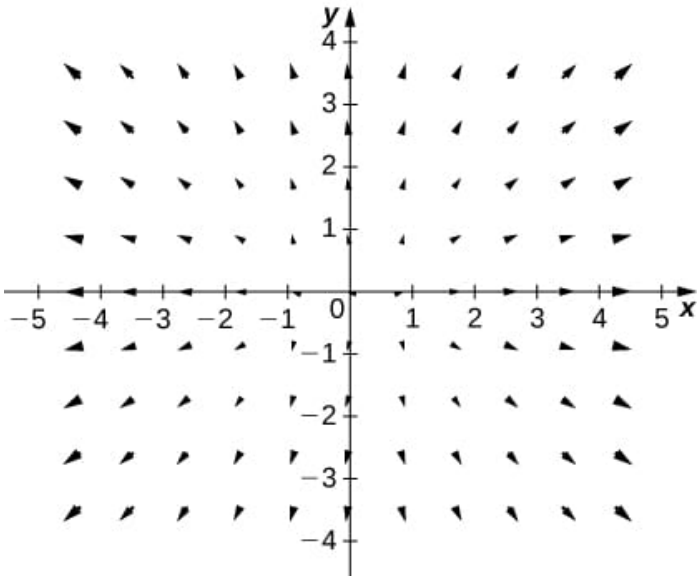
False

4. Vector field $\vecs F = \dfrac{\langle y, x\rangle}{\sqrt{x^2 + y^2}}$ is neither a radial field nor a rotation field.

For exercises 5 - 13, describe each vector field by drawing some of its vectors.

5. [T] $\vecs F(x, y) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$

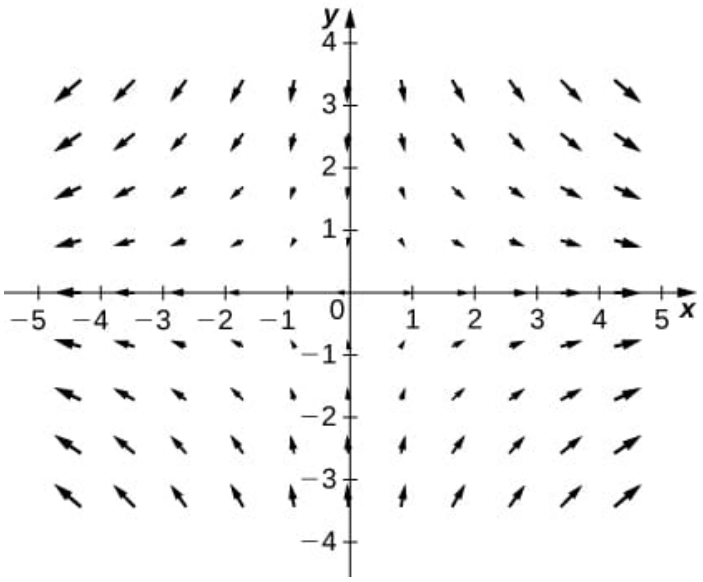
Answer



6. [T] $\vecs F(x, y) = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$

7. [T] $\vecs F(x, y) = x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}$

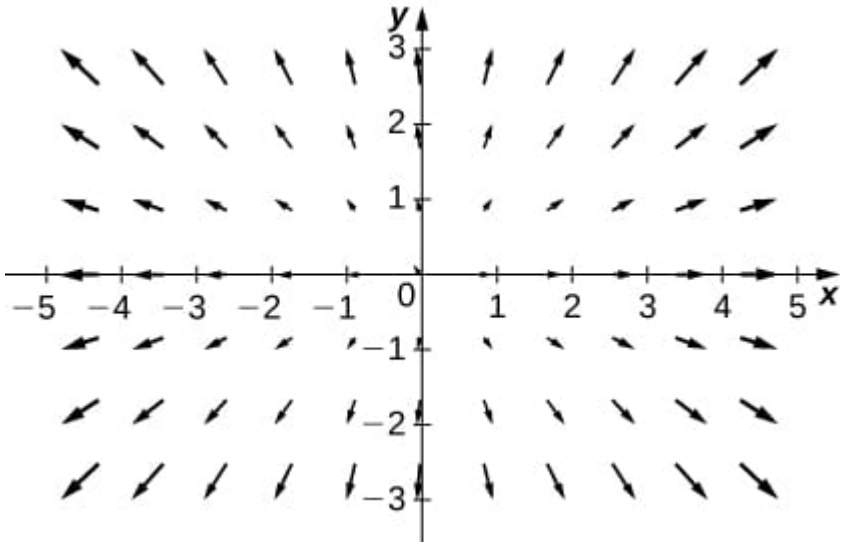
Answer



8. [T] $\vecs{F}(x,y) = \hat{\mathbf{i}} + \hat{\mathbf{j}}$

9. [T] $\vecs{F}(x,y) = 2x\ \hat{\mathbf{i}} + 3y\ \hat{\mathbf{j}}$

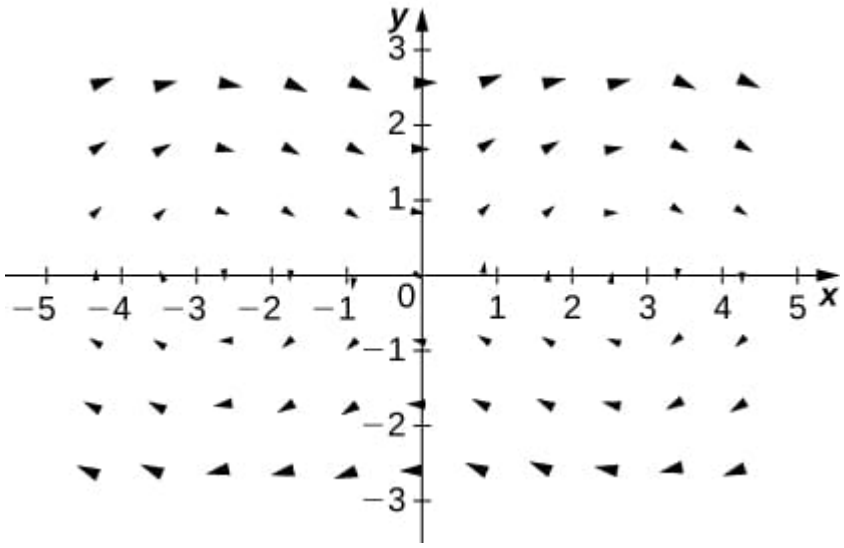
Answer



10. [T] $\vecs{F}(x,y) = 3\ \hat{\mathbf{i}} + x\ \hat{\mathbf{j}}$

11. [T] $\vecs{F}(x,y) = y\ \hat{\mathbf{i}} + \sin x\ \hat{\mathbf{j}}$

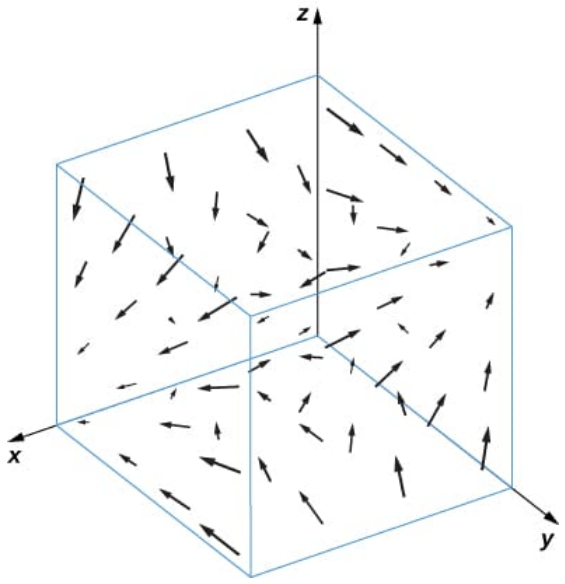
Answer



12. [T] $\vecs{F}(x, y, z) = x\ \hat{\mathbf{i}} + y\ \hat{\mathbf{j}} + z\ \hat{\mathbf{k}}$

13. [T] $\vecs{F}(x, y, z) = 2x\ \hat{\mathbf{i}} - 2y\ \hat{\mathbf{j}} - 2z\ \hat{\mathbf{k}}$

Answer



14. [T] $\vecs{F}(x, y, z) = yz\ \hat{\mathbf{i}} - xz\ \hat{\mathbf{j}}$

For exercises 15 - 20, find the gradient vector field of each function f.

15. $f(x, y) = x \sin y + \cos y$

Answer

$$\vecs{F}(x, y) = \sin(y)\ \hat{\mathbf{i}} + (x \cos y - \sin y)\ \hat{\mathbf{j}}$$

16. $f(x, y, z) = ze^{-xy}$

17. $f(x, y, z) = x^2y + xy + y^2z$

Answer

$$\vecs F(x,y,z) = (2xy + y)\ \hat{\mathbf{i}} + (x^2 + x + 2yz)\ \hat{\mathbf{j}} + y^2\ \hat{\mathbf{k}}$$

18. $f(x,y) = x^2 \sin(5y)$

19. $f(x,y) = \ln(1 + x^2 + 2y^2)$

Answer

$$\vecs F(x,y) = \frac{2x}{1 + x^2 + 2y^2}\ \hat{\mathbf{i}} + \frac{4y}{1 + x^2 + 2y^2}\ \hat{\mathbf{j}}$$

20. $f(x,y,z) = x \cos\left(\frac{y}{z}\right)$

21. What is vector field $\vecs F(x,y)$ with a value at (x,y) that is of unit length and points toward $(1,0)$?

Answer

$$\vecs F(x,y) = \frac{(1-x)\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}}{\sqrt{(1-x)^2 + y^2}}$$

For exercises 22 - 24, write formulas for the vector fields with the given properties.

22. All vectors are parallel to the x-axis and all vectors on a vertical line have the same magnitude.

23. All vectors point toward the origin and have constant length.

Answer

$$\vecs F(x,y) = \frac{(y\,\hat{\mathbf{i}} - x\,\hat{\mathbf{j}})}{\sqrt{x^2 + y^2}}$$

24. All vectors are of unit length and are perpendicular to the position vector at that point.

25. Give a formula $\vecs F(x,y) = M(x,y)\ \hat{\mathbf{i}} + N(x,y)\ \hat{\mathbf{j}}$ for the vector field in a plane that has the properties that $\vecs F = \vecs 0$ at $(0,0)$ and that at any other point (a,b), $\vecs F$ is tangent to circle $x^2 + y^2 = a^2 + b^2$ and points in the clockwise direction with magnitude $\|\vecs F\| = \sqrt{a^2 + b^2}$.

Answer

$$\vecs F(x,y) = y\ \hat{\mathbf{i}} - x\ \hat{\mathbf{j}}$$

26. Is vector field $\vecs F(x,y) = (P(x,y), Q(x,y)) = (\sin x + y)\ \hat{\mathbf{i}} + (\cos y + x)\ \hat{\mathbf{j}}$ a gradient field?

27. Find a formula for vector field $\vecs F(x,y) = M(x,y)\ \hat{\mathbf{i}} + N(x,y)\ \hat{\mathbf{j}}$ given the fact that for all points (x,y), $\vecs F$ points toward the origin and $\|\vecs F\| = \dfrac{10}{x^2 + y^2}$.

Answer

$$\vecs F(x,y) = \frac{-10}{(x^2 + y^2)^{3/2}}(x\ \hat{\mathbf{i}} + y\ \hat{\mathbf{j}})$$

For exercises 28 - 29, assume that an electric field in the xy-plane caused by an infinite line of charge along the x-axis is a gradient field with potential function $V(x,y) = c \ln\left(\frac{r_0}{\sqrt{x^2+y^2}}\right)$, where $c > 0$ is a constant and r_0 is a reference distance at which the potential is assumed to be zero.

28. Find the components of the electric field in the x- and y-directions, where $\vecs E(x,y) = -\vecs \nabla V(x,y)$.

29. Show that the electric field at a point in the xy-plane is directed outward from the origin and has magnitude $\|\vecs E\| = \dfrac{c}{r}$, where $r = \sqrt{x^2 + y^2}$.

Answer

$$\|\vecs{E}\| = \frac{c}{|r|^2} r = \frac{c}{|r|} \frac{r}{|r|}$$

A ***flow line*** (or *streamline*) of a vector field $\vecs{F}$ is a curve $\vecs{r}(t)$ such that $d\vecs{r}/dt = \vecs{F}(\vecs{r}(t))$. If $\vecs{F}$ represents the velocity field of a moving particle, then the flow lines are paths taken by the particle. Therefore, flow lines are tangent to the vector field.

For exercises 30 and 31, show that the given curve $\vecs{c}(t)$ is a flow line of the given velocity vector field $\vecs{F}(x, y, z)$.

30. $\vecs{c}(t) = \langle e^{2t}, \ln|t|, \frac{1}{t} \rangle, \ t \neq 0; \quad \vecs{F}(x, y, z) = \langle 2x, z, -z^2 \rangle$

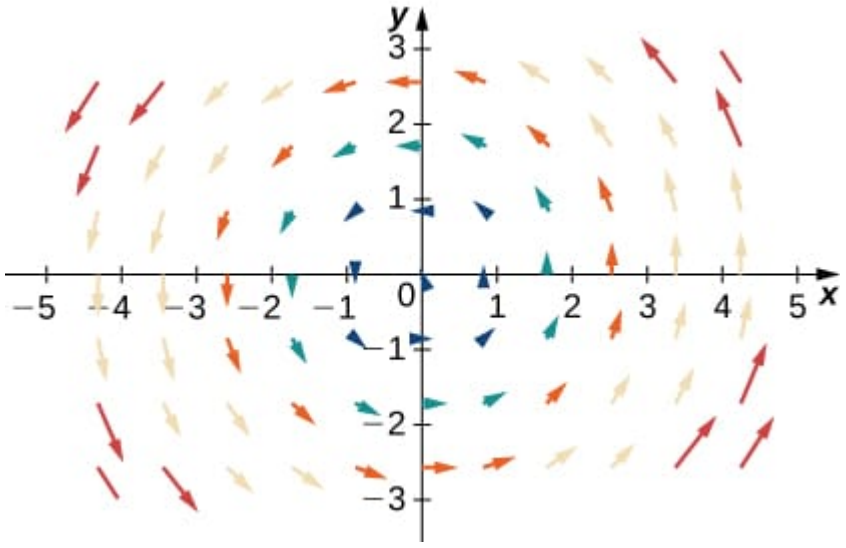
31. $\vecs{c}(t) = \langle \sin t, \cos t, e^t \rangle; \quad \vecs{F}(x, y, z) = \ \langle y, -x, z \rangle$

Answer

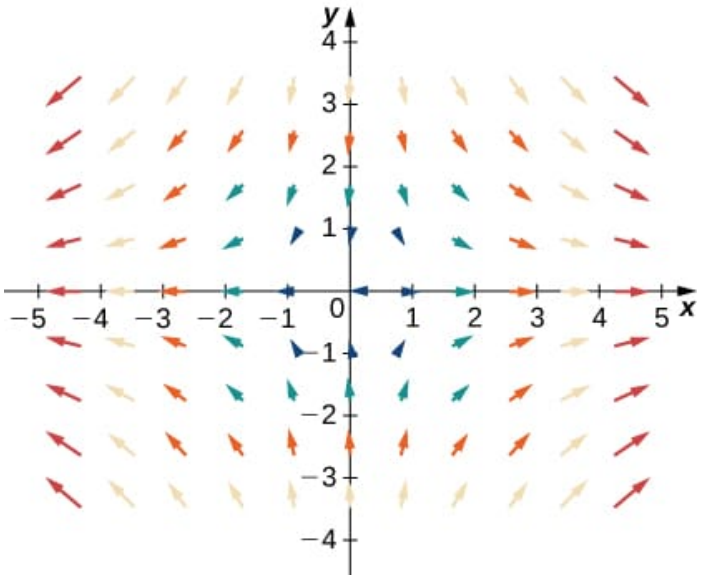
$$\vecs{c}\,'(t) = \langle \cos t, -\sin t, e^{-t} \rangle = \vecs{F}(\vecs{c}(t))$$

For exercises 32 - 34, let $\vecs{F} = x\ \hat{\mathbf{i}} + y\ \hat{\mathbf{j}}$, $\vecs{G} = -y\ \hat{\mathbf{i}} + x\ \hat{\mathbf{j}}$, and $\vecs{H} = x\ \hat{\mathbf{i}} - y\ \hat{\mathbf{j}}$. Match $\vecs{F}$, $\vecs{G}$, and $\vecs{H}$ with their graphs.

32.



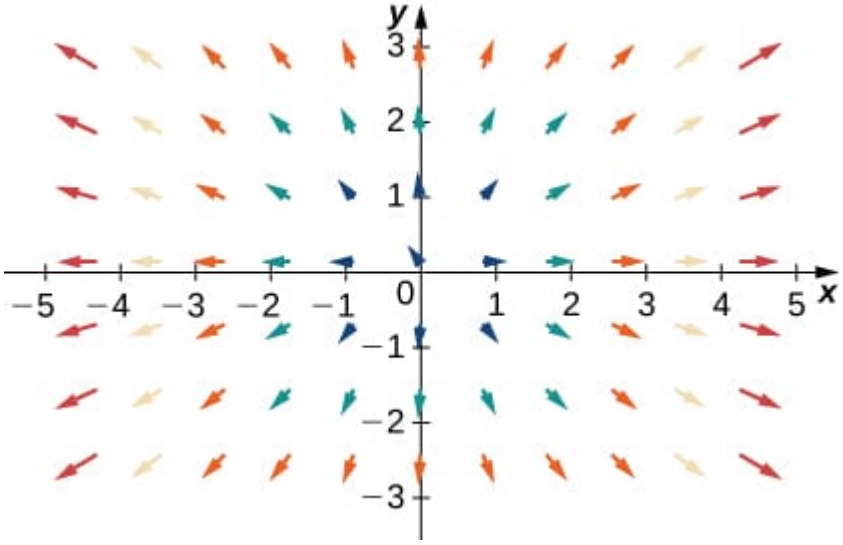
33.



Answer

$\vecs{H}$

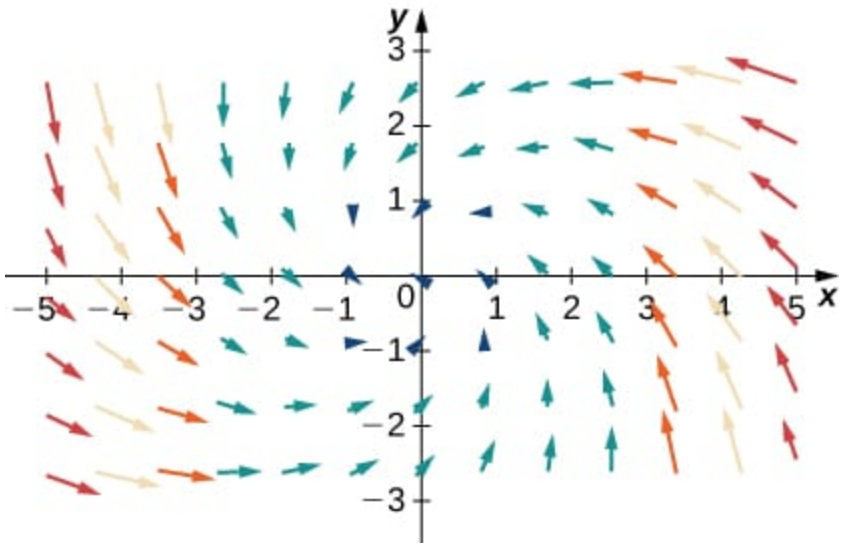
34.



For exercises 35 - 38, let $\vecs{F} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$, $\vecs{G} = -y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}$, and $\vecs{H} = -x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$. Match the vector fields with their graphs in (I)–(IV).

a. $\vecs{F} + \vecs{G}$
b. $\vecs{F} + \vecs{H}$
c. $\vecs{G} + \vecs{H}$
d. $-\vecs{F} + \vecs{G}$

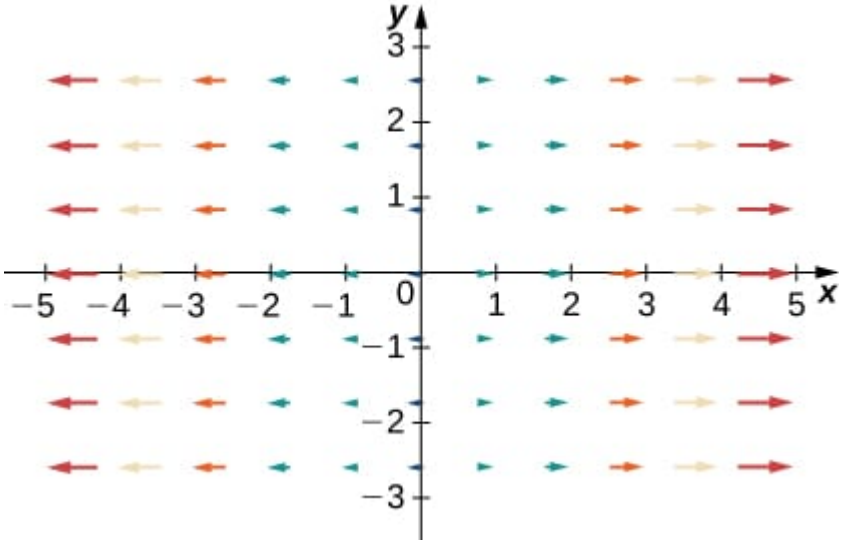
35.



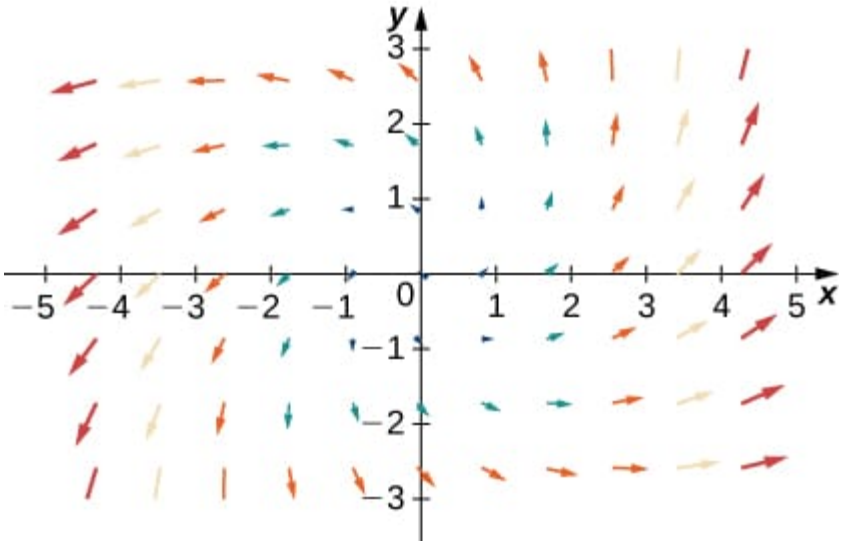
Answer

d. $-\vecs{F}+\vecs{G}$

36.



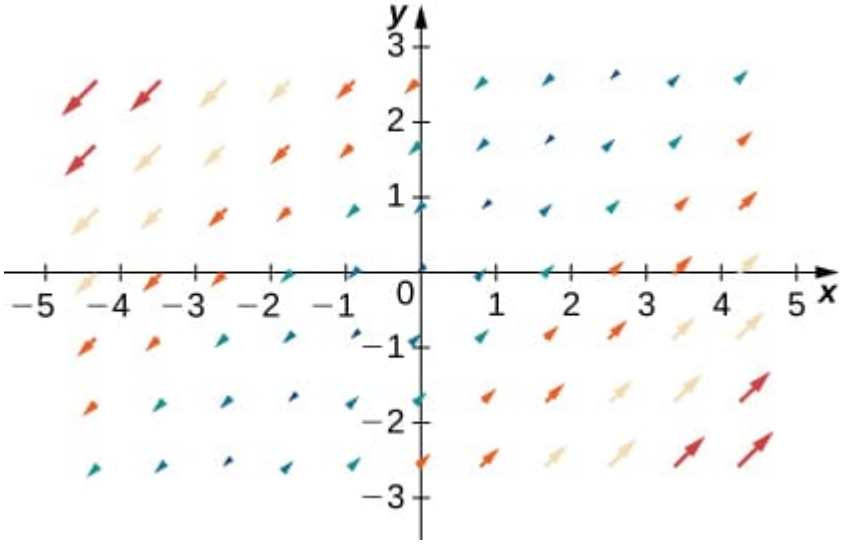
37.



Answer

a. $\vecs{F} + \vecs{G}$

38.



Contributors

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15.2: LINE INTEGRALS

Learning Objectives

- Calculate a scalar line integral along a curve.
- Calculate a vector line integral along an oriented curve in space.
- Use a line integral to compute the work done in moving an object along a curve in a vector field.
- Describe the flux and circulation of a vector field.

We are familiar with single-variable integrals of the form $\int_a^b f(x)\ dx$, where the domain of integration is an interval $[a, b]$. Such an interval can be thought of as a curve in the xy-plane, since the interval defines a line segment with endpoints $(a, 0)$ and $(b, 0)$—in other words, a line segment located on the x-axis. Suppose we want to integrate over *any* curve in the plane, not just over a line segment on the x-axis. Such a task requires a new kind of integral, called a *line integral*.

Line integrals have many applications to engineering and physics. They also allow us to make several useful generalizations of the Fundamental Theorem of Calculus. And, they are closely connected to the properties of vector fields, as we shall see.

Scalar Line Integrals

A line integral gives us the ability to integrate multivariable functions and vector fields over arbitrary curves in a plane or in space. There are two types of line integrals: scalar line integrals and vector line integrals. Scalar line integrals are integrals of a scalar function over a curve in a plane or in space. Vector line integrals are integrals of a vector field over a curve in a plane or in space. Let's look at scalar line integrals first.

A scalar line integral is defined just as a single-variable integral is defined, except that for a scalar line integral, the integrand is a function of more than one variable and the domain of integration is a curve in a plane or in space, as opposed to a curve on the x-axis.

For a scalar line integral, we let C be a smooth curve in a plane or in space and let f be a function with a domain that includes C. We chop the curve into small pieces. For each piece, we choose point P in that piece and evaluate f at P. (We can do this because all the points in the curve are in the domain of f.) We multiply $f(P)$ by the arc length of the piece Δs, add the product $f(P)\Delta s$ over all the pieces, and then let the arc length of the pieces shrink to zero by taking a limit. The result is the scalar line integral of the function over the curve.

For a formal description of a scalar line integral, let C be a smooth curve in space given by the parameterization $\vecs r(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$. Let $f(x, y, z)$ be a function with a domain that includes curve C. To define the line integral of the function f over C, we begin as most definitions of an integral begin: we chop the curve into small pieces. Partition the parameter interval $[a, b]$ into n subintervals $[t_{i-l}, t_i]$ of equal width for $1 \le i \le n$, where $t_0 = a$ and $t_n = b$ (Figure 15.2.1). Let t_i^* be a value in the i^{th} interval $[t_{i-l}, t_i]$. Denote the endpoints of $\vecs r(t_0)$, $\vecs r(t_1)$,..., $\vecs r(t_n)$ by P_0,..., P_n. Points P_i divide curve C into n pieces C_1, C_2,..., C_n,with lengths Δs_1, Δs_2,..., Δs_n, respectively. Let P_i^* denote the endpoint of $\vecs r(t_i^*)$ for $1 \le i \le n$. Now, we evaluate the function f at point P_i^* for $1 \le i \le n$. Note that P_i^* is in piece C_1, and therefore P_i^* is in the domain of f. Multiply $f(P_i^*)$ by the length Δs_1 of C_1, which gives the area of the "sheet" with base C_1, and height $f(P_i^*)$. This is analogous to using rectangles to approximate area in a single-variable integral. Now, we form the sum $\sum_{i=1}^{n} f(P_i^*)\ \Delta s_i$.

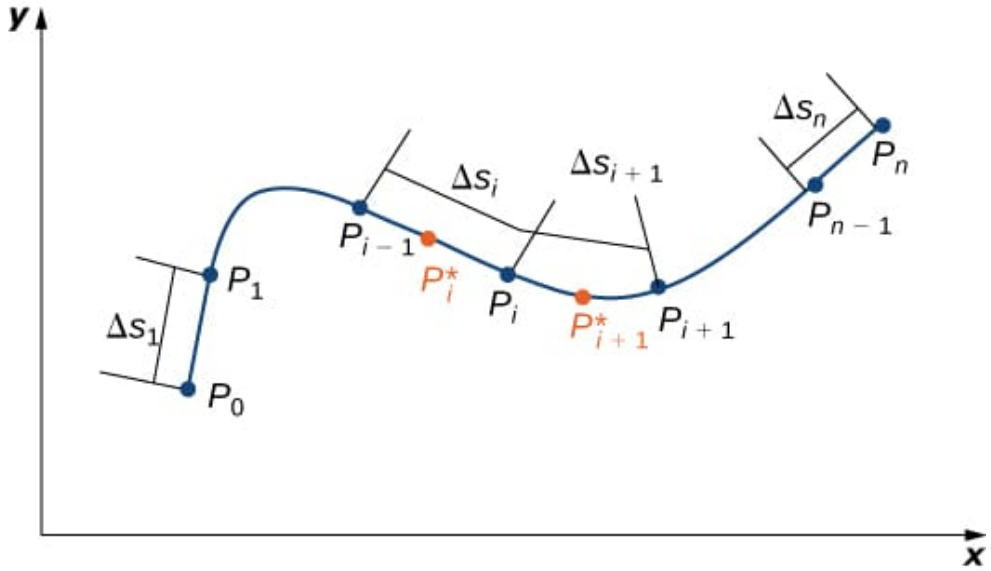


Figure 15.2.1: Curve C has been divided into n pieces, and a point inside each piece has been chosen.

Note the similarity of this sum versus a Riemann sum; in fact, this definition is a generalization of a Riemann sum to arbitrary curves in space. Just as with Riemann sums and integrals of form $\int_a^b g(x)\ dx$, we define an integral by letting the width of the pieces of the curve shrink to zero by taking a limit. The result is the scalar line integral of f along C.

You may have noticed a difference between this definition of a scalar line integral and a single-variable integral. In this definition, the arc lengths Δs_1, Δs_2,..., Δs_n aren't necessarily the same; in the definition of a single-variable integral, the curve in the x-axis is partitioned into pieces of equal length. This difference does not have any effect in the limit. As we shrink the arc lengths to zero, their values become close enough that any small difference becomes irrelevant.

DEFINITION: Scalar Line Integral

Let f be a function with a domain that includes the smooth curve C that is parameterized by $\vecs r(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$. The *scalar line integral* of f along C is

$$\int_C f(x,y,z)\,ds = \lim_{n\to\infty}\sum_{i=1}^{n} f(P_i^*)\,\Delta s_i \tag{15.2.1}$$

if this limit exists (t_i^* and Δs_i are defined as in the previous paragraphs). If C is a planar curve, then C can be represented by the parametric equations $x = x(t)$, $y = y(t)$, and $a \le t \le b$. If C is smooth and $f(x,y)$ is a function of two variables, then the scalar line integral of f along C is defined similarly as

$$\int_C f(x,y)\,ds = \lim_{n\to\infty}\sum_{i=1}^{n} f(P_i^*)\,\Delta s_i, \tag{15.2.2}$$

if this limit exists.

If f is a continuous function on a smooth curve C, then $\int_C f\ ds$ always exists. Since $\int_C f\ ds$ is defined as a limit of Riemann sums, the continuity of f is enough to guarantee the existence of the limit, just as the integral $\int_a^b g(x)\ dx$ exists if g is continuous over $[a,b]$.

Before looking at how to compute a line integral, we need to examine the geometry captured by these integrals. Suppose that $f(x,y) \ge 0$ for all points (x,y) on a smooth planar curve C. Imagine taking curve C and projecting it "up" to the surface defined by $f(x,y)$, thereby creating a new curve C' that lies in the graph of $f(x,y)$ (Figure 15.2.2). Now we drop a "sheet" from C' down to

the xy-plane. The area of this sheet is $\int_C f(x,y)ds$. If $f(x,y) \leq 0$ for some points in C, then the value of $\int_C f(x,y)\ ds$ is the area above the xy-plane less the area below the xy-plane. (Note the similarity with integrals of the form $\int_a^b g(x)\ dx$.)

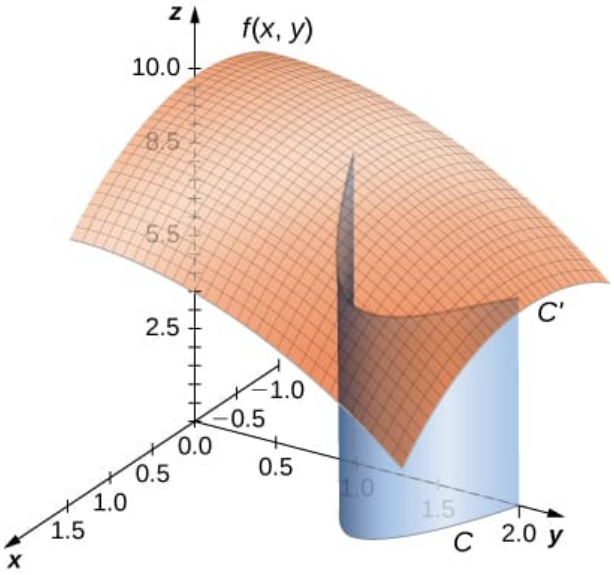


Figure 15.2.2: The area of the blue sheet is $\int_C f(x,y)\ ds$.

From this geometry, we can see that line integral $\int_C f(x,y)\ ds$ does not depend on the parameterization $\vecs{r}(t)$ of C. As long as the curve is traversed exactly once by the parameterization, the area of the sheet formed by the function and the curve is the same. This same kind of geometric argument can be extended to show that the line integral of a three-variable function over a curve in space does not depend on the parameterization of the curve.

Example 15.2.1: Finding the Value of a Line Integral

Find the value of integral $\int_C 2\ ds$, where C is the upper half of the unit circle.

Solution

The integrand is $f(x,y) = 2$. Figure 15.2.3 shows the graph of $f(x,y) = 2$, curve C, and the sheet formed by them. Notice that this sheet has the same area as a rectangle with width π and length 2. Therefore, $\int_C 2\ ds = 2\pi\ \text{units}^2$.

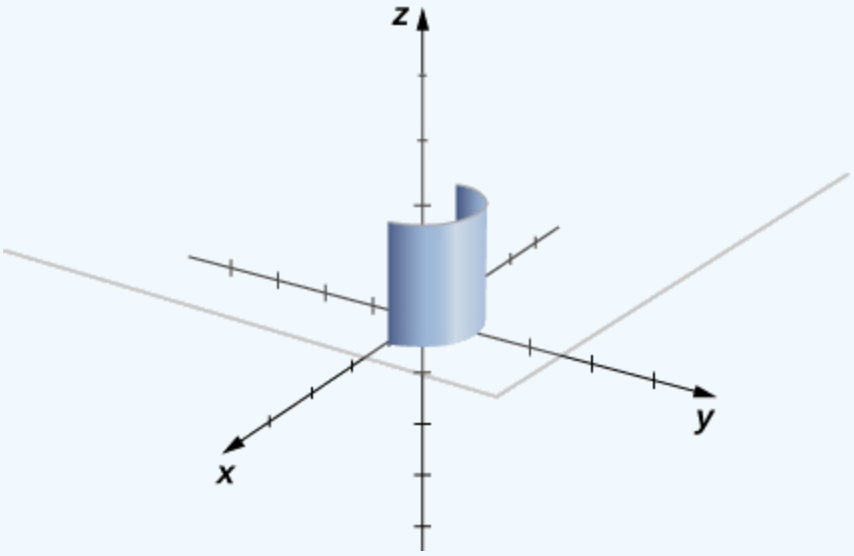


Figure 15.2.3: The sheet that is formed by the upper half of the unit circle in a plane and the graph of $f(x,y)=2$.

To see that $\int_C 2\ ds = 2\pi$ using the definition of line integral, we let $\vecs r(t)$ be a parameterization of C. Then, $f(\vecs r(t_i)) = 2$ for any number t_i in the domain of $\vecs r$. Therefore,

$$\begin{aligned}\int_C f\,ds &= \lim_{n\to\infty}\sum_{i=1}^{n} f(\vecs r(t_i^*))\,\Delta s_i \\ &= \lim_{n\to\infty}\sum_{i=1}^{n} 2\,\Delta s_i \\ &= 2\lim_{n\to\infty}\sum_{i=1}^{n} \Delta s_i \\ &= 2(\text{length of } C) \\ &= 2\pi \text{ units}^2.\end{aligned}$$

Exercise 15.2.1

Find the value of $\int_C (x+y)\ ds$, where C is the curve parameterized by $x=t$, $y=t$, $0\le t\le 1$.

Hint available at the end of this section.

Note that in a scalar line integral, the integration is done with respect to arc length s, which can make a scalar line integral difficult to calculate. To make the calculations easier, we can translate $\int_C f\ ds$ to an integral with a variable of integration that is t.

Let $\vecs r(t) = \langle x(t), y(t), z(t)\rangle$ for $a\le t\le b$ be a parameterization of C. Since we are assuming that C is smooth, $\vecs r\,'(t) = \langle x\,'(t), y\,'(t), z\,'(t)\rangle$ is continuous for all t in $[a,b]$. In particular, $x\,'(t)$, $y\,'(t)$, and $z\,'(t)$ exist for all t in $[a,b]$. According to the arc length formula, we have

$$\text{length}(C_i) = \Delta s_i = \int_{t_{i-1}}^{t_i} \|\vecs r'(t)\|\,dt.$$

If width $\Delta t_i = t_i - t_{i-1}$ is small, then function $\int_{t_{i-1}}^{t_i} \|\vecs r\,'(t)\|\ dt \ \approx\ \|\vecs r\,'(t_i^*)\|\ \Delta t_i$, $\|\vecs r\,'(t)\|$ is almost constant over the interval $[t_{i-1}, t_i]$.Therefore,

$$\int_{t_{i-1}}^{t_i} \|\vecs r'(t)\|\,dt \approx \|\vecs r'(t_i^*)\|\,\Delta t_i, \tag{15.2.3}$$

and we have

$$\sum_{i=1}^{n} f(\vecs r(t_i^*))\,\Delta s_i \approx \sum_{i=1}^{n} f(\vecs r(t_i^*))\|\vecs r'(t_i^*)\|\,\Delta t_i.$$

See Figure 15.2.4.

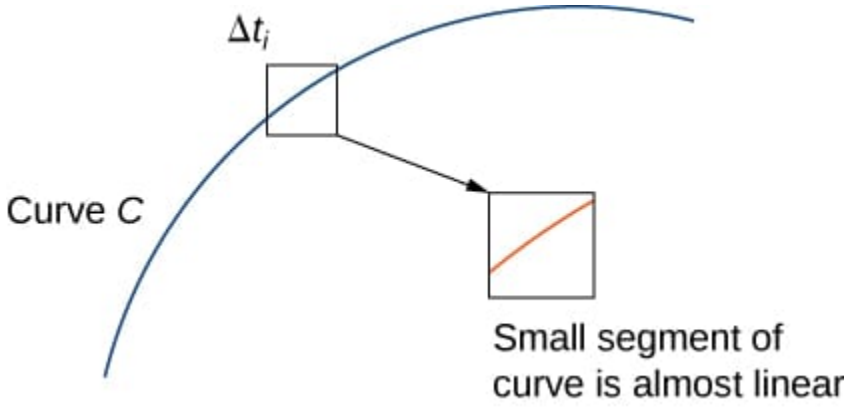


Figure 15.2.4: If we zoom in on the curve enough by making Δt_i very small, then the corresponding piece of the curve is approximately linear.

Note that

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(\vecs r(t_i^*))\|\vecs r'(t_i^*)\|\,\Delta t_i = \int_a^b f(\vecs r(t))\|\vecs r'(t)\|\,dt.$$

In other words, as the widths of intervals $[t_{i-1}, t_i]$ shrink to zero, the sum $\sum_{i=1}^{n} f(\vecs r(t_i^*))\|\vecs r\,'(t_i^*)\|\,\Delta t_i$ converges to the integral $\int_a^b f(\vecs r(t))\|\vecs r\,'(t)\|\,dt$. Therefore, we have the following theorem.

Theorem: Evaluating a Scalar Line Integral

Let f be a continuous function with a domain that includes the smooth curve C with parameterization $\vecs r(t)$, $a \le t \le b$. Then

$$\int_C f\,ds = \int_a^b f(\vecs r(t))\|\vecs r'(t)\|\,dt. \tag{15.2.4}$$

Although we have labeled Equation 15.2.3 as an equation, it is more accurately considered an approximation because we can show that the left-hand side of Equation 15.2.3 approaches the right-hand side as $n \to \infty$. In other words, letting the widths of the pieces shrink to zero makes the right-hand sum arbitrarily close to the left-hand sum. Since

$$\|\vecs r'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2},$$

we obtain the following theorem, which we use to compute scalar line integrals.

Theorem: Scalar Line Integral Calculation

Let f be a continuous function with a domain that includes the smooth curve C with parameterization $\vecs r(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$. Then

$$\int_C f(x,y,z)\,ds = \int_a^b f(\vecs{r}(t))\sqrt{(x\prime(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt.$$

Similarly,

$$\int_C f(x,y)\,ds = \int_a^b f(\vecs{r}(t))\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

if C is a planar curve and f is a function of two variables.

Note that a consequence of this theorem is the equation $ds = \|\vecs{r}\,'(t)\|\,dt$. In other words, the change in arc length can be viewed as a change in the t-domain, scaled by the magnitude of vector $\vecs{r}\,'(t)$.

Example 15.2.2: Evaluating a Line Integral

Find the value of integral $\int_C (x^2 + y^2 + z)\,ds$, where C is part of the helix parameterized by $\vecs{r}(t) = \langle \cos t, \sin t, t\rangle$, $0 \le t \le 2\pi$.

Solution

To compute a scalar line integral, we start by converting the variable of integration from arc length s to t. Then, we can use Equation 15.2.1 to compute the integral with respect to t. Note that

$$f(\vecs{r}(t)) = \cos^2 t + \sin^2 t + t = 1 + t$$

and

$$\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} = \sqrt{(-\sin(t))^2 + \cos^2(t) + 1} = \sqrt{2}.$$

Therefore,

$$\int_C (x^2 + y^2 + z)\,ds = \int_0^{2\pi} (1 + t)\sqrt{2}\,dt.$$

Notice that Equation 15.2.1 translated the original difficult line integral into a manageable single-variable integral. Since

$$\begin{aligned}\int_0^{2\pi} (1 + t)\sqrt{2}\,dt &= \left[\sqrt{2}t + \frac{\sqrt{2}t^2}{2}\right]_0^{2\pi} \\ &= 2\sqrt{2}\pi + 2\sqrt{2}\pi^2,\end{aligned}$$

we have

$$\int_C (x^2 + y^2 + z)\,ds = 2\sqrt{2}\pi + 2\sqrt{2}\pi^2.$$

Exercise 15.2.2

Evaluate $\int_C (x^2 + y^2 + z)ds$, where C is the curve with parameterization $\vecs{r}(t) = \langle \sin(3t), \cos(3t)\rangle$, $0 \le t \le \frac{\pi}{4}$.

Hint available at the end of this section.

Example 15.2.3: Independence of Parameterization

Find the value of integral $\int_C (x^2 + y^2 + z)\ ds$, where C is part of the helix parameterized by $\vecs{r}(t) = \langle \cos(2t), \sin(2t), 2t \rangle$, $0 \leq t \leq \pi$. Notice that this function and curve are the same as in the previous example; the only difference is that the curve has been reparameterized so that time runs twice as fast.

Solution

As with the previous example, we use Equation 15.2.1 to compute the integral with respect to t. Note that $f(\vecs{r}(t)) = \cos^2(2t) + \sin^2(2t) + 2t = 2t + 1$ and

$$\begin{aligned}\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} &= \sqrt{(-\sin t + \cos t + 4)} \\ &= 22\end{aligned}$$

so we have

$$\begin{aligned}\int_C (x^2 + y^2 + z) ds &= 2\sqrt{2} \int_0^{\pi} (1 + 2t) dt \\ &= 2\sqrt{2} \Big[t + t^2 \Big]_0^{\pi} \\ &= 2\sqrt{2}(\pi + \pi^2).\end{aligned}$$

Notice that this agrees with the answer in the previous example. Changing the parameterization did not change the value of the line integral. Scalar line integrals are independent of parameterization, as long as the curve is traversed exactly once by the parameterization.

Exercise 15.2.3

Evaluate line integral $\int_C (x^2 + yz)\ ds$, where C is the line with parameterization $\vecs{r}(t) = \langle 2t, 5t, -t \rangle$, $0 \leq t \leq 10$. Reparameterize C with parameterization $s(t) = \langle 4t, 10t, -2t \rangle$, $0 \leq t \leq 5$, recalculate line integral $\int_C (x^2 + yz)\ ds$, and notice that the change of parameterization had no effect on the value of the integral.

Hint available at the end of this section.

Now that we can evaluate line integrals, we can use them to calculate arc length. If $f(x, y, z) = 1$, then

$$\begin{aligned}\int_C f(x, y, z)\, ds &= \lim_{n \to \infty} \sum_{i=1}^{n} f(t_i^*)\, \Delta s_i \\ &= \lim_{n \to \infty} \sum_{i=1}^{n} \Delta s_i \\ &= \lim_{n \to \infty} \text{length}(C) \\ &= \text{length}(C).\end{aligned}$$

Therefore, $\int_C 1\ ds$ is the arc length of C.

Example 15.2.4: Calculating Arc Length

A wire has a shape that can be modeled with the parameterization $\vecs{r}(t) = \langle \cos t, \sin t, \frac{2}{3} t^{3/2} \rangle$, $0 \leq t \leq 4\pi$. Find the length of the wire.

Solution

The length of the wire is given by $\int_C 1\ ds$, where C is the curve with parameterization $\vecs r$. Therefore,

$$\begin{aligned}\text{The length of the wire} &= \int_C 1\, ds \\ &= \int_0^{4\pi} ||\vecs r'(t)||\, dt \\ &= \int_0^{4\pi} \sqrt{(-\sin t)^2 + \cos^2 t + t}dt \\ &= \int_0^{4\pi} \sqrt{1+t}dt \\ &= \frac{2(1+t)^{\frac{3}{2}}}{3}\Bigg|_0^{4\pi} \\ &= \frac{2}{3}\left((1+4\pi)^{3/2} - 1\right).\end{aligned}$$

Exercise 15.2.4

Find the length of a wire with parameterization $\vecs r(t) = \langle 3t+1, 4-2t, 5+2t\rangle$, $0 \le t \le 4$.

Hint available at the end of this section.

Answers

Exercise 15.2.1

Find the shape formed by C and the graph of function $f(x,y) = x + y$.

$\sqrt{2}$

Exercise 15.2.2

Use the two-variable version of scalar line integral definition (Equation 15.2.2).

$$\frac{1}{3} + \frac{\sqrt{2}}{6} + \frac{3\pi}{4}$$

Exercise 15.2.3

Use Equation 15.2.1.

Both line integrals equal $-\frac{1000\sqrt{30}}{3}$.

Exercise 15.2.4

Find the line integral of one over the corresponding curve.

$4\sqrt{17}$

15.2: Line Integrals - Part 2

15.2E: Line Integrals (Exercises)

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15.2: Line Integrals - Part 2

In this part of the section, we are going to develop a second type of line integral called a vector line integral. Although it's result is also a scalar, it is defined a little differently than the scalar line integral defined earlier in this section. The previous definition is repeated below for easy reference.

DEFINITION: Scalar Line Integral

Let f be a function with a domain that includes the smooth curve C that is parameterized by $\vecs{r}(t) = \langle x(t), y(t), z(t) \rangle$, $a \leq t \leq b$. The *scalar line integral* of f along C is

$$\int_C f(x, y, z)\, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(P_i^*)\, \Delta s_i \tag{15.2.1}$$

if this limit exists t_i^* and Δs_i are defined as in the previous paragraphs). If C is a planar curve, then C can be represented by the parametric equations $x = x(t)$, $y = y(t)$, and $a \leq t \leq b$. If C is smooth and $f(x, y)$ is a function of two variables, then the scalar line integral of f along C is defined similarly as

$$\int_C f(x, y)\, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(P_i^*)\, \Delta s_i, \tag{15.2.2}$$

if this limit exists.

Vector Line Integrals

The second type of line integrals are vector line integrals, in which we integrate along a curve through a vector field. For example, let

$$\vecs{F}(x, y, z) = P(x, y, z)\, \hat{\mathbf{i}} + Q(x, y, z)\, \hat{\mathbf{j}} + R(x, y, z)\, \hat{\mathbf{k}}$$

be a continuous vector field in $\mathbb{R}^3$ that represents a force on a particle, and let C be a smooth curve in $\mathbb{R}^3$ contained in the domain of $\vecs{F}$. How would we compute the work done by $\vecs{F}$ in moving a particle along C?

To answer this question, first note that a particle could travel in two directions along a curve: a forward direction and a backward direction. The work done by the vector field depends on the direction in which the particle is moving. Therefore, we must specify a direction along curve C; such a specified direction is called an orientation of a curve. The specified direction is the positive direction along C; the opposite direction is the negative direction along C. When C has been given an orientation, C is called an oriented curve (Figure $15.2.5$). The work done on the particle depends on the direction along the curve in which the particle is moving.

A closed curve is one for which there exists a parameterization $\vecs{r}(t)$, $a \leq t \leq b$, such that $\vecs{r}(a) = \vecs{r}(b)$, and the curve is traversed exactly once. In other words, the parameterization is one-to-one on the domain (a, b).

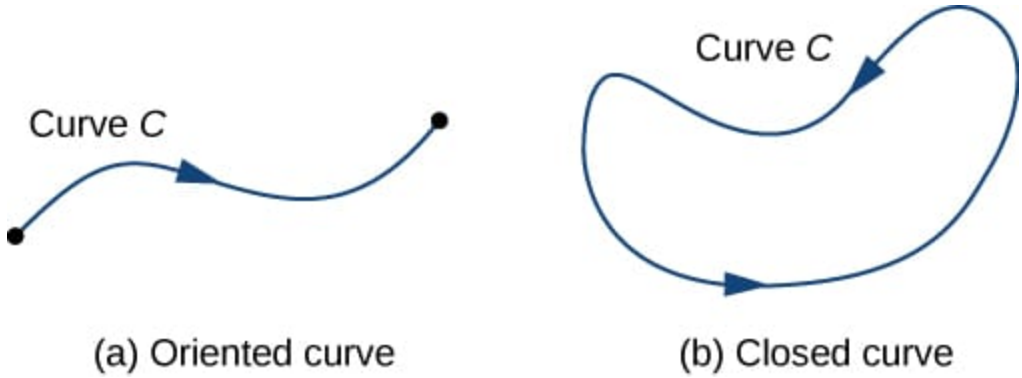


Figure 15.2.5: (a) An oriented curve between two points. (b) A closed oriented curve.

Let $\vecs r(t)$ be a parameterization of C for $a \le t \le b$ such that the curve is traversed exactly once by the particle and the particle moves in the positive direction along C. Divide the parameter interval $[a, b]$ into n subintervals $[t_{i-1}, t_i]$, $0 \le i \le n$, of equal width. Denote the endpoints of $r(t_0)$, $r(t_1)$,..., $r(t_n)$ by P_0,...,P_n. Points P_i divide C into n pieces. Denote the length of the piece from P_{i-1} to P_i by Δs_i. For each i, choose a value t_i^* in the subinterval $[t_{i-1}, t_i]$. Then, the endpoint of $\vecs r(t_i^*)$ is a point in the piece of C between P_{i-1} and P_i (Figure 15.2.6). If Δs_i is small, then as the particle moves from P_{i-1} to P_i along C, it moves approximately in the direction of $\vecs T(P_i)$, the unit tangent vector at the endpoint of $\vecs r(t_i^*)$. Let P_i^* denote the endpoint of $\vecs r(t_i^*)$. Then, the work done by the force vector field in moving the particle from P_{i-1} to P_i is $\vecs F(P_i^*) \cdot (\Delta s_i \vecs T(P_i^*))$, so the total work done along C is

$$\sum_{i=1}^{n} \vecs F(P_i^*) \cdot (\Delta s_i \vecs T(P_i^*)) = \sum_{i=1}^{n} \vecs F(P_i^*) \cdot \vecs T(P_i^*)\, \Delta s_i.$$

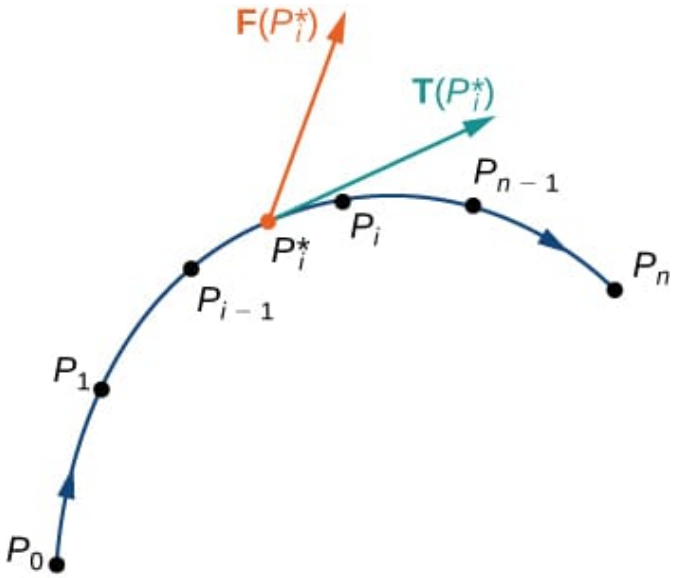


Figure 15.2.6: Curve C is divided into n pieces, and a point inside each piece is chosen. The dot product of any tangent vector in the ith piece with the corresponding vector $\vecs F$ is approximated by $\vecs F(P_i^*) \cdot \vecs T(P_i^*)$.

Letting the arc length of the pieces of C get arbitrarily small by taking a limit as $n \to \infty$ gives us the work done by the field in moving the particle along C. Therefore, the work done by $\vecs F$ in moving the particle in the positive direction along C is defined as

$$W = \int_C \vecs F \cdot \vecs T\, ds,$$

which gives us the concept of a vector line integral.

DEFINITION: Line Integral of a Vector Field

The vector line integral of vector field $\vecs F$ along oriented smooth curve C is

$$\int_C \vecs F \cdot \vecs T\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} \vecs F(P_i^*) \cdot \vecs T(P_i^*) \Delta s_i$$

if that limit exists.

With scalar line integrals, neither the orientation nor the parameterization of the curve matters. As long as the curve is traversed exactly once by the parameterization, the value of the line integral is unchanged. With vector line integrals, the orientation of the curve does matter. If we think of the line integral as computing work, then this makes sense: if you hike up a mountain, then the gravitational force of Earth does negative work on you. If you walk down the mountain by the exact same path, then Earth's gravitational force does positive work on you. In other words, reversing the path changes the work value from negative to positive in this case. Note that if C is an oriented curve, then we let $-C$ represent the same curve but with opposite orientation.

As with scalar line integrals, it is easier to compute a vector line integral if we express it in terms of the parameterization function $\vecs r$ and the variable t. To translate the integral $\int_C \vecs F \cdot \vecs T ds$ in terms of t, note that unit tangent vector $\vecs T$ along C is given by $\vecs T = \dfrac{\vecs r'(t)}{\|\vecs r'(t)\|}$ (assuming $\|\vecs r\,'(t)\| \neq 0$). Since $ds = \|\vecs r\,'(t)\|\, dt$, as we saw when discussing scalar line integrals, we have

$$\vecs F \cdot \vecs T\, ds = \vecs F(\vecs r(t)) \cdot \frac{\vecs r'(t)}{\|\vecs r'(t)\|} \|\vecs r'(t)\| dt = \vecs F(\vecs r(t)) \cdot \vecs r'(t)\, dt.$$

Thus, we have the following formula for computing vector line integrals:

$$\int_C \vecs F \cdot \vecs T\, ds = \int_a^b \vecs F(\vecs r(t)) \cdot \vecs r'(t)\, dt. \tag{15.2.3}$$

Because of Equation 15.2.3, we often use the notation $\int_C \vecs F \cdot d\vecs r$ for the line integral $\int_C \vecs F \cdot \vecs T\ ds$.

If $\vecs r(t) = \langle x(t), y(t), z(t)\rangle$, then $\dfrac{d\vecs r}{dt}$ denotes vector $\langle x\,'(t), y\,'(t), z\,'(t)\rangle$, and $d\vecs r = \vecs r'(t)\ dt$.

Example 15.2.5: Evaluating a Vector Line Integral

Find the value of integral $\int_C \vecs F \cdot d\vecs r$, where C is the semicircle parameterized by $\vecs r(t) = \langle \cos t, \sin t\rangle$, $0 \le t \le \pi$ and $\vecs F = \langle -y, x\rangle$.

Solution

We can use Equation 15.2.3 to convert the variable of integration from s to t. We then have

$$\vecs F(\vecs r(t)) = \langle -\sin t, \cos t\rangle \text{ and } \vecs r'(t) = \langle -\sin t, \cos t\rangle.$$

Therefore,

$$\begin{aligned}\int_C \vecs F \cdot d\vecs r &= \int_0^\pi \langle -\sin t, \cos t\rangle \cdot \langle -\sin t, \cos t\rangle\, dt \\ &= \int_0^\pi \sin^2 t + \cos^2 t\, dt \\ &= \int_0^\pi 1\, dt = \pi.\end{aligned}$$

See Figure 15.2.7.

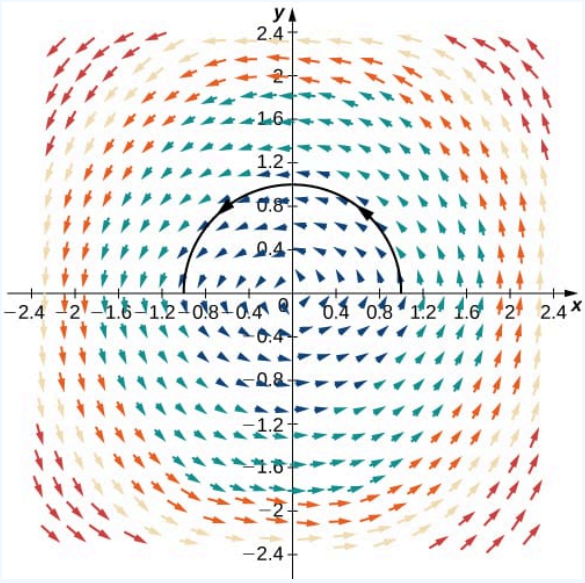


Figure 15.2.7: This figure shows curve $\vecs r(t) = \langle \cos t,\ \sin t\rangle$, $0 \le t \le \pi$ in vector field $\vecs F = \langle -y,\ x\rangle$.

Example 15.2.6: Reversing Orientation

Find the value of integral $\displaystyle\int_C \vecs F \cdot d\vecs r$, where C is the semicircle parameterized by $\vecs r(t) = \langle \cos(t+\pi), \sin t\rangle$, $0 \le t \le \pi$ and $\vecs F = \langle -y, x\rangle$.

Solution

Notice that this is the same problem as Example 15.2.5, except the orientation of the curve has been traversed. In this example, the parameterization starts at $\vecs r(0) = \langle -1, 0\rangle$ and ends at $\vecs r(\pi) = \langle 1, 0\rangle$. By Equation 15.2.3,

$$\begin{aligned}\int_C \vecs F \cdot d\vecs r &= \int_0^\pi \langle -\sin t, \cos(t+\pi)\rangle \cdot \langle -\sin(t+\pi), \cos t\rangle dt \\ &= \int_0^\pi \langle -\sin t, -\cos t\rangle \cdot \langle \sin t, \cos t\rangle dt \\ &= \int_0^\pi (-\sin^2 t - \cos^2 t) dt \\ &= \int_0^\pi -1 dt \\ &= -\pi.\end{aligned}$$

Notice that this is the negative of the answer in Example 15.2.5. It makes sense that this answer is negative because the orientation of the curve goes against the "flow" of the vector field.

Let C be an oriented curve and let $-C$ denote the same curve but with the orientation reversed. Then, the previous two examples illustrate the following fact:

$$\int_{-C} \vecs F \cdot d\vecs r = -\int_C \vecs F \cdot d\vecs r.$$

That is, reversing the orientation of a curve changes the sign of a line integral.

Exercise 15.2.6

Let $\vecs{F} = x\ \hat{\mathbf{i}} + y\ \hat{\mathbf{j}}$ be a vector field and let C be the curve with parameterization $\langle t, t^2 \rangle$ for $0 \le t \le 2$. Which is greater: $\int_C \vecs{F} \cdot \vecs{T}\ ds$ or $\int_{-C} \vecs{F} \cdot \vecs{T}\ ds$?

Hint available at the end of this section.

Another standard notation for integral $\int_C \vecs{F} \cdot d\vecs{r}$ is $\int_C P\ dx + Q\ dy + R\ dz$. In this notation, P, Q, and R are functions, and we think of $d\vecs{r}$ as vector $\langle dx, dy, dz \rangle$. To justify this convention, recall that $d\vecs{r} = \vecs{T}\ ds = \vecs{r}\,'(t)\ dt = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle\ dt$. Therefore,

$$\vecs{F} \cdot d\vecs{r} = \langle P, Q, R \rangle \cdot \langle dx, dy, dz \rangle = P\,dx + Q\,dy + R\,dz.$$

If $d\vecs{r} = \langle dx, dy, dz \rangle$, then $\frac{dr}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$, which implies that $d\vecs{r} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle\ dt$. Therefore

$$\int_C \vecs{F} \cdot d\vecs{r} = \int_C P\,dx + Q\,dy + R\,dz \tag{15.2.4}$$

$$= \int_a^b \left(P\big(\vecs{r}(t)\big)\frac{dx}{dt} + Q\big(\vecs{r}(t)\big)\frac{dy}{dt} + R\big(\vecs{r}(t)\big)\frac{dz}{dt} \right) dt. \tag{15.2.5}$$

Example 15.2.7: Finding the Value of an Integral of the Form $\int_C P\ dx + Q\ dy + R\ dz$

Find the value of integral $\int_C z\ dx + x\ dy + y\ dz$, where C is the curve parameterized by $\vecs{r}(t) = \langle t^2, \sqrt{t}, t \rangle$, $1 \le t \le 4$.

Solution

As with our previous examples, to compute this line integral we should perform a change of variables to write everything in terms of t. In this case, Equation 15.2.5 allows us to make this change:

$$\begin{aligned}
\int_C z\,dx + x\,dy + y\,dz &= \int_1^4 \left(t(2t) + t^2\left(\frac{1}{2\sqrt{t}}\right) + \sqrt{t} \right) dt \\
&= \int_1^4 \left(2t^2 + \frac{t^{3/2}}{2} + \sqrt{t} \right) dt \\
&= \left[\frac{2t^3}{3} + \frac{t^{5/2}}{5} + \frac{2t^{3/2}}{3} \right]_{t=1}^{t=4} \\
&= \frac{793}{15}.
\end{aligned}$$

Exercise 15.2.7

Find the value of $\int_C 4x\ dx + z\ dy + 4y^2\ dz$, where C is the curve parameterized by $\vecs{r}(t) = \langle 4\cos(2t), 2\sin(2t), 3 \rangle$, $0 \le t \le \frac{\pi}{4}$.

Hint available at the end of this section.

We have learned how to integrate smooth oriented curves. Now, suppose that C is an oriented curve that is not smooth, but can be written as the union of finitely many smooth curves. In this case, we say that C is a piecewise smooth curve. To be precise, curve C is piecewise smooth if C can be written as a union of n smooth curves C_1, C_2,..., C_n such that the endpoint of C_i is the

starting point of C_{i+1} (Figure 15.2.8). When curves C_i satisfy the condition that the endpoint of C_i is the starting point of C_{i+1}, we write their union as $C_1 + C_2 + \cdots + C_n$.

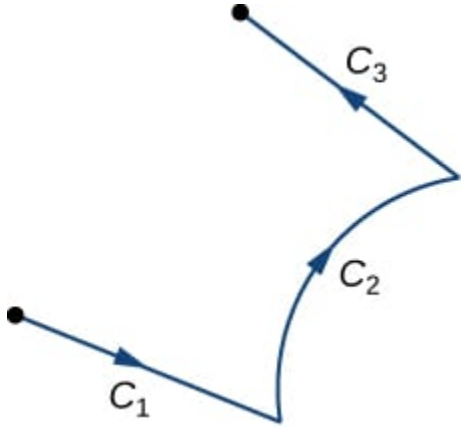


Figure 15.2.8: The union of C_1, C_2, C_3 is a piecewise smooth curve.

The next theorem summarizes several key properties of vector line integrals.

Theorem: Properties of Vector Line Integrals

Let $\vecs F$ and $\vecs G$ be continuous vector fields with domains that include the oriented smooth curve C. Then

1. $\displaystyle\int_C (\vecs F + \vecs G) \cdot d\vecs r = \int_C \vecs F \cdot d\vecs r + \int_C \vecs G \cdot d\vecs r$
2. $\displaystyle\int_C k\vecs F \cdot d\vecs r = k\int_C \vecs F \cdot d\vecs r$, where k is a constant
3. $\displaystyle\int_{-C} \vecs F \cdot d\vecs r = -\int_C \vecs F \cdot d\vecs r$
4. Suppose instead that C is a piecewise smooth curve in the domains of $\vecs F$ and $\vecs G$, where $C = C_1 + C_2 + \cdots + C_n$ and $C_1, C_2, \ldots, C_n$ are smooth curves such that the endpoint of C_i is the starting point of C_{i+1}. Then

$$\int_C \vecs F \cdot d\vecs r = \int_{C_1} \vecs F \cdot d\vecs r + \int_{C_2} \vecs F \cdot d\vecs r + \cdots + \int_{C_n} \vecs F \cdot d\vecs r.$$

Notice the similarities between these items and the properties of single-variable integrals. Properties i. and ii. say that line integrals are linear, which is true of single-variable integrals as well. Property iii. says that reversing the orientation of a curve changes the sign of the integral. If we think of the integral as computing the work done on a particle traveling along C, then this makes sense. If the particle moves backward rather than forward, then the value of the work done has the opposite sign. This is analogous to the equation $\displaystyle\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$. Finally, if $[a_1, a_2]$, $[a_2, a_3]$,..., $[a_{n-1}, a_n]$ are intervals, then

$$\int_{a_1}^{a_n} f(x)\,dx = \int_{a_1}^{a_2} f(x)\,dx + \int_{a_1}^{a_3} f(x)\,dx + \cdots + \int_{a_{n-1}}^{a_n} f(x)\,dx,$$

which is analogous to property iv.

Example 15.2.8: Using Properties to Compute a Vector Line Integral

Find the value of integral $\displaystyle\int_C \vecs F \cdot \vecs T\ ds$, where C is the rectangle (oriented counterclockwise) in a plane with vertices $(0,0)$, $(2,0)$, $(2,1)$, and $(0,1)$, and where $\vecs F = \langle x - 2y, y - x\rangle$ (Figure 15.2.9).

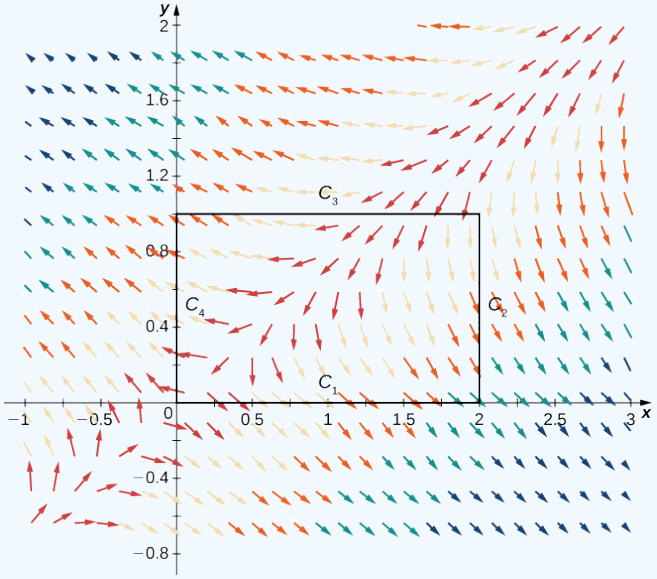


Figure 15.2.9: Rectangle and vector field for Example 15.2.8.

