BASIC MATH (GRADE 6)



Illustrative Mathematics

OpenUp Resources



OpenUp Resources

Basic Math (Grade 6)

Illustrative Mathematics

This text is disseminated via the Open Education Resource (OER) LibreTexts Project (https://LibreTexts.org) and like the thousands of other texts available within this powerful platform, it is freely available for reading, printing, and "consuming."

The LibreTexts mission is to bring together students, faculty, and scholars in a collaborative effort to provide and accessible, and comprehensive platform that empowers our community to develop, curate, adapt, and adopt openly licensed resources and technologies; through these efforts we can reduce the financial burden born from traditional educational resource costs, ensuring education is more accessible for students and communities worldwide.

Most, but not all, pages in the library have licenses that may allow individuals to make changes, save, and print this book. Carefully consult the applicable license(s) before pursuing such effects. Instructors can adopt existing LibreTexts texts or Remix them to quickly build course-specific resources to meet the needs of their students. Unlike traditional textbooks, LibreTexts' web based origins allow powerful integration of advanced features and new technologies to support learning.



LibreTexts is the adaptable, user-friendly non-profit open education resource platform that educators trust for creating, customizing, and sharing accessible, interactive textbooks, adaptive homework, and ancillary materials. We collaborate with individuals and organizations to champion open education initiatives, support institutional publishing programs, drive curriculum development projects, and more.

The LibreTexts libraries are Powered by NICE CXone Expert and was supported by the Department of Education Open Textbook Pilot Project, the California Education Learning Lab, the UC Davis Office of the Provost, the UC Davis Library, the California State University Affordable Learning Solutions Program, and Merlot. This material is based upon work supported by the National Science Foundation under Grant No. 1246120, 1525057, and 1413739.

Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation nor the US Department of Education.

Have questions or comments? For information about adoptions or adaptions contact info@LibreTexts.org or visit our main website at https://LibreTexts.org.

This text was compiled on 06/09/2025

TABLE OF CONTENTS

Licensing

1: Area and Surface Area

- Section 1: Lessons Reasoning to Find Area
  - 1.1: Tiling the Plane
  - 1.2: Finding Area by Decomposing and Rearranging
  - 1.3: Reasoning to Find Area
- Section 2: Parallelograms
  - 2.1: Parallelograms
  - 2.2: Bases and Heights of Parallelograms
  - 2.3: Area of Parallelograms
- Section 3: Triangles
  - 3.1: From Parallelograms to Triangles
  - 3.2: Area of Triangles
  - 3.3: Formula for the Area of a Triangle
  - 3.4: Bases and Heights of Triangles
- Section 4: Polygons
  - 4.1: Polygons
- Section 5: Surface Area
  - 5.1: Polyhedra
  - 5.2: Nets and Surface Area
  - 5.3: More Nets, More Surface Area
  - 5.4: Distinguishing Between Surface Area and Volume
- Section 6: Squares and Cubes
  - 6.1: Squares and Cubes
  - 6.2: Surface Area of a Cube
- Section 7: Surface Area
  - 7.1: What is Surface Area?
- Section 8: Let's Put it to Work
  - 8.1: Designing a Tent

2: Introducing Ratios

- Section 9: What are Ratios?
  - 9.1: Introducing Ratios and Ratio Language
  - 9.2: Representing Ratios with Diagrams
- Section 10: Equivalent Ratios
  - 10.1: Recipes
  - 10.2: Color Mixtures
  - 10.3: Defining Equivalent Ratios
- Section 11: Representing Equivalent Ratios
  - 11.1: Introducing Double Number Line Diagrams
  - 11.2: Creating Double Number Line Diagrams
  - 11.3: Constant Speed

- 11.4: Comparing Situations by Examining Ratios
- Section 12: Solving Ratio and Rate Problems
  - 12.1: Representing Ratios with Tables
  - 12.2: Navigating a Table of Equivalent Ratios
  - 12.3: Tables and Double Number Line Diagrams
  - 12.4: Solving Equivalent Ratio Problems
- Section 13: Part-Part-Whole Ratios
  - 13.1: Part-Part-Whole Ratios
  - 13.2: Solving More Ratio Problems
- Section 14: Let's Put it to Work
  - 14.1: A Fermi Problem
  - 14.2: How Much for One?

3: Unit Rates and Percentages

- Section 15: Unit of Measurement
  - 15.1: The Burj Khalifa
- Section 16: Unit Conversion
  - 16.1: Anchoring Units of Measurement
  - 16.2: Measuring with Different-Sized Units
  - 16.3: Converting Units
- Section 17: Rates
  - 17.1: Comparing Speeds and Prices
  - 17.2: Interpreting Rates
  - 17.3: Equivalent Ratios Have the Same Unit Rates
  - 17.4: More about Constant Speed
  - 17.5: Solving Rate Problems
- Section 18: Percentages
  - 18.1: Percentages and Double Number Lines
  - 18.2: Percentages and Tape Diagrams
  - 18.3: Benchmark Percentages
  - 18.4: Solving Percentage Problems
  - 18.5: Finding This Percent of That
  - 18.6: Finding the Percentage
- Section 19: Let's Put it to Work
  - 19.1: Painting a Room
- Section 20: New Page
  - 20.1: What Are Percentages?

4: Dividing Fractions

- Section 21: Making Sense of Division
  - 21.1: Size of Divisor and Size of Quotient
  - 21.2: Meanings of Division
  - 21.3: Interpreting Division Situations
- Section 22: Meanings of Fraction Division
  - 22.1: How Many Groups? (Part 1)
  - 22.2: How Many Groups? (Part 2)
  - 22.3: Using Diagrams to Find the Number of Groups

- 22.4: What Fraction of a Group?
- 22.5: How Much in Each Group? (Part 1)
- 22.6: How Much in Each Group? (Part 2)
- Section 23: Algorithm for Fraction Division
  - 23.1: Dividing by Unit and Non-Unit Fractions
  - 23.2: Using an Algorithm to Divide Fractions
- Section 24: Fractions in Lengths, Areas, and Volumes
  - 24.1: Fractional Lengths
  - 24.2: Rectangles with Fractional Side Lengths
  - 24.3: Fractional Lengths in Triangles and Prisms
  - 24.4: Volume of Prisms
- Section 25: Let's Put It to Work
  - 25.1: Solving Problems Involving Fractions
  - 25.2: Fitting Boxes into Boxes

5: Arithmetic in Base Ten

- Section 26: Warming Up to Decimals
  - 26.1: Using Decimals in a Shopping Context
- Section 27: Adding and Subtracting Decimals
  - 27.1: Using Diagrams to Represent Addition and Subtraction
  - 27.2: Adding and Subtracting Decimals with Few Non-Zero Digits
  - 27.3: Adding and Subtracting Decimals with Many Non-Zero Digits
- Section 28: Multiplying Decimals
  - 28.1: Decimal Points in Products
  - 28.2: Methods for Multiplying Decimals
  - 28.3: Using Diagrams to Represent Multiplication
  - 28.4: Calculating Products of Decimals
- Section 29: Dividing Decimals
  - 29.1: Using the Partial Quotients Method
  - 29.2: Using Long Division
  - 29.3: Dividing Numbers that Result in Decimals
  - 29.4: Dividing Decimals by Whole Numbers
  - 29.5: Dividing Decimals by Decimals
- Section 30: Let's Put It to Work
  - 30.1: Using Operations on Decimals to Solve Problems
  - 30.2: Making and Measuring Boxes

6: Expressions and Equations

- Section 31: Equations in One Variable
  - 31.1: Tape Diagrams and Equations
  - 31.2: Truth and Equations
  - 31.3: Staying in Balance
  - 31.4: Practice Solving Equations and Representing Situations with Equations
  - 31.5: A New Way to Interpret a over b
- Section 32: Equal and Equivalent
  - 32.1: Write Expressions Where Letters Stand for Numbers
  - 32.2: Revisit Percentages
  - 32.3: Equal and Equivalent

- 32.4: The Distributive Property, Part 1
- 32.5: The Distributive Property, Part 2
- 32.6: The Distributive Property, Part 3
- Section 33: Expressions with Exponents
  - 33.1: Meaning of Exponents
  - 33.2: Expressions with Exponents
  - 33.3: Evaluating Expressions with Exponents
  - 33.4: Equivalent Exponential Expressions
- Section 34: Relationship Between Quantities
  - 34.1: Two Related Quantities, Part 1
  - 34.2: Two Related Quantities, Part 2
  - 34.3: More Relationships
- Section 35: Let's Put it to Work
  - 35.1: Tables, Equations, and Graphs, Oh My!

7: Rational Numbers

- Section 36: Negative Numbers and Absolute Value
  - 36.1: Positive and Negative Numbers
  - 36.2: Points on the Number Line
  - 36.3: Comparing Positive and Negative Numbers
  - 36.4: Ordering Rational Numbers
  - 36.5: Using Negative Numbers to Make Sense of Contexts
  - 36.6: Absolute Value of Numbers
  - 36.7: Comparing Numbers and Distance from Zero
- Section 37: Inequalities
  - 37.1: Writing and Graphing Inequalities
  - 37.2: Solutions of Inequalities
  - 37.3: Interpreting Inequalities
- Section 38: The Coordinate Plane
  - 38.1: Points on the Coordinate Plane
  - 38.2: Constructing the Coordinate Plane
  - 38.3: Interpreting Points on a Coordinate Plane
  - 38.4: Distances on a Coordinate Plane
  - 38.5: Shapes on the Coordinate Plane
- Section 39: Common Factors and Common Multiples
  - 39.1: Common Factors
  - 39.2: Common Multiples
  - 39.3: Using Common Multiples and Common Factors
- Section 40: Let's Put it to Work
  - 40.1: Drawing on the Coordinate Plane

8: Data Sets and Distributions

- Section 41: Data, Variability, and Statistical Questions
  - 41.1: Got Data?
  - 41.2: Statistical Questions
- Section 42: Dot Plots and Histograms
  - 42.1: Representing Data Graphically
  - 42.2: Dot Plots

- 42.3: Using Dot Plots to Answer Statistical Questions
- 42.4: Interpreting Histograms
- 42.5: Using Histograms to Answer Statistical Questions
- 42.6: Describing Distributions on Histograms
- Section 43: Measures of Center and Variability
  - 43.1: Mean
  - 43.2: Finding and Interpreting the Mean as the Balance Point
  - 43.3: Variability and MAD
  - 43.4: Using Mean and MAD to Make Comparisons
- Section 44: Median and IQR
  - 44.1: Median
  - 44.2: Comparing Mean and Median
  - 44.3: Quartiles and Interquartile Range
  - 44.4: Box Plots
    - 44.4.1: Using Box Plots
- Section 45: Let's Put it to Work
  - 45.1: Using Data to Solve Problems

9: Putting it All Together

- Section 46: Making Connections
  - 46.1: Fermi Problems
  - 46.2: If Our Class Were the World
  - 46.3: Rectangle Madness
- Section 47: Voting
  - 47.1: More than Two Choices
  - 47.2: Picking Representatives
- Section 48: Let's Put it to Work
  - 48.1: How Do We Choose?

Index

Glossary

Detailed Licensing

Licensing

*A detailed breakdown of this resource's licensing can be found in **Back Matter/Detailed Licensing**.*

SECTION OVERVIEW

1: Area and Surface Area

Section 1: Lessons Reasoning to Find Area

1.1: Tiling the Plane
1.2: Finding Area by Decomposing and Rearranging
1.3: Reasoning to Find Area

Section 2: Parallelograms

2.1: Parallelograms
2.2: Bases and Heights of Parallelograms
2.3: Area of Parallelograms

Section 3: Triangles

3.1: From Parallelograms to Triangles
3.2: Area of Triangles
3.3: Formula for the Area of a Triangle
3.4: Bases and Heights of Triangles

Section 4: Polygons

4.1: Polygons

Section 5: Surface Area

5.1: Polyhedra
5.2: Nets and Surface Area
5.3: More Nets, More Surface Area
5.4: Distinguishing Between Surface Area and Volume

Section 6: Squares and Cubes

6.1: Squares and Cubes
6.2: Surface Area of a Cube

Section 7: Surface Area

7.1: What is Surface Area?

Section 8: Let's Put it to Work

8.1: Designing a Tent

This page titled 1: Area and Surface Area is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

Section 1: Lessons Reasoning to Find Area

This page titled Section 1: Lessons Reasoning to Find Area is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

1.1: Tiling the Plane

Lesson

Let's look at tiling patterns and think about area.

Exercise 1.1.1: Which One Doesn't Belong: Tilings

Which pattern doesn’t belong?

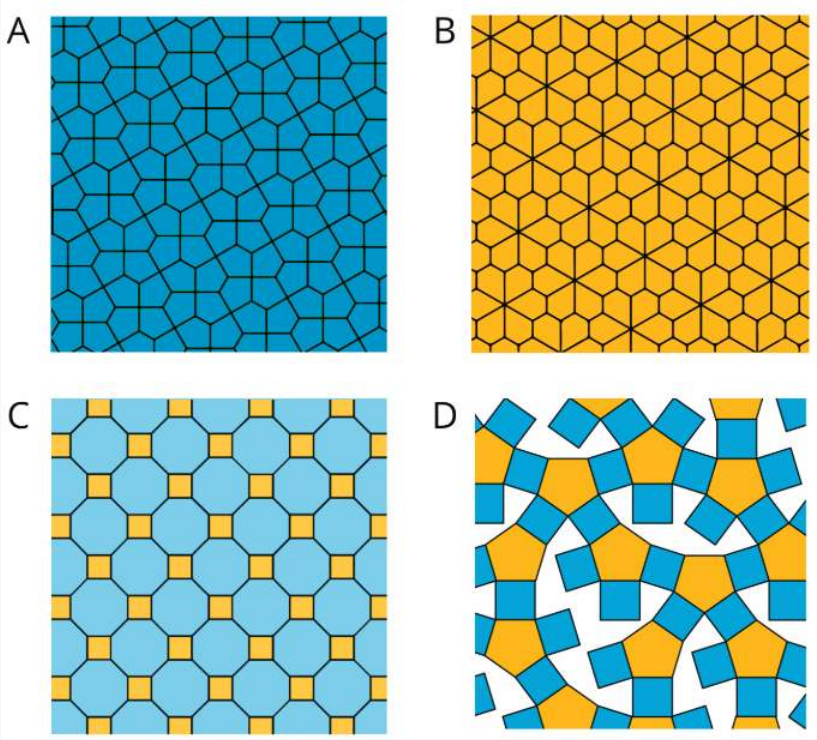


Figure 1.1.1: Four patterns of tiles labeled A, B, C, and D. Pattern A is all blue tiles, pattern B is all yellow tiles, pattern C is a combination of blue and yellow tiles, and pattern D is a combination of blue tiles, yellow tiles, and blank spaces.

Exercise 1.1.2: More Red, Green, or Blue?

Your teacher will assign you to look at Pattern A or Pattern B.

In your pattern, which shapes cover more of the plane: blue rhombuses, red trapezoids, or green triangles? Explain how you know.

You may use the sliders and the shapes in this applet to help. Explore what you can see or hide, and what you can move or turn.

Are you ready for more?

On graph paper, create a tiling pattern so that:

- The pattern has at least two different shapes.
- The same amount of the plane is covered by each type of shape.

Summary

In this lesson, we learned about *tiling* the plane, which means covering a two-dimensional region with copies of the same shape or shapes such that there are no gaps or overlaps.

Then, we compared tiling patterns and the shapes in them. In thinking about which patterns and shapes cover more of the plane, we have started to reason about **area**.

We will continue this work, and to learn how to use mathematical tools strategically to help us do mathematics.

Glossary Entries

Definition: Area

Area is the number of square units that cover a two-dimensional region, without any gaps or overlaps.

For example, the area of region A is 8 square units. The area of the shaded region of B is $\frac{1}{2}$ square unit.

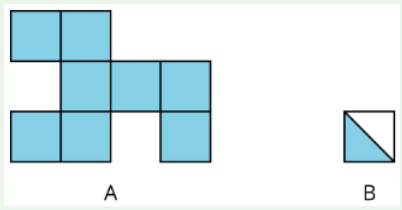


Figure 1.1.2

Definition: Region

A region is the space inside of a shape. Some examples of two-dimensional regions are inside a circle or inside a polygon. Some examples of three-dimensional regions are the inside of a cube or the inside of a sphere.

Practice

Exercise 1.1.3

Which square—large, medium, or small—covers more of the plane? Explain your reasoning.

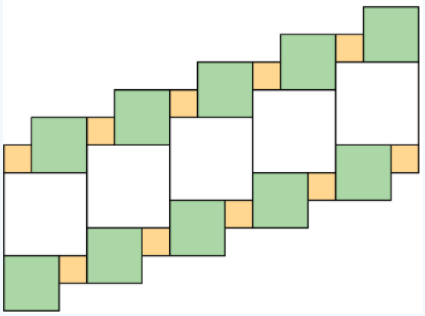


Figure 1.1.3

Exercise 1.1.4

Draw three different quadrilaterals, each with an area of 12 square units.



Figure 1.1.4

Exercise 1.1.5

Use copies of the rectangle to show how a rectangle could:

a. tile the plane.

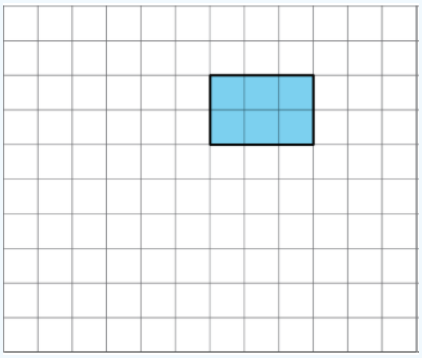


Figure 1.1.5

b. *not* tile the plane.

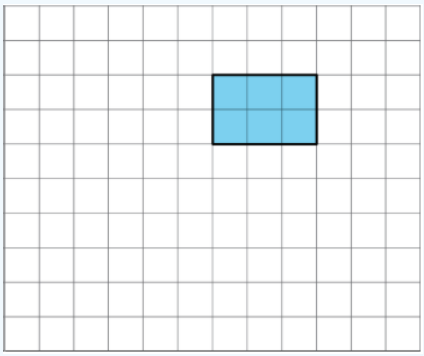


Figure 1.1.6

Exercise 1.1.6

The area of this shape is 24 square units. Which of these statements is true about the area? Select **all** that apply.

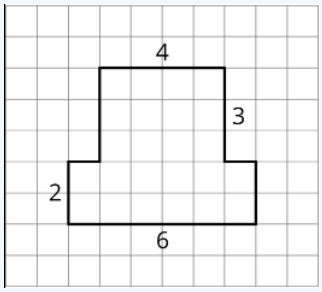


Figure 1.1.7: A figure on a grid. Top side of 4 units and a bottom side of 6 units. The lower portion of the figure has sides that are 2 units tall, and the upper portion has sides that 3 units tall. All angles are right angles.

A. The area can be found by counting the number of squares that touch the edge of the shape.
B. It takes 24 grid squares to cover the shape without gaps and overlaps.
C. The area can be found by multiplying the sides lengths that are 6 units and 4 units.
D. The area can be found by counting the grid squares inside the shape.
E. The area can be found by adding 4×3 and 6×2.

Exercise 1.1.7

Here are two copies of the same figure. Show two different ways for finding the area of the shaded region. All angles are right angles.

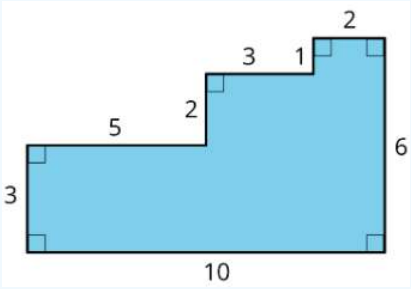


Figure 1.1.8: A figure with a bottom of 10 units, a right side of six units, and a left side that rises 3 units, then goes across 5 units, then goes up another 2 units, then across another 3 units, then up another 1 unit, and across another 2 units to connect to the right side. All angles are right angles.

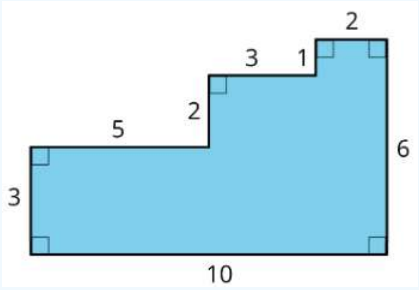


Figure 1.1.9: A figure with a bottom of 10 units, a right side of six units, and a left side that rises 3 units, then goes across 5 units, then goes up another 2 units, then across another 3 units, then up another 1 unit, and across another 2 units to connect to the right side. All angles are right angles.

Exercise 1.1.8

Which shape has a larger area: a rectangle that is 7 inches by $\frac{3}{4}$ inch, or a square with side length of $2\frac{1}{2}$ inches? Show your reasoning.

This page titled 1.1: Tiling the Plane is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

1.2: Finding Area by Decomposing and Rearranging

Lesson

Let's create shapes and find their areas.

Exercise 1.2.1: What is Area?

You may recall that the term **area** tells us something about the number of squares inside a two-dimensional shape.

1. Here are four drawings that each show squares inside a shape. Select **all** drawings whose squares could be used to find the area of the shape. Be prepared to explain your reasoning.

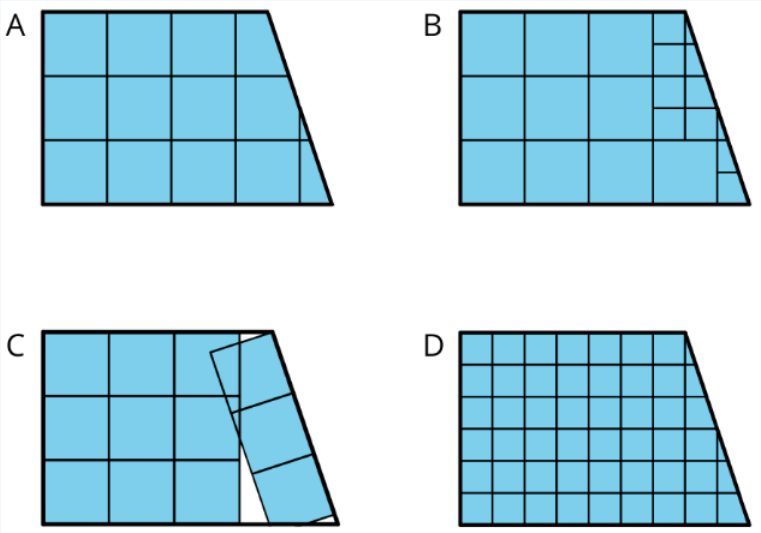


Figure 1.2.1: Four drawings that each show squares inside a shape. Shape A is broken up into large squares, Shape B is broken up into a combination of large and small squares, Shape C is broken up into a combination of large squares and white space, and Shape D is broken up into small squares.

2. Write a definition of area that includes all the information that you think is important.

Exercise 1.2.2: Composing Shapes

This applet has one square and some small, medium, and large right triangles. The area of the square is 1 square unit.

Click on a shape and drag to move it. Grab the point at the vertex and drag to turn it.

1. Notice that you can put together two small triangles to make a square. What is the area of the square composed of two small triangles? Be prepared to explain your reasoning.
2. Use your shapes to create a new shape with an area of 1 square unit that is not a square. Draw your shape on paper and label it with its area.
3. Use your shapes to create a new shape with an area of 2 square units. Draw your shape and label it with its area.
4. Use your shapes to create a *different* shape with an area of 2 square units. Draw your shape and label it with its area.
5. Use your shapes to create a new shape with an area of 4 square units. Draw your shape and label it with its area.

Are you ready for more?

Find a way to use all of your pieces to compose a single large square. What is the area of this large square?

Exercise 1.2.3: Tangram Triangles

Recall that the area of the square you saw earlier is 1 square unit. Complete each statement and explain your reasoning.

1. The area of the small triangle is _____ square units. I know this because . . .
2. The area of the medium triangle is _____ square units. I know this because . . .

3. The area of the large triangle is _____ square units. I know this because . . .

Summary

Here are two important principles for finding **area**:

1. If two figures can be placed one on top of the other so that they match up exactly, then they have the *same area*.
2. We can **decompose** a figure (break a figure into pieces) and **rearrange** the pieces (move the pieces around) to find its area.

Here are illustrations of the two principles.

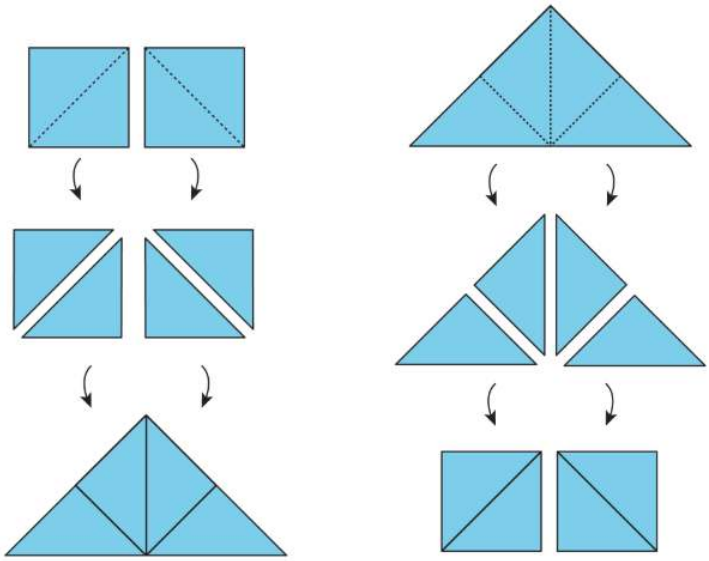


Figure 1.2.2: An image of two squares, which are then decomposed into four triangles, which are then rearranged into one large triangle. Another image of a large triangle that is decomposed into four smaller triangles, and then rearranged into two squares.

- Each square on the left can be decomposed into 2 triangles. These triangles can be rearranged into a large triangle. So the large triangle has the *same area* as the 2 squares.
- Similarly, the large triangle on the right can be decomposed into 4 equal triangles. The triangles can be rearranged to form 2 squares. If each square has an area of 1 square unit, then the area of the large triangle is 2 square units. We also can say that each small triangle has an area of $\frac{1}{2}$ square unit.

Glossary Entries

Definition: Area

Area is the number of square units that cover a two-dimensional region, without any gaps or overlaps.

For example, the area of region A is 8 square units. The area of the shaded region of B is $\frac{1}{2}$ square unit.

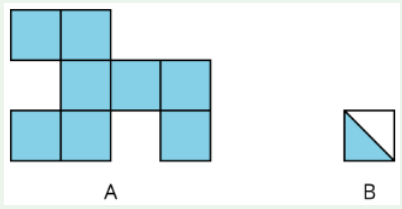


Figure 1.2.3

Definition: Compose

Compose means “put together.” We use the word *compose* to describe putting more than one figure together to make a new shape.

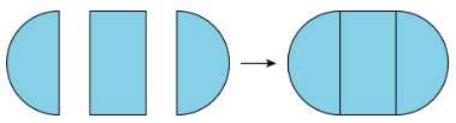


Figure 1.2.4

Definition: Decompose

Decompose means “take apart.” We use the word *decompose* to describe taking a figure apart to make more than one new shape.

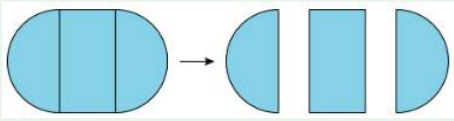


Figure 1.2.5

Definition: Region

A region is the space inside of a shape. Some examples of two-dimensional regions are inside a circle or inside a polygon. Some examples of three-dimensional regions are the inside of a cube or the inside of a sphere.

Practice

Exercise 1.2.4

The diagonal of a rectangle is shown.

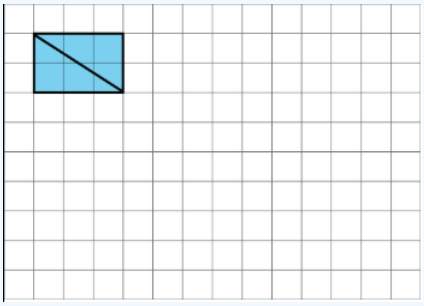


Figure 1.2.6

1. Decompose the rectangle along the diagonal, and recompose the two pieces to make a *different* shape.
2. How does the area of this new shape compare to the area of the original rectangle? Explain how you know.

Exercise 1.2.5

Priya decomposed a square into 16 smaller, equal-size squares and then cut out 4 of the small squares and attached them around the outside of original square to make a new figure.

How does the area of her new figure compare with that of the original square?

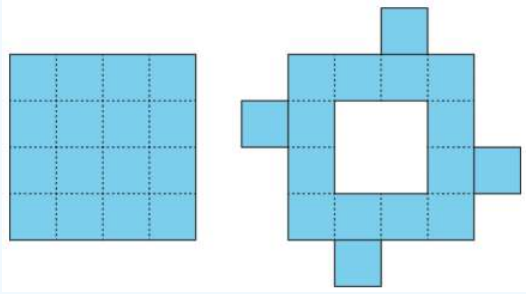


Figure 1.2.7

A. The area of the new figure is greater.
B. The two figures have the same area.
C. The area of the original square is greater.
D. We don't know because neither the side length nor the area of the original square is known.

Exercise 1.2.6

The area of the square is 1 square unit. Two small triangles can be put together to make a square or to make a medium triangle.

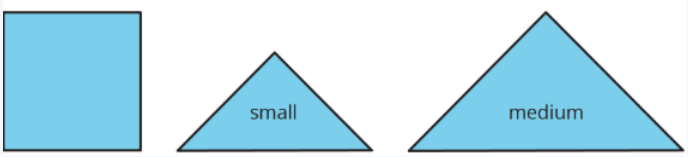


Figure 1.2.8

Which figure also has an area of $1\frac{1}{2}$ square units? Select **all** that apply.

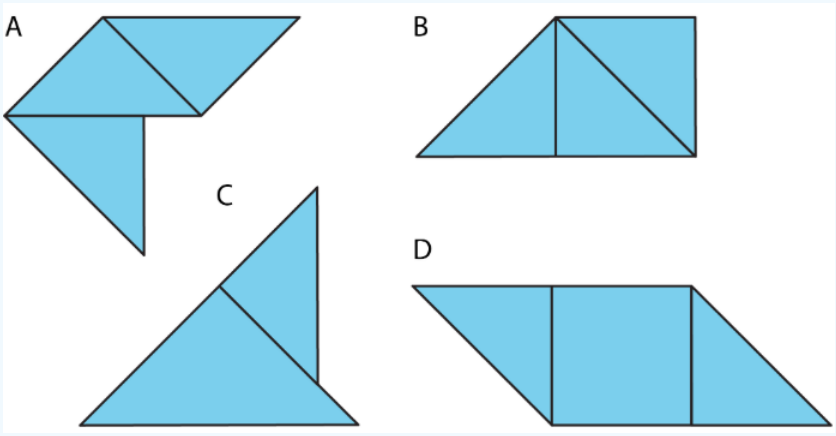


Figure A is composed of three small triangles, figure B is composed of three small triangles in a different arrangement, figure C is composed of one medium triangle and one small triangle, and figure D is composed of two small triangles and one square.

A. Figure A
B. Figure B
C. Figure C
D. Figure D

Exercise 1.2.7

The area of a rectangular playground is 78 square meters. If the length of the playground is 13 meters, what is its width?

(From Unit 1.1.1)

Exercise 1.2.8

A student said, "We can't find the area of the shaded region because the shape has many different measurements, instead of just a length and a width that we could multiply."

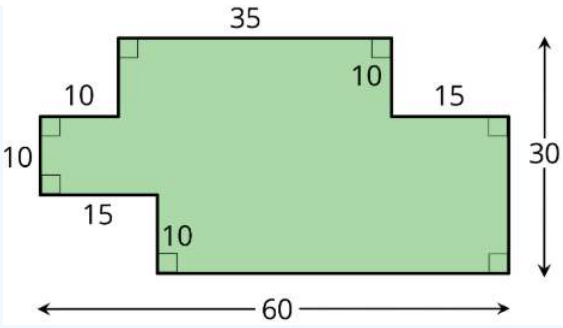


Figure 1.2.10: A multi-sided figure. The sides on top measure 10 units, 35 units, and 15 units. Two of the three sides on the left measure 10 units. One of the two sides on the right measures 10 units. One of the two sides on the bottom measure 15 units. The total width of the figure is 60 units, and the total height is 30 units. All angles are right angles.

Explain why the student's statement about area is incorrect.

(From Unit 1.1.1)

This page titled 1.2: Finding Area by Decomposing and Rearranging is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

1.3: Reasoning to Find Area

Lesson

Let's decompose and rearrange shapes to find their areas.

Exercise 1.3.1: Comparing Regions

Is the area of Figure A greater than, less than, or equal to the area of the shaded region in Figure B? Be prepared to explain your reasoning.

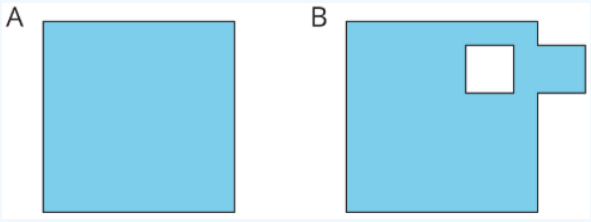


Figure 1.3.1

Exercise 1.3.2: On the Grid

Each grid square is 1 square unit. Find the area, in square units, of each shaded region without counting every square. Be prepared to explain your reasoning.

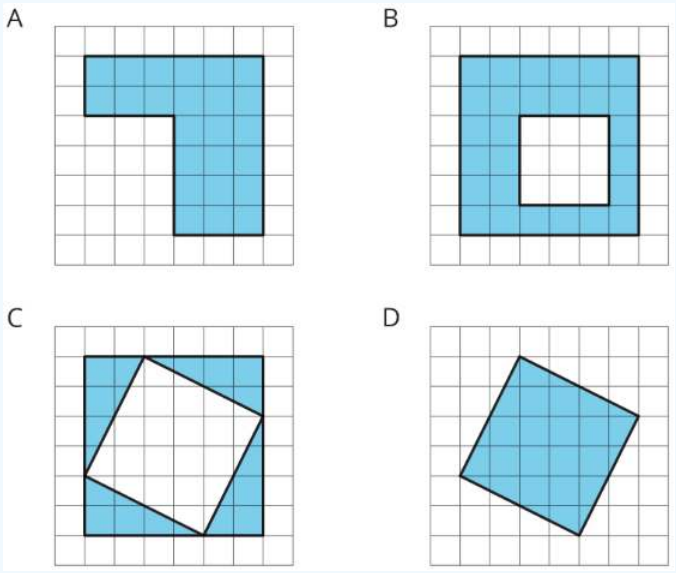


Figure D is a green tilted square.

Are you ready for more?

Rearrange the triangles from Figure C so they fit inside Figure D. Draw and color a diagram of your work.

Exercise 1.3.3: Off the Grid

Find the area of the shaded region(s) of each figure. Explain or show your reasoning.

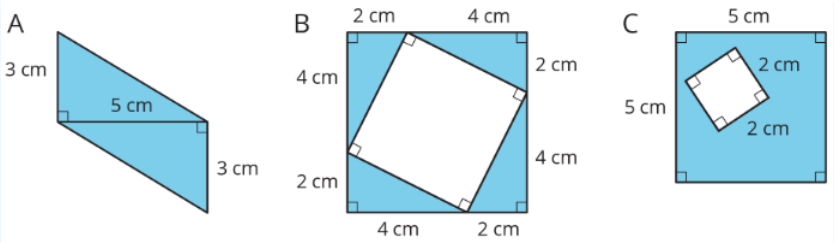


Figure C, square with side length = 5 centimeters. tilted white square with side length = 2 centimeters inside of square. Area around the white square is shaded.

Summary

There are different strategies we can use to find the area of a region. We can:

- Decompose it into shapes whose areas you know how to calculate; find the area of each of those shapes, and then add the areas.

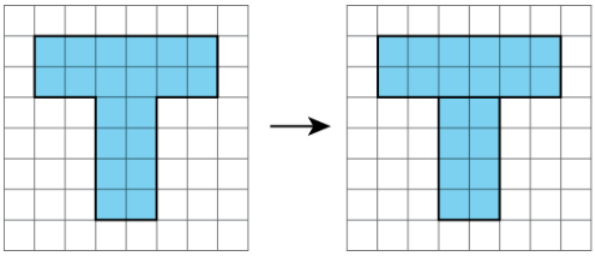


Figure 1.3.4: Two images of t-shaped objects on a grids. The upper portion is 2 units tall and 6 units wide. The stem of the t is 4 units tall and 2 units wide. The second image is the same, except there is a line separating the upper portion and lower portion into two rectangles.

- Decompose it and rearrange the pieces into shapes whose areas you know how to calculate; find the area of each of those shapes, and then add the areas.

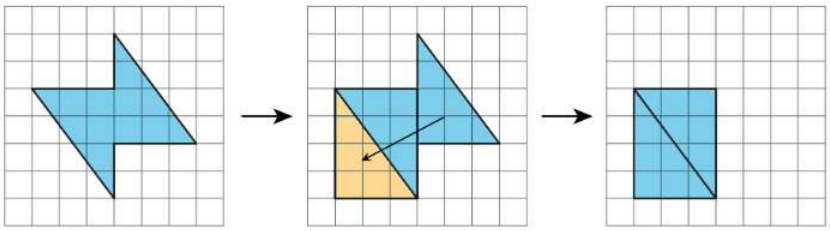


Figure 1.3.5: 3 figures on grids with arrows pointing to the right between figures 1 and 2 and figures 2 and 3. On the left is an irregular figure. In the middle, the figure has been divided into two triangles with base = 3 units, height = 4 units. the triangle on the right has been moved 3 units left and 2 units down. On the right, rectangle made up of the two triangles from the figures on the left and in the middle. base = 3 units, height = 4 units.

- Consider it as a shape with a missing piece; calculate the area of the shape and the missing piece, and then subtract the area of the piece from the area of the shape.

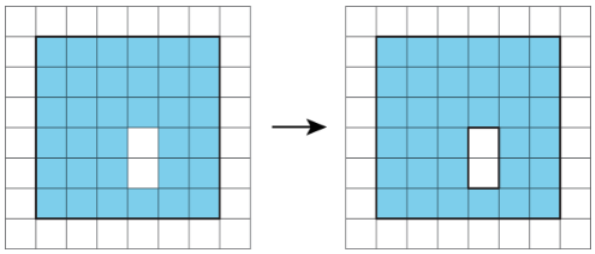


Figure 1.3.6

The area of a figure is always measured in square units. When both side lengths of a rectangle are given in centimeters, then the area is given in square centimeters. For example, the area of this rectangle is 32 square centimeters.

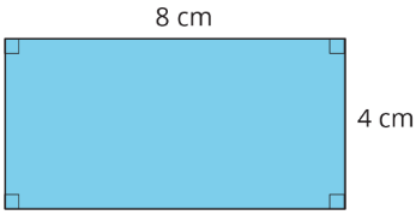


Figure 1.3.7

Glossary Entries

Definition: Area

Area is the number of square units that cover a two-dimensional region, without any gaps or overlaps.

For example, the area of region A is 8 square units. The area of the shaded region of B is $\frac{1}{2}$ square unit.

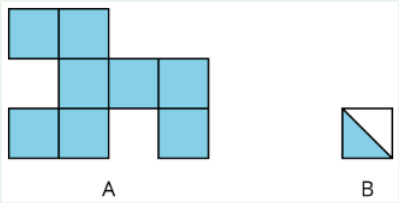


Figure 1.3.8

Definition: Compose

Compose means "put together." We use the word *compose* to describe putting more than one figure together to make a new shape.

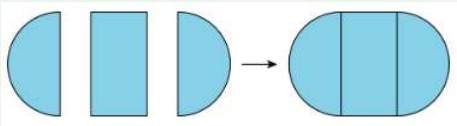


Figure 1.3.9

Definition: Decompose

Decompose means "take apart." We use the word *decompose* to describe taking a figure apart to make more than one new shape.

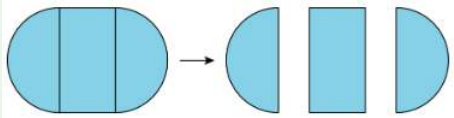


Figure 1.3.10

Definition: Region

A region is the space inside of a shape. Some examples of two-dimensional regions are inside a circle or inside a polygon. Some examples of three-dimensional regions are the inside of a cube or the inside of a sphere.

Practice

Exercise 1.3.4

Find the area of each shaded region. Show your reasoning.

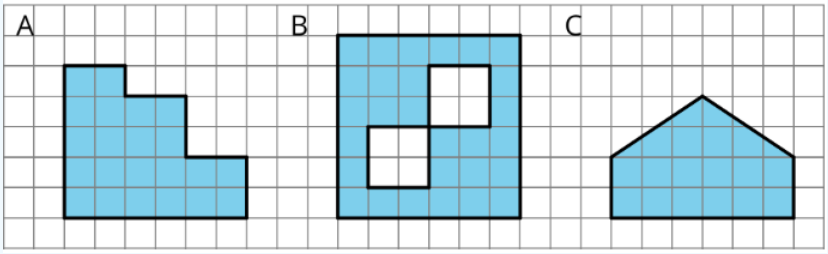


Figure 1.3.11

Exercise 1.3.5

Find the area of each shaded region. Show or explain your reasoning.

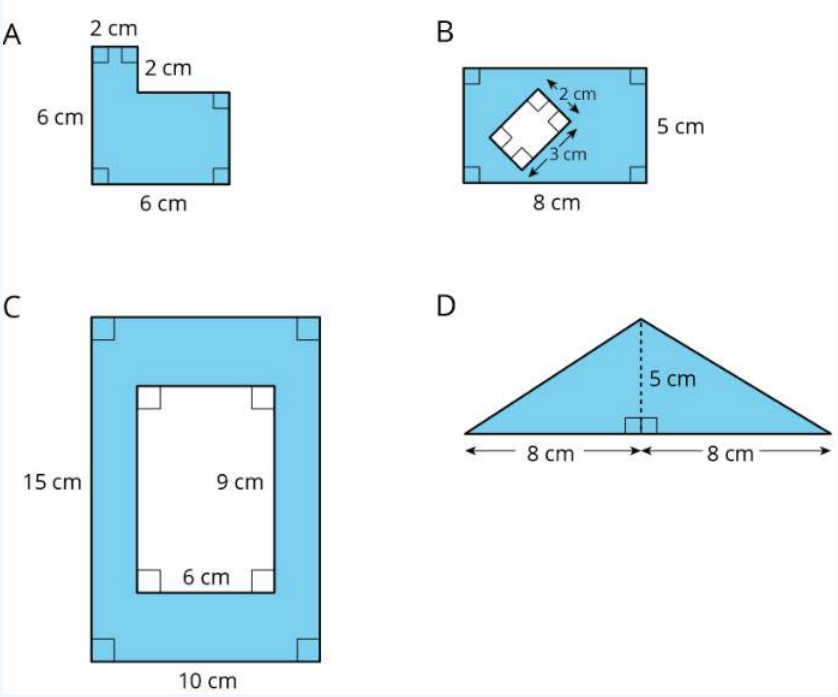


Figure D, triangle, base = 16 centimeters, height = 5 centimeters. Triangle is symmetric.

Exercise 1.3.6

Two plots of land have very different shapes. Noah said that both plots of land have the same area.

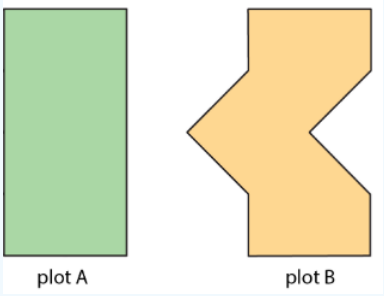


Figure 1.3.13: Two shapes labeled plot A and plot B. Plot A is a rectangle and plot B is the same height, but has a triangular shape removed from the right side, and an identical triangle shape added to the left side.

Do you agree with Noah? Explain your reasoning.

Exercise 1.3.7

A homeowner is deciding on the size of tiles to use to fully tile a rectangular wall in her bathroom that is 80 inches by 40 inches. The tiles are squares and come in three side lengths: 8 inches, 4 inches, and 2 inches. State if you agree with each statement about the tiles. Explain your reasoning.

1. Regardless of the size she chooses, she will need the same number of tiles.
2. Regardless of the size she chooses, the area of the wall that is being tiled is the same.
3. She will need two 2-inch tiles to cover the same area as one 4-inch tile.
4. She will need four 4-inch tiles to cover the same area as one 8-inch tile.
5. If she chooses the 8-inch tiles, she will need a quarter as many tiles as she would with 2-inch tiles.

(From Unit 1.1.2)

This page titled 1.3: Reasoning to Find Area is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

Section 2: Parallelograms

This page titled Section 2: Parallelograms is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

2.1: Parallelograms

Lesson

Let's investigate the features and areas of parallelograms.

Exercise 2.1.1: Features of a Parallelogram

Figures A, B, and C are **parallelograms**. Figures D, E, and F are *not* parallelograms.

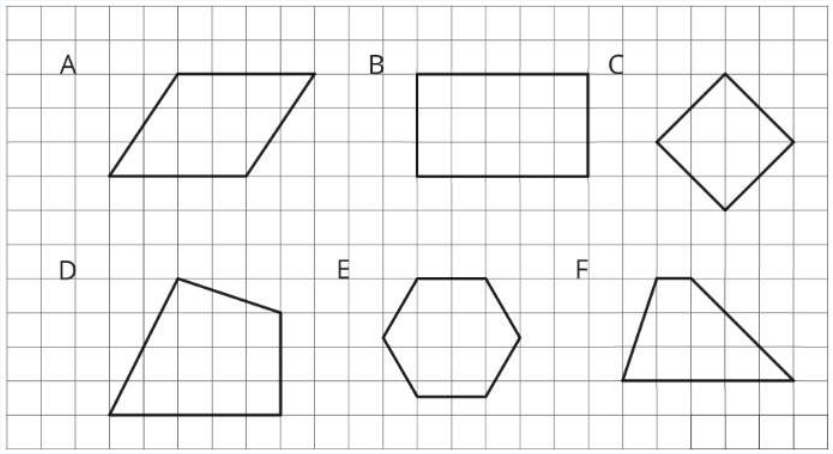


Figure 2.1.1

Study the examples and non-examples. What do you notice about:

1. the number of sides that a parallelogram has?
2. opposite sides of a parallelogram?
3. opposite angles of a parallelogram?

Exercise 2.1.2: Area of a Parallelogram

1. Find the area of the parallelogram and explain your reasoning.
2. Change the parallelogram by dragging the green points at its vertices. Find its area and explain your reasoning.
3. If you used the polygons on the side, how were they helpful? If you did not, could you use one or more of the polygons to show another way to find the area of the parallelogram?

Exercise 2.1.3: Lots of Parallelograms

Find the area of each parallelogram. Show your reasoning.

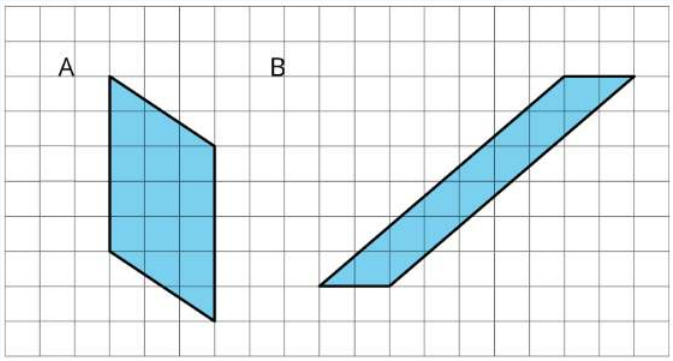


Figure 2.1.2

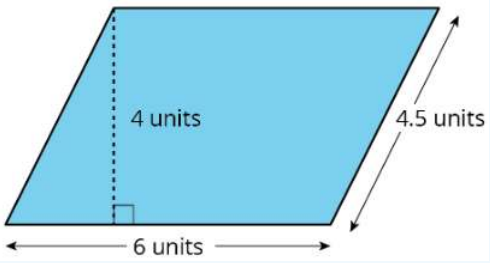


Figure 2.1.3

Summary

A **parallelogram** is a quadrilateral (it has four sides). The opposite sides of a parallelogram are parallel. It is also true that the opposite sides of a parallelogram have equal length, and the opposite angles of a parallelogram have equal measure.

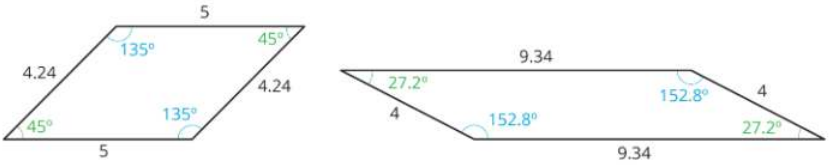


Figure 2.1.4: Two parallelograms with the angles and side lengths provided. On the left, Top and bottom sides = 5 units. Left and right sides = 4.24 units. Top left and bottom right angles = 135 degrees. Top right and bottom left angles = 45 degrees. On the right, Top and bottom sides = 9.34 units. Left and right sides = 4 units. Top left and bottom right angles = 27.2 degrees. Top right and bottom left angles = 152.8 degrees.

There are several strategies for finding the area of a **parallelogram**.

- We can decompose and rearrange a parallelogram to form a rectangle. Here are three ways:

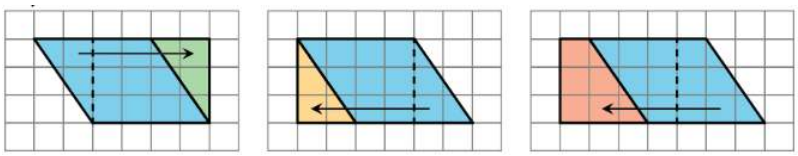


Figure 2.1.5: Three identical parallelograms on separate grids, each has a base of four units and a height of three units. First parallelogram, vertical dashed segment extending from the bottom left vertex to the opposite side, forming a triangle. An arrow extends from the triangle to the opposite side of the parallelogram to create a rectangle 4 units wide and 3 units high. Second parallelogram, vertical dashed segment extending from the top right vertex to the opposite side, forming a triangle. An arrow extends from the triangle to the opposite side of the parallelogram to create a rectangle 4 units wide and 3 units high. Third parallelogram, vertical dashed segment through the middle of the parallelogram. An arrow extends from the resulting shape to the opposite side of the parallelogram to create a rectangle 4 units wide and 3 units high.

- We can enclose the parallelogram and then subtract the area of the two triangles in the corner.

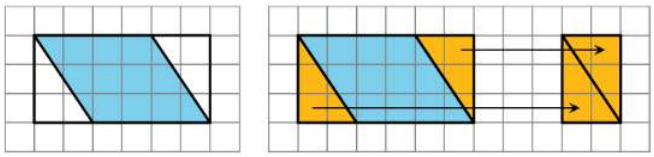


Figure 2.1.6: Two drawings of parallelograms on grids. On the left, a triangle in the bottom left and top right corner is in white and the center is colored blue. On the right, the image from the left is repeated, but the triangles in the corners are colored yellow. Arrows are drawn from the triangles to the right. The triangles two triangles are joined on the right to form a rectangle.

Both of these ways will work for any parallelogram. However, for some parallelograms the process of decomposing and rearranging requires a lot more steps than if we enclose the parallelogram with a rectangle and subtract the combined area of the two triangles in the corners.

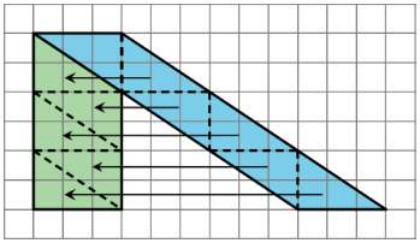


Figure 2.1.7: A shaded parallelogram on a grid. Base of three units. Slanted sides that decline 6 vertical units over 9 horizontal units. Parallelogram decomposed by dashed segments into six equal right triangles. Each triangle has a vertical side of 2 units and horizontal side of 3 units. Arrows extend to the left from each of the lower 5 triangles. The resulting shape is a rectangle that is 6 units tall by 3 units wide.

Glossary Entries

Definition: Parallelogram

A parallelogram is a type of quadrilateral that has two pairs of parallel sides.

Here are two examples of parallelograms.

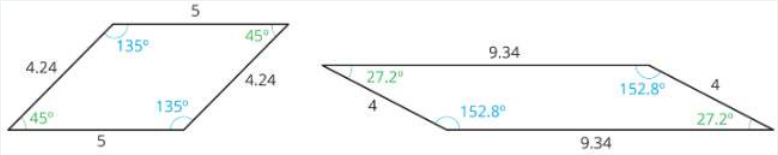


Figure 2.1.8: Two parallelograms with the angles and side lengths provided. On the left, Top and bottom sides = 5 units. Left and right sides = 4.24 units. Top left and bottom right angles = 135 degrees. Top right and bottom left angles = 45 degrees. On the right, Top and bottom sides = 9.34 units. Left and right sides = 4 units. Top left and bottom right angles = 27.2 degrees. Top right and bottom left angles = 152.8 degrees.

Definition: Quadrilateral

A quadrilateral is a type of polygon that has 4 sides. A rectangle is an example of a quadrilateral. A pentagon is not a quadrilateral, because it has 5 sides.

Practice

Exercise 2.1.4

Select **all** of the parallelograms. For each figure that is *not* selected, explain how you know it is not a parallelogram.

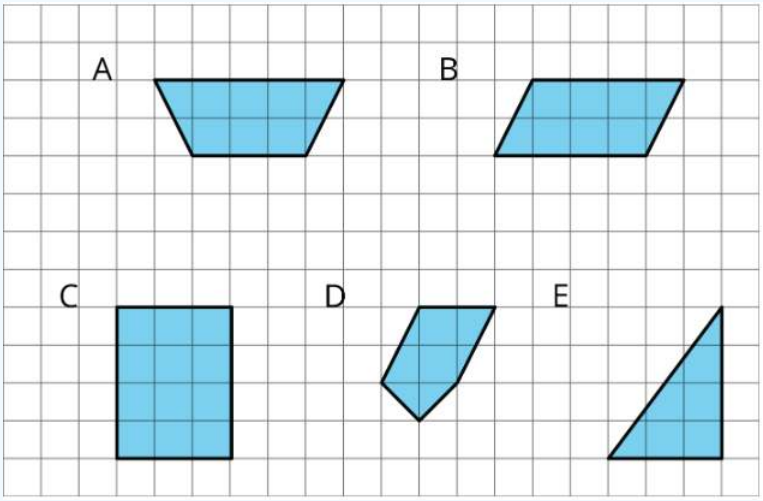


Figure E is a right triangle.

Exercise 2.1.5

1. Decompose and rearrange this parallelogram to make a rectangle.

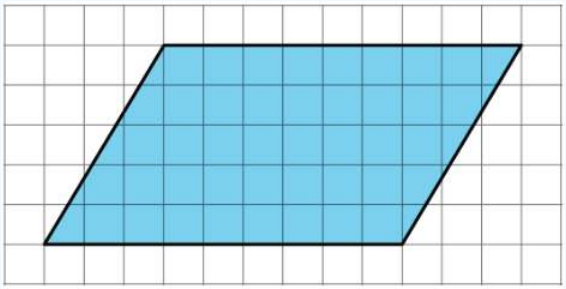


Figure 2.1.10

2. What is the area of the parallelogram? Explain or your reasoning.

Exercise 2.1.6

Find the area of the parallelogram.

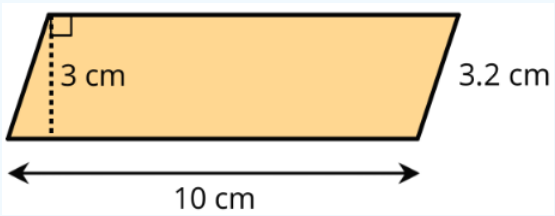


Figure 2.1.11

Exercise 2.1.7

Explain why this quadrilateral is *not* a parallelogram.

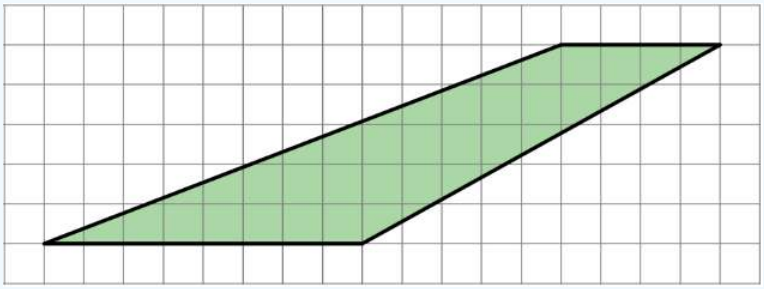


Figure 2.1.12: A quadrilateral on a grid. Bottom side length of 8 units. Top side length of 4 units. The left side ascends 5 units while moving right 13 units, and the right side ascends 5 units while moving right 9 units.

Exercise 2.1.8

Find the area of each shape. Show your reasoning.

https://math.libretexts.org/@go/page/39635

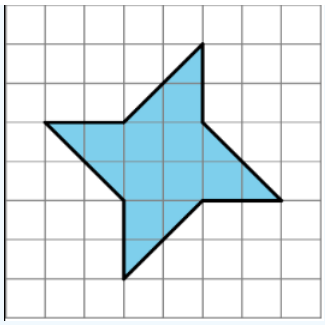


Figure 2.1.13: A shape with eight sides drawn on a grid. Four sides are straight sides and extend left, right, up and, down for 2 units each. The remaining sides are angled sides connecting each of the straight sides to the next. The shape is a total of 6 units tall and 6 units wide.

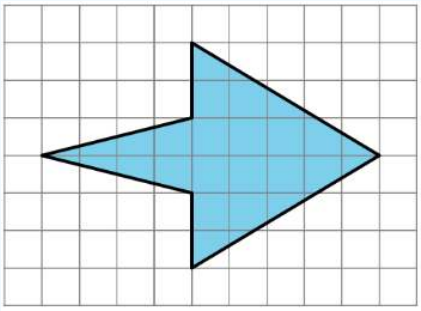


Figure 2.1.14

(From Unit 1.1.3)

Exercise 2.1.9

Find the area of the rectangle with each set of side lengths.

1. 5 in and $\frac{1}{3}$ in
2. 5 in and $\frac{4}{3}$ in
3. $\frac{5}{2}$ in and $\frac{4}{3}$ in
4. $\frac{7}{6}$ in and $\frac{6}{7}$ in

(From Unit 1.1.1)

This page titled 2.1: Parallelograms is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

2.2: Bases and Heights of Parallelograms

Lesson

Let's investigate the area of parallelograms some more.

Exercise 2.2.1: A Parallelogram and Its Rectangles

Elena and Tyler were finding the area of this parallelogram:

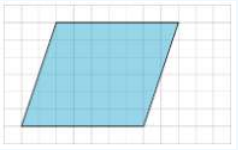


Figure 2.2.1

Move the slider to see how Tyler did it:

Move the slider to see how Elena did it:

How are the two strategies for finding the area of a parallelogram the same? How they are different?

Exercise 2.2.2: The Right Height?

Study the examples and non-examples of **bases** and **heights** of parallelograms.

- Examples: The dashed segments in these drawings represent the corresponding height for the given base.

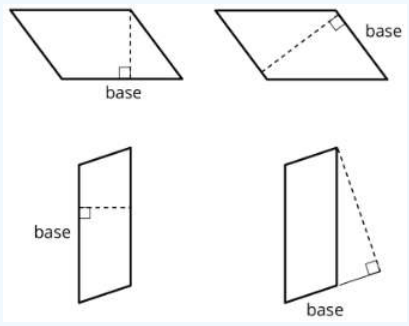


Figure 2.2.2

- Non-examples: The dashed segments in these drawings do *not* represent the corresponding height for the given base.

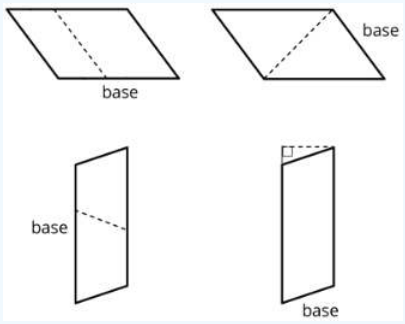


Figure 2.2.3

1. Select **all** the statements that are true about bases and heights in a parallelogram.
   a. Only a horizontal side of a parallelogram can be a base.
   b. Any side of a parallelogram can be a base.
   c. A height can be drawn at any angle to the side chosen as the base.
   d. A base and its corresponding height must be perpendicular to each other.
   e. A height can only be drawn inside a parallelogram.

f. A height can be drawn outside of the parallelogram, as long as it is drawn at a 90-degree angle to the base.
g. A base cannot be extended to meet a height.

2. Five students labeled a base b and a corresponding height h for each of these parallelograms. Are all drawings correctly labeled? Explain how you know.

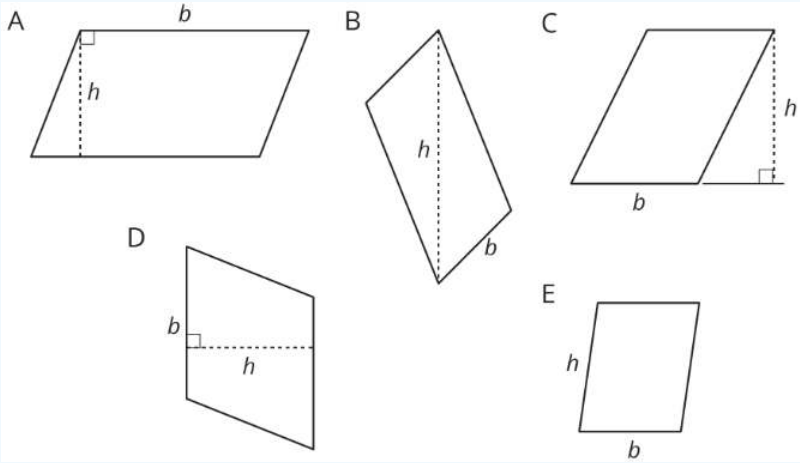


Figure 2.2.4

Are you ready for more?

In the applet, the parallelogram is made of solid line segments, and the height and supporting lines are made of dashed line segments. A base b and corresponding height h are labeled.

Exercise 2.2.3: Finding the Formula for Area of Parallelograms

For each parallelogram:

- Identify a base and a corresponding height, and record their lengths in the table.
- Find the area of the parallelogram and record it in the last column of the table.

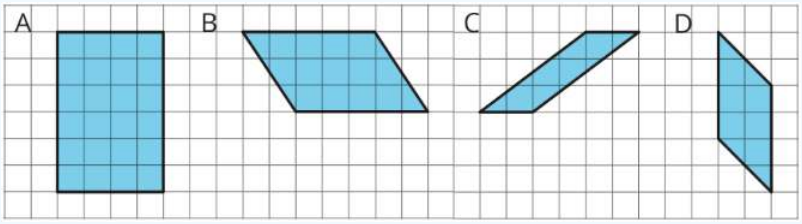


Figure 2.2.5

Table 2.2.1

parallelogram	base (units)	height (units)	area (sq units)
A			
B			
C			
D			
any parallelogram	b	h	

In the last row, write an expression for the area of any parallelogram, using b and h.

Are you ready for more?

1. What happens to the area of a parallelogram if the height doubles but the base is unchanged? If the height triples? If the height is 100 times the original?

2. What happens to the area if both the base and the height double? Both triple? Both are 100 times their original lengths?

Summary

- We can choose any of the four sides of a parallelogram as the **base**. Both the side (the segment) and its length (the measurement) are called the base.
- If we draw any perpendicular segment from a point on the base to the opposite side of the parallelogram, that segment will always have the same length. We call that value the **height**. There are infinitely many segments that can represent the height!

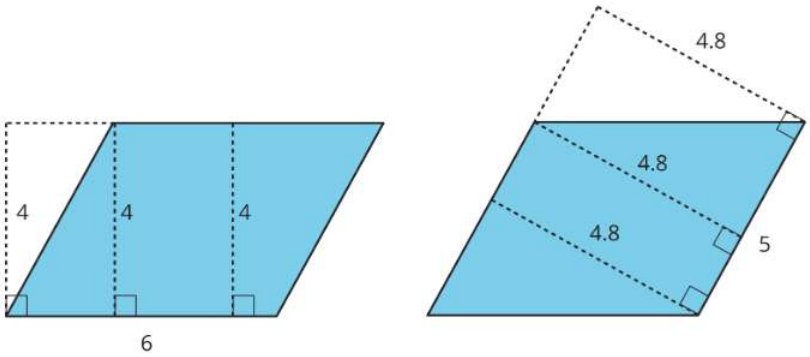


Figure 2.2.6: 2 copies of the same parallelogram. On the left, base = 6 units. Corresponding height = 4 units. On the right, base = 5 units. Corresponding height = 4.8 units. For both, 3 different segments are shown to represent the height.

Here are two copies of the same parallelogram. On the left, the side that is the base is 6 units long. Its corresponding height is 4 units. On the right, the side that is the base is 5 units long. Its corresponding height is 4.8 units. For both, three different segments are shown to represent the height. We could draw in many more!

No matter which side is chosen as the base, the area of the parallelogram is the product of that base and its corresponding height. We can check this:

$$4 \times 6 = 24 \qquad \text{and} \qquad 4.8 \times 5 = 24$$

We can see why this is true by decomposing and rearranging the parallelograms into rectangles.

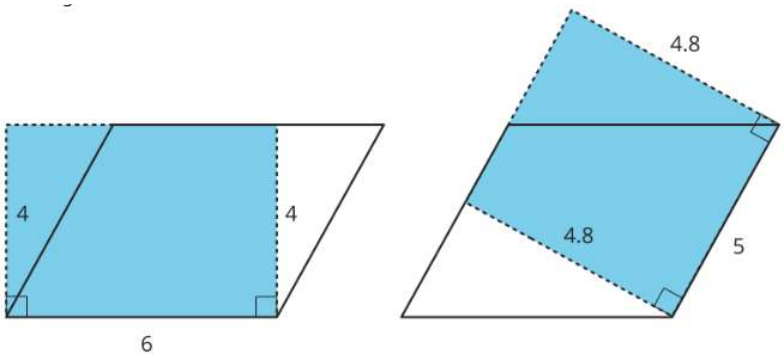


Figure 2.2.7

Notice that the side lengths of each rectangle are the base and height of the parallelogram. Even though the two rectangles have different side lengths, the products of the side lengths are equal, so they have the same area! And both rectangles have the same area as the parallelogram.

We often use letters to stand for numbers. If b is base of a parallelogram (in units), and h is the corresponding height (in units), then the area of the parallelogram (in square units) is the product of these two numbers. $b \cdot h$

Notice that we write the multiplication symbol with a small dot instead of a $\times$ symbol. This is so that we don't get confused about whether $\times$ means multiply, or whether the letter x is standing in for a number.

In high school, you will be able to prove that a perpendicular segment from a point on one side of a parallelogram to the opposite side will always have the same length.

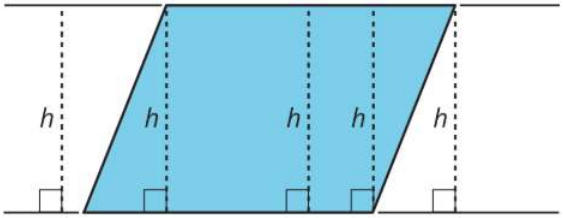


Figure 2.2.8

You can see this most easily when you draw a parallelogram on graph paper. For now, we will just use this as a fact.

Glossary Entries

Definition: base (of a parallelogram or triangle)

We can choose any side of a parallelogram or triangle to be the shape's base. Sometimes we use the word *base* to refer to the length of this side.

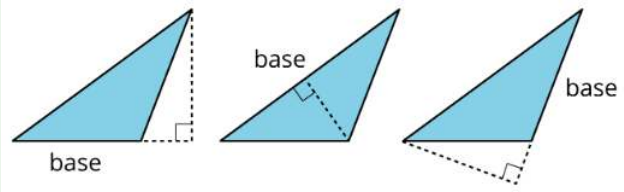


Figure 2.2.9

Definition: height (of a parallelogram or triangle)

The height is the shortest distance from the base of the shape to the opposite side (for a parallelogram) or opposite vertex (for a triangle).

We can show the height in more than one place, but it will always be perpendicular to the chosen base.

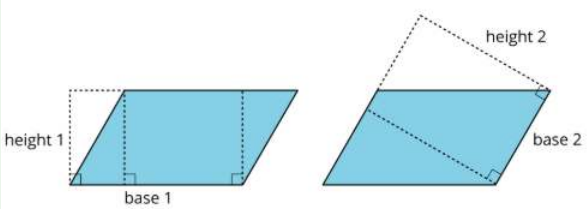


Figure 2.2.10

Definition: Parallelogram

A parallelogram is a type of quadrilateral that has two pairs of parallel sides.

Here are two examples of parallelograms.

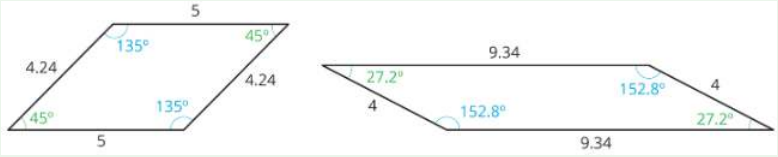


Figure 2.2.11: Two parallelograms with the angles and side lengths provided. On the left, Top and bottom sides = 5 units. Left and right sides = 4.24 units. Top left and bottom right angles = 135 degrees. Top right and bottom left angles = 45 degrees. On the right, Top and bottom sides = 9.34 units. Left and right sides = 4 units. Top left and bottom right angles = 27.2 degrees. Top right and bottom left angles = 152.8 degrees.

Definition: Quadrilateral

A quadrilateral is a type of polygon that has 4 sides. A rectangle is an example of a quadrilateral. A pentagon is not a quadrilateral, because it has 5 sides.

Practice

Exercise 2.2.4

Select **all** parallelograms that have a correct height labeled for the given base.

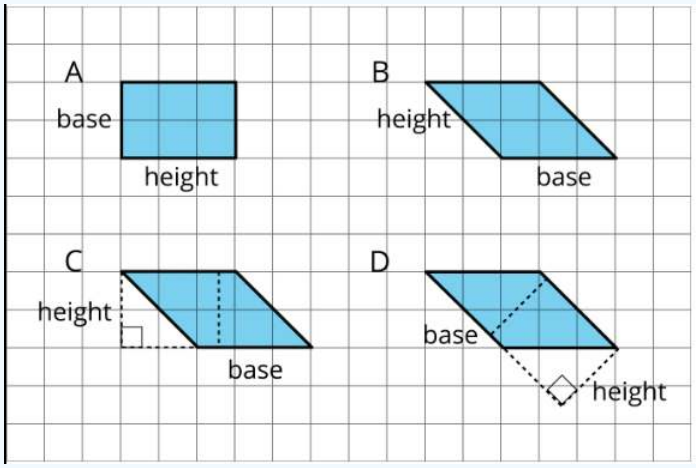


Figure 2.2.12: 4 parallelograms on a grid, labeled A, B, C, D. Parallelogram A, base = 3 units, height = 2 units. Parallelogram B, base = 3 units, height = 2 units. Parallelogram C, base = 3 units, height = 2 units. Parallelogram D, base = diagonal across two unit squares, height = 3 units.

A. A
B. B
C. C
D. D

Exercise 2.2.5

The side labeled b has been chosen as the base for this parallelogram.

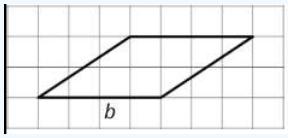


Figure 2.2.13

Draw a segment showing the height corresponding to that base.

Exercise 2.2.6

Find the area of each parallelogram.

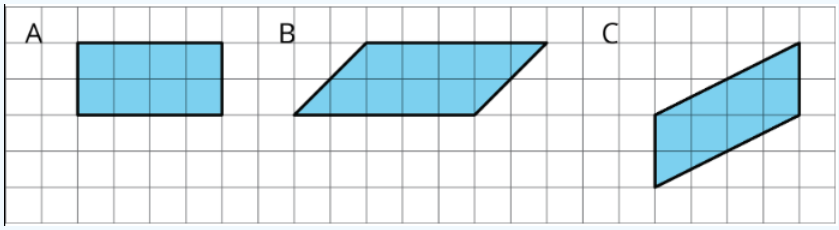


Figure 2.2.14: 3 parallelograms on a grid, labeled A, B, C. Parallelogram A, base = 4 units, height = 2 units. Parallelogram B, base = 5 units, height = 2 units. Parallelogram C, base = 2 units, height = 4 units.

Exercise 2.2.7

If the side that is 6 units long is the base of this parallelogram, what is its corresponding height?

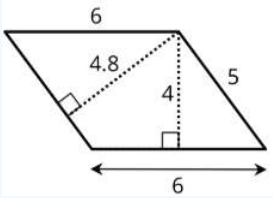


Figure 2.2.15: A parallelogram with its bottom and top sides labeled 6 and its right side labeled 5. A dashed line perpendicular to the right side is labeled 4.8, and a dashed line perpendicular to the bottom side is labeled 4.

A. 6 units
B. 4.8 units
C. 4 units
D. 5 units

Exercise 2.2.8

Find the area of each parallelogram.

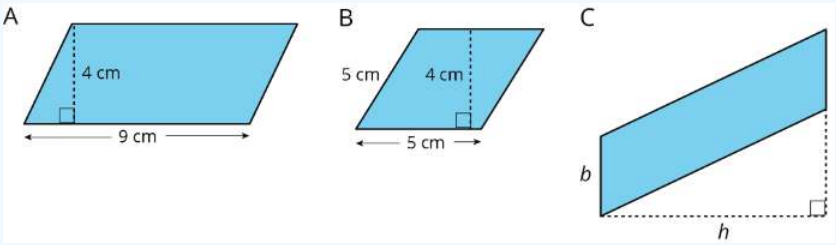


Figure 2.2.16: 3 parallelograms labeled A, B, C. Parallelogram A, base = 9 centimeters, height = 4 centimeters. Parallelogram B, base = 5 centimeters, height = 4 centimeters. Parallelogram C, base = b, height = h.

Exercise 2.2.9

Do you agree with each of these statements? Explain your reasoning.

1. A parallelogram has six sides.
2. Opposite sides of a parallelogram are parallel.
3. A parallelogram can have one pair or two pairs of parallel sides.
4. All sides of a parallelogram have the same length.
5. All angles of a parallelogram have the same measure.

(From Unit 1.2.1)

Exercise 2.2.10

A square with an area of 1 square meter is decomposed into 9 identical small squares. Each small square is decomposed into two identical triangles.

1. What is the area, in square meters, of 6 triangles? If you get stuck, consider drawing a diagram.
2. How many triangles are needed to compose a region that is $1\frac{1}{2}$ square meters?

(From Unit 1.1.2)

This page titled 2.2: Bases and Heights of Parallelograms is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

2.3: Area of Parallelograms

Lesson

Let's practice finding the area of parallelograms.

Exercise 2.3.1: Missing Dots



Figure 2.3.1

How many dots are in the image?

How do you see them?

Exercise 2.3.2: More Areas of Parallelograms

1. a. Calculate the area of the given figure in the applet. Then, check if your area calculation is correct by clicking the *Show Area* checkbox.
   b. Uncheck the Area checkbox. Move one of the vertices of the parallelogram to create a new parallelogram. When you get a parallelogram that you like, sketch it and calculate the area. Then, check if your calculation is correct by using the *Show Area* button again.
   c. Repeat this process two more times. Draw and label each parallelogram with its measurements and the area you calculated.
2. Here is Parallelogram B. What is the corresponding height for the base that is 10 cm long? Explain or show your reasoning.

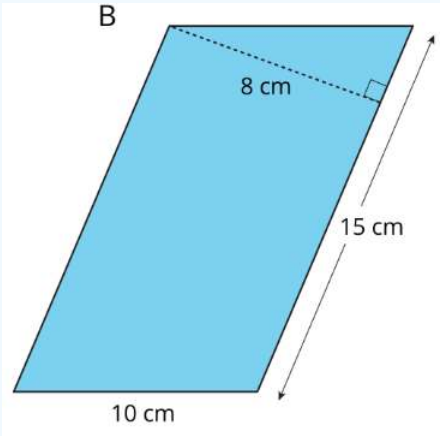


Figure 2.3.2

3. Here are two different parallelograms with the same area.
   a. Explain why their areas are equal.
   b. Drag points to create two new parallelograms that are not identical copies of each other but that have the same area as each other. Sketch your parallelograms and explain or show how you know their areas are equal. Then, click on the *Check* button to see if the two areas are indeed equal.

Are you ready for more?

Here is a parallelogram composed of smaller parallelograms. The shaded region is composed of four identical parallelograms. All lengths are in inches.

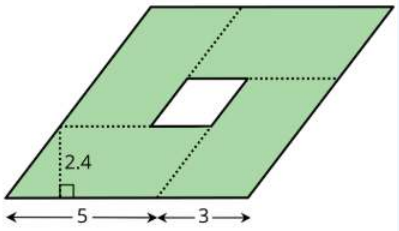


Figure 2.3.3: Parallelogram composed of smaller parallelograms. The shaded region is composed of four identical parallelograms, for each, base = 5, height = 2 and 4 tenths, slant length = 3.

What is the area of the unshaded parallelogram in the middle? Explain or show your reasoning.

Summary

Any pair of base and corresponding height can help us find the area of a parallelogram, but some base-height pairs are more easily identified than others.

When a parallelogram is drawn on a grid and has *horizontal* sides, we can use a horizontal side as the base. When it has *vertical* sides, we can use a vertical side as the base. The grid can help us find (or estimate) the lengths of the base and of the corresponding height.

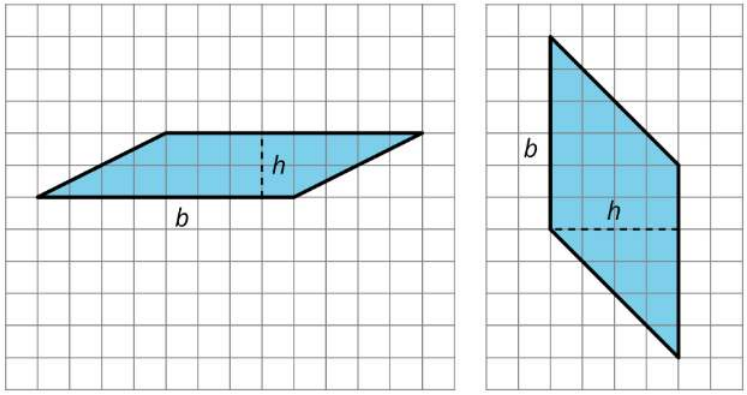


Figure 2.3.4: Two parallelograms drawn on two grids. First parallelogram, 2 horizontal sides each 8 units long, 2 slanted sides that rise 2 vertical units over 4 horizontal units. Bottom horizontal side labeled, b. A 2-unit perpendicular segment labeled, h, connects the horizontal sides. Second parallelogram, 2 vertical sides each 6 units long, 2 slanted sides that rise 4 vertical units over 4 horizontal units. The left vertical side is labeled, b. A 4-unit perpendicular segment labeled, h, connects one vertex of the vertical side to a point on the other vertical side.

When a parallelogram is *not* drawn on a grid, we can still find its area if a base and a corresponding height are known.

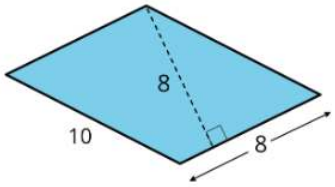


Figure 2.3.5

In this parallelogram, the corresponding height for the side that is 10 units long is not given, but the height for the side that is 8 units long is given. This base-height pair can help us find the area.

Regardless of their shape, parallelograms that have the same base and the same height will have the same area; the product of the base and height will be equal. Here are some parallelograms with the same pair of base-height measurements.

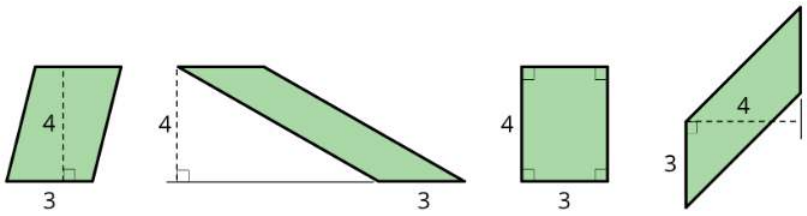


Figure 2.3.6

Glossary Entries

Definition: base (of a parallelogram or triangle)

We can choose any side of a parallelogram or triangle to be the shape's base. Sometimes we use the word *base* to refer to the length of this side.

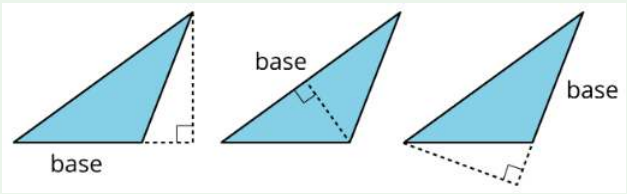


Figure 2.3.7

Definition: height (of a parallelogram or triangle)

The height is the shortest distance from the base of the shape to the opposite side (for a parallelogram) or opposite vertex (for a triangle).

We can show the height in more than one place, but it will always be perpendicular to the chosen base.

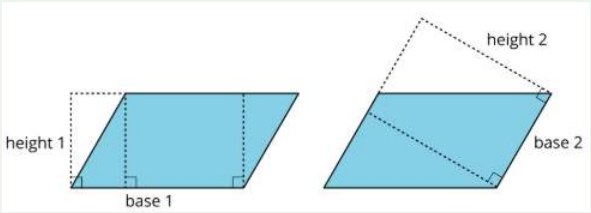


Figure 2.3.8

Definition: Parallelogram

A parallelogram is a type of quadrilateral that has two pairs of parallel sides.

Here are two examples of parallelograms.

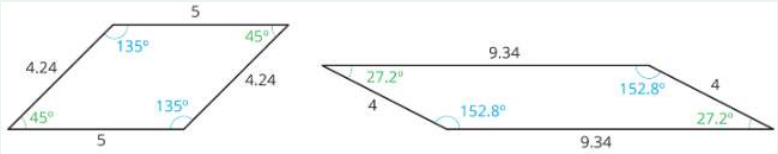


Figure 2.3.9: Two parallelograms with the angles and side lengths provided. On the left, Top and bottom sides = 5 units. Left and right sides = 4.24 units. Top left and bottom right angles = 135 degrees. Top right and bottom left angles = 45 degrees. On the right, Top and bottom sides = 9.34 units. Left and right sides = 4 units. Top left and bottom right angles = 27.2 degrees. Top right and bottom left angles = 152.8 degrees.

Definition: Quadrilateral

A quadrilateral is a type of polygon that has 4 sides. A rectangle is an example of a quadrilateral. A pentagon is not a quadrilateral, because it has 5 sides.

Practice

Exercise 2.3.3

Which three of these parallelograms have the same area as each other?

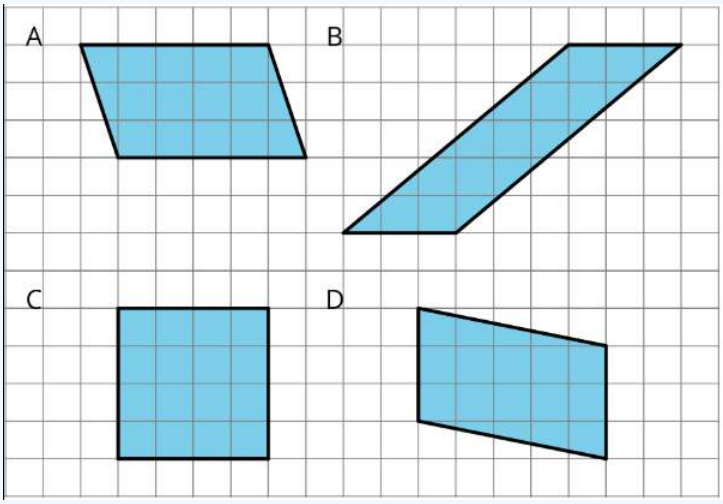


Figure D, the left and right sides are each 3 units, and the top and bottom sides descend 1 unit as they move right 5 units.

A. A
B. B
C. C
D. D

Exercise 2.3.4

Which pair of base and height produces the greatest area? All measurements are in centimeters.

A. $b = 4, h = 3.5$
B. $b = 0.8, h = 20$
C. $b = 6, h = 2.25$
D. $b = 10, h = 1.4$

Exercise 2.3.5

Here are the areas of three parallelograms. Use them to find the missing length (labeled with a "?") on each parallelogram.

A. 10 square units
B. 21 square units
C. 25 square units

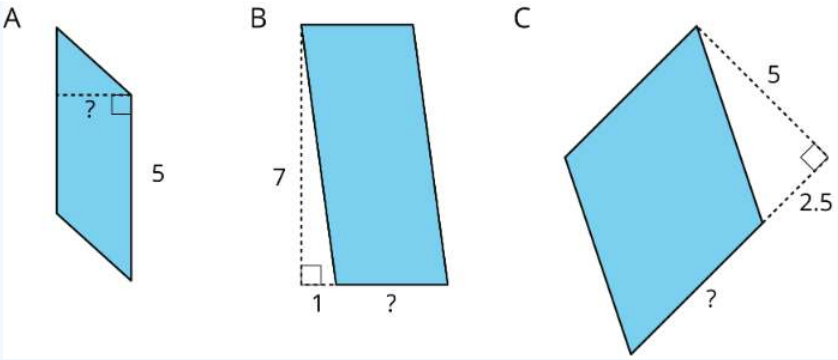


Figure 2.3.11: Three parallelograms labeled A, B, C. Parallelogram A, right side = 5 units, height = question mark. Parallelogram B, bottom side = question mark. Right triangle drawn outside of parallelogram, base = 1 unit, height = 7 units. Parallelogram C, bottom side = question mark. Right triangle drawn outside of parallelogram, base = 2.5 units, height = 5 units.

Exercise 2.3.6

The Dockland Building in Hamburg, Germany is shaped like a parallelogram.



Figure 2.3.12: Architecture Port Elbe Hamburg Building, by Max Pixel. Public Domain. freegreatpicture.com. Source.

If the length of the building is 86 meters and its height is 55 meters, what is the area of this face of the building?

Exercise 2.3.7

Select **all** segments that could represent a corresponding height if the side m is the base.

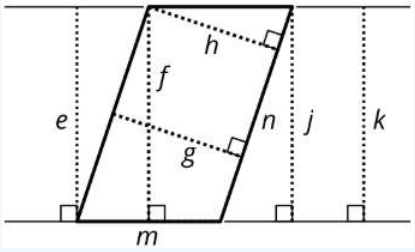


Figure 2.3.13

A. e
B. f
C. g
D. h
E. j
F. k
G. n

(From Unit 1.2.2)

Exercise 2.3.8

Find the area of the shaded region. All measurements are in centimeters. Show your reasoning.

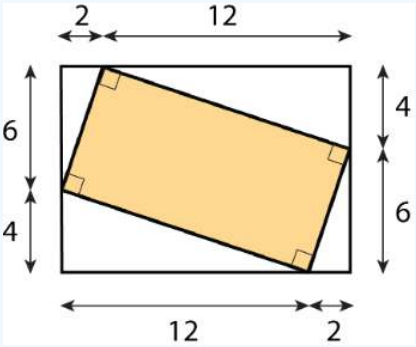


Figure 2.3.14

(From Unit 1.1.3)

This page titled 2.3: Area of Parallelograms is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

Section 3: Triangles

This page titled Section 3: Triangles is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

3.1: From Parallelograms to Triangles

Lesson

Let's compare parallelograms and triangles.

Exercise 3.1.1: Same Parallelograms, Different Bases

Here are two copies of a parallelogram. Each copy has one side labeled as the base b and a segment drawn for its corresponding height and labeled h.

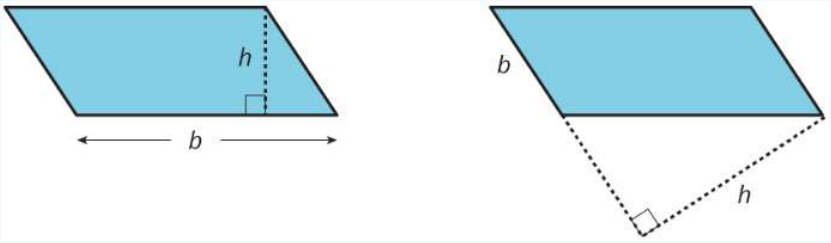


Figure 3.1.1

1. The base of the parallelogram on the left is 2.4 centimeters; its corresponding height is 1 centimeter. Find its area in square centimeters.
2. The height of the parallelogram on the right is 2 centimeters. How long is the base of that parallelogram? Explain your reasoning.

Exercise 3.1.2: A Tale of Two Triangles (Part 1)

Two polygons are identical if they match up exactly when placed one on top of the other.

1. Draw *one* line to decompose each polygon into two identical triangles, if possible. If you choose to, you can also draw the triangles.
2. Which quadrilaterals can be decomposed into two identical triangles?
   Pause here for a small-group discussion.
3. Study the quadrilaterals that were, in fact, decomposable into two identical triangles. What do you notice about them? Write a couple of observations about what these quadrilaterals have in common.

Are you ready for more?

Draw some other types of quadrilaterals that are not already shown. Try to decompose them into two identical triangles. Can you do it? Come up with a general rule about what must be true if a quadrilateral can be decomposed into two identical triangles.

GeoGebra Applet XpEZQfqp

Exercise 3.1.3: A Tale of Two Triangles (Part 2)

This applet has eight pairs of triangles. Each group member should choose 1–2 pairs of triangles. Use them to help you answer the following questions.

1. a. Which pair(s) of triangles do you have?
   b. Can each pair be composed into a rectangle? A parallelogram?
2. Discuss your responses to the first question with your group. Then, complete each of the following statements with *all*, *some*, or *none*. Sketch 1–2 examples to illustrate each completed statement.
   a. _____ of these pairs of identical triangles can be composed into a *rectangle*.
   b. _____ of these pairs of identical triangles can be composed into a *parallelogram*.

Summary

A parallelogram can always be decomposed into two identical triangles by a segment that connects opposite vertices.

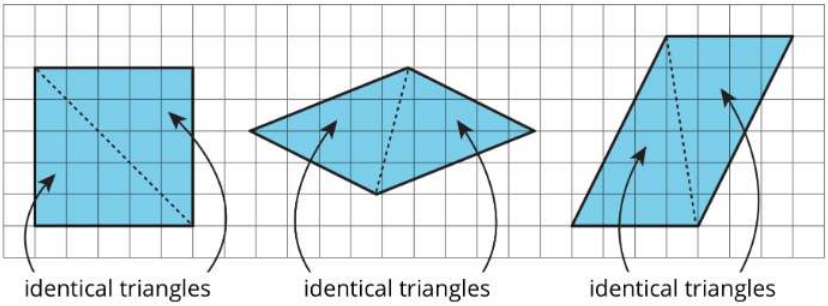


Figure 3.1.2

Going the other way around, two identical copies of a triangle can always be arranged to form a parallelogram, regardless of the type of triangle being used.

To produce a parallelogram, we can join a triangle and its copy along any of the three sides, so the same pair of triangles can make different parallelograms.

Here are examples of how two copies of both Triangle A and Triangle F can be composed into three different parallelograms.

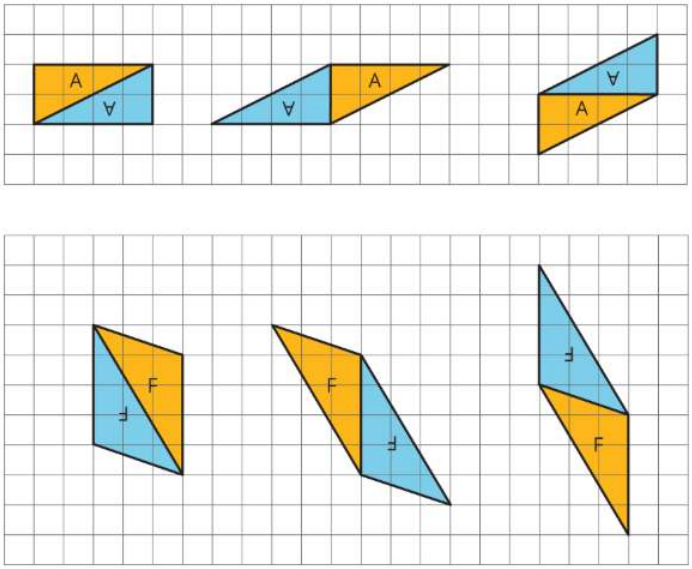


Figure 3.1.3

This special relationship between triangles and parallelograms can help us reason about the area of any triangle.

Practice

Exercise 3.1.4

To decompose a quadrilateral into two identical shapes, Clare drew a dashed line as shown in the diagram.

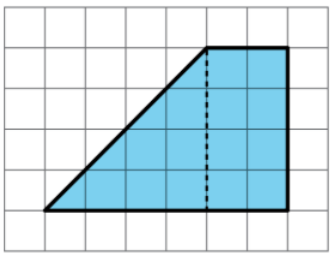


Figure 3.1.4

1. She said the that two resulting shapes have the same area. Do you agree? Explain your reasoning.
2. Did Clare partition the figure into two identical shapes? Explain your reasoning.

Exercise 3.1.5

Triangle R is a right triangle. Can we use two copies of Triangle R to compose a parallelogram that is not a square?



Figure 3.1.5

If so, explain how or sketch a solution. If not, explain why not.

Exercise 3.1.6

Two copies of this triangle are used to compose a parallelogram. Which parallelogram *cannot* be a result of the composition? If you get stuck, consider using tracing paper.

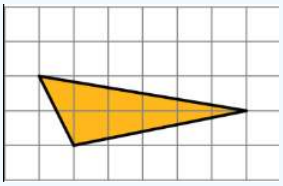


Figure 3.1.6

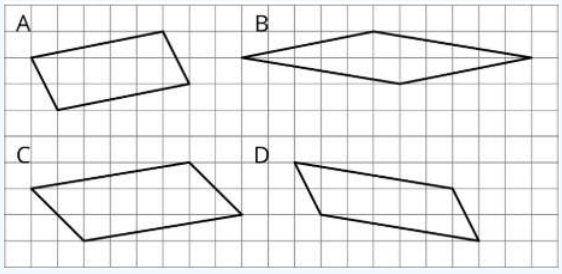


Figure 3.1.7

A. A
B. B
C. C
D. D

Exercise 3.1.7

1. On the grid, draw at least three different quadrilaterals that can each be decomposed into two identical triangles with a single cut (show the cut line). One or more of the quadrilaterals should have non-right angles.



Figure 3.1.8

2. Identify the type of each quadrilateral.

Exercise 3.1.8

1. A parallelogram has a base of 9 units and a corresponding height of $\frac{2}{3}$ units. What is its area?
2. A parallelogram has a base of 9 units and an area of 12 square units. What is the corresponding height for that base?
3. A parallelogram has an area of 7 square units. If the height that corresponds to a base is $\frac{1}{4}$ unit, what is the base?

(From Unit 1.2.3)

Exercise 3.1.9

Select **all** the segments that could represent the height if side n is the base.

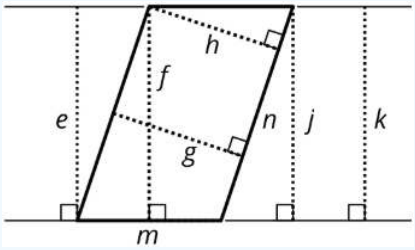


Figure 3.1.9

A. e
B. f
C. g
D. h
E. m
F. n
G. j
H. k

(From Unit 1.2.2)

This page titled 3.1: From Parallelograms to Triangles is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

3.2: Area of Triangles

Lesson

Let's use what we know about parallelograms to find the area of triangles.

Exercise 3.2.1: Composing Parallelograms

Here is Triangle M.

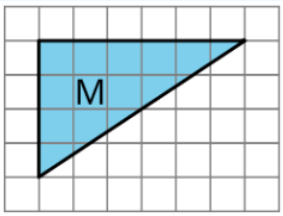


Figure 3.2.1

Han made a copy of Triangle M and composed three different parallelograms using the original M and the copy, as shown here.

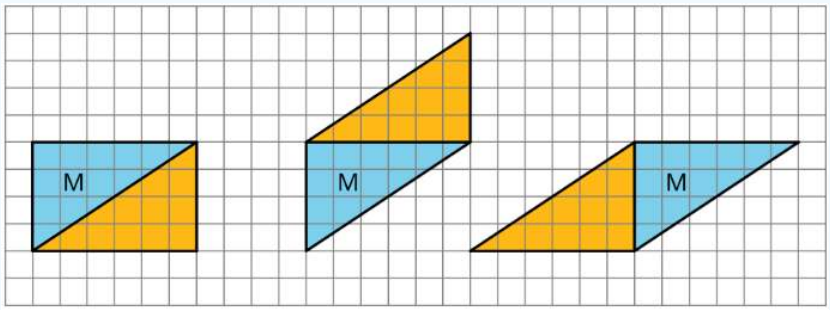


Figure 3.2.2: 3 different parallelograms on a grid composed of Triangle M and a copy. First parallelogram, Triangle M and a copy along the slanted side of the triangle. Second parallelogram, Triangle M and a copy along the horizontal side of the triangle. Third parallelogram, Triangle M and a copy along the vertical side of the triangle.

1. For each parallelogram Han composed, identify a base and a corresponding height, and write the measurements on the drawing.
2. Find the area of each parallelogram Han composed. Show your reasoning.

Exercise 3.2.2: More Triangles

Find the areas of at least two of these triangles. Show your reasoning.

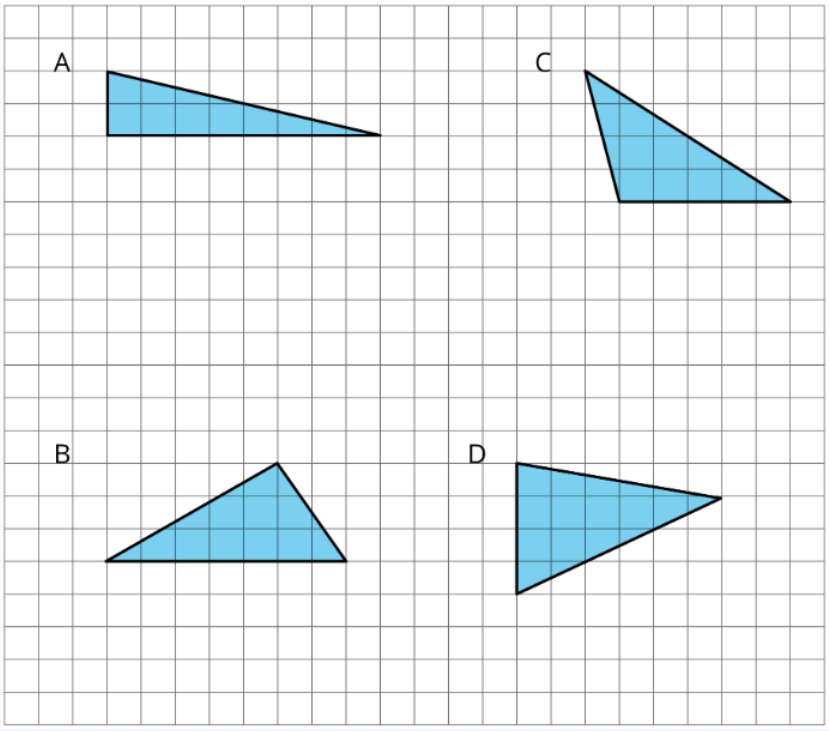


Figure 3.2.3

Exercise 3.2.3: Decomposing a Parallelogram

1. Your teacher will give you two copies of a parallelogram. Glue or tape *one* copy of your parallelogram here and find its area. Show your reasoning.
2. Decompose the second copy of your parallelogram by cutting along the dotted lines. Take *only* the small triangle and the trapezoid, and rearrange these two pieces into a different parallelogram. Glue or tape the newly composed parallelogram on your paper.
3. Find the area of the new parallelogram you composed. Show your reasoning.
4. What do you notice about the relationship between the area of this new parallelogram and the original one?
5. How do you think the area of the large triangle compares to that of the new parallelogram: Is it larger, the same, or smaller? Why is that?
6. Glue or tape the remaining large triangle to your paper. Use any part of your work to help you find its area. Show your reasoning.

Are you ready for more?

Can you decompose this triangle and rearrange its parts to form a rectangle? Describe how it could be done.

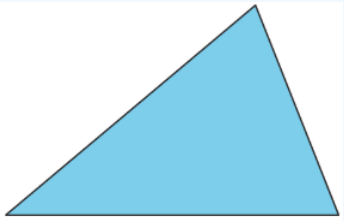


Figure 3.2.4

Summary

We can reason about the area of a triangle by using what we know about parallelograms. Here are three general ways to do this:

- Make a copy of the triangle and join the original and the copy along an edge to create a parallelogram. Because the two triangles have the same area, one copy of the triangle has half the area of that parallelogram.

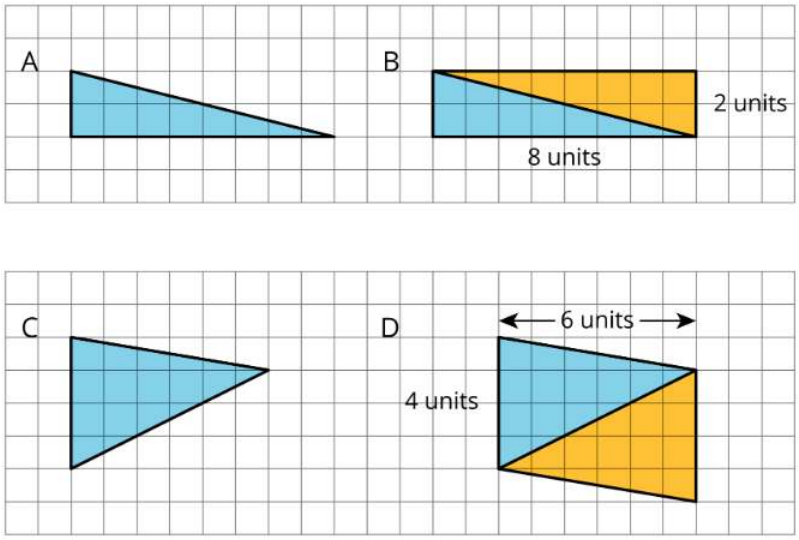


Figure C is another triangle, and figure D is the same triangle as figure C, with a copy along the edge of the original to create parallelogram. The left base of the parallelogram is labeled 4 units, and the height is labeled 6 units.

The area of Parallelogram B is 16 square units because the base is 8 units and the height 2 units. The area of Triangle A is half of that, which is 8 square units. The area of Parallelogram D is 24 square units because the base is 4 units and the height 6 units. The area of Triangle C is half of that, which is 12 square units.

- Decompose the triangle into smaller pieces and compose them into a parallelogram.

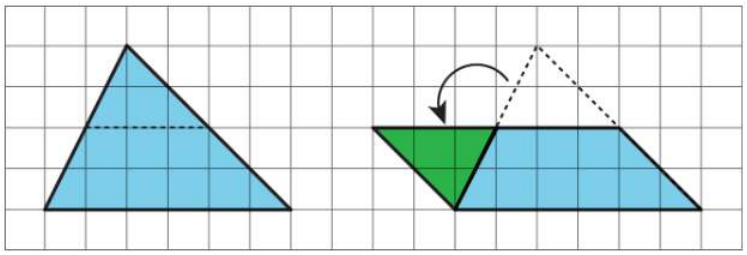


Figure 3.2.6: Two images of a triangle. Image on right has a dashed line cutting off the top portion. Image on left has the cut off portion moved next to the bottom of the triangle to create a parallelogram. An arrow indicating that the cut off portion from other image was moved.

In the new parallelogram, $b = 6$, $h = 2$, and $6 \cdot 2 = 12$, so its area is 12 square units. Because the original triangle and the parallelogram are composed of the same parts, the area of the original triangle is also 12 square units.

- Draw a rectangle around the triangle. Sometimes the triangle has half of the area of the rectangle.

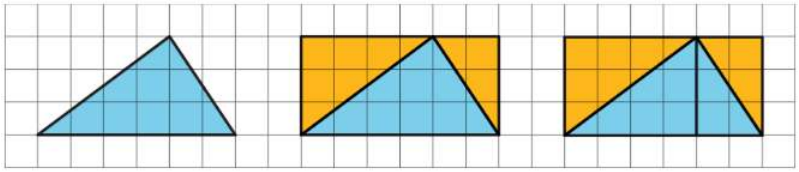


Figure 3.2.7

The large rectangle can be decomposed into smaller rectangles. The one on the left has area $4 \cdot 3$ or 12 square units; the one on the right has area $2 \cdot 3$ or 6 square units. The large triangle is also decomposed into two right triangles. Each of the right triangles is half of a smaller rectangle, so their areas are 6 square units and 3 square units. The large triangle has area 9 square units.

Sometimes, the triangle is half of what is left of the rectangle after removing two copies of the smaller right triangles.

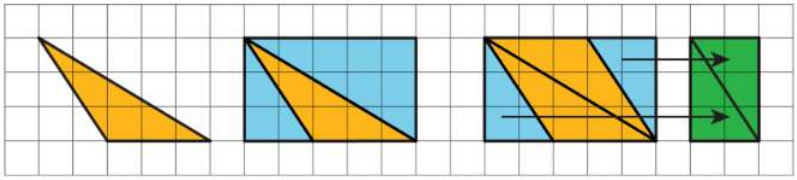


Figure 3.2.8: Three images of the same triangle. The first image is the triangle alone. The second is the triangle surrounded by a rectangle. The third image is of the triangle now with a copy composed into a parallelogram within the rectangle, with arrows drawing the remaining parts of the rectangle into a smaller rectangle.

The right triangles being removed can be composed into a small rectangle with area $(2 \cdot 3)$ square units. What is left is a parallelogram with area $5 \cdot 3 - 2 \cdot 3$, which equals $15 - 6$ or 9 square units. Notice that we can compose the same parallelogram with two copies of the original triangle! The original triangle is half of the parallelogram, so its area is $\frac{1}{2} \cdot 9$ or 4.5 square units.