Solution

Note that curve C is the union of its four sides, and each side is smooth. Therefore C is piecewise smooth. Let C_1 represent the side from $(0,0)$ to $(2,0)$, let C_2 represent the side from $(2,0)$ to $(2,1)$, let C_3 represent the side from $(2,1)$ to $(0,1)$, and let C_4 represent the side from $(0,1)$ to $(0,0)$ (Figure 15.2.9). Then,

$$\int_C \vecs{F} \cdot \vecs{T}\, dr = \int_{C_1} \vecs{F} \cdot \vecs{T}\, dr + \int_{C_2} \vecs{F} \cdot \vecs{T}\, dr + \int_{C_3} \vecs{F} \cdot \vecs{T}\, dr + \int_{C_4} \vecs{F} \cdot \vecs{T}\, dr.$$

We want to compute each of the four integrals on the right-hand side using Equation 15.2.1. Before doing this, we need a parameterization of each side of the rectangle. Here are four parameterizations (note that they traverse C counterclockwise):

$$\begin{aligned} C_1 &: \langle t, 0\rangle, 0 \le t \le 2 \\ C_2 &: \langle 2, t\rangle, 0 \le t \le 1 \\ C_3 &: \langle 2-t, 1\rangle, 0 \le t \le 2 \\ C_4 &: \langle 0, 1-t\rangle, 0 \le t \le 1. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{C_1} \vecs{F} \cdot \vecs{T}\, dr &= \int_0^2 \vecs{F}(\vecs{r}(t)) \cdot \vecs{r}'(t)\, dt \\ &= \int_0^2 \langle t - 2(0), 0 - t\rangle \cdot \langle 1, 0\rangle\, dt = \int_0^2 t\, dt \\ &= \left[\tfrac{t^2}{2}\right]_0^2 = 2. \end{aligned}$$

Notice that the value of this integral is positive, which should not be surprising. As we move along curve C_1 from left to right, our movement flows in the general direction of the vector field itself. At any point along C_1, the tangent vector to the curve and the corresponding vector in the field form an angle that is less than 90°. Therefore, the tangent vector and the force vector have a positive dot product all along C_1, and the line integral will have positive value.

The calculations for the three other line integrals are done similarly:

$$\int_{C_2} \vecs{F} \cdot d\vecs{r} = \int_0^1 \langle 2-2t, t-2\rangle \cdot \langle 0,1\rangle \, dt$$
$$= \int_0^1 (t-2)\, dt$$
$$= \left[\tfrac{t^2}{2} - 2t\right]_0^1 = -\frac{3}{2},$$

$$\int_{C_3} \vecs{F} \cdot \vecs{T}\, ds = \int_0^2 \langle (2-t)-2, 1-(2-t)\rangle \cdot \langle -1,0\rangle \, dt$$
$$= \int_0^2 t\, dt = 2,$$

and

$$\int_{C_4} \vecs{F} \cdot d\vecs{r} = \int_0^1 \langle -2(1-t), 1-t\rangle \cdot \langle 0,-1\rangle \, dt$$
$$= \int_0^1 (t-1)\, dt$$
$$= \left[\tfrac{t^2}{2} - t\right]_0^1 = -\frac{1}{2}.$$

Thus, we have $\int_C \vecs{F} \cdot d\vecs{r} = 2$.

Exercise 15.2.8

Calculate line integral $\int_C \vecs{F} \cdot d\vecs{r}$, where $\vecs{F}$ is vector field $\langle y^2, 2xy+1\rangle$ and C is a triangle with vertices $(0,0)$, $(4,0)$, and $(0,5)$, oriented counterclockwise.

Hint available at the end of this section.

Applications of Line Integrals

Scalar line integrals have many applications. They can be used to calculate the length or mass of a wire, the surface area of a sheet of a given height, or the electric potential of a charged wire given a linear charge density. Vector line integrals are extremely useful in physics. They can be used to calculate the work done on a particle as it moves through a force field, or the flow rate of a fluid across a curve. Here, we calculate the mass of a wire using a scalar line integral and the work done by a force using a vector line integral.

Suppose that a piece of wire is modeled by curve C in space. The mass per unit length (the linear density) of the wire is a continuous function $\rho(x,y,z)$. We can calculate the total mass of the wire using the scalar line integral $\int_C \rho(x,y,z)\ ds$. The reason is that mass is density multiplied by length, and therefore the density of a small piece of the wire can be approximated by $\rho(x^*,y^*,z^*)\ \Delta s$ for some point (x^*,y^*,z^*) in the piece. Letting the length of the pieces shrink to zero with a limit yields the line integral $\int_C \rho(x,y,z)\ ds$.

Example 15.2.9: Calculating the Mass of a Wire

Calculate the mass of a spring in the shape of a curve parameterized by $\langle t, 2\cos t, 2\sin t\rangle$, $0 \le t \le \frac{\pi}{2}$, with a density function given by $\rho(x,y,z) = e^x + yz$ kg/m (Figure 15.2.10).

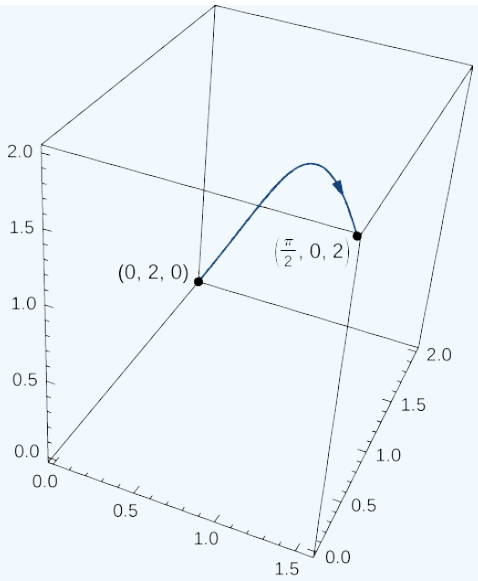


Figure 15.2.10: The wire from Example 15.2.9.

Solution

To calculate the mass of the spring, we must find the value of the scalar line integral $\int_C (e^x + yz)\ ds$, where C is the given helix. To calculate this integral, we write it in terms of t using Equation 15.2.1:

$$\begin{aligned}\int_C (e^x + yz)\ ds &= \int_0^{\frac{\pi}{2}} \left((e^t + 4\cos t \sin t)\sqrt{1 + (-2\cos t)^2 + (2\sin t)^2}\right)\ dt \\ &= \int_0^{\frac{\pi}{2}} \left((e^t + 4\cos t \sin t)\sqrt{5}\right)\ dt \\ &= \sqrt{5}\Big[e^t + 2\sin^2 t\Big]_{t=0}^{t=\pi/2} \\ &= \sqrt{5}(e^{\pi/2} + 1).\end{aligned}$$

Therefore, the mass is $\sqrt{5}(e^{\pi/2} + 1)$ kg.

Exercise 15.2.9

Calculate the mass of a spring in the shape of a helix parameterized by $\vecs{r}(t) = \langle \cos t, \sin t, t\rangle$, $0 \le t \le 6\pi$, with a density function given by $\rho(x, y, z) = x + y + z$ kg/m.

Hint available at the end of this section.

When we first defined vector line integrals, we used the concept of work to motivate the definition. Therefore, it is not surprising that calculating the **work done by a vector field** representing a force is a standard use of vector line integrals. Recall that if an object moves along curve C in force field $\vecs{F}$, then the work required to move the object is given by $\int_C \vecs{F} \cdot d\vecs{r}$.

Example 15.2.10: Calculating Work

How much work is required to move an object in vector force field $\vecs{F} = \langle yz, xy, xz\rangle$ along path $\vecs{r}(t) = \langle t^2, t, t^4\rangle$, $0 \le t \le 1$? See Figure 15.2.11.

Solution

Let C denote the given path. We need to find the value of $\int_C \vecs{F} \cdot d\vecs{r}$. To do this, we use Equation 15.2.3 :

$$\begin{aligned}\int_C \vecs{F} \cdot d\vecs{r} &= \int_0^1 (\langle t^5, t^3, t^6\rangle \cdot \langle 2t, 1, 4t^3\rangle)\, dt \\ &= \int_0^1 (2t^6 + t^3 + 4t^9)\, dt \\ &= \left[\frac{2t^7}{7} + \frac{t^4}{4} + \frac{2t^{10}}{5}\right]_{t=0}^{t=1} = \frac{131}{140} \text{ units of work.}\end{aligned}$$

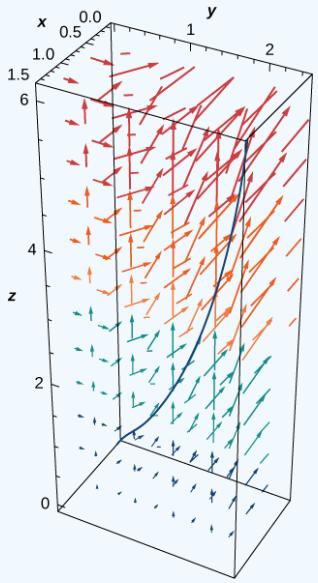


Figure 15.2.11: The curve and vector field for Example 15.2.10.

Flux

We close this section by discussing two key concepts related to line integrals: flux across a plane curve and circulation along a plane curve. Flux is used in applications to calculate fluid flow across a curve, and the concept of circulation is important for characterizing conservative gradient fields in terms of line integrals. Both these concepts are used heavily throughout the rest of this chapter. The idea of flux is especially important for Green's theorem, and in higher dimensions for Stokes' theorem and the divergence theorem.

Let C be a plane curve and let $\vecs{F}$ be a vector field in the plane. Imagine C is a membrane across which fluid flows, but C does not impede the flow of the fluid. In other words, C is an idealized membrane invisible to the fluid. Suppose $\vecs{F}$ represents the velocity field of the fluid. How could we quantify the rate at which the fluid is crossing C?

Recall that the line integral of $\vecs{F}$ along C is $\int_C \vecs{F} \cdot \vecs{T}\, ds$—in other words, the line integral is the dot product of the vector field with the unit tangential vector with respect to arc length. If we replace the unit tangential vector with unit normal vector $\vecs{N}(t)$ and instead compute integral $\int_C \vecs{F} \cdot \vecs{N}\, ds$, we determine the flux across C. To be precise, the definition of integral $\int_C \vecs{F} \cdot \vecs{N}\, ds$ is the same as integral $\int_C \vecs{F} \cdot \vecs{T}\, ds$, except the $\vecs{T}$ in the Riemann sum is replaced with $\vecs{N}$. Therefore, the flux across C is defined as

$$\int_C \vecs{F} \cdot \vecs{N}\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} \vecs{F}(P_i^*) \cdot \vecs{N}(P_i^*)\, \Delta s_i,$$

where P_i^* and Δs_i are defined as they were for integral $\int_C \vecs{F} \cdot \vecs{T}\, ds$. Therefore, a flux integral is an integral that is *perpendicular* to a vector line integral, because $\vecs{N}$ and $\vecs{T}$ are perpendicular vectors.

If $\vecs F$ is a velocity field of a fluid and C is a curve that represents a membrane, then the flux of $\vecs F$ across C is the quantity of fluid flowing across C per unit time, or the rate of flow.

More formally, let C be a plane curve parameterized by $\vecs r(t) = \langle x(t),\ y(t)\rangle$, $a \le t \le b$. Let $\vecs n(t) = \langle y\,'(t),\ -x\,'(t)\rangle$ be the vector that is normal to C at the endpoint of $\vecs r(t)$ and points to the right as we traverse C in the positive direction (Figure 15.2.12). Then, $\vecs N(t) = \dfrac{\vecs n(t)}{\|\vecs n(t)\|}$ is the unit normal vector to C at the endpoint of $\vecs r(t)$ that points to the right as we traverse C.

DEFINITION: Flux

The *flux* of $\vecs F$ across C is line integral

$$\int_C \vecs F \cdot \frac{\vecs n(t)}{\|\vecs n(t)\|}\, ds.$$

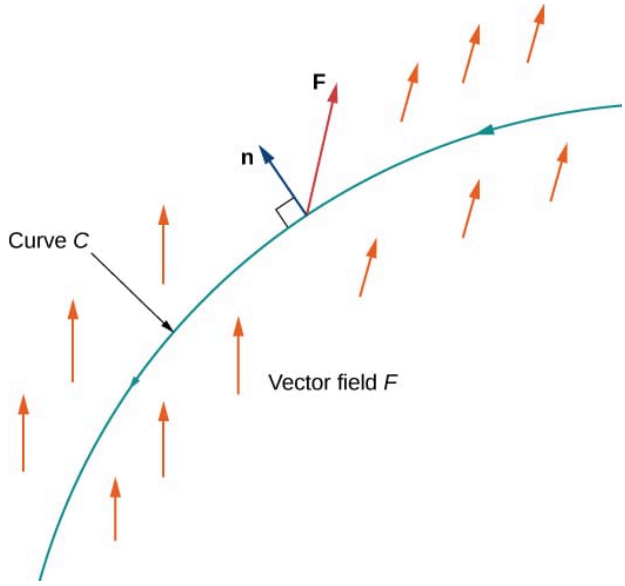


Figure 15.2.12: The flux of vector field $\vecs F$ *across curve* C *is computed by an integral similar to a vector line integral.*

We now give a formula for calculating the flux across a curve. This formula is analogous to the formula used to calculate a vector line integral (see Equation 15.2.3).

Theorem: Calculating Flux Across a Curve

Let $\vecs F$ be a vector field and let C be a smooth curve with parameterization $r(t) = \langle x(t), y(t)\rangle$, $a \le t \le b$.Let $\vecs n(t) = \langle y\,'(t), -x\,'(t)\rangle$. The flux of $\vecs F$ across C is

$$\int_C \vecs F \cdot \vecs N\, ds = \int_a^b \vecs F(\vecs r(t)) \cdot \vecs n(t)\, dt. \tag{15.2.6}$$

Proof

Before deriving the formula, note that

$$\|\vecs n(t)\| = \|\langle y'(t), -x'(t)\rangle\| = \sqrt{(y'(t))^2 + (x'(t))^2} = \|\vecs r'(t)\|.$$

Therefore,

$$\int_C \vecs{F} \cdot \vecs{N}\, ds = \int_C \vecs{F} \cdot \frac{\vecs{n}(t)}{\|\vecs{n}(t)\|}\, ds$$
$$= \int_a^b \vecs{F} \cdot \frac{\vecs{n}(t)}{\|\vecs{n}(t)\|} \|\vecs{r}'(t)\|\, dt$$
$$= \int_a^b \vecs{F}(\vecs{r}(t)) \cdot \vecs{n}(t)\, dt.$$

□

Example 15.2.11: Flux across a Curve

Calculate the flux of $\vecs{F} = \langle 2x, 2y \rangle$ across a unit circle oriented counterclockwise (Figure 15.2.13).

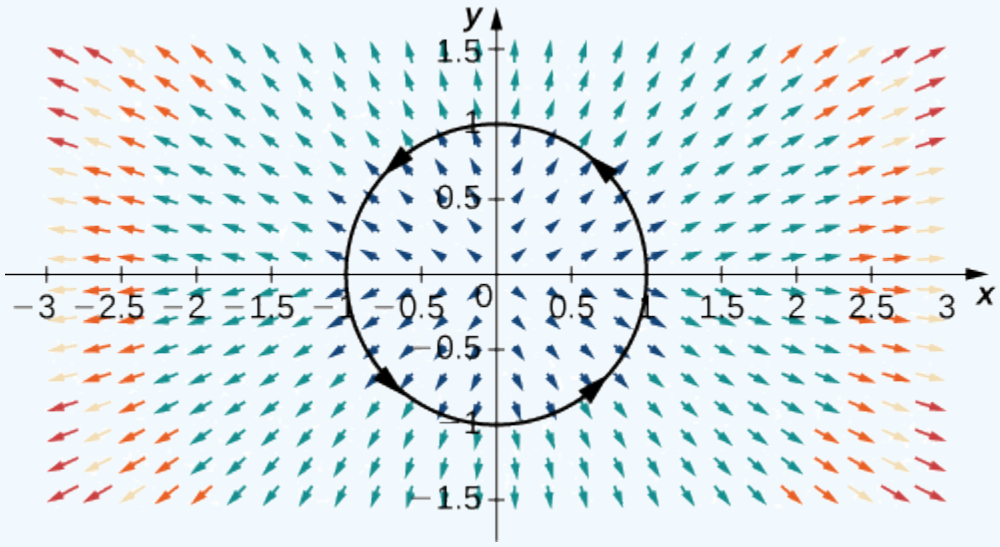


Figure 15.2.13: A unit circle in vector field $\vecs{F} = \langle 2x,\ 2y \rangle$.

Solution

To compute the flux, we first need a parameterization of the unit circle. We can use the standard parameterization $\vecs{r}(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le 2\pi$. The normal vector to a unit circle is $\langle \cos t, \sin t \rangle$. Therefore, the flux is

$$\int_C \vecs{F} \cdot \vecs{N}\, ds = \int_0^{2\pi} \langle 2\cos t, 2\sin t \rangle \cdot \langle \cos t, \sin t \rangle\, dt$$
$$= \int_0^{2\pi} (2\cos^2 t + 2\sin^2 t)\, dt$$
$$= 2\int_0^{2\pi} (\cos^2 t + \sin^2 t)\, dt$$
$$= 2\int_0^{2\pi} dt = 4\pi.$$

Exercise 15.2.11

Calculate the flux of $\vecs{F} = \langle x + y, 2y \rangle$ across the line segment from $(0, 0)$ to $(2, 3)$, where the curve is oriented from left to right.

Hint available at the end of this section.

Let $\vecs F(x,y) = \langle P(x,y), Q(x,y)\rangle$ be a two-dimensional vector field. Recall that integral $\int_C \vecs F \cdot \vecs T \; ds$ is sometimes written as $\int_C P \; dx + Q \; dy$. Analogously, flux $\int_C \vecs F \cdot \vecs N \; ds$ is sometimes written in the notation $\int_C -Q \; dx + P \; dy$, because the unit normal vector $\vecs N$ is perpendicular to the unit tangent $\vecs T$. Rotating the vector $d\vecs r = \langle dx, dy\rangle$ by 90° results in vector $\langle dy, -dx\rangle$. Therefore, the line integral in Example 15.2.8 can be written as $\int_C -2y \; dx + 2x \; dy$.

Circulation

Now that we have defined flux, we can turn our attention to circulation. The line integral of vector field $\vecs F$ along an oriented closed curve is called the circulation of $\vecs F$ along C. Circulation line integrals have their own notation: $\oint_C \vecs F \cdot \vecs T \; ds$. The circle on the integral symbol denotes that C is "circular" in that it has no endpoints. Example 15.2.5 shows a calculation of circulation.

To see where the term *circulation* comes from and what it measures, let $\vecs v$ represent the velocity field of a fluid and let C be an oriented closed curve. At a particular point P, the closer the direction of $\vecs v(P)$ is to the direction of $\vecs T(P)$, the larger the value of the dot product $\vecs v(P) \cdot \vecs T(P)$. The maximum value of $\vecs v(P) \cdot \vecs T(P)$ occurs when the two vectors are pointing in the exact same direction; the minimum value of $\vecs v(P) \cdot \vecs T(P)$ occurs when the two vectors are pointing in opposite directions. Thus, the value of the circulation $\oint_C \vecs v \cdot \vecs T \; ds$ measures the tendency of the fluid to move in the direction of C.

Example 15.2.12: Calculating Circulation

Let $\vecs F = \langle -y, \; x\rangle$ be the vector field from Example 15.2.3 and let C represent the unit circle oriented counterclockwise. Calculate the circulation of $\vecs F$ along C.

Solution

We use the standard parameterization of the unit circle: $\vecs r(t) = \langle \cos t, \sin t\rangle$, $0 \le t \le 2\pi$. Then, $\vecs F(\vecs r(t)) = \langle -\sin t, \cos t\rangle$ and $\vecs r\,'(t) = \langle -\sin t, \cos t\rangle$. Therefore, the circulation of $\vecs F$ along C is

$$\begin{aligned} \oint_C \vecs F \cdot \vecs T \, ds &= \int_0^{2\pi} \langle -\sin t, \cos t\rangle \cdot \langle -\sin t, \cos t\rangle \, dt \\ &= \int_0^{2\pi} (\sin^2 t + \cos^2 t) \, dt \\ &= \int_0^{2\pi} dt = 2\pi \text{ units of work.} \end{aligned}$$

Notice that the circulation is positive. The reason for this is that the orientation of C "flows" with the direction of $\vecs F$. At any point along the circle, the tangent vector and the vector from $\vecs F$ form an angle of less than 90°, and therefore the corresponding dot product is positive.

In Example 15.2.12, what if we had oriented the unit circle clockwise? We denote the unit circle oriented clockwise by $-C$. Then

$$\oint_{-C} \vecs F \cdot \vecs T \, ds = -\oint_C \vecs F \cdot \vecs T \, ds = -2\pi \text{ units of work.}$$

Notice that the circulation is negative in this case. The reason for this is that the orientation of the curve flows against the direction of $\vecs F$.

Exercise 15.2.12

Calculate the circulation of $\vecs F(x,y) = \langle -\dfrac{y}{x^2+y^2}, \; \dfrac{x}{x^2+y^2}\rangle$ along a unit circle oriented counterclockwise.

Hint available at the end of this section.

Example 15.2.13: Calculating Work

Calculate the work done on a particle that traverses circle C of radius 2 centered at the origin, oriented counterclockwise, by field $\vecs{F}(x,y) = \langle -2,\ y\rangle$. Assume the particle starts its movement at $(1,\ 0)$.

Solution

The work done by $\vecs{F}$ on the particle is the circulation of $\vecs{F}$ along C: $\oint_C \vecs{F} \cdot \vecs{T}\ ds$. We use the parameterization $\vecs{r}(t) = \langle 2\cos t,\ 2\sin t\rangle$, $0 \le t \le 2\pi$ for C. Then, $\vecs{r}\,'(t) = \langle -2\sin t,\ 2\cos t\rangle$ and $\vecs{F}(\vecs{r}(t)) = \langle -2,\ 2\sin t\rangle$. Therefore, the circulation of $\vecs{F}$ along C is

$$\begin{aligned}\oint_C \vecs{F}\cdot\vecs{T}\,ds &= \int_0^{2\pi} \langle -2, 2\sin t\rangle \cdot \langle -2\sin t, 2\cos t\rangle\, dt \\ &= \int_0^{2\pi} (4\sin t + 4\sin t\cos t)\, dt \\ &= \Big[-4\cos t + 4\sin^2 t\Big]_0^{2\pi} \\ &= \big(-4\cos(2\pi) + 2\sin^2(2\pi)\big) - \big(-4\cos(0) + 4\sin^2(0)\big) \\ &= -4 + 4 = 0 \text{ units of work.}\end{aligned}$$

The force field does zero work on the particle.

Notice that the circulation of $\vecs{F}$ along C is zero. Furthermore, notice that since $\vecs{F}$ is the gradient of $f(x,y) = -2x + \dfrac{y^2}{2}$, $\vecs{F}$ is conservative. We prove in a later section that under certain broad conditions, the circulation of a conservative vector field along a closed curve is zero.

Exercise 15.2.14

Calculate the work done by field $\vecs{F}(x,y) = \langle 2x,\ 3y\rangle$ on a particle that traverses the unit circle. Assume the particle begins its movement at $(-1,\ 0)$.

Hint available at the end of this section.

Key Concepts

- Line integrals generalize the notion of a single-variable integral to higher dimensions. The domain of integration in a single-variable integral is a line segment along the x-axis, but the domain of integration in a line integral is a curve in a plane or in space.
- If C is a curve, then the length of C is $\displaystyle\int_C ds$.
- There are two kinds of line integral: scalar line integrals and vector line integrals. Scalar line integrals can be used to calculate the mass of a wire; vector line integrals can be used to calculate the work done on a particle traveling through a field.
- Scalar line integrals can be calculated using Equation 15.2.1; vector line integrals can be calculated using Equation 15.2.3.
- Two key concepts expressed in terms of line integrals are flux and circulation. Flux measures the rate that a field crosses a given line; circulation measures the tendency of a field to move in the same direction as a given closed curve.

Key Equations

- **Calculating a scalar line integral**

$$\int_C f(x,y,z)\ ds = \int_a^b f(\vecs{r}(t))\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\ dt$$

- **Calculating a vector line integral**

$$\int_C \vecs{F}\cdot d\vecs{r} = \int_C \vecs{F}\cdot\vecs{T}\ ds = \int_a^b \vecs{F}(\vecs{r}(t))\cdot\vecs{r}\,'(t)\ dt$$

or

$$\int_C P\ dx + Q\ dy + R\ dz = \int_a^b \left(P\big(\vecs{r}(t)\big)\frac{dx}{dt} + Q\big(\vecs{r}(t)\big)\frac{dy}{dt} + R\big(\vecs{r}(t)\big)\frac{dz}{dt}\right)\ dt$$

- **Calculating flux**

$$\int_C \vecs{F}\cdot\frac{\vecs{n}(t)}{\|\vecs{n}(t)\|}\ ds = \int_a^b \vecs{F}(\vecs{r}(t))\cdot\vecs{n}(t)\ dt$$

Glossary

circulation
: the tendency of a fluid to move in the direction of curve C. If C is a closed curve, then the circulation of $\vecs{F}$ along C is line integral $\int_C \vecs{F}\cdot\vecs{T}\ ds$, which we also denote $\oint_C \vecs{F}\cdot\vecs{T}\ ds$.

closed curve
: a curve for which there exists a parameterization $\vecs{r}(t), a\le t\le b$, such that $\vecs{r}(a)=\vecs{r}(b)$, and the curve is traversed exactly once

flux
: the rate of a fluid flowing across a curve in a vector field; the flux of vector field $\vecs{F}$ across plane curve C is line integral $\int_C \vecs{F}\cdot\frac{\vecs{n}(t)}{\|\vecs{n}(t)\|}\ ds$

line integral
: the integral of a function along a curve in a plane or in space

orientation of a curve
: the orientation of a curve C is a specified direction of C

piecewise smooth curve
: an oriented curve that is not smooth, but can be written as the union of finitely many smooth curves

scalar line integral
: the scalar line integral of a function f along a curve C with respect to arc length is the integral $\int_C f\ ds$, it is the integral of a scalar function f along a curve in a plane or in space; such an integral is defined in terms of a Riemann sum, as is a single-variable integral

vector line integral
: the vector line integral of vector field $\vecs{F}$ along curve C is the integral of the dot product of $\vecs{F}$ with unit tangent vector $\vecs{T}$ of C with respect to arc length, $\int_C \vecs{F}\cdot\vecs{T}\ ds$; such an integral is defined in terms of a Riemann sum, similar to a single-variable integral

16.2: Line Integrals

Answers

Exercise 15.2.6

Imagine moving along the path and computing the dot product $\vecs{F}\cdot\vecs{T}$ as you go.

$$\int_C \vecs{F}\cdot\vecs{T}\,ds$$

Exercise 15.2.7

Write the integral in terms of t using Equation 15.2.5.

-26

Exercise 15.2.8

Write the triangle as a union of its three sides, then calculate three separate line integrals.

0

Exercise 15.2.9

Calculate the line integral of ρ over the curve with parameterization $\vecs r$.

$18\sqrt{2}\pi^2$ kg

Exercise 15.2.11

Use Equation 15.2.6.

$3/2$

Exercise 15.2.12

Use Equation 15.2.6.

2π units of work

Exercise 15.2.14

Use Equation 15.2.6.

0 units of work

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15.2E: Line Integrals (Exercises)

1. *True or False?* Line integral $\int_C f(x,y)\ ds$ is equal to a definite integral if C is a smooth curve defined on $[a,b]$ and if function f is continuous on some region that contains curve C.

Answer

True

2. *True or False?* Vector functions $\vecs r_1 = t\ \hat{\mathbf{i}} + t^2\ \hat{\mathbf{j}}, \quad 0 \le t \le 1$, and $\vecs r_2 = (1-t)\ \hat{\mathbf{i}} + (1-t)^2\ \hat{\mathbf{j}}, \quad 0 \le t \le 1$, define the same oriented curve.

3. *True or False?* $\int_{-C}(P\ dx + Q\ dy) = \int_C (P\ dx - Q\ dy)$

Answer

False

4. *True or False?* A piecewise smooth curve C consists of a finite number of smooth curves that are joined together end to end.

5. *True or False?* If C is given by $x(t) = t, \quad y(t) = t, \quad 0 \le t \le 1$, then $\int_C xy\ ds = \int_0^1 t^2\ dt$.

Answer

False

For the following exercises, use a computer algebra system (CAS) to evaluate the line integrals over the indicated path.

6. [T] $\int_C (x+y)\ ds$

$C: x = t, y = (1-t), z = 0$ from $(0,1,0)$ to $(1,0,0)$

7. [T] $\int_C (x-y)ds$

$C: \vecs r(t) = 4t\ \hat{\mathbf{i}} + 3t\ \hat{\mathbf{j}}$ when $0 \le t \le 2$

Answer

$$\int_C (x-y)\ ds = 10$$

8. [T] $\int_C (x^2+y^2+z^2)\ ds$

$C: \vecs r(t) = \sin t\ \hat{\mathbf{i}} + \cos t\ \hat{\mathbf{j}} + 8t\ \hat{\mathbf{k}}$ when $0 \le t \le \dfrac{\pi}{2}$

9. [T] Evaluate $\int_C xy^4\ ds$, where C is the right half of circle $x^2+y^2=16$ and is traversed in the clockwise direction.

Answer

$$\int_C xy^4\ ds = \frac{8192}{5}$$

10. [T] Evaluate $\int_C 4x^3 ds$, where C is the line segment from $(-2,-1)$ to $(1,2)$.

For the following exercises, find the work done.

11. Find the work done by vector field $\vecs F(x,y,z) = x\ \hat{\mathbf{i}} + 3xy\ \hat{\mathbf{j}} - (x+z)\ \hat{\mathbf{k}}$ on a particle moving along a line segment that goes from $(1,4,2)$ to $(0,5,1)$.

Answer

$W = 8$ units of work

12. Find the work done by a person weighing 150 lb walking exactly one revolution up a circular, spiral staircase of radius 3 ft if the person rises 10 ft.

13. Find the work done by force field $\vecs F(x,y,z) = -\frac{1}{2}x\ \hat{\mathbf{i}} - \frac{1}{2}y\ \hat{\mathbf{j}} + \frac{1}{4}\ \hat{\mathbf{k}}$ on a particle as it moves along the helix $\vecs r(t) = \cos t\ \hat{\mathbf{i}} + \sin t\ \hat{\mathbf{j}} + t\ \hat{\mathbf{k}}$ from point $(1,0,0)$ to point $(-1,0,3\pi)$.

Answer

$W = \frac{3\pi}{4}$ units of work

14. Find the work done by vector field $\vecs F(x,y) = y\ \hat{\mathbf{i}} + 2x\ \hat{\mathbf{j}}$ in moving an object along path C, which joins points $(1,0)$ and $(0,1)$.

15. Find the work done by force $\vecs F(x,y) = 2y\ \hat{\mathbf{i}} + 3x\ \hat{\mathbf{j}} + (x+y)\ \hat{\mathbf{k}}$ in moving an object along curve $\vecs r(t) = \cos(t)\ \hat{\mathbf{i}} + \sin(t)\ \hat{\mathbf{j}} + 16\ \hat{\mathbf{k}}$, where $0 \le t \le 2\pi$.

Answer

$W = \pi$ units of work

16. Find the mass of a wire in the shape of a circle of radius 2 centered at (3, 4) with linear mass density $\rho(x,y) = y^2$.

For the following exercises, evaluate the line integrals.

17. Evaluate $\int_C \vecs F \cdot d\vecs r$, where $\vecs F(x,y) = -1\ \hat{\mathbf{j}}$, and C is the part of the graph of $y = \frac{1}{2}x^3 - x$ from $(2,2)$ to $(-2,-2)$.

Answer

$\int_C \vecs F \cdot d\vecs r = 4$ units of work

18. Evaluate $\int_\gamma (x^2+y^2+z^2)^{-1} ds$, where γ is the helix $x = \cos t, y = \sin t, z = t$, with $0 \le t \le T$.

19. Evaluate $\int_C yz\ dx + xz\ dy + xy\ dz$ over the line segment from $(1,1,1)$ to $(3,2,0)$.

Answer

$\int_C yz\ dx + xz\ dy + xy\ dz = -1$

20. Let C be the line segment from point (0, 1, 1) to point (2, 2, 3). Evaluate line integral $\int_C y\ ds$.

21. [T] Use a computer algebra system to evaluate the line integral $\int_C y^2\ dx + x\ dy$, where C is the arc of the parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$.

Answer

$\int_C (y^2)\ dx + (x)\ dy = \dfrac{245}{6}$

22. [T] Use a computer algebra system to evaluate the line integral $\int_C (x+3y^2)\ dy$ over the path C given by $x = 2t, y = 10t$, where $0 \le t \le 1$.

23. [T] Use a CAS to evaluate line integral $\int_C xy\ dx + y\ dy$ over path C given by $x = 2t, y = 10t$, where $0 \le t \le 1$.

Answer

$$\int_C xy\ dx + y\ dy = \frac{190}{3}$$

24. Evaluate line integral $\int_C (2x - y)\ dx + (x + 3y)\ dy$, where C lies along the x-axis from $x = 0$ to $x = 5$.

26. [T] Use a CAS to evaluate $\int_C \frac{y}{2x^2 - y^2}\ ds$, where C is defined by the parametric equations $x = t, y = t$, for $1 \le t \le 5$.

Answer

$$\int_C \frac{y}{2x^2 - y^2}\ ds = \sqrt{2} \ln 5$$

27. [T] Use a CAS to evaluate $\int_C xy\ ds$, where C is defined by the parametric equations $x = t^2, y = 4t$, for $0 \le t \le 1$.

In the following exercises, find the work done by force field $\vecs F$ on an object moving along the indicated path.

28. $\vecs F(x, y) = -x\ \hat{\mathbf{i}} - 2y\ \hat{\mathbf{j}}$

$C : y = x^3$ from $(0, 0)$ to $(2, 8)$

Answer

$W = -66$ units of work

29. $\vecs F(x, y) = 2x\ \hat{\mathbf{i}} + y\ \hat{\mathbf{j}}$

<C: counterclockwise around the triangle with vertices $(0, 0), (1, 0)$, and $(1, 1)$

30. $\vecs F(x, y, z) = x\ \hat{\mathbf{i}} + y\ \hat{\mathbf{j}} - 5z\ \hat{\mathbf{k}}$

$C : \vecs r(t) = 2 \cos t\ \hat{\mathbf{i}} + 2 \sin t\ \hat{\mathbf{j}} + t\ \hat{\mathbf{k}}, \quad 0 \le t \le 2\pi$

Answer

$W = -10\pi^2$ units of work

31. Let $\vecs F$ be vector field $\vecs F(x, y) = (y^2 + 2xe^y + 1)\ \hat{\mathbf{i}} + (2xy + x^2e^y + 2y)\ \hat{\mathbf{j}}$. Compute the work of integral $\int_C \vecs F \cdot d\vecs r$, where C is the path $\vecs r(t) = \sin t\ \hat{\mathbf{i}} + \cos t\ \hat{\mathbf{j}}, \quad 0 \le t \le \frac{\pi}{2}$.

32. Compute the work done by force $\vecs F(x, y, z) = 2x\ \hat{\mathbf{i}} + 3y\ \hat{\mathbf{j}} - z\ \hat{\mathbf{k}}$ along path $\vecs r(t) = t\ \hat{\mathbf{i}} + t^2\ \hat{\mathbf{j}} + t^3\ \hat{\mathbf{k}}$, where $0 \le t \le 1$.

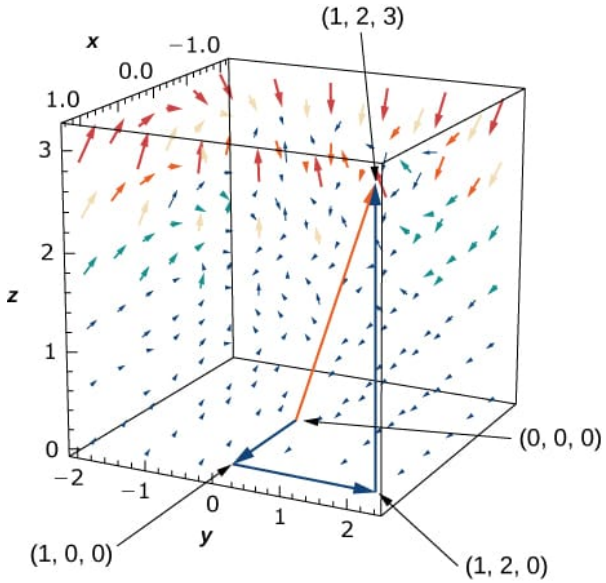
Answer

$W = 2$ units of work

33. Evaluate $\int_C \vecs F \cdot d\vecs r$, where $\vecs F(x, y) = \frac{1}{x + y}\ \hat{\mathbf{i}} + \frac{1}{x + y}\ \hat{\mathbf{j}}$ and C is the segment of the unit circle going counterclockwise from $(1, 0)$ to $(0, 1)$.

34. Force $\vecs F(x, y, z) = zy\ \hat{\mathbf{i}} + x\ \hat{\mathbf{j}} + z^2x\ \hat{\mathbf{k}}$ acts on a particle that travels from the origin to point $(1, 2, 3)$. Calculate the work done if the particle travels:

a. along the path $(0, 0, 0) \to (1, 0, 0) \to (1, 2, 0) \to (1, 2, 3)$ along straight-line segments joining each pair of endpoints;
b. along the straight line joining the initial and final points.
c. Is the work the same along the two paths?



Answer

a. $W = 11$ units of work;
b. $W = \dfrac{39}{4} = 9\frac{3}{4}$ units of work;
c. No

35. Find the work done by vector field $\vecs F(x, y, z) = x\ \hat{\mathbf{i}} + 3xy\ \hat{\mathbf{j}} - (x + z)\ \hat{\mathbf{k}}$ on a particle moving along a line segment that goes from $(1, 4, 2)$ to $(0, 5, 1)$.

36. How much work is required to move an object in vector field $\vecs F(x, y) = y\ \hat{\mathbf{i}} + 3x\ \hat{\mathbf{j}}$ along the upper part of ellipse $\dfrac{x^2}{4} + y^2 = 1$ from $(2, 0)$ to $(-2, 0)$?

Answer

$W = 2\pi$ units of work

37. A vector field is given by $\vecs F(x, y) = (2x + 3y)\ \hat{\mathbf{i}} + (3x + 2y)\ \hat{\mathbf{j}}$. Evaluate the line integral of the field around a circle of unit radius traversed in a clockwise fashion.

38. Evaluate the line integral of scalar function xy along parabolic path $y = x^2$ connecting the origin to point $(1, 1)$.

Answer

$$\int_C f\ ds = \frac{25\sqrt{5} + 1}{120}$$

39. Find $\displaystyle\int_C y^2\ dx + (xy - x^2)\ dy$ along $C : y = 3x$ from $(0, 0)$ to $(1, 3)$.

40. Find $\displaystyle\int_C y^2\ dx + (xy - x^2)\ dy$ along $C : y^2 = 9x$ from $(0, 0)$ to $(1, 3)$.

Answer

$$\int_C y^2\ dx + (xy - x^2)\ dy = 6.15$$

For the following exercises, use a CAS to evaluate the given line integrals.

41. [T] Evaluate $\vecs F(x, y, z) = x^2 z\ \hat{\mathbf{i}} + 6y\ \hat{\mathbf{j}} + yz^2\ \hat{\mathbf{k}}$, where C is represented by $\vecs r(t) = t\ \hat{\mathbf{i}} + t^2\ \hat{\mathbf{j}} + \ln t\ \hat{\mathbf{k}}, 1 \le t \le 3$.

42. [T] Evaluate line integral $\displaystyle\int_\gamma xe^y\ ds$ where, γ is the arc of curve $x = e^y$ from $(1, 0)$ to $(e, 1)$.

Answer

$$\int_\gamma xe^y \; ds \approx 7.157$$

43. [T] Evaluate the integral $\int_\gamma xy^2 \; ds$, where γ is a triangle with vertices $(0,1,2), (1,0,3)$, and $(0,-1,0)$.

44. [T] Evaluate line integral $\int_\gamma (y^2 - xy) \; dx$, where γ is curve $y = \ln x$ from $(1,0)$ toward $(e,1)$.

Answer

$$\int_\gamma (y^2 - xy) \; dx \approx -1.379$$

45. [T] Evaluate line integral $\int_\gamma xy^4 \; ds$, where γ is the right half of circle $x^2 + y^2 = 16$.

46. [T] Evaluate $\int C\vecsF \cdot d\vecsr$, $\int C\vecsF \cdot d\vecsr$, where $\vecsF(x,y,z) = x^2y \; \hat{\mathbf{i}} + (x - z) \; \hat{\mathbf{j}} + xyz \; \hat{\mathbf{k}}$ and

$C : \vecsr(t) = t \; \hat{\mathbf{i}} + t^2 \; \hat{\mathbf{j}} + 2 \; \hat{\mathbf{k}}, 0 \le t \le 1.$

Answer

$$\int_C \vecsF \cdot d\vecsr \approx -1.133 \text{ units of work}$$

47. Evaluate $\int_C \vecsF \cdot d\vecsr$, where $\vecsF(x,y) = 2x \sin y \; \hat{\mathbf{i}} + (x^2 \cos y - 3y^2) \; \hat{\mathbf{j}}$ and

C is any path from $(-1,0)$ to $(5,1)$.

48. Find the line integral of $\vecsF(x,y,z) = 12x^2 \; \hat{\mathbf{i}} - 5xy \; \hat{\mathbf{j}} + xz \; \hat{\mathbf{k}}$ over path C defined by $y = x^2, z = x^3$ from point $(0,0,0)$ to point $(2,4,8)$.

Answer

$$\int_C \vecsF \cdot d\vecsr \approx 22.857 \text{ units of work}$$

49. Find the line integral of $\int_C (1 + x^2y) \; ds$, where C is ellipse $\vecsr(t) = 2\cos t \; \hat{\mathbf{i}} + 3 \sin t \; \hat{\mathbf{j}}$ from $0 \le t \le \pi$.

For the following exercises, find the flux.

50. Compute the flux of $\vecsF = x^2 \; \hat{\mathbf{i}} + y \; \hat{\mathbf{j}}$ across a line segment from $(0,0)$ to $(1,2)$.

Answer

$$\text{flux} = -\tfrac{1}{3}$$

51. Let $\vecsF = 5 \; \hat{\mathbf{i}}$ and let C be curve $y = 0$, with $0 \le x \le 4$. Find the flux across C.

52. Let $\vecsF = 5 \; \hat{\mathbf{j}}$ and let C be curve $y = 0$, with $0 \le x \le 4$. Find the flux across C.

Answer

$$\text{flux} = -20$$

53. Let $\vecsF = -y \; \hat{\mathbf{i}} + x \; \hat{\mathbf{j}}$ and let $C : \vecsr(t) = \cos t \; \hat{\mathbf{i}} + \sin t \; \hat{\mathbf{j}}$ for $0 \le t \le 2\pi$. Calculate the flux across C.

54. Let $\vecsF = (x^2 + y^3) \; \hat{\mathbf{i}} + (2xy) \; \hat{\mathbf{j}}$. Calculate flux $\vecsF$ orientated counterclockwise across the curve $C : x^2 + y^2 = 9$.

Answer

$$\text{flux} = 0$$

Complete the rest of the exercises as stated.

55. Find the line integral of $\int_C z^2\ dx + y\ dy + 2y\ dz$, where C consists of two parts: C_1 and C_2. C_1 is the intersection of cylinder $x^2 + y^2 = 16$ and plane $z = 3$ from $(0, 4, 3)$ to $(-4, 0, 3)$. C_2 is a line segment from $(-4, 0, 3)$ to $(0, 1, 5)$.

56. A spring is made of a thin wire twisted into the shape of a circular helix $x = 2\cos t,\ \ y = 2\sin t,\ \ z = t$. Find the mass of two turns of the spring if the wire has a constant mass density of ρ grams per cm.

Answer

$m = 4\pi\rho\sqrt{5}$ grams

57. A thin wire is bent into the shape of a semicircle of radius a. If the linear mass density at point P is directly proportional to its distance from the line through the endpoints, find the mass of the wire.

58. An object moves in force field $\vecs{F}(x, y, z) = y^2\ \hat{\mathbf{i}} + 2(x + 1)y\ \hat{\mathbf{j}}$ counterclockwise from point $(2, 0)$ along elliptical path $x^2 + 4y^2 = 4$ to $(-2, 0)$, and back to point $(2, 0)$ along the x-axis. How much work is done by the force field on the object?

Answer

$W = 0$ units of work

59. Find the work done when an object moves in force field $\vecs{F}(x, y, z) = 2x\ \hat{\mathbf{i}} - (x + z)\ \hat{\mathbf{j}} + (y - x)\ \hat{\mathbf{k}}$ along the path given by $\vecs{r}(t) = t^2\ \hat{\mathbf{i}} + (t^2 - t)\ \hat{\mathbf{j}} + 3\ \hat{\mathbf{k}},\ \ 0 \le t \le 1$.

60. If an inverse force field $\vecs{F}$ is given by $\vecs{F}(x, y, z) = \dfrac{k}{\|r\|^3}r$, where k is a constant, find the work done by $\vecs{F}$ as its point of application moves along the x-axis from $A(1, 0, 0)$ to $B(2, 0, 0)$.

Answer

$W = \frac{k}{2}$ units of work

61. David and Sandra plan to evaluate line integral $\int_C \vecs{F} \cdot d\vecs{r}$ along a path in the xy-plane from $(0, 0)$ to $(1, 1)$. The force field is $\vecs{F}(x, y) = (x + 2y)\ \hat{\mathbf{i}} + (-x + y^2)\ \hat{\mathbf{j}}$. David chooses the path that runs along the x-axis from $(0, 0)$ to $(1, 0)$ and then runs along the vertical line $x = 1$ from $(1, 0)$ to the final point $(1, 1)$. Sandra chooses the direct path along the diagonal line $y = x$ from $(0, 0)$ to $(1, 1)$. Whose line integral is larger and by how much?

Contributors

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15.3: CONSERVATIVE VECTOR FIELDS

Learning Objectives

- Describe simple and closed curves; define connected and simply connected regions.
- Explain how to find a potential function for a conservative vector field.
- Use the Fundamental Theorem for Line Integrals to evaluate a line integral in a vector field.
- Explain how to test a vector field to determine whether it is conservative.

In this section, we continue the study of conservative vector fields. We examine the Fundamental Theorem for Line Integrals, which is a useful generalization of the Fundamental Theorem of Calculus to line integrals of conservative vector fields. We also discover show how to test whether a given vector field is conservative, and determine how to build a potential function for a vector field known to be conservative.

Curves and Regions

Before continuing our study of conservative vector fields, we need some geometric definitions. The theorems in the subsequent sections all rely on integrating over certain kinds of curves and regions, so we develop the definitions of those curves and regions here. We first define two special kinds of curves: closed curves and simple curves. As we have learned, a closed curve is one that begins and ends at the same point. A simple curve is one that does not cross itself. A curve that is both closed and simple is a simple closed curve (Figure $15.3.1$).

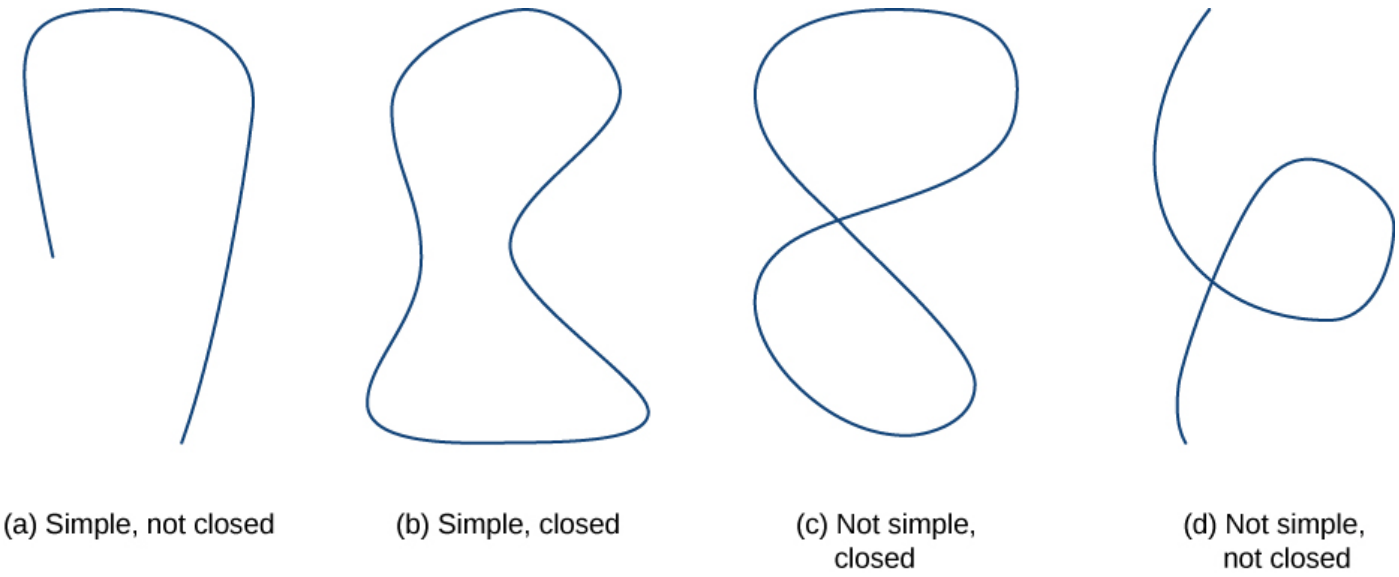


Figure $15.3.1$. Types of curves that are simple or not simple and closed or not closed.

DEFINITION: Closed Curves

Curve C is a *closed curve* if there is a parameterization $\vecs{r}(t)$, $a \leq t \leq b$ of C such that the parameterization traverses the curve exactly once and $\vecs{r}(a) = \vecs{r}(b)$. Curve C is a simple curve if C does not cross itself. That is, C is simple if there exists a parameterization $\vecs{r}(t)$, $a \leq t \leq b$ of C such that $\vecs{r}$ is one-to-one over (a, b). It is possible for $\vecs{r}(a) = \vecs{r}(b)$, meaning that the simple curve is also closed.

Example 15.3.1: Determining Whether a Curve Is Simple and Closed

Is the curve with parameterization $\vecs r(t) = \left\langle \cos t, \frac{\sin(2t)}{2} \right\rangle$, $0 \le t \le 2\pi$ a simple closed curve?

Solution

Note that $\vecs r(0) = \langle 1, 0 \rangle = \vecs r(2\pi)$; therefore, the curve is closed. The curve is not simple, however. To see this, note that $\vecs r\left(\frac{\pi}{2}\right) = \langle 0, 0 \rangle = \vecs r\left(\frac{3\pi}{2}\right)$, and therefore the curve crosses itself at the origin (Figure 15.3.2).

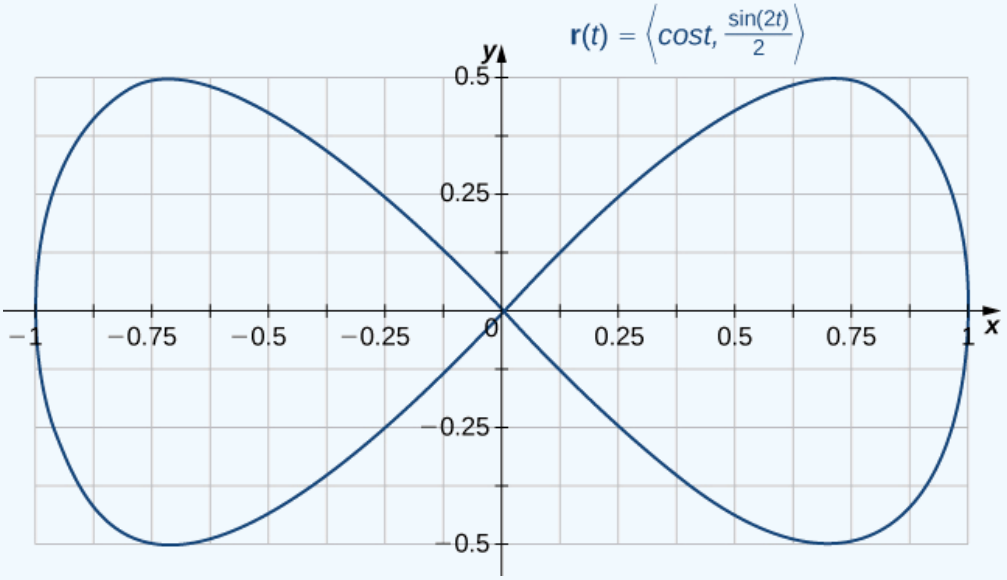


Figure 15.3.2. *A curve that is closed but not simple.*

Exercise 15.3.1

Is the curve given by parameterization $\vecs r(t) = \langle 2\cos t, 3\sin t \rangle$, $0 \le t \le 6\pi$, a simple closed curve?

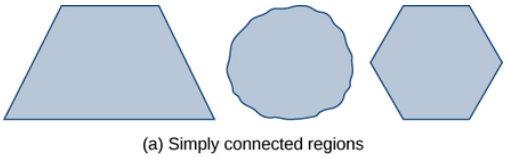
Hint available at the end of this section.

Many of the theorems in this chapter relate an integral over a region to an integral over the boundary of the region, where the region's boundary is a simple closed curve or a union of simple closed curves. To develop these theorems, we need two geometric definitions for regions: that of a connected region and that of a simply connected region. A connected region is one in which there is a path in the region that connects any two points that lie within that region. A simply connected region is a connected region that does not have any holes in it. These two notions, along with the notion of a simple closed curve, allow us to state several generalizations of the Fundamental Theorem of Calculus later in the chapter. These two definitions are valid for regions in any number of dimensions, but we are only concerned with regions in two or three dimensions.

DEFINITION: connected regions

A region D is a *connected region* if, for any two points P_1 and P_2, there is a path from P_1 to P_2 with a trace contained entirely inside D. A region D is a simply connected region if D is connected for any simple closed curve C that lies inside D, and curve C can be shrunk continuously to a point while staying entirely inside D. In two dimensions, a region is simply connected if it is connected and has no holes.

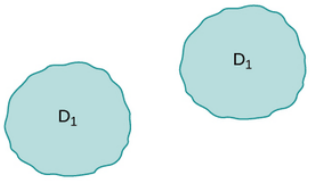
All simply connected regions are connected, but not all connected regions are simply connected (Figure 15.3.3).



(a) Simply connected regions



(b) Connected regions that are not simply connected

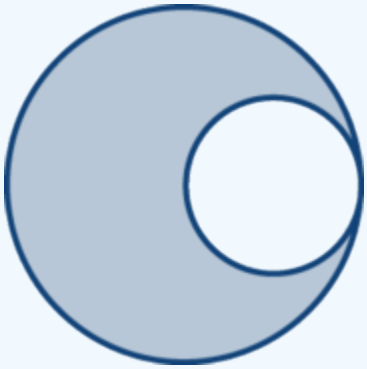


(c) A region that is not connected

Figure 15.3.3: Not all connected regions are simply connected. (a) Simply connected regions have no holes. (b) Connected regions that are not simply connected may have holes but you can still find a path in the region between any two points. (c) A region that is not connected has some points that cannot be connected by a path in the region.

Exercise 15.3.2

Is the region in the below image connected? Is the region simply connected?



Hint available at the end of this section.

Fundamental Theorem for Line Integrals

Now that we understand some basic curves and regions, let's generalize the Fundamental Theorem of Calculus to line integrals. Recall that the Fundamental Theorem of Calculus says that if a function f has an antiderivative F, then the integral of f from a to b depends only on the values of F at a and at b—that is,

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

If we think of the gradient as a derivative, then the same theorem holds for vector line integrals. We show how this works using a motivational example.

Example 15.3.2: Evaluating a Line Integral and the Antiderivatives of the Endpoints

Let $\vecs F(x,y) = \langle 2x, 4y\rangle$. Calculate $\int_C \vecs F \cdot d\vecs r$, where C is the line segment from $(0,0)$ to $(2,2)$ (Figure 15.3.4).

Solution

We use the method from the previous section to calculate $\int_C \vecs F \cdot d\vecs r$. Curve C can be parameterized by $\vecs r(t) = \langle 2t, 2t\rangle$, $0 \le t \le 1$. Then, $\vecs F(\vecs r(t)) = \langle 4t, 8t\rangle$ and $\vecs r\,'(t) = \langle 2, 2\rangle$, which implies that

$$\begin{aligned}\int_C \vecs F \cdot d\vecs r &= \int_0^1 \langle 4t, 8t\rangle \cdot \langle 2, 2\rangle dt \\ &= \int_0^1 (8t + 16t)dt = \int_0^1 24t dt \\ &= \left[12t^2\right]_0^1 = 12.\end{aligned}$$

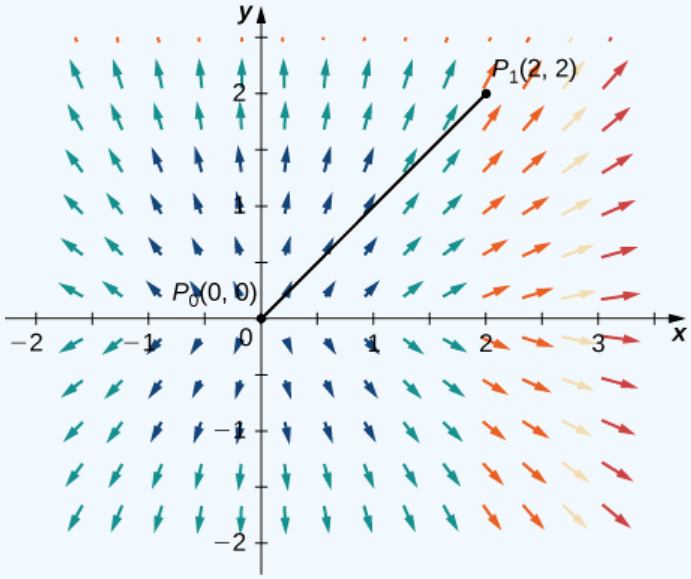


Figure 15.3.4: The value of line integral $\int_C \vecs F \cdot d\vecs r$ *depends only on the value of the potential function of* $\vecs F$ *at the endpoints of the curve.*

Notice that $\vecs F = \vecs \nabla f$, where $f(x,y) = x^2 + 2y^2$. If we think of the gradient as a derivative, then f is an "antiderivative" of $\vecs F$. In the case of single-variable integrals, the integral of derivative $g\,'(x)$ is $g(b) - g(a)$, where *a* is the start point of the interval of integration and *b* is the endpoint. If vector line integrals work like single-variable integrals, then we would expect integral $\vecs F$ to be $f(P_1) - f(P_0)$, where P_1 is the endpoint of the curve of integration and P_0 is the start point. Notice that this is the case for this example:

$$\int_C \vecs F \cdot d\vecs r = \int_C \vecs \nabla f \cdot d\vecs r = 12$$

and

$$f(2,2) - f(0,0) = 4 + 8 - 0 = 12.$$

In other words, the integral of a "derivative" can be calculated by evaluating an "antiderivative" at the endpoints of the curve and subtracting, just as for single-variable integrals.

The following theorem says that, under certain conditions, what happened in the previous example holds for any gradient field. The same theorem holds for vector line integrals, which we call the **Fundamental Theorem for Line Integrals**.

Theorem: THE FUNDAMENTAL THEOREM FOR LINE INTEGRALS

Let C be a piecewise smooth curve with parameterization $\vecs r(t)$, $a \le t \le b$. Let f be a function of two or three variables with first-order partial derivatives that exist and are continuous on C. Then,

$$\int_C \vecs\nabla f \cdot d\vecs r = f(\vecs r(b)) - f(\vecs r(a)). \tag{15.3.1}$$

Proof

First,

$$\int_C \vecs\nabla f \cdot d\vecs r = \int_a^b \vecs\nabla f(\vecs r(t)) \cdot \vecs r'(t)\, dt.$$

By the chain rule,

$$\frac{d}{dt}(f(\vecs r(t)) = \vecs\nabla f(\vecs r(t)) \cdot \vecs r'(t)$$

Therefore, by the Fundamental Theorem of Calculus,

$$\begin{aligned}\int_C \vecs\nabla f \cdot d\vecs r &= \int_a^b \vecs\nabla f(\vecs r(t)) \cdot \vecs r'(t) dt \\ &= \int_a^b \frac{d}{dt}(f(\vecs r(t)) dt \\ &= \left[f(\vecs r(t))\right]_{t=a}^{t=b} \\ &= f(\vecs r(b)) - f(\vecs r(a)).\end{aligned}$$

□

We know that if $\vecs F$ is a conservative vector field, there is a potential function f such that $\vecs\nabla f = \vecs F$. Therefore

$$\int_C \vecs F \cdot d\vecs r = \int_C \vecs\nabla f \cdot d\vecs r = f(\vecs r(b)) - f(\vecs r(a)).$$

In other words, just as with the Fundamental Theorem of Calculus, computing the line integral $\int_C \vecs F \cdot d\vecs r$, where $\vecs F$ is conservative, is a two-step process:

1. Find a potential function ("antiderivative") f for $\vecs F$ and
2. Compute the value of f at the endpoints of C and calculate their difference $f(\vecs r(b)) - f(\vecs r(a))$.

Keep in mind, however, there is one major difference between the Fundamental Theorem of Calculus and the Fundamental Theorem for Line Integrals:
A function of one variable that is continuous must have an antiderivative. However, a vector field, even if it is continuous, does not need to have a potential function.

Example 15.3.3: Applying the Fundamental Theorem

Calculate integral $\int_C \vecs F \cdot d\vecs r$, where $\vecs F(x, y, z) = \langle 2x \ln y, \frac{x^2}{y} + z^2, 2yz\rangle$ and C is a curve with parameterization $\vecs r(t) = \langle t^2, t, t\rangle$, $1 \le t \le e$

a. without using the Fundamental Theorem of Line Integrals and
b. using the Fundamental Theorem of Line Integrals.

Solution

1. First, let's calculate the integral without the Fundamental Theorem for Line Integrals and instead use the method we learned in the previous section:

$$\begin{aligned}\int_C \vecs{F}\cdot dr &= \int_1^e \vecs{F}(\vecs{r}(t))\cdot \vecs{r}'(t)\,dt \\ &= \int_1^e \langle 2t^2\ln t, \frac{t^4}{t}+t^2, 2t^2\rangle \cdot \langle 2t,1,1\rangle\,dt \\ &= \int_1^e (4t^3\ln t + t^3 + 3t^2)\,dt \\ &= \int_1^e 4t^3\ln t\,dt + \int_1^e (t^3+3t^2)\,dt \\ &= \int_1^e 4t^3\ln t\,dt + \left[\frac{t^4}{4}+t^3\right]_1^e \\ &= \int_1^e 4t^3\ln t\,dt + \frac{e^4}{4}+e^3-\frac{1}{4}-1 \\ &= \int_1^e 4t^3\ln t\,dt + \frac{e^4}{4}+e^3-\frac{5}{4}\end{aligned}$$

Integral $\int_1^e t^3\ln t\ dt$ requires integration by parts. Let $u=\ln t$ and $dv=t^3$. Then $u=\ln t$, $dv=t^3$

and

$$du=\frac{1}{t}\,dt,\quad v=\frac{t^4}{4}.$$

Therefore,

$$\begin{aligned}\int_1^e t^3\ln t\,dt &= \left[\frac{t^4}{4}\ln t\right]_1^e - \frac{1}{4}\int_1^e t^3\,dt \\ &= \frac{e^4}{4}-\frac{1}{4}\left(\frac{e^4}{4}-\frac{1}{4}\right).\end{aligned}$$

Thus,

$$\begin{aligned}\int_C \vecs{F}\cdot d\vecs{r} &= 4\int_1^e t^3\ln t\,dt \quad + \quad \frac{e^4}{4}+e^3-\frac{5}{4} \\ &= 4\left(\frac{e^4}{4}-\frac{1}{4}\left(\frac{e^4}{4}-\frac{1}{4}\right)\right)+\frac{e^4}{4}+e^3-\frac{5}{4} \\ &= e^4-\frac{e^4}{4}+\frac{1}{4}+\frac{e^4}{4}+e^3-\frac{5}{4} \\ &= e^4+e^3-1.\end{aligned}$$

2. Given that $f(x,y,z)=x^2\ln y+yz^2$ is a potential function for $\vecs{F}$, let's use the Fundamental Theorem for Line Integrals to calculate the integral. Note that

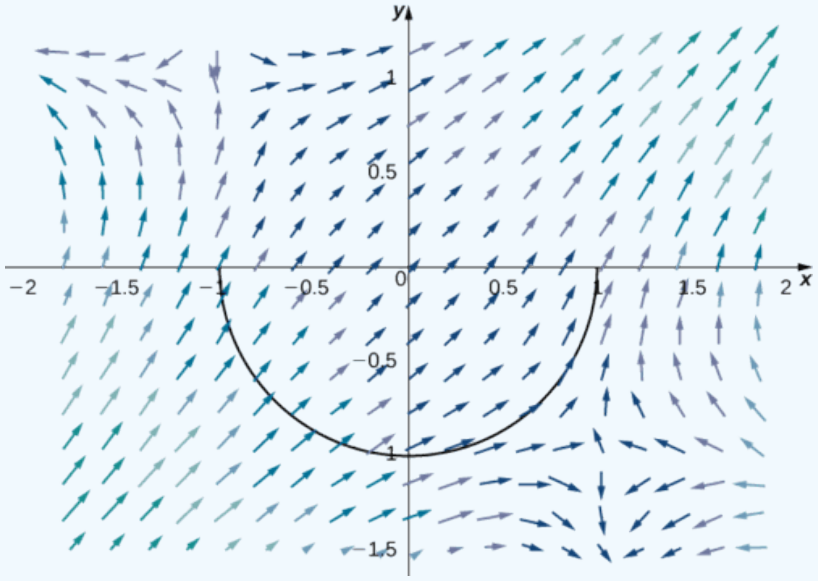
$$\begin{aligned}\int_C \vecs{F}\cdot d\vecs{r} &= \int_C \vecs{\nabla} f\cdot d\vecs{r} \\ &= f(\vecs{r}(e))-f(\vecs{r}(1)) \\ &= f(e^2,e,e)-f(1,1,1) \\ &= e^4+e^3-1.\end{aligned}$$

This calculation is much more straightforward than the calculation we did in (a). As long as we have a potential function, calculating a line integral using the Fundamental Theorem for Line Integrals is much easier than calculating without the theorem.

Example 15.3.3 illustrates a nice feature of the Fundamental Theorem of Line Integrals: it allows us to calculate more easily many vector line integrals. As long as we have a potential function, calculating the line integral is only a matter of evaluating the potential function at the endpoints and subtracting.

Exercise 15.3.3

Given that $f(x,y) = (x-1)^2 y + (y+1)^2 x$ is a potential function for $\vecs F(x,y) = \langle 2xy - 2y + (y+1)^2, (x-1)^2 + 2yx + 2x\rangle$, calculate integral $\int_C \vecs F \cdot d\vecs r$, where C is the lower half of the unit circle oriented counterclockwise.



Hint available at the end of this section.

The Fundamental Theorem for Line Integrals has two important consequences. The first consequence is that if $\vecs F$ is conservative and C is a closed curve, then the circulation of $\vecs F$ along C is zero—that is, $\int_C \vecs F \cdot d\vecs r = 0$. To see why this is true, let f be a potential function for $\vecs F$. Since C is a closed curve, the terminal point $\vecs r(b)$ of C is the same as the initial $\vecs r(a)$ of C—that is, $\vecs r(a) = \vecs r(b)$. Therefore, by the Fundamental Theorem for Line Integrals,

$$\oint_C \vecs F \cdot d\vecs r = \oint_C \vecs \nabla f \cdot d\vecs r \tag{15.3.2}$$

$$= f(\vecs r(b)) - f(\vecs r(a)) \tag{15.3.3}$$

$$= f(\vecs r(b)) - f(\vecs r(b)) \tag{15.3.4}$$

$$= 0. \tag{15.3.5}$$

Recall that the reason a conservative vector field $\vecs F$ is called "conservative" is because such vector fields model forces in which energy is conserved. We have shown gravity to be an example of such a force. If we think of vector field $\vecs F$ in integral $\oint_C \vecs F \cdot d\vecs r$ as a gravitational field, then the equation $\oint_C \vecs F \cdot d\vecs r = 0$ follows. If a particle travels along a path that starts and ends at the same place, then the work done by gravity on the particle is zero.

The second important consequence of the Fundamental Theorem for Line Integrals (Equation 15.3.1) is that line integrals of conservative vector fields are independent of path—meaning, they depend only on the endpoints of the given curve, and do not depend on the path between the endpoints.

DEFINITION: Path Independence

Let $\vecs F$ be a vector field with domain D; it is independent of path (or path independent) if

$$\int_{C_1} \vecs F \cdot d\vecs r = \int_{C_2} \vecs F \cdot d\vecs r$$

for any paths C_1 and C_2 in D with the same initial and terminal points.

The second consequence is stated formally in the following theorem.

Theorem: CONSERVATIVE FIELDS

If $\vecs F$ is a *conservative vector field*, then $\vecs F$ is independent of path.

Proof

Let D denote the domain of $\vecs F$ and let C_1 and C_2 be two paths in D with the same initial and terminal points (Figure $15.3.5$). Call the initial point P_1 and the terminal point P_2. Since $\vecs F$ is conservative, there is a potential function f for $\vecs F$. By the Fundamental Theorem for Line Integrals,

$$\int_{C_1} \vecs F \cdot d\vecs r = f(P_2) - f(P_1) = \int_{C_2} \vecs F \cdot d\vecs r.$$

Therefore, $\int_{C_1} \vecs F \cdot d\vecs r = \int_{C_2} \vecs F \cdot d\vecs r$ and $\vecs F$ is independent of path.

□

To visualize what independence of path means, imagine three hikers climbing from base camp to the top of a mountain. Hiker 1 takes a steep route directly from camp to the top. Hiker 2 takes a winding route that is not steep from camp to the top. Hiker 3 starts by taking the steep route but halfway to the top decides it is too difficult for him. Therefore he returns to camp and takes the non-steep path to the top. All three hikers are traveling along paths in a gravitational field. Since gravity is a force in which energy is conserved, the gravitational field is conservative. By independence of path, the total amount of work done by gravity on each of the hikers is the same because they all started in the same place and ended in the same place. The work done by the hikers includes other factors such as friction and muscle movement, so the total amount of energy each one expended is not the same, but the net energy expended against gravity is the same for all three hikers.

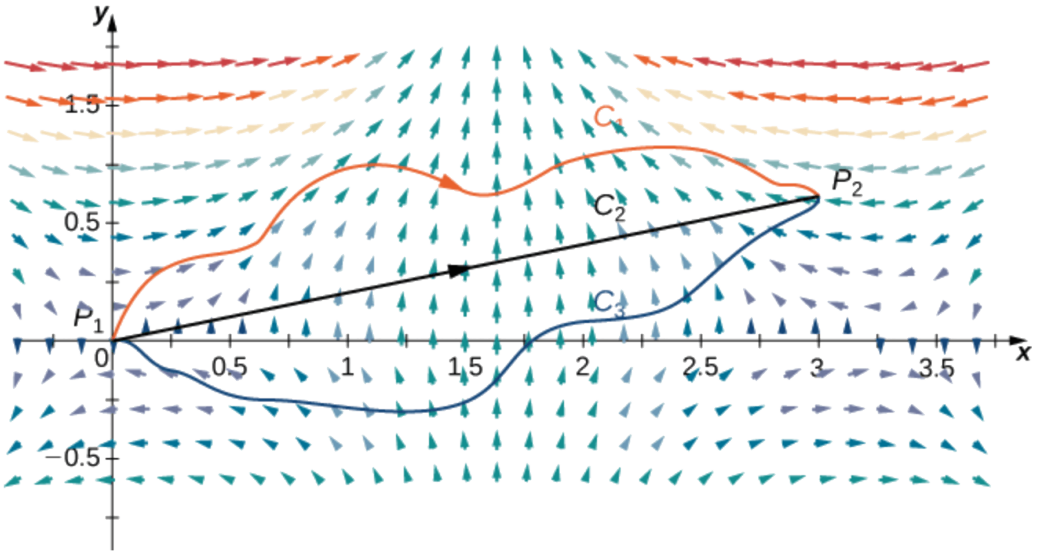


Figure 15.3.4: The vector field is conservative, and therefore independent of path.

We have shown that if $\vecs F$ is conservative, then $\vecs F$ is independent of path. It turns out that if the domain of $\vecs F$ is open and connected, then the converse is also true. That is, if $\vecs F$ is independent of path and the domain of $\vecs F$ is open and connected, then $\vecs F$ is conservative. Therefore, the set of conservative vector fields on open and connected domains is precisely the set of vector fields independent of path.

Theorem: THE PATH INDEPENDENCE TEST FOR CONSERVATIVE FIELDS

If $\vecs F$ is a continuous vector field that is independent of path and the domain D of $\vecs F$ is open and connected, then $\vecs F$ is conservative.

Proof

We prove the theorem for vector fields in $\mathbb{R}^2$. The proof for vector fields in $\mathbb{R}^3$ is similar. To show that $\vecs F = \langle P, Q\rangle$ is conservative, we must find a potential function f for $\vecs F$. To that end, let X be a fixed point in D. For any point (x, y) in D, let C be a path from X to (x, y). Define $f(x, y)$ by $f(x, y) = \int_C \vecs F \cdot d\vecs r$. (Note that this definition of f makes sense only because $\vecs F$ is independent of path. If $\vecs F$ was not independent of path, then it might be possible to find another path C' from X to (x, y) such that $\int_C \vecs F \cdot d\vecs r \neq \int_{C'} \vecs F \cdot d\vecs r$, and in such a case $f(x, y)$ would not be a function.) We want to show that f has the property $\vecs \nabla f = \vecs F$.

Since domain D is open, it is possible to find a disk centered at (x, y) such that the disk is contained entirely inside D. Let (a, y) with $a < x$ be a point in that disk. Let C be a path from X to (x, y) that consists of two pieces: C_1 and C_2. The first piece, C_1, is any path from C to (a, y) that stays inside D; C_2 is the horizontal line segment from (a, y) to (x, y) (Figure 15.3.6). Then

$$f(x, y) = \int_{C_1} \vecs F \cdot d\vecs r + \int_{C_2} \vecs F \cdot d\vecs r.$$

The first integral does not depend on x, so

$$f_x(x, y) = \frac{\partial}{\partial x} \int_{C_2} \vecs F \cdot d\vecs r.$$

If we parameterize C_2 by $\vecs r(t) = \langle t, y\rangle$, $a \leq t \leq x$, then

$$\begin{aligned} f_x(x,y) &= \frac{\partial}{\partial x}\int_{C_2} \vecs{F}\cdot d\vecs{r} \\ &= \frac{\partial}{\partial x}\int_a^x \vecs{F}(\vecs{r}(t))\cdot \vecs{r}'(t)\,dt \\ &= \frac{\partial}{\partial x}\int_a^x \vecs{F}(\vecs{r}(t))\cdot \frac{d}{dt}(\langle t,y\rangle)\,dt \\ &= \frac{\partial}{\partial x}\int_a^x \vecs{F}(\vecs{r}(t))\cdot \langle 1,0\rangle\,dt \\ &= \frac{\partial}{\partial x}\int_a^x P(t,y)\,dt. \end{aligned}$$

By the Fundamental Theorem of Calculus (part 1),

$$f_x(x,y) = \frac{\partial}{\partial x}\int_a^x P(t,y)\,dt = P(x,y).$$

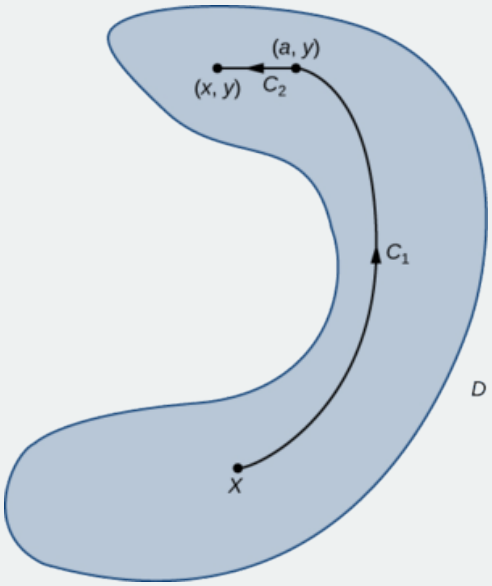


Figure 15.3.6. Here, C_1 is any path from C to (a,y) that stays inside D, and C_2 is the horizontal line segment from (a,y) to (x,y).

A similar argument using a vertical line segment rather than a horizontal line segment shows that $f_y(x,y) = Q(x,y)$.

Therefore $\vecs{\nabla} f = \vecs{F}$ and $\vecs{F}$ is conservative.

□

We have spent a lot of time discussing and proving the theorems above, but we can summarize them simply: a vector field $\vecs{F}$ on an open and connected domain is conservative if and only if it is independent of path. This is important to know because conservative vector fields are extremely important in applications, and these theorems give us a different way of viewing what it means to be conservative using path independence.

Example 15.3.4: Showing That a Vector Field Is Not Conservative

Use path independence to show that vector field $\vecs{F}(x,y) = \langle x^2 y, y+5\rangle$ is not conservative.

Solution

We can indicate that $\vecs{F}$ is not conservative by showing that $\vecs{F}$ is not path independent. We do so by giving two different paths, C_1 and C_2, that both start at $(0,0)$ and end at $(1,1)$, and yet $\int_{C_1} \vecs{F}\cdot d\vecs{r} \neq \int_{C_2} \vecs{F}\cdot d\vecs{r}$.

Let C_1 be the curve with parameterization $\vecs r_1(t) = \langle t,\ t\rangle$, $0 \le t \le 1$ and let C_2 be the curve with parameterization $\vecs r_2(t) = \langle t,\ t^2\rangle$, $0 \le t \le 1$ (Figure 15.3.7.). Then

$$\begin{aligned}\int_{C_1} \vecs F \cdot d\vecs r &= \int_0^1 \vecs F(\vecs r_1(t)) \cdot \vecs r_1{}'(t)\, dt \\ &= \int_0^1 \langle t^3, t+5\rangle \cdot \langle 1,1\rangle\, dt = \int_0^1 (t^3 + t + 5)\, dt \\ &= \left[\frac{t^4}{4} + \frac{t^2}{2} + 5t\right]_0^1 = \frac{23}{4}\end{aligned}$$

and

$$\begin{aligned}\int_{C_2} \vecs F \cdot d\vecs r &= \int_0^1 \vecs F(\vecs r_2(t)) \cdot \vecs r_2{}'(t)\, dt \\ &= \int_0^1 \langle t^4, t^2+5\rangle \cdot \langle 1, 2t\rangle\, dt = \int_0^1 (t^4 + 2t^3 + 10t)\, dt \\ &= \left[\frac{t^5}{5} + \frac{t^4}{2} + 5t^2\right]_0^1 = \frac{57}{10}.\end{aligned}$$

Since $\int_{C_1} \vecs F \cdot d\vecs r \ne \int_{C_2} \vecs F \cdot d\vecs r$, the value of a line integral of $\vecs F$ depends on the path between two given points. Therefore, $\vecs F$ is not independent of path, and $\vecs F$ is not conservative.

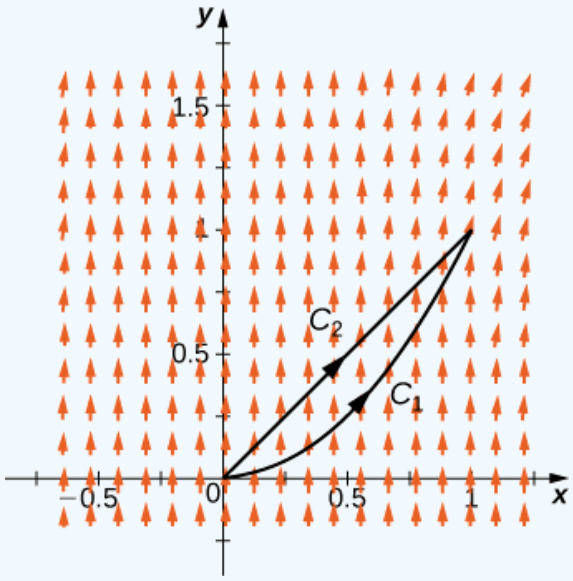


Figure 15.3.7: Curves C_1 and C_2 are both oriented from left to right.

Exercise 15.3.4

Show that $\vecs F(x,y) = \langle xy,\ x^2y^2\rangle$ is not path independent by considering the line segment from $(0,0)$ to $(0,2)$ and the piece of the graph of $y = \dfrac{x^2}{2}$ that goes from $(0,0)$ to $(0,2)$.

Hint available at the end of this section.

Conservative Vector Fields and Potential Functions

As we have learned, the Fundamental Theorem for Line Integrals says that if $\vecs F$ is conservative, then calculating $\int_C \vecs F \cdot d\vecs r$ has two steps: first, find a potential function f for $\vecs F$ and, second, calculate $f(P_1) - f(P_0)$, where P_1 is the endpoint

of C and P_0 is the starting point. To use this theorem for a conservative field $\vecs F$, we must be able to find a potential function f for $\vecs F$. Therefore, we must answer the following question: Given a conservative vector field $\vecs F$, how do we find a function f such that $\vecs\nabla f = \vecs F$? Before giving a general method for finding a potential function, let's motivate the method with an example.

Example 15.3.5: Finding a Potential Function

Find a potential function for $\vecs F(x,y) = \langle 2xy^3, 3x^2y^2 + \cos(y)\rangle$, thereby showing that $\vecs F$ is conservative.

Solution

Suppose that $f(x,y)$ is a potential function for $\vecs F$. Then, $\vecs\nabla f = \vecs F$, and therefore

$$f_x(x,y) = 2xy^3 \quad \text{and} \quad f_y(x,y) = 3x^2y^2 + \cos y.$$

Integrating the equation $f_x(x,y) = 2xy^3$ with respect to x yields the equation

$$f(x,y) = x^2y^3 + h(y).$$

Notice that since we are integrating a two-variable function with respect to x, we must add a constant of integration that is a constant with respect to x, but may still be a function of y. The equation $f(x,y) = x^2y^3 + h(y)$ can be confirmed by taking the partial derivative with respect to x:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2y^3) + \frac{\partial}{\partial x}(h(y)) = 2xy^3 + 0 = 2xy^3.$$

Since f is a potential function for $\vecs F$,

$$f_y(x,y) = 3x^2y^2 + \cos(y),$$

and therefore

$$3x^2y^2 + g'(y) = 3x^2y^2 + \cos(y).$$

This implies that $h'(y) = \cos y$, so $h(y) = \sin y + C$. Therefore, *any* function of the form $f(x,y) = x^2y^3 + \sin(y) + C$ is a potential function. Taking, in particular, $C = 0$ gives the potential function $f(x,y) = x^2y^3 + \sin(y)$.

To verify that f is a potential function, note that $\vecs\nabla f(x,y) = \langle 2xy^3, 3x^2y^2 + \cos y\rangle = \vecs F$.

Exercise 15.3.5

Find a potential function for $\vecs F(x,y) = \langle e^xy^3 + y, 3e^xy^2 + x\rangle$.

Hint available at the end of this section.

The logic of the previous example extends to finding the potential function for any conservative vector field in $\mathbb{R}^2$. Thus, we have the following problem-solving strategy for finding potential functions:

PROBLEM-SOLVING STRATEGY: FINDING A POTENTIAL FUNCTION FOR A CONSERVATIVE VECTOR FIELD $\vecs F(x,y) = \langle P(x,y), Q(x,y)\rangle$

1. Integrate P with respect to x. This results in a function of the form $g(x,y) + h(y)$, where $h(y)$ is unknown.
2. Take the partial derivative of $g(x,y) + h(y)$ with respect to y, which results in the function $gy(x,y) + h'(y)$.
3. Use the equation $gy(x,y) + h'(y) = Q(x,y)$ to find $h'(y)$.
4. Integrate $h'(y)$ to find $h(y)$.
5. Any function of the form $f(x,y) = g(x,y) + h(y) + C$, where C is a constant, is a potential function for $\vecs F$.

We can adapt this strategy to find potential functions for vector fields in $\mathbb{R}^3$, as shown in the next example.

Example 15.3.6: Finding a Potential Function in $\mathbb{R}^3$

Find a potential function for $F(x,y,z) = \langle 2xy, x^2 + 2yz^3, 3y^2z^2 + 2z\rangle$, thereby showing that $\vecs F$ is conservative.

Solution

Suppose that f is a potential function. Then, $\vecs\nabla f = \vecs F$ and therefore $f_x(x,y,z) = 2xy$. Integrating this equation with respect to x yields the equation $f(x,y,z) = x^2y + g(y,z)$ for some function g. Notice that, in this case, the constant of integration with respect to x is a function of y and z.

Since f is a potential function,

$$x^2 + 2yz^3 = f_y(x,y,z) = x^2 + g_y(y,z).$$

Therefore,

$$g_y(y,z) = 2yz^3.$$

Integrating this function with respect to y yields

$$g(y,z) = y^2z^3 + h(z)$$

for some function $h(z)$ of z alone. (Notice that, because we know that g is a function of only y and z, we do not need to write $g(y,z) = y^2z^3 + h(x,z)$.) Therefore,

$$f(x,y,z) = x^2y + g(y,z) = x^2y + y^2z^3 + h(z).$$

To find f, we now must only find h. Since f is a potential function,

$$3y^2z^2 + 2z = g_z(y,z) = 3y^2z^2 + h'(z).$$

This implies that $h\,'(z) = 2z$, so $h(z) = z^2 + C$. Letting $C = 0$ gives the potential function

$$f(x,y,z) = x^2y + y^2z^3 + z^2.$$

To verify that f is a potential function, note that $\vecs\nabla f(x,y,z) = \langle 2xy, x^2 + 2yz^3, 3y^2z^2 + 2z\rangle = \vecs F(x,y,z)$.

Exercise 15.3.6

Find a potential function for $\vecs F(x,y,z) = \langle 12x^2, \cos y\cos z, 1 - \sin y\sin z\rangle$.

Hint available at the end of this section.

We can apply the process of finding a potential function to a gravitational force. Recall that, if an object has unit mass and is located at the origin, then the gravitational force in $\mathbb{R}^2$ that the object exerts on another object of unit mass at the point (x,y) is given by vector field

$$\vecs F(x,y) = -G\left\langle \frac{x}{(x^2+y^2)^{3/2}}, \frac{y}{(x^2+y^2)^{3/2}}\right\rangle,$$

where G is the universal gravitational constant. In the next example, we build a potential function for $\vecs F$, thus confirming what we already know: that gravity is conservative.

Example 15.3.7: Finding a Potential Function

Find a potential function f for $\vecs{F}(x,y) = -G\left\langle \dfrac{x}{(x^2+y^2)^{3/2}}, \dfrac{y}{(x^2+y^2)^{3/2}} \right\rangle$.

Solution

Suppose that f is a potential function. Then, $\vecs{\nabla} f = \vecs{F}$ and therefore

$$f_x(x,y) = \frac{-Gx}{(x^2+y^2)^{3/2}}.$$

To integrate this function with respect to x, we can use u-substitution. If $u = x^2+y^2$, then $\dfrac{du}{2} = x\ dx$, so

$$\begin{aligned} \int \frac{-Gx}{(x^2+y^2)^{3/2}}\, dx &= \int \frac{-G}{2u^{3/2}}\, du \\ &= \frac{G}{\sqrt{u}} + h(y) \\ &= \frac{G}{\sqrt{x^2+y^2}} + h(y) \end{aligned}$$

for some function $h(y)$. Therefore,

$$f(x,y) = \frac{G}{\sqrt{x^2+y^2}} + h(y).$$

Since f is a potential function for $\vecs{F}$,

$$f_y(x,y) = \frac{-Gy}{(x^2+y^2)^{3/2}}$$

.

Since $f(x,y) = \dfrac{G}{\sqrt{x^2+y^2}} + h(y)$, $f_y(x,y)$ also equals $\dfrac{-Gy}{(x^2+y^2)^{3/2}} + h\,'(y)$.

Therefore,

$$\frac{-Gy}{(x^2+y^2)^{3/2}} + h'(y) = \frac{-Gy}{(x^2+y^2)^{3/2}},$$

which implies that $h\,'(y) = 0$. Thus, we can take $h(y)$ to be any constant; in particular, we can let $h(y) = 0$. The function

$$f(x,y) = \frac{G}{\sqrt{x^2+y^2}}$$

is a potential function for the gravitational field $\vecs{F}$. To confirm that f is a potential function, note that

$$\begin{aligned} \vecs{\nabla} f(x,y) &= \langle -\frac{1}{2}\frac{G}{(x^2+y^2)^{3/2}}(2x), -\frac{1}{2}\frac{G}{(x^2+y^2)^{3/2}}(2y)\rangle \\ &= \langle \frac{-Gx}{(x^2+y^2)^{3/2}}, \frac{-Gy}{(x^2+y^2)^{3/2}}\rangle \\ &= \vecs{F}(x,y). \end{aligned}$$

Exercise 15.3.7

Find a potential function f for the three-dimensional gravitational force $\vecs{F}(x,y,z) = \left\langle \frac{-Gx}{(x^2+y^2+z^2)^{3/2}}, \frac{-Gy}{(x^2+y^2+z^2)^{3/2}}, \frac{-Gz}{(x^2+y^2+z^2)^{3/2}} \right\rangle$.

Hint available at the end of this section.

Testing a Vector Field

Until now, we have worked with vector fields that we know are conservative, but if we are not told that a vector field is conservative, we need to be able to test whether it is conservative. Recall that, if $\vecs{F}$ is conservative, then $\vecs{F}$ has the cross-partial property (see The Cross-Partial Property of Conservative Vector Fields). That is, if $\vecs{F} = \langle P, Q, R \rangle$ is conservative, then $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$. So, if $\vecs{F}$ has the cross-partial property, then is $\vecs{F}$ conservative? If the domain of $\vecs{F}$ is open and simply connected, then the answer is yes.

Theorem: THE CROSS-PARTIAL TEST FOR CONSERVATIVE FIELDS

If $\vecs{F} = \langle P, Q, R \rangle$ is a vector field on an open, simply connected region D and $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$ throughout D, then $\vecs{F}$ is conservative.

Although a proof of this theorem is beyond the scope of the text, we can discover its power with some examples. Later, we see why it is necessary for the region to be simply connected.

Combining this theorem with the cross-partial property, we can determine whether a given vector field is conservative:

Theorem: CROSS-PARTIAL PROPERTY OF CONSERVATIVE FIELDS

Let $\vecs{F} = \langle P, Q, R \rangle$ be a vector field on an open, simply connected region D. Then $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$ throughout D if and only if $\vecs{F}$ is conservative.

The version of this theorem in $\mathbb{R}^2$ is also true. If $\vecs{F}(x,y) = \langle P, Q \rangle$ is a vector field on an open, simply connected domain in $\mathbb{R}^2$, then $\vecs{F}$ is conservative if and only if $P_y = Q_x$.

Example 15.3.8: Determining Whether a Vector Field Is Conservative

Determine whether vector field $\vecs{F}(x,y,z) = \langle xy^2z, x^2yz, z^2 \rangle$ is conservative.

Solution

Note that the domain of $\vecs{F}$ is all of $\mathbb{R}^2$ and $\mathbb{R}^3$ is simply connected. Therefore, we can use **The Cross-Partial Property of Conservative Vector Fields** to determine whether $\vecs{F}$ is conservative. Let

$$P(x,y,z) = xy^2z$$

$$Q(x,y,z) = x^2yz$$

and

$$R(x,y,z) = z^2.$$

Since $Q_z(x,y,z) = x^2y$ and $R_y(x,y,z) = 0$, the vector field is not conservative.

Example 15.3.9: Determining Whether a Vector Field Is Conservative

Determine vector field $\vecs F(x,y) = \langle x\ln(y),\ \frac{x^2}{2y}\rangle$ is conservative.

Solution

Note that the domain of $\vecs F$ is the part of $\mathbb{R}^2$ in which $y > 0$. Thus, the domain of $\vecs F$ is part of a plane above the x -axis, and this domain is simply connected (there are no holes in this region and this region is connected). Therefore, we can use **The Cross-Partial Property of Conservative Vector Fields** to determine whether $\vecs F$ is conservative. Let

$$P(x,y) = x\ln(y) \text{ and } Q(x,y) = \frac{x^2}{2y}.$$

Then $P_y(x,y) = \frac{x}{y} = Q_x(x,y)$ and thus $\vecs F$ is conservative.

Exercise 15.3.8

Determine whether $\vecs F(x,y) = \langle \sin x\cos y,\ \cos x\sin y\rangle$ is conservative.

Hint available at the end of this section.

When using **The Cross-Partial Property of Conservative Vector Fields**, it is important to remember that a theorem is a tool, and like any tool, it can be applied only under the right conditions. In the case of **The Cross-Partial Property of Conservative Vector Fields**, the theorem can be applied only if the domain of the vector field is simply connected.

To see what can go wrong when misapplying the theorem, consider the vector field from Example 15.3.4:

$$\vecs F(x,y) = \frac{y}{x^2+y^2}\,\hat{\mathbf{i}} + \frac{-x}{x^2+y^2}\,\hat{\mathbf{j}}.$$

This vector field satisfies the cross-partial property, since

$$\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \frac{(x^2+y^2)-y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

and

$$\frac{\partial}{\partial x}\left(\frac{-x}{x^2+y^2}\right) = \frac{-(x^2+y^2)+x(2x)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}.$$

Since $\vecs F$ satisfies the cross-partial property, we might be tempted to conclude that $\vecs F$ is conservative. However, $\vecs F$ is not conservative. To see this, let

$$\vecs r(t) = \langle \cos t, \sin t\rangle,\ \ 0 \le t \le \pi$$

be a parameterization of the upper half of a unit circle oriented counterclockwise (denote this C_1) and let

$$\vecs s(t) = \langle \cos t, -\sin t\rangle,\ \ 0 \le t \le \pi$$

be a parameterization of the lower half of a unit circle oriented clockwise (denote this C_2). Notice that C_1 and C_2 have the same starting point and endpoint. Since $\sin^2 t + \cos^2 t = 1$,

$$\vecs F(\vecs r(t)) \cdot \vecs r'(t) = \langle \sin(t), -\cos(t)\rangle \cdot \langle -\sin(t), \cos(t)\rangle = -1$$

and

$$\vecs F(\vecs s(t)) \cdot \vecs s'(t) = \langle -\sin t, -\cos t\rangle \cdot \langle -\sin t, -\cos t\rangle = \sin^2 t + \cos^2 t = 1.$$

Therefore,

$$\int_{C_1} \vecs{F} \cdot d\vecs{r} = \int_0^{\pi} -1 \, dt = -\pi$$

and

$$\int_{C_2} \vecs{F} \cdot d\vecs{r} = \int_0^{\pi} 1 \, dt = \pi.$$

Thus, C_1 and C_2 have the same starting point and endpoint, but $\int_{C_1} \vecs{F} \cdot d\vecs{r} \neq \int_{C_2} \vecs{F} \cdot d\vecs{r}$. Therefore, $\vecs{F}$ is not independent of path and $\vecs{F}$ is not conservative.

To summarize: $\vecs{F}$ satisfies the cross-partial property and yet $\vecs{F}$ is not conservative. What went wrong? Does this contradict **The Cross-Partial Property of Conservative Vector Fields**? The issue is that the domain of $\vecs{F}$ is all of $\mathbb{R}^2$ except for the origin. In other words, the domain of $\vecs{F}$ has a hole at the origin, and therefore the domain is not simply connected. Since the domain is not simply connected, **The Cross-Partial Property of Conservative Vector Fields** does not apply to $\vecs{F}$.

We close this section by looking at an example of the usefulness of the Fundamental Theorem for Line Integrals. Now that we can test whether a vector field is conservative, we can always decide whether the Fundamental Theorem for Line Integrals can be used to calculate a vector line integral. If we are asked to calculate an integral of the form $\int_C \vecs{F} \cdot d\vecs{r}$, then our first question should be: Is $\vecs{F}$ conservative? If the answer is yes, then we should find a potential function and use the Fundamental Theorem for Line Integrals to calculate the integral. If the answer is no, then the Fundamental Theorem for Line Integrals cannot help us and we have to use other methods, such as using the method from the previous section (using $\vecs{F}(\vecs{r}(t))$ and $\vecs{r}'(t)$).

Example 15.3.10: Using the Fundamental Theorem for Line Integrals

Calculate line integral $\int_C \vecs{F} \cdot d\vecs{r}$, where $\vecs{F}(x, y, z) = \langle 2xe^y z + e^x z, \; x^2 e^y z, \; x^2 e^y + e^x \rangle$ and C is any smooth curve that goes from the origin to $(1, 1, 1)$.

Solution

Before trying to compute the integral, we need to determine whether $\vecs{F}$ is conservative and whether the domain of $\vecs{F}$ is simply connected. The domain of $\vecs{F}$ is all of $\mathbb{R}^3$, which is connected and has no holes. Therefore, the domain of $\vecs{F}$ is simply connected. Let

$$P(x, y, z) = 2xe^y z + e^x z, \;\; Q(x, y, z) = x^2 e^y z, \;\; \text{and} \;\; R(x, y, z) = x^2 e^y + e^x$$

so that $\vecs{F}(x, y, z) = \langle P, Q, R \rangle$. Since the domain of $\vecs{F}$ is simply connected, we can check the cross partials to determine whether $\vecs{F}$ is conservative. Note that

$$\begin{aligned} P_y(x, y, z) &= 2xe^y z = Q_x(x, y, z) \\ P_z(x, y, z) &= 2xe^y + e^x = R_x(x, y, z) \\ Q_z(x, y, z) &= x^2 e^y = R_y(x, y, z). \end{aligned}$$

Therefore, $\vecs{F}$ is conservative.

To evaluate $\int_C \vecs{F} \cdot d\vecs{r}$ using the Fundamental Theorem for Line Integrals, we need to find a potential function f for $\vecs{F}$. Let f be a potential function for $\vecs{F}$. Then, $\vecs{\nabla} f = \vecs{F}$, and therefore $f_x(x, y, z) = 2xe^y z + e^x z$. Integrating this equation with respect to x gives $f(x, y, z) = x^2 e^y z + e^x z + h(y, z)$ for some function h. Differentiating this equation with respect to y gives $x^2 e^y z + h_y(y, z) = Q(x, y, z) = x^2 e^y z$, which implies that $h_y(y, z) = 0$. Therefore, h is a function of z only, and $f(x, y, z) = x^2 e^y z + e^x z + h(z)$. To find h, note that $f_z = x^2 e^y + e^x + h'(z) = R = x^2 e^y + e^x$. Therefore, $h'(z) = 0$ and we can take $h(z) = 0$. A potential function for $\vecs{F}$ is $f(x, y, z) = x^2 e^y z + e^x z$.

Now that we have a potential function, we can use the Fundamental Theorem for Line Integrals to evaluate the integral. By the theorem,

$$\begin{aligned}\int_C \vecs{F}\cdot d\vecs{r} &= \int_C \vecs{\nabla} f \cdot d\vecs{r} \\ &= f(1,1,1) - f(0,0,0) \\ &= 2e.\end{aligned}$$

Analysis

Notice that if we hadn't recognized that $\vecs{F}$ is conservative, we would have had to parameterize C and use the method from the previous section. Since curve C is unknown, using the Fundamental Theorem for Line Integrals is much simpler.

Exercise 15.3.9

Calculate integral $\int_C \vecs{F}\cdot d\vecs{r}$, where $\vecs{F}(x,y) = \langle \sin x \sin y, 5 - \cos x \cos y\rangle$ and C is a semicircle with starting point $(0,\pi)$ and endpoint $(0,-\pi)$.

Hint available at the end of this section.

Example 15.3.11: Work Done on a Particle

Let $\vecs{F}(x,y) = \langle 2xy^2, 2x^2y\rangle$ be a force field. Suppose that a particle begins its motion at the origin and ends its movement at any point in a plane that is not on the x-axis or the y-axis. Furthermore, the particle's motion can be modeled with a smooth parameterization. Show that $\vecs{F}$ does positive work on the particle.

Solution

We show that $\vecs{F}$ does positive work on the particle by showing that $\vecs{F}$ is conservative and then by using the Fundamental Theorem for Line Integrals.

To show that $\vecs{F}$ is conservative, suppose $f(x,y)$ were a potential function for $\vecs{F}$. Then, $\vecs{\nabla} f(x,y) = \vecs{F}(x,y) = \langle 2xy^2, 2x^2y\rangle$ and therefore $f_x(x,y) = 2xy^2$ and $f_y(x,y) = 2x^2y$. The equation $fx(x,y) = 2xy^2$ implies that $f(x,y) = x^2y^2 + h(y)$. Deriving both sides with respect to y yields $f_y(x,y) = 2x^2y + h\,'(y)$. Therefore, $h\,'(y) = 0$ and we can take $h(y) = 0$.

If $f(x,y) = x^2y^2$, then note that $\vecs{\nabla} f(x,y) = \langle 2xy^2, 2x^2y\rangle = \vecs{F}$, and therefore f is a potential function for $\vecs{F}$.

Let (a,b) be the point at which the particle stops is motion, and let C denote the curve that models the particle's motion. The work done by $\vecs{F}$ on the particle is $\int_C \vecs{F}\cdot d\vecs{r}$. By the Fundamental Theorem for Line Integrals,

$$\begin{aligned}\int_C \vecs{F}\cdot d\vecs{r} &= \int_C \nabla f \cdot d\vecs{r} \\ &= f(a,b) - f(0,0) \\ &= a^2b^2.\end{aligned}$$

Since $a \neq 0$ and $b \neq 0$, by assumption, $a^2b^2 > 0$. Therefore, $\int_C \vecs{F}\cdot d\vecs{r} > 0$, and $\vecs{F}$ does positive work on the particle.

Analysis

Notice that this problem would be much more difficult without using the Fundamental Theorem for Line Integrals. To apply the tools we have learned, we would need to give a curve parameterization and use the method from the previous section. Since the path of motion C can be as exotic as we wish (as long as it is smooth), it can be very difficult to parameterize the motion of the particle.

Exercise 15.3.10

Let $\vecs F(x,y) = \langle 4x^3y^4, 4x^4y^3 \rangle$, and suppose that a particle moves from point $(4,4)$ to $(1,1)$ along any smooth curve. Is the work done by $\vecs F$ on the particle positive, negative, or zero?

Hint available at the end of this section.

Key Concepts

- The theorems in this section require curves that are closed, simple, or both, and regions that are connected or simply connected.
- The line integral of a conservative vector field can be calculated using the Fundamental Theorem for Line Integrals. This theorem is a generalization of the Fundamental Theorem of Calculus in higher dimensions. Using this theorem usually makes the calculation of the line integral easier.
- Conservative fields are independent of path. The line integral of a conservative field depends only on the value of the potential function at the endpoints of the domain curve.
- Given vector field $\vecs F$, we can test whether $\vecs F$ is conservative by using the cross-partial property. If $\vecs F$ has the cross-partial property and the domain is simply connected, then $\vecs F$ is conservative (and thus has a potential function). If $\vecs F$ is conservative, we can find a potential function by using the Problem-Solving Strategy.
- The circulation of a conservative vector field on a simply connected domain over a closed curve is zero.

Key Equations

- **Fundamental Theorem for Line Integrals**
  $\displaystyle\int_C \vecs\nabla f \cdot d\vecs r = f(\vecs r(b)) - f(\vecs r(a))$
- **Circulation of a conservative field over curve *C* that encloses a simply connected region**
  $\displaystyle\oint_C \vecs\nabla f \cdot d\vecs r = 0$

Glossary

closed curve
a curve that begins and ends at the same point

connected region
a region in which any two points can be connected by a path with a trace contained entirely inside the region

Fundamental Theorem for Line Integrals
the value of line integral $\displaystyle\int_C \vecs\nabla f \cdot d\vecs r$ depends only on the value of f at the endpoints of C: $\displaystyle\int_C \vecs\nabla f \cdot d\vecs r = f(\vecs r(b)) - f(\vecs r(a))$

independence of path
a vector field $\vecs F$ has path independence if $\displaystyle\int_{C_1} \vecs F \cdot d\vecs r = \int_{C_2} \vecs F \cdot d\vecs r$ for any curves C_1 and C_2 in the domain of $\vecs F$ with the same initial points and terminal points

simple curve
a curve that does not cross itself

simply connected region
a region that is connected and has the property that any closed curve that lies entirely inside the region encompasses points that are entirely inside the region

Answers

Exercise 15.3.1

Sketch the curve.

Yes

Exercise 15.3.2

Consider the definitions.

The region in the figure is connected. The region in the figure is not simply connected.

Exercise 15.3.3

The Fundamental Theorem for Line Intervals says this integral depends only on the value of f at the endpoints of C.

2

Exercise 15.3.4

Calculate the corresponding line integrals.

If C_1 and C_2 represent the two curves, then

$$\int_{C_1} \vecs{F} \cdot d\vecs{r} \neq \int_{C_2} \vecs{F} \cdot d\vecs{r}.$$

Exercise 15.3.5

Follow the steps in Example 15.3.5.

$f(x, y) = e^x y^3 + xy$

Exercise 15.3.6

Following Example 15.3.6, begin by integrating with respect to x.

$f(x, y, z) = 4x^3 + \sin y \cos z + z$

Exercise 15.3.7

Follow the Problem-Solving Strategy.

$f(x, y, z) = \dfrac{G}{\sqrt{x^2 + y^2 + z^2}}$

Exercise 15.3.8

Use **The Cross-Partial Property of Conservative Vector Fields** from the previous section.

It is conservative.

Exercise 15.3.9

Use the Fundamental Theorem for Line Integrals.

-10π

Exercise 15.3.10

Use the Fundamental Theorem for Line Integrals.

Negative

15.3E: Conservative Vector Fields (Exercises)

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15.3E: Conservative Vector Fields (Exercises)

1. *True* or *False?* If vector field $\vecs F$ is conservative on the open and connected region D, then line integrals of $\vecs F$ are path independent on D, regardless of the shape of D.

Answer

True

2. *True* or *False?* Function $\vecs r(t) = \vecs a + t(\vecs b - \vecs a)$, where $0 \le t \le 1$, parameterizes the straight-line segment from $\vecs a$ to $\vecs b$.

Answer

True

3. *True* or *False?* Vector field $\vecs F(x, y, z) = (y \sin z)\ \hat{\mathbf{i}} + (x \sin z)\ \hat{\mathbf{j}} + (xy \cos z)\ \hat{\mathbf{k}}$ is conservative.

Answer

True

4. *True* or *False?* Vector field $\vecs F(x, y, z) = y\ \hat{\mathbf{i}} + (x + z)\ \hat{\mathbf{j}} - y\ \hat{\mathbf{k}}$ is conservative.

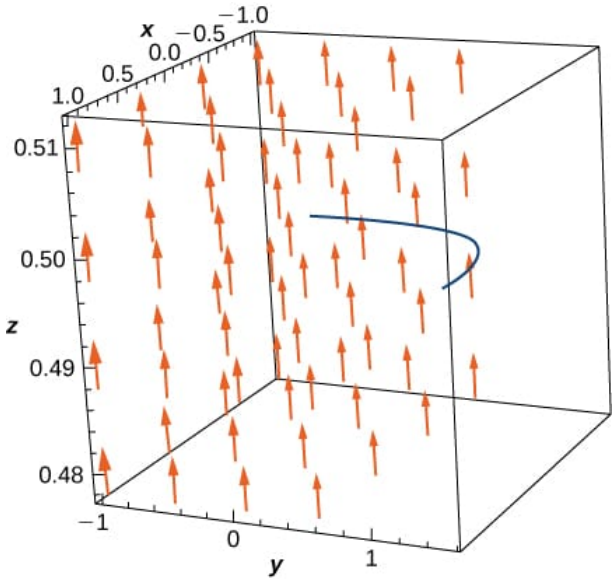
5. Justify the Fundamental Theorem of Line Integrals for $\int_C \vecs F \cdot d\vecs r$ in the case when $\vecs F(x, y) = (2x + 2y)\ \hat{\mathbf{i}} + (2x + 2y)\ \hat{\mathbf{j}}$ and C is a portion of the positively oriented circle $x^2 + y^2 = 25$ from $(5, 0)$ to $(3, 4)$.

Answer

$\int_C \vecs F \cdot d\vecs r = 24$ units of work

6. [T] Find $\int_C \vecs F \cdot d\vecs r$, where $\vecs F(x, y) = (ye^{xy} + \cos x)\ \hat{\mathbf{i}} + \left(xe^{xy} + \frac{1}{y^2+1}\right)\ \hat{\mathbf{j}}$ and C is a portion of curve $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{2}$.

7. [T] Evaluate line integral $\int_C \vecs F \cdot d\vecs r$, where $\vecs F(x, y) = (e^x \sin y - y)\ \hat{\mathbf{i}} + (e^x \cos y - x - 2)\ \hat{\mathbf{j}}$, and C is the path given by $\vecs r(t) = (t^3 \sin \frac{\pi t}{2})\ \hat{\mathbf{i}} - (\frac{\pi}{2} \cos(\frac{\pi t}{2} + \frac{\pi}{2}))\ \hat{\mathbf{j}}$ for $0 \le t \le 1$.



Answer

$$\int_C \vecs{F} \cdot d\vecs{r} = \left(e - \frac{3\pi}{2}\right) \text{ units of work}$$

For the following exercises, determine whether the vector field is conservative and, if it is, find the potential function.

8. $\vecs{F}(x, y) = 2xy^3 \ \hat{\mathbf{i}} + 3y^2x^2 \ \hat{\mathbf{j}}$

9. $\vecs{F}(x, y) = (-y + e^x \sin y) \ \hat{\mathbf{i}} + ((x + 2)e^x \cos y) \ \hat{\mathbf{j}}$

Answer

Not conservative

10. $\vecs{F}(x, y) = (e^{2x} \sin y) \ \hat{\mathbf{i}} + (e^{2x} \cos y) \ \hat{\mathbf{j}}$

11. $\vecs{F}(x, y) = (6x + 5y) \ \hat{\mathbf{i}} + (5x + 4y) \ \hat{\mathbf{j}}$

Answer

Conservative, $f(x, y) = 3x^2 + 5xy + 2y^2 + k$

12. $\vecs{F}(x, y) = (2x \cos(y) - y \cos(x)) \ \hat{\mathbf{i}} + (-x^2 \sin(y) - \sin(x)) \ \hat{\mathbf{j}}$

13. $\vecs{F}(x, y) = (ye^x + \sin(y)) \ \hat{\mathbf{i}} + (e^x + x \cos(y)) \ \hat{\mathbf{j}}$

Answer

Conservative, $f(x, y) = ye^x + x \sin(y) + k$

For the following exercises, evaluate the line integrals using the Fundamental Theorem of Line Integrals.

14. $\oint_C (y \ \hat{\mathbf{i}} + x \ \hat{\mathbf{j}}) \cdot d\vecs{r}$, where C is any path from $(0, 0)$ to $(2, 4)$

15. $\oint_C (2y \ dx + 2x \ dy)$, where C is the line segment from $(0, 0)$ to $(4, 4)$

Answer

$$\oint_C (2y \ dx + 2x \ dy) = 32 \text{ units of work}$$

16. [T] $\oint_C \left[\arctan \frac{y}{x} - \frac{xy}{x^2 + y^2}\right] \ dx + \left[\frac{x^2}{x^2 + y^2} + e^{-y}(1 - y)\right] \ dy$, where C is any smooth curve from $(1, 1)$ to $(-1, 2)$.

17. Find the conservative vector field for the potential function $f(x, y) = 5x^2 + 3xy + 10y^2$.

Answer

$$\vecs{F}(x, y) = (10x + 3y) \ \hat{\mathbf{i}} + (3x + 20y) \ \hat{\mathbf{j}}$$

For the following exercises, determine whether the vector field is conservative and, if so, find a potential function.

18. $\vecs{F}(x, y) = (12xy) \ \hat{\mathbf{i}} + 6(x^2 + y^2) \ \hat{\mathbf{j}}$

19. $\vecs{F}(x, y) = (e^x \cos y) \ \hat{\mathbf{i}} + 6(e^x \sin y) \ \hat{\mathbf{j}}$

Answer

$\vecs{F}$ is not conservative.

20. $\vecs{F}(x, y) = (2xye^{x^2y}) \ \hat{\mathbf{i}} + 6(x^2e^{x^2y}) \ \hat{\mathbf{j}}$

21. $\vecs{F}(x, y, z) = (ye^z) \ \hat{\mathbf{i}} + (xe^z) \ \hat{\mathbf{j}} + (xye^z) \ \hat{\mathbf{k}}$

Answer

$\vecs F$ is conservative and a potential function is $f(x, y, z) = xye^z + k$.

22. $\vecs F(x, y, z) = (\sin y)\ \hat{\mathbf{i}} - (x \cos y)\ \hat{\mathbf{j}} + \ \hat{\mathbf{k}}$

23. $\vecs F(x, y, z) = \dfrac{1}{y}\ \hat{\mathbf{i}} - \dfrac{x}{y^2}\ \hat{\mathbf{j}} + (2z - 1)\ \hat{\mathbf{k}}$

Answer

$\vecs F$ is conservative and a potential function is $f(x, y, z) = \dfrac{x}{y} + z^2 - z + k$.

24. $\vecs F(x, y, z) = 3z^2\ \hat{\mathbf{i}} - \cos y\ \hat{\mathbf{j}} + 2xz\ \hat{\mathbf{k}}$

25. $\vecs F(x, y, z) = (2xy)\ \hat{\mathbf{i}} + (x^2 + 2yz)\ \hat{\mathbf{j}} + y^2\ \hat{\mathbf{k}}$

Answer

$\vecs F$ is conservative and a potential function is $f(x, y, z) = x^2y + y^2z + k$.

For the following exercises, determine whether the given vector field is conservative and find a potential function.

26. $\vecs F(x, y) = (e^x \cos y)\ \hat{\mathbf{i}} + 6(e^x \sin y)\ \hat{\mathbf{j}}$

27. $\vecs F(x, y) = (2xye^{x^2y})\ \hat{\mathbf{i}} + (x^2e^{x^2y})\ \hat{\mathbf{j}}$

Answer

$\vecs F$ is conservative and a potential function is $f(x, y) = e^{x^2y} + k$

For the following exercises, evaluate the integral using the Fundamental Theorem of Line Integrals.

28. Evaluate $\displaystyle\int_C \vecs\nabla f \cdot d\vecs r$, where $f(x, y, z) = \cos(\pi x) + \sin(\pi y) - xyz$ and C is any path that starts at $(1, 12, 2)$ and ends at $(2, 1, -1)$.

29. [T] Evaluate $\displaystyle\int_C \vecs\nabla f \cdot d\vecs r$, where $f(x, y) = xy + e^x$ and C is a straight line from $(0, 0)$ to $(2, 1)$.

Answer

$\displaystyle\int_C \vecs F \cdot d\vecs r = \left(e^2 + 1\right)$ units of work

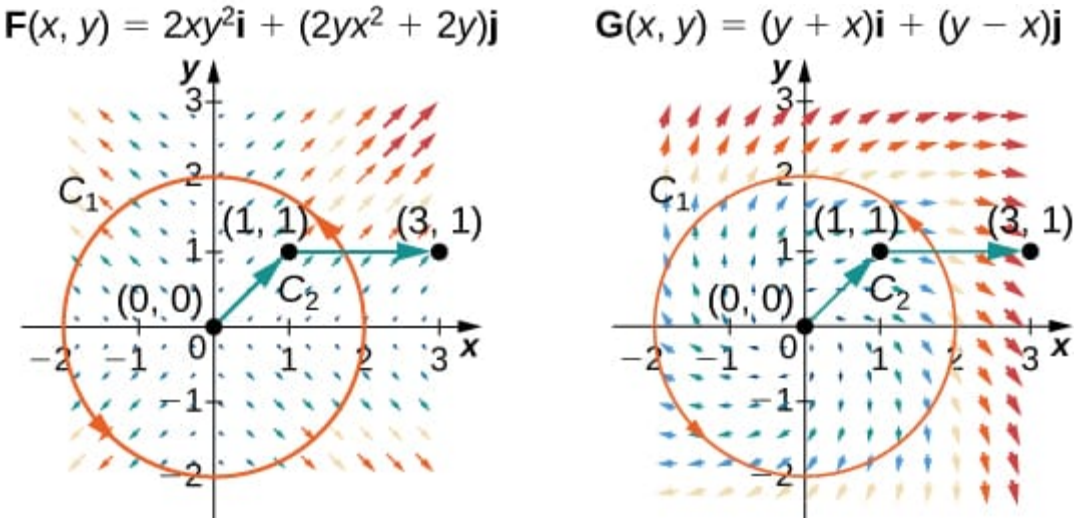
30. [T] Evaluate $\displaystyle\int_C \vecs\nabla f \cdot d\vecs r$, where $f(x, y) = x^2y - x$ and C is any path in a plane from (1, 2) to (3, 2).

31. Evaluate $\displaystyle\int_C \vecs\nabla f \cdot d\vecs r$, where $f(x, y, z) = xyz^2 - yz$ and C has initial point $(1, 2, 3)$ and terminal point $(3, 5, 2)$.

Answer

$\displaystyle\int_C \vecs F \cdot d\vecs r = 38$ units of work

For the following exercises, let $\vecs F(x, y) = 2xy^2\ \hat{\mathbf{i}} + (2yx^2 + 2y)\ \hat{\mathbf{j}}$ and $G(x, y) = (y + x)\ \hat{\mathbf{i}} + (y - x)\ \hat{\mathbf{j}}$, and let C_1 be the curve consisting of the circle of radius 2, centered at the origin and oriented counterclockwise, and C_2 be the curve consisting of a line segment from $(0, 0)$ to $(1, 1)$ followed by a line segment from $(1, 1)$ to $(3, 1)$.



32. Calculate the line integral of $\vecs{F}$ over C_1.

33. Calculate the line integral of $\vecs{G}$ over C_1.

Answer

$\oint_{C_1} \vecs{G} \cdot d\vecs{r} = -8\pi$ units of work

34. Calculate the line integral of $\vecs{F}$ over C_2.

35. Calculate the line integral of $\vecs{G}$ over C_2.

Answer

$\oint_{C_2} \vecs{F} \cdot d\vecs{r} = 7$ units of work

36. [T] Let $\vecs{F}(x, y, z) = x^2\ \hat{\mathbf{i}} + z\sin(yz)\ \hat{\mathbf{j}} + y\sin(yz)\ \hat{\mathbf{k}}$. Calculate $\oint_C \vecs{F} \cdot d\vecs{r}$, where C is a path from $A = (0, 0, 1)$ to $B = (3, 1, 2)$.

37. [T] Find line integral $\oint_C \vecs{F} \cdot dr$ of vector field $\vecs{F}(x, y, z) = 3x^2z\ \hat{\mathbf{i}} + z^2\ \hat{\mathbf{j}} + (x^3 + 2yz)\ \hat{\mathbf{k}}$ along curve C parameterized by $\vecs{r}(t) = (\frac{\ln t}{\ln 2})\ \hat{\mathbf{i}} + t^{3/2}\ \hat{\mathbf{j}} + t\cos(\pi t), 1 \le t \le 4$.

Answer

$\int_C \vecs{F} \cdot d\vecs{r} = 159$ units of work

For exercises 38 - 40, show that the following vector fields are conservative. Then calculate $\int_C \vecs{F} \cdot d\vecs{r}$ for the given curve.

38. $\vecs{F}(x, y) = (xy^2 + 3x^2y)\ \hat{\mathbf{i}} + (x + y)x^2\ \hat{\mathbf{j}}$; C is the curve consisting of line segments from $(1, 1)$ to $(0, 2)$ to $(3, 0)$.

39. $\vecs{F}(x, y) = \dfrac{2x}{y^2 + 1}\ \hat{\mathbf{i}} - \dfrac{2y(x^2 + 1)}{(y^2 + 1)^2}\ \hat{\mathbf{j}}$; C is parameterized by $x = t^3 - 1,\ \ y = t^6 - t$, for $0 \le t \le 1$.

Answer

$\int_C \vecs{F} \cdot d\vecs{r} = -1$ units of work

40. [T] $\vecs{F}(x, y) = [\cos(xy^2) - xy^2\sin(xy^2)]\ \hat{\mathbf{i}} - 2x^2y\sin(xy^2)\ \hat{\mathbf{j}}$; C is the curve $\langle e^t, e^{t+1}\rangle$, for $-1 \le t \le 0$.

41. The mass of Earth is approximately $6 \times 10^{27} g$ and that of the Sun is 330,000 times as much. The gravitational constant is $6.7 \times 10^{-8} cm^3/s^2 \cdot g$. The distance of Earth from the Sun is about $1.5 \times 10^{12} cm$. Compute, approximately, the work necessary to increase the distance of Earth from the Sun by $1\ cm$.

Answer

4×10^{31} erg

42. [T] Let $\vecs{F}(x, y, z) = (e^x \sin y)\ \hat{\mathbf{i}} + (e^x \cos y)\ \hat{\mathbf{j}} + z^2\ \hat{\mathbf{k}}$. Evaluate the integral $\int_C \vecs{F} \cdot d\vecs{r}$, where $\vecs{r}(t) = \langle \sqrt{t}, t^3, e^{\sqrt{t}} \rangle$, for $0 \le t \le 1$.

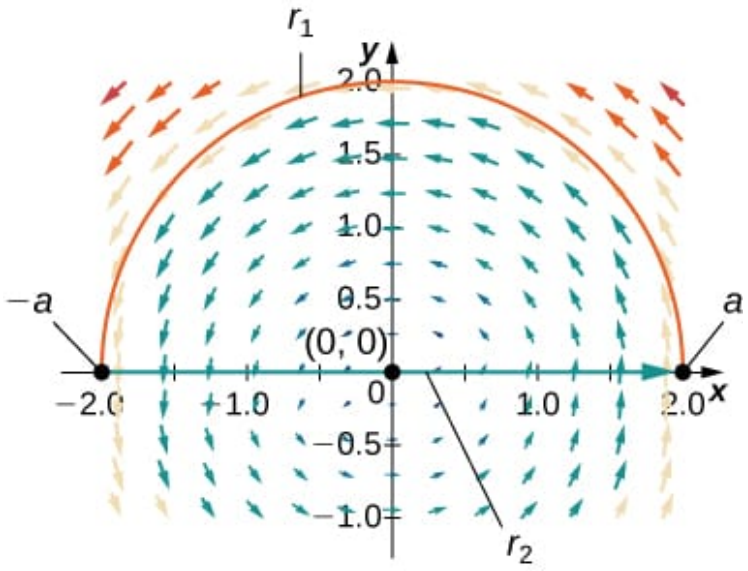
43. [T] Let $C : [1, 2] \to \mathbb{R}^2$ be given by $x = e^{t-1}, y = \sin\left(\frac{\pi}{t}\right)$. Use a computer to compute the integral $\int_C \vecs{F} \cdot d\vecs{r} = \int_C 2x \cos y\ dx - x^2 \sin y\ dy$, where $\vecs{F}(x, y) = (2x \cos y)\ \hat{\mathbf{i}} - (x^2 \sin y)\ \hat{\mathbf{j}}$.

Answer

$\int_C \vecs{F} \cdot d\vecs{s} = 0.4687$ units of work

44. [T] Use a computer algebra system to find the mass of a wire that lies along the curve $\vecs{r}(t) = (t^2 - 1)\ \hat{\mathbf{j}} + 2t\ \hat{\mathbf{k}}$, where $0 \le t \le 1$, if the density is given by $d(t) = \frac{3}{2}t$.

45. Find the circulation and flux of field $\vecs{F}(x, y) = -y\ \hat{\mathbf{i}} + x\ \hat{\mathbf{j}}$ around and across the closed semicircular path that consists of semicircular arch $\vecs{r}_1(t) = (a \cos t)\ \hat{\mathbf{i}} + (a \sin t)\ \hat{\mathbf{j}}, \quad 0 \le t \le \pi$, followed by line segment $\vecs{r}_2(t) = t\ \hat{\mathbf{i}}, \quad -a \le t \le a$.



Answer

circulation $= \pi a^2$ and flux $= 0$

46. Compute $\int_C \cos x \cos y\ dx - \sin x \sin y\ dy$, where $\vecs{r}(t) = \langle t, t^2 \rangle, \quad 0 \le t \le 1$.

47. Complete the proof of the theorem titled THE PATH INDEPENDENCE TEST FOR CONSERVATIVE FIELDS by showing that $f_y = Q(x, y)$.

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15.4: GREEN'S THEOREM

Learning Objectives

- Apply the circulation form of Green's theorem.
- Apply the flux form of Green's theorem.
- Calculate circulation and flux on more general regions.

In this section, we examine Green's theorem, which is an extension of the Fundamental Theorem of Calculus to two dimensions. Green's theorem has two forms: a circulation form and a flux form, both of which require region D in the double integral to be simply connected. However, we will extend Green's theorem to regions that are not simply connected.

Put simply, Green's theorem relates a line integral around a simply closed plane curve C and a double integral over the region enclosed by C. The theorem is useful because it allows us to translate difficult line integrals into more simple double integrals, or difficult double integrals into more simple line integrals.

Extending the Fundamental Theorem of Calculus

Recall that the Fundamental Theorem of Calculus says that

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

As a geometric statement, this equation says that the integral over the region below the graph of $F\,'(x)$ and above the line segment $[a, b]$ depends only on the value of F at the endpoints a and b of that segment. Since the numbers a and b are the boundary of the line segment $[a, b]$, the theorem says we can calculate integral $\int_a^b F\,'(x)\;dx$ based on information about the boundary of line segment $[a, b]$ (Figure 15.4.1). The same idea is true of the Fundamental Theorem for Line Integrals:

$$\int_C \vecs{\nabla} f \cdot d\vecs{r} = f(\vecs{r}(b)) - f(\vecs{r}(a)).$$

When we have a potential function (an "antiderivative"), we can calculate the line integral based solely on information about the boundary of curve C.

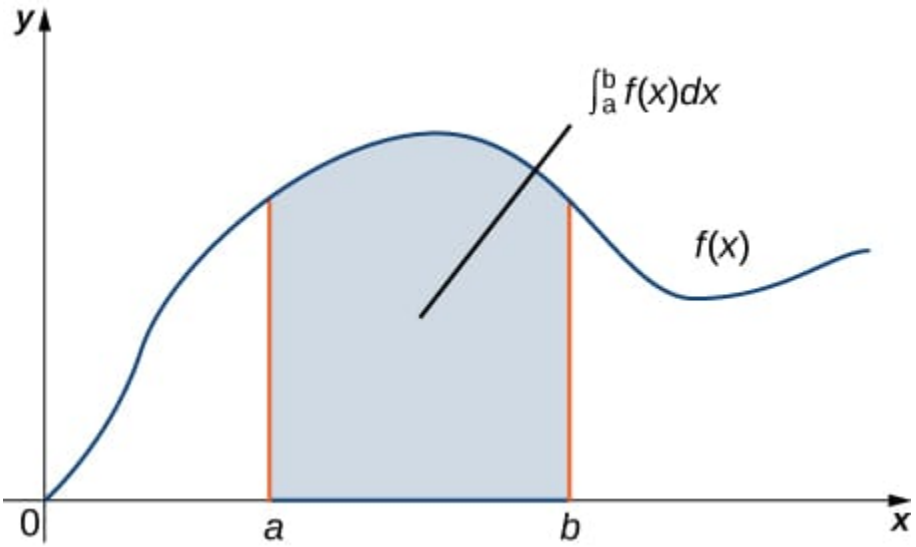


Figure 15.4.1: The Fundamental Theorem of Calculus says that the integral over line segment $[a, b]$ depends only on the values of the antiderivative at the endpoints of $[a, b]$.

Green's theorem takes this idea and extends it to calculating double integrals. Green's theorem says that we can calculate a double integral over region D based solely on information about the boundary of D. Green's theorem also says we can calculate a line integral over a simple closed curve C based solely on information about the region that C encloses. In particular, Green's theorem connects a double integral over region D to a line integral around the boundary of D.

Circulation Form of Green's Theorem

The first form of Green's theorem that we examine is the circulation form. This form of the theorem relates the vector line integral over a simple, closed plane curve C to a double integral over the region enclosed by C. Therefore, the circulation of a vector field along a simple closed curve can be transformed into a double integral and vice versa.

GREEN'S THEOREM (CIRCULATION FORM)

Let D be an open, simply connected region with a boundary curve C that is a piecewise smooth, simple closed curve oriented counterclockwise (Figure 15.4.2). Let $\vecs{F} = \langle P, Q \rangle$ be a vector field with component functions that have continuous partial derivatives on D. Then,

$$\oint_C \vecs{F} \cdot d\vecs{r} = \oint_C P\,dx + Q\,dy \tag{15.4.1}$$

$$= \iint_D (Q_x - P_y)\,dA. \tag{15.4.2}$$

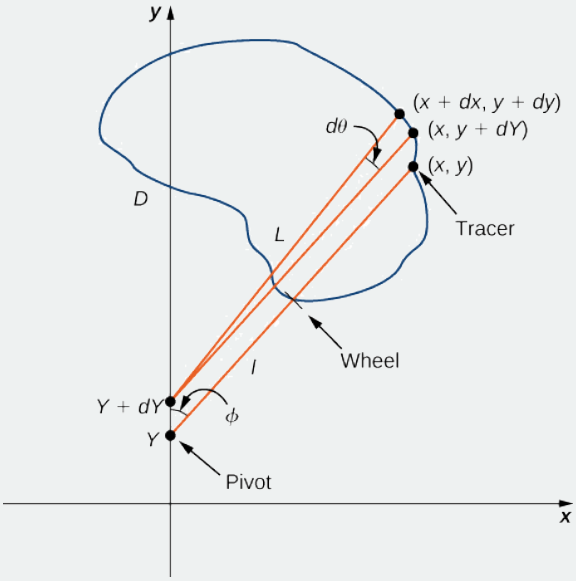


Figure 15.4.2: The circulation form of Green's theorem relates a line integral over curve C to a double integral over region D.

Notice that Green's theorem can be used only for a two-dimensional vector field $\vecs{F}$. If $\vecs{F}$ is a three-dimensional field, then Green's theorem does not apply. Since

$$\int_C P\,dx + Q\,dy = \int_C \vecs{F} \cdot \vecs{T}\,ds$$

this version of Green's theorem is sometimes referred to as the ***tangential form* of Green's theorem.**

The proof of Green's theorem is rather technical, and beyond the scope of this text. Here we examine a proof of the theorem in the special case that D is a rectangle. For now, notice that we can quickly confirm that the theorem is true for the special case in which $\vecs{F} = \langle P, Q \rangle$ is conservative. In this case,

$$\oint_C P\,dx + Q\,dy = 0$$

because the circulation is zero in conservative vector fields. $\vecs F$ satisfies the cross-partial condition, so $P_y = Q_x$. Therefore,

$$\iint_D (Q_x - P_y)\, dA = \int_D 0\, dA = 0 = \oint_C P\, dx + Q\, dy$$

which confirms Green's theorem in the case of **conservative vector fields**.

Proof

Let's now prove that the circulation form of Green's theorem is true when the region D is a rectangle. Let D be the rectangle $[a, b] \times [c, d]$ oriented counterclockwise. Then, the boundary C of D consists of four piecewise smooth pieces C_1, C_2, C_3, and C_4 (Figure 15.4.3). We parameterize each side of D as follows:

$$C_1 : \vecs r_1(t) = \langle t, c\rangle,\ a \le t \le b$$

$$C_2 : \vecs r_2(t) = \langle b, t\rangle,\ c \le t \le d$$

$$-C_3 : \vecs r_3(t) = \langle t, d\rangle,\ a \le t \le b$$

$$-C_4 : \vecs r_4(t) = \langle a, t\rangle,\ c \le t \le d.$$

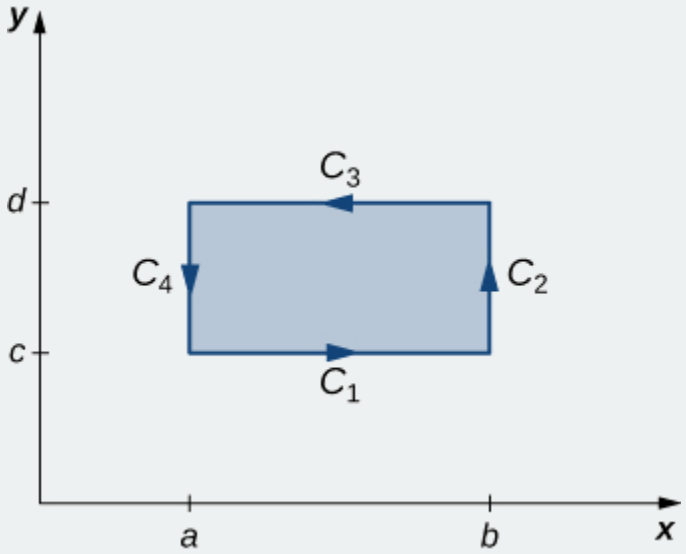


Figure 15.4.3: Rectangle D is oriented counterclockwise.

Then,

$$\begin{aligned}
\int_C \vecs F \cdot d\vecs r &= \int_{C_1} \vecs F \cdot d\vecs r + \int_{C_2} \vecs F \cdot d\vecs r + \int_{C_3} \vecs F \cdot d\vecs r + \int_{C_4} \vecs F \cdot d\vecs r \\
&= \int_{C_1} \vecs F \cdot d\vecs r + \int_{C_2} \vecs F \cdot d\vecs r - \int_{-C_3} \vecs F \cdot d\vecs r - \int_{-C_4} \vecs F \cdot d\vecs r \\
&= \int_a^b \vecs F(\vecs r_1(t)) \cdot \vecs r_1'(t)\, dt + \int_c^d \vecs F(\vecs r_2(t)) \cdot \vecs r_2'(t)\, dt - \int_a^b \vecs F(\vecs r_3(t)) \cdot \vecs r_3'(t)\, dt - \int_c^d \vecs F(\vecs r_4(t)) \cdot \vecs r_4'(t)\, dt \\
&= \int_a^b P(t, c)\, dt + \int_c^d Q(b, t)\, dt - \int_a^b P(t, d)\, dt - \int_c^d Q(a, t)\, dt \\
&= \int_a^b (P(t, c) - P(t, d))\, dt + \int_c^d (Q(b, t) - Q(a, t))\, dt \\
&= -\int_a^b (P(t, d) - P(t, c))\, dt + \int_c^d (Q(b, t) - Q(a, t))\, dt.
\end{aligned}$$

By the Fundamental Theorem of Calculus,

$$P(t,d)-P(t,c)=\int_c^d \frac{\partial}{\partial y}P(t,y)dy$$

and

$$Q(b,t)-Q(a,t)=\int_a^b \frac{\partial}{\partial x}Q(x,t)\,dx.$$

Therefore,

$$-\int_a^b (P(t,d)-P(t,c))\,dt+\int_c^d (Q(b,t)-Q(a,t))\,dt=-\int_a^b\int_c^d \frac{\partial}{\partial y}P(t,y)\,dy\,dt+\int_c^d\int_a^b \frac{\partial}{\partial x}Q(x,t)\,dx\,dt.$$

But,

$$\begin{aligned}-\int_a^b\int_c^d \frac{\partial}{\partial y}P(t,y)\,dy\,dt+\int_c^d\int_a^b \frac{\partial}{\partial x}Q(x,t)\,dx\,dt&=-\int_a^b\int_c^d \frac{\partial}{\partial y}P(x,y)\,dy\,dx+\int_c^d\int_a^b \frac{\partial}{\partial x}Q(x,y)\,dx\,dy\\&=\int_a^b\int_c^d (Q_x-P_y)\,dy\,dx\\&=\iint_D (Q_x-P_y)\,dA.\end{aligned}$$

Therefore, $\int_C \vecs{F}\cdot d\vecs{r}=\iint_D (Q_x-P_y)\ dA$ and we have proved Green's theorem in the case of a rectangle.

□

To prove Green's theorem over a general region D, we can decompose D into many tiny rectangles and use the proof that the theorem works over rectangles. The details are technical, however, and beyond the scope of this text.

Example 15.4.1: Applying Green's Theorem over a Rectangle

Calculate the line integral

$$\oint_C x^2y\,dx+(y-3)\,dy,$$

where C is a rectangle with vertices $(1,1)$, $(4,1)$, $(4,5)$, and $(1,5)$ oriented counterclockwise.

Solution

Let $\vecs{F}(x,y)=\langle P(x,y),Q(x,y)\rangle=\langle x^2y,y-3\rangle$. Then, $Q_x(x,y)=0$ and $P_y(x,y)=x^2$. Therefore, $Q_x-P_y=-x^2$.

Let D be the rectangular region enclosed by C (Figure 15.4.4). By Green's theorem,

$$\begin{aligned}\oint_C x^2y\,dx+(y-3)\,dy&=\iint_D (Q_x-P_y)\,dA\\&=\iint_D -x^2\,dA=\int_1^5\int_1^4 -x^2\,dx\,dy\\&=\int_1^5 -\frac{x^3}{3}\Big|_1^4\,dy\\&=\int_1^5\left(-\frac{64}{3}+\frac{1}{3}\right)\,dy\\&=\int_1^5 -21\,dy=-84.\end{aligned}$$

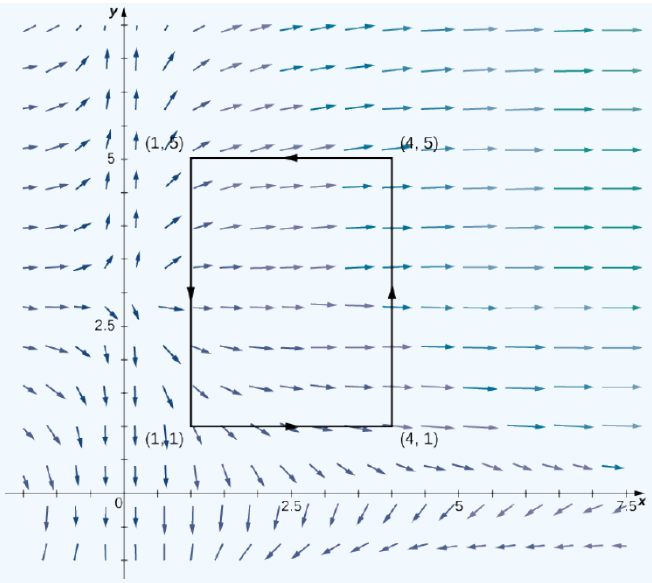


Figure 15.4.4: The line integral over the boundary of the rectangle can be transformed into a double integral over the rectangle.