Practice

Exercise 3.2.4

To find the area of this right triangle, Diego and Jada used different strategies. Diego drew a line through the midpoints of the two longer sides, which decomposes the triangle into a trapezoid and a smaller triangle. He then rearranged the two shapes into a parallelogram.

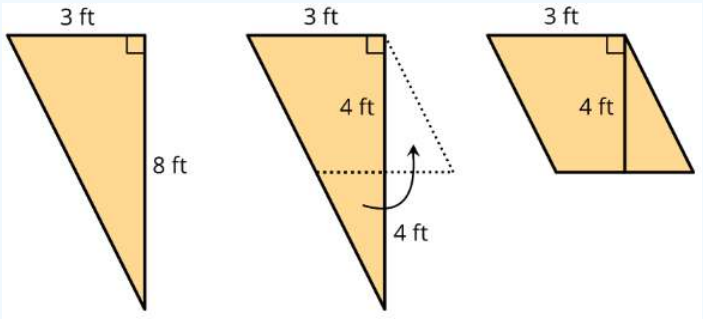


Figure 3.2.9: A triangle with one side labeled 3 feet and another side labeled 8 feet. A second image displays the same triangle with a dashed line bisecting the triangle so the side that was labeled 8 feet is now two pieces, each labeled 4 feet. An arrow indicates that the resulting smaller portion is rotated to create a parallelogram with a base of 3 feet and a height of 4 feet.

Jada made a copy of the triangle, rotated it, and lined it up against one side of the original triangle so that the two triangles make a parallelogram.

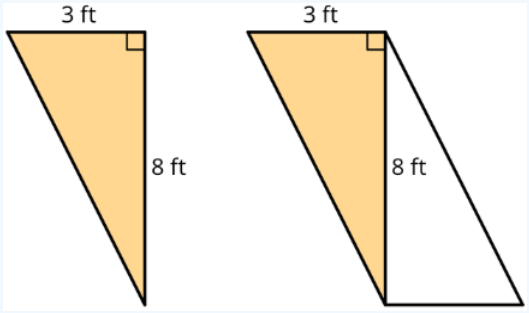


Figure 3.2.10

1. Explain how Diego might use his parallelogram to find the area of the triangle.
2. Explain how Jada might use her parallelogram to find the area of the triangle.

Exercise 3.2.5

Find the area of the triangle. Explain or show your reasoning.

a.

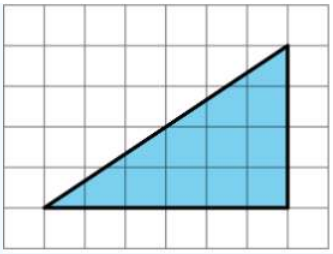


Figure 3.2.11

b.

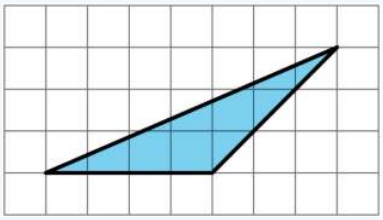


Figure 3.2.12

Exercise 3.2.6

Which of the three triangles has the greatest area? Show your reasoning. If you get stuck, try using what you know about the area of parallelograms.

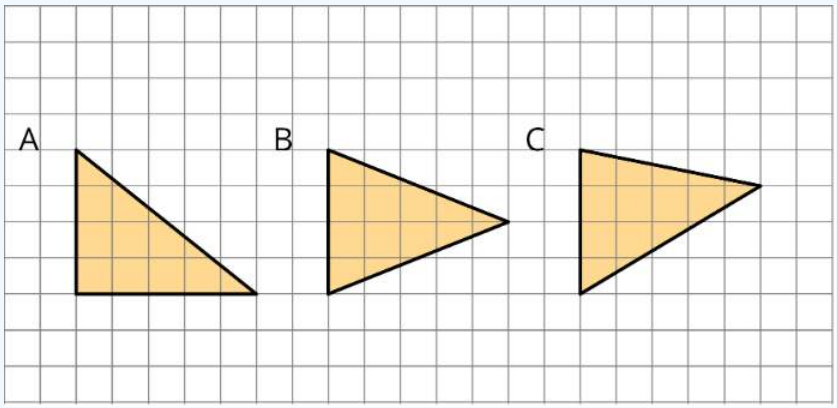


Figure 3.2.13

Exercise 3.2.7

Draw an identical copy of each triangle such that the two copies together form a parallelogram. If you get stuck, consider using tracing paper.

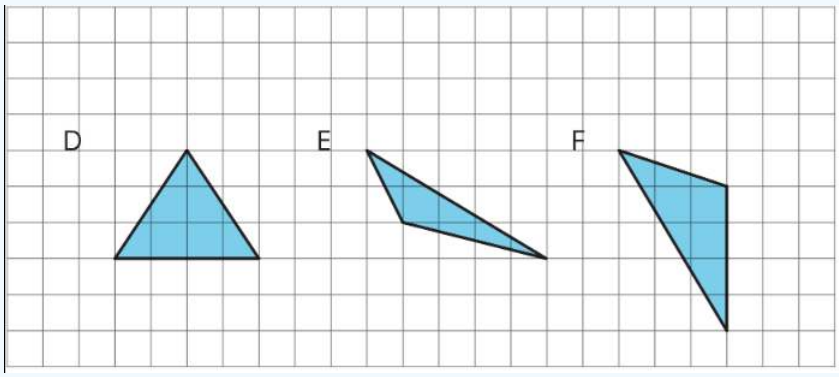


Figure 3.2.14

(From Unit 1.3.1)

Exercise 3.2.8

1. A parallelogram has a base of 3.5 units and a corresponding height of 2 units. What is its area?
2. A parallelogram has a base of 3 units and an area of 1.8 square units. What is the corresponding height for that base?
3. A parallelogram has an area of 20.4 square units. If the height that corresponds to a base is 4 units, what is the base?

(From Unit 1.2.3)

This page titled 3.2: Area of Triangles is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

3.3: Formula for the Area of a Triangle

Lesson

Let's write and use a formula to find the area of a triangle.

Exercise 3.3.1: Bases and Heights of a Triangle

Study the examples and non-examples of **bases** and **heights** in a triangle.

- Examples: These dashed segments represent heights of the triangle.

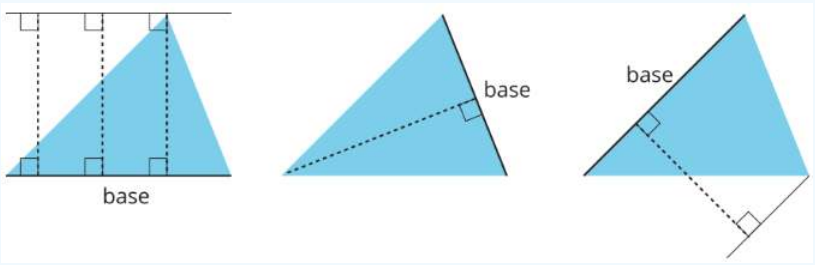


Figure 3.3.1

- Non-examples: These dashed segments do *not* represent heights of the triangle.

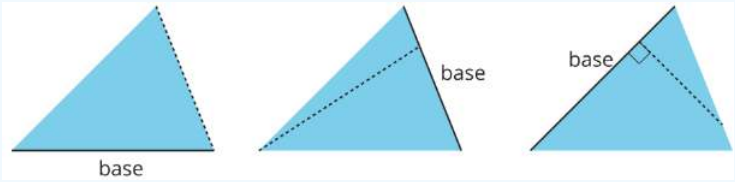


Figure 3.3.2

Select **all** the statements that are true about bases and heights in a triangle.

1. Any side of a triangle can be a base.
2. There is only one possible height.
3. A height is always one of the sides of a triangle.
4. A height that corresponds to a base must be drawn at an acute angle to the base.
5. A height that corresponds to a base must be drawn at a right angle to the base.
6. Once we choose a base, there is only one segment that represents the corresponding height.
7. A segment representing a height must go through a vertex.

Exercise 3.3.2: Finding a Formula for Area of a Triangle

For each triangle:

- Identify a base and a corresponding height, and record their lengths in the table.
- Find the area of the triangle and record it in the last column of the table.

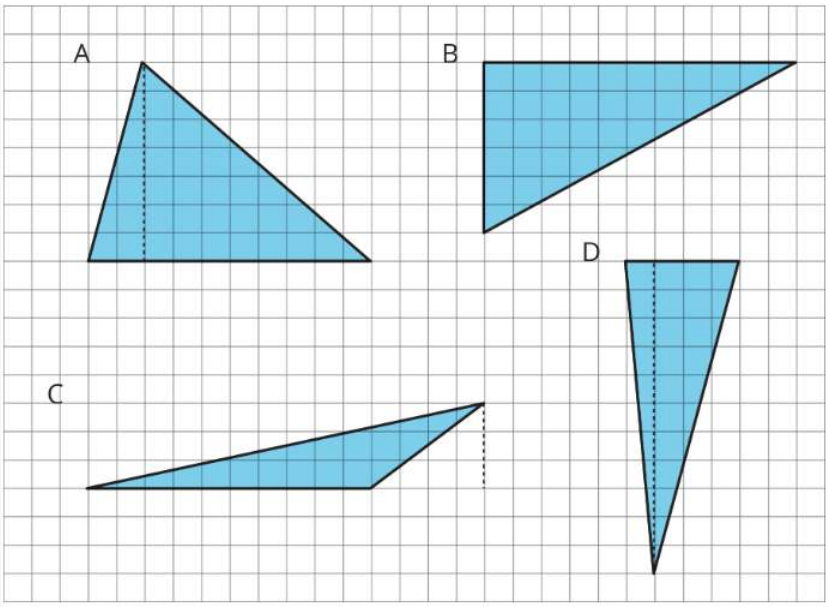


Figure 3.3.3

Table 3.3.1

triangle	base (units)	height (units)	area (square units)
A			
B			
C			
D			
any triangle	b	h	

In the last row, write an expression for the area of any triangle, using b and h.

Exercise 3.3.3: Applying the Formula for Area of Triangles

For each triangle, circle a base measurement that you can use to find the area of the triangle. Then, find the area of any *three* triangles. Show your reasoning.

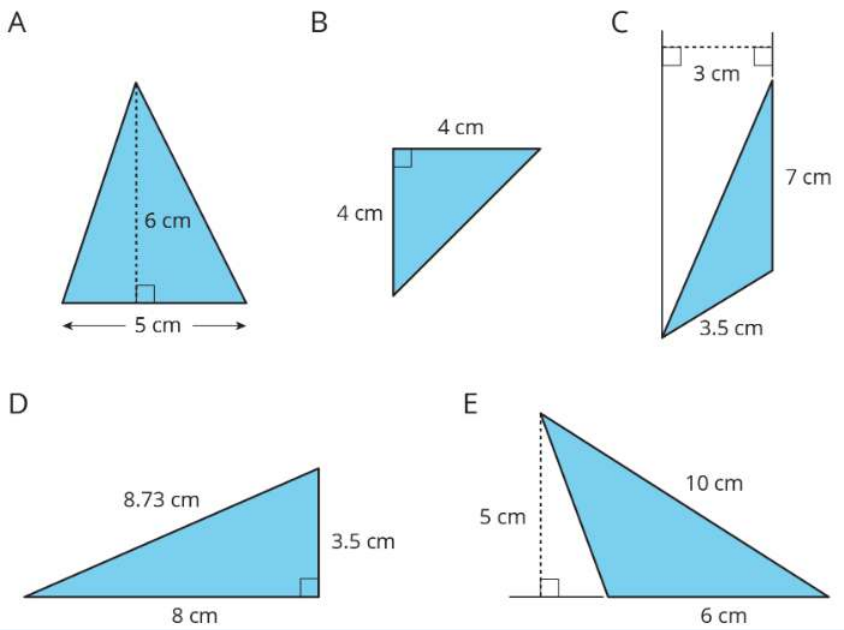


Figure 3.3.4

Summary

- We can choose any of the three sides of a triangle to call the **base**. The term "base" refers to both the side and its length (the measurement).
- The corresponding **height** is the length of a perpendicular segment from the base to the vertex opposite of it. The **opposite vertex** is the vertex that is *not* an endpoint of the base.

Here are three pairs of bases and heights for the same triangle. The dashed segments in the diagrams represent heights.

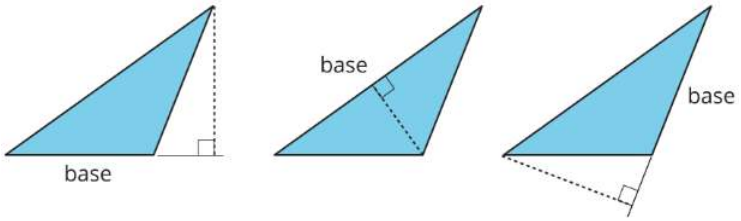


Figure 3.3.5

A segment showing a height must be drawn at a right angle to the base, but it can be drawn in more than one place. It does not have to go through the opposite vertex, as long as it connects the base and a line that is parallel to the base and goes through the opposite vertex, as shown here.

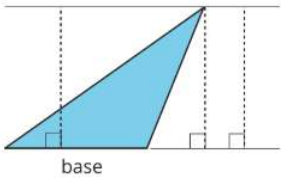


Figure 3.3.6

The base-height pairs in a triangle are closely related to those in a parallelogram. Recall that two copies of a triangle can be composed into one or more parallelograms. Each parallelogram shares at least one base with the triangle.

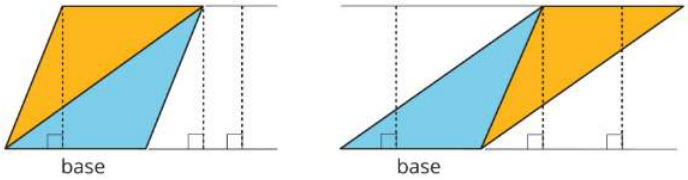


Figure 3.3.7: Two identical triangles, each with a copy composing the triangle into two different parallelograms. In each parallelogram has the bottom side labeled base and dashed lines at right angles to the base indicating the height of the parallelogram.

For any base that they share, the corresponding height is also shared, as shown by the dashed segments.

We can use the base-height measurements and our knowledge of parallelograms to find the area of any triangle.

- The formula for the area of a parallelogram with base b and height h is $b \cdot h$.
- A triangle takes up half of the area of a parallelogram with the same base and height. We can therefore express the area A of a triangle as: $A = \frac{1}{2} \cdot b \cdot h$

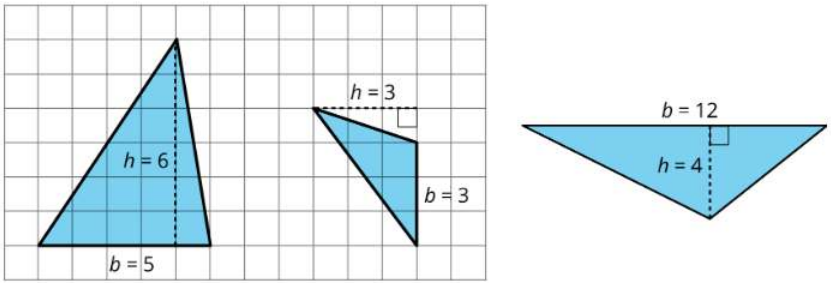


Figure 3.3.8

- The area of Triangle A is 15 square units because $\frac{1}{2} \cdot 5 \cdot 6 = 15$.
- The area of Triangle B is 4.5 square units because $\frac{1}{2} \cdot 3 \cdot 3 = 4.5$.
- The area of Triangle C is 24 square units because $\frac{1}{2} \cdot 12 \cdot 4 = 24$.

In each case, one side of the triangle is the base but neither of the other sides is the height. This is because the angle between them is not a right angle.

In right triangles, however, the two sides that are perpendicular can be a base and a height.

The area of this triangle is 18 square units whether we use 4 units or 9 units for the base.

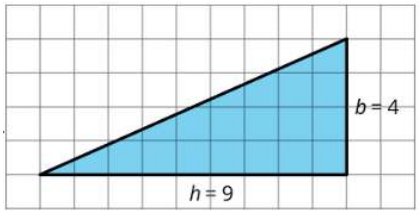


Figure 3.3.9

Glossary Entries

Definition: Opposite Vertex

For each side of a triangle, there is one vertex that is not on that side. This is the opposite vertex.

For example, point A is the opposite vertex to side BC.

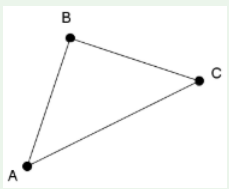


Figure 3.3.10

Practice

Exercise 3.3.4

Select **all** drawings in which a corresponding height h for a given base b is correctly identified.

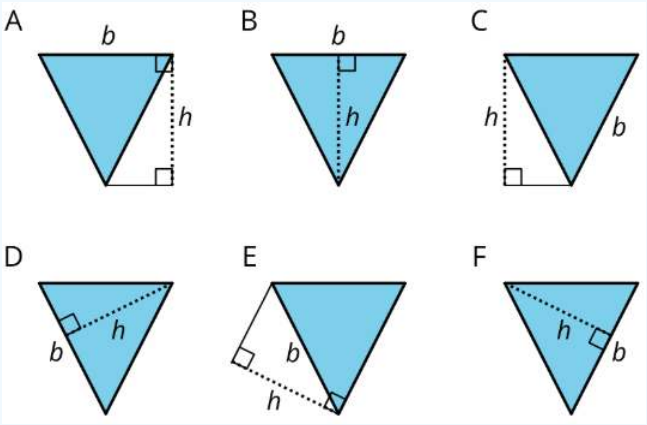


Figure 3.3.11: Six images of the same triangle, labeled A, B, C, D, E, and F. Triangle A, top side is labeled, b, and a dashed line labeled, h, extends straight down from the right vertex. Triangle B, top side is labeled b and a dashed line labeled, h, extends from the center of the top side to the opposite vertex. Triangle C, the right side is labeled, b, and a dashed line extends from the left top vertex straight down to the level of the bottom vertex labeled, h. Triangle D the left side is labeled, b, and a perpendicular line labeled, h, extends to the opposite vertex. Triangle E, the left side is labeled, b, and a dashed line labeled, h, extends out from the bottom vertex at a right angle to the right side. Triangle F, the right side is labeled, b, and a perpendicular dashed line labeled, h, extends from the side labeled, b, to the opposite vertex.

A. A
B. B
C. C
D. D
E. E
F. F

Exercise 3.3.5

For each triangle, a base and its corresponding height are labeled.

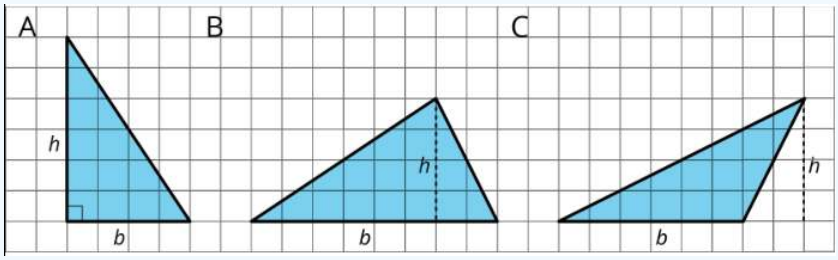


Figure 3.3.12: 3 triangles on a grid labeled A, B, C. A, base = 4, height = 6. B, base = 8, height = 4. C, base = 6, height = 4.

1. Find the area of each triangle.
2. How is the area related to the base and its corresponding height?

Exercise 3.3.6

Here is a right triangle. Name a corresponding height for each base.

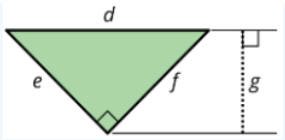


Figure 3.3.13

1. Side d
2. Side e
3. Side f

Exercise 3.3.7

Find the area of the shaded triangle. Show your reasoning.

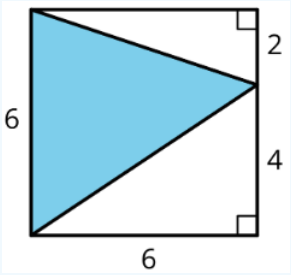


Figure 3.3.14: A square with a shaded triangle contained inside it. The left and bottom sides of the square are labeled 6, and the right side is labeled 2 above the point where vertex of the shaded triangle meets the side, and 4 below the point where the vertex meets the side.

(From Unit 1.3.2)

Exercise 3.3.8

Andre drew a line connecting two opposite corners of a parallelogram. Select **all** true statements about the triangles created by the line Andre drew.

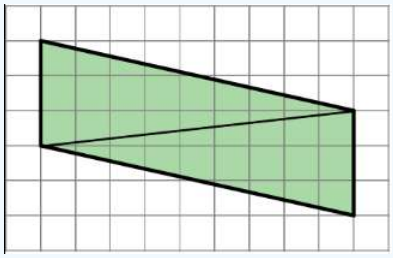


Figure 3.3.15

A. Each triangle has two sides that are 3 units long.
B. Each triangle has a side that is the same length as the diagonal line.
C. Each triangle has one side that is 3 units long.
D. When one triangle is placed on top of the other and their sides are aligned, we will see that one triangle is larger than the other.
E. The two triangles have the same area as each other.

(From Unit 1.3.1)

Exercise 3.3.9

Here is an octagon. (Note: The diagonal sides of the octagon are *not* 4 inches long.)

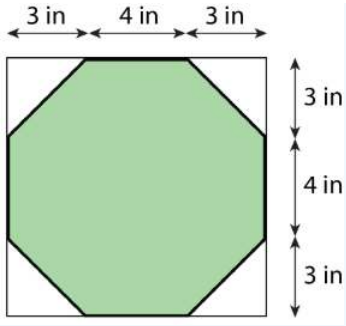


Figure 3.3.16

1. While estimating the area of the octagon, Lin reasoned that it must be less than 100 square inches. Do you agree? Explain your reasoning.
2. Find the exact area of the octagon. Show your reasoning.

(From Unit 1.1.3)

This page titled 3.3: Formula for the Area of a Triangle is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

3.4: Bases and Heights of Triangles

Lesson

Let's use different base-height pairs to find the area of a triangle.

Exercise 3.4.1: An Area of 12

Draw one triangle with an area of 12 square units. Try to draw a non-right triangle. Be prepared to explain how you know the area of your triangle is 12 square units.

Exercise 3.4.2: Hunting for Heights

1. Here are three copies of the same triangle. The triangle is rotated so that the side chosen as the base is at the bottom and is horizontal. Draw a height that corresponds to each base. Use an index card to help you.

Side a as the base:

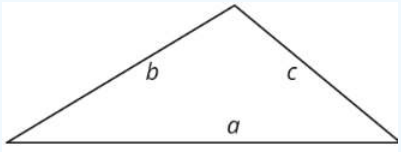


Figure 3.4.1

Side b as the base:

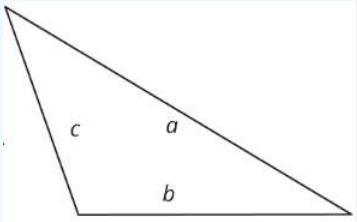


Figure 3.4.2

Side c as the base:

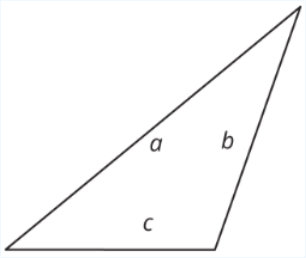


Figure 3.4.3

Pause for your teacher's instructions before moving to the next question.

2. Draw a line segment to show the height for the chosen base in each triangle.

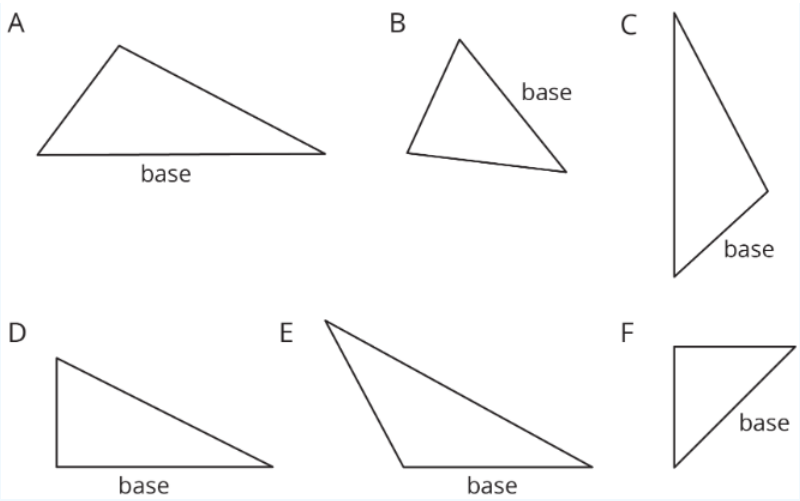


Figure 3.4.4

Exercise 3.4.3: Some Bases Are Better Than Others

For each triangle, identify and label a base and height. If needed, draw a line segment to show the height.

Then, find the area of the triangle. Show your reasoning. (The side length of each square on the grid is 1 unit.)

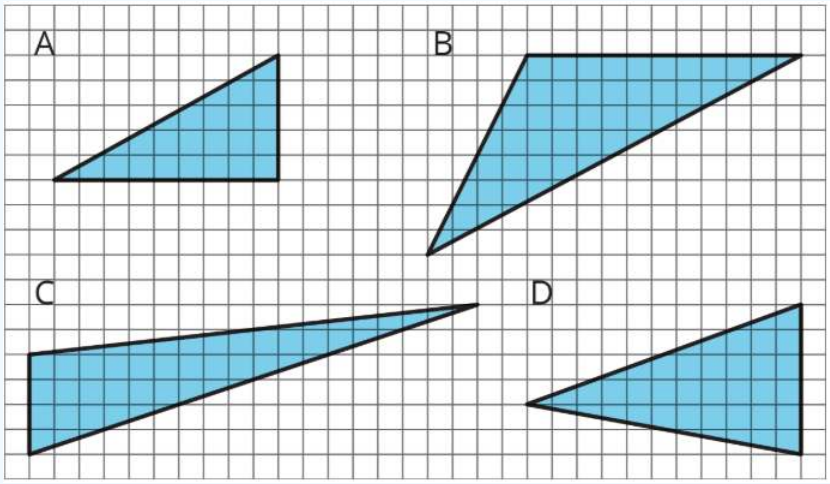


Figure 3.4.5

Are you ready for more?

Find the area of this triangle. Show your reasoning.

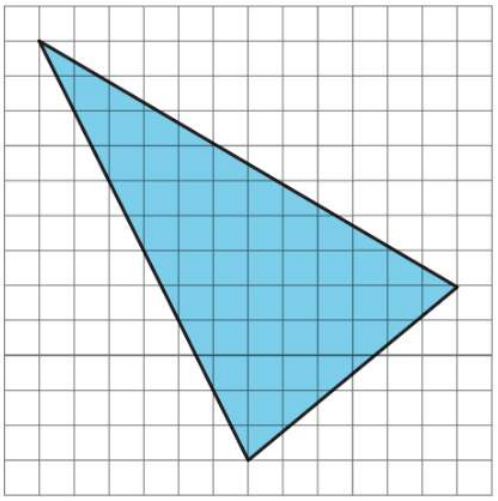


Figure 3.4.6

Summary

A height of a triangle is a perpendicular segment between the side chosen as the base and the opposite vertex. We can use tools with right angles to help us draw height segments.

An index card (or any stiff paper with a right angle) is a handy tool for drawing a line that is perpendicular to another line.

1. Choose a side of a triangle as the base. Identify its opposite vertex.
2. Line up one edge of the index card with that base.
3. Slide the card along the base until a perpendicular edge of the card meets the opposite vertex.
4. Use the card edge to draw a line from the vertex to the base. That segment represents the height.

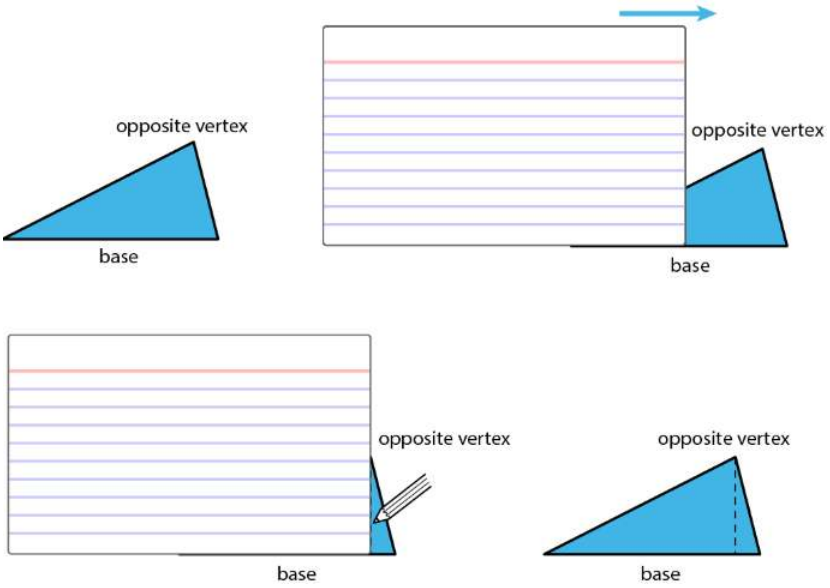


Figure 3.4.7

Sometimes we may need to extend the line of the base to identify the height, such as when finding the height of an obtuse triangle, or whenever the opposite vertex is not directly over the base. In these cases, the height segment is typically drawn *outside* of the triangle.

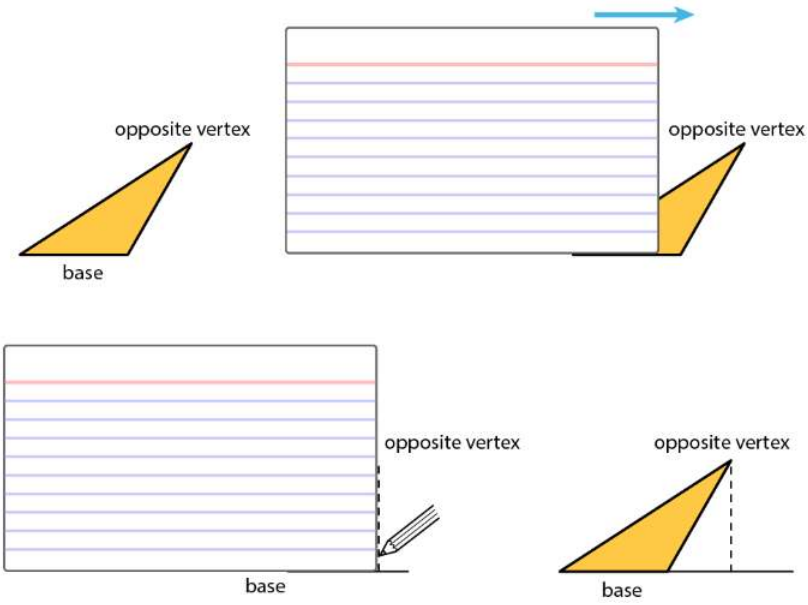


Figure 3.4.8

Even though any side of a triangle can be a base, some base-height pairs can be more easily determined than others, so it helps to choose strategically.

For example, when dealing with a right triangle, it often makes sense to use the two sides that make the right angle as the base and the height because one side is already perpendicular to the other.

If a triangle is on a grid and has a horizontal or a vertical side, you can use that side as a base and use the grid to find the height, as in these examples:

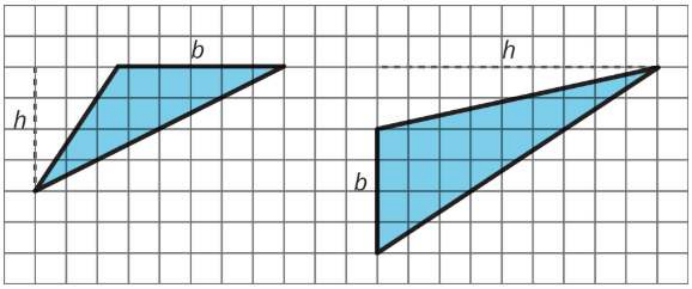


Figure 3.4.9

Glossary Entries

Definition: Edge

Each straight side of a polygon is called an edge.

For example, the edges of this polygon are segments AB, BC, CD, DE, and EA.

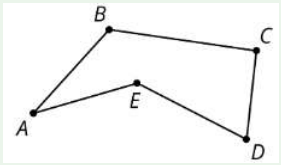


Figure 3.4.10

Definition: Opposite Vertex

For each side of a triangle, there is one vertex that is not on that side. This is the opposite vertex.

For example, point A is the opposite vertex to side BC.

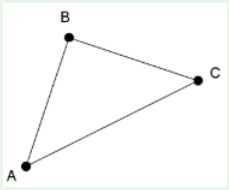


Figure 3.4.11

Definition: Vertex

A vertex is a point where two or more edges meet. When we have more than one vertex, we call them vertices.

The vertices in this polygon are labeled A, B, C, D, and E.

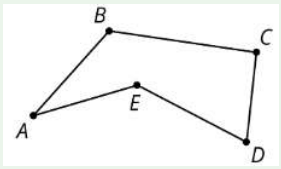


Figure 3.4.12

Practice

Exercise 3.4.4

For each triangle, a base is labeled b. Draw a line segment that shows its corresponding height. Use an index card to help you draw a straight line.

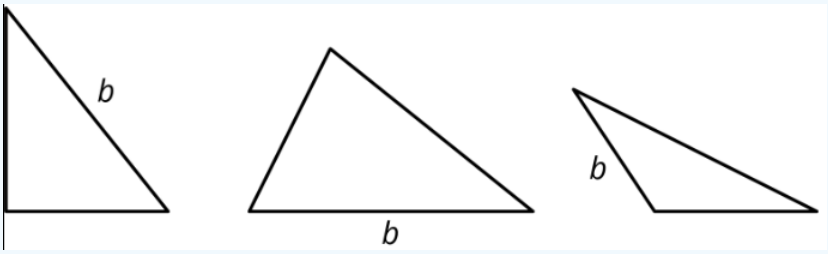


Figure 3.4.13

Exercise 3.4.5

Select all triangles that have an area of 8 square units. Explain how you know.

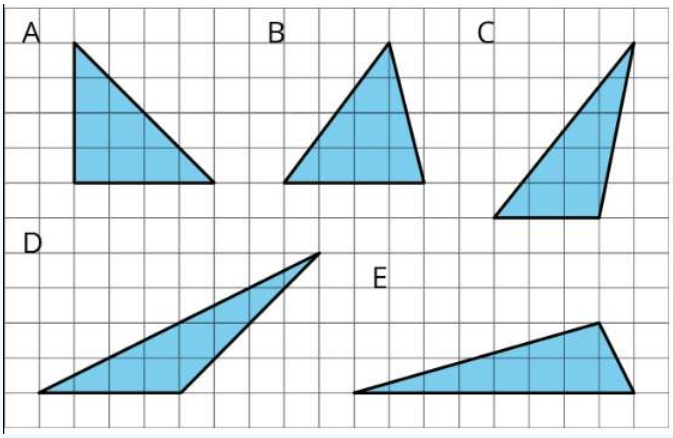


Figure 3.4.14: 5 triangles on grid labeled A, B, C, D, E. A base = 4, height = 4. B base = 4, height = 4. C base = 3, height = 5. D base = 4, height = 4. E base = 8, height = 2.

Exercise 3.4.6

Find the area of the triangle. Show your reasoning.

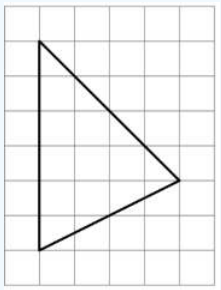


Figure 3.4.15

If you get stuck, carefully consider which side of the triangle to use as the base.

Exercise 3.4.7

Can side d be the base for this triangle? If so, which length would be the corresponding height? If not, explain why not.

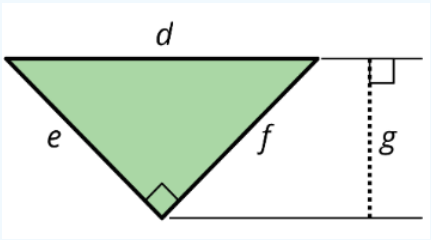


Figure 3.4.16

Exercise 3.4.8

Find the area of this shape. Show your reasoning.

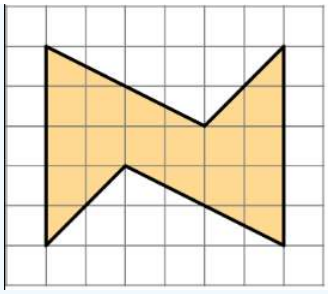


Figure 3.4.17

(From Unit 1.1.3)

Exercise 3.4.9

On the grid, sketch two different parallelograms that have equal area. Label a base and height of each and explain how you know the areas are the same.



Figure 3.4.18

(From Unit 1.2.3)

This page titled 3.4: Bases and Heights of Triangles is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

Section 4: Polygons

This page titled Section 4: Polygons is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

4.1: Polygons

Lesson

Let's investigate polygons and their areas.

Exercise 4.1.1: Which One Doesn't Belong: Bases and Heights

Which one doesn't belong?

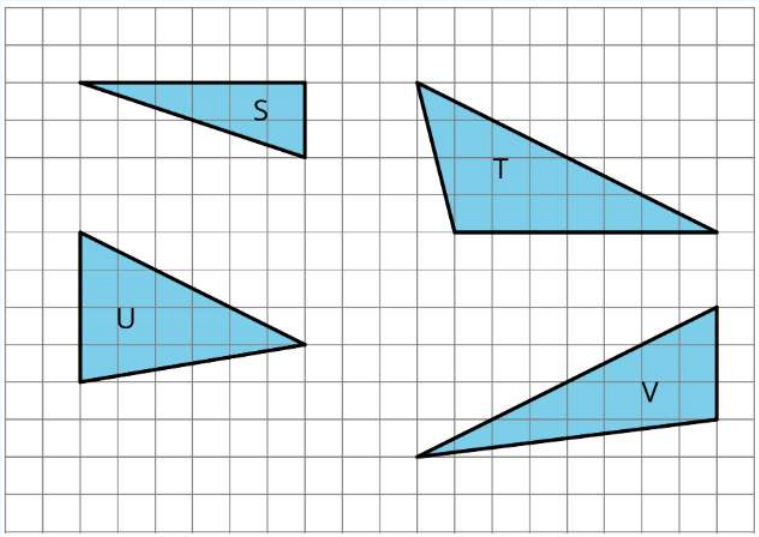


Figure 4.1.1

Exercise 4.1.2: What Are Polygons?

Here are five **polygons**:

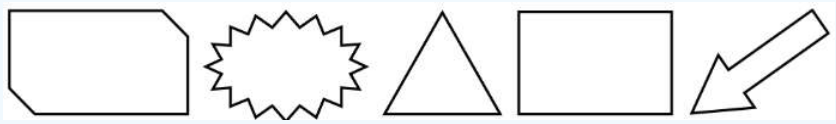


Figure 4.1.2

Here are six figures that are *not* polygons:

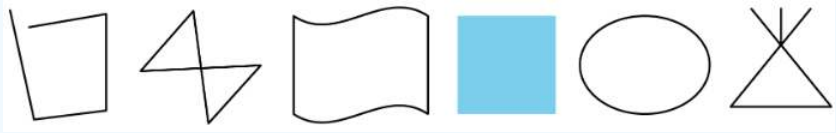


Figure 4.1.3

1. Circle the figures that are polygons.

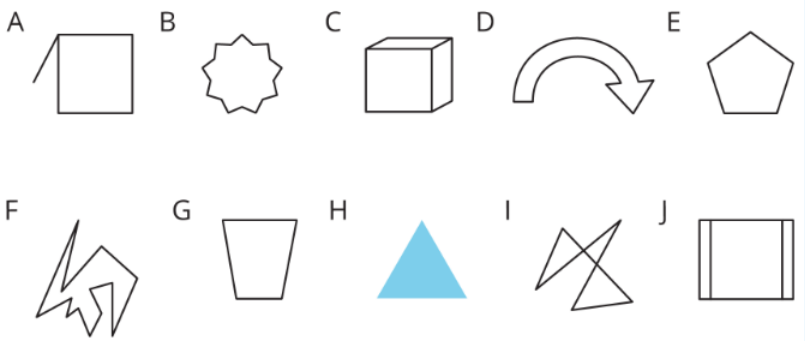


Figure 4.1.4

2. What do the figures you circled have in common? What characteristics helped you decide whether a figure was a polygon?

Exercise 4.1.3: Quadrilateral Strategies

Find the area of *two* **quadrilaterals** of your choice. Show your reasoning.

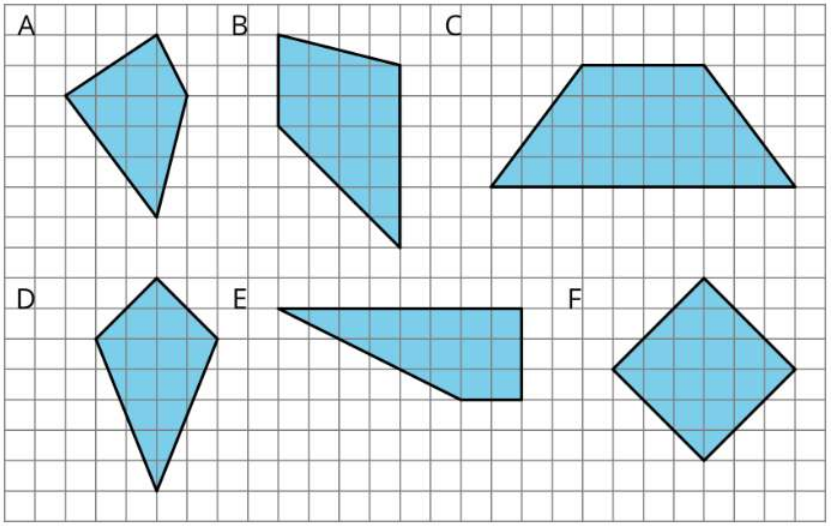


Figure 4.1.5

Are you ready for more?

Here is a trapezoid. a and b represent the lengths of its bottom and top sides. The segment labeled h represents its height; it is perpendicular to both the top and bottom sides.

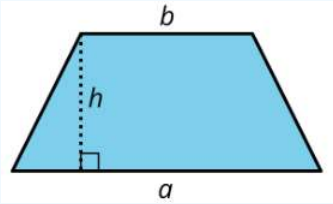


Figure 4.1.6

Apply area-reasoning strategies—decomposing, rearranging, duplicating, etc.—on the trapezoid so that you have one or more shapes with areas that you already know how to find. Use the shapes to help you write a formula for the area of a trapezoid. Show your reasoning.

Exercise 4.1.4: Pinwheel

Find the area of the shaded region in square units. Show your reasoning.

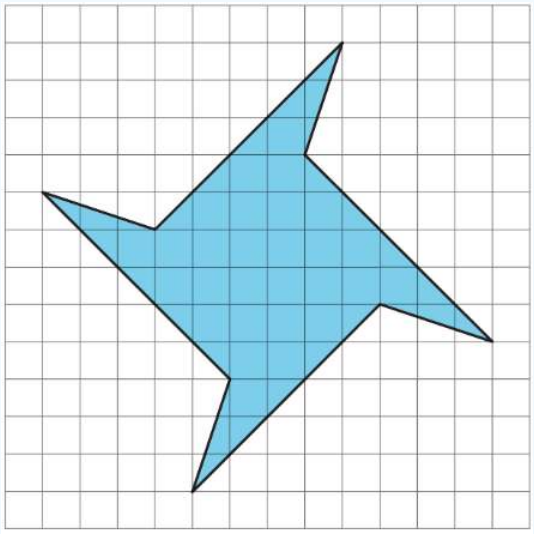


Figure 4.1.7

Summary

A **polygon** is a two-dimensional figure composed of straight line segments.

- Each end of a line segment connects to one other line segment. The point where two segments connect is a **vertex**. The plural of vertex is vertices.
- The segments are called the **edges** or **sides** of the polygon. The sides never cross each other. There are always an equal number of vertices and sides.

Here is a polygon with 5 sides. The vertices are labeled A, B, C, D, and E.

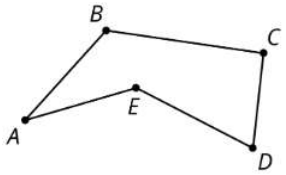


Figure 4.1.8

A polygon encloses a region. To find the area of a polygon is to find the area of the region inside it.

We can find the area of a polygon by decomposing the region inside it into triangles and rectangles.

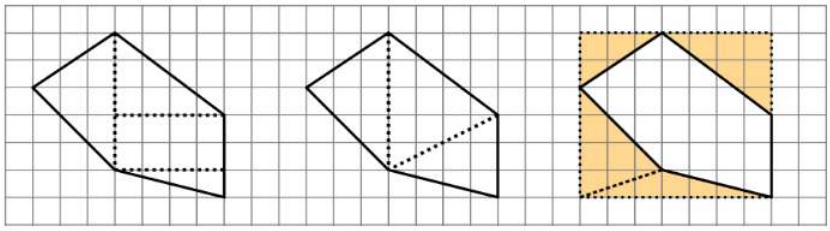


Figure 4.1.9

The first two diagrams show the polygon decomposed into triangles and rectangles; the sum of their areas is the area of the polygon. The last diagram shows the polygon enclosed with a rectangle; subtracting the areas of the triangles from the area of the rectangle gives us the area of the polygon.

Glossary Entries

Definition: Polygon

A polygon is a closed, two-dimensional shape with straight sides that do not cross each other.

Figure $ABCDE$ is an example of a polygon.

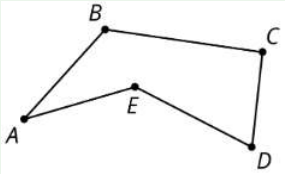


Figure 4.1.10

Practice

Exercise 4.1.5

Select **all** the polygons.

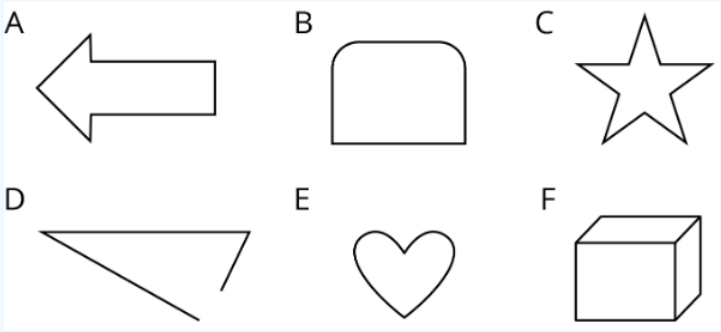


Figure 4.1.11

A. A
B. B
C. C
D. D
E. E
F. F

Exercise 4.1.6

Mark each vertex with a large dot. How many edges and vertices does this polygon have?

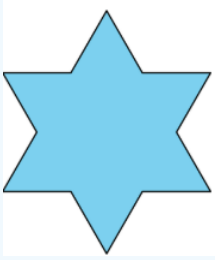


Figure 4.1.12

Exercise 4.1.7

Find the area of this trapezoid. Explain or show your strategy.

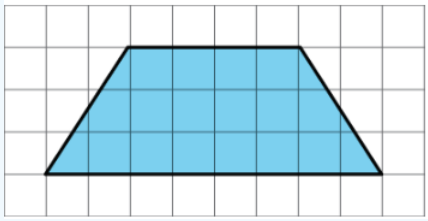


Figure 4.1.13

Exercise 4.1.8

Lin and Andre used different methods to find the area of a regular hexagon with 6-inch sides. Lin decomposed the hexagon into six identical, equilateral triangles. Andre decomposed the hexagon into a rectangle and two triangles.

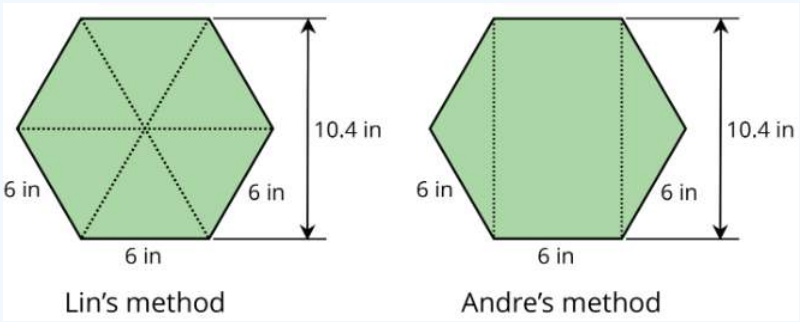


Figure 4.1.14: 2 identical hexagons labeled Lin's method and Andre's method. Each hexagon has three sides labeled 6 inches and a height labeled 10.4 inches. Lin's method, hexagon divided into 6 identical, equilateral triangles. Andre's method, hexagon decomposed into a rectangle and a triangle on either side of the rectangle.

Find the area of the hexagon using each person's method. Show your reasoning.

Exercise 4.1.9

1. Identify a base and a corresponding height that can be used to find the area of this triangle. Label the base b and the corresponding height h.

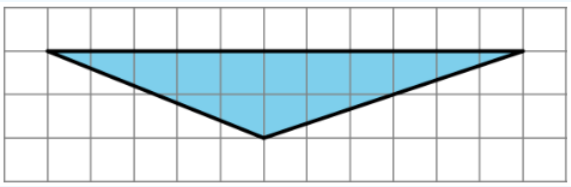


Figure 4.1.15: A triangle on a grid that has two vertices 11 units apart from one another horizontally. The third vertex is 2 units below the horizontal side, 5 units to the right of the left vertex, and 6 units to the left of the right vertex.

2. Find the area of the triangle. Show your reasoning.

(From Unit 1.3.3)

Exercise 4.1.10

On the grid, draw three different triangles with an area of 8 square units. Label the base and height of each triangle.

https://math.libretexts.org/@go/page/39642

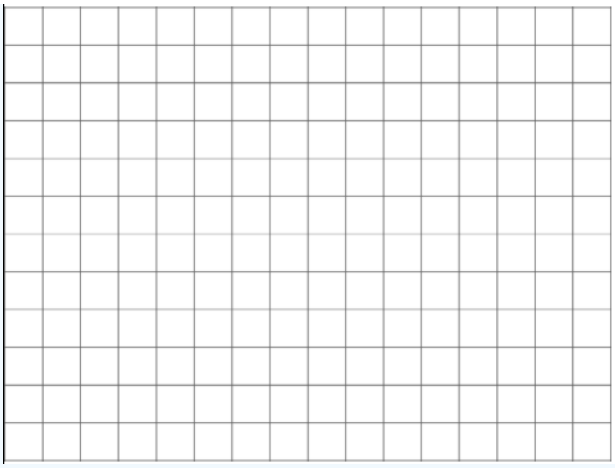


Figure 4.1.16

(From Unit 1.3.4)

This page titled 4.1: Polygons is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

Section 5: Surface Area

This page titled Section 5: Surface Area is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

5.1: Polyhedra

Lesson

Let's investigate polyhedra.

Exercise 5.1.1: What are Polyhedra?

Here are pictures that represent **polyhedra**:

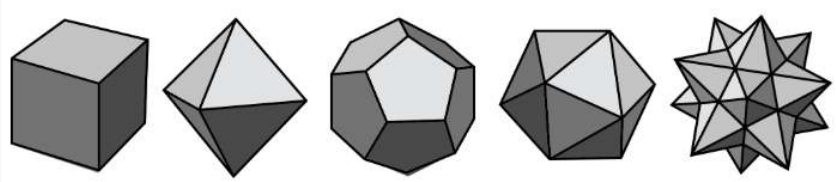


Figure 5.1.1

Here are pictures that do *not* represent polyhedra:

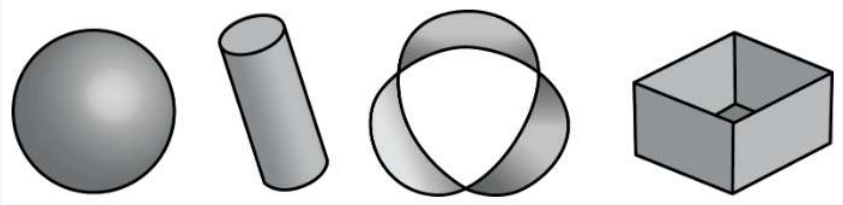


Figure 5.1.2

1. Your teacher will give you some figures or objects. Sort them into polyhedra and non-polyhedra.
2. What features helped you distinguish the polyhedra from the other figures?

Exercise 5.1.2: Prisms and Pyramids

1. Here are some polyhedra called **prisms**.

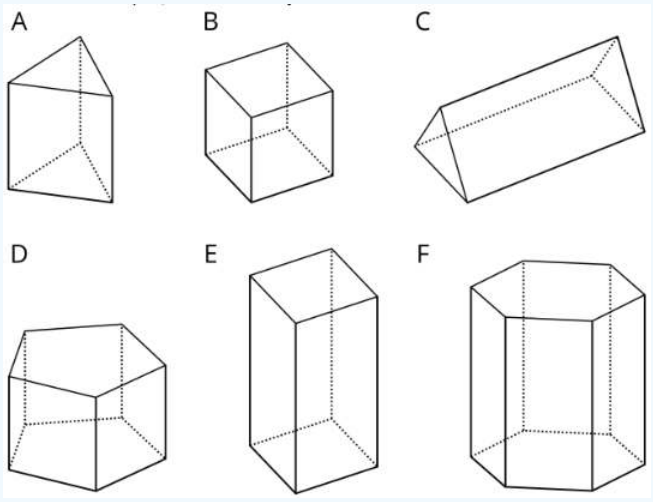


Figure 5.1.3

Here are some polyhedra called **pyramids**.

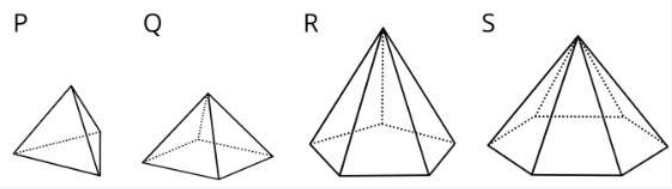


Figure 5.1.4

a. Look at the prisms. What are their characteristics or features?
b. Look at the pyramids. What are their characteristics or features?

2. Which of these **nets** can be folded into Pyramid P? Select all that apply.

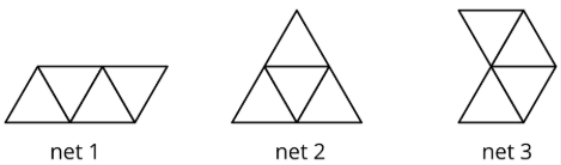


Figure 5.1.5: Three figures labeled net 1, net 2, and net 3. Net 1 has four small triangles arranged horizontally to create a parallelogram, net two has four small triangles arranged to make a larger triangle, and net 3 has two four small triangles which all meet at their vertices.

3. Your teacher will give your group a set of polygons and assign a polyhedron.
   a. Decide which polygons are needed to compose your assigned polyhedron. List the polygons and how many of each are needed.
   b. Arrange the cut-outs into a net that, if taped and folded, can be assembled into the polyhedron. Sketch the net. If possible, find more than one way to arrange the polygons (show a different net for the same polyhedron).

Are you ready for more?

What is the smallest number of faces a polyhedron can possibly have? Explain how you know.

Exercise 5.1.3: Assembling Polyhedra

1. Your teacher will give you the net of a polyhedron. Cut out the net, and fold it along the edges to assemble a polyhedron. Tape or glue the flaps so that there are no unjoined edges.
2. How many **vertices**, **edges**, and **faces** are in your polyhedron?

Summary

A **polyhedron** is a three-dimensional figure composed of faces. Each face is a filled-in polygon and meets only one other face along a complete edge. The ends of the edges meet at points that are called vertices.

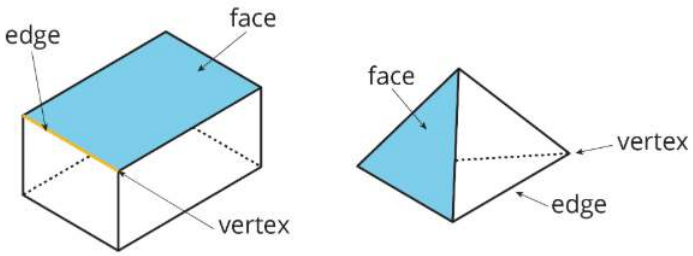


Figure 5.1.6

A polyhedron always encloses a three-dimensional region.

The plural of polyhedron is polyhedra. Here are some drawings of polyhedra:

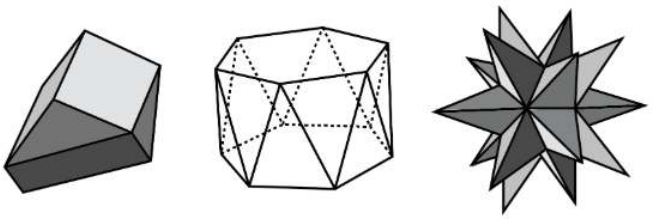


Figure 5.1.7

A **prism** is a type of polyhedron with two identical faces that are parallel to each other and that are called **bases**. The bases are connected by a set of rectangles (or sometimes parallelograms).

A prism is named for the shape of its bases. For example, if the base is a pentagon, then it is called a "pentagonal prism."

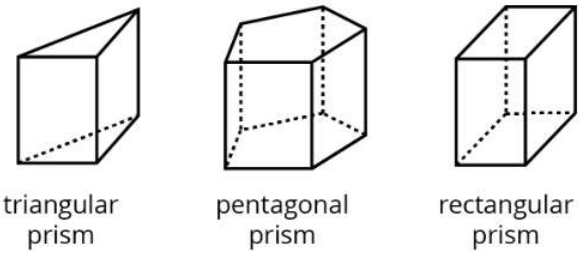


Figure 5.1.8

A **pyramid** is a type of polyhedron that has one special face called the base. All of the other faces are triangles that all meet at a single vertex.

A pyramid is named for the shape of its base. For example, if the base is a pentagon, then it is called a "pentagonal pyramid."

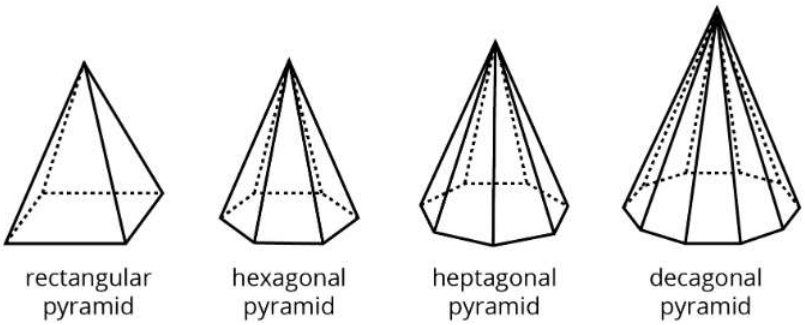


Figure 5.1.9

A **net** is a two-dimensional representation of a polyhedron. It is composed of polygons that form the faces of a polyhedron.

A cube has 6 square faces, so its net is composed of six squares, as shown here.

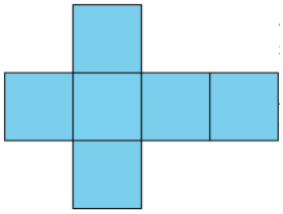


Figure 5.1.10

A net can be cut out and folded to make a model of the polyhedron.

In a cube, every face shares its edges with 4 other squares. In a net of a cube, not all edges of the squares are joined with another edge. When the net is folded, however, each of these open edges will join another edge.

It takes practice to visualize the final polyhedron by just looking at a net.

Definition: Base (of a Prism or Pyramid)

The word *base* can also refer to a face of a polyhedron.

A prism has two identical bases that are parallel. A pyramid has one base.

A prism or pyramid is named for the shape of its base.

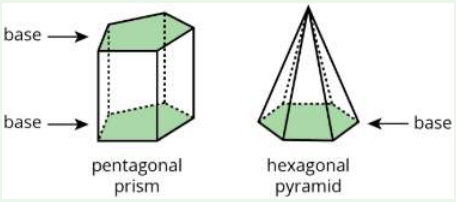


Figure 5.1.11: The figure on the left is labeled pentagonal prism. There are two identical pentagons on the top and bottom. Each vertex of a pentagon is connected by a vertical segment to the corresponding vertex of the other pentagons. The pentagons are each shaded, with the word base pointing to each. The figure on the right is labeled hexagonal pyramid. There is a hexagon on the bottom shaded green. From a point above the hexagon extend 6 segments, each connected to a vertex of the hexagon.

Definition: Face

Each flat side of a polyhedron is called a face. For example, a cube has 6 faces, and they are all squares.

Definition: Net

A net is a two-dimensional figure that can be folded to make a polyhedron.

Here is a net for a cube.

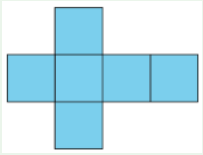


Figure 5.1.12

Definition: Polyhedron

A polyhedron is a closed, three-dimensional shape with flat sides. When we have more than one polyhedron, we call them polyhedra.

Here are some drawings of polyhedra.

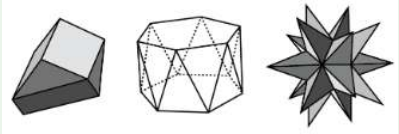


Figure 5.1.13

Definition: Prism

A prism is a type of polyhedron that has two bases that are identical copies of each other. The bases are connected by rectangles or parallelograms.

Here are some drawings of prisms.

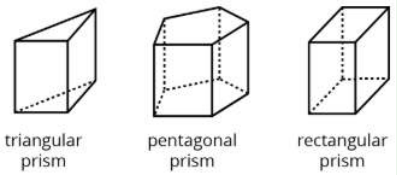


Figure 5.1.14

Definition: Pyramid

A pyramid is a type of polyhedron that has one base. All the other faces are triangles, and they all meet at a single vertex.

Here are some drawings of pyramids.

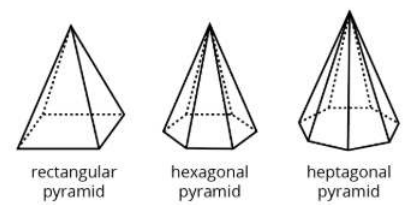


Figure 5.1.15

Definition: Surface Area

The surface area of a polyhedron is the number of square units that covers all the faces of the polyhedron, without any gaps or overlaps.

For example, if the faces of a cube each have an area of 9 cm^2, then the surface area of the cube is $6 \cdot 9$, or 54 cm^2.

Practice

Exercise 5.1.4

Select **all** the polyhedra.

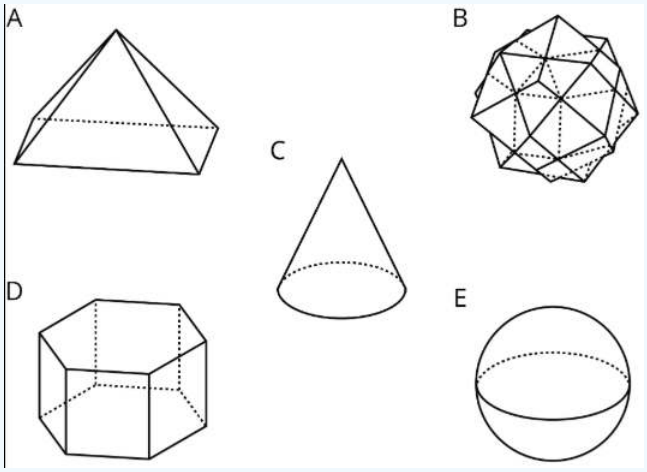


Figure 5.1.16

A. A
B. B
C. C
D. D
E. E

Exercise 5.1.5

1. Is this polyhedron a prism, a pyramid, or neither? Explain how you know.

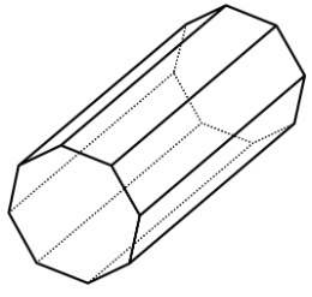


Figure 5.1.17

2. How many faces, edges, and vertices does it have?

Exercise 5.1.6

Tyler said this net cannot be a net for a square prism because not all the faces are squares.

Do you agree with Tyler? Explain your reasoning.

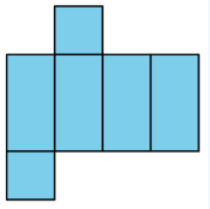


Figure 5.1.18

Exercise 5.1.7

Explain why each of these triangles has an area of 9 square units.

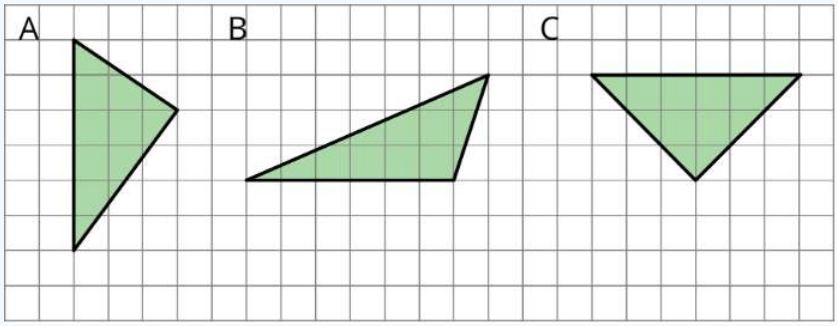


Figure 5.1.19

(From Unit 1.3.2)

Exercise 5.1.8

1. A parallelogram has a base of 12 meters and a height of 1.5 meters. What is its area?
2. A triangle has a base of 16 inches and a height of $\frac{1}{8}$ inches. What is its area?
3. A parallelogram has an area of 28 square feet and a height of 4 feet. What is its base?
4. A triangle has an area of 32 square millimeters and a base of 8 millimeters. What is its height?

(From Unit 1.3.3)

Exercise 5.1.9

Find the area of the shaded region. Show or explain your reasoning.

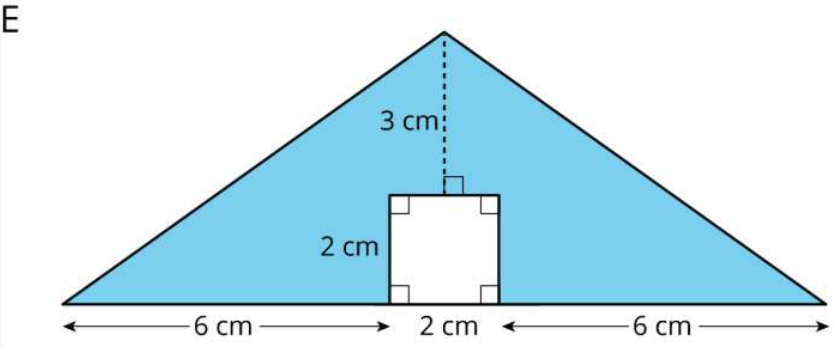


Figure 5.1.20: 2 congruent right triangles touching height to height. On each, base = 7 centimeters, height = 5 centimeters. square with side length = 2 centimeters cut out from the center of the figure. area around the square shaded.

(From Unit 1.1.3)

This page titled 5.1: Polyhedra is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

5.2: Nets and Surface Area

Lesson

Let's use nets to find the surface area of polyhedra.

Exercise 5.2.1: Matching Nets

Each of the nets can be assembled into a polyhedron. Match each net with its corresponding polyhedron, and name the polyhedron. Be prepared to explain how you know the net and polyhedron go together.

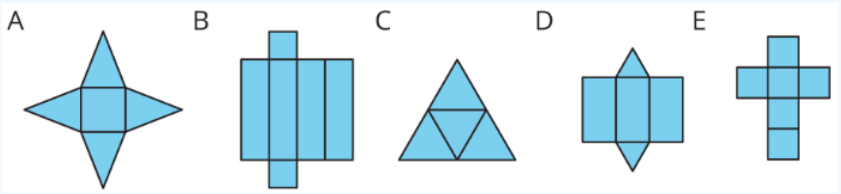


Figure 5.2.1

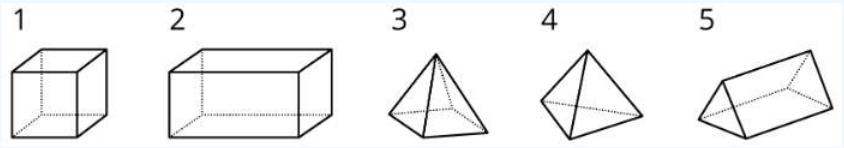


Figure 5.2.2

Exercise 5.2.2: Using Nets to Find Surface Area

1. Name the polyhedron that each net would form when assembled.

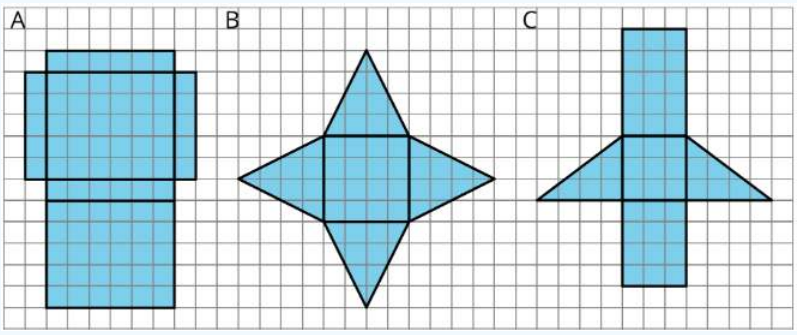


Figure 5.2.3: Three nets on a grid, labeled A, B, and C. Net A is composed of two rectangles that are 5 units tall by 6 units wide, two that are 5 units high and one unit wide, and two that are one unit high and six units wide. Net B is a square with a side length of 4 units and is surrounded by triangles that are four units wide at the base and four units high. Net C is a square with a side length of 3, a rectangle 3 units wide and 5 units high, another rectangle that is 3 units wide and 4 units tall, and two triangles, one on either side, that are three units tall by four units across.

2. Your teacher will give you the nets of three polyhedra. Cut out the nets and assemble the three-dimensional shapes.
3. Find the **surface area** of each polyhedron. Explain your reasoning clearly.

Are you ready for more?

1. For each net, decide if it can be assembled into a rectangular prism

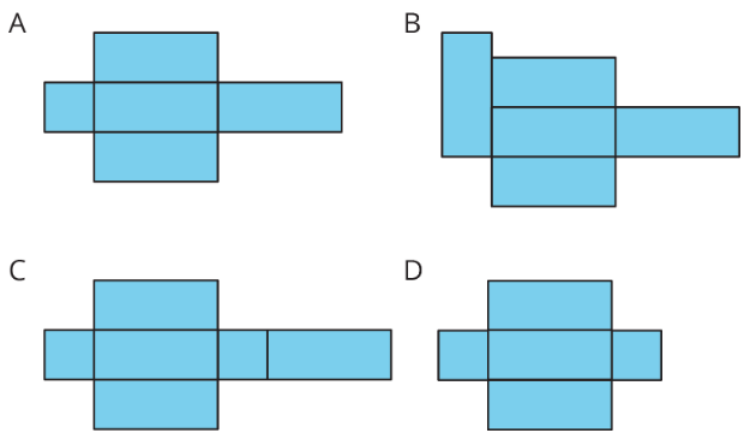


Figure 5.2.4

2. For each net, decide if it can be folded into a triangular prism.

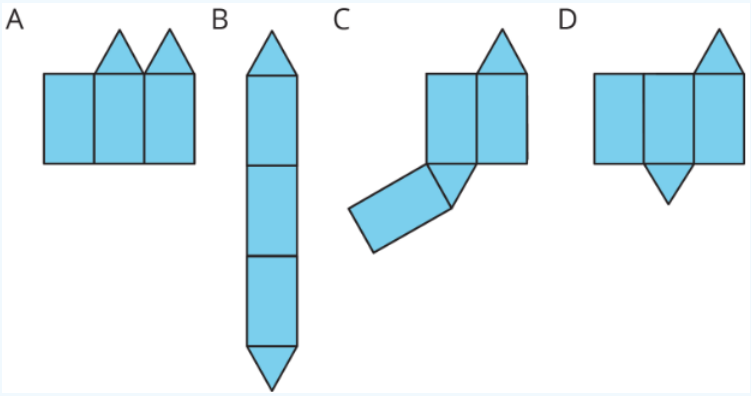


Figure 5.2.5

Summary

A net of a *pyramid* has one polygon that is the base. The rest of the polygons are triangles. A pentagonal pyramid and its net are shown here.

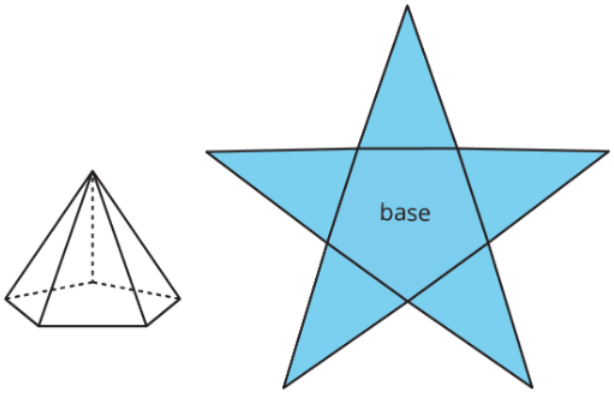


Figure 5.2.6

A net of a *prism* has two copies of the polygon that is the base. The rest of the polygons are rectangles. A pentagonal prism and its net are shown here.

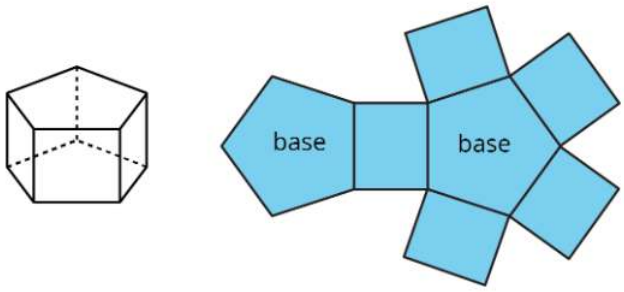


Figure 5.2.7

In a rectangular prism, there are three pairs of parallel and identical rectangles. Any pair of these identical rectangles can be the bases.

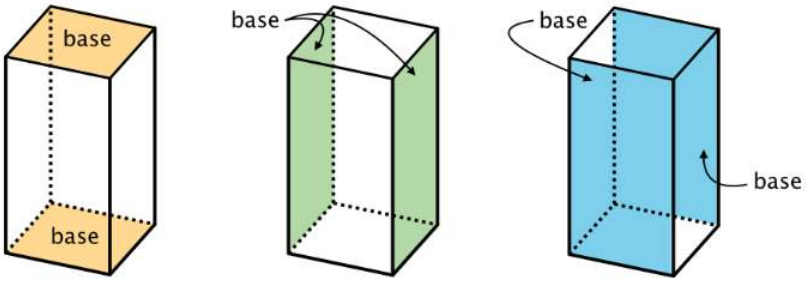


Figure 5.2.8

Because a net shows all the faces of a polyhedron, we can use it to find its surface area. For instance, the net of a rectangular prism shows three pairs of rectangles: 4 units by 2 units, 3 units by 2 units, and 4 units by 3 units.

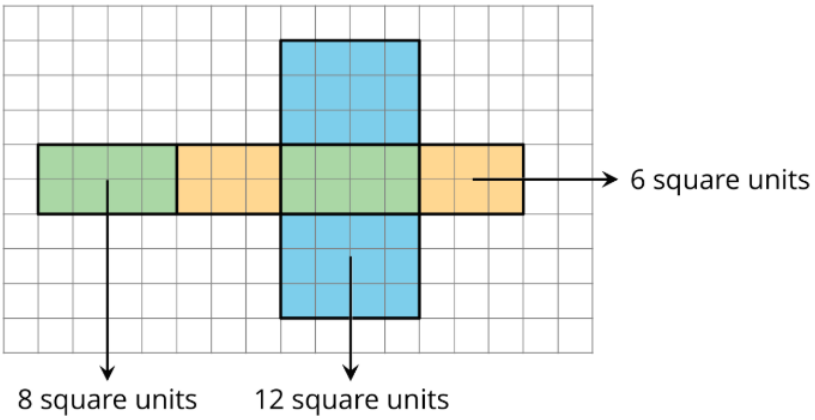


Figure 5.2.9

The **surface area** of the rectangular prism is 52 square units because $8+8+6+6+12+12=52$.

Glossary Entries

Definition: Base (of a Prism or Pyramid)

The word *base* can also refer to a face of a polyhedron.

A prism has two identical bases that are parallel. A pyramid has one base.

A prism or pyramid is named for the shape of its base.

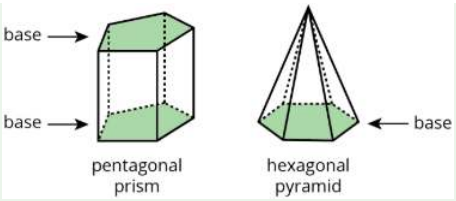


Figure 5.2.10: The figure on the left is labeled pentagonal prism. There are two identical pentagons on the top and bottom. Each vertex of a pentagon is connected by a vertical segment to the corresponding vertex of the other pentagons. The pentagons are each shaded, with the word base pointing to each. The figure on the right is labeled hexagonal pyramid. There is a hexagon on the bottom shaded green. From a point above the hexagon extend 6 segments, each connected to a vertex of the hexagon.

Definition: Face

Each flat side of a polyhedron is called a face. For example, a cube has 6 faces, and they are all squares.

Definition: Net

A net is a two-dimensional figure that can be folded to make a polyhedron.

Here is a net for a cube.

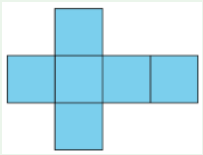


Figure 5.2.11

Definition: Polyhedron

A polyhedron is a closed, three-dimensional shape with flat sides. When we have more than one polyhedron, we call them polyhedra.

Here are some drawings of polyhedra.

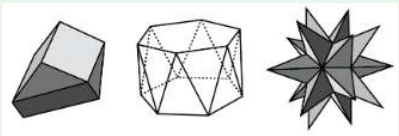


Figure 5.2.12

Definition: Prism

A prism is a type of polyhedron that has two bases that are identical copies of each other. The bases are connected by rectangles or parallelograms.

Here are some drawings of prisms.

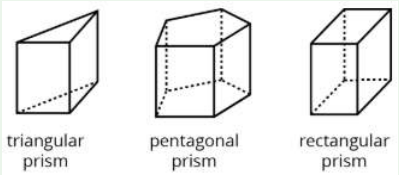


Figure 5.2.13

Definition: Pyramid

A pyramid is a type of polyhedron that has one base. All the other faces are triangles, and they all meet at a single vertex.

Here are some drawings of pyramids.

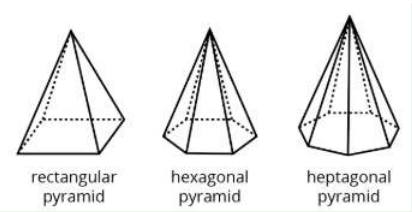


Figure 5.2.14

Definition: Surface Area

The surface area of a polyhedron is the number of square units that covers all the faces of the polyhedron, without any gaps or overlaps.

For example, if the faces of a cube each have an area of 9 cm^2, then the surface area of the cube is $6 \cdot 9$, or 54 cm^2.

Practice

Exercise 5.2.3

Can this net be assembled into a cube? Explain how you know. Label parts of the net with letters or numbers if it helps your explanation.

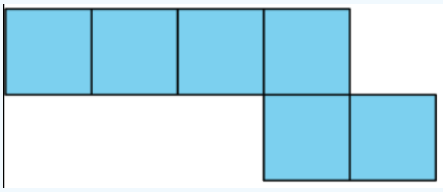


Figure 5.2.15

Exercise 5.2.4

1. What polyhedron can be assembled from this net? Explain how you know.

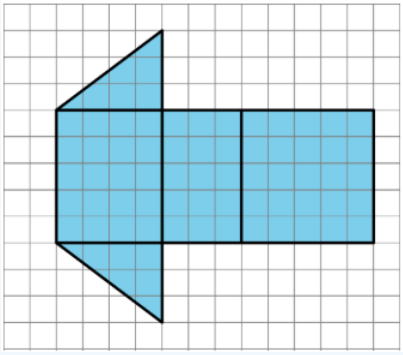


Figure 5.2.16: Net on a grid. 3 adjacent rectangles, from left to right 4 by 5 units, 3 by 5 units, 5 by 5 units. above and below first rectangle are triangles with base = 4 units, height = 3 units.

2. Find the surface area of this polyhedron. Show your reasoning.

Exercise 5.2.5

Here are two nets. Mai said that both nets can be assembled into the same triangular prism. Do you agree? Explain or show your reasoning.

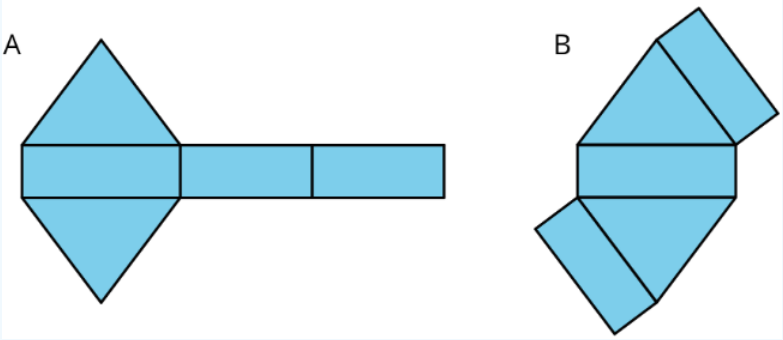


Figure 5.2.17

Exercise 5.2.6

Here are two three-dimensional figures.

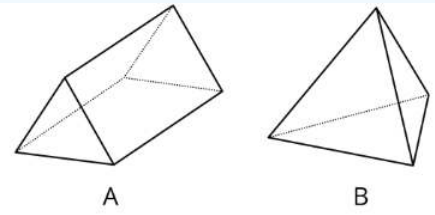


Figure 5.2.18

Tell whether each of the following statements describes Figure A, Figure B, both, or neither.

1. This figure is a polyhedron.
2. This figure has triangular faces.
3. There are more vertices than edges in this figure.
4. This figure has rectangular faces.
5. This figure is a pyramid.
6. There is exactly one face that can be the base for this figure.
7. The base of this figure is a triangle.
8. This figure has two identical and parallel faces that can be the base.

(From Unit 1.5.2)

Exercise 5.2.7

Select **all** units that can be used for surface area.

A. square meters
B. feet
C. centimeters
D. cubic inches
E. square inches
F. square feet

(From Unit 1.5.1)

Exercise 5.2.8

Find the area of this polygon. Show your reasoning.

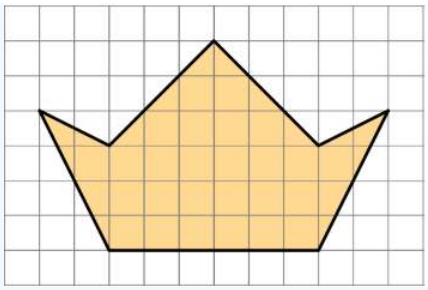


Figure 5.2.19

(From Unit 1.4.1)

This page titled 5.2: Nets and Surface Area is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

5.3: More Nets, More Surface Area

Lesson

Let's draw nets and find the surface area of polyhedra.

Exercise 5.3.1: Notice and Wonder: Wrapping Paper

Kiran is wrapping this box of sports cards as a present for a friend.

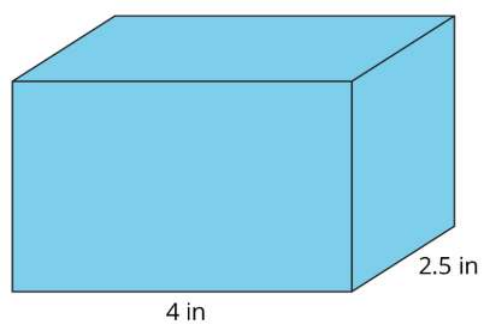


Figure 5.3.1

What do you notice? What do you wonder?

Exercise 5.3.2: Building Prisms and Pyramids

Your teacher will give you a drawing of a polyhedron. You will draw its net and calculate its surface area.

1. What polyhedron do you have?
2. Study your polyhedron. Then, draw its net on graph paper. Use the side length of a grid square as the unit.
3. Label each polygon on the net with a name or number.
4. Find the surface area of your polyhedron. Show your thinking in an organized manner so that it can be followed by others.

Exercise 5.3.3: Comparing Boxes

Here are the nets of three cardboard boxes that are all rectangular prisms. The boxes will be packed with 1-centimeter cubes. All lengths are in centimeters.

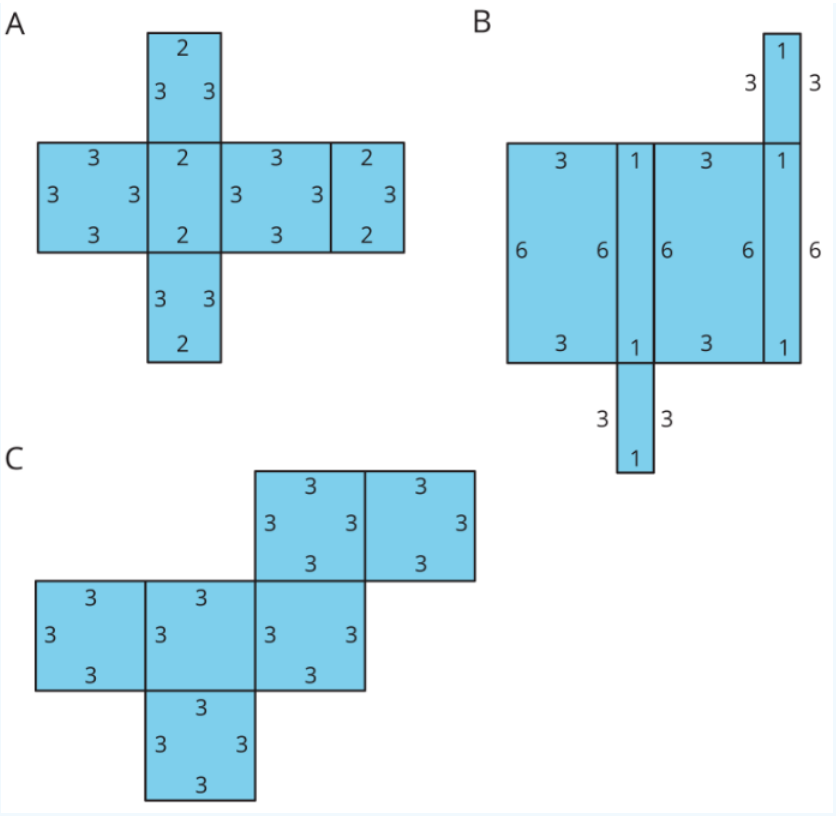


Figure 5.3.2

1. Compare the surface areas of the boxes. Which box will use the least cardboard? Show your reasoning.
2. Now compare the volumes of these boxes in cubic centimeters. Which box will hold the most 1-centimeter cubes? Show your reasoning.

Are you ready for more?

Figure C shows a net of a cube. Draw a different net of a cube. Draw another one. And then another one. How many different nets can be drawn and assembled into a cube?

Summary

The surface area of a polyhedron is the sum of the areas of all of the faces. Because a net shows us all faces of a polyhedron at once, it can help us find the surface area. We can find the areas of all polygons in the net and add them.

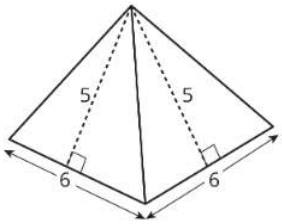


Figure 5.3.3

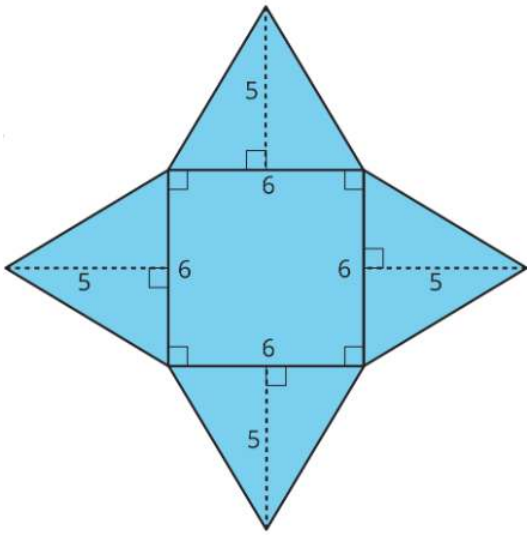


Figure 5.3.4

A square pyramid has a square and four triangles for its faces. Its surface area is the sum of the areas of the square base and the four triangular faces:

$$(6 \cdot 6) + 4 \cdot (\tfrac{1}{2} \cdot 5 \cdot 6) = 96$$

The surface area of this square pyramid is 96 square units.

Glossary Entries

Definition: Base (of a Prism or Pyramid)

The word *base* can also refer to a face of a polyhedron.

A prism has two identical bases that are parallel. A pyramid has one base.

A prism or pyramid is named for the shape of its base.

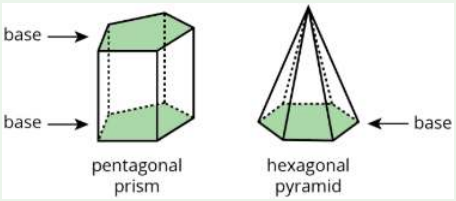


Figure 5.3.5: The figure on the left is labeled pentagonal prism. There are two identical pentagons on the top and bottom. Each vertex of a pentagon is connected by a vertical segment to the corresponding vertex of the other pentagons. The pentagons are each shaded, with the word base pointing to each. The figure on the right is labeled hexagonal pyramid. There is a hexagon on the bottom shaded green. From a point above the hexagon extend 6 segments, each connected to a vertex of the hexagon.

Definition: Face

Each flat side of a polyhedron is called a face. For example, a cube has 6 faces, and they are all squares.

Definition: Net

A net is a two-dimensional figure that can be folded to make a polyhedron.

Here is a net for a cube.

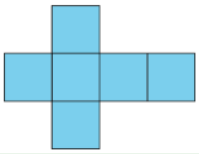


Figure 5.3.6

Definition: Polyhedron

A polyhedron is a closed, three-dimensional shape with flat sides. When we have more than one polyhedron, we call them polyhedra.

Here are some drawings of polyhedra.

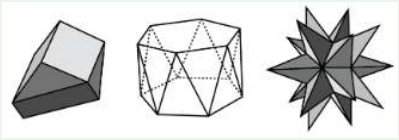


Figure 5.3.7

Definition: Prism

A prism is a type of polyhedron that has two bases that are identical copies of each other. The bases are connected by rectangles or parallelograms.

Here are some drawings of prisms.

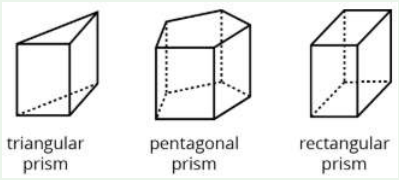


Figure 5.3.8

Definition: Pyramid

A pyramid is a type of polyhedron that has one base. All the other faces are triangles, and they all meet at a single vertex.

Here are some drawings of pyramids.

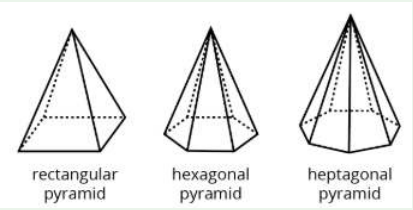


Figure 5.3.9

Definition: Surface Area

The surface area of a polyhedron is the number of square units that covers all the faces of the polyhedron, without any gaps or overlaps.

For example, if the faces of a cube each have an area of 9 cm^2, then the surface area of the cube is $6 \cdot 9$, or 54 cm^2.

Practice

Exercise 5.3.4

Jada drew a net for a polyhedron and calculated its surface area.

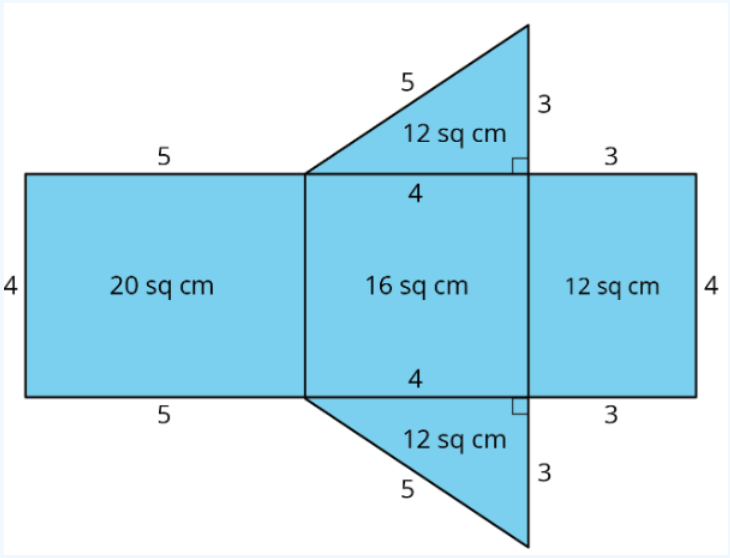


Figure 5.3.10: Triangular prism, Faces are: 5 by 4 rectangle, area 20 square centimeters. 4 by 4 square, area 16 square centimeters, 2 triangles, bases of 5, heights of 3, areas of 12 square centimeters each, and 3 by 4 rectangle, area =12 square centimeters.

1. What polyhedron can be assembled from this net?
2. Jada made some mistakes in her area calculation. What were the mistakes?
3. Find the surface area of the polyhedron. Show your reasoning.

Exercise 5.3.5

A cereal box is 8 inches by 2 inches by 12 inches. What is its surface area? Show your reasoning. If you get stuck, consider drawing a sketch of the box or its net and labeling the edges with their measurements.

Exercise 5.3.6

Twelve cubes are stacked to make this figure.

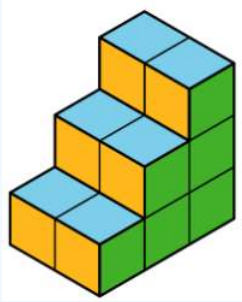


Figure 5.3.11

1. What is its surface area?
2. How would the surface area change if the top two cubes are removed?

(From Unit 1.5.1)

Exercise 5.3.7

Here are two polyhedra and their nets. Label all edges in the net with the correct lengths.

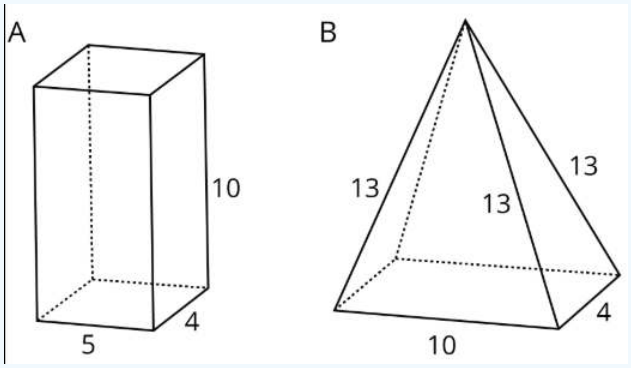


Figure 5.3.12

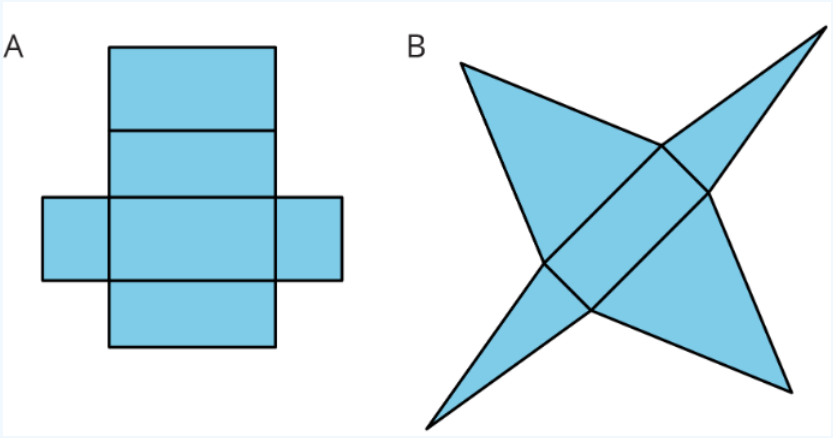


Figure 5.3.13

Exercise 5.3.8

1. What three-dimensional figure can be assembled from the net?

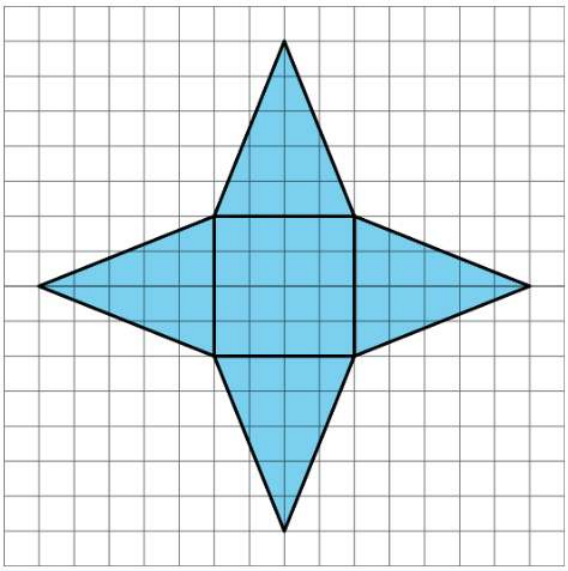


Figure 5.3.14

2. What is the surface area of the figure? (One grid square is 1 square unit.)

(From Unit 1.5.3)

This page titled 5.3: More Nets, More Surface Area is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

5.4: Distinguishing Between Surface Area and Volume

Lesson

Let's contrast surface area and volume.

Exercise 5.4.1: Attributes and Their Measures

For each quantity, choose one or more appropriate units of measurement.

For the last two, think of a quantity that could be appropriately measured with the given units.

Quantities

1. Perimeter of a parking lot:
2. Volume of a semi truck:
3. Surface area of a refrigerator:
4. Length of an eyelash:
5. Area of a state:
6. Volume of an ocean:
7. _____: miles
8. _____: cubic meters

Units

- millimeters (mm)
- feet (ft)
- meters (m)
- square inches (sq in)
- square feet (sq ft)
- square miles (sq mi)
- cubic kilometers (cu km)
- cubic yards (cu yd)

Exercise 5.4.2: Building with 8 Cubes

This applet has 16 cubes in its hidden stack. Build two different shapes using 8 cubes for each.

For each shape, determine the following information and write it on a sticky note.

- Give a name or a label (e.g., Mae's First Shape or Eric's Steps).
- Determine its volume.
- Determine its surface area.

Exercise 5.4.3: Comparing Prisms Without Building Them

Three rectangular prisms each have a height of 1 cm.

- Prism A has a base that is 1 cm by 11 cm.
- Prism B has a base that is 2 cm by 7 cm.
- Prism C has a base that is 3 cm by 5 cm.

1. Find the surface area and volume of each prism. Use the dot paper to draw the prisms, if needed.

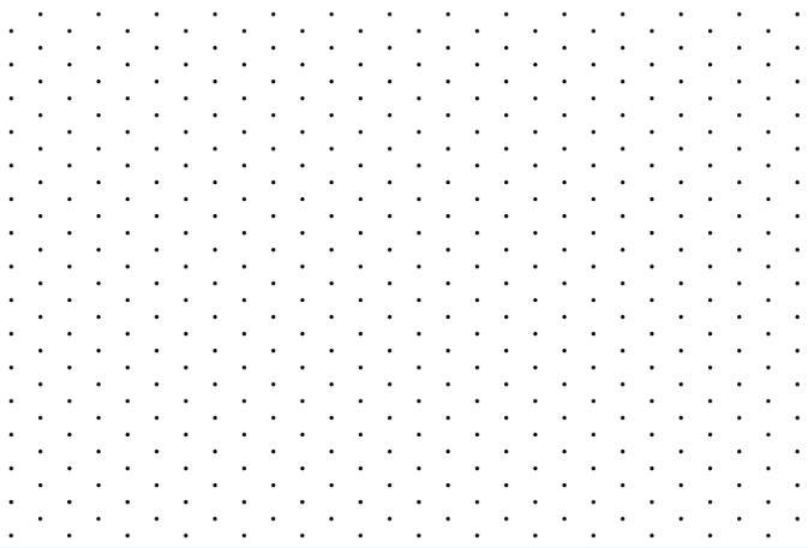


Figure 5.4.1

2. Analyze the volumes and surface areas of the prisms. What do you notice? Write 1 or 2 observations about them.

Are you ready for more?

Can you find more examples of prisms that have the same surface areas but different volumes? How many can you find?

Summary

Length is a one-dimensional attribute of a geometric figure. We measure lengths using units like millimeters, centimeters, meters, kilometers, inches, feet, yards, and miles.

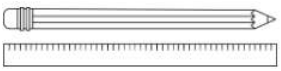


Figure 5.4.2

Area is a two-dimensional attribute. We measure area in square units. For example, a square that is 1 centimeter on each side has an area of 1 square centimeter.

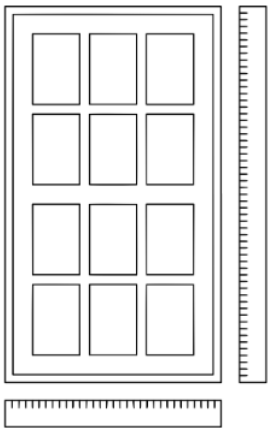


Figure 5.4.3

Volume is a three-dimensional attribute. We measure volume in cubic units. For example, a cube that is 1 kilometer on each side has a volume of 1 cubic kilometer.

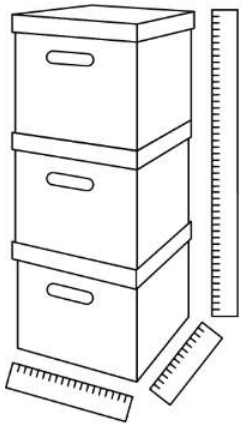


Figure 5.4.4

Surface area and volume are different attributes of three-dimensional figures. Surface area is a two-dimensional measure, while volume is a three-dimensional measure.

Two figures can have the same volume but different surface areas. For example:

- A rectangular prism with side lengths of 1 cm, 2 cm, and 2 cm has a volume of 4 cu cm and a surface area of 16 sq cm.
- A rectangular prism with side lengths of 1 cm, 1 cm, and 4 cm has the same volume but a surface area of 18 sq cm.

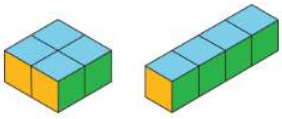


Figure 5.4.5: On left, rectangular prism with side lengths of 1 centimeter, 2 centimeters, and 2 centimeters has a volume of 4 cubic centimeters and a surface area of 16 square centimeters. On right, rectangular prism with side lengths of 1 centimeter, 1 centimeter, and 4 centimeters has the same volume but a surface area of 18 square centimeters.

Similarly, two figures can have the same surface area but different volumes.

- A rectangular prism with side lengths of 1 cm, 1 cm, and 5 cm has a surface area of 22 sq cm and a volume of 5 cu cm.
- A rectangular prism with side lengths of 1 cm, 2 cm, and 3 cm has the same surface area but a volume of 6 cu cm.

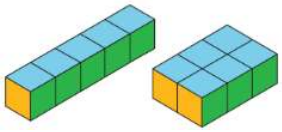


Figure 5.4.6: On left, rectangular prism with side lengths of 1 centimeter, 1 centimeter, and 5 centimeters has a surface area of 22 square centimeters and a volume of 5 cubic centimeters. On right, rectangular prism with side lengths of 1 centimeter, 2 centimeters, and 3 centimeters has the same surface area but a volume of 6 cubic centimeters.

Glossary Entries

Definition: Base (of a Prism or Pyramid)

The word *base* can also refer to a face of a polyhedron.

A prism has two identical bases that are parallel. A pyramid has one base.

A prism or pyramid is named for the shape of its base.

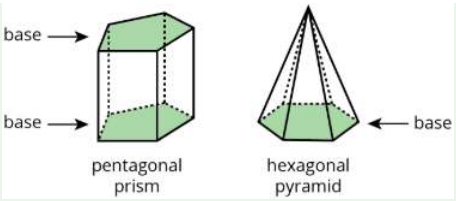


Figure 5.4.7: The figure on the left is labeled pentagonal prism. There are two identical pentagons on the top and bottom. Each vertex of a pentagon is connected by a vertical segment to the corresponding vertex of the other pentagons. The pentagons are each shaded, with the word base pointing to each. The figure on the right is labeled hexagonal pyramid. There is a hexagon on the bottom shaded green. From a point above the hexagon extend 6 segments, each connected to a vertex of the hexagon.

Definition: Face

Each flat side of a polyhedron is called a face. For example, a cube has 6 faces, and they are all squares.

Definition: Net

A net is a two-dimensional figure that can be folded to make a polyhedron.

Here is a net for a cube.

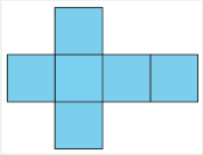


Figure 5.4.9

Definition: Polyhedron

A polyhedron is a closed, three-dimensional shape with flat sides. When we have more than one polyhedron, we call them polyhedra.

Here are some drawings of polyhedra.

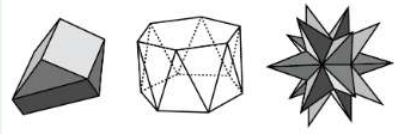


Figure 5.4.10

Definition: Prism

A prism is a type of polyhedron that has two bases that are identical copies of each other. The bases are connected by rectangles or parallelograms.

Here are some drawings of prisms.

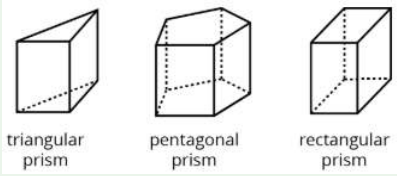


Figure 5.4.11

Definition: Pyramid

A pyramid is a type of polyhedron that has one base. All the other faces are triangles, and they all meet at a single vertex.