Analysis

If we were to evaluate this line integral without using Green's theorem, we would need to parameterize each side of the rectangle, break the line integral into four separate line integrals, and use the methods from the section titled **Line Integrals** to evaluate each integral. Furthermore, since the vector field here is not conservative, we cannot apply the Fundamental Theorem for Line Integrals. Green's theorem makes the calculation much simpler.

Example 15.4.2: Applying Green's Theorem to Calculate Work

Calculate the work done on a particle by force field

$$\vecs{F}(x,y) = \langle y + \sin x, e^y - x \rangle$$

as the particle traverses circle $x^2 + y^2 = 4$ exactly once in the counterclockwise direction, starting and ending at point $(2, 0)$.

Solution

Let C denote the circle and let D be the disk enclosed by C. The work done on the particle is

$$W = \oint_C (y + \sin x)\, dx + (e^y - x)\, dy.$$

As with Example 15.4.1, this integral can be calculated using tools we have learned, but it is easier to use the double integral given by Green's theorem (Figure 15.4.5).

Let $\vecs{F}(x,y) = \langle P(x,y), Q(x,y) \rangle = \langle y + \sin x, e^y - x \rangle$. Then, $Q_x = -1$ and $P_y = 1$. Therefore, $Q_x - P_y = -2$.

By Green's theorem,

$$\begin{aligned} W &= \oint_C (y + \sin(x))dx + (e^y - x)\, dy \\ &= \iint_D (Q_x - P_y)\, dA \\ &= \iint_D -2\, dA \\ &= -2(area(D)) = -2\pi(2^2) = -8\pi. \end{aligned}$$

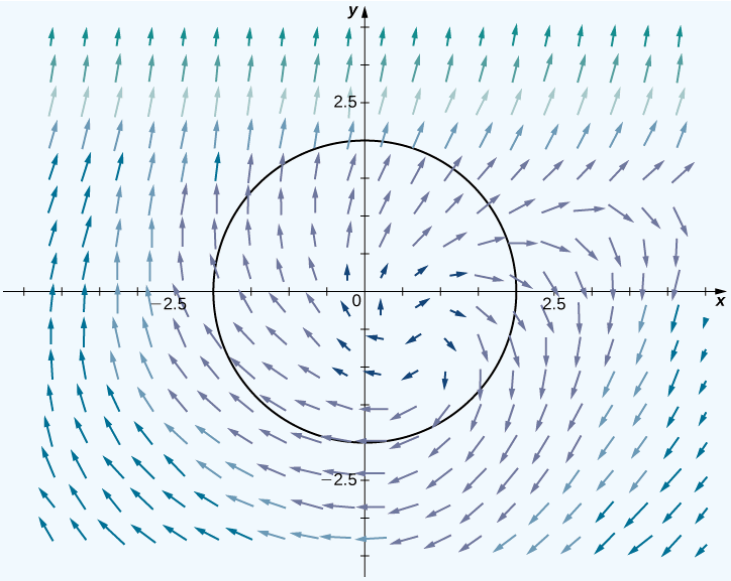


Figure 15.4.5: The line integral over the boundary circle can be transformed into a double integral over the disk enclosed by the circle.

Exercise 15.4.2

Use Green's theorem to calculate line integral

$$\oint_C \sin(x^2)\,dx + (3x - y)\,dy.$$

where C is a right triangle with vertices $(-1, 2)$, $(4, 2)$, and $(4, 5)$ oriented counterclockwise.

Hint available at the end of this section.

In the preceding two examples, the double integral in Green's theorem was easier to calculate than the line integral, so we used the theorem to calculate the line integral. In the next example, the double integral is more difficult to calculate than the line integral, so we use Green's theorem to translate a double integral into a line integral.

Example 15.4.3: Applying Green's Theorem over an Ellipse

Calculate the area enclosed by ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (Figure 15.4.6).

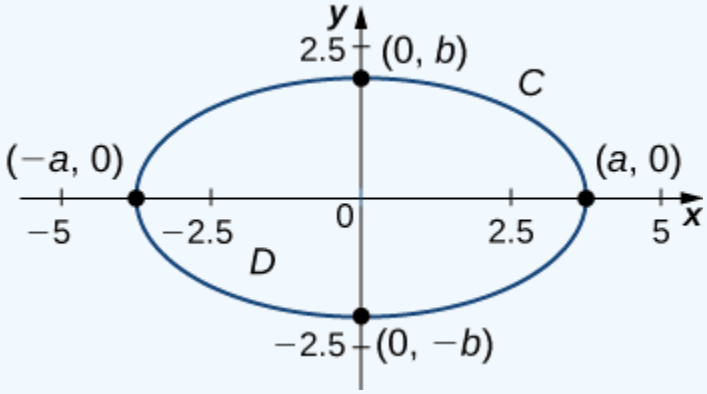


Figure 15.4.6: Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is denoted by C.

Solution

Let C denote the ellipse and let D be the region enclosed by C. Recall that ellipse C can be parameterized by

- $x = a\cos t$,
- $y = b\sin t$,
- $0 \le t \le 2\pi$.

Calculating the area of D is equivalent to computing double integral $\iint_D \, dA$. To calculate this integral without Green's theorem, we would need to divide D into two regions: the region above the *x*-axis and the region below. The area of the ellipse is

$$\int_{-a}^{a}\int_{0}^{\sqrt{b^2-(bx/a)^2}} dy\,dx + \int_{-a}^{a}\int_{-\sqrt{b^2-(bx/a)^2}}^{0} dy\,dx.$$

These two integrals are not straightforward to calculate (although when we know the value of the first integral, we know the value of the second by symmetry). Instead of trying to calculate them, we use Green's theorem to transform $\iint_D \, dA$ into a line integral around the boundary C.

Consider vector field

$$\vecs{F}(x,y) = \langle P, Q\rangle = \langle -\frac{y}{2}, \frac{x}{2}\rangle.$$

Then, $Q_x = \frac{1}{2}$ and $P_y = -\frac{1}{2}$, and therefore $Q_x - P_y = 1$. Notice that $\vecs{F}$ was chosen to have the property that $Q_x - P_y = 1$. Since this is the case, Green's theorem transforms the line integral of $\vecs{F}$ over C into the double integral of 1 over D.

By Green's theorem,

$$\begin{aligned}\iint_D dA &= \iint_D (Q_x - P_y)\,dA \\ &= \int_C \vecs{F}\cdot d\vecs{r} = \frac{1}{2}\int_C -y\,dx + x\,dy \\ &= \frac{1}{2}\int_0^{2\pi} -b\sin t(-a\sin t) + a(\cos t)b\cos t\,dt \\ &= \frac{1}{2}\int_0^{2\pi} ab\cos^2 t + ab\sin^2 t\,dt \\ &= \frac{1}{2}\int_0^{2\pi} ab\,dt = \pi ab.\end{aligned}$$

Therefore, the area of the ellipse is $\pi ab \text{ units}^2$.

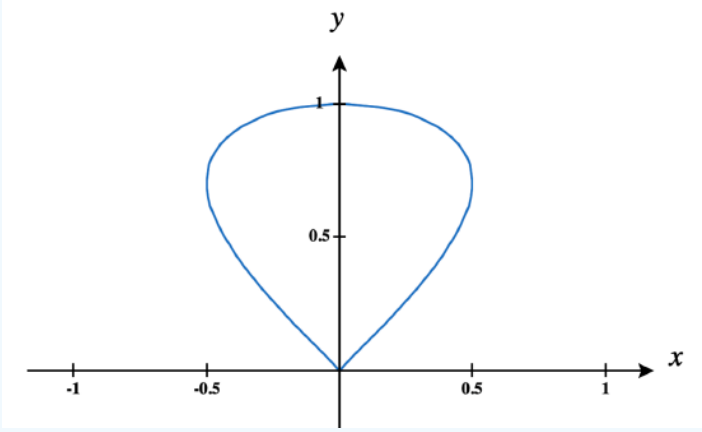
In Example 15.4.3, we used vector field $\vecs{F}(x,y) = \langle P, Q\rangle = \langle -\frac{y}{2}, \frac{x}{2}\rangle$ to find the area of any ellipse. The logic of the previous example can be extended to derive a formula for the area of any region D. Let D be any region with a boundary that is a simple closed curve C oriented counterclockwise. If $\vecs{F}(x,y) = \langle P, Q\rangle = \langle -\frac{y}{2}, \frac{x}{2}\rangle$, then $Q_x - P_y = 1$. Therefore, by the same logic as in Example 15.4.3,

$$\text{area of } D = \iint_D dA = \frac{1}{2}\oint_C -y\,dx + x\,dy. \tag{15.4.3}$$

It's worth noting that if $\vecs{F} = \langle P, Q\rangle$ is any vector field with $Q_x - P_y = 1$, then the logic of the previous paragraph works. So, Equation 15.4.3 is not the only equation that uses a vector field's mixed partials to get the area of a region.

Exercise 15.4.3

Find the area of the region enclosed by the curve with parameterization $\vecs{r}(t) = \langle \sin t\cos t, \sin t\rangle$, $0 \le t \le \pi$.



Hint available at the end of this section.

Flux Form of Green's Theorem

The circulation form of Green's theorem relates a double integral over region D to line integral $\oint_C \vecs{F} \cdot \vecs{T} ds$, where C is the boundary of D. The flux form of Green's theorem relates a double integral over region D to the flux across boundary C. The flux of a fluid across a curve can be difficult to calculate using the flux line integral. This form of Green's theorem allows us to translate a difficult flux integral into a double integral that is often easier to calculate.

GREEN'S THEOREM (FLUX FORM)

Let D be an open, simply connected region with a boundary curve C that is a piecewise smooth, simple closed curve that is oriented counterclockwise (Figure 15.4.7). Let $\vecs{F} = \langle P, Q \rangle$ be a vector field with component functions that have continuous partial derivatives on an open region containing D. Then,

$$\oint_C \vecs{F} \cdot \vecs{N}\, ds = \iint_D P_x + Q_y \, dA. \tag{15.4.4}$$

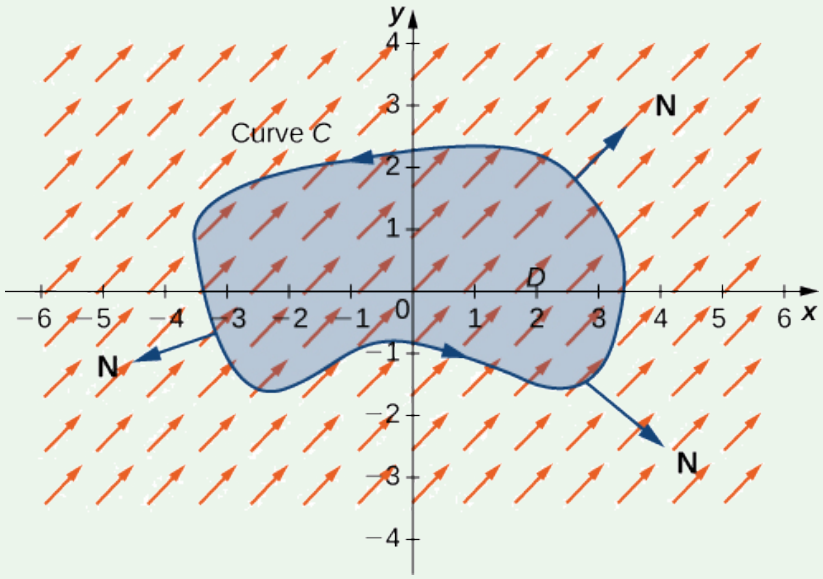


Figure 15.4.7: The flux form of Green's theorem relates a double integral over region D to the flux across curve C.

Because this form of Green's theorem contains unit normal vector $\vecs{N}$, it is sometimes referred to as the *normal form* of Green's theorem.

Proof

Recall that $\oint_C \vecs{F} \cdot \vecs{N}\ ds = \oint_C -Q\ dx + P\ dy$. Let $M = -Q$ and $N = P$. By the circulation form of Green's theorem,

$$\begin{aligned} \oint_C -Q\,dx + P\,dy &= \oint_C M\,dx + N\,dy \\ &= \iint_D N_x - M_y\,dA \\ &= \iint_D P_x - (-Q)_y\,dA \\ &= \iint_D P_x + Q_y\,dA. \end{aligned}$$

□

Example 15.4.4A: Applying Green's Theorem for Flux across a Circle

Let C be a circle of radius r centered at the origin (Figure 15.4.8) and let $\vecs{F}(x, y) = \langle x, y \rangle$. Calculate the flux across C.

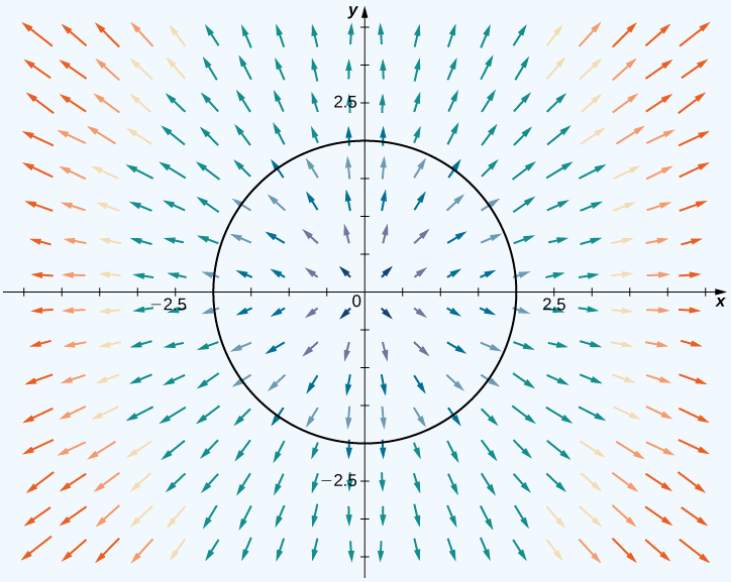


Figure 15.4.8: Curve C is a circle of radius r centered at the origin.

Solution

Let D be the disk enclosed by C. The flux across C is $\oint_C \vecs{F} \cdot \vecs{N}\ ds$. We could evaluate this integral using tools we have learned, but Green's theorem makes the calculation much more simple. Let $P(x, y) = x$ and $Q(x, y) = y$ so that $\vecs{F} = \langle P, Q \rangle$. Note that $P_x = 1 = Q_y$, and therefore $P_x + Q_y = 2$. By Green's theorem,

$$\int_C \vecs{F} \cdot \vecs{N}\,ds = \iint_D 2\,dA = 2 \iint_D dA.$$

Since $\iint_D dA$ is the area of the circle, $\iint_D dA = \pi r^2$. Therefore, the flux across C is $2\pi r^2$.

Example $15.4.4B$: Applying Green's Theorem for Flux across a Triangle

Let S be the triangle with vertices $(0,0)$, $(1,0)$, and $(0,3)$ oriented clockwise (Figure $15.4.9$). Calculate the flux of $\vecs{F}(x,y) = \langle P(x,y), Q(x,y)\rangle = \langle x^2 + e^y, x + y\rangle$ across S.

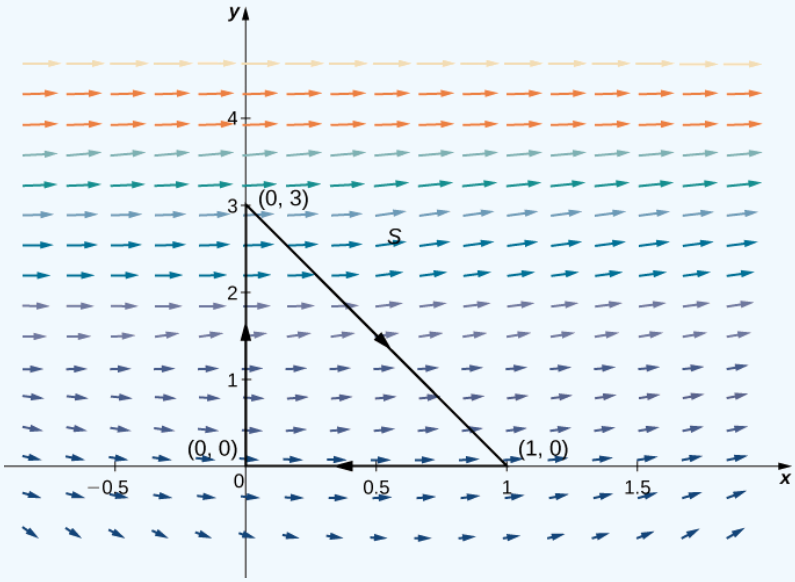


Figure $15.4.9$: Curve S is a triangle with vertices $(0,0)$, $(1,0)$, and $(0,3)$ oriented clockwise.

Solution

To calculate the flux without Green's theorem, we would need to break the flux integral into three line integrals, one integral for each side of the triangle. Using Green's theorem to translate the flux line integral into a single double integral is much more simple.

Let D be the region enclosed by S. Note that $P_x = 2x$ and $Q_y = 1$; therefore, $P_x + Q_y = 2x + 1$. Green's theorem applies only to simple closed curves oriented counterclockwise, but we can still apply the theorem because $\oint_C \vecs{F} \cdot \vecs{N}\ ds = -\oint_{-S} \vecs{F} \cdot \vecs{N}\ ds$ and $-S$ is oriented counterclockwise. By Green's theorem, the flux is

$$\begin{aligned} \oint_C \vecs{F} \cdot \vecs{N}\, ds &= \oint_{-S} \vecs{F} \cdot \vecs{N}\, ds \\ &= -\iint_D (P_x + Q_y)\, dA \\ &= -\iint_D (2x+1)\, dA. \end{aligned}$$

Notice that the top edge of the triangle is the line $y = -3x + 3$. Therefore, in the iterated double integral, the y-values run from $y = 0$ to $y = -3x + 3$, and we have

$$
\begin{aligned}
-\iint_D (2x+1)\,dA &= -\int_0^1 \int_0^{-3x+3} (2x+1)\,dy\,dx \\
&= -\int_0^1 (2x+1)y\Big|_0^{-3x+3}\,dx \\
&= -\int_0^1 (2x+1)(-3x+3)\,dx \\
&= -\int_0^1 (-6x^2+3x+3)\,dx \\
&= -\left[-2x^3+\frac{3x^2}{2}+3x\right]\Big|_0^1 \\
&= -\frac{5}{2}.
\end{aligned}
$$

Exercise 15.4.4

Calculate the flux of $\vecs F(x,y) = \langle x^3, y^3\rangle$ across a unit circle oriented counterclockwise.

Hint available at the end of this section.

Example 15.4.5: Applying Green's Theorem for Water Flow across a Rectangle

Water flows from a spring located at the origin. The velocity of the water is modeled by vector field $\vecs v(x,y) = \langle 5x+y, x+3y\rangle$ m/sec. Find the amount of water per second that flows across the rectangle with vertices $(-1,-2)$, $(1,-2)$, $(1,3)$,and $(-1,3)$, oriented counterclockwise (Figure 15.4.10).

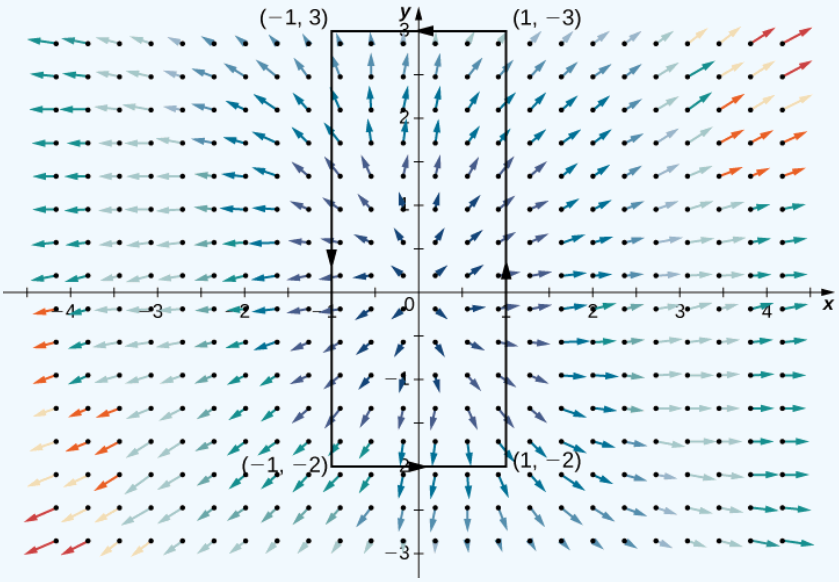


Figure 15.4.10: Water flows across the rectangle with vertices $(-1,-2)$, $(1,-2)$, $(1,3)$,and $(-1,3)$, oriented counterclockwise.

Solution

Let C represent the given rectangle and let D be the rectangular region enclosed by C. To find the amount of water flowing across C, we calculate flux $\int_C \vecs v \cdot d\vecs r$. Let $P(x,y) = 5x+y$ and $Q(x,y) = x+3y$ so that $\vecs v = \langle P, Q\rangle$. Then, $P_x = 5$ and $Q_y = 3$. By Green's theorem,

$$\int_C \vecs{v} \cdot d\vecs{r} = \iint_D (P_x + Q_y)\, dA$$
$$= \iint_D 8\, dA$$
$$= 8(area\ of\ D) = 80.$$

Therefore, the water flux is 80 m^2/sec.

Recall that if vector field $\vecs{F}$ is conservative, then $\vecs{F}$ does no work around closed curves—that is, the circulation of $\vecs{F}$ around a closed curve is zero. In fact, if the domain of $\vecs{F}$ is simply connected, then $\vecs{F}$ is conservative if and only if the circulation of $\vecs{F}$ around any closed curve is zero. If we replace "circulation of $\vecs{F}$" with "flux of $\vecs{F}$," then we get a definition of a source-free vector field. The following statements are all equivalent ways of defining a source-free field $\vecs{F} = \langle P, Q \rangle$ on a simply connected domain (note the similarities with properties of conservative vector fields):

1. The flux $\oint_C \vecs{F} \cdot \vecs{N}\ ds$ across any closed curve C is zero.
2. If C_1 and C_2 are curves in the domain of $\vecs{F}$ with the same starting points and endpoints, then $\int_{C_1} \vecs{F} \cdot \vecs{N}\ ds = \int_{C_2} \vecs{F} \cdot \vecs{N}\ ds$. In other words, flux is independent of path.
3. There is a stream function $g(x, y)$ for $\vecs{F}$. A stream function for $\vecs{F} = \langle P, Q \rangle$ is a function g such that $P = g_y$ and $Q = -g_x$.Geometrically, $\vecs{F} = \langle a, b \rangle$ is tangential to the level curve of g at (a, b). Since the gradient of g is perpendicular to the level curve of g at (a, b), stream function g has the property $\vecs{F}(a, b) \cdot \vecs{\nabla} g(a, b) = 0$ for any point (a, b) in the domain of g. (Stream functions play the same role for source-free fields that potential functions play for conservative fields.)
4. $P_x + Q_y = 0$

Example 15.4.6: Finding a Stream Function

Verify that rotation vector field $\vecs{F}(x, y) = \langle y, -x \rangle$ is source free, and find a stream function for $\vecs{F}$.

Solution

Note that the domain of $\vecs{F}$ is all of $\mathbb{R}^2$, which is simply connected. Therefore, to show that $\vecs{F}$ is source free, we can show any of items 1 through 4 from the previous list to be true. In this example, we show that item 4 is true. Let $P(x, y) = y$ and $Q(x, y) = -x$. Then $P_x = 0 = Q_y$, and therefore $P_x + Q_y = 0$. Thus, $\vecs{F}$ is source free.

To find a stream function for $\vecs{F}$, proceed in the same manner as finding a potential function for a conservative field. Let g be a stream function for $\vecs{F}$. Then $g_y = y$, which implies that

$$g(x, y) = \frac{y^2}{2} + h(x).$$

Since $-g_x = Q = -x$, we have $h\,'(x) = x$. Therefore,

$$h(x) = \frac{x^2}{2} + C.$$

Letting $C = 0$ gives stream function

$$g(x, y) = \frac{x^2}{2} + \frac{y^2}{2}.$$

To confirm that g is a stream function for $\vecs{F}$, note that $g_y = y = P$ and $-g_x = -x = Q$.

Notice that source-free rotation vector field $\vecs{F}(x, y) = \langle y, -x \rangle$ is perpendicular to conservative radial vector field $\vecs{\nabla} g = \langle x, y \rangle$ (Figure 15.4.11).

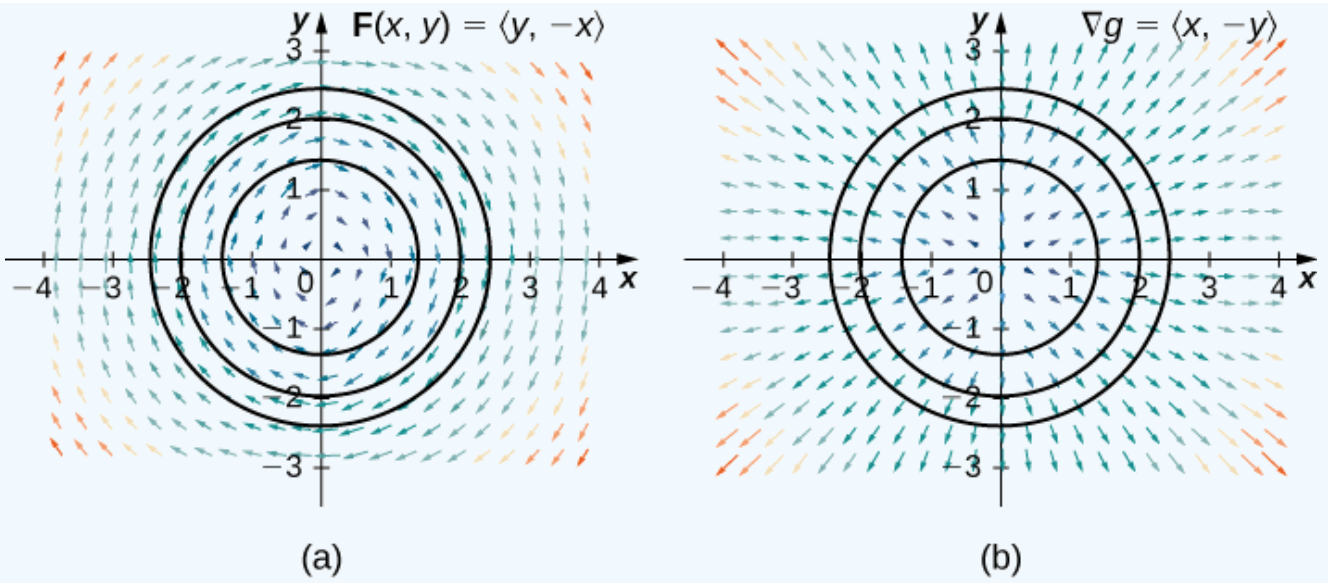


Figure 15.4.11: (a) In this image, we see the three-level curves of g and vector field $\vecs F$. Note that the $\vecs F$ vectors on a given level curve are tangent to the level curve. (b) In this image, we see the three-level curves of g and vector field $\vecs \nabla g$. The gradient vectors are perpendicular to the corresponding level curve. Therefore, $\vecs F(a,b) \cdot \vecs \nabla g(a,b) = 0$ for any point in the domain of g.

Exercise 15.4.6

Find a stream function for vector field $\vecs F(x,y) = \langle x \sin y, \cos y \rangle$.

Hint available at the end of this section.

Vector fields that are both conservative and source free are important vector fields. One important feature of conservative and source-free vector fields on a simply connected domain is that any potential function f of such a field satisfies Laplace's equation $f_{xx} + f_{yy} = 0$. Laplace's equation is foundational in the field of partial differential equations because it models such phenomena as gravitational and magnetic potentials in space, and the velocity potential of an ideal fluid. A function that satisfies Laplace's equation is called a *harmonic* function. Therefore any potential function of a conservative and source-free vector field is harmonic.

To see that any potential function of a conservative and source-free vector field on a simply connected domain is harmonic, let f be such a potential function of vector field $\vecs F = \langle P, Q \rangle$. Then, $f_x = P$ and $f_x = Q$ because $\vecs \nabla f = \vecs F$. Therefore, $f_{xx} = P_x$ and $f_{yy} = Q_y$. Since $\vecs F$ is source free, $f_{xx} + f_{yy} = P_x + Q_y = 0$, and we have that f is harmonic.

Example 15.4.7: Satisfying Laplace's Equation

For vector field $\vecs F(x,y) = \langle e^x \sin y, e^x \cos y \rangle$, verify that the field is both conservative and source free, find a potential function for $\vecs F$, and verify that the potential function is harmonic.

Solution

Let $P(x,y) = e^x \sin y$ and $Q(x,y) = e^x \cos y$. Notice that the domain of $\vecs F$ is all of two-space, which is simply connected. Therefore, we can check the cross-partials of $\vecs F$ to determine whether $\vecs F$ is conservative. Note that $P_y = e^x \cos y = Q_x$, so $\vecs F$ is conservative. Since $P_x = e^x \sin y$ and $Q_y = e^x \sin y$, $P_x + Q_y = 0$ and the field is source free.

To find a potential function for $\vecs F$, let f be a potential function. Then, $\vecs \nabla f = \vecs F$, so $f_x(x,y) = e^x \sin y$. Integrating this equation with respect to *x* gives $f(x,y) = e^x \sin y + h(y)$. Since $f_y(x,y) = e^x \cos y$, differentiating f with respect to *y* gives $e^x \cos y = e^x \cos y + h\,'(y)$. Therefore, we can take $h(y) = 0$, and $f(x,y) = e^x \sin y$ is a potential function for f.

To verify that f is a harmonic function, note that $f_{xx}(x,y) = \dfrac{\partial}{\partial x}(e^x \sin y) = e^x \sin y$ and

$f_{yy}(x,y) = \dfrac{\partial}{\partial x}(e^x \cos y) = -e^x \sin y$. Therefore, $f_{xx} + f_{yy} = 0$, and f satisfies Laplace's equation.

Exercise 15.4.7

Is the function $f(x,y) = e^{x+5y}$ harmonic?

Hint available at the end of this section.

Green's Theorem on General Regions

Green's theorem, as stated, applies only to regions that are simply connected—that is, Green's theorem as stated so far cannot handle regions with holes. Here, we extend Green's theorem so that it does work on regions with finitely many holes (Figure 15.4.12).

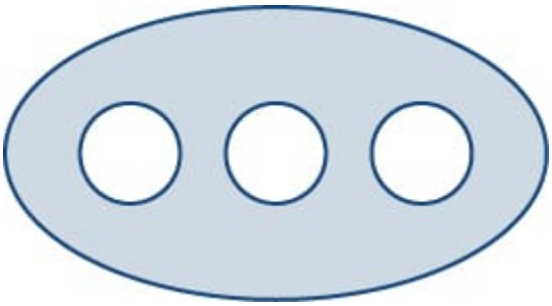


Figure 15.4.12: Green's theorem, as stated, does not apply to a nonsimply connected region with three holes like this one.

Before discussing extensions of Green's theorem, we need to go over some terminology regarding the boundary of a region. Let D be a region and let C be a component of the boundary of D. We say that C is *positively oriented* if, as we walk along C in the direction of orientation, region D is always on our left. Therefore, the counterclockwise orientation of the boundary of a disk is a positive orientation, for example. Curve C is *negatively oriented* if, as we walk along C in the direction of orientation, region D is always on our right. The clockwise orientation of the boundary of a disk is a negative orientation, for example.

Let D be a region with finitely many holes (so that D has finitely many boundary curves), and denote the boundary of D by ∂D (Figure 15.4.13). To extend Green's theorem so it can handle D, we divide region D into two regions, D_1 and D_2 (with respective boundaries ∂D_1 and ∂D_2), in such a way that $D = D_1 \cup D_2$ and neither D_1 nor D_2 has any holes (Figure 15.4.13).

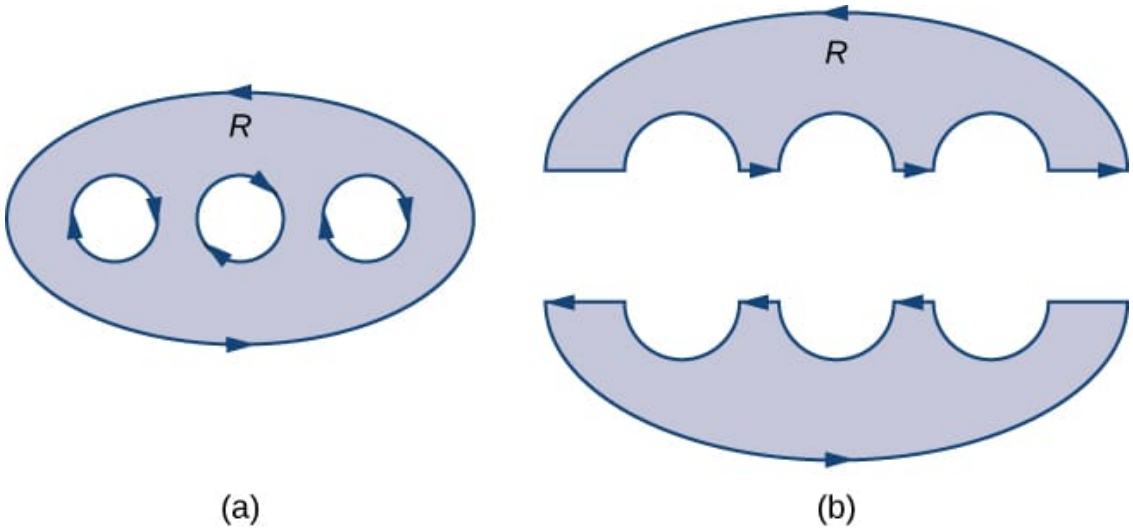


Figure 15.4.13: (a) Region D with an oriented boundary has three holes. (b) Region D split into two simply connected regions has no holes.

Assume the boundary of D is oriented as in the figure, with the inner holes given a negative orientation and the outer boundary given a positive orientation. The boundary of each simply connected region D_1 and D_2 is positively oriented. If $\vecs{F}$ is a vector field defined on D, then Green's theorem says that

$$\oint_{\partial D} \vecs{F} \cdot d\vecs{r} = \oint_{\partial D_1} \vecs{F} \cdot d\vecs{r} + \oint_{\partial D_2} \vecs{F} \cdot d\vecs{r} \tag{15.4.5}$$

$$= \iint_{D_1} Q_x - P_y \, dA + \iint_{D_2} Q_x - P_y \, dA \tag{15.4.6}$$

$$= \iint_{D} (Q_x - P_y) \, dA. \tag{15.4.7}$$

Therefore, Green's theorem still works on a region with holes.

To see how this works in practice, consider annulus D in Figure 15.4.14 and suppose that $F = \langle P, Q \rangle$ is a vector field defined on this annulus. Region D has a hole, so it is not simply connected. Orient the outer circle of the annulus counterclockwise and the inner circle clockwise (Figure 15.4.14) so that, when we divide the region into D_1 and D_2, we are able to keep the region on our left as we walk along a path that traverses the boundary. Let D_1 be the upper half of the annulus and D_2 be the lower half. Neither of these regions has holes, so we have divided D into two simply connected regions.

We label each piece of these new boundaries as P_i for some i, as in Figure 15.4.14. If we begin at P and travel along the oriented boundary, the first segment is P_1, then P_2, P_3, and P_4. Now we have traversed D_1 and returned to P. Next, we start at P again and traverse D_2. Since the first piece of the boundary is the same as P_4 in D_1, but oriented in the opposite direction, the first piece of D_2 is $-P_4$. Next, we have P_5, then $-P_2$, and finally P_6.

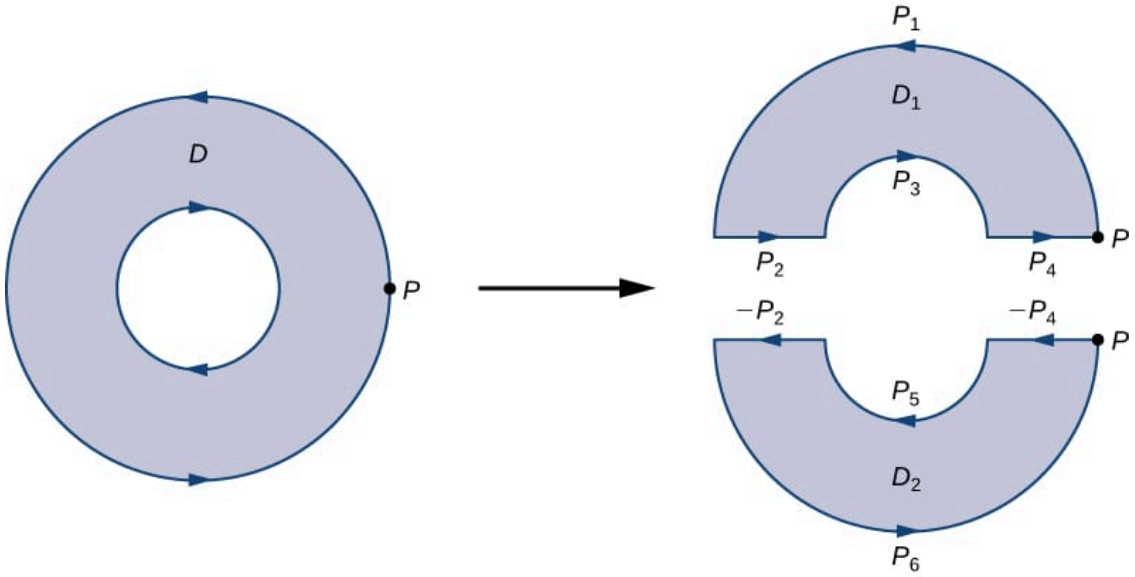


Figure 15.4.14: Breaking the annulus into two separate regions gives us two simply connected regions. The line integrals over the common boundaries cancel out.

Figure 15.4.14 shows a path that traverses the boundary of D. Notice that this path traverses the boundary of region D_1, returns to the starting point, and then traverses the boundary of region D_2. Furthermore, as we walk along the path, the region is always on our left. Notice that this traversal of the P_i paths covers the entire boundary of region D. If we had only traversed one portion of the boundary of D, then we cannot apply Green's theorem to D.

The boundary of the upper half of the annulus, therefore, is $P_1 \cup P_2 \cup P_3 \cup P_4$ and the boundary of the lower half of the annulus is $-P_4 \cup P_5 \cup -P_2 \cup P_6$. Then, Green's theorem implies

$$\begin{aligned}\oint_{\partial D} \vecs{F}\cdot d\vecs{r} &= \int_{P_1} \vecs{F}\cdot d\vecs{r} + \int_{P_2} \vecs{F}\cdot d\vecs{r} + \int_{P_3} \vecs{F}\cdot d\vecs{r} + \int_{P_4} \vecs{F}\cdot d\vecs{r} + \int_{-P_4} \vec{}\\ &= \int_{P_1} \vecs{F}\cdot d\vecs{r} + \int_{P_2} \vecs{F}\cdot d\vecs{r} + \int_{P_3} \vecs{F}\cdot d\vecs{r} + \int_{P_4} \vecs{F}\cdot d\vecs{r} + \int_{P_4} \vecs\\ &= \int_{P_1} \vecs{F}\cdot d\vecs{r} + \int_{P_3} \vecs{F}\cdot d\vecs{r} + \int_{P_5} \vecs{F}\cdot d\vecs{r} + \int_{P_6} \vecs{F}\cdot d\vecs{r}\\ &= \oint_{\partial D_1} \vecs{F}\cdot d\vecs{r} + \oint_{\partial D_2} \vecs{F}\cdot d\vecs{r}\\ &= \iint_{D_1} (Q_x - P_y)\, dA + \iint_{D_2} (Q_x - P_y)\, dA\\ &= \iint_{D} (Q_x - P_y)\, dA.\end{aligned}$$

Therefore, we arrive at the equation found in Green's theorem—namely,

$$\oint_{\partial D} \vecs{F}\cdot d\vecs{r} = \iint_{D} (Q_x - P_y)\, dA.$$

The same logic implies that the flux form of Green's theorem can also be extended to a region with finitely many holes:

$$\oint_{C} \vecs{F}\cdot \vecs{N}\, ds = \iint_{D} (P_x + Q_y)\, dA.$$

Example 15.4.8A: Using Green's Theorem on a Region with Holes

Calculate the integral

$$\oint_{\partial D} (\sin x - \frac{y^3}{3})dx + (\frac{y^3}{3} + \sin y)dy,$$

where D is the annulus given by the polar inequalities $1 \le r \le 2$, $0 \le \theta \le 2\pi$.

Solution

Although D is not simply connected, we can use the extended form of Green's theorem to calculate the integral. Since the integration occurs over an annulus, we convert to polar coordinates:

$$\begin{aligned}\oint_{\partial D} (\sin x - \frac{y^3}{3})\, dx + (\frac{x^3}{3} + \sin y)\, dy &= \iint_{D} (Q_x - P_y)\, dA\\ &= \iint_{D} (x^2 + y^2)\, dA\\ &= \int_0^{2\pi}\int_1^2 r^3\, dr d\theta = \int_0^{2\pi} \frac{15}{4}\, d\theta\\ &= \frac{15\pi}{2}.\end{aligned}$$

Example 15.4.8B: Using the Extended Form of Green's Theorem

Let $\vecs{F} = \langle P, Q\rangle = \left\langle \frac{y}{x^2+y^2}, -\frac{x}{x^2+y^2}\right\rangle$ and let C be any simple closed curve in a plane oriented counterclockwise. What are the possible values of $\oint_C \vecs{F}\cdot d\vecs{r}$?

Solution

We use the extended form of Green's theorem to show that $\oint_C \vecs{F} \cdot d\vecs{r}$ is either 0 or -2π—that is, no matter how crazy curve C is, the line integral of $\vecs{F}$ along C can have only one of two possible values. We consider two cases: the case when C encompasses the origin and the case when C does not encompass the origin.

Case 1: C Does Not Encompass the Origin

In this case, the region enclosed by C is simply connected because the only hole in the domain of $\vecs{F}$ is at the origin. We showed in our discussion of cross-partials that $\vecs{F}$ satisfies the cross-partial condition. If we restrict the domain of $\vecs{F}$ just to C and the region it encloses, then $\vecs{F}$ with this restricted domain is now defined on a simply connected domain. Since $\vecs{F}$ satisfies the cross-partial property on its restricted domain, the field $\vecs{F}$ is conservative on this simply connected region and hence the circulation $\oint_C \vecs{F} \cdot d\vecs{r}$ is zero.

Case 2: C Does Encompass the Origin

In this case, the region enclosed by C is not simply connected because this region contains a hole at the origin. Let C_1 be a circle of radius a centered at the origin so that C_1 is entirely inside the region enclosed by C (Figure $15.4.15$). Give C_1 a clockwise orientation.

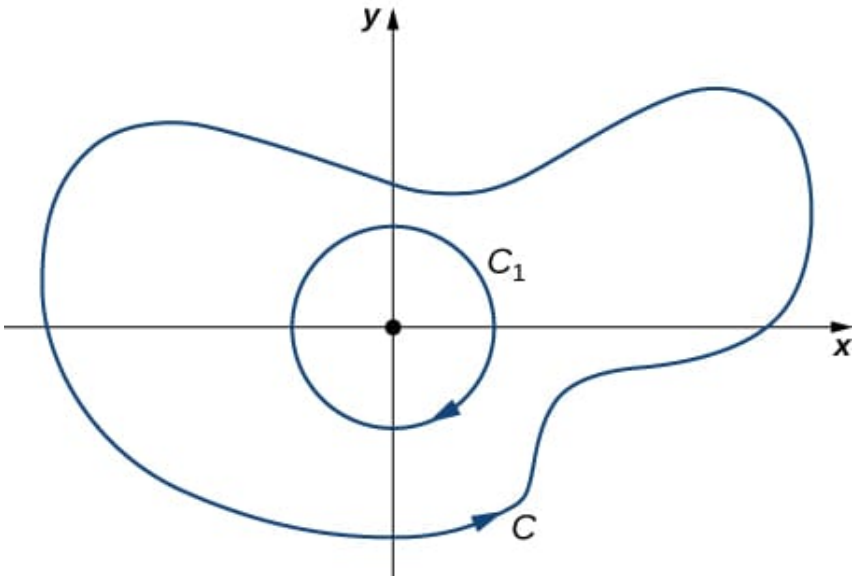


Figure $15.4.15$: Choose circle C_1 centered at the origin that is contained entirely inside C.

Let D be the region between C_1 and C, and C is orientated counterclockwise. By the extended version of Green's theorem,

$$\begin{aligned}\int_C \vecs{F} \cdot d\vecs{r} + \int_{C_1} \vecs{F} \cdot d\vecs{r} &= \iint_D Qx\_P_y \, dA \\ &= \iint_D -\frac{y^2 - x^2}{(x^2+y^2)^2} + \frac{y^2 - x^2}{(x^2+y^2)^2} dA \\ &= 0,\end{aligned}$$

and therefore

$$\int_C \vecs{F} \cdot d\vecs{r} = -\int_{C_1} \vecs{F} \cdot d\vecs{r}.$$

Since C_1 is a specific curve, we can evaluate $\int_{C_1} \vecs{F} \cdot d\vecs{r}$. Let

$$x = a\cos t, \quad y = a\sin t, \quad 0 \le t \le 2\pi$$

be a parameterization of C_1. Then,

$$\begin{aligned}\int_{C_1} \vecs{F}\cdot d\vecs{r} &= \int_0^{2\pi} \vecs{F}(\vecs{r}(t))\cdot \vecs{r}'(t)\,dt \\ &= \int_0^{2\pi} \langle -\frac{\sin(t)}{a}, -\frac{\cos(t)}{a}\rangle \cdot \langle -a\sin(t), -a\cos(t)\rangle\,dt \\ &= \int_0^{2\pi} \sin^2(t)+\cos^2(t)\,dt \\ &= \int_0^{2\pi} dt = 2\pi.\end{aligned}$$

Therefore, $\int_C \vecs{F}\cdot d\vecs{r} = -2\pi$.

Exercise 15.4.8

Calculate integral $\oint_{\partial D} \vecs{F}\cdot d\vecs{r}$, where D is the annulus given by the polar inequalities $2 \le r \le 5$, $0 \le \theta \le 2\pi$, and $\vecs{F}(x,y) = \langle x^3, 5x + e^y \sin y\rangle$.

Hint available at the end of this section.

MEASURING AREA FROM A BOUNDARY: THE PLANIMETER

Imagine you are a doctor who has just received a magnetic resonance image of your patient's brain. The brain has a tumor (Figure 15.4.16). How large is the tumor? To be precise, what is the area of the red region? The red cross-section of the tumor has an irregular shape, and therefore it is unlikely that you would be able to find a set of equations or inequalities for the region and then be able to calculate its area by conventional means. You could approximate the area by chopping the region into tiny squares (a Riemann sum approach), but this method always gives an answer with some error.

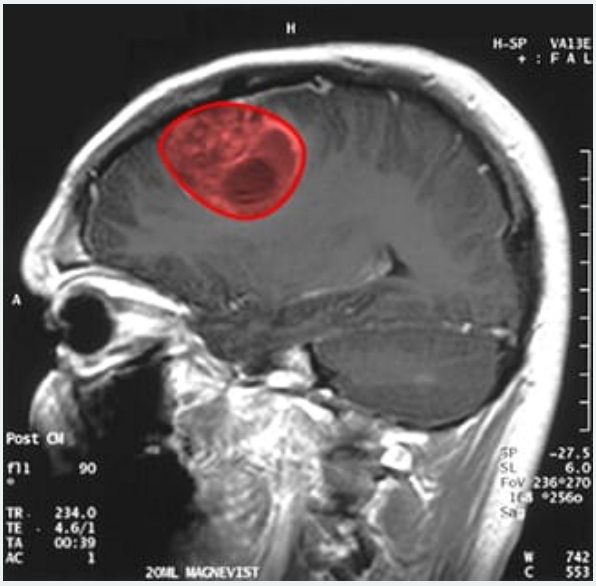


Figure 15.4.16: This magnetic resonance image of a patient's brain shows a tumor, which is highlighted in red. (credit: modification of work by Christaras A, Wikimedia Commons)

Instead of trying to measure the area of the region directly, we can use a device called a *rolling planimeter* to calculate the area of the region exactly, simply by measuring its boundary. In this project you investigate how a planimeter works, and you use Green's theorem to show the device calculates area correctly.

A rolling planimeter is a device that measures the area of a planar region by tracing out the boundary of that region (Figure 15.4.17). To measure the area of a region, we simply run the tracer of the planimeter around the boundary of the region. The planimeter measures the number of turns through which the wheel rotates as we trace the boundary; the area of the shape is proportional to this number of wheel turns. We can derive the precise proportionality equation using Green's theorem. As the tracer moves around the boundary of the region, the tracer arm rotates and the roller moves back and forth (but does not rotate).

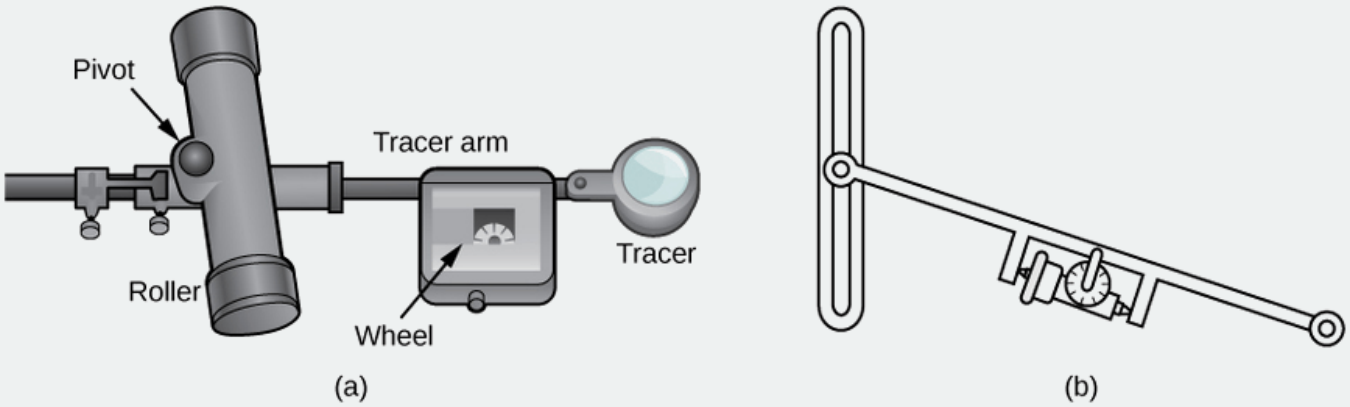


Figure 15.4.17: (a) A rolling planimeter. The pivot allows the tracer arm to rotate. The roller itself does not rotate; it only moves back and forth. (b) An interior view of a rolling planimeter. Notice that the wheel cannot turn if the planimeter is moving back and forth with the tracer arm perpendicular to the roller.

Let C denote the boundary of region D, the area to be calculated. As the tracer traverses curve C, assume the roller moves along the *y*-axis (since the roller does not rotate, one can assume it moves along a straight line). Use the coordinates (x, y) to represent points on boundary C, and coordinates $(0, Y)$ to represent the position of the pivot. As the planimeter traces C, the pivot moves along the *y*-axis while the tracer arm rotates on the pivot.

Watch a short animation of a planimeter in action.

Begin the analysis by considering the motion of the tracer as it moves from point (x, y) counterclockwise to point $(x + dx, y + dy)$ that is close to (x, y) (Figure 15.4.18). The pivot also moves, from point $(0, Y)$ to nearby point $(0, Y + dY)$. How much does the wheel turn as a result of this motion? To answer this question, break the motion into two parts. First, roll the pivot along the *y*-axis from $(0, Y)$ to $(0, Y + dY)$ without rotating the tracer arm. The tracer arm then ends up at point $(x, y + dY)$ while maintaining a constant angle ϕ with the *x*-axis. Second, rotate the tracer arm by an angle $d\theta$ without moving the roller. Now the tracer is at point $(x + dx, y + dy)$. Let ll be the distance from the pivot to the wheel and let L be the distance from the pivot to the tracer (the length of the tracer arm).

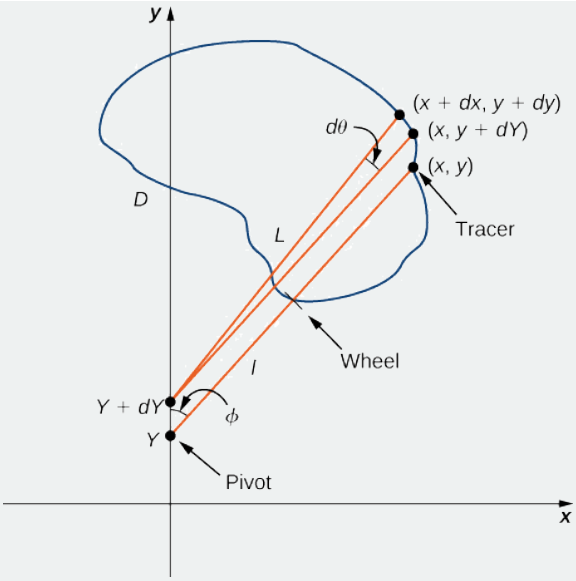


Figure 15.4.18: Mathematical analysis of the motion of the planimeter.

1. Explain why the total distance through which the wheel rolls the small motion just described is $\sin\phi dY + ld\theta = \frac{x}{L}dY + ld\theta$.
2. Show that $\oint_C d\theta = 0$.
3. Use step 2 to show that the total rolling distance of the wheel as the tracer traverses curve C is
   Total wheel roll $= \frac{1}{L}\oint_C x dY$.
   Now that you have an equation for the total rolling distance of the wheel, connect this equation to Green's theorem to calculate area D enclosed by C.
4. Show that $x^2 + (y - Y)^2 = L^2$.
5. Assume the orientation of the planimeter is as shown in Figure 15.4.18. Explain why $Y \leq y$, and use this inequality to show there is a unique value of Y for each point (x, y): $Y = y = \sqrt{L^2 - x^2}$.
6. Use step 5 to show that $dY = dy + \frac{x}{L^2 - x^2}dx$.
7. Use Green's theorem to show that $\oint_C \frac{x}{L^2 - x^2}dx = 0$.
8. Use step 7 to show that the total wheel roll is
   $$\text{Total wheel roll} \quad = \quad 1L\oint_C x\, dy.$$
   It took a bit of work, but this equation says that the variable of integration *Y* in step 3 can be replaced with *y*.
9. Use Green's theorem to show that the area of D is $\oint_C x\ dy$. The logic is similar to the logic used to show that the area of $D = 12\oint_C -y\ dx + x\ dy$.
10. Conclude that the area of D equals the length of the tracer arm multiplied by the total rolling distance of the wheel.

You now know how a planimeter works and you have used Green's theorem to justify that it works. To calculate the area of a planar region D, use a planimeter to trace the boundary of the region. The area of the region is the length of the tracer arm multiplied by the distance the wheel rolled.

Key Concepts

- Green's theorem relates the integral over a connected region to an integral over the boundary of the region. Green's theorem is a version of the Fundamental Theorem of Calculus in one higher dimension.
- Green's Theorem comes in two forms: a circulation form and a flux form. In the circulation form, the integrand is $\vecs{F}$

$\cdot \vecs{T}$. In the flux form, the integrand is $\vecs{F} \cdot \vecs{N}$.

- Green's theorem can be used to transform a difficult line integral into an easier double integral, or to transform a difficult double integral into an easier line integral.
- A vector field is source free if it has a stream function. The flux of a source-free vector field across a closed curve is zero, just as the circulation of a conservative vector field across a closed curve is zero.

Key Equations

- **Green's theorem, circulation form**
  $\oint_C P\ dx + Q\ dy = \iint_D Q_x - P_y\ dA$, where C is the boundary of D
- **Green's theorem, flux form**
  $\oint_C \vecs{F} \cdot \vecs{N}\ ds = \iint_D P_x + Q_y\ dA$, where C is the boundary of D
- **Green's theorem, extended version**
  $\oint_{\partial D} \vecs{F} \cdot d\vecs{r} = \iint_D Q_x - P_y\ dA$

Glossary

Green's theorem
: relates the integral over a connected region to an integral over the boundary of the region

stream function
: if $\vecs{F} = \langle P, Q \rangle$ is a source-free vector field, then stream function g is a function such that $P = g_y$ and $Q = -g_x$

Contributions

Paul Seeburger created a corrected image and the worked out solution for Exercise $15.4.3$.

Answers

Exercise $15.4.2$

Transform the line integral into a double integral.

$\frac{45}{2}$

Exercise $15.4.3$

Use Equation $15.4.3$.

$\frac{2}{3}$ units^2.

For convenience, let's use the same vector field we used in Example $15.4.3$, where $Q_x - P_y = 1$.

$$\vecs{F}(x, y) = \langle P, Q \rangle = \langle -\frac{y}{2}, \frac{x}{2} \rangle.$$

For the curve $r(t) = \langle \sin t \cos t, \sin t \rangle$ with $0 \le t \le \pi$, we have

$$\vecs{r}'(t) = \langle \cos^2 t - \sin^2 t, \cos t \rangle \text{ and}$$
$$\vecs{F}(t) = \left\langle -\frac{\sin t}{2}, \frac{\sin t \cos t}{2} \right\rangle$$

By Green's theorem,

$$
\begin{aligned}
\text{Area of the region } D = \iint_D dA &= \iint_D (Q_x - P_y)\, dA \\
&= \int_C \vecs{F} \cdot d\vecs{r} = \frac{1}{2}\int_C -y\, dx + x\, dy \\
&= \frac{1}{2}\int_0^{\pi} \Big(-\sin t(\cos^2 t - \sin^2 t) + \sin t \cos^2 t \Big)\, dt \\
&= \frac{1}{2}\int_0^{\pi} \Big(-\sin t \cos^2 t + \sin t \sin^2 t + \sin t \cos^2 t \Big)\, dt \\
&= \frac{1}{2}\int_0^{\pi} \Big(\cancel{-\sin t \cos^2 t} + \sin t(1 - \cos^2 t) + \cancel{\sin t \cos^2 t} \Big)\, dt \\
&= \frac{1}{2}\int_0^{\pi} \Big(\sin t - \cos^2 t \sin t \Big)\, dt \\
&= \frac{1}{2}\left[-\cos t + \frac{\cos^3 t}{3} \right]\Bigg|_0^{\pi} \\
&= \frac{1}{2}\left[-\cos \pi + \frac{\cos^3 \pi}{3} + \cos 0 - \frac{\cos^3 0}{3} \right] \\
&= \frac{1}{2}\left[1 - \frac{1}{3} + 1 - \frac{1}{3} \right] \\
&= \frac{1}{2}\left[2 - \frac{2}{3} \right] \\
&= \frac{1}{2}\left[\frac{6}{3} - \frac{2}{3} \right] \\
&= \frac{1}{2}\left[\frac{4}{3} \right] \\
&= \frac{2}{3} \text{ units}^2
\end{aligned}
$$

Therefore the area of this region is $\frac{2}{3}$ units^2.

Exercise 15.4.4

Apply Green's theorem and use polar coordinates.

$\frac{3\pi}{2}$

Exercise 15.4.6

Follow the outline provided in the previous example.

$g(x, y) = -x \cos y$

Exercise 15.4.7

Determine whether the function satisfies Laplace's equation.

No

Exercise 15.4.8

Use the extended version of Green's theorem.

105π

15.4E: Green's Theorem (Exercises)

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15.4E: Green's Theorem (Exercises)

For the following exercises, evaluate the line integrals by applying Green's theorem.

1. $\int_C 2xy\ dx + (x+y)\ dy$, where C is the path from $(0,0)$ to $(1,1)$ along the graph of $y = x^3$ and from $(1,1)$ to $(0,0)$ along the graph of $y = x$ oriented in the counterclockwise direction

2. $\int_C 2xy\ dx + (x+y)\ dy$, where C is the boundary of the region lying between the graphs of $y = 0$ and $y = 4 - x^2$ oriented in the counterclockwise direction

Answer

$$\int_C 2xy\ dx + (x+y)\ dy = \frac{32}{3} \text{ units of work}$$

3. $\int_C 2\arctan\left(\frac{y}{x}\right)\ dx + \ln(x^2+y^2)\ dy$, where C is defined by $x = 4 + 2\cos\theta,\ \ y = 4\sin\theta$ oriented in the counterclockwise direction

4. $\int_C \sin x\cos y\ dx + (xy + \cos x\sin y)\ dy$, where C is the boundary of the region lying between the graphs of $y = x$ and $y = \sqrt{x}$ oriented in the counterclockwise direction

Answer

$$\int_C \sin x\cos y\ dx + (xy + \cos x\sin y)\ dy = \frac{1}{12} \text{ units of work}$$

5. $\int_C xy\ dx + (x+y)\ dy$, where C is the boundary of the region lying between the graphs of $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$ oriented in the counterclockwise direction

6. $\oint_C (-y\ dx + x\ dy)$, where C consists of line segment C_1 from $(-1,0)$ to $(1,0)$, followed by the semicircular arc C_2 from $(1,0)$ back to $(-1,0)$

Answer

$$\oint_C (-y\ \ dx + x\ \ dy) = \pi \text{ units of work}$$

For the following exercises, use Green's theorem.

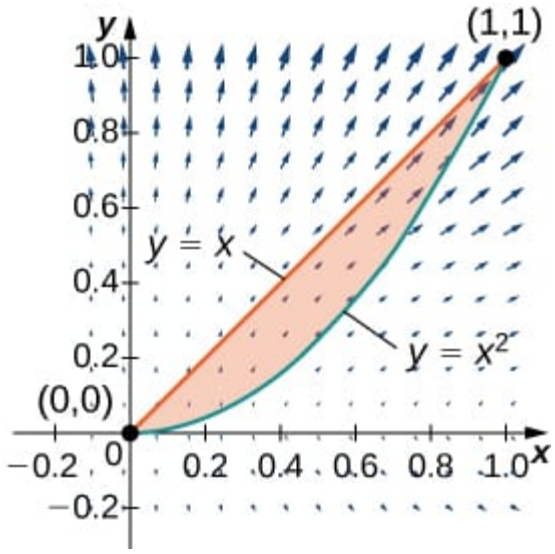
7. Let C be the curve consisting of line segments from $(0,0)$ to $(1,1)$ to $(0,1)$ and back to $(0,0)$. Find the value of $\int_C xy\ dx + \sqrt{y^2+1}\ dy$.

8. Evaluate line integral $\int_C xe^{-2x}\ dx + (x^4 + 2x^2y^2)\ dy$, where C is the boundary of the region between circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, and is a positively oriented curve.

Answer

$$\int_C xe^{-2x}\ dx + (x^4 + 2x^2y^2)\ dy = 0 \text{ units of work}$$

9. Find the counterclockwise circulation of field $\vecs{F}(x,y) = xy\ \hat{\mathbf{i}} + y^2\ \hat{\mathbf{j}}$ around and over the boundary of the region enclosed by curves $y = x^2$ and $y = x$ in the first quadrant and oriented in the counterclockwise direction.

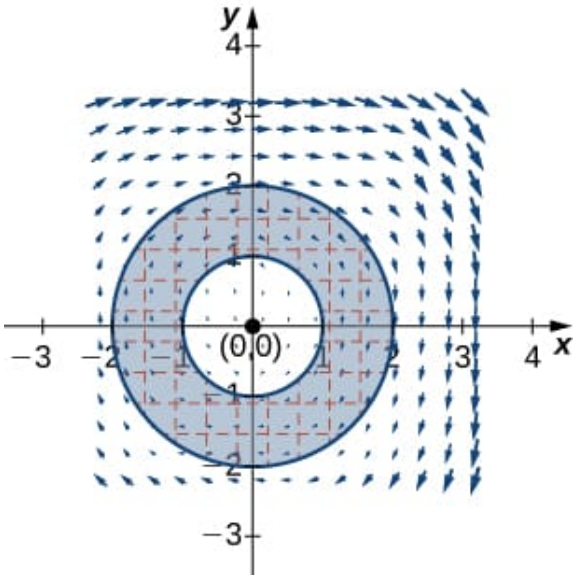


10. Evaluate $\oint_C y^3\ dx - x^3y^2\ dy$, where C is the positively oriented circle of radius 2 centered at the origin.

Answer

$\oint_C y^3\ dx - x^3y^2\ dy = -20\pi$ units of work

11. Evaluate $\oint_C y^3\ dx - x^3\ dy$, where C includes the two circles of radius 2 and radius 1 centered at the origin, both with positive orientation.



12. Calculate $\oint_C -x^2y\ dx + xy^2\ dy$, where C is a circle of radius 2 centered at the origin and oriented in the counterclockwise direction.

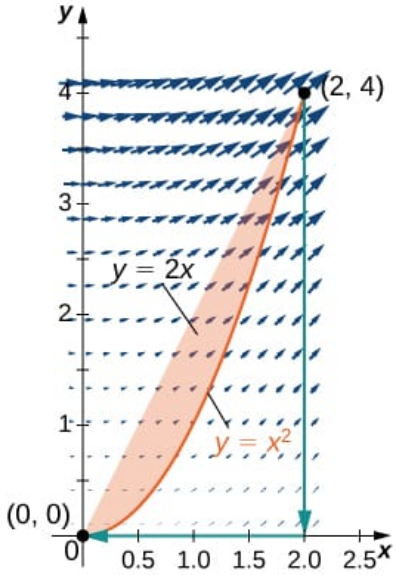
Answer

$\oint_C -x^2y\ dx + xy^2\ dy = 8\pi$ units of work

13. Calculate integral $\oint_C 2[y + x\sin(y)]\ dx + [x^2\cos(y) - 3y^2]\ dy$ along triangle C with vertices $(0,0),\ (1,0)$ and $(1,1)$, oriented

counterclockwise, using Green's theorem.

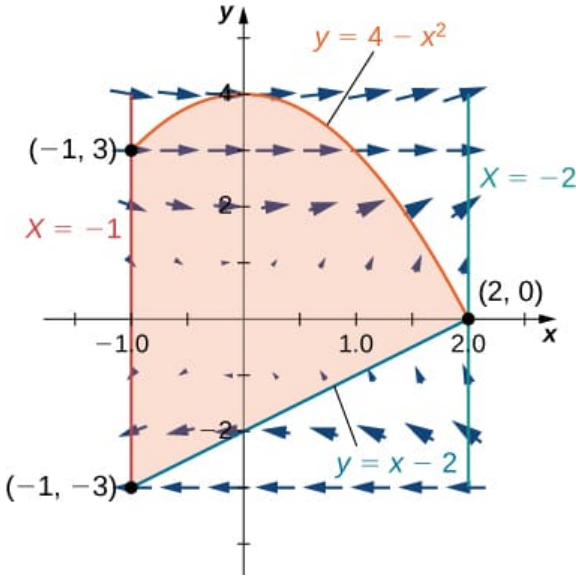
14. Evaluate integral $\oint_C (x^2 + y^2)\ dx + 2xy\ dy$, where C is the curve that follows parabola $y = x^2$ from $(0, 0),\ (2, 4)$, then the line from $(2, 4)$ to $(2, 0)$, and finally the line from $(2, 0)$ to $(0, 0)$.



Answer

$\oint_C (x^2 + y^2)\ dx + 2xy\ dy = 0$ units of work

15. Evaluate line integral $\oint_C (y - \sin(y)\cos(y))\ dx + 2x\sin^2(y)\ dy$, where C is oriented in a counterclockwise path around the region bounded by $x = -1,\ x = 2,\ y = 4 - x^2$, and $y = x - 2$.



For the following exercises, use Green's theorem to find the area.

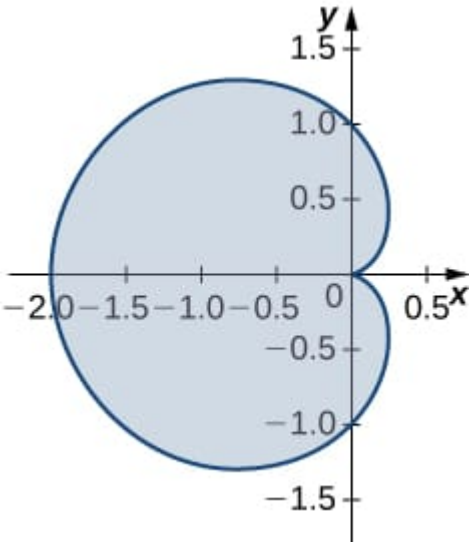
16. Find the area between ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and circle $x^2 + y^2 = 25$.