Here are some drawings of pyramids.

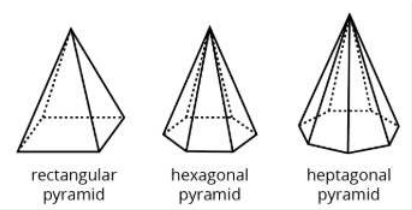


Figure 5.4.12

Definition: Surface Area

The surface area of a polyhedron is the number of square units that covers all the faces of the polyhedron, without any gaps or overlaps.

For example, if the faces of a cube each have an area of 9 cm^2, then the surface area of the cube is $6 \cdot 9$, or 54 cm^2.

Definition: Volume

Volume is the number of cubic units that fill a three-dimensional region, without any gaps or overlaps.

For example, the volume of this rectangular prism is 60 units^3, because it is composed of 3 layers that are each 20 units^3.

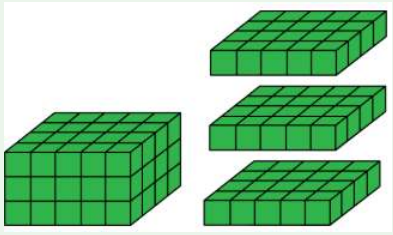


Figure 5.4.13

Practice

Exercise 5.4.4

Match each quantity with an appropriate unit of measurement.

A. The surface area of a tissue box
B. The amount of soil in a planter box
C. The area of a parking lot
D. The length of a soccer field
E. The volume of a fish tank

1. Square meters
2. Yards
3. Cubic inches
4. Cubic feet
5. Square centimeters

Exercise 5.4.5

Here is a figure built from snap cubes.



Figure 5.4.14

1. Find the volume of the figure in cubic units.
2. Find the surface area of the figure in square units.
3. True or false: If we double the number of cubes being stacked, both the volume and surface area will double. Explain or show how you know.

Exercise 5.4.6

Lin said, "Two figures with the same volume also have the same surface area."

1. Which two figures suggest that her statement is true?
2. Which two figures could show that her statement is *not* true?

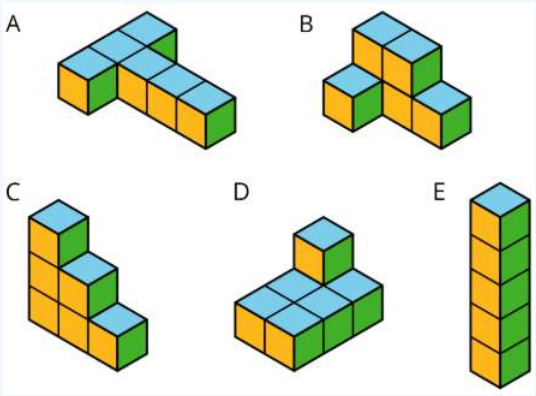


Figure 5.4.15: 5 figures composed of unit cubes, labeled A, B, C, D, E. A is a T shape, 3 cubes across, 3 cubes down. B, 2 cubes stacked on top of 3 cubes, 1 additional cube in front. C, 1 cube stacked on top of 2 cubes stacked on top of 3 cubes. D. 2 by 4 cubes laying flat with 1 additional cube stacked on top. E. 6 cubes stacked on top of one another, 1 by 1 by 6.

Exercise 5.4.7

Draw a pentagon (five-sided polygon) that has an area of 32 square units. Label all relevant sides or segments with their measurements, and show that the area is 32 square units.

(From Unit 1.4.1)

Exercise 5.4.8

1. Draw a net for this rectangular prism.

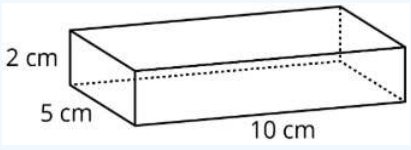


Figure 5.4.16

2. Find the surface area of the rectangular prism.

(From Unit 1.5.4)

This page titled 5.4: Distinguishing Between Surface Area and Volume is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

Section 6: Squares and Cubes

This page titled Section 6: Squares and Cubes is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

6.1: Squares and Cubes

Lesson

Let's investigate perfect squares and perfect cubes.

Exercise 6.1.1: Perfect Squares

1. The number 9 is a perfect **square**. Find four numbers that are perfect squares and two numbers that are not perfect squares.
2. A square has side length 7 in. What is its area?
3. The area of a square is 64 sq cm. What is its side length?

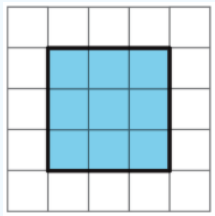


Figure 6.1.1

Exercise 6.1.2: Building with 32 Cubes

Use the 32 snap cubes in the applet's hidden stack to build the largest single cube you can. Each small cube has side length of 1 unit.

1. How many snap cubes did you use?
2. What is the side length of the cube you built?
3. What is the area of each face of the built cube? Show your reasoning.
4. What is the volume of the built cube? Show your reasoning.

Are you ready for more?

This applet has a total of 64 snap cubes. Build the largest single cube you can.

GeoGebra Applet XFx3bD7h

1. How many snap cubes did you use?
2. What is the edge length of the new cube you built?
3. What is the area of each face of this built cube? Show your reasoning.
4. What is the volume of this built cube? Show your reasoning.

Exercise 6.1.3: Perfect Cubes

1. The number 27 is a perfect **cube**. Find four other numbers that are perfect cubes and two numbers that are *not* perfect cubes.
2. A cube has side length 4 cm. What is its volume?
3. A cube has side length 10 inches. What is its volume?
4. A cube has side length units. What is its volume?

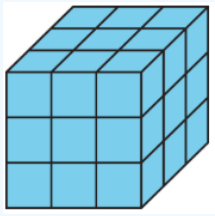


Figure 6.1.2

Exercise 6.1.4: Introducing Exponents

Make sure to include correct units of measure as part of each answer.

1. A square has side length 10 cm. Use an **exponent** to express its area.
2. The area of a square is 7^2 sq in. What is its side length?
3. The area of a square is 81 m^2. Use an exponent to express this area.
4. A cube has edge length 5 in. Use an exponent to express its volume.
5. The volume of a cube is 6^3 cm^3. What is its edge length?
6. A cube has edge length s units. Use an exponent to write an expression for its volume.

Are you ready for more?

The number 15,625 is both a perfect square and a perfect cube. It is a perfect square because it equals 125^2. It is also a perfect cube because it equals 25^3. Find another number that is both a perfect square and a perfect cube. How many of these can you find?

Summary

When we multiply two of the same numbers together, such as $5 \cdot 5$, we say we are **squaring** the number. We can write it like this: 5^2

Because $5 \cdot 5 = 25$, we write $5^2 = 25$ and we say, "5 squared is 25."

When we multiply three of the same numbers together, such as $4 \cdot 4 \cdot 4$, we say we are **cubing** the number. We can write it like this: 4^3

Because $4 \cdot 4 \cdot 4 \cdot = 64$, we write $4^3 = 64$ and we say, "4 cubed is 64."

We also use this notation for square and cubic units.

- A square with side length 5 inches has area 25 in^2.
- A cube with edge length 4 cm has volume 64 cm^3.

To read 25 in^2, we say "25 square inches," just like before.

The area of a square with side length 7 kilometers is 7 km^2. The volume of a cube with edge length 2 millimeters is 2^3 mm^3.

In general, the area of a square with side length s is s^2, and the volume of a cube with edge length s is s^3.

Glossary Entries

Definition: Cubed

We use the word *cubed* to mean "to the third power." This is because a cube with side length s has a volume of $s \cdot s \cdot s$, or s^3.

Definition: Exponent

In expressions like 5^3 and 8^2, the 3 and the 2 are called exponents. They tell you how many factors to multiply. For example, $5^3 = 5 \cdot 5 \cdot 5$, and $8^2 = 8 \cdot 8$.

Definition: Squared

We use the word *squared* to mean "to the second power." This is because a square with side length s has an area of $s \cdot s$, or s^2.

Practice

Exercise 6.1.5

What is the volume of this cube?

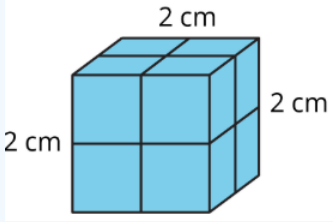


Figure 6.1.3

Exercise 6.1.6

1. Decide if each number on the list is a perfect square.
   - 16
   - 20
   - 25
   - 100
   - 125
   - 144
   - 225
   - 10,000
2. Write a sentence that explains your reasoning.

Exercise 6.1.7

1. Decide if each number on the list is a perfect cube.
   - 1
   - 3
   - 8
   - 9
   - 27
   - 64
   - 100
   - 125
2. Explain what a perfect cube is.

Exercise 6.1.8

1. A square has side length 4 cm. What is its area?
2. The area of a square is 49 m^2. What is its side length?
3. A cube has edge length 3 in. What is its volume?

Exercise 6.1.9

Prism A and Prism B are rectangular prisms.

- Prism A is 3 inches by 2 inches by 1 inch.
- Prism B is 1 inch by 1 inch by 6 inches.

Select **all** statements that are true about the two prisms.

A. They have the same volume.
B. They have the same number of faces.
C. More inch cubes can be packed into Prism A than into Prism B.
D. The two prisms have the same surface area.
E. The surface area of Prism B is greater than that of Prism A.

(From Unit 1.5.5)

Exercise 6.1.10

1. What polyhedron can be assembled from this net?

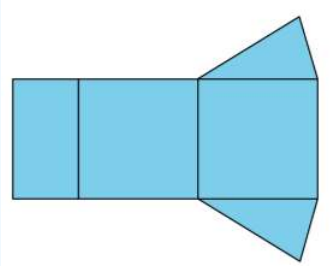


Figure 6.1.4

2. What information would you need to find its surface area? Be specific, and label the diagram as needed.

(From Unit 1.5.3)

Exercise 6.1.11

Find the surface area of this triangular prism. All measurements are in meters.

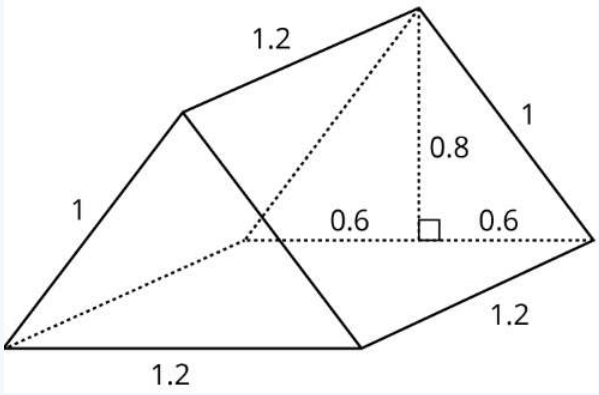


Figure 6.1.5: Triangular prism. front and back triangular faces, base = 1 and 2 tenths, height = 8 tenths. left and right rectangular sides, length = 1 and 2 tenths, width = 1. bottom rectangular, length = 1 and 2 tenths, width = 1 and 2 tenths.

(From Unit 1.5.4)

This page titled 6.1: Squares and Cubes is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

6.2: Surface Area of a Cube

Lesson

Let's write a formula to find the surface area of a cube.

Exercise 6.2.1: Exponent Review

Select the greater expression of each pair without calculating the value of each expression. Be prepared to explain your choices.

- $10 \cdot 3$ or 10^3
- 13^2 or $12 \cdot 12$
- $97+97+97+97+97+97$ or $5 \cdot 97$

Exercise 6.2.2: The Net of a Cube

1. A cube has edge length 5 inches.
   a. Draw a net for this cube, and label its sides with measurements.
   b. What is the shape of each face?
   c. What is the area of each face?
   d. What is the surface area of this cube?
   e. What is the volume of this cube?
2. A second cube has edge length 17 units.
   a. Draw a net for this cube, and label its sides with measurements.
   b. Explain why the area of each face of this cube is 17^2 square units.
   c. Write an expression for the surface area, in square units.
   d. Write an expression for the volume, in cubic units.

Exercise 6.2.3: Every Cube in the Whole World

A cube has edge length s.

1. Draw a net for the cube.
2. Write an expression for the area of each face. Label each face with its area.
3. Write an expression for the surface area.
4. Write an expression for the volume.

Summary

The volume of a cube with edge length s is s^3.

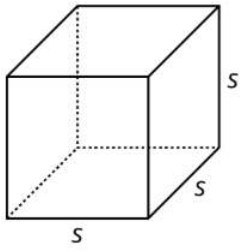


Figure 6.2.1

A cube has 6 faces that are all identical squares. The surface area of a cube with edge length s is $6 \cdot s^2$.

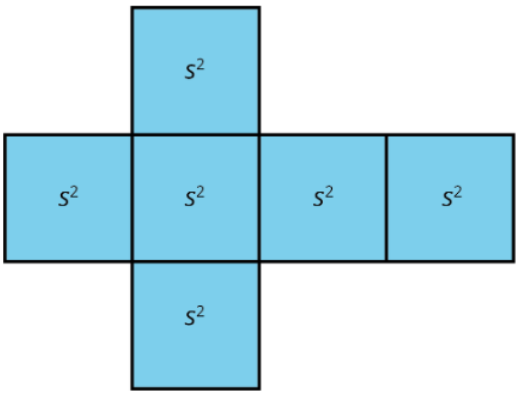


Figure 6.2.2

Glossary Entries

Definition: Cubed

We use the word *cubed* to mean "to the third power." This is because a cube with side length s has a volume of $s \cdot s \cdot s$, or s^3.

Definition: Exponent

In expressions like 5^3 and 8^2, the 3 and the 2 are called exponents. They tell you how many factors to multiply. For example, $5^3 = 5 \cdot 5 \cdot 5$, and $8^2 = 8 \cdot 8$.

Definition: Squared

We use the word *squared* to mean "to the second power." This is because a square with side length s has an area of $s \cdot s$, or s^2.

Practice

Exercise 6.2.4

1. What is the volume of a cube with edge length 8 in?
2. What is the volume of a cube with edge length $\frac{1}{3}$ cm?
3. A cube has a volume of 8 ft^3. What is its edge length?

Exercise 6.2.5

1. What three-dimensional figure can be assembled from this net?

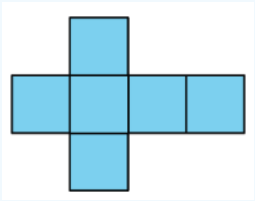


Figure 6.2.3

2. If each square has a side length of 61 cm, write an expression for the surface area and another for the volume of the figure.

Exercise 6.2.6

1. Draw a net for a cube with edge length x cm.
2. What is the surface area of this cube?
3. What is the volume of this cube?

Exercise 6.2.7

Here is a net for a rectangular prism that was not drawn accurately.

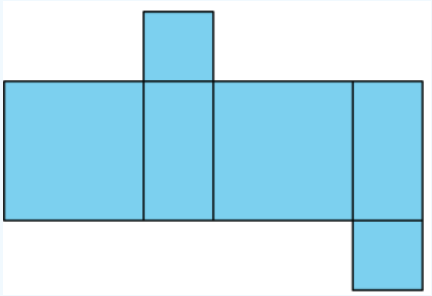


Figure 6.2.4

1. Explain what is wrong with the net.
2. Draw a net that can be assembled into a rectangular prism.
3. Create another net for the same prism.

(From Unit 1.5.3)

Exercise 6.2.8

State whether each figure is a polyhedron. Explain how you know.

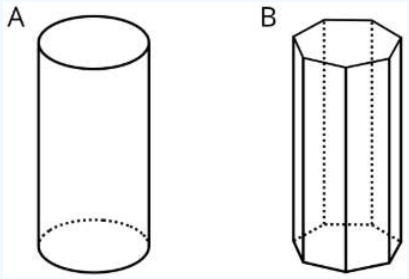


Figure 6.2.5

(From Unit 1.5.2)

Exercise 6.2.9

Here is Elena's work for finding the surface area of a rectangular prism that is 1 foot by 1 foot by 2 feet.

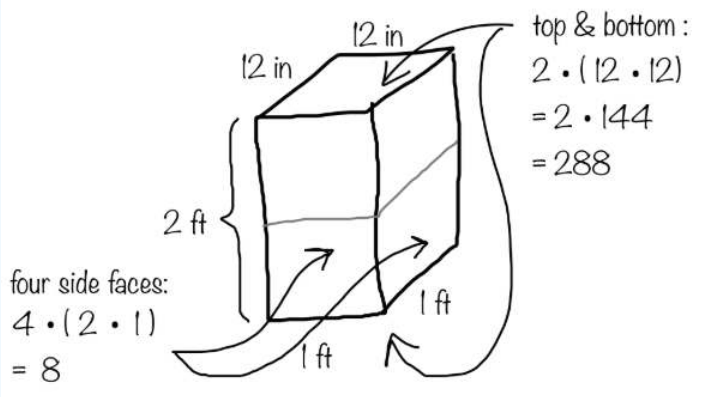


Figure 6.2.6: Rectangular prism. written top & bottom: 2 times 12 times 12 = 2 times 144 = 288. four side faces: 4 times 2 times 1 = 8. top face 12 inches by 12 inches. Bottom face 1 foot by 1 foot. height 2 feet.

She concluded that the surface area of the prism is 296 square feet. Do you agree with her? Explain your reasoning.

(From Unit 1.5.1)

This page titled 6.2: Surface Area of a Cube is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

Section 7: Surface Area

This page was auto-generated because a user created a sub-page to this page.

Section 7: Surface Area is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

7.1: What is Surface Area?

Lesson

Let's cover the surfaces of some three-dimensional objects.

Exercise 7.1.1: Covering the Cabinet (Part 1)

Your teacher will show you a video about a cabinet or some pictures of it.

Estimate an answer to the question: How many sticky notes would it take to cover the cabinet, excluding the bottom?

Exercise 7.1.2: Covering the Cabinet (Part 2)

Earlier, you learned about a cabinet being covered with sticky notes.

1. How could you find the actual number of sticky notes it will take to cover the cabinet, excluding the bottom? What information would you need to know?
2. Use the information you have to find the number of sticky notes to cover the cabinet. Show your reasoning.

Are you ready for more?

How many sticky notes are needed to cover the outside of 2 cabinets pushed together (including the bottom)? What about 3 cabinets? 20 cabinets?

Exercise 7.1.3: Building with Snap Cubes

Here is a rectangular prism built from 12 cubes:

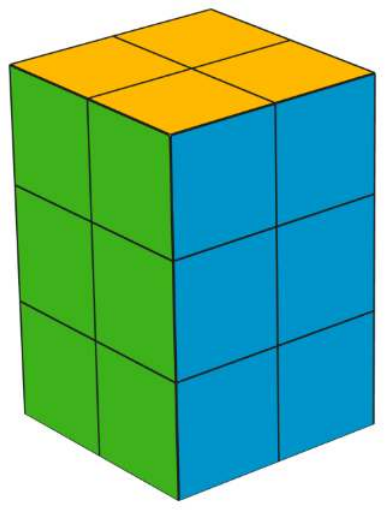


Figure 7.1.1: Rectangular prism. 3 faces shown. face 1, green, side of shape, base = 2 units, height = 3 units. face 2, blue, side of shape, base = 2 units, height = 3 units. face 3, top of shape, square, side length = 2 units.

It has six **faces**, but you can only see three of them in the sketch. It has a **surface area** of 32 square units.

The applet has 12 blocks, too. They are all in the same spot on the screen, like a hidden stack of blocks. You will always know where the stack is because it sits on a gray square. To use a block, drag the red point to move it. Click on the red points to change from left/right movement to up/down movement.

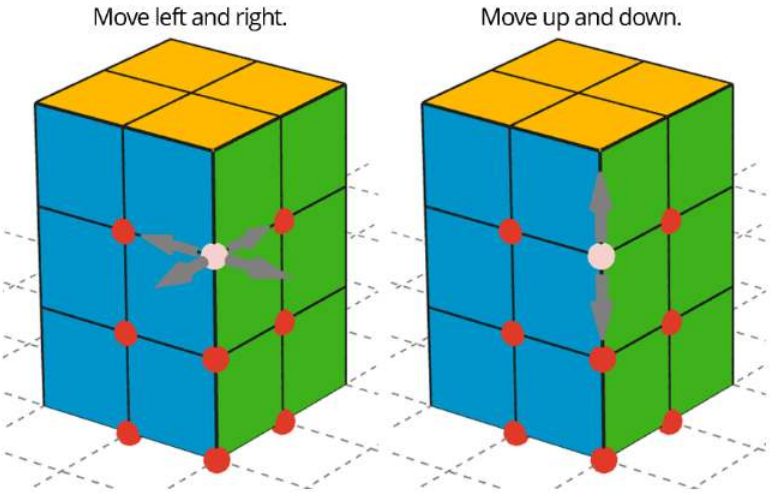


Figure 7.1.2: Two rectangular prisms. Prism on left labeled move left and right, arrows showing left and right movement within the prism. Prism on right labeled move up and down, arrows showing up and down movement within the prism. For each prism, 3 faces shown. face 1, green, side of shape, base = 2 units, height = 3 units. face 2, blue, side of shape, base = 2 units, height = 3 units. face 3, top of shape, square, side length = 2 units.

Use all 12 cubes to build a different rectangular prism (with different edge lengths than shown in the prism here). You can turn the view to see all of the faces of your figure.

1. How many faces does your figure have?
2. What is the surface area of your figure in square units?

Summary

- The **surface area** of a figure (in square units) is the number of unit squares it takes to cover the entire surface without gaps or overlaps.
- If a three-dimensional figure has flat sides, the sides are called **faces**.
- The surface area is the total of the areas of the faces.

For example, a rectangular prism has six faces. The surface area of the prism is the total of the areas of the six rectangular faces.

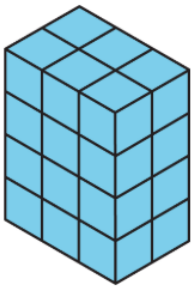


Figure 7.1.3

So the surface area of a rectangular prism that has edge-lengths 2 cm, 3 cm, and 4 cm has a surface area of $(2 \cdot 3) + (2 \cdot 3) + (2 \cdot 4) + (2 \cdot 4) + (3 \cdot 4) + (3 \cdot 4)$ or 52 square centimeters.

Glossary Entries

Definition: Face

Each flat side of a polyhedron is called a face. For example, a cube has 6 faces, and they are all squares.

Definition: Surface Area

The surface area of a polyhedron is the number of square units that covers all the faces of the polyhedron, without any gaps or overlaps.

For example, if the faces of a cube each have an area of 9 cm^2, then the surface area of the cube is $6 \cdot 9$, or 54 cm^2.

Practice

Exercise 7.1.4

What is the surface area of this rectangular prism?

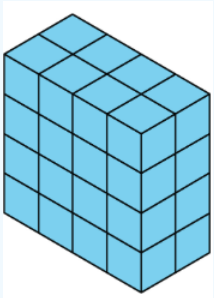


Figure 7.1.4

A. 16 square units
B. 32 square units
C. 48 square units
D. 64 square units

Exercise 7.1.5

Which description can represent the surface area of this trunk?



Figure 7.1.5: Trunk 1, by Wikimedia Commons. Public Domain. Wikimedia Commons. Source.

A. The number of square inches that cover the top of the trunk.
B. The number of square feet that cover all the outside faces of the trunk.
C. The number of square inches of horizontal surface inside the trunk.
D. The number of cubic feet that can be packed inside the trunk.

Exercise 7.1.6

Which figure has a greater surface area?

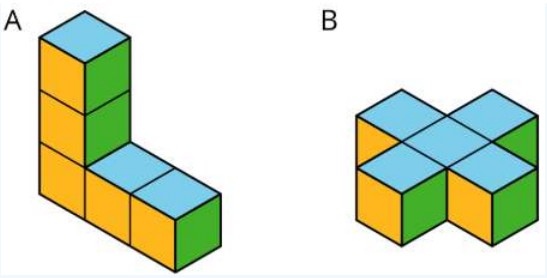


Figure 7.1.6

Exercise 7.1.7

A rectangular prism is 4 units high, 2 units wide, and 6 units long. What is its surface area in square units? Explain or show your reasoning.

Exercise 7.1.8

Draw an example of each of these triangles on the grid.

1. A right triangle with an area of 6 square units.
2. An acute triangle with an area of 6 square units.
3. An obtuse triangle with an area of 6 square units.



Figure 7.1.7

(From Unit 1.3.3)

Exercise 7.1.9

Find the area of triangle MOQ in square units. Show your reasoning.

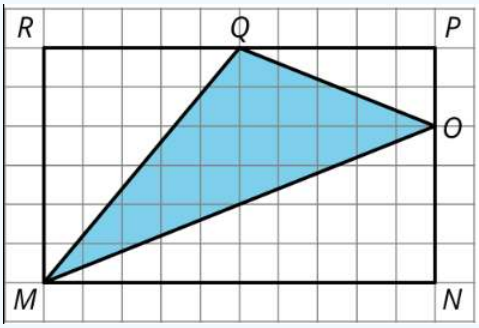


Figure 7.1.8: Triangle MOQ is enclosed in rectangle MNPR. Sides RM and PN of the rectangle are 6 units high and sides RP and MN are 10 units long. Vertex Q is on side RP, 5 units from R and vertex O is on side PN, 2 units down from P.

(From Unit 1.3.4)

https://math.libretexts.org/@go/page/39647

Exercise 7.1.10

Find the area of this shape. Show your reasoning.

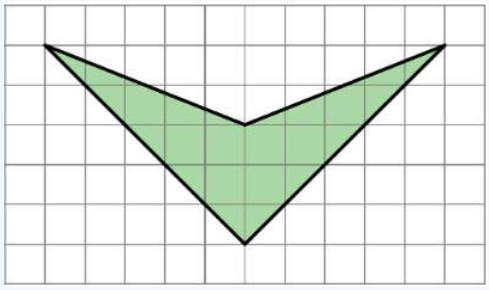


Figure 7.1.9

(From Unit 1.1.3)

This page titled 7.1: What is Surface Area? is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

Section 8: Let’s Put it to Work

This page titled Section 8: Let’s Put it to Work is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

8.1: Designing a Tent

Lesson

Let's design some tents.

Exercise 8.1.1: Tent Design - Part 1

Have you ever been camping?

You might know that sleeping bags are all about the same size, but tents come in a variety of shapes and sizes.

Your task is to design a tent to accommodate up to four people, and estimate the amount of fabric needed to make your tent. Your design and estimate must be based on the information given and have mathematical justification.

First, look at these examples of tents, the average specifications of a camping tent, and standard sleeping bag measurements. Talk to a partner about:

- Similarities and differences among the tents
- Information that will be important in your designing process
- The pros and cons of the various designs

Tent Styles

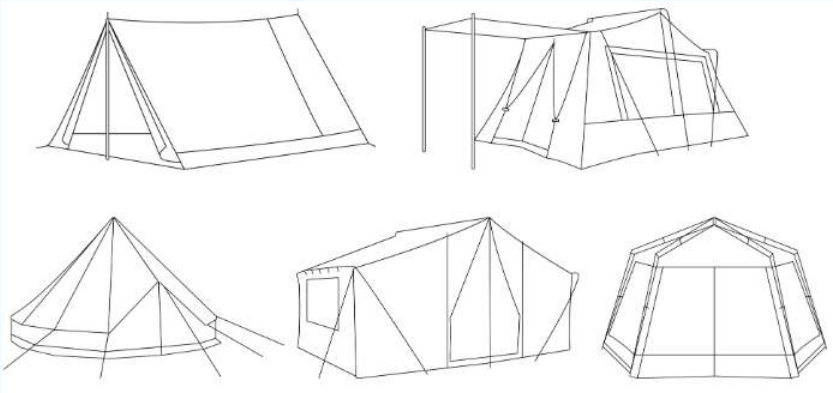


Figure 8.1.1

Tent Height Specifications

Table 8.1.1

height description	height of tent	notes
sitting height	3 feet	Campers are able to sit, lie, or crawl inside tent.
kneeling height	4 feet	Campers are able to kneel inside tent. Found mainly in 3–4 person tents.
stooping height	5 feet	Campers are able to move around on their feet inside tent, but most campers will not be able to stand upright.
standing height	6 feet	Most adult campers are able to stand upright inside tent.
roaming height	7 feet	Adult campers are able to stand upright and walk around inside tent.

Sleeping Bag Measurements

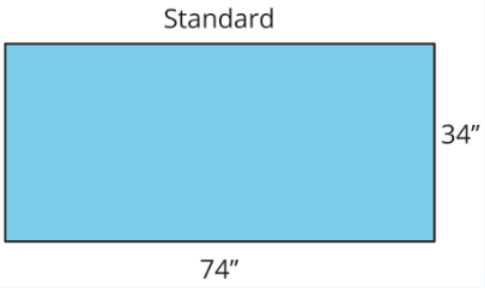


Figure 8.1.2

1. Create and sketch your tent design. The tent must include a floor.
2. What decisions were important when choosing your tent design?
3. How much fabric do you estimate will be necessary to make your tent? Show your reasoning and provide mathematical justification.

Exercise 8.1.2: Tent Design - Part 2

1. Explain your tent design and fabric estimate to your partner or partners. Be sure to explain why you chose this design and how you found your fabric estimate.
2. Compare the estimated fabric necessary for each tent in your group. Discuss the following questions:
   - Which tent design used the least fabric? Why?
   - Which tent design used the most fabric? Why?
   - Which change in design most impacted the amount of fabric needed for the tent? Why?

This page titled 8.1: Designing a Tent is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

SECTION OVERVIEW

2: Introducing Ratios

Section 9: What are Ratios?

9.1: Introducing Ratios and Ratio Language
9.2: Representing Ratios with Diagrams

Section 10: Equivalent Ratios

10.1: Recipes
10.2: Color Mixtures
10.3: Defining Equivalent Ratios

Section 11: Representing Equivalent Ratios

11.1: Introducing Double Number Line Diagrams
11.2: Creating Double Number Line Diagrams
11.3: Constant Speed
11.4: Comparing Situations by Examining Ratios

Section 12: Solving Ratio and Rate Problems

12.1: Representing Ratios with Tables
12.2: Navigating a Table of Equivalent Ratios
12.3: Tables and Double Number Line Diagrams
12.4: Solving Equivalent Ratio Problems

Section 13: Part-Part-Whole Ratios

13.1: Part-Part-Whole Ratios
13.2: Solving More Ratio Problems

Section 14: Let's Put it to Work

14.1: A Fermi Problem
14.2: How Much for One?

This page titled 2: Introducing Ratios is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

Section 9: What are Ratios?

This page titled Section 9: What are Ratios? is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

9.1: Introducing Ratios and Ratio Language

Lesson

Let's describe two quantities at the same time.

Exercise 9.1.1: What Kind and How Many?

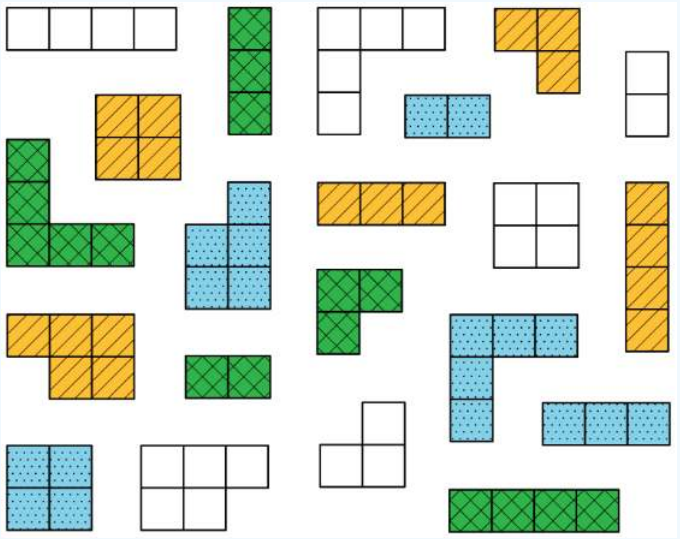


Figure 9.1.1

Think of different ways you could sort these figures. What categories could you use? How many groups would you have?

Exercise 9.1.2: The Teacher's Collection

1. Think of a way to sort your teacher's collection into two or three categories. Count the items in each category, and record the information in the table.

Table 9.1.1

category name			
category amount			

Pause here so your teacher can review your work.

2. Write at least two sentences that describe **ratios** in the collection. Remember, there are many ways to write a ratio:
   - The ratio of *one category* to *another category* is _____ to _____.
   - The ratio of *one category* to *another category* is _____ : _____.
   - There are _____ of *one category* for every _____ of *another category*.

Exercise 9.1.3: The Student's Collection

1. Sort your collection into three categories. You can experiment with different ways of arranging these categories. Then, count the items in each category, and record the information in the table.

Table 9.1.2

category name			
category amount			

2. Write at least two sentences that describe **ratios** in the collection. Remember, there are many ways to write a ratio:

- The ratio of *one category* to *another category* is _____ to _____.
- The ratio of *one category* to *another category* is _____ : _____.
- There are _____ of *one category* for every _____ of *another category*.

Pause here so your teacher can review your sentences.

3. Make a visual display of your items that clearly shows one of your statements. Be prepared to share your display with the class.

Are you ready for more?

1. Use two colors to shade the rectangle so there are 2 square units of one color for every 1 square unit of the other color.
2. The rectangle you just colored has an area of 24 square units. Draw a different shape that does *not* have an area of 24 square units, but that can also be shaded with two colors in a $2 : 1$ ratio. Shade your new shape using two colors.

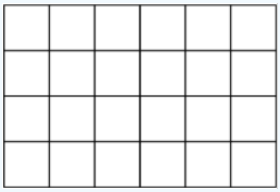


Figure 9.1.2

Summary

A **ratio** is an association between two or more quantities. There are many ways to describe a situation in terms of ratios. For example, look at this collection:

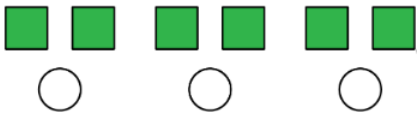


Figure 9.1.3

Here are some correct ways to describe the collection:

- The ratio of squares to circles is $6 : 3$.
- The ratio of circles to squares is 3 to 6.

Notice that the shapes can be arranged in equal groups, which allow us to describe the shapes using other numbers.

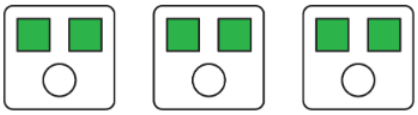


Figure 9.1.4

- There are 2 squares for every 1 circle.
- There is 1 circle for every 2 squares.

Glossary Entries

Definition: Ratio

A ratio is an association between two or more quantities.

For example, the ratio $3 : 2$ could describe a recipe that uses 3 cups of flour for every 2 eggs, or a boat that moves 3 meters every 2 seconds. One way to represent the ratio $3 : 2$ is with a diagram that has 3 blue squares for every 2 green squares.

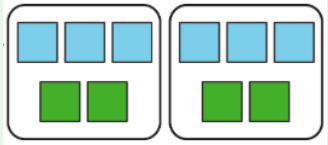


Figure 9.1.5

Practice

Exercise 9.1.4

In a fruit basket there are 9 bananas, 4 apples, and 3 plums.

1. The ratio of bananas to apples is _____ : _____.
2. The ratio of plums to apples is _____ to _____.
3. For every _____ apples, there are _____ plums.
4. For every 3 bananas there is one _____.

Exercise 9.1.5

Complete the sentences to describe this picture.



Figure 9.1.6

1. The ratio of dogs to cats is _____.
2. For every _____ dogs, there are _____ cats.

Exercise 9.1.6

Write two different sentences that use ratios to describe the number of eyes and legs in this picture.

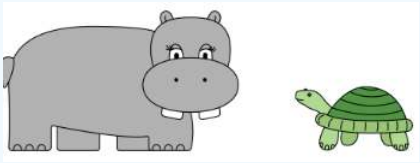


Figure 9.1.7

Exercise 9.1.7

Choose an appropriate unit of measurement for each quantity.

1. area of a rectangle
2. volume of a prism
3. side of a square
4. area of a square
5. volume of a cube

- cm
- cm^3
- cm^2

(From Unit 1.6.1)

Exercise 9.1.8

Find the volume and surface area of each prism.

1. Prism A: 3 cm by 3 cm by 3 cm



Figure 9.1.8

2. Prism B: 5 cm by 5 cm by 1 cm



Figure 9.1.9

3. Compare the volumes of the prisms and then their surface areas. Does the prism with the greater volume also have the greater surface area?

(From Unit 1.5.5)

Exercise 9.1.9

Which figure is a triangular prism? Select **all** that apply.

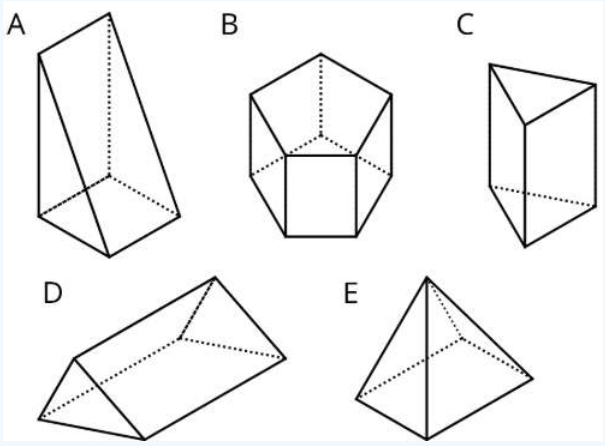


Figure 9.1.10: Five shapes. Shape A has a square flat bottom and 2 triangular sides connected by rectangular sides. Shape B has a hexagon flat bottom, a hexagon flat top, and five flat rectangular sides. Shape C has a triangle flat bottom, a triangle flat top, and three rectangular sides. Shape D has a rectangle flat bottom and two sides shaped like a triangle connected by rectangular sides. Shape E has a rectangle flat bottom and four triangle sides that connect at a top point.

A. A
B. B
C. C
D. D
E. E

(From Unit 1.5.2)

This page titled 9.1: Introducing Ratios and Ratio Language is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

9.2: Representing Ratios with Diagrams

Lesson

Let's use diagrams to represent ratios.

Exercise 9.2.1: Number Talk: Dividing 4 and Multiplying by $\frac{1}{4}$

Find the value of each expression mentally.

1. $24 \div 4$
2. $\frac{1}{4} \cdot 24$
3. $24 \cdot \frac{1}{4}$
4. $5 \div 4$

Exercise 9.2.2: A Collection of Snap Cubes

Here is a collection of snap cubes.



Figure 9.2.1

1. Choose two of the colors in the image, and draw a diagram showing the number of snap cubes for these two colors.
2. Trade papers with a partner. On their paper, write a sentence to describe a ratio shown in their diagram. Your partner will do the same for your diagram.
3. Return your partner's paper. Read the sentence written on your paper. If you disagree, explain your thinking.

Exercise 9.2.3: Blue Paint and Art Paste

Elena mixed 2 cups of white paint with 6 tablespoons of blue paint.

Here is a diagram that represents this situation.

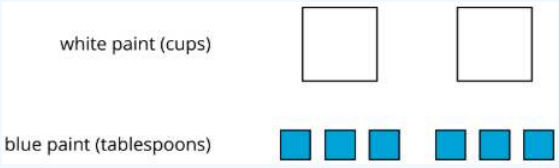


Figure 9.2.2: A discrete diagram of squares that represent the amount of paint. The top row is labeled white paint, in cups and contains 2 large squares. The bottom row is labeled blue paint, in tablespoons and contains 6 small squares.

1. Discuss each statement, and circle **all** those that correctly describe this situation. Make sure that both you and your partner agree with each circled answer.
   a. The ratio of cups of white paint to tablespoons of blue paint is $2 : 6$.
   b. For every cup of white paint, there are 2 tablespoons of blue paint.
   c. There is 1 cup of white paint for every 3 tablespoons of blue paint.
   d. There are 3 tablespoons of blue paint for every cup of white paint.
   e. For each tablespoon of blue paint, there are 3 cups of white paint.
   f. For every 6 tablespoons of blue paint, there are 2 cups of white paint.
   g. The ratio of tablespoons of blue paint to cups of white paint is 6 to 2.

2. Jada mixed 8 cups of flour with 2 pints of water to make paste for an art project.
    a. Draw a diagram that represents the situation.
    b. Write at least two sentences describing the ratio of flour and water.

Exercise 9.2.4: Card Sort: Spaghetti Sauce

Your teacher will give you cards describing different recipes for spaghetti sauce. In the diagrams:

- a circle represents a cup of tomato sauce
- a square represents a tablespoon of oil
- a triangle represents a teaspoon of oregano



Figure 9.2.3: Spaghetti Sauce, by eat quiche. CC BY 2.0. Flickr. Source.

1. Take turns with your partner to match a sentence with a diagram.
    a. For each match that you find, explain to your partner how you know it's a match.
    b. For each match that your partner finds, listen carefully to their explanation. If you disagree, discuss your thinking and work to reach an agreement.
2. After you and your partner have agreed on all of the matches, check your answers with the answer key. If there are any errors, discuss why and revise your matches.
3. There were two diagrams that each matched with two different sentences. Which were they?
    - Diagram _____ matched with both sentences _____ and _____.
    - Diagram _____ matched with both sentences _____ and _____.
4. Select one of the other diagrams and invent another sentence that could describe the ratio shown in the diagram.

Are you ready for more?

Create a diagram that represents any of the ratios in a recipe of your choice. Is it possible to include more than 2 ingredients in your diagram?

Summary

Ratios can be represented using diagrams. The diagrams do not need to include realistic details. For example, a recipe for lemonade says, "Mix 2 scoops of lemonade powder with 6 cups of water."

Instead of this:

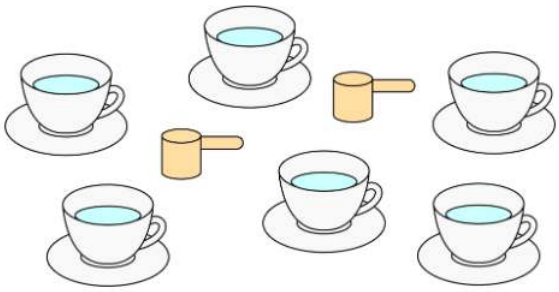


Figure 9.2.4

We can draw something like this:

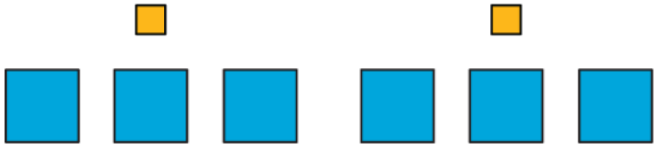


Figure 9.2.5

This diagram shows that the ratio of cups of water to scoops of lemonade powder is 6 to 2. We can also see that for every scoop of lemonade powder, there are 3 cups of water.

Glossary Entries

Definition: Ratio

A ratio is an association between two or more quantities.

For example, the ratio $3 : 2$ could describe a recipe that uses 3 cups of flour for every 2 eggs, or a boat that moves 3 meters every 2 seconds. One way to represent the ratio $3 : 2$ is with a diagram that has 3 blue squares for every 2 green squares.

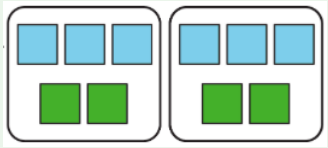


Figure 9.2.6

Practice

Exercise 9.2.5

Here is a diagram that describes the cups of green and white paint in a mixture.

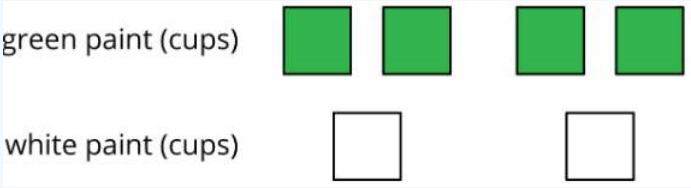


Figure 9.2.7: A diagram of squares that represent the number of cups of paint. The top row is labeled green paint, in cups and contains 4 green squares. The bottom row is labeled white paint, in cups and contains 2 white squares.

Select **all** the statements that correctly describe this diagram.

A. The ratio of cups of white paint to cups of green paint is 2 to 4.
B. For every cup of green paint, there are two cups of white paint.
C. The ratio of cups of green paint to cups of white paint is $4 : 2$.
D. For every cup of white paint, there are two cups of green paint.
E. The ratio of cups of green paint to cups of white paint is $2 : 4$.

Exercise 9.2.6

To make a snack mix, combine 2 cups of raisins with 4 cups of pretzels and 6 cups of almonds.

1. Create a diagram to represent the quantities of each ingredient in this recipe.
2. Use your diagram to complete each sentence.
   - The ratio of _____ to _____ to _____ is _____ : _____ : _____.
   - There are _____ cups of pretzels for every cup of raisins.
   - There are _____ cups of almonds for every cup of raisins.

Exercise 9.2.7

1. A square is 3 inches by 3 inches. What is its area?
2. A square has a side length of 5 feet. What is its area?
3. The area of a square is 36 square centimeters. What is the length of each side of the square?

(From Unit 1.6.1)

Exercise 9.2.8

Find the area of this quadrilateral. Explain or show your strategy.

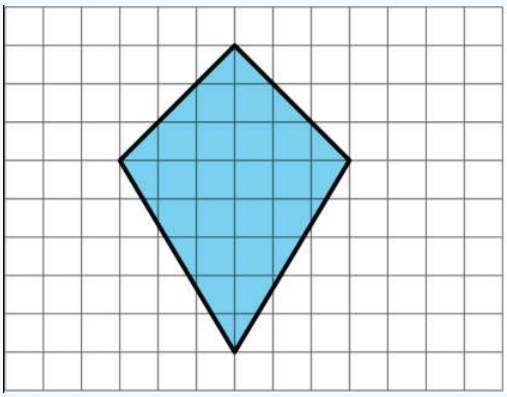


Figure 9.2.8

(From Unit 1.4.1)

Exercise 9.2.9

Complete each equation with a number that makes it true.

1. $\frac{1}{8} \cdot 8 =$ ____
2. $\frac{3}{8} \cdot 8 =$ ____

1. $\frac{1}{8} \cdot 7 =$ ____
2. $\frac{3}{8} \cdot 7 =$ ____

(From Unit 2.1.1)

This page titled 9.2: Representing Ratios with Diagrams is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

Section 10: Equivalent Ratios

This page titled Section 10: Equivalent Ratios is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

10.1: Recipes

Lesson

Let's explore how ratios affect the way a recipe tastes.

Exercise 10.1.1: Flower Pattern

This flower is made up of yellow hexagons, red trapezoids, and green triangles.

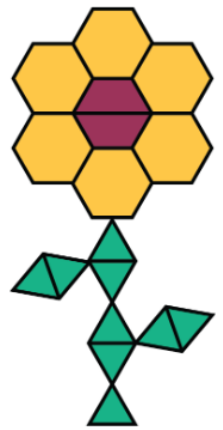


Figure 10.1.1

1. Write sentences to describe the ratios of the shapes that make up this pattern.
2. How many of each shape would be in two copies of this flower pattern?

Exercise 10.1.2: Powdered Drink Mix

Here are diagrams representing three mixtures of powdered drink mix and water:

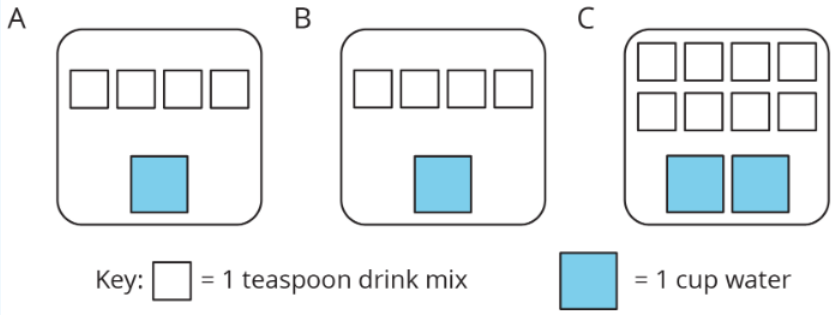


Figure 10.1.2: A discrete diagram for three quantities, labeled A, B, and C. Diagram A has 4 white squares and 1 blue square. Diagram B has 4 white squares and 1 blue square. Diagram C has 8 white squares and 2 blue squares. Below diagram Key shows 1 white square equals 1 teaspoon drink mix and 1 blue square equals 1 cup water.

1. How would the taste of Mixture A compare to the taste of Mixture B?
2. Use the diagrams to complete each statement:
   a. Mixture B uses _____ cups of water and _____ teaspoons of drink mix. The ratio of cups of water to teaspoons of drink mix in Mixture B is _____.
   b. Mixture C uses _____ cups of water and _____ teaspoons of drink mix. The ratio of cups of water to teaspoons of drink mix in Mixture C is _____.
3. How would the taste of Mixture B compare to the taste of Mixture C?

Are you ready for more?

Sports drinks use sodium (better known as salt) to help people replenish electrolytes. Here are the nutrition labels of two sports drinks.

A

Nutrition Facts

Serving Size 8 fl oz (240 mL)
Serving Per Container 4

Amount Per Serving
Calories 50

		% Daily Value*
Total Fat	0 g	0%
Sodium	110 mg	5%
Potassium	30 mg	1%
Total Carbohydrate	14 g	5%
Sugars	14 g	
Protein	0 g	

% Daily Value are based on a 2,000 calorie diet.

B

Nutrition Facts

Serving Size 12 fl oz (355 mL)
Serving Per Container about 2.5

Amount Per Serving
Calories 80

		% Daily Value*
Total Fat	0 g	0%
Sodium	150 mg	6%
Potassium	35 mg	1%
Total Carbohydrate	21 g	7%
Sugars	20 g	
Protein	0 g	

% Daily Value are based on a 2,000 calorie diet.

Figure 10.1.3: Two nutrition fact labels, A and B. Drink A: serving size 8 fluid ounces. Servings per container, 4. Amount per serving: calories, 50. Total Fat, 0 grams; Percent Daily Value, 0. Sodium, 110 milligrams; Percent Daily Value, 5. Potassium, 130 milligrams; Percent Daily Value, 1. Total Carbohydrate, 14 grams; Percent Daily Value, 5. Sugars, 14 grams. Protein, 0 grams. Percent Daily Values are based on a 2,000 calorie diet. Drink B: serving size 12 fluid ounces. Servings per container, 2.5. Amount per serving: calories, 80. Total Fat, 0 grams; Percent Daily Value, 0. Sodium, 150 milligrams; Percent Daily Value, 6. Potassium, 35 milligrams; Percent Daily Value, 1. Total Carbohydrate, 21 grams; Percent Daily Value, 7. Sugars, 20 grams. Protein, 0 grams. Percent Daily Values are based on a 2,000 calorie diet.

1. Which of these drinks is saltier? Explain how you know.
2. If you wanted to make sure a sports drink was less salty than both of the ones given, what ratio of sodium to water would you use?

Exercise 10.1.3: Batches of Cookies

A recipe for one batch of cookies calls for 5 cups of flour and 2 teaspoons of vanilla.

1. Draw a diagram that shows the amount of flour and vanilla needed for *two* batches of cookies.
2. How many batches can you make with 15 cups of flour and 6 teaspoons of vanilla? Show the additional batches by adding more ingredients to your diagram.
3. How much flour and vanilla would you need for 5 batches of cookies?
4. Whether the ratio of cups of flour to teaspoons of vanilla is $5 : 2$, $10 : 4$, or $15 : 6$, the recipes would make cookies that taste the same. We call these **equivalent ratios**.
   a. Find another ratio of cups of flour to teaspoons of vanilla that is equivalent to these ratios.
   b. How many batches can you make using this new ratio of ingredients?

Summary

A recipe for fizzy juice says, “Mix 5 cups of cranberry juice with 2 cups of soda water.”

To double this recipe, we would use 10 cups of cranberry juice with 4 cups of soda water. To triple this recipe, we would use 15 cups of cranberry juice with 6 cups of soda water.

This diagram shows a single batch of the recipe, a double batch, and a triple batch:

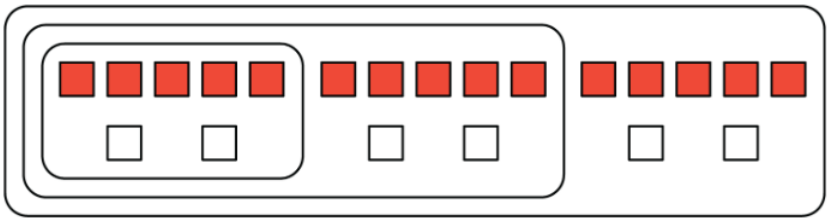


Figure 10.1.4: Equivalent ratios of a recipe for fizzy juice using cranberry juice and soda water. One batch equals 5 cups cranberry juice and 2 cups soda water. Two batches equal 10 cups cranberry juice and 4 cups soda water. Three batches equal 15 cups cranberry juice and 6 cups soda water.

We say that the ratios $5 : 2$, $10 : 4$, and $15 : 6$ are **equivalent**. Even though the amounts of each ingredient within a single, double, or triple batch are not the same, they would make fizzy juice that tastes the same.

Practice

Exercise 10.1.4

A recipe for 1 batch of spice mix says, “Combine 3 teaspoons of mustard seeds, 5 teaspoons of chili powder, and 1 teaspoon of salt.” How many batches are represented by the diagram? Explain or show your reasoning.

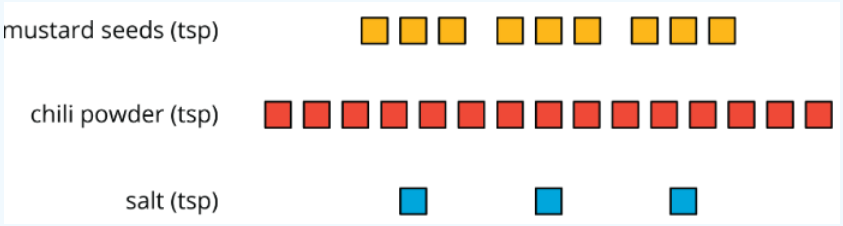


Figure 10.1.5: A discrete diagram for three quantities labeled mustard seeds, in teaspoons, chili powder, in teaspoons, and salt, in teaspoons. The data are as follows: mustard seeds, 9 yellow squares. chili powder, 15 red squares. salt, 3 blue squares.

Exercise 10.1.5

Priya makes chocolate milk by mixing 2 cups of milk and 5 tablespoons of cocoa powder. Draw a diagram that clearly represents two batches of her chocolate milk.

Exercise 10.1.6

In a recipe for fizzy grape juice, the ratio of cups of sparkling water to cups of grape juice concentrate is 3 to 1.

1. Find two more ratios of cups of sparkling water to cups of juice concentrate that would make a mixture that tastes the same as this recipe.
2. Describe another mixture of sparkling water and grape juice that would taste different than this recipe.

Exercise 10.1.7

Write the missing number under each tick mark on the number line.

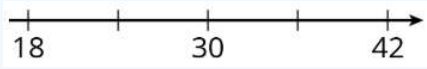


Figure 10.1.6

(From Unit 2.1.1)

Exercise 10.1.8

At the kennel, there are 6 dogs for every 5 cats.

1. The ratio of dogs to cats is _____ to _____.
2. The ratio of cats to dogs is _____ to _____.
3. For every _____ dogs there are _____ cats.
4. The ratio of cats to dogs is _____ : _____.

(From Unit 2.1.1)

Exercise 10.1.9

Elena has 80 unit cubes. What is the volume of the largest cube she can build with them?

(From Unit 1.6.1)

Exercise 10.1.10

Fill in the blanks to make each equation true.

1. $3 \cdot \frac{1}{3} =$ _____
2. $10 \cdot \frac{1}{10} =$ _____
3. $19 \cdot \frac{1}{19} =$ _____
4. $a \cdot \frac{1}{a} =$ _____

(As long as a does not equal 0.)

1. $5 \cdot$ _____ $= 1$
2. $17 \cdot$ _____ $= 1$
3. $b \cdot$ _____ $= 1$

(From Unit 2.1.1)

This page titled 10.1: Recipes is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

10.2: Color Mixtures

Lesson

Let's see what color-mixing has to do with ratios.

Exercise 10.2.1: Number Talk: Adjusting a Factor

Find the value of each product mentally.

$6 \cdot 15$

$12 \cdot 15$

$6 \cdot 45$

$13 \cdot 45$

Exercise 10.2.2: Turning Green

1. In the left cylinder, mix 5 ml of blue and 15 ml of yellow. This is a *single batch* of green.
2. Suppose you have one batch of green but want to make more. Which of the following would produce the same shade of green?
   If you're unsure, try creating the mixture in the right cylinder. Start with the amounts in a single batch (5 ml of blue and 15 ml of yellow) and . . .
   a. add 20 ml of blue and 20 ml of yellow
   b. double the amount of blue and the amount of yellow
   c. triple the amount of blue and the amount of yellow
   d. mix a single batch with a double batch
   e. double the amount of blue and triple the amount of yellow
3. For one of the mixtures that produces the same shade, write down the number of ml of blue and yellow used in the mixture.
4. For the same mixture that produces the same shade, draw a diagram of the mixture. Make sure your diagram shows the number of milliliters of blue, yellow, and the number of batches.
5. Someone was trying to make the same shade as the original single batch, but started by adding 20 ml of blue and 20 ml of yellow. How can they add more but still create the same shade of green?
6. Invent a recipe for a *bluer* shade of green. Write down the amounts of yellow and blue that you used, and draw a diagram. Explain how you know it will be bluer than the original single batch of green before testing it out.

Are you ready for more?

Someone has made a shade of green by using 17 ml of blue and 13 ml of yellow. They are sure it cannot be turned into the original shade of green by adding more blue or yellow. Either explain how more can be added to create the original green shade, or explain why this is impossible.

Exercise 10.2.3: Perfect Purple Water

The recipe for Perfect Purple Water says, “Mix 8 ml of blue water with 3 ml of red water.”

Jada mixes 24 ml of blue water with 9 ml of red water. Andre mixes 16 ml of blue water with 9 ml of red water.

1. Which person will get a color mixture that is the same shade as Perfect Purple Water? Explain or show your reasoning.
2. Find another combination of blue water and red water that will also result in the same shade as Perfect Purple Water. Explain or show your reasoning.

Summary

When mixing colors, doubling or tripling the amount of each color will create the same shade of the mixed color. In fact, you can always multiply the amount of *each* color by *the same number* to create a different amount of the same mixed color.

For example, a batch of dark orange paint uses 4 ml of red paint and 2 ml of yellow paint.

- To make two batches of dark orange paint, we can mix 8 ml of red paint with 4 ml of yellow paint.
- To make three batches of dark orange paint, we can mix 12 ml of red paint with 6 ml of yellow paint.

Here is a diagram that represents 1, 2, and 3 batches of this recipe.

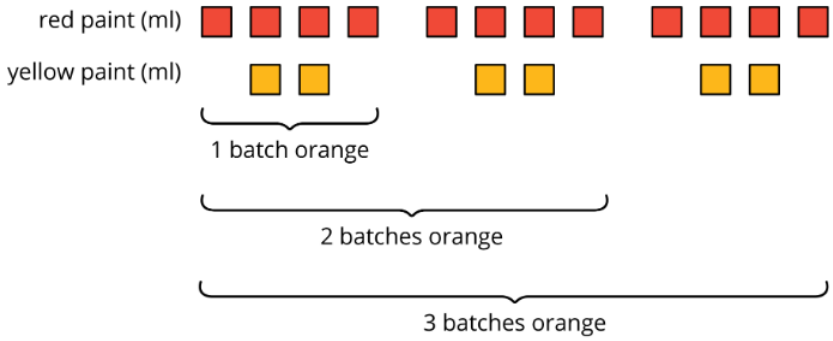


Figure 10.2.1: A discrete diagram that contain red and yellow squares. The top row is labeled red paint, in ml and contain 12 red squares that are grouped by 4s. The bottom row is labeled yellow paint, in ml and contain 6 yellow squares grouped by 2s. There is a brace that contains 4 red squares and 2 yellow squares labeled 1 batch orange. A second brace contains 8 red squares and 4 yellow squares labeled 2 batches orange. A third brace contains 12 red squares and 6 yellow squares labeled 3 batches orange.

We say that the ratios $4:2$, $8:4$, and $12:6$ are *equivalent* because they describe the same color mixture in different numbers of batches, and they make the same shade of orange.

Practice

Exercise 10.2.4

Here is a diagram showing a mixture of red paint and green paint needed for 1 batch of a particular shade of brown.

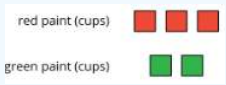


Figure 10.2.2

Add to the diagram so that it shows 3 batches of the same shade of brown paint.

Exercise 10.2.5

Diego makes green paint by mixing 10 tablespoons of yellow paint and 2 tablespoons of blue paint. Which of these mixtures produce the same shade of green paint as Diego's mixture? Select **all** that apply.

A. For every 5 tablespoons of blue paint, mix in 1 tablespoon of yellow paint.
B. Mix tablespoons of blue paint and yellow paint in the ratio $1:5$.
C. Mix tablespoons of yellow paint and blue paint in the ratio 15 to 3.
D. Mix 11 tablespoons of yellow paint and 3 tablespoons of blue paint.
E. For every tablespoon of blue paint, mix in 5 tablespoons of yellow paint.

Exercise 10.2.6

To make 1 batch of sky blue paint, Clare mixes 2 cups of blue paint with 1 gallon of white paint.

1. Explain how Clare can make 2 batches of sky blue paint.
2. Explain how to make a mixture that is a darker shade of blue than the sky blue.
3. Explain how to make a mixture that is a lighter shade of blue than the sky blue.

Exercise 10.2.7

A smoothie recipe calls for 3 cups of milk, 2 frozen bananas and 1 tablespoon of chocolate syrup.

1. Create a diagram to represent the quantities of each ingredient in the recipe.
2. Write 3 different sentences that use a ratio to describe the recipe.

(From Unit 2.1.2)

Exercise 10.2.8

Write the missing number under each tick mark on the number line.

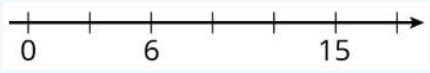


Figure 10.2.3

(From Unit 2.1.1)

Exercise 10.2.9

Find the area of the parallelogram. Show your reasoning.

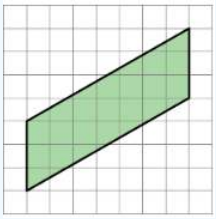


Figure 10.2.4

(From Unit 1.2.1)

Exercise 10.2.10

Complete each equation with a number that makes it true.

1. $11 \cdot \frac{1}{4} =$ ____
2. $7 \cdot \frac{1}{4} =$ ____
3. $13 \cdot \frac{1}{27} =$ ____

1. $13 \cdot \frac{1}{99} =$ ____
2. $x \cdot \frac{1}{y} =$ ____

(As long as y does not equal 0.)

(From Unit 2.1.1)

This page titled 10.2: Color Mixtures is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

10.3: Defining Equivalent Ratios

Lesson

Let's investigate equivalent ratios some more.

Exercise 10.3.1: Dots and Half Dots

Dot Pattern 1:



Figure 10.3.1

Dot Pattern 2:

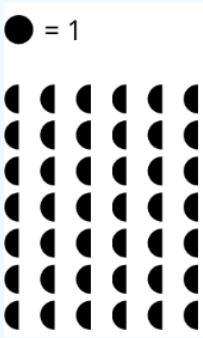


Figure 10.3.2

Exercise 10.3.2: Tuna Casserole

Here is a recipe for tuna casserole.

Ingredients

- 3 cups cooked elbow-shaped pasta
- 6 ounce can tuna, drained
- 10 ounce can cream of chicken soup
- 1 cup shredded cheddar cheese
- cups French fried onions



Figure 10.3.3: Sausage Egg Casserole, by RBerteig. CC BY 2.0. Flickr. Source.

Instructions

Combine the pasta, tuna, soup, and half of the cheese. Transfer into a 9 inch by 18 inch baking dish. Put the remaining cheese on top. Bake 30 minutes at 350 degrees. During the last 5 minutes, add the French fried onions. Let sit for 10 minutes before

serving.

1. What is the ratio of the ounces of soup to the cups of shredded cheese to the cups of pasta in one batch of casserole?
2. How much of each of these 3 ingredients would be needed to make:
   a. twice the amount of casserole?
   b. half the amount of casserole?
   c. five times the amount of casserole?
   d. one-fifth the amount of casserole?
3. What is the ratio of cups of pasta to ounces of tuna in one batch of casserole?
4. How many batches of casserole would you make if you used the following amounts of ingredients?
   a. 9 cups of pasta and 18 ounces of tuna?
   b. 36 ounces of tuna and 18 cups of pasta?
   c. 1 cup of pasta and 2 ounces of tuna?

Are you ready for more?

The recipe says to use a 9 inch by 18 inch baking dish. Determine the length and width of a baking dish with the same height that could hold:

1. Twice the amount of casserole
2. Half the amount of casserole
3. Five times the amount of casserole
4. One-fifth the amount of casserole

Exercise 10.3.3: What Are Equivalent Ratios?

The ratios $5 : 3$ and $10 : 6$ are **equivalent ratios**.

1. Is the ratio $15 : 12$ equivalent to these? Explain your reasoning.
2. Is the ratio $30 : 18$ equivalent to these? Explain your reasoning.
3. Give two more examples of ratios that are equivalent to $5 : 3$.
4. How do you know when ratios are equivalent and when they are *not* equivalent?
5. Write a definition of *equivalent ratios*.
   Pause here so your teacher can review your work and assign you a ratio to use for your visual display.
6. Create a visual display that includes:
   - the title “Equivalent Ratios”
   - your best definition of *equivalent ratios*
   - the ratio your teacher assigned to you
   - at least two examples of ratios that are equivalent to your assigned ratio
   - an explanation of how you know these examples are equivalent
   - at least one example of a ratio that is *not* equivalent to your assigned ratio
   - an explanation of how you know this example is *not* equivalent

Be prepared to share your display with the class.

Summary

All ratios that are **equivalent** to $a : b$ can be made by multiplying both a and b by the same number.

For example, the ratio $18 : 12$ is equivalent to $9 : 6$ because both 9 and 6 are multiplied by the same number: 2.

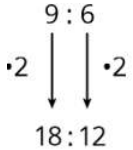


Figure 10.3.4

$3 : 2$ is also equivalent to $9 : 6$, because both 9 and 6 are multiplied by the same number: $\frac{1}{3}$.

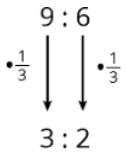


Figure 10.3.5

Is $18 : 15$ equivalent to $9 : 6$?

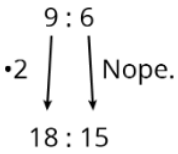


Figure 10.3.6

No, because 18 is $9 \cdot 2$, but 15 is *not* $6 \cdot 2$.

Glossary Entries

Definition: Equivalent Ratios

Two ratios are equivalent if you can multiply each of the numbers in the first ratio by the same factor to get the numbers in the second ratio. For example, $8 : 6$ is equivalent to $4 : 3$, because $8 \cdot \frac{1}{2} = 4$ and $6 \cdot \frac{1}{2} = 3$.

A recipe for lemonade says to use 8 cups of water and 6 lemons. If we use 4 cups of water and 3 lemons, it will make half as much lemonade. Both recipes taste the same, because and are equivalent ratios.

Table 10.3.1

cups of water	number of lemons
8	6
4	3

Practice

Exercise 10.3.4

Each of these is a pair of equivalent ratios. For each pair, explain why they are equivalent ratios or draw a diagram that shows why they are equivalent ratios.

1. $4 : 5$ and $8 : 10$
2. $18 : 3$ and $6 : 1$

1. $2 : 7$ and $10,000 : 35,000$

Exercise 10.3.5

Explain why $6 : 4$ and $18 : 8$ are not equivalent ratios.

Exercise 10.3.6

Are the ratios $3 : 6$ and $6 : 3$ equivalent? Why or why not?

Exercise 10.3.7

This diagram represents 3 batches of light yellow paint. Draw a diagram that represents 1 batch of the same shade of light yellow paint.

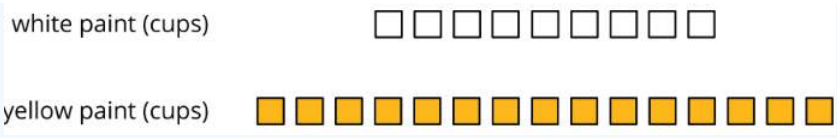


Figure 10.3.7

(From Unit 2.2.2)

Exercise 10.3.8

In the fruit bowl there are 6 bananas, 4 apples, and 3 oranges.

1. For every 4 _____, there are 3 _____.
2. The ratio of _____ to _____ is $6 : 3$.
3. The ratio of _____ to _____ is 4 to 6.
4. For every 1 orange, there are _____ bananas.

(From Unit 2.1.1)

Exercise 10.3.9

Write fractions for points A and B on the number line.

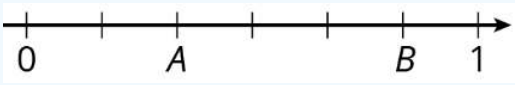


Figure 10.3.8

(From Unit 2.1.1)

This page titled 10.3: Defining Equivalent Ratios is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

Section 11: Representing Equivalent Ratios

This page titled Section 11: Representing Equivalent Ratios is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

11.1: Introducing Double Number Line Diagrams

Lesson

Let's use number lines to represent equivalent ratios.

Exercise 11.1.1: Number Talk: Adjusting Another Factor

Find the value of each product mentally.

$(4.5) \cdot 4$

$(4.5) \cdot 8$

$\frac{1}{10} \cdot 65$

$\frac{2}{10} \cdot 65$

Exercise 11.1.2: Drink Mix on a Double Number Line

The other day, we made drink mixtures by mixing 4 teaspoons of powdered drink mix for every cup of water. Here are two ways to represent multiple batches of this recipe:

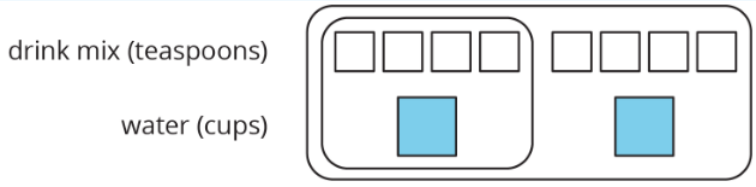


Figure 11.1.1: A discrete diagram for two quantities labeled drink mix, in teaspoons and water in cups. For the number of teaspoons of drink mix, there are 8 small squares. For the number of cups of water, there are 2 large squares. Two groups of 4 small squares and 1 large square are indicated and one group of 4 small squares and 1 large square is circled.

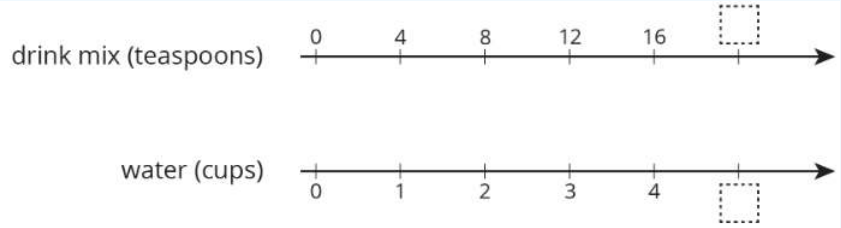


Figure 11.1.2: A double number line with 6 evenly spaced tick marks. The top number line is labeled Drink mix, in teaspoons& and the numbers 0, 4, 8, 12, and 16 are indicated. The last tick mark is blank. The bottom number line is labeled Water, in cups and the numbers 0, 1, 2, 3, 4 are indicated. The last tick mark is blank.

1. How can we tell that $4 : 1$ and $12 : 3$ are equivalent ratios?
2. How are these representations the same? How are these representations different?
3. How many teaspoons of drink mix should be used with 3 cups of water?
4. How many cups of water should be used with 16 teaspoons of drink mix?
5. What numbers should go in the empty boxes on the **double number line diagram**? What do these numbers mean?

Are you ready for more?

Recall that a *perfect square* is a number of objects that can be arranged into a square. For example, 9 is a perfect square because 9 objects can be arranged into 3 rows of 3. 16 is also a perfect square, because 16 objects can be arranged into 4 rows of 4. In contrast, 12 is not a perfect square because you can't arrange 12 objects into a square.

1. How many whole numbers starting with 1 and ending with 100 are perfect squares?
2. What about whole numbers starting with 1 and ending with 1,000?

Exercise 11.1.3: Blue Paint on a Double Number Line

Here is a diagram showing Elena's recipe for light blue paint.

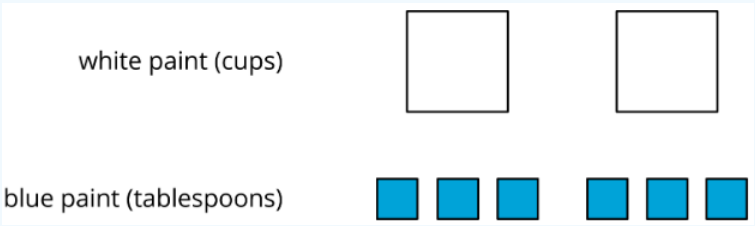


Figure 11.1.3: A discrete diagram of squares that represent the amount of paint. The top row is labeled white paint, in cups and contains 2 large squares. The bottom row is labeled blue paint, in tablespoons and contains 6 small squares.

1. Complete the double number line diagram to show the amounts of white paint and blue paint in different-sized batches of light blue paint.

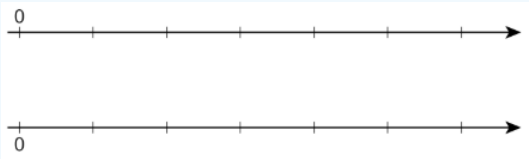


Figure 11.1.4

2. Compare your double number line diagram with your partner. Discuss your thinking. If needed, revise your diagram.
3. How many cups of white paint should Elena mix with 12 tablespoons of blue paint? How many batches would this make?
4. How many tablespoons of blue paint should Elena mix with 6 cups of white paint? How many batches would this make?
5. Use your double number line diagram to find another amount of white paint and blue paint that would make the same shade of light blue paint.
6. How do you know that these mixtures would make the same shade of light blue paint?

Summary

You can use a **double number line diagram** to find many equivalent ratios. For example, a recipe for fizzy juice says, "Mix 5 cups of cranberry juice with 2 cups of soda water." The ratio of cranberry juice to soda water is $5 : 2$. Multiplying both ingredients by the same number creates equivalent ratios.

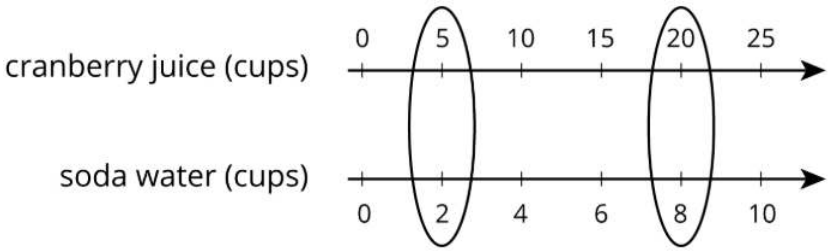


Figure 11.1.5: A double number line with 6 evenly spaced tick marks. The top number line is labeled cranberry juice, in cups and the numbers 0, 2, 4, 6, 8, and 10 are indicated. The bottom number line is labeled soda water, in cups and the numbers 0, 2, 4, 6, 8, and 10 are indicated. There is a circle drawn around the 5 and the 2 and another circle drawn around the 20 and the 8.

This double number line shows that the ratio $20 : 8$ is equivalent to $5 : 2$. If you mix 20 cups of cranberry juice with 8 cups of soda water, it makes 4 times as much fizzy juice that tastes the same as the original recipe.

Glossary Entries

Definition: Double Number Line Diagram

A double number line diagram uses a pair of parallel number lines to represent equivalent ratios. The locations of the tick marks match on both number lines. The tick marks labeled 0 line up, but the other numbers are usually different.

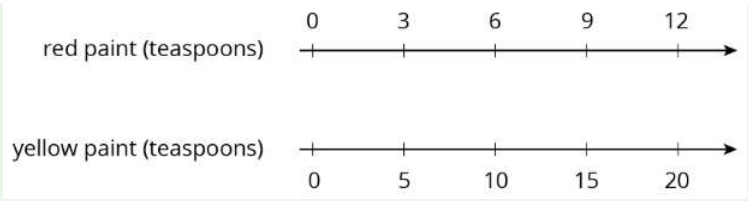


Figure 11.1.6

Practice

Exercise 11.1.4

A particular shade of orange paint has 2 cups of yellow paint for every 3 cups of red paint. On the double number line, circle the numbers of cups of yellow and red paint needed for 3 batches of orange paint.

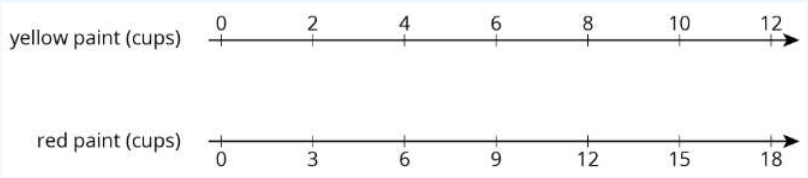


Figure 11.1.7

Exercise 11.1.5

This double number line diagram shows the amount of flour and eggs needed for 1 batch of cookies.

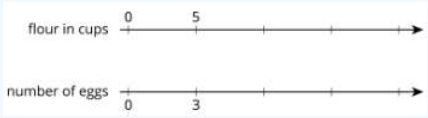


Figure 11.1.8: Double number line, 5 evenly spaced tick marks. Top line, flour, cups. Beginning at first tick mark, labels: 0, 5, blank, blank, blank. Bottom line, number of eggs. Beginning at first tick mark, labeled 0, 3, blank, blank, blank.

1. Complete the diagram to show the amount of flour and eggs needed for 2, 3, and 4 batches of cookies.
2. What is the ratio of cups of flour to eggs?
3. How much flour and how many eggs are used in 4 batches of cookies?

1. How much flour is used with 6 eggs?
2. How many eggs are used with 15 cups of flour?

Exercise 11.1.6

Here is a representation showing the amount of red and blue paint that make 2 batches of purple paint.

1. On the double number line, label the tick marks to represent amounts of red and blue paint used to make batches of this shade of purple paint.

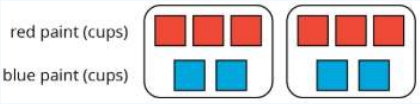


Figure 11.1.9: A discrete diagram for two quantities labeled red paint, cups and blue paint, cups. For the number of cups of red paint, there are 6 red squares. For the number of cups of blue paint, there are 4 blue squares. Two groups of 3 red squares squares and 2 blue squares are circled.

1. How many batches are made with 12 cups of red paint?
2. How many batches are made with 6 cups of blue paint?

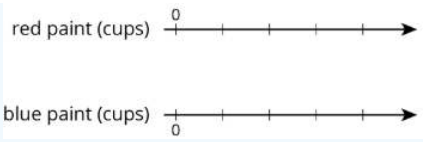


Figure 11.1.10

Exercise 11.1.7

Diego estimates that there will need to be 3 pizzas for every 7 kids at his party. Select **all** the statements that express this ratio.

A. The ratio of kids to pizzas is $7 : 3$.
B. The ratio of pizzas to kids is 3 to 7.
C. The ratio of kids to pizzas is $3 : 7$.
D. The ratio of pizzas to kids is 7 to 3.
E. For every 7 kids there need to be 3 pizzas.

(From Unit 2.1.1)

Exercise 11.1.8

1. Draw a parallelogram that is not a rectangle that has an area of 24 square units. Explain or show how you know the area is 24 square units.

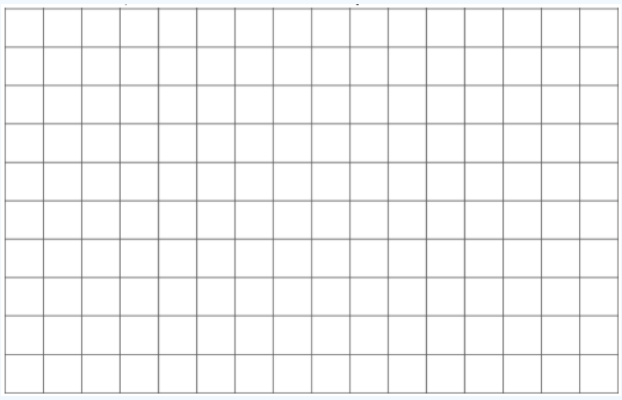


Figure 11.1.11

2. Draw a triangle that has an area of 24 square units. Explain or show how you know the area is 24 square units.

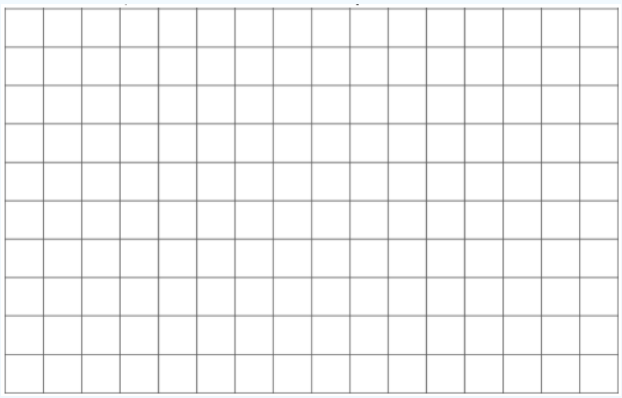


Figure 11.1.12

(From Unit 1.2.3)

11.1: Introducing Double Number Line Diagrams is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

11.2: Creating Double Number Line Diagrams

Lesson

Let's draw double number line diagrams like a pro.

Exercise 11.2.1: Orderling on a Number Line

1. Locate and label the following numbers on the number line.

$\frac{1}{2}$ $\quad$ $\frac{1}{4}$ $\quad$ $1\frac{3}{4}$ $\quad$ 1.5 $\quad$ 1.75

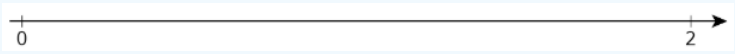


Figure 11.2.1

2. Based on where you placed the numbers, locate and label four more fractions or decimals on the number line.

Exercise 11.2.2: Just a Little Green

The other day, we made green water by mixing 5 ml of blue water with 15 ml of yellow water. We want to make a very small batch of the same shade of green water. We need to know how much yellow water to mix with only 1 ml of blue water.

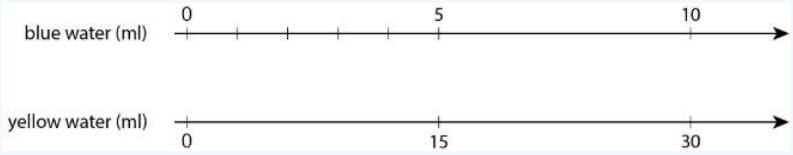


Figure 11.2.2: Double number line, 3 evenly spaced tick marks. Top line, blue water, milliliters. Beginning at first mark, labels: 0, 5, 10. There are 4 evenly spaced tick marks between 0 and 5. Bottom line, yellow water, milliliters. Beginning at first mark, labels: 0, 15, 30.

1. On the number line for blue water, label the four tick marks shown.
2. On the number line for yellow water, draw and label tick marks to show the amount of yellow water needed for each amount of blue water.
3. How much yellow water should be used for 1 ml of blue water? Circle where you can see this on the double number line.
4. How much yellow water should be used for 11 ml of blue water?
5. How much yellow water should be used for 8 ml of blue water?
6. Why is it useful to know how much yellow water should be used with 1 ml of blue water?

Exercise 11.2.3: Art Paste on a Double Number Line

A recipe for art paste says “For every 2 pints of water, mix in 8 cups of flour.”

1. Follow the instructions to draw a double number line diagram representing the recipe for art paste.
   a. Use a ruler to draw two parallel lines.
   b. Label the first line “pints of water.” Label the second line “cups of flour.”
   c. Draw at least 6 equally spaced tick marks that line up on both lines.
   d. Along the water line, label the tick marks with the amount of water in 0, 1, 2, 3, 4, and 5 batches of art paste.
   e. Along the flour line, label the tick marks with the amount of flour in 0, 1, 2, 3, 4, and 5 batches of art paste.
2. Compare your double number line diagram with your partner’s. Discuss your thinking. If needed, revise your diagram.
3. Next, use your double number line to answer these questions:
   a. How much flour should be used with 10 pints of water?
   b. How much water should be used with 24 cups of flour?
   c. How much flour **per** pint of water does this recipe use?

Are you ready for more?

A square with side of 10 units overlaps a square with side of 8 units in such a way that its corner B is placed exactly at the center of the smaller square. As a result of the overlapping, the two sides of the large square intersect the two sides of the small

square exactly at points C and E, as shown. The length of CD is 6 units.

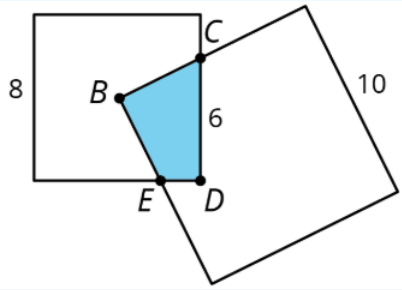


Figure 11.2.3: A square with side of 10 units overlaps a square with side of 8 units. A corner of the larger square, labeled B, is placed exactly at the center of the smaller square. The two sides of the large square intersect the two sides of the small square at C and E. The corner of the small square that is inside the large square is labeled D. The length of the segment from C to D is 6 units.

What is the area of the overlapping region $CDEB$?

Exercise 11.2.4: Revisiting Tuna Casserole

The other day, we looked at a recipe for tuna casserole that called for 10 ounces of cream of chicken soup for every 3 cups of elbow-shaped pasta.

1. Draw a double number line diagram that represents the amounts of soup and pasta in different-sized batches of this recipe.
2. If you made a large amount of tuna casserole by mixing 40 ounces of soup with 15 cups of pasta, would it taste the same as the original recipe? Explain or show your reasoning.
3. The original recipe called for 6 ounces of tuna for every 3 cups of pasta. Add a line to your diagram to represent the amount of tuna in different batches of casserole.
4. How many ounces of soup should you mix with 30 ounces of tuna to make a casserole that tastes the same as the original recipe?

Summary

Here are some guidelines to keep in mind when drawing a double number line diagram:

- The two parallel lines should have labels that describe what the numbers represent.
- The tick marks and numbers should be spaced at equal intervals.
- Numbers that line up vertically make equivalent ratios.

For example, the ratio of the number of eggs to cups of milk in a recipe is $4 : 1$. Here is a double number line that represents the situation:

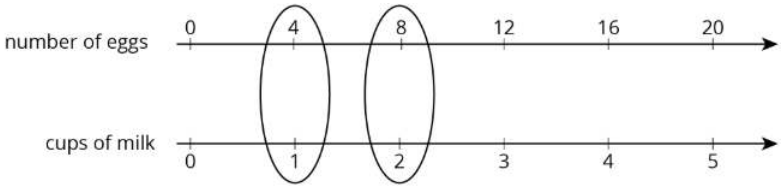


Figure 11.2.4

We can also say that this recipe uses “4 eggs per cup of milk” because the word **per** means “for each.”

Glossary Entries

Definition: Double Number Line Diagram

A double number line diagram uses a pair of parallel number lines to represent equivalent ratios. The locations of the tick marks match on both number lines. The tick marks labeled 0 line up, but the other numbers are usually different.

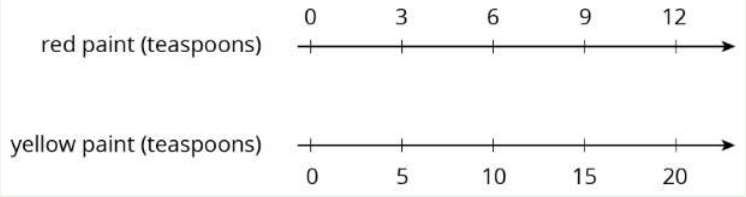


Figure 11.2.5

Definition: Per

The word *per* means "for each." For example, if the price is \$5 per ticket, that means you will pay \$5 *for each* ticket. Buying 4 tickets would cost \$20, because $4 \cdot 5 = 20$

Practice

Exercise 11.2.5

A recipe for cinnamon rolls uses 2 tablespoons of sugar per teaspoon of cinnamon for the filling. Complete the double number line diagram to show the amount of cinnamon and sugar in 3, 4, and 5 batches.

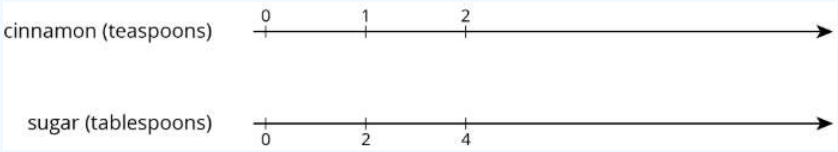


Figure 11.2.6: Double number line, 3 evenly spaced tick marks. Top line, cinnamon, teaspoons. Beginning at first tick mark, labels: 0, 1, 2. Bottom line, sugar, tablespoons. Beginning at first tick mark, labels: 0, 2, 4.

Exercise 11.2.6

One batch of meatloaf contains 2 pounds of beef and $\frac{1}{2}$ cup of bread crumbs. Complete the double number line diagram to show the amounts of beef and bread crumbs needed for 1, 2, 3, and 4 batches of meatloaf.

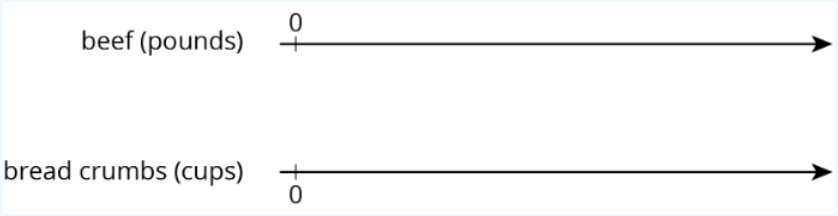


Figure 11.2.7: Double number line, 1 tick mark on the far left. Top line, beef, pounds. Tick mark is labeled 0. Bottom line, bread crumbs, cups. Tick mark is labeled 0.

Exercise 11.2.7

A recipe for tropical fruit punch says, "Combine 4 cups of pineapple juice with 5 cups of orange juice."

1. Create a double number showing the amount of each type of juice in 1, 2, 3, 4, and 5 batches of the recipe.
2. If 12 cups of pineapple juice are used with 20 cups of orange juice, will the recipe taste the same? Explain your reasoning.
3. The recipe also calls for $\frac{1}{3}$ cup of lime juice for every 5 cups of orange juice. Add a line to your diagram to represent the amount of lime juice in different batches of tropical fruit punch.

Exercise 11.2.8

One batch of pink paint uses 2 cups of red paint and 7 cups of white paint. Mai made a large amount of pink paint using 14 cups of red paint.

1. How many batches of pink paint did she make?
2. How many cups of white paint did she use?

(From Unit 2.2.2)

Exercise 11.2.9

1. Find three different ratios that are equivalent to the ratio $3 : 11$.
2. Explain why your ratios are equivalent.

(From Unit 2.2.3)

Exercise 11.2.10

Here is a diagram that represents the pints of red and yellow paint in a mixture.

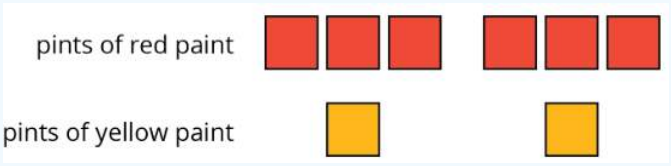


Figure 11.2.8

Select **all** statements that accurately describe the diagram.

A. The ratio of yellow paint to red paint is 2 to 6.
B. For every 3 pints of red paint, there is 1 pint of yellow paint.
C. For every pint of yellow paint, there are 3 pints of red paint.
D. For every pint of yellow paint there are 6 pints of red paint.
E. The ratio of red paint to yellow paint is $6 : 2$.

(From Unit 2.1.2)

11.2: Creating Double Number Line Diagrams is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

11.3: Constant Speed

Lesson

Let's use ratios to work with how fast things move.

Exercise 11.3.1: Number Talk: Dividing Powers by 10

Find the quotient mentally.

$30 \div 10$

$34 \div 10$

$3.4 \div 10$

$34 \div 100$

Exercise 11.3.2: Moving 10 Meters

Your teacher will set up a straight path with a 1-meter warm-up zone and a 10-meter measuring zone. Follow the following instructions to collect the data.

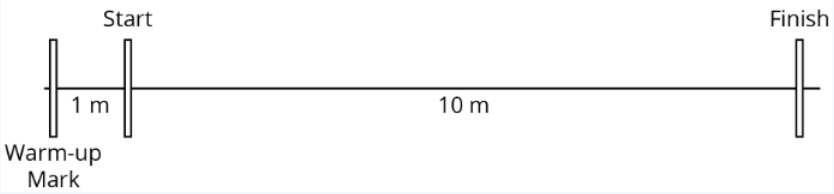


Figure 11.3.1: A diagram of a line with three markings. The first mark is labeled Warm-up Mark, the second mark is labeled Start, and the third mark is labeled Finish. The distance between the first and second mark is labeled 1m. The distance between the second and third mark is labeled 10m.

1. a. The person with the stopwatch (the "timer") stands at the finish line. The person being timed (the "mover") stands at the warm-up line.
   b. On the first round, the mover starts moving *at a slow, steady speed* along the path. When the mover reaches the start line, they say, "Start!" and the timer starts the stopwatch.
   c. The mover keeps moving steadily along the path. When they reach the finish line, the timer stops the stopwatch and records the time, rounded to the nearest second, in the table.
   d. On the second round, the mover follows the same instructions, but this time, moving *at a quick, steady speed*. The timer records the time the same way.
   e. Repeat these steps until each person in the group has gone twice: once at a slow, steady speed, and once at a quick, steady speed.

Table 11.3.1

your slow moving time (seconds)	your fast moving time (seconds)

2. After you finish collecting the data, use the double number line diagrams to answer the questions. Use the times your partner collected while you were moving.

Moving slowly:

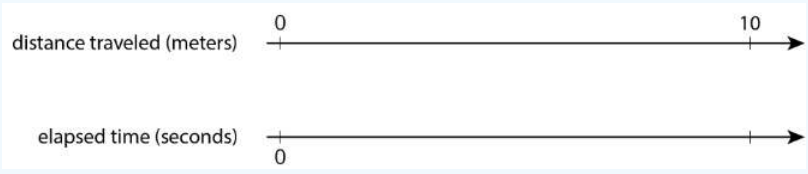


Figure 11.3.2: Double number line titled, Moving Slowly. 2 evenly spaced tick marks. Top line, distance traveled, meters. Beginning at first tick mark, labels: 0, 10. Bottom line, elapsed time, seconds. Beginning at first tick mark, labels: 0, blank.

Moving quickly:

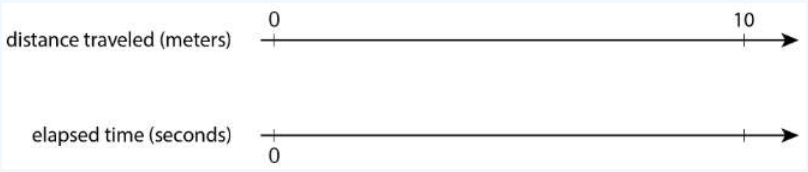


Figure 11.3.3: Double number line titled, Moving Slowly. 2 evenly spaced tick marks. Top line, distance traveled, meters. Beginning at first tick mark, labels: 0, 10. Bottom line, elapsed time, seconds. Beginning at first tick mark, labels: 0, blank.

a. Estimate the distance in meters you traveled in 1 second when moving slowly.
b. Estimate the distance in meters you traveled in 1 second when moving quickly.
c. Trade diagrams with someone who is not your partner. How is the diagram representing someone moving slowly different from the diagram representing someone moving quickly?

Exercise 11.3.3: Moving for 10 Seconds

Lin and Diego both ran for 10 seconds, each at a constant speed. Lin ran 40 meters and Diego ran 55 meters.

1. Who was moving faster? Explain your reasoning.
2. How far did each person move in 1 second? If you get stuck, consider drawing double number line diagrams to represent the situations.
3. Use your data from the previous activity to find how far *you* could travel in 10 seconds at your quicker speed.
4. Han ran 100 meters in 20 seconds at a constant speed. Is this speed faster, slower, or the same as Lin's? Diego's? Yours?

Are you ready for more?

Lin and Diego want to run a race in which they will both finish when the timer reads exactly 30 seconds. Who should get a head start, and how long should the head start be?

Summary

Suppose a train traveled 100 meters in 5 seconds at a constant speed. To find its speed in **meters per second,** we can create a double number line:

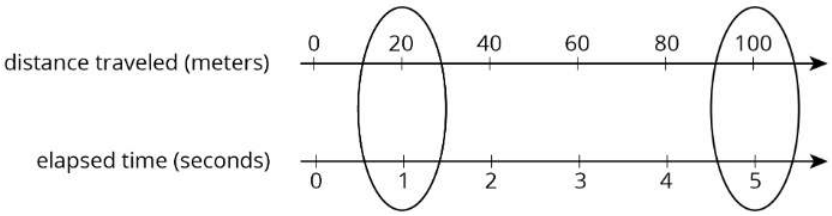


Figure 11.3.4

The double number line shows that the train's speed was 20 meters per second. We can also find the speed by dividing: $100 \div 5 = 20$.

Once we know the speed in meters per second, many questions about the situation become simpler to answer because we can multiply the amount of time an object travels by the speed to get the distance. For example, at this rate, how far would the train go in 30 seconds? Since $20 \cdot 30 = 600$, the train would go 600 meters in 30 seconds.

Glossary Entries

Definition: Double Number Line Diagram

A double number line diagram uses a pair of parallel number lines to represent equivalent ratios. The locations of the tick marks match on both number lines. The tick marks labeled 0 line up, but the other numbers are usually different.

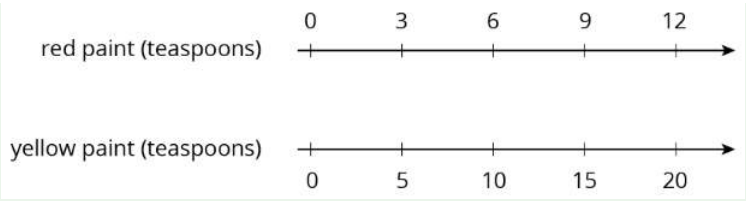


Figure 11.3.5

Definition: Meters per Second

Meters per second is a unit for measuring speed. It tells how many meters an object goes in one second.

For example, a person walking 3 meters per second is going faster than another person walking 2 meters per second.

Definition: Per

The word *per* means "for each." For example, if the price is \$5 per ticket, that means you will pay \$5 *for each* ticket. Buying 4 tickets would cost \$20, because $4 \cdot 5 = 20$

Definition: Unit Price

The unit price is the cost for one item or for one unit of measure. For example, if 10 feet of chain link fencing cost \$150, then the unit price is $150 \div 10$, or \$15 per foot.

Practice

Exercise 11.3.4

Han ran 10 meters in 2.7 seconds. Priya ran 10 meters in 2.4 seconds.

1. Who ran faster? Explain how you know.
2. At this rate, how long would it take each person to run 50 meters? Explain or show your reasoning.

Exercise 11.3.5

A scooter travels 30 feet in 2 seconds at a constant speed.

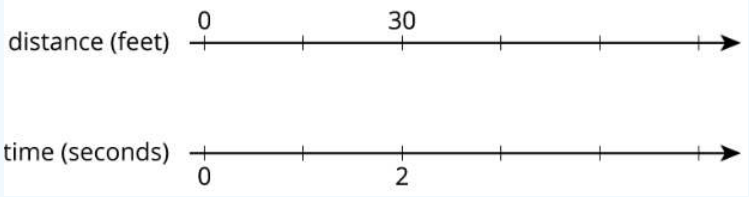


Figure 11.3.6: Double number line, 6 evenly spaced tick marks. Top line, distance, feet. Beginning at first tick mark, labels: 0, blank, 30, blank, blank, blank. Bottom line, time, seconds. Beginning at first tick mark, labels: 0, blank, 2, blank,blank, blank.

1. What is the speed of the scooter in feet per second?
2. Complete the double number line to show the distance the scooter travels after 1, 3, 4, and 5 seconds.
3. A skateboard travels 55 feet in 4 seconds. Is the skateboard going faster, slower, or the same speed as the scooter?

Exercise 11.3.6

A cargo ship traveled 150 nautical miles in 6 hours at a constant speed. How far did the cargo ship travel in one hour?

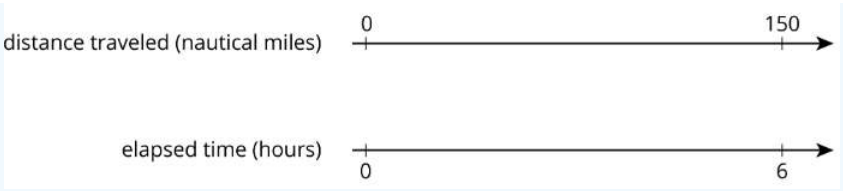


Figure 11.3.7: Double number line, one tick mark at beginning and one at end. Top line, distance traveled, nautical miles. Beginning at first tick mark, labels: 0 and 150. Bottom line, elapsed time, hours. Beginning at first tick mark, labeled 0 and 6.

Exercise 11.3.7

A recipe for pasta dough says, “Use 150 grams of flour per large egg.”

1. How much flour is needed if 6 large eggs are used?
2. How many eggs are needed if 450 grams of flour are used?

(From Unit 2.2.1)

Exercise 11.3.8

The grocery store is having a sale on frozen vegetables. 4 bags are being sold for $11.96. At this rate, what is the cost of:

1. 1 bag
2. 9 bags

(From Unit 2.3.3)

Exercise 11.3.9

A pet owner has 5 cats. Each cat has 2 ears and 4 paws.

1. Complete the double number line to show the numbers of ears and paws for 1, 2, 3, 4, and 5 cats.
2. If there are 3 cats in the room, what is the ratio of ears to paws?

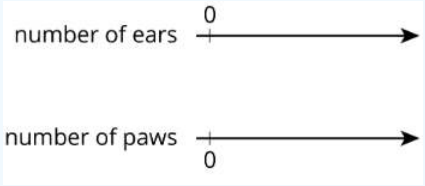


Figure 11.3.8

1. If there are 4 cats in the room, what is the ratio of paws to ears?
2. If all 5 cats are in the room, how many more paws are there than ears?

(From Unit 2.3.2)

Exercise 11.3.10

Each of these is a pair of equivalent ratios. For each pair, explain why they are equivalent ratios or draw a representation that shows why they are equivalent ratios.

1. $5 : 1$ and $15 : 3$
2. $25 : 5$ and $10 : 2$
3. $198 : 1,287$ and $2 : 13$

(From Unit 2.2.3)

This page titled 11.3: Constant Speed is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

11.4: Comparing Situations by Examining Ratios

Lesson

Let's use ratios to compare situations.

Exercise 11.4.1: Treadmills

Mai and Jada each ran on a treadmill. The treadmill display shows the distance, in miles, each person ran and the amount of time it took them, in minutes and seconds.

Here is Mai's treadmill display:



Figure 11.4.1: A picture of Mais treadmill display for the distance and time ran. Data on the treadmill are: Total Time: 24 minutes, 0 seconds. Distance, 3.0 miles. Pace, 8 minutes, 0 seconds. Calories, 481. Incline, 10. Level, 12. Pulse, 125.



Figure 11.4.2: A picture of Jadas treadmill display for distance and time ran. Data on the treadmill are: Total Time: 30 minutes, 0 seconds. Distance, 3.0 miles. Pace, 10 minutes, 0 seconds. Calories, 411. Incline, 10. Level, 12. Pulse, 125.

1. What is the same about their workouts? What is different about their workouts?
2. If each person ran at a constant speed the entire time, who was running faster? Explain your reasoning.

Exercise 11.4.2: Concert Tickets

Diego paid $47 for 3 tickets to a concert. Andre paid $141 for 9 tickets to a concert. Did they pay at the **same rate**? Explain your reasoning.

Exercise 11.4.3: Sparkling Orange Juice

Lin and Noah each have their own recipe for making sparkling orange juice.

- Lin mixes 3 liters of orange juice with 4 liters of soda water.
- Noah mixes 4 liters of orange juice with 5 liters of soda water.

How do the two mixtures compare in taste? Explain your reasoning.

Are you ready for more?

1. How can Lin make her sparkling orange juice taste the same as Noah's just by adding more of one ingredient? How much will she need?
2. How can Noah make his sparkling orange juice taste the same as Lin's just by adding more of one ingredient? How much will he need?

Summary

Sometimes we want to know whether two situations are described by the **same rate**. To do that, we can write an equivalent ratio for one or both situations so that one part of their ratios has the same value. Then we can compare the other part of the ratios.

For example, do these two paint mixtures make the same shade of orange?

- Kiran mixes 9 teaspoons of red paint with 15 teaspoons of yellow paint.
- Tyler mixes 7 teaspoons of red paint with 10 teaspoons of yellow paint.

Here is a double number line that represents Kiran's paint mixture. The ratio $9 : 15$ is equivalent to the ratios $3 : 5$ and $6 : 10$.

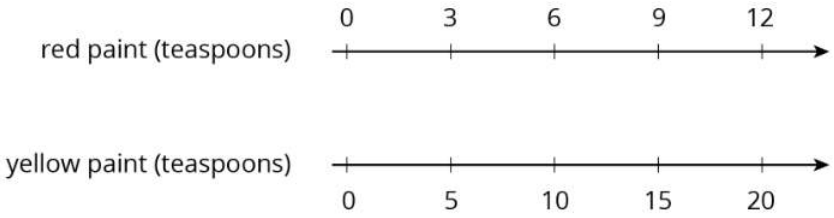


Figure 11.4.3

For 10 teaspoons of yellow paint, Kiran would mix in 6 teaspoons of red paint. This is less red paint than Tyler mixes with 10 teaspoons of yellow paint. The ratios and are not equivalent, so these two paint mixtures would not be the same shade of orange.