Answer

$A = 19\pi \text{ units}^2$

17. Find the area of the region enclosed by parametric equation

$$\vecs{p}(\theta) = (\cos(\theta) - \cos^2(\theta))\ \hat{\mathbf{i}} + (\sin(\theta) - \cos(\theta)\sin(\theta))\ \hat{\mathbf{j}} \text{ for } 0 \le \theta \le 2\pi.$$



18. Find the area of the region bounded by hypocycloid $\vecs{r}(t) = \cos^3(t)\ \hat{\mathbf{i}} + \sin^3(t)\ \hat{\mathbf{j}}$. The curve is parameterized by $t \in [0, 2\pi]$.

Answer

$A = \frac{3}{8\pi} \text{ units}^2$

19. Find the area of a pentagon with vertices $(0, 4),\ (4, 1),\ (3, 0),\ (-1, -1)$, and $(-2, 2)$.

20. Use Green's theorem to evaluate $\int_{C^+} (y^2 + x^3)\ dx + x^4\ dy$, where C^+ is the perimeter of square $[0, 1] \times [0, 1]$ oriented counterclockwise.

Answer

$$\int_{C^+} (y^2 + x^3)\ dx + x^4\ dy = 0$$

21. Use Green's theorem to prove the area of a disk with radius a is $A = \pi a^2 \text{ units}^2$.

22. Use Green's theorem to find the area of one loop of a four-leaf rose $r = 3\sin 2\theta$. (*Hint*: $x\ dy - y\ dx = r^2\ d\theta$).

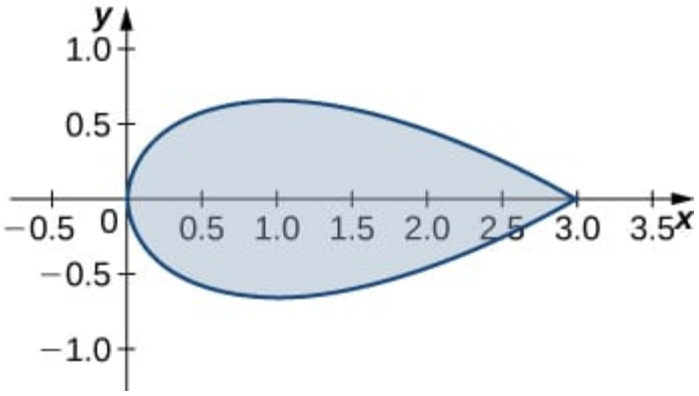
Answer

$A = \frac{9\pi}{8} \text{ units}^2$

23. Use Green's theorem to find the area under one arch of the cycloid given by the parametric equations: $x = t - \sin t,\ \ y = 1 - \cos t,\ \ t \ge 0$.

24. Use Green's theorem to find the area of the region enclosed by curve

$$\vecs{r}(t) = t^2\ \hat{\mathbf{i}} + \left(\tfrac{t^3}{3} - t\right)\ \hat{\mathbf{j}}, \text{ for } -\sqrt{3} \le t \le \sqrt{3}.$$



Answer

$A = \frac{8\sqrt{3}}{5}$ units^2

25. [T] Evaluate Green's theorem using a computer algebra system to evaluate the integral $\int_C xe^y\ dx + e^x\ dy$, where C is the circle given by $x^2 + y^2 = 4$ and is oriented in the counterclockwise direction.

26. Evaluate $\int_C (x^2y - 2xy + y^2)\ ds$, where C is the boundary of the unit square $0 \le x \le 1,\ \ 0 \le y \le 1$, traversed counterclockwise.

Answer

$$\int_C (x^2y - 2xy + y^2)\ ds = 3$$

27. Evaluate $\int_C \frac{-(y+2)\,dx + (x-1)\,dy}{(x-1)^2 + (y+2)^2}$, where C is any simple closed curve with an interior that does not contain point $(1, -2)$ traversed counterclockwise.

28. Evaluate $\int_C \frac{x\,dx + y\,dy}{x^2 + y^2}$, where C is any piecewise, smooth simple closed curve enclosing the origin, traversed counterclockwise.

Answer

$$\int_C \frac{x\,dx + y\,dy}{x^2 + y^2} = 2\pi$$

For the following exercises, use Green's theorem to calculate the work done by force $\vecs{F}$ on a particle that is moving counterclockwise around closed path C.

29. $\vecs{F}(x, y) = xy\ \hat{\mathbf{i}} + (x + y)\ \hat{\mathbf{j}}, \quad C : x^2 + y^2 = 4$

30. $\vecs{F}(x, y) = (x^{3/2} - 3y)\ \hat{\mathbf{i}} + (6x + 5\sqrt{y})\ \hat{\mathbf{j}}, \quad C$: boundary of a triangle with vertices $(0, 0),\ (5, 0)$, and $(0, 5)$

Answer

$W = \frac{225}{2}$ units of work

31. Evaluate $\int_C (2x^3 - y^3)\ dx + (x^3 + y^3)\ dy$, where C is a unit circle oriented in the counterclockwise direction.

32. A particle starts at point $(-2, 0)$, moves along the x-axis to $(2, 0)$, and then travels along semicircle $y = \sqrt{4 - x^2}$ to the starting point. Use Green's theorem to find the work done on this particle by force field $\vecs{F}(x, y) = x\ \hat{\mathbf{i}} + (x^3 + 3xy^2)\ \hat{\mathbf{j}}$.

Answer

$W = 12\pi$ units of work

33. David and Sandra are skating on a frictionless pond in the wind. David skates on the inside, going along a circle

of radius 2 in a counterclockwise direction. Sandra skates once around a circle of radius 3, also in the counterclockwise direction. Suppose the force of the wind at point (x, y) is $\vecs F(x, y) = (x^2y + 10y)\ \hat{\mathbf{i}} + (x^3 + 2xy^2)\ \hat{\mathbf{j}}$. Use Green's theorem to determine who does more work.

34. Use Green's theorem to find the work done by force field $\vecs F(x, y) = (3y - 4x)\ \hat{\mathbf{i}} + (4x - y)\ \hat{\mathbf{j}}$ when an object moves once counterclockwise around ellipse $4x^2 + y^2 = 4$.

Answer

$W = 2\pi$ units of work

35. Use Green's theorem to evaluate line integral $\oint_C e^{2x} \sin 2y\ dx + e^{2x} \cos 2y\ dy$, where C is ellipse $9(x - 1)^2 + 4(y - 3)^2 = 36$ oriented counterclockwise.

36. Evaluate line integral $\oint_C y^2\ dx + x^2\ dy$, where C is the boundary of a triangle with vertices $(0, 0),\ (1, 1)$, and $(1, 0)$, with the counterclockwise orientation.

Answer

$\oint_C y^2\ dx + x^2\ dy = \dfrac{1}{3}$ units of work

37. Use Green's theorem to evaluate line integral $\int_C \vecs h \cdot d\vecs r$ if $\vecs h(x, y) = e^y\ \hat{\mathbf{i}} - \sin \pi x\ \hat{\mathbf{j}}$, where C is a triangle with vertices $(1, 0),\ (0, 1)$, and $(-1, 0)$, traversed counterclockwise.

38. Use Green's theorem to evaluate line integral $\int_C \sqrt{1 + x^3}\ dx + 2xy\ dy$ where C is a triangle with vertices $(0, 0),\ (1, 0)$, and $(1, 3)$ oriented clockwise.

Answer

$\int_C \sqrt{1 + x^3}\ dx + 2xy\ dy = 3$ units of work

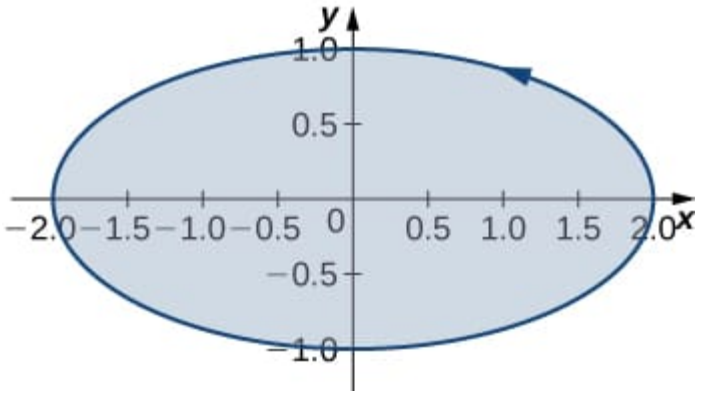
39. Use Green's theorem to evaluate line integral $\int_C x^2y\ dx - xy^2\ dy$ where C is a circle $x^2 + y^2 = 4$ oriented counterclockwise.

40. Use Green's theorem to evaluate line integral $\int_C \left(3y - e^{\sin x}\right)\ dx + \left(7x + \sqrt{y^4 + 1}\right)\ dy$ where C is circle $x^2 + y^2 = 9$ oriented in the counterclockwise direction.

Answer

$\int_C \left(3y - e^{\sin x}\right)\ dx + \left(7x + \sqrt{y^4 + 1}\right)\ dy = 36\pi$ units of work

41. Use Green's theorem to evaluate line integral $\int_C (3x - 5y)\ dx + (x - 6y)\ dy$, where C is ellipse $\frac{x^2}{4} + y^2 = 1$ and is oriented in the counterclockwise direction.



42. Let C be a triangular closed curve from $(0,0)$ to $(1,0)$ to $(1,1)$ and finally back to $(0,0)$. Let $\vecs F(x,y) = 4y\ \hat{\mathbf{i}} + 6x^2\ \hat{\mathbf{j}}$. Use Green's theorem to evaluate $\oint_C \vecs F \cdot d\vecs r$.

Answer

$\oint_C \vecs F \cdot d\vecs r = 2$ units of work

43. Use Green's theorem to evaluate line integral $\oint_C y\ dx - x\ dy$, where C is circle $x^2 + y^2 = a^2$ oriented in the clockwise direction.

44. Use Green's theorem to evaluate line integral $\oint_C (y + x)\ dx + (x + \sin y)\ dy$, where C is any smooth simple closed curve joining the origin to itself oriented in the counterclockwise direction.

Answer

$\oint_C (y + x)\ dx + (x + \sin y)\ dy = 0$ units of work

45. Use Green's theorem to evaluate line integral $\oint_C \left(y - \ln(x^2 + y^2)\right)\ dx + \left(2\arctan\dfrac{y}{x}\right)\ dy$, where C is the positively oriented circle $(x-2)^2 + (y-3)^2 = 1$.

46. Use Green's theorem to evaluate $\oint_C xy\ dx + x^3y^3\ dy$, where C is a triangle with vertices $(0,0)$, $(1,0)$, and $(1,2)$ with positive orientation.

Answer

$\oint_C xy\ dx + x^3y^3\ dy = 2221$ units of work

47. Use Green's theorem to evaluate line integral $\int_C \sin y\ dx + x\cos y\ dy$, where C is ellipse $x^2 + xy + y^2 = 1$ oriented in the counterclockwise direction.

48. Let $\vecs F(x,y) = (\cos(x^5) - 13y^3)\ \hat{\mathbf{i}} + 13x^3\ \hat{\mathbf{j}}$. Find the counterclockwise circulation $\oint_C \vecs F \cdot d\vecs r$, where C is a curve consisting of the line segment joining $(-2,0)$ and $(-1,0)$, half circle $y = \sqrt{1 - x^2}$, the line segment joining $(1,0)$ and $(2,0)$, and half circle $y = \sqrt{4 - x^2}$.

Answer

$\oint_C \vecs F \cdot d\vecs r = 15\pi^4$ units of work

49. Use Green's theorem to evaluate line integral $\int_C \sin(x^3)\ dx + 2ye^{x^2}\ dy$, where C is a triangular closed curve that connects the points $(0,0)$, $(2,2)$, and $(0,2)$ counterclockwise.

50. Let C be the boundary of square $0 \le x \le \pi,\ \ 0 \le y \le \pi$, traversed counterclockwise. Use Green's theorem to find $\int_C \sin(x + y)\ dx + \cos(x + y)\ dy$.

Answer

$\int_C \sin(x + y)\ dx + \cos(x + y)\ dy = 4$ units of work

51. Use Green's theorem to evaluate line integral $\int_C \vecs{F} \cdot d\vecs{r}$, where $\vecs{F}(x, y) = (y^2 - x^2)\ \hat{\mathbf{i}} + (x^2 + y^2)\ \hat{\mathbf{j}}$, and C is a triangle bounded by $y = 0,\ \ x = 3$, and $y = x$, oriented counterclockwise.

52. Use Green's Theorem to evaluate integral $\int_C \vecs{F} \cdot d\vecs{r}$, where $\vecs{F}(x, y) = (xy^2)\ \hat{\mathbf{i}} + x\ \hat{\mathbf{j}}$, and C is a unit circle oriented in the counterclockwise direction.

Answer

$\int_C \vecs{F} \cdot d\vecs{r} = \pi$ units of work

53. Use Green's theorem in a plane to evaluate line integral $\oint_C (xy + y^2)\ dx + x^2\ dy$, where C is a closed curve of a region bounded by $y = x$ and $y = x^2$ oriented in the counterclockwise direction.

54. Calculate the outward flux of $\vecs{F}(x, y) = -x\ \hat{\mathbf{i}} + 2y\ \hat{\mathbf{j}}$ over a square with corners $(\pm 1,\ \pm 1)$, where the unit normal is outward pointing and oriented in the counterclockwise direction.

Answer

$\oint_C \vecs{F} \cdot \vecs{N}\ ds = 4$

55. [T] Let C be circle $x^2 + y^2 = 4$ oriented in the counterclockwise direction. Evaluate $\oint_C \left[(3y - e^{\arctan x})\, dx + \left(7x + \sqrt{y^4 + 1}\right)\ dy\right]$ using a computer algebra system.

56. Find the flux of field $\vecs{F}(x, y) = -x\ \hat{\mathbf{i}} + y\ \hat{\mathbf{j}}$ across $x^2 + y^2 = 16$ oriented in the counterclockwise direction.

Answer

$\oint_C \vecs{F} \cdot \vecs{N}\ ds = 32\pi$

57. Let $\vecs{F} = (y^2 - x^2)\ \hat{\mathbf{i}} + (x^2 + y^2)\ \hat{\mathbf{j}}$, and let C be a triangle bounded by $y = 0,\ \ x = 3$, and $y = x$ oriented in the counterclockwise direction. Find the outward flux of $\vecs{F}$ through C.

58. [T] Let C be unit circle $x^2 + y^2 = 1$ traversed once counterclockwise. Evaluate $\int_C \left[-y^3 + \sin(xy) + xy\cos(xy)\right]\ \ dx + \left[x^3 + x^2\cos(xy)\right]\ \ dy$ by using a computer algebra system.

Answer

$\int_C \left[-y^3 + \sin(xy) + xy\cos(xy)\right]\ \ dx + \left[x^3 + x^2\cos(xy)\right]\ \ dy = 4.7124$ units of work

59. [T] Find the outward flux of vector field $\vecs{F}(x, y) = xy^2\ \hat{\mathbf{i}} + x^2y\ \hat{\mathbf{j}}$ across the boundary of annulus $R = \{(x, y) : 1 \le x^2 + y^2 \le 4\} = \{(r, \theta) : 1 \le r \le 2,\ \ 0 \le \theta \le 2\pi\}$ using a computer algebra system.

60. Consider region R bounded by parabolas $y = x^2$ and $x = y^2$. Let C be the boundary of R oriented counterclockwise. Use Green's theorem to evaluate $\oint_C \left(y + e^{\sqrt{x}}\right)\ \ dx + \left(2x + \cos(y^2)\right)\ \ dy$.

Answer

$\oint_C \left(y + e^{\sqrt{x}}\right)\ \ dx + \left(2x + \cos(y^2)\right)\ \ dy = 13$ units of work

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15.5: DIVERGENCE AND CURL

Learning Objectives

- Determine divergence from the formula for a given vector field.
- Determine curl from the formula for a given vector field.
- Use the properties of curl and divergence to determine whether a vector field is conservative.

In this section, we examine two important operations on a vector field: divergence and curl. They are important to the field of calculus for several reasons, including the use of curl and divergence to develop some higher-dimensional versions of the Fundamental Theorem of Calculus. In addition, curl and divergence appear in mathematical descriptions of fluid mechanics, electromagnetism, and elasticity theory, which are important concepts in physics and engineering. We can also apply curl and divergence to other concepts we already explored. For example, under certain conditions, a vector field is conservative if and only if its curl is zero.

In addition to defining curl and divergence, we look at some physical interpretations of them, and show their relationship to conservative and source-free vector fields.

Divergence

Divergence is an operation on a vector field that tells us how the field behaves toward or away from a point. Locally, the divergence of a vector field $\vecs F$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ at a particular point P is a measure of the "outflowing-ness" of the vector field at P. If $\vecs F$ represents the velocity of a fluid, then the divergence of $\vecs F$ at P measures the net rate of change with respect to time of the amount of fluid flowing away from P (the tendency of the fluid to flow "out of" P). In particular, if the amount of fluid flowing into P is the same as the amount flowing out, then the divergence at P is zero.

Definition: divergence in $\mathbb{R}^3$

If $\vecs F = \langle P, Q, R\rangle$ is a vector field in $\mathbb{R}^3$ and $P_x,\ Q_y$, and R_z all exist, then the divergence of $\vecs F$ is defined by

$$\operatorname{div} F = P_x + Q_y + R_z \tag{15.5.1}$$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}. \tag{15.5.2}$$

Note the divergence of a vector field is not a vector field, but a scalar function. In terms of the gradient operator

$$\vecs\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle$$

divergence can be written symbolically as the dot product

$$\operatorname{div} \vecs F = \vecs\nabla \cdot \vecs F.$$

Note this is merely helpful notation, because the dot product of a vector of operators and a vector of functions is not meaningfully defined given our current definition of dot product.

If $\vecs F = \langle P, Q\rangle$ is a vector field in $\mathbb{R}^2$, and P_x and Q_y both exist, then the divergence of $\vecs F$ is defined similarly as

$$\begin{aligned}\operatorname{div} \vecs F &= P_x + Q_y \\ &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \\ &= \vecs\nabla \cdot \vecs F.\end{aligned}$$

To illustrate this point, consider the two vector fields in Figure 15.5.1. At any particular point, the amount flowing in is the same as the amount flowing out, so at every point the "outflowing-ness" of the field is zero. Therefore, we expect the divergence of both fields to be zero, and this is indeed the case, as

$$\text{div}(\langle 1,2\rangle) = \frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(2) = 0$$

and

$$\text{div}(\langle -y,x\rangle) = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0.$$

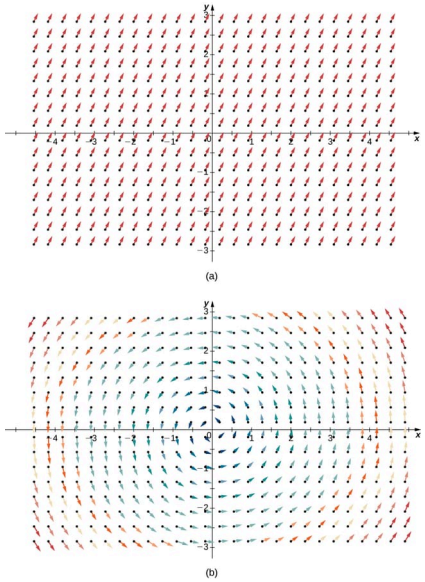


Figure 15.5.1: (a) Vector field $\langle 1,2\rangle$ has zero divergence. (b) Vector field $\langle -y,x\rangle$ also has zero divergence.

By contrast, consider radial vector field $\vecs{R}(x,y) = \langle -x,-y\rangle$ in Figure 15.5.2. At any given point, more fluid is flowing in than is flowing out, and therefore the "outgoingness" of the field is negative. We expect the divergence of this field to be negative, and this is indeed the case, as

$$\text{div}(\vecs{R}) = \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(-y) = -2.$$

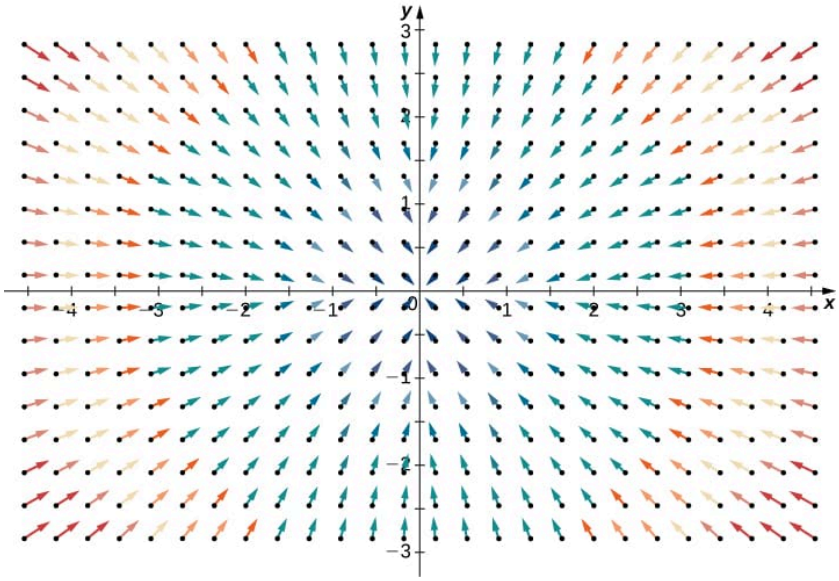


Figure 15.5.2: This vector field has negative divergence.

To get a global sense of what divergence is telling us, suppose that a vector field in $\mathbb{R}^2$ represents the velocity of a fluid. Imagine taking an elastic circle (a circle with a shape that can be changed by the vector field) and dropping it into a fluid. If the circle maintains its exact area as it flows through the fluid, then the divergence is zero. This would occur for both vector fields in Figure 15.5.1. On the other hand, if the circle's shape is distorted so that its area shrinks or expands, then the divergence is not zero. Imagine dropping such an elastic circle into the radial vector field in Figure 15.5.2 so that the center of the circle lands at point $(3, 3)$. The circle would flow toward the origin, and as it did so the front of the circle would travel more slowly than the back, causing the circle to "scrunch" and lose area. This is how you can see a negative divergence.

Example 15.5.1: Calculating Divergence at a Point

If $\vecs F(x, y, z) = e^x\hat{\imath} + yz\hat{\jmath} - yz^2\hat{k}$, then find the divergence of $\vecs F$ at $(0, 2, -1)$.

Solution

The divergence of $\vecs F$ is

$$\frac{\partial}{\partial x}(e^x) + \frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(yz^2) = e^x + z - 2yz.$$

Therefore, the divergence at $(0, 2, -1)$ is $e^0 - 1 + 4 = 4$. If $\vecs F$ represents the velocity of a fluid, then more fluid is flowing out than flowing in at point $(0, 2, -1)$.

Exercise 15.5.1

Find $\operatorname{div}\ \vecs F$ for

$$\vecs F(x, y, z) = \langle xy,\ 5 - z^2,\ x^2 + y^2\rangle.$$

Hint available at the end of this section.

One application for divergence occurs in physics, when working with magnetic fields. A magnetic field is a vector field that models the influence of electric currents and magnetic materials. Physicists use divergence in Gauss's law for magnetism, which states that if $\vecs B$ is a magnetic field, then $\vecs\nabla \cdot \vecs B = 0$; in other words, the divergence of a magnetic field is zero.

Example 15.5.2: Determining Whether a Field Is Magnetic

Is it possible for $\vecs F(x, y) = \langle x^2y,\ \ y - xy^2\rangle$ to be a magnetic field?

Solution

If $\vecs F$ were magnetic, then its divergence would be zero. The divergence of $\vecs F$ is

$$\frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(y - xy^2) = 2xy + 1 - 2xy = 1$$

and therefore $\vecs F$ cannot model a magnetic field (Figure 15.5.3).

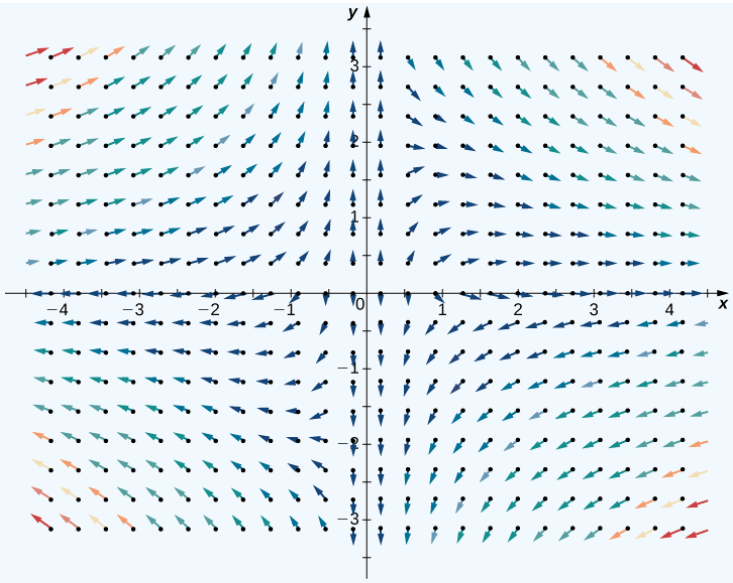


Figure 15.5.3: The divergence of vector field $\vecs{F}(x,y) = \langle x^2y,\ y - xy^2 \rangle$ is one, so it cannot model a magnetic field.

Another application for divergence is detecting whether a field is source free. Recall that a source-free field is a vector field that has a stream function; equivalently, a source-free field is a field with a flux that is zero along any closed curve. The next two theorems say that, under certain conditions, source-free vector fields are precisely the vector fields with zero divergence.

Theorem: Divergence of a Source-Free Vector Field

If $\vecs{F} = \langle P, Q \rangle$ is a source-free continuous vector field with differentiable component functions, then $\text{div } \vecs{F} = 0$.

Proof

Since $\vecs{F}$ is source free, there is a function $g(x,y)$ with $g_y = P$ and $-g_x = Q$. Therefore, $\vecs{F} = \langle g_y, -g_x \rangle$ and $\text{div } \vecs{F} = g_{yx} - g_{xy} = 0$ by Clairaut's theorem.

□

The converse of Divergence of a Source-Free Vector Field is true on simply connected regions, but the proof is too technical to include here. Thus, we have the following theorem, which can test whether a vector field in $\mathbb{R}^2$ is source free.

Theorem: Divergence Test for Source-Free Vector Fields

Let $\vecs{F} = \langle P, Q \rangle$ be a continuous vector field with differentiable component functions with a domain that is simply connected. Then, $\text{div } \vecs{F} = 0$ if and only if $\vecs{F}$ is source free.

Example 15.5.3: Determining Whether a Field Is Source Free

Is field $\vecs{F}(x,y) = \langle x^2y,\ 5 - xy^2 \rangle$ source free?

Solution

Note the domain of $\vecs{F}$ is $\mathbb{R}^2$ which is simply connected. Furthermore, $\vecs{F}$ is continuous with differentiable component functions. Therefore, we can use the Divergence Test for Source-Free Vector Fields to analyze $\vecs{F}$. The divergence of $\vecs{F}$ is

$$\frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(5 - xy^2) = 2xy - 2xy = 0.$$

Therefore, $\vecs F$ is source free by the Divergence Test for Source-Free Vector Fields.

Exercise 15.5.2

Let $\vecs F(x, y) = \langle -ay, bx\rangle$ be a rotational field where a and b are positive constants. Is $\vecs F$ source free?

Hint available at the end of this section.

Recall that the flux form of Green's theorem says that

$$\oint_C \vecs F \cdot \vecs N \, ds = \iint_D P_x + Q_y \, dA,$$

where C is a simple closed curve and D is the region enclosed by C. Since $P_x + Q_y = \text{div } \vecs F$, Green's theorem is sometimes written as

$$\oint_C \vecs F \cdot \vecs N \, ds = \iint_D \text{div}\, \vecs F \, dA.$$

Therefore, Green's theorem can be written in terms of divergence. If we think of divergence as a derivative of sorts, then Green's theorem says the "derivative" of $\vecs F$ on a region can be translated into a line integral of $\vecs F$ along the boundary of the region. This is analogous to the Fundamental Theorem of Calculus, in which the derivative of a function f on a line segment $[a, b]$ can be translated into a statement about f on the boundary of $[a, b]$. Using divergence, we can see that Green's theorem is a higher-dimensional analog of the Fundamental Theorem of Calculus.

We can use all of what we have learned in the application of divergence. Let $\vecs v$ be a vector field modeling the velocity of a fluid. Since the divergence of $\vecs v$ at point P measures the "outflowing-ness" of the fluid at P, $\text{div } v(P) > 0$ implies that more fluid is flowing out of P than flowing in. Similarly, $\text{div } v(P) < 0$ implies the more fluid is flowing in to P than is flowing out, and $\text{div } \vecs v(P) = 0$ implies the same amount of fluid is flowing in as flowing out.

Example 15.5.4: Determining Flow of a Fluid

Suppose $\vecs v(x, y) = \langle -xy, y\rangle, \ y > 0$ models the flow of a fluid. Is more fluid flowing into point $(1, 4)$ than flowing out?

Solution

To determine whether more fluid is flowing into $(1, 4)$ than is flowing out, we calculate the divergence of $\vecs v$ at $(1, 4)$:

$$div(\vecs v) = \frac{\partial}{\partial x}(-xy) + \frac{\partial}{\partial y}(y) = -y + 1.$$

To find the divergence at $(1, 4)$ substitute the point into the divergence: $-4 + 1 = -3$. Since the divergence of $\vecs v$ at $(1, 4)$ is negative, more fluid is flowing in than flowing out (Figure 15.5.4).

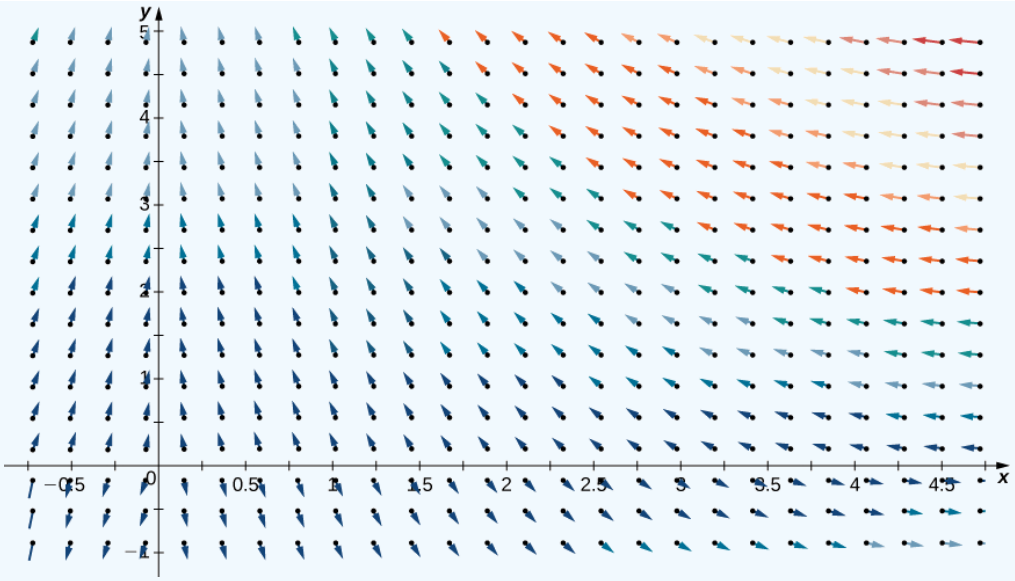


Figure 15.5.4: Vector field $\vecs{v}(x,y) = \langle -xy, y\rangle$ has negative divergence at $(1,4)$

Exercise 15.5.3

For vector field $\vecs{v}(x,y) = \langle -xy, y\rangle,\ y > 0$, find all points P such that the amount of fluid flowing in to P equals the amount of fluid flowing out of P.

Hint available at the end of this section.

Curl

The second operation on a vector field that we examine is the curl, which measures the extent of rotation of the field about a point. Suppose that $\vecs{F}$ represents the velocity field of a fluid. Then, the curl of $\vecs{F}$ at point P is a vector that measures the tendency of particles near P to rotate about the axis that points in the direction of this vector. The magnitude of the curl vector at P measures how quickly the particles rotate around this axis. In other words, the curl at a point is a measure of the vector field's "spin" at that point. Visually, imagine placing a paddlewheel into a fluid at P, with the axis of the paddlewheel aligned with the curl vector (Figure 15.5.5). The curl measures the tendency of the paddlewheel to rotate.

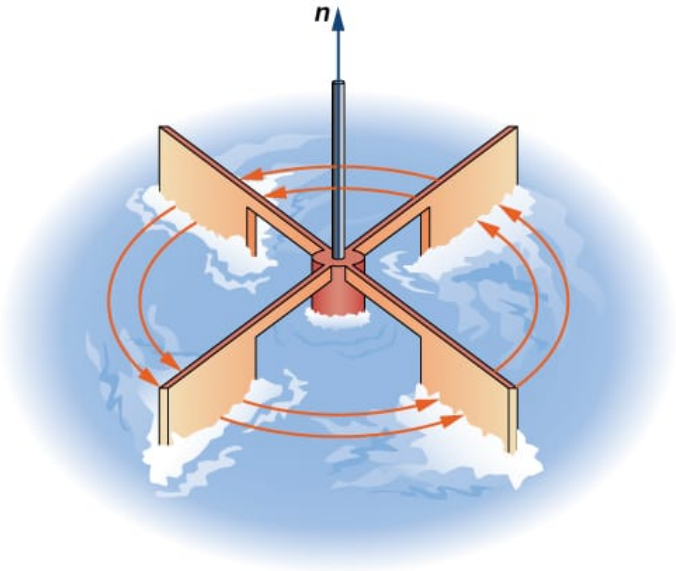


Figure 15.5.5: To visualize curl at a point, imagine placing a small paddlewheel into the vector field at a point.

Consider the vector fields in Figure 15.5.1. In part (a), the vector field is constant and there is no spin at any point. Therefore, we expect the curl of the field to be zero, and this is indeed the case. Part (b) shows a rotational field, so the field has spin. In particular, if you place a paddlewheel into a field at any point so that the axis of the wheel is perpendicular to a plane, the wheel rotates counterclockwise. Therefore, we expect the curl of the field to be nonzero, and this is indeed the case (the curl is $2\,\hat{\mathbf{k}}$).

To see what curl is measuring globally, imagine dropping a leaf into the fluid. As the leaf moves along with the fluid flow, the curl measures the tendency of the leaf to rotate. If the curl is zero, then the leaf doesn't rotate as it moves through the fluid.

Definition: Curl

If $\vecs F = \langle P, Q, R\rangle$ is a vector field in $\mathbb{R}^3$, and $P_x,\ Q_y$, and R_z all exist, then the curl of $\vecs F$ is defined by

$$\operatorname{curl}\vecs F = (R_y - Q_z)\,\hat{\mathbf{i}} + (P_z - R_x)\,\hat{\mathbf{j}} + (Q_x - P_y)\,\hat{\mathbf{k}} \tag{15.5.3}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\mathbf{k}}. \tag{15.5.4}$$

Note that the curl of a vector field is a vector field, in contrast to divergence.

The definition of curl can be difficult to remember. To help with remembering, we use the notation $\vecs\nabla \times \vecs F$ to stand for a "determinant" that gives the curl formula:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}.$$

The determinant of this matrix is

$$(R_y - Q_z)\,\hat{\mathbf{i}} - (R_x - P_z)\,\hat{\mathbf{j}} + (Q_x - P_y)\,\hat{\mathbf{k}} = (R_y - Q_z)\,\hat{\mathbf{i}} + (P_z - R_x)\,\hat{\mathbf{j}} + (Q_x - P_y)\,\hat{\mathbf{k}} = \operatorname{curl}\vecs F.$$

Thus, this matrix is a way to help remember the formula for curl. Keep in mind, though, that the word *determinant* is used very loosely. A determinant is not really defined on a matrix with entries that are three vectors, three operators, and three functions.

If $\vecs F = \langle P, Q\rangle$ is a vector field in $\mathbb{R}^2$, then the curl of $\vecs F$, by definition, is

$$\text{curl}\,\vecs{F} = (Q_x - P_y)\,\hat{\mathbf{k}} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\mathbf{k}}.$$

Example 15.5.5: Finding the Curl of a Three-Dimensional Vector Field

Find the curl of $\vecs{F}(x,y,z) = \langle P, Q, R\rangle = \langle x^2 z, e^y + xz, xyz\rangle$.

Solution

The curl is

$$\begin{aligned}\text{curl}\,\vecs{F} &= \vecs{\nabla} \times \vecs{F} \\ &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} \\ &= (R_y - Q_z)\,\hat{\mathbf{i}} + (P_z - R_x)\,\hat{\mathbf{j}} + (Q_x - P_y)\,\hat{\mathbf{k}} \\ &= (xz - x)\,\hat{\mathbf{i}} + (x^2 - yz)\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}.\end{aligned}$$

Exercise 15.5.4

Find the curl of $\vecs{F}(x,y,z) = \langle \sin x \;\; \cos z, \;\; \sin y \;\; \sin z, \;\; \cos x \;\; \cos y\rangle$ at point $\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)$.

Hint available at the end of this section.

Example 15.5.6: Finding the Curl of a Two-Dimensional Vector Field

Find the curl of $\vecs{F}(x,y) = \langle P, Q\rangle = \langle y, 0\rangle$.

Solution

Notice that this vector field consists of vectors that are all parallel. In fact, each vector in the field is parallel to the *x*-axis. This fact might lead us to the conclusion that the field has no spin and that the curl is zero. To test this theory, note that

$$\text{curl}\,\vecs{F} = (Q_x - P_y)\,\hat{\mathbf{k}} = -\hat{\mathbf{k}} \neq \vecs{0}.$$

Therefore, this vector field does have spin. To see why, imagine placing a paddlewheel at any point in the first quadrant (Figure 15.5.6). The larger magnitudes of the vectors at the top of the wheel cause the wheel to rotate. The wheel rotates in the clockwise (negative) direction, causing the coefficient of the curl to be negative.

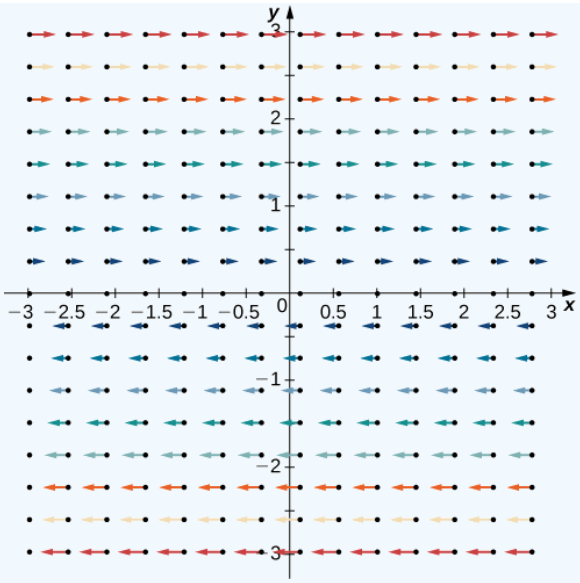


Figure 15.5.6: Vector field $\vecs{F}(x,y) = \langle y, 0\rangle$ consists of vectors that are all parallel.

Note that if $\vecs{F} = \langle P, Q\rangle$ is a vector field in a plane, then $\text{curl } \vecs{F} \cdot \hat{\mathbf{k}} = (Q_x - P_y)\ \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = Q_x - P_y$. Therefore, the circulation form of Green's theorem is sometimes written as

$$\oint_C \vecs{F} \cdot d\vecs{r} = \iint_D \text{curl}\, \vecs{F} \cdot\ \hat{\mathbf{k}}\, dA,$$

where C is a simple closed curve and D is the region enclosed by C. Therefore, the circulation form of Green's theorem can be written in terms of the curl. If we think of curl as a derivative of sorts, then Green's theorem says that the "derivative" of $\vecs{F}$ on a region can be translated into a line integral of $\vecs{F}$ along the boundary of the region. This is analogous to the Fundamental Theorem of Calculus, in which the derivative of a function f on line segment $[a,b]$ can be translated into a statement about f on the boundary of $[a,b]$. Using curl, we can see the circulation form of Green's theorem is a higher-dimensional analog of the Fundamental Theorem of Calculus.

We can now use what we have learned about curl to show that gravitational fields have no "spin." Suppose there is an object at the origin with mass m_1 at the origin and an object with mass m_2. Recall that the gravitational force that object 1 exerts on object 2 is given by field

$$\vecs{F}(x,y,z) = -Gm_1m_2 \left\langle \frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}} \right\rangle.$$

Example 15.5.7: Determining the Spin of a Gravitational Field

Show that a gravitational field has no spin.

Solution

To show that $\vecs{F}$ has no spin, we calculate its curl. Let

- $P(x,y,z) = \dfrac{x}{(x^2+y^2+z^2)^{3/2}}$,
- $Q(x,y,z) = \dfrac{y}{(x^2+y^2+z^2)^{3/2}}$, and
- $R(x,y,z) = \dfrac{z}{(x^2+y^2+z^2)^{3/2}}$.

Then,

$$\begin{aligned}\text{curl}\,\vec{F} &= -Gm_1m_2[(R_y-Q_z)\hat{\mathbf{i}}+(P_z-R_x)\hat{\mathbf{j}}+(Q_x-P_y)\hat{\mathbf{k}}]\\ &= -Gm_1m_2\begin{pmatrix}\left(\frac{-3yz}{(x^2+y^2+z^2)^{5/2}}-\left(\frac{-3yz}{(x^2+y^2+z^2)^{5/2}}\right)\right)\hat{\mathbf{i}}\\ +\left(\frac{-3xz}{(x^2+y^2+z^2)^{5/2}}-\left(\frac{-3xz}{(x^2+y^2+z^2)^{5/2}}\right)\right)\hat{\mathbf{j}}\\ +\left(\frac{-3xy}{(x^2+y^2+z^2)^{5/2}}-\left(\frac{-3xy}{(x^2+y^2+z^2)^{5/2}}\right)\right)\hat{\mathbf{k}}\end{pmatrix}\\ &= \vec{0}.\end{aligned}$$

Since the curl of the gravitational field is zero, the field has no spin.

Exercise 15.5.7

Field $\vec{v}(x,y)=\langle -\frac{y}{x^2+y^2},\frac{x}{x^2+y^2}\rangle$ models the flow of a fluid. Show that if you drop a leaf into this fluid, as the leaf moves over time, the leaf does not rotate.

Hint available at the end of this section.

Using Divergence and Curl

Now that we understand the basic concepts of divergence and curl, we can discuss their properties and establish relationships between them and conservative vector fields.

If $\vec{F}$ is a vector field in $\mathbb{R}^3$ then the curl of $\vec{F}$ is also a vector field in $\mathbb{R}^3$. Therefore, we can take the divergence of a curl. The next theorem says that the result is always zero. This result is useful because it gives us a way to show that some vector fields are not the curl of any other field. To give this result a physical interpretation, recall that divergence of a velocity field $\vec{v}$ at point P measures the tendency of the corresponding fluid to flow out of P. Since $\text{div}(\text{curl}\ \vec{v})=0$, the net rate of flow in vector field $\text{curl}\ \vec{v}$ at any point is zero. Taking the curl of vector field $\vec{F}$ eliminates whatever divergence was present in $\vec{F}$.

Theorem: Divergence of the Curl

Let $\vec{F}=\langle P,Q,R\rangle$ be a vector field in $\mathbb{R}^3$ such that the component functions all have continuous second-order partial derivatives. Then,

$$\text{div}(\text{curl}\,\vec{F})=\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F})=0.$$

Proof

By the definitions of divergence and curl, and by Clairaut's theorem,

$$\begin{aligned}\text{div}(\text{curl}\,\vec{F}) &= \text{div}[(R_y-Q_z)\,\hat{\mathbf{i}}+(P_z-R_x)\,\hat{\mathbf{j}}+(Q_x-P_y)\,\hat{\mathbf{k}}]\\ &= R_{yx}-Q_{xz}+P_{yz}-R_{yx}+Q_{zx}-P_{zy}\\ &= 0.\end{aligned}$$

□

Example 15.5.8: Showing That a Vector Field Is Not the Curl of Another

Show that $\vecs F(x,y,z) = e^x\,\hat{\mathbf{i}} + yz\,\hat{\mathbf{j}} + xz^2\,\hat{\mathbf{k}}$ is not the curl of another vector field. That is, show that there is no other vector $\vecs G$ with $\text{curl}\ \vecs G = \vecs F$.

Solution

Notice that the domain of $\vecs F$ is all of $\mathbb{R}^3$ and the second-order partials of $\vecs F$ are all continuous. Therefore, we can apply the previous theorem to $\vecs F$.

The divergence of $\vecs F$ is $e^x + z + 2xz$. If $\vecs F$ were the curl of vector field $\vecs G$, then $\text{div}\ \vecs F = \text{div}(\text{curl}\ \vecs G) = 0$. But, the divergence of $\vecs F$ is not zero, and therefore $\vecs F$ is not the curl of any other vector field.

Exercise 15.5.8

Is it possible for $\vecs G(x,y,z) = \langle \sin x,\ \cos y,\ \sin(xyz)\rangle$ to be the curl of a vector field?

Hint available at the end of this section.

With the next two theorems, we show that if $\vecs F$ is a conservative vector field then its curl is zero, and if the domain of $\vecs F$ is simply connected then the converse is also true. This gives us another way to test whether a vector field is conservative.

Theorem: Curl of a Conservative Vector Field

If $\vecs F = \langle P, Q, R\rangle$ is conservative, then $\text{curl}\ \vecs F = \vecs 0$.

Proof

Since conservative vector fields satisfy the cross-partials property, all the cross-partials of $\vecs F$ are equal. Therefore,

$$\begin{aligned}\text{curl}\,\vecs F &= (R_y - Q_z)\,\hat{\mathbf{i}} + (P_z - R_x)\,\hat{\mathbf{j}} + (Q_x - P_y)\,\hat{\mathbf{k}} \\ &= \vecs 0.\end{aligned}$$

□

The same theorem is true for vector fields in a plane.

Since a conservative vector field is the gradient of a scalar function, the previous theorem says that $\text{curl}\ (\vecs\nabla f) = \vecs 0$ for any scalar function f. In terms of our curl notation, $\vecs\nabla \times \vecs\nabla(f) = \vecs 0$. This equation makes sense because the cross product of a vector with itself is always the zero vector. Sometimes equation $\vecs\nabla \times \vecs\nabla(f) = \vecs 0$ is simplified as $\vecs\nabla \times \vecs\nabla = \vecs 0$.

Theorem: Curl Test for a Conservative Field

Let $\vecs F = \langle P, Q, R\rangle$ be a vector field in space on a simply connected domain. If $\text{curl}\ \vecs F = \vecs 0$, then $\vecs F$ is conservative.

Proof

Since $\text{curl}\ \vecs{F} = \vecs{0}$, we have that $R_y = Q_z$, $P_z = R_x$, and $Q_x = P_y$. Therefore, $\vecs{F}$ satisfies the cross-partials property on a simply connected domain, and the Cross-Partial Property of Conservative Fields implies that $\vecs{F}$ is conservative.

□

The same theorem is also true in a plane. Therefore, if $\vecs{F}$ is a vector field in a plane or in space and the domain is simply connected, then $\vecs{F}$ is conservative if and only if $\text{curl}\ \vecs{F} = \vecs{0}$.

Example 15.5.9: Testing Whether a Vector Field Is Conservative

Use the curl to determine whether $\vecs{F}(x, y, z) = \langle yz, xz, xy \rangle$ is conservative.

Solution

Note that the domain of $\vecs{F}$ is all of $\mathbb{R}^3$ which is simply connected (Figure 15.5.7). Therefore, we can test whether $\vecs{F}$ is conservative by calculating its curl.

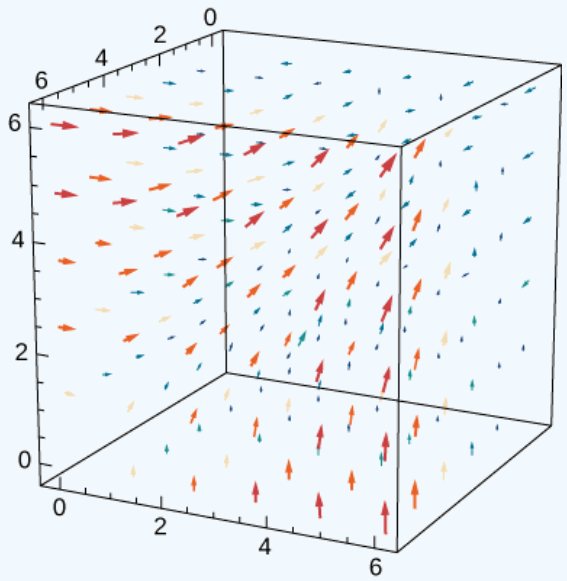


Figure 15.5.7: The curl of vector field $\vecs{F}(x, y, z) = \langle yz, xz, xy \rangle$ is zero.

The curl of $\vecs{F}$ is

$$\left(\frac{\partial}{\partial y}xy - \frac{\partial}{\partial z}xz\right)\hat{\mathbf{i}} + \left(\frac{\partial}{\partial y}yz - \frac{\partial}{\partial z}xy\right)\hat{\mathbf{j}} + \left(\frac{\partial}{\partial y}xz - \frac{\partial}{\partial z}yz\right)\hat{\mathbf{k}} = (x - x)\hat{\mathbf{i}} + (y - y)\hat{\mathbf{j}} + (z - z)\hat{\mathbf{k}} = \vecs{0}.$$

Thus, $\vecs{F}$ is conservative.

We have seen that the curl of a gradient is zero. What is the divergence of a gradient? If f is a function of two variables, then $\text{div}(\vecs{\nabla} f) = \vecs{\nabla} \cdot (\vecs{\nabla} f) = f_{xx} + f_{yy}$. We abbreviate this "double dot product" as $\vecs{\nabla}^2$. This operator is called the *Laplace operator,* and in this notation Laplace's equation becomes $\vecs{\nabla}^2 f = 0$. Therefore, a harmonic function is a function that becomes zero after taking the divergence of a gradient.

Similarly, if f is a function of three variables then

$$\text{div}(\vecs{\nabla} f) = \vecs{\nabla} \cdot (\vecs{\nabla} f) = f_{xx} + f_{yy} + f_{zz}.$$

Using this notation we get Laplace's equation for harmonic functions of three variables:

$$\vecs{\nabla}^2 f = 0.$$

Harmonic functions arise in many applications. For example, the potential function of an electrostatic field in a region of space that has no static charge is harmonic.

Example 15.5.10: Finding a Potential Function

Is it possible for $f(x,y) = x^2 + x - y$ to be the potential function of an electrostatic field that is located in a region of $\mathbb{R}^2$ free of static charge?

Solution

If f were such a potential function, then f would be harmonic. Note that $f_{xx} = 2$ and $f_{yy} = 0$, and so $f_{xx} + f_{yy} \neq 0$. Therefore, f is not harmonic and f cannot represent an electrostatic potential.

Exercise 15.5.10

Is it possible for function $f(x,y) = x^2 - y^2 + x$ to be the potential function of an electrostatic field located in a region of $\mathbb{R}^2$ free of static charge?

Hint available at the end of this section.

Key Concepts

- The divergence of a vector field is a scalar function. Divergence measures the "outflowing-ness" of a vector field. If $\vecs{v}$ is the velocity field of a fluid, then the divergence of $\vecs{v}$ at a point is the outflow of the fluid less the inflow at the point.
- The curl of a vector field is a vector field. The curl of a vector field at point P measures the tendency of particles at P to rotate about the axis that points in the direction of the curl at P.
- A vector field with a simply connected domain is conservative if and only if its curl is zero.

Key Equations

- **Curl**

$$\vecs{\nabla} \times \vecs{F} = (R_y - Q_z)\,\hat{\mathbf{i}} + (P_z - R_x)\,\hat{\mathbf{j}} + (Q_x - P_y)\,\hat{\mathbf{k}}$$

- **Divergence**

$$\vecs{\nabla} \cdot \vecs{F} = P_x + Q_y + R_z$$

- **Divergence of curl is zero**

$$\vecs{\nabla} \cdot (\vecs{\nabla} \times \vecs{F}) = 0$$

- **Curl of a gradient is the zero vector**

$$\vecs{\nabla} \times (\vecs{\nabla} f) = 0$$

Glossary

curl
the curl of vector field $\vecs{F} = \langle P, Q, R \rangle$, denoted $\vecs{\nabla} \times \vecs{F}$ is the "determinant" of the matrix

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}.$$

and is given by the expression $(R_y - Q_z)\ \hat{\mathbf{i}} + (P_z - R_x)\ \hat{\mathbf{j}} + (Q_x - P_y)\ \hat{\mathbf{k}}$; it measures the tendency of particles at a point to rotate about the axis that points in the direction of the curl at the point

divergence
: the divergence of a vector field $\vecs{F} = \langle P, Q, R \rangle$, denoted $\vecs{\nabla} \times \vecs{F}$, is $P_x + Q_y + R_z$; it measures the "outflowing-ness" of a vector field

Check Your Understanding

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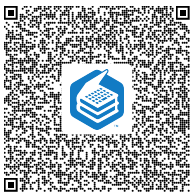
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Query 15.5.1

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Query 15.5.4

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Answers

Exercise 15.5.1

Follow Example 15.5.1.

$\text{div } \vecs{F} = y$

Exercise 15.5.2

Calculate the divergence.

Yes

Exercise 15.5.3

Find where the divergence is zero.

All points on line $y = 1$.

Exercise 15.5.4

Find the determinant of matrix $\vecs{\nabla} \times \vecs{F}$.

$-\hat{i}$

Exercise 15.5.7

Calculate the curl.

$\text{curl } \vecs{v} = \vecs{0}$

Exercise 15.5.8

Find the divergence of $\vecs{G}$.

No.

Exercise 15.5.10

Determine whether the function is harmonic.

Yes.

15.5E: Divergence and Curl (Exercises)

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15.5E: Divergence and Curl (Exercises)

For the following exercises, determine whether the statement is *True* or *False*.

1. If the coordinate functions of $\vecs F : \mathbb{R}^3 \to \mathbb{R}^3$ have continuous second partial derivatives, then $\text{curl } (\text{div } \vecs F)$ equals zero.

2. $\vecs \nabla \cdot (x\ \hat{\mathbf{i}} + y\ \hat{\mathbf{j}} + z\ \hat{\mathbf{k}}) = 1$.

Answer

False

3. All vector fields of the form $\vecs F(x, y, z) = f(x)\ \hat{\mathbf{i}} + g(y)\ \hat{\mathbf{j}} + h(z)\ \hat{\mathbf{k}}$ are conservative.

4. If $\text{curl } \vecs F = \vecs 0$, then $\vecs F$ is conservative.

Answer

True

5. If $\vecs F$ is a constant vector field then $\text{div } \vecs F = 0$.

6. If $\vecs F$ is a constant vector field then $\text{curl } \vecs F = \vecs 0$.

Answer

True

For the following exercises, find the curl of $\vecs F$.

7. $\vecs F(x, y, z) = xy^2z^4\ \hat{\mathbf{i}} + (2x^2y + z)\ \hat{\mathbf{j}} + y^3z^2\ \hat{\mathbf{k}}$

8. $\vecs F(x, y, z) = x^2z\ \hat{\mathbf{i}} + y^2x\ \hat{\mathbf{j}} + (y + 2z)\ \hat{\mathbf{k}}$

Answer

$\text{curl } \vecs F = \ \hat{\mathbf{i}} + x^2\ \hat{\mathbf{j}} + y^2\ \hat{\mathbf{k}}$

9. $\vecs F(x, y, z) = 3xyz^2\ \hat{\mathbf{i}} + y^2 \sin z\ \hat{\mathbf{j}} + xe^{2z}\ \hat{\mathbf{k}}$

10. $\vecs F(x, y, z) = x^2yz\ \hat{\mathbf{i}} + xy^2z\ \hat{\mathbf{j}} + xyz^2\ \hat{\mathbf{k}}$

Answer

$\text{curl } \vecs F = (xz^2 - xy^2)\ \hat{\mathbf{i}} + (x^2y - yz^2)\ \hat{\mathbf{j}} + (y^2z - x^2z)\ \hat{\mathbf{k}}$

11. $\vecs F(x, y, z) = (x\ \ \cos y)\ \hat{\mathbf{i}} + xy^2\ \hat{\mathbf{j}}$

12. $\vecs F(x, y, z) = (x - y)\ \hat{\mathbf{i}} + (y - z)\ \hat{\mathbf{j}} + (z - x)\ \hat{\mathbf{k}}$

Answer

$\text{curl } \ \vecs F = \ \hat{\mathbf{i}} + \ \hat{\mathbf{j}} + \ \hat{\mathbf{k}}$

13. $\vecs F(x, y, z) = xyz\ \hat{\mathbf{i}} + x^2y^2z^2\ \hat{\mathbf{j}} + y^2z^3\ \hat{\mathbf{k}}$

14. $\vecs F(x, y, z) = xy\ \hat{\mathbf{i}} + yz\ \hat{\mathbf{j}} + xz\ \hat{\mathbf{k}}$

Answer

$\text{curl } \ \vecs F = -y\ \hat{\mathbf{i}} - z\ \hat{\mathbf{j}} - x\ \hat{\mathbf{k}}$

15. $\vecs F(x,y,z) = x^2\ \hat{\mathbf{i}} + y^2\ \hat{\mathbf{j}} + z^2\ \hat{\mathbf{k}}$

16. $\vecs F(x,y,z) = ax\ \hat{\mathbf{i}} + by\ \hat{\mathbf{j}} + c\ \hat{\mathbf{k}}$ for constants $a,\ b,\ c.$

Answer

$$\text{curl}\ \ \vecs F = \vecs 0$$

For the following exercises, find the divergence of $\vecs F$.

17. $\vecs F(x,y,z) = x^2 z\ \hat{\mathbf{i}} + y^2 x\ \hat{\mathbf{j}} + (y+2z)\ \hat{\mathbf{k}}$

18. $\vecs F(x,y,z) = 3xyz^2\ \hat{\mathbf{i}} + y^2 \sin z\ \hat{\mathbf{j}} + xe^2 z\ \hat{\mathbf{k}}$

Answer

$$\text{div}\ \vecs F = 3yz^2 + 2y\ \ \sin z + 2xe^{2z}$$

19. $\vecs F(x,y) = (\sin x)\ \hat{\mathbf{i}} + (\cos y)\ \hat{\mathbf{j}}$

20. $\vecs F(x,y,z) = x^2\ \hat{\mathbf{i}} + y^2\ \hat{\mathbf{j}} + z^2\ \hat{\mathbf{k}}$

Answer

$$\text{div}\ \vecs F = 2(x+y+z)$$

21. $\vecs F(x,y,z) = (x-y)\ \hat{\mathbf{i}} + (y-z)\ \hat{\mathbf{j}} + (z-x)\ \hat{\mathbf{k}}$

22. $\vecs F(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}\ \hat{\mathbf{i}} + \dfrac{y}{\sqrt{x^2+y^2}}\ \hat{\mathbf{j}}$

Answer

$$\text{div}\ \vecs F = \frac{1}{\sqrt{x^2+y^2}}$$

23. $\vecs F(x,y) = x\ \hat{\mathbf{i}} - y\ \hat{\mathbf{j}}$

24. $\vecs F(x,y,z) = ax\ \hat{\mathbf{i}} + by\ \hat{\mathbf{j}} + c\ \hat{\mathbf{k}}$ for constants $a,\ b,\ c.$

Answer

$$\text{div}\ \vecs F = a + b$$

25. $\vecs F(x,y,z) = xyz\ \hat{\mathbf{i}} + x^2 y^2 z^2\ \hat{\mathbf{j}} + y^2 z^3\ \hat{\mathbf{k}}$

26. $\vecs F(x,y,z) = xy\ \hat{\mathbf{i}} + yz\ \hat{\mathbf{j}} + xz\ \hat{\mathbf{k}}$

Answer

$$\text{div}\ \vecs F = x + y + z$$

For exercises 27 & 28, determine whether each of the given scalar functions is harmonic.

27. $u(x,y,z) = e^{-x}(\cos y - \sin y)$

28. $w(x,y,z) = (x^2+y^2+z^2)^{-1/2}$

Answer

Harmonic

29. If $\vecs F(x,y,z) = 2\ \hat{\mathbf{i}} + 2xj + 3yk$ and $\vecs G(x,y,z) = x\ \hat{\mathbf{i}} - y\ \hat{\mathbf{j}} + z\ \hat{\mathbf{k}}$, find $\text{curl}\ (\vecs F \times \vecs G)$.

30. If $\vecs F(x,y,z) = 2\ \hat{\mathbf{i}} + 2xj + 3yk$ and $\vecs G(x,y,z) = x\ \hat{\mathbf{i}} - y\ \hat{\mathbf{j}} + z\ \hat{\mathbf{k}}$, find $\text{div}\ (\vecs F \times \vecs G)$.

Answer

$$\text{div } (\vecs{F} \times \vecs{G}) = 2z + 3x$$

31. Find $\text{div } \vecs{F}$, given that $\vecs{F} = \vecs{\nabla} f$, where $f(x, y, z) = xy^3z^2$.

32. Find the divergence of $\vecs{F}$ for vector field $\vecs{F}(x, y, z) = (y^2 + z^2)(x + y)\ \hat{\mathbf{i}} + (z^2 + x^2)(y + z)\ \hat{\mathbf{j}} + (x^2 + y^2)(z + x)\ \hat{\mathbf{k}}$.

Answer

$$\text{div } \vecs{F} = 2r^2$$

33. Find the divergence of $\vecs{F}$ for vector field $\vecs{F}(x, y, z) = f_1(y, z)\ \hat{\mathbf{i}} + f_2(x, z)\ \hat{\mathbf{j}} + f_3(x, y)\ \hat{\mathbf{k}}$.

For exercises 34 - 36, use $r = |\vecs{r}|$ and $\vecs{r}(x, y, z) = \langle x, y, z \rangle$.

34. Find the $\text{curl } \vecs{r}$

Answer

$$\text{curl } \vecs{r} = \vecs{0}$$

35. Find the $\text{curl } \dfrac{\vecs{r}}{r}$.

36. Find the $\text{curl } \dfrac{\vecs{r}}{r^3}$.

Answer

$$\text{curl } \frac{\vecs{r}}{r^3} = \vecs{0}$$

37. Let $\vecs{F}(x, y) = \dfrac{-y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}}{x^2 + y^2}$, where $\vecs{F}$ is defined on $\{(x, y)^{\text{int}}\mathbb{R} | (x, y) \neq (0, 0)\}$. Find $\text{curl } \vecs{F}$.

For the following exercises, use a computer algebra system to find the curl of the given vector fields.

38. [T] $\vecs{F}(x, y, z) = \arctan\left(\dfrac{x}{y}\right)\ \hat{\mathbf{i}} + \ln\sqrt{x^2 + y^2}\ \hat{\mathbf{j}} + \ \hat{\mathbf{k}}$

Answer

$$\text{curl } \vecs{F} = \frac{2x}{x^2 + y^2}\ \hat{\mathbf{k}}$$

39. [T] $\vecs{F}(x, y, z) = \sin(x - y)\ \hat{\mathbf{i}} + \sin(y - z)\ \hat{\mathbf{j}} + \sin(z - x)\ \hat{\mathbf{k}}$

For the following exercises, find the divergence of $\vecs{F}$ at the given point.

40. $\vecs{F}(x, y, z) = \ \hat{\mathbf{i}} + \ \hat{\mathbf{j}} + \ \hat{\mathbf{k}}$ at $(2, -1, 3)$

Answer

$$\text{div } \vecs{F} = 0$$

41. $\vecs{F}(x, y, z) = xyz\ \hat{\mathbf{i}} + y\ \hat{\mathbf{j}} + z\ \hat{\mathbf{k}}$ at $(1, 2, 3)$

42. $\vecs{F}(x, y, z) = e^{-xy}\ \hat{\mathbf{i}} + e^{xz}\ \hat{\mathbf{j}} + e^{yz}\ \hat{\mathbf{k}}$ at $(3, 2, 0)$

Answer

$$\text{div } \vecs{F} = 2 - 2e^{-6}$$

43. $\vecs{F}(x, y, z) = xyz\ \hat{\mathbf{i}} + y\ \hat{\mathbf{j}} + z\ \hat{\mathbf{k}}$ at $(1, 2, 1)$

44. $\vecs{F}(x, y, z) = e^x \sin y\ \hat{\mathbf{i}} - e^x \cos y\ \hat{\mathbf{j}}$ at $(0, 0, 3)$

Answer

$$\text{div } \vecs{F} = 0$$

For exercises 45- 49, find the curl of $\vecs{F}$ at the given point.

45. $\vecs{F}(x,y,z) = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ at $(2,-1,3)$

46. $\vecs{F}(x,y,z) = xyz\ \hat{\mathbf{i}} + y\ \hat{\mathbf{j}} + x\ \hat{\mathbf{k}}$ at $(1,2,3)$

Answer

$$\text{curl } \vecs{F} = \hat{\mathbf{j}} - 3\ \hat{\mathbf{k}}$$

47. $\vecs{F}(x,y,z) = e^{-xy}\ \hat{\mathbf{i}} + e^{xz}\ \hat{\mathbf{j}} + e^{yz}\ \hat{\mathbf{k}}$ at $(3,2,0)$

48. $\vecs{F}(x,y,z) = xyz\ \hat{\mathbf{i}} + y\ \hat{\mathbf{j}} + z\ \hat{\mathbf{k}}$ at $(1,2,1)$

Answer

$$\text{curl } \vecs{F} = 2\ \hat{\mathbf{j}} - \hat{\mathbf{k}}$$

49. $\vecs{F}(x,y,z) = e^x \sin y\ \hat{\mathbf{i}} - e^x \cos y\ \hat{\mathbf{j}}$ at $(0,0,3)$

50. Let $\vecs{F}(x,y,z) = (3x^2y + az)\ \hat{\mathbf{i}} + x^3\ \hat{\mathbf{j}} + (3x + 3z^2)\ \hat{\mathbf{k}}$. For what value of a is $\vecs{F}$ conservative?

Answer

$$a = 3$$

51. Given vector field $\vecs{F}(x,y) = \dfrac{1}{x^2+y^2}\langle -y, x\rangle$ on domain $D = \dfrac{\mathbb{R}^2}{\{(0,0)\}} = \{(x,y)^{\text{int}}\mathbb{R}^2 | (x,y) \neq (0,0)\}$, is $\vecs{F}$ conservative?

52. Given vector field $\vecs{F}(x,y) = \dfrac{1}{x^2+y^2}\langle x, y\rangle$ on domain $D = \dfrac{\mathbb{R}^2}{\{(0,0)\}}$, is $\vecs{F}$ conservative?

Answer

$\vecs{F}$ is conservative.

53. Find the work done by force field $\vecs{F}(x,y) = e^{-y}\ \hat{\mathbf{i}} - xe^{-y}\ \hat{\mathbf{j}}$ in moving an object from$P(0,1)$ to $Q(2,0)$. Is the force field conservative?

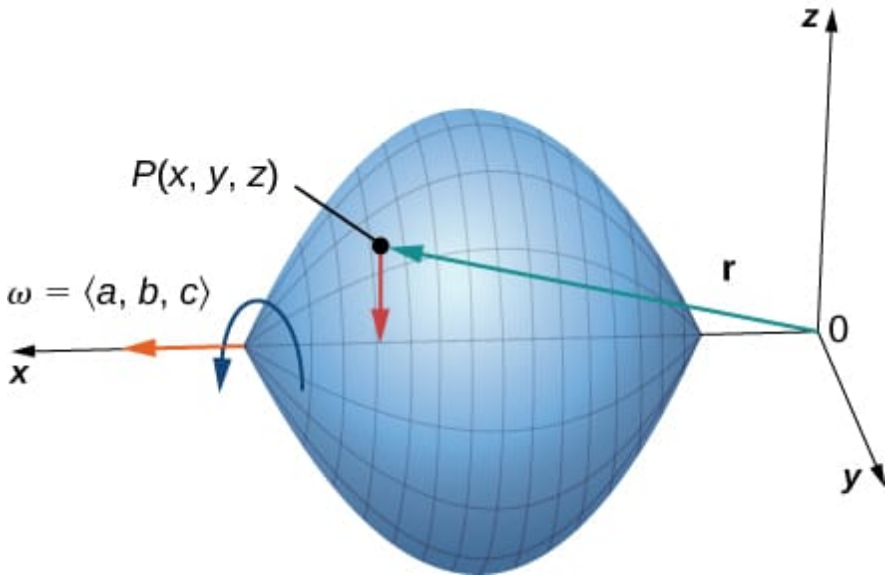
54. Compute divergence $\vecs{F}(x,y,z) = (\sinh x)\ \hat{\mathbf{i}} + (\cosh y)\ \hat{\mathbf{j}} - xyz\ \hat{\mathbf{k}}$.