When we talk about two things happening at the same rate, we mean that the ratios of the quantities in the two situations are equivalent. There is also something specific about the situation that is the same.

- If two ladybugs are moving at the same rate, then they are traveling at the *same constant speed*.
- If two bags of apples are selling for the same rate, then they have the *same unit price*.
- If we mix two kinds of juice at the same rate, then the mixtures have the *same taste*.
- If we mix two colors of paint at the same rate, then the mixtures have the *same shade*.

Glossary Entries

Definition: Double Number Line Diagram

A double number line diagram uses a pair of parallel number lines to represent equivalent ratios. The locations of the tick marks match on both number lines. The tick marks labeled 0 line up, but the other numbers are usually different.

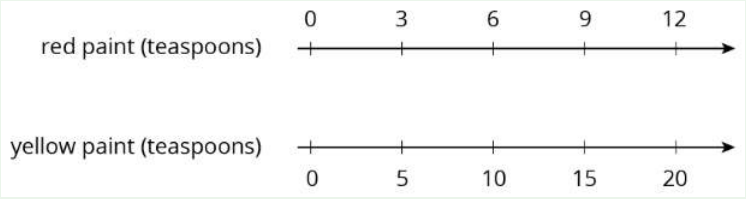


Figure 11.4.4

Definition: Meters per Second

Meters per second is a unit for measuring speed. It tells how many meters an object goes in one second.

For example, a person walking 3 meters per second is going faster than another person walking 2 meters per second.

Definition: Per

The word *per* means “for each.” For example, if the price is \$5 per ticket, that means you will pay \$5 *for each* ticket. Buying 4 tickets would cost \$20, because $4 \cdot 5 = 20$

Definition: Same Rate

We use the words *same rate* to describe two situations that have equivalent ratios.

For example, a sink is filling with water at a rate of 2 gallons per minute. If a tub is also filling with water at a rate of 2 gallons per minute, then the sink and the tub are filling at the same rate.

Definition: Unit Price

The unit price is the cost for one item or for one unit of measure. For example, if 10 feet of chain link fencing cost $150, then the unit price is $150 \div 10$, or $15 per foot.

Practice

Exercise 11.4.4

A slug travels 3 centimeters in 3 seconds. A snail travels 6 centimeters in 6 seconds. Both travel at constant speeds. Mai says, "The snail was traveling faster because it went a greater distance." Do you agree with Mai? Explain or show your reasoning.

Exercise 11.4.5

If you blend 2 scoops of chocolate ice cream with 1 cup of milk, you get a milkshake with a stronger chocolate flavor than if you blended 3 scoops of chocolate ice cream with 2 cups of milk. Explain or show why.

Exercise 11.4.6

There are 2 mixtures of light purple paint.

- Mixture A is made with 5 cups of purple paint and 2 cups of white paint.
- Mixture B is made with 15 cups of purple paint and 8 cups of white paint.

Which mixture is a lighter shade of purple? Explain your reasoning.

Exercise 11.4.7

Tulip bulbs are on sale at store A, at 5 for $11.00, and the regular price at store B is 6 for $13. Is each store pricing tulip bulbs at the same rate? Explain how you know.

Exercise 11.4.8

A plane travels at a constant speed. It takes 6 hours to travel 3,360 miles.

1. What is the plane's speed in miles per hour?
2. At this rate, how many miles can it travel in 10 hours?

(From Unit 2.3.4)

Exercise 11.4.9

A pound of ground beef costs $5. At this rate, what is the cost of:

1. 3 pounds?
2. $\frac{1}{2}$ pound?
3. $\frac{1}{4}$ pound?
4. $\frac{3}{4}$ pound?
5. $3\frac{3}{4}$ pounds?

(From Unit 2.3.3)

Exercise 11.4.10

In a triple batch of a spice mix, there are 6 teaspoons of garlic powder and 15 teaspoons of salt. Answer the following questions about the mix. If you get stuck, create a double number line.

1. How much garlic powder is used with 5 teaspoons of salt?
2. How much salt is used with 8 teaspoons of garlic powder?
3. If there are 14 teaspoons of spice mix, how much salt is in it?
4. How much more salt is there than garlic powder if 6 teaspoons of garlic powder are used?

(From Unit 2.3.2)

This page titled 11.4: Comparing Situations by Examining Ratios is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

Section 12: Solving Ratio and Rate Problems

This page titled Section 12: Solving Ratio and Rate Problems is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

12.1: Representing Ratios with Tables

Lesson

Let's use tables to represent equivalent ratios.

Exercise 12.1.1: How Is It Growing?

Look for a pattern in the figures.

1. How many total tiles will be in:
   a. the 4th figure?
   b. the 5th figure?
   c. the 10th figure?
2. How do you see it growing?

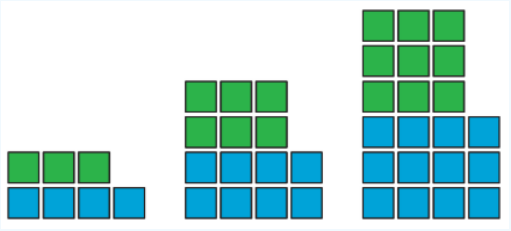


Figure 12.1.1: A growing pattern of tiles arranged in rows. The first figure: row 1, 3 green tiles; row 2, 4 blue tiles. The second figure: row 1 and row 2, 3 green tiles; row 3 and row 4, 4 blue tiles. The third figure: row 1, row 2, and row 3, 3 green tiles; row 4, row 5, and row 6, 4 blue tiles.

Exercise 12.1.2: A Huge Amount of Sparkling Orange Juice

Noah's recipe for one batch of sparkling orange juice uses 4 liters of orange juice and 5 liters of soda water.

1. Use the double number line to show how many liters of each ingredient to use for different-sized batches of sparkling orange juice.
2. If someone mixes 36 liters of orange juice and 45 liters of soda water, how many batches would they make?
3. If someone uses 400 liters of orange juice, how much soda water would they need?
4. If someone uses 455 liters of soda water, how much orange juice would they need?
5. Explain the trouble with using a double number line diagram to answer the last two questions.

Exercise 12.1.3: Bacthes of Trail Mix

A recipe for trail mix says: "Mix 7 ounces of almonds with 5 ounces of raisins." Here is a **table** that has been started to show how many ounces of almonds and raisins would be in different-sized batches of this trail mix.

Table 12.1.1

almonds (oz)	raisins (oz)
7	5
28	
	10
3.5	
	250
56	

1. Complete the table so that ratios represented by each row are equivalent.

2. What methods did you use to fill in the table?
3. How do you know that each row shows a ratio that is equivalent to $7 : 5$? Explain your reasoning.

Are you ready for more?

You have created a best-selling recipe for chocolate chip cookies. The ratio of sugar to flour is $2 : 5$.

Create a table in which each entry represents amounts of sugar and flour that might be used at the same time in your recipe.

- One entry should have amounts where you have fewer than 25 cups of flour.
- One entry should have amounts where you have between 20–30 cups of sugar.
- One entry can have any amounts using more than 500 units of flour.

Summary

A **table** is a way to organize information. Each horizontal set of entries is called a *row*, and each vertical set of entries is called a *column*. (The table shown has 2 columns and 5 rows.) A table can be used to represent a collection of equivalent ratios.

Here is a double number line diagram and a table that both represent the situation: "The price is $2 for every 3 mangos."

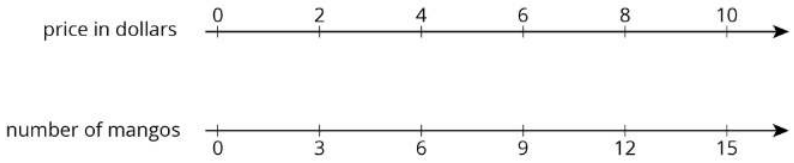


Figure 12.1.2

column column

	price in dollars	number of mangos
row →	2	3
row →	4	6
row →	6	9
row →	8	12
row →	10	15

Figure 12.1.3

Glossary Entries

Definition: Table

A table organizes information into horizontal *rows* and vertical *columns*. The first row or column usually tells what the numbers represent.

For example, here is a table showing the tail lengths of three different pets. This table has four rows and two columns.

Table 12.1.2

pet	tail length (inches)
dog	22
cat	12
mouse	2

Practice

Exercise 12.1.4

Complete the table to show the amounts of yellow and red paint needed for different-sized batches of the same shade of orange paint.

Table 12.1.3

yellow paint (quarts)	red paint (quarts)
5	6

Explain how you know that these amounts of yellow paint and red paint will make the same shade of orange as the mixture in the first row of the table.

Exercise 12.1.5

A car travels at a constant speed, as shown on the double number line.

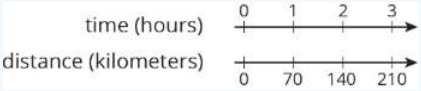


Figure 12.1.4: Double number line, 4 evenly spaced tick marks. Top line, time, hours. Beginning at first tick mark, labels: 0, 1, 2, 3. Bottom line, distance, kilometers. Beginning at first tick mark, labels: 0, 70, 140, 210.

How far does the car travel in 14 hours? Explain or show your reasoning.

Exercise 12.1.6

The olive trees in an orchard produce 3,000 pounds of olives a year. It takes 20 pounds of olives to make 3 liters of olive oil. How many liters of olive oil can this orchard produce in a year? If you get stuck, consider using the table.

Table 12.1.4

olives (pounds)	olive oil (liters)
20	3
100	
3, 000	

Exercise 12.1.7

At a school recess, there needs to be a ratio of 2 adults for every 24 children on the playground. The double number line represents the number of adults and children on the playground at recess.

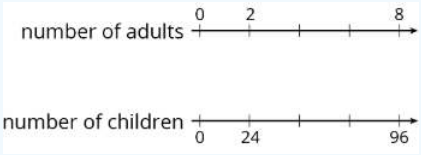


Figure 12.1.5: Double number line, 5 evenly spaced tick marks. Top line, number of adults. Beginning at first tick mark, labels: 0, 2, blank, blank, 8. Bottom line, number of children. Beginning at first tick mark, labeled 0, 24, blank, blank, 96.

1. Label each remaining tick mark with its value.
2. How many adults are needed if there are 72 children? Circle your answer on the double number line.

(From Unit 2.3.1)

Exercise 12.1.8

While playing basketball, Jada's heart rate goes up to 160 beats per minute. While jogging, her heart beats 25 times in 10 seconds. Assuming her heart beats at a constant rate while jogging, which of these activities resulted in a higher heart rate? Explain your reasoning.

(From Unit 2.3.5)

Exercise 12.1.9

A shopper bought the following items at the farmer's market:

1. 6 ears of corn for $1.80. What was the cost per ear?
2. 12 apples for $2.88. What was the cost per apple?
3. 5 tomatoes for $3.10. What was the cost per tomato?

(From Unit 2.3.3)

12.1: Representing Ratios with Tables is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

12.2: Navigating a Table of Equivalent Ratios

Lesson

Let's use a table of equivalent ratios like a pro.

Exercise 12.2.1: Number Talk: Multiplying by a Unit Fraction

Find the product mentally.

$\frac{1}{3} \cdot 21$

$\frac{1}{6} \cdot 21$

$(5.6) \cdot \frac{1}{8}$

$\frac{1}{4} \cdot (5.6)$

Exercise 12.2.2: Comparing Taco Prices

Use the table to help you solve these problems. Explain or show your reasoning.

1. Noah bought 4 tacos and paid $6. At this rate, how many tacos could he buy for $15?
2. Jada's family bought 50 tacos for a party and paid $72. Were Jada's tacos the same price as Noah's tacos?

Table 12.2.1

number of tacos	price in dollars

Exercise 12.2.3: Hourly Wages

Lin is paid $90 for 5 hours of work. She used the table to calculate how much she would be paid at this rate for 8 hours of work.

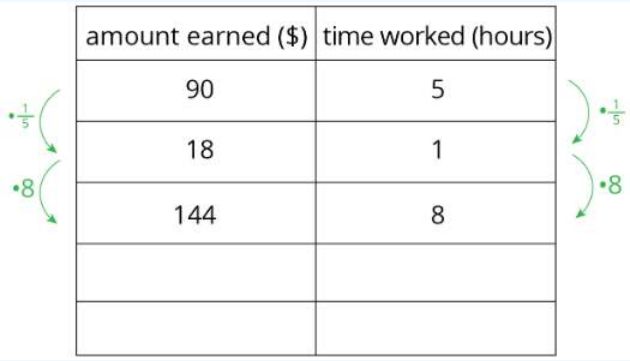


Figure 12.2.1: A 2-column table with 5 rows of data. The first column is labeled amount earned, in dollars and the second column is labeled time worked, in hours. Row 1: 90, 5; Row 2: 18, 1; Row 3: 144, 8; Row 4 and 5: blank. An arrow pointing from row 1 to row 2 is labeled times one-fifth. Another arrow pointing from row 2 to row 3 is labeled times 8.

1. What is the meaning of the 18 that appears in the table?
2. Why was the number $\frac{1}{5}$ used as a multiplier?
3. Explain how Lin used this table to solve the problem.
4. At this rate, how much would Lin be paid for 3 hours of work? For 2.1 hours of work?

Exercise 12.2.4: Zeno's Memory Card

In 2016, 128 gigabytes (GB) of portable computer memory cost $32.

1. Here is a double number line that represents the situation:

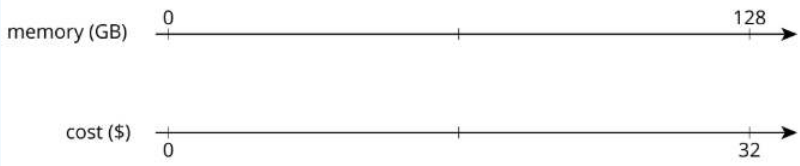


Figure 12.2.2

One set of tick marks has already been drawn to show the result of multiplying 128 and 32 each by $\frac{1}{2}$. Label the amount of memory and the cost for these tick marks.

Next, keep multiplying by $\frac{1}{2}$ and drawing and labeling new tick marks, until you can no longer clearly label each new tick mark with a number.

2. Here is a table that represents the situation. Find the cost of 1 gigabyte. You can use as many rows as you need.

Table 12.2.2

memory (gigabytes)	cost (dollars)
128	32

3. Did you prefer the double number line or the table for solving this problem? Why?

Are you ready for more?

A kilometer is 1,000 meters because *kilo* is a prefix that means 1,000. The prefix *mega* means 1,000,000 and *giga* (as in gigabyte) means 1,000,000,000. One byte is the amount of memory needed to store one letter of the alphabet. About how many of each of the following would fit on a 1-gigabyte flash drive?

1. letters
2. pages
3. books
4. movies
5. songs

Summary

Finding a row containing a “1” is often a good way to work with tables of equivalent ratios. For example, the price for 4 lbs of granola is $5. At that rate, what would be the price for 62 lbs of granola?

Here are tables showing two different approaches to solving this problem. Both of these approaches are correct. However, one approach is more efficient.

- Less efficient



Figure 12.2.3: A 2-column table with 6 rows of data. First column labeled granola, in pounds, second column labeled price, in dollars. The data is as follows: Row 1: 4, 5; Row 2: 8, 10; Row 3: 16, 20; Row 4: 32, 40; Row 5: 64, 80; Row 6: 62, 77.50. An arrow pointing from row 1 to row 2, then 2 to 3, then 3 to 4, then 4 to 5 is labeled times 2. The last arrow from row 5 to 6 is labeled subtract 2 pounds on the left of the table and is labeled subtract $2.50 on the right.

- More efficient

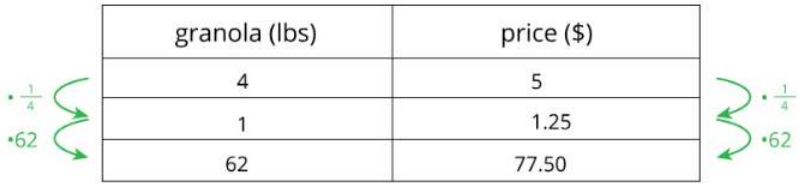


Figure 12.2.4: 2-column table, 3 rows of data. First column labeled granola, in pounds, second column labeled price, in dollars. The data is as follows: Row 1: 4, 5; Row 2: 1, 1.25; Row 3: 62, 77.50. An arrow pointing from row 1 to row 2 is labeled times one-fourth and another arrow pointing from row 2 to row 3 is labeled times 62.

Notice how the more efficient approach starts by finding the price for 1 lb of granola.

Remember that dividing by a whole number is the same as multiplying by a unit fraction. In this example, we can divide by 4 or multiply by $\frac{1}{4}$ to find the unit price.

Glossary Entries

Definition: Table

A table organizes information into horizontal *rows* and vertical *columns*. The first row or column usually tells what the numbers represent.

For example, here is a table showing the tail lengths of three different pets. This table has four rows and two columns.

Table 12.2.3

pet	tail length (inches)
dog	22
cat	12
mouse	2

Practice

Exercise 12.2.5

Priya collected 2,400 grams of pennies in a fundraiser. Each penny has a mass of 2.5 grams. How much money did Priya raise? If you get stuck, consider using the table.

Table 12.2.4

number of pennies	mass in grams
1	2.5

number of pennies	mass in grams
	2, 400

Exercise 12.2.6

Kiran reads 5 pages in 20 minutes. He spends the same amount of time per page. How long will it take him to read 11 pages? If you get stuck, consider using the table.

Table 12.2.5

time in minutes	number of pages
20	5
	1
	11

Exercise 12.2.7

Mai is making personal pizzas. For 4 pizzas, she uses 10 ounces of cheese.

Table 12.2.6

number of pizzas	ounces of cheese
4	10

a. How much cheese does Mai use per pizza?
b. b. At this rate, how much cheese will she need to make 15 pizzas?

Exercise 12.2.8

Clare is paid $90 for 5 hours of work. At this rate, how many seconds does it take for her to earn 25 cents?

Exercise 12.2.9

A car that travels 20 miles in $\frac{1}{2}$ hour at constant speed is traveling at the same speed as a car that travels 30 miles in $\frac{3}{4}$ hour at a constant speed. Explain or show why.

(From Unit 2.3.5)

Exercise 12.2.10

Lin makes her favorite juice blend by mixing cranberry juice with apple juice in the ratio shown on the double number line. Complete the diagram to show smaller and larger batches that would taste the same as Lin's favorite blend.

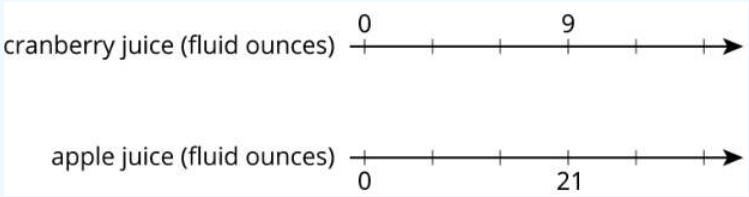


Figure 12.2.5: Double number line, 6 evenly spaced tick marks. Top line, cranberry juice, fluid ounces. Beginning at first tick mark, labels: 0, blank, blank, 9, blank, blank. Bottom line, apple juice, fluid ounces. Beginning at first tick mark, labeled 0, blank, blank, 21, blank, blank.

https://math.libretexts.org/@go/page/39932

(From Unit 2.3.1)

Exercise 12.2.11

Each of these is a pair of equivalent ratios. For each pair, explain why they are equivalent ratios or draw a representation that shows why they are equivalent ratios.

1. $600 : 450$ and $60 : 45$
2. $60 : 45$ and $4 : 3$
3. $600 : 450$ and $4 : 3$

(From Unit 2.2.3)

12.2: Navigating a Table of Equivalent Ratios is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

12.3: Tables and Double Number Line Diagrams

Lesson

Let's contrast double number lines and tables.

Exercise 12.3.1: Number Talk: Constant Divident

Find the quotients mentally.

$150 \div 2$

$150 \div 4$

$150 \div 8$

Locate and label the quotients on the number line.



Figure 12.3.1

Exercise 12.3.2: Moving 3,000 Meters

The other day, we saw that Han can run 100 meters in 20 seconds.

Han wonders how long it would take him to run 3,000 meters at this rate. He made a table of equivalent ratios.

1. Do you agree that this table represents the situation? Explain your reasoning.

Table 12.3.1

20	100
10	50
1	5
3,000	

2. Complete the last row with the missing number.
3. What question about the situation does this number answer?
4. What could Han do to improve his table?
5. Priya can bike 150 meters in 20 seconds. At this rate, how long would it take her to bike 3,000 meters?

Table 12.3.2

6. Priya's neighbor has a dirt bike that can go 360 meters in 15 seconds. At this rate, how long would it take them to ride 3,000 meters?

Exercise 12.3.3: The International Space Station

The International Space Station orbits around the Earth at a constant speed. Your teacher will give you either a double number line or a table that represents this situation. Your partner will get the other representation.



Figure 12.3.2: International Space Station, by NASA. Public Domain. NASA. Source.

1. Complete the parts of your representation that you can figure out for sure.
2. Share information with your partner, and use the information that your partner shares to complete your representation.
3. What is the speed of the International Space Station?
4. Place the two completed representations side by side. Discuss with your partner some ways in which they are the same and some ways in which they are different.
5. Record at least one way that they are the same and one way they are different.

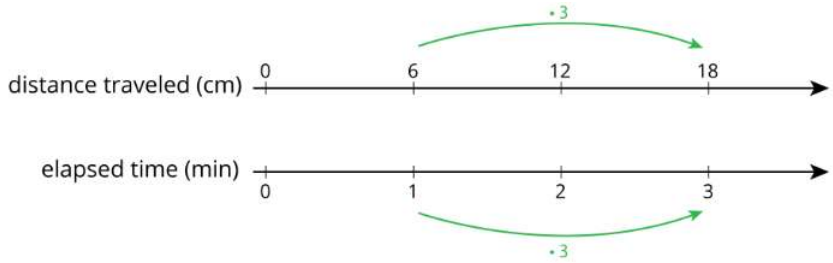
Are you ready for more?

Earth's circumference is about 40,000 kilometers and the orbit of the International Space Station is just a bit more than this. About how long does it take for the International Space Station to orbit Earth?

Summary

On a double number line diagram, we put labels in front of each line to tell what the numbers represent. On a table, we put labels at the top of each column to tell what the numbers represent.

Here are two different ways we can represent the situation: "A snail is moving at a constant speed down a sidewalk, traveling 6 centimeters per minute."



distance traveled (cm)	elapsed time (min)
12	2
6	1
60	10
18	3

•3 •3

Figure 12.3.3: A double number line with 4 evenly spaced tick marks. For distance traveled, in centimeters the numbers 0, 6, 12, and 18 are indicated and an arrow labeled times 3 is drawn from 6 to 18. For elapsed time, in minutes the numbers 0, 1, 2, and 3 are indicated and an arrow labeled times 3 is drawn from 1 to 3. A 2-column table with 4 rows of data. The first column is labeled distance traveled, in centimeters and the second column is labeled elapsed time, in minutes. Row 1: 12, 2; Row 2: 6, 1; Row 3: 60, 10; Row 4: 18, 3. An arrow labeled times 3 is drawn from Rows 2 to 4 to the left of the first column and to the right of the second column.

Both double number lines and tables can help us use multiplication to make equivalent ratios, but there is an important difference between the two representations.

On a double number line, the numbers on each line are listed in order. With a table, you can write the ratios in any order. For this reason, sometimes a table is easier to use to solve a problem.

For example, what if we wanted to know how far the snail travels in 10 minutes? Notice that 60 centimeters in 10 minutes is shown on the table, but there is not enough room for this information on the double number line.

Glossary Entries

Definition: Table

A table organizes information into horizontal *rows* and vertical *columns*. The first row or column usually tells what the numbers represent.

For example, here is a table showing the tail lengths of three different pets. This table has four rows and two columns.

Table 12.3.3

pet	tail length (inches)
dog	22
cat	12
mouse	2

Practice

Exercise 12.3.4

The double number line shows how much water and how much lemonade powder to mix to make different amounts of lemonade.

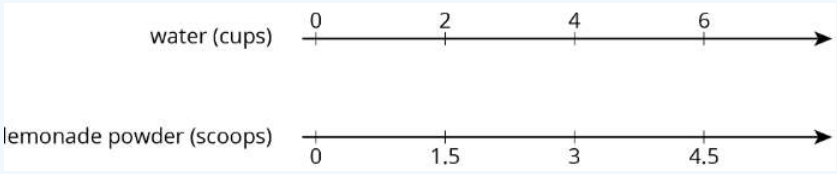


Figure 12.3.4

Make a table that represents the same situation.

Exercise 12.3.5

A bread recipe uses 3 tablespoons of olive oil for every 2 cloves of crushed garlic.

1. Complete the table to show different-sized batches of bread that taste the same as the recipe.
2. Draw a double number line that represents the same situation.
3. Which representation do you think works better in this situation? Explain why.

Table 12.3.4

olive oil (tablespoons)	crushed garlic (cloves)
3	2
1	
2	
5	
10	

Exercise 12.3.6

Clare travels at a constant speed, as shown on the double number line.

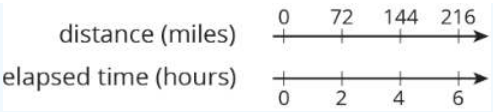


Figure 12.3.5: Double number line, 4 evenly spaced tick marks. Top line, distance, miles. Beginning at first tick mark, labels: 0, 72, 144, 216. Bottom line, elapsed time, hours. Beginning at first tick mark, labels: 0, 2, 4, 6.

At this rate, how far does she travel in each of these intervals of time? Explain or show your reasoning. If you get stuck, consider using a table.

1. 1 hour
2. 3 hours
3. 6.5 hours

Exercise 12.3.7

Lin and Diego travel in cars on the highway at constant speeds. In each case, decide who was traveling faster and explain how you know.

1. During the first half hour, Lin travels 23 miles while Diego travels 25 miles.
2. After stopping for lunch, they travel at different speeds. To travel the next 60 miles, it takes Lin 65 minutes and it takes Diego 70 minutes.

(From Unit 2.3.4)

Exercise 12.3.8

A sports drink recipe calls for $\frac{5}{3}$ tablespoons of powdered drink mix for every 12 ounces of water. How many batches can you make with 5 tablespoons of drink mix and 36 ounces of water? Explain your reasoning.

(From Unit 2.2.1)

Exercise 12.3.9

In this cube, each small square has side length 1 unit.

1. What is the surface area of this cube?
2. What is the volume of this cube?

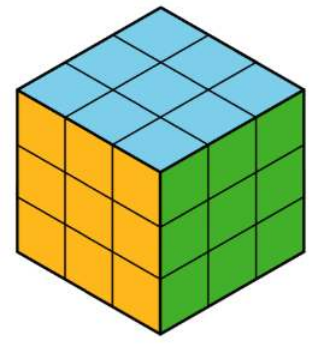


Figure 12.3.6

(From Unit 1.6.2)

12.3: Tables and Double Number Line Diagrams is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

12.4: Solving Equivalent Ratio Problems

Lesson

Let's practice getting information from our partner.

Exercise 12.4.1: What Do You Want to Know?

Consider the problem: A red car and a blue car enter the highway at the same time and travel at a constant speed. How far apart are they after 4 hours?

What information would you need to be able to solve the problem?

Exercise 12.4.2: Info Gap: Hot Chocolate and Potatoes

Your teacher will give you either a *problem card* or a *data card*. Do not show or read your card to your partner.

If your teacher gives you the *problem card*:

1. Silently read your card and think about what information you need to be able to answer the question.
2. Ask your partner for the specific information that you need.
3. Explain how you are using the information to solve the problem.
   Continue to ask questions until you have enough information to solve the problem.
4. Share the *problem card* and solve the problem independently.
5. Read the *data card* and discuss your reasoning.

If your teacher gives you the *data card*:

1. Silently read your card.
2. Ask your partner "*What specific information do you need?*" and wait for them to *ask* for information.
   If your partner asks for information that is not on the card, do not do the calculations for them. Tell them you don't have that information.
3. Before sharing the information, ask "*Why do you need that information?*" Listen to your partner's reasoning and ask clarifying questions.
4. Read the *problem card* and solve the problem independently.
5. Share the *data card* and discuss your reasoning.

Pause here so your teacher can review your work. Ask your teacher for a new set of cards and repeat the activity, trading roles with your partner.

Exercise 12.4.3: Comparing Reading Rates

- Lin read the first 54 pages from a 270-page book in the last 3 days.
- Diego read the first 100 pages from a 320-page book in the last 4 days.
- Elena read the first 160 pages from a 480-page book in the last 5 days.

If they continue to read every day at these rates, who will finish first, second, and third? Explain or show your reasoning.

Are you ready for more?

The ratio of cats to dogs in a room is $2:3$. Five more cats enter the room, and then the ratio of cats to dogs is $9:11$. How many cats and dogs were in the room to begin with?

Summary

To solve problems about something happening at the same rate, we often need:

- Two pieces of information that allow us to write a ratio that describes the situation.
- A third piece of information that gives us one number of an equivalent ratio. Solving the problem often involves finding the other number in the equivalent ratio.

Suppose we are making a large batch of fizzy juice and the recipe says, "Mix 5 cups of cranberry juice with 2 cups of soda water." We know that the ratio of cranberry juice to soda water is $5 : 2$, and that we need 2.5 cups of cranberry juice per cup of soda water.

We still need to know something about the size of the large batch. If we use 16 cups of soda water, what number goes with 16 to make a ratio that is equivalent to $5 : 2$?

To make this large batch taste the same as the original recipe, we would need to use 40 cups of cranberry juice.

Table 12.4.1

cranberry juice (cups)	soda water (cups)
5	2
2.5	1
40	16

Glossary Entries

Definition: Table

A table organizes information into horizontal *rows* and vertical *columns*. The first row or column usually tells what the numbers represent.

For example, here is a table showing the tail lengths of three different pets. This table has four rows and two columns.

Table 12.4.2

pet	tail length (inches)
dog	22
cat	12
mouse	2

Practice

Exercise 12.4.4

A chef is making pickles. He needs 15 gallons of vinegar. The store sells 2 gallons of vinegar for $3.00 and allows customers to buy any amount of vinegar. Decide whether each of the following ratios correctly represents the price of vinegar.

1. 4 gallons to $3.00
2. 1 gallon to $1.50
3. 30 gallons to $45.00
4. $2.00 to 30 gallons
5. $1.00 to $\frac{2}{3}$ gallon

Exercise 12.4.5

A caterer needs to buy 21 pounds of pasta to cater a wedding. At a local store, 8 pounds of pasta cost $12. How much will the caterer pay for the pasta there?

1. Write a ratio for the given information about the cost of pasta.
2. Would it be more helpful to write an equivalent ratio with 1 pound of pasta as one of the numbers, or with $1 as one of the numbers? Explain your reasoning, and then write that equivalent ratio.
3. Find the answer and explain or show your reasoning.

Exercise 12.4.6

Lin is reading a 47-page book. She read the first 20 pages in 35 minutes.

1. If she continues to read at the same rate, will she be able to complete this book in under 1 hour?
2. If so, how much time will she have left? If not, how much more time is needed? Explain or show your reasoning.

Exercise 12.4.7

Diego can type 140 words in 4 minutes.

1. At this rate, how long will it take him to type 385 words?
2. How many words can he type in 15 minutes?

If you get stuck, consider creating a table.

Exercise 12.4.8

A train that travels 30 miles in $\frac{1}{3}$ hour at a constant speed is going faster than a train that travels 20 miles in $\frac{1}{2}$ hour at a constant speed. Explain or show why.

(From Unit 2.3.5)

Exercise 12.4.9

Find the surface area of the polyhedron that can be assembled from this net. Show your reasoning.

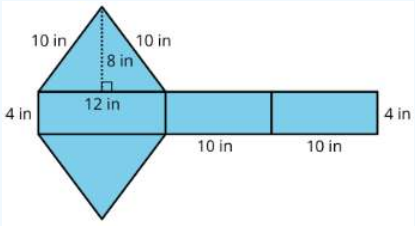


Figure 12.4.1: Polyhedron net. Three rectangles side by side. First rectangle, height of 4 inches and length of 12 inches. Second rectangle, height of 4 inches and length of 10 inches. Third rectangle, height of 4 inches and length of 10 inches. Two isosceles triangles whose bases are attached to top and bottom of first rectangle. Triangle has height of 8 inches and sides of 10 inches.

(From Unit 1.5.3)

12.4: Solving Equivalent Ratio Problems is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

Section 13: Part-Part-Whole Ratios

This page titled Section 13: Part-Part-Whole Ratios is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

13.1: Part-Part-Whole Ratios

Lesson

Let's look at situations where you can add the quantities in a ratio together.

Exercise 13.1.1: True or False: Mulltiplying by a Unit Fraction

True or false?

$\frac{1}{5} \cdot 45 = \frac{45}{5}$

$\frac{1}{5} \cdot 20 = \frac{1}{4} \cdot 24$

$42 \cdot \frac{1}{6} = \frac{1}{6} \cdot 42$

$486 \cdot \frac{1}{12} = \frac{480}{12} + \frac{6}{12}$

Exercise 13.1.2: Cubes of Paint

A recipe for maroon paint says, "Mix 5 ml of red paint with 3 ml of blue paint."

1. Use snap cubes to represent the amounts of red and blue paint in the recipe. Then, draw a sketch of your snap-cube representation of the maroon paint.
   a. What amount does each cube represent?
   b. How many milliliters of maroon paint will there be?
2. a. Suppose each cube represents 2 ml. How much of each color paint is there?
   Red: _____ ml
   Blue: _____ ml
   Maroon: _____ ml
   b. Suppose each cube represents 5 ml. How much of each color paint is there?
   Red: _____ ml
   Blue: _____ ml
   Maroon: _____ ml
3. a. Suppose you need 80 ml of maroon paint. How much red and blue paint would you mix? Be prepared to explain your reasoning.
   Red: _____ ml
   Blue: _____ ml
   Maroon: 80 ml
   b. If the original recipe is for one batch of maroon paint, how many batches are in 80 ml of maroon paint?

Exercise 13.1.3: Sneakers, Chicken, and Fruit Juice

Solve each of the following problems and show your thinking. If you get stuck, consider drawing a **tape diagram** to represent the situation.

1. The ratio of students wearing sneakers to those wearing boots is 5 to 6. If there are 33 students in the class, and all of them are wearing either sneakers or boots, how many of them are wearing sneakers?
2. A recipe for chicken marinade says, "Mix 3 parts oil with 2 parts soy sauce and 1 part orange juice." If you need 42 cups of marinade in all, how much of each ingredient should you use?
3. Elena makes fruit punch by mixing 4 parts cranberry juice to 3 parts apple juice to 2 parts grape juice. If one batch of fruit punch includes 30 cups of apple juice, how large is this batch of fruit punch?

Are you ready for more?

Using the recipe from earlier, how much fruit punch can you make if you have 50 cups of cranberry juice, 40 cups of apple juice, and 30 cups of grape juice?

Exercise 13.1.4: Invent Your Own Ratio Problem

1. Invent another ratio problem that can be solved with a tape diagram and solve it. If you get stuck, consider looking back at the problems you solved in the earlier activity.
2. Create a visual display that includes:
   - The new problem that you wrote, without the solution.
   - Enough work space for someone to show a solution.
3. Trade your display with another group, and solve each other's problem. Include a tape diagram as part of your solution. Be prepared to share the solution with the class.
4. When the solution to the problem you invented is being shared by another group, check their answer for accuracy.

Summary

A **tape diagram** is another way to represent a ratio. All the parts of the diagram that are the same size have the same value.

For example, this tape diagram represents the ratio of ducks to swans in a pond, which is $4 : 5$.

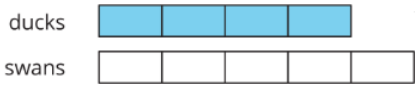


Figure 13.1.1

The first tape represents the number of ducks. It has 4 parts.

The second tape represents the number of swans. It has 5 parts.

There are 9 parts in all, because $4 + 5 = 9$.

Suppose we know there are 18 of these birds in the pond, and we want to know how many are ducks.

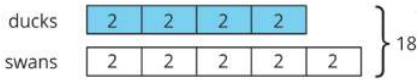


Figure 13.1.2

The 9 equal parts on the diagram need to represent 18 birds in all. This means that each part of the tape diagram represents 2 birds, because $18 \div 9 = 2$.

There are 4 parts of the tape representing ducks, and $4 \cdot 2 = 8$, so there are 8 ducks in the pond.

Glossary Entries

Definition: Tape Diagram

A tape diagram is a group of rectangles put together to represent a relationship between quantities.

For example, this tape diagram shows a ratio of 30 gallons of yellow paint to 50 gallons of blue paint.

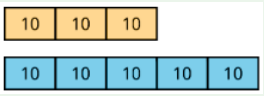


Figure 13.1.3

If each rectangle were labeled 5, instead of 10, then the same picture could represent the equivalent ratio of 15 gallons of yellow paint to 25 gallons of blue paint.

Practice

Exercise 13.1.5

Here is a tape diagram representing the ratio of red paint to yellow paint in a mixture of orange paint.

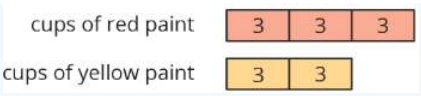


Figure 13.1.4

1. What is the ratio of yellow paint to red paint?
2. How many total cups of orange paint will this mixture yield?

Exercise 13.1.6

At the kennel, the ratio of cats to dogs is $4 : 5$. There are 27 animals in all. Here is a tape diagram representing this ratio.

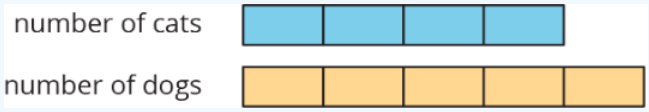


Figure 13.1.5

1. What is the value of each small rectangle?
2. How many dogs are at the kennel?
3. How many cats are at the kennel?

Exercise 13.1.7

Last month, there were 4 sunny days for every rainy day. If there were 30 days in the month, how many days were rainy? Explain your reasoning. If you get stuck, consider using a tape diagram.

Exercise 13.1.8

Noah entered a 100-mile bike race. He knows he can ride 32 miles in 160 minutes. At this rate, how long will it take him to finish the race? Use each table to find the answer. Next, explain which table you think works better in finding the answer.

Table A:

Table 13.1.1

distance (miles)	elapsed time (minutes)
32	160
1	
100	

Table B:

Table 13.1.2

distance (miles)	elapsed time (minutes)
32	160
96	
4	
100	

(From Unit 2.4.2)

https://math.libretexts.org/@go/page/39935

Exercise 13.1.9

A cashier worked an 8-hour day, and earned $58.00. The double number line shows the amount she earned for working different numbers of hours. For each question, explain your reasoning.

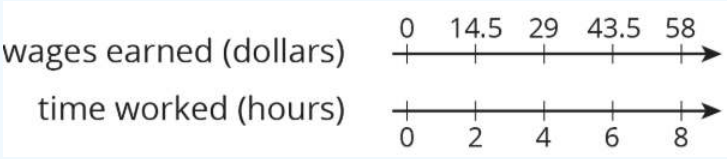


Figure 13.1.6

1. How much does the cashier earn per hour?
2. How much does the cashier earn if she works 3 hours?

(From Unit 2.4.3)

Exercise 13.1.10

A grocery store sells bags of oranges in two different sizes.

- The 3-pound bags of oranges cost $4.
- The 8-pound bags of oranges for $9.

Which oranges cost less per pound? Explain or show your reasoning.

(From Unit 2.3.5)

13.1: Part-Part-Whole Ratios is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

13.2: Solving More Ratio Problems

Lesson

Let's compare all our strategies for solving ratio problems.

Exercise 13.2.1: You Tell the Story

Describe a situation with two quantities that this tape diagram could represent.

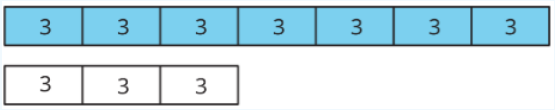


Figure 13.2.1

Exercise 13.2.2: A Trip to the Aquarium

Consider the problem: A teacher is planning a class trip to the aquarium. The aquarium requires 2 chaperones for every 15 students. The teacher plans accordingly and orders a total of 85 tickets. How many tickets are for chaperones, and how many are for students?

1. Solve this problem in *one* of three ways:

Use a triple number line.

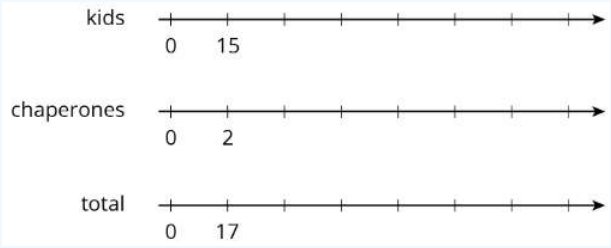


Figure 13.2.2: A triple number line with 8 evenly spaced tick marks. For kids the numbers 0 and 15 are indicated on the first 2 tick marks. For chaperones the numbers 0 and 2 are indicated on the first 2 tick marks. For total the numbers 0 and 17 are indicated on the first 2 tick marks.

Use a table.
(Fill rows as needed.)

Table 13.2.1

kids	chaperones	total
15	2	17

Use a tape diagram.

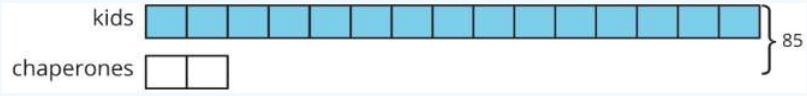


Figure 13.2.3

2. After your class discusses all three strategies, which do you prefer for this problem and why?

Are you ready for more?

Use the digits 1 through 9 to create three equivalent ratios. Use each digit only one time.

□: □ is equivalent to □□: □ and □□: □□

Exercise 13.2.3: Salad Dressing and Moving Boxes

Solve each problem, and show your thinking. Organize it so it can be followed by others. If you get stuck, consider drawing a double number line, table, or tape diagram.

1. A recipe for salad dressing calls for 4 parts oil for every 3 parts vinegar. How much oil should you use to make a total of 28 teaspoons of dressing?
2. Andre and Han are moving boxes. Andre can move 4 boxes every half hour. Han can move 5 boxes every half hour. How long will it take Andre and Han to move all 72 boxes?

Summary

When solving a problem involving equivalent ratios, it is often helpful to use a diagram. Any diagram is fine as long as it correctly shows the mathematics and you can explain it.

Let's compare three different ways to solve the same problem: The ratio of adults to kids in a school is $2:7$. If there is a total of 180 people, how many of them are adults?

- *Tape diagrams* are especially useful for this type of problem because both parts of the ratio have the same units ("number of people") and we can see the total number of parts.

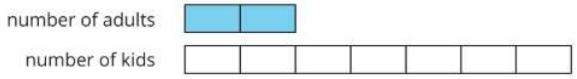


Figure 13.2.4

This tape diagram has 9 equal parts, and they need to represent 180 people total. That means each part represents $180 \div 9$, or 20 people.

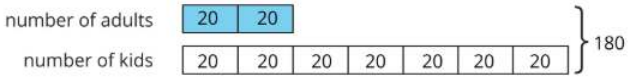


Figure 13.2.5: A tape diagram. For number of adults 2 equal parts and the number 20 is labeled in each part. For number of kids 7 equal parts and the number 20 is labeled in each equal part. A brace from number of adults to number of kids is labeled 180.

Two parts of the tape diagram represent adults. There are 40 adults in the school because $2 \cdot 20 = 40$.

- *Double or triple number lines* are useful when we want to see how far apart the numbers are from one another. They are harder to use with very big or very small numbers, but they could support our reasoning.

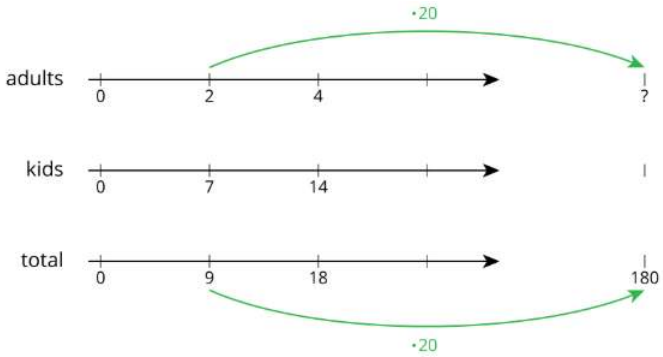


Figure 13.2.6: A triple number line with 4 evenly spaced tick marks. For adults the numbers 0, 2, 4, are indicated and the last tick mark is blank. For kids the numbers 0, 7, 14 are indicated and the last tick mark is blank. For total the numbers 0, 9, 18 are indicated and the last tick mark is blank. An arrow pointing from the 2nd tick mark of adults to a blank tick mark beyond the number line is labeled times 20. A second arrow pointing from the 2nd tick mark of total to a tick mark beyond the number line marked 180 is labeled times 20.

- *Tables* are especially useful when the problem has very large or very small numbers.

adults	kids	total
2	7	9
?		180

×20 ×20

Figure 13.2.7: A 3-column table with 2 rows of data. Column 1 is labeled adults, column 2 is labeled kids, and column 3 is labeled total. Row 1: 2, 7, 9; Row 2: blank, blank, 180. An arrow from row 1 to row 2 is labeled times 20.

We ask ourselves, “9 times what is 180?” The answer is 20. Next, we multiply 2 by 20 to get the total number of adults in the school.

Another reason to make diagrams is to communicate our thinking to others. Here are some good habits when making diagrams:

- Label each part of the diagram with what it represents.
- Label important amounts.
- Make sure you read what the question is asking and answer it.
- Make sure you make the answer easy to find.
- Include units in your answer. For example, write “4 cups” instead of just “4.”
- Double check that your ratio language is correct and matches your diagram.

Glossary Entries

Definition: Tape Diagram

A tape diagram is a group of rectangles put together to represent a relationship between quantities.

For example, this tape diagram shows a ratio of 30 gallons of yellow paint to 50 gallons of blue paint.

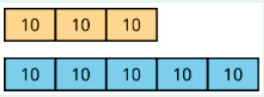


Figure 13.2.8

If each rectangle were labeled 5, instead of 10, then the same picture could represent the equivalent ratio of 15 gallons of yellow paint to 25 gallons of blue paint.

Practice

Exercise 13.2.4

Describe a situation that could be represented with this tape diagram.

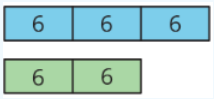


Figure 13.2.9

Exercise 13.2.5

There are some nickels, dimes, and quarters in a large piggy bank. For every 2 nickels there are 3 dimes. For every 2 dimes there are 5 quarters. There are 500 coins total.

1. How many nickels, dimes, and quarters are in the piggy bank? Explain your reasoning.
2. How much are the coins in the piggy bank worth?

Exercise 13.2.6

Two horses start a race at the same time. Horse A gallops at a steady rate of 32 feet per second and Horse B gallops at a steady rate of 28 feet per second. After 5 seconds, how much farther will Horse A have traveled? Explain or show your reasoning.

Exercise 13.2.7

Andre paid $13 for 3 books. Diego bought 12 books priced at the same rate. How much did Diego pay for the 12 books? Explain your reasoning.

(From Unit 2.3.5)

Exercise 13.2.8

Which polyhedron can be assembled from this net?

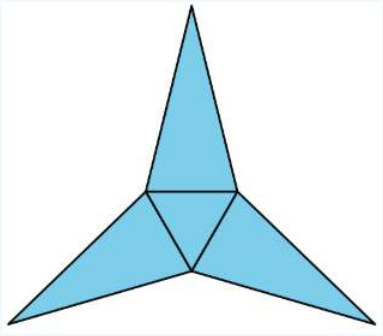


Figure 13.2.10

A. A triangular pyramid
B. A trapezoidal prism
C. A rectangular pyramid
D. A triangular prism

(From Unit 1.5.4)

Exercise 13.2.9

Find the area of the triangle. Show your reasoning. If you get stuck, consider drawing a rectangle around the triangle.

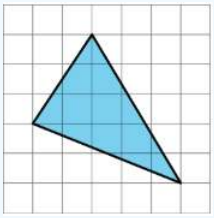


Figure 13.2.11: A slanted triangle drawn on a 7 by 7 grid. The bottom left vertice is 1 unit over and 3 units up from the bottom left corner of the grid. The top vertice is 3 units over and 6 units up from the bottom left corner of the grid. The bottom right vertice is 6 units over and 1 unit up from the bottom left corner of the grid.

(From Unit 1.3.4)

13.2: Solving More Ratio Problems is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

Section 14: Let's Put it to Work

This page titled Section 14: Let's Put it to Work is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

14.1: A Fermi Problem

Lesson

Let's solve a Fermi problem.

Exercise 14.1.1: Fix It!

Andre likes a hot cocoa recipe with 1 cup of milk and 3 tablespoons of cocoa. He poured 1 cup of milk but accidentally added 5 tablespoons of cocoa.

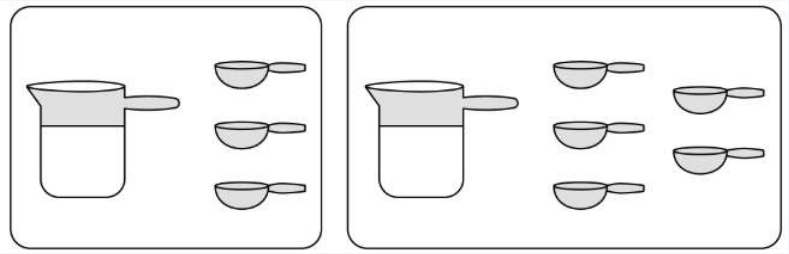


Figure 14.1.1

1. How can you fix Andre's mistake and make his hot cocoa taste like the recipe?
2. Explain how you know your adjustment will make Andre's hot cocoa taste the same as the one in the recipe.

Exercise 14.1.2: Who Was Fermi?

1. Record the Fermi question that your class will explore together.
2. Make an estimate of the answer. If making an estimate is too hard, consider writing down a number that would definitely be too low and another number that would definitely be too high.
3. What are some smaller sub-questions we would need to figure out to reasonably answer our bigger question?
4. Think about how the smaller questions above should be organized to answer the big question. Label each smaller question with a number to show the order in which they should be answered. If you notice a gap in the set of sub-questions (i.e., there is an unlisted question that would need to be answered before the next one could be tackled), write another question to fill the gap.

Exercise 14.1.3: Researching Your Own Fermi Problem

1. Brainstorm at least five Fermi problems that you want to research and solve. If you get stuck, consider starting with "How much would it cost to . . .?" or "How long would it take to . . .?"
2. Pause here so your teacher can review your questions and approve one of them.
3. Use the graphic organizer to break your problem down into sub-questions.

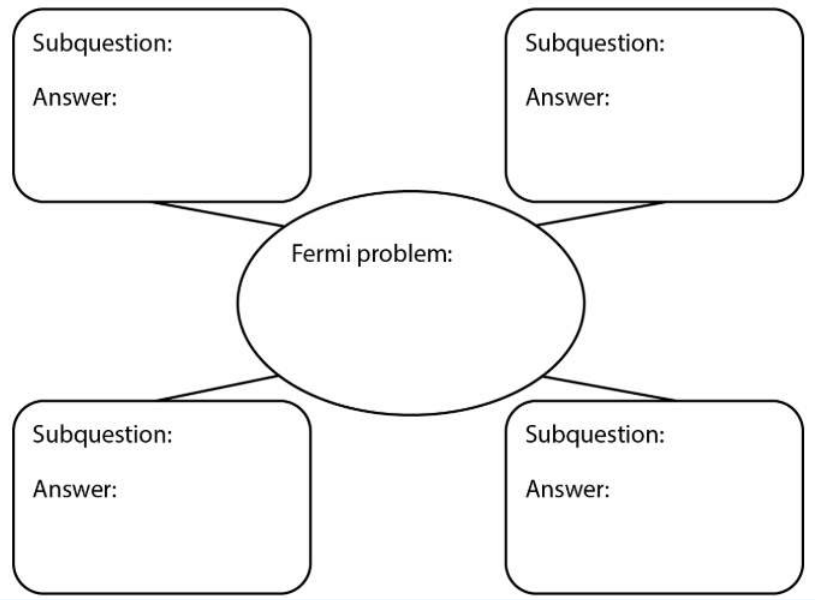


Figure 14.1.2

4. Find the information you need to get closer to answering your question. Measure, make estimates, and perform any necessary calculations. If you get stuck, consider using tables or double number line diagrams.
5. Create a visual display that includes your Fermi problem and your solution. Organize your thinking so it can be followed by others.

This page titled 14.1: A Fermi Problem is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

14.2: How Much for One?

Lesson

Let's use ratios to describe how much things cost.

Exercise 14.2.1: Number Talk: Remainders in Division

Find the quotient mentally.

$246 \div 12$

Exercise 14.2.2: Grocery Shopping

Answer each question and explain or show your reasoning.

1. Eight avocados cost $4.
   a. How much do 16 avocados cost?
   b. How much do 20 avocados cost?
   c. How much do 9 avocados cost?
2. Twelve large bottles of water cost $9.
   a. How many bottles can you buy for $3?
   b. What is the cost per bottle of water?
   c. How much would 7 bottles of water cost?
3. A 10-pound sack of flour costs $8.
   a. How much does 40 pounds of flour cost?
   b. What is the cost per pound of flour?

Here is an applet you can use if you choose to.

Are you ready for more?

It is commonly thought that buying larger packages or containers, sometimes called *buying in bulk*, is a great way to save money. For example, a 6-pack of soda might cost $3 while a 12-pack of the same brand costs $5.

Find 3 different cases where it is not true that buying in bulk saves money. You may use the internet or go to a local grocery store and take photographs of the cases you find. Make sure the products are the same brand. For each example that you find, give the quantity or size of each, and describe how you know that the larger size is not a better deal.

Exercise 14.2.3: More Shopping

1. Four bags of chips cost $6.
   a. What is the cost per bag?
   b. At this rate, how much will 7 bags of chips cost?
2. At a used book sale, 5 books cost $15.
   a. What is the cost per book?
   b. At this rate, how many books can you buy for $21?
3. Neon bracelets cost $1 for 4.
   a. What is the cost per bracelet?
   b. At this rate, how much will 11 neon bracelets cost?
   Pause here so your teacher can review your work.
4. Your teacher will assign you one of the problems. Create a visual display that shows your solution to the problem. Be prepared to share your solution with the class.

Here is an applet you may use if you choose to.

Summary

The **unit price** is the price of 1 thing—for example, the price of 1 ticket, 1 slice of pizza, or 1 kilogram of peaches.

If 4 movie tickets cost \$28, then the unit price would be the cost *per* ticket. We can create a double number line to find the unit price.

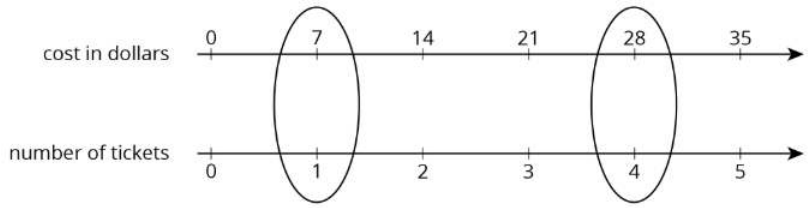


Figure 14.2.1: Double number line, 6 evenly spaced tick marks. Top line, cost in dollars. Beginning with the first tick mark, labels: 0, 7, 14, 21, 28, 35. Bottom line, number of tickets. Beginning with the first tick mark, labels: 0, 1, 2, 3, 4, 5. There is a circle around 7 and 1 and a second circle around 28 and 4.

This double number line shows that the cost for 1 ticket is \$7. We can also find the unit price by dividing, $28 \div 4 = 7$, or by multiplying, $28 \cdot \frac{1}{4} = 7$.

Glossary Entries

Definition: Double Number Line Diagram

A double number line diagram uses a pair of parallel number lines to represent equivalent ratios. The locations of the tick marks match on both number lines. The tick marks labeled 0 line up, but the other numbers are usually different.

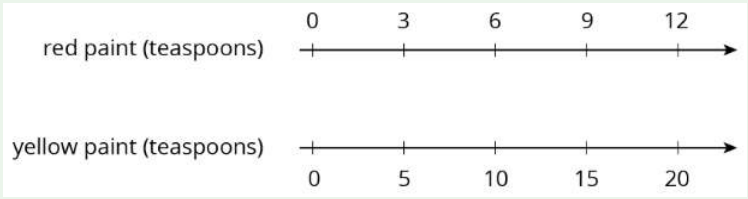


Figure 14.2.2

Definition: Per

The word *per* means "for each." For example, if the price is \$5 per ticket, that means you will pay \$5 *for each* ticket. Buying 4 tickets would cost \$20, because $4 \cdot 5 = 20$

Definition: Unit Price

The unit price is the cost for one item or for one unit of measure. For example, if 10 feet of chain link fencing cost \$150, then the unit price is $150 \div 10$, or \$15 per foot.

Practice

Exercise 14.2.4

In 2016, the cost of 2 ounces of pure gold was \$2,640. Complete the double number line to show the cost for 1, 3, and 4 ounces of gold.

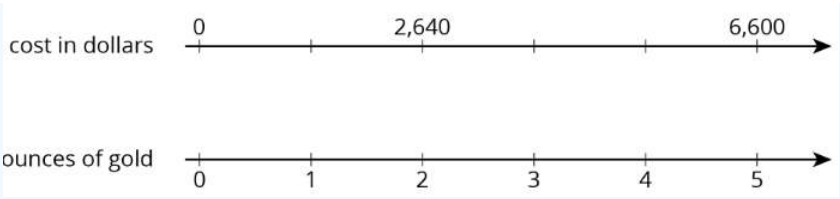


Figure 14.2.3: A double number line with 6 evenly spaced tick marks. The top number line is labeled cost in dollars; the first tick mark is labeled 0, the third tick mark is labeled 2,640 and the sixth tick mark is labeled 6600. The bottom number line is labeled ounces of gold and starting with the first tick mark 0, 1, 2, 3, 4 and 5 are labeled.

Exercise 14.2.5

The double number line shows that 4 pounds of tomatoes cost $14. Draw tick marks and write labels to show the prices of 1, 2, and 3 pounds of tomatoes.

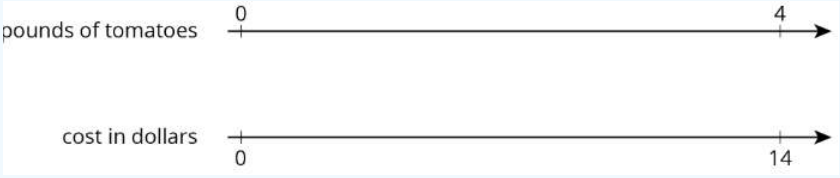


Figure 14.2.4: A double number line with one tick mark at the beginning of the line and one at the end. The top number line is labeled pounds of tomatoes with the first tick mark labeled 0 and the last tick mark labeled 4. The bottom number line is labeled cost in dollars with the first tick mark labeled 0 and the last tick mark labeled 14.

Exercise 14.2.6

4 movie tickets cost $48. At this rate, what is the cost of:

1. 5 movie tickets?
2. 11 movie tickets?

Exercise 14.2.7

Priya bought these items at the grocery store. Find each unit price.

1. 12 eggs for $3. How much is the cost per egg?
2. 3 pounds of peanuts for $7.50. How much is the cost per pound?
3. 4 rolls of toilet paper for $2. How much is the cost per roll?
4. 10 apples for $3.50. How much is the cost per apple?

Exercise 14.2.8

Clare made a smoothie with 1 cup of yogurt, 3 tablespoons of peanut butter, 2 teaspoons of chocolate syrup, and 2 cups of crushed ice.

1. Kiran tried to double this recipe. He used 2 cups of yogurt, 6 tablespoons of peanut butter, 5 teaspoons of chocolate syrup, and 4 cups of crushed ice. He didn't think it tasted right. Describe how the flavor of Kiran's recipe compares to Clare's recipe.
2. How should Kiran change the quantities that he used so that his smoothie tastes just like Clare's?

(From Unit 2.2.1)

Exercise 14.2.9

A drama club is building a wooden stage in the shape of a trapezoidal prism. The height of the stage is 2 feet. Some measurements of the stage are shown here.

https://math.libretexts.org/@go/page/39927

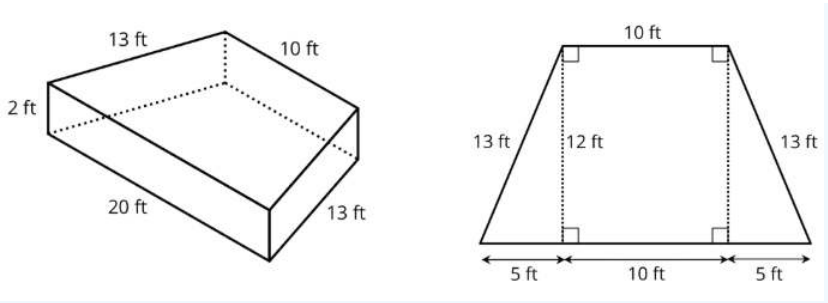


Figure 14.2.5: Trapezoidal prism is shown on left. Trapezoid is shown on right. Trapezoidal prism has bottom base of 20 feet, top base of 10 feet, legs of 13 feet, and depth 2 feet. The trapezoid has top base 10 feet, legs 13 feet, and bottom base 20 feet. Two vertical dashed lines are drawn from the top two vertices to the bottom base of the trapezoid creating two right triangles and a rectangle. The rectangle has top and bottom base of 10 feet and vertical sides 12 feet. In the two triangles, horizontal legs 5 feet, vertical legs 12 feet, and hypotenuse 13 feet.

What is the area of all the faces of the stage, excluding the bottom? Show your reasoning. If you get stuck, consider drawing a net of the prism.

(From Unit 1.5.4)

14.2: How Much for One? is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

SECTION OVERVIEW

3: Unit Rates and Percentages

Section 15: Unit of Measurement

15.1: The Burj Khalifa

Section 16: Unit Conversion

16.1: Anchoring Units of Measurement
16.2: Measuring with Different-Sized Units
16.3: Converting Units

Section 17: Rates

17.1: Comparing Speeds and Prices
17.2: Interpreting Rates
17.3: Equivalent Ratios Have the Same Unit Rates
17.4: More about Constant Speed
17.5: Solving Rate Problems

Section 18: Percentages

18.1: Percentages and Double Number Lines
18.2: Percentages and Tape Diagrams
18.3: Benchmark Percentages
18.4: Solving Percentage Problems
18.5: Finding This Percent of That
18.6: Finding the Percentage

Section 19: Let's Put it to Work

19.1: Painting a Room

Section 20: New Page

20.1: What Are Percentages?

This page titled 3: Unit Rates and Percentages is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

Section 15: Unit of Measurement

This page titled Section 15: Unit of Measurement is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

15.1: The Burj Khalifa

Lesson

Let's investigate the Burj Khalifa building.

Exercise 15.1.1: Estimating Height

Use the picture to estimate the height of Hyperion, the tallest known tree.

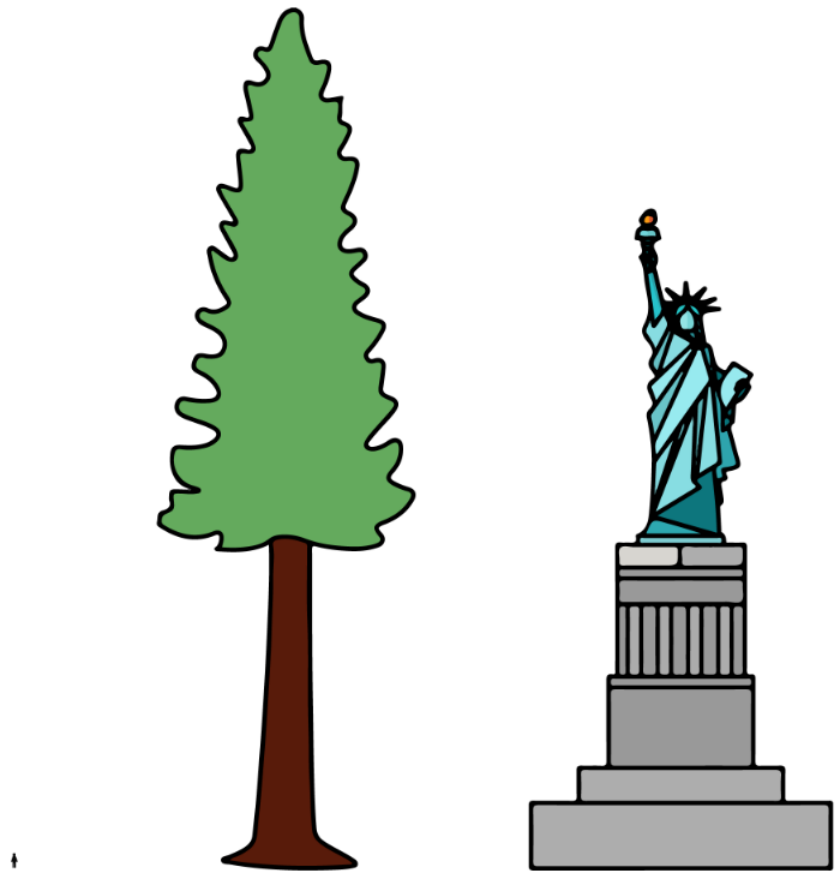


Figure 15.1.1

Exercise 15.1.2: Window Washing



Figure 15.1.2: Window Washers, by sneakerdog. CC BY 2.0. Flickr. Source.

A window-washing crew can finish 15 windows in 18 minutes.

If this crew was assigned to wash all the windows on the outside of the Burj Khalifa, how long will the crew be washing at this rate?

Exercise 15.1.3: Climbing the Burj Khalifa

In 2011, a professional climber scaled the outside of the Burj Khalifa, making it all the way to 828 meters (the highest point on which a person can stand) in 6 hours.

Assuming they climbed at the same rate the whole way:

1. How far did they climb in the first 2 hours?
2. How far did they climb in 5 hours?
3. How far did they climb in the final 15 minutes?

Are you ready for more?

Have you ever seen videos of astronauts on the Moon jumping really high? An object on the Moon weighs less than it does on Earth because the Moon has much less mass than Earth.

1. A person who weighs 100 pounds on Earth weighs 16.5 pounds on the Moon. If a boy weighs 60 pounds on Earth, how much does he weigh on the Moon?
2. Every 100 pounds on Earth are the equivalent to 38 pounds on Mars. If the same boy travels to Mars, how much would he weigh there?

Summary

There are many real-world situations in which something keeps happening at the same rate. For example:

- a bus stop that is serviced by 4 buses per hour
- a washing machine that takes 45 minutes per load of laundry
- a school cafeteria that serves 15 students per minute

In situations like these, we can use equivalent ratios to predict how long it will take for something to happen some number of times, or how many times it will happen in a particular length of time.

For example, how long will it take the school cafeteria to serve 600 students?

The table shows that it will take the cafeteria 40 minutes to serve 600 students.

Table 15.1.1

number of students	time in minutes

number of students	time in minutes
15	1
60	4
600	40

How many students can the cafeteria serve in 1 hour?

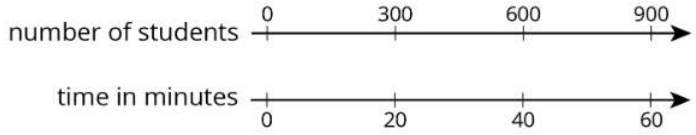


Figure 15.1.3: Double number line, 4 evenly spaced tick marks. Top line, number of students. Beginning at first tick mark, labels: 0, 300, 600, 900. Bottom line, time in minutes. Beginning at first tick mark, labels: 0, 20, 40, 60.

The double number line shows that the cafeteria can serve 900 students in 1 hour.

Practice

Exercise 15.1.4

An elevator travels 310 feet in 10 seconds. At that speed, how far can this elevator travel in 12 seconds? Explain your reasoning.

Exercise 15.1.5

Han earns $33.00 for babysitting 4 hours. At this rate, how much will he earn if he babysits for 7 hours? Explain your reasoning.

Exercise 15.1.6

The cost of 5 cans of dog food is $4.35. At this price, how much do 11 cans of dog food cost? Explain your reasoning.

Exercise 15.1.7

A restaurant has 26 tables in its dining room. It takes the waitstaff 10 minutes to clear and set 4 tables. At this rate, how long will it take the waitstaff to clear and set all the tables in the dining room? Explain or show your reasoning.

Exercise 15.1.8

A sandwich shop serves 4 ounces of meat and 3 ounces of cheese on each sandwich. After making sandwiches for an hour, the shop owner has used 91 combined ounces of meat and cheese.

1. How many combined ounces of meat and cheese are used on each sandwich?
2. How many sandwiches were made in the hour?
3. How many ounces of meat were used?
4. How many ounces of cheese were used?

(From Unit 2.5.2)

Exercise 15.1.9

Here is a flower made up of yellow hexagons, red trapezoids, and green triangles.

1. How many copies of this flower pattern could you build if you had 30 yellow hexagons, 50 red trapezoids, and 60 green triangles?

2. Of which shape would you have the most left over?

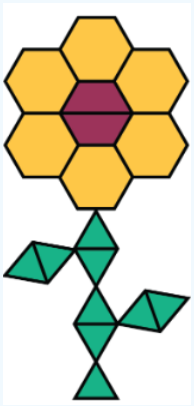


Figure 15.1.4

(From Unit 2.4.4)

Exercise 15.1.10

Match each quantity in the first list with an appropriate unit of measurement from the second list.

A. the perimeter of a baseball field
B. the area of a bed sheet
C. the volume of a refrigerator
D. the surface area of a tissue box
E. the length of a spaghetti noodle
F. the volume of a large lake
G. the surface area of the the moon

1. centimeters (cm)
2. cubic feet (cu ft)
3. cubic kilometers (cu km)
4. meters (m)
5. square feet (sq ft)
6. square inches (sq in)
7. square kilometers (sq km)

(From Unit 1.5.5)

This page titled 15.1: The Burj Khalifa is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

Section 16: Unit Conversion

This page titled Section 16: Unit Conversion is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

16.1: Anchoring Units of Measurement

Lesson

Let's see how big different things are.

Exercise 16.1.1: Estimating Volume

Estimate the volume of the tiny salt shaker.



Figure 16.1.1

Exercise 16.1.2: Cutting Strong

Your teacher will assign you one of the following lengths:

1 centimeter, 1 foot, 1 inch, 1 meter, or 1 yard.

Estimate and cut a piece of string as close to your assigned length as you can without using a measurement tool.

Exercise 16.1.3: Card Sort: Measurements

Your teacher will give you some cards with the names of different units of measurement and other cards with pictures of objects.

1. Sort the units of measurement into groups based on the attribute they measure. Pause here so your teacher can review your groups.
2. Match each picture card that has “L” in the top right corner with the closest unit to the length of the object.
3. Match each picture card that has “V” in the top right corner with the closest unit to the volume of the object.
4. Match each picture card that has “WM” in the top right corner with the closest unit to the weight or mass of the object.

Your teacher will assign you a new group to discuss how you matched the objects. If you disagree, work to reach an agreement.

Summary

We can use everyday objects to estimate standard units of measurement.

For units of length:

- 1 millimeter is about the thickness of a dime.
- 1 centimeter is about the width of a pinky finger.
- 1 inch is about the length from the tip of your thumb to the first knuckle.
- 1 foot is the length of a football.
- 1 yard is about the length of a baseball bat.
- 1 meter is about the length of a baseball bat and ball.
- 1 kilometer is about the distance someone walks in ten minutes.
- 1 mile is about the distance someone runs in ten minutes.

For units of volume:

- 1 milliliter is about the volume of a raindrop.
- 1 cup is about the volume of a school milk carton.

- 1 quart is about the volume of a large sports drink bottle.
- 1 liter is about the volume of a reusable water bottle.
- 1 gallon is about the volume of a large milk jug.

For units of weight and mass:

- 1 gram is about the mass of a raisin.
- 1 ounce is about the weight of a slice of bread.
- 1 pound is about the weight of a loaf of bread.
- 1 kilogram is about the mass of a textbook.
- 1 ton is about the weight of a small car.

Practice

Exercise 16.1.4

Select the unit from the list that you would use to measure each object.

1. The length of a pencil
2. The weight or mass of a pencil
3. The volume of a pencil
4. The weight or mass of a hippopotamus
5. The length of a hippopotamus
6. The length of a fingernail clipping
7. The weight or mass of a fingernail clipping
8. The volume of a sink
9. The volume of a bowl
10. The length of a chalkboard or whiteboard
11. The weight or mass of a chalkboard or whiteboard
12. The length of the border between the United States and Canada

- centimeters
- cups
- feet
- gallons
- grams
- inches
- kilograms
- kilometers
- liters
- meters
- miles
- milliliters
- millimeters
- ounces
- pounds
- quarts
- tons
- yards

Exercise 16.1.5

When this pet hamster is placed on a digital scale, the scale reads 1.5.



Figure 16.1.2: “Hamster in hand”, by Keith Pomakis . CC BY-SA 2.0. Wikimedia Commons. Source.

What could be the units?

Exercise 16.1.6

Circle the larger unit of measure. Then, determine if the unit measures distance, volume, or weight (or mass).

1. meter or kilometer
2. yard or foot
3. cup or quart
4. pound or ounce
5. liter or milliliter
6. gram or kilogram

Exercise 16.1.7

Elena mixes 5 cups of apple juice with 2 cups of sparkling water to make sparkling apple juice. For a party, she wants to make 35 cups of sparkling apple juice. How much of each ingredient should Elena use? Explain or show your reasoning.

(From Unit 2.5.1)

Exercise 16.1.8

Lin bought 3 hats for $22.50. At this rate, how many hats could she buy with $60.00? If you get stuck, consider using the table.

Table 16.1.1

number of hats	price in dollars

(From Unit 2.4.2)

Exercise 16.1.9

Light travels about 180 million kilometers in 10 minutes. How far does it travel in 1 minute? How far does it travel in 1 second? Show your reasoning.

(From Unit 2.3.4)

16.1: Anchoring Units of Measurement is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

16.2: Measuring with Different-Sized Units

Lesson

Let's measure things.

Exercise 16.2.1: Width of a Paper

Your teacher will show you two rods. Does it take more green rods or blue rods lined up end to end to measure the width of a piece of printer paper?

Exercise 16.2.2: Measurement Stations

Station 1

- Each large cube is 1 cubic inch. Count how many cubic inches completely pack the box without gaps.
- Each small cube is 1 cubic centimeter. Each rod is composed of 10 cubic centimeters. Count how many cubic centimeters completely fill the box.

Table 16.2.1

	cubic inches	cubic centimeters
volume of the box		

Station 2

Your teacher showed you a length.

- Use the meter stick to measure the length to the nearest meter.
- Use a ruler to measure the length to the nearest foot.

Table 16.2.2

	meters	feet
length of _____		

Station 3

Watch the video.

- Count how many times you can fill the quart bottle from the gallon jug.
- Count how many times you can fill the liter bottle from the gallon jug.

Video *Quarts and Liters* available at player.vimeo.com/video/304136597.

Table 16.2.3

	quarts	liters
1 gallon of water		

Station 4

Use the applet to record the weights of different objects in different units.

Record their weights in ounces, pounds, grams, and kilograms.

Table 16.2.4

object	ounces	pounds	grams	kilograms

Station 5

- Count how many level teaspoons of salt it takes to fill the graduated cylinder to 20 milliliters, 40 milliliters, and 50 milliliters.
- Pour the salt back into the original container.

Table 16.2.5

	milliliters	teaspoons
small amount of salt	20	
medium amount of salt	40	
large amount of salt	50	

After you finish all five stations, answer the following questions with your group.

1. Which is larger, a cubic inch or a cubic centimeter? Did more cubic inches or cubic centimeters fit in the cardboard box? Why?
2. Did it take more feet or meters to measure the indicated length? Why?
3. Which is bigger, a quart or a liter? Explain your reasoning.
4. Use the data from Station 4 to put the units of weight and mass in order from smallest to largest. Explain your reasoning.
5. a. About how many teaspoons of salt would it take to fill the graduated cylinder to 100 milliliters?
   b. If you poured 15 teaspoons of salt into an empty graduated cylinder, about how many milliliters would it fill?
   c. How many milliliters per teaspoon are there?
   d. How many teaspoons per milliliter are there?

Are you ready for more?

People in the medical field use metric measurements when working with medicine. For example, a doctor might prescribe medication in 10 mg tablets.

Brainstorm a list of reasons why healthcare workers would do this. Organize your thinking so it can be followed by others.

Summary

The size of the unit we use to measure something affects the measurement.

If we measure the same quantity with different units, it will take more of the smaller unit and fewer of the larger unit to express the measurement. For example, a room that measures 4 yards in length will measure 12 feet.

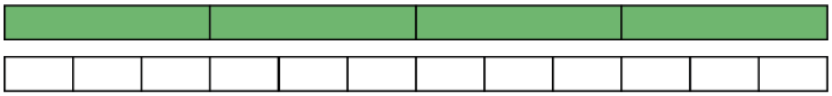


Figure 16.2.1

There are 3 feet in a yard, so one foot is $\frac{1}{3}$ of a yard.

- It takes 3 times as many feet to measure the same length as it does with yards.
- It takes $\frac{1}{3}$ as many yards to measure the same length as it does with feet.

Practice

Exercise 16.2.3

Decide if each is a measurement of length, area, volume, or weight (or mass).

1. How many centimeters across a handprint
2. How many square inches of paper needed to wrap a box
3. How many gallons of water in a fish tank
4. How many pounds in a bag of potatoes
5. How many feet across a swimming pool

6. How many ounces in a bag of grapes
7. How many liters in a punch bowl
8. How many square feet of grass in a lawn

(From Unit 3.2.1)

Exercise 16.2.4

Clare says, “This classroom is 11 meters long. A meter is longer than a yard, so if I measure the length of this classroom in yards, I will get less than 11 yards.” Do you agree with Clare? Explain your reasoning.

Exercise 16.2.5

Tyler’s height is 57 inches. What could be his height in centimeters?

A. 22.4
B. 57
C. 144.8
D. $3,551$

Exercise 16.2.6

A large soup pot holds 20 quarts. What could be its volume in liters?

A. 7.57
B. 19
C. 21
D. 75.7

Exercise 16.2.7

Clare wants to mail a package that weighs $4\frac{1}{2}$ pounds. What could this weight be in kilograms?

A. 2.04
B. 4.5
C. 9.92
D. $4,500$

Exercise 16.2.8

Noah bought 15 baseball cards for $9.00. Assuming each baseball card costs the same amount, answer the following questions.

1. At this rate, how much will 30 baseball cards cost? Explain your reasoning.
2. At this rate, how much will 12 baseball cards cost? Explain your reasoning.
3. Do you think this information would be better represented using a table or a double number line? Explain your reasoning.

(From Unit 2.4.3)

Exercise 16.2.9

Jada traveled 135 miles in 3 hours. Andre traveled 228 miles in 6 hours. Both Jada and Andre traveled at a constant speed.

1. How far did Jada travel in 1 hour?
2. How far did Andre travel in 1 hour?
3. Who traveled faster? Explain or show your reasoning.

(From Unit 2.3.4)

16.2: Measuring with Different-Sized Units is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

16.3: Converting Units

Lesson

Let's convert measurements to different units.

Exercise 16.3.1: Number Talk: Fractions of Number

Find the values mentally.

$\frac{1}{4}$ of 32

$\frac{3}{4}$ of 32

$\frac{3}{8}$ of 32

$\frac{3}{8}$ of 64

Exercise 16.3.2: Road Trip

Elena and her mom are on a road trip outside the United States. Elena sees this road sign.

Elena's mom is driving 75 miles per hour when she gets pulled over for speeding.



Figure 16.3.1

1. The police officer explains that 8 kilometers is approximately 5 miles.
   a. How many kilometers are in 1 mile?
   b. How many miles are in 1 kilometer?
2. If the speed limit is 80 kilometers per hour, and Elena's mom was driving 75 miles per hour, was she speeding? By how much?

Exercise 16.3.3: Veterinary Weights

A veterinarian uses weights in kilograms to figure out what dosages of medicines to prescribe for animals. For every 10 kilograms, there are 22 pounds.

1. Calculate each animal's weight in kilograms. Explain or show your reasoning. If you get stuck, consider drawing a double number line or table.
   a. Fido the Labrador weighs 88 pounds.
   b. Spot the Beagle weighs 33 pounds.
   c. Bella the Chihuahua weighs $5\frac{1}{2}$ pounds.
2. A certain medication says it can only be given to animals over 25 kilograms. How much is this in pounds?

Exercise 16.3.4: Cooking with a Tablespoon

Diego is trying to follow a recipe, but he cannot find any measuring cups! He only has a tablespoon. In the cookbook, it says that 1 cup equals 16 tablespoons.

1. How could Diego use the tablespoon to measure out these ingredients?
   $\frac{1}{2}$ cup almonds
   $1\frac{1}{4}$ cup of oatmeal
   $2\frac{3}{4}$ cup of flour

2. Diego also adds the following ingredients. How many cups of each did he use?
   28 tablespoons of sugar
   6 tablespoons of cocoa powder

Summary

When we measure something in two different units, the measurements form an equivalent ratio. We can reason with these equivalent ratios to convert measurements from one unit to another.

Suppose you cut off 20 inches of hair. Your Canadian friend asks how many centimeters of hair that was. Since 100 inches equal 254 centimeters, we can use equivalent ratios to find out how many centimeters equal 20 inches.

Using a double number line:

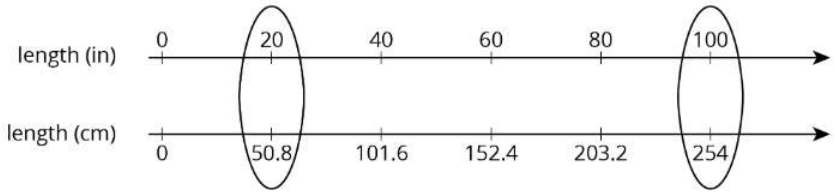


Figure 16.3.2

Using a table:

Table 16.3.1

length (in)	length (cm)
100	254
1	2.54
20	50.8

One quick way to solve the problem is to start by finding out how many centimeters are in 1 inch. We can then multiply 2.54 and 20 to find that 20 inches equal 50.8 centimeters.

Practice

Exercise 16.3.5

Priya's family exchanged 250 dollars for 4,250 pesos. Priya bought a sweater for 510 pesos. How many dollars did the sweater cost?

Table 16.3.2

pesos	dollars
4, 250	250
	25
	1
	3
510	

Exercise 16.3.6

There are 3,785 milliliters in 1 gallon, and there are 4 quarts in 1 gallon. For each question, explain or show your reasoning.

1. How many milliliters are in 3 gallons?
2. How many milliliters are in 1 quart?

Exercise 16.3.7

Lin knows that there are 4 quarts in a gallon. She wants to convert 6 quarts to gallons, but cannot decide if she should multiply 6 by 4 or divide 6 by 4 to find her answer. What should she do? Explain or show your reasoning. If you get stuck, consider drawing a double number line or using a table.

Exercise 16.3.8

Tyler has a baseball bat that weighs 28 ounces. Find this weight in kilograms and in grams. (Note: 1 kilogram ≈ 35 ounces)

Exercise 16.3.9

Identify whether each unit measures length, volume, or weight (or mass).

- Mile
- Cup
- Pound
- Centimeter
- Liter
- Gram
- Pint
- Yard
- Kilogram
- Teaspoon
- Milliliter

(From Unit 3.1.1)

Exercise 16.3.10

A recipe for trail mix uses 7 ounces of almonds with 5 ounces of raisins. (Almonds and raisins are the only ingredients.) How many ounces of almonds would be in a one-pound bag of this trail mix? Explain or show your reasoning.

(From Unit 2.4.1)

Exercise 16.3.11

An ant can travel at a constant speed of 980 inches every 5 minutes.

1. How far does the ant travel in 1 minute?
2. At this rate, how far can the ant travel in 7 minutes?

(From Unit 2.3.4)

16.3: Converting Units is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

Section 17: Rates

This page titled Section 17: Rates is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

17.1: Comparing Speeds and Prices

Lesson

Let's compare some speeds and some prices.

Exercise 17.1.1: Closest Quotient

Is the value of each expression closer to $\frac{1}{2}$, 1, or $1\frac{1}{2}$?

1. $20 \div 18$
2. $9 \div 20$
3. $7 \div 5$

Exercise 17.1.2: More Treadmills

Some students did treadmill workouts, each one running at a constant speed. Answer the questions about their workouts. Explain or show your reasoning.

- Tyler ran 4,200 meters in 30 minutes.
- Kiran ran 6,300 meters in hour.
- Mai ran 6.3 kilometers in 45 minutes.

1. What is the same about the workouts done by:
   a. Tyler and Kiran?
   b. Kiran and Mai?
   c. Mai and Tyler?
2. At what rate did each of them run?
3. How far did Mai run in her first 30 minutes on the treadmill?

Are you ready for more?

Tyler and Kiran each started running at a constant speed at the same time. Tyler ran 4,200 meters in 30 minutes and Kiran ran 6,300 meters in $\frac{1}{2}$ hour. Eventually, Kiran ran 1 kilometer more than Tyler. How much time did it take for this to happen?

Exercise 17.1.3: The Best Deal on Beans

Four different stores posted ads about special sales on 15-oz cans of baked beans.

1. Which store is offering the best deal? Explain your reasoning.

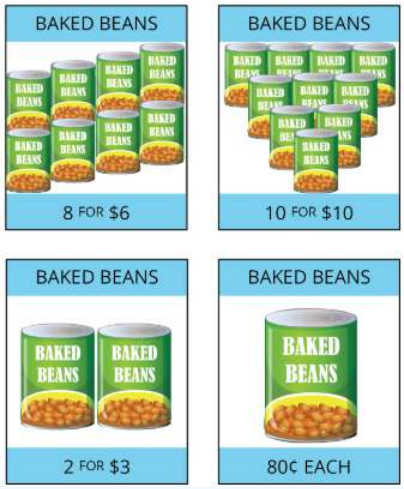


Figure 17.1.1: Four different images of ads for baked beans. The first ad has 8 cans of beans and is labeled 8 for 6 dollars. The second ad shows 10 cans of beans and is labeled 10 for 10 dollars. The third ad shows 2 cans of beans and is labeled 2 for 3 dollars. The fourth ad shows 1 can of beans and is labeled 80 cents each.

2. The last store listed is also selling 28-oz cans of baked beans for $1.40 each. How does that price compare to the other prices?

Summary

Diego ran 3 kilometers in 20 minutes. Andre ran 2,550 meters in 17 minutes. Who ran faster? Since neither their distances nor their times are the same, we have two possible strategies:

- Find the time each person took to travel the *same distance*. The person who traveled that distance in less time is faster.
- Find the distance each person traveled in the *same time*. The person who traveled a longer distance in the same amount of time is faster.

It is often helpful to compare distances traveled in *1 unit* of time (1 minute, for example), which means finding the speed such as meters per minute.

Let's compare Diego and Andre's speeds in meters per minute.

Table 17.1.1

distance (meters)	time (minutes)
3,000	20
1,500	10
150	1

Table 17.1.2

distance (meters)	time (minutes)
2,550	17
150	1

Both Diego and Andre ran 150 meters per minute, so they ran at the same speed.

Finding ratios that tell us how much of quantity A per 1 unit of quantity B is an efficient way to compare rates in different situations. Here are some familiar examples:

- Car speeds in *miles per hour*.
- Fruit and vegetable prices in *dollars per pound*.

Glossary Entries

Definition: Unit Price

The unit price is the cost for one item or for one unit of measure. For example, if 10 feet of chain link fencing cost $150, then the unit price is $150 \div 10$, or $15 per foot.

Practice

Exercise 17.1.4

Mai and Priya were on scooters. Mai traveled 15 meters in 6 seconds. Priya travels 22 meters in 10 seconds. Who was moving faster? Explain your reasoning.

Exercise 17.1.5

Here are the prices for cans of juice that are the same brand and the same size at different stores. Which store offers the best deal? Explain your reasoning.

Store X: 4 cans for $2.48

Store Y: 5 cans for $3.00

Store Z: 59 cents per can

Exercise 17.1.6

Costs of homes can be very different in different parts of the United States.

1. A 450-square-foot apartment in New York City costs $540,000. What is the price per square foot? Explain or show your reasoning.
2. A 2,100-square-foot home in Cheyenne, Wyoming, costs $110 per square foot. How much does this home cost? Explain or show your reasoning.

Exercise 17.1.7

There are 33.8 fluid ounces in a liter. There are 128 fluid ounces in a gallon. About how many liters are in a gallon?

1. 2
2. 3
3. 4
4. 5

Is your estimate larger or smaller than the actual number of liters in a gallon? Explain how you know.

(From Unit 3.2.3)

Exercise 17.1.8

Diego is 165 cm tall. Andre is 1.7 m tall. Who is taller, Diego or Andre? Explain your reasoning.

(From Unit 3.2.2)

Exercise 17.1.9

Name an object that could be about the same length as each measurement.

1. 4 inches
2. 6 feet
3. 1 meter

4. 5 yards

1. 6 centimeters
2. 2 millimeters
3. 3 kilometers

(From Unit 3.2.1)

17.1: Comparing Speeds and Prices is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

17.2: Interpreting Rates

Lesson

Let's explore unit rates.

Exercise 17.2.1: Something per Something

1. Think of two things you have heard described in terms of “something per something.”
2. Share your ideas with your group, and listen to everyone else’s idea. Make a group list of all unique ideas. Be prepared to share these with the class.

Exercise 17.2.2: Cooking Oatmeal

Priya, Han, Lin, and Diego are all on a camping trip with their families. The first morning, Priya and Han make oatmeal for the group. The instructions for a large batch say, “Bring 15 cups of water to a boil, and then add 6 cups of oats.”

1. Priya says, “The ratio of the cups of oats to the cups of water is . That’s 0.4 cups of oats per cup of water.”
2. Han says, “The ratio of the cups of water to the cups of oats is . That’s 2.5 cups of water per cup of oats.”

1. Who is correct? Explain your reasoning. If you get stuck, consider using the table.

Table 17.2.1

water (cups)	oats (cups)
15	6
1	
	1

2. The next weekend after the camping trip, Lin and Diego each decide to cook a large batch of oatmeal to have breakfasts ready for the whole week.
   a. Lin decides to cook 5 cups of oats. How many cups of water should she boil?
   b. Diego boils 10 cups of water. How many cups of oats should he add into the water?
3. Did you use Priya’s rate (0.4 cups of oats per cup of water) or Han’s rate (2.5 cups of water per cup of oats) to help you answer each of the previous two questions? Why?

Exercise 17.2.3: Cheesecake, Milk, and Raffle Tickets

For each situation, find the **unit rates**.

1. A cheesecake recipe says, “Mix 12 oz of cream cheese with 15 oz of sugar.”
   - How many ounces of sugar is that for every ounce of cream cheese?
   - How many ounces of cream cheese are there for every ounce of sugar?
2. Mai’s family drinks a total of 10 gallons of milk every 6 weeks.
   - How many gallons of milk does the family drink per week?
   - How many weeks does it take the family to consume 1 gallon of milk?
3. Tyler paid $16 for 4 raffle tickets.
   - What is the price per ticket?
   - How many tickets is that per dollar?
4. For each problem, decide which unit rate from the previous situations you prefer to use. Next, solve the problem, and show your thinking.
   a. If Lin wants to make extra cheesecake filling, how much cream cheese will she need to mix with 35 ounces of sugar?
   b. How many weeks will it take Mai’s family to finish 3 gallons of milk?
   c. How much would all 1,000 raffle tickets cost?

Are you ready for more?

Write a “deal” on tickets for Tyler’s raffle that sounds good, but is actually a little worse than just buying tickets at the normal price.

Summary

Suppose a farm lets us pick 2 pounds of blueberries for 5 dollars. We can say:

Table 17.2.2

blueberries (pounds)	price (dollars)
2	5
1	$\frac{5}{2}$
$\frac{2}{5}$	1

- We get $\frac{2}{5}$ pound of blueberries per dollar.
- The blueberries cost $\frac{5}{2}$ dollars per pound.

The “cost per pound” and the “number of pounds per dollar” are the two *unit rates* for this situation.

A **unit rate** tells us how much of one quantity for 1 of the other quantity. Each of these numbers is useful in the right situation.

If we want to find out how much 8 pounds of blueberries will cost, it helps to know how much 1 pound of blueberries will cost.

Table 17.2.3

blueberries (pounds)	price (dollars)
1	$\frac{5}{2}$
8	$8 \cdot \frac{5}{2}$

If we want to find out how many pounds we can buy for 10 dollars, it helps to know how many pounds we can buy for 1 dollar.

Table 17.2.4

blueberries (pounds)	price (dollars)
$\frac{2}{5}$	1
$10 \cdot \frac{2}{5}$	10

Which unit rate is most useful depends on what question we want to answer, so be ready to find either one!

Glossary Entries

Definition: Unit Price

The unit price is the cost for one item or for one unit of measure. For example, if 10 feet of chain link fencing cost \$150, then the unit price is $150 \div 10$, or \$15 per foot.

Definition: Unit Rate

A unit rate is a rate per 1.

For example, 12 people share 2 pies equally. One unit rate is 6 people per pie, because $12 \div 2 = 6$. The other unit rate is $\frac{1}{6}$ of a pie per person, because $2 \div 12 = \frac{1}{6}$.

Practice

Exercise 17.2.4

A pink paint mixture uses 4 cups of white paint for every 3 cups of red paint. The table shows different quantities of red and white paint for the same shade of pink. Complete the table.

Table 17.2.5

white paint (cups)	red paint (cups)
4	3
	1
1	
	4
5	

Exercise 17.2.5

A farm lets you pick 3 pints of raspberries for $12.00.

1. What is the cost per pint?
2. How many pints do you get per dollar?
3. At this rate, how many pints can you afford for $20.00?
4. At this rate, how much will 8 pints of raspberries cost?

Exercise 17.2.6

Han and Tyler are following a polenta recipe that uses 5 cups of water for every 2 cups of cornmeal.

- Han says, “I am using 3 cups of water. I will need $1\frac{1}{5}$ cups of cornmeal.”
- Tyler says, “I am using 3 cups of cornmeal. I will need $7\frac{1}{2}$ cups of water.”

Do you agree with either of them? Explain your reasoning.

Exercise 17.2.7

A large art project requires enough paint to cover 1,750 square feet. Each gallon of paint can cover 350 square feet. Each square foot requires $\frac{1}{350}$ of a gallon of paint.

Andre thinks he should use the rate $\frac{1}{350}$ gallons of paint per square foot to find how much paint they need. Do you agree with Andre? Explain or show your reasoning.

Exercise 17.2.8

Andre types 208 words in 4 minutes. Noah types 342 words in 6 minutes. Who types faster? Explain your reasoning.

(From Unit 3.3.1)

Exercise 17.2.9

A corn vendor at a farmer's market was selling a bag of 8 ears of corn for $2.56. Another vendor was selling a bag of 12 for $4.32. Which bag is the better deal? Explain or show your reasoning.

(From Unit 3.3.1)

Exercise 17.2.10

A soccer field is 100 meters long. What could be its length in yards?

A. 33.3
B. 91
C. 100
D. 109

(From Unit 3.2.2)

17.2: Interpreting Rates is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

17.3: Equivalent Ratios Have the Same Unit Rates

Lesson

Let's revisit equivalent ratios.

Exercise 17.3.1: Which One Doesn't Belong: Comparing Speeds

Which one doesn't belong? Be prepared to explain your reasoning.

5 miles in 15 minutes

3 minutes per mile

20 miles per hour

32 kilometers per hour

Exercise 17.3.2: Price of Burritos

1. Two burritos cost $14. Complete the table to show the cost for 4, 5, and 10 burritos at that rate. Next, find the cost for a single burrito in each case.

Table 17.3.1

number of burritos	cost in dollars	unit price (dollars per burrito)
2	14	
4		
5		
10		
b		

2. What do you notice about the values in this table?
3. Noah bought b burritos and paid c dollars. Lin bought twice as many burritos as Noah and paid twice the cost he did. How much did Lin pay per burrito?

Table 17.3.2

	number of burritos	cost in dollars	unit price (dollars per burrito)
Noah	b	c	$\frac{c}{b}$
Lin	$2 \cdot b$	$2 \cdot c$	

4. Explain why, if you can buy b burritos for c dollars, or buy $2 \cdot b$ burritos for $2 \cdot c$ dollars, the cost per item is the same in either case.

Exercise 17.3.3: Making Bracelets

1. Complete the table. Then, explain the strategy you used to do so.

Table 17.3.3

time in hours	number of bracelets	speed (bracelets per hour)
2		6
5		6

time in hours	number of bracelets	speed (bracelets per hour)
7		6
	66	6
	100	6



Figure 17.3.1: Leather Lanyard Bracelet, by Maegan Tintari. CC BY 2.0. Flickr. Source.

2. Here is a partially filled table from an earlier activity. Use the same strategy you used for the bracelet problem to complete this table.

Table 17.3.4

number of burritos	cost in dollars	unit price (dollars per burrito)
	14	7
	28	7
5		7
10		7

3. Next, compare your results with those in the first table in the previous activity. Do they match? Explain why or why not.

Exercise 17.3.4: How Much Applesauce?

It takes 4 pounds of apples to make 6 cups of applesauce.

1. At this rate, how much applesauce can you make with:
   a. 7 pounds of apples?
   b. 10 pounds of apples?
2. How many pounds of apples would you need to make:
   a. 9 cups of applesauce?
   b. 20 cups of applesauce?

Table 17.3.5

pounds of apples	cups of applesauce
4	6
7	
10	
	9
	20

Are you ready for more?

1. Jada eats 2 scoops of ice cream in 5 minutes. Noah eats 3 scoops of ice cream in 5 minutes. How long does it take them to eat 1 scoop of ice cream working together (if they continue eating ice cream at the same rate they do individually)?
2. The garden hose at Andre's house can fill a 5-gallon bucket in 2 minutes. The hose at his next-door neighbor's house can fill a 10-gallon bucket in 8 minutes. If they use both their garden hoses at the same time, and the hoses continue working at the same rate they did when filling a bucket, how long will it take to fill a 750-gallon pool?

Summary

The table shows different amounts of apples selling at the same rate, which means all of the ratios in the table are equivalent. In each case, we can find the *unit price* in dollars per pound by dividing the price by the number of pounds.

Table 17.3.6

apples (pounds)	price (dollars)	unit price (dollars per pound)
4	10	$10 \div 4 = 2.50$
8	20	$20 \div 8 = 2.50$
20	50	$50 \div 20 = 2.50$

The unit price is always the same. Whether we buy 4 pounds of apples for 10 dollars or 8 pounds of apples for 20 dollars, the apples cost 2.50 dollars per pound.

We can also find the number of pounds of apples we can buy per dollar by dividing the number of pounds by the price.

Table 17.3.7

apples (pounds)	price (dollars)	pounds per dollar
4	10	$4 \div 10 = 0.4$
8	20	$8 \div 20 = 0.4$
20	50	$20 \div 50 = 0.4$

The number of pounds we can buy for a dollar is the same as well! Whether we buy 4 pounds of apples for 10 dollars or 8 pounds of apples for 20 dollars, we are getting 0.4 pounds per dollar.

This is true in all contexts: when two ratios are equivalent, their unit rates will be equal.

Table 17.3.8

quantity x	quantity y	unit rate 1	unit rate 2
a	b	$\frac{a}{b}$	$\frac{b}{a}$
$s \cdot a$	$s \cdot b$	$\frac{s \cdot a}{s \cdot b} = \frac{a}{b}$	$\frac{s \cdot b}{s \cdot a} = \frac{b}{a}$

Glossary Entries

Definition: Unit Price

The unit price is the cost for one item or for one unit of measure. For example, if 10 feet of chain link fencing cost \$150, then the unit price is $150 \div 10$, or \$15 per foot.

Definition: Unit Rate

A unit rate is a rate per 1.

For example, 12 people share 2 pies equally. One unit rate is 6 people per pie, because $12 \div 2 = 6$. The other unit rate is $\frac{1}{6}$ of a pie per person, because $2 \div 12 = \frac{1}{6}$.

Practice

Exercise 17.3.5

A car travels 55 miles per hour for 2 hours. Complete the table.

Table 17.3.9

time (hours)	distance (miles)	miles per hour
1	55	55
$\frac{1}{2}$		
$1\frac{1}{2}$		
	110	

Exercise 17.3.6

The table shows the amounts of onions and tomatoes in different-sized batches of a salsa recipe.

Table 17.3.10

onions (ounces)	tomatoes (ounces)
2	16
4	32
6	48

Elena notices that if she takes the number in the tomatoes column and divides it by the corresponding number in the onions column, she always gets the same result.

What is the meaning of the number that Elena has calculated?

Exercise 17.3.7

A restaurant is offering 2 specials: 10 burritos for $12, or 6 burritos for $7.50. Noah needs 60 burritos for his party. Should he buy 6 orders of the 10-burrito special or 10 orders of the 6-burrito special? Explain your reasoning.

Exercise 17.3.8

Complete the table so that the cost per banana remains the same.

Table 17.3.11

number of bananas	cost in dollars	unit price (dollars per banana)
4		0.50
6		0.50
7		0.50
10		0.50
	10.00	0.50
	16.50	0.50

Exercise 17.3.9

Two planes travel at a constant speed. Plane A travels 2,800 miles in 5 hours. Plane B travels 3,885 miles in 7 hours. Which plane is faster? Explain your reasoning.

(From Unit 3.3.1)

Exercise 17.3.10

A car has 15 gallons of gas in its tank. The car travels 35 miles per gallon of gas. It uses $\frac{1}{35}$ of a gallon of gas to go 1 mile.

- How far can the car travel with 15 gallons? Show your reasoning.
- How much gas does the car use to go 100 miles? Show your reasoning.

(From Unit 3.3.2)

Exercise 17.3.11

A box of cereal weighs 600 grams. How much is this weight in pounds? Explain or show your reasoning. (Note: 1 kilogram = 2.2 pounds)

(From Unit 3.2.3)

17.3: Equivalent Ratios Have the Same Unit Rates is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

17.4: More about Constant Speed

Lesson

Let's investigate constant speed some more.

Exercise 17.4.1: Back on the Treadmill Again

While training for a race, Andre's dad ran 12 miles in 75 minutes on a treadmill. If he runs at that rate:

1. How long would it take him to run 8 miles?
2. How far could he run in 30 minutes?

Exercise 17.4.2: Picnics on the Rail Trail



Figure 17.4.1: Danvers Rail Trail, by massmatt. CC BY 2.0. Flickr. Source.

Kiran and Clare live 24 miles away from each other along a rail trail. One Saturday, the two friends started walking toward each other along the trail at 8:00 a.m. with a plan to have a picnic when they meet.

Kiran walks at a **speed** of 3 miles per hour while Clare walks 3.4 miles per hour.

1. After one hour, how far apart will they be?
2. Make a table showing how far apart the two friends are after 0 hours, 1 hour, 2 hours, and 3 hours.
3. At what time will the two friends meet and have their picnic?
4. Kiran says "If I walk 3 miles per hour toward you, and you walk 3.4 miles per hour toward me, it's the same as if you stay put and I jog 6.4 miles per hour." What do you think Kiran means by this? Is he correct?
5. Several months later, they both set out at 8:00 a.m. again, this time with Kiran jogging and Clare still walking at 3.4 miles per hour. This time, they meet at 10:30 a.m. How fast was Kiran jogging?

Are you ready for more?

1. On his trip to meet Clare, Kiran brought his dog with him. At the same time Kiran and Clare started walking, the dog started running 6 miles per hour. When it got to Clare it turned around and ran back to Kiran. When it got to Kiran, it turned around and ran back to Clare, and continued running in this fashion until Kiran and Clare met. How far did the dog run?
2. The next Saturday, the two friends leave at the same time again, and Kiran jogs twice as fast as Clare walks. Where on the rail trail do Kiran and Clare meet?

Exercise 17.4.3: Swimming and Biking

Jada bikes 2 miles in 12 minutes. Jada's cousin swims 1 mile in 24 minutes.

1. Who is moving faster? How much faster?
2. One day Jada and her cousin line up on the end of a swimming pier on the edge of a lake. At the same time, they start swimming and biking in opposite directions.
   a. How far apart will they be after 15 minutes?
   b. How long will it take them to be 5 miles apart?

Summary

When two objects are each moving at a constant speed and their distance-to-time ratios are equivalent, we say that they are moving at the *same speed*. If their time-distance ratios are not equivalent, they are not moving at the same speed.

We describe **speed** in units of distance per unit of time, like *miles per hour* or *meters per second*.

- A snail that crawls 5 centimeters in 2 minutes is traveling at a rate of 2.5 centimeters per minute.
- A toddler that walks 9 feet in 6 seconds is traveling at a rate of 1.5 feet per second.
- A cyclist who bikes 20 kilometers in 2 hours is traveling at a rate of 10 kilometers per hour.

We can also use **pace** to describe distance and time. We measure pace in units such as *hours per mile* or *seconds per meter*.

- A snail that crawls 5 centimeters in 2 minutes has a pace of 0.4 minutes per centimeter.
- A toddler walking 9 feet in 6 seconds has a pace of $\frac{2}{3}$ seconds per foot.
- A cyclist who bikes 20 kilometers in 2 hours has a pace of 0.1 hours per kilometer.

Speed and pace are reciprocals. Both can be used to compare whether one object is moving faster or slower than another object.

- An object with the higher speed is *faster* than one with a lower speed because the former travels a greater distance in the same amount of time.
- An object with the greater pace is *slower* than one with a smaller pace because the former takes more time to travel the same distance.

Because speed is a *rate per 1 unit of time* for ratios that relate distance and time, we can multiply the amount of time traveled by the speed to find the distance traveled.

Table 17.4.1

time (minutes)	distance (centimeters)
2	5
1	2.5
4	$4 \cdot (2.5)$

Glossary Entries

Definition: Pace

Pace is one way to describe how fast something is moving. Pace tells how much time it takes the object to travel a certain distance.