Answer

$$\text{div } \vecs{F} = \cosh x + \sinh y - xy$$

55. Compute $\text{curl } \vecs{F} = (\sinh x)\ \hat{\mathbf{i}} + (\cosh y)\ \hat{\mathbf{j}} - xyz\ \hat{\mathbf{k}}$.

For the following exercises, consider a rigid body that is rotating about the x-axis counterclockwise with constant angular velocity $\vecs{\omega} = \langle a, b, c\rangle$. If P is a point in the body located at $\vecs{r} = x\ \hat{\mathbf{i}} + y\ \hat{\mathbf{j}} + z\ \hat{\mathbf{k}}$, the velocity at P is given by vector field $\vecs{F} = \vecs{\omega} \times \vecs{r}$.



56. Express $\vecs{F}$ in terms of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ vectors.

Answer

$$\vecs{F} = (bz - cy)\ \hat{\mathbf{i}} + (cx - az)\ \hat{\mathbf{j}} + (ay - bx)\ \hat{\mathbf{k}}$$

57. Find $\text{div}\ F$.

58. Find $\text{curl}\ F$

Answer

$$\text{curl}\ \ \vecs{F} = 2\vecs{\omega}$$

In the following exercises, suppose that $\vecs{\nabla} \cdot \vecs{F} = 0$ and $\vecs{\nabla} \cdot \vecs{G} = 0$.

59. Does $\vecs{F} + \vecs{G}$ necessarily have zero divergence?

60. Does $\vecs{F} \times \vecs{G}$ necessarily have zero divergence?

Answer

$\vecs{F} \times \vecs{G}$ does not have zero divergence.

In the following exercises, suppose a solid object in $\mathbb{R}^3$ has a temperature distribution given by $T(x, y, z)$. The heat flow vector field in the object is $\vecs{F} = -k\vecs{\nabla} T$, where $k > 0$ is a property of the material. The heat flow vector points in the direction opposite to that of the gradient, which is the direction of greatest temperature decrease. The divergence of the heat flow vector is $\vecs{\nabla} \cdot \vecs{F} = -k\vecs{\nabla} \cdot \vecs{\nabla} T = -k\vecs{\nabla}^2 T$.

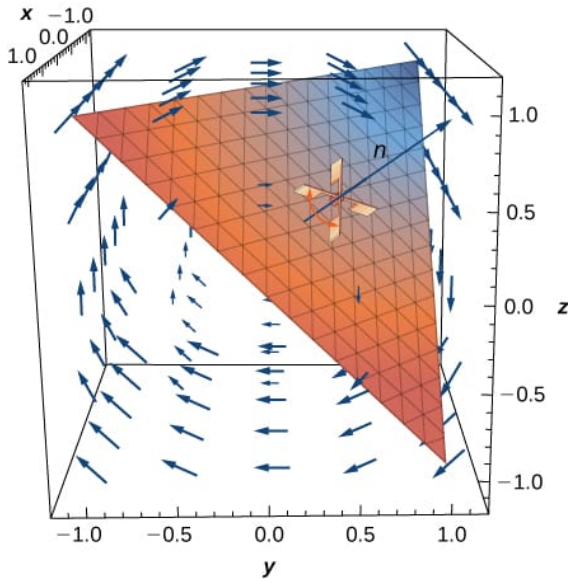
61. Compute the heat flow vector field.

62. Compute the divergence.

Answer

$$\vecs{\nabla} \cdot \vecs{F} = -200k[1 + 2(x^2 + y^2 + z^2)]e^{-x^2+y^2+z^2}$$

63. [T] Consider rotational velocity field $\vecs{v} = \langle 0, 10z, -10y \rangle$. If a paddlewheel is placed in plane $x + y + z = 1$ with its axis normal to this plane, using a computer algebra system, calculate how fast the paddlewheel spins in revolutions per unit time.



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15.6: Surface Integrals

Learning Objectives

- Find the parametric representations of a cylinder, a cone, and a sphere.
- Describe the surface integral of a scalar-valued function over a parametric surface.
- Use a surface integral to calculate the area of a given surface.
- Explain the meaning of an oriented surface, giving an example.
- Describe the surface integral of a vector field.
- Use surface integrals to solve applied problems.

We have seen that a line integral is an integral over a path in a plane or in space. However, if we wish to integrate over a surface (a two-dimensional object) rather than a path (a one-dimensional object) in space, then we need a new kind of integral that can handle integration over objects in higher dimensions. We can extend the concept of a line integral to a surface integral to allow us to perform this integration.

Surface integrals are important for the same reasons that line integrals are important. They have many applications to physics and engineering, and they allow us to develop higher dimensional versions of the Fundamental Theorem of Calculus. In particular, surface integrals allow us to generalize Green's theorem to higher dimensions, and they appear in some important theorems we discuss in later sections.

Parametric Surfaces

A surface integral is similar to a line integral, except the integration is done over a surface rather than a path. In this sense, surface integrals expand on our study of line integrals. Just as with line integrals, there are two kinds of surface integrals: a surface integral of a scalar-valued function and a surface integral of a vector field.

However, before we can integrate over a surface, we need to consider the surface itself. Recall that to calculate a scalar or vector line integral over curve C, we first need to parameterize C. In a similar way, to calculate a surface integral over surface S, we need to parameterize S. That is, we need a working concept of a *parameterized surface* (or a *parametric surface*), in the same way that we already have a concept of a parameterized curve.

A parameterized surface is given by a description of the form

$$\vecs{r}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle.$$

Notice that this parameterization involves two parameters, u and v, because a surface is two-dimensional, and therefore two variables are needed to trace out the surface. The parameters u and v vary over a region called the parameter domain, or parameter space—the set of points in the uv-plane that can be substituted into $\vecs{r}$. Each choice of u and v in the parameter domain gives a point on the surface, just as each choice of a parameter t gives a point on a parameterized curve. The entire surface is created by making all possible choices of u and v over the parameter domain.

Definition: Parameter Domain

Given a parameterization of surface

$$\vecs{r}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle.$$

the *parameter domain* of the parameterization is the set of points in the uv-plane that can be substituted into $\vecs{r}$.

Example 15.6.1: Parameterizing a Cylinder

Describe surface S parameterized by

$$\vecs{r}(u,v) = \langle \cos u,\ \sin u,\ v\rangle,\ -\infty < u < \infty,\ -\infty < v < \infty.$$

Solution

To get an idea of the shape of the surface, we first plot some points. Since the parameter domain is all of $\mathbb{R}^2$, we can choose any value for *u* and *v* and plot the corresponding point. If $u = v = 0$, then $\vecs{r}(0,0) = \langle 1,0,0\rangle$, so point (1, 0, 0) is on S. Similarly, points $\vecs{r}(\pi,2) = (-1,0,2)$ and $\vecs{r}\left(\frac{\pi}{2},4\right) = (0,1,4)$ are on S.

Although plotting points may give us an idea of the shape of the surface, we usually need quite a few points to see the shape. Since it is time-consuming to plot dozens or hundreds of points, we use another strategy. To visualize S, we visualize two families of curves that lie on S. In the first family of curves we hold u constant; in the second family of curves we hold v constant. This allows us to build a "skeleton" of the surface, thereby getting an idea of its shape.

- Suppose that u is a constant K. Then the curve traced out by the parameterization is $\langle \cos K,\ \sin K,\ v\rangle$, which gives a vertical line that goes through point $(\cos K, \sin K, v\rangle$ in the xy-plane.
- Suppose that v is a constant K. Then the curve traced out by the parameterization is $\langle \cos u,\ \sin u,\ K\rangle$, which gives a circle in plane $z = K$ with radius 1 and center $(0,0,K)$.

If u is held constant, then we get vertical lines; if v is held constant, then we get circles of radius 1 centered around the vertical line that goes through the origin. Therefore the surface traced out by the parameterization is cylinder $x^2 + y^2 = 1$ (Figure 15.6.1).

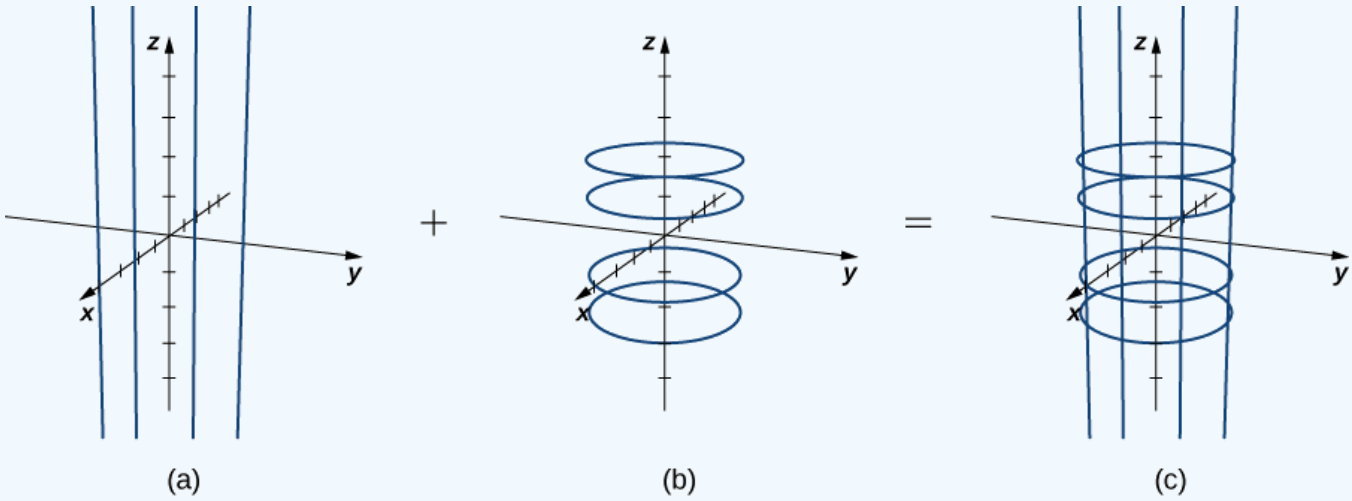


Figure 15.6.1: (a) Lines $\langle \cos K,\ \sin K, v\rangle$ for $K = 0,\ \frac{\pi}{2},\ \pi$, and $\frac{3\pi}{2}$. (b) Circles $\langle \cos u,\ \sin u,\ K\rangle$ for $K = -2,-1,1$, and 2 (c) The lines and circles together. As u and v vary, they describe a cylinder.

Notice that if $x = \cos u$ and $y = \sin u$, then $x^2 + y^2 = 1$, so points from S do indeed lie on the cylinder. Conversely, each point on the cylinder is contained in some circle $\langle \cos u,\ \sin u,\ k\rangle$ for some k, and therefore each point on the cylinder is contained in the parameterized surface (Figure 15.6.2).

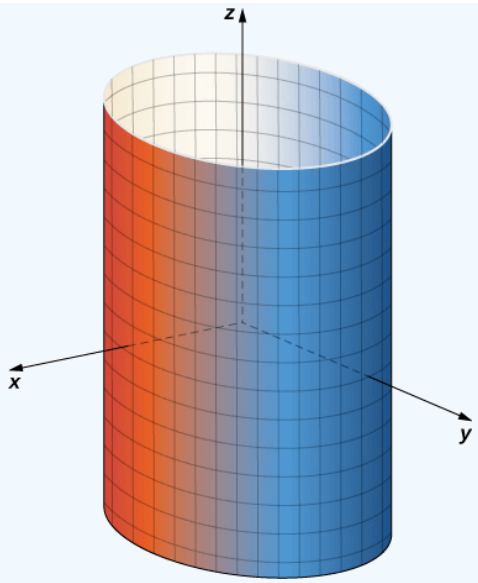


Figure 15.6.2: Cylinder $x^2 + y^2 = r^2$ has parameterization $\vecs{r}(u, v) = \langle r\ \cos u,\ r\ \sin u,\ v\rangle,\ 0 \le u \le 2\pi,\ -\infty < v < \infty$.

Analysis

Notice that if we change the parameter domain, we could get a different surface. For example, if we restricted the domain to $0 \le u \le \pi,\ -\infty < v < 6$, then the surface would be a half-cylinder of height 6.

Exercise 15.6.1

Describe the surface with parameterization

$$\vecs{r}(u, v) = \langle 2\ \cos u,\ 2\ \sin u,\ v\rangle,\ 0 \le u \le 2\pi,\ -\infty < v < \infty$$

Hint available at the end of this section.

Answer available at the end of this section.

It follows from Example 15.6.1 that we can parameterize all cylinders of the form $x^2 + y^2 = R^2$. If S is a cylinder given by equation $x^2 + y^2 = R^2$, then a parameterization of S is $\vecs{r}(u, v) = \langle R\ \cos u,\ R\ \sin u,\ v\rangle,\ 0 \le u \le 2\pi,\ -\infty < v < \infty$.

We can also find different types of surfaces given their parameterization, or we can find a parameterization when we are given a surface.

Example 15.6.2: Describing a Surface

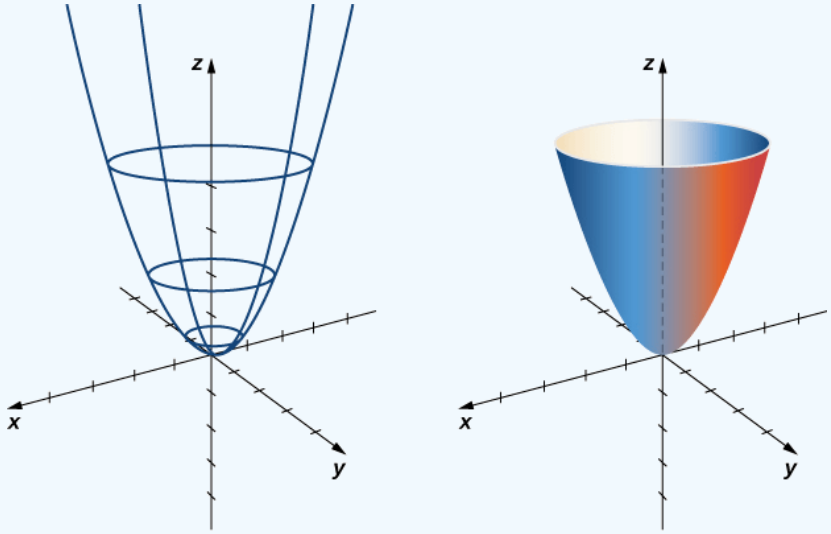
Describe surface S parameterized by $\vecs{r}(u, v) = \langle u\ \cos v,\ u\ \sin v,\ u^2\rangle,\ 0 \le u < \infty,\ 0 \le v < 2\pi$.

Solution

Notice that if u is held constant, then the resulting curve is a circle of radius u in plane $z = u$. Therefore, as u increases, the radius of the resulting circle increases. If v is held constant, then the resulting curve is a vertical parabola. Therefore, we expect the surface to be an elliptic paraboloid. To confirm this, notice that

$$\begin{aligned} x^2 + y^2 &= (u\,\cos v)^2 + (u\,\sin v)^2 \\ &= u^2\cos^2 v + u^2 sin^2 v \\ &= u^2 \\ &= z \end{aligned}$$

Therefore, the surface is the elliptic paraboloid $x^2 + y^2 = z$ (Figure 15.6.3).



(a) (b)

Figure 15.6.3: (a) Circles arise from holding u constant; the vertical parabolas arise from holding v constant. (b) An elliptic paraboloid results from all choices of u and v in the parameter domain.

Exercise 15.6.2

Describe the surface parameterized by $\vecs r(u,v) = \langle u\ \cos v,\ u\ \sin v,\ u\rangle,\ -\infty < u < \infty,\ 0 \le v < 2\pi$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 15.6.3: Finding a Parameterization

Give a parameterization of the cone $x^2 + y^2 = z^2$ lying on or above the plane $z = -2$.

Solution

The horizontal cross-section of the cone at height $z = u$ is circle $x^2 + y^2 = u^2$. Therefore, a point on the cone at height u has coordinates $(u\ \cos v,\ u\ \sin v,\ u)$ for angle v. Hence, a parameterization of the cone is $\vecs r(u,v) = \langle u\ \cos v,\ u\ \sin v,\ u\rangle$. Since we are not interested in the entire cone, only the portion on or above plane $z = -2$, the parameter domain is given by $-2 < u < \infty,\ 0 \le v < 2\pi$ (Figure 15.6.4).

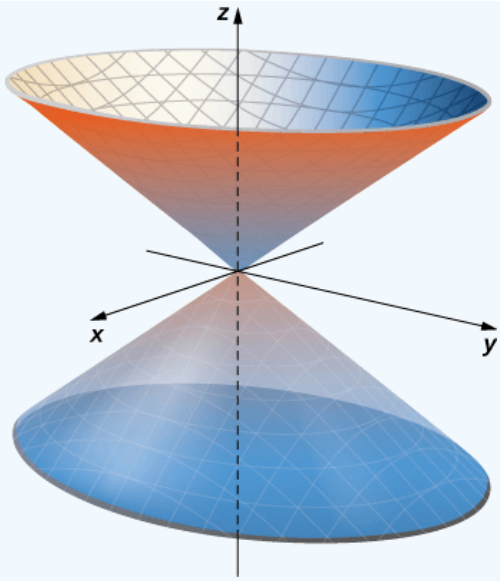


Figure 15.6.4: Cone $x^2 + y^2 = z^2$ has parametrization $\vecs r(u, v) = \langle u\ \cos v,\ u\ \sin v \rangle,\ -\infty < u < \infty,\ 0 \le v \le 2\pi.$

Exercise 15.6.3

Give a parameterization for the portion of cone $x^2 + y^2 = z^2$ lying in the first octant.

Hint available at the end of this section.

Answer available at the end of this section.

We have discussed parameterizations of various surfaces, but two important types of surfaces need a separate discussion: spheres and graphs of two-variable functions. To parameterize a sphere, it is easiest to use spherical coordinates. The sphere of radius ρ centered at the origin is given by the parameterization

$$\vecs r(\phi, \theta) = \langle \rho\ \cos\theta\ \sin\phi,\ \rho\ \sin\theta\ \sin\phi,\ \rho\ \cos\phi \rangle,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi.$$

The idea of this parameterization is that as ϕ sweeps downward from the positive z-axis, a circle of radius $\rho\ \sin\phi$ is traced out by letting θ run from 0 to 2π. To see this, let ϕ be fixed. Then

$$\begin{aligned} x^2 + y^2 &= (\rho\ \cos\theta\ \sin\phi)^2 + (\rho\ \sin\theta\ \sin\phi)^2 \\ &= \rho^2\ \sin^2\phi(\cos^2\theta + \sin^2\theta) \\ &= \rho^2\ \sin^2\phi \\ &= (\rho\ \sin\phi)^2. \end{aligned}$$

This results in the desired circle (Figure 15.6.5).

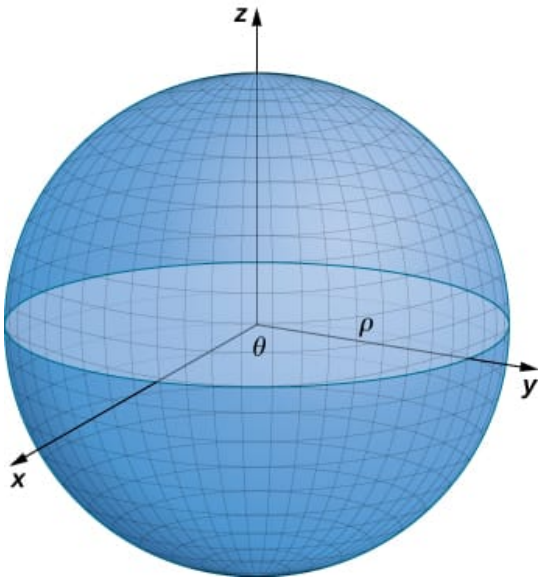


Figure 15.6.5: The sphere of radius ρ has parameterization $\vecs r(\phi,\theta)=\langle \rho\ \cos\theta\ \sin\phi,\ \rho\ \sin\theta\ \sin\phi,\ \rho\ \cos\phi\rangle,\ 0\le\theta\le 2\pi,\ 0\le\phi\le\pi.$

Finally, to parameterize the graph of a two-variable function, we first let $z=f(x,y)$ be a function of two variables. The simplest parameterization of the graph of f is $\vecs r(x,y)=\langle x,y,f(x,y)\rangle$, where x and y vary over the domain of f (Figure 15.6.6). For example, the graph of $f(x,y)=x^2y$ can be parameterized by $\vecs r(x,y)=\langle x,y,x^2y\rangle$, where the parameters x and y vary over the domain of f. If we only care about a piece of the graph of f - say, the piece of the graph over rectangle $[1,3]\times[2,5]$ - then we can restrict the parameter domain to give this piece of the surface:

$$\vecs r(x,y)=\langle x,y,x^2y\rangle,\ 1\le x\le 3,\ 2\le y\le 5.$$

Similarly, if S is a surface given by equation $x=g(y,z)$ or equation $y=h(x,z)$, then a parameterization of S is $\vecs r(y,z)=\langle g(y,z),\ y,z\rangle$ or $\vecs r(x,z)=\langle x,h(x,z),z\rangle$, respectively. For example, the graph of paraboloid $2y=x^2+z^2$ can be parameterized by $\vecs r(x,y)=\left\langle x,\dfrac{x^2+z^2}{2},z\right\rangle,\ 0\le x<\infty,\ 0\le z<\infty$. Notice that we do not need to vary over the entire domain of y because x and z are squared.

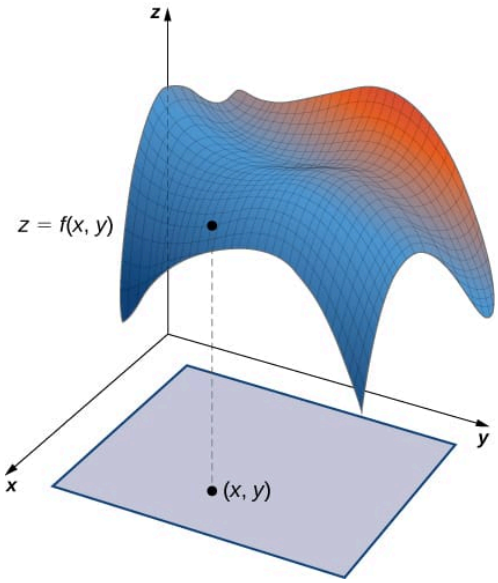


Figure 15.6.6: The simplest parameterization of the graph of a function is $\vecs r(x,y)=\langle x,y,f(x,y)\rangle$.

Let's now generalize the notions of smoothness and regularity to a parametric surface. Recall that curve parameterization $\vecs r(t),\ a\le t\le b$ is regular (or **smooth**) if $\vecs r'(t)\neq\vecs 0$ for all t in $[a,b]$. For a curve, this condition ensures that the image of $\vecs r$ really is a curve, and not just a point. For example, consider curve parameterization $\vecs r(t)=\langle 1,2\rangle,\ 0\le t\le 5$. The

image of this parameterization is simply point $(1, 2)$, which is not a curve. Notice also that $\vecs r'(t) = \vecs 0$. The fact that the derivative is the zero vector indicates we are not actually looking at a curve.

Analogously, we would like a notion of regularity (or smoothness) for surfaces so that a surface parameterization really does trace out a surface. To motivate the definition of regularity of a surface parameterization, consider the parameterization

$$\vecs r(u, v) = \langle 0,\ \cos v,\ 1\rangle,\ 0 \le u \le 1,\ 0 \le v \le \pi.$$

Although this parameterization appears to be the parameterization of a surface, notice that the image is actually a line (Figure 15.6.7). How could we avoid parameterizations such as this? Parameterizations that do not give an actual surface? Notice that $\vecs r_u = \langle 0, 0, 0\rangle$ and $\vecs r_v = \langle 0, -\sin v, 0\rangle$, and the corresponding cross product is zero. The analog of the condition $\vecs r'(t) = \vecs 0$ is that $\vecs r_u \times \vecs r_v$ is not zero for point (u, v) in the parameter domain, which is a regular parameterization.

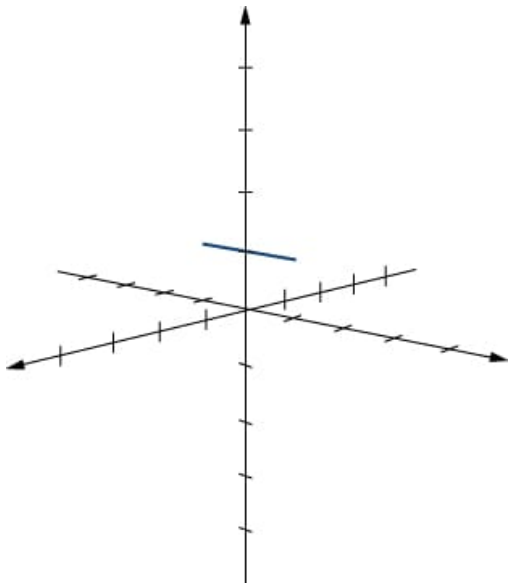


Figure 15.6.7: The image of parameterization $\vecs r(u, v) = \langle 0, \cos v, 1\rangle,\ \ 0 \le u \le 1,\ \ 0 \le v \le \pi$ is a line.

Definition: Regular Parameterization

Parameterization $\vecs r(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$ is a **regular parameterization** if $\vecs r_u \times \vecs r_v$ is not zero for point (u, v) in the parameter domain.

If parameterization $\vec{r}$ is regular, then the image of $\vec{r}$ is a two-dimensional object, as a surface should be. Throughout this chapter, parameterizations $\vecs r(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$are assumed to be regular.

Recall that curve parameterization $\vecs r(t),\ \ a \le t \le b$ is smooth if $\vecs r'(t)$ is continuous and $\vecs r'(t) \ne \vecs 0$ for all t in $[a, b]$. Informally, a curve parameterization is smooth if the resulting curve has no sharp corners. The definition of a smooth surface parameterization is similar. Informally, a surface parameterization is *smooth* if the resulting surface has no sharp corners.

Definition: Smooth Surfaces

A surface parameterization $\vecs r(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$ is *smooth* if vector $\vecs r_u \times \vecs r_v$ is not zero for any choice of u and v in the parameter domain.

A surface may also be *piecewise smooth* if it has smooth faces but also has locations where the directional derivatives do not exist.

Example 15.6.4: Identifying Smooth and Nonsmooth Surfaces

Which of the figures in Figure 15.6.8 is smooth?

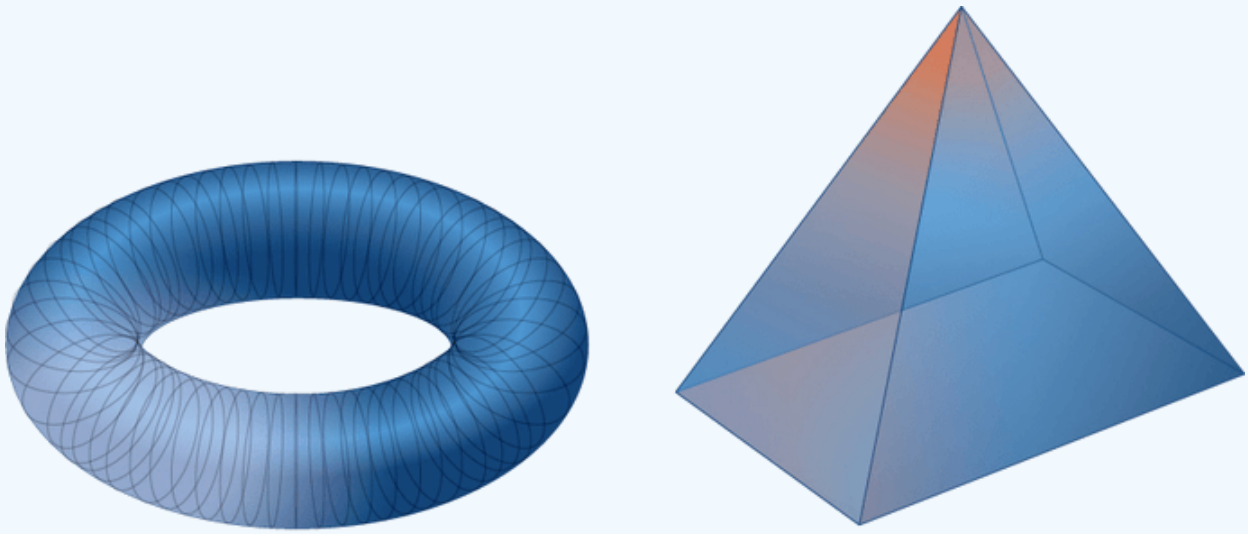


Figure 15.6.8: (a) This surface is smooth. (b) This surface is piecewise smooth.

Solution

The surface in Figure 15.6.8a can be parameterized by

$$\vecs{r}(u,v) = \langle (2+\cos v)\cos u,\ (2+\cos v)\sin u,\ \sin v\rangle,\ 0 \le u < 2\pi,\ 0 \le v < 2\pi$$

(we can use technology to verify). Notice that vectors

$$\vecs{r}_u = \langle -(2+\cos v)\sin u,\ (2+\cos v)\cos u, 0\rangle$$

and

$$\vecs{r}_v = \langle -\sin v\ \cos u,\ -\sin v\ \sin u,\ \cos v\rangle$$

exist for any choice of u and v in the parameter domain, and

$$\begin{aligned}\vecs{r}_u \times \vecs{r}_v &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -(2+\cos v)\sin u & (2+\cos v)\cos u & 0 \\ -\sin v\ \cos u & -\sin v\ \sin u & \cos v \end{vmatrix} \\ &= [(2+\cos v)\cos u\ \cos v]\hat{\mathbf{i}} + [2+\cos v)\sin u\ \cos v]\hat{\mathbf{j}} + [(2+\cos v)\sin v\ \sin^2 u + (2+\cos v)\sin v\ \cos^2 u]\hat{\mathbf{k}} \\ &= [(2+\cos v)\cos u\ \cos v]\hat{\mathbf{i}} + [(2+\cos v)\sin u\ \cos v]\hat{\mathbf{j}} + [(2+\cos v)\sin v]\hat{\mathbf{k}}.\end{aligned}$$

The $\hat{\mathbf{k}}$ component of this vector is zero only if $v = 0$ or $v = \pi$. If $v = 0$ or $v = \pi$, then the only choices for u that make the $\hat{\mathbf{j}}$ component zero are $u = 0$ or $u = \pi$. But, these choices of u do not make the $\hat{\mathbf{i}}$ component zero. Therefore, $\vecs{r}_u \times \vecs{r}_v$ is not zero for any choice of u and v in the parameter domain, and the parameterization is smooth. Notice that the corresponding surface has no sharp corners.

In the pyramid in Figure 15.6.8b, the sharpness of the corners ensures that directional derivatives do not exist at those locations. Therefore, the pyramid has no smooth parameterization. However, the pyramid consists of four smooth faces, and thus this surface is piecewise smooth.

Exercise 15.6.4

Is the surface parameterization $\vecs r(u,v) = \langle u^{2v}, v+1,\ \sin u\rangle,\ 0 \le u \le 2,\ 0 \le v \le 3$ smooth?

Hint available at the end of this section.

Answer available at the end of this section.

Surface Area of a Parametric Surface

Our goal is to define a surface integral, and as a first step we have examined how to parameterize a surface. The second step is to define the surface area of a parametric surface. The notation needed to develop this definition is used throughout the rest of this chapter.

Let S be a surface with parameterization $\vecs r(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$ over some parameter domain D. We assume here and throughout that the surface parameterization $\vecs r(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$ is continuously differentiable—meaning, each component function has continuous partial derivatives. Assume for the sake of simplicity that D is a rectangle (although the following material can be extended to handle nonrectangular parameter domains). Divide rectangle D into subrectangles D_{ij} with horizontal width Δu and vertical length Δv. Suppose that *i* ranges from 1 to *m* and *j* ranges from 1 to *n* so that D is subdivided into *mn* rectangles. This division of D into subrectangles gives a corresponding division of surface S into pieces S_{ij}. Choose point P_{ij} in each piece S_{ij}. Point P_{ij} corresponds to point (u_i, v_j) in the parameter domain.

Note that we can form a grid with lines that are parallel to the u-axis and the v-axis in the uv-plane. These grid lines correspond to a set of grid curves on surface S that is parameterized by $\vecs r(u,v)$. Without loss of generality, we assume that P_{ij} is located at the corner of two grid curves, as in Figure 15.6.9. If we think of $\vecs r$ as a mapping from the uv-plane to $\mathbb{R}^3$, the grid curves are the image of the grid lines under $\vecs r$. To be precise, consider the grid lines that go through point (u_i, v_j). One line is given by $x = u_i,\ y = v$; the other is given by $x = u,\ y = v_j$. In the first grid line, the horizontal component is held constant, yielding a vertical line through (u_i, v_j). In the second grid line, the vertical component is held constant, yielding a horizontal line through (u_i, v_j). The corresponding grid curves are $\vecs r(u_i, v)$ and (u, v_j) and these curves intersect at point P_{ij}.

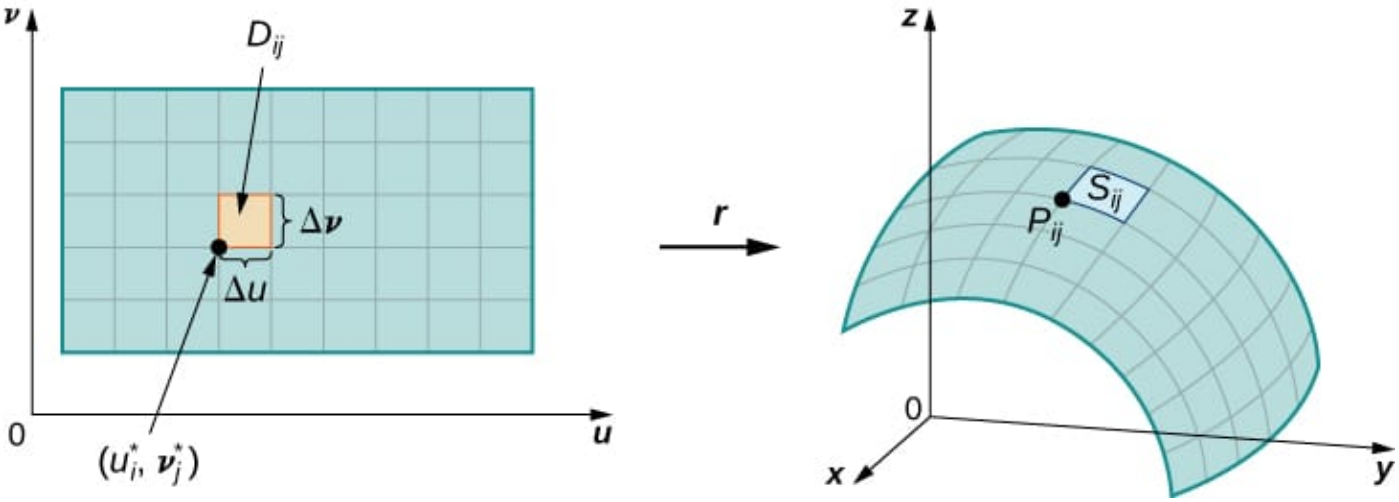


Figure 15.6.9: Grid lines on a parameter domain correspond to grid curves on a surface.

Now consider the vectors that are tangent to these grid curves. For grid curve $\vecs r(u_i, v)$, the tangent vector at P_{ij} is

$$\vecs t_v(P_{ij}) = \vecs r_v(u_i, v_j) = \langle x_v(u_i, v_j),\ y_v(u_i, v_j),\ z_v(u_i, v_j)\rangle.$$

For grid curve $\vecs r(u, v_j)$, the tangent vector at P_{ij} is

$$\vecs t_u(P_{ij}) = \vecs r_u(u_i, v_j) = \langle x_u(u_i, v_j),\ y_u(u_i, v_j),\ z_u(u_i, v_j)\rangle.$$

If vector $\vecs N = \vecs t_u(P_{ij}) \times \vecs t_v(P_{ij})$ exists and is not zero, then the tangent plane at P_{ij} exists (Figure 15.6.10). If piece S_{ij} is small enough, then the tangent plane at point P_{ij} is a good approximation of piece S_{ij}.

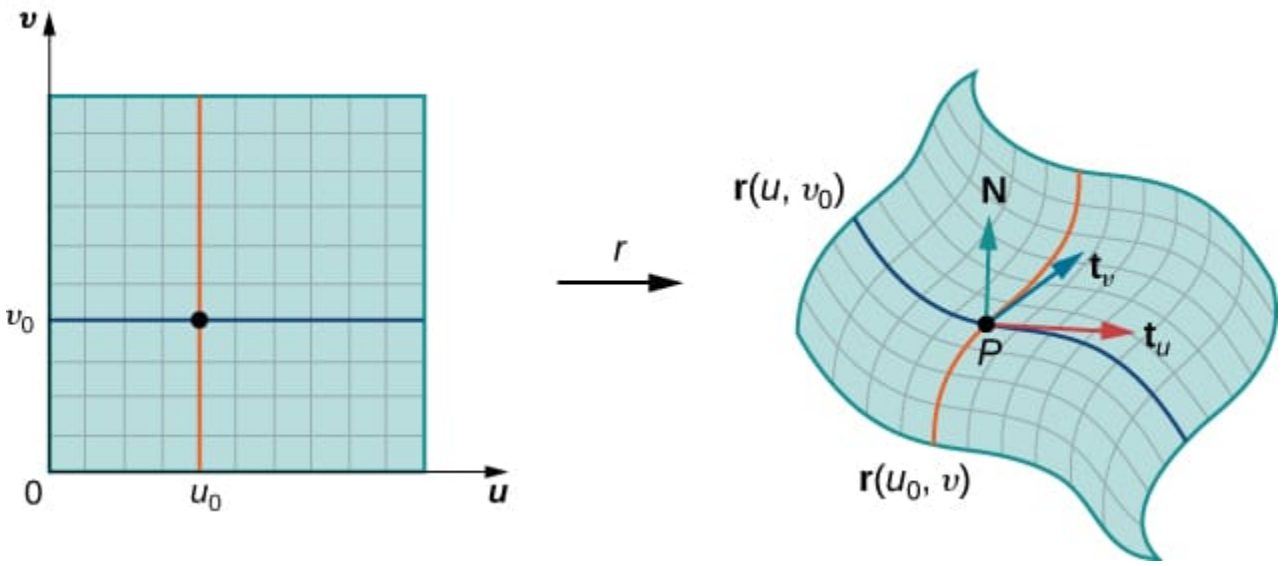


Figure 15.6.10: If the cross product of vectors $\vecs t_u$ and $\vecs t_v$ exists, then there is a tangent plane.

The tangent plane at P_{ij} contains vectors $\vecs t_u(P_{ij})$ and $\vecs t_v(P_{ij})$ and therefore the parallelogram spanned by $\vecs t_u(P_{ij})$ and $\vecs t_v(P_{ij})$ is in the tangent plane. Since the original rectangle in the uv-plane corresponding to S_{ij} has width Δu and length Δv, the parallelogram that we use to approximate S_{ij} is the parallelogram spanned by $\Delta u \vecs t_u(P_{ij})$ and $\Delta v \vecs t_v(P_{ij})$. In other words, we scale the tangent vectors by the constants Δu and Δv to match the scale of the original division of rectangles in the parameter domain. Therefore, the area of the parallelogram used to approximate the area of S_{ij} is

$$\Delta S_{ij} \approx ||(\Delta u \vecs t_u(P_{ij})) \times (\Delta v \vecs t_v(P_{ij}))|| = ||\vecs t_u(P_{ij}) \times \vecs t_v(P_{ij})|| \Delta u \, \Delta v.$$

Varying point P_{ij} over all pieces S_{ij} and the previous approximation leads to the following definition of surface area of a parametric surface (Figure 15.6.11).

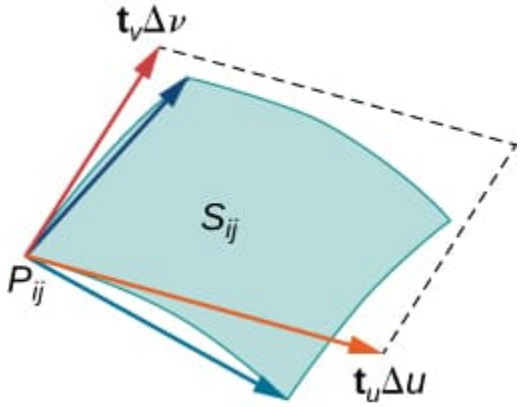


Figure 15.6.11: The parallelogram spanned by $\vecs t_u$ and $\vecs t_v$ approximates the piece of surface S_{ij}.

Definition: Smooth Parameterization of Surface

Let $\vecs r(u, v) = \langle x(u, v), \ y(u, v), \ z(u, v) \rangle$ with parameter domain D be a **smooth parameterization of surface** S. Furthermore, assume that S is traced out only once as (u, v) varies over D. The surface area of S is

$$\iint_D ||\vecs t_u \times \vecs t_v|| \, dA, \tag{15.6.1}$$

where $\vecs{t}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$

and

$$\vecs{t}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle.$$

Example 15.6.5: Calculating Surface Area

Calculate the lateral surface area (the area of the "side," not including the base) of the right circular cone with height h and radius r.

Solution

Before calculating the surface area of this cone using Equation 15.6.1, we need a parameterization. We assume this cone is in $\mathbb{R}^3$ with its vertex at the origin (Figure 15.6.12). To obtain a parameterization, let α be the angle that is swept out by starting at the positive z-axis and ending at the cone, and let $k = \tan\alpha$. For a height value v with $0 \le v \le h$, the radius of the circle formed by intersecting the cone with plane $z = v$ is kv. Therefore, a parameterization of this cone is

$$\vecs{s}(u, v) = \langle kv\cos u,\ kv\sin u,\ v\rangle,\ 0 \le u < 2\pi,\ 0 \le v \le h.$$

The idea behind this parameterization is that for a fixed v-value, the circle swept out by letting u vary is the circle at height v and radius kv. As v increases, the parameterization sweeps out a "stack" of circles, resulting in the desired cone.

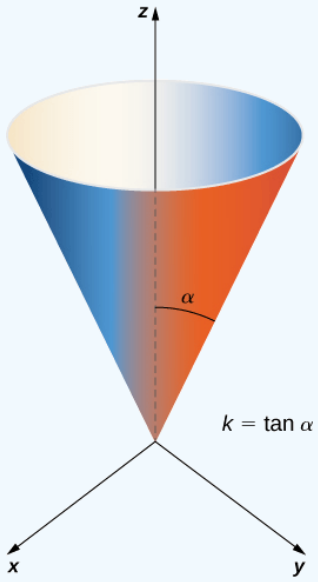


Figure 15.6.12: The right circular cone with radius $r = kh$ and height h has parameterization $\vecs{s}(u, v) = \langle kv\ \cos u,\ kv\ \sin u,\ v\rangle,\ 0 \le u < 2\pi,\ 0 \le v \le h$.

With a parameterization in hand, we can calculate the surface area of the cone using Equation 15.6.1. The tangent vectors are $\vecs{t}_u = \langle -kv\ \sin u,\ kv\ \cos u,\ 0\rangle$ and $\vecs{t}_v = \langle k\ \cos u,\ k\ \sin u,\ 1\rangle$. Therefore,

$$\begin{aligned}\vecs{t}_u \times \vecs{t}_v &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -kv\sin u & kv\cos u & 0 \\ k\cos u & k\sin u & 1 \end{vmatrix} \\ &= \langle kv\cos u,\ kv\sin u,\ -k^2v\sin^2 u - k^2v\cos^2 u\rangle \\ &= \langle kv\cos u,\ kv\sin u,\ -k^2v\rangle.\end{aligned}$$

The magnitude of this vector is

$$||\langle kv\cos u,\ kv\sin u,\ -k^2v\rangle|| = \sqrt{k^2v^2\cos^2 u + k^2v^2\sin^2 u + k^4v^2}$$
$$= \sqrt{k^2v^2 + k^4v^2}$$
$$= kv\sqrt{1+k^2}.$$

By Equation 15.6.1, the surface area of the cone is

$$\iint_D ||\vecs{t}_u \times \vecs{t}_v||\,dA = \int_0^h \int_0^{2\pi} kv\sqrt{1+k^2}\,du\,dv$$
$$= 2\pi k\sqrt{1+k^2}\int_0^h v\,dv$$
$$= 2\pi k\sqrt{1+k^2}\left[\frac{v^2}{2}\right]_0^h$$
$$= \pi k h^2\sqrt{1+k^2}.$$

Since $k = \tan\alpha = r/h$,

$$\pi k h^2\sqrt{1+k^2} = \pi\frac{r}{h}h^2\sqrt{1+\frac{r^2}{h^2}}$$
$$= \pi r h\sqrt{1+\frac{r^2}{h^2}}$$
$$= \pi r\sqrt{h^2 + h^2\left(\frac{r^2}{h^2}\right)}$$
$$= \pi r\sqrt{h^2+r^2}.$$

Therefore, the lateral surface area of the cone is $\pi r\sqrt{h^2+r^2}$.

Analysis

The surface area of a right circular cone with radius r and height h is usually given as $\pi r^2 + \pi r\sqrt{h^2+r^2}$. The reason for this is that the circular base is included as part of the cone, and therefore the area of the base πr^2 is added to the lateral surface area $\pi r\sqrt{h^2+r^2}$ that we found.

Exercise 15.6.5

Find the surface area of the surface with parameterization $\vecs{r}(u,v) = \langle u+v,\ u^2,\ 2v\rangle,\ 0\le u\le 3,\ 0\le v\le 2$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 15.6.6: Calculating Surface Area

Show that the surface area of the sphere $x^2+y^2+z^2 = r^2$ is $4\pi r^2$.

Solution

The sphere has parameterization

$$r\ \cos\theta\ \sin\phi,\ \ r\ \sin\theta\ \sin\phi,\ \ r\ \cos\phi\rangle,\ \ 0\le\theta<2\pi,\ \ 0\le\phi\le\pi.$$

The tangent vectors are

$$\vecs{t}_\theta = \langle -r\ \sin\theta\ \sin\phi,\ r\ \cos\theta\ \sin\phi,\ 0\rangle$$

and

$$\vecs{t}_\phi = \langle r\ \cos\theta\ \cos\phi,\ r\ \sin\theta\ \cos\phi,\ -r\ \sin\phi\rangle.$$

Therefore,

$$\begin{aligned}\vecs{t}_\phi \times \vecs{t}_\theta &= \langle r^2\cos\theta\ \sin^2\phi,\ r^2\sin\theta\ \sin^2\phi,\ r^2\sin^2\theta\ \sin\phi\ \cos\phi + r^2\cos^2\theta\ \sin\phi\ \cos\phi\rangle \\ &= \langle r^2\cos\theta\ \sin^2\phi,\ r^2\sin\theta\ \sin^2\phi,\ r^2\sin\phi\ \cos\phi\rangle.\end{aligned}$$

Now,

$$\begin{aligned}||\vecs{t}_\phi \times \vecs{t}_\theta|| &= \sqrt{r^4\sin^4\phi\ \cos^2\theta + r^4\sin^4\phi\ \sin^2\theta + r^4\sin^2\phi\ \cos^2\phi} \\ &= \sqrt{r^4\sin^4\phi + r^4\sin^2\phi\ \cos^2\phi} \\ &= r^2\sqrt{\sin^2\phi} \\ &= r\ \sin\phi.\end{aligned}$$

Notice that $\sin\phi \geq 0$ on the parameter domain because $0 \leq \phi < \pi$, and this justifies equation $\sqrt{\sin^2\phi} = \sin\phi$. The surface area of the sphere is

$$\int_0^{2\pi}\int_0^{\pi} r^2\sin\phi\,d\phi\,d\theta = r^2\int_0^{2\pi} 2\,d\theta = 4\pi r^2.$$

We have derived the familiar formula for the surface area of a sphere using surface integrals.

Exercise 15.6.6

Show that the surface area of cylinder $x^2 + y^2 = r^2,\ 0 \leq z \leq h$ is $2\pi rh$. Notice that this cylinder does not include the top and bottom circles.

Hint available at the end of this section.

Answer available at the end of this section.

In addition to parameterizing surfaces given by equations or standard geometric shapes such as cones and spheres, we can also parameterize surfaces of revolution. Therefore, we can calculate the surface area of a surface of revolution by using the same techniques. Let $y = f(x) \geq 0$ be a positive single-variable function on the domain $a \leq x \leq b$ and let S be the surface obtained by rotating f about the x-axis (Figure 15.6.13). Let θ be the angle of rotation. Then, S can be parameterized with parameters x and θ by

$$\vecs{r}(x,\theta) = \langle x, f(x)\ \cos\theta,\ f(x)\sin\theta\rangle,\ a \leq x \leq b,\ 0 \leq x \leq 2\pi.$$

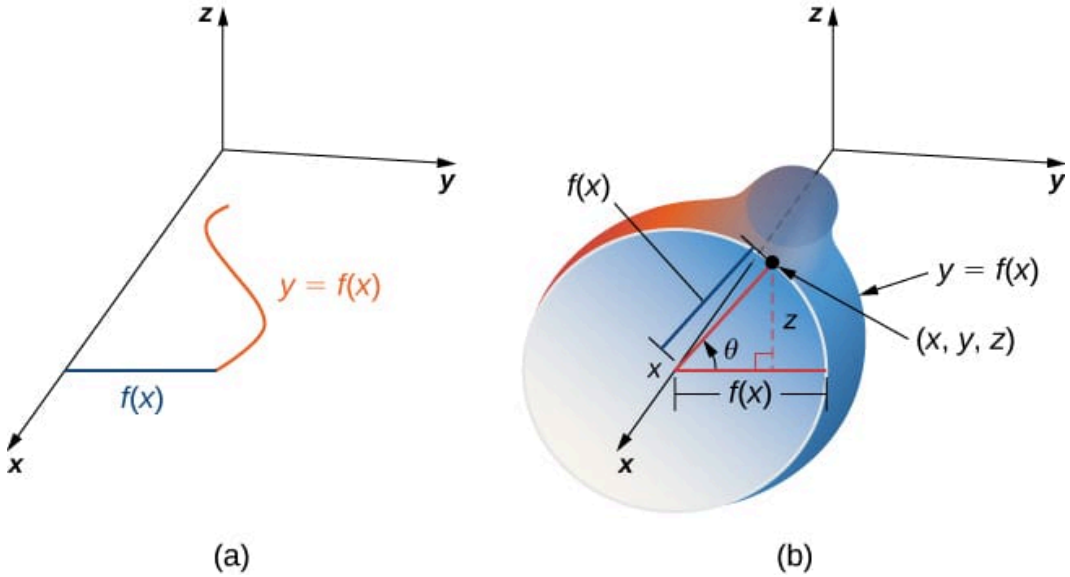


Figure 15.6.13: We can parameterize a surface of revolution by $\vecs{r}(x,\theta) = \langle x, f(x)\cos\theta, f(x)\sin\theta\rangle,\ a \le x \le b,\ 0 \le x < 2\pi$.

Example 15.6.7: Calculating Surface Area

Find the area of the surface of revolution obtained by rotating $y = x^2,\ 0 \le x \le b$ about the x-axis (Figure 15.6.14).

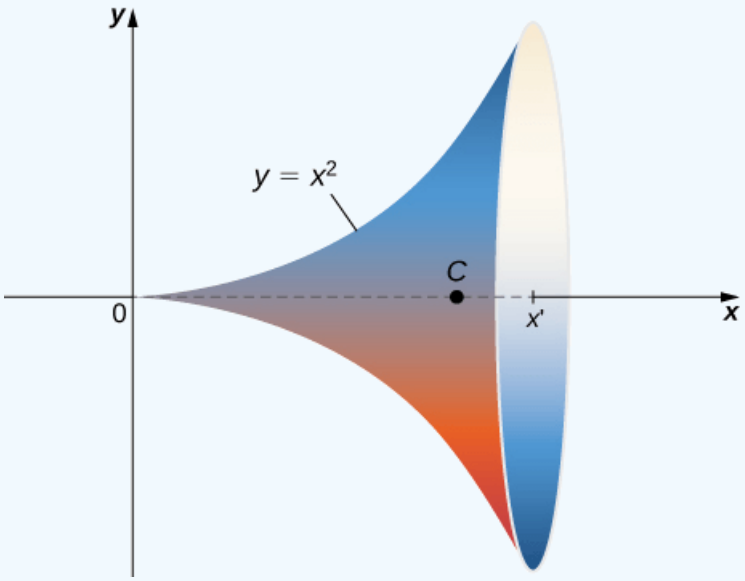


Figure 15.6.14: A surface integral can be used to calculate the surface area of this solid of revolution.

Solution

This surface has parameterization $\vecs{r}(x,\theta) = \langle x,\ x^2\cos\theta,\ x^2\sin\theta\rangle,\ 0 \le x \le b,\ 0 \le x < 2\pi$.

The tangent vectors are $\vecs{t}_x = \langle 1,\ 2x\ \cos\theta,\ 2x\ \sin\theta\rangle$ and $\vecs{t}_\theta = \langle 0,\ -x^2\sin\theta,\ -x^2\cos\theta\rangle$.

Therefore,

$$\begin{aligned}\vecs{t}_x \times \vecs{t}_\theta &= \langle 2x^3\cos^2\theta + 2x^3\sin^2\theta,\ -x^2\cos\theta,\ -x^2\sin\theta\rangle \\ &= \langle 2x^3,\ -x^2\cos\theta,\ -x^2\sin\theta\rangle\end{aligned}$$

and

$$\begin{aligned}\vecs{t}_x \times \vecs{t}_\theta &= \sqrt{4x^6 + x^4\cos^2\theta + x^4\sin^2\theta} \\ &= \sqrt{4x^6 + x^4} \\ &= x^2\sqrt{4x^2+1}\end{aligned}$$

The area of the surface of revolution is

$$\begin{aligned}\int_0^b \int_0^{2\pi} x^2\sqrt{4x^2+1}\,d\theta\,dx &= 2\pi\int_0^b x^2\sqrt{4x^2+1}\,dx \\ &= 2\pi\left[\frac{1}{64}\left(2\sqrt{4x^2+1}(8x^3+x)\,\sinh^{-1}(2x)\right)\right]_0^b \\ &= 2\pi\left[\frac{1}{64}\left(2\sqrt{4b^2+1}(8b^3+b)\,\sinh^{-1}(2b)\right)\right].\end{aligned}$$

Exercise 15.6.7

Use Equation 15.6.1 to find the area of the surface of revolution obtained by rotating curve $y = \sin x,\ 0 \le x \le \pi$ about the x -axis.

Hint available at the end of this section.

Answer available at the end of this section.

Surface Integral of a Scalar-Valued Function

Now that we can parameterize surfaces and we can calculate their surface areas, we are able to define surface integrals. First, let's look at the surface integral of a scalar-valued function. Informally, the surface integral of a scalar-valued function is an analog of a scalar line integral in one higher dimension. The domain of integration of a scalar line integral is a parameterized curve (a one-dimensional object); the domain of integration of a scalar surface integral is a parameterized surface (a two-dimensional object). Therefore, the definition of a surface integral follows the definition of a line integral quite closely. For scalar line integrals, we chopped the domain curve into tiny pieces, chose a point in each piece, computed the function at that point, and took a limit of the corresponding Riemann sum. For scalar surface integrals, we chop the domain *region* (no longer a curve) into tiny pieces and proceed in the same fashion.

Let S be a piecewise smooth surface with parameterization $\vecs{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$ with parameter domain D and let $f(x,y,z)$ be a function with a domain that contains S. For now, assume the parameter domain D is a rectangle, but we can extend the basic logic of how we proceed to any parameter domain (the choice of a rectangle is simply to make the notation more manageable). Divide rectangle D into subrectangles D_{ij} with horizontal width Δu and vertical length Δv. Suppose that i ranges from 1 to m and j ranges from 1 to n so that D is subdivided into mn rectangles. This division of D into subrectangles gives a corresponding division of S into pieces S_{ij}. Choose point P_{ij} in each piece S_{ij} evaluate P_{ij} at f, and multiply by area S_{ij} to form the Riemann sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij})\,\Delta S_{ij}.$$

To define a surface integral of a scalar-valued function, we let the areas of the pieces of S shrink to zero by taking a limit.

Definition: Surface Integral of a Scalar-Valued Function

The **surface integral of a scalar-valued function** of f over a piecewise smooth surface S is

$$\iint_S f(x,y,z)dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij})\Delta S_{ij}.$$

Again, notice the similarities between this definition and the definition of a scalar line integral. In the definition of a line integral we chop a curve into pieces, evaluate a function at a point in each piece, and let the length of the pieces shrink to zero by taking the limit of the corresponding Riemann sum. In the definition of a surface integral, we chop a surface into pieces, evaluate a function at a point in each piece, and let the area of the pieces shrink to zero by taking the limit of the corresponding Riemann sum. Thus, a surface integral is similar to a line integral but in one higher dimension.

The definition of a scalar line integral can be extended to parameter domains that are not rectangles by using the same logic used earlier. The basic idea is to chop the parameter domain into small pieces, choose a sample point in each piece, and so on. The exact shape of each piece in the sample domain becomes irrelevant as the areas of the pieces shrink to zero.

Scalar surface integrals are difficult to compute from the definition, just as scalar line integrals are. To develop a method that makes surface integrals easier to compute, we approximate surface areas ΔS_{ij} with small pieces of a tangent plane, just as we did in the previous subsection. Recall the definition of vectors $\vecs t_u$ and $\vecs t_v$:

$$\vecs t_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \text{ and } \vecs t_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle.$$

From the material we have already studied, we know that

$$\Delta S_{ij} \approx ||\vecs t_u(P_{ij}) \times \vecs t_v(P_{ij})|| \, \Delta u \, \Delta v.$$

Therefore,

$$\iint_S f(x,y,z)\, dS \approx \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}) ||\vecs t_u(P_{ij}) \times \vecs t_v(P_{ij})|| \, \Delta u \, \Delta v.$$

This approximation becomes arbitrarily close to $\lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}) \Delta S_{ij}$ as we increase the number of pieces S_{ij} by letting m and n go to infinity. Therefore, we have the following equation to calculate **scalar surface integrals**:

$$\iint_S f(x,y,z)\, dS = \iint_D f(\vecs r(u,v)) ||\vecs t_u \times \vecs t_v||\, dA. \tag{15.6.2}$$

Equation 15.6.2 allows us to calculate a surface integral by transforming it into a double integral. This equation for surface integrals is analogous to the equation for line integrals:

$$\iint_C f(x,y,z)\, ds = \int_a^b f(\vecs r(t)) ||\vecs r'(t)||\, dt.$$

In this case, vector $\vecs t_u \times \vecs t_v$ is perpendicular to the surface, whereas vector $\vecs r'(t)$ is tangent to the curve.

Example 15.6.8: Calculating a Surface Integral

Calculate surface integral

$$\iint_S 5\, dS,$$

where S is the surface with parameterization $\vecs r(u,v) = \langle u,\ u^2,\ v \rangle$ for $0 \le u \le 2$ and $0 \le v \le u$.

Solution

Notice that this parameter domain D is a triangle, and therefore the parameter domain is not rectangular. This is not an issue though, because Equation 15.6.2 does not place any restrictions on the shape of the parameter domain.

To use Equation 15.6.2 to calculate the surface integral, we first find vectors $\vecs t_u$ and $\vecs t_v$. Note that $\vecs t_u = \langle 1, 2u, 0 \rangle$ and $\vecs t_v = \langle 0, 0, 1 \rangle$. Therefore,

$$\vecs{t}_u \times \vecs{t}_v = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 2u, -1, 0 \rangle$$

and

$$||\vecs{t}_u \times \vecs{t}_v|| = \sqrt{1+4u^2}.$$

By Equation 15.6.2,

$$\begin{aligned} \iint_S 5\, dS &= 5 \iint_D \sqrt{1+4u^2}\, dA \\ &= 5 \int_0^2 \int_0^u \sqrt{1+4u^2}\, dv\, du = 5 \int_0^2 u\sqrt{1+4u^2}\, du \\ &= 5 \left[\frac{(1+4u^2)^{3/2}}{3} \right]_0^2 \\ &= \frac{5(17^{3/2}-1)}{3} \approx 115.15. \end{aligned}$$

Example 15.6.9: Calculating the Surface Integral of a Cylinder

Calculate surface integral

$$\iint_S (x+y^2)\, dS,$$

where S is cylinder $x^2+y^2=4,\ 0 \le z \le 3$ (Figure 15.6.15).

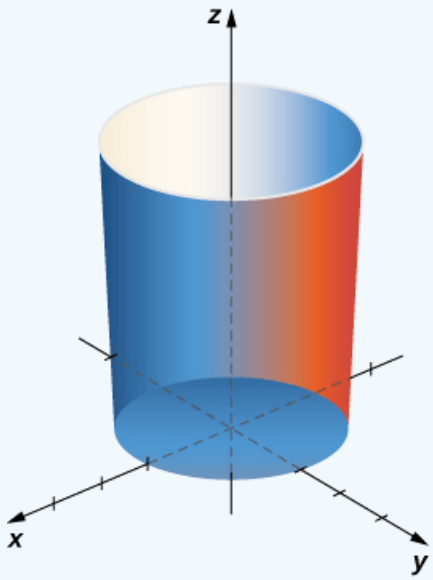


Figure 15.6.15: Integrating function $f(x,y,z) = x+y^2$ over a cylinder.

Solution

To calculate the surface integral, we first need a parameterization of the cylinder. A parameterization is $\vecs{r}(u,v) = \langle \cos u,\ \sin u,\ v \rangle, 0 \le u \le 2\pi,\ 0 \le v \le 3$.

The tangent vectors are $\vecs{t}_u = \langle \sin u,\ \cos u,\ 0 \rangle$ and $\vecs{t}_v = \langle 0, 0, 1 \rangle$. Then,

$$\vecs{t}_u \times \vecs{t}_v = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \cos u,\ \sin u,\ 0 \rangle$$

and $||\vecs{t}_u \times \vecs{t}_v|| = \sqrt{\cos^2 u + \sin^2 u} = 1$. By Equation 15.6.2,

$$\begin{aligned} \iint_S f(x,y,z)dS &= \iint_D f(\vecs{r}(u,v))||\vecs{t}_u \times \vecs{t}_v||\, dA \\ &= \int_0^3 \int_0^{2\pi} (\cos u + \sin^2 u)\, du\, dv \\ &= \int_0^3 \left[\sin u + \frac{u}{2} - \frac{\sin(2u)}{4} \right]_0^{2\pi} dv \\ &= \int_0^3 \pi\, dv = 3\pi. \end{aligned}$$

Exercise 15.6.8

Calculate

$$\iint_S (x^2 - z)\, dS,$$

where S is the surface with parameterization $\vecs{r}(u,v) = \langle v,\ u^2 + v^2,\ 1 \rangle,\ 0 \le u \le 2,\ 0 \le v \le 3$.

Hint available at the end of this section.

Answer available at the end of this section.

Example 15.6.10: Calculating the Surface Integral of a Piece of a Sphere

Calculate surface integral

$$\iint_S f(x,y,z)\, dS,$$

where $f(x,y,z) = z^2$ and S is the surface that consists of the piece of sphere $x^2 + y^2 + z^2 = 4$ that lies on or above plane $z = 1$ and the disk that is enclosed by intersection plane $z = 1$ and the given sphere (Figure 15.6.16).

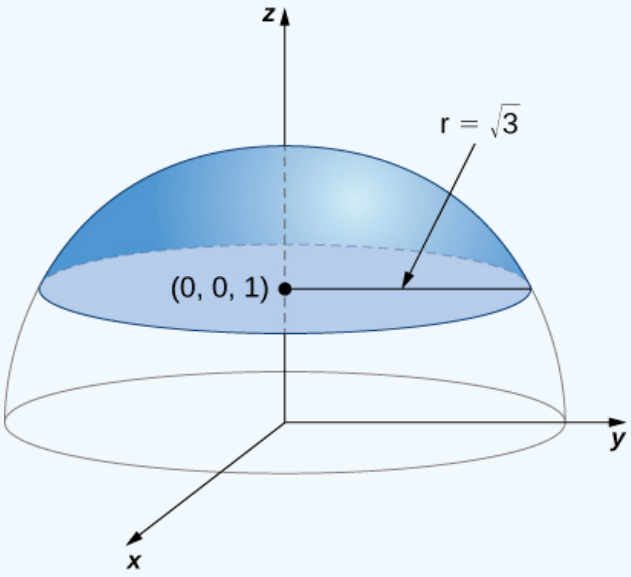


Figure 15.6.16: Calculating a surface integral over surface S.

Solution

Notice that S is not smooth but is piecewise smooth; S can be written as the union of its base S_1 and its spherical top S_2, and both S_1 and S_2 are smooth. Therefore, to calculate

$$\iint_S z^2 dS,$$

we write this integral as

$$\iint_{S_1} z^2\, dS + \iint_{S_2} z^2\, dS$$

and we calculate integrals

$$\iint_{S_1} z^2\, dS$$

and

$$\iint_{S_2} Z^2\, dS.$$

First, we calculate $\iint_{S_1} z^2\ dS$. To calculate this integral we need a parameterization of S_1. This surface is a disk in plane $z = 1$ centered at $(0, 0, 1)$. To parameterize this disk, we need to know its radius. Since the disk is formed where plane $z = 1$ intersects sphere $x^2 + y^2 + z^2 = 4$, we can substitute $z = 1$ into equation $x^2 + y^2 + z^2 = 4$:

$$x^2 + y^2 + 1 = 4 \Rightarrow x^2 + y^2 = 3.$$

Therefore, the radius of the disk is $\sqrt{3}$ and a parameterization of S_1 is $\vecs r(u, v) = \langle u\ \cos v,\ u\ \sin v,\ 1\rangle,\ 0 \le u \le \sqrt{3},\ 0 \le v \le 2\pi$. The tangent vectors are $\vecs t_u = \langle \cos v,\ \sin v,\ 0\rangle$ and $\vecs t_v = \langle -u\ \sin v,\ u\ \cos v,\ 0\rangle$, and thus

$$\vecs t_u \times \vecs t_v = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \langle 0,\ 0, u\ \cos^2 v + u\ \sin^2 v\rangle = \langle 0, 0, u\rangle.$$

The magnitude of this vector is u. Therefore,

$$\begin{aligned} \iint_{S_1} z^2\, dS &= \int_0^{\sqrt{3}} \int_0^{2\pi} f(r(u,v))||t_u \times t_v||\, dv\, du \\ &= \int_0^{\sqrt{3}} \int_0^{2\pi} u\, dv\, du \\ &= 2\pi \int_0^{\sqrt{3}} u\, du \\ &= 2\pi\sqrt{3}. \end{aligned}$$

Now we calculate

$$\iint_{S_2} dS.$$

To calculate this integral, we need a parameterization of S_2. The parameterization of full sphere $x^2 + y^2 + z^2 = 4$ is

$$\vecs r(\phi, \theta) = \langle 2\ \cos\theta\ \sin\phi,\ 2\ \sin\theta\ \sin\phi,\ 2\ \cos\phi\rangle,\ 0 \le \theta \le 2\pi, 0 \le \phi \le \pi.$$

Since we are only taking the piece of the sphere on or above plane $z = 1$, we have to restrict the domain of ϕ. To see how far this angle sweeps, notice that the angle can be located in a right triangle, as shown in Figure 15.6.17 (the $\sqrt{3}$ comes from the fact that the base of S is a disk with radius $\sqrt{3}$). Therefore, the tangent of ϕ is $\sqrt{3}$, which implies that ϕ is $\pi/6$. We now have a parameterization of S_2:

$$\vecs{r}(\phi, \theta) = \langle 2\ \cos\theta\ \sin\phi,\ 2\ \sin\theta\ \sin\phi,\ 2\ \cos\phi\rangle,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi/3.$$

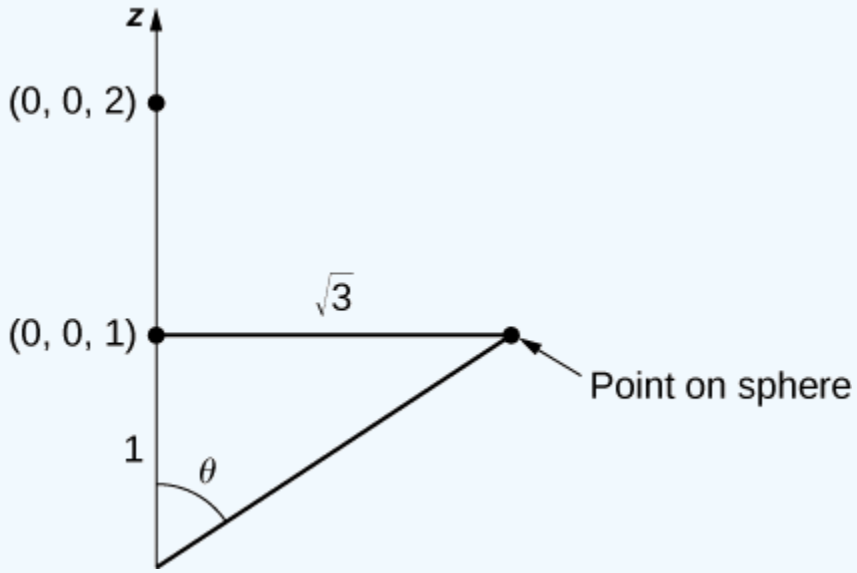


Figure 15.6.17: The maximum value of ϕ has a tangent value of $\sqrt{3}$.