For example, Diego walks at a pace of 10 minutes per mile. Elena walks at a pace of 11 minutes per mile. Elena walks slower than Diego, because it takes her more time to travel the same distance.

Definition: Speed

Speed is one way to describe how fast something is moving. Speed tells how much distance the object travels in a certain amount of time.

For example, Tyler walks at a speed of 4 miles per hour. Priya walks at a speed of 5 miles per hour. Priya walks faster than Tyler, because she travels more distance in the same amount of time.

Definition: Unit Price

The unit price is the cost for one item or for one unit of measure. For example, if 10 feet of chain link fencing cost \$150, then the unit price is $150 \div 10$, or \$15 per foot.

Definition: Unit Rate

A unit rate is a rate per 1.

For example, 12 people share 2 pies equally. One unit rate is 6 people per pie, because $12 \div 2 = 6$. The other unit rate is $\frac{1}{6}$ of a pie per person, because $2 \div 12 = \frac{1}{6}$.

Practice

Exercise 17.4.4

A kangaroo hops 2 kilometers in 3 minutes. At this rate:

1. How long does it take the kangaroo to travel 5 kilometers?
2. How far does the kangaroo travel in 2 minutes?

Exercise 17.4.5

Mai runs around a 400-meter track at a constant speed of 250 meters per minute. How many minutes does it take Mai to complete 4 laps of the track? Explain or show your reasoning.

Exercise 17.4.6

At 10:00 a.m., Han and Tyler both started running toward each other from opposite ends of a 10-mile path along a river. Han runs at a pace of 12 minutes per mile. Tyler runs at a pace of 15 minutes per mile.

1. How far does Han run after a half hour? After an hour?
2. Do Han and Tyler meet on the path within 1 hour? Explain or show your reasoning.

Exercise 17.4.7

Two skateboarders start a race at the same time. Skateboarder A travels at a steady rate of 15 feet per second. Skateboarder B travels at a steady rate of 22 feet per second. After 4 minutes, how much farther will Skateboarder B have traveled? Explain your reasoning.

(From Unit 2.5.2)

Exercise 17.4.8

There are 4 tablespoons in $\frac{1}{4}$ cup. There are 2 cups in 1 pint. How many tablespoons are there in 1 pint? If you get stuck, consider drawing a double number line or making a table.

(From Unit 3.2.3)

Exercise 17.4.9

Two larger cubes are made out of unit cubes. Cube A is 2 by 2 by 2. Cube B is 4 by 4 by 4. The side length of Cube B is twice that of Cube A.

1. Is the surface area of Cube B also twice that of Cube A? Explain or show your reasoning.
2. Is the volume of Cube B also twice that of Cube A? Explain or show your reasoning.

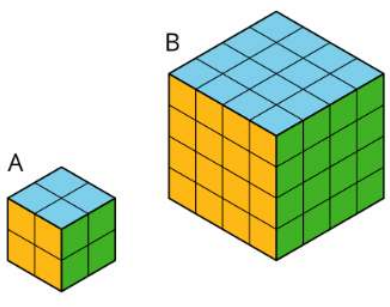


Figure 17.4.2

(From Unit 1.5.1)

17.4: More about Constant Speed is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

17.5: Solving Rate Problems

Lesson

Let's use unit rates like a pro.

Exercise 17.5.1: Grid of 100

How much is shaded in each one?

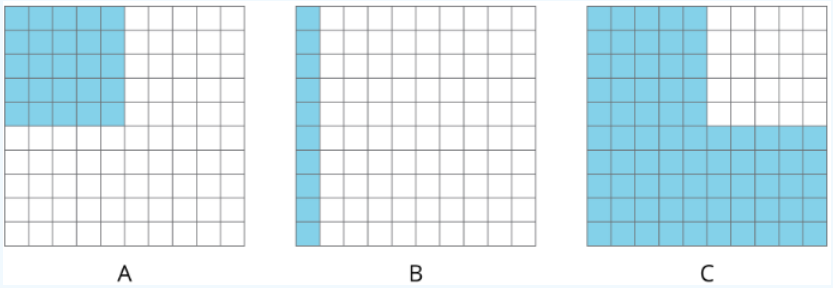


Figure 17.5.1: Three 10 unit by 10 unit square grids labeled A, B, and C. Grid A has the first 5 units in each column shaded in columns 1 through 5. Grid B has 10 units shaded in column 1. Grid C has 10 units shaded in columns 1 through 5 and the last 5 units in each column shaded in columns 6 through 10.

Exercise 17.5.2: Card Sort: Is it a Deal?

Your teacher will give you a set of cards showing different offers.

1. Find card A and work with your partner to decide whether the offer on card A is a good deal. Explain or show your reasoning.
2. Next, split cards B–E so you and your partner each have two.
   a. Decide individually if your two cards are good deals. Explain your reasoning.
   b. For each of your cards, explain to your partner if you think it is a good deal and why. Listen to your partner's explanations for their cards. If you disagree, explain your thinking.
   c. Revise any decisions about your cards based on the feedback from your partner.
3. When you and your partner are in agreement about cards B–E, place all the cards you think are a good deal in one stack and all the cards you think are a bad deal in another stack. Be prepared to explain your reasoning.

Are you ready for more?

Time to make your own deal! Read the information on card F and then decide what you would charge if you were the clerk. When your teacher signals, trade cards with another group and decide whether or not you would take the other group's offer.

Keep in mind that you may offer a fair deal or an unfair deal, but the goal is to set a price close enough to the value it should be that the other group cannot immediately tell if the deal you offer is a good one.

Exercise 17.5.3: The Fastest of All

Wild animals from around the world wanted to hold an athletic competition, but no one would let them on an airplane. They decided to just measure how far each animal could sprint in one minute and send the results to you to decide the winner.

You look up the following information about converting units of length:

1 inch = 2.54 centimeters

Table 17.5.1

animal	sprint distance
cougar	1,408 yards
antelope	1 mile

animal	sprint distance
hare	49,632 inches
kangaroo	1,073 meters
ostrich	1.15 kilometers
coyote	3,773 feet

1. Which animal sprinted the farthest?
2. What are the place rankings for all of the animals?

Summary

Sometimes we can find and use more than one unit rate to solve a problem.

Suppose a grocery store is having a sale on shredded cheese. A small bag that holds 8 ounces is sold for $2. A large bag that holds 2 kilograms is sold for $16. How do you know which is a better deal?

Here are two different ways to solve this problem:

Compare dollars per kilogram.

- The large bag costs $8 per kilogram, because $16 \div 2 = 8$.
- The small bag holds $\frac{1}{2}$ pound of cheese, because there are 16 ounces in 1 pound, and $8 \div 16 = \frac{1}{2}$.
  The small bag costs $4 per pound, because $2 \div \frac{1}{2} = 4$. This is about $8.80 per kilogram, because there are about 2.2 pounds in 1 kilogram, and $4.00 \cdot 2.2 = 8.80$.

The large bag is a better deal, because it costs less money for the same amount of cheese.

Compare ounces per dollar.

- With the small bag, we get 4 ounces per dollar, because $8 \div 2 = 4$.
- The large bag holds 2,000 grams of cheese. There are 1,000 grams in 1 kilogram, and $2 \cdot 1,000 = 2,000$. This means 125 grams per dollar, because $2,000 \div 16 = 125$.
  There are about 28.35 grams in 1 ounce, and $125 \div 28.35 \approx 4.4$, so this is about 4.4 ounces per dollar.

The large bag is a better deal, because you get more cheese for the same amount of money.

Another way to solve the problem would be to compare the unit prices of each bag in dollars per ounce. Try it!

Glossary Entries

Definition: Pace

Pace is one way to describe how fast something is moving. Pace tells how much time it takes the object to travel a certain distance.

For example, Diego walks at a pace of 10 minutes per mile. Elena walks at a pace of 11 minutes per mile. Elena walks slower than Diego, because it takes her more time to travel the same distance.

Definition: Speed

Speed is one way to describe how fast something is moving. Speed tells how much distance the object travels in a certain amount of time.

For example, Tyler walks at a speed of 4 miles per hour. Priya walks at a speed of 5 miles per hour. Priya walks faster than Tyler, because she travels more distance in the same amount of time.

Definition: Unit Price

The unit price is the cost for one item or for one unit of measure. For example, if 10 feet of chain link fencing cost $150, then the unit price is $150 \div 10$, or $15 per foot.

Definition: Unit Rate

A unit rate is a rate per 1.

For example, 12 people share 2 pies equally. One unit rate is 6 people per pie, because $12 \div 2 = 6$. The other unit rate is $\frac{1}{6}$ of a pie per person, because $2 \div 12 = \frac{1}{6}$.

Practice

Exercise 17.5.4

This package of sliced cheese costs $2.97.

How much would a package with 18 slices cost at the same price per slice? Explain or show your reasoning.

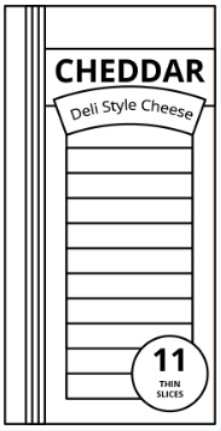


Figure 17.5.2

Exercise 17.5.5

A copy machine can print 480 copies every 4 minutes. For each question, explain or show your reasoning.

1. How many copies can it print in 10 minutes?
2. A teacher printed 720 copies. How long did it take to print?

Exercise 17.5.6

Order these objects from heaviest to lightest.

(Note: 1 pound = 16 ounces, 1 kilogram $\approx$ 2.2 pounds, and 1 ton = 2,000 pounds)

Table 17.5.2

item	weight
school bus	9 tons
horse	1, 100 pounds
elephant	5, 500 kilograms
grand piano	15, 840 ounces

Exercise 17.5.7

Andre sometimes mows lawns on the weekend to make extra money. Two weeks ago, he mowed a neighbor's lawn for $\frac{1}{2}$ hour and earned \$10. Last week, he mowed his uncle's lawn for $\frac{3}{2}$ hours and earned \$30. This week, he mowed the lawn of a community center for 2 hours and earned \$30.

Which jobs paid better than others? Explain your reasoning.

(From Unit 3.3.1)

Exercise 17.5.8

Calculate and express your answer in decimal form.

1. $\frac{1}{2} \cdot 17$
2. $\frac{3}{4} \cdot 200$
3. $(0.2) \cdot 40$
4. $(0.35) \cdot 60$

(From Unit 3.1.1)

Exercise 17.5.9

Here is a polygon.

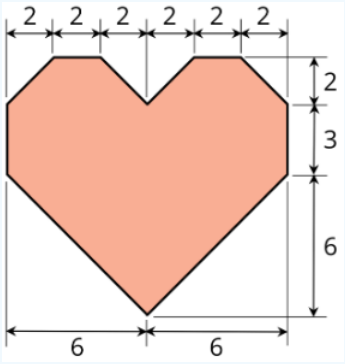


Figure 17.5.3: A heart-shaped polygon. Arrows along the bottom indicate the width of each half of the polygon is 6. Arrows along the top indicate that each of the six sides at the top span 2 units. Arrows along the right side indicate that the top portion of the polygon is 2 units, the middle portion is 3 units, and the bottom portion is 6 units.

1. Decompose this polygon so that its area can be calculated. All measurements are in centimeters.
2. Calculate its area. Organize your work so that it can be followed by others.

(From Unit 1.5.1)

17.5: Solving Rate Problems is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

Section 18: Percentages

This page titled Section 18: Percentages is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

18.1: Percentages and Double Number Lines

Lesson

Let's use double number lines to represent percentages.

Exercise 18.1.1: Fundraising Goal

Each of three friends—Lin, Jada, and Andre—had the goal of raising $40. How much money did each person raise? Be prepared to explain your reasoning.

1. Lin raised 100% of her goal.
2. Jada raised 50% of her goal.
3. Andre raised 150% of his goal.

Exercise 18.1.2: Three-Day Biking Trip

Elena biked 8 miles on Saturday. Use the double number line to answer the questions. Be prepared to explain your reasoning.

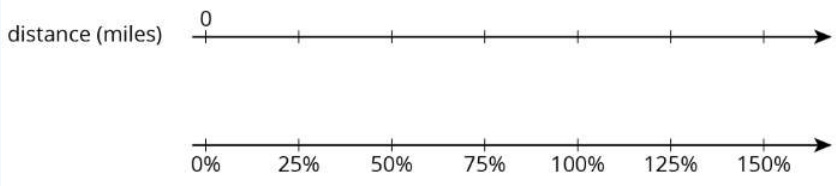


Figure 18.1.1: A double number line for distance in miles with 7 evenly spaced tick marks. The top number line has the number 0 on the first tick mark and the remaining tick marks are blank. The bottom number line starting with the first tick mark, 0 percent, 25 percent, 50 percent, 75 percent, 100 percent, 125 percent and 150 percent are labeled.

1. What is 100% of her Saturday distance?
2. On Sunday, she biked 75% of her Saturday distance. How far was that?
3. On Monday, she biked 125% of her Saturday distance. How far was that?

Exercise 18.1.3: Puppies Grow Up

1. Jada has a new puppy that weighs 9 pounds. The vet says that the puppy is now at about 20% of its adult weight. What will be the adult weight of the puppy?

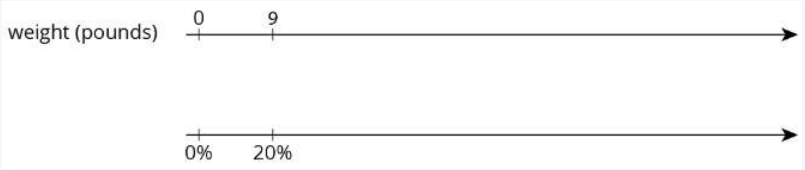


Figure 18.1.2

2. Andre also has a puppy that weighs 9 pounds. The vet says that this puppy is now at about 30% of its adult weight. What will be the adult weight of Andre's puppy?

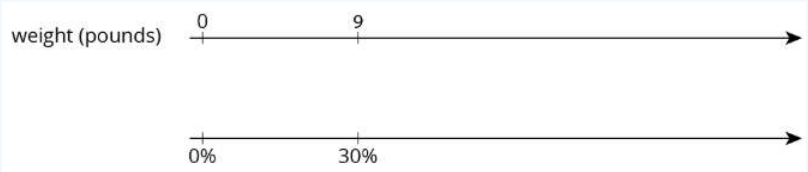


Figure 18.1.3

3. What is the same about Jada and Andre's puppies? What is different?

Are you ready for more?

A loaf of bread costs $2.50 today. The same size loaf cost 20 cents in 1955.

1. What percentage of today's price did someone in 1955 pay for bread?
2. A job pays $10.00 an hour today. If the same percentage applies to income as well, how much would that job have paid in 1955?

Summary

We can use a double number line to solve problems about percentages. For example, what is 30% of 50 pounds? We can draw a double number line like this:

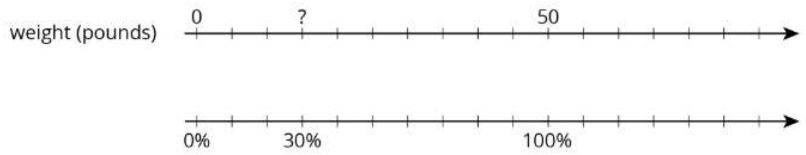


Figure 18.1.4

We divide the distance between 0% and 100% and that between 0 and 50 pounds into ten equal parts. We label the tick marks on the top line by counting by 5s ($50 \div 10 = 5$) and on the bottom line counting by 10% ($100 \div 10 = 10$). We can then see that 30% of 50 pounds is 15 pounds.

We can also use a table to solve this problem.



Figure 18.1.5: A 2-column table with 3 rows of data. First column, weight, pounds. Data are as follows: 50, 5, 15. Second column, percentage. Data are as follows: 100, 10, 30. Arrow between first and second row says times the fraction 1 over 10. Arrow between second and third row says times 3.

Suppose we know that 140% of an amount is $28. What is 100% of that amount? Let's use a double number line to find out.

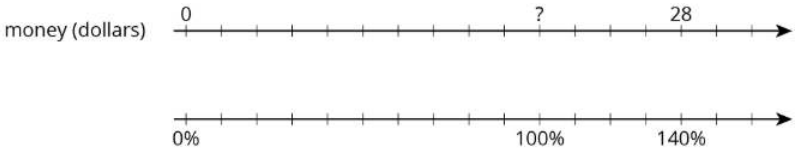


Figure 18.1.6: A double number line with 17 evenly spaced tick marks. The top number line is labeled money, in dollars and the first tick mark is labeled 0, the eleventh tick mark is labeled with a question mark, and the fifteenth tick mark is labeled 28. The bottom number line is unlabeled and the first tick mark is labeled 0 percent, the eleventh tick mark is labeled 100 percent, and the fifteenth tick mark is labeled 140 percent.

We divide the distance between 0% and 140% and that between $0 and $28 into fourteen equal intervals. We label the tick marks on the top line by counting by 2s and on the bottom line counting by 10%. We would then see that 100% is $20.

Or we can use a table as shown.

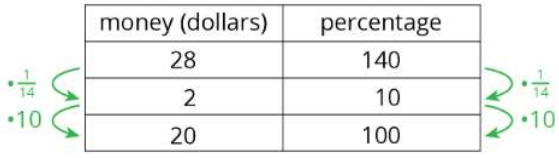


Figure 18.1.7: A two column table with three rows of data. The first column has the heading weight, in pounds. The second column has the heading percentage. Row 1; 28, 140. Row 2; 2, 10. Row 3; 20, 100. Arrows on both sides of the table from row 1 to row 2 are labeled multiply by one fourteenth. Arrows on both sides of the table from row 2 to row 3 are labeled multiply by 10.

Glossary Entries

Definition: Percent

The word *percent* means "for each 100." The symbol for percent is %.

For example, a quarter is worth 25 cents, and a dollar is worth 100 cents. We can say that a quarter is worth 25% of a dollar.



Figure 18.1.8



Figure 18.1.9: A diagram of two bars with different lengths. The top bar is labeled 1 Quarter and 25 cents is labeled inside the bar. The bottom bar is labeled 1 Dollar. It is 4 times longer than the top bar and 100 cents is labeled inside the bar.

Definition: Percentage

A percentage is a rate per 100.

For example, a fish tank can hold 36 liters. Right now there is 27 liters of water in the tank. The percentage of the tank that is full is 75%.

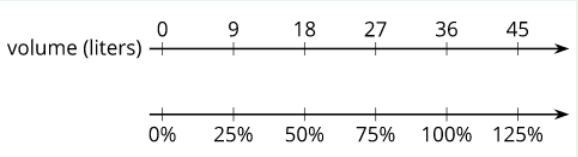


Figure 18.1.10

Practice

Exercise 18.1.4

Solve each problem. If you get stuck, consider using the double number lines.

1. During a basketball practice, Mai attempted 40 free throws and was successful on 25% of them. How many successful free throws did she make?

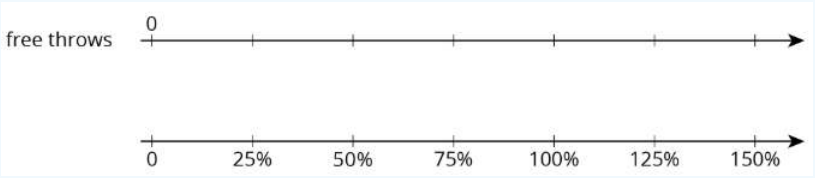


Figure 18.1.11: A double number line with 7 evenly spaced tick marks. The top number line is labeled free throws and the first tick mark is labeled 0. The other tick marks are unlabeled. The bottom number line is not labeled and starting with the first tick mark 0, 25 percent, 50 percent, 75 percent, 100 percent, 125 percent, and 150 percent are labeled.

2. Yesterday, Priya successfully made 12 free throws. Today, she made 150% as many. How many successful free throws did Priya make today?

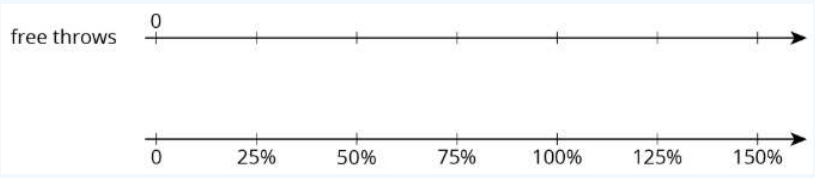


Figure 18.1.12: A double number line with 7 evenly spaced tick marks. The top number line is labeled free throws and the first tick mark is labeled 0. The other tick marks are unlabeled. The bottom number line is not labeled and starting with the first tick mark 0, 25 percent, 50 percent, 75 percent, 100 percent, 125 percent, and 150 percent are labeled.

Exercise 18.1.5

A 16-ounce bottle of orange juice says it contains 200 milligrams of vitamin C, which is 250% of the daily recommended allowance of vitamin C for adults. What is 100% of the daily recommended allowance of vitamin C for adults?

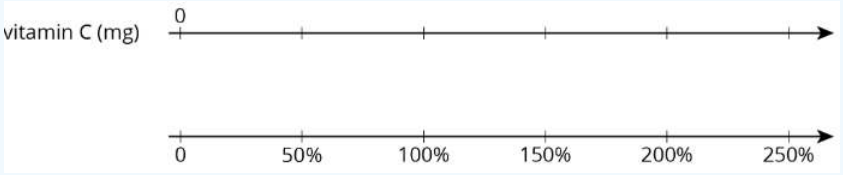


Figure 18.1.13

Exercise 18.1.6

At a school, 40% of the sixth-grade students said that hip-hop is their favorite kind of music. If 100 sixth-grade students prefer hip hop music, how many sixth-grade students are at the school? Explain or show your reasoning.

Exercise 18.1.7

Diego has a skateboard, scooter, bike, and go-cart. He wants to know which vehicle is the fastest. A friend records how far Diego travels on each vehicle in 5 seconds. For each vehicle, Diego travels as fast as he can along a straight, level path.

Table 18.1.1

vehicle	distance traveled
skateboard	90 feet
scooter	1, 020 inches
bike	4, 800 centimeters
go-cart	0.03 kilometers

1. What is the distance each vehicle traveled in centimeters?
2. Rank the vehicles in order from fastest to slowest.

(From Unit 3.3.5)

Exercise 18.1.8

It takes 10 pounds of potatoes to make 15 pounds of mashed potatoes. At this rate:

1. How many pounds of mashed potatoes can they make with 15 pounds of potatoes?
2. How many pounds of potatoes are needed to make 50 pounds of mashed potatoes?

(From Unit 3.3.3)

18.1: Percentages and Double Number Lines is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

18.2: Percentages and Tape Diagrams

Lesson

Let's use tape diagrams to understand percentages.

Exercise 18.2.1: Notice and Wonder: Tape Diagrams

What do you notice? What do you wonder?

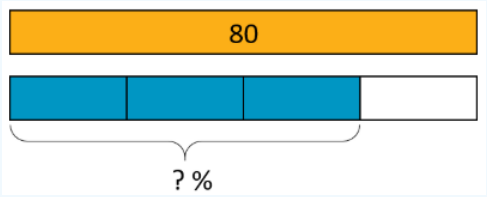


Figure 18.2.1

Exercise 18.2.2: Revisiting Jada's Puppy

Jada has a new puppy that weighs 9 pounds. It is now at about 20% of its adult weight.

1. Here is a diagram that Jada drew about the weight of her puppy.

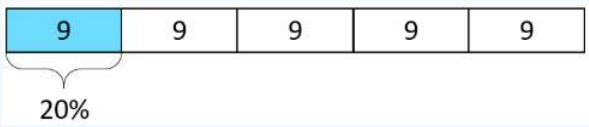


Figure 18.2.2

   a. The adult weight of the puppy will be 45 pounds. How can you see that in the diagram?
   b. What fraction of its adult weight is the puppy now? How can you see that in the diagram?

2. Jada's friend has a dog that weighs 90 pounds. Here is a diagram Jada drew that represents the weight of her friend's dog and the weight of her puppy.

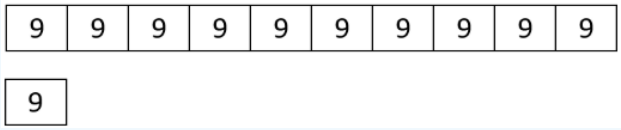


Figure 18.2.3

   a. How many times greater is the dog's weight than the puppy's?
   b. Compare the weight of the puppy and the dog using fractions.
   c. Compare the weight of the puppy and the dog using percentages.

Exercise 18.2.3: 5 Dollars

Noah has $5.

1. a. Elena has 40% as much as Noah. How much does Elena have?
   b. Compare Elena's and Noah's money using fractions. Draw a diagram to illustrate.
2. a. Diego has 150% as much as Noah. How much does Diego have?
   b. Compare Diego's and Noah's money using fractions. Draw a diagram to illustrate.

Exercise 18.2.4: Staying Hydrated

During the first part of a hike, Andre drank 1.5 liters of the water he brought.

1. If this is 50% of the water he brought, how much water did he bring?

2. If he drank 80% of his water on his entire hike, how much did he drink?

Are you ready for more?

Decide if each scenario is possible.

1. Andre plans to bring his dog on his next hike, along with 150% as much water as he brought on this hike.
2. Andre plans to drink 150% of the water he brought on his hike.

Summary

Tape diagrams can help us make sense of percentages.

Consider two problems that we solved earlier using double number lines and tables: “What is 30% of 50 pounds?” and “What is 100% of a number if 140% of it is 28?”

Here is a tape diagram that shows that 30% of 50 pounds is 15 pounds.

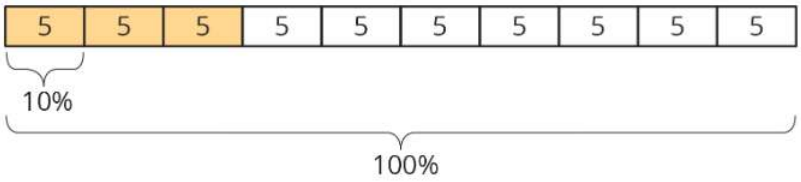


Figure 18.2.4

This diagram shows that if 140% of some number is 28, then that number must be 20.

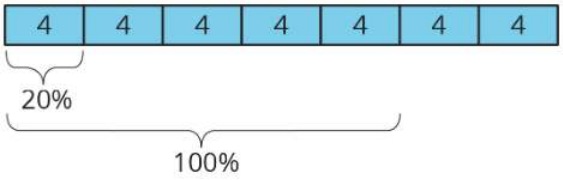


Figure 18.2.5

Glossary Entries

Definition: Percent

The word *percent* means “for each 100.” The symbol for percent is %.

For example, a quarter is worth 25 cents, and a dollar is worth 100 cents. We can say that a quarter is worth 25% of a dollar.



Figure 18.2.6

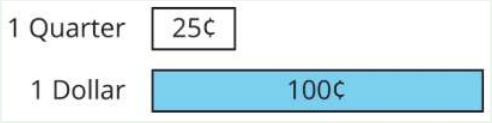


Figure 18.2.7: A diagram of two bars with different lengths. The top bar is labeled 1 Quarter and 25 cents is labeled inside the bar. The bottom bar is labeled 1 Dollar. It is 4 times longer than the top bar and 100 cents is labeled inside the bar.

Definition: Percentage

A percentage is a rate per 100.

For example, a fish tank can hold 36 liters. Right now there is 27 liters of water in the tank. The percentage of the tank that is full is 75%.

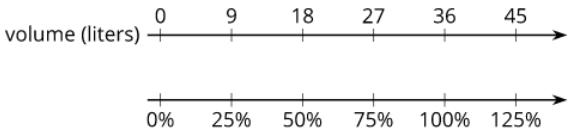


Figure 18.2.8

Practice

Exercise 18.2.5

Here is a tape diagram that shows how far two students walked.

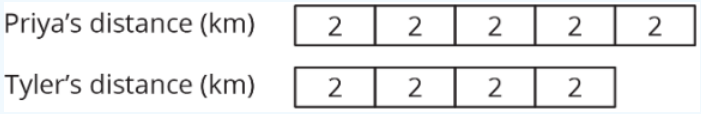


Figure 18.2.9

1. What percentage of Priya's distance did Tyler walk?
2. What percentage of Tyler's distance did Priya walk?

Exercise 18.2.6

A bakery makes 40 different flavors of muffins. 25% of the flavors have chocolate as one of the ingredients. Draw a tape diagram to show how many flavors have chocolate and how many don't.

Exercise 18.2.7

There are 70 students in the school band. 40% of them are sixth graders, 20% are seventh graders, and the rest are eighth graders.

1. How many band members are sixth graders?
2. How many band members are seventh graders?
3. What percentage of the band members are eighth graders? Explain your reasoning.

Exercise 18.2.8

Jada has a monthly budget for her cell phone bill. Last month she spent 120% of her budget, and the bill was $60. What is Jada's monthly budget? Explain or show your reasoning.

(From Unit 3.4.2)

Exercise 18.2.9

Which is a better deal, 5 tickets for $12.50 or 8 tickets for $20.16? Explain your reasoning.

(From Unit 3.3.5)

Exercise 18.2.10

An athlete runs 8 miles in 50 minutes on a treadmill. At this rate:

1. How long will it take the athlete to run 9 miles?
2. How far can the athlete run in 1 hour?

(From Unit 3.3.4)

18.2: Percentages and Tape Diagrams is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

18.3: Benchmark Percentages

Lesson

Let's contrast percentages and fractions.

Exercise 18.3.1: What Percentage is Shaded?

What percentage of each diagram is shaded?

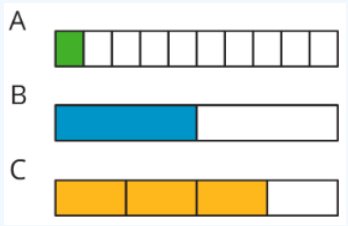


Figure 18.3.1

Exercise 18.3.2: Liters, Meters, and Hours

1. a. How much is 50% of 10 liters of milk?
   b. How far is 50% of a 2,000-kilometer trip?
   c. How long is 50% of a 24-hour day?
   d. How can you find 50% of any number?
2. a. How far is 10% of a 2,000-kilometer trip?
   b. How much is 10% of 10 liters of milk?
   c. How long is 10% of a 24-hour day?
   d. How can you find 10% of any number?
3. a. How long is 75% of a 24-hour day?
   b. How far is 75% of a 2,000-kilometer trip?
   c. How much is 75% of 10 liters of milk?
   d. How can you find 75% of any number?

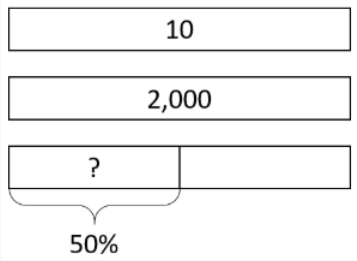


Figure 18.3.2

Exercise 18.3.3: Nine is ...

Explain how you can calculate each value mentally.

1. 9 is 50% of what number?
2. 9 is 25% of what number?
3. 9 is 10% of what number?
4. 9 is 75% of what number?
5. 9 is 150% of what number?

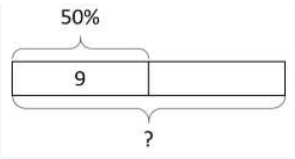


Figure 18.3.3

Exercise 18.3.4: Matching the Percentage

Match the percentage that describes the relationship between each pair of numbers. One percentage will be left over. Be prepared to explain your reasoning.

1. 7 is what percentage of 14?
2. 5 is what percentage of 20?
3. 3 is what percentage of 30?
4. 6 is what percentage of 8?
5. 20 is what percentage of 5?

- 4%
- 10%
- 25%
- 50%
- 75%
- 400%

Are you ready for more?

1. What percentage of the world's current population is under the age of 14?
2. How many people is that?
3. How many people are 14 or older?

Summary

Certain percentages are easy to think about in terms of fractions.

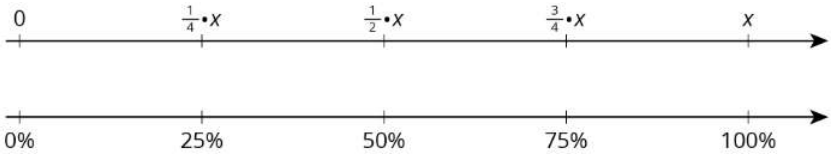


Figure 18.3.4: A double number line with 5 evenly spaced tick marks. The tick marks on the top number line are labeled 0, one fourth times x, one half times x, three fourths times x, and x. The tick marks on the bottom number line are labeled 0 percent, 25 percent, 50 percent, 75 percent, and 100 percent.

- 25% of a number is always $\frac{1}{4}$ of that number.
  For example, 25% of 40 liters is $\frac{1}{4} \cdot 40$ or 10 liters.
- 50% of a number is always $\frac{1}{2}$ of that number.
  For example, 50% of 82 kilometers $\frac{1}{2} \cdot 82$ or 41 kilometers.
- 75% of a number is always $\frac{3}{4}$ of that number.
  For example, 75% of 1 pound is $\frac{3}{4}$ pound.
- 10% of a number is always $\frac{1}{10}$ of that number.
  For example, 10% of 95 meters is 9.5 meters.
- We can also find multiples of 10% using tenths.
  For example, 70% of a number is always $\frac{7}{10}$ of that number, so 70% of 30 days is $\frac{7}{10} \cdot 30$ or 21 days.

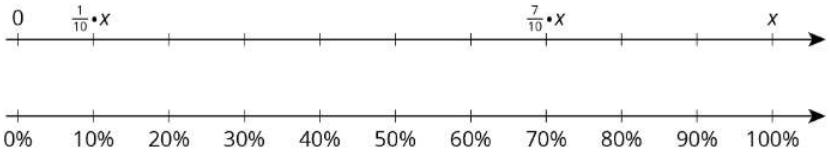


Figure 18.3.5: A double number line with 11 evenly spaced tick marks. For the top number line the number 0 is on the first tick mark, one tenth times x on the second, seven tenths times x on the eighth, and x on the eleventh. The remaining tick marks are blank. On the bottom number line starting from the first tick mark, zero percent, 10 percent, 20 percent, 30 percent, 40 percent, 50 percent, 60 percent, 70 percent, 80 percent, 90 percent, and 100 percent are labeled.

Glossary Entries

Definition: Percent

The word *percent* means "for each 100." The symbol for percent is %.

For example, a quarter is worth 25 cents, and a dollar is worth 100 cents. We can say that a quarter is worth 25% of a dollar.



Figure 18.3.6

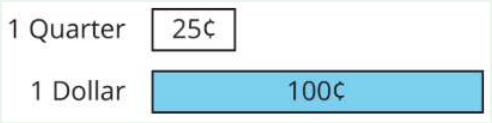


Figure 18.3.7: A diagram of two bars with different lengths. The top bar is labeled 1 Quarter and 25 cents is labeled inside the bar. The bottom bar is labeled 1 Dollar. It is 4 times longer than the top bar and 100 cents is labeled inside the bar.

Definition: Percentage

A percentage is a rate per 100.

For example, a fish tank can hold 36 liters. Right now there is 27 liters of water in the tank. The percentage of the tank that is full is 75%.

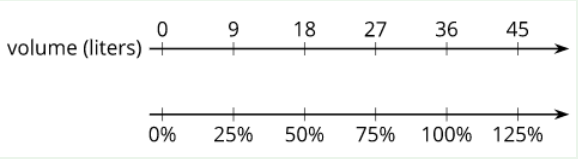


Figure 18.3.8

Practice

Exercise 18.3.5

1. How can you find 50% of a number quickly in your head?
2. Andre lives 1.6 km from school. What is 50% of 1.6 km?
3. Diego lives $\frac{1}{2}$ mile from school. What is 50% of $\frac{1}{2}$ mile?

Exercise 18.3.6

There is a 10% off sale on laptop computers. If someone saves $35 on a laptop, what was its original cost? If you get stuck, consider using the table.

Table 18.3.1

savings (dollars)	percentage

savings (dollars)	percentage
35	10
?	100

Exercise 18.3.7

Explain how to calculate these mentally.

1. 15 is what percentage of 30?
2. 3 is what percentage of 12?
3. 6 is what percentage of 10?

Exercise 18.3.8

Noah says that to find 20% of a number he divides the number by 5. For example, 20% of 60 is 12, because $60 \div 5 = 12$. Does Noah's method always work? Explain why or why not.

Exercise 18.3.9

Diego has 75% of \$10. Noah has 25% of \$30. Diego thinks he has more money than Noah, but Noah thinks they have an equal amount of money. Who is right? Explain your reasoning.

(From Unit 3.4.1)

Exercise 18.3.10

Lin and Andre start walking toward each other at the same time from opposite ends of 22-mile walking trail. Lin walks at a speed of 2.5 miles per hour. Andre walks at a speed of 3 miles per hour.

Here is a table showing the distances traveled and how far apart Lin and Andre were over time. Use the table to find how much time passes before they meet.

Table 18.3.2

elapsed time (hour)	Lin's distance (miles)	Andre's distance (miles)	distance apart (miles)
0	0	0	22
1	2.5	3	16.5

(From Unit 3.3.4)

18.3: Benchmark Percentages is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

18.4: Solving Percentage Problems

Lesson

Let's solve more percentage problems.

Exercise 18.4.1: Number Talk: Multiplication with Decimals

Find the products mentally.

$6 \cdot (0.8) \cdot 2$

$(4.5) \cdot (0.6) \cdot 4$

Exercise 18.4.2: Coupons

Han and Clare go shopping, and they each have a coupon. Answer each question and show your reasoning.

1. Han buys an item with a normal price of $15, and uses a 10% off coupon. How much does he save by using the coupon?



Figure 18.4.1: Coupons 5, by Hobbies on a Budget. CC BY 2.0. Flickr. Source.

2. Clare buys an item with a normal price of $24, but saves $6 by using a coupon. For what percentage off is this coupon?

Are you ready for more?

Clare paid full price for an item. Han bought the same item for 80% of the full price. Clare said, “I can’t believe I paid 125% of what you paid, Han!” Is what she said true? Explain.

Exercise 18.4.3: Info Gap: Music Devices

Your teacher will give you either a *problem card* or a *data card*. Do not show or read your card to your partner.

If your teacher gives you the *problem card*:

1. Silently read your card and think about what information you need to be able to answer the question.
2. Ask your partner for the specific information that you need.
3. Explain how you are using the information to solve the problem.
   Continue to ask questions until you have enough information to solve the problem.
4. Share the *problem card* and solve the problem independently.
5. Read the *data card* and discuss your reasoning.

If your teacher gives you the *data card*:

1. Silently read your card.
2. Ask your partner “*What specific information do you need?*” and wait for them to *ask* for information.
   If your partner asks for information that is not on the card, do not do the calculations for them. Tell them you don’t have that information.
3. Before sharing the information, ask “*Why do you need that information?*” Listen to your partner’s reasoning and ask clarifying questions.
4. Read the *problem card* and solve the problem independently.
5. Share the *data card* and discuss your reasoning.

Summary

A pot can hold 36 liters of water. What percentage of the pot is filled when it contains 9 liters of water?

Here are two different ways to solve this problem:

- Using a double number line:

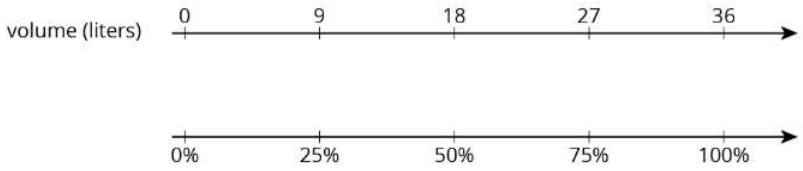


Figure 18.4.2

We can divide the distance between 0 and 36 into four equal intervals, so 9 is $\frac{1}{4}$ of 36, or 9 is 25% of 36.

- Using a table:

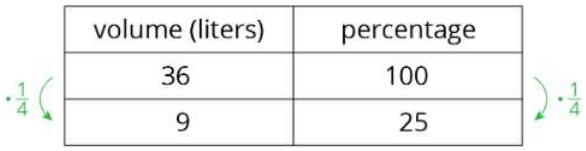


Figure 18.4.3: A table with two columns. First column, volume in liters, 36, 9. Second column, percentage, 100, 25. Outside the table, an arrow on each side labeled times fraction one over four points from the first data row to the second data row.

Glossary Entries

Definition: Percent

The word *percent* means "for each 100." The symbol for percent is %.

For example, a quarter is worth 25 cents, and a dollar is worth 100 cents. We can say that a quarter is worth 25% of a dollar.



Figure 18.4.4

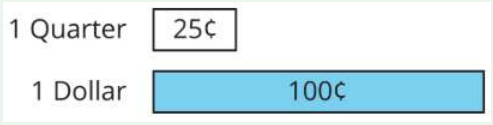


Figure 18.4.5: A diagram of two bars with different lengths. The top bar is labeled 1 Quarter and 25 cents is labeled inside the bar. The bottom bar is labeled 1 Dollar. It is 4 times longer than the top bar and 100 cents is labeled inside the bar.

Definition: Percentage

A percentage is a rate per 100.

For example, a fish tank can hold 36 liters. Right now there is 27 liters of water in the tank. The percentage of the tank that is full is 75%.

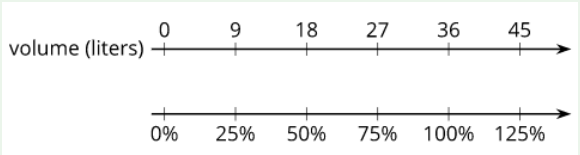


Figure 18.4.6

Practice

Exercise 18.4.4

For each problem, explain or show your reasoning.

1. 160 is what percentage of 40?
2. 40 is 160% of what number?
3. What number is 40% of 160?

Exercise 18.4.5

A store is having a 20%-off sale on all merchandise. If Mai buys one item and saves $13, what was the original price of her purchase? Explain or show your reasoning.

Exercise 18.4.6

The original price of a scarf was $16. During a store-closing sale, a shopper saved $12 on the scarf. What percentage discount did she receive? Explain or show your reasoning.

Exercise 18.4.7

Select **all** the expressions whose value is larger than 100.

A. 120% of 100
B. 50% of 150
C. 150% of 50
D. 20% of 800
E. 200% of 30
F. 500% of 400
G. 1% of 1,000

Exercise 18.4.8

An ant travels at a constant rate of 30 cm every 2 minutes.

1. At what pace does the ant travel per centimeter?
2. At what speed does the ant travel per minute?

(From Unit 3.3.4)

Exercise 18.4.9

Is $3\frac{1}{2}$ cups more or less than 1 liter? Explain or show your reasoning. (Note: 1 cup $\approx$ 236.6 milliliters)

(From Unit 3.2.3)

Exercise 18.4.10

Name a unit of measurement that is about the same size as each object.

1. The distance of a doorknob from the floor is about 1 _____.
2. The thickness of a fingernail is about 1 _____.
3. The volume of a drop of honey is about 1 _____.
4. The weight or mass of a pineapple is about 1 _____.
5. The thickness of a picture book is about 1 _____.
6. The weight or mass of a buffalo is about 1 _____.
7. The volume of a flower vase is about 1 _____.

8. The weight or mass of 20 staples is about 1 _____.
9. The volume of a melon is about 1 _____.
10. The length of a piece of printer paper is about 1 _____.

(From Unit 3.2.1)

18.4: Solving Percentage Problems is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

18.5: Finding This Percent of That

Lesson

Let's solve percentage problems like a pro.

Exercise 18.5.1: Number Talk: Decimals

Find the value of each expression mentally.

$(0.23) \cdot 100$

$50 \div 100$

$145 \cdot \frac{1}{100}$

$7 \div 100$

Exercise 18.5.2: Audience Size

A school held several evening activities last month—a music concert, a basketball game, a drama play, and literacy night. The music concert was attended by 250 people. How many people came to each of the other activities?

1. Attendance at a basketball game was 30% of attendance at the concert.
2. Attendance at the drama play was 140% of attendance at the concert.
3. Attendance at literacy night was 44% of attendance at the concert.

Are you ready for more?

50% of the people who attended the drama play also attended the music concert. What percentage of the people who attended the music concert also attended the drama play?

Exercise 18.5.3: Everything is On Sale

During a sale, every item in a store is 80% of its regular price.

1. If the regular price of a T-shirt is $10, what is its sale price?
2. The regular prices of five items are shown here. Find the sale price of each item.

Table 18.5.1

	item 1	item 2	item 3	item 4	item 5
regular price	$1	$4	$10	$55	$120
sale price					

3. You found 80% of many values. Was there a process you repeated over and over to find the sale prices? If so, describe it.

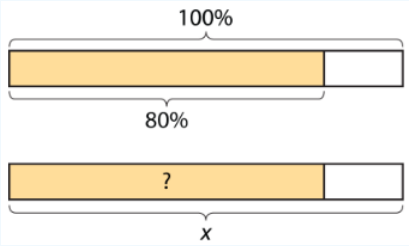


Figure 18.5.1

4. Select **all** of the expressions that could be used to find 80% of x. Be prepared to explain your reasoning.

$\frac{8}{100} \cdot x$ $\quad \frac{8}{10} \cdot x$ $\quad \frac{8}{5} \cdot x$ $\quad 80 \cdot x$ $\quad (0.8) \cdot x$

$\frac{80}{100} \cdot x$ $\quad \frac{4}{10} \cdot x$ $\quad \frac{4}{5} \cdot x$ $\quad 8 \cdot x$ $\quad (0.08) \cdot x$

Summary

To find 49% of a number, we can multiply the number by $\frac{49}{100}$ or 0.49.

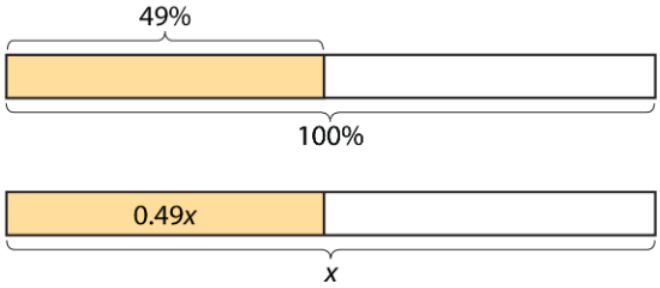


Figure 18.5.2

To find 135% of a number, we can multiply the number by $\frac{135}{100}$ or 1.35.

To find 6% of a number, we can multiply the number by $\frac{6}{100}$ or 0.06.

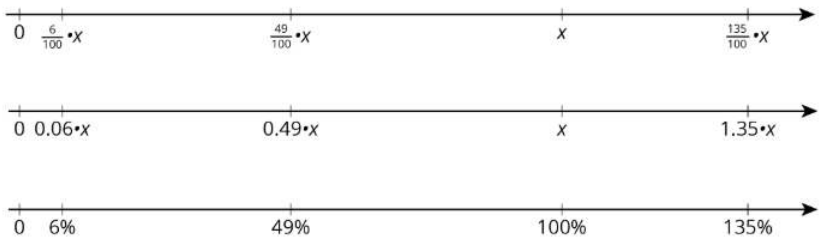


Figure 18.5.3: A triple number line with 5 tick marks. For the top number line, starting with the first tick mark, the numbers 0, the fraction 6 over 100, end fraction, times x, the fraction 49 over 100, end fraction, times x, x, and the fraction 135 over 100, end fraction, times x, are labeled. For the middle number line, starting with the first tick mark, the numbers 0, 0 point 0 6, times x, 0 point 4 9, times x, x, and 1 point 3 5, times x are labeled. For the bottom number line, starting with the first tick mark, the numbers 0, 6 percent, 49 percent, 100 percent, and 135 percent are labeled.

In general, to find P% of x, we can multiply: $\frac{P}{100} \cdot x$

Glossary Entries

Definition: Percent

The word *percent* means "for each 100." The symbol for percent is %.

For example, a quarter is worth 25 cents, and a dollar is worth 100 cents. We can say that a quarter is worth 25% of a dollar.



Figure 18.5.4

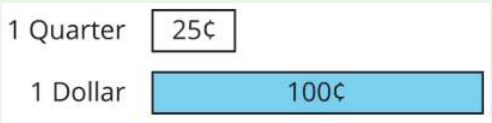


Figure 18.5.5: A diagram of two bars with different lengths. The top bar is labeled 1 Quarter and 25 cents is labeled inside the bar. The bottom bar is labeled 1 Dollar. It is 4 times longer than the top bar and 100 cents is labeled inside the bar.

Definition: Percentage

A percentage is a rate per 100.

For example, a fish tank can hold 36 liters. Right now there is 27 liters of water in the tank. The percentage of the tank that is full is 75%.

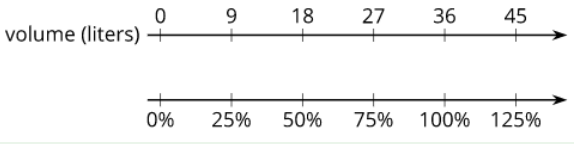


Figure 18.5.6

Practice

Exercise 18.5.4

1. To find 40% of 75, Priya calculates . Does her calculation give the correct value for 40% of 75? Explain or show how you know.
2. If represents a number, does always represent 40% of that number? Explain your reasoning.

Exercise 18.5.5

Han spent 75 minutes practicing the piano over the weekend. For each question, explain or show your reasoning.

1. Priya practiced the violin for 152% as much time as Han practiced the piano. How long did she practice?
2. Tyler practiced the clarinet for 64% as much time as Han practiced the piano. How long did he practice?

Exercise 18.5.6

Last Sunday 1,575 people visited the amusement park. 56% of the visitors were adults, 16% were teenagers, and 28% were children ages 12 and under. Find the number of adults, teenagers, and children that visited the park.

Exercise 18.5.7

Order from greatest to least:

- 55% of 180
- 300% of 26
- 12% of 700

Exercise 18.5.8

Complete each statement.

- 20% of 60 is _____
- 25% of _____ is 6
- _____% of 100 is 14
- 50% of 90 is _____
- 10% of _____ is 7
- 30% of 70 is _____

(From Unit 3.4.5)

Exercise 18.5.9

A shopper needs 24 sandwich rolls. The store sells identical rolls in 2 differently sized packages. They sell a six-pack for $5.28 and a four-pack for $3.40. Should the shopper buy 4 six-packs or 6 four-packs? Explain your reasoning.

(From Unit 3.3.5)

Exercise 18.5.10

On a field trip, there are 3 chaperones for every 20 students. There are 92 people on the trip. Answer these questions. If you get stuck, consider using a tape diagram.

1. How many chaperones are there?
2. How many children are there?

(From Unit 2.5.1)

18.5: Finding This Percent of That is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

18.6: Finding the Percentage

Lesson

Let's find percentages in general.

Exercise 18.6.1: True or False: Percentages

Is each statement true or false? Be prepared to explain your reasoning.

1. 25% of 512 is equal to $\frac{1}{4} \cdot 500$.
2. 90% of 133 is equal to $(0.9) \cdot 133$.
3. 30% of 44 is equal to 3% of 440.
4. The percentage 21 is of 28 is equal to the percentage 30 is of 40.

Exercise 18.6.2: Jumping Rope

A school held a jump-roping contest. Diego jumped rope for 20 minutes.

1. Jada jumped rope for 15 minutes. What percentage of Diego's time is that?
2. Lin jumped rope for 24 minutes. What percentage of Diego's time is that?
3. Noah jumped rope for 9 minutes. What percentage of Diego's time is that?
4. Record your answers in this table. Write the quotients in the last column as decimals.

Table 18.6.1

	time (minutes)	percentage	time ÷ 20
Diego	20	100	$\frac{20}{20} = 1$
Jada	15		$\frac{15}{20} =$
Lin	24		$\frac{24}{20} =$
Noah	9		$\frac{9}{20} =$

5. What do you notice about the numbers in the last two columns of the table?

Exercise 18.6.3: Restaurant Cpacity

A restaurant has a sign by the front door that says, "Maximum occupancy: 75 people." Answer each question and explain or show your reasoning.

1. What percentage of its capacity is 9 people?
2. What percentage of its capacity is 51 people?
3. What percentage of its capacity is 84 people?

Are you ready for more?

Water makes up about 71% of Earth's surface, while the other 29% consists of continents and islands. 96% of all Earth's water is contained within the oceans as salt water, while the remaining 4% is fresh water located in lakes, rivers, glaciers, and the polar ice caps.

If the total volume of water on Earth is 1,386 million cubic kilometers, what is the volume of salt water? What is the volume of fresh water?

Summary

What percentage of 90 kg is 36 kg? One way to solve this problem is to first find what percentage 1 kg is of 90, and then multiply by 36.

mass (kg)	percentage
90	100
1	$\frac{1}{90}\cdot 100$
36	$\frac{36}{90}\cdot 100$

Figure 18.6.1: A table with two columns. The first column is labeled mass in kilograms. The second column is labeled percentage. The data are as follows: row 1: 90 kilograms, 100 percent; row 2: one kilogram, the fraction 1 over 90, end fraction, times 100; row 3: 36 kilograms, the fraction 36 over 90, end fraction, times 100. Arrows on both sides of the table from row 1 to row 2 are labeled multiply by the fraction 1 over 90. Arrows on both sides of the table from row 2 to row 3 are labeled multiply by 36.

From the table we can see that 1 kg is $\left(\frac{1}{90}\cdot 100\right)\%$, so 36 kg is $\left(\frac{36}{90}\cdot 100\right)\%$ or 40% of 90. We can confirm this on a double number line:

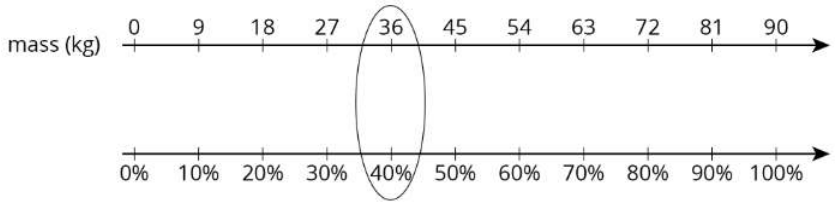


Figure 18.6.2

In general, to find what percentage a number C is of another number B is to calculate $\frac{C}{B}$ of 100%. We can find do that by multiplying: $\frac{C}{B}\cdot 100$

Suppose a school club has raised \$88 for a project but needs a total of \$160. What percentage of its goal has the club raised?

To find what percentage \$88 is of \$160, we find $\frac{88}{160}$ of 100% or $\frac{88}{160}\cdot 100$, which equals $\frac{11}{20}\cdot 100$ or 55. The club has raised 55% of its goal.

Glossary Entries

Definition: Percent

The word *percent* means "for each 100." The symbol for percent is %.

For example, a quarter is worth 25 cents, and a dollar is worth 100 cents. We can say that a quarter is worth 25% of a dollar.



Figure 18.6.3

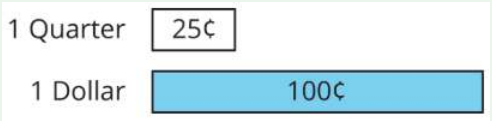


Figure 18.6.4: A diagram of two bars with different lengths. The top bar is labeled 1 Quarter and 25 cents is labeled inside the bar. The bottom bar is labeled 1 Dollar. It is 4 times longer than the top bar and 100 cents is labeled inside the bar.

Definition: Percentage

A percentage is a rate per 100.

For example, a fish tank can hold 36 liters. Right now there is 27 liters of water in the tank. The percentage of the tank that is full is 75%.

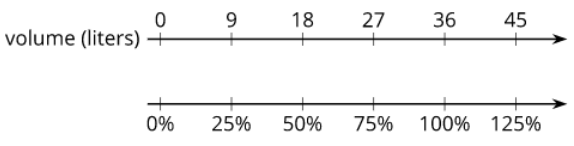


Figure 18.6.5

Practice

Exercise 18.6.4

A sign in front of a roller coaster says "You must be 40 inches tall to ride." What percentage of this height is:

1. 34 inches?
2. 54 inches?

Exercise 18.6.5

At a hardware store, a tool set normally costs $80. During a sale this week, the tool set costs $12 less than usual. What percentage of the usual price is the savings? Explain or show your reasoning.

Exercise 18.6.6

A bathtub can hold 80 gallons of water. The faucet flows at a rate of 4 gallons per minute. What percentage of the tub will be filled after 6 minutes?

Exercise 18.6.7

The sale price of every item in a store is 85% of its usual price.

1. The usual price of a backpack is $30, what is its sale price?
2. The usual price of a sweatshirt is $18, what is its sale price?
3. The usual price of a soccer ball is $24.80, what is its sale price?

(From Unit 3.4.6)

Exercise 18.6.8

A shopper needs 48 hot dogs. The store sells identical hot dogs in 2 differently sized packages. They sell a six-pack of hot dogs for $2.10, and an eight-pack of hot dogs for $3.12. Should the shopper buy 8 six-packs, or 6 eight-packs? Explain your reasoning.

(From Unit 3.3.5)

Exercise 18.6.9

Elena is 56 inches tall.

1. What is her height in centimeters? (Note: 100 inches = 254 centimeters)
2. What is her height in meters?

(From Unit 3.2.3)

18.6: Finding the Percentage is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

Section 19: Let's Put it to Work

This page titled Section 19: Let's Put it to Work is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

19.1: Painting a Room

Lesson

Let's see what it takes to paint a room.

Exercise 19.1.1: Getting Ready to Paint

What are some tools that are helpful when painting a room?

Exercise 19.1.2: How Much It Costs to Paint

Here is the floor plan for a bedroom:

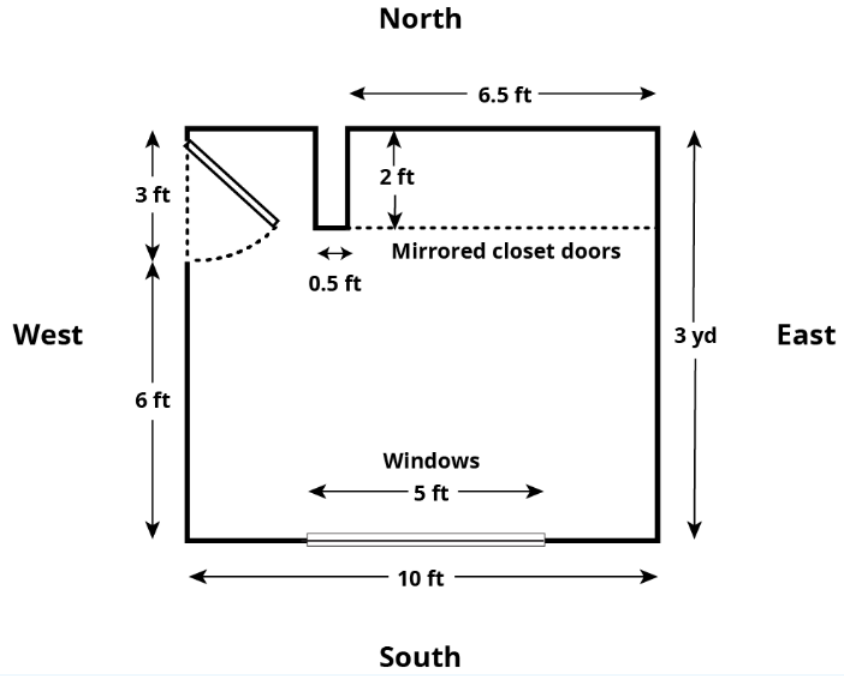


Figure 19.1.1: A floor plan for a bedroom that closely resembles the shape of a rectangle with the top side labeled North, the bottom side labeled South, the right side labeled East, and the left side labeled West.

Imagine you are planning to repaint all the walls in this room, including inside the closet.

- The east wall is 3 yards long.
- The south wall is 10 feet long but has a window, 5 feet by 3 feet, that does not need to be painted.
- The west wall is 3 yards long but has a door, 7 feet tall and 3 feet wide, that does not need to be painted.
- The north wall includes a closet, 6.5 feet wide, with floor-to-ceiling mirrored doors that do not need to be painted. There is, however, a smaller wall between the west wall and the closet that does need to be painted on all sides. The wall is 0.5 feet wide and extends 2 feet into the room.
- The ceiling in this room is 8 feet high.
- All of the corners are right angles.

1. If you paint all the walls in the room, how many square feet do you need to cover?
2. An advertisement about the paint that you want to use reads: “Just 2 quarts covers 175 square feet!” If you need to apply two coats of paint on all the walls, how much paint do you need to buy?
3. Paint can only be purchased in 1-quart, 1-gallon, and 5-gallon containers. How much will all supplies for the project cost if the cans of paint cost $10.90 for a quart, $34.90 for a gallon, and $165.00 for 5 gallons?
4. You have a coupon for 20% off all quart-sized paint cans. How does that affect the cost of the project?

Exercise 19.1.3: How Long It Takes to Paint

After buying the supplies, you start painting the east wall. It takes you 96 minutes to put two coats of paint on that wall (not including a lunch break between the two coats).

1. Your friend stops by to see how you are doing and comments that you are 25% finished with the painting. Are they correct?
2. Your friend offers to help you with the rest of the painting. It takes the two of you 150 more minutes of painting time to finish the entire room. How much time did your friend save you?

19.1: Painting a Room is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

Section 20: New Page

This page was auto-generated because a user created a sub-page to this page.

Section 20: New Page is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

20.1: What Are Percentages?

Lesson

Let's learn about percentages.

Exercise 20.1.1: Dollars and Cents

Find each answer mentally.

1. A sticker costs 25 cents. How many dollars is that?
2. A pen costs 1.50 dollars. How many cents is that?
3. How many cents are in one dollar?
4. How many dollars are in one cent?

Exercise 20.1.2: Coins

1. Complete the table to show the values of these U.S. coins.



Figure 20.1.1: By United States Mint. Public Domain. United States Mint. Source.

Table 20.1.1

coin	penny	nickel	dime	quarter	half dollar	dollar
value (cents)						

The value of a quarter is 25% of the value of a dollar because there are 25 cents for every 100 cents.

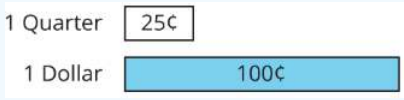


Figure 20.1.2: A diagram of two bars with different lengths. The top bar is labeled 1 Quarter and 25 cents is labeled inside the bar. The bottom bar is labeled 1 Dollar. It is 4 times longer than the top bar and 100 cents is labeled inside the bar.

2. Write the name of the coin that matches each expression.
   - 25% of a dollar
   - 5% of a dollar
   - 1% of a dollar
   - 100% of a dollar
   - 10% of a dollar
   - 50% of a dollar
3. The value of 6 dimes is what **percent** of the value of a dollar?
4. The value of 6 quarters is what percent of the value of a dollar?

Are you ready for more?

Find two different sets of coins that each make 120% of a dollar, where no type of coin is in both sets.

Exercise 20.1.3: Coins on a Number Line



Figure 20.1.3

A $1 coin is worth 100% of the value of a dollar. Here is a double number line that shows this.

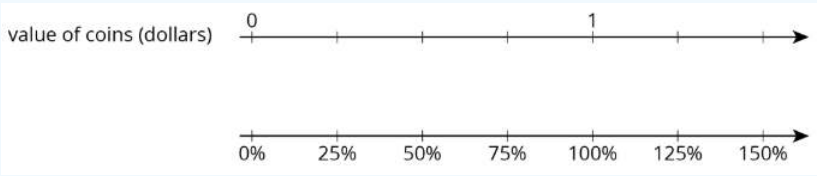


Figure 20.1.4

1. The coins in Jada's pocket are worth 75% of a dollar. How much are they worth (in dollars)?
2. The coins in Diego's pocket are worth 150% of a dollar. How much are they worth (in dollars)?
3. Elena has 3 quarters and 5 dimes. What percentage of a dollar does she have?

Summary

A **percentage** is a *rate per 100*.

We can find percentages of $10 using a double number line where 10 and 100% are aligned, as shown here:

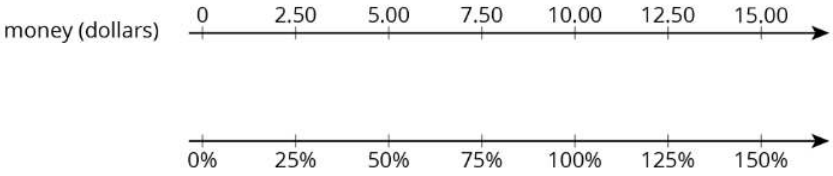


Figure 20.1.5

Looking at the double number line, we can see that $5.00 is 50% of $10.00 and that $12.50 is 125% of $10.00.

Glossary Entries

Definition: Percent

The word *percent* means "for each 100." The symbol for percent is %.

For example, a quarter is worth 25 cents, and a dollar is worth 100 cents. We can say that a quarter is worth 25% of a dollar.



Figure 20.1.6

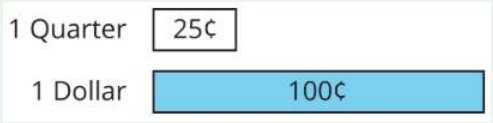


Figure 20.1.7: A diagram of two bars with different lengths. The top bar is labeled 1 Quarter and 25 cents is labeled inside the bar. The bottom bar is labeled 1 Dollar. It is 4 times longer than the top bar and 100 cents is labeled inside the bar.

Definition: Percentage

A percentage is a rate per 100.

For example, a fish tank can hold 36 liters. Right now there is 27 liters of water in the tank. The percentage of the tank that is full is 75%.

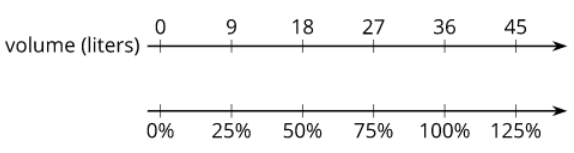


Figure 20.1.8

Practice

Exercise 20.1.4

What percentage of a dollar is the value of each coin combination?

1. 4 dimes
2. 1 nickel and 3 pennies
3. 5 quarters and 1 dime

Exercise 20.1.5

1. List three different combinations of coins, each with a value of 30% of a dollar.
2. List two different combinations of coins, each with a value of 140% of a dollar.

Exercise 20.1.6

The United States government used to make coins of many different values. For each coin, state its worth as a percentage of \$1.

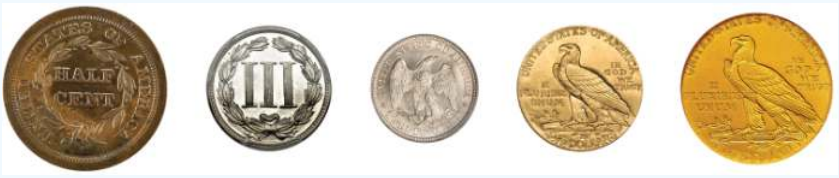


Figure 20.1.9

$\frac{1}{2}$ cent 3 cents 20 cents $\$2\frac{1}{2}$ \$5

Exercise 20.1.7

Complete the double number to line show percentages of \$50.

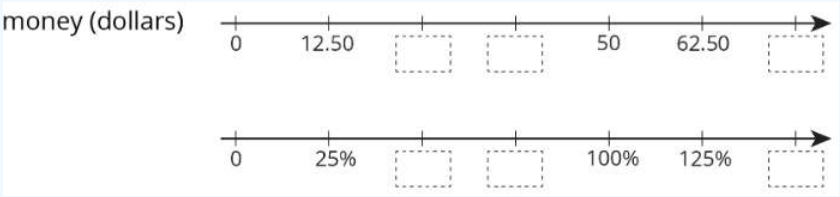


Figure 20.1.10

Exercise 20.1.8

Elena bought 8 tokens for \$4.40. At this rate:

1. How many tokens could she buy with \$6.05?
2. How much do 19 tokens cost?

(From Unit 3.3.5)

Exercise 20.1.9

A snail travels 10 cm in 4 minutes. At this rate:

1. How long will it take the snail to travel 24 cm?
2. How far does the snail travel in 6 minutes?

(From Unit 3.3.4)

Exercise 20.1.10

1. 3 tacos cost $18. Complete the table to show the cost of 4, 5, and 6 tacos at the same rate.

Table 20.1.2

number of tacos	cost in dollars	rate in dollars per taco
3	18	
4		
5		
6		

2. If you buy t tacos for c dollars, what is the unit rate?

(From Unit 3.3.3)

20.1: What Are Percentages? is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

SECTION OVERVIEW

4: Dividing Fractions

Section 21: Making Sense of Division

21.1: Size of Divisor and Size of Quotient
21.2: Meanings of Division
21.3: Interpreting Division Situations

Section 22: Meanings of Fraction Division

22.1: How Many Groups? (Part 1)
22.2: How Many Groups? (Part 2)
22.3: Using Diagrams to Find the Number of Groups
22.4: What Fraction of a Group?
22.5: How Much in Each Group? (Part 1)
22.6: How Much in Each Group? (Part 2)

Section 23: Algorithm for Fraction Division

23.1: Dividing by Unit and Non-Unit Fractions
23.2: Using an Algorithm to Divide Fractions

Section 24: Fractions in Lengths, Areas, and Volumes

24.1: Fractional Lengths
24.2: Rectangles with Fractional Side Lengths
24.3: Fractional Lengths in Triangles and Prisms
24.4: Volume of Prisms

Section 25: Let's Put It to Work

25.1: Solving Problems Involving Fractions
25.2: Fitting Boxes into Boxes

This page titled 4: Dividing Fractions is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

Section 21: Making Sense of Division

This page titled Section 21: Making Sense of Division is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

21.1: Size of Divisor and Size of Quotient

Lesson

Let's explore quotients of different sizes.

Exercise 21.1.1: Number Talk: Size of Dividend and Divisor

Find the value of each expression mentally.

$5,000 \div 5$

$5,000 \div 2,500$

$5,000 \div 10,000$

$5,000 \div 500,000$

Exercise 21.1.2: All Stacked Up

1. Here are several types of objects. For each type of object, estimate how many are in a stack that is 5 feet high. Be prepared to explain your reasoning.

Cardboard boxes



Figure 21.1.1: Cardboard Boxes, by FrancisRomeo. Public Domain. Pixabay. Source.

Notebooks



Figure 21.1.2

Bricks



Figure 21.1.3

Coins



Figure 21.1.4

2. A stack of books is 72 inches tall. Each book is 2 inches thick. Which expression tells us how many books are in the stack? Be prepared to explain your reasoning.
   - $72 \cdot 2$
   - $72 - 2$
   - $2 \div 72$
   - $72 \div 2$
3. Another stack of books is 43 inches tall. Each book is $\frac{1}{2}$-inch thick. Write an expression that represents the number of books in the stack.

Exercise 21.1.3: All in Order

Your teacher will give you two sets of papers with division expressions.

1. Without computing, estimate the quotients in each set and order them from greatest to least. Be prepared to explain your reasoning.
   Pause here for a class discussion.
   Record the expressions in each set in order from the greatest value to the least.
   a. Set 1
   b. Set 2
2. Without computing, estimate the quotients and sort them into the following three groups. Be prepared to explain your reasoning.

$30 \div \frac{1}{2}$	$9 \div 10$	$18 \div 19$	$15,000 \div 1,500,000$
$30 \div 0.45$	$9 \div 10,000$	$18 \div 0.18$	$15,000 \div 14,500$

- Close to 0
- Close to 1
- Much larger than 1

Are you ready for more?

Write 10 expressions of the form $12 \div ?$ in a list ordered from least to greatest. Can you list expressions that have value near 1 without equaling 1? How close can you get to the value 1?

Summary

Here is a division expression: $60 \div 4$. In this division, we call 60 the *dividend* and 4 the *divisor*. The result of the division is the quotient. In this example, the quotient is 15, because $60 \div 4 = 15$.

We don't always have to make calculations to have a sense of what a quotient will be. We can reason about it by looking at the size of the dividend and the divisor. Let's look at some examples.

- In $100 \div 11$ and in $18 \div 2.9$ the dividend is larger than the divisor. $100 \div 11$ is very close to $99 \div 11$, which is 9. The quotient $18 \div 2.9$ is close to $18 \div 3$ or 6.
  In general, when a larger number is divided by a smaller number, the quotient is greater than 1.
- In $99 \div 101$ and in $7.5 \div 7.4$ the dividend and divisor are very close to each other. $99 \div 101$ is very close to $99 \div 100$, which is $\frac{99}{100}$ or 0.99. The quotient $7.5 \div 7.4$ is close to $7.5 \div 7,5$, which is 1.
  In general, when we divide two numbers that are nearly equal to each other, the quotient is close to 1.

- In $10 \div 101$ and in $50 \div 198$ the dividend is smaller than the divisor. $10 \div 101$ is very close to $10 \div 100$, which is $\frac{10}{100}$ or 0.1. The division $50 \div 198$ is close to $50 \div 200$, which is $\frac{1}{4}$ or 0.25.
  In general, when a smaller number is divided by a larger number, the quotient is less than 1.

Practice

Exercise 21.1.4

Order from smallest to largest:

- Number of pennies in a stack that is 1 ft high
- Number of books in a stack that is 1 ft high
- Number of dollar bills in a stack that is 1 ft high
- Number of slices of bread in a stack that is 1 ft high

Exercise 21.1.5

Use each of the numbers 4, 40, and 4000 once to complete the sentences.

1. The value of ____ $\div 40.01$ is close to 1.
2. The value of ____ $\div 40.01$ is much less than 1.
3. The value of ____ $\div 40.01$ is much greater than 1.

Exercise 21.1.6

Without computing, decide whether the value of each expression is much smaller than 1, close to 1, or much greater than 1.

1. $100 \div \frac{1}{1000}$
2. $50\frac{1}{3} \div 50\frac{1}{4}$
3. $4.7 \div 5.2$
4. $2 \div 7335$
5. $2,000,001 \div 9$
6. $0.002 \div 2,000$

Exercise 21.1.7

A rocking horse has a weight limit of 60 pounds.

1. What percentage of the weight limit is 33 pounds?
2. What percentage of the weight limit is 114 pounds?
3. What weight is 95% of the limit?

(From Unit 3.4.7)

Exercise 21.1.8

Compare using $>$, $=$, or $<$.

1. 0.7 _____ 0.70
2. $0.03 + \frac{6}{10}$ _____ $0.30 + \frac{6}{100}$
3. 0.9 _____ 0.12

(From Unit 3.4.5)

Exercise 21.1.9

Diego has 90 songs on his playlist. How many songs are there for each genre?

1. 40% rock
2. 10% country
3. 30% hip-hop
4. The rest is electronica

(From Unit 3.4.4)

Exercise 21.1.10

A garden hose emits 9 quarts of water in 6 seconds. At this rate:

1. How long will it take the hose to emit 12 quarts?
2. How much water does the hose emit in 10 seconds?

(From Unit 3.3.4)

This page titled 21.1: Size of Divisor and Size of Quotient is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

21.2: Meanings of Division

Lesson

Let's explore ways to think about division.

Exercise 21.2.1: A Division Expression

Here is an expression: $20 \div 4$.

What are some ways to think about this expression? Describe at least two meanings you think it could have.

Exercise 21.2.2: Bags of Almonds

A baker has 12 pounds of almonds. She puts them in bags, so that each bag has the same weight.

Clare and Tyler drew diagrams and wrote equations to show how they were thinking about $12 \div 6$.

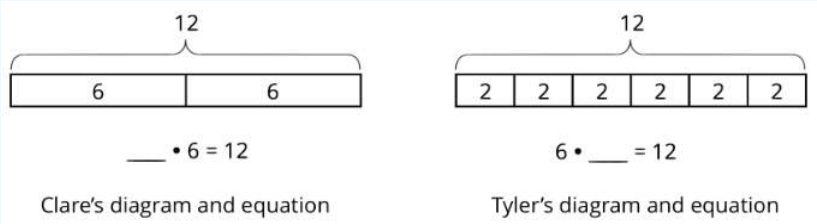


Figure 21.2.1: Two tape diagrams. On left, Clare's diagram and equation. 2 equal parts labeled 6. Total, 12. Equation blank times 6 = 12. On right, On right, Tyler's diagram and equation. 6 equal parts labeled 2. Total, 12. Equation 6 times blank = 12.

1. How do you think Clare and Tyler thought about $12 \div 6$? Explain what each diagram and the parts of each equation could mean about the situation with the bags of almonds. Make sure to include the meaning of the missing number.
   Pause here for a class discussion.
2. Explain what each division expression could mean about the situation with the bags of almonds. Then draw a diagram and write a multiplication equation to show how you are thinking about the expression.
   a. $12 \div 4$
   b. $12 \div 2$
   c. $12 \div \frac{1}{2}$

Are you ready for more?

A loaf of bread is cut into slices.

1. If each slice is $\frac{1}{2}$ of a loaf, how many slices are there?
2. If each slice is $\frac{1}{5}$ of a loaf, how many slices are there?
3. What happens to the number of slices as each slice gets smaller?
4. What would dividing by 0 mean in this situation about slicing bread?

Summary

Suppose 24 bagels are being distributed into boxes. The expression $24 \div 3$ could be understood in two ways:

- 24 bagels are distributed equally into 3 boxes, as represented by this diagram:

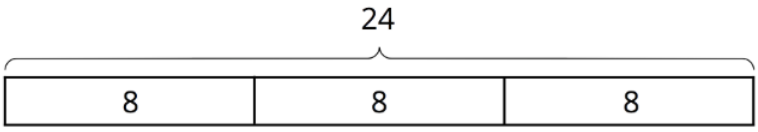


Figure 21.2.2

- 24 bagels are distributed into boxes, 3 bagels in each box, as represented by this diagram:

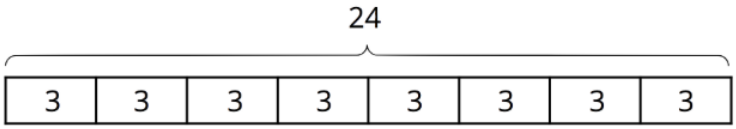


Figure 21.2.3

In both interpretations, the quotient is the same ($24 \div 3 = 8$), but it has different meanings in each case. In the first case, the 8 represents the number of bagels in each of the 3 boxes. In the second, it represents the number of boxes that were formed with 3 bagels in each box.

These two ways of seeing division are related to how 3, 8, and 24 are related in a multiplication. Both $3 \cdot 8$ and $8 \cdot 3$ equal 24.

- $3 \cdot 8 = 24$ can be read as "3 groups of 8 make 24."
- $8 \cdot 3 = 24$ can be read as "8 groups of 3 make 24."

If 3 and 24 are the only numbers given, the multiplication equations would be: $3 \cdot ? = 24$ $? \cdot 3 = 24$

In both cases, the division $24 \div 3$ can be used to find the value of the "?" But now we see that it can be interpreted in more than one way, because the "?" can refer to *the size of a group* (as in "3 groups of what number make 24?"), or to *the number of groups* (as in "How many groups of 3 make 24?").

Practice

Exercise 21.2.3

Twenty pounds of strawberries are being shared equally by a group of friends. The equation $20 \div 5 = 4$ represents the division of strawberries.

1. If the 5 represents the number of people, what does the 4 represent?
2. If the 5 represents the pounds of strawberries per person, what does the 4 represent?

Exercise 21.2.4

A sixth-grade science club needs \$180 to pay for the tickets to a science museum. All tickets cost the same amount.

What could $180 \div 15$ mean in this situation? Describe two different possible meanings of this expression. Then, find the quotient and explain what it means in each case.

Exercise 21.2.5

Write a multiplication equation that corresponds to each division equation.

1. $10 \div 5 = ?$
2. $4.5 \div 3 = ?$
3. $\frac{1}{2} \div 4 = ?$

Exercise 21.2.6

Write a division or multiplication equation that represents each situation. Use a "?" for the unknown quantity.

1. 2.5 gallons of water are poured into 5 equally sized bottles. How much water is in each bottle?
2. A large bucket of 200 golf balls is divided into 4 smaller buckets. How many golf balls are in each small bucket?
3. Sixteen socks are put into pairs. How many pairs are there?

Exercise 21.2.7

Find a value for a that makes each statement true.

1. $a \div 6$ is greater than 1
2. $a \div 6$ is equal to 1

3. $a \div 6$ is less than 1
4. $a \div 6$ is equal to a whole number

(From Unit 4.1.1)

Exercise 21.2.8

Complete the table. Write each percentage as a percent of 1.

Table 21.2.1

fraction	decimal	percentage
$\frac{1}{4}$	0.25	25% of 1
	0.1	
		75% of 1
$\frac{1}{5}$		
	1.5	
		140% of 1

(From Unit 3.4.5)

Exercise 21.2.9

Jada walks at a speed of 3 miles per hour. Elena walks at a speed of 2.8 miles per hour. If they both begin walking along a walking trail at the same time, how much farther will Jada walk after 3 hours? Explain your reasoning.

(From Unit 3.3.4)

21.2: Meanings of Division is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

21.3: Interpreting Division Situations

Lesson

Let's explore situations that involve division.

Exercise 21.3.1: Dot Image: Properties of Multiplication

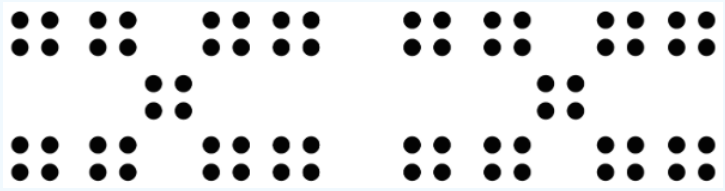


Figure 21.3.1

Exercise 21.3.2: Homemade Jams

Draw a diagram and write a multiplication equation to represent each situation. Then answer the question.

1. Mai had 4 jars. In each jar, she put $2\frac{1}{4}$ cups of homemade blueberry jam. Altogether, how many cups of jam are in the jars?
2. Priya filled 5 jars, using a total of $7\frac{1}{2}$ cups of strawberry jam. How many cups of jam are in each jar?
3. Han had some jars. He put $\frac{3}{4}$ cup of grape jam in each jar, using a total of $6\frac{3}{4}$ cups. How many jars did he fill?

Here is an applet to use if you choose to.

The toolbar includes buttons that represent 1 whole and fractional parts, as shown here. Click a button to choose a quantity, and then click in the work space of the applet window to drop it. When you're done choosing pieces, use the Move tool (the arrow) to drag them into the jars. You can always go back and get more pieces, or delete them with the Trash Can tool.



Figure 21.3.2: Image of fraction bars applet toolbar. Toolbar contains 5 images: arrow cursor, blue rectangle labeled 1, green rectangle labeled one half, yellow rectangle labeled one third, red rectangle labeled one fourth.

The jars in this applet are shown as stacked to make it easier to combine the jam and find out how much you have.

Here are the questions again.

1. Mai had 4 jars. In each jar, she put $2\frac{1}{4}$ cups of homemade blueberry jam. Altogether, how many cups of jam are in the jars?
2. Priya filled 5 jars, using a total of $7\frac{1}{2}$ cups of strawberry jam. How many cups of jam are in each jar?
3. Han had some jars. He put $\frac{3}{4}$ cup of grape jam in each jar, using a total of $6\frac{3}{4}$ cups. How many jars did he fill?

Exercise 21.3.3: Making Granola

1. Consider the problem: To make 1 batch of granola, Kiran needs 26 ounces of oats. The only measuring tool he has is a 4-ounce scoop. How many scoops will it take to measure 26 ounces of oats?
   a. Will the answer be more than 1 or less than 1?
   b. Write a multiplication equation and a division equation that represent this situation. Use "?" to represent the unknown quantity.
   c. Find the unknown quantity. If you get stuck, consider drawing a diagram.
2. The recipe calls for 14 ounces of mixed nuts. To get that amount, Kiran uses 4 bags of mixed nuts.
   a. Write a mathematical question that might be asked about this situation.
   b. What might the equation $14 \div 4 = ?$ represent in Kiran's situation?
   c. Find the quotient. Show your reasoning. If you get stuck, consider drawing a diagram.

Summary

If a situation involves equal-sized groups, it is helpful to make sense of it in terms of the number of groups, the size of each group, and the total amount. Here are three examples to help us better understand such situations.

- Suppose we have 3 bottles with $6\frac{1}{2}$ ounces of water in each, and the total amount of water is not given. Here we have 3 groups, $6\frac{1}{2}$ ounces in each group, and an unknown total, as shown in this diagram:

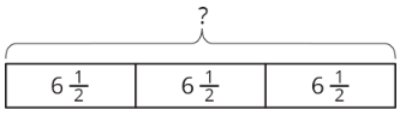


Figure 21.3.3

We can express this situation as a multiplication problem. The unknown is the product, so we can simply multiply the 2 known numbers to find it. $3 \cdot 6\frac{1}{2} = ?$

- Next, suppose we have 20 ounces of water to fill 6 equal-sized bottles, and the amount in each bottle is not given. Here we have 6 groups, an unknown amount in each, and a total of 20. We can represent it like this:

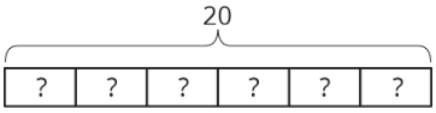


Figure 21.3.4

This situation can also be expressed using multiplication, but the unknown is a factor, rather than the product: $6 \cdot ? = 20$

To find the unknown, we cannot simply multiply, but we can think of it as a division problem: $20 \div 6 = ?$

- Now, suppose we have 40 ounces of water to pour into bottles, 12 ounces in each bottle, but the number of bottles is not given. Here we have an unknown number of groups, 12 in each group, and a total of 40.

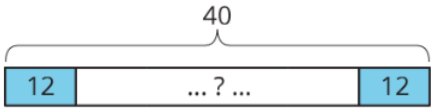


Figure 21.3.5: A tape diagram with 3 parts. The first part and third part are equal and are labeled 12. The second part is labeled with a question mark. A brace from the beginning of the diagram to the end of the diagram is labeled 40.

Again, we can think of this in terms of multiplication, with a different factor being the unknown: $40 \div 12 = ?$

Likewise, we can use division to find the unknown:

Whenever we have a multiplication situation, one factor tells us *how many groups* there are, and the other factor tells us *how much is in each group*.

Sometimes we want to find the total. Sometimes we want to find how many groups there are. Sometimes we want to find how much is in each group. Anytime we want to find out how many groups there are or how much is in each group, we can represent the situation using division.

Practice

Exercise 21.3.4

Write a multiplication equation and a division equation that this diagram could represent.

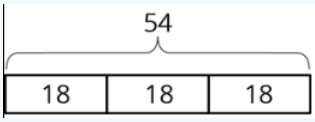


Figure 21.3.6

Exercise 21.3.5

Consider the problem: Mai has $36 to spend on movie tickets. Each movie ticket costs $4.50. How many tickets can she buy?

1. Write a multiplication equation and a division equation to represent this situation.
2. Find the answer. Draw a diagram, if needed.
3. Use the multiplication equation to check your answer.

Exercise 21.3.6

Kiran said that this diagram can show the solution to $16 \div 8 = ?$ or $16 \div 2 = ?$, depending on how we think about the equations and the "?".

Explain or show how Kiran is correct.

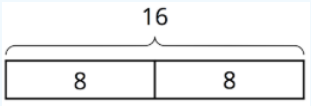


Figure 21.3.7

Exercise 21.3.7

Write a sentence describing a situation that could be represented by the equation $4 \div 1\frac{1}{3} = ?$.

(From Unit 4.1.2)

Exercise 21.3.8

Noah said, "When you divide a number by a second number, the result will always be smaller than the first number."

Jada said, "I think the result could be larger or smaller, depending on the numbers."

Do you agree with either of them? Explain or show your reasoning.

(From Unit 4.1.1)

Exercise 21.3.9

Mini muffins cost $3.00 per dozen.

- Andre says, "I have $2.00, so I can afford 8 muffins."
- Elena says, "I want to get 16 muffins, so I'll need to pay $4.00."

Do you agree with either of them? Explain your reasoning.

(From Unit 3.3.3)

Exercise 21.3.10

A family has a monthly budget of $2,400. How much money is spent on each category?

1. 44% is spent on housing.
2. 23% is spent on food.
3. 6% is spent on clothing.
4. 17% is spent on transportation.
5. The rest is put into savings.

(From Unit 3.4.6)

21.3: Interpreting Division Situations is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

Section 22: Meanings of Fraction Division

This page titled Section 22: Meanings of Fraction Division is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

22.1: How Many Groups? (Part 1)

Lesson

Let's play with blocks and diagrams to think about division with fractions.

Exercise 22.1.1: Equal-Sized Groups

Write a multiplication equation and a division equation for each sentence or diagram.

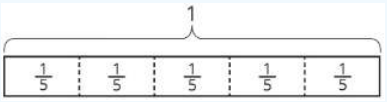
1. Eight $5 bills are worth $40.
2. There are 9 thirds in 3 ones.
3. 

Figure 22.1.1

Exercise 22.1.2: Reasoning with Pattern Blocks

Use the pattern blocks in the applet to answer the questions. (If you need help aligning the pieces, you can turn on the grid.)

1. If a hexagon represents 1 whole, what fraction do each of the following shapes represent? Be prepared to show or explain your reasoning.
   a. 1 triangle
   b. 1 rhombus
   c. 1 trapezoid
   d. 4 triangles
   e. 3 rhombuses
   f. 2 hexagons
   g. 1 hexagon and 1 trapezoid
2. Here are Elena's diagrams for $2 \cdot \frac{1}{2} = 1$ and $6 \cdot \frac{1}{3} = 2$. Do you think these diagrams represent the equations? Explain or show your reasoning.

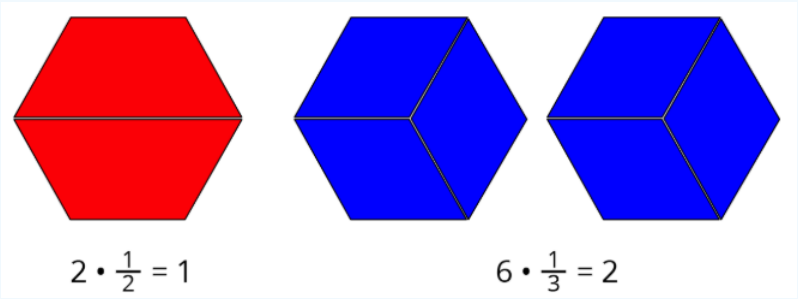


Figure 22.1.2: Two diagrams of pattern blocks: The first is of one red hexagon composed of two red trapezoids. The diagram is labeled with the equation 2 times one half equals 1. The second diagram is of two blue hexagons where each hexagon is composed of three blue rhombuses. The second diagram is labeled with the equation 6 times, one third, equals 2.

3. Use pattern blocks to represent each multiplication equation. Remember that a hexagon represents 1 whole.
   a. $3 \cdot \frac{1}{6} = \frac{1}{2}$
   b. $2 \cdot \frac{3}{2} = 3$
4. Answer the questions. If you get stuck, consider using pattern blocks.
   a. How many $\frac{1}{2}$s are in 4?
   b. How many $\frac{2}{3}$s are in 2?
   c. How many $\frac{1}{6}$s are in $1\frac{1}{2}$?

Summary

Some problems that involve equal-sized groups also involve fractions. Here is an example: "How many $\frac{1}{6}$ are in 2?" We can express this question with multiplication and division equations.

$$? \cdot \frac{1}{6} = 2$$

$$2 \div \frac{1}{6} = ?$$

Pattern-block diagrams can help us make sense of such problems. Here is a set of pattern blocks.

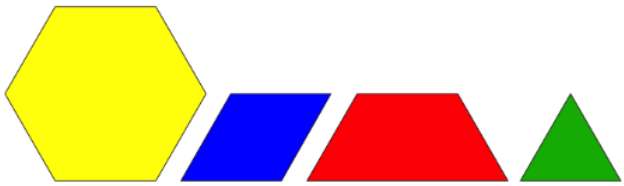


Figure 22.1.3

If the hexagon represents 1 whole, then a triangle must represent $\frac{1}{6}$, because 6 triangles make 1 hexagon. We can use the triangle to represent the $\frac{1}{6}$ in the problem.

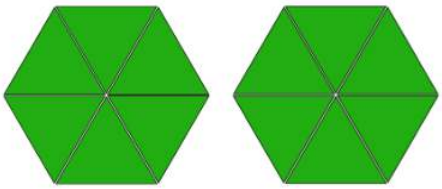


Figure 22.1.4

Twelve triangles make 2 hexagons, which means there are 12 groups of $\frac{1}{6}$ in 2.

If we write the 12 in the place of the "?" in the original equations, we have:

$$12 \cdot \frac{1}{6} = 2$$

$$2 \div \frac{1}{6} = 12$$

Practice

Exercise 22.1.3

Consider the problem: A shopper buys cat food in bags of 3 lbs. Her cat eats $\frac{3}{4}$ lb each week. How many weeks does one bag last?

1. Draw a diagram to represent the situation and label your diagram so it can be followed by others. Answer the question.
2. Write a multiplication or division equation to represent the situation.
3. Multiply your answer in the first question (the number of weeks) by $\frac{3}{4}$. Did you get 3 as a result? If not, revise your previous work.

Exercise 22.1.4

Use the diagram to answer the question: How many $\frac{1}{3}$s are in $1\frac{2}{3}$? The hexagon represents 1 whole. Explain or show your reasoning.

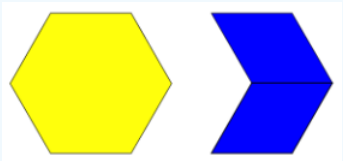


Figure 22.1.5

Exercise 22.1.5

Which question can be represented by the equation $? \cdot \frac{1}{8} = 3$?

A. How many 3s are in $\frac{1}{8}$?
B. What is 3 groups of $\frac{1}{8}$?
C. How many $\frac{1}{8}$s are in 3?
D. What is $\frac{1}{8}$ of 3?

Exercise 22.1.6

Write two division equations for each multiplication equation.

1. $15 \cdot \frac{2}{5} = 6$
2. $6 \cdot \frac{4}{3} = 8$
3. $16 \cdot \frac{7}{8} = 14$

Exercise 22.1.7

Noah and his friends are going to an amusement park. The total cost of admission for 8 students is $100, and all students share the cost equally. Noah brought $13 for his ticket. Did he bring enough money to get into the park? Explain your reasoning.

(From Unit 4.1.2)

Exercise 22.1.8

Write a division expression with a quotient that is:

1. greater than $8 \div 0.001$
2. less than $8 \div 0.001$
3. between $8 \div 0.001$ and $8 \div \frac{1}{10}$

(From Unit 4.1.1)

Exercise 22.1.9

Find each unknown number.

1. 12 is 150% of what number?
2. 5 is 50% of what number?
3. 10% of what number is 300?
4. 5% of what number is 72?
5. 20 is 80% of what number?

(From Unit 3.4.5)

22.1: How Many Groups? (Part 1) is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

22.2: How Many Groups? (Part 2)

Lesson

Let's use blocks and diagrams to understand more about division with fractions.

Exercise 22.2.1: Reasoning with Fraction Strips

Write a fraction or whole number as an answer for each question. If you get stuck, use the fraction strips. Be prepared to share your reasoning.

1. How many $\frac{1}{2}$s are in 2?
2. How many $\frac{1}{5}$s are in 3?
3. How many $\frac{1}{8}$s are in $1\frac{1}{4}$?
4. $1 \div \frac{2}{6} = ?$
5. $2 \div \frac{2}{9} = ?$
6. $4 \div \frac{2}{10} = ?$

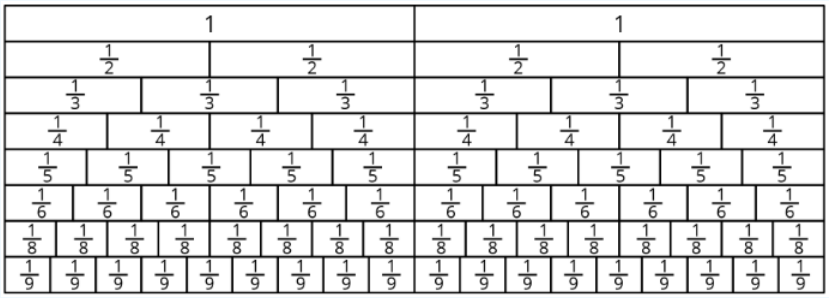


Figure 22.2.1: Fraction strips depicting 2 in 8 different ways, by rows. First row, two 1s. Second row, 4 of the fraction one over two. Third row, 6 of the fraction one over three. Fourth row, 8 of the fraction one over four. Fifth row, 10 of the fraction one over five. Sixth row, 12 of the fraction one over six. Seventh row, 16 of the fraction one over eight. Eighth row, 18 of the fraction one over nine.

Exercise 22.2.2: More Reasoning with Pattern Blocks

Use the pattern blocks in the applet to answer the questions. (If you need help aligning the pieces, you can turn on the grid.)

1. If the trapezoid represents 1 whole, what do each of these other shapes represent? Be prepared to explain or show your reasoning.
   a. 1 triangle
   b. 1 rhombus
   c. 1 hexagon
2. Use pattern blocks to represent each multiplication equation. Use the trapezoid to represent 1 whole.
   a. $3 \cdot \frac{1}{3} = 1$
   b. $3 \cdot \frac{2}{3} = 2$
3. Diego and Jada were asked "How many rhombuses are in a trapezoid?"
   - Diego says, "$1\frac{1}{3}$. If I put 1 rhombus on a trapezoid, the leftover shape is a triangle, which is $\frac{1}{3}$ of the trapezoid."
   - Jada says, "I think it's $1\frac{1}{2}$. Since we want to find out 'how many rhombuses,' we should compare the leftover triangle to a rhombus. A triangle is $\frac{1}{2}$ of a rhombus."

     Do you agree with either of them? Explain or show your reasoning.
4. Select **all** the equations that can be used to answer the question: "How many rhombuses are in a trapezoid?"
   - $\frac{2}{3} \div ? = 1$
   - $? \cdot \frac{2}{3} = 1$
   - $1 \div \frac{2}{3} = ?$
   - $1 \cdot \frac{2}{3} = ?$

- $? \div \frac{2}{3} = 1$

Exercise 22.2.3: Drawing Diagrams to Show Equal-sized Groups

For each situation, draw a diagram for the relationship of the quantities to help you answer the question. Then write a multiplication equation or a division equation for the relationship. Be prepared to share your reasoning.

1. The distance around a park is $\frac{3}{2}$ miles. Noah rode his bicycle around the park for a total of 3 miles. How many times around the park did he ride?
2. You need $\frac{3}{4}$ yard of ribbon for one gift box. You have 3 yards of ribbon. How many gift boxes do you have ribbon for?
3. The water hose fills a bucket at $\frac{1}{3}$ gallon per minute. How many minutes does it take to fill a 2-gallon bucket?

Are you ready for more?

How many heaping teaspoons are in a heaping tablespoon? How would the answer depend on the shape of the spoons?

Summary

Suppose one batch of cookies requires $\frac{2}{3}$ cup flour. How many batches can be made with 4 cups of flour?

We can think of the question as being: "How many $\frac{2}{3}$ are in 4?" and represent it using multiplication and division equations.

$$? \cdot \frac{2}{3} = 4$$

$$4 \div \frac{2}{3} = ?$$

Let's use pattern blocks to visualize the situation and say that a hexagon is 1 whole.

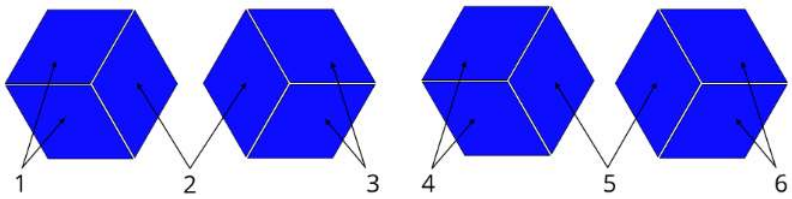


Figure 22.2.2: Diagram of 4 hexagons. Each hexagon is made up of 3 rhombuses using pattern blocks. 6 pairs of rhombuses are shown by pairs of arrows and labeled 1, 2, 3, 4, 5, and 6.

Since 3 rhombuses make a hexagon, 1 rhombus represents $\frac{1}{3}$ and 2 rhombuses represent $\frac{2}{3}$. We can see that 6 pairs of rhombuses make 4 hexagons, so there are 6 groups of $\frac{2}{3}$ in 4.

Other kinds of diagrams can also help us reason about equal-sized groups involving fractions. This example shows how we might reason about the same question from above: "How many $\frac{2}{3}$-cups are in 4 cups?"

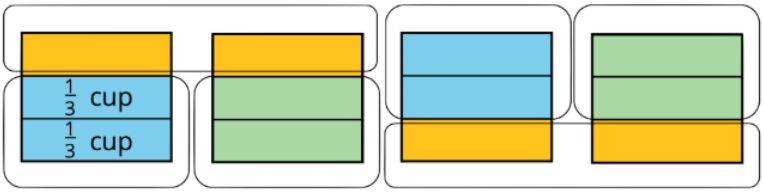


Figure 22.2.3: Diagram with 4 rectangles, partitioned into thirds. In the first rectangle two parts are each labeled one third. Across all 4 rectangles, 6 pairs of one third pieces are circled.

We can see each "cup" partitioned into thirds, and that there are 6 groups of $\frac{2}{3}$-cup in 4 cups. In both diagrams, we see that the unknown value (or the "?" in the equations) is 6. So we can now write:

$$6 \cdot \frac{2}{3} = 4$$

$$4 \div \frac{2}{3} = 6$$

Practice

Exercise 22.2.4

Use the tape diagram to find the value of $\frac{1}{2} \div \frac{1}{3}$. Show your reasoning.

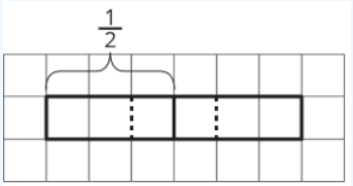


Figure 22.2.4: A tape diagram on a square grid, composed of 6 squares and is partitioned by a solid line into two equal parts. The tape diagram is also partitioned by two vertical dashed lines resulting in three equal parts. A brace extends over the first 3 squares and is labeled one half.

Exercise 22.2.5

What is the value of $\frac{1}{2} \div \frac{1}{3}$? Use pattern blocks to represent and find this value. The yellow hexagon represents 1 whole. Explain or show your reasoning.



Figure 22.2.5

Exercise 22.2.6

Use a standard inch ruler to answer each question. Then, write a multiplication equation and a division equation that answer the question.

1. How many $\frac{1}{2}$s are in 7?
2. How many $\frac{3}{8}$s are in 6?
3. How many $\frac{5}{16}$s are in $1\frac{7}{8}$?

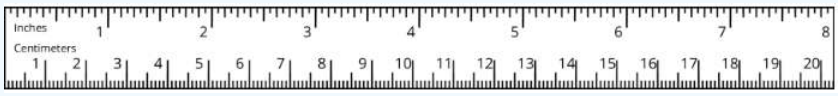


Figure 22.2.6

Exercise 22.2.7

Use the tape diagram to answer the question: How many $\frac{2}{5}$s are in $1\frac{1}{2}$? Show your reasoning.

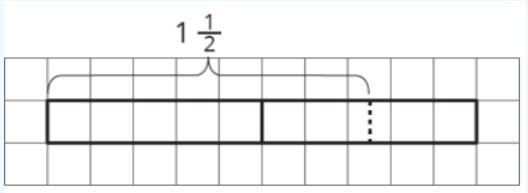


Figure 22.2.7

Exercise 22.2.8

Write a multiplication equation and a division equation to represent each sentence or diagram.

1. There are 12 fourths in 3.
2.

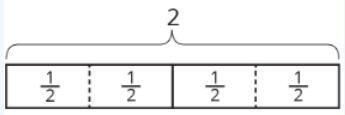


Figure 22.2.8

3. How many $\frac{2}{3}$s are in 6?
4.

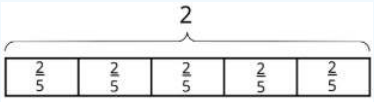


Figure 22.2.9

(From Unit 4.2.1)

Exercise 22.2.9

At a farmer's market, two vendors sell fresh milk. One vendor sells 2 liters for $3.80, and another vendor sells 1.5 liters for $2.70. Which is the better deal? Explain your reasoning.

(From Unit 3.3.1)

Exercise 22.2.10

A recipe uses 5 cups of flour for every 2 cups of sugar.

1. How much sugar is used for 1 cup of flour?
2. How much flour is used for 1 cup of sugar?
3. How much flour is used with 7 cups of sugar?
4. How much sugar is used with 6 cups of flour?

(From Unit 3.3.2)

22.2: How Many Groups? (Part 2) is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

22.3: Using Diagrams to Find the Number of Groups

Lesson

Let's draw tape diagrams to think about division with fractions.

Exercise 22.3.1: How Many of These in That?

1. We can think of the division expression $10 \div 2\frac{1}{2}$ as the question: "How many groups of $2\frac{1}{2}$ are in 10?" Complete the tape diagram to represent this question. Then find the answer.

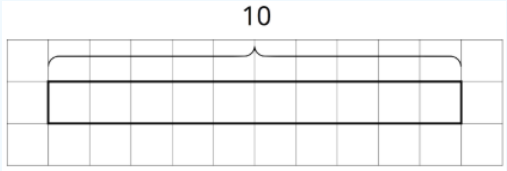


Figure 22.3.1

2. Complete the tape diagram to represent the question: "How many groups of 2 are in 7?" Then find the answer.

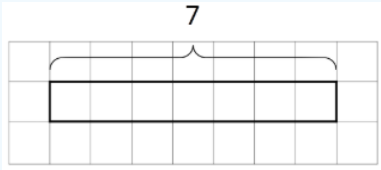


Figure 22.3.2

Exercise 22.3.2: Representing Groups of Fractions with Tape Diagrams

To make sense of the question "How many $\frac{2}{3}$s are in 1?," Andre wrote equations and drew a tape diagram.

$? \cdot \frac{2}{3} = 1$

$1 \div \frac{2}{3} = ?$

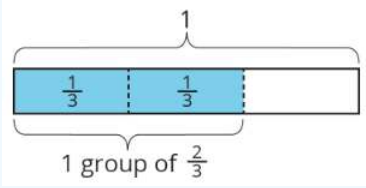


Figure 22.3.3: A tape diagram with three equal parts. The first two parts are shaded and are each labeled one third, total 1. A bracket is labeled 1 group of two thirds, and contains the first two parts.

1. In an earlier task, we used pattern blocks to help us solve the equation $1 \div \frac{2}{3} = ?$. Explain how Andre's tape diagram can also help us solve the equation.
2. Write a multiplication equation and a division equation for each question. Then, draw a tape diagram and find the answer.
   a. How many $\frac{3}{4}$s are in 1?

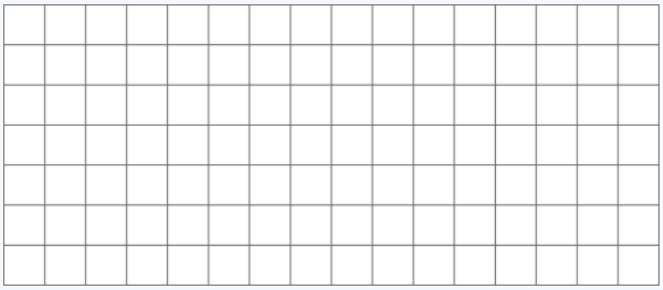


Figure 22.3.4

b. How many $\frac{2}{3}$s are in 3?

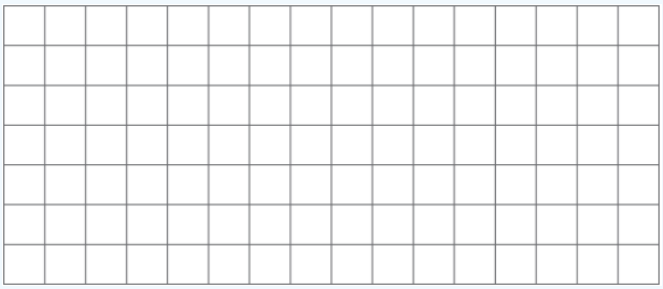


Figure 22.3.5

c. How many $\frac{3}{2}$s are in 5?

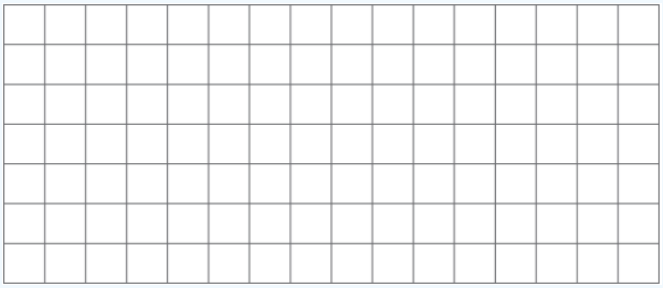


Figure 22.3.6

Exercise 22.3.3: Finding Number of Groups

1. Write a multiplication equation or a division equation for each question. Then, find the answer and explain or show your reasoning.
   a. How many $\frac{3}{8}$-inch thick books make a stack that is 6 inches tall?
   b. How many groups of $\frac{1}{2}$ pound are in $2\frac{3}{4}$ pounds?
2. Write a question that can be represented by the division equation $5 \div 1\frac{1}{2} = ?$. Then, find the answer and explain or show your reasoning.

Summary

A baker used 2 kilograms of flour to make several batches of a pastry recipe. The recipe called for $\frac{2}{5}$ kilogram of flour per batch. How many batches did she make?

We can think of the question as: "How many groups of $\frac{2}{5}$ kilogram make 2 kilograms?" and represent that question with the equations:

$$? \cdot \frac{2}{5} = 2$$

$$2 \div \frac{2}{5} = ?$$

To help us make sense of the question, we can draw a tape diagram. This diagram shows 2 whole kilograms, with each kilogram partitioned into fifths.

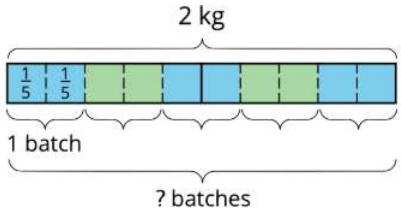


Figure 22.3.7: Fraction bar diagram. 10 equal parts. Each part labeled the fraction 1 over 5. Total labeled 2 kilograms and question mark batches. Every two parts labeled 1 batch.

We can see there are 5 groups of $\frac{2}{5}$ in 2. Multiplying 5 and $\frac{2}{5}$ allows us to check this answer: $5 \cdot \frac{2}{5} = \frac{10}{5}$ and $\frac{10}{5} = 2$, so the answer is correct.

Notice the number of groups that result from $2 \div \frac{2}{5}$ is a whole number. Sometimes the number of groups we find from dividing may not be a whole number. Here is an example:

Suppose one serving of rice is $\frac{3}{4}$ cup. How many servings are there in $3\frac{1}{2}$ cups?

$$? \cdot \frac{3}{4} = 3\frac{1}{2}$$

$$3\frac{1}{2} \div \frac{3}{4} = ?$$

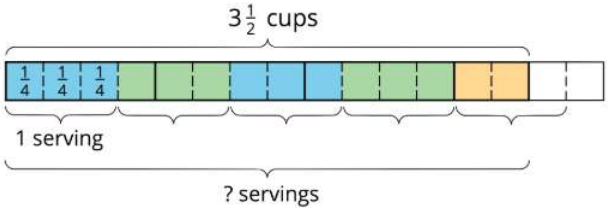


Figure 22.3.8: Fraction bar diagram. 16 equal parts. 14 parts shaded. 14 parts labeled unknown number of groups and 3 and one half cups. Each part labeled the fraction 1 over 4. First 3 parts labeled 1 serving.

Looking at the diagram, we can see there are 4 full groups of $\frac{3}{4}$, plus 2 fourths. If 3 fourths make a whole group, then 2 fourths make $\frac{2}{3}$ of a group. So the number of servings (the "?" in each equation) is $4\frac{2}{3}$. We can check this by multiplying $4\frac{2}{3}$ and $\frac{3}{4}$. $4\frac{2}{3} \cdot \frac{3}{4} = \frac{14}{3} \cdot \frac{3}{4}$, and $\frac{14}{3} \cdot \frac{3}{4} = \frac{14}{4}$, which is indeed equivalent to $3\frac{1}{2}$.

Practice

Exercise 22.3.4

We can think of $3 \div \frac{1}{4}$ as the question "How many groups of $\frac{1}{4}$ are in 3?" Draw a tape diagram to represent this question. Then find the answer.

Exercise 22.3.5

Describe how to draw a tape diagram to represent and answer $3 \div \frac{3}{5} = ?$ for a friend who was absent.

Exercise 22.3.6

How many groups of $\frac{1}{2}$ day are in 1 week?

1. Write a multiplication equation or a division equation to represent the question.
2. Draw a tape diagram to show the relationship between the quantities and to answer the question. Use graph paper, if needed.

Exercise 22.3.7

Diego said that the answer to the question "How many groups of $\frac{5}{6}$ are in 1?" is $\frac{6}{5}$ or $1\frac{1}{5}$. Do you agree with him? Explain or show your reasoning.

Exercise 22.3.8

Select **all** the equations that can represent the question: "How many groups of $\frac{4}{5}$ are in 1?"

A. $? \cdot 1 = \frac{4}{5}$
B. $1 \cdot \frac{4}{5} = ?$
C. $\frac{4}{5} \div 1 = ?$
D. $? \cdot \frac{4}{5} = 1$

E. $1 \div \frac{4}{5} = ?$

(From Unit 4.2.2)

Exercise 22.3.9

Calculate each percentage mentally.

1. What is 10% of 70?
2. What is 10% of 110?
3. What is 25% of 160?
4. What is 25% of 48?
5. What is 50% of 90?
6. What is 50% of 350?
7. What is 75% of 300?
8. What is 75% of 48?

(From Unit 3.4.5)

22.3: Using Diagrams to Find the Number of Groups is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

22.4: What Fraction of a Group?

Lesson

Let's think about dividing things into groups when we can't even make one whole group.

Exercise 22.4.1: Estimating a Fraction of a Number

1. Estimate the quantities:
   a. What is $\frac{1}{3}$ of 7?
   b. What is $\frac{4}{5}$ of $9\frac{2}{3}$?
   c. What is $2\frac{4}{7}$ of $10\frac{1}{9}$?
2. Write a multiplication expression for each of the previous questions.

Exercise 22.4.2: Fractions of Ropes

The segments in the applet represent 4 different lengths of rope. Compare one rope to another, moving the rope by dragging the open circle at one endpoint. You can use the yellow pins to mark off lengths.

1. Complete each sentence comparing the lengths of the ropes. Then, use the measurements shown on the grid to write a multiplication equation and a division equation for each comparison.
   a. Rope B is _____ times as long as rope A.
   b. Rope C is _____ times as long as rope A.
   c. Rope D is _____ times as long as rope A.
2. Each equation can be used to answer a question about Ropes C and D. What could each question be?
   a. $? \cdot 3 = 9$ and $9 \div 3 = ?$
   b. $? \cdot 9 = 3$ and $3 \div 9 = ?$

Exercise 22.4.3: Fractional Batches of Ice Cream

One batch of an ice cream recipe uses 9 cups of milk. A chef makes different amounts of ice cream on different days. Here are the amounts of milk she used:

- Monday: 12 cups
- Tuesday: $22\frac{1}{2}$ cups
- Thursday: 6 cups
- Friday: $7\frac{1}{2}$ cups

1. How many batches of ice cream did she make on these days? For each day, write a division equation, draw a tape diagram, and find the answer.
   a. Monday

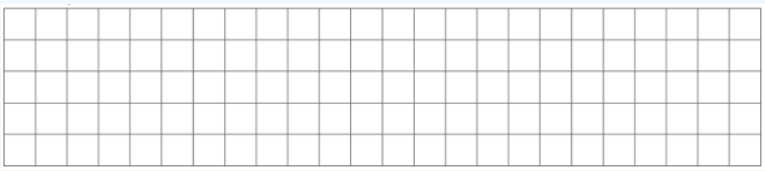


Figure 22.4.1

   b. Tuesday

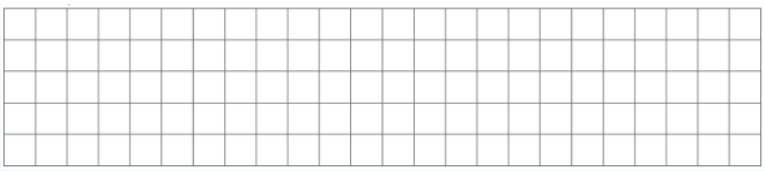


Figure 22.4.2

2. What fraction of a batch of ice cream did she make on these days? For each day, write a division equation, draw a tape diagram, and find the answer.

    a. Thursday

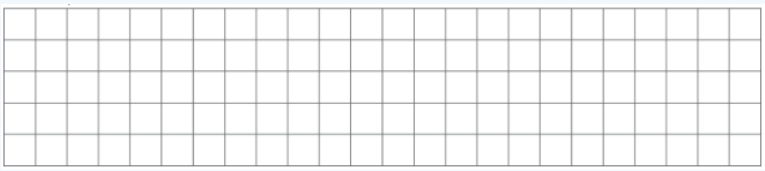


Figure 22.4.3

    b. Friday

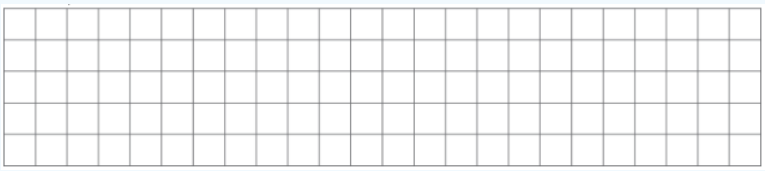


Figure 22.4.4

3. For each question, write a division equation, draw a tape diagram, and find the answer.

    a. What fraction of 9 is 3?

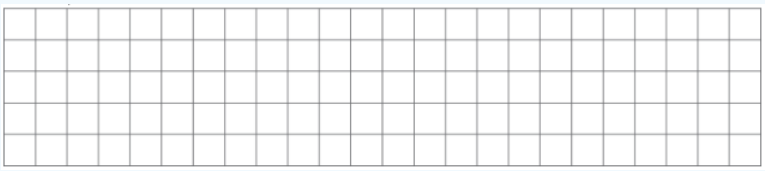


Figure 22.4.5

    b. What fraction of 5 is $\frac{1}{2}$?

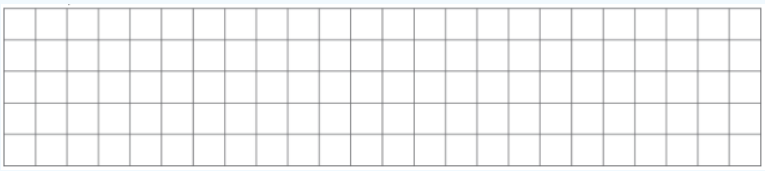


Figure 22.4.6

Summary

It is natural to think about groups when we have more than one group, but we can also have a *fraction of a group*.

To find the amount in a fraction of a group, we can multiply the fraction by the amount in the whole group. If a bag of rice weighs 5 kg, $\frac{3}{4}$ of a bag would weigh $\left(\frac{3}{4} \cdot 5\right)$ kg.

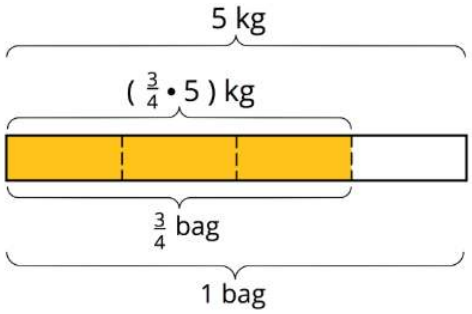


Figure 22.4.7: Fraction bar diagram. 4 equal parts. 3 parts shaded. Total labeled 1 bag and 5 kilograms. 3 parts labeled the fraction 3 over 4 bag and parentheses the fraction 3 over 4 times 5 kilograms.

Sometimes we need to find what fraction of a group an amount is. Suppose a full bag of flour weighs 6 kg. A chef used 3 kg of flour. What fraction of a full bag was used? In other words, what fraction of 6 kg is 3 kg?

This question can be represented by a multiplication equation and a division equation, as well as by a diagram.

$$? \cdot 6 = 3$$

$$3 \div 6 = ?$$

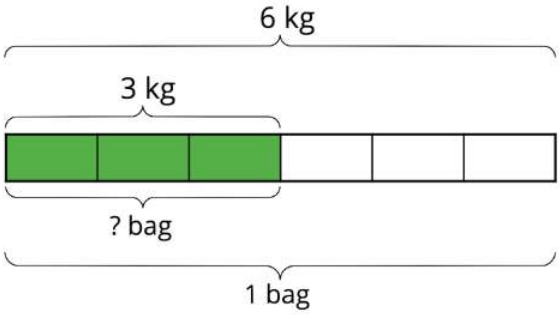


Figure 22.4.8: A tape diagram of 6 equal parts. Above the diagram, a brace from the beginning of the diagram to the end of the diagram is labeled 6 kilograms. Below the diagram, a brace from the beginning of the diagram to the end of the diagram is labeled 1 bag. A third brace that contains the first three parts is labeled three kilograms. Below the diagram, a fourth brace which also contains the first three parts is labeled question mark bag.

We can see from the diagram that 3 is $\frac{1}{2}$ of 6, and we can check this answer by multiplying: $\frac{1}{2} \cdot 6 = 3$.

In *any* situation where we want to know what fraction one number is of another number, we can write a division equation to help us find the answer.

For example, "What fraction of 3 is $2\frac{1}{4}$?" can be expressed as $? \cdot 3 = 2\frac{1}{4}$, which can also be written as $2\frac{1}{4} \div 3 = ?$.

The answer to "What is $2\frac{1}{4} \div 3$?" is also the answer to the original question.

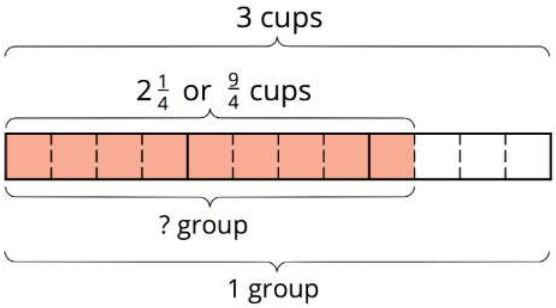


Figure 22.4.9: Fraction bar diagram. 12 equal parts. 9 parts shaded. Total labeled 1 group and 3 cups. 9 parts labeled unknown quantity group and 2 and the fraction 1 over 4 or the fraction 9 over 4.

The diagram shows that 3 wholes contain 12 fourths, and $2\frac{1}{4}$ contains 9 fourths, so the answer to this question is $\frac{9}{12}$, which is equivalent to $\frac{3}{4}$.

We can use diagrams to help us solve other division problems that require finding a fraction of a group. For example, here is a diagram to help us answer the question: "What fraction of $\frac{9}{4}$ is $\frac{3}{2}$?," which can be written as $\frac{3}{2} \div \frac{9}{4} = ?$.

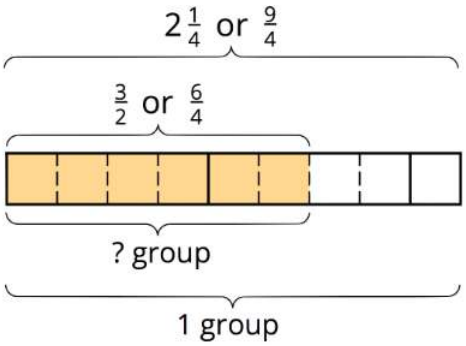


Figure 22.4.10: Fraction bar diagram. 9 equal parts. 6 parts shaded. Total labeled 1 group and 2 and the fraction 1 over 4 or the fraction 9 over 4. 6 parts labeled unknown quantity group and the fraction 3 over 2 or the fraction 6 over 4.

We can see that the quotient is $\frac{6}{9}$, which is equivalent to $\frac{2}{3}$. To check this, let's multiply. $\frac{2}{3} \cdot \frac{9}{4} = \frac{18}{12}$, and $\frac{18}{12}$ is, indeed, equal to $\frac{3}{2}$.

Practice

Exercise 22.4.4

A recipe calls for $\frac{1}{2}$ lb of flour for 1 batch. How many batches can be made with each of these amounts?

1. 1 lb
2. $\frac{3}{4}$ lb
3. $\frac{1}{4}$ lb

Exercise 22.4.5

Whiskers the cat weighs $2\frac{2}{3}$ kg. Piglio weighs 4 kg. For each question, write a multiplication equation and a division equation, decide whether the answer is greater than 1 or less than 1, and then find the answer.

1. How many times as heavy as Piglio is Whiskers?
2. How many times as heavy as Whiskers is Piglio?

Exercise 22.4.6

Andre is walking from his home to a festival that is $1\frac{5}{8}$ kilometers away. He walks $\frac{1}{3}$ kilometer and then takes a quick rest. Which question can be represented by the equation $? \cdot 1\frac{5}{8} = \frac{1}{3}$ in this situation?

A. What fraction of the trip has Andre completed?
B. What fraction of the trip is left?
C. How many more kilometers does Andre have to walk to get to the festival?
D. How many kilometers is it from home to the festival and back home?

Exercise 22.4.7

Draw a tape diagram to represent the question: What fraction of $2\frac{1}{2}$ is $\frac{4}{5}$?
Then find the answer.

Exercise 22.4.8

How many groups of $\frac{3}{4}$ are in each of these quantities?

1. $\frac{11}{4}$
2. $6\frac{1}{2}$

(From Unit 4.2.3)

Exercise 22.4.9

Which question can be represented by the equation $4 \div \frac{2}{7} = ?$

A. What is 4 groups of $\frac{2}{7}$?
B. How many $\frac{2}{7}$s are in 4?
C. What is $\frac{2}{7}$ of 4?
D. How many 4s are in $\frac{2}{7}$?

(From Unit 4.2.1)

22.4: What Fraction of a Group? is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

22.5: How Much in Each Group? (Part 1)

Lesson

Let's look at division problems that help us find the size of one group.

Exercise 22.5.1: Inventing a Situation

1. Think of a situation with a question that can be represented by the equation $12 \div \frac{2}{3} = ?$ Describe the situation and the question.
2. Trade descriptions with your partner, and answer your partner's question.

Exercise 22.5.2: How Much in One Batch?

To make 5 batches of cookies, 10 cups of flour are required. Consider the question: How many cups of flour does each batch require?

We can write equations and draw a diagram to represent this situation.

$$5 \cdot ? = 10$$

$$10 \div 5 = ?$$

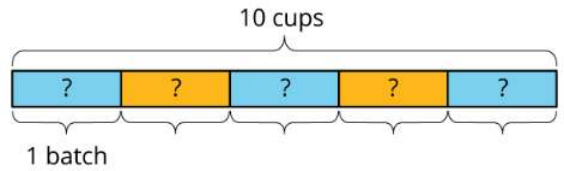


Figure 22.5.1: A tape diagram of 5 equal parts with each part labeled with a question mark. Above the diagram is a brace that represents the total labeled 10 cups. Below the diagram, 5 braces are indicated where each brace contains 1 equal part. The first of the braces is labeled 1 batch.

This helps us see that each batch requires 2 cups of flour.

For each question, write a multiplication equation and a division equation, draw a diagram, and find the answer.

1. To make 4 batches of cupcakes, it takes 6 cups of flour. How many cups of flour are needed for 1 batch?
2. To make $\frac{1}{2}$ batch of rolls, it takes $\frac{5}{4}$ cups of flour. How many cups of flour are needed for 1 batch?
3. Two cups of flour make $\frac{2}{3}$ batch of bread. How many cups of flour make 1 batch?

Exercise 22.5.3: One Container and One Section of Highway

Here are three tape diagrams that represent situations about filling containers of water.

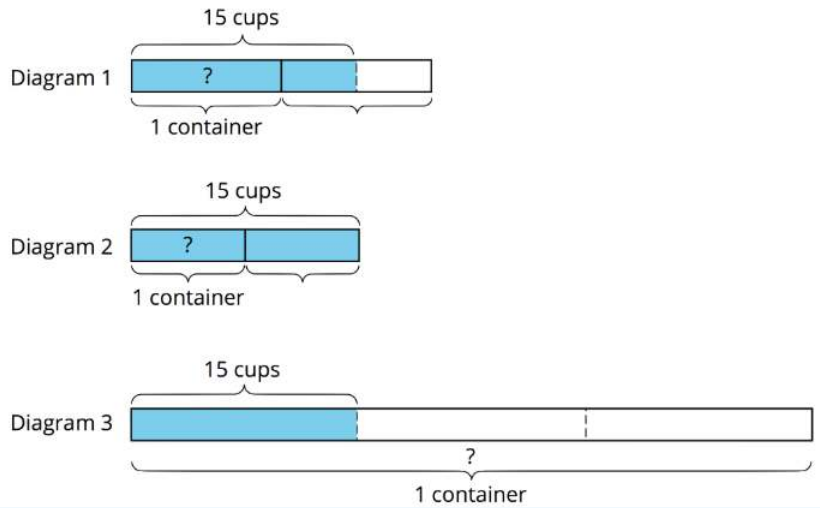


Figure 22.5.2: 3 fraction bar diagrams. Diagram 1, diagram 2, and diagram 3. Diagram 1 2 equal parts. 1 and the fraction 1 over 2 parts shaded. 1 part labeled unknown quantity and 1 container. One and the fraction 1 over 2 parts labeled 15 cups. Diagram 2, 2 equal parts. Both parts shaded. 1 part labeled unknown quantity and 1 container. Total labeled 15 cups. Diagram 3, 3 equal parts. 1 part shaded and labeled 15 cups. Total labeled the unknown quantity and 1 container.

Match each situation to a diagram and use the diagram to help you answer the question. Then, write a multiplication equation and a division equation to represent the situation.

1. Tyler poured a total of 15 cups of water into 2 equal-sized bottles and filled each bottle. How much water was in each bottle?
2. Kiran poured a total of 15 cups of water into equal-sized pitchers and filled $1\frac{1}{2}$ pitchers. How much water was in the full pitcher?
3. It takes 15 cups of water to fill $\frac{1}{3}$ pail. How much water is needed to fill 1 pail?

Here are tape diagrams that represent situations about cleaning sections of highway.

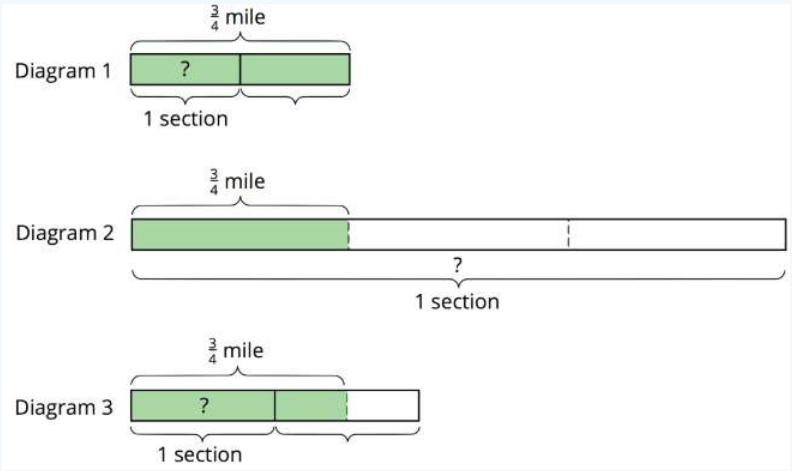


Figure 22.5.3: 3 fraction bar diagrams. Diagram 1, diagram 2, and diagram 3. Diagram 1, 2 equal parts both shaded. 1 part labeled unknown quantity and 1 section. Total labeled the fraction 3 over 4 mile. Diagram 2, 3 equal parts. 1 part shaded. 1 part labeled the fraction 3 over 4 mile. Total labeled unknown quantity and 1 section. Diagram 3, 2 equal parts. One and the fraction 1 over 2 parts shaded. 1 part labeled unknown quantity and 1 section. One and the fraction 1 over 2 parts labeled the fraction 3 over 4 mile.

Match each situation to a diagram and use the diagram to help you answer the question. Then, write a multiplication equation and a division equation to represent the situation.

4. Priya's class has adopted two equal sections of a highway to keep clean. The combined length is $\frac{3}{4}$ of a mile. How long is each section?
5. Lin's class has also adopted some sections of highway to keep clean. If $1\frac{1}{2}$ sections are $\frac{3}{4}$ mile long, how long is each section?

6. A school has adopted a section of highway to keep clean. If $\frac{1}{3}$ of the section is $\frac{3}{4}$ mile long, how long is the section?

Are you ready for more?

To make a Cantor ternary set:

- Start with a tape diagram of length 1 unit. This is step 1.
- Color in the middle third of the tape diagram. This is step 2.
- Do the same to each remaining segment that is not colored in. This is step 3.
- Keep repeating this process.

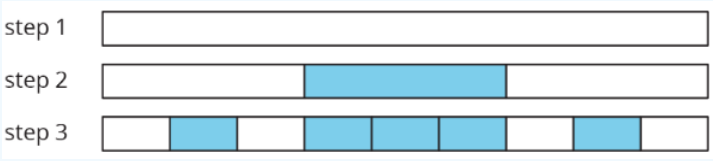


Figure 22.5.4: Three tape diagrams of equal length, step 1, step 2, and step 3. step 1 is a solid white bar. step 2 is divided into 3 equal parts. Second part is shaded blue. step 3 is divided into 9 equal parts. The second, fourth, fifth, sixth, and eighth parts are shaded blue.

1. How much of the diagram is colored in after step 2? Step 3? Step 10?
2. If you continue this process, how much of the tape diagram will you color?
3. Can you think of a different process that will give you a similar result? For example, color the first fifth instead of the middle third of each strip.

Summary

Sometimes we know the amount for *multiple* groups, but we don't know how much is in one group. We can use division to find out.

For example, if 5 people share $8\frac{1}{2}$ pounds of cherries equally, how many pounds of cherries does each person get?

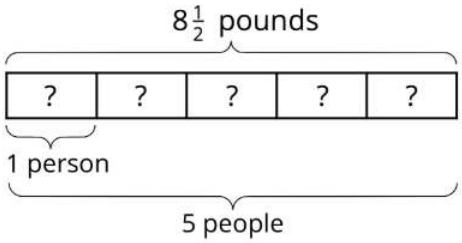


Figure 22.5.5: A fraction bar diagram. 5 equal parts. Each part labeled unknown quantity. 1 part labeled 1 person. Total labeled 8 and one half pounds and 5 people.

We can represent this situation with a multiplication equation and a division equation:

$$5 \cdot ? = 8\frac{1}{2}$$

$$8\frac{1}{2} \div 5 = ?$$

$8\frac{1}{2} \div 5$ can be written as $\frac{17}{2} \div 5$. Dividing by 5 is equivalent to multiplying by $\frac{1}{5}$, and $\frac{17}{2} \cdot \frac{1}{5} = \frac{17}{10}$. This means each person gets $1\frac{7}{10}$ pounds.

Other times, we know the amount for *a fraction* of a group, but we don't know the size of one whole group. We can also use division to find out.

For example, Jada poured 5 cups of iced tea in a pitcher and filled $\frac{2}{3}$ of the pitcher. How many cups of iced tea fill the entire pitcher?

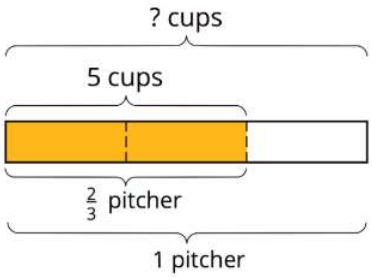


Figure 22.5.6: A fraction bar diagram. 3 equal parts. 2 parts shaded and labeled 5 cups and the fraction 2 over 3 pitcher. Total labeled unknown quantity cups and 1 pitcher.

We can represent this situation with a multiplication equation and a division equation:

$$\frac{2}{3} \cdot ? = 5$$

$$5 \div \frac{2}{3} = ?$$

The diagram can help us reason about the answer. If $\frac{2}{3}$ of a pitcher is 5 cups, then $\frac{1}{3}$ of a pitcher is half of 5, which is $\frac{5}{2}$. Because there are 3 thirds in 1 whole, there would be $(3 \cdot \frac{5}{2})$ or $\frac{15}{2}$ cups in one whole pitcher. We can check our answer by multiplying: $\frac{2}{3} \cdot \frac{15}{2} = \frac{30}{6}$, and $\frac{30}{6} = 5$.

Notice that in the first example, the number of groups is greater than 1 (5 people) and in the second, the number of groups is less than 1 ($\frac{2}{3}$ of a pitcher), but the division and multiplication equations for both situations have the same structures.

Practice

Exercise 22.5.4

For each situation, complete the tape diagram to represent and answer the question.

1. Mai has picked 1 cup of strawberries for a cake, which is enough for $\frac{3}{4}$ of the cake. How many cups does she need for the whole cake?

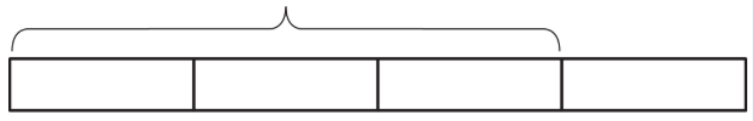


Figure 22.5.7

2. Priya has picked $1\frac{1}{2}$ cups of raspberries, which is enough for $\frac{3}{4}$ of a cake. How many cups does she need for the whole cake?

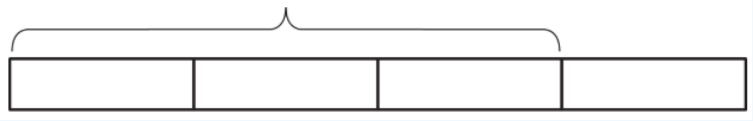


Figure 22.5.8

Exercise 22.5.5

Consider the problem: Tyler painted $\frac{9}{2}$ square yards of wall area with 3 gallons of paint. How many gallons of paint does it take to paint each square yard of wall?

1. Write multiplication and division equations to represent the situation.
2. Draw a diagram to represent and answer the question.

Exercise 22.5.6

Consider the problem: After walking $\frac{1}{4}$ mile from home, Han is $\frac{1}{3}$ of his way to school. What is the distance between his home and school?

1. Write multiplication and division equations to represent this situation.
2. Complete the diagram to represent and answer the question.

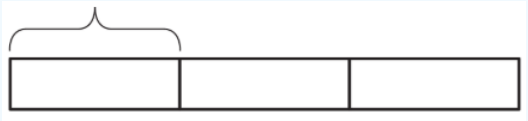


Figure 22.5.9

Exercise 22.5.7

Here is a division equation: $\frac{4}{5} \div \frac{2}{3} = ?$

1. Write a multiplication equation that corresponds to the division equation.
2. Draw a diagram to represent and answer the question.

(From Unit 4.2.4)

Exercise 22.5.8

Consider the problem: A set of books that are each 1.5 inches wide are being organized on a bookshelf that is 36 inches wide. How many books can fit on the shelf?

1. Write multiplication and division equations to represent the situation.
2. Find the answer. Draw a diagram, if needed.
3. Use the multiplication equation to check your answer.

(From Unit 4.1.3)

Exercise 22.5.9

1. Without calculating, order the quotients from smallest or largest.

$$56 \div 8 \qquad 56 \div 8,000,000 \qquad 56 \div 0.000008$$

2. Explain how you decided the order of the three expressions.
3. Find a number n so that $56 \div n$ is greater than 1 but less than 7.

(From Unit 4.1.1)

22.5: How Much in Each Group? (Part 1) is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

22.6: How Much in Each Group? (Part 2)

Lesson

Let's practice dividing fractions in different situations.

Exercise 22.6.1: Number Talk: Greater Than 1 or Less Than 1?

Decide whether each quotient is greater than 1 or less than 1.

$\frac{1}{2} \div \frac{1}{4}$

$1 \div \frac{3}{4}$

$\frac{2}{3} \div \frac{7}{8}$

$2\frac{7}{8} \div 2\frac{3}{5}$

Exercise 22.6.2: Two Water Containers



Figure 22.6.1

1. After looking at these pictures, Lin says, “I see the fraction $\frac{2}{5}$.” Jada says, “I see the fraction $\frac{3}{4}$.” What quantities are Lin and Jada referring to?
2. Consider the problem: How many liters of water fit in the water dispenser?
   a. Write a multiplication equation and a division equation for the question.
   b. Find the answer and explain your reasoning. If you get stuck, consider drawing a diagram.
   c. Check your answer using the multiplication equation.

Exercise 22.6.3: Amount in One Group

Write a multiplication equation and a division equation and draw a diagram to represent each situation. Then, find the answer and explain your reasoning.

1. Jada bought $3\frac{1}{2}$ yards of fabric for $21. How much did each yard cost?
2. $\frac{4}{9}$ kilogram of baking soda costs $2. How much does 1 kilogram of baking soda cost?
3. Diego can fill $1\frac{1}{5}$ bottles with 3 liters of water. How many liters of water fill 1 bottle?
4. $\frac{5}{4}$ gallons of water fill $\frac{5}{6}$ of a bucket. How many gallons of water fill the entire bucket?

Are you ready for more?

The largest sandwich ever made weighed 5,440 pounds. If everyone on Earth shares the sandwich equally, how much would you get? What fraction of a regular sandwich does this represent?

Exercise 22.6.4: Inventing Another Situation

1. Think of a situation with a question that can be represented by $\frac{1}{3} \div \frac{1}{4} = ?$. Describe the situation and the question.
2. Trade descriptions with a partner.
   - Review each other's description and discuss whether each question matches the equation.
   - Revise your description based on the feedback from your partner.
3. Find the answer to your question. Explain or show your reasoning. If you get stuck, consider drawing a diagram.

Summary

Sometimes we have to think carefully about how to solve a problem that involves multiplication and division. Diagrams and equations can help us.

For example, $\frac{3}{4}$ of a pound of rice fills $\frac{2}{5}$ of a container. There are two whole amounts to keep track of here: 1 whole pound and 1 whole container. The equations we write and the diagram we draw depend on what question we are trying to answer.

- How many pounds fill 1 container?

$$\frac{2}{5} \cdot ? = \frac{3}{4}$$

$$\frac{3}{4} \div \frac{2}{5} = ?$$

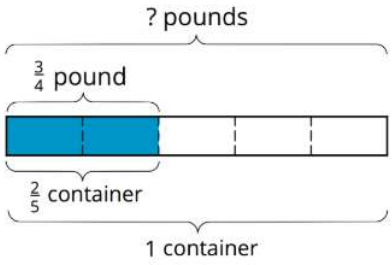


Figure 22.6.2: Fraction bar diagram. 5 equal parts. 2 parts shaded. First 2 parts labeled the fraction 3 over 4 pound and the fraction 2 over 5 container. Total labeled unknown pounds and 1 container.

If $\frac{2}{5}$ of a container is filled with $\frac{3}{4}$ pound, then $\frac{1}{5}$ of a container is filled with half of $\frac{3}{4}$, or $\frac{3}{8}$, pound. One whole container then has $5 \cdot \frac{3}{8}$ (or $\frac{15}{8}$) pounds.

- What fraction of a container does 1 pound fill?

$$\frac{3}{4} \cdot ? = \frac{2}{5}$$

$$\frac{2}{5} \div \frac{3}{4} = ?$$

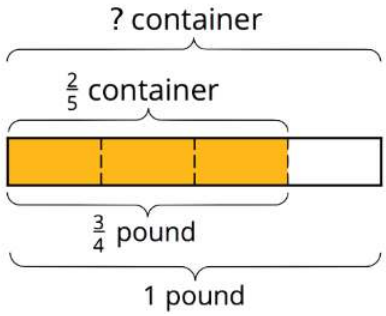


Figure 22.6.3: Fraction bar diagram. 4 equal parts. 3 parts shaded. First 3 parts labeled the fraction 2 over 5 container and the fraction 3 over 4 pound. Total labeled unknown quantity container and 1 pound.

If $\frac{3}{4}$ pound fills $\frac{2}{5}$ of a container, then $\frac{1}{4}$ pound fills a third of $\frac{2}{5}$, or $\frac{2}{15}$, of a container. One whole pound then fills $4 \cdot \frac{2}{15}$ (or $\frac{8}{15}$) of a container.

Practice

Exercise 22.6.5

A group of friends is sharing $2\frac{1}{2}$ pounds of berries.

1. If each friend received $\frac{5}{4}$ of a pound of berries, how many friends are sharing the berries?
2. If 5 friends are sharing the berries, how many pounds of berries does each friend receive?

Exercise 22.6.6

$\frac{2}{5}$ kilogram of soil fills $\frac{1}{3}$ of a container. Can 1 kilogram of soil fit in the container? Explain or show your reasoning.

Exercise 22.6.7

After raining for $\frac{3}{4}$ of an hour, a rain gauge is $\frac{2}{5}$ filled. If it continues to rain at that rate for 15 more minutes, what fraction of the rain gauge will be filled?

1. To help answer this question, Diego wrote the equation $\frac{3}{4} \div \frac{2}{5} = ?$. Explain why this equation does *not* represent the situation.
2. Write a multiplication equation and a division equation that do represent the situation.



Figure 22.6.4: Rain Gauge, by Bidgee (Own Work). CC BY 3.0. Wikimedia Commons. Source.

Exercise 22.6.8

3 tickets to the museum cost $12.75. At this rate, what is the cost of:

1. 1 ticket?
2. 5 tickets?

(From Unit 2.3.3)

Exercise 22.6.9

Elena went 60 meters in 15 seconds. Noah went 50 meters in 10 seconds. Elena and Noah both moved at a constant speed.

1. How far did Elena go in 1 second?
2. How far did Noah go in 1 second?
3. Who went faster? Explain or show your reasoning.

(From Unit 2.3.4)

Exercise 22.6.10

The first row in the table shows a recipe for 1 batch of trail mix. Complete the table to show recipes for 2, 3, and 4 batches of the same type of trail mix.

Table 22.6.1

number of batches	cups of cereal	cups of almonds	cups of raisins
1	2	$\frac{1}{3}$	$\frac{1}{4}$
2			
3			
4			

(From Unit 2.4.1)

22.6: How Much in Each Group? (Part 2) is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

Section 23: Algorithm for Fraction Division

This page titled Section 23: Algorithm for Fraction Division is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

23.1: Dividing by Unit and Non-Unit Fractions

Lesson

Let's look for patterns when we divide by a fraction.

Exercise 23.1.1: Dividing by a Whole Number

Work with a partner. One person solves the problems labeled “Partner A” and the other person solves those labeled “Partner B.” Write an equation for each question. If you get stuck, consider drawing a diagram.

1. Partner A:

How many 3s are in 12?
Division equation:

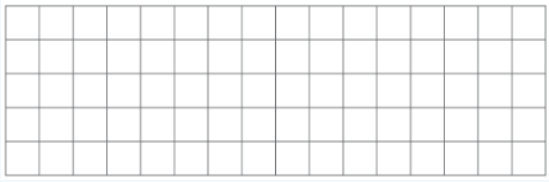


Figure 23.1.1

How many 4s are in 12?
Division equation:

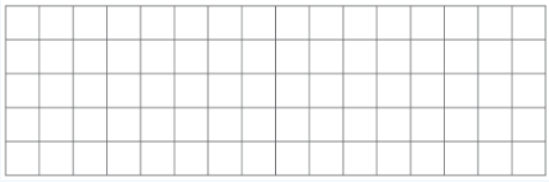


Figure 23.1.2

How many 6s are in 12?
Division equation:

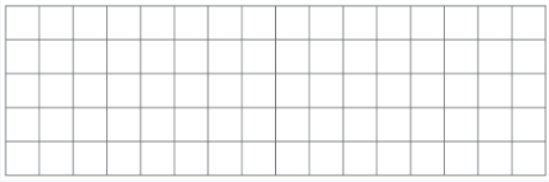


Figure 23.1.3

Partner B:

What is 12 groups of $\frac{1}{3}$?
Multiplication equation:

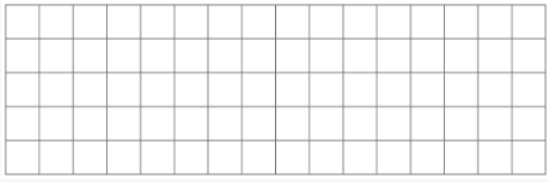


Figure 23.1.4

What is 12 groups of $\frac{1}{4}$?
Multiplication equation:

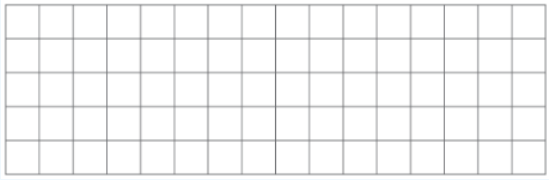


Figure 23.1.5

What is 12 groups of $\frac{1}{6}$?
Multiplication equation:

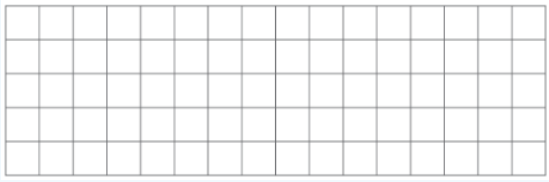


Figure 23.1.6

2. What do you notice about the diagrams and equations? Discuss with your partner.
3. Complete this sentence based on what you noticed:
   Dividing by a whole number a produces the same result as multiplying by _____.

Exercise 23.1.2: Dividing by Unit Fractions

To find the value of $6 \div \frac{1}{2}$, Elena thought, "How many $\frac{1}{2}$s are in 6?" and then she drew this tape diagram. It shows 6 ones, with each one partitioned into 2 equal pieces.

$$6 \div \frac{1}{2}$$

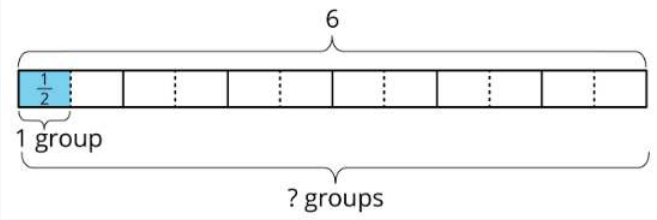


Figure 23.1.7: Fraction bar diagram. 12 equal parts. 1 part shaded and labeled one half and 1 person. Total labeled 6 and unknown quantity groups.

1. For each division expression, complete the diagram using the same method as Elena. Then, find the value of the expression.

   a. $6 \div \frac{1}{3}$

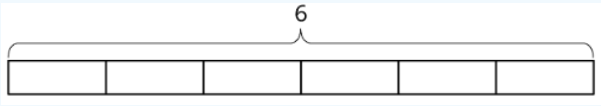


Figure 23.1.8

Value of the expression: _____

   b. $6 \div \frac{1}{4}$

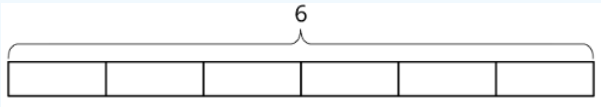


Figure 23.1.9

Value of the expression: _____

   c. $6 \div \frac{1}{6}$

https://math.libretexts.org/@go/page/40247

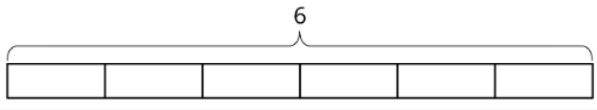


Figure 23.1.10

Value of the expression: _____

2. Examine the expressions and answers more closely. Look for a pattern. How could you find how many halves, thirds, fourths, or sixths were in 6 without counting all of them? Explain your reasoning.
3. Use the pattern you noticed to find the values of these expressions. If you get stuck, consider drawing a diagram.
   a. $6 \div \frac{1}{8}$
   b. $6 \div \frac{1}{10}$
   c. $6 \div \frac{1}{25}$
   d. $6 \div \frac{1}{b}$
4. Find the value of each expression.
   a. $8 \div \frac{1}{4}$
   b. $12 \div \frac{1}{5}$
   c. $a \div \frac{1}{2}$
   d. $a \div \frac{1}{b}$

Exercise 23.1.3: Dividing by Non-unit Fractions

1. To find the value of $6 \div \frac{2}{3}$, Elena started by drawing a diagram the same way she did for $6 \div \frac{1}{3}$.

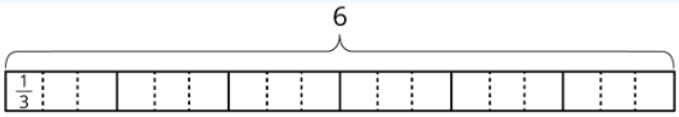


Figure 23.1.11: Fraction bar diagram. 18 equal parts. Every third part has a solid line. All other lines are dashed. First part labeled the fraction 1 over 3. Total labeled 6.

   a. Complete the diagram to show how many $\frac{2}{3}$s are in 6.
   b. Elena says, "To find $6 \div \frac{2}{3}$, I can just take the value of $6 \div \frac{1}{3}$ and then either multiply it by $\frac{1}{2}$ or divide it by 2." Do you agree with her? Explain your reasoning.

2. For each division expression, complete the diagram using the same method as Elena. Then, find the value of the expression. Think about how you could find that value without counting all the pieces in your diagram.
   a. $6 \div \frac{3}{4}$

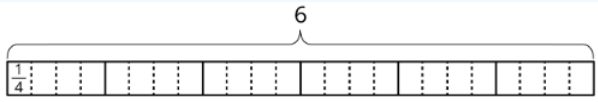


Figure 23.1.12: Fraction bar diagram. 24 equal parts. Every fourth part has a solid line. All other lines are dashed. First part labeled the fraction 1 over 4. Total labeled 6.

Value of the expression:_____

   b. $6 \div \frac{4}{3}$

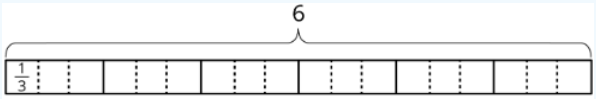


Figure 23.1.13: Fraction bar diagram. 18 equal parts. Every third part has a solid line. All other lines are dashed. First part labeled the fraction 1 over 3. Total labeled 6.

Value of the expression:_____

   c. $6 \div \frac{4}{6}$

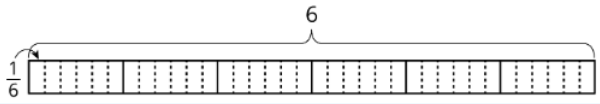


Figure 23.1.14: Fraction bar diagram. 36 equal parts. Every sixth part has a solid line. All other lines are dashed. First part labeled the fraction 1 over 6. Total labeled 6.

Value of the expression:_____

3. Elena examined her diagrams and noticed that she always took the same two steps to show division by a fraction on a tape diagram. She said:

"My first step was to divide each 1 whole into as many parts as the number in the denominator. So if the expression is $6 \div \frac{3}{4}$, I would break each 1 whole into 4 parts. Now I have 4 times as many parts.

My second step was to put a certain number of those parts into one group, and that number is the numerator of the divisor. So if the fraction is $\frac{3}{4}$, I would put 3 of the $\frac{1}{4}$s into one group. Then I could tell how many $\frac{3}{4}$s are in 6."

Which expression represents how many $\frac{3}{4}$s Elena would have after these two steps? Be prepared to explain your reasoning.

- $6 \div 4 \cdot 3$
- $6 \div 4 \div 3$
- $6 \cdot 4 \div 3$
- $6 \cdot 4 \cdot 3$

4. Use the pattern Elena noticed to find the values of these expressions. If you get stuck, consider drawing a diagram.

   a. $6 \div \frac{2}{7}$
   b. $6 \div \frac{3}{10}$
   c. $6 \div \frac{6}{25}$

Are you ready for more?

Find the missing value.

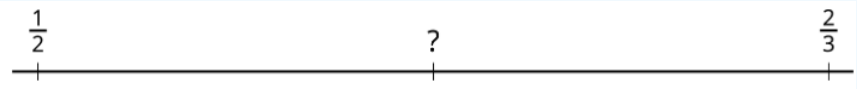


Figure 23.1.15

Summary

To answer the question "How many $\frac{1}{3}$s are in 4?" or "What is $4 \div \frac{1}{3}$?", we can reason that there are 3 thirds in 1, so there are $(4 \cdot 3)$ thirds in 4.

In other words, dividing 4 by $\frac{1}{3}$ has the same result as multiplying 4 by 3.

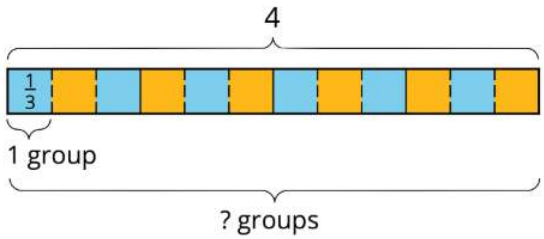


Figure 23.1.16: Fraction bar diagram. 12 equal parts. Each part labeled the fraction 1 over 3. One part labeled 1 group. Total labeled 4 and unknown quantity groups.

$$4 \div \frac{1}{3} = 4 \cdot 3$$

In general, dividing a number by a unit fraction $\frac{1}{b}$ is the same as multiplying the number by b, which is the **reciprocal** of $\frac{1}{b}$.

How can we reason about $4 \div \frac{2}{3}$?

We already know that there are $(4 \cdot 3)$ or 12 groups of $\frac{1}{3}$s in 4. To find how many $\frac{2}{3}$s are in 4, we need to put together every 2 of the $\frac{1}{3}$s into a group. Doing this results in half as many groups, which is 6 groups. In other words:

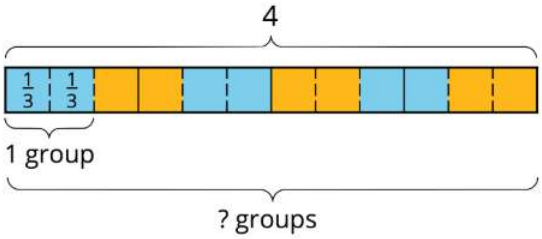


Figure 23.1.17: Fraction bar diagram. 12 equal parts. Each part labeled the fraction 1 over 3. Two parts labeled 1 group. Total labeled 4 and unknown quantity groups.

$$4 \div \frac{2}{3} = (4 \cdot 3) \div 2$$

or

$$4 \div \frac{2}{3} = (4 \cdot 3) \cdot \frac{1}{2}$$

In general, dividing a number by $\frac{a}{b}$, is the same as multiplying the number by b and then dividing by a, or multiplying the number by b and then by $\frac{1}{a}$.

Glossary Entries

Definition: Reciprocal

Dividing 1 by a number gives the reciprocal of that number. For example, the reciprocal of 12 is $\frac{1}{12}$, and the reciprocal of $\frac{2}{5}$ is $\frac{5}{2}$.

Practice

Exercise 23.1.4

Priya is sharing 24 apples equally with some friends. She uses division to determine how many people can have a share if each person gets a particular number of apples. For example, $24 \div 4 = 6$ means that if each person gets 4 apples, then 6 people can have apples. Here are some other calculations:

$$24 \div 4 = 6 \qquad 24 \div 2 = 12 \qquad 24 \div 1 = 24 \qquad 24 \div \frac{1}{2} = ?$$

1. Priya thinks the "?" represents a number less than 24. Do you agree? Explain or show your reasoning.
2. In the case of $24 \div \frac{1}{2} = ?$, how many people can have apples?

Exercise 23.1.5

Here is a centimeter ruler.

1. Use the ruler to find $1 \div \frac{1}{10}$ and $4 \div \frac{1}{10}$.
2. What calculation did you do each time?
3. Use this pattern to find $18 \div \frac{1}{10}$.
4. Explain how you could find $4 \div \frac{2}{10}$ and $4 \div \frac{8}{10}$.



Figure 23.1.18: A portion of a ruler with the top labeled inches and the bottom of the ruler labeled centimeters. The top of the ruler has the numbers 1 and 2 indicated. There are 15 evenly spaced tick marks between the beginning of the ruler and 1 and between 1 and 2. The bottom of the ruler has the numbers 1 through 5 indicated. There are 9 evenly spaced tick marks between the beginning of the ruler and 1, 1 and 2, 2 and 3, 3 and 4, and 4 and 5.

Exercise 23.1.6

Find each quotient.

1. $5 \div \frac{1}{10}$
2. $5 \div \frac{3}{10}$
3. $5 \div \frac{9}{10}$

Exercise 23.1.7

Use the fact that $2\frac{1}{2} \div \frac{1}{8} = 20$ to find $2\frac{1}{2} \div \frac{5}{8}$. Explain or show your reasoning.

Exercise 23.1.8

Consider the problem: It takes one week for a crew of workers to pave $\frac{3}{5}$ kilometer of a road. At that rate, how long will it take to pave 1 kilometer?

Write a multiplication equation and a division equation to represent the question. Then find the answer and show your reasoning.

(From Unit 4.2.6)

Exercise 23.1.9

A box contains $1\frac{3}{4}$ pounds of pancake mix. Jada used $\frac{7}{8}$ pound for a recipe. What fraction of the pancake mix in the box did she use? Explain or show your reasoning. Draw a diagram, if needed.

(From Unit 4.2.4)

Exercise 23.1.10

Calculate each percentage mentally.

1. 25% of 400
2. 50% of 90
3. 75% of 200
4. 10% of $8,000$
5. 5% of 20

(From Unit 3.4.4)

23.1: Dividing by Unit and Non-Unit Fractions is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

23.2: Using an Algorithm to Divide Fractions

Lesson

Let's divide fractions using the rule we learned.

Exercise 23.2.1: Multiplying Fractions

Evaulate each expression.

1. $\frac{2}{3} \cdot 27$
2. $\frac{1}{2} \cdot \frac{2}{3}$
3. $\frac{2}{9} \cdot \frac{3}{5}$
4. $\frac{27}{100} \cdot \frac{200}{9}$
5. $\left(1\frac{3}{4}\right) \cdot \frac{5}{7}$

Exercise 23.2.2: Dividing a Fraction by a Fraction

Work with a partner. One person works on the questions labeled "Partner A" and the other person works on those labeled "Partner B."

1. Partner A:

Find the value of each expression by completing the diagram.

a. $\frac{3}{4} \div \frac{1}{8}$

How many $\frac{1}{8}$s in $\frac{3}{4}$?

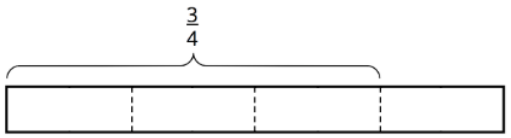


Figure 23.2.1

b. $\frac{9}{10} \div \frac{3}{5}$

How many $\frac{3}{5}$s in $\frac{9}{10}$?

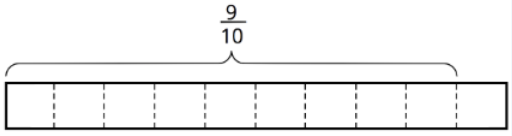


Figure 23.2.2

Use the applet to confirm your answers and explore your own examples.

Partner B:

Elena said: "If I want to divide 4 by $\frac{2}{5}$, I can multiply 4 by 5 and then divide it by 2 or multiply it by $\frac{1}{2}$."

Find the value of each expression using the strategy Elena described.

a. $\frac{3}{4} \div \frac{1}{8}$
b. $\frac{9}{10} \div \frac{3}{5}$

2. What do you notice about the diagrams and expressions? Discuss with your partner.
3. Complete this sentence based on what you noticed:
   To divide a number n by a fraction $\frac{a}{b}$, we can multiply n by _____ and then divide the product by _____.
4. Select **all** equations that represent the statement you completed.
   - $n \div \frac{a}{b} = n \cdot b \div a$

- $n \div \frac{a}{b} = n \cdot a \div b$
- $n \div \frac{a}{b} = n \cdot \frac{a}{b}$
- $n \div \frac{a}{b} = n \cdot \frac{b}{a}$

Exercise 23.2.3: Using an Algorithm to Divide Fractions

Calculate each quotient. Show your thinking and be prepared to explain your reasoning.

1. $\frac{8}{9} \div 4$
2. $\frac{3}{4} \div \frac{1}{2}$
3. $3\frac{1}{3} \div \frac{2}{9}$
4. $\frac{9}{2} \div \frac{3}{8}$
5. $6\frac{2}{5} \div 3$
6. After biking $5\frac{1}{2}$ miles, Jada has traveled $\frac{2}{3}$ of the length of her trip. How long (in miles) is the entire length of her trip? Write an equation to represent the situation, and then find the answer.

Are you ready for more?

Suppose you have a pint of grape juice and a pint of milk. You pour 1 tablespoon of the grape juice into the milk and mix it up. Then you pour 1 tablespoon of this mixture back into the grape juice. Which liquid is more contaminated?

Summary

The division $a \div \frac{3}{4} = ?$ is equivalent to $\frac{3}{4} \cdot ? = a$, so we can think of it as meaning "$\frac{3}{4}$ of what number is a?" and represent it with a diagram as shown. The length of the entire diagram represents the unknown number.

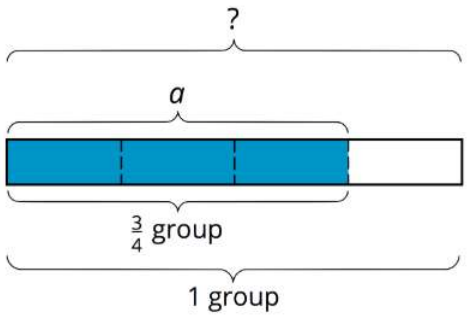


Figure 23.2.3: Fraction bar diagram. 4 equal parts. 3 parts shaded. First 3 parts labeled the fraction 3 over 4 and a. Total labeled unknown quantity and 1 group.

If $\frac{3}{4}$ of a number is a, then to find the number, we can first divide a by 3 to find $\frac{1}{4}$ of the number. Then we multiply the result by 4 to find the number.

The steps above can be written as: $a \div 3 \cdot 4$. Dividing by 3 is the same as multiplying by $\frac{1}{3}$, so we can also write the steps as: $a \cdot \frac{1}{3} \cdot 4$.

In other words: $a \div 3 \cdot 4 = a \cdot \frac{1}{3} \cdot 4$. And $a \cdot \frac{1}{3} \cdot 4 = a \cdot \frac{4}{3}$, so we can say that:

$$a \div \frac{3}{4} = a \cdot \frac{4}{3}$$

In general, dividing a number by a fraction $\frac{c}{d}$ is the same as multiplying the number by $\frac{d}{c}$, which is the reciprocal of the fraction.

Glossary Entries

Definition: Reciprocal

Dividing 1 by a number gives the reciprocal of that number. For example, the reciprocal of 12 is $\frac{1}{12}$, and the reciprocal of $\frac{2}{5}$ is $\frac{5}{2}$.

Practice

Exercise 23.2.4

Select **all** the statements that show correct reasoning for finding $\frac{14}{15} \div \frac{7}{5}$.

A. Multiplying $\frac{14}{15}$ by 5 and then by $\frac{1}{7}$.
B. Dividing $\frac{14}{15}$ by 5, and then multiplying by $\frac{1}{7}$.
C. Multiplying $\frac{14}{15}$ by 7, and then multiplying by $\frac{1}{5}$.
D. Multiplying $\frac{14}{15}$ by 5 and then dividing by 7.
E. Multiplying $\frac{15}{14}$ by 7 and then dividing by 5.

Exercise 23.2.5

Clare said that $\frac{4}{3} \div \frac{5}{2}$ is $\frac{10}{3}$. She reasoned: $\frac{4}{3} \cdot 5 = \frac{20}{3}$ and $\frac{20}{3} \div 2 = \frac{10}{3}$.

Explain why Clare's answer and reasoning are incorrect. Find the correct quotient.

Exercise 23.2.6

Find the value of $\frac{15}{4} \div \frac{5}{8}$. Show your reasoning.

Exercise 23.2.7

Consider the problem: Kiran has $2\frac{3}{4}$ pounds of flour. When he divides the flour into equal-sized bags, he fills $4\frac{1}{8}$ bags. How many pounds fit in each bag?

Write a multiplication equation and a division equation to represent the question. Then, find the answer and show your reasoning.

Exercise 23.2.8

Divide $4\frac{1}{2}$ by each of these unit fractions.

1. $\frac{1}{8}$
2. $\frac{1}{4}$
3. $\frac{1}{6}$

(From Unit 4.3.1)

Exercise 23.2.9

Consider the problem: After charging for $\frac{1}{3}$ of an hour, a phone is at $\frac{2}{5}$ of its full power. How long will it take the phone to charge completely?

Decide whether each equation can represent the situation.

1. $\frac{1}{3} \cdot ? = \frac{2}{5}$
2. $\frac{1}{3} \div \frac{2}{5} = ?$
3. $\frac{2}{5} \div \frac{1}{3} = ?$
4. $\frac{2}{5} \cdot ? = \frac{1}{3}$

(From Unit 4.2.6)

Exercise 23.2.10

Elena and Noah are each filling a bucket with water. Noah's bucket is $\frac{2}{5}$ full and the water weighs $2\frac{1}{2}$ pounds. How much does Elena's water weigh if her bucket is full and her bucket is identical to Noah's?

1. Write multiplication and division equations to represent the question.
2. Draw a diagram to show the relationship between the quantities and to find the answer.

(From Unit 4.2.5)

23.2: Using an Algorithm to Divide Fractions is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

Section 24: Fractions in Lengths, Areas, and Volumes

This page titled Section 24: Fractions in Lengths, Areas, and Volumes is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

24.1: Fractional Lengths

Lesson

Let's solve problems about fractional lengths.

Exercise 24.1.1: Number Talk: Multiplication Strategies

Find the product mentally.

$$19 \cdot 14$$

Exercise 24.1.2: Info Gap: How Many Would It Take?

Your teacher will give you either a *problem card* or a *data card*. Do not show or read your card to your partner.

If your teacher gives you the *problem card*:

1. Silently read your card and think about what information you need to be able to answer the question.
2. Ask your partner for the specific information that you need.
3. Explain how you are using the information to solve the problem.
   Continue to ask questions until you have enough information to solve the problem.
4. Share the *problem card* and solve the problem independently.
5. Read the *data card* and discuss your reasoning.

If your teacher gives you the *data card*:

1. Silently read your card.
2. Ask your partner "*What specific information do you need?*" and wait for them to *ask* for information.
   If your partner asks for information that is not on the card, do not do the calculations for them. Tell them you don't have that information.
3. Before sharing the information, ask "*Why do you need that information?*" Listen to your partner's reasoning and ask clarifying questions.
4. Read the *problem card* and solve the problem independently.
5. Share the *data card* and discuss your reasoning.

Are you ready for more?

Lin has a work of art that is 14 inches by 20 inches. She wants to frame it with large paper clips laid end to end.

1. If each paper clip is $1\frac{3}{4}$ inch long, how many paper clips would she need? Show your reasoning and be sure to think about potential gaps and overlaps. Consider making a sketch that shows how the paper clips could be arranged.
2. How many paper clips are needed if the paper clips are spaced $\frac{1}{4}$ inch apart? Describe the arrangement of the paper clips at the corners of the frame.

Exercise 24.1.3: How Many Times as Tall or as Far?

1. A second-grade student is 4 feet tall. Her teacher is $5\frac{2}{3}$ feet tall.
   a. How many times as tall as the student is the teacher?
   b. What fraction of the teacher's height is the student's height?
2. Find each quotient. Show your reasoning and check your answer.
   a. $9 \div \frac{3}{5}$
   b. $1\frac{7}{8} \div \frac{3}{4}$
3. Write a division equation that can help answer each of these questions. Then find the answer. If you get stuck, consider drawing a diagram.
   a. A runner ran $1\frac{4}{5}$ miles on Monday and $6\frac{3}{10}$ miles on Tuesday. How many times her Monday's distance was her Tuesday's distance?

b. A cyclist planned to ride $9\frac{1}{2}$ miles but only managed to travel $3\frac{7}{8}$ miles. What fraction of his planned trip did he travel?

Exercise 24.1.4: Comparing Paper Rolls

The photo shows a situation that involves fractions.



Figure 24.1.1

1. Complete the sentences. Be prepared to explain your reasoning.
   a. The length of the long tube is about _____ times the length of a short tube.
   b. The length of a short tube is about _____ times the length of the long tube.
2. If the length of the long paper roll is $11\frac{1}{4}$ inches, what is the length of each short paper roll?

Summary

Division can help us solve comparison problems in which we find out how many times as large or as small one number is compared to another. For example, a student is playing two songs for a music recital. The first song is $1\frac{1}{2}$ minutes long. The second song is $3\frac{3}{4}$ minutes long.

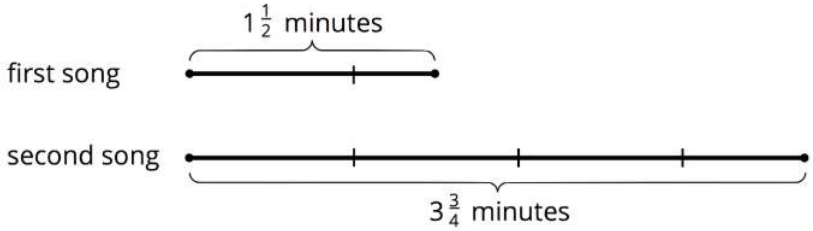


Figure 24.1.2

We can ask two different comparison questions and write different multiplication and division equations to represent each question.

- How many times as long as the first song is the second song?

$$? \cdot 1\frac{1}{2} = 3\frac{3}{4}$$

$$3\frac{3}{4} \div 1\frac{1}{2} = ?$$

- What fraction of the second song is the first song?

$$? \cdot 3\frac{3}{4} = 1\frac{1}{2}$$

$$1\frac{1}{2} \div 3\frac{3}{4} = ?$$

We can use the algorithm we learned to calculate the quotients.

$$= \frac{15}{4} \div \frac{3}{2}$$
$$= \frac{15}{4} \cdot \frac{2}{3}$$
$$= \frac{30}{12}$$
$$= \frac{5}{2}$$

This means the second song is $2\frac{1}{2}$ times as long as the first song.

$$= \frac{3}{2} \div \frac{15}{4}$$
$$= \frac{3}{2} \cdot \frac{4}{15}$$
$$= \frac{12}{30}$$
$$= \frac{2}{5}$$

This means the first song is $\frac{2}{5}$ times as long as the second song.

Practice

Exercise 24.1.5

One inch is around $2\frac{11}{20}$ centimeters.

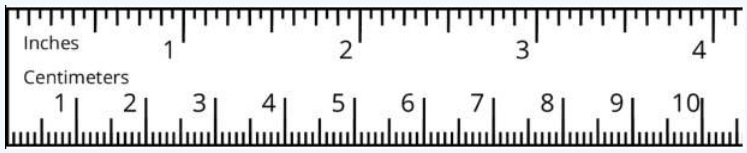


Figure 24.1.3: Ruler. Top of ruler, inches. Scale 0 to 4, by 1s. Each inch is divided into 16 equal parts with tick marks. Bottom of ruler, centimeters. Scale 0 to 10 by 1s. Each centimeter is divided into 10 equal parts with tick marks.

1. How many centimeters long is 3 inches? Show your reasoning.
2. What fraction of an inch is 1 centimeter? Show your reasoning.
3. What question can be answered by finding $10 \div 2\frac{11}{20}$ in this situation?

Exercise 24.1.6

A zookeeper is $6\frac{1}{4}$ feet tall. A young giraffe in his care is $9\frac{3}{8}$ feet tall.

1. How many times as tall as the zookeeper is the giraffe?
2. What fraction of the giraffe's height is the zookeeper's height?

Exercise 24.1.7

A rectangular bathroom floor is covered with square tiles that are $1\frac{1}{2}$ feet by $1\frac{1}{2}$ feet. The length of the bathroom floor is $10\frac{1}{2}$ feet and the width is $6\frac{1}{2}$ feet.

1. How many tiles does it take to cover the length of the floor?
2. How many tiles does it take to cover the width of the floor?

Exercise 24.1.8

The Food and Drug Administration (FDA) recommends a certain amount of nutrient intake per day called the "daily value." Food labels usually show percentages of the daily values for several different nutrients—calcium, iron, vitamins, etc.

Consider the problem: In $\frac{3}{4}$ cup of oatmeal, there is $\frac{1}{10}$ of the recommended daily value of iron. What fraction of the daily recommended value of iron is in 1 cup of oatmeal?

Write a multiplication equation and a division equation to represent the question. Then find the answer and show your reasoning.

(From Unit 4.3.2)

Exercise 24.1.9

What fraction of $\frac{1}{2}$ is $\frac{1}{3}$? Draw a tape diagram to represent and answer the question. Use graph paper if needed.

(From Unit 4.2.4)

Exercise 24.1.10

Noah says, "There are $2\frac{1}{2}$ groups of $\frac{4}{5}$ in 2." Do you agree with him? Draw a tape diagram to show your reasoning. Use graph paper, if needed.

(From Unit 4.2.3)

24.1: Fractional Lengths is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

24.2: Rectangles with Fractional Side Lengths

Lesson

Let's explore rectangles that have fractional measurements.

Exercise 24.2.1: Areas of Squares

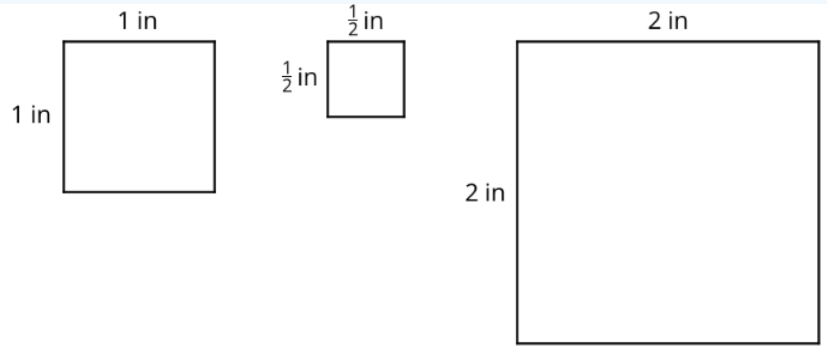


Figure 24.2.1

1. What do you notice about the areas of the squares?
2. Kiran says "A square with side lengths of $\frac{1}{3}$ inch has an area of $\frac{1}{3}$ square inches." Do you agree? Explain or show your reasoning.

Exercise 24.2.2: Areas of Squares and Rectangles

Your teacher will give you graph paper and a ruler.

1. On the graph paper, draw a square with side lengths of 1 inch. Inside this square, draw another square with side lengths of $\frac{1}{4}$ inch.
   Use your drawing to answer the questions.
   a. How many squares with side lengths of $\frac{1}{4}$ inch can fit in a square with side lengths of 1 inch?
   b. What is the area of a square with side lengths of $\frac{1}{4}$ inch? Explain or show your reasoning.
2. On the graph paper, draw a rectangle that is $3\frac{1}{2}$ inches by $2\frac{1}{4}$ inches.
   For each question, write a division expression and then find the answer.
   a. How many $\frac{1}{4}$-inch segments are in a length of $3\frac{1}{2}$ inches?
   b. How many $\frac{1}{4}$-inch segments are in a length of $2\frac{1}{4}$ inches?
3. Use your drawing to show that a rectangle that is $3\frac{1}{2}$ inches by $2\frac{1}{4}$ inches has an area of $7\frac{7}{8}$ square inches.

Exercise 24.2.3: Areas of Rectangles

Each of these multiplication expressions represents the area of a rectangle.

$$2 \cdot 4 \qquad 2\frac{1}{2} \cdot 4 \qquad 2 \cdot 4\frac{3}{4} \qquad 2\frac{1}{2} \cdot 4\frac{3}{4}$$

1. All regions shaded in light blue have the same area. Match each diagram to the expression that you think represents its area. Be prepared to explain your reasoning.

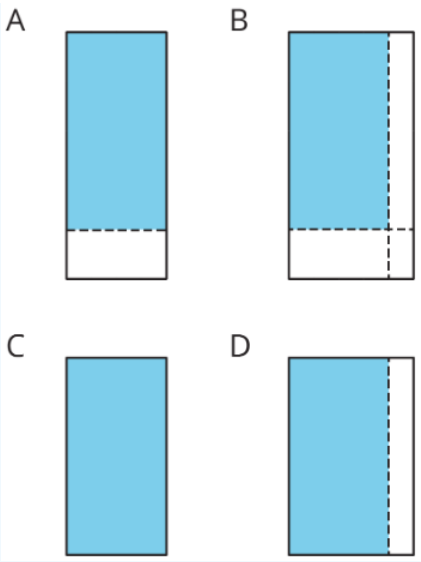


Figure 24.2.2: Four vertically oriented rectangles labeled A, B, C, D. Rectangle A has a horizontal dotted line about three quarters of the way down its width. The top portion is shaded blue. Rectangle B has a horizontal dotted line about three quarters of the way down its width and a vertical dotted line about three quarters of the way to the right of its length. The top left portion of the rectangle is shaded blue. Rectangle C is shaded blue. Rectangle D has a vertical dotted line about three quarters of the way to the right of its length. The left portion of the rectangle is shaded blue.

2. Use the diagram that matches $2\frac{1}{2} \cdot 4\frac{3}{4}$ to show that the value of $2\frac{1}{2} \cdot 4\frac{3}{4}$ is $11\frac{7}{8}$.

Are you ready for more?

The following rectangles are composed of squares, and each rectangle is constructed using the previous rectangle. The side length of the first square is 1 unit.



Figure 24.2.3: A sequence of rectangles composed of squares. The first is a unit square. The second is composed of two unit squares, stacked vertically. The third is composed of two unit squares, stacked vertically, and a 2 by 2 unit square on the right. The fourth is composed of two unit squares, stacked vertically, a 2 by 2 unit square on the right, and a 3 by 3 unit square at the bottom. The fifth is composed of two unit squares, stacked vertically, a 2 by 2 unit square on the right, a 3 by 3 unit square at the bottom, and a 4 by 4 unit square on the left.

1. Draw the next four rectangles that are constructed in the same way. Then complete the table with the side lengths of the rectangle and the fraction of the longer side over the shorter side.

Table 24.2.1

short side	long side	$\frac{\text{long side}}{\text{short side}}$
1		
1		
2		
3		

2. Describe the values of the fraction of the longer side over the shorter side. What happens to the fraction as the pattern continues?

Exercise 24.2.4: How Many Would it Take? (Part 2)

Noah would like to cover a rectangular tray with rectangular tiles. The tray has a width of $11\frac{1}{4}$ inches and an area of $50\frac{5}{8}$ square inches.

1. Find the length of the tray in inches.
2. If the tiles are $\frac{3}{4}$ inch by $\frac{9}{16}$ inch, how many would Noah need to cover the tray completely, without gaps or overlaps? Explain or show your reasoning.
3. Draw a diagram to show how Noah could lay the tiles. Your diagram should show how many tiles would be needed to cover the length and width of the tray, but does not need to show every tile.

Summary

If a rectangle has side lengths a units and b units, the area is $a \cdot b$ square units. For example, if we have a rectangle with $\frac{1}{2}$-inch side lengths, its area is $\frac{1}{2} \cdot \frac{1}{2}$ or $\frac{1}{4}$ square inches.

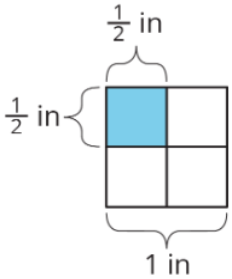


Figure 24.2.4: A large square evenly divided into 4 smaller squares. The large square has bottom horizontal side length of 1 inch. Of the four smaller squares, the top left square is shaded blue. It has side lengths labeled one half inch.

This means that if we know the *area* and *one side length* of a rectangle, we can divide to find the *other* side length.

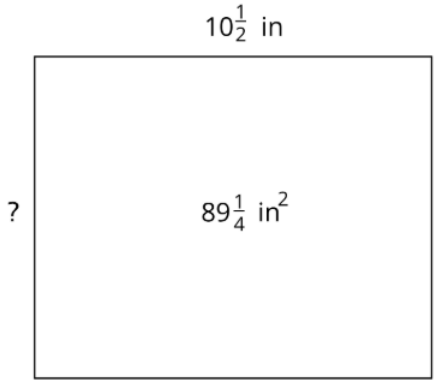


Figure 24.2.5

If one side length of a rectangle is $10\frac{1}{2}$ in and its area is $89\frac{1}{4}$ in^2, we can write this equation to show their relationship: $10\frac{1}{2} \cdot ? = 89\frac{1}{4}$

Then, we can find the other side length, in inches, using division: $89\frac{1}{4} \div 10\frac{1}{2} = ?$

Practice

Exercise 24.2.5

1. Find the unknown side length of the rectangle if its area is 11 m^2. Show your reasoning.

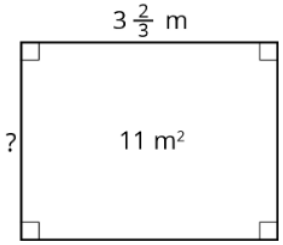


Figure 24.2.6: A rectangle that has area labeled 11 meters squared. The side length of one side of the rectangle is labeled three and two thirds meters and the side length of the other side is labeled with a question mark. Each angle has a right angle symbol indicated.

2. Check your answer by multiplying it by the given side length ($3\frac{2}{3}$). Is the resulting product 11? If not, revise your previous work.

Exercise 24.2.6

A worker is tiling the floor of a rectangular room that is 12 feet by 15 feet. The tiles are square with side lengths $1\frac{1}{3}$ feet. How many tiles are needed to cover the entire floor? Show your reasoning.

Exercise 24.2.7

A television screen has length $16\frac{1}{2}$ inches, width w inches, and area 462 square inches. Select **all** the equations that represent the relationship of the side lengths and area of the television.

A. $w \cdot 462 = 16\frac{1}{2}$
B. $16\frac{1}{2} \cdot w = 462$
C. $462 \div 16\frac{1}{2} = w$
D. $462 \div w = 16\frac{1}{2}$
E. $16\frac{1}{2} \cdot 462 = w$

Exercise 24.2.8

The area of a rectangle is $17\frac{1}{2}$ in^2 and its shorter side is $3\frac{1}{2}$ in. Draw a diagram that shows this information. What is the length of the longer side?

Exercise 24.2.9

A bookshelf is 42 inches long.

1. How many books of length $1\frac{1}{2}$ inches will fit on the bookshelf? Explain your reasoning.
2. A bookcase has 5 of these bookshelves. How many feet of shelf space is there? Explain your reasoning.

(From Unit 4.4.1)

Exercise 24.2.10

Find the value of $\frac{5}{32} \div \frac{25}{4}$. Show your reasoning.

(From Unit 4.3.2)

Exercise 24.2.11

How many groups of $1\frac{2}{3}$ are in each of these quantities?

1. $1\frac{5}{6}$
2. $4\frac{1}{3}$

3. $\frac{5}{6}$

(From Unit 4.2.3)

Exercise 24.2.12

It takes $1\frac{1}{4}$ minutes to fill a 3-gallon bucket of water with a hose. At this rate, how long does it take to fill a 50-gallon tub? If you get stuck, consider using a table.

(From Unit 2.4.4)

24.2: Rectangles with Fractional Side Lengths is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

24.3: Fractional Lengths in Triangles and Prisms

Lesson

Let's explore area and volume when fractions are involved.

Exercise 24.3.1: Area of Triangle

Find the area of Triangle A in square centimeters. Show your reasoning.

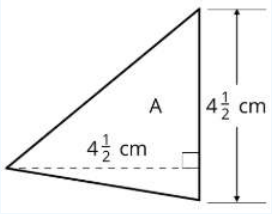


Figure 24.3.1: A triangle labeled A with a vertical side. One vertex is to the left of the vertical side. A dashed horizontal line is drawn from the first vertex to the vertical side of the triangle and a right angle symbol is indicated. The dashed line and the vertical side are both labeled 4 and one half centimeters.

Exercise 24.3.2: Bases and Heights of Triangles

1. The area of Triangle B is 8 square units. Find the length of b. Show your reasoning.

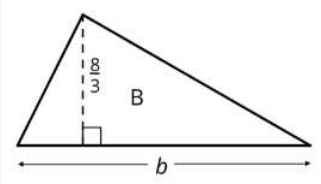


Figure 24.3.2: A triangle labeled B has a horizontal side on the bottom of the triangle and a vertex above the horizontal side. A dashed line is drawn from the vertex to the horizontal side and a right angle symbol is indicated. The horizontal side is labeled lowercase b and the dashed line is labeled eight thirds.

2. The area of Triangle C is $\frac{54}{5}$ square units. What is the length of h? Show your reasoning.

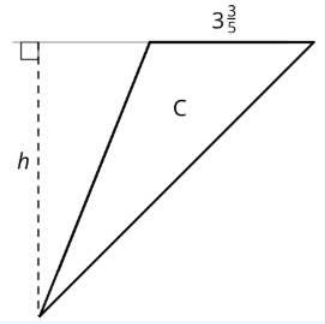


Figure 24.3.3: A triangle labeled C has a horizontal side at the top of the triangle and a vertex below the horizontal side and to the left. A horizontal line extends from the horizontal side and to the left. A vertical dashed line is drawn from the bottom vertex to the extended horizontal line and a right angle symbol is indicated. The dashed line is labeled h and the horizontal side of the triangle is labeled 3 and three fifths.

Exercise 24.3.3: Volumes of Cubes and Prisms

Use cubes or the applet to help you answer the following questions.

1. Here is a drawing of a cube with edge lengths of 1 inch.
   a. How many cubes with edge lengths of $\frac{1}{2}$ inch are needed to fill this cube?
   b. What is the volume, in cubic inches, of a cube with edge lengths of $\frac{1}{2}$ inch? Explain or show your reasoning.

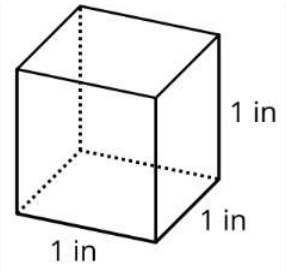


Figure 24.3.4

2. Four cubes are piled in a single stack to make a prism. Each cube has an edge length of $\frac{2}{2}$ inch. Sketch the prism, and find its volume in cubic inches.
3. Use cubes with an edge length of $\frac{1}{2}$ inch to build prisms with the lengths, widths, and heights shown in the table.
   a. For each prism, record in the table how many $\frac{1}{2}$-inch cubes can be packed into the prism and the volume of the prism.

Table 24.3.1

prism length (in)	prism width (in)	prism height (in)	number of $\frac{1}{2}$-inch cubes in prism	volume of prism (in^3)
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$		
1	1	$\frac{1}{2}$		
2	1	$\frac{1}{2}$		
2	2	1		
4	2	$\frac{3}{2}$		
5	4	2		
5	4	$2\frac{1}{2}$		

   b. Examine the values in the table. What do you notice about the relationship between the edge lengths of each prism and its volume?
4. What is the volume of a rectangular prism that is $1\frac{1}{2}$ inches by $2\frac{1}{4}$ inches by 4 inches? Show your reasoning.

Are you ready for more?

A unit fraction has a 1 in the numerator.

- These are unit fractions: $\frac{1}{3}, \frac{1}{100}, \frac{1}{1}$.
- These are *not* unit fractions: $\frac{2}{9}, \frac{8}{1}, 2\frac{1}{5}$.

1. Find three unit fractions whose sum is $\frac{1}{2}$. An example is: $\frac{1}{8}+\frac{1}{8}+\frac{1}{4}=\frac{1}{2}$ How many examples like this can you find?
2. Find a box whose surface area in square units equals its volume in cubic units. How many like this can you find?

Summary

If a rectangular prism has edge lengths of 2 units, 3 units, and 5 units, we can think of it as 2 layers of unit cubes, with each layer having $(3 \cdot 5)$ unit cubes in it. So the volume, in cubic units, is: $2 \cdot 3 \cdot 5$

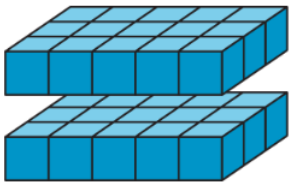


Figure 24.3.5

To find the volume of a rectangular prism with fractional edge lengths, we can think of it as being built of cubes that have a unit fraction for their edge length. For instance, if we build a prism that is $\frac{1}{2}$-inch tall, $\frac{3}{2}$-inch wide, and 4 inches long using cubes with a $\frac{1}{2}$-inch edge length, we would have:

- A height of 1 cube, because $1 \cdot \frac{1}{2} = \frac{1}{2}$.
- A width of 3 cubes, because $3 \cdot \frac{1}{2} = \frac{3}{2}$.
- A length of 8 cubes, because $8 \cdot \frac{1}{2} = 4$.

The volume of the prism would be $1 \cdot 3 \cdot 8$, or 24 cubic units. How do we find its volume in cubic inches? We know that each cube with a $\frac{1}{2}$-inch edge length has a volume of $\frac{1}{8}$ cubic inch, because $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$. Since the prism is built using 24 of these cubes, its volume, in cubic inches, would then be $24 \cdot \frac{1}{8}$, or 3 cubic inches.

The volume of the prism, in cubic inches, can also be found by multiplying the fractional edge lengths in inches: $\frac{1}{2} \cdot \frac{3}{2} \cdot 4 = 3$

Practice

Exercise 24.3.4

Clare is using little wooden cubes with edge length $\frac{1}{2}$ inch to build a larger cube that has edge length 4 inches. How many little cubes does she need? Explain your reasoning.

Exercise 24.3.5

The triangle has an area of $7\frac{7}{8}$ cm^2 and a base of $5\frac{1}{4}$ cm.

What is the length of h? Explain your reasoning.

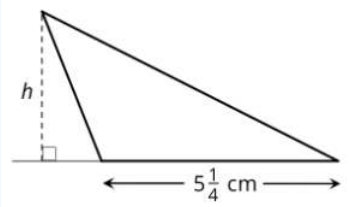


Figure 24.3.6: A triangle with a horizontal base labeled five and one fourth centimeters. A vertex is above and to the left of the horizontal side and a horizontal line is extended to the left from the base. A vertical dashed line is drawn from the top vertex to the extended base and a right angle symbol is indicated. The vertical dashed line is labeled h.

Exercise 24.3.6

1. Which expression can be used to find how many cubes with edge length of $\frac{1}{3}$ unit fit in a prism that is 5 units by 5 units by 8 units? Explain or show your reasoning.
   - $\left(5 \cdot \frac{1}{3}\right) \cdot \left(5 \cdot \frac{1}{3}\right) \cdot \left(8 \cdot \frac{1}{3}\right)$
   - $5 \cdot 5 \cdot 8$
   - $(5 \cdot 3) \cdot (5 \cdot 3) \cdot (8 \cdot 3)$
   - $(5 \cdot 5 \cdot 8) \cdot \left(\frac{1}{3}\right)$
2. Mai says that we can also find the answer by multiplying the edge lengths of the prism and then multiplying the result by 27. Do you agree with her? Explain your reasoning.

Exercise 24.3.7

A builder is building a fence with $6\frac{1}{4}$-inch-wide wooden boards, arranged side-by-side with no gaps or overlaps. How many boards are needed to build a fence that is 150 inches long? Show your reasoning.

(From Unit 4.4.1)

https://math.libretexts.org/@go/page/40249

Exercise 24.3.8

Find the value of each expression. Show your reasoning and check your answer.

1. $2\frac{1}{7} \div \frac{2}{7}$
2. $\frac{17}{20} \div \frac{1}{4}$

(From Unit 4.4.1)

Exercise 24.3.9

Consider the problem: A bucket contains $11\frac{2}{3}$ gallons of water and is $\frac{5}{6}$ full. How many gallons of water would be in a full bucket?

Write a multiplication and a division equation to represent the situation. Then, find the answer and show your reasoning.

(From Unit 4.3.2)

Exercise 24.3.10

There are 80 kids in a gym. 75% are wearing socks. How many are *not* wearing socks? If you get stuck, consider using a tape diagram.

(From Unit 3.4.3)

Exercise 24.3.11

1. Lin wants to save $75 for a trip to the city. If she has saved $37.50 so far, what percentage of her goal has she saved? What percentage remains?
2. Noah wants to save $60 so that he can purchase a concert ticket. If he has saved $45 so far, what percentage of his goal has he saved? What percentage remains?

(From Unit 3.4.2)

24.3: Fractional Lengths in Triangles and Prisms is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

24.4: Volume of Prisms

Lesson

Let's look a the volume of prisms that have fractional measurements.

Exercise 24.4.1: A Box of Cubes

1. How many cubes with an edge length of 1 inch fill this box?

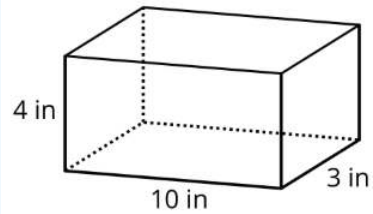


Figure 24.4.1

2. If the cubes had an edge length of 2 inches, would you need more or fewer cubes to fill the box? Explain your reasoning.
3. If the cubes had an edge length of $\frac{1}{2}$ inch, would you need more or fewer cubes to fill the box? Explain your reasoning.

Exercise 24.4.2: Cubes with Fractional Edge Lengths

1. Diego says that 108 cubes with an edge length of $\frac{1}{3}$ inch are needed to fill a rectangular prism that is 3 inches by 1 inch by $1\frac{1}{3}$ inch.
   a. Explain or show how this is true. If you get stuck, consider drawing a diagram.
   b. What is the volume, in cubic inches, of the rectangular prism? Explain or show your reasoning.
2. Lin and Noah are packing small cubes into a larger cube with an edge length of $1\frac{1}{2}$ inches. Lin is using cubes with an edge length of $\frac{1}{2}$ inch, and Noah is using cubes with an edge length of $\frac{1}{4}$ inch.
   a. Who would need more cubes to fill the $1\frac{1}{2}$-inch cube? Be prepared to explain your reasoning.
   b. If Lin and Noah each use their small cubes to find the volume of the larger $1\frac{1}{2}$-inch cube, will they get the same answer? Explain or show your reasoning.

Exercise 24.4.3: Fish Tank and Baking Pan

1. A nature center has a fish tank in the shape of a rectangular prism. The tank is 10 feet long, $8\frac{1}{4}$ feet wide, and 6 feet tall.
   a. What is the volume of the tank in cubic feet? Explain or show your reasoning.



Figure 24.4.2: Aquarium récifal, by Serge Talfer (Self-photographed). Public Domain. Wikimedia Commons. Source.

   b. The nature center's caretaker filled $\frac{4}{5}$ of the tank with water. What was the volume of the water in the tank, in cubic feet? What was the height of the water in the tank? Explain or show your reasoning.
   c. Another day, the tank was filled with 330 cubic feet of water. The height of the water was what fraction of the height of the tank? Show your reasoning.
2. Clare's recipe for banana bread won't fit in her favorite pan. The pan is $8\frac{1}{2}$ inches by 11 inches by 2 inches. The batter fills the pan to the very top, and when baking, the batter spills over the sides. To avoid spills, there should be about an inch between the top of the batter and the rim of the pan.

Clare has another pan that is 9 inches by 9 inches by $2\frac{1}{2}$ inches. If she uses this pan, will the batter spill over during baking?

Are you ready for more?

1. Find the area of a rectangle with side lengths $\frac{1}{2}$ and $\frac{2}{3}$.
2. Find the volume of a rectangular prism with side lengths $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$.
3. What do you think happens if we keep multiplying fractions $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdot \frac{5}{6} \ldots$?
4. Find the area of a rectangle with side lengths $\frac{1}{1}$ and $\frac{2}{1}$.
5. Find the volume of a rectangular prism with side lengths $\frac{1}{1}$, $\frac{2}{1}$, and $\frac{1}{3}$.
6. What do you think happens if we keep multiplying fractions $\frac{1}{1} \cdot \frac{2}{1} \cdot \frac{1}{3} \cdot \frac{4}{1} \cdot \frac{1}{5} \ldots$?

Summary

If a rectangular prism has edge lengths a units, b units, and c units, the volume is the product of a, b, and c. $V = a \cdot b \cdot c$

This means that if we know the *volume* and *two edge lengths*, we can divide to find the *third* edge length.

Suppose the volume of a rectangular prism is $400\frac{1}{2}$ cm^3, one edge length is $\frac{11}{2}$ cm, another is 6 cm, and the third edge length is unknown. We can write a multiplication equation to represent the situation: $\frac{11}{2} \cdot 6 \cdot ? = 400\frac{1}{2}$

We can find the third edge length by dividing: $400\frac{1}{2} \div \left(\frac{11}{2} \cdot 6\right) = ?$

Practice

Exercise 24.4.4

A pool in the shape of a rectangular prism is being filled with water. The length and width of the pool is 24 feet and 15 feet. If the height of the water in the pool is $1\frac{1}{3}$ feet, what is the volume of the water in cubic feet?

Exercise 24.4.5

A rectangular prism measures $2\frac{2}{5}$ inches by $3\frac{1}{5}$ inches by 2 inch.

1. Priya said, "It takes more cubes with edge length $\frac{2}{5}$ inch than cubes with edge length $\frac{1}{5}$ inch to pack the prism." Do you agree with Priya? Explain or show your reasoning.
2. How many cubes with edge length $\frac{1}{5}$ inch fit in the prism? Show your reasoning.
3. Explain how you can use your answer in the previous question to find the volume of the prism in cubic inches.

Exercise 24.4.6

1. Here is a right triangle. What is its area?
2. What is the height h for the base that is $\frac{5}{4}$ units long? Show your reasoning.

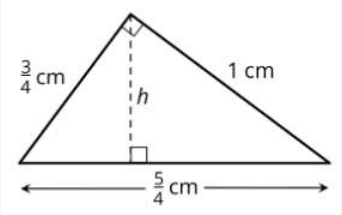


Figure 24.4.3: A right triangle with horizontal base on the bottom, labeled five fourths centimeter. One side is labeled three fourths centimeter and the other side is labeled 1 centimeter. A vertical dashed line is drawn from the right angle to the horizontal base and labeled h.

(From Unit 4.4.3)

Exercise 24.4.7

To give their animals essential minerals and nutrients, farmers and ranchers often have a block of salt—called “salt lick”—available for their animals to lick.

1. A rancher is ordering a box of cube-shaped salt licks. The edge lengths of each salt lick are $\frac{5}{12}$ foot. Is the volume of one salt lick greater or less than 1 cubic foot? Explain your reasoning.



Figure 24.4.4: Salt-lick 4 Beentree, by Beentree (Own work). CC BY-SA 4.0. Wikimedia Commons. Source.

2. The box that contains the salt lick is $1\frac{1}{4}$ feet by $1\frac{2}{3}$ feet by $\frac{5}{6}$ feet. How many cubes of salt lick fit in the box? Explain or show your reasoning.

Exercise 24.4.8

1. How many groups of $\frac{1}{3}$ inch are in $\frac{3}{4}$ inch?
2. How many inches are in $1\frac{2}{5}$ groups of $1\frac{2}{3}$ inches?

(From Unit 4.4.1)

Exercise 24.4.9

Here is a table that shows the ratio of flour to water in an art paste. Complete the table with values in equivalent ratios.

Table 24.4.1

cups of flour	cups of water
1	$\frac{1}{2}$
4	
	3
$\frac{1}{2}$	

(From Unit 2.4.2)

24.4: Volume of Prisms is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

Section 25: Let's Put It to Work

This page titled Section 25: Let's Put It to Work is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

25.1: Solving Problems Involving Fractions

Lesson

Let's add, subtract, multiply, and divide fractions.

Exercise 25.1.1: Operations with Fractions

Without calculating, order the expressions according to their values from least to greatest. Be prepared to explain your reasoning.

$$\frac{3}{4}+\frac{2}{3} \qquad \frac{3}{4}-\frac{2}{3} \qquad \frac{3}{4}\cdot\frac{2}{3} \qquad \frac{3}{4}\div\frac{2}{3}$$

Exercise 25.1.2: Situations with $\frac{3}{4}$ and $\frac{1}{2}$

Here are four situations that involve $\frac{3}{4}$ and $\frac{1}{2}$.

- Before calculating, decide if each answer is greater than 1 or less than 1.
- Write a multiplication equation or division equation for the situation.
- Answer the question. Show your reasoning. Draw a tape diagram, if needed.

1. There was $\frac{3}{4}$ liter of water in Andre's water bottle. Andre drank $\frac{1}{2}$ of the water. How many liters of water did he drink?
2. The distance from Han's house to his school is $\frac{3}{4}$ kilometers. Han walked $\frac{1}{2}$ kilometers. What fraction of the distance from his house to the school did Han walk?
3. Priya's goal was to collect $\frac{1}{2}$ kilograms of trash. She collected $\frac{3}{4}$ kilograms of trash. How many times her goal was the amount of trash she collected?
4. Mai's class volunteered to clean a park with an area of $\frac{1}{2}$ square mile. Before they took a lunch break, the class had cleaned $\frac{3}{4}$ of the park. How many square miles had they cleaned before lunch?

Exercise 25.1.3: Pairs of Problems

1. Work with a partner to write equations for the following questions. One person works on the questions labeled A1, B1, . . . , E1 and the other person works on those labeled A2, B2, . . . , E2.

   A1. Lin's bottle holds $3\frac{1}{4}$ cups of water. She drank 1 cup of water. What fraction of the water in the bottle did she drink?

   B1. Plant A is $\frac{16}{3}$ feet tall. This is $\frac{4}{5}$ as tall as Plant B. How tall is Plant B?

   C1. $\frac{8}{9}$ kilogram of berries is put into a container that already has $\frac{7}{3}$ kilogram of berries. How many kilograms are in the container?

   D1. The area of a rectangle is $14\frac{1}{2}$ sq cm and one side is $4\frac{1}{2}$ cm. How long is the other side?

   E1. A stack of magazines is $4\frac{2}{5}$ inches high. The stack needs to fit into a box that is $2\frac{1}{8}$ inches high. How many inches too high is the stack?

   A2. Lin's bottle holds $3\frac{1}{4}$ cups of water. After she drank some, there were $1\frac{1}{2}$ cups of water in the bottle. How many cups did she drink?

   B2. Plant A is $\frac{16}{3}$ feet tall. Plant C is $\frac{4}{5}$ as tall as Plant A. How tall is Plant C?

   C2. A container with $\frac{8}{9}$ kilogram of berries is $\frac{2}{3}$ full. How many kilograms can the container hold?

   D2. The side lengths of a rectangle are $4\frac{1}{2}$ cm and $2\frac{2}{5}$ cm. What is the area of the rectangle?

   E2. A stack of magazines is $4\frac{2}{5}$ inches high. Each magazine is $\frac{2}{5}$-inch thick. How many magazines are in the stack?

2. Trade papers with your partner, and check your partner's equations. If you disagree, work to reach an agreement.
3. Your teacher will assign 2 or 3 questions for you to answer. For each question:
   a. Estimate the answer before calculating it.

b. Find the answer, and show your reasoning.

Exercise 25.1.4: Baking Cookies

Mai, Kiran, and Clare are baking cookies together. They need $\frac{3}{4}$ cup of flour and $\frac{1}{3}$ cup of butter to make a batch of cookies. They each brought the ingredients they had at home.

- Mai brought 2 cups of flour and $\frac{1}{4}$ cup of butter.
- Kiran brought 1 cup of flour and $\frac{1}{2}$ cup of butter.
- Clare brought $1\frac{1}{4}$ cups of flour and $\frac{3}{4}$ cup of butter.

If the students have plenty of the other ingredients they need (sugar, salt, baking soda, etc.), how many whole batches of cookies can they make? Explain your reasoning.

Summary

We can add, subtract, multiply, and divide both whole numbers and fractions. Here is a summary of how we add, subtract, multiply, and divide fractions.

- To add or subtract fractions, we often look for a common denominator so the pieces involved are the same size. This makes it easy to add or subtract the pieces.

$$\frac{3}{2} - \frac{4}{5} = \frac{15}{10} - \frac{8}{10}$$

- To multiply fractions, we often multiply the numerators and the denominators.

$$\frac{3}{8} \cdot \frac{5}{9} = \frac{3 \cdot 5}{8 \cdot 9}$$

- To divide a number by a fraction $\frac{a}{b}$, we can multiply the number by $\frac{b}{a}$, which is the reciprocal of $\frac{a}{b}$.

$$\frac{4}{7} \div \frac{5}{3} = \frac{4}{7} \cdot \frac{3}{5}$$

Practice

Exercise 25.1.5

An orange has about $\frac{1}{4}$ cup of juice. How many oranges are needed to make $2\frac{1}{2}$ cups of juice? Select **all** the equations that represent this question.

A. $? \cdot \frac{1}{4} = 2\frac{1}{2}$
B. $\frac{1}{4} \div 2\frac{1}{2} = ?$
C. $? \cdot 2\frac{1}{2} = \frac{1}{4}$
D. $2\frac{1}{2} \div \frac{1}{4} = ?$

Exercise 25.1.6

Mai, Clare, and Tyler are hiking from a parking lot to the summit of a mountain. They pass a sign that gives distances.

Parking lot: $\frac{3}{4}$ mile

Summit: $1\frac{1}{2}$ miles

- Mai says: “We are one third of the way there.”
- Clare says: “We have to go twice as far as we have already gone.”
- Tyler says: “The total hike is three times as long as what we have already gone.”

Do you agree with any of them? Explain your reasoning.

Exercise 25.1.7

Priya's cat weighs $5\frac{1}{2}$ pounds and her dog weighs $8\frac{1}{4}$ pounds. First, estimate the number that would comlpete each sentence. Then, calculate the answer. If any of your estimates were not close to the answer, explain why that may be.

1. The cat is _____ as heavy as the dog.
2. Their combined weight is _____ pounds.
3. The dog is _____ pounds heavier than the cat.

Exercise 25.1.8

Before refrigerators existed, some people had blocks of ice delivered to their homes. A delivery wagon had a storage box in the shape of a rectangular prism that was $7\frac{1}{2}$ feet by 6 feet by 6 feet. The cubic ice blocks stored in the box had side lengths $1\frac{1}{2}$ feet. How many ice blocks fit in the storage box?

A. 270
B. $3\frac{3}{8}$
C. 80
D. 180

(From Unit 4.4.4)

Exercise 25.1.9

Fill in the blanks with 0.001, 0.1, 10, or 1000 so that the value of each quotient is in the correct column.

Close to $\frac{1}{100}$

- ____ $\div 9$
- $12 \div$ ____

Close to 1

- ____ $\div 0.12$
- $\frac{1}{8} \div$ ____

Greater than 100

- ____ $\div \frac{1}{3}$
- $700.7 \div$ ____

(From Unit 4.1.1)

Exercise 25.1.10

A school club sold 300 shirts. 31% were sold to fifth graders, 52% were sold to sixth graders, and the rest were sold to teachers. How many shirts were sold to each group—fifth graders, sixth graders, and teachers? Explain or show your reasoning.

(From Unit 3.4.6)

Exercise 25.1.11

Jada has some pennies and dimes. The ratio of Jada's pennies to dimes is 2 to 3.

1. From the information given, can you determine how many coins Jada has?
2. If Jada has 55 coins, how many of each kind of coin does she have?
3. How much are her coins worth?

(From Unit 2.5.1)

25.1: Solving Problems Involving Fractions is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

25.2: Fitting Boxes into Boxes

Lesson

Let's use what we learned about fractions to find shipping costs.

Exercise 25.2.1: Determining Shipping Costs (Part 1)

An artist makes necklaces. She packs each necklace in a small jewelry box that is $1\frac{3}{4}$ inches by $2\frac{1}{4}$ inches by $\frac{3}{4}$ inch.

A department store ordered 270 necklaces. The artist plans to ship the necklaces to the department store using flat-rate shipping boxes from the post office.

1. Consider the problem: Which of the flat-rate boxes should she use to minimize her shipping cost? What other information would you need to be able to solve the problem?
2. Discuss this information with your group. Make a plan for using this information to find the most inexpensive way to ship the jewelry boxes. Once you have agreed on a plan, write down the main steps.

Exercise 25.2.2: Determining Shipping Costs (Part 2)

Work with your group to find the best plan for shipping the boxes of necklaces. Each member of your group should select a different type of flat-rate shipping box and answer the following questions. Recall that each jewelry box is $1\frac{3}{4}$ inches by $2\frac{1}{4}$ inches by $\frac{3}{4}$ inch, and that there are 270 jewelry boxes to be shipped.

For each type of flat-rate shipping box:

1. Find how many jewelry boxes can fit into the box. Explain or show how the jewelry boxes can be packed in the shipping box. Draw a sketch to show your thinking, if needed.
2. Calculate the total cost of shipping all 270 jewelry boxes in shipping boxes of that type. Show your reasoning and organize your work so it can be followed by others.