The tangent vectors are $\vecs{t}_\phi = \langle 2\ \cos\theta\ \cos\phi,\ 2\ \sin\theta\ \cos\phi,\ -2\ \sin\phi\rangle$ and $\vecs{t}_\theta = \langle -2\sin\theta\sin\phi,\ u\cos\theta\sin\phi,\ 0\rangle$, and thus

$$\begin{aligned}\vecs{t}_\phi \times \vecs{t}_\theta &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2\cos\theta\cos\phi & 2\sin\theta\cos\phi & -2\sin\phi \\ -2\sin\theta\sin\phi & 2\cos\theta\sin\phi & 0 \end{vmatrix} \\ &= \langle 4\ \cos\theta\ \sin^2\phi,\ 4\ \sin\theta\ \sin^2\phi,\ 4\ \cos^2\theta\ \cos\phi\ \sin\phi + 4\ \sin^2\theta\ \cos\phi\ \sin\phi\rangle \\ &= \langle 4\ \cos\theta\ \sin^2\phi,\ 4\ \sin\theta\ \sin^2\phi,\ 4\ \cos\phi\ \sin\phi\rangle.\end{aligned}$$

The magnitude of this vector is

$$\begin{aligned}\vecs{t}_\phi \times \vecs{t}_\theta &= \sqrt{16\ \cos^2\theta\ \sin^4\phi + 16\ \sin^2\theta\ \sin^4\phi + 16\ \cos^2\phi\ \sin^2\phi} \\ &= 4\sqrt{\sin^4\phi + \cos^2\phi\ \sin^2\phi}.\end{aligned}$$

Therefore,

$$\begin{aligned}\iint_{S_2} z\,dS &= \int_0^{\pi/6}\int_0^{2\pi} f(\vecs r(\phi,\theta))||\vecs t_\phi \times \vecs t_\theta||\,d\theta\,d\phi \\ &= \int_0^{\pi/6}\int_0^{2\pi} 16\,\cos^2\phi\sqrt{\sin^4\phi+\cos^2\phi\,\sin^2\phi}\,d\theta\,d\phi \\ &= 32\pi\int_0^{\pi/6}\cos^2\phi\sqrt{\sin^4\phi+\cos^2\phi\,\sin^2\phi}\,d\phi \\ &= 32\pi\int_0^{\pi/6}\cos^2\phi\,\sin\phi\sqrt{\sin^2\phi+\cos^2\phi}\,d\phi \\ &= 32\pi\int_0^{\pi/6}\cos^2\phi\,\sin\phi\,d\phi \\ &= 32\pi\left[-\frac{\cos^3\phi}{3}\right]_0^{\pi/6} \\ &= 32\pi\left[\frac{1}{3}-\frac{\sqrt{3}}{8}\right] = \frac{32\pi}{3}-4\sqrt{3}.\end{aligned}$$

Since

$$\iint_S z^2\,dS = \iint_{S_1} z^2\,dS + \iint_{S_2} z^2\,dS,$$

we have

$$\iint_S z^2\,dS = (2\pi-4)\sqrt{3}+\frac{32\pi}{3}.$$

Analysis

In this example we broke a surface integral over a piecewise surface into the addition of surface integrals over smooth subsurfaces. There were only two smooth subsurfaces in this example, but this technique extends to finitely many smooth subsurfaces.

Exercise 15.6.9

Calculate line integral $\iint_S (x-y)\;dS$, where S is cylinder $x^2+y^2=1,\;\;0\le z\le 2$, including the circular top and bottom.

Hint available at the end of this section.

Answer available at the end of this section.

Scalar surface integrals have several real-world applications. Recall that scalar line integrals can be used to compute the mass of a wire given its density function. In a similar fashion, we can use scalar surface integrals to compute the mass of a sheet given its density function. If a thin sheet of metal has the shape of surface S and the density of the sheet at point (x,y,z) is $\rho(x,y,z)$ then mass m of the sheet is

$$m = \iint_S \rho(x,y,z)\,dS. \tag{15.6.3}$$

Example 15.6.11: Calculating the Mass of a Sheet

A flat sheet of metal has the shape of surface $z=1+x+2y$ that lies above rectangle $0\le x\le 4$ and $0\le y\le 2$. If the density of the sheet is given by $\rho(x,y,z)=x^2yz$, what is the mass of the sheet?

Solution

Let S be the surface that describes the sheet. Then, the mass of the sheet is given by $m = \iint_S x^2yx \ dS$. To compute this surface integral, we first need a parameterization of S. Since S is given by the function $f(x,y) = 1 + x + 2y$, a parameterization of S is $\vecs r(x,y) = \langle x, \ y, \ 1 + x + 2y \rangle, \ 0 \le x \le 4, \ 0 \le y \le 2$.

The tangent vectors are $\vecs t_x = \langle 1, 0, 1 \rangle$ and $\vecs t_y = \langle 1, 0, 2 \rangle$. Therefore, $\vecs t_x + \vecs t_y = \langle -1, -2, 1 \rangle$ and $||\vecs t_x \times \vecs t_y|| = \sqrt{6}$.

By the definition of the line integral (Section 16.2),

$$\begin{aligned} m &= \iint_S x^2yz \, dS \\ &= \sqrt{6} \int_0^4 \int_0^2 x^2y(1 + x + 2y) \, dy \, dx \\ &= \sqrt{6} \int_0^4 \frac{22x^2}{3} + 2x^3 \, dx \\ &= \frac{2560\sqrt{6}}{9} \approx 696.74. \end{aligned}$$

Exercise 15.6.10

A piece of metal has a shape that is modeled by paraboloid $z = x^2 + y^2, \ 0 \le z \le 4$, and the density of the metal is given by $\rho(x,y,z) = z + 1$. Find the mass of the piece of metal.

Hint available at the end of this section.

Answer available at the end of this section.

Orientation of a Surface

Recall that when we defined a scalar line integral, we did not need to worry about an orientation of the curve of integration. The same was true for scalar surface integrals: we did not need to worry about an "orientation" of the surface of integration.

On the other hand, when we defined vector line integrals, the curve of integration needed an orientation. That is, we needed the notion of an oriented curve to define a vector line integral without ambiguity. Similarly, when we define a surface integral of a vector field, we need the notion of an oriented surface. An oriented surface is given an "upward" or "downward" orientation or, in the case of surfaces such as a sphere or cylinder, an "outward" or "inward" orientation.

Let S be a smooth surface. For any point (x,y,z) on S, we can identify two unit normal vectors $\vecs N$ and $-\vecs N$. If it is possible to choose a unit normal vector $\vecs N$ at every point (x,y,z) on S so that $\vecs N$ varies continuously over S, then S is "*orientable*." Such a choice of unit normal vector at each point gives the **orientation of a surface** S. If you think of the normal field as describing water flow, then the side of the surface that water flows toward is the "negative" side and the side of the surface at which the water flows away is the "positive" side. Informally, a choice of orientation gives S an "outer" side and an "inner" side (or an "upward" side and a "downward" side), just as a choice of orientation of a curve gives the curve "forward" and "backward" directions.

Closed surfaces such as spheres are orientable: if we choose the outward normal vector at each point on the surface of the sphere, then the unit normal vectors vary continuously. This is called the *positive orientation of the closed surface* (Figure 15.6.18). We also could choose the inward normal vector at each point to give an "inward" orientation, which is the negative orientation of the surface.

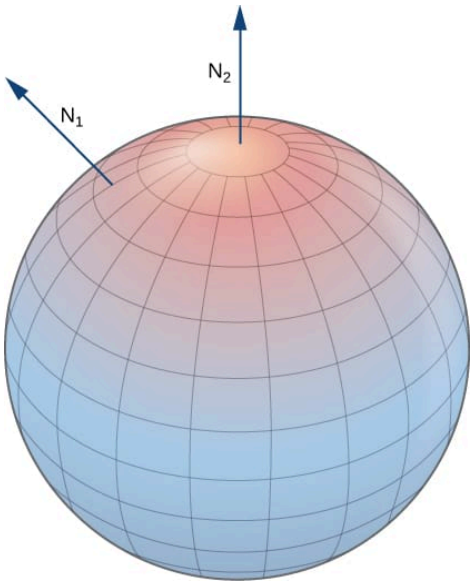


Figure 15.6.18: An oriented sphere with positive orientation.

A portion of the graph of any smooth function $z = f(x, y)$ is also orientable. If we choose the unit normal vector that points "above" the surface at each point, then the unit normal vectors vary continuously over the surface. We could also choose the unit normal vector that points "below" the surface at each point. To get such an orientation, we parameterize the graph of f in the standard way: $\vecs r(x, y) = \langle x,\ y,\ f(x, y)\rangle$, where x and y vary over the domain of f. Then, $\vecs t_x = \langle 1, 0, f_x\rangle$ and $\vecs t_y = \langle 0, 1, f_y\rangle$, and therefore the cross product $\vecs t_x \times \vecs t_y$ (which is normal to the surface at any point on the surface) is $\langle -f_x,\ -f_y,\ 1\rangle$Since the z-component of this vector is one, the corresponding unit normal vector points "upward," and the upward side of the surface is chosen to be the "positive" side.

Let S be a smooth orientable surface with parameterization $\vecs r(u, v)$. For each point $\vecs r(a, b)$ on the surface, vectors $\vecs t_u$ and $\vecs t_v$ lie in the tangent plane at that point. Vector $\vecs t_u \times \vecs t_v$ is normal to the tangent plane at $\vecs r(a, b)$ and is therefore normal to S at that point. Therefore, the choice of unit normal vector

$$\vecs N = \frac{\vecs t_u \times \vecs t_v}{||\vecs t_u \times \vecs t_v||}$$

gives an orientation of surface S.

Example 15.6.12: Choosing an Orientation

Give an orientation of cylinder $x^2 + y^2 = r^2,\ 0 \le z \le h$.

Solution

This surface has parameterization $\vecs r(u, v) = \langle r\ \cos u,\ r\ \sin u,\ v\rangle,\ 0 \le u < 2\pi,\ 0 \le v \le h$.

The tangent vectors are $\vecs t_u = \langle -r\ \sin u,\ r\ \cos u,\ 0\rangle$ and $\vecs t_v = \langle 0, 0, 1\rangle$. To get an orientation of the surface, we compute the unit normal vector

$$\vecs N = \frac{\vecs t_u \times \vecs t_v}{||\vecs t_u \times \vecs t_v||}$$

In this case, $\vecs t_u \times \vecs t_v = \langle r\ \cos u,\ r\ \sin u,\ 0\rangle$ and therefore

$$||\vecs t_u \times \vecs t_v|| = \sqrt{r^2 \cos^2 u + r^2 \sin^2 u} = r.$$

An orientation of the cylinder is

$$\vecs N(u,v) = \frac{\langle r\cos u,\ r\sin u,\ 0\rangle}{r} = \langle \cos u,\ \sin u,\ 0\rangle.$$

Notice that all vectors are parallel to the xy-plane, which should be the case with vectors that are normal to the cylinder. Furthermore, all the vectors point outward, and therefore this is an outward orientation of the cylinder (Figure 15.6.19).

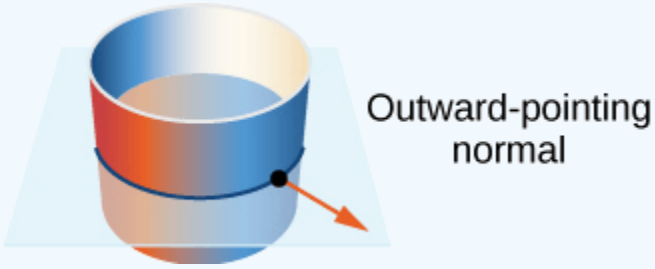


Figure 15.6.19: If all the vectors normal to a cylinder point outward, then this is an outward orientation of the cylinder.

Exercise 15.6.11

Give the "upward" orientation of the graph of $f(x,y) = xy$.

Hint available at the end of this section.

Answer available at the end of this section.

Since every curve has a "forward" and "backward" direction (or, in the case of a closed curve, a clockwise and counterclockwise direction), it is possible to give an orientation to any curve. Hence, it is possible to think of every curve as an oriented curve. This is not the case with surfaces, however. Some surfaces cannot be oriented; such surfaces are called *nonorientable*. Essentially, a surface can be oriented if the surface has an "inner" side and an "outer" side, or an "upward" side and a "downward" side. Some surfaces are twisted in such a fashion that there is no well-defined notion of an "inner" or "outer" side.

The classic example of a nonorientable surface is the Möbius strip. To create a Möbius strip, take a rectangular strip of paper, give the piece of paper a half-twist, and the glue the ends together (Figure 15.6.20). Because of the half-twist in the strip, the surface has no "outer" side or "inner" side. If you imagine placing a normal vector at a point on the strip and having the vector travel all the way around the band, then (because of the half-twist) the vector points in the opposite direction when it gets back to its original position. Therefore, the strip really only has one side.



Figure 15.6.20: The construction of a Möbius strip.

Since some surfaces are *nonorientable*, it is not possible to define a vector surface integral on all piecewise smooth surfaces. This is in contrast to vector line integrals, which can be defined on any piecewise smooth curve.

Surface Integral of a Vector Field

With the idea of orientable surfaces in place, we are now ready to define a surface integral of a vector field. The definition is analogous to the definition of the flux of a vector field along a plane curve. Recall that if $\vecs F$ is a two-dimensional vector field and C is a plane curve, then the definition of the flux of $\vecs F$ along C involved chopping C into small pieces, choosing a point inside each piece, and calculating $\vecs F \cdot \vecs N$ at the point (where $\vecs N$ is the unit normal vector at the point). The definition of a surface integral of a vector field proceeds in the same fashion, except now we chop surface S into small pieces, choose a point in the small (two-dimensional) piece, and calculate $\vecs F \cdot \vecs N$ at the point.

To place this definition in a real-world setting, let S be an oriented surface with unit normal vector $\vecs N$. Let $\vecs v$ be a velocity field of a fluid flowing through S, and suppose the fluid has density $\rho(x,y,z)$ Imagine the fluid flows through S, but S is completely permeable so that it does not impede the fluid flow (Figure 15.6.21). The mass flux of the fluid is the rate of mass flow per unit area. The mass flux is measured in mass per unit time per unit area. How could we calculate the mass flux of the fluid across S?

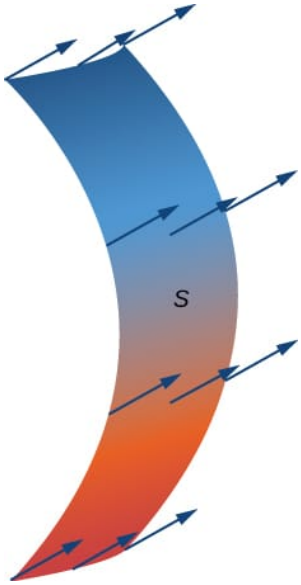


Figure 15.6.21: Fluid flows across a completely permeable surface S.

The rate of flow, measured in mass per unit time per unit area, is $\rho\vecs N$. To calculate the mass flux across S, chop S into small pieces S_{ij}. If S_{ij} is small enough, then it can be approximated by a tangent plane at some point P in S_{ij}. Therefore, the unit normal vector at P can be used to approximate $\vecs N(x,y,z)$ across the entire piece S_{ij} because the normal vector to a plane does not change as we move across the plane. The component of the vector ρv at P in the direction of $\vecs N$ is $\rho\vecs v\cdot\vecs N$ at P. Since S_{ij} is small, the dot product $\rho v\cdot N$ changes very little as we vary across S_{ij} and therefore $\rho\vecs v\cdot\vecs N$ can be taken as approximately constant across S_{ij}. To approximate the mass of fluid per unit time flowing across S_{ij} (and not just locally at point P), we need to multiply $(\rho\vecs v\cdot\vecs N)(P)$ by the area of S_{ij}. Therefore, the mass of fluid per unit time flowing across S_{ij} in the direction of $\vecs N$ can be approximated by $(\rho\vecs v\cdot\vecs N)\Delta S_{ij}$ where $\vecs N$, ρ and $\vecs v$ are all evaluated at P (Figure 15.6.22). This is analogous to the flux of two-dimensional vector field $\vecs F$ across plane curve C, in which we approximated flux across a small piece of C with the expression $(\vecs F\cdot\vecs N)\ \Delta s$. To approximate the mass flux across S, form the sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(\rho\vecs v\cdot\vecs N)\Delta S_{ij}.$$

As pieces S_{ij} get smaller, the sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(\rho\vecs v\cdot\vecs N)\Delta S_{ij}$$

gets arbitrarily close to the mass flux. Therefore, the mass flux is

$$\iint_S \rho\vecs v\cdot\vecs N\,dS=\lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}(\rho\vecs v\cdot\vecs N)\Delta S_{ij}.$$

This is a surface integral of a vector field. Letting the vector field $\rho\vecs v$ be an arbitrary vector field $\vecs F$ leads to the following definition.

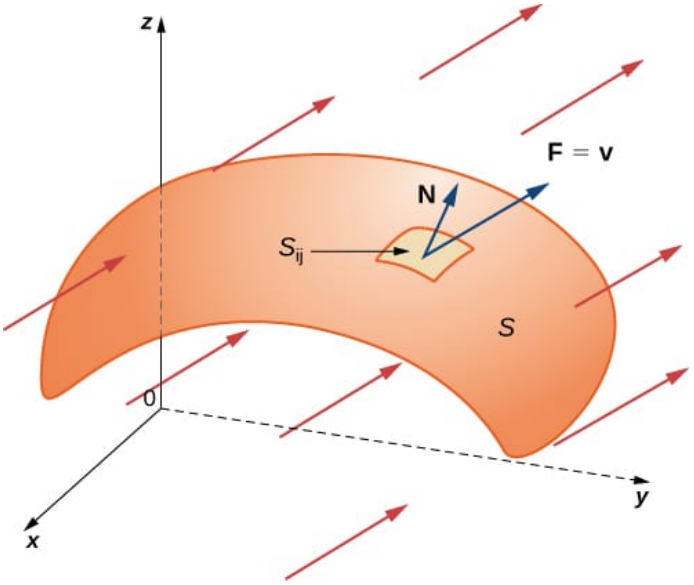


Figure 15.6.22: The mass of fluid per unit time flowing across S_{ij} in the direction of $\vecs N$ can be approximated by $(\rho \vecs v \cdot \vecs N)\Delta S_{ij}$.

Definition: Surface Integrals

Let $\vecs F$ be a continuous vector field with a domain that contains oriented surface S with unit normal vector $\vecs N$. The surface integral of $\vecs F$ over S is

$$\iint_S \vecs F \cdot d\vecs S = \iint_S \vecs F \cdot \vecs N \, dS. \tag{15.6.4}$$

Notice the parallel between this definition and the definition of vector line integral $\int_C \vecs F \cdot \vecs N \; dS$. A surface integral of a vector field is defined in a similar way to a flux line integral across a curve, except the domain of integration is a surface (a two-dimensional object) rather than a curve (a one-dimensional object). Integral $\iint_S \vecs F \cdot \vecs N \; dS$ is called the *flux of* $\vecs F$ *across* S, just as integral $\int_C \vecs F \cdot \vecs N \; dS$ is the flux of $\vecs F$ across curve C. A surface integral over a vector field is also called a **flux integral**.

Just as with vector line integrals, surface integral $\iint_S \vecs F \cdot \vecs N \; dS$ is easier to compute after surface S has been parameterized. Let $\vecs r(u, v)$ be a parameterization of S with parameter domain D. Then, the unit normal vector is given by $\vecs N = \dfrac{\vecs t_u \times \vecs t_v}{||\vecs t_u \times \vecs t_v||}$ and, from Equation 15.6.4, we have

$$\begin{aligned}
\int_C \vecs F \cdot \vecs N \, dS &= \iint_S \vecs F \cdot \frac{\vecs t_u \times \vecs t_v}{||\vecs t_u \times \vecs t_v||} \, dS \\
&= \iint_D \left(\vecs F(\vecs r(u, v)) \cdot \frac{\vecs t_u \times \vecs t_v}{||\vecs t_u \times \vecs t_v||} \right) ||\vecs t_u \times \vecs t_v|| \, dA \\
&= \iint_D (\vecs F(\vecs r(u, v)) \cdot (\vecs t_u \times \vecs t_v)) \, dA.
\end{aligned}$$

Therefore, to compute a surface integral over a vector field we can use the equation

$$\iint_S \vecs F \cdot \vecs N \, dS = \iint_D (\vecs F(\vecs r(u, v)) \cdot (\vecs t_u \times \vecs t_v)) \, dA. \tag{15.6.5}$$

Example 15.6.13: Calculating a Surface Integral

Calculate the surface integral

$$\iint_S \vecs{F} \cdot \vecs{N}\, dS,$$

where $\vecs{F} = \langle -y, x, 0\rangle$ and S is the surface with parameterization

$$\vecs{r}(u, v) = \langle u, v^2 - u,\ u + v\rangle,\ 0 \le u \le 3,\ 0 \le v \le 4.$$

Solution

The tangent vectors are $\vecs{t}_u = \langle 1, -1, 1\rangle$ and $\vecs{t}_v = \langle 0, 2v, 1\rangle$. Therefore,

$$\vecs{t}_u \times \vecs{t}_v = \langle -1 - 2v, -1, 2v\rangle.$$

By Equation 15.6.5,

$$\begin{aligned}
\iint_S \vecs{F} \cdot d\vecs{S} &= \int_0^4 \int_0^3 F(\vecs{r}(u, v)) \cdot (\vecs{t}_u \times \vecs{t}_v)\, du\, dv \\
&= \int_0^4 \int_0^3 \langle u - v^2,\ u,\ 0\rangle \cdot \langle -1 - 2v,\ -1,\ 2v\rangle\, du\, dv \\
&= \int_0^4 \int_0^3 [(u - v^2)(-1 - 2v) - u]\, du\, dv \\
&= \int_0^4 \int_0^3 (2v^3 + v^2 - 2uv - 2u)\, du\, dv \\
&= \int_0^4 [2v^3 u + v^2 u - vu^2 - u^2]\Big|_0^3\, dv \\
&= \int_0^4 (6v^3 + 3v^2 - 9v - 9)\, dv \\
&= \left[\frac{3v^4}{2} + v^3 - \frac{9v^2}{2} - 9v\right]_0^4 \\
&= 340.
\end{aligned}$$

Therefore, the flux of $\vecs{F}$ across S is 340.

Exercise 15.6.12

Calculate surface integral

$$\iint_S \vecs{F} \cdot \vecs{N}\, dS,$$

where $\vecs{F} = \langle 0, -z, y\rangle$ and S is the portion of the unit sphere in the first octant with outward orientation.

Hint available at the end of this section.

Answer available at the end of this section.

Example 15.6.14: Calculating Mass Flow Rate

Let $\vecs{v}(x,y,z) = \langle 2x,\ 2y,\ z\rangle$ represent a velocity field (with units of meters per second) of a fluid with constant density 80 kg/m^3. Let S be hemisphere $x^2+y^2+z^2=9$ with $z \le 0$ such that S is oriented outward. Find the mass flow rate of the fluid across S.

Solution

A parameterization of the surface is

$$\vecs{r}(\phi,\theta) = \langle 3\cos\theta\sin\phi,\ 3\sin\theta\sin\phi,\ 3\cos\phi\rangle,\ 0\le\theta\le 2\pi,\ 0\le\phi\le\pi/2.$$

As in Example 15.6.7, the tangent vectors are $\vecs{t}_\theta = \langle -3\ \sin\theta\ \sin\phi,\ 3\ \cos\theta\ \sin\phi,\ 0\rangle$ and $\vecs{t}_\phi = \langle 3\ \cos\theta\ \cos\phi,\ 3\ \sin\theta\ \cos\phi,\ -3\ \sin\phi\rangle$, and their cross product is

$$\vecs{t}_\phi \times \vecs{t}_\theta = \langle 9\cos\theta\sin^2\phi,\ 9\sin\theta\sin^2\phi,\ 9\sin\phi\cos\phi\rangle.$$

Notice that each component of the cross product is positive, and therefore this vector gives the outward orientation. Therefore we use the orientation

$$\vecs{N} = \langle 9\ \cos\theta\ \sin^2\phi,\ 9\ \sin\theta\ \sin^2\phi,\ 9\ \sin\phi\ \cos\phi\rangle$$

for the sphere.

By Equation 15.6.4,

$$\begin{aligned}
\iint_S \rho v\cdot dS &= 80\int_0^{2\pi}\int_0^{\pi/2} v(r(\phi,\theta))\cdot(t_\phi\times t_\theta)\,d\phi\,d\theta \\
&= 80\int_0^{2\pi}\int_0^{\pi/2} \langle 6\cos\theta\sin\phi,\ 6\sin\theta\sin\phi,\ 3\cos\phi\rangle\cdot\langle 9\cos\theta\sin^2\phi,\ 9\sin\theta\sin^2\phi,\ 9\sin\phi\cos\phi\rangle\,d\phi\,d\theta \\
&= 80\int_0^{2\pi}\int_0^{\pi/2} 54\sin^3\phi + 27\cos^2\phi\sin\phi\,d\phi\,d\theta \\
&= 80\int_0^{2\pi}\int_0^{\pi/2} 54(1-\cos^2\phi)\sin\phi + 27\cos^2\phi\sin\phi\,d\phi\,d\theta \\
&= 80\int_0^{2\pi}\int_0^{\pi/2} 54\sin\phi - 27\cos^2\phi\sin\phi\,d\phi\,d\theta \\
&= 80\int_0^{2\pi}\Big[-54\cos\phi + 9\cos^3\phi\Big]_{\phi=0}^{\phi=2\pi} d\theta \\
&= 80\int_0^{2\pi} 45\,d\theta \\
&= 7200\pi.
\end{aligned}$$

Therefore, the mass flow rate is 7200π kg/sec/m^2.

Exercise 15.6.13

Let $\vecs{v}(x,y,z) = \langle x^2+y^2,\ z,\ 4y\rangle$ m/sec represent a velocity field of a fluid with constant density 100 kg/m^3. Let S be the half-cylinder $\vecs{r}(u,v) = \langle \cos u,\ \sin u,\ v\rangle,\ 0\le u\le\pi,\ 0\le v\le 2$ oriented outward. Calculate the mass flux of the fluid across S.

Hint available at the end of this section.

Answer available at the end of this section.

In Example 15.6.14, we computed the mass flux, which is the rate of mass flow per unit area. If we want to find the flow rate

(measured in volume per time) instead, we can use flux integral

$$\iint_S \vecs{v} \cdot \vecs{N}\, dS,$$

which leaves out the density. Since the flow rate of a fluid is measured in volume per unit time, flow rate does not take mass into account. Therefore, we have the following characterization of the flow rate of a fluid with velocity $\vecs{v}$ across a surface S:

$$\text{Flow rate of fluid across S} = \iint_S \vecs{v} \cdot d\vecs{S}.$$

To compute the flow rate of the fluid in Example 15.6.14, we simply remove the density constant, which gives a flow rate of 90π m^3/sec.

Both mass flux and flow rate are important in physics and engineering. Mass flux measures how much mass is flowing across a surface; flow rate measures how much volume of fluid is flowing across a surface.

In addition to modeling fluid flow, surface integrals can be used to model heat flow. Suppose that the temperature at point (x, y, z) in an object is $T(x, y, z)$. Then the heat flow is a vector field proportional to the negative temperature gradient in the object. To be precise, the heat flow is defined as vector field $F = -k\nabla T$, where the constant *k* is the *thermal conductivity* of the substance from which the object is made (this constant is determined experimentally). The rate of heat flow across surface S in the object is given by the flux integral

$$\iint_S \vecs{F} \cdot d\vecs{S} = \iint_S -k\vecs{\nabla T} \cdot d\vecs{S}.$$

Example 15.6.15: Calculating Heat Flow

A cast-iron solid cylinder is given by inequalities $x^2 + y^2 \leq 1,\ \ 1 \leq z \leq 4$. The temperature at point (x, y, z) in a region containing the cylinder is $T(x, y, z) = (x^2 + y^2)z$. Given that the thermal conductivity of cast iron is 55, find the heat flow across the boundary of the solid if this boundary is oriented outward.

Solution

Let S denote the boundary of the object. To find the heat flow, we need to calculate flux integral

$$\iint_S -k\vecs{\nabla T} \cdot d\vecs{S}.$$

Notice that S is not a smooth surface but is piecewise smooth, since S is the union of three smooth surfaces (the circular top and bottom, and the cylindrical side). Therefore, we calculate three separate integrals, one for each smooth piece of S. Before calculating any integrals, note that the gradient of the temperature is $\vecs{\nabla T} = \langle 2xz,\ \ 2yz,\ \ x^2 + y^2\rangle$.

First we consider the circular bottom of the object, which we denote S_1. We can see that S_1 is a circle of radius 1 centered at point $(0, 0, 1)$ sitting in plane $z = 1$. This surface has parameterization $\vecs{r}(u, v) = \langle v\ \cos u,\ \ v\ \sin u,\ \ 1\rangle,\ \ 0 \leq u < 2\pi,\ \ 0 \leq v \leq 1$.

Therefore,

$\vecs{t}_u = \langle -v\ \sin u,\ \ v\ \cos u,\ \ 0\rangle$ and $\vecs{t}_v = \langle \cos u,\ \ v\ \sin u,\ \ 0\rangle$, and $\vecs{t}_u \times \vecs{t}_v = \langle 0,\ \ 0, -v\ \sin^2 u - v\ \cos^2 u\rangle = \langle 0,\ \ 0, -v\rangle$.

Since the surface is oriented outward and S_1 is the bottom of the object, it makes sense that this vector points downward. By Equation 15.6.5 the heat flow across S_1 is

$$\begin{aligned}\iint_{S_1} -k\vecs{\nabla T}\cdot d\vecs{S} &= -55\int_0^{2\pi}\int_0^1 \vecs{\nabla T}(u,v)\cdot(\vecs{t}_u\times\vecs{t}_v)\,dv\,du\\ &= -55\int_0^{2\pi}\int_0^1 \langle 2v\cos u,\ 2v\sin u,\ v^2\cos^2 u+v^2\sin^2 u\rangle\cdot\langle 0,0,-v\rangle\,dv\,du\\ &= -55\int_0^{2\pi}\int_0^1 \langle 2v\cos u,\ 2v\sin u,\ v^2\rangle\cdot\langle 0,0,-v\rangle\,dv\,du\\ &= -55\int_0^{2\pi}\int_0^1 -v^3\,dv\,du\\ &= -55\int_0^{2\pi} -\frac{1}{4}du\\ &= \frac{55\pi}{2}.\end{aligned}$$

Now let's consider the circular top of the object, which we denote S_2. We see that S_2 is a circle of radius 1 centered at point $(0,0,4)$, sitting in plane $z=4$. This surface has parameterization $\vecs{r}(u,v)=\langle v\ \cos u,\ v\ \sin u,\ 4\rangle,\ 0\le u<2\pi,\ 0\le v\le 1$.

Therefore, $\vecs{t}_u=\langle -v\ \sin u,\ v\ \cos u,\ 0\rangle$ and $\vecs{t}_v=\langle\cos u,\ v\ \sin u,\ 0\rangle$, and $\vecs{t}_u\times\vecs{t}_v=\langle 0,\ 0,-v\ \sin^2 u-v\ \cos^2 u\rangle=\langle 0,0,-v\rangle$.

Since the surface is oriented outward and S_1 is the top of the object, we instead take vector $\vecs{t}_v\times\vecs{t}_u=\langle 0,0,v\rangle$. By Equation 15.6.5, the heat flow across S_1 is

$$\begin{aligned}\iint_{S_2} -k\vecs{\nabla T}\cdot d\vecs{S} &= -55\int_0^{2\pi}\int_0^1 \vecs{\nabla T}(u,v)\cdot\ (\vecs{t}_u\times\vecs{t}_v)\,dv\,du\\ &= -55\int_0^{2\pi}\int_0^1 \langle 8v\cos u,\ 8v\sin u,\ v^2\cos^2 u+v^2\sin^2 u\rangle\cdot\langle 0,0,-v\rangle\,dv\,du\\ &= -55\int_0^{2\pi}\int_0^1 \langle 8v\cos u,\ 8v\sin u,\ v^2\rangle\cdot\langle 0,0,-v\rangle\ dv\,du\\ &= -55\int_0^{2\pi}\int_0^1 -v^3\,dv\,du = -55\int_0^{2\pi} -\frac{1}{4}\,du = -\frac{55\pi}{2}.\end{aligned}$$

Last, let's consider the cylindrical side of the object. This surface has parameterization $\vecs{r}(u,v)=\langle\cos u,\ \sin u,\ v\rangle,\ 0\le u<2\pi,\ 1\le v\le 4$. By Example 15.6.9, we know that $\vecs{t}_u\times\vecs{t}_v=\langle\cos u,\ \sin u,\ 0\rangle$. By Equation 15.6.5,

$$\begin{aligned}\iint_{S_3} -k\vecs{\nabla T}\cdot d\vecs{S} &= -55\int_0^{2\pi}\int_1^4 \vecs{\nabla T}(u,v)\cdot(\vecs{t}_u\times\vecs{t}_v)\,dv\,du\\ &= -55\int_0^{2\pi}\int_1^4 \langle 2v\cos u,\ 2v\sin u,\ \cos^2 u+\sin^2 u\rangle\cdot\langle\cos u,\ \sin u,\ 0\rangle\,dv\,du\\ &= -55\int_0^{2\pi}\int_0^1 \langle 2v\cos^2 u,\ 2v\sin u,\ 1\rangle\cdot\langle\cos u,\ \sin u,\ 0\rangle\,dv\ du\\ &= -55\int_0^{2\pi}\int_0^1 (2v\cos^2 u+2v\sin^2 u)\,dv\,du\\ &= -55\int_0^{2\pi}\int_0^1 2v\,dv\,du\\ &= -55\int_0^{2\pi} du\\ &= -110\pi.\end{aligned}$$

Therefore, the rate of heat flow across S is

$$\frac{55\pi}{2}-\frac{55\pi}{2}-110\pi=-110\pi.$$

Exercise 15.6.14

A cast-iron solid ball is given by inequality $x^2 + y^2 + z^2 \leq 1$. The temperature at a point in a region containing the ball is $T(x, y, z) = \frac{1}{3}(x^2 + y^2 + z^2)$. Find the heat flow across the boundary of the solid if this boundary is oriented outward.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- Surfaces can be parameterized, just as curves can be parameterized. In general, surfaces must be parameterized with two parameters.
- Surfaces can sometimes be oriented, just as curves can be oriented. Some surfaces, such as a Möbius strip, cannot be oriented.
- A surface integral is like a line integral in one higher dimension. The domain of integration of a surface integral is a surface in a plane or space, rather than a curve in a plane or space.
  - The integrand of a surface integral can be a scalar function or a vector field. To calculate a surface integral with an integrand that is a function, use Equation 15.6.2. To calculate a surface integral with an integrand that is a vector field, use Equation 15.6.5.
- If S is a surface, then the area of S is

$$\iint_S dS.$$

Key Equations

- **Scalar surface integral**

$$\iint_S f(x, y, z)\, dS = \iint_D f(\vecs{r}(u, v))||\vecs{t}_u \times \vecs{t}_v||\, dA$$

- **Flux integral**

$$\iint_S \vecs{F} \cdot \vecs{N}\, dS = \iint_S \vecs{F} \cdot d\vecs{S} = \iint_D \vecs{F}(\vecs{r}(u, v)) \cdot (\vecs{t}_u \times \vecs{t}_v)\, dA$$

Glossary

flux integral
another name for a surface integral of a vector field; the preferred term in physics and engineering

grid curves
curves on a surface that are parallel to grid lines in a coordinate plane

heat flow
a vector field proportional to the negative temperature gradient in an object

mass flux
the rate of mass flow of a fluid per unit area, measured in mass per unit time per unit area

orientation of a surface
if a surface has an “inner” side and an “outer” side, then an orientation is a choice of the inner or the outer side; the surface could also have “upward” and “downward” orientations

parameter domain (parameter space)
the region of the uv-plane over which the parameters u and v vary for parameterization $\vecs{r}(u, v) = \langle x(u, v),$

$y(u,v),\ z(u,v)\rangle$

parameterized surface (parametric surface)
a surface given by a description of the form $\vecs r(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$, where the parameters u and v vary over a parameter domain in the uv-plane

regular parameterization
parameterization $\vecs r(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v)\rangle$ such that $r_u \times r_v$ is not zero for point (u,v) in the parameter domain

surface area
the area of surface S given by the surface integral

$$\iint_S dS$$

surface integral
an integral of a function over a surface

surface integral of a scalar-valued function
a surface integral in which the integrand is a scalar function

surface integral of a vector field
a surface integral in which the integrand is a vector field

Answers

Exercise 15.6.1
Hold u and v constant, and see what kind of curves result.

Exercise 15.6.1
Cylinder $x^2 + y^2 = 4$

Exercise 15.6.2
Hold u constant and see what kind of curves result. Imagine what happens as u increases or decreases.

Exercise 15.6.2
Cone $x^2 + y^2 = z^2$

Exercise 15.6.3
Consider the parameter domain for this surface.

Exercise 15.6.3
$\vecs r(u,v) = \langle u\ \cos v,\ u\ \sin v,\ u\rangle,\ 0 < u < \infty,\ 0 \le v < \dfrac{\pi}{2}$

Exercise 15.6.4
Investigate the cross product $\vecs r_u \times \vecs r_v$.

Exercise 15.6.4
Yes

Exercise 15.6.5
Use Equation 15.6.1.

Exercise 15.6.5
≈ 43.02

Exercise 15.6.6
Use the standard parameterization of a cylinder and follow the previous example.

Exercise 15.6.6
With the standard parameterization of a cylinder, Equation 15.6.1 shows that the surface area is $2\pi rh$.

Exercise 15.6.7
Use the parameterization of surfaces of revolution given before Example 15.6.7.

Exercise 15.6.7
$2\pi(\sqrt{2} + \sinh^{-1}(1))$

Exercise 15.6.8
Use Equation 15.6.2.

Exercise 15.6.8
24

Exercise 15.6.9

Break the integral into three separate surface integrals.

Exercise 15.6.9

0

Exercise 15.6.10

The mass of a sheet is given by Equation 15.6.3. A useful parameterization of a paraboloid was given in a previous example.

Exercise 15.6.10

$38.401\pi \approx 120.640$

Exercise 15.6.11

Parameterize the surface and use the fact that the surface is the graph of a function.

Exercise 15.6.11

$$\vecs{N}(x,y) = \left\langle \frac{-y}{\sqrt{1+x^2+y^2}}, \frac{-x}{\sqrt{1+x^2+y^2}}, \frac{1}{\sqrt{1+x^2+y^2}} \right\rangle$$

Exercise 15.6.12

Use Equation 15.6.5.

Exercise 15.6.12

0

Exercise 15.6.13

Use Equation 15.6.4.

Exercise 15.6.13

400 kg/sec/m

Exercise 15.6.14

Follow the steps of Example 15.6.15.

Exercise 15.6.14

$-\frac{440\pi}{3}$

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15.7: Stokes' Theorem

Learning Objectives

- Explain the meaning of Stokes' theorem.
- Use Stokes' theorem to evaluate a line integral.
- Use Stokes' theorem to calculate a surface integral.
- Use Stokes' theorem to calculate a curl.

In this section, we study Stokes' theorem, a higher-dimensional generalization of Green's theorem. This theorem, like the Fundamental Theorem for Line Integrals and Green's theorem, is a generalization of the Fundamental Theorem of Calculus to higher dimensions. Stokes' theorem relates a vector surface integral over surface S in space to a line integral around the boundary of S. Therefore, just as the theorems before it, Stokes' theorem can be used to reduce an integral over a geometric object S to an integral over the boundary of S. In addition to allowing us to translate between line integrals and surface integrals, Stokes' theorem connects the concepts of curl and circulation. Furthermore, the theorem has applications in fluid mechanics and electromagnetism. We use Stokes' theorem to derive Faraday's law, an important result involving electric fields.

Stokes' Theorem

Stokes' theorem says we can calculate the flux of $curl\ \vecs{F}$ across surface S by knowing information only about the values of $\vecs{F}$ along the boundary of S. Conversely, we can calculate the line integral of vector field $\vecs{F}$ along the boundary of surface S by translating to a double integral of the curl of $\vecs{F}$ over S.

Let S be an oriented smooth surface with unit normal vector $\vecs{N}$. Furthermore, suppose the boundary of S is a simple closed curve C. The orientation of S induces the positive orientation of C if, as you walk in the positive direction around C with your head pointing in the direction of $\vecs{N}$, the surface is always on your left. With this definition in place, we can state *Stokes' theorem*.

Theorem 15.7.1: Stokes' Theorem

Let S be a piecewise smooth oriented surface with a boundary that is a simple closed curve C with positive orientation (Figure 15.7.1). If $\vecs{F}$ is a vector field with component functions that have continuous partial derivatives on an open region containing S, then

$$\int_C \vecs{F} \cdot d\vecs{r} = \iint_S curl\, \vecs{F} \cdot d\vecs{S}. \tag{15.7.1}$$

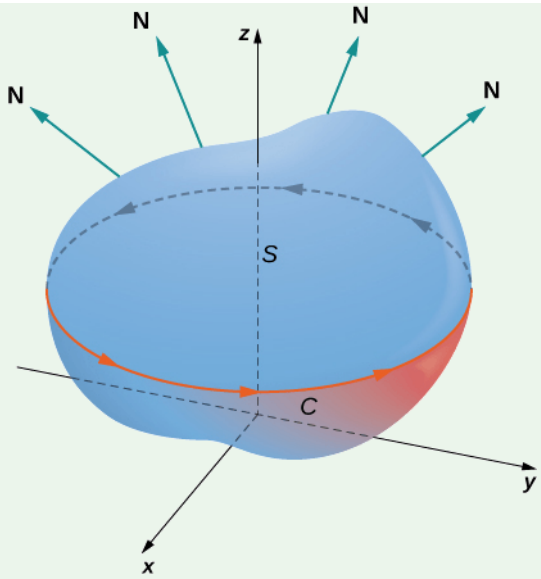


Figure 15.7.1: Stokes' theorem relates the flux integral over the surface to a line integral around the boundary of the surface. Note that the orientation of the curve is positive.

Suppose surface S is a flat region in the xy-plane with upward orientation. Then the unit normal vector is $\vecs{k}$ and surface integral

$$\iint_S curl\,\vecs{F}\cdot d\vecs{S}$$

is actually the double integral

$$\iint_S curl\,\vecs{F}\cdot\vecs{k}\,dA.$$

In this special case, Stokes' theorem gives

$$\int_C \vecs{F}\cdot d\vecs{r}=\iint_S curl\,\vecs{F}\cdot\vecs{k}\,dA.$$

However, this is the flux form of Green's theorem, which shows us that Green's theorem is a special case of Stokes' theorem. Green's theorem can only handle surfaces in a plane, but Stokes' theorem can handle surfaces in a plane or in space.

The complete proof of Stokes' theorem is beyond the scope of this text. We look at an intuitive explanation for the truth of the theorem and then see proof of the theorem in the special case that surface S is a portion of a graph of a function, and S, the boundary of S, and $\vecs{F}$ are all fairly tame.

Proof

First, we look at an informal proof of the theorem. This proof is not rigorous, but it is meant to give a general feeling for why the theorem is true. Let S be a surface and let D be a small piece of the surface so that D does not share any points with the boundary of S. We choose D to be small enough so that it can be approximated by an oriented square E. Let D inherit its orientation from S, and give E the same orientation. This square has four sides; denote them $E_l,\ E_r,\ E_u$, and E_d for the left, right, up, and down sides, respectively. On the square, we can use the flux form of Green's theorem:

$$\int_{E_l+E_d+E_r+E_u}\vecs{F}\cdot d\vecs{r}=\iint_E curl\,\vecs{F}\cdot\vecs{N}\,dS=\iint_E curl\,\vecs{F}\cdot d\vecs{S}.$$

To approximate the flux over the entire surface, we add the values of the flux on the small squares approximating small pieces of the surface (Figure 15.7.2).

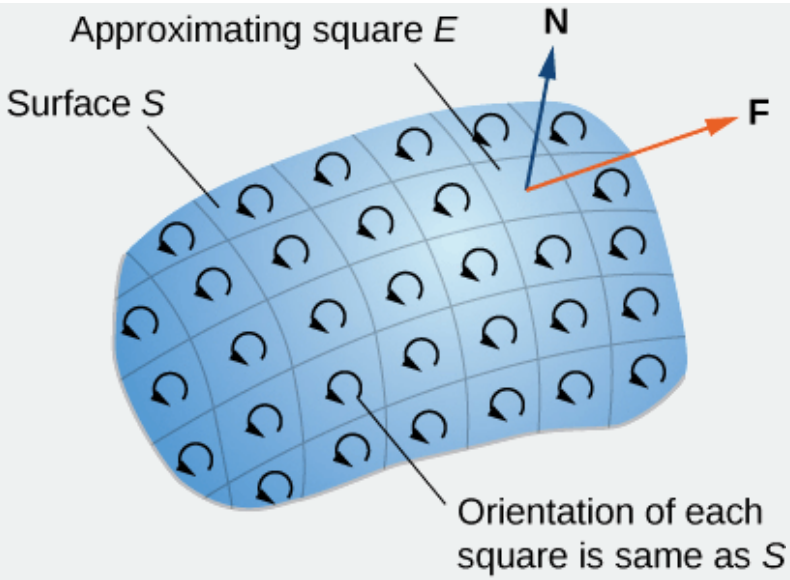


Figure 15.7.2: Chop the surface into small pieces. The pieces should be small enough that they can be approximated by a square.

By Green's theorem, the flux across each approximating square is a line integral over its boundary. Let F be an approximating square with an orientation inherited from S and with a right side E_l (so F is to the left of E). Let F_r denote the right side of F; then, $E_l = -F_r$. In other words, the right side of F is the same curve as the left side of E, just oriented in the opposite direction. Therefore,

$$\int_{E_l} \vecs{F} \cdot d\vecs{r} = -\int_{F_r} \vecs{F} \cdot d\vecs{r}.$$

As we add up all the fluxes over all the squares approximating surface S, line integrals

$$\int_{E_l} \vecs{F} \cdot d\vecs{r}$$

and

$$\int_{F_r} \vecs{F} \cdot d\vecs{r}$$

cancel each other out. The same goes for the line integrals over the other three sides of E. These three line integrals cancel out with the line integral of the lower side of the square above E, the line integral over the left side of the square to the right of E, and the line integral over the upper side of the square below E (Figure 15.7.3). After all this cancelation occurs over all the approximating squares, the only line integrals that survive are the line integrals over sides approximating the boundary of S. Therefore, the sum of all the fluxes (which, by Green's theorem, is the sum of all the line integrals around the boundaries of approximating squares) can be approximated by a line integral over the boundary of S. In the limit, as the areas of the approximating squares go to zero, this approximation gets arbitrarily close to the flux.

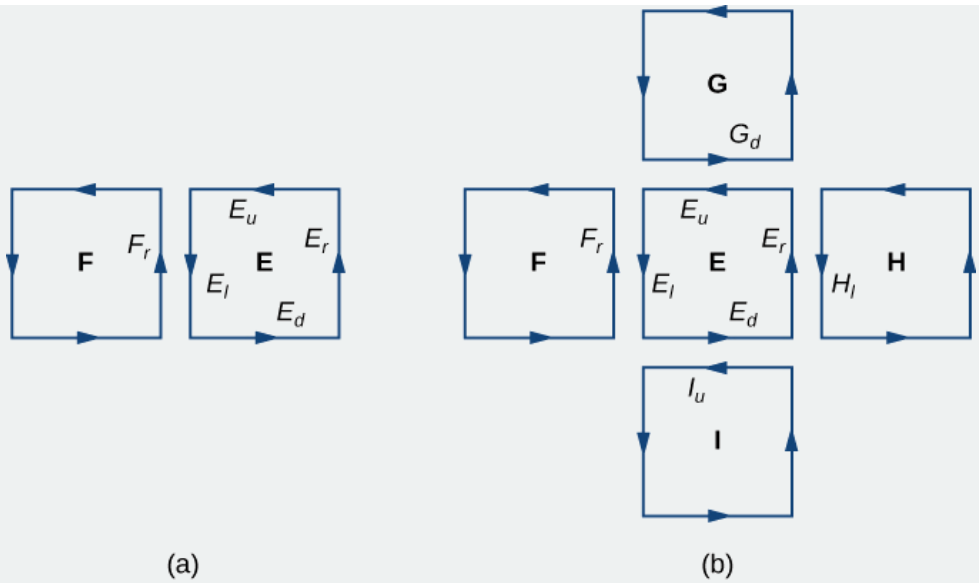


Figure 15.7.3: (a) The line integral along E_l cancels out the line integral along F_r because $E_l = -F_r$. (b) The line integral along any of the sides of E cancels out with the line integral along a side of an adjacent approximating square.

Let's now look at a rigorous proof of the theorem in the special case that S is the graph of function $z = f(x, y)$, where x and y vary over a bounded, simply connected region D of finite area (Figure 15.7.4). Furthermore, assume that f has continuous second-order partial derivatives. Let C denote the boundary of S and let C' denote the boundary of D. Then, D is the "shadow" of S in the plane and C' is the "shadow" of C. Suppose that S is oriented upward. The counterclockwise orientation of C is positive, as is the counterclockwise orientation of C'. Let $\vecs F(x, y, z) = \langle P, Q, R\rangle$ be a vector field with component functions that have continuous partial derivatives.

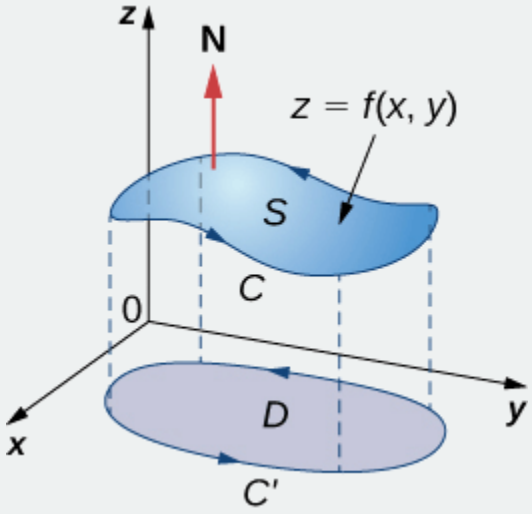


Figure 15.7.4: D is the "shadow," or projection, of S in the plane and C' is the projection of C.

We take the standard parameterization of S : $x = x,\ y = y,\ z = g(x, y)$. The tangent vectors are $\vecs t_x = \langle 1, 0, g_x\rangle$ and $\vecs t_y = \langle 0, 1, g_y\rangle$, and therefore $\vecs t_x \times \vecs t_y = \langle -g_x,\ -g_y,\ 1\rangle$.

$$\iint_S curl\, \vecs F \cdot d\vecs S = \iint_D [-(R_y - Q_z)z_x - (P_z - R_x)z_y + (Q_x - P_y)]\, dA,$$

where the partial derivatives are all evaluated at $(x, y, g(x, y))$, making the integrand depend on x and y only. Suppose $\langle x(t),\ y(t)\rangle,\ a \le t \le b$ is a parameterization of C'. Then, a parameterization of C is $\langle x(t),\ y(t),\ g(x(t),\ y(t))\rangle,\ a \le t \le b$. Armed with these parameterizations, the Chain rule, and Green's theorem, and keeping in mind that P, Q and R are all functions of x and y, we can evaluate line integral

$$\begin{aligned}\int_C \vecs{F}\cdot d\vecs{r} &= \int_a^b (Px'(t)+Qy'(t)+Rz'(t))\,dt \\ &= \int_a^b \left[Px'(t)+Qy'(t)+R\left(\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}\right)\right]dt \\ &= \int_a^b \left[\left(P+R\frac{\partial z}{\partial x}\right)x'(t)+\left(Q+R\frac{\partial z}{\partial y}\right)y'(t)\right]dt \\ &= \int_{C'}\left(P+R\frac{\partial z}{\partial x}\right)dx+\left(Q+R\frac{\partial z}{\partial y}\right)dy \\ &= \iint_D\left[\frac{\partial}{\partial x}\left(Q+R\frac{\partial z}{\partial y}\right)-\frac{\partial}{\partial y}\left(P+R\frac{\partial z}{\partial x}\right)\right]dA \\ &= \iint_D\left(\frac{\partial Q}{\partial x}+\frac{\partial Q}{\partial z}\frac{\partial z}{\partial x}+\frac{\partial R}{\partial x}\frac{\partial z}{\partial y}+\frac{\partial R}{\partial z}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}+R\frac{\partial^2 z}{\partial x\partial y}\right)-\left(\frac{\partial P}{\partial y}+\frac{\partial P}{\partial z}\frac{\partial z}{\partial y}+\frac{\partial R}{\partial z}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x}+R\frac{\partial^2 z}{\partial y\partial x}\right)\end{aligned}$$

By Clairaut's theorem,

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}$$

Therefore, four of the terms disappear from this double integral, and we are left with

$$\iint_D\left[-(R_y-Q_z)Z_x-(P_z-R_x)z_y+(Q_x-P_y)\right]dA,$$

which equals

$$\iint_S curl\,\vecs{F}\cdot d\vecs{S}.$$

□

We have shown that Stokes' theorem is true in the case of a function with a domain that is a simply connected region of finite area. We can quickly confirm this theorem for another important case: when vector field $\vecs{F}$ is a conservative field. If $\vecs{F}$ is conservative, the curl of $\vecs{F}$ is zero, so

$$\iint_S curl\,\vecs{F}\cdot d\vecs{S}=0.$$

Since the boundary of S is a closed curve, the integral

$$\int_C \vecs{F}\cdot d\vecs{r}.$$

is also zero.

Example 15.7.1: Verifying Stokes' Theorem for a Specific Case

Verify that Stokes' theorem is true for vector field $\vecs{F}(x,y)=\langle -z,x,0\rangle$ and surface S, where S is the hemisphere, oriented outward, with parameterization $\vecs{r}(\phi,\theta)=\langle \sin\phi\ \cos\theta,\ \sin\phi\ \sin\theta,\ \cos\phi\rangle,\ 0\le\theta\le\pi,\ 0\le\phi\le\pi$ as shown in Figure 15.7.5.

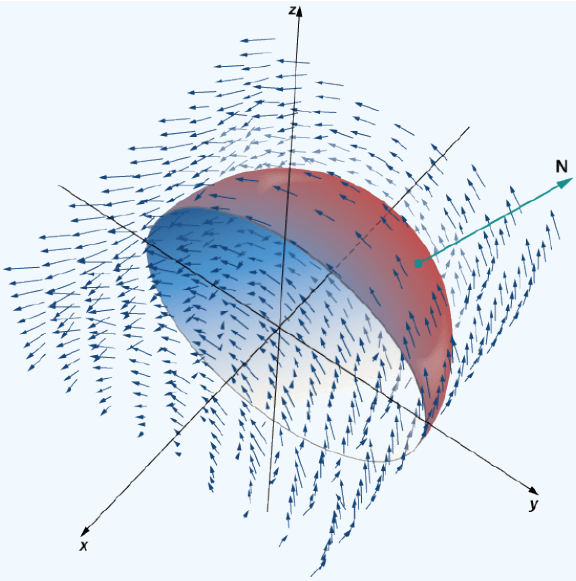


Figure 15.7.5: Verifying Stokes' theorem for a hemisphere in a vector field.

Solution

Let C be the boundary of S. Note that C is a circle of radius 1, centered at the origin, sitting in plane $y = 0$. This circle has parameterization $\langle \cos t,\ 0,\ \sin t \rangle,\ 0 \le t \le 2\pi$. the equation for scalar surface integrals

$$\begin{aligned} \int_C \vecs{F} \cdot d\vecs{r} &= \int_0^{2\pi} \langle -\sin t,\ \cos t,\ 0 \rangle \cdot \langle -\sin t,\ 0,\ \cos t \rangle \, dt \\ &= \int_0^{2\pi} \sin^2 t \, dt \\ &= \pi. \end{aligned}$$

By the equation for vector line integrals,

$$\begin{aligned} \iint_S curl\, \vecs{F} \cdot d\vecs{S} &= \iint_D curl\, \vecs{F}(\vecs{r}(\phi, \theta)) \cdot (\vecs{t}_\phi \times \vecs{t}_\theta)\, dA \\ &= \iint_D \langle 0, -1, 1 \rangle \cdot \langle \cos\theta \sin^2\phi,\ \sin\theta \sin^2\phi,\ \sin\phi \cos\phi \rangle \, dA \\ &= \int_0^{\pi} \int_0^{\pi} (\sin\phi \cos\phi - \sin\theta \sin^2\phi)\, d\phi d\theta \\ &= \frac{\pi}{2} \int_0^{\pi} \sin\theta \, d\theta \\ &= \pi. \end{aligned}$$

Therefore, we have verified Stokes' theorem for this example.

Exercise 15.7.1

Verify that Stokes' theorem is true for vector field $\vecs{F}(x, y, z) = \langle y, x, -z \rangle$ and surface S, where S is the upwardly oriented portion of the graph of $f(x, y) = x^2 y$ over a triangle in the xy-plane with vertices $(0, 0),\ (2, 0)$, and $(0, 2)$.

Hint available at the end of this section.

Answer available at the end of this section.

Applying Stokes' Theorem

Stokes' theorem translates between the flux integral of surface S to a line integral around the boundary of S. Therefore, the theorem allows us to compute surface integrals or line integrals that would ordinarily be quite difficult by translating the line integral into a surface integral or vice versa. We now study some examples of each kind of translation.

Example 15.7.2: Calculating a Surface Integral

Calculate surface integral

$$\iint_S curl\, \vecs{F} \cdot d\vecs{S},$$

where S is the surface, oriented outward, in Figure 15.7.6 and $\vecs{F} = \langle z,\ 2xy,\ x + y\rangle$.

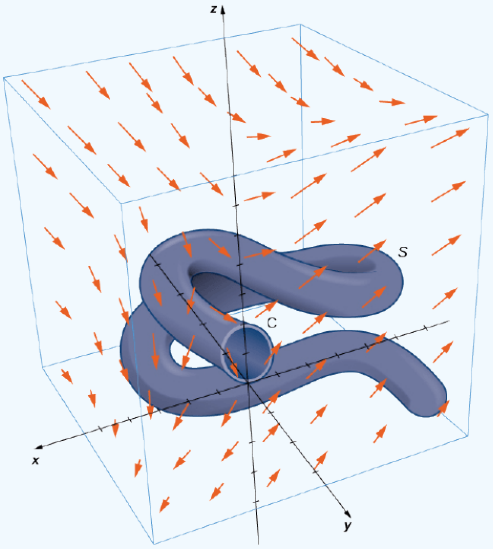


Figure 15.7.6: A complicated surface in a vector field.

Solution

Note that to calculate

$$\iint_S curl\, \vecs{F} \cdot d\vecs{S}$$

without using Stokes' theorem, we would need the equation for scalar surface integrals. Use of this equation requires a parameterization of S. Surface S is complicated enough that it would be extremely difficult to find a parameterization. Therefore, the methods we have learned in previous sections are not useful for this problem. Instead, we use Stokes' theorem, noting that the boundary C of the surface is merely a single circle with radius 1.

The curl of $\vecs{F}$ is $\langle 1, 1, 2y\rangle$. By Stokes' theorem,

$$\iint_S curl\, \vecs{F} \cdot d\vecs{S} = \int_C \vecs{F} \cdot d\vecs{r},$$

where C has parameterization $\langle \cos t,\ \sin t,\ 1\rangle, 0 \le t \le 2\pi$. By the equation for vector line integrals,

$$\begin{aligned}\iint_S curl\, F \cdot d\vecs S &= \int_C \vecs F \cdot d\vecs r \\ &= \int_0^2 \langle 1,\, \sin t\, \cos t,\, \cos t + \sin t\rangle \cdot \langle -\sin t,\, \cos t,\, 0\rangle\, dt \\ &= \int_0^{2\pi} (-\sin t + 2\, \sin t\, \cos^2 t)\, dt \\ &= \left[\cos t - \frac{2\, \cos^3 t}{3}\right]_0^{2\pi} \\ &= \cos(2\pi) - \frac{2\, \cos^3(2\pi)}{3} - \left(\cos(0) - \frac{2\, \cos^3(0)}{3}\right) \\ &= 0.\end{aligned}$$

An amazing consequence of Stokes' theorem is that if S' is any other smooth surface with boundary C and the same orientation as S, then

$$\iint_S curl\, \vecs F \cdot d\vecs S = \int_C \vecs F \cdot d\vecs r = 0$$

because Stokes' theorem says the surface integral depends on the line integral around the boundary only.

In Example 15.7.2, we calculated a surface integral simply by using information about the boundary of the surface. In general, let S_1 and S_2 be smooth surfaces with the same boundary C and the same orientation. By Stokes' theorem,

$$\iint_{S_1} curl\, \vecs F \cdot d\vecs S = \int_C \vecs F \cdot d\vecs r = \iint_{S_2} curl\, \vecs F \cdot d\vecs S. \tag{15.7.2}$$

Therefore, if

$$\iint_{S_1} curl\, \vecs F \cdot d\vecs S$$

is difficult to calculate but

$$\iint_{S_2} curl\, \vecs F \cdot d\vecs S$$

is easy to calculate, Stokes' theorem allows us to calculate the easier surface integral. In Example 15.7.2, we could have calculated

$$\iint_S curl\, \vecs F \cdot d\vecs S$$

by calculating

$$\iint_{S'} curl\, \vecs F \cdot d\vecs S,$$

where $\vecs S'$ is the disk enclosed by boundary curve C (a much more simple surface with which to work).

Equation 15.7.2 shows that flux integrals of curl vector fields are surface independent in the same way that line integrals of gradient fields are path independent. Recall that if $\vecs F$ is a two-dimensional conservative vector field defined on a simply connected domain, f is a *potential function* for $\vecs F$, and C is a curve in the domain of $\vecs F$, then

$$\int_C \vecs F \cdot d\vecs r$$

depends only on the endpoints of C. Therefore if C' is any other curve with the same starting point and endpoint as C (that is, C' has the same orientation as C), then

$$\int_C \vecs{F} \cdot d\vecs{r} = \int_{C'} \vecs{F} \cdot d\vecs{r}$$

In other words, the value of the integral depends on the boundary of the path only; it does not really depend on the path itself.

Analogously, suppose that S and S' are surfaces with the same boundary and same orientation, and suppose that $\vecs{G}$ is a three-dimensional vector field that can be written as the curl of another vector field $\vecs{F}$ (so that $\vecs{F}$ is like a "potential field" of $\vecs{G}$). By Equation 15.7.2,

$$\iint_S \vecs{G} \cdot d\vecs{S} = \iint_S curl\, \vecs{F} \cdot d\vecs{S} = \int_C \vecs{F} \cdot d\vecs{r} = \iint_{S'} curl\, \vecs{F} \cdot d\vecs{S} = \iint_{S'} \vecs{G} \cdot d\vecs{S}.$$

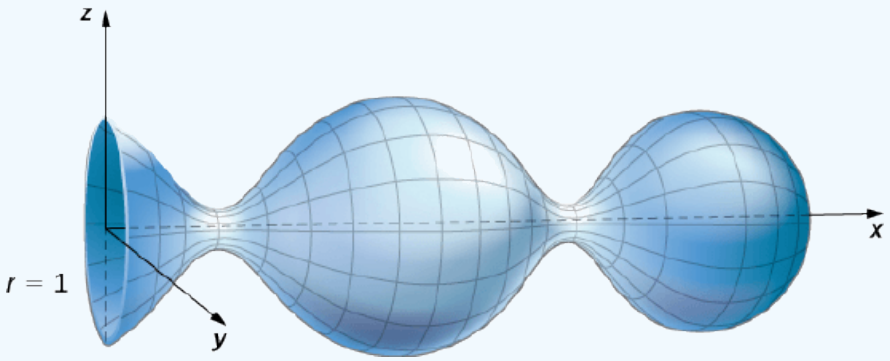
Therefore, the flux integral of $\vecs{G}$ does not depend on the surface, only on the boundary of the surface. Flux integrals of vector fields that can be written as the curl of a vector field are surface independent in the same way that line integrals of vector fields that can be written as the gradient of a scalar function are path independent.

Exercise 15.7.2

Use Stokes' theorem to calculate surface integral

$$\iint_S curl\, \vecs{F} \cdot d\vecs{S},$$

where $\vecs{F}(x, y, z) = \langle z, x, y \rangle$ and S is the surface as shown in the following figure. The boundary curve, C is oriented clockwise when looking along the positive y-axis.



Hint available at the end of this section.

Answer available at the end of this section.

Example 15.7.3: Calculating a Line Integral

Calculate the line integral

$$\int_C \vecs{F} \cdot d\vecs{r},$$

where $\vecs{F} = \langle xy,\ x^2 + y^2 + z^2,\ yz \rangle$ and C is the boundary of the parallelogram with vertices $(0, 0, 1),\ (0, 1, 0),\ (2, 0, -1)$, and $(2, 1, -2)$.

Solution

To calculate the line integral directly, we need to parameterize each side of the parallelogram separately, calculate four separate line integrals, and add the result. This is not overly complicated, but it is time-consuming.

By contrast, let's calculate the line integral using Stokes' theorem. Let S denote the surface of the parallelogram. Note that S is the portion of the graph of $z = 1 - x - y$ for (x, y) varying over the rectangular region with vertices $(0, 0),\ (0, 1),\ (2, 0)$, and $(2, 1)$ in the xy-plane. Therefore, a parameterization of S is $\langle x, y,\ 1 - x - y\rangle,\ 0 \le x \le 2,\ 0 \le y \le 1$. The curl of $\vecs F$ is $\langle -z,\ 0,\ x\rangle$,and Stokes' theorem and the equation for scalar surface integrals

$$\begin{aligned}\int_C \vecs F \cdot d\vecs r &= \iint_S curl\, \vecs F \cdot d\vecs S \\ &= \int_0^2 \int_0^1 curl\, \vecs F(x, y) \cdot (\vecs t_x \times \vecs t_y)\, dy\, dx \\ &= \int_0^2 \int_0^1 \langle -(1 - x - y),\ 0,\ x\rangle \cdot (\langle 1,\ 0, -1\rangle \times \langle 0,\ 1,\ -1\rangle)\, dy\, dx \\ &= \int_0^2 \int_0^1 \langle x + y - 1,\ 0,\ x\rangle \cdot \langle 1, 1, 1\rangle\, dy\, dx \\ &= \int_0^2 \int_0^1 2x + y - 1\, dy\, dx \\ &= 3.\end{aligned}$$

Exercise 15.7.3

Use Stokes' theorem to calculate line integral

$$\int_C \vecs F \cdot d\vecs r,$$

where $\vecs F = \langle z, x, y\rangle$ and C is the boundary of a triangle with vertices $(0, 0, 1),\ (3, 0, -2)$, and $(0, 1, 2)$.

Hint available at the end of this section.

Answer available at the end of this section.

Interpretation of Curl

In addition to translating between line integrals and flux integrals, Stokes' theorem can be used to justify the physical interpretation of curl that we have learned. Here we investigate the relationship between curl and circulation, and we use Stokes' theorem to state Faraday's law—an important law in electricity and magnetism that relates the curl of an electric field to the rate of change of a magnetic field.

Recall that if C is a closed curve and $\vecs F$ is a vector field defined on C, then the circulation of $\vecs F$ around C is line integral

$$\int_C \vecs F \cdot d\vecs r.$$

If $\vecs F$ represents the velocity field of a fluid in space, then the circulation measures the tendency of the fluid to move in the direction of C.