Exercise 25.2.3: Determining Shipping Costs (Part 3)

1. Share and discuss your work with the other members of your group. Your teacher will display questions to guide your discussion. Note the feedback from your group so you can use it to revise your work.
2. Using the feedback from your group, revise your work to improve its correctness, clarity, and accuracy. Correct any errors. You may also want to add notes or diagrams, or remove unnecessary information.
3. Which shipping boxes should the artist use? As a group, decide which boxes you recommend for shipping 270 jewelry boxes. Be prepared to share your reasoning.

25.2: Fitting Boxes into Boxes is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

SECTION OVERVIEW

5: Arithmetic in Base Ten

Section 26: Warming Up to Decimals

26.1: Using Decimals in a Shopping Context

Section 27: Adding and Subtracting Decimals

27.1: Using Diagrams to Represent Addition and Subtraction
27.2: Adding and Subtracting Decimals with Few Non-Zero Digits
27.3: Adding and Subtracting Decimals with Many Non-Zero Digits

Section 28: Multiplying Decimals

28.1: Decimal Points in Products
28.2: Methods for Multiplying Decimals
28.3: Using Diagrams to Represent Multiplication
28.4: Calculating Products of Decimals

Section 29: Dividing Decimals

29.1: Using the Partial Quotients Method
29.2: Using Long Division
29.3: Dividing Numbers that Result in Decimals
29.4: Dividing Decimals by Whole Numbers
29.5: Dividing Decimals by Decimals

Section 30: Let's Put It to Work

30.1: Using Operations on Decimals to Solve Problems
30.2: Making and Measuring Boxes

This page titled 5: Arithmetic in Base Ten is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

Section 26: Warming Up to Decimals

This page titled Section 26: Warming Up to Decimals is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

26.1: Using Decimals in a Shopping Context

Lesson

Let's use what we know about decimals to make shopping decisions.

Exercise 26.1.1: Snacks from the Concession Stand

Clare went to a concession stand that sells pretzels for $3.25, drinks for $1.85, and bags of popcorn for $0.99 each. She bought at least one of each item and spent no more than $10.



Figure 26.1.1: By m01229. CC BY 2.0. Flickr. Source.

1. Could Clare have purchased 2 pretzels, 2 drinks, and 2 bags of popcorn? Explain your reasoning.
2. Could she have bought 1 pretzel, 1 drink, and 5 bags of popcorn? Explain your reasoning.

Exercise 26.1.2: Planning a Dinner Party

You are planning a dinner party with a budget of $50 and a menu that consists of 1 main dish, 2 side dishes, and 1 dessert. There will be 8 guests at your party.

Choose your menu items and decide on the quantities to buy so you stay on budget. If you choose meat, fish, or poultry for your main dish, plan to buy at least 0.5 pound per person.

1. The budget is $ _____ per guest.
2. Use the worksheet to record your choices and estimated costs. Then find the estimated total cost and cost per person. See examples in the first two rows.

Table 26.1.1

item	quantity needed	advertised price	estimated subtotal ($)	estimated cost per person ($)
example main dish: fish	4 pounds	$6.69 per pound	$4 \cdot 7 = 28$	$28 \div 8 = 3.50$
example dessert: cupcakes	8 cupcakes	$2.99 per 6 cupcakes	$2 \cdot 3 = 6$	$6 \div = 0.75$
main dish:				
side dish 1:				
side dish 2:				
dessert:				
estimated total				

3. Is your estimated total close to your budget? If so, continue to the next question. If not, revise your menu choices until your estimated total is close to the budget.
4. Calculate the actual costs of the two most expensive items and add them. Show your reasoning.
5. How will you know if your total cost for all menu items will or will not exceed your budget? Is there a way to predict this without adding all the exact costs? Explain your reasoning.

Are you ready for more?

How much would it cost to plant the grass on a football field? Explain or show your reasoning.

Summary

We often use decimals when dealing with money. In these situations, sometimes we round and make estimates, and other times we calculate the numbers more precisely.

There are many different ways we can add, subtract, multiply, and divide decimals. When we perform these calculations, it is helpful to understand the meanings of the digits in a number and the properties of operations. We will investigate how these understandings help us work with decimals in upcoming lessons.

Practice

Exercise 26.1.3

Mai had $14.50. She spent $4.35 at the snack bar and $5.25 at the arcade. What is the exact amount of money Mai has left?

A. $9.60
B. $10.60
C. $4.90
D. $5.90

Exercise 26.1.4

A large cheese pizza costs $7.50. Diego has $40 to spend on pizzas. How many large cheese pizzas can he afford? Explain or show your reasoning.

Exercise 26.1.5

Tickets to a show cost $5.50 for adults and $4.25 for students. A family is purchasing 2 adult tickets and 3 student tickets.

1. Estimate the total cost.
2. What is the exact cost?
3. If the family pays $25, what is the exact amount of change they should receive?

Exercise 26.1.6

Chicken costs $3.20 per pound, and beef costs $4.59 per pound. Answer each question and show your reasoning.

1. What is the exact cost of 3 pounds of chicken?
2. What is the exact cost of 3 pound of beef?
3. How much more does 3 pounds of beef cost than 3 pounds of chicken?

Exercise 26.1.7

1. How many $\frac{1}{5}$-liter glasses can Lin fill with a $1\frac{1}{2}$-liter bottle of water?
2. How many $1\frac{1}{2}$-liter bottles of water does it take to fill a 16-liter jug?

(From Unit 4.5.1)

Exercise 26.1.8

Let the side length of each small square on the grid represents 1 unit. Draw two different triangles, each with base $5\frac{1}{2}$ units and area $19\frac{1}{4}$ units^2.

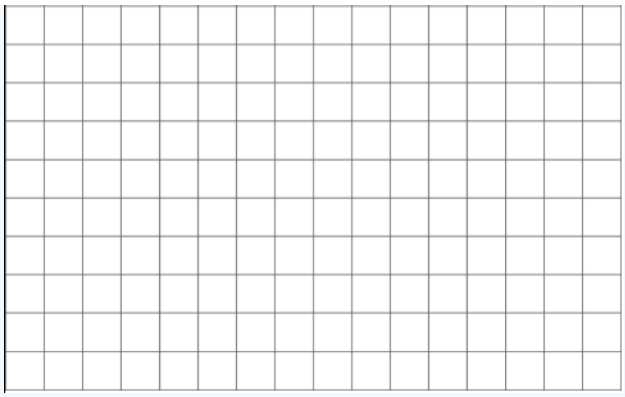


Figure 26.1.2

Why does each of your triangles have area $19\frac{1}{4}$ units^2? Explain or show your reasoning.

(From Unit 4.4.3)

Exercise 26.1.9

Find each quotient.

1. $\frac{5}{6} \div \frac{1}{6}$
2. $1\frac{1}{6} \div \frac{1}{12}$
3. $\frac{10}{6} \div \frac{1}{24}$

(From Unit 4.3.1)

This page titled 26.1: Using Decimals in a Shopping Context is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

Section 27: Adding and Subtracting Decimals

This page titled Section 27: Adding and Subtracting Decimals is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

27.1: Using Diagrams to Represent Addition and Subtraction

Lesson

Let's represent addition and subtraction of decimals.

Exercise 27.1.1: Changing Values

1. Here is a rectangle.



Figure 27.1.1

What number does the rectangle represent if each small square represents:

a. 1
b. 0.1
c. 0.01
d. 0.001

2. Here is a square.

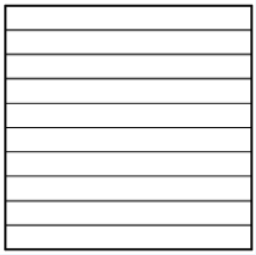


Figure 27.1.2

What number does the square represent if each small rectangle represents:

a. 10
b. 0.1
c. 0.00001

Exercise 27.1.2: Squares and Rectangles

You may be familiar with base-ten blocks that represent ones, tens, and hundreds. Here are some diagrams that we will use to represent digital base-ten units. A large square represents 1 one. A rectangle represents 1 tenth. A small square represents 1 hundredth.

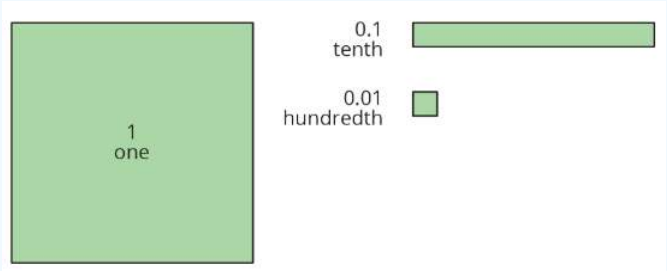


Figure 27.1.3

The applet has tools that create each of the base-ten blocks.

Select a Block tool, and then click on the screen to place it.

One



Figure 27.1.4

Tenth

Figure 27.1.5

Hundredth



Figure 27.1.6

Click on the Move tool when you are done choosing blocks.



Figure 27.1.7

1. Here is the diagram that Priya drew to represent 0.13. Draw a different diagram that represents 0.13. Explain why your diagram and Priya's diagram represent the same number.



Figure 27.1.8

2. Here is the diagram that Han drew to represent 0.25. Draw a different diagram that represents 0.25. Explain why your diagram and Han's diagram represent the same number.

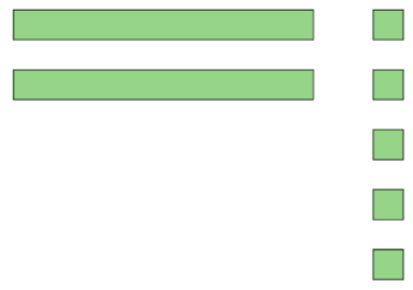


Figure 27.1.9

3. For each of these numbers, draw or describe two different diagrams that represent it.
   a. 0.1
   b. 0.02
   c. 0.43
4. Use diagrams of base-ten units to represent the following sums and find their values. Think about how you could use as few units as possible to represent each number.
   a. $0.03 + 0.05$
   b. $0.06 + 0.07$
   c. $0.4 + 0.7$

Exercise 27.1.3: Finding Sums in Different Ways

1. Here are two ways to calculate the value of $0.26 + 0.07$. In the diagram, each rectangle represents 0.1 and each square represents 0.01.

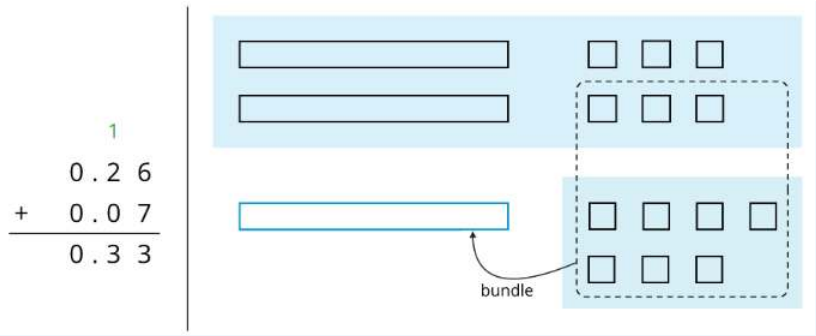


Figure 27.1.10: A diagram of two strategies used to calculate addition expression. The left strategy, a vertical equation of 0 point 2 6 plus 0 point 0 7 results in 0 point 3 3. A 1 is written above the 2 in the tenths column. The right strategy, a base ten diagram. There are 2 large rectangles and 6 small squares indicated. Directly below, the squares are an additional 7 small squares indicated. A dashed circle contains 10 of the small squares with an arrow, labeled bundle, pointing to a third large rectangle. The third rectangle is drawn under the other two existing large rectangles.

Use what you know about base-ten units and addition of base-ten numbers to explain:

a. Why ten squares can be “bundled” into a rectangle.
b. How this “bundling” is reflected in the computation.

The applet has tools that create each of the base-ten blocks. Select a Block tool, and then click on the screen to place it.

One



Figure 27.1.11

Tenth

Figure 27.1.12

Hundredth



Figure 27.1.13

Click on the Move tool when you are done choosing blocks.



Figure 27.1.14

2. Find the value of $0.38 + 0.69$ by drawing a diagram. Can you find the sum without bundling? Would it be useful to bundle some pieces? Explain your reasoning.
3. Calculate $0.38 + 0.69$. Check your calculation against your diagram in the previous question.
4. Find each sum. The larger square represents 1, the rectangle represents 0.1, and the smaller square represents 0.01.

a.

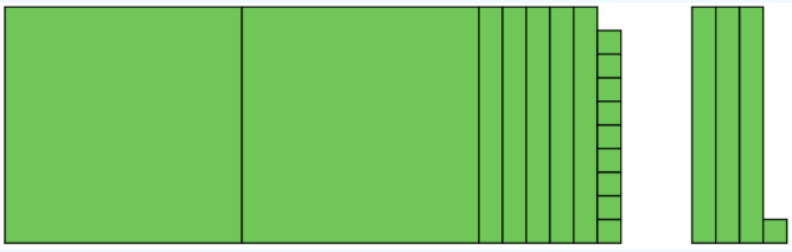


Figure 27.1.15

b.

$$\begin{array}{r} 6.0\,3 \\ +\ 0.0\,9\,8 \\ \hline \end{array}$$

Figure 27.1.16

Are you ready for more?

A distant, magical land uses jewels for their bartering system. The jewels are valued and ranked in order of their rarity. Each jewel is worth 3 times the jewel immediately below it in the ranking. The ranking is red, orange, yellow, green, blue, indigo, and violet. So a red jewel is worth 3 orange jewels, a green jewel is worth 3 blue jewels, and so on.

1. If you had 500 violet jewels and wanted to trade so that you carried as few jewels as possible, which jewels would you have?
2. Suppose you have 1 orange jewel, 2 yellow jewels, and 1 indigo jewel. If you're given 2 green jewels and 1 yellow jewels, what is the fewest number of jewels that could represent the value of the jewels you have?

Exercise 27.1.4: Representing Subtraction

1. Here are diagrams that represent differences. Removed pieces are marked with Xs. The larger rectangle represents 1 tenth. For each diagram, write a numerical subtraction expression and determine the value of the expression.

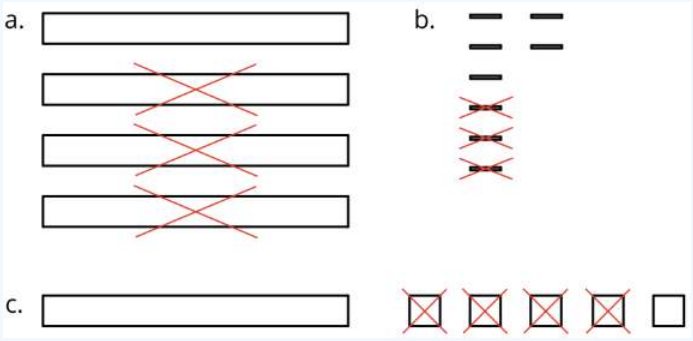


Figure 27.1.17

2. Express each subtraction in words.
   a. $0.05 - 0.02$
   b. $0.024 - 0.003$
   c. $1.26 - 0.14$
3. Find each difference by drawing a diagram and by calculating with numbers. Make sure the answers from both methods match. If not, check your diagram and your numerical calculation.
   a. $0.05 - 0.02$
   b. $0.024 - 0.003$
   c. $1.26 - 0.14$

Summary

Base-ten diagrams represent collections of base-ten units—tens, ones, tenths, hundredths, etc. We can use them to help us understand sums of decimals.

Suppose we are finding $0.08 + 0.13$. Here is a diagram where a square represents 0.01 and a rectangle (made up of ten squares) represents 0.1.

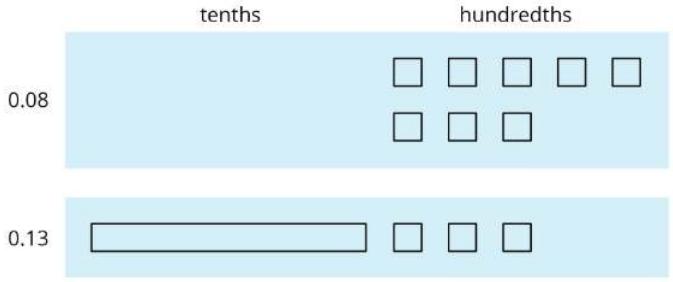


Figure 27.1.18: Base ten diagram. First row, 0 point 0 8. No rods in the tenths column. 8 small squares in the hundredths column. Second row, 0 point 13. 1 rod in the tenths column. 3 small squares in the hundredths column.

To find the sum, we can "bundle" (or compose) 10 hundredths as 1 tenth.

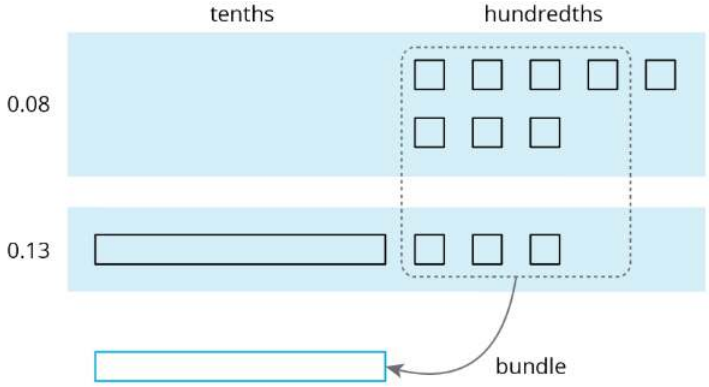


Figure 27.1.19: Base ten diagram. First row, 0 point 0 8. No rods in the tenths column. 8 small squares in the hundredths column. Second row, 0 point 13. 1 rod in the tenths column. 3 small squares in the hundredths column. A square is drawn around the 10 small squares. An arrow is drawn to a tenths rod outside the diagram. The arrow is labeled bundle.

We now have 2 tenths and 1 hundredth, so $0.08 + 0.13 = 0.21$.

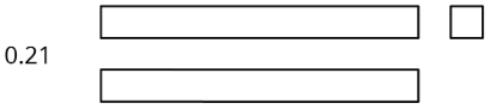


Figure 27.1.20

We can also use vertical calculation to find $0.08 + 0.13$.

$$\begin{array}{r} {\scriptstyle 1} \\ 0.1\ 3 \\ +\ \ 0.0\ 8 \\ \hline 0.2\ 1 \end{array}$$

Figure 27.1.21

Notice how this representation also shows 10 hundredths are bundled (or composed) as 1 tenth.

This works for any decimal place. Suppose we are finding $0.008 + 0.013$. Here is a diagram where a small rectangle represents 0.001.

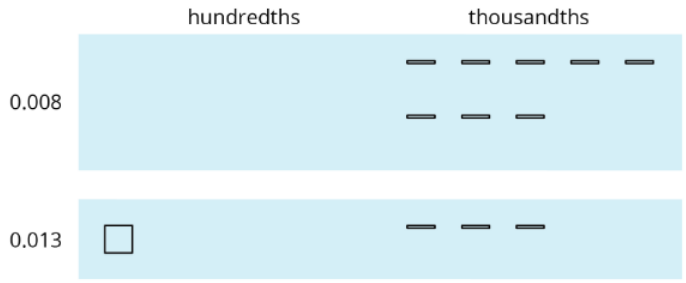


Figure 27.1.22: Base ten diagram. First row, 0 point 0 0 8. No squares in the hundredths column. 8 small rectangles in the thousandths column. Second row, 0 point 0 1 3. 1 small square in the hundredths column. 3 small rectangles in the thousandths column.

We can "bundle" (or compose) 10 thousandths as 1 hundredth.

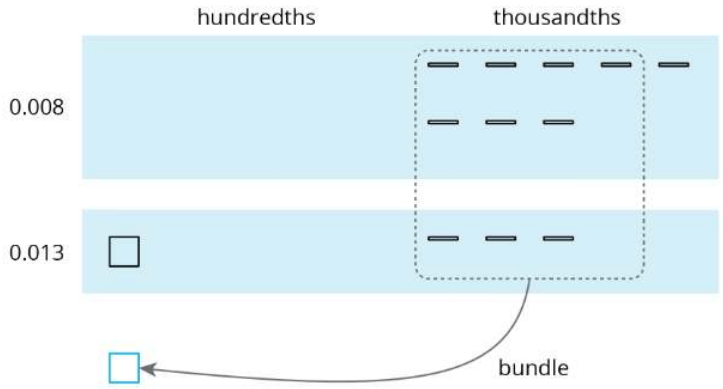


Figure 27.1.23: Base ten diagram. First row, 0 point 0 0 8. No squares in the hundredths column. 8 small rectangles in the thousandths column. Second row, 0 point 0 1 3. 1 small square in the hundredths column. 3 small rectangles in the thousandths column. A square is drawn around the 10 small rectangles. An arrow is drawn to a small square outside the diagram. The arrow is labeled bundle.

The sum is 2 hundredths and 1 thousandth.



Figure 27.1.24

Here is a vertical calculation of $0.008 + 0.013$

$$\begin{array}{r} {\scriptstyle 1} \\ 0.013 \\ +\ 0.008 \\ \hline 0.021 \end{array}$$

Figure 27.1.25: Vertical addition. First line. 0 point 0 13. Second line. Plus 0 point 0 0 8. Horizontal line. Third line. 0 point 0 2 1. Above the 1 in the first line is 1.

Practice

Exercise 27.1.5

Use the given key to answer the questions.

https://math.libretexts.org/@go/page/40301

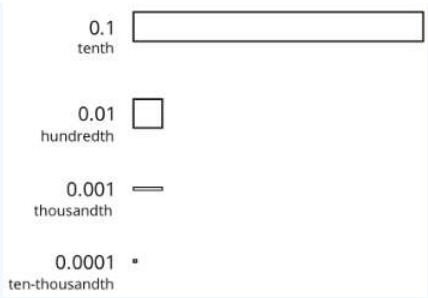


Figure 27.1.26: Base-ten figures. 1 large rectangle labeled, 0 point 1, tenth, 1 square labeled, 0 point 0 1, hundredth, 1 small rectangle labeled, 0 point 0 0 1, thousandth, 1 small square labeled, 0 point 0 0 0 1, ten-thousandth.

1. What number does this diagram represent?



Figure 27.1.27

2. Draw a diagram that represents 0.216.
3. Draw a diagram that represents 0.304.

Exercise 27.1.6

Here are diagrams that represent 0.137 and 0.284.

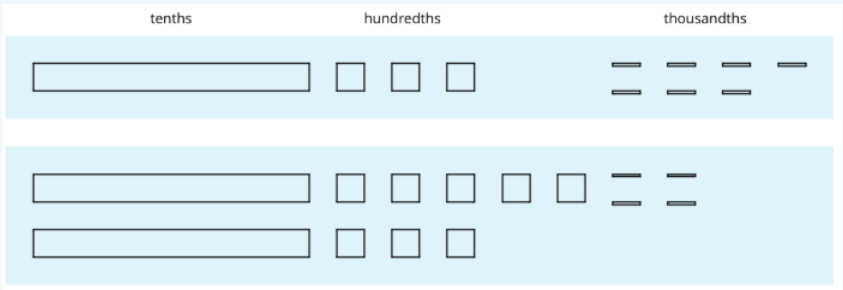


Figure 27.1.28: Two base-ten diagrams representing 0 point 1 3 7 and 0 point 2 8 4. First diagram, 1 rectangle labeled, tenths, 3 squares labeled, hundredths, 7 small rectangles labeled, thousandths. Second diagram, 2 rectangles labeled, tenths, 8 squares labeled, hundredths, 4 small rectangles labeled, thousandths.

1. Use the diagram to find the value of $0.137 + 0.284$. Explain your reasoning.
2. Calculate the sum vertically.

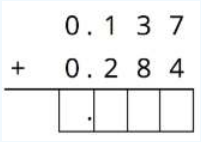


Figure 27.1.29

3. How was your reasoning about $0.137 + 0.284$ the same with the two methods? How was it different?

Exercise 27.1.7

For the first two problems, circle the vertical calculation where digits of the same kind are lined up. Then, finish the calculation and find the sum. For the last two problems, find the sum using vertical calculation.

1. $3.25 + 1$

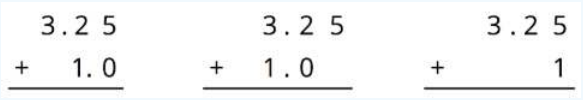


Figure 27.1.30: 3 vertical calculations of 3 point 2 5 plus 1. First calculation, 3 point 2 5 plus 1 point 0, the decimal point for 1 point 0 is shifted one place to the right. Second calculation, 3 point 2 5 plus 1 point 0, the decimal points are aligned. Third calculation, 3 point 2 5 plus 1, the 1 is shifted two places to the right.

2. $0.5 + 1.15$

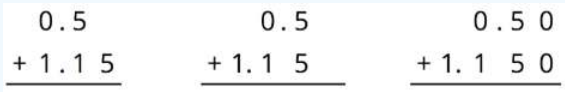


Figure 27.1.31: 3 vertical calculations of 0 point 5 plus 1 point 1 5. First calculation, 0 point 5 plus 1 point 1 5, decimal points are aligned. Second calculation, 0 point 5 plus 1 point 1 5, decimal point of 1 point 1 5 shifted one place to the left. Third calculation, 0 point 5 0 plus 1 point 1 5 0, decimal point of 1 point 1 5 0 shifted one place to the left.

3. $10.6 + 1.7$
4. $123 + 0.2$

Exercise 27.1.8

Andre has been practicing his math facts. He can now complete 135 multiplication facts in 90 seconds.

1. If Andre is answering questions at a constant rate, how many facts can he answer per second?
2. Noah also works at a constant rate, and he can complete 75 facts in 1 minute. Who is working faster? Explain or show your reasoning.

(From Unit 2.3.4)

27.1: Using Diagrams to Represent Addition and Subtraction is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

27.2: Adding and Subtracting Decimals with Few Non-Zero Digits

Lesson

Let's add and subtract decimals.

Exercise 27.2.1: Do the Zeros Matter?

1. Evaluate mentally: $1.009 + 0.391$
2. Decide if each equation is true or false. Be prepared to explain your reasoning.
   a. $34.56000 = 34.56$
   b. $25 = 25.0$
   c. $2.405 = 2.45$

Exercise 27.2.2: Calculating Sums

1. Andre and Jada drew base-ten diagrams to represent $0.007 + 0.004$ Andre drew 11 small rectangles. Jada drew only two figures: a square and a small rectangle.

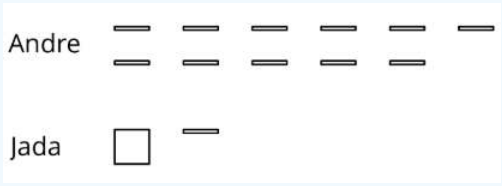


Figure 27.2.1

   a. If both students represented the sum correctly, what value does each small rectangle represent? What value does each square represent?
   b. Draw or describe a diagram that could represent the sum $0.008 + 0.07$.

2. Here are two calculations of $0.2 + 0.05$. Which is correct? Explain why one is correct and the other is incorrect.

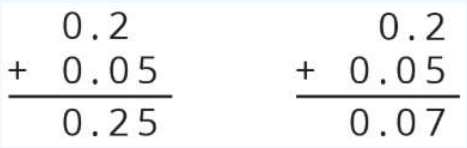


Figure 27.2.2: Two calculations of zero point 2 plus zero point five are indicated. The calculation on the left adds zero point 2 and zero point zero 5 by aligning the ones units, tenths unit, and hundredths unit. The sum is zero point 2 5. The calculation on the right adds zero point 2 and zero point zero five by aligning the hundredths unit under the tenths unit. The sum is zero point zero 7.

3. Compute each sum. If you get stuck, consider drawing base-ten diagrams to help you.
   a.

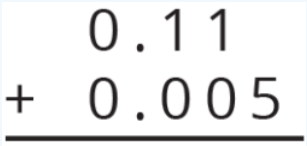


Figure 27.2.3

   b. $0.209 + 0.01$
   c. $10.2 + 1.1456$

- The applet has tools that create each of the base-ten blocks. This time you need to decide the value of each block before you begin.
- Select a Block tool, and then click on the screen to place it.
- Click on the Move tool (the arrow) when you are done choosing blocks.
- Subtract by deleting with the delete tool (the trash can), not by crossing out.

Exercise 27.2.3: Subtracting Decimals of Different Lengths

To represent $0.4 - 0.03$, Diego and Noah drew different diagrams. Each rectangle represented 0.1. Each square represented 0.01.

- Diego started by drawing 4 rectangles to represent 0.4. He then replaced 1 rectangle with 10 squares and crossed out 3 squares to represent subtraction of 0.03, leaving 3 rectangles and 7 squares in his diagram.

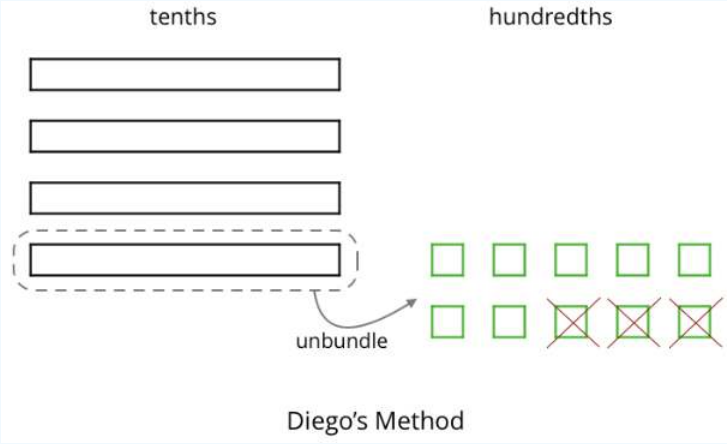


Figure 27.2.4: A base-ten diagram labeled Diego's Method. There are 2 columns for the diagram. The first column header is labeled tenths and there are 4 rectangles. The second column header is labeled hundredths and there are 10 squares in that column. The last rectangle is circled with a dashed line and an arrow pointing from the rectangle to the column of squares is labeled unbundle. The last three squares are crossed out.

- Noah started by drawing 4 rectangles to represent 0.4. He then crossed out 3 of rectangles to represent the subtraction, leaving 1 rectangle in his diagram.

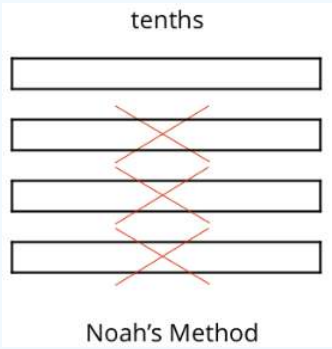


Figure 27.2.5

1. Do you agree that either diagram correctly represents $0.4 - 0.03$? Discuss your reasoning with a partner.
2. To represent $0.4 - 0.03$, Elena drew another diagram. She also started by drawing 4 rectangles. She then replaced all 4 rectangles with 40 squares and crossed out 3 squares to represent subtraction of 0.03, leaving 37 squares in her diagram. Is her diagram correct? Discuss your reasoning with a partner.

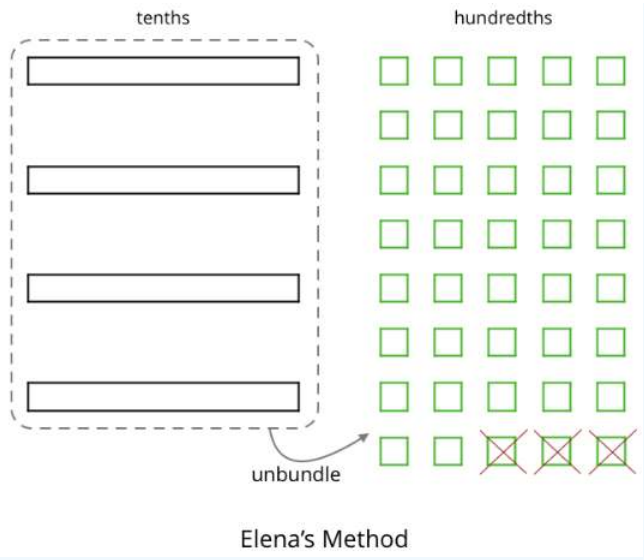


Figure 27.2.6:A base-ten diagram labeled Elena's Method. There are 2 columns for the diagram. The first column header is labeled tenths and there are 4 rectangles. The second column header is labeled hundredths and there are 40 squares in that column. All four rectangles are circled with a dashed line and an arrow pointing from the rectangles to the column of squares is labeled unbundle. The last three squares are crossed out.

3. Find each difference. If you get stuck, you can use the applet to represent each expression and find its value.
   a. $0.3 - 0.05$
   b. $2.1 - 0.4$
   c. $1.03 - 0.06$
   d. $0.02 - 0.007$

Be prepared to explain your reasoning.

- The applet has tools that create each of the base-ten blocks. This time you need to decide the value of each block before you begin.
- Select a Block tool, and then click on the screen to place it.
- Click on the Move tool (the arrow) when you are done choosing blocks.
- Subtract by deleting with the delete tool (the trash can), not by crossing out.

Are you ready for more?

A distant, magical land uses jewels for their bartering system. The jewels are valued and ranked in order of their rarity. Each jewel is worth 3 times the jewel immediately below it in the ranking. The ranking is red, orange, yellow, green, blue, indigo, and violet. So a red jewel is worth 3 orange jewels, a green jewel is worth 3 blue jewels, and so on.

At the Auld Shoppe, a shopper buys items that are worth 2 yellow jewels, 2 green jewels, 2 blue jewels, and 1 indigo jewel. If they came into the store with 1 red jewel, 1 yellow jewel, 2 green jewels, 1 blue jewel, and 2 violet jewels, what jewels do they leave with? Assume the shopkeeper gives them their change using as few jewels as possible.

Summary

Base-ten diagrams can help us understand subtraction as well. Suppose we are finding $0.23 - 0.07$. Here is a diagram showing 0.23, or 2 tenths and 3 hundredths.

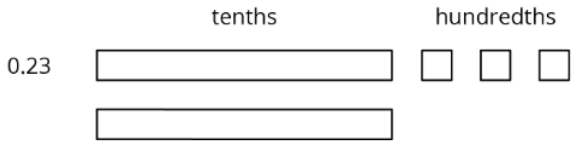


Figure 27.2.7

Subtracting 7 hundredths means removing 7 small squares, but we do not have enough to remove. Because 1 tenth is equal to 10 hundredths, we can “unbundle” (or decompose) one of the tenths (1 rectangle) into 10 hundredths (10 small squares).

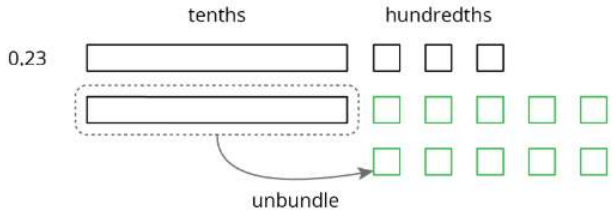


Figure 27.2.8: Base ten diagram. 0 point 23. Two rectangles in the tenths column. 3 small squares in the hundredths column. A dotted rectangle is drawn around one of the rectangles with an arrow to 10 small squares. The arrow is labeled unbundle.

We now have 1 tenth and 13 hundredths, from which we can remove 7 hundredths.



Figure 27.2.9: Base ten diagram. 0 point 23. One rectangle in the tenths column. 13 small squares in the hundredths column. 7 small squares have an X through them. The words subtract 0 point 0 7 is below the small squares.

We have 1 tenth and 6 hundredths remaining, so $0.23 - 0.07 = 0.16$.

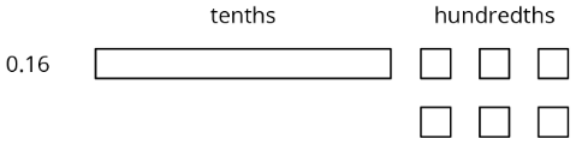


Figure 27.2.10

Here is a vertical calculation of $0.23 - 0.07$.

$$\begin{array}{r} {\scriptstyle 1}{\scriptstyle 13} \\ 0.\cancel{2}\,\cancel{3} \\ -\quad 0.0\;7 \\ \hline 0.1\;6 \end{array}$$

Figure 27.2.11: Vertical subtraction. First line. 0 point 23. The 2 is crossed out and has a 1 above it. The 3 is crossed out and has 13 above it. Second line. Minus 0 point 0 0 7. Horizontal line. Third line. 0 point 17.

Notice how this representation also shows a tenth is unbundled (or decomposed) into 10 hundredths in order to subtract 7 hundredths.

This works for any decimal place. Suppose we are finding $0.023 - 0.007$. Here is a diagram showing 0.023.

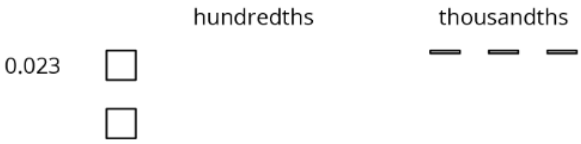


Figure 27.2.12

We want to remove 7 thousandths (7 small rectangles). We can “unbundle” (or decompose) one of the hundredths into 10 thousandths.

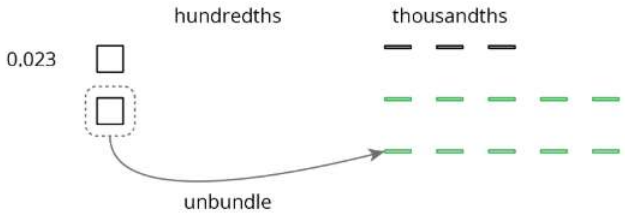


Figure 27.2.13: Base 10 diagram. 0 point 0 2 3. Two small squares in the hundredths column. Three small rectangles in the thousandths column. A square is drawn around 1 small square. An arrow is drawn to 10 small rectangles. The arrow is labeled bundle.

Now we can remove 7 thousandths.

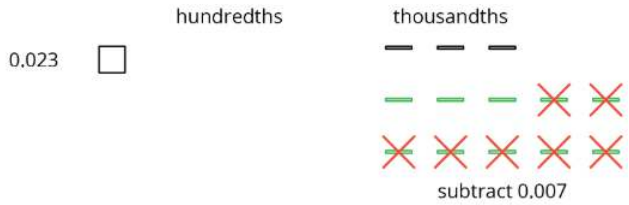


Figure 27.2.14: Base 10 diagram. 0 point 0 2 3. One small square in the hundredths column. 13 small rectangles in the thousandths column. 7 small rectangles have an X through them. Below the small rectangles are the words subtract 0 point 0 0 7.

We have 1 hundredth and 6 thousandths remaining, so $0.023 - 0.007 = 0.016$

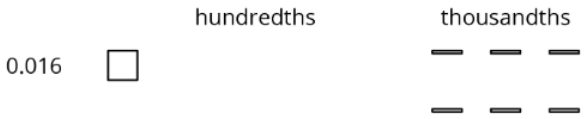


Figure 27.2.15

Here is a vertical calculation of $0.023 - 0.007$.

Figure 27.2.16: Vertical subtraction. First line. 0 point 0 2 3. The 2 is crossed out and has a 1 above it. The 3 is crossed out and has 13 above it. Second line. Minus 0 point 0 0 7. Horizontal line. Third line. 0 point 0 1 6.

Practice

Exercise 27.2.4

Here is a base-ten diagram that represents 1.13. Use the diagram to find $1.13 - 0.46$.

Explain or show your reasoning.

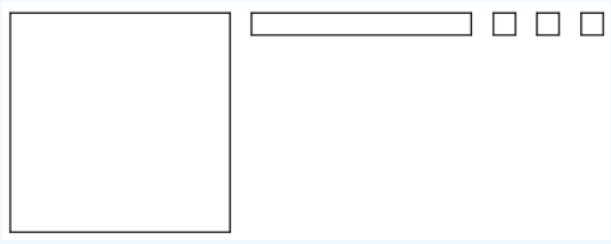


Figure 27.2.17

Exercise 27.2.5

Compute the following sums. If you get stuck, consider drawing base-ten diagrams.

1. $0.027 + 0.004$
2. $0.203 + 0.01$
3. $1.2 + 0.145$

Exercise 27.2.6

A student said we cannot subtract 1.97 from 20 because 1.97 has two decimal digits and 20 has none. Do you agree with him? Explain or show your reasoning.

Exercise 27.2.7

Decide which calculation shows the correct way to find $0.3 - 0.006$ and explain your reasoning.

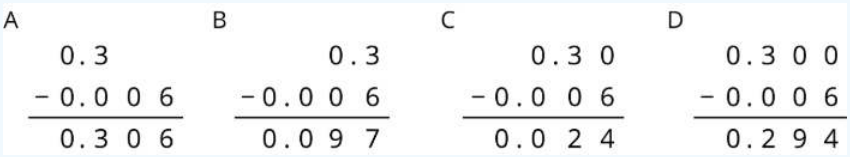


Figure 27.2.18: 4 subtraction problems. problem a. 3 tenths - 6 thousandths = 306 thousandths. problem b. 3 tenths - 6 thousandths = 97 thousandths. problem c. 3 tenths - 6 thousandths = 24 thousandths. problem d. 3 tenths - 6 thousandths = 294 thousandths.

Exercise 27.2.8

Complete the calculations so that each shows the correct difference.

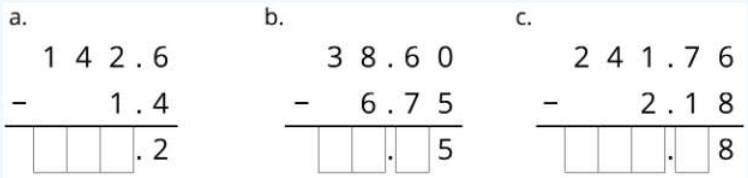


Figure 27.2.19: 3 decimal subtraction problems. Problem a. 14 and 6 tenths -1 and 4 tenths =blank blank blank point 2. Problem b. 38 and 60 hundredths -6 and 75 hundredths = blank blank point blank 5. Problem c. 241 and 76 hundredths -2 and 18 hundredths = blank blank blank point blank 8.

Exercise 27.2.9

The school store sells pencils for $0.30 each, hats for $14.50 each, and binders for $3.20 each. Elena would like to buy 3 pencils, a hat, and 2 binders. She estimated that the cost will be less than $20.

1. Do you agree with her estimate? Explain your reasoning.
2. Estimate the number of pencils could she buy with $5. Explain or show your reasoning.

(From Unit 5.1.1)

Exercise 27.2.10

A rectangular prism measures $7\frac{1}{2}$ cm by 12 cm by $15\frac{1}{2}$ cm.

1. Calculate the number of cubes with edge length $\frac{1}{2}$ cm that fit in this prism.
2. What is the volume of the prism in cm^3? Show your reasoning. If you are stuck, think about how many cubes with $\frac{1}{2}$-cm edge lengths fit into 1 cm^2.

(From Unit 4.4.4)

Exercise 27.2.11

At a constant speed, a car travels 75 miles in 60 minutes. How far does the car travel in 18 minutes? If you get stuck, consider using the table.

Table 27.2.1

minutes	distance in miles

minutes	distance in miles
60	75
6	
18	

(From Unit 2.4.2)

27.2: Adding and Subtracting Decimals with Few Non-Zero Digits is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

27.3: Adding and Subtracting Decimals with Many Non-Zero Digits

Lesson

Let's practice adding and subtracting decimals.

Exercise 27.3.1: The Cost of a Photo Print

1. Here are three ways to write a subtraction calculation. What do you notice? What do you wonder?

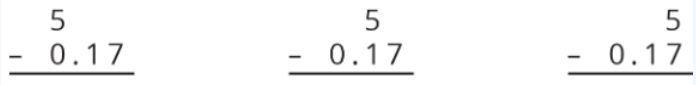


Figure 27.3.1

2. Clare bought a photo for 17 cents and paid with a $5 bill. Look at the previous question. Which way of writing the numbers could Clare use to find the change she should receive? Be prepared to explain how you know.
3. Find the amount of change that Clare should receive. Show your reasoning, and be prepared to explain how you calculate the difference of 0.17 and 5.

Exercise 27.3.2: Decimals All Around

1. Find the value of each expression. Show your reasoning.
   a. $11.3 - 9.5$
   b. $318.8 - 94.63$
   c. $0.02 - 0.0116$
2. Discuss with a partner:
   - Which method or methods did you use in the previous question? Why?
   - In what ways were your methods effective? Was there an expression for which your methods did not work as well as expected?
3. Lin's grandmother ordered needles that were 0.3125 inches long to administer her medication, but the pharmacist sent her needles that were 0.6875 inches long. How much longer were these needles than the ones she ordered? Show your reasoning.
4. There is 0.162 liter of water in a 1-liter bottle. How much more water should be put in the bottle so it contains exactly 1 liter? Show your reasoning.
5. One micrometer is 1 millionth of a meter. A red blood cell is about 7.5 micrometers in diameter. A coarse grain of sand is about 70 micrometers in diameter. Find the difference between the two diameters in *meters*. Show your reasoning.

Exercise 27.3.3: Missing Numbers

Write the missing digits in each calculation so that the value of each sum or difference is correct. Be prepared to explain your reasoning.

1.

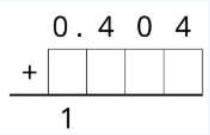


Figure 27.3.2: 404 thousandth + blank, blank, blank, blank = 1, blank, blank, blank.

2.

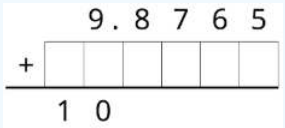


Figure 27.3.3: 9 and 8765 ten thousandths + blank, blank, blank, blank, blank, blank = 1, 0, blank, blank, blank, blank.

3.

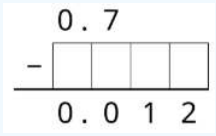


Figure 27.3.4: Third calculation, 0 point 7 minus an unknown quantity with four digits is equal to 0 point 0 1 2.

4.

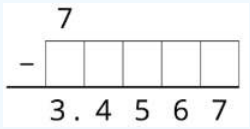


Figure 27.3.5: Fourth calculation, 7 minus an unknown quantity with five digits is equal to 3 point 4 5 6 7.

5.

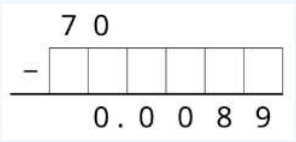


Figure 27.3.6: Fifth calculation, 70 minus an unknown quantity with six digits is equal to 0 point 0 0 8 9.

Are you ready for more?

In a crypt arithmetic puzzle, the digits 0-9 are represented using the first 10 letters of the alphabet. Use your understanding of decimal addition to determine which digits go with the letters A, B, C, D, E, F, G, H, I, and J. How many possibilities can you find?

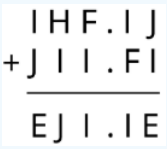


Figure 27.3.7

Summary

Base-ten diagrams work best for representing subtraction of numbers with few non-zero digits, such as $0.16 - 0.09$. For numbers with many non-zero digits, such as $.25103 - 0.04671$, it would take a long time to draw the base-ten diagram. With vertical calculations, we can find this difference efficiently.

Thinking about base-ten diagrams can help us make sense of this calculation.

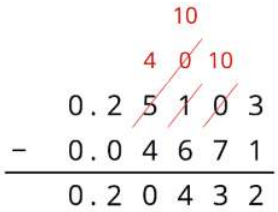


Figure 27.3.8: A setup for the subtraction calculation 0 point 2 5 1 0 3 subtract 0 point 0 4 6 7 1 results in 0 point 1 0 4 3 2. The number 0 point 2 5 1 0 3 is on top with the subtract 0 point 0 4 6 7 1 beneath, and the 0 from the first number lines up vertically with the 0 from the second number, the 2 from the first number lines up vertically with the 0 from the second, the 5 from the first number lines up vertically with the 4 from the second, and so on. The 1 in the thousandths place of the first number is unbundled to make ten groups of ten thousandths. The five in the hundredths place has 1 unbundled to make 4 hundredths and 10 thousandths.

The thousandth in 0.25103 is unbundled (or decomposed) to make 10 ten-thousandths so that we can subtract 7 ten-thousandths. Similarly, one of the hundredths in 0.25103 is unbundled (or decomposed) to make 10 thousandths.

Practice

Exercise 27.3.4

For each subtraction problem, circle the correct calculation.

1. $7.2 - 3.67$
2. $16 - 1.4$

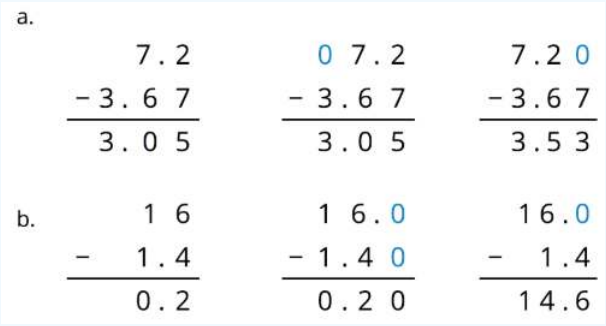


Figure 27.3.9: Option a and option b, three decimal subtraction problems for each option. Option a, on left, 7 and 2 tenths - 3 and 67 hundredths = 3 and 5 hundredths. in middle, 0, 7 and 2 tenths - 3 and 67 hundredths = 3 and 5 hundredths. on right, 7 and 20 hundredths - 3 and 67 hundredths = 3 and 53 hundredths. Option b, on left, 16 - 1 and 4 tenths = 2 tenths. in middle, 16 and 0 tenths - 1 and 40 hundredths = 20 hundredths. on right, 16 and 0 tenths - 1 and 4 tenths = 14 and 6 tenths.

Exercise 27.3.5

Explain how you could find the difference of 1 and 0.1978.

Exercise 27.3.6

A bag of chocolates is labeled to contain 0.384 pound of chocolates. The actual weight of the chocolates is 0.3798 pound.

1. Are the chocolates heavier or lighter than the weight stated on the label? Explain how you know.
2. How much heavier or lighter are the chocolates than stated on the label? Show your reasoning.

Exercise 27.3.7

Complete the calculations so that each shows the correct sum.

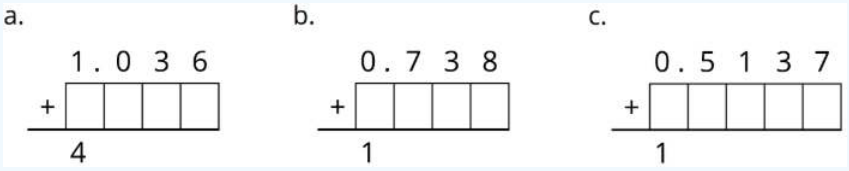


Figure 27.3.10: 3 calculations with missing digits, a, b, and c. a, 1 point 0 3 6 plus an unknown quantity with four digits is equal to 4. b, 0 point 7 3 8 plus an unknown quantity with four digits is equal to 1. c, 0 point 5 1 3 7 plus an unknown quantity with five digits is equal to 1.

Exercise 27.3.8

A shipping company is loading cube-shaped crates into a larger cube-shaped container. The smaller cubes have side lengths of $2\frac{1}{2}$ feet, and the larger shipping container has side lengths of 10 feet. How many crates will fit in the large shipping container? Explain your reasoning.

(From Unit 4.4.3)

https://math.libretexts.org/@go/page/40299

Exercise 27.3.9

For every 9 customers, the chef prepares 2 loaves of bread.

1. Here is double number line showing varying numbers of customers and the loaves prepared. Complete the missing information.

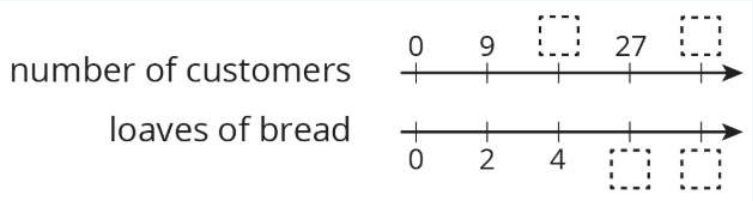


Figure 27.3.11

2. The same information is shown on a table. Complete the missing information.

Table 27.3.1

customers	loaves
9	2
	4
27	
	14
1	

Use either representation to answer these questions.

- How many loaves are needed for 63 customers?
- How many customers are there if the chef prepares 20 loaves?
- How much of a loaf is prepared for each customer?

(From Unit 2.4.3)

27.3: Adding and Subtracting Decimals with Many Non-Zero Digits is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

Section 28: Multiplying Decimals

This page titled Section 28: Multiplying Decimals is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

28.1: Decimal Points in Products

Lesson

Let's look at products that are decimals.

Exercise 28.1.1: Multiplying by 10

1. In which equation is the value of x the largest?

$$x \cdot 10 = 810 \qquad x \cdot 10 = 81 \qquad x \cdot 10 = 8.1 \qquad x \cdot 10 = 0.81$$

2. How many times the size of 0.81 is 810?

Exercise 28.1.2: Fractionally Speaking: Powers of Ten

Work with a partner. One person solves the problems labeled "Partner A" and the other person solves those labeled "Partner B." Then compare your results.

1. Find each product or quotient. Be prepared to explain your reasoning.

   Partner A

   a. $250 \cdot \frac{1}{10}$
   b. $250 \cdot \frac{1}{100}$
   c. $48 \div 10$
   d. $48 \div 100$

   Partner B

   a. $250 \div 10$
   b. $250 \div 100$
   c. $48 \cdot \frac{1}{10}$
   d. $48 \cdot \frac{1}{100}$

2. Use your work in the previous problems to find $720 \cdot (0.1)$ and $720 \cdot (0.01)$. Explain your reasoning.
   Pause here for a class discussion.
3. Find each product. Show your reasoning.

   a. $36 \cdot (0.1)$
   b. $(24.5) \cdot (0.1)$
   c. $(1.8) \cdot (0.1)$
   d. $54 \cdot (0.01)$
   e. $(9.2) \cdot (0.01)$

4. Jada says: "If you multiply a number by 0.001, the decimal point of the number moves three places to the left." Do you agree with her? Explain your reasoning.

Exercise 28.1.3: Fractionally Speaking: Multiples of Powers of Ten

1. Select **all** expressions that are equivalent to $(0.6) \cdot (0.5)$. Be prepared to explain your reasoning.

   a. $6 \cdot (0.1) \cdot 5 \cdot (0.1)$
   b. $6 \cdot (0.01) \cdot 5 \cdot (0.1)$
   c. $6 \cdot \frac{1}{10} \cdot 5 \cdot \frac{1}{10}$
   d. $6 \cdot \frac{1}{1{,}000} \cdot 5 \cdot \frac{1}{100}$
   e. $6 \cdot (0.001) \cdot 5 \cdot (0.01)$
   f. $6 \cdot 5 \cdot \frac{1}{10} \cdot \frac{1}{10}$
   g. $\frac{6}{10} \cdot \frac{5}{10}$

2. Find the value of $(0.6) \cdot (0.5)$. Show your reasoning.

3. Find the value of each product by writing and reasoning with an equivalent expression with fractions.
   a. $(0.3) \cdot (0.02)$
   b. $(0.7) \cdot (0.05)$

Are you ready for more?

Ancient Romans used the letter I for 1, V for 5, X for 10, L for 50, C for 100, D for 500, and M for 1,000. Write a problem involving merchants at an agora, an open-air market, that uses multiplication of numbers written with Roman numerals.

Summary

We can use fractions like $\frac{1}{10}$ and $\frac{1}{100}$ to reason about the location of the decimal point in a product of two decimals.

Let's take $24 \cdot (0.1)$ as an example. There are several ways to find the product:

- We can interpret it as 24 groups of 1 tenth (or 24 tenths), which is 2.4.
- We can think of it as $24 \cdot \frac{1}{10}$, which is equal to $\frac{24}{10}$ (and also equal to 2.4).
- Multiplying by $\frac{1}{10}$ has the same result as dividing by 10, so we can also think of the product as $24 \div 0$, which is equal to 2.4.

Similarly, we can think of $(0.7) \cdot (0.09)$ as 7 tenths times 9 hundredths, and write:

$$\left(7 \cdot \frac{1}{10}\right) \cdot \left(9 \cdot \frac{1}{100}\right)$$

We can rearrange whole numbers and fractions:

$$(7 \cdot 9) \cdot \left(\frac{1}{10} \cdot \frac{1}{100}\right)$$

This tells us that $(0.7) \cdot (0.09) = 0.063$.

$$63 \cdot \frac{1}{1{,}000} = \frac{63}{1{,}000}$$

Here is another example: To find $(1.5) \cdot (0.43)$, we can think of 1.5 as 15 tenths and 0.43 as 43 hundredths. We can write the tenths and hundredths as fractions and rearrange the factors.

$$\left(15 \cdot \frac{1}{10}\right) \cdot \left(43 \cdot \frac{1}{100}\right) = 15 \cdot 43 \cdot \frac{1}{1{,}000}$$

Multiplying 15 and 43 gives us 645, and multiplying $\frac{1}{10}$ and $\frac{1}{100}$ gives us $\frac{1}{1{,}000}$. So $(1.5) \cdot (0.43)$ is $645 \cdot \frac{1}{1{,}000}$, which is 0.645.

Practice

Exercise 28.1.4

1. Find the product of each number and $\frac{1}{100}$

   $122.1 \qquad 11.8 \qquad 1350.1 \qquad 1.704$

2. What happens to the decimal point of the original number when you multiply it by $\frac{1}{100}$? Why do you think that is? Explain your reasoning.

Exercise 28.1.5

Which expression has the same value as $(0.06) \cdot (0.154)$? Select **all** that apply.

A. $6 \cdot \frac{1}{100} \cdot 154 \cdot \frac{1}{1{,}000}$
B. $6 \cdot 154 \cdot \frac{1}{100{,}000}$
C. $6 \cdot (0.1) \cdot 154 \cdot (0.01)$
D. $6 \cdot 154 \cdot (0.00001)$
E. 0.00924

Exercise 28.1.6

Calculate the value of each expression by writing the decimal factors as fractions, then writing their product as a decimal. Show your reasoning.

1. $(0.01) \cdot (0.02)$
2. $(0.3) \cdot (0.2)$
3. $(1.2) \cdot 5$
4. $(0.9) \cdot (1.1)$
5. $(1.5) \cdot 2$

Exercise 28.1.7

Write three numerical expressions that are equivalent to $(0.0004) \cdot (0.005)$.

Exercise 28.1.8

Calculate each sum.

1. $33.1 + 1.95$
2. $1.075 + 27.105$
3. $0.401 + 9.28$

(From Unit 5.2.2)

Exercise 28.1.9

Calculate each difference. Show your reasoning.

1. $13.2 - 1.78$
2. $23.11 - 0.376$
3. $0.9 - 0.245$

(From Unit 5.2.3)

Exercise 28.1.10

On the grid, draw a quadrilateral *that is not a rectangle* that has an area of 18 square units. Show how you know the area is 18 square units.

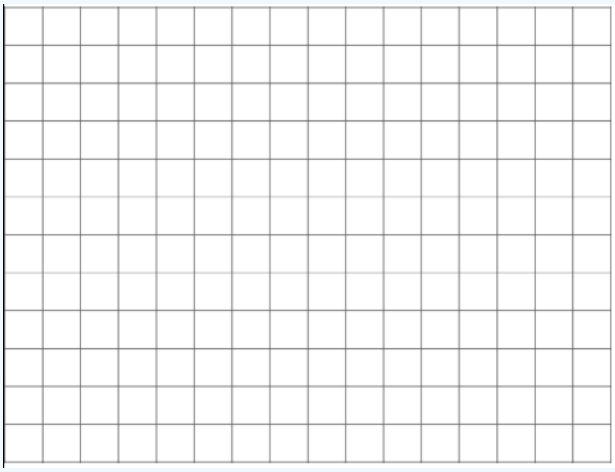


Figure 28.1.1

(From Unit 1.1.3)

28.1: Decimal Points in Products is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

28.2: Methods for Multiplying Decimals

Lesson

Let's look at some ways we can represent multiplication of decimals.

Exercise 28.2.1: Equivalent Expressions

Write as many expressions as you can think of that are equal to 0.6. Do not use addition or subtraction.

Exercise 28.2.2: Using Properties of Numbers to Reason about Multiplication

Elena and Noah used different methods to compute $(0.23) \cdot (1.5)$. Both calcuations were correct.

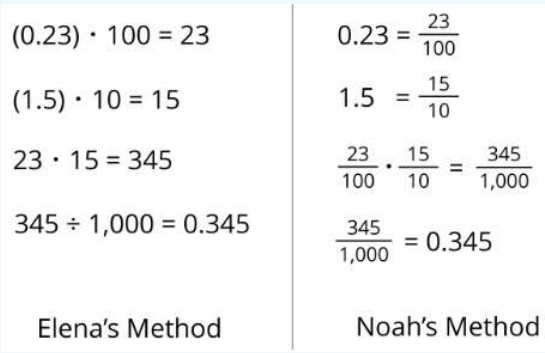


Figure 28.2.1: Elena's method. 0 point 2 3 times 100 equals 23. 1 point 5 times 10 equals 15. 23 times 15 equals 345. 345 divided by 1000 equals 0 point 3 4 5. Noah's method. 0 point 2 3 equals the fraction 23 over 100. 1 point 5 equals the fraction 15 over 10. The fraction 23 over 100 times the fraction 15 over 10 equals the fractions 345 over 1000. The fraction 345 over 1000 equals 0 point 3 4 5.

1. Analyze the two methods, then discuss these questions with your partner.
   - Which method makes more sense to you? Why?
   - What might Elena do to compute $(0.16) \cdot (0.03)$? What might Noah do to compute $(0.16) \cdot (0.03)$? Will the two methods result in the same value?
2. Compute each product using the equation $21 \cdot 47 = 987$ and what you know about fractions, decimals, and place value. Explain or show your reasoning.
   a. $(2.1) \cdot (4.7)$
   b. $21 \cdot (0.047)$
   c. $(0.021) \cdot (4.7)$

Exercise 28.2.3: Using Area Diagrams to Reason about Multiplication

1. In the diagram, the side length of each square is 0.1 unit.
   a. Explain why the area of each square is *not* 0.1 square unit.

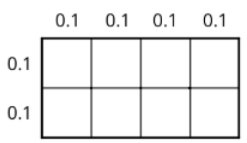


Figure 28.2.2

   b. How can you use the area of each square to find the area of the rectangle? Explain or show your reasoning.
   c. Explain how the diagram shows that the equation $(0.4) \cdot (0.2) = 0.08$ is true.
2. Label the squares with their side lengths so the area of this rectangle represents $40 \cdot 20$.

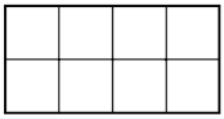


Figure 28.2.3

a. What is the area of each square?
b. Use the squares to help you find $40 \cdot 20$. Explain or show your reasoning.

3. Label the squares with their side lengths so the area of this rectangle represents $(0.04) \cdot (0.02)$. Next, use the diagram to help you find $(0.04) \cdot (0.02)$. Explain or show your reasoning.

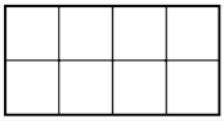


Figure 28.2.4

Summary

Here are three other ways to calculate a product of two decimals such as $(0.04) \cdot (0.07)$.

- First, we can multiply each decimal by the same power of 10 to obtain whole-number factors.

$$\begin{aligned}(0.04) \cdot 100 &= 4\\ (0.07) \cdot 100 &= 7\\ 4 \cdot 7 &= 28\end{aligned}$$

Because we multiplied both 0.04 and 0.07 by 100 to get 4 and 7, the product 28 is $(100 \cdot 100)$ times the original product, so we need to divide 28 by 10,000.

$$28 \div 10,000 = 0.0028$$

- Second, we can write each decimal as a fraction, $0.04 = \frac{4}{100}$ and $0.07 = \frac{7}{100}$, and multiply them.

$$\frac{4}{100} \cdot \frac{7}{100} = \frac{28}{10,000} = 0.0028$$

- Third, we can use an area model. The product $(0.04) \cdot (0.07)$ can be thought of as the area of a rectangle with side lengths of 0.04 unit and 0.07 unit.

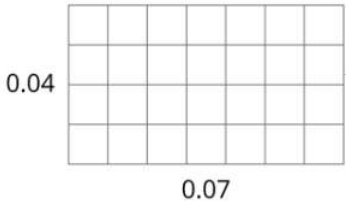


Figure 28.2.5

In this diagram, each small square is 0.01 unit by 0.01 unit. The area of each square, in square units, is therefore $\left(\frac{1}{100} \cdot \frac{1}{100}\right)$, which is $\frac{1}{10,000}$.

Because the rectangle is composed of 28 small squares, the area of the rectangle, in square units, must be:

$$28 \cdot \frac{1}{10,000} = \frac{28}{10,000} = 0.0028$$

All three calculations show that $(0.04) \cdot (0.07) = 0.0028$

Practice

Exercise 28.2.4

Find each product. Show your reasoning.

1. $(1.2) \cdot (0.11)$

2. $(0.34) \cdot (0.02)$
3. $120 \cdot (0.002)$

Exercise 28.2.5

You can use a rectangle to represent $(0.3) \cdot (0.5)$.

1. What must the side length of each square represent for the rectangle to correctly represent $(0.3) \cdot (0.5)$?
2. What area is represented by each square?
3. What is $(0.3) \cdot (0.5)$? Show your reasoning.

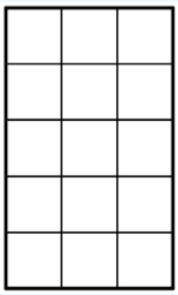


Figure 28.2.6

Exercise 28.2.6

One gallon of gasoline in Buffalo, New York costs \$2.29. In Toronto, Canada, one liter of gasoline costs \$0.91. There are 3.8 liters in one gallon.

1. How much does one gallon of gas cost in Toronto? Round your answer to the nearest cent.
2. Is the cost of gas greater in Buffalo or in Toronto? How much greater?

Exercise 28.2.7

Calculate each sum or difference.

$$10.3 + 3.7 \qquad 20.99 - 4.97 \qquad 15.99 + 23.51 \qquad 1.893 - 0.353$$

(From Unit 5.2.1)

Exercise 28.2.8

Find the value of $\frac{49}{50} \div \frac{7}{6}$ using any method.

(From Unit 4.3.2)

Exercise 28.2.9

Find the area of the shaded region. All angles are right angles. Show your reasoning.

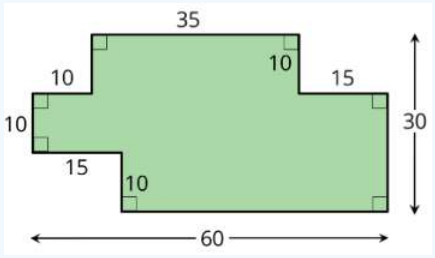


Figure 28.2.7: A multi-sided figure. The sides on top measure 10 units, 35 units, and 15 units. Two of the three sides on the left measure 10 units. One of the two sides on the right measures 10 units. One of the two sides on the bottom measure 15 units. The total width of the figure is 60 units, and the total height is 30 units. All angles are right angles.

(From Unit 1.1.1)

Exercise 28.2.10

1. Priya finds $(1.05) \cdot (2.8)$ by calculating $105 \cdot 28$, then moving the decimal point three places to the left. Why does Priya's method make sense?
2. Use Priya's method to calculate $(1.05) \cdot (2.8)$. You can use the fact that $105 \cdot 28 = 2,940$.
3. Use Priya's method to calculate $(0.0015) \cdot (0.024)$.

28.2: Methods for Multiplying Decimals is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

28.3: Using Diagrams to Represent Multiplication

Lesson

Let's use area diagrams to find products.

Exercise 28.3.1: Estimate the Product

For each of the following products, choose the best estimate of its value. Be prepared to explain your reasoning.

1. $(6.8) \cdot (2.3)$
   - 1.40
   - 14
   - 140
2. $74 \cdot (8.1)$
   - 5.6
   - 56
   - 560
3. $166 \cdot (0.09)$
   - 1.66
   - 16.6
   - 166
4. $(3.4) \cdot (1.9)$
   - 6.5
   - 65
   - 650

Exercise 28.3.2: Connecting Area Diagrams to Calculations with Whole Numbers

1. Here are three ways of finding the area of a rectangle that is 24 units by 13 units.

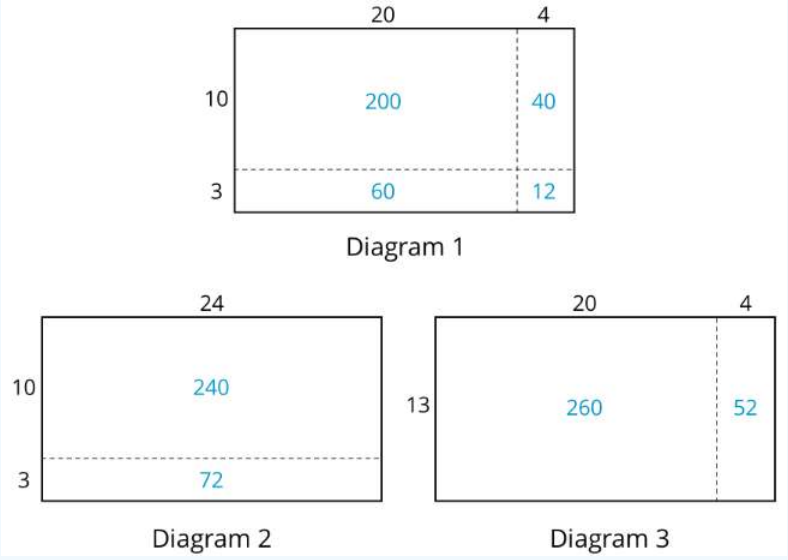


Figure 28.3.1: Diagram 1, rectangle partitioned vertically and horizontally into 4 rectangles. Top left rectangle, vertical side, 10, horizontal side, 20, area, 200. Top right rectangle, horizontal side, 4, area, 40. Bottom left rectangle, vertical side, 3, area 60. Bottom right rectangle, area, 12. Diagram 2, rectangle partitioned horizontally into 2 rectangles. Top rectangle, vertical side, 10, horizontal side, 24, area, 240. Bottom rectangle, vertical side 3, area, 72. Diagram 3, rectangle partitioned vertically into 2 rectangles. Left rectangle, vertical side, 13, horizontal side, 20, area, 260. Right rectangle, horizontal side, 4, area, 52.

Discuss with your partner:

a. What do the diagrams have in common? How are they alike?
b. How are they different?
c. If you were to find the area of a rectangle that is 37 units by 19 units, which of the three ways of decomposing the rectangle would you use? Why?

2. You may be familiar with different ways to write multiplication calculations. Here are two ways to calculate 24 times 13.

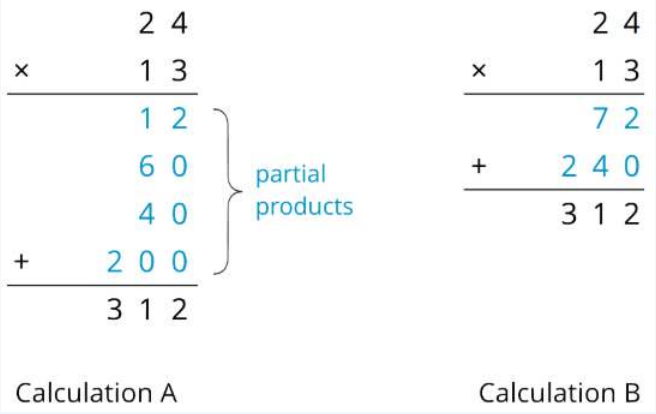


Figure 28.3.2: Two vertical calculations of 24 times 13. Calculation A, 7 rows. First row: 24. Second row: multiplication symbol, 13. Horizontal line. Third row: 12. Fourth row: 60. Fifth row: 40. Sixth row: plus 200. Horizontal line. Seventh row: 312. Calculation B, 5 rows. First row: 24. Second row: multiplication symbol, 13. Horizontal line. Third row: 72. Fourth row: plus 240. Horizontal line. Fifth row: 312.

Discuss with your partner:

a. In Calculation A, how are each of the partial products obtained? For instance, where does the 12 come from?
b. In Calculation B, how are the 72 and 240 obtained?
c. Look at the diagrams in the first question. Which diagram corresponds to Calculation A? Which one corresponds to Calculation B?
d. How are the partial products in Calculation A and the 72 and 240 in Calculation B related to the numbers in the diagrams?

3. Use the two following methods to find the product of 18 and 14, then compare the values obtained.

a. Calculate numerically.

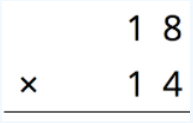


Figure 28.3.3

b. Here is a rectangle that is 18 units by 14 units. Find its area, in square units by decomposing it. Show your reasoning.

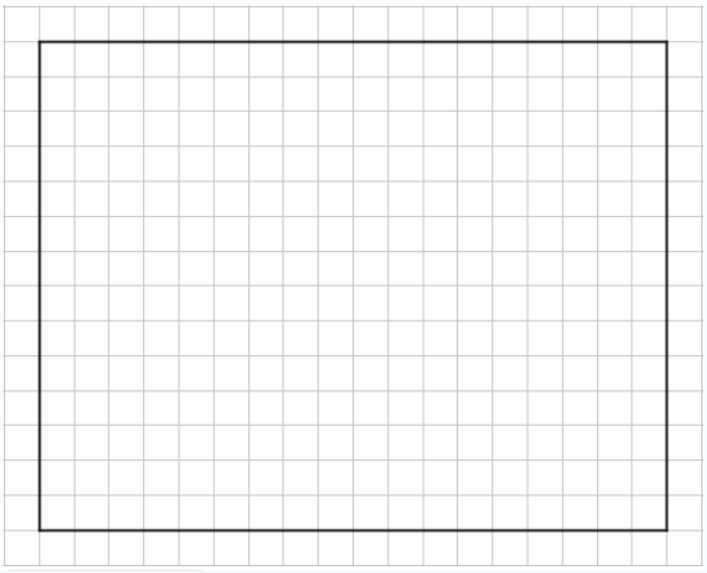


Figure 28.3.4

4. Compare the values of $18 \cdot 14$ that you obtained using the two methods. If they are not the same, check your work.
   - Use the applet to verify your answers and explore your own scenarios. To adjust the values, move the dots on the ends of the segments.

Exercise 28.3.3: Connecting Area Diagrams to Calculations with Decimals

1. You can use area diagrams to represent products of decimals. Here is an area diagram that represents $(2.4) \cdot (1.3)$.

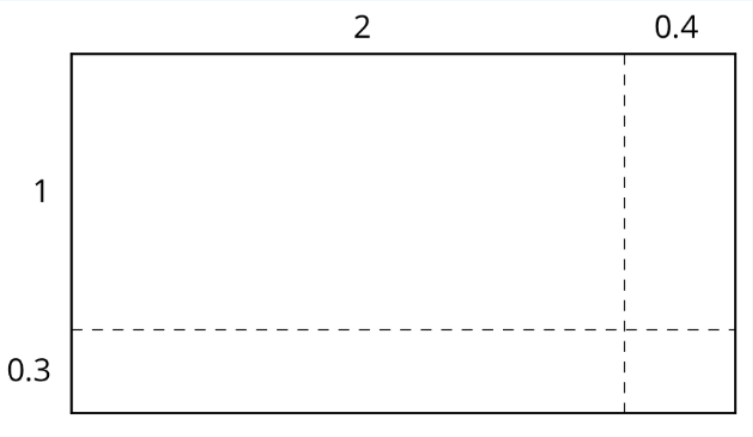


Figure 28.3.5: A rectangle partitioned vertically and horizontally into 4 rectangles. Top left rectangle, vertical side, 1, horizontal side, 2. Top right rectangle, vertical side, 1, horizontal side, 0 point 4. Bottom left rectangle, vertical side, 0 point 3, horizontal side, 2. Bottom right rectangle, vertical side, 0 point 3, horizontal side, 0 point 4.

   a. Find the region that represents $(0.4) \cdot (0.3)$? Label that region with its area of 0.12.
   b. Label each of the other regions with their respective areas.
   c. Find the value of $(2.4) \cdot (1.3)$. Show your reasoning.

2. Here are two ways of calculating 2.4 times 1.3.

https://math.libretexts.org/@go/page/40303

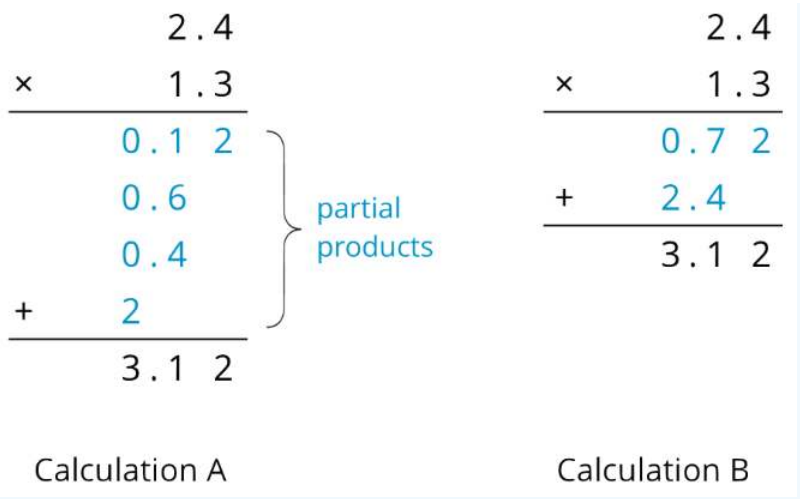


Figure 28.3.6: Two vertical calculations of 2 point 4 times 1 point 3. Calculation A, 7 rows. First row: 2 point 4. Second row: multiplication symbol, 1 point 3. Horizontal line. Third row: 0 point 1 2. Fourth row: 0 point 6. Fifth row: 0 point 4. Sixth row: plus 2. Horizontal line. Seventh row: 3 point 1 2. Calculation B, 5 rows. First row: 2 point 4. Second row: multiplication symbol, 1 point 3. Horizontal line. Third row: 0 point 7 2. Fourth row: plus 2 point 4. Horizontal line. Fifth row: 3 point 1 2.

Analyze the calculations and discuss with a partner:

a. In Calculation A, where does the 0.12 and other partial products come from? In Calculation B, where do the 0.72 and 2.4 come from? How are the other numbers in blue calculated?
b. In each calculation, why are the numbers in blue lined up vertically the way they are?

3. Find the product of $(3.1) \cdot (1.5)$ by drawing and labeling an area diagram. Show your reasoning.
4. Show how to calculate $(3.1) \cdot (1.5)$ using numbers without a diagram. Be prepared to explain your reasoning. If you are stuck, use the examples in a previous question to help you.
5. Use the applet to verify your answers and explore your own scenarios. To adjust the values, move the dots on the ends of the segments.

Are you ready for more?

How many hectares is the property of your school? How many morgens is that?

Exercise 28.3.4: Using the Partial Products Method

1. Label the area diagram to represent $(2.5) \cdot (1.2)$ and to find that product.

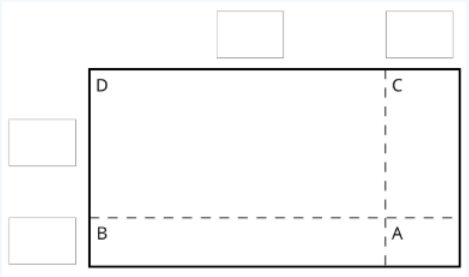


Figure 28.3.7

a. Decompose each number into its base-ten units (ones, tenths, etc.) and write them in the boxes on each side of the rectangle.
b. Label Regions A, B, C, and D with their areas. Show your reasoning.
c. Find the product that the area diagram represents. Show your reasoning.

2. Here are two ways to calculate $(2.5) \cdot (1.2)$. Each number with a box gives the area of one or more regions in the area diagram.

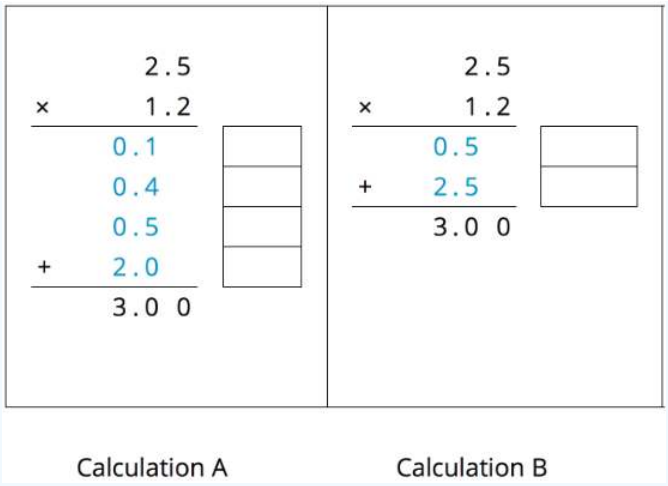


Figure 28.3.8: Two vertical calculations of 2 point 5 times 1 point 2. Calculation A, 7 rows. First row: 2 point 5. Second row: multiplication symbol, 1 point 2. Horizontal line. Third row: 0 point 1. Fourth row: 0 point 4. Fifth row: 0 point 5. Sixth row: plus 2. Horizontal line. Seventh row: 3 point 0 0. Calculation B, 5 rows. First row: 2 point 5. Second row: multiplication symbol, 1 point 2. Horizontal line. Third row: 0 point 5. Fourth row: plus 2 point 5. Horizontal line. Fifth row: 3 point 0 0.

a. In the boxes next to each number, write the letter(s) of the corresponding region(s).
b. In Calculation B, which two numbers are being multiplied to obtain 0.5?
   Which numbers are being multiplied to obtain 2.5?

Summary

Suppose that we want to calculate the product of two numbers that are written in base ten. To explain how, we can use what we know about base-ten numbers and areas of rectangles.

Here is a diagram of a rectangle with side lengths 3.4 units and 1.2 units.

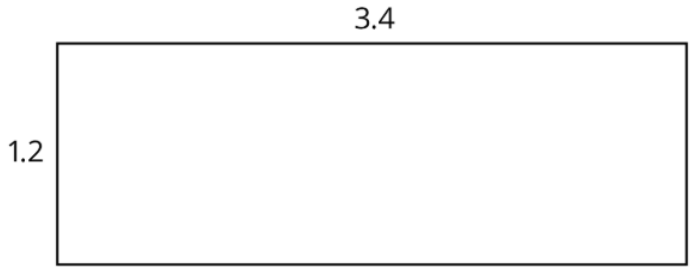


Figure 28.3.9

Its area, in square units, is the product

$$(3.4)\cdot(1.2)$$

To calculate this product and find the area of the rectangle, we can decompose each side length into its base-ten units, $3.4 = 3 + 0.4$ and $1.2 = 1 + 0.2$, decomposing the rectangle into four smaller sub-rectangles.

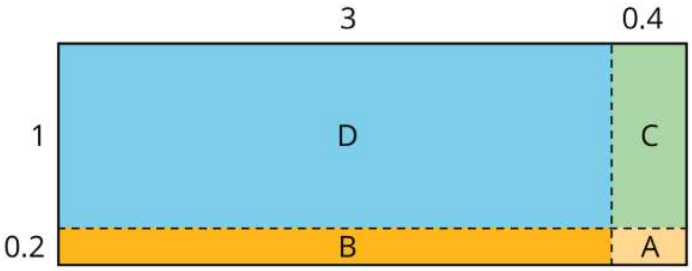


Figure 28.3.10: Area diagram. A rectangle partitioned into 4 rectangles, A, B, C, D. D, vertical side, 1, horizontal side, 3. C, vertical side, 1, horizontal side, 0 point 4. B, vertical side, 0 point 2, horizontal side, 3. A, vertical side, 0 point 2, horizontal side, 0 point 4.

We can rewrite the product and expand it twice:

$$\begin{aligned}(3.4)\cdot(1.2) &= (3+0.4)\cdot(1+0.2)\\ &= (3+0.4)\cdot 1+(3+0.4)\cdot 0.2\\ &= 3\cdot 1+3\cdot(0.2)+(0.4)\cdot 1+(0.4)\cdot(0.2)\end{aligned}$$

In the last expression, each of the four terms is called a partial product. Each partial product gives the area of a sub-rectangle in the diagram. The sum of the four partial products gives the area of the entire rectangle.

We can show the horizontal calculations above as two vertical calculations.

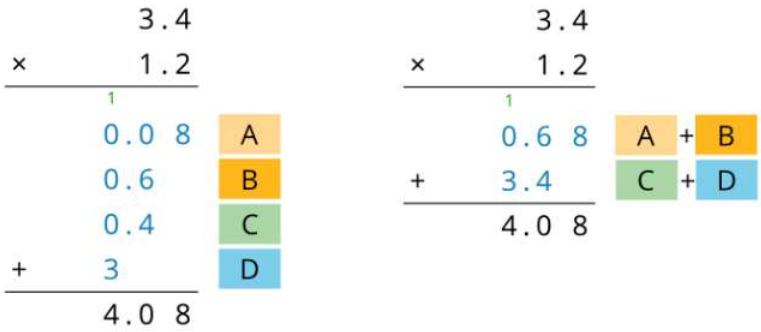


Figure 28.3.11: Two vertical calculations of 3 point 4 times 1 point 2. First calculation, 7 rows. First row: 3 point 4. Second row: multiplication symbol, 1 point 2. Horizontal line. Third row: 0 point 0 8, A. Fourth row: 0 point 6, B. Fifth row: 0 point 4, C. Sixth row: plus 3, D. Horizontal line. Seventh row: 4 point 0 8. Second calculation, 5 rows. First row: 3 point 4. Second row: multiplication symbol, 1 point 2. Horizontal line. Third row: 0 point 6 8, A plus B. Fourth row: plus 3 point 4, C plus D. Horizontal line. Fifth row: 4 point 0 8.

The calculation on the left is an example of the partial products method. It shows the values of each partial product and the letter of the corresponding sub-rectangle. Each partial product gives an area:

- A is 0.2 unit by 0.4 unit, so its area is 0.08 square unit.
- B is 3 unit by 0.2 unit, so its area is 0.6 square unit.
- C is 0.4 unit by 1 unit, so its area is 0.4 square unit.
- D is 3 units by 1 unit, so its area is 3 square units.
- The sum of the partial products is $0.08+0.6+0.4+3$, so the area of the rectangle is 4.08 square units.

The calculation on the right shows the values of two products. Each value gives a combined area of two sub-rectangles:

- The combined regions of A and B have an area of 0.68 square units; 0.68 is the value of $(3+0.4)\cdot 0.2$.
- The combined regions of C and D have an area of 3.4 square units; 3.4 is the value of $(3+0.4)\cdot 1$.
- The sum of the values of two products is $0.68+3.4$, so the area of the rectangle is 4.08 square units.

Practice

Exercise 28.3.5

Here is a rectangle that has been partitioned into four smaller rectangles.

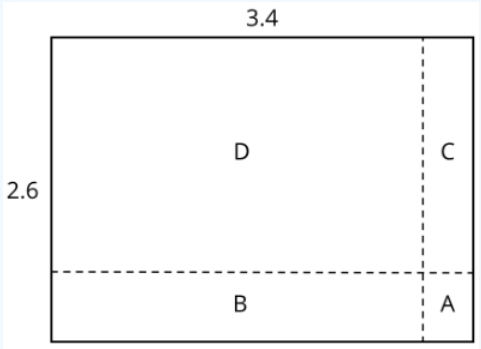


Figure 28.3.12

For each expression, choose the sub-rectangle whose area, in square units, matches the expression.

1. $3 \cdot (0.6)$
2. $(0.4) \cdot 2$
3. $(0.4) \cdot (0.6)$
4. $3 \cdot 2$

Exercise 28.3.6

Here is an area diagram that represents $(3.1) \cdot (1.4)$.

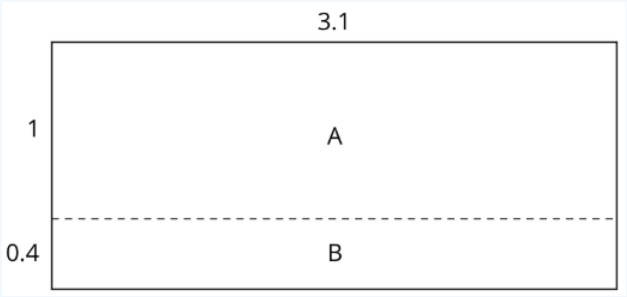


Figure 28.3.13

1. Find the areas of sub-rectangles A and B.
2. What is the area of the 3.1 by 1.4 rectangle?

Exercise 28.3.7

Draw an area diagram to find $(0.36) \cdot (0.53)$. Label and organize your work so that it can be followed by others.

Exercise 28.3.8

Find each product. Show your reasoning.

1. $(2.5) \cdot (1.4)$
2. $(0.64) \cdot (0.81)$

Exercise 28.3.9

Complete the calculations so that each shows the correct sum.

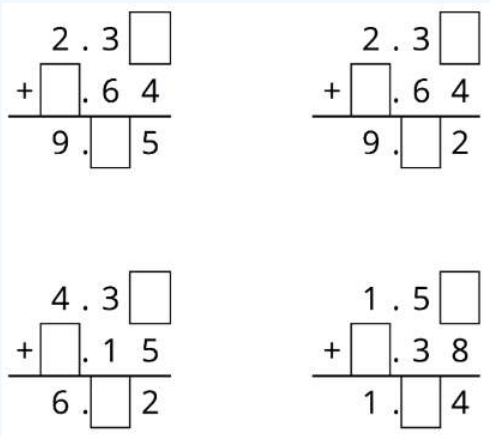


Figure 28.3.14: Four calculations with missing digits. First calculation, 2 point 3 blank plus blank point 6 4 is equal to 9 point blank five. Second calculation, 2 point 3 blank plus blank point 6 4 is equal to 9 point blank 2. Third calculation, 4 point 3 blank plus blank point 1 5 is equal to 6 point blank 2. Fourth calculation, 1 point 5 blank plus blank point 3 8 is equal to 1 point blank 4.

(From Unit 5.2.2)

Exercise 28.3.10

Diego bought 12 mini muffins for $4.20.

1. At this rate, how much would Diego pay for 4 mini muffins?
2. How many mini muffins could Diego buy with $3.00? Explain or show your reasoning. If you get stuck, consider using the table.

Table 28.3.1

number of mini muffins	price in dollars
12	4.20

(From Unit 2.4.2)

28.3: Using Diagrams to Represent Multiplication is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

28.4: Calculating Products of Decimals

Lesson

Let's multiply decimals.

Exercise 28.4.1: Number Talk: Twenty Times a Number

Evaluate mentally.

$20 \cdot 5$

$20 \cdot (0.8)$

$20 \cdot (0.04)$

$20 \cdot (5.84)$

Exercise 28.4.2: Calculating Products of Decimals

1. A common way to find a product of decimals is to calculate a product of whole numbers, then place the decimal point in the product.

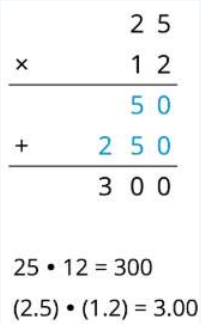


Figure 28.4.1: Vertical calculation of 25 times 12, 5 rows. First row: 25. Second row: multiplication symbol, 12. Horizontal line. Third row: 50. Fourth row: plus 250. Horizontal line. Fifth row: 300. Horizontal calculation, 25 times 12 is equal to 300. Horizontal calculation, 2 point 5 times 1 point 2 is equal to 3 point 0 0.

Here is an example for $(2.5) \cdot (1.2)$.

Use what you know about decimals and place value to explain why the decimal point of the product is placed where it is.

2. Use the method shown in the first equation to calculate each product.
   a. $(4.6) \cdot (0.9)$
   b. $(16.5) \cdot (0.7)$
3. Use area diagrams to check your earlier calculations. For each problem:
   - Decompose each number into its base-ten units and write them in the boxes on each side of the rectangle.
   - Write the area of each lettered region in the diagram. Then find the area of the entire rectangle. Show your reasoning.

   a. $(4.6) \cdot (0.9)$

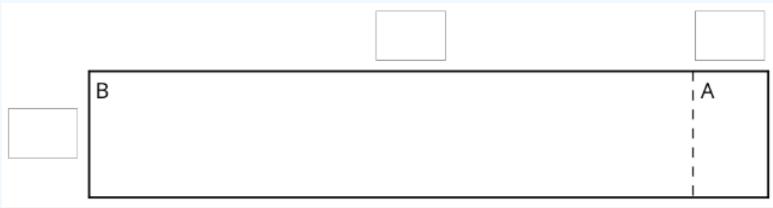


Figure 28.4.2

   b. $(16.5) \cdot (0.7)$

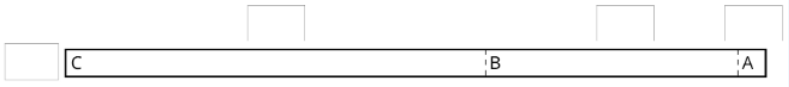


Figure 28.4.3

4. About how many centimeters are in 6.25 inches if 1 inch is about 2.5 centimeters? Show your reasoning.

Exercise 28.4.3: Practicing Multiplication of Decimals

1. Calculate each product. Show your reasoning. If you get stuck, consider drawing an area diagram to help.
   a. $(5.6) \cdot (1.8)$
   b. $(0.008) \cdot (7.2)$
2. A rectangular playground is 18.2 meters by 12.75 meters.
   a. Find its area in square meters. Show your reasoning.
   b. If 1 meter is approximately 3.28 feet, what are the approximate side lengths of the playground in feet? Show your reasoning.

Are you ready for more?

1. Write the following expressions as decimals.
   a. $1 - 0.1$
   b. $1 - 0.1 + 10 - 0.01$
   c. $1 - 0.1 + 10 - 0.01 + 100 - 0.001$
2. Describe the decimal that results as this process continues.
3. What would happen to the decimal if all of the addition and subtraction symbols became multiplication symbols? Explain your reasoning.

Summary

We can use $84 \cdot 43$ and what we know about place value to find $(8.4) \cdot (4.3)$.

Since 8.4 is 84 tenths and 4.3 is 43 tenths, then:

$$(8.4) \cdot (4.3) = \frac{84}{10} \cdot \frac{43}{10} = \frac{84 \cdot 43}{100}$$

That means we can compute $84 \cdot 43$ and then divide by 100 to find $(8.4) \cdot (4.3)$.

$$84 \cdot 43 = 3612$$

$$(8.4) \cdot (4.3) = 36.12$$

Using fractions such as $\frac{1}{10}$, $\frac{1}{100}$, and $\frac{1}{1{,}000}$ allows us to find the product of two decimals using the following steps:

- Write each decimal factor as a product of a whole number and a fraction.
- Multiply the whole numbers.
- Multiply the fractions.
- Multiply the products of the whole numbers and fractions.

We know multiplying by fractions such as $\frac{1}{10}$, $\frac{1}{100}$, and $\frac{1}{1{,}000}$ is the same as dividing by 10, 100, and 1,000, respectively. This means we can move the decimal point in the whole-number product to the left the appropriate number of spaces to correctly place the decimal point.

Practice

Exercise 28.4.4

Here are an unfinished calculation of $(0.54) \cdot (3.8)$ and a 0.54-by-3.8 rectangle.

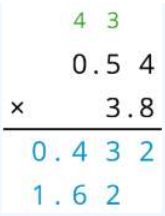


Figure 28.4.4

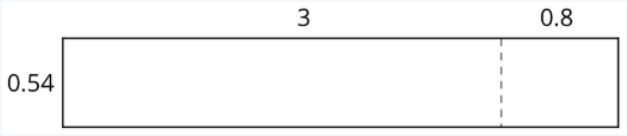


Figure 28.4.5

1. Which part of the rectangle has an area of 0.432? Which part of the rectangle has an area of 1.62? Show your reasoning.
2. What is $(0.54) \cdot (3.8)$?

Exercise 28.4.5

Explain how the product of 3 and 65 could be used to find $(0.03) \cdot (0.65)$.

Exercise 28.4.6

Use vertical calculation to find each product.

1. $(5.4) \cdot (2.4)$
2. $(1.67) \cdot (3.5)$

Exercise 28.4.7

A pound of blueberries costs \$3.98 and a pound of clementines costs \$2.49. What is the combined cost of 0.6 pound of blueberries and 1.8 pounds of clementines? Round your answer to the nearest cent.

Exercise 28.4.8

Complete the calculations so that each shows the correct sum or difference.

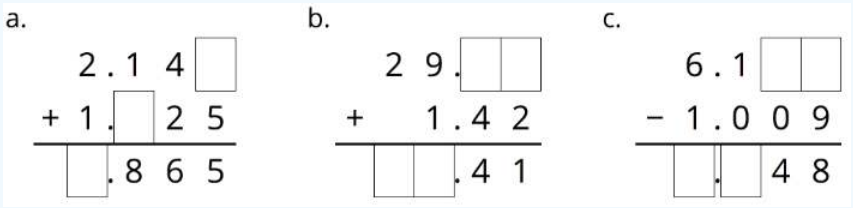


Figure 28.4.6: 2 decimal addition problems and 1 decimal subtraction problem. Problem a. 2 point 1, 4, blank + 1 and blank, 2, 5 = blank and 865 thousandths. Problem b. 29 point blank, blank + 1 and 42 hundredths = blank blank point 4, 1. Problem c. 6 point 1, blank, blank - 1 and 9 thousandths = blank point blank, 4, 8.

(From Unit 5.2.2)

Exercise 28.4.9

Which has a greater value: $7.4 - 0.0022$ or $7.39 - 0.0012$? Show your reasoning.

(From Unit 5.2.3)

https://math.libretexts.org/@go/page/40302

Exercise 28.4.10

Andre is planting saplings (baby trees). It takes him 30 minutes to plant 3 saplings. If each sapling takes the same amount of time to plant, how long will it take Andre to plant 14 saplings? If you get stuck, consider using the table.

Table 28.4.1

number of saplings	time in minutes
3	30
1	
14	

(From Unit 2.4.2)

28.4: Calculating Products of Decimals is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

Section 29: Dividing Decimals

This page titled Section 29: Dividing Decimals is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

29.1: Using the Partial Quotients Method

Lesson

Let's divide whole numbers.

Exercise 29.1.1: Using Base-Ten Diagrams to Calculate Quotients

Elena used base-ten diagrams to find $372 \div 3$. She started by representing 372.

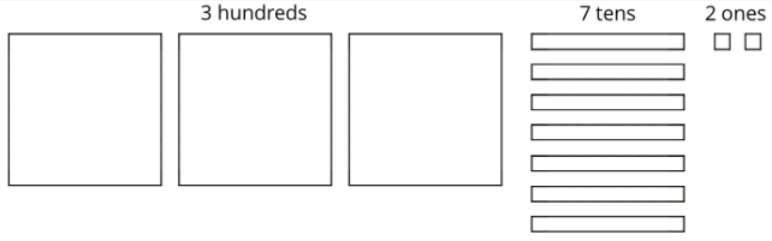


Figure 29.1.1

She made 3 groups, each with 1 hundred. Then, she put the tens and ones in each of the 3 groups. Here is her diagram for $372 \div 3$.

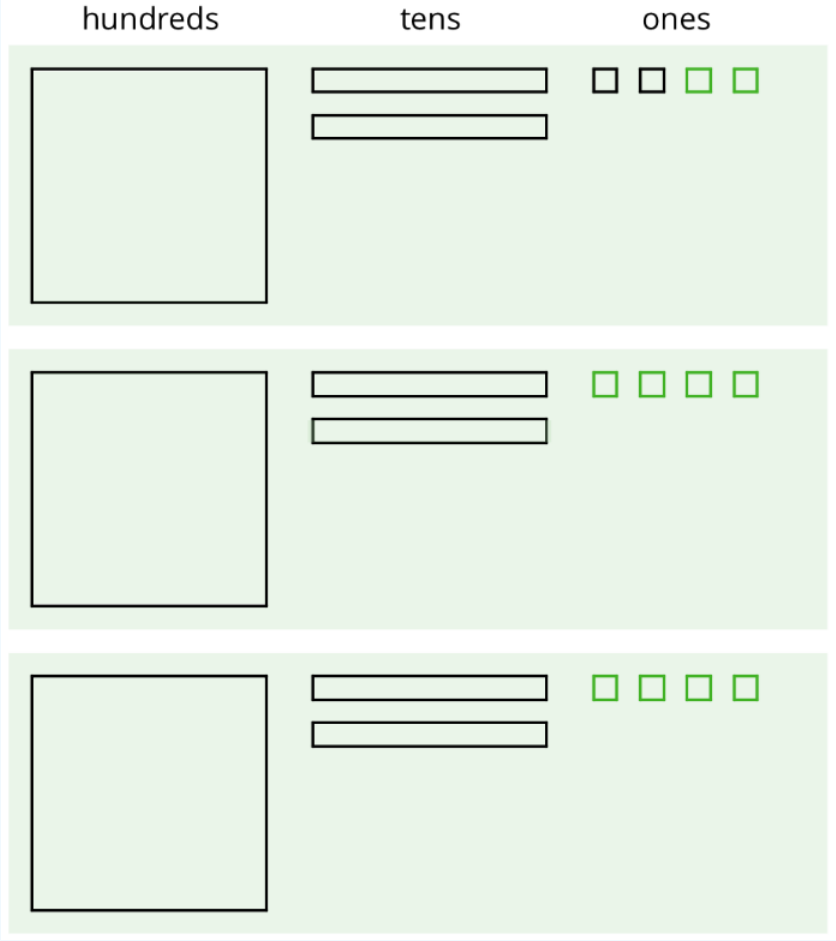


Figure 29.1.2

Discuss with a partner:

- Elena's diagram for 372 has 7 tens. The one for $372 \div 3$ has only 6 tens. Why?
- Where did the extra ones (small squares) come from?

Exercise 29.1.2: Using the Partial Quotients Method to Calculate Quotients

1. Andre calculated $657 \div 3$ using a method that was different from Elena's.

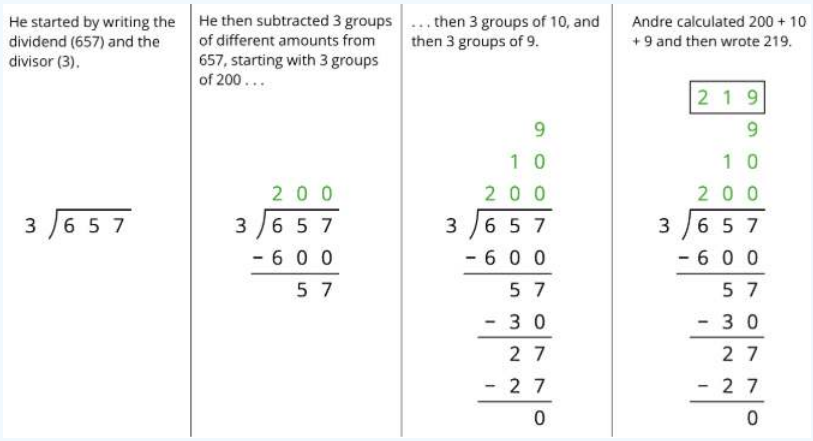


Figure 29.1.3: Method of calculating 657 divided by 3, 4 steps. First step, 1 row, 3, long division symbol with 657 inside. Second step, 4 rows. First row: 200. Second row. 3, long division symbol with 657 inside. Third row: minus 600. Horizontal line. Fourth row: 57. Third step, 10 rows. First row: 9. Second row: 10. Third row: 200. Fourth row: 3, long division symbol with 657 inside. Fifth row: minus 600. Horizontal line. Sixth row: 57. Seventh row: minus 30. Horizontal line. Eighth row: 27. Ninth row: minus 27. Horizontal line. Tenth row: 0. Fourth step: Same as third step, but with 219 at the top row.

   a. Andre subtracted 600 from 657. What does the 600 represent?
   b. Andre wrote 10 above the 200, and then subtracted 30 from 57. How is the 30 related to the 10?
   c. What do the numbers 200, 10, and 9 represent?
   d. What is the meaning of the 0 at the bottom of Andre's work?

2. How might Andre calculate $896 \div 4$? Explain or show your reasoning.

Exercise 29.1.3: What's the Quotient?

1. Find the quotient of $1,332 \div 9$ using one of the methods you have seen so far. Show your reasoning.
2. Find each quotient and show your reasoning. Use the partial quotients method at least once.
   a. $1,115 \div 5$
   b. $665 \div 7$
   c. $432 \div 16$

Summary

We can find the quotient $345 \div 3$ in different ways.

One way is to use a base-ten diagram to represent the hundreds, tens, and ones and to create equal-sized groups.

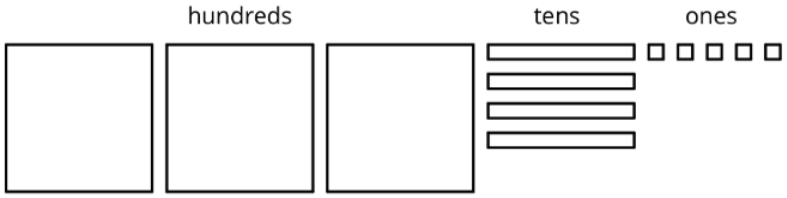


Figure 29.1.4

We can think of the division by 3 as splitting up 345 into 3 equal groups.