Let $\vecs F$ be a continuous vector field and let D_τ be a small disk of radius r with center P_0 (Figure 15.7.7). If D_τ is small enough, then $(curl\ \vecs F)(P) \approx (curl\ \vecs F)(P_0)$ for all points P in D_τ because the curl is continuous. Let C_τ be the boundary circle of D_τ: By Stokes' theorem,

$$\int_{C_\tau} \vecs F \cdot d\vecs r = \iint_{D_\tau} curl\, \vecs F \cdot \vecs N\, dS \approx \iint_{D_\tau} (curl\, \vecs F)(P_0) \cdot \vecs N(P_0)\, dS.$$

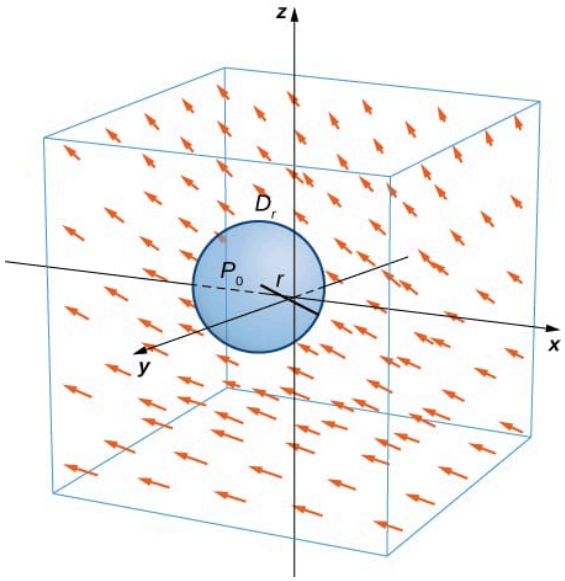


Figure 15.7.7: Disk D_τ is a small disk in a continuous vector field.

The quantity $(curl\ \vecs F)(P_0)\cdot\vecs N(P_0)$ is constant, and therefore

$$\iint_{D_\tau}(curl\,\vecs F)(P_0)\cdot\vecs N(P_0)\,dS=\pi r^2[(curl\,\vecs F)(P_0)\cdot\vecs N(P_0)].$$

Thus

$$\int_{C_\tau}\vecs F\cdot d\vecs r\approx\pi r^2[(curl\,\vecs F)(P_0)\cdot\vecs N(P_0)],$$

and the approximation gets arbitrarily close as the radius shrinks to zero. Therefore Stokes' theorem implies that

$$(curl\,\vecs F)(P_0)\cdot\vecs N(P_0)=\lim_{r\to0^+}\frac{1}{\pi r^2}\int_{C_\tau}\vecs F\cdot d\vecs r.$$

This equation relates the curl of a vector field to the circulation. Since the area of the disk is πr^2, this equation says we can view the curl (in the limit) as the circulation per unit area. Recall that if $\vecs F$ is the velocity field of a fluid, then circulation

$$\oint_{C_\tau}\vecs F\cdot d\vecs r=\oint_{C_\tau}\vecs F\cdot\vecs T\,ds$$

is a measure of the tendency of the fluid to move around C_τ: The reason for this is that $\vecs F\cdot\vecs T$ is a component of $\vecs F$ in the direction of $\vecs T$, and the closer the direction of $\vecs F$ is to $\vecs T$, the larger the value of $\vecs F\cdot\vecs T$ (remember that if $\vecs a$ and $\vecs b$ are vectors and $\vecs b$ is fixed, then the dot product $\vecs a\cdot\vecs b$ is maximal when $\vecs a$ points in the same direction as $\vecs b$). Therefore, if $\vecs F$ is the velocity field of a fluid, then $curl\ \vecs F\cdot\vecs N$ is a measure of how the fluid rotates about axis $\vecs N$. The effect of the curl is largest about the axis that points in the direction of $\vecs N$, because in this case $curl\ \vecs F\cdot\vecs N$ is as large as possible.

To see this effect in a more concrete fashion, imagine placing a tiny paddlewheel at point P_0 (Figure 15.7.8). The paddlewheel achieves its maximum speed when the axis of the wheel points in the direction of curl $\vecs F$. This justifies the interpretation of the curl we have learned: curl is a measure of the rotation in the vector field about the axis that points in the direction of the normal vector $\vecs N$, and Stokes' theorem justifies this interpretation.

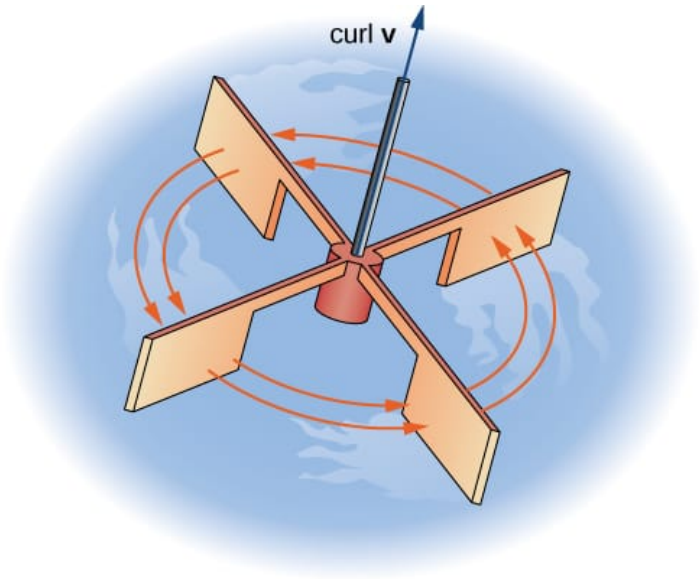


Figure 15.7.8: To visualize curl at a point, imagine placing a tiny paddlewheel at that point in the vector field.

Now that we have learned about Stokes' theorem, we can discuss applications in the area of electromagnetism. In particular, we examine how we can use Stokes' theorem to translate between two equivalent forms of Faraday's law. Before stating the two forms of Faraday's law, we need some background terminology.

Let C be a closed curve that models a thin wire. In the context of electric fields, the wire may be moving over time, so we write $C(t)$ to represent the wire. At a given time t, curve $C(t)$ may be different from original curve C because of the movement of the wire, but we assume that $C(t)$ is a closed curve for all times t. Let $D(t)$ be a surface with $C(t)$ as its boundary, and orient $C(t)$ so that $D(t)$ has positive orientation. Suppose that $C(t)$is in a magnetic field $\vecs{B}(t)$ that can also change over time. In other words, $\vecs{B}$ has the form

$$\vecs{B}(x,y,z) = \langle P(x,y,z),\, Q(x,y,z),\, R(x,y,z)\rangle,$$

where P, Q, and R can all vary continuously over time. We can produce current along the wire by changing field $\vecs{B}(t)$ (this is a consequence of Ampere's law). Flux $\phi(t) = \iint_{D(t)} \vecs{B}(t)\cdot d\vecs{S}$ creates electric field $\vecs{E}(t)$ that does work. The integral form of Faraday's law states that

$$Work = \int_{C(t)} \vecs{E}(t)\cdot d\vecs{r} = -\frac{\partial \phi}{\partial t}.$$

In other words, the work done by $\vecs{E}$ is the line integral around the boundary, which is also equal to the rate of change of the flux with respect to time. The differential form of Faraday's law states that

$$curl\, \vecs{E} = -\frac{\partial \vecs{B}}{\partial t}.$$

Using Stokes' theorem, we can show that the differential form of Faraday's law is a consequence of the integral form. By Stokes' theorem, we can convert the line integral in the integral form into surface integral

$$-\frac{\partial \phi}{\partial t} = \int_{C(t)} \vecs{E}(t)\cdot d\vecs{r} = \iint_{D(t)} curl\, \vecs{E}(t)\cdot d\vecs{S}.$$

Since

$$\phi(t) = \iint_{D(t)} B(t)\cdot d\vecs{S},$$

then as long as the integration of the surface does not vary with time we also have

$$-\frac{\partial \phi}{\partial t} = \iint_{D(t)} -\frac{\partial \vecs{B}}{\partial t} \cdot d\vecs{S}.$$

Therefore,

$$\iint_{D(t)} -\frac{\partial \vecs{B}}{\partial t} \cdot d\vecs{S} = \iint_{D(t)} curl\, \vecs{E} \cdot d\vecs{S}.$$

To derive the differential form of Faraday's law, we would like to conclude that $curl\ \vecs{E} = -\dfrac{\partial \vecs{B}}{\partial t}$: In general, the equation

$$\iint_{D(t)} -\frac{\partial \vecs{B}}{\partial t} \cdot d\vecs{S} = \iint_{D(t)} curl\, \vecs{E} \cdot d\vecs{S}$$

is not enough to conclude that $curl\ \vecs{E} = -\dfrac{\partial \vecs{B}}{\partial t}$: The integral symbols do not simply "cancel out," leaving equality of the integrands. To see why the integral symbol does not just cancel out in general, consider the two single-variable integrals $\int_0^1 x\ dx$ and $\int_0^1 f(x)\ dx$, where

$$f(x) = \begin{cases} 1, & \text{if } 0 \le x \le 1/2 \\ 0, & \text{if } 1/2 \le x \le 1. \end{cases}$$

Both of these integrals equal $\frac{1}{2}$, so $\int_0^1 x\ dx = \int_0^1 f(x)\ dx$.

However, $x \neq f(x)$. Analogously, with our equation

$$\iint_{D(t)} -\frac{\partial \vecs{B}}{\partial t} \cdot d\vecs{S} = \iint_{D(t)} curl\, \vecs{E} \cdot d\vecs{S},$$

we cannot simply conclude that $curl\ \vecs{E} = -\dfrac{\partial \vecs{B}}{\partial t}$ just because their integrals are equal. However, in our context, equation

$$\iint_{D(t)} -\frac{\partial \vecs{B}}{\partial t} \cdot d\vecs{S} = \iint_{D(t)} curl\, \vecs{E} \cdot d\vecs{S}$$

is true for *any* region, however small (this is in contrast to the single-variable integrals just discussed). If $\vecs{F}$ and $\vecs{G}$ are three-dimensional vector fields such that

$$\iint_S \vecs{F} \cdot d\vecs{S} = \iint_S \vecs{G} \cdot d\vecs{S}$$

for any surface S, then it is possible to show that $\vecs{F} = \vecs{G}$ by shrinking the area of S to zero by taking a limit (the smaller the area of S, the closer the value of $\iint_S \vecs{F} \cdot d\vecs{S}$ to the value of $\vecs{F}$ at a point inside S). Therefore, we can let area $D(t)$ shrink to zero by taking a limit and obtain the differential form of Faraday's law:

$$curl\, \vecs{E} = -\frac{\partial \vecs{B}}{\partial t}.$$

In the context of electric fields, the curl of the electric field can be interpreted as the negative of the rate of change of the corresponding magnetic field with respect to time.

Example 15.7.4: Using Faraday's Law

Calculate the curl of electric field $\vecs{E}$ if the corresponding magnetic field is constant field $\vecs{B}(t) = \langle 1, -4, 2 \rangle$.

Solution

Since the magnetic field does not change with respect to time, $-\frac{\partial \vecs{B}}{\partial t} = \vecs{0}$. By Faraday's law, the curl of the electric field is therefore also zero.

Analysis

A consequence of Faraday's law is that the curl of the electric field corresponding to a constant magnetic field is always zero.

Exercise 15.7.4

Calculate the curl of electric field $\vecs{E}$ if the corresponding magnetic field is $\vecs{B}(t) = \langle tx,\ ty,\ -2tz \rangle,\ 0 \le t < \infty$.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- Stokes' theorem relates a flux integral over a surface to a line integral around the boundary of the surface. Stokes' theorem is a higher dimensional version of Green's theorem, and therefore is another version of the Fundamental Theorem of Calculus in higher dimensions.
- Stokes' theorem can be used to transform a difficult surface integral into an easier line integral, or a difficult line integral into an easier surface integral.
- Through Stokes' theorem, line integrals can be evaluated using the simplest surface with boundary C.
- Faraday's law relates the curl of an electric field to the rate of change of the corresponding magnetic field. Stokes' theorem can be used to derive Faraday's law.

Key Equations

- **Stokes' theorem**

$$\int_C \vecs{F} \cdot d\vecs{r} = \iint_S curl\, \vecs{F} \cdot d\vecs{S}$$

Glossary

Stokes' theorem
relates the flux integral over a surface S to a line integral around the boundary C of the surface S

surface independent
flux integrals of curl vector fields are surface independent if their evaluation does not depend on the surface but only on the boundary of the surface

Answers

Exercise 15.7.1
Calculate the double integral and line integral separately.

Exercise 15.7.1
Both integrals give $-\frac{136}{45}$:

Exercise 15.7.2
Parameterize the boundary of S and translate to a line integral.

Exercise 15.7.2
$-\pi$

Exercise 15.7.3
This triangle lies in plane $z = 1 - x + y$.

Exercise 15.7.3

$\frac{3}{2}$

Exercise 15.7.4

- Use the differential form of Faraday's law.
- Notice that the curl of the electric field does not change over time, although the magnetic field does change over time.

Exercise 15.7.4

$curl\ \vecs{E} = \langle x,\ y,\ -2z \rangle$

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15.8: The Divergence Theorem

Learning Objectives

- Explain the meaning of the divergence theorem.
- Use the divergence theorem to calculate the flux of a vector field.
- Apply the divergence theorem to an electrostatic field.

We have examined several versions of the Fundamental Theorem of Calculus in higher dimensions that relate the integral around an oriented boundary of a domain to a "derivative" of that entity on the oriented domain. In this section, we state the divergence theorem, which is the final theorem of this type that we will study. The divergence theorem has many uses in physics; in particular, the divergence theorem is used in the field of partial differential equations to derive equations modeling heat flow and conservation of mass. We use the theorem to calculate flux integrals and apply it to electrostatic fields.

Overview of Theorems

Before examining the divergence theorem, it is helpful to begin with an overview of the versions of the Fundamental Theorem of Calculus we have discussed:

1. The Fundamental Theorem of Calculus:
$$\int_a^b f'(x)\,dx = f(b) - f(a).$$
This theorem relates the integral of derivative f' over line segment $[a, b]$ along the x-axis to a difference of f evaluated on the boundary.
2. The Fundamental Theorem for Line Integrals:
$$\int_C \vecs{\nabla} f \cdot d\vecs{r} = f(P_1) - f(P_0),$$
where P_0 is the initial point of C and P_1 is the terminal point of C. The Fundamental Theorem for Line Integrals allows path C to be a path in a plane or in space, not just a line segment on the x-axis. If we think of the gradient as a derivative, then this theorem relates an integral of derivative $\vecs{\nabla} f$ over path C to a difference of f evaluated on the boundary of C.
3. Green's theorem, circulation form:
$$\iint_D (Q_x - P_y)\,dA = \int_C \vecs{F} \cdot d\vecs{r}.$$
Since $Q_x - P_y = \text{curl}\ \vecs{F} \cdot \hat{\mathbf{k}}$ and curl is a derivative of sorts, Green's theorem relates the integral of derivative curl $\vecs{F}$ over planar region D to an integral of $\vecs{F}$ over the boundary of D.
4. Green's theorem, flux form:
$$\iint_D (P_x + Q_y)\,dA = \int_C \vecs{F} \cdot \vecs{N}\,dS.$$
Since $P_x + Q_y = \text{div}\ \vecs{F}$ and divergence is a derivative of sorts, the flux form of Green's theorem relates the integral of derivative div $\vecs{F}$ over planar region D to an integral of $\vecs{F}$ over the boundary of D.
5. Stokes' theorem:
$$\iint_S curl\,\vecs{F} \cdot d\vecs{S} = \int_C \vecs{F} \cdot d\vecs{r}.$$
If we think of the curl as a derivative of sorts, then Stokes' theorem relates the integral of derivative curl $\vecs{F}$ over surface S (not necessarily planar) to an integral of $\vecs{F}$ over the boundary of S.

Stating the Divergence Theorem

The divergence theorem follows the general pattern of these other theorems. If we think of divergence as a derivative of sorts, then the divergence theorem relates a triple integral of derivative div $\vecs F$ over a solid to a flux integral of $\vecs F$ over the boundary of the solid. More specifically, the divergence theorem relates a flux integral of vector field $\vecs F$ over a closed surface S to a triple integral of the divergence of $\vecs F$ over the solid enclosed by S.

The Divergence Theorem

Let S be a piecewise, smooth closed surface that encloses solid E in space. Assume that S is oriented outward, and let $\vecs F$ be a vector field with continuous partial derivatives on an open region containing E (Figure 15.8.1). Then

$$\iiint_E \text{div } \vecs F \, dV = \iint_S \vecs F \cdot d\vecs S. \tag{15.8.1}$$

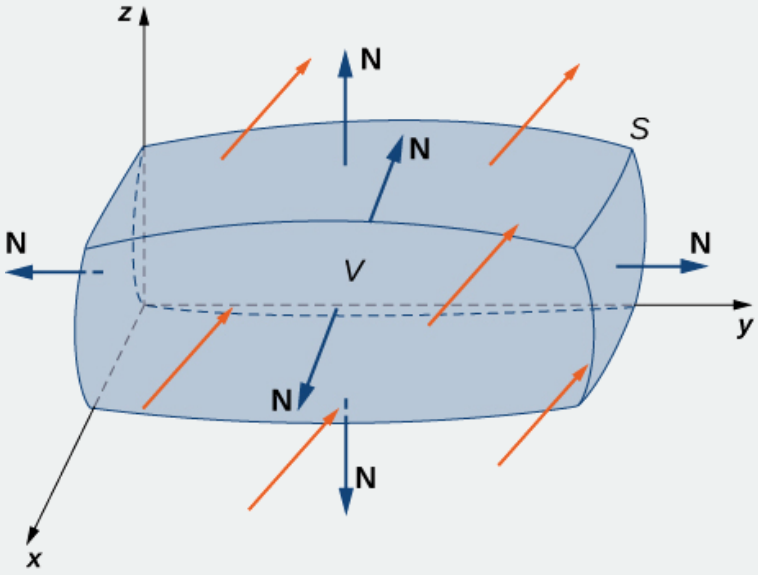


Figure 15.8.1: The divergence theorem relates a flux integral across a closed surface S to a triple integral over solid E enclosed by the surface.

Recall that the flux form of Green's theorem states that

$$\iint_D \text{div } \vecs F \, dA = \int_C \vecs F \cdot \vecs N \, dS.$$

Therefore, the divergence theorem is a version of Green's theorem in one higher dimension.

The proof of the divergence theorem is beyond the scope of this text. However, we look at an informal proof that gives a general feel for why the theorem is true, but does not prove the theorem with full rigor. This explanation follows the informal explanation given for why Stokes' theorem is true.

Proof

Let B be a small box with sides parallel to the coordinate planes inside E (Figure 15.8.2a). Let the center of B have coordinates (x, y, z) and suppose the edge lengths are Δx, Δy, and Δz. (Figure 15.8.1b). The normal vector out of the top of the box is $\hat{\mathbf{k}}$ and the normal vector out of the bottom of the box is $-\hat{\mathbf{k}}$. The dot product of $\vecs F = \langle P, Q, R \rangle$ with $\hat{\mathbf{k}}$ is R and the dot product with $-\hat{\mathbf{k}}$ is $-R$. The area of the top of the box (and the bottom of the box) ΔS is $\Delta x \Delta y$.

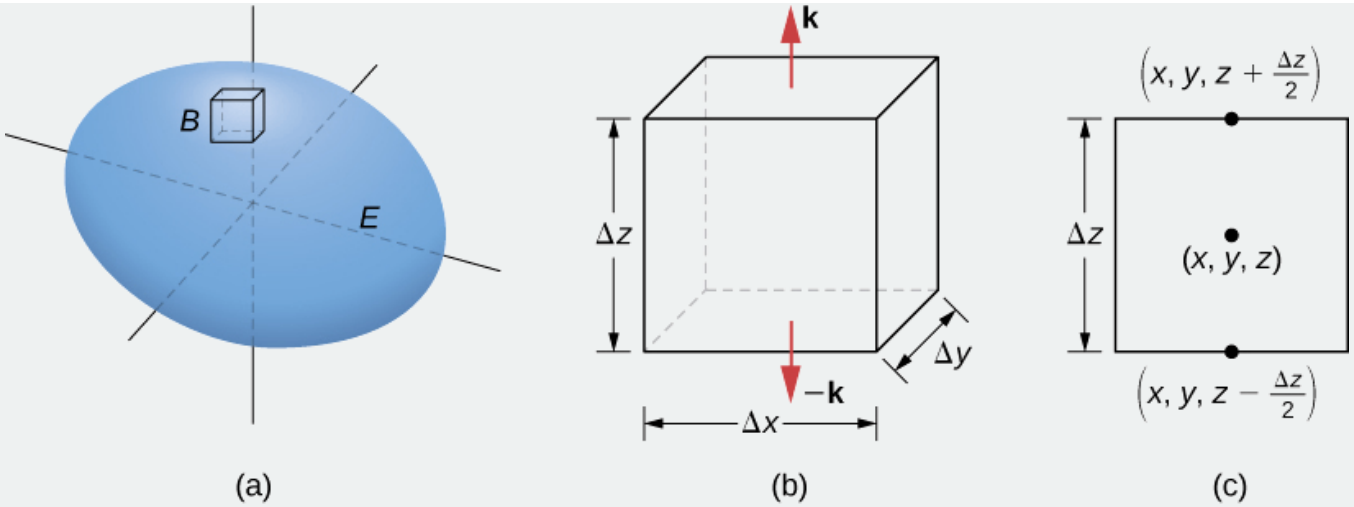


Figure 15.8.2: (a) A small box B inside surface E has sides parallel to the coordinate planes. (b) Box B has side lengths$\Delta x,\ \Delta y$, and Δz (c) If we look at the side view of B, we see that, since (x, y, z) is the center of the box, to get to the top of the box we must travel a vertical distance of $\frac{\Delta z}{2}$ up from (x, y, z). Similarly, to get to the bottom of the box we must travel a distance $\frac{\Delta z}{2}$ down from (x, y, z).

The flux out of the top of the box can be approximated by $R\left(x,\ y,\ z+\frac{\Delta z}{2}\right)\ \Delta x\ \Delta y$ (Figure 15.8.2c) and the flux out of the bottom of the box is $-R\left(x,\ y,\ z-\frac{\Delta z}{2}\right)\ \Delta x\ \Delta y$. If we denote the difference between these values as ΔR, then the net flux in the vertical direction can be approximated by $\Delta R\ \Delta x\ \Delta y$. However,

$$\Delta R\,\Delta x\,\Delta y = \left(\frac{\Delta R}{\Delta z}\right)\Delta x\,\Delta y\Delta z \approx \left(\frac{\partial R}{\partial z}\right)\Delta V.$$

Therefore, the net flux in the vertical direction can be approximated by $\left(\frac{\partial R}{\partial z}\right)\Delta V$. Similarly, the net flux in the x-direction can be approximated by $\left(\frac{\partial P}{\partial x}\right)\ \Delta V$ and the net flux in the y-direction can be approximated by $\left(\frac{\partial Q}{\partial y}\right)\ \Delta V$. Adding the fluxes in all three directions gives an approximation of the total flux out of the box:

$$\text{Total flux } \approx \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)\Delta V = \text{div } \vecs{F}\,\Delta V.$$

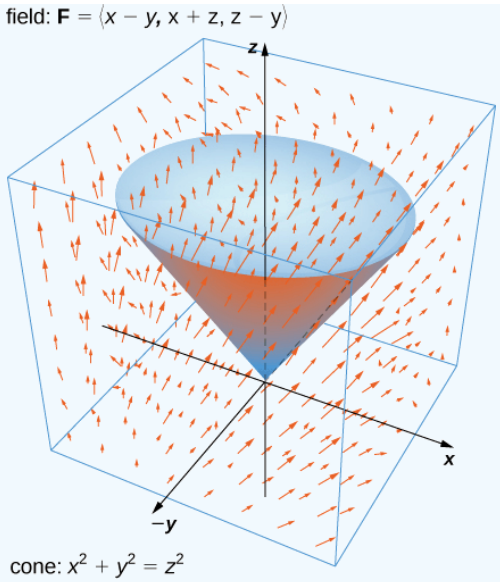
This approximation becomes arbitrarily close to the value of the total flux as the volume of the box shrinks to zero.

The sum of $\text{div } \vecs{F}\ \Delta V$ over all the small boxes approximating E is approximately $\iiint_E \text{div } \vecs{F}\ dV$. On the other hand, the sum of $\text{div } \vecs{F}\ \Delta V$ over all the small boxes approximating E is the sum of the fluxes over all these boxes. Just as in the informal proof of Stokes' theorem, adding these fluxes over all the boxes results in the cancelation of a lot of the terms. If an approximating box shares a face with another approximating box, then the flux over one face is the negative of the flux over the shared face of the adjacent box. These two integrals cancel out. When adding up all the fluxes, the only flux integrals that survive are the integrals over the faces approximating the boundary of E. As the volumes of the approximating boxes shrink to zero, this approximation becomes arbitrarily close to the flux over S.

□

Example 15.8.1: Verifying the Divergence Theorem

Verify the divergence theorem for vector field $\vecs{F} = \langle x-y,\ x+z,\ z-y\rangle$ and surface S that consists of cone $x^2+y^2=z^2$, $0\le z\le 1$, and the circular top of the cone (see the following figure). Assume this surface is positively oriented.



Solution

Let E be the solid cone enclosed by S. To verify the theorem for this example, we show that

$$\iiint_E \text{div } \vecs{F}\, dV = \iint_S \vecs{F} \cdot d\vecs{S}$$

by calculating each integral separately.

To compute the triple integral, note that $\text{div } \vecs{F} = P_x + Q_y + R_z = 2$, and therefore the triple integral is

$$\begin{aligned}\iiint_E \text{div } \vecs{F}\, dV &= 2 \iiint_E dV \\ &= 2\,(volume\ of\ E).\end{aligned}$$

The volume of a right circular cone is given by $\pi r^2 \frac{h}{3}$. In this case, $h = r = 1$. Therefore,

$$\iiint_E \text{div } \vecs{F}\, dV = 2\,(volume\ of\ E) = \frac{2\pi}{3}.$$

To compute the flux integral, first note that S is piecewise smooth; S can be written as a union of smooth surfaces. Therefore, we break the flux integral into two pieces: one flux integral across the circular top of the cone and one flux integral across the remaining portion of the cone. Call the circular top S_1 and the portion under the top S_2. We start by calculating the flux across the circular top of the cone. Notice that S_1 has parameterization

$$\vecs{r}(u, v) = \langle u\, \cos v,\ u\, \sin v,\ 1\rangle,\ 0 \le u \le 1,\ 0 \le v \le 2\pi.$$

Then, the tangent vectors are $\vecs{t}_u = \langle \cos v,\ \sin v,\ 0\rangle$ and $\vecs{t}_v = \langle -u\ \sin v,\ u\ \cos v, 0\rangle$. Therefore, the flux across S_1 is

$$\begin{aligned}\iint_{S_1} \vecs{F} \cdot d\vecs{S} &= \int_0^1 \int_0^{2\pi} \vecs{F}(\vecs{r}(u,v)) \cdot (\vecs{t}_u \times \vecs{t}_v)\, dA \\ &= \int_0^1 \int_0^{2\pi} \langle u\, \cos v - u\, \sin v,\ u\, \cos v + 1,\ 1 - u\, \sin v\rangle \cdot \langle 0, 0, u\rangle\, dv\, du \\ &= \int_0^1 \int_0^{2\pi} u - u^2 \sin v\, dv du \\ &= \pi.\end{aligned}$$

We now calculate the flux over S_2. A parameterization of this surface is

$$\vecs r(u,v) = \langle u\cos v,\ u\sin v,\ u\rangle,\ 0\le u\le 1,\ 0\le v\le 2\pi.$$

The tangent vectors are $\vecs t_u = \langle \cos v,\ \sin v,\ 1\rangle$ and $\vecs t_v = \langle -u\ \sin v,\ u\ \cos v, 0\rangle$, so the cross product is

$$\vecs t_u \times \vecs t_v = \langle -u\cos v,\ -u\sin v,\ u\rangle.$$

Notice that the negative signs on the x and y components induce the negative (or inward) orientation of the cone. Since the surface is positively oriented, we use vector $\vecs t_v \times \vecs t_u = \langle u\ \cos v,\ u\ \sin v,\ -u\rangle$ in the flux integral. The flux across S_2 is then

$$\begin{aligned}\iint_{S_2} \vecs F\cdot d\vecs S &= \int_0^1\int_0^{2\pi} \vecs F(\vecs r(u,v))\cdot(\vecs t_u\times\vecs t_v)\,dA\\ &= \int_0^1\int_0^{2\pi} \langle u\cos v - u\sin v,\ u\cos v + u,\ u - u\sin v\rangle\cdot\langle u\cos v,\ u\sin v,\ -u\rangle\,dv\,du\\ &= \int_0^1\int_0^{2\pi} u^2\cos^2 v + 2u^2\sin v - u^2\,dv\,du\\ &= -\frac{\pi}{3}\end{aligned}$$

The total flux across S is

$$\iint_S \vecs F\cdot d\vecs S = \iint_{S_1}\vecs F\cdot d\vecs S + \iint_{S_2}\vecs F\cdot d\vecs S = \frac{2\pi}{3} = \iiint_E \operatorname{div}\vecs F\,dV,$$

and we have verified the divergence theorem for this example.

Exercise 15.8.1

Verify the divergence theorem for vector field $\vecs F(x,y,z) = \langle x+y+z,\ y,\ 2x-y\rangle$ and surface S given by the cylinder $x^2+y^2=1,\ 0\le z\le 3$ plus the circular top and bottom of the cylinder. Assume that S is positively oriented.

Hint available at the end of this section.

Answer available at the end of this section.

Recall that the divergence of continuous field $\vecs F$ at point P is a measure of the "outflowing-ness" of the field at P. If $\vecs F$ represents the velocity field of a fluid, then the divergence can be thought of as the rate per unit volume of the fluid flowing out less the rate per unit volume flowing in. The divergence theorem confirms this interpretation. To see this, let P be a point and let B_τ be a ball of small radius r centered at P (Figure 15.8.3). Let S_τ be the boundary sphere of B_τ. Since the radius is small and $\vecs F$ is continuous, $\operatorname{div}\vecs F(Q)\approx\operatorname{div}\vecs F(P)$ for all other points Q in the ball. Therefore, the flux across S_τ can be approximated using the divergence theorem:

$$\iint_{S_\tau}\vecs F\cdot d\vecs S = \iiint_{B_\tau}\operatorname{div}\vecs F\,dV \approx \iiint_{B_\tau}\operatorname{div}\vecs F(P)\,dV.$$

Since $divF(P)$ is a constant,

$$\iiint_{B_\tau}\operatorname{div}\vecs F(P)\,dV = \operatorname{div}\vecs F(P)\,V(B_\tau).$$

Therefore, flux

$$\iint_{S_\tau}\vecs F\cdot d\vecs S$$

can be approximated by $\vecs F(P)\ V(B_\tau)$. This approximation gets better as the radius shrinks to zero, and therefore

$$\operatorname{div}\vecs F(P) = \lim_{\tau\to 0}\frac{1}{V(B_\tau)}\iint_{S_\tau}\vecs F\cdot d\vecs S.$$

This equation says that the divergence at P is the net rate of outward flux of the fluid per unit volume.

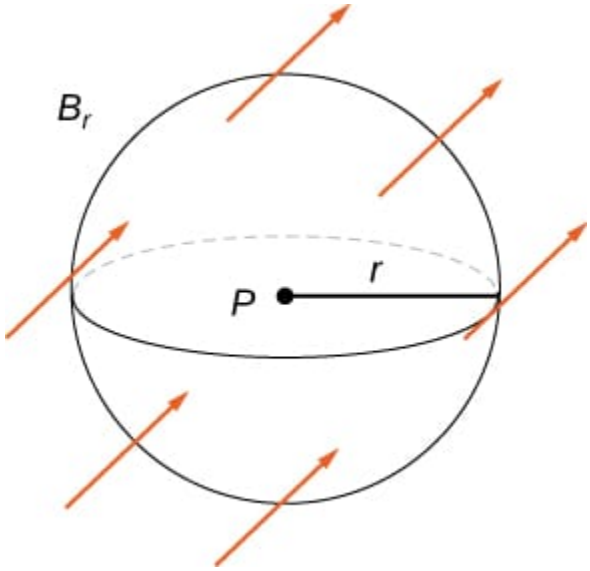


Figure 15.8.3: Ball B_r of small radius r centered at P.

Using the Divergence Theorem

The divergence theorem translates between the flux integral of closed surface S and a triple integral over the solid enclosed by S. Therefore, the theorem allows us to compute flux integrals or triple integrals that would ordinarily be difficult to compute by translating the flux integral into a triple integral and vice versa.

Example 15.8.2: Applying the Divergence Theorem

Calculate the surface integral

$$\iint_S \vecs{F} \cdot d\vecs{S},$$

where S is cylinder $x^2 + y^2 = 1$, $0 \leq z \leq 2$, including the circular top and bottom, and $\vecs{F} = \left\langle \frac{x^3}{3} + yz,\ \frac{y^3}{3} - \sin(xz),\ z - x - y \right\rangle$.

Solution

We could calculate this integral without the divergence theorem, but the calculation is not straightforward because we would have to break the flux integral into three separate integrals: one for the top of the cylinder, one for the bottom, and one for the side. Furthermore, each integral would require parameterizing the corresponding surface, calculating tangent vectors and their cross product..

By contrast, the divergence theorem allows us to calculate the single triple integral

$$\iiint_E \operatorname{div} \vecs{F}\, dV,$$

where E is the solid enclosed by the cylinder. Using the divergence theorem (Equation 15.8.1) and converting to cylindrical coordinates, we have

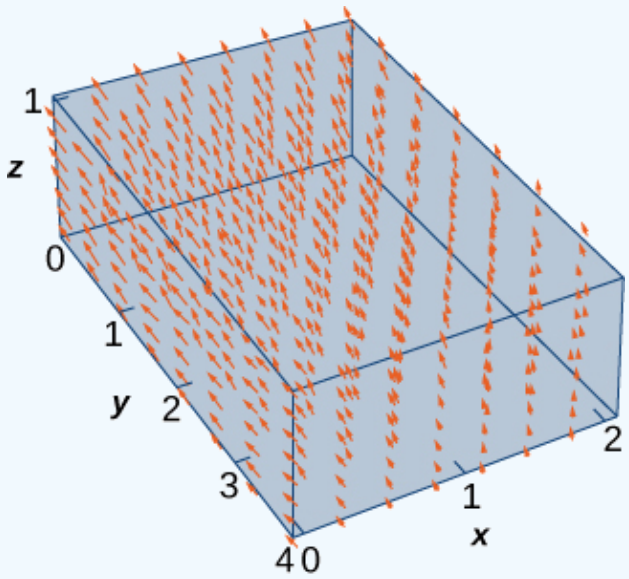
$$\begin{aligned}\iint_S \vecs{F}\cdot d\vecs{S} &= \iiint_E \operatorname{div}\vecs{F}\,dV,\\ &= \iiint_E (x^2+y^2+1)\,dV\\ &= \int_0^{2\pi}\int_0^1\int_0^2 (r^2+1)\,r\,dz\,dr\,d\theta\\ &= \frac{3}{2}\int_0^{2\pi} d\theta\\ &= 3\pi.\end{aligned}$$

Exercise 15.8.2

Use the divergence theorem to calculate flux integral

$$\iint_S \vecs{F}\cdot d\vecs{S},$$

where S is the boundary of the box given by $0\le x\le 2,\ 0\le y\le 4,\ 0\le z\le 1$ and $\vecs{F}=\langle x^2+yz,\ y-z,\ 2x+2y+2z\rangle$ (see the following figure).



Hint available at the end of this section.

Answer available at the end of this section.

Example 15.8.3: Applying the Divergence Theorem

Let $\vecs{v}=\langle -\frac{y}{z},\ \frac{x}{z},\ 0\rangle$ be the velocity field of a fluid. Let C be the solid cube given by $1\le x\le 4,\ 2\le y\le 5,\ 1\le z\le 4$, and let S be the boundary of this cube (see the following figure). Find the flow rate of the fluid across S.

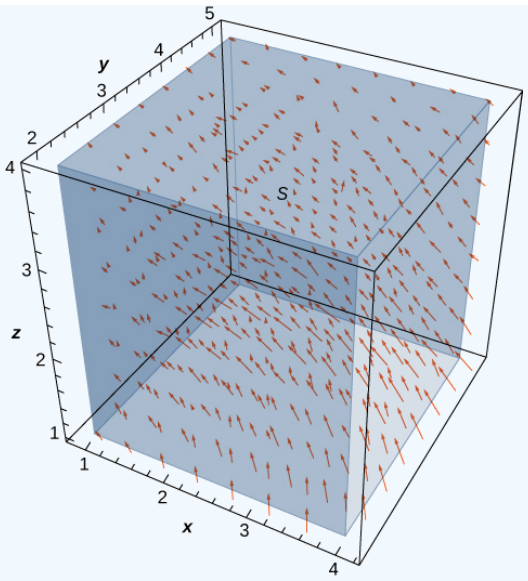


Figure 15.8.4: Vector field $\vecs{v} = \langle -\frac{y}{z}, \frac{x}{z}, 0 \rangle$.

Solution

The flow rate of the fluid across S is $\iint_S \vecs{v} \cdot d\vecs{S}$. Before calculating this flux integral, let's discuss what the value of the integral should be. Based on Figure 15.8.4, we see that if we place this cube in the fluid (as long as the cube doesn't encompass the origin), then the rate of fluid entering the cube is the same as the rate of fluid exiting the cube. The field is rotational in nature and, for a given circle parallel to the xy-plane that has a center on the z-axis, the vectors along that circle are all the same magnitude. That is how we can see that the flow rate is the same entering and exiting the cube. The flow into the cube cancels with the flow out of the cube, and therefore the flow rate of the fluid across the cube should be zero.

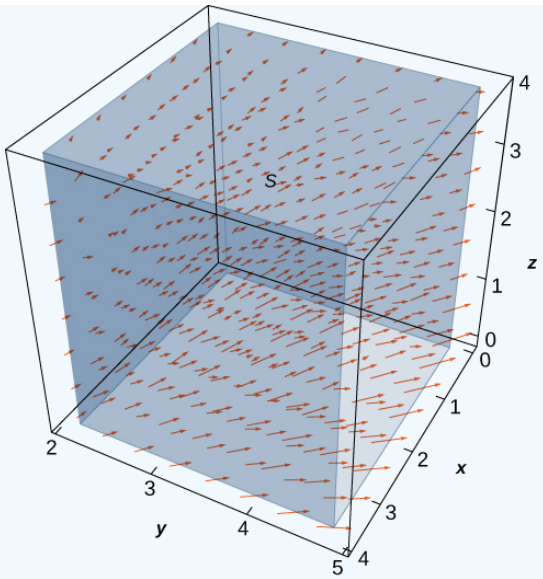
To verify this intuition, we need to calculate the flux integral. Calculating the flux integral directly requires breaking the flux integral into six separate flux integrals, one for each face of the cube. We also need to find tangent vectors, compute their cross product. However, using the divergence theorem makes this calculation go much more quickly:

$$\begin{aligned} \iint_S \vecs{v} \cdot d\vecs{S} &= \iiint_C \operatorname{div} \vecs{v}\, dV \\ &= \iiint_C 0\, dV = 0. \end{aligned}$$

Therefore the flux is zero, as expected.

Exercise 15.8.3

Let $\vecs{v} = \langle \frac{x}{z}, \frac{y}{z}, 0 \rangle$ be the velocity field of a fluid. Let C be the solid cube given by $1 \le x \le 4,\ 2 \le y \le 5,\ 1 \le z \le 4$, and let S be the boundary of this cube (see the following figure). Find the flow rate of the fluid across S.



Hint available at the end of this section.

Answer available at the end of this section.

Example illustrates a remarkable consequence of the divergence theorem. Let S be a piecewise, smooth closed surface and let $\vecs F$ be a vector field defined on an open region containing the surface enclosed by S. If $\vecs F$ has the form $F = \langle f(y,z),\ g(x,z),\ h(x,y)\rangle$, then the divergence of $\vecs F$ is zero. By the divergence theorem, the flux of $\vecs F$ across S is also zero. This makes certain flux integrals incredibly easy to calculate. For example, suppose we wanted to calculate the flux integral $\iint_S \vecs F \cdot d\vecs S$ where S is a cube and

$$\vecs F = \langle \sin(y)\, e^{yz},\ x^2 z^2,\ \cos(xy)\, e^{\sin x}\rangle.$$

Calculating the flux integral directly would be difficult, if not impossible, using techniques we studied previously. At the very least, we would have to break the flux integral into six integrals, one for each face of the cube. But, because the divergence of this field is zero, the divergence theorem immediately shows that the flux integral is zero.

We can now use the divergence theorem to justify the physical interpretation of divergence that we discussed earlier. Recall that if $\vecs F$ is a continuous three-dimensional vector field and P is a point in the domain of $\vecs F$, then the divergence of $\vecs F$ at P is a measure of the "outflowing-ness" of $\vecs F$ at P. If $\vecs F$ represents the velocity field of a fluid, then the divergence of $\vecs F$ at P is a measure of the net flow rate out of point P (the flow of fluid out of P less the flow of fluid in to P). To see how the divergence theorem justifies this interpretation, let B_τ be a ball of very small radius r with center P, and assume that B_τ is in the domain of $\vecs F$. Furthermore, assume that B_τ has a positive, outward orientation. Since the radius of B_τ is small and $\vecs F$ is continuous, the divergence of $\vecs F$ is approximately constant on B_τ. That is, ifv P' is any point in B_τ, then $\operatorname{div} \vecs F(P) \approx \operatorname{div} \vecs F(P')$. Let S_τ denote the boundary sphere of B_τ. We can approximate the flux across S_τ using the divergence theorem as follows:

$$\begin{aligned}\iint_{S_\tau} \vecs F \cdot d\vecs S &= \iiint_{B_\tau} \operatorname{div} \vecs F\, dV \\ &\approx \iiint_{B_\tau} \operatorname{div} \vecs F(P)\, dV \\ &= \operatorname{div} \vecs F(P)\, V(B_\tau).\end{aligned}$$

As we shrink the radius r to zero via a limit, the quantity $\operatorname{div} \vecs F(P)\ V(B_\tau)$ gets arbitrarily close to the flux. Therefore,

$$\operatorname{div} \vecs F(P) = \lim_{\tau \to 0} \frac{1}{V(B_\tau)} \iint_{S_\tau} \vecs F \cdot d\vecs S$$

and we can consider the divergence at P as measuring the net rate of outward flux per unit volume at P. Since "outflowing-ness"

is an informal term for the net rate of outward flux per unit volume, we have justified the physical interpretation of divergence we discussed earlier, and we have used the divergence theorem to give this justification.

Application to Electrostatic Fields

The divergence theorem has many applications in physics and engineering. It allows us to write many physical laws in both an integral form and a differential form (in much the same way that Stokes' theorem allowed us to translate between an integral and differential form of Faraday's law). Areas of study such as fluid dynamics, electromagnetism, and quantum mechanics have equations that describe the conservation of mass, momentum, or energy, and the divergence theorem allows us to give these equations in both integral and differential forms.

One of the most common applications of the divergence theorem is to **electrostatic fields**. An important result in this subject is **Gauss' law**. This law states that if S is a closed surface in electrostatic field $\vecs{E}$, then the flux of $\vecs{E}$ across S is the total charge enclosed by S (divided by an electric constant). We now use the divergence theorem to justify the special case of this law in which the electrostatic field is generated by a stationary point charge at the origin.

If (x, y, z) is a point in space, then the distance from the point to the origin is $r = \sqrt{x^2 + y^2 + z^2}$. Let $\vecs{F}_\tau$ denote radial vector field $\vecs{F}_\tau = \frac{1}{\tau^2}\left\langle \frac{x}{\tau}, \frac{y}{\tau}, \frac{z}{\tau} \right\rangle$.The vector at a given position in space points in the direction of unit radial vector $\left\langle \frac{x}{\tau}, \frac{y}{\tau}, \frac{z}{\tau} \right\rangle$ and is scaled by the quantity $1/\tau^2$. Therefore, the magnitude of a vector at a given point is inversely proportional to the square of the vector's distance from the origin. Suppose we have a stationary charge of q Coulombs at the origin, existing in a vacuum. The charge generates electrostatic field $\vecs{E}$ given by

$$\vecs{E} = \frac{q}{4\pi\epsilon_0}\vecs{F}_\tau,$$

where the approximation $\epsilon_0 = 8.854 \times 10^{-12}$ farad (F)/m is an electric constant. (The constant ϵ_0 is a measure of the resistance encountered when forming an electric field in a vacuum.) Notice that $\vecs{E}$ is a radial vector field similar to the gravitational field described in [link]. The difference is that this field points outward whereas the gravitational field points inward. Because

$$\vecs{E} = \frac{q}{4\pi\epsilon_0}\vecs{F}_\tau = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\tau^2}\left\langle \frac{x}{\tau}, \frac{y}{\tau}, \frac{z}{\tau} \right\rangle\right),$$

we say that electrostatic fields obey an inverse-square law. That is, the electrostatic force at a given point is inversely proportional to the square of the distance from the source of the charge (which in this case is at the origin). Given this vector field, we show that the flux across closed surface S is zero if the charge is outside of S, and that the flux is $q/epsilon_0$ if the charge is inside of S . In other words, the flux across S is the charge inside the surface divided by constant ϵ_0. This is a special case of Gauss' law, and here we use the divergence theorem to justify this special case.

To show that the flux across S is the charge inside the surface divided by constant ϵ_0, we need two intermediate steps. First we show that the divergence of $\vecs{F}_\tau$ is zero and then we show that the flux of $\vecs{F}_\tau$ across any smooth surface S is either zero or 4π. We can then justify this special case of Gauss' law.

Example 15.8.4: The Divergence of F_τ is Zero

Verify that the divergence of $\vecs{F}_\tau$ is zero where $\vecs{F}_\tau$ is defined (away from the origin).

Solution

Since $\tau = \sqrt{x^2 + y^2 + z^2}$, the quotient rule gives us

$$\frac{\partial}{\partial x}\left(\frac{x}{\tau^3}\right) = \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right)$$
$$= \frac{(x^2+y^2+z^2)^{3/2} - x\left[\frac{3}{2}(x^2+y^2+z^2)^{1/2}2x\right]}{(x^2+y^2+z^2)^3}$$
$$= \frac{\tau^3 - 3x^2\tau}{\tau^6} = \frac{\tau^2 - 3x^2}{\tau^5}.$$

Similarly,

$$\frac{\partial}{\partial y}\left(\frac{y}{\tau^3}\right) = \frac{\tau^2 - 3y^2}{\tau^5} \; and \; \frac{\partial}{\partial z}\left(\frac{z}{\tau^3}\right) = \frac{\tau^2 - 3z^2}{\tau^5}.$$

Therefore,

$$\text{div } \vecs{F}_\tau = \frac{\tau^2 - 3x^2}{\tau^5} + \frac{\tau^2 - 3y^2}{\tau^5} + \frac{\tau^2 - 3z^2}{\tau^5}$$
$$= \frac{3\tau^2 - 3(x^2+y^2+z^2)}{\tau^5}$$
$$= \frac{3\tau^2 - 3\tau^2}{\tau^5} = 0.$$

Notice that since the divergence of $\vecs{F}_\tau$ is zero and $\vecs{E}$ is $\vecs{F}_\tau$ scaled by a constant, the divergence of electrostatic field $\vecs{E}$ is also zero (except at the origin).

Flux across a Smooth Surface

Let S be a connected, piecewise smooth closed surface and let $\vecs{F}_\tau = \frac{1}{\tau^2}\left\langle \frac{x}{\tau}, \frac{y}{\tau}, \frac{z}{\tau} \right\rangle$. Then,

$$\iint_S \vecs{F}_\tau \cdot d\vecs{S} = \begin{cases} 0, & \text{if } S \text{ does not encompass the origin} \\ 4\pi, & \text{if } S \text{ encompasses the origin.} \end{cases}$$

In other words, this theorem says that the flux of $\vecs{F}_\tau$ across any piecewise smooth closed surface S depends only on whether the origin is inside of S.

Proof

The logic of this proof follows the logic of [link], only we use the divergence theorem rather than Green's theorem.

First, suppose that S does not encompass the origin. In this case, the solid enclosed by S is in the domain of $\vecs{F}_\tau$, and since the divergence of $\vecs{F}_\tau$ is zero, we can immediately apply the divergence theorem and find that

$$\iint_S \vecs{F} \cdot d\vecs{S}$$

is zero.

Now suppose that S does encompass the origin. We cannot just use the divergence theorem to calculate the flux, because the field is not defined at the origin. Let S_a be a sphere of radius a inside of S centered at the origin. The outward normal vector field on the sphere, in spherical coordinates, is

$$\vecs{t}_\phi \times \vecs{t}_\theta = \langle a^2 \cos\theta \sin^2\phi, \; a^2 \sin\theta \sin^2\phi, \; a^2 \sin\phi \cos\phi \rangle$$

(see [link]). Therefore, on the surface of the sphere, the dot product $\vecs{F}_\tau \cdot \vecs{N}$ (in spherical coordinates) is

$$\begin{aligned}\vecs{F}_\tau \cdot \vecs{N} &= \left\langle \frac{\sin\phi\cos\theta}{a^2}, \frac{\sin\phi\sin\theta}{a^2}, \frac{\cos\phi}{a^2} \right\rangle \cdot \langle a^2\cos\theta\sin^2\phi, a^2\sin\theta\sin^2\phi, a^2\sin\phi\cos\phi\rangle \\ &= \sin\phi(\langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle \cdot \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle) \\ &= \sin\phi.\end{aligned}$$

The flux of $\vecs{F}_\tau$ across S_a is

$$\iint_{S_a} \vecs{F}_\tau \cdot \vecs{N} dS = \int_0^{2\pi}\int_0^{\pi} \sin\phi \, d\phi \, d\theta = 4\pi.$$

Now, remember that we are interested in the flux across S, not necessarily the flux across S_a. To calculate the flux across S, let E be the solid between surfaces S_a and S. Then, the boundary of E consists of S_a and S. Denote this boundary by $S - S_a$ to indicate that S is oriented outward but now S_a is oriented inward. We would like to apply the divergence theorem to solid E. Notice that the divergence theorem, as stated, can't handle a solid such as E because E has a hole. However, the divergence theorem can be extended to handle solids with holes, just as Green's theorem can be extended to handle regions with holes. This allows us to use the divergence theorem in the following way. By the divergence theorem,

$$\begin{aligned}\iint_{S-S_a} \vecs{F}_\tau \cdot d\vecs{S} &= \iint_S \vecs{F}_\tau \cdot d\vecs{S} - \iint_{S_a} \vecs{F}_\tau \cdot d\vecs{S} \\ &= \iiint_E \operatorname{div} \vecs{F}_\tau \, dV \\ &= \iiint_E 0 \, dV = 0.\end{aligned}$$

Therefore,

$$\iint_S \vecs{F}_\tau \cdot d\vecs{S} = \iint_{S_a} \vecs{F}_\tau \cdot d\vecs{S} = 4\pi,$$

and we have our desired result.

□

Now we return to calculating the flux across a smooth surface in the context of electrostatic field $\vecs{E} = \frac{q}{4\pi\epsilon_0}\vecs{F}_\tau$ of a point charge at the origin. Let S be a piecewise smooth closed surface that encompasses the origin. Then

$$\begin{aligned}\iint_S \vecs{E} \cdot d\vecs{S} &= \iint_S \frac{q}{4\pi\epsilon_0}\vecs{F}_\tau \cdot d\vecs{S} \\ &= \frac{q}{4\pi\epsilon_0}\iint_S \vecs{F}_\tau \cdot d\vecs{S} \\ &= \frac{q}{\epsilon_0}.\end{aligned}$$

If S does not encompass the origin, then

$$\iint_S \vecs{E} \cdot d\vecs{S} = \frac{q}{4\pi\epsilon_0}\iint_S \vecs{F}_\tau \cdot d\vecs{S} = 0.$$

Therefore, we have justified the claim that we set out to justify: the flux across closed surface S is zero if the charge is outside of S, and the flux is q/ϵ_0 if the charge is inside of S.

This analysis works only if there is a single point charge at the origin. In this case, Gauss' law says that the flux of $\vecs{E}$ across S is the total charge enclosed by S. Gauss' law can be extended to handle multiple charged solids in space, not just a single point charge at the origin. The logic is similar to the previous analysis, but beyond the scope of this text. In full generality, Gauss' law states that if S is a piecewise smooth closed surface and Q is the total amount of charge inside of S, then the flux of $\vecs{E}$ across S is Q/ϵ_0.

Example 15.8.5: Using Gauss' law

Suppose we have four stationary point charges in space, all with a charge of 0.002 Coulombs (C). The charges are located at $(0,0,1)$, $(1,1,4)$, $(-1,0,0)$, and $(-2,-2,2)$. Let $\vecs{E}$ denote the electrostatic field generated by these point charges. If S is the sphere of radius 2 oriented outward and centered at the origin, then find

$$\iint_S \vecs{E} \cdot d\vecs{S}.$$

Solution

According to Gauss' law, the flux of $\vecs{E}$ across S is the total charge inside of S divided by the electric constant. Since S has radius 2, notice that only two of the charges are inside of S: the charge at $(0,1,1)$ and the charge at $(-1,0,0)$. Therefore, the total charge encompassed by S is 0.004 and, by Gauss' law,

$$\iint_S \vecs{E} \cdot d\vecs{S} = \frac{0.004}{8.854 \times 10^{-12}} \approx 4.418 \times 10^9\, V - m.$$

Exercise 15.8.4

Work the previous example for surface S that is a sphere of radius 4 centered at the origin, oriented outward.

Hint available at the end of this section.

Answer available at the end of this section.

Key Concepts

- The divergence theorem relates a surface integral across closed surface S to a triple integral over the solid enclosed by S. The divergence theorem is a higher dimensional version of the flux form of Green's theorem, and is therefore a higher dimensional version of the Fundamental Theorem of Calculus.
- The divergence theorem can be used to transform a difficult flux integral into an easier triple integral and vice versa.
- The divergence theorem can be used to derive Gauss' law, a fundamental law in electrostatics.

Key Equations

- **Divergence theorem**

$$\iiint_E \text{div } \vecs{F}\, dV = \iint_S \vecs{F} \cdot d\vecs{S}$$

Glossary

divergence theorem
: a theorem used to transform a difficult flux integral into an easier triple integral and vice versa

Gauss' law
: if S is a piecewise, smooth closed surface in a vacuum and Q is the total stationary charge inside of S, then the flux of electrostatic field $\vecs{E}$ across S is Q/ϵ_0

inverse-square law
: the electrostatic force at a given point is inversely proportional to the square of the distance from the source of the charge

Answers

Exercise 15.8.1

Calculate both the flux integral and the triple integral with the divergence theorem and verify they are equal.

Exercise 15.8.1

Both integrals equal 6π.

Exercise 15.8.2

Calculate the corresponding triple integral.

Exercise 15.8.2

40

Exercise 15.8.3

Use the divergence theorem and calculate a triple integral

Exercise 15.8.3

$9\ \ln(16)$

Exercise 15.8.4

Use Gauss' law.

Exercise 15.8.4

$\approx 6.777 \times 10^9$

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Chapter 15 Review Exercises

***True or False?* Justify your answer with a proof or a counterexample.**

1. The vector field $\vecs F(x,y) = x^2y\ \hat{\mathbf{i}} + y^2x\ \hat{\mathbf{j}}$ is conservative.

Answer

False

2. For vector field $\vecs F(x,y) = P(x,y)\ \hat{\mathbf{i}} + Q(x,y)\ \hat{\mathbf{j}}$, if $P_y(x,y) = Q_z(x,y)$ in open region D, then $\int_{\partial D} P\ dx + Q\ dy = 0$.

3. The divergence of a vector field is a vector field.

Answer

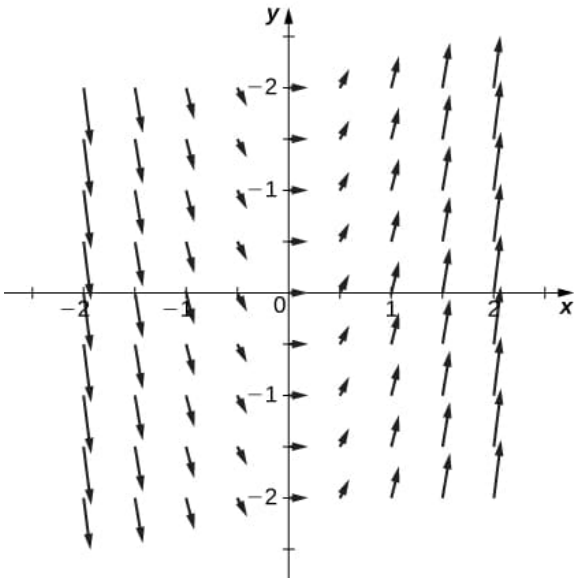
False

4. If $curl\ \vecs F = \vecs 0$, then $\vecs F$ is a conservative vector field.

Draw the following vector fields.

5. $\vecs F(x,y) = \frac{1}{2}\ \hat{\mathbf{i}} + 2x\ \hat{\mathbf{j}}$

Answer



6. $\vecs F(x,y) = \sqrt{\frac{y\hat{\mathbf{i}} + 3x\hat{\mathbf{j}}}{x^2+y^2}}$

Are the following the vector fields conservative? If so, find the potential function $\vecs F$ such that $\vecs F = \vecs \nabla f$.

7. $\vecs F(x,y) = y\ \hat{\mathbf{i}} + (x - 2e^y)\ \hat{\mathbf{j}}$

Answer

Conservative, $f(x,y) = xy - 2e^y$

8. $\vecs F(x,y) = (6xy)\ \hat{\mathbf{i}} + (3x^2 - ye^y)\ \hat{\mathbf{j}}$

9. $\vecs{F}(x,y) = (2xy + z^2)\ \hat{\mathbf{i}} + (x^2 + 2yz)\ \hat{\mathbf{j}} + (2xz + y^2)\ \hat{\mathbf{k}}$

Answer

Conservative, $f(x,y,z) = x^2y + y^2z + z^2x$

10. $\vecs{F}(x,y,z) = (e^x y)\ \hat{\mathbf{i}} + (e^x + z)\ \hat{\mathbf{j}} + (e^x + y^2)\ \hat{\mathbf{k}}$

Evaluate the following integrals.

11. $\int_C x^2\ dy + (2x - 3xy)\ dx$, along $C : y = \frac{1}{2}x$ from $(0,0)$ to $(4,2)$

Answer

$-\frac{16}{3}$

12. $\int_C y\ dx + xy^2\ dy$, where $C : x = \sqrt{t},\ \ y = t - 1,\ \ 0 \le t \le 1$

13. $\iint_S xy^2\ dS$, where S is the surface $z = x^2 - y,\ \ 0 \le x \le 1,\ \ 0 \le y \le 4$

Answer

$\frac{32\sqrt{2}}{9}(3\sqrt{3} - 1)$

Find the divergence and curl for the following vector fields.

14. $\vecs{F}(x,y,z) = 3xyz\ \hat{\mathbf{i}} + xye^x\ \hat{\mathbf{j}} - 3xy\ \hat{\mathbf{k}}$

15. $\vecs{F}(x,y,z) = e^x\ \hat{\mathbf{i}} + e^{xy}\ \hat{\mathbf{j}} - e^{xyz}\ \hat{\mathbf{k}}$

Answer

Divergence: $e^x + x\ e^{xy} + xy\ e^{xyz}$
Curl: $xze^{xyz}\ \hat{\mathbf{i}} - yze^{xyz}\ \hat{\mathbf{j}} + ye^{xy}\ \hat{\mathbf{k}}$

Use Green's theorem to evaluate the following integrals.

16. $\int_C 3xy\ dx + 2xy^2\ dy$, where C is a square with vertices $(0,0),\ \ (0,2),\ \ (2,2)$ and $(2,0)$.

17. $\oint_C 3y\ dx + (x + e^y)\ dy$, where C is a circle centered at the origin with radius 3.

Answer

-2π

Use Stokes' theorem to evaluate $\iint_S curl\ \vecs{F} \cdot dS$.

18. $\vecs{F}(x,y,z) = y\ \hat{\mathbf{i}} - x\ \hat{\mathbf{j}} + z\ \hat{\mathbf{k}}$, where S is the upper half of the unit sphere

19. $\vecs{F}(x,y,z) = y\ \hat{\mathbf{i}} + xyz\ \hat{\mathbf{j}} - 2zx\ \hat{\mathbf{k}}$, where S is the upward-facing paraboloid $z = x^2 + y^2$ lying in cylinder $x^2 + y^2 = 1$

Answer

$-\pi$

Use the divergence theorem to evaluate $\iint_S \vecs{F} \cdot dS$.

20. $\vecs{F}(x,y,z) = (x^3y)\ \hat{\mathbf{i}} + (3y - e^x)\ \hat{\mathbf{j}} + (z + x)\ \hat{\mathbf{k}}$, over cube S defined by $-1 \le x \le 1,\ \ 0 \le y \le 2,\ \ 0 \le z \le 2$

21. $\vecs{F}(x,y,z) = (2xy)\ \hat{\mathbf{i}} + (-y^2)\ \hat{\mathbf{j}} + (2z^3)\ \hat{\mathbf{k}}$, where S is bounded by paraboloid $z = x^2 + y^2$ and plane $z = 2$

Answer

$31\pi/2$

22. Find the amount of work performed by a 50-kg woman ascending a helical staircase with radius 2 m and height 100 m. The woman completes five revolutions during the climb.

23. Find the total mass of a thin wire in the shape of a semicircle with radius $\sqrt{2}$, and a density function of $\rho(x,y) = y + x^2$.

Answer

$\sqrt{2}(2+\pi)$

24. Find the total mass of a thin sheet in the shape of a hemisphere with radius 2 for $z \geq 0$ with a density function $\rho(x,y,z) = x + y + z$.

25. Use the divergence theorem to compute the value of the flux integral over the unit sphere with $\vecs{F}(x,y,z) = 3z\ \hat{\mathbf{i}} + 2y\ \hat{\mathbf{j}} + 2x\ \hat{\mathbf{k}}$.

Answer

$2\pi/3$

Contributors

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INDEX

C

calcplot:yes
- 14.1: Iterated Integrals and Area
- 14.1E: Iterated Integrals and Area (Exercises)
- 14.2a: Double Integrals Over Rectangular Regions
- 14.2aE: Double Integrals Part 1 (Exercises)
- 14.2b: Double Integrals Over General Regions
- 14.2bE: Double Integrals Part 2 (Exercises)
- 14.3: Double Integration with Polar Coordinates
- 14.3b: Double Integrals in Polar Coordinates
- 14.3E: Double Integrals in Polar Coordinates (Exercises)
- 14.4: Triple Integrals
- 14.4E: Triple Integrals (Exercises 2)
- 14.4E: Triple Integrals (Exercises)
- 14.5: Triple Integrals in Cylindrical and Spherical Coordinates
- 14.6: Calculating Centers of Mass and Moments of Inertia
- 14.7: Change of Variables in Multiple Integrals (Jacobians)
- 15.0: Introduction to Vector Field Chapter
- 15.1: Vector Fields
- 15.1E: Vector Fields (Exercises)
- 15.2: Line Integrals
- 15.2E: Line Integrals (Exercises)
- 15.3: Conservative Vector Fields
- 15.3E: Conservative Vector Fields (Exercises)
- 15.4: Green's Theorem
- 15.4E: Green's Theorem (Exercises)
- 15.5: Divergence and Curl
- 15.5E: Divergence and Curl (Exercises)
- 15.6: Surface Integrals
- 15.7: Stokes' Theorem
- 15.8: The Divergence Theorem
- Chapter 14 Review Exercises
- Chapter 14: Multiple Integration
- Chapter 15: Vector Fields, Line Integrals, and Vector Theorems

curl
- 15.5: Divergence and Curl

D

divergence
- 15.5: Divergence and Curl

divergence theorem
- 15.8: The Divergence Theorem

DOUBLE INTEGRAL IN POLAR COORDINATES
- 14.3: Double Integration with Polar Coordinates

H

hidetop:solutions
- 14.1: Iterated Integrals and Area
- 14.1E: Iterated Integrals and Area (Exercises)
- 14.2a: Double Integrals Over Rectangular Regions
- 14.2aE: Double Integrals Part 1 (Exercises)
- 14.2b: Double Integrals Over General Regions
- 14.2bE: Double Integrals Part 2 (Exercises)
- 14.3: Double Integration with Polar Coordinates
- 14.3b: Double Integrals in Polar Coordinates
- 14.3E: Double Integrals in Polar Coordinates (Exercises)
- 14.4: Triple Integrals
- 14.4E: Triple Integrals (Exercises 2)
- 14.4E: Triple Integrals (Exercises)
- 14.5: Triple Integrals in Cylindrical and Spherical Coordinates
- 14.6: Calculating Centers of Mass and Moments of Inertia
- 14.7: Change of Variables in Multiple Integrals (Jacobians)
- 15.0: Introduction to Vector Field Chapter
- 15.1: Vector Fields
- 15.1E: Vector Fields (Exercises)
- 15.2: Line Integrals
- 15.2E: Line Integrals (Exercises)
- 15.3: Conservative Vector Fields
- 15.3E: Conservative Vector Fields (Exercises)
- 15.4: Green's Theorem
- 15.4E: Green's Theorem (Exercises)
- 15.5: Divergence and Curl
- 15.5E: Divergence and Curl (Exercises)
- 15.6: Surface Integrals
- 15.7: Stokes' Theorem
- 15.8: The Divergence Theorem
- Chapter 14 Review Exercises
- Chapter 14: Multiple Integration
- Chapter 15: Vector Fields, Line Integrals, and Vector Theorems

I

iterated integration
- 14.1: Iterated Integrals and Area

M

Multiple Integration
- 14.1: Iterated Integrals and Area

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  - Front Matter - *Undeclared*
    - TitlePage - *Undeclared*
    - InfoPage - *Undeclared*
    - Table of Contents - *Undeclared*
    - Licensing - *Undeclared*
  - Chapter 14: Multiple Integration - *CC BY-NC-SA 4.0*
    - 14.1: Iterated Integrals and Area - *CC BY-NC-SA 3.0*
      - 14.1E: Iterated Integrals and Area (Exercises) - *CC BY-NC-SA 4.0*
    - 14.2a: Double Integrals Over Rectangular Regions - *CC BY-NC-SA 4.0*
      - 14.2aE: Double Integrals Part 1 (Exercises) - *CC BY-NC-SA 4.0*
    - 14.2b: Double Integrals Over General Regions - *CC BY-NC-SA 4.0*
      - 14.2bE: Double Integrals Part 2 (Exercises) - *CC BY-NC-SA 4.0*
    - 14.3: Double Integration with Polar Coordinates - *CC BY-NC-SA 3.0*
      - 14.3b: Double Integrals in Polar Coordinates - *CC BY-NC-SA 4.0*
      - 14.3E: Double Integrals in Polar Coordinates (Exercises) - *CC BY-NC-SA 4.0*
    - 14.4: Triple Integrals - *CC BY-NC-SA 4.0*
      - 14.4E: Triple Integrals (Exercises 2) - *CC BY-NC-SA 4.0*
      - 14.4E: Triple Integrals (Exercises) - *CC BY-NC-SA 4.0*
    - 14.5: Triple Integrals in Cylindrical and Spherical Coordinates - *CC BY-NC-SA 4.0*
    - 14.6: Calculating Centers of Mass and Moments of Inertia - *CC BY-NC-SA 4.0*
    - 14.7: Change of Variables in Multiple Integrals (Jacobians) - *CC BY-NC-SA 4.0*
    - Chapter 14 Review Exercises - *CC BY-NC-SA 4.0*
  - Chapter 15: Vector Fields, Line Integrals, and Vector Theorems - *CC BY-NC-SA 4.0*
    - 15.0: Introduction to Vector Field Chapter - *CC BY-NC-SA 4.0*
    - 15.1: Vector Fields - *CC BY-NC-SA 4.0*
      - 15.1E: Vector Fields (Exercises) - *CC BY-NC-SA 4.0*
    - 15.2: Line Integrals - *CC BY-NC-SA 4.0*
      - 15.2: Line Integrals - Part 2 - *Undeclared*
      - 15.2E: Line Integrals (Exercises) - *CC BY-NC-SA 4.0*
    - 15.3: Conservative Vector Fields - *CC BY-NC-SA 4.0*
      - 15.3E: Conservative Vector Fields (Exercises) - *CC BY-NC-SA 4.0*
    - 15.4: Green's Theorem - *CC BY-NC-SA 4.0*
      - 15.4E: Green's Theorem (Exercises) - *CC BY-NC-SA 4.0*
    - 15.5: Divergence and Curl - *CC BY-NC-SA 4.0*
      - 15.5E: Divergence and Curl (Exercises) - *CC BY-NC-SA 4.0*
    - 15.6: Surface Integrals - *CC BY-NC-SA 4.0*
    - 15.7: Stokes' Theorem - *CC BY-NC-SA 4.0*
    - 15.8: The Divergence Theorem - *CC BY-NC-SA 4.0*
    - Chapter 15 Review Exercises - *Undeclared*
  - Back Matter - *Undeclared*
    - Index - *Undeclared*
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