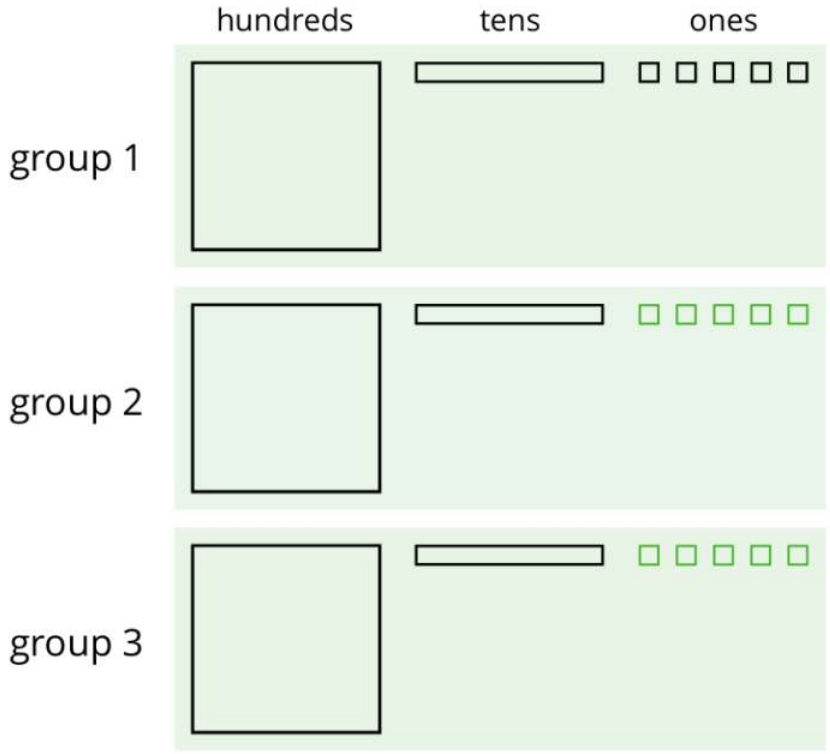


Figure 29.1.5

Each group has 1 hundred, 1 ten, and 5 ones, so $345 \div 3 = 115$. Notice that in order to split 345 into 3 equal groups, one of the tens had to be unbundled or decomposed into 10 ones.

Another way to divide 345 by 3 is by using the partial quotients method, in which we keep subtracting 3 groups of some amount from 345.

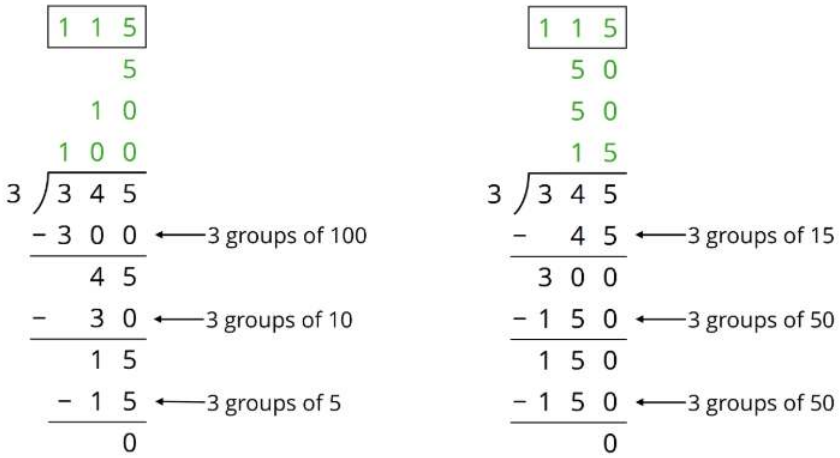


Figure 29.1.6: 2 partial quotients methods of 345 divided by 3. First calculation, 11 rows. First row: 115. Second row: 5. Third row: 10. Fourth row: 100. Fifth row: 3, long division symbol with 345 inside. Sixth row: minus 300, 3 groups of 100. Horizontal line. Seventh row: 45. Eighth row: minus 30, 3 groups of 10. Horizontal line. Ninth row: 15. Tenth row: minus 15, 3 groups of 5. Horizontal line. Eleventh row: 0. Second calculation, 11 rows. First row: 115. Second row: 50. Third row: 50. Fourth row: 15. Fifth row: 3, long division symbol with 345 inside. Sixth row: minus 45, 3 groups of 15. Horizontal line. Seventh row: 300. Eighth row: minus 150, 3 groups of 50. Horizontal line. Ninth row: 150. Tenth row: minus 150, 3 groups of 50. Horizontal line. Eleventh row: 0

- In the calculation on the left, first we subtract 3 groups of 100, then 3 groups of 10, and then 3 groups of 5. Adding up the partial quotients $(100 + 10 + 5)$ gives us 115.
- The calculation on the right shows a different amount per group subtracted each time (3 groups of 15, 3 groups of 50, and 3 more groups of 50), but the total amount in each of the 3 groups is still 115. There are other ways of calculating $345 \div 3$ using the partial quotients method.

Both the base-ten diagrams and partial quotients methods are effective. If, however, the dividend and divisor are large, as in $1,248 \div 26$, then the base-ten diagrams will be time-consuming.

Practice

Exercise 29.1.4

Here is one way to find $2,105 \div 5$ using partial quotients. Show a different way of using partial quotients to divide 2,105 by 5.

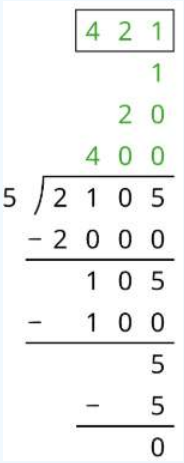


Figure 29.1.7: Partial quotient method of 2,105 divided by 5, 11 rows. First row: 421. Second row: 1. Third row: 20. Fourth row: 400. Fifth row: 5, long division symbol with 2,105 inside. Sixth row: minus 2,000. Horizontal line. Seventh row: 105. Eighth row: minus 100. Horizontal line. Ninth row: 5. Tenth row: minus 5. Horizontal line. Eleventh row: 0.

Exercise 29.1.5

Andre and Jada both found $657 \div 3$ using the partial quotients method, but they did the calculations differently, as shown here.

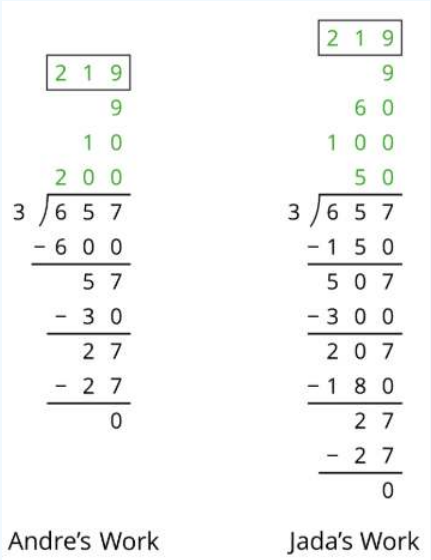


Figure 29.1.8: Andre and Jada's calculation of a partial product problem, 657 divided by 3. Andre's calculation. 657 inside the division bar, divided by 3. 3 divides into 657, 200 times. Subtract 600 from 657 for a remainder of 57. 3 divides into 57, 10 times. Subtract 30 from 57 for a remainder of 27. 3 divides into 27, 9 times. Subtract 27 from 27 for a remainder of 0. Add all of Andre's partial quotients of 200, 10 and 9 for a total of 219. Jada's calculation. 657 inside the division bar, divided by 3. 3 divides into 657, 50 times. Subtract 150 from 657 for a remainder of 507. 3 divides into 507, 100 times. Subtract 300 from 507 for a remainder of 207. 3 divides into 207, 60 times. Subtract 180 from 207 for a remainder of 27. 3 divides into 27, 9 times. Subtract 27 from 27 for a remainder of 0. Add all of Jada's partial quotients of 50, 100, 60 and 9 for a total of 219.

1. How is Jada's work the same as Andre's work? How is it different?
2. Explain why they have the same answer.

Exercise 29.1.6

Which might be a better way to evaluate $1,150 \div 46$: drawing base-ten diagrams or using the partial quotients method? Explain your reasoning.

Exercise 29.1.7

Here is an incomplete calculation of $534 \div 6$.

Write the missing numbers (marked with "?") that would make the calculation complete.

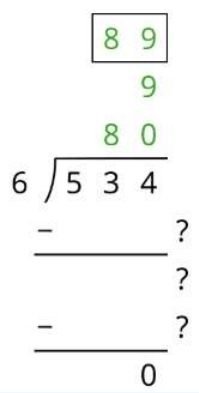


Figure 29.1.9: An incomplete calculation of 534 divided by 6, 8 rows. First row: 89. Second row: 9. Third row: 80. Fourth row: 6, long division symbol with 534 inside. Fifth row: minus question mark symbol. Horizontal line. Sixth row: question mark symbol. Seventh row: minus question mark symbol. Horizontal line. Eighth row: 0.

Exercise 29.1.8

Use the partial quotients method to find $1,032 \div 43$.

Exercise 29.1.9

Which of the polygons has the greatest area?

A. A rectangle that is 3.25 inches wide and 6.1 inches long.
B. A square with side length of 4.6 inches.
C. A parallelogram with a base of 5.875 inches and a height of 3.5 inches.
D. A triangle with a base of 7.18 inches and a height of 5.4 inches.

(From Unit 5.3.4)

Exercise 29.1.10

One micrometer is a millionth of a meter. A certain spider web is 4 micrometers thick. A fiber in a shirt is 1 hundred-thousandth of a meter thick.

1. Which is wider, the spider web or the fiber? Explain your reasoning.
2. How many meters wider?

(From Unit 5.2.3)

29.1: Using the Partial Quotients Method is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

29.2: Using Long Division

Lesson

Let's use long division.

Exercise 29.2.1: Number Talk: Estimating Quotients

Estimate these quotients mentally.

$500 \div 7$

$1,394 \div 9$

Exercise 29.2.2: Lin Uses Long Division

Lin has a method of calculating quotients that is different from Elena's method and Andre's method. Here is how she found the quotient of $657 \div 3$:

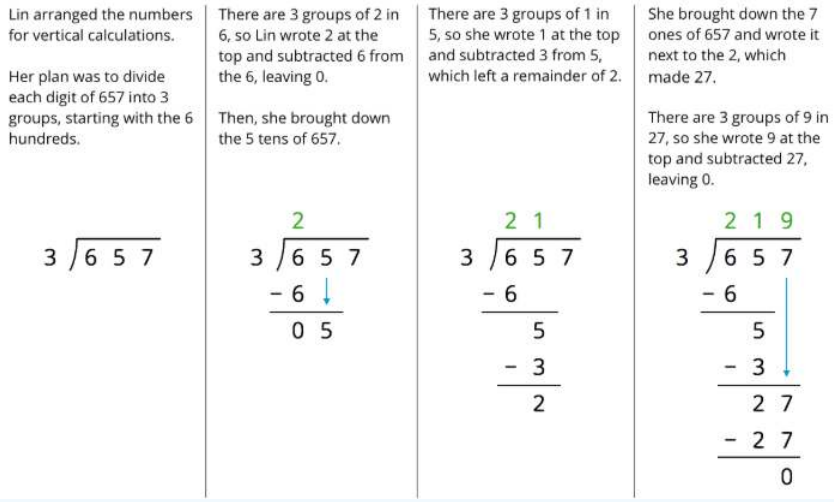


Figure 29.2.1: Long division calculation of 657 divided by 3, 4 steps. Step 1, 1 row. First row: 3, long division symbol with 657 inside. Step 2, 4 rows. First row: 2, second row: 3, long division symbol with 657 inside, third row: minus 6, horizontal line, fourth row: 0 5. Step 3, 6 rows. First row: 2 1, second row: 3, long division symbol with 657 inside, third row: minus 6, horizontal line, fourth row: 5, fifth row: minus 3, horizontal line, sixth row: 2. Step 4, 8 rows. First row: 219, second row: 3, long division symbol with 657 inside, third row: minus 6, horizontal line, fourth row: 5, fifth row: minus 3, horizontal line, sixth row: 27, seventh row: minus 27, horizontal line, eighth row: 0.

1. Discuss with your partner how Lin's method is similar to and different from drawing base-ten diagrams or using the partial quotients method.
   - Lin subtracted $3 \cdot 2$ then $3 \cdot 1$, and lastly $3 \cdot 9$. Earlier, Andre subtracted $3 \cdot 200$ then $3 \cdot 10$, and lastly $3 \cdot 9$. Why did they have the same quotient?
   - In the third step, why do you think Lin wrote the 7 next to the remainder of 2 rather than adding 7 and 2 to get 9?
2. Lin's method is called **long division**. Use this method to find the following quotients. Check your answer by multiplying it by the divisor.
   a. $846 \div 3$
   b. $1,816 \div 4$
   c. $768 \div 12$

Exercise 29.2.3: Dividing Whole Numbers

1. Find each quoteint.
   a. $633 \div 3$
   b. $1001 \div 7$

c. $2996 \div 14$

2. Here is Priya's calculation of $906 \div 3$

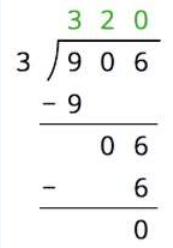


Figure 29.2.2: Long division of 906 divided by 3. 6 rows. First row: 320. Second row: 3, long division symbol with 906 inside. Third row: minus 9. Horizontal line. Fourth row: 0 6. Fifth row: minus 6. Horizontal line. Sixth row: 0.

a. Priya wrote 320 for the value of $906 \div 3$. Check her answer by multiplying it by 3. What product do you get and what does it tell you about Priya's answer?

b. Describe Priya's mistake, then show the correct calculation and answer.

Summary

Long division is another method for calculating quotients. It relies on place value to perform and record the division.

When we use long division, we work from left to right and with one digit at a time, starting with the leftmost digit of the dividend. We remove the largest group possible each time, using the placement of the digit to indicate the size of each group. Here is an example of how to find $948 \div 3$ using long division.

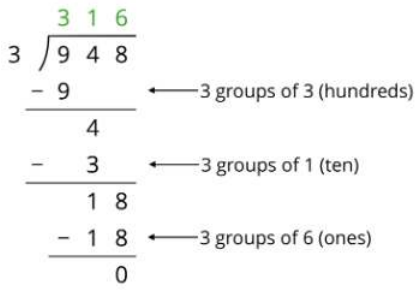


Figure 29.2.3: Long division of 948 divided by 3. 8 rows. First row: 316. Second row: 3, long division symbol with 948 inside. Third row: minus 9, 3 groups of 3 hundreds. Horizontal line. Fourth row: 4. Fifth row: minus 3, 3 groups of 1 ten. Horizontal line. Sixth row: 18. Seventh row: minus 18, 3 groups of 6 ones. Horizontal line. Eight row: 0.

- We start by dividing 9 hundreds into 3 groups, which means 3 hundreds in each group. Instead of writing 300, we simply write 3 in the hundreds place, knowing that it means 3 hundreds.
- There are no remaining hundreds, so we work with the tens. We can make 3 groups of 1 ten in 4 tens, so we write 1 in the tens place above the 4 of 948. Subtracting 3 tens from 4 tens, we have a remainder of 1 ten.
- We know that 1 ten is 10 ones. Combining these with the 8 ones from 948, we have 18 ones. We can make 3 groups of 6, so we write 6 in the ones place.

In total, there are 3 groups of 3 hundreds, 1 ten, and 6 ones in 948, so $948 \div 3 = 316$.

Glossary Entries

Definition: Long Division

Long division is a way to show the steps for dividing numbers in decimal form. It finds the quotient one digit at a time, from left to right.

For example, here is the long division for $57 \div 4$.

$$\begin{array}{r} 14.25 \\ 4\overline{)57.00} \\ \underline{-4} \\ 17 \\ \underline{-16} \\ 10 \\ \underline{-8} \\ 20 \\ \underline{-20} \\ 0 \end{array}$$

Figure 29.2.4

Practice

Exercise 29.2.4

Kiran is using long division to find $696 \div 12$.

$$12\overline{)696}$$

Figure 29.2.5

He starts by dividing 69 by 12. In which decimal place should Kiran place the first digit of the quotient (5)?

A. Hundreds
B. Tens
C. Ones
D. Tenths

Exercise 29.2.5

Here is a long-division calculation of $917 \div 7$.

$$\begin{array}{r} 131 \\ 7\overline{)917} \\ \underline{-7} \\ 21 \\ \underline{-21} \\ 7 \\ \underline{-7} \\ 0 \end{array}$$

Figure 29.2.6: Example division problem 917 divided by 7. Inside the division bar is 917. 7 divides into 9, one time. Subtract 7 for a remainder of 2. Bring down the 1. 7 divides into 21, three times. Subtract 21 from 21 for a remainder of 0. Bring down the 7. 7 divides into 7, one time. Subtract 7 for a remainder of 0. The quotient is 131.

1. There is a 7 under the 9 of 917. What does this 7 represent?
2. What does the subtraction of 7 from 9 mean?
3. Why is a 1 written next to the 2 from $9 - 7$?

Exercise 29.2.6

Han's calculation of $972 \div 9$ is shown here.

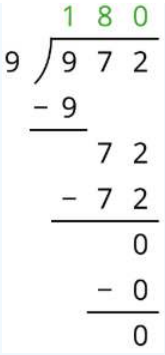


Figure 29.2.7: Division problem. Under the division house is 972. To the left of the house is 9. Above the house is 180. Below 972, row 1 is minus 9. horizontal line, row 2, 7. row 3, minus 0, horizontal line, Row 4, 72. Row 5 minus 72. horizontal line. Row 6. 0.

1. Find $180 \cdot 9$.
2. Use your calculation of $180 \cdot 9$ to explain how you know Han has made a mistake.
3. Identify and correct Han's mistake.

Exercise 29.2.7

Find each quotient.

a. $5\overline{)465}$

b. $12\overline{)924}$

c. $3\overline{)1107}$

Figure 29.2.8

Exercise 29.2.8

One ounce of a yogurt contains of 1.2 grams of sugar. How many grams of sugar are in 14.25 ounces of yogurt?

A. 0.171 grams
B. 1.71 grams
C. 17.1 grams
D. 171 grams

(From Unit 5.3.3)

Exercise 29.2.9

The mass of one coin is 16.718 grams. The mass of a second coin is 27.22 grams. How much greater is the mass of the second coin than the first? Show your reasoning.

(From Unit 5.2.3)

29.2: Using Long Division is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

29.3: Dividing Numbers that Result in Decimals

Lesson

Let's find quotients that are not whole numbers.

Exercise 29.3.1: Number Talk: Evaluationg Quotients

Find the quotients mentally.

$400 \div 8$

$80 \div 8$

$16 \div 8$

$496 \div 8$

Exercise 29.3.2: Keep Dividing

Mai used base-ten diagrams to calculate $62 \div 5$. She started by representing 62.

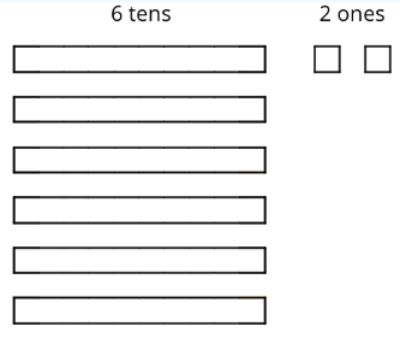


Figure 29.3.1

She then made 5 groups, each with 1 ten. There was 1 ten left. She unbundled it into 10 ones and distributed the ones across the 5 groups.

Here is Mai's diagram for $62 \div 5$.



Figure 29.3.2

1. Discuss these questions with a partner and write down your answers:
   a. Mai should have a total of 12 ones, but her diagram shows only 10. Why?
   b. She did not originally have tenths, but in her diagram each group has 4 tenths. Why?

c. What value has Mai found for $62 \div 5$? Explain your reasoning.

2. Find the quotient of $511 \div 5$ by drawing base-ten diagrams or by using the partial quotients method. Show your reasoning. If you get stuck, work with your partner to find a solution.
3. Four students share a $271 prize from a science competition. How much does each student get if the prize is shared equally? Show your reasoning.

Exercise 29.3.3: Using Long Division to Calculate Quotients

Here is how Lin calculated $62 \div 5$.

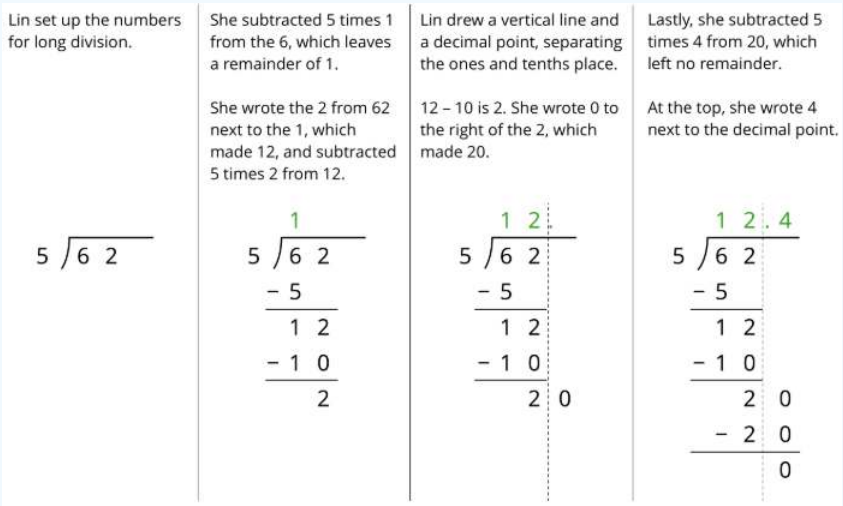


Figure 29.3.3: Long division calculation of 62 divided by 5, 4 steps. Step 1, 1 row. First row: 5, long division symbol with 62 inside. Step 2, 6 rows. First row: 1, second row: 5, long division symbol with 62 inside, third row: minus 5, horizontal line, fourth row: 12, fifth row: minus 10, horizontal line, sixth row: 2. Step 3, 6 rows. First row: 12, second row: 5, long division symbol with 62 inside, third row: minus 5, horizontal line, fourth row: 12, fifth row: minus 10, horizontal line, sixth row: 2 point 0. Step 4, 8 rows. First row: 12 point 4, second row: 5, long division symbol with 62 inside, third row: minus 5, horizontal line, fourth row: 12, fifth row: minus 10, horizontal line, sixth row: 2 point 0, seventh row: minus 2 point 0, horizontal line, eighth row: 0.

1. Discuss with your partner:
   - Lin put a 0 after the remainder of 2. Why? Why does this 0 not change the value of the quotient?
   - Lin subtracted 5 groups of 4 from 20. What value does the 4 in the quotient represent?
   - What value did Lin find for $62 \div 5$?
2. Use long division to find the value of each expression. Then pause so your teacher can review your work.
   a. $126 \div 8$
   b. $90 \div 12$
3. Use long division to show that:
   a. $5 \div 4$, or $\frac{5}{4}$, is 1.25.
   b. $4 \div 5$, or $\frac{4}{5}$, is 0.8.
   c. $1 \div 8$, or $\frac{1}{8}$, is 0.125.
   d. $1 \div 25$, or $\frac{1}{25}$, is 0.04.
4. Noah said we cannot use long division to calculate $10 \div 3$ because there will always be a remainder.
   a. What do you think Noah meant by “there will always be a remainder”?
   b. Do you agree with him? Explain your reasoning.

Summary

Dividing a whole number by another whole number does not always produce a whole-number quotient. Let’s look at $86 \div 4$, which we can think of as dividing 86 into 4 equal groups.

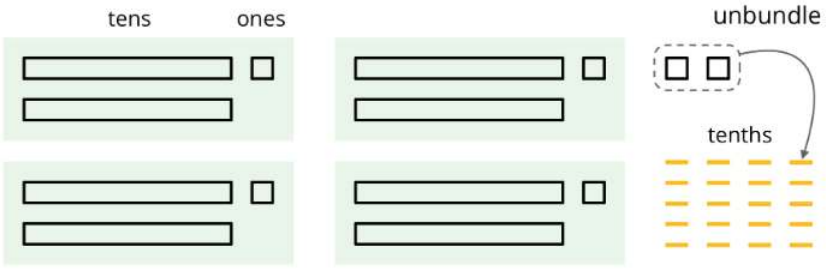


Figure 29.3.4: 4 groups of base-ten diagrams. Each group contains 2 rectangles labeled, tens, and 1 square labeled, ones. 2 squares are unbundled into 20 small rectangles labeled, tenths.

We can see in the base-ten diagram that there are 4 groups of 21 in 86 with 2 ones left over. To find the quotient, we need to distribute the 2 ones into the 4 groups. To do this, we can unbundle or decompose the 2 ones into 20 tenths, which enables us to put 5 tenths in each group.

Once the 20 tenths are distributed, each group will have 2 tens, 1 one, and 5 tenths, so $86 \div 4 = 21.5$.

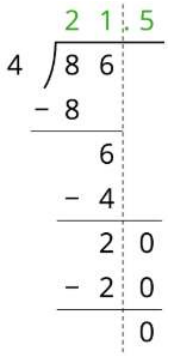


Figure 29.3.5: Long division calculation of 86 divided by 4. 8 rows. First row: 21 point 5, Second row: 4, long division symbol with 86 inside. Third row: minus 8. Horizontal line. Fourth row: 6. Fifth row: minus 4. Horizontal line. Sixth row: 2 point 0. Seventh row: minus 2 point 0. Horizontal line. Eighth row: 0.

We can also calculate $86 \div 4$ using long division.

The calculation shows that, after removing 4 groups of 21, there are 2 ones remaining. We can continue dividing by writing a 0 to the right of the 2 and thinking of that remainder as 20 tenths, which can then be divided into 4 groups.

To show that the quotient we are working with now is in the tenth place, we put a decimal point to the right of the 1 (which is in the ones place) at the top. It may also be helpful to draw a vertical line to separate the ones and the tenths.

There are 4 groups of 5 tenths in 20 tenths, so we write 5 in the tenths place at the top. The calculation likewise shows $86 \div 4 = 21.5$.

Glossary Entries

Definition: Long Division

Long division is a way to show the steps for dividing numbers in decimal form. It finds the quotient one digit at a time, from left to right.

For example, here is the long division for $57 \div 4$.

$$\begin{array}{r} 14.25 \\ 4\overline{)57.00} \\ \underline{-4} \\ 17 \\ \underline{-16} \\ 10 \\ \underline{-8} \\ 20 \\ \underline{-20} \\ 0 \end{array}$$

Figure 29.3.6

Practice

Exercise 29.3.4

Use long division to show that the fraction and decimal in each pair are equal.

$\frac{3}{4}$ and 0.75 $\frac{3}{50}$ and 0.06 $\frac{7}{25}$ and 0.28

Exercise 29.3.5

Mai walked $\frac{1}{8}$ of a 30-mile walking trail. How many miles did Mai walk? Explain or show your reasoning.

Exercise 29.3.6

Use long division to find each quotient. Write your answer as a decimal.

1. $99 \div 12$
2. $216 \div 5$
3. $1,988 \div 8$

Exercise 29.3.7

Tyler reasoned: "$\frac{9}{25}$ is equivalent to $\frac{18}{50}$ and to $\frac{36}{100}$, so the decimal of $\frac{9}{25}$ is 0.36."

1. Use long division to show that Tyler is correct.
2. Is the decimal of $\frac{18}{50}$ also 0.36? Use long division to support your answer.

Exercise 29.3.8

Complete the calculations so that each shows the correct difference.

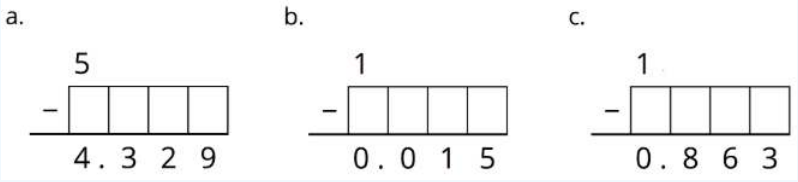


Figure 29.3.7: Fill in the blank subtraction problem a. 5 minus blank blank blank blank equals 4 point 329. Fill in the blank subtraction problem b. 1 minus blank blank blank blank equals 0 point 015. Fill in the blank subtraction problem c. 1 minus blank blank blank blank equals 0 point 863.

(From Unit 5.2.3)

Exercise 29.3.9

Use the equation $124 \cdot 15 = 1,860$ and what you know about fractions, decimals, and place value to explain how to place the decimal point when you compute $(1.24) \cdot (0.15)$

(From Unit 5.3.2)

29.3: Dividing Numbers that Result in Decimals is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

29.4: Dividing Decimals by Whole Numbers

Lesson

Let's divide decimals by whole numbers.

Exercise 29.4.1: Number Talk: Dividing by 4

Find each quotient mentally.

$80 \div 4$

$12 \div 4$

$1.2 \div 4$

$81.2 \div 4$

Exercise 29.4.2: Using Diagrams to Represent Division

To find $53.8 \div 4$ using diagrams, Elena began by representing 53.8.

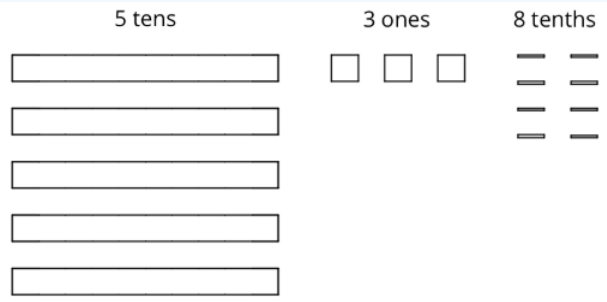


Figure 29.4.1

She placed 1 ten into each group, unbundled the remaining 1 ten into 10 ones, and went on distributing the units.

This diagram shows Elena's initial placement of the units and the unbundling of 1 ten.

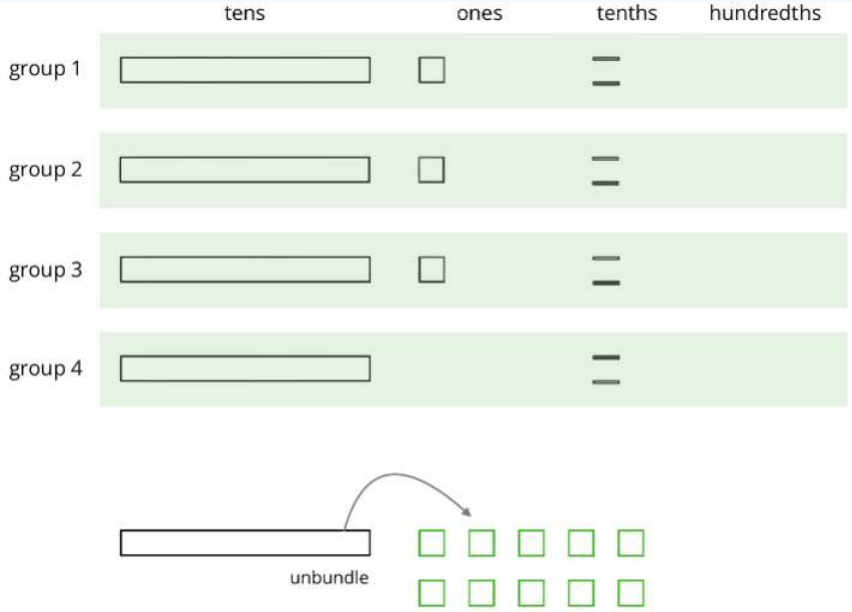


Figure 29.4.2: 4 groups of base-ten diagrams. Group 1, 1 rectangle labeled, tens, 1 square labeled, ones, 2 small rectangles labeled, tenths. Group 2, 1 rectangle labeled, tens, 1 square labeled, ones, 2 small rectangles labeled, tenths. Group 3, 1 rectangle labeled, tens, 1 square labeled, ones, 2 small rectangles labeled, tenths. Group 4, 1 rectangle labeled, tens, 2 small rectangles labeled, tenths. Below, 1 rectangle is unbundled into 10 squares.

1. Complete the diagram by continuing the division process. How would you use the available units to make 4 equal groups? As the units get placed into groups, show them accordingly and cross out those pieces from the bottom. If you unbundle a unit, draw the resulting pieces.
2. What value did you find for $53.8 \div 4$? Be prepared to explain your reasoning.
3. Use long division to find $53.8 \div 4$. Check your answer by multiplying it by the divisor 4.
4. Use long division to find $77.4 \div 5$. If you get stuck, you can draw diagrams or use another method.

Are you ready for more?

A distant, magical land uses jewels for their bartering system. The jewels are valued and ranked in order of their rarity. Each jewel is worth 3 times the jewel immediately below it in the ranking. The ranking is red, orange, yellow, green, blue, indigo, and violet. So a red jewel is worth 3 orange jewels, a green jewel is worth 3 blue jewels, and so on.

A group of 4 craftsmen are paid 1 of each jewel. If they split the jewels evenly amongst themselves, which jewels does each craftsman get?

Exercise 29.4.3: Dividends and Divisors

Analyze the dividends, divisors, and quotients in the calculations, and then answer the questions.

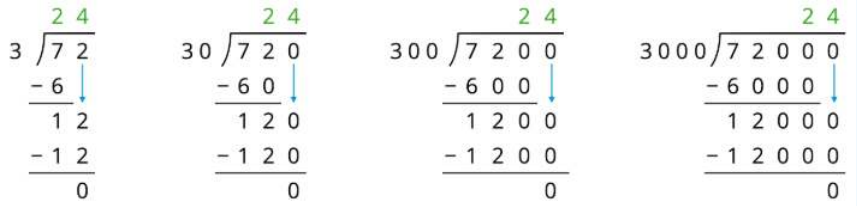


Figure 29.4.3: 4 long division calculations. 72 divided by 3, 720 divided by 30, 7,200 divided by 300, and 72,000 divided by 3,000. First calculation, 6 rows. First row: 24. Second row: 3, long division symbol with 72 inside. Third row: minus 6. Horizontal line. Fourth row: 12. Fifth row: minus 12. Horizontal line. Sixth row: 0. Second calculation, 6 rows. First row: 24. Second row: 30, long division symbol with 720 inside. Third row: minus 60. Horizontal line. Fourth row: 120. Fifth row: minus 120. Horizontal line. Sixth row: 0. Third calculation, 6 rows. First row: 24. Second row: 300, long division symbol with 7,200 inside. Third row: minus 600. Horizontal line. Fourth row: 1,200. Fifth row: minus 1,200. Horizontal line. Sixth row: 0. Fourth calculation, 6 rows. First row: 24. Second row: 3,000, long division symbol with 72,000 inside. Third row: minus 6,000. Horizontal line. Fourth row: 12,000. Fifth row: minus 12,000. Horizontal line. Sixth row: 0.

1. Complete each sentence. In the calculations shown:
   - Each dividend is _____ times the dividend to the left of it.
   - Each divisor is _____ times the divisor to the left of it.
   - Each quotient is _____ the quotient to the left of it.
2. Suppose we are writing a calculation to the right of $72,000 \div 3,000$. Which expression has a quotient of 24? Be prepared to explain your reasoning.
   a. $72,000 \div 30,000$
   b. $720,000 \div 300,000$
   c. $720,000 \div 30,000$
   d. $720,000 \div 3,000$
3. Suppose we are writing a calculation to the left of $72 \div 3$. Write an expression that would also give a quotient of 24. Be prepared to explain your reasoning.
4. Decide which of the following expressions would have the same value as $250 \div 10$. Be prepared to share your reasoning.
   a. $250 \div 0.1$
   b. $25 \div 1$
   c. $2.5 \div 1$
   d. $2.5 \div 0.1$
   e. $2,500 \div 100$
   f. $0.25 \div 0.01$

Summary

We know that fractions such as $\frac{6}{4}$ and $\frac{60}{40}$ are equivalent because:

- The numerator and denominator of are each 10 times those of $\frac{6}{4}$.
- Both fractions can be simplified to $\frac{3}{2}$.
- 600 divided by 400 is 1.5, and 60 divided by 40 is also 1.5.

Just like fractions, division expressions can be equivalent. For example, the expressions $540 \div 90$ and $5,400 \div 900$ are both equivalent to $54 \div 9$ because:

- They all have a quotient of 6.
- The dividend and the divisor in $540 \div 90$ are each 10 times the dividend and divisor in $54 \div 9$. Those in $5,400 \div 900$ are each 100 times the dividend and divisor in $54 \div 90$. In both cases, the quotient does not change.

This means that an expression such as $5.4 \div 0.9$ also has the same value as $54 \div 9$. Both the dividend and divisor of $5.4 \div 0.9$ are $\frac{1}{10}$ of those in $54 \div 9$.

In general, multiplying a dividend and a divisor by the same number does not change the quotient. Multiplying by powers of 10 (e.g., 10, 100, 1,000, etc.) can be particularly useful for dividing decimals, as we will see in an upcoming lesson.

Glossary Entries

Definition: Long Division

Long division is a way to show the steps for dividing numbers in decimal form. It finds the quotient one digit at a time, from left to right.

For example, here is the long division for $57 \div 4$.

$$\begin{array}{r} 14.25 \\ 4\overline{)57.00} \\ \underline{-4} \\ 17 \\ \underline{-16} \\ 10 \\ \underline{-8} \\ 20 \\ \underline{-20} \\ 0 \end{array}$$

Figure 29.4.4

Practice

Exercise 29.4.4

Here is a diagram representing a base-ten number. The large rectangle represents a unit that is 10 times the value of the square. The square represents a unit that is 10 times the value of the small rectangle.



Figure 29.4.5

Here is a diagram showing the number being divided into 5 equal groups.



Figure 29.4.6

1. If a large rectangle represents 1,000, what division problem did the second diagram show? What is its answer?
2. If a large rectangle represents 100, what division problem did the second diagram show? What is its answer?
3. If a large rectangle represents 10, what division problem did the second diagram show? What is its answer?

Exercise 29.4.5

1. Explain why all of these expressions have the same value.

$$4.5 \div 0.09 \qquad 45 \div 0.9 \qquad 450 \div 9 \qquad 4500 \div 90$$

2. What is the common value?

Exercise 29.4.6

Use long division to find each quotient.

1. $7.89 \div 2$
2. $39.54 \div 3$
3. $0.176 \div 5$

Exercise 29.4.7

Four students set up a lemonade stand. At the end of the day, their profit is $17.52. How much money do they each have when the profit is split equally? Show or explain your reasoning.

Exercise 29.4.8

1. A standard sheet of paper in the United States is 11 inches long and 8.5 inches wide. Each inch is 2.54 centimeters. How long and wide is a standard sheet of paper in centimeters?
2. A standard sheet of paper in Europe is 21.0 cm wide and 29.7 cm long. Which has the greater area, the standard sheet of paper in the United States or the standard sheet of paper in Europe? Explain your reasoning.

(From Unit 5.3.4)

29.4: Dividing Decimals by Whole Numbers is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

29.5: Dividing Decimals by Decimals

Lesson

Let's divide decimals by decimals.

Exercise 29.5.1: Same Values

1. Use long division to find the value of $5.04 \div 7$.
2. Select **all** of the quotients that have the same value as $5.04 \div 7$. Be prepared to explain how you know.
   a. $5.04 \div 70$
   b. $50.4 \div 70$
   c. $504,000 \div 700$
   d. $504,000 \div 700,000$

Exercise 29.5.2: Placing Decimal Points in Quotients

1. Think of one or more ways to find $3 \div 0.12$. Show your reasoning.
2. Find $1.8 \div 0.004$. Show your reasoning. If you get stuck, think about what equivalent division expression you could write.
3. Diego said, "To divide decimals, we can start by moving the decimal point in both the dividend and divisor by the same number of places and in the same direction. Then we find the quotient of the resulting numbers."
   Do you agree with Diego? Use the division expression $7.5 \div 1.25$ to support your answer.

Are you ready for more?

Can we create an equivalent division expression by multiplying both the dividend and divisor by a number that is *not* a multiple of 10 (for example: 4, 20, or $\frac{1}{2}$)? Would doing so produce the same quotient? Explain or show your reasoning.

Exercise 29.5.3: Two Ways to Calculate Quotients of Decimals

1. Here are two calculations of $48.78 \div 9$. Work with your partner to answer the following questions.

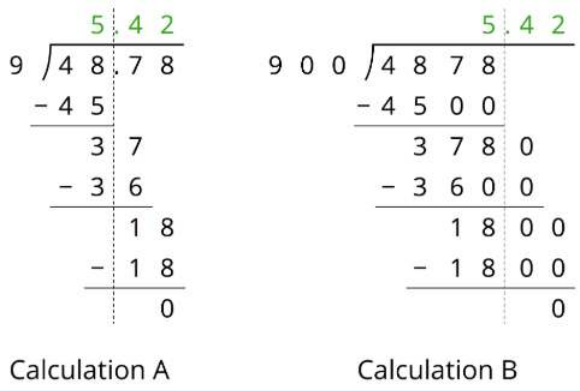


Figure 29.5.1: 2 long division calculations of 48 point 7 8 divided by 9. Calculation A, 8 rows. First row: 5 point 4 2. Second row: 9, long division symbol with 48 point 7 8 inside. Third row: minus 45. Horizontal line. Fourth row: 3 point 7. Fifth row: minus 3 point 6. Horizontal line. Sixth row: point 1 8. Seventh row: minus point 1 8. Horizontal line. Eighth row: 0. Calculation B, 8 rows. First row: 5 point 4 2. Second row: 900, long division symbol with 4,878 inside. Third row: minus 4,500. Horizontal line. Fourth row: 378 point 0. Fifth row: minus 360 point 0. Horizontal line. Sixth row: 18 point 0 0. Seventh row: minus 18 point 0 0. Horizontal line. Eighth row: 0.

   a. How are the two calculations the same? How are they different?
   b. Look at Calculation A. Explain how you can tell that the 36 means "36 tenths" and the 18 means "18 hundredths."
   c. Look at Calculation B. What do the 3600 and 1800 mean?
   d. We can think of $48.78 \div 9 = 5.42$ as saying, "There are 9 groups of 5.42 in 48.78." We can think of $4878 \div 900 = 5.42$ as saying, "There are 900 groups of 5.42 in 4878." How might we show that both statements are true?

2. a. Explain why $51.2 \div 6.4$ has the same value as $5.12 \div 0.64$.
   b. Write a division expression that has the same value as $51.2 \div 6.4$ but is easier to use to find the value. Then, find the value using long division.

Exercise 29.5.4: Practicing Division with Decimals

Find each quotient. Discuss your quotients with your group and agree on the correct answers. Consult your teacher if the group can't agree.

1. $106.5 \div 3$
2. $58.8 \div 0.7$
3. $257.4 \div 1.1$
4. Mai is making friendship bracelets. Each bracelet is made from 24.3 cm of string. If she has 170.1 cm of string, how many bracelets can she make? Explain or show your reasoning.

Summary

One way to find a quotient of two decimals is to multiply each decimal by a power of 10 so that both products are whole numbers.

If we multiply both decimals by the same power of 10, this does not change the value of the quotient. For example, the quotient $7.65 \div 1.2$ can be found by multiplying the two decimals by 10 (or by 100) and instead finding $76.5 \div 12$ or $765 \div 120$.

To calculate $765 \div 120$, which is equivalent to $76.5 \div 12$, we could use base-ten diagrams, partial quotients, or long division. Here is the calculation with long division:

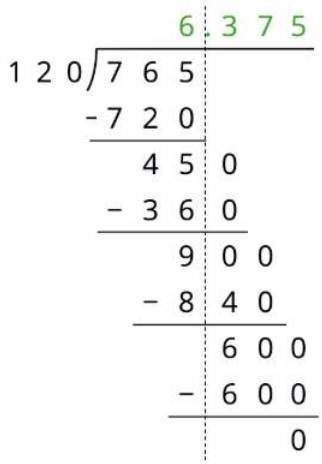


Figure 29.5.2: Long division of 765 divided by 120. 10 rows. Top row: 6 point 3 7 5. Second row: 120, long division symbol with 765 inside. Third row: minus 720. Horizontal line. Fourth row: 45 point 0. Fifth row: minus 36 point 0. Horizontal line. Sixth row: 9 point 0 0. Seventh row: minus 8 point 4 0. Horizontal line. Eighth row: point 6 0 0. Ninth row: minus point 6 0 0. Horizontal line. Tenth row: 0.

Glossary Entries

Definition: Long Division

Long division is a way to show the steps for dividing numbers in decimal form. It finds the quotient one digit at a time, from left to right.

For example, here is the long division for $57 \div 4$.

$$\begin{array}{r} 14.25 \\ 4\overline{)57.00} \\ \underline{-4} \\ 17 \\ \underline{-16} \\ 10 \\ \underline{-8} \\ 20 \\ \underline{-20} \\ 0 \end{array}$$

Figure 29.5.3

Practice

Exercise 29.5.5

A student said, "To find the value of $109.2 \div 6$, I can divide 1,092 by 60."

1. Do you agree with her? Explain your reasoning.
2. Calculate the quotient of $109.2 \div 6$ using any method of your choice.

Exercise 29.5.6

Here is how Han found $31.59 \div 13$:

$$\begin{array}{r} 2.43 \\ 13\overline{)31.59} \\ \underline{-26} \\ 55 \\ \underline{-52} \\ 39 \\ \underline{-39} \\ 0 \end{array}$$

Figure 29.5.4: Long division of 31 point 5 9 divided by 13. 8 rows. Top row: 2 point 4 3. Second row: 13, long division symbol with 31 point 5 9 inside. Third row: minus 26. Horizontal line. Fourth row: 55. Fifth row: minus 52. Horizontal line. Sixth row: 39. Seventh row: minus 39. Horizontal line. Eighth row: 0.

1. At the second step, Han subtracts 52 from 55. How do you know that these numbers represent tenths?
2. At the third step, Han subtracts 39 from 39. How do you know that these numbers represent hundredths?
3. Check that Han's answer is correct by calculating the product of 2.43 and 13.

Exercise 29.5.7

1. Write two division expressions that have the same value as $61.12 \div 3.2$.
2. Find the value of $61.12 \div 3.2$. Show your reasoning.

Exercise 29.5.8

A bag of pennies weighs 5.1 kilograms. Each penny weighs 2.5 grams. About how many pennies are in the bag?

A. 20
B. 200
C. $2,000$
D. $20,000$

Exercise 29.5.9

Find each difference. If you get stuck, consider drawing a diagram.

$$2.5 - 1.6 \qquad 0.72 - 0.4 \qquad 11.3 - 1.75 \qquad 73 - 1.3$$

(From Unit 5.2.3)

Exercise 29.5.10

Plant B is $6\frac{2}{3}$ inches tall. Plant C is $4\frac{4}{15}$ inches tall. Complete the sentences and show your reasoning.

1. Plant C is _____ times as tall as Plant B.
2. Plant C is _____ inches _____ (taller or shorter) than Plant B.

(From Unit 4.4.1)

Exercise 29.5.11

At a school, 460 of the students walk to school.

1. The number of students who take public transit is 20% of the number of students who walk. How many students take public transit?
2. The number of students who bike to school is 5% of the number of students who walk. How many students bike to school?
3. The number of students who ride the school bus is 110% of the number of students who walk. How many students ride the school bus?

(From Unit 3.4.6)

29.5: Dividing Decimals by Decimals is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

Section 30: Let's Put It to Work

This page titled Section 30: Let's Put It to Work is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

30.1: Using Operations on Decimals to Solve Problems

Lesson

Let's solve some problems using decimals.

Exercise 30.1.1: Close Estimates

For each expression, choose the best estimate of its value.

1. $76.2 \div 15$
   - 0.5
   - 5
   - 50
2. $56.34 \div 48$
   - 1
   - 10
   - 100
3. $124.3 \div 20$
   - 6
   - 60
   - 600

Exercise 30.1.2: Applying Division with Decimals

Your teacher will assign to you either Problem A or Problem B. Work together as a group to answer the questions. Be prepared to create a visual display to show your reasoning with the class.

Problem A:

A piece of rope is 5.75 meters in length.

1. If it is cut into 20 equal pieces, how long will each piece be?
2. If it is cut into 0.05-meter pieces, how many pieces will there be?

Problem B:

A tortoise travels 0.945 miles in 3.5 hours.

1. If it moves at a constant speed, how many miles per hour is it traveling?
2. At this rate, how long will it take the tortoise to travel 4.86 miles?



Figure 30.1.1: By skeeze. Public Domain. Pixabay. Source.

Exercise 30.1.3: Distance between Hurdles

There are 10 equally-spaced hurdles on a race track. The first hurdle is 13.72 meters from the start line. The final hurdle is 14.02 meters from the finish line. The race track is 110 meters long.



Figure 30.1.2: August 3rd London Olympics 2012 stadium hurdles, by Steve Flair. CC BY 2.0. Wikimedia Commons. Source.

1. Draw a diagram that shows the hurdles on the race track. Label all known measurements.
2. How far are the hurdles from one another? Explain or show your reasoning.
3. A professional runner takes 3 strides between each pair of hurdles. The runner leaves the ground 2.2 meters *before* the hurdle and returns to the ground 1 meter *after* the hurdle.
   About how long are each of the runner's strides between the hurdles? Show your reasoning.

Exercise 30.1.4: Examining a Tennis Court

Here is a diagram of a tennis court.

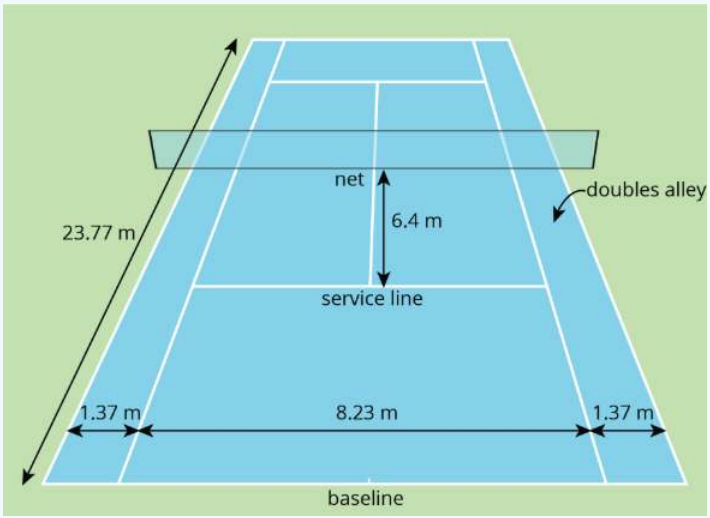


Figure 30.1.3: Diagram of a tennis court. Length of court is 23 point 7 7 m, baseline consists of lengths 1 point 3 7 m, 8 point 2 3 m, and 1 point 3 7 m. Distance from service line to net is 6 point 4 m.

The full tennis court, used for doubles, is a rectangle. All of the angles made by the line segments in the diagram are right angles.

1. The net partitions the tennis court into two halves. Is each half a square? Explain your reasoning.
2. Is the service line halfway between the net and the baseline? Explain your reasoning.
3. Lines painted on a tennis court are 5 cm wide. A painter made markings to show the length and width of the court, then painted the lines to the outside of the markings.
   a. Did the painter's mistake increase or decrease the overall size of the tennis court? Explain how you know.
   b. By how many square meters did the court's size change? Explain your reasoning.

Summary

Diagrams can help us communicate and model mathematics. A clearly-labeled diagram helps us visualize what is happening in a problem and accurately communicate the information we need.

Sports offer great examples of how diagrams can help us solve problems. For example, to show the placement of the running hurdles in a diagram, we needed to know what the distances 13.72 and 14.02 meters tell us and the number of hurdles to draw. An accurate diagram not only helped us set up and solve the problem correctly, but also helped us see that there are only *nine* spaces between ten hurdles.

To communicate information clearly and solve problems correctly, it is also important to be precise in our measurements and calculations, especially when they involve decimals.

In tennis, for example, the length of the court is 23.77 meters. Because the boundary lines on a tennis court have a significant width, we would want to know whether this measurement is taken between the inside of the lines, the center of the lines, or the outside of the lines. Diagrams can help us attend to this detail, as shown here.

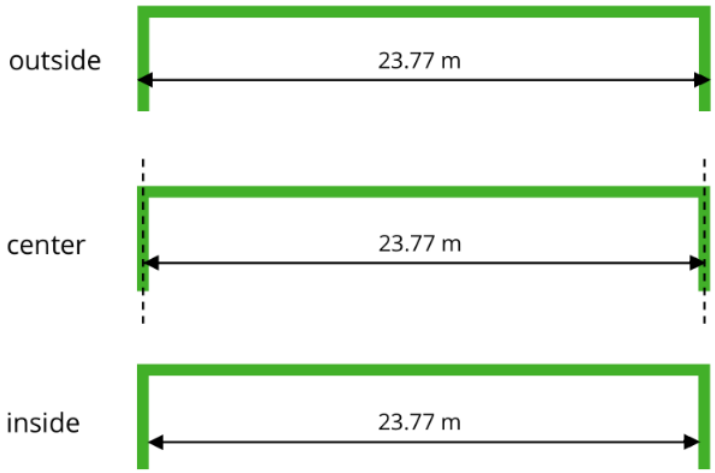


Figure 30.1.4: Three diagrams, labeled outside, center, and inside, representing the length of a tennis court. Outside, line labeled 23 point 7 7 m with arrows on both ends, tips of arrows touch the outside of the green boundary lines. Center, line labeled 23 point 7 7 m with arrows on both ends, tips of arrows touch the center of the green boundary lines. Inside, line labeled 23 point 7 7 m with arrows on both ends, tips of arrows touch the inside of the green boundary lines.

The accuracy of this measurement matters to the tennis players who use the court, so it matters to those who paint the boundaries as well. The tennis players practice their shots to be on or within certain lines. If the tennis court on which they play is not precisely measured, their shots may not land as intended in relation to the boundaries. Court painters usually need to be sure their measurements are accurate to within $\frac{1}{100}$ of a meter or one centimeter.

Practice

Exercise 30.1.5

A roll of ribbon was 12 meters long. Diego cut 9 pieces of ribbon that were 0.4 meter each to tie some presents. He then used the remaining ribbon to make some wreaths. Each wreath required 0.6 meter. For each question, explain your reasoning.

1. How many meters of ribbon were available for making wreaths?
2. How many wreaths could Diego make with the available ribbon?

Exercise 30.1.6

The Amazon rainforest covered 6.42 million square kilometers in 1994. In 2014, it covered only $\frac{50}{59}$ as much. Which is closest to the area of the Amazon forest in 2014? Explain how you know without calculating the exact area.

A. 6.4 million km^2
B. 5.4 million km^2
C. 4.4 million km^2
D. 3.4 million km^2
E. 2.4 million km^2

Exercise 30.1.7

To get an A in her math class, Jada needs to have at least 90% of the total number of points possible. The table shows Jada's results before the final test in the class.

Table 30.1.1

	Jada's points	total points possible
Homework	141	150
Test 1	87	100
Test 2	81	100
Test 3	91	100

1. Does Jada have 90% of the total possible points *before* the final test? Explain how you know.
2. Jada thinks that if she gets at least 92 out of 100 on the final test, she will get an A in the class. Do you agree? Explain.

Exercise 30.1.8

Find the following differences. Show your reasoning.

- $0.151 - 0.028$
- $0.106 - 0.0315$
- $3.572 - 2.6014$

(From Unit 5.2.3)

Exercise 30.1.9

Find these quotients. Show your reasoning.

- $24.2 \div 1.1$
- $13.25 \div 0.4$
- $170.28 \div 0.08$

(From Unit 5.4.5)

30.1: Using Operations on Decimals to Solve Problems is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

30.2: Making and Measuring Boxes

Lesson

Let's use what we know about decimals to make and measure boxes.

Exercise 30.2.1: Folding Paper Boxes

Your teacher will demonstrate how to make an open-top box by folding a sheet of paper. Your group will receive 3 or more sheets of square paper. Each person in your group will make 1 box. Before you begin folding:

1. Record the side lengths of your papers, from the smallest to the largest.
   - Paper for Box 1: _____ cm
   - Paper for Box 2: _____ cm
   - Paper for Box 3: _____ cm
2. Compare the side lengths of the square sheets of paper. Be prepared to explain how you know.
   a. The side length of the paper for Box 2 is _____ times the side length of the paper for Box 1.
   b. The side length of the paper for Box 3 is _____ times the side length of the paper for Box 1.
3. Make some predictions about the measurements of the three boxes your group will make:
   - The surface area of Box 3 will be _____ as large as that of Box 1.
   - Box 2 will be _____ times as tall as Box 1.
   - Box 3 will be _____ times as tall as Box 1.

Now you are ready to fold your paper into a box!

Exercise 30.2.2: Sizing Up Paper Boxes

Now that you have made your boxes, you will measure them and check your predictions about how their heights and surface areas compare.

1. a. Measure the length and height of each box to the nearest tenth of a centimeter. Record the measurements in the table.

Table 30.2.1

	side length of paper (cm)	length of box (cm)	height of box (cm)	surface area (sq cm)
Box 1				
Box 2				
Box 3				

   b. Calculate the surface area of each box. Show your reasoning and decide on an appropriate level of precision for describing the surface area (Is it the nearest 10 square centimeters, nearest square centimeter, or something else?). Record your answers in the table.
2. To see how many times as large one measurement is when compared to another, we can compute their quotient. Divide each measurement of Box 2 by the corresponding measurement for Box 1 to complete the following statements.
   a. The length of Box 2 is _____ times the length of Box 1.
   b. The height of Box 2 is _____ times the height of Box 1.
   c. The surface area of Box 2 is _____ times the surface area of Box 1.
3. Find out how the dimensions of Box 3 compare to those of Box 1 by computing quotients of their lengths, heights, and surface areas. Show your reasoning.
   a. The length of Box 3 is _____ times the length of Box 1.
   b. The height of Box 3 is _____ times the height of Box 1.
   c. The surface area of Box 3 is _____ times the surface area of Box 1.

4. Record your results in the table.

Table 30.2.2

	side length of paper	length of box	height of box	surface area
Box 2 compared to Box 1				
Box 3 compared to Box 1				

5. Earlier, in the first activity, you made predictions about how the heights and surface areas of the two larger boxes would compare to those of the smallest box. Discuss with your group:
   - How accurate were your predictions? Were they close to the results you found by performing calculations?
   - Let's say you had another piece of square paper to make Box 4. If the side length of this paper is 4 times the side length of the paper for Box 1, predict how the length, height, and surface area of Box 4 would compare to those of Box 1. How did you make your prediction?

30.2: Making and Measuring Boxes is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

SECTION OVERVIEW

6: Expressions and Equations

Section 31: Equations in One Variable

31.1: Tape Diagrams and Equations
31.2: Truth and Equations
31.3: Staying in Balance
31.4: Practice Solving Equations and Representing Situations with Equations
31.5: A New Way to Interpret a over b

Section 32: Equal and Equivalent

32.1: Write Expressions Where Letters Stand for Numbers
32.2: Revisit Percentages
32.3: Equal and Equivalent
32.4: The Distributive Property, Part 1
32.5: The Distributive Property, Part 2
32.6: The Distributive Property, Part 3

Section 33: Expressions with Exponents

33.1: Meaning of Exponents
33.2: Expressions with Exponents
33.3: Evaluating Expressions with Exponents
33.4: Equivalent Exponential Expressions

Section 34: Relationship Between Quantities

34.1: Two Related Quantities, Part 1
34.2: Two Related Quantities, Part 2
34.3: More Relationships

Section 35: Let's Put it to Work

35.1: Tables, Equations, and Graphs, Oh My!

This page titled 6: Expressions and Equations is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

Section 31: Equations in One Variable

This page titled Section 31: Equations in One Variable is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

31.1: Tape Diagrams and Equations

Lesson

Let's see how tape diagrams and equations can show relationships between amounts.

Exercise 31.1.1: Which Diagram is Which?

1. Here are two diagrams. One represents $2+5=7$. The other represents $5 \cdot 2 = 10$. Which is which? Label the length of each diagram.

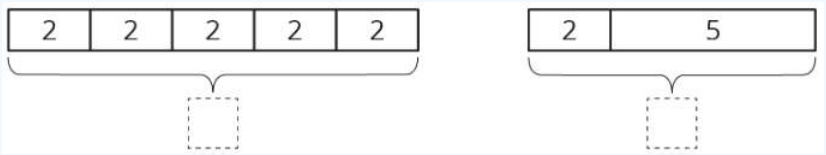


Figure 31.1.1

2. Draw a diagram that represents each equation.

$$4+3=7 \qquad 4 \cdot 3 = 12$$

Exercise 31.1.2: Match Equations and Tape Diagrams

Here are two tape diagrams. Match each equation to one of the tape diagrams.



Figure 31.1.2

1. $4+x=12$
2. $12 \div 4 = x$
3. $4 \cdot x = 12$
4. $12 = 4+x$
5. $12-x=4$
6. $12 = 4 \cdot x$
7. $12-4=x$
8. $x = 12-4$
9. $x+x+x+x=12$

Exercise 31.1.3: Draw Diagrams for Equations

For each equation, draw a diagram and find the value of the unknown that makes the equation true.

1. $18 = 3+x$
2. $18 = 3 \cdot y$

Are you ready for more?

You are walking down a road, seeking treasure. The road branches off into three paths. A guard stands in each path. You know that only one of the guards is telling the truth, and the other two are lying. Here is what they say:

- Guard 1: The treasure lies down this path.
- Guard 2: No treasure lies down this path; seek elsewhere.
- Guard 3: The first guard is lying.

Which path leads to the treasure?

Summary

Tape diagrams can help us understand relationships between quantities and how operations describe those relationships.

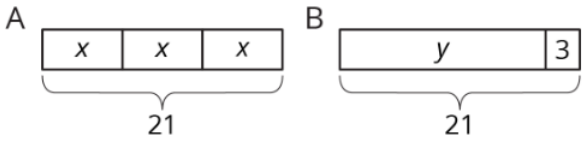


Figure 31.1.3

Diagram A has 3 parts that add to 21. Each part is labeled with the same letter, so we know the three parts are equal. Here are some equations that all represent diagram A:

$$\begin{aligned} x+x+x &= 12 \\ 3\cdot x &= 21 \\ x &= 21\div 3 \\ x &= \frac{1}{3}\cdot 21 \end{aligned}$$

Notice that the number 3 is not seen in the diagram; the 3 comes from counting 3 boxes representing 3 equal parts in 21.

We can use the diagram or any of the equations to reason that the value of x is 7.

Diagram B has 2 parts that add to 21. Here are some equations that all represent diagram B:

$$\begin{aligned} y+3 &= 21 \\ y &= 21-3 \\ 3 &= 21-y \end{aligned}$$

We can use the diagram or any of the equations to reason that the value of y is 18.

Practice

Exercise 31.1.4

Here is an equation: $x+4=17$

1. Draw a tape diagram to represent the equation.
2. Which part of the diagram shows the quantity x? What about 4? What about 17?
3. How does the diagram show that $x+4$ has the same value as 17?

Exercise 31.1.5

Diego is trying to find the value of x in $5\cdot x=25$. He draws this diagram but is not certain how to proceed.

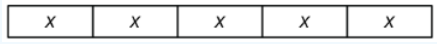


Figure 31.1.4

1. Complete the tape diagram so it represents the equation $5\cdot x=35$.
2. Find the value of x.

Exercise 31.1.6

Match each equation to one of the two tape diagrams.

1. $x+3=9$
2. $3\cdot x=9$
3. $9=3\cdot x$
4. $3+x=9$
5. $x=9-3$
6. $x=9\div 3$
7. $x+x+x=9$

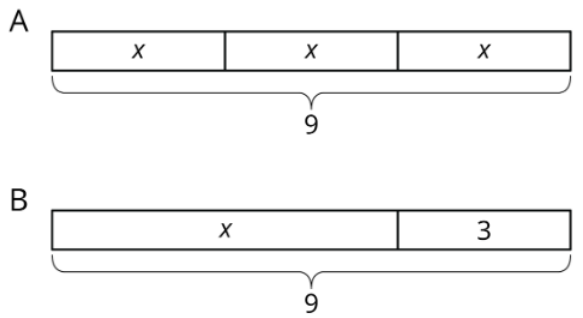


Figure 31.1.5

Exercise 31.1.7

For each equation, draw a tape diagram and find the unknown value.

1. $x + 9 = 16$
2. $4 \cdot x = 28$

Exercise 31.1.8

A shopper paid $2.52 for 4.5 pounds of potatoes, $7.75 for 2.5 pounds of broccoli, and $2.45 for 2.5 pounds of pears. What is the unit price of each item she bought? Show your reasoning.

(From Unit 5.4.5)

Exercise 31.1.9

A sports drink bottle contains 16.9 fluid ounces. Andre drank 80% of the bottle. How many fluid ounces did Andre drink? Show your reasoning.

(From Unit 3.4.5)

Exercise 31.1.10

The daily recommended allowance of calcium for a sixth grader is 1,200 mg. One cup of milk has 25% of the recommended daily allowance of calcium. How many milligrams of calcium are in a cup of milk? If you get stuck, consider using the double number line.

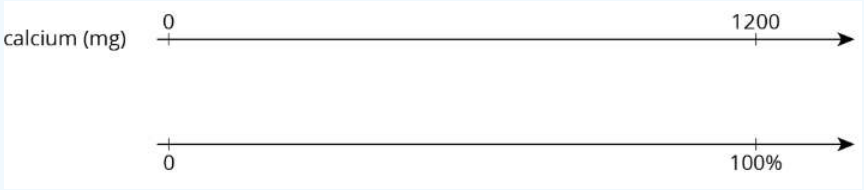


Figure 31.1.6: A double number line with 2 tick marks at either end of the line. The top number line is labeled calcium in milligrams and the tick marks are labeled 0 and 1200. The bottom number line is not labeled and the tick marks are labeled 0 and 100 percent.

(From Unit 3.4.2)

This page titled 31.1: Tape Diagrams and Equations is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

31.2: Truth and Equations

Lesson

Let's use equations to represent stories and see what it means to solve equations.

Exercise 31.2.1: Three Letters

1. The equation $a+b=c$ could be true or false.
   a. If a is 3, b is 4, and c is 5, is the equation true or false?
   b. Find new values of a, b, and c that make the equation true.
   c. Find new values of a, b, and c that make the equation false.
2. The equation $x \cdot y = z$ could be true or false.
   a. If x is 3, y is 4, and z is 12, is the equation true or false?
   b. Find new values of x, y, and z that make the equation true.
   c. Find new values of x, y, and z that make the equation false.

Exercise 31.2.2: Storytime

Here are three situations and six equations. Which equation best represents each situation? If you get stuck, consider drawing a diagram.

$$x+5=20 \qquad x=20+5 \qquad 5x=20$$
$$x+20=5 \qquad 5 \cdot 20 = x \qquad 20x=5$$

1. After Elena ran 5 miles on Friday, she had run a total of 20 miles for the week. She ran x miles before Friday.
2. Andre's school has 20 clubs, which is five times as many as his cousin's school. His cousin's school has x clubs.
3. Jada volunteers at the animal shelter. She divided 5 cups of cat food equally to feed 20 cats. Each cat received x cups of food.

Exercise 31.2.3: Using Structures to Find Solutions

Here are some equations that contain a **variable** and a list of values. Think about what each equation means and find a **solution** in the list of values. If you get stuck, consider drawing a diagram. Be prepared to explain why your solution is correct.

1. $1000-a=400$
2. $12.6=b+4.1$
3. $8c=8$
4. $\frac{2}{3} \cdot d = \frac{10}{9}$
5. $10e=1$
6. $10=0.5f$
7. $0.99=1-g$
8. $h+\frac{3}{7}=1$

List:

$\frac{1}{8}$	$\frac{3}{7}$	$\frac{4}{7}$	$\frac{3}{5}$	$\frac{5}{3}$	$\frac{7}{3}$	0.01	0.1	0.5
1	2	8.5	9.5	16.7	20	400	600	1400

Are you ready for more?

One solution to the equation $a+b+c=10$ is $a=2, b=5, c=3$.

How many different whole-number solutions are there to the equation $a+b+c=10$? Explain or show your reasoning.

Summary

An equation can be true or false. An example of a true equation is $7+1=4\cdot 2$. An example of a false equation is $7+1=9$.

An equation can have a letter in it, for example, $u+1=8$. This equation is false if u is 3, because $3+1$ does not equal 8. This equation is true if u is 7, because $7+1=8$.

A letter in an equation is called a **variable**. In $u+1=8$, the variable is u. A number that can be used in place of the variable that makes the equation true is called a **solution** to the equation. In $u+1=8$, the solution is 7.

When a number is written next to a variable, the number and the variable are being multiplied. For example, $7x=21$ means the same thing as $7\cdot x=21$. A number written next to a variable is called a **coefficient**. If no coefficient is written, the coefficient is 1. For example, in the equation $p+3=5$, the coefficient of p is 1.

Glossary Entries

Definition: Coefficient

A coefficient is a number that is multiplied by a variable.

For example, in the expression $3x+5$, the coefficient of x is 3. In the expression $y+5$, the coefficient of y is 1, because $y=1\cdot y$.

Definition: Solution to an Equation

A solution to an equation is a number that can be used in place of the variable to make the equation true.

For example, 7 is the solution to the equation $m+1=8$, because it is true that $7+1=8$. The solution to $m+1=8$ is not 9, because $9+1\neq 8$.

Definition: Variable

A variable is a letter that represents a number. You can choose different numbers for the value of the variable.

For example, in the expression $10-x$, the variable is x. If the value of x is 3, then $10-x=7$, because $10-3=7$. If the value of x is 6, then $10-x=4$, because $10-6=4$.

Practice

Exercise 31.2.4

Select **all** the true equations.

A. $5+0=0$
B. $15\cdot 0=0$
C. $1.4+2.7=4.1$
D. $\frac{2}{3}\cdot\frac{5}{9}=\frac{7}{12}$
E. $4\frac{2}{3}=5-\frac{1}{3}$

Exercise 31.2.5

Mai's water bottle had 24 ounces in it. After she drank x ounces of water, there were 10 ounces left. Select **all** the equations that represent this situation.

A. $24\div 10=x$
B. $24+10=x$
C. $24-10=x$
D. $x+10=24$
E. $10x=24$

Exercise 31.2.6

Priya has 5 pencils, each x inches in length. When she lines up the pencils end to end, they measure 34.5 inches. Select **all** the equations that represent this situation.

A. $5 + x = 34.5$
B. $5x = 34.5$
C. $34.5 \div 5 = x$
D. $34.5 - 5 = x$
E. $x = (34.5) \cdot 5$

Exercise 31.2.7

Match each equation with a solution from the list of values.

A. $2a = 4.6$
B. $b + 2 = 4.6$
C. $c \div 2 = 4.6$
D. $d - 2 = 4.6$
E. $e + \frac{3}{8} = 2$
F. $\frac{1}{8} f = 3$
G. $g \div \frac{8}{5} = 1$

1. $\frac{8}{5}$
2. $1\frac{5}{8}$
3. 2.3
4. 2.6
5. 6.6
6. 9.2
7. 24

Exercise 31.2.8

The daily recommended allowance of vitamin C for a sixth grader is 45 mg. 1 orange has about 75% of the recommended daily allowance of vitamin C. How many milligrams are in 1 orange? If you get stuck, consider using the double number line.

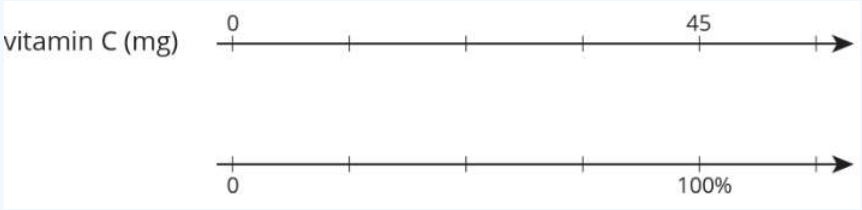


Figure 31.2.1: Double number line, 6 evenly spaced tick marks. Top line, vitamin C, mg. Beginning at first tick mark, labels: 0, blank, blank, blank, 45, blank. Bottom line. Beginning at first tick mark, labeled 0, blank, blank, blank, 100 percent, blank.

(From Unit 3.4.2)

Exercise 31.2.9

There are 90 kids in the band. 20% of the kids own their own instruments, and the rest rent them.

1. How many kids own their own instruments?
2. How many kids rent instruments?
3. What percentage of kids rent their instruments?

(From Unit 3.4.3)

Exercise 31.2.10

Find each product.

1. $(0.25) \cdot (1.4)$
2. $(0.061) \cdot (0.43)$
3. $(1.017) \cdot (0.072)$
4. $(5.226) \cdot (0.037)$

(From Unit 5.3.4)

31.2: Truth and Equations is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

31.3: Staying in Balance

Lesson

Let's use balanced hangers to help us solve equations.

Exercise 31.3.1: Hanging Around

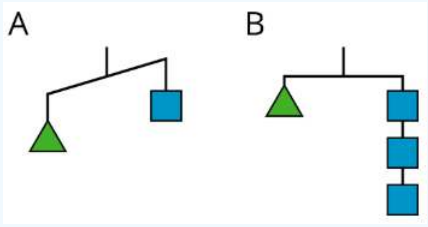


Figure 31.3.1: Two hangers, A and B. A, unbalanced. Left side, 1 triangle, right side, 1 square. Left lower than right. B, balanced. Left side, 1 triangle, right side, 3 identical squares.

For diagram A, find:

1. One thing that *must* be true
2. One thing that *could* be true or false
3. One thing that *cannot possibly* be true

For diagram B, find:

1. One thing that *must* be true
2. One thing that *could* be true or false
3. One thing that *cannot possibly* be true

Exercise 31.3.2: Match Equations and Hangers

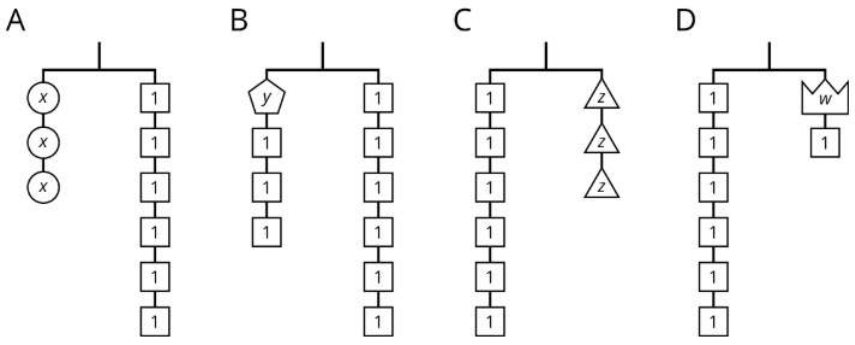


Figure 31.3.2: Four balanced hangers, A, B, C, and D. A, left side, 3 identical circles, x, right side, 6 identical squares, 1. B, left side, 1 pentagon, y, 3 identical squares, 1, right side, 6 identical squares, 1. C, left side, 6 identical squares, 1, right side, 3 identical triangles, z. D, left side, 6 identical squares, 1, right side, 1 crown, w, 1 square, 1.

1. Match each hanger to an equation. Complete the equation by writing x, y, z, or w in the empty box.

$\square + 3 = 6$ $\quad 3 \cdot \square = 6$ $\quad 6 = \square + 1$ $\quad 6 = 3 \cdot \square$

Figure 31.3.3

2. Find a solution to each equation. Use the hangers to explain what each solution means.

Exercise 31.3.3: Connecting Diagrams to Equations and Solutions

Here are some balanced hangers. Each piece is labeled with its weight.

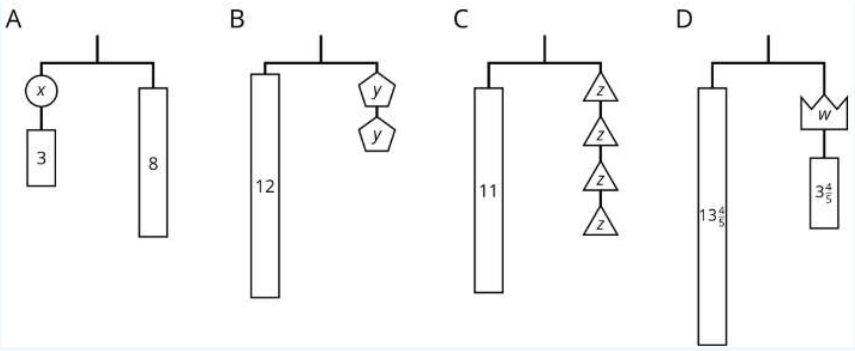


Figure 31.3.4: Four balanced hangers, A, B, C, and D. A, left side, 1 circle, x, 1 rectangle, 3, right side, 1 rectangle, 8. B, left side, 1 rectangle, 12, right side, 2 identical pentagons, y. C, left side, 1 rectangle, 11, right side, four identical triangles, z. D, left side, 1 rectangle, 13 and four fifths, right side, 1 crown, w, 1 rectangle, 3 and four fifths.

For each diagram:

1. Write an equation.
2. Explain how to reason with the *diagram* to find the weight of a piece with a letter.
3. Explain how to reason with the *equation* to find the weight of a piece with a letter.

Are you ready for more?

When you have the time, visit the site https://solveme.edc.org/Mobiles.html to solve some trickier puzzles that use hanger diagrams like the ones in this lesson. You can even build new ones. (If you want to do this during class, check with your teacher first!)

Summary

A hanger stays balanced when the weights on both sides are equal. We can change the weights and the hanger will stay balanced as long as both sides are changed in the same way. For example, adding 2 pounds to each side of a balanced hanger will keep it balanced. Removing half of the weight from each side will also keep it balanced.

An equation can be compared to a balanced hanger. We can change the equation, but for a true equation to remain true, the same thing must be done to both sides of the equal sign. If we add or subtract the same number on each side, or multiply or divide each side by the same number, the new equation will still be true.

This way of thinking can help us find solutions to equations. Instead of checking different values, we can think about subtracting the same amount from each side or dividing each side by the same number.

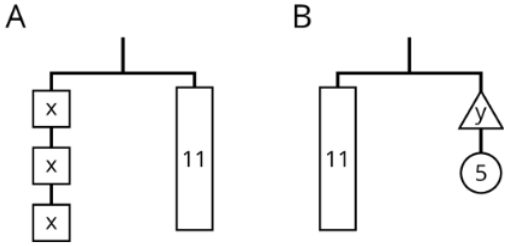


Figure 31.3.5: Two balanced hangers, A and B. A, left side, 3 identical squares, x, right side, 1 rectangle, 11. B, left side 1 rectangle, 11, right side, 1 triangle, y, and 1 circle, 5.

Diagram A can be represented by the equation $3x = 11$.

If we break the 11 into 3 equal parts, each part will have the same weight as a block with an x.

Splitting each side of the hanger into 3 equal parts is the same as dividing each side of the equation by 3.

- $3x$ divided by 3 is x.
- 11 divided by 3 is $\frac{11}{3}$.
- If $3x = 11$ is true, then $x = \frac{11}{3}$ is true.
- The solution to $3x = 11$ is $\frac{11}{3}$.

Diagram B can be represented with the equation $11 = y + 5$.

If we remove a weight of 5 from each side of the hanger, it will stay in balance.

Removing 5 from each side of the hanger is the same as subtracting 5 from each side of the equation.

- $11 - 5$ is 6.
- $y + 5 - 5$ is y.
- If $11 = y + 5$ is true, then $6 = y$ is true.
- The solution to $11 = y + 5$ is 6.

Glossary Entries

Definition: Coefficient

A coefficient is a number that is multiplied by a variable.

For example, in the expression $3x + 5$, the coefficient of x is 3. In the expression $y + 5$, the coefficient of y is 1, because $y = 1 \cdot y$.

Definition: Solution to an Equation

A solution to an equation is a number that can be used in place of the variable to make the equation true.

For example, 7 is the solution to the equation $m + 1 = 8$, because it is true that $7 + 1 = 8$. The solution to $m + 1 = 8$ is not 9, because $9 + 1 \neq 8$.

Definition: Variable

A variable is a letter that represents a number. You can choose different numbers for the value of the variable.

For example, in the expression $10 - x$, the variable is x. If the value of x is 3, then $10 - x = 7$, because $10 - 3 = 7$. If the value of x is 6, then $10 - x = 4$, because $10 - 6 = 4$.

Practice

Exercise 31.3.4

Select **all** the equations that represent the hanger.

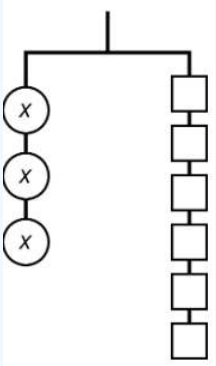


Figure 31.3.6

A. $x + x + x = 1 + 1 + 1 + 1 + 1 + 1$
B. $x \cdot x \cdot x = 6$
C. $3x = 6$
D. $x + 3 = 6$
E. $x \cdot x \cdot x = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1$

Exercise 31.3.5

Write an equation to represent each hanger.

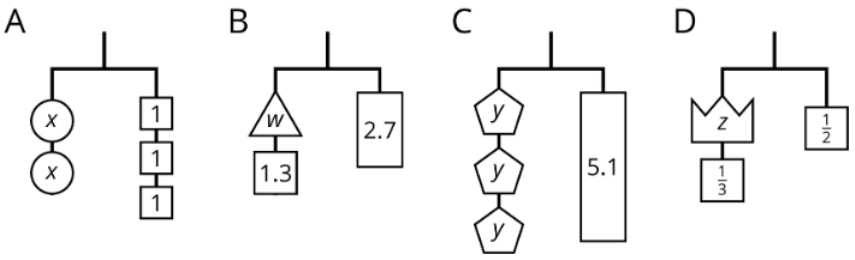


Figure 31.3.7: Four balanced hangers, A, B, C, and D. A, left side, 2 identical circles, x, right side, 3 identical squares, 1. B, left side, 1 triangle, w, 1 square, 1 point 3, right side, 1 rectangle, 2 point 7. C, left side, 3 identical pentagons, y, right side, 1 rectangle, 5 point 1. D, left side, 1 crown, z, 1 square, one third, right side, 1 square, one half.

Exercise 31.3.6

1. Write an equation to represent the hanger.
2. Explain how to reason with the hanger to find the value of x.
3. Explain how to reason with the equation to find the value of x.

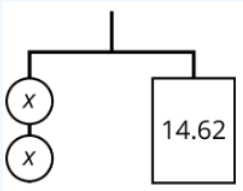


Figure 31.3.8

Exercise 31.3.7

Andre says that x is 7 because he can move the two 1s with the x to the other side.

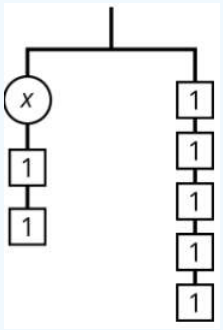


Figure 31.3.9

Do you agree with Andre? Explain your reasoning.

Exercise 31.3.8

Match each equation to one of the diagrams.

1. $12 - m = 4$
2. $12 = 4 \cdot m$
3. $m - 4 = 12$
4. $\frac{m}{4} = 12$

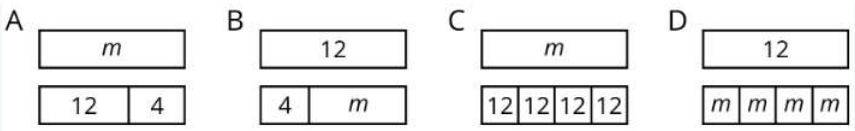


Figure 31.3.10: Four tape diagrams labeled A, B, C, and D. Tape diagram A has 2 bars of equal length. The top bar is labeled m. The bottom bar is partitioned into 2 parts labeled 12 and 4. Tape diagram B has 2 bars of equal length. The top bar is labeled 12. The bottom bar is partitioned into 2 parts labeled 4 and m. Tape diagram C has 2 bars of equal length. The top bar is labeled m. The bottom bar is partitioned into 4 parts labeled 12, 12, 12, and 12. Tape diagram D has 2 bars of equal length. The top bar is labeled 12. The bottom bar is partitioned into 4 parts labeled m, m, m, and m.

(From Unit 6.1.1)

Exercise 31.3.9

The area of a rectangle is 14 square units. It has side lengths x and y. Given each value for x, find y.

1. $x = 2\frac{1}{3}$
2. $x = 4\frac{1}{5}$
3. $x = \frac{7}{6}$

(From Unit 4.4.2)

Exercise 31.3.10

Lin needs to save up $20 for a new game. How much money does she have if she has saved each percentage of her goal. Explain your reasoning.

1. 25%
2. 75%
3. 125%

(From Unit 3.4.2)

31.3: Staying in Balance is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

31.4: Practice Solving Equations and Representing Situations with Equations

Lesson

Let's solve equations by doing the same to each side.

Exercise 31.4.1: Number Talk: Subtracting From Five

Find the value of each expression mentally.

$5 - 2$

$5 - 2.1$

$5 - 2.17$

$5 - 2\frac{7}{8}$

Exercise 31.4.2: Row Game: Solving Equations Practice

Solve the equations in one column. Your partner will work on the other column.

Check in with your partner after you finish each row. Your answers in each row should be the same. If your answers aren't the same, work together to find the error and correct it.

Table 31.4.1

column A	column B
$18 = 2x$	$36 = 4x$
$17 = x + 9$	$13 = x + 5$
$8x = 56$	$3x = 21$
$21 = \frac{1}{4}x$	$28 = \frac{1}{3}x$
$6x = 45$	$8x = 60$
$x + 4\frac{5}{6} = 9$	$x + 3\frac{5}{5} = 8$
$\frac{5}{7}x = 55$	$\frac{3}{7}x = 33$
$\frac{1}{5} = 6x$	$\frac{1}{3} = 10x$
$2.17 + x = 5$	$6.17 + x = 9$
$\frac{20}{3} - \frac{10}{9}x$	$\frac{14}{5} = \frac{7}{15}x$
$14.88 + x = 17.05$	$3.91 + x = 6.08$
$3\frac{3}{4}x = 1\frac{1}{4}$	$\frac{7}{5}x = \frac{7}{15}$

Exercise 31.4.3: Choosing Equations to Match Situations

Circle **all** of the equations that describe each situation. If you get stuck, consider drawing a diagram. Then find the solution for each situation.

1. Clare has 8 fewer books than Mai. If Mai has 26 books, how many books does Clare have?
   - $26 - x = 8$
   - $x = 26 + 8$
   - $26 - 8 = x$
   - $x =$ ____

2. A coach formed teams of 8 from all the players in a soccer league. There are 14 teams. How many players are in the league?
   - $y = 14 \div 8$
   - $\frac{y}{8} = 14$
   - $\frac{1}{8}y = 14$
   - $y = 14 \cdot 8$
   - $y =$ ____
3. Kiran scored 223 more points in a computer game than Tyler. If Kiran scored 409 points, how many points did Tyler score?
   - $223 = 409 - z$
   - $409 - 223 = z$
   - $409 + 223 = z$
   - $409 = 223 + z$
   - $z =$ ____
4. Mai ran 27 miles last week, which was three times as far as Jada ran. How far did Jada run?
   - $3w = 27$
   - $w = \frac{1}{3} \cdot 27$
   - $w = 27 \div 3$
   - $w = 3 \cdot 27$
   - $w =$ ____

Are you ready for more?

Mai's mother was 28 when Mai was born. Mai is now 12 years old. In how many years will Mai's mother be twice Mai's age? How old will they be then?

Summary

Writing and solving equations can help us answer questions about situations.

Suppose a scientist has 13.68 liters of acid and needs 16.05 liters for an experiment. How many more liters of acid does she need for the experiment?

- We can represent this situation with the equation:

$$13.68 + x = 16.05$$

- When working with hangers, we saw that the solution can be found by subtracting 13.68 from each side. This gives us some new equations that also represent the situation:

$$x = 16.05 - 13.68$$

- Finding a solution in this way leads to a variable on one side of the equal sign and a number on the other. We can easily read the solution—in this case, 2.37—from an equation with a letter on one side and a number on the other. We often write solutions in this way.

$$x = 2.37$$

Let's say a food pantry takes a 54-pound bag of rice and splits it into portions that each weigh $\frac{3}{4}$ of a pound. How many portions can they make from this bag?

- We can represent this situation with the equation:

$$\frac{3}{4}x = 54$$

- We can find the value of x by dividing each side by $\frac{3}{4}$. This gives us some new equations that represent the same situation:

$$x = 54 \div \frac{3}{4}$$

- The solution is 72 portions.

$$x = 72$$

Glossary Entries

Definition: Coefficient

A coefficient is a number that is multiplied by a variable.

For example, in the expression $3x+5$, the coefficient of x is 3. In the expression $y+5$, the coefficient of y is 1, because $y=1\cdot y$.

Definition: Solution to an Equation

A solution to an equation is a number that can be used in place of the variable to make the equation true.

For example, 7 is the solution to the equation $m+1=8$, because it is true that $7+1=8$. The solution to $m+1=8$ is not 9, because $9+1\neq 8$.

Definition: Variable

A variable is a letter that represents a number. You can choose different numbers for the value of the variable.

For example, in the expression $10-x$, the variable is x. If the value of x is 3, then $10-x=7$, because $10-3=7$. If the value of x is 6, then $10-x=4$, because $10-6=4$.

Practice

Exercise 31.4.4

Select **all** the equations that describe each situation and then find the solution.

1. Kiran's backpack weighs 3 pounds less than Clare's backpack. Clare's backpack weighs 14 pounds. How much does Kiran's backpack weigh?
   - $x+3=14$
   - $3x=14$
   - $x=14-3$
   - $x=14\div 3$
2. Each notebook contains 60 sheets of paper. Andre has 5 notebooks. How many sheets of paper do Andre's notebooks contain?
   - $y=60\div 5$
   - $y=5\cdot 60$
   - $\frac{y}{5}=60$
   - $5y=60$

Exercise 31.4.5

Solve each equation.

1. $2x=5$
2. $y+1.8=14.7$
3. $6=\frac{1}{2}z$
4. $3\frac{1}{4}=\frac{1}{2}+w$
5. $2.5t=10$

Exercise 31.4.6

For each equation, draw a tape diagram that represents the equation.

1. $3\cdot x=18$
2. $3+x=18$

3. $17 - 6 = x$

(From Unit 6.1.1)

Exercise 31.4.7

Find each product.

$$(21.2) \cdot (0.02) \qquad (2.05) \cdot (0.004)$$

(From Unit 5.3.4)

Exercise 31.4.8

For a science experiment, students need to find 25% of 60 grams.

- Jada says, "I can find this by calculating $\frac{1}{4}$ of 60."
- Andre says, "25% of 60 means $\frac{25}{100} \cdot 60$."

Do you agree with either of them? Explain your reasoning.

(From Unit 3.4.4)

31.4: Practice Solving Equations and Representing Situations with Equations is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

31.5: A New Way to Interpret a over b

Lesson

Let's investigate what a fraction means when the numerator and denominator are not whole numbers.

Exercise 31.5.1: Recalling Ways of Solving

Solve each equation. Be prepared to explain your reasoning.

$$0.07 = 10m \qquad 10.1 = t + 7.2$$

Exercise 31.5.2: Interpreting $\frac{a}{b}$

Solve each equation.

1. $35 = 7x$
2. $35 = 11x$
3. $7x = 7.7$
4. $0.3x = 2.1$
5. $\frac{2}{5} = \frac{1}{2}x$

Are you ready for more?

Solve the equation. Try to find some shortcuts.

$$\frac{1}{6} \cdot \frac{3}{20} \cdot \frac{5}{42} \cdot \frac{7}{72} \cdot x = \frac{1}{384}$$

Exercise 31.5.3: Storytime Again

Take turns with your partner telling a story that might be represented by each equation. Then, for each equation, choose one story, state what quantity x describes, and solve the equation. If you get stuck, consider drawing a diagram.

$$0.7 + x = 12 \qquad \frac{1}{4}x = \frac{3}{2}$$

Summary

In the past, you learned that a fraction such as $\frac{4}{5}$ can be thought of in a few ways.

- $\frac{4}{5}$ is a number you can locate on the number line by dividing the section between 0 and 1 into 5 equal parts and then counting 4 of those parts to the right of 0.
- $\frac{4}{5}$ is the share that each person would have if 4 wholes were shared equally among 5 people. This means that $\frac{4}{5}$ is the result of *dividing* 4 by 5.

We can extend this meaning of *a fraction as a quotient* to fractions whose numerators and denominators are not whole numbers. For example, we can represent 4.5 pounds of rice divided into portions that each weigh 1.5 pounds as: $\frac{4.5}{1.5} = 4.5 \div 1.5 = 3$. In other words, $\frac{4.5}{1.5} = 3$ because the quotient of 4.5 and 1.5 is 3.

Fractions that involve non-whole numbers can also be used when we solve equations.

Suppose a road under construction is $\frac{3}{8}$ finished and the length of the completed part is $\frac{4}{3}$ miles. How long will the road be when completed?

We can write the equation $\frac{3}{8}x = \frac{4}{3}$ to represent the situation and solve the equation.

The completed road will be $3\frac{5}{9}$ or about 3.6 miles long.

$$\frac{3}{8}x = \frac{4}{3}$$
$$x = \frac{\frac{4}{3}}{\frac{3}{8}}$$
$$x = \frac{4}{3} \cdot \frac{8}{3}$$
$$x = \frac{32}{9} = 3\frac{5}{9}$$

Glossary Entries

Definition: Coefficient

A coefficient is a number that is multiplied by a variable.

For example, in the expression $3x+5$, the coefficient of x is 3. In the expression $y+5$, the coefficient of y is 1, because $y = 1 \cdot y$.

Definition: Solution to an Equation

A solution to an equation is a number that can be used in place of the variable to make the equation true.

For example, 7 is the solution to the equation $m+1=8$, because it is true that $7+1=8$. The solution to $m+1=8$ is not 9, because $9+1 \neq 8$.

Definition: Variable

A variable is a letter that represents a number. You can choose different numbers for the value of the variable.

For example, in the expression $10-x$, the variable is x. If the value of x is 3, then $10-x=7$, because $10-3=7$. If the value of x is 6, then $10-x=4$, because $10-6=4$.

Practice

Exercise 31.5.4

Select **all** the expressions that equal $\frac{3.15}{0.45}$.

A. $(3.15) \cdot (0.45)$
B. $(3.15) \div (0.45)$
C. $(3.15) \cdot \frac{1}{0.45}$
D. $(3.15) \div \frac{45}{100}$
E. $(3.15) \cdot \frac{100}{45}$
F. $(\frac{0.45}{3.15}$

Exercise 31.5.5

Which expressions are solutions to the equation $\frac{3}{4}x = 15$? Select **all** that apply.

A. $\frac{15}{\frac{3}{4}}$
B. $\frac{15}{\frac{4}{3}}$
C. $\frac{4}{3} \cdot 15$
D. $\frac{3}{4} \cdot 15$
E. $15 \div \frac{3}{4}$

Exercise 31.5.6

Solve each equation.

$$4a = 32 \qquad 4 = 32b \qquad 10c = 26 \qquad 26 = 100d$$

Exercise 31.5.7

For each equation, write a story problem represented by the equation. For each equation, state what quantity x represents. If you get stuck, consider drawing a diagram.

1. $\frac{3}{4} + x = 2$
2. $1.5x = 6$

Exercise 31.5.8

Write as many mathematical expressions or equations as you can about the image. Include a fraction, a decimal number, or a percentage in each.



Figure 31.5.1: A fundraiser thermometer labeled Fundraiser, our goal, 2 hundred 50 thousand dollars. The numbers 50 thousand through 2 hundred 50 thousand, in increments of 50 thousand dollars, are indicated. There are 4 evenly spaced tick marks between each indicated dollar value. Starting from the bottom, the thermometer is shaded to the first tick mark above 1 hundred thousand dollars.

(From Unit 3.4.4)

Exercise 31.5.9

In a lilac paint mixture, 40% of the mixture is white paint, 20% is blue, and the rest is red. There are 4 cups of blue paint used in a batch of lilac paint.

1. How many cups of white paint are used?
2. How many cups of red paint are used?
3. How many cups of lilac paint will this batch yield?

If you get stuck, consider using a tape diagram.

(From Unit 3.4.3)

Exercise 31.5.10

Triangle P has a base of 12 inches and a corresponding height of 8 inches. Triangle Q has a base of 15 inches and a corresponding height of 6.5 inches. Which triangle has a greater area? Show your reasoning.

(From Unit 1.3.3)

31.5: A New Way to Interpret a over b is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

Section 32: Equal and Equivalent

This page titled Section 32: Equal and Equivalent is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

32.1: Write Expressions Where Letters Stand for Numbers

Lesson

Let's use expressions with variables to describe situations.

Exercise 32.1.1: Algebra Talk: When x is 6

If x is 6, what is:

$x + 4$

$7 - x$

x^2

$\frac{1}{3}x$

Exercise 32.1.2: Lemonade Sales and Heights

1. Lin set up a lemonade stand. She sells the lemonade for $0.50 per cup.
   a. Complete the table to show how much money she would collect if she sold each number of cups.

Table 32.1.1

lemonade sold (number of cups)	12	183	c
money collected (dollars)			

   b. How many cups did she sell if she collected $127.50? Be prepared to explain your reasoning.

2. Elena is 59 inches tall. Some other people are taller than Elena.
   a. Complete the table to show the height of each person.

Table 32.1.2

person	**Andre**	**Lin**	**Noah**
how much taller than Elena (inches)	4	$6\frac{1}{2}$	d
person's height (inches)			

   b. If Noah is $64\frac{3}{4}$ inches tall, how much taller is he than Elena?

Exercise 32.1.3: Building Expressions

1. Clare is 5 years older than her cousin.
   a. How old would Clare be if her cousin is:
      10 years old?
      2 years old?
      x years old?
   b. Clare is 12 years old. How old is Clare's cousin?
2. Diego has 3 times as many comic books as Han.
   a. How many comic books does Diego have if Han has:
      6 comic books?
      n books?
   b. Diego has 27 comic books. How many comic books does Han have?
3. Two fifths of the vegetables in Priya's garden are tomatoes.

a. How many tomatoes are there if Priya's garden has:
   20 vegetables?
   x vegetables?
b. Priya's garden has 6 tomatoes. How many total vegetables are there?

4. A school paid $31.25 for each calculator.
   a. If the school bought x calculators, how much did they pay?
   b. The school spent $500 on calculators. How many did the school buy?

Are you ready for more?

Kiran, Mai, Jada, and Tyler went to their school carnival. They all won chips that they could exchange for prizes. Kiran won $\frac{2}{3}$ as many chips as Jada. Mai won 4 times as many chips as Kiran. Tyler won half as many chips as Mai.

1. Write an expression for the number of chips Tyler won. You should only use one variable: J, which stands for the number of chips Jada won.
2. If Jada won 42 chips, how many chips did Tyler, Kiran, and Mai each win?

Summary

Suppose you share a birthday with a neighbor, but she is 3 years older than you. When you were 1, she was 4. When you were 9, she was 12. When you are 42, she will be 45.

If we let a represent your age at any time, your neighbor's age can be expressed $a+3$.

your age	1	9	42	a
neighbor's age	4	12	45	$a+3$

Table 32.1.3

We often use a letter such as x or a as a placeholder for a number in expressions. These are called *variables* (just like the letters we used in equations, previously). Variables make it possible to write expressions that represent a calculation even when we don't know all the numbers in the calculation.

How old will you be when your neighbor is 32? Since your neighbor's age is calculated with the expression $a+3$, we can write the equation $a+3=32$. When your neighbor is 32 you will be 29, because $a+3=32$ is true when a is 29.

Practice

Exercise 32.1.4

Instructions for a craft project say that the length of a piece of red ribbon should be 7 inches less than the length of a piece of blue ribbon.

1. How long is the red ribbon if the length of the blue ribbon is:
   10 inches?
   27 inches?
   x inches?
2. How long is the blue ribbon if the red ribbon is 12 inches?

Exercise 32.1.5

Tyler has 3 times as many books as Mai.

1. How many books does Mai have if Tyler has:
   15 books?
   21 books?
   x books?
2. Tyler has 18 books. How many books does Mai have?

Exercise 32.1.6

A bottle holds 24 ounces of water. It has x ounces of water in it.

1. What does $24 - x$ represent in this situation?
2. Write a question about this situation that has $24 - x$ for the answer.

Exercise 32.1.7

Write an equation represented by this tape diagram using each of these operations.

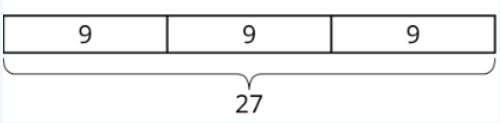


Figure 32.1.1

1. addition
2. subtraction
3. multiplication
4. division

(From Unit 6.1.1)

Exercise 32.1.8

Select **all** the equations that describe each situation and then find the solution.

1. Han's house is 450 meters from school. Lin's house is 135 meters closer to school. How far is Lin's house from school?
   - $z = 450 + 135$
   - $z = 450 - 135$
   - $z - 135 = 450$
   - $z + 135 = 450$
2. Tyler's playlist has 36 songs. Noah's playlist has one quarter as many songs as Tyler's playlist. How many songs are on Noah's playlist?
   - $w = 4 \cdot 36$
   - $w = 36 \div 4$
   - $4w = 36$
   - $\frac{w}{4} = 36$

(From Unit 6.1.4)

Exercise 32.1.9

You had $50. You spent 10% of the money on clothes, 20% on games, and the rest on books. How much money was spent on books?

(From Unit 3.4.3)

Exercise 32.1.10

A trash bin has a capacity of 50 gallons. What percentage of its capacity is each amount? Show your reasoning.

1. 5 gallons
2. 30 gallons
3. 45 gallons
4. 100 gallons

(From Unit 3.4.5)

32.1: Write Expressions Where Letters Stand for Numbers is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

32.2: Revisit Percentages

Lesson

Lets use equations to find percentages.

Exercise 32.2.1: Number Talk: Percentages

Solve each problem mentally.

1. Bottle A contains 4 ounces of water, which is 25% of the amount of water in Bottle B. How much water is there in Bottle B?
2. Bottle C contains 150% of the water in Bottle B. How much water is there in Bottle C?
3. Bottle D contains 12 ounces of water. What percentage of the amount of water in Bottle B is this?

Exercise 32.2.2: Representing a Percentage Problem with an Equation

1. Answer each question and show your reasoning.
   a. Is 60% of 400 equal to 87?
   b. Is 60% of 200 equal to 87?
   c. Is 60% of 120 equal to 87?
2. 60% of x is equal to 87. Write an equation that expresses the relationship between 60%, x, and 87. Solve your equation.
3. Write an equation to help you find the value of each variable. Solve the equation.
   60% of c is 43.2.
   38% of e is 190.

Exercise 32.2.3: Puppies Grow Up, Revisited

1. Puppy A weighs 8 pounds, which is about 25% of its adult weight. What will be the adult weight of Puppy A?
2. Puppy B weighs 8 pounds, which is about 75% of its adult weight. What will be the adult weight of Puppy B?
3. If you haven't already, write an equation for each situation. Then, show how you could find the adult weight of each puppy by solving the equation.

Are you ready for more?

Diego wants to paint his room purple. He bought one gallon of purple paint that is 30% red paint and 70% blue paint. Diego wants to add more blue to the mix so that the paint mixture is 20% red, 80% blue.

1. How much blue paint should Diego add? Test the following possibilities: 0.2 gallons, 0.3 gallons, 0.4 gallons, 0.5 gallons.
2. Write an equation in which x represents the amount of paint Diego should add.
3. Check that the amount of paint Diego should add is a solution to your equation.

Summary

If we know that 455 students are in school today and that number represents 70% attendance, we can write an equation to figure out how many students go to the school.

The number of students in school today is known in two different ways: as 70% of the students in the school, and also as 455. If s represents the total number of students who go to the school, then 70% of s, or $\frac{70}{100}s$, represents the number of students that are in school today, which is 455.

We can write and solve the equation:

$$\frac{70}{100}s = 455$$
$$s = 455 \div \frac{70}{100}$$
$$3 = 455 \cdot \frac{100}{70}$$
$$s = 650$$

There are 650 students in the school.

In general, equations can help us solve problems in which one amount is a percentage of another amount.

Exercise 32.2.4

A crew has paved $\frac{3}{4}$ of a mile of road. If they have completed 50% of the work, how long is the road they are paving?

Exercise 32.2.5

40% of x is 35.

1. Write an equation that shows the relationship of 40%, x, and 35.
2. Use your equation to find x. Show your reasoning.

Exercise 32.2.6

Priya has completed 9 exam questions. This is 60% of the questions on the exam.

1. Write an equation representing this situation. Explain the meaning of any variables you use.
2. How many questions are on the exam? Show your reasoning.

Exercise 32.2.7

Answer each question. Show your reasoning.

20% of a is 11. What is a?

75% of b is 12. What is b?

80% of c is 20. What is c?

200% of d is 18. What is d?

Exercise 32.2.8

For the equation $2n - 3 = 7$

1. What is the variable?
2. What is the coefficient of the variable?
3. Which of these is the solution to the equation? $2, 3, 5, 7, n$

(From Unit 6.1.2)

Exercise 32.2.9

Which of these is a solution to the equation $\frac{1}{8} = \frac{2}{5} \cdot x$?

A. $\frac{2}{40}$
B. $\frac{5}{16}$
C. $\frac{11}{40}$
D. $\frac{17}{40}$

(From Unit 6.1.2)

Exercise 32.2.10

Find the quotients.

1. $0.009 \div 0.001$
2. $0.009 \div 0.002$
3. $0.0045 \div 0.001$
4. $0.0045 \div 0.002$

(From Unit 5.4.5)

32.2: Revisit Percentages is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

32.3: Equal and Equivalent

Lesson

Let's use diagrams to figure out which expressions are equivalent and which are just sometimes equal.

Exercise 32.3.1: Algebra Talk: Solving Equations by Seeing Structure

Find a solution to each equation mentally.

$3 + x = 8$

$10 = 12 - x$

$x^2 = 49$

$\frac{1}{3}x = 6$

Exercise 32.3.2: Using Diagrams to Show That Expressions are Equivalent

Here is a diagram of $x + 2$ and $3x$ when x is 4. Notice that the two diagrams are lined up on their left sides.

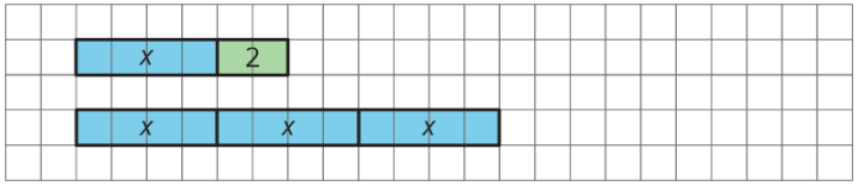


Figure 32.3.1: Two tape diagrams on a grid. Top diagram 2 parts, x, 2. The part labeled x is composed of four unit squares. The part labeled 2 is composed of two unit squares. Bottom diagram 3 parts x,x,x, each part is composed of four unit squares.

In each of your drawings below, line up the diagrams on one side.

1. Draw a diagram of $x + 2$, and a separate diagram of $3x$, when x is 3.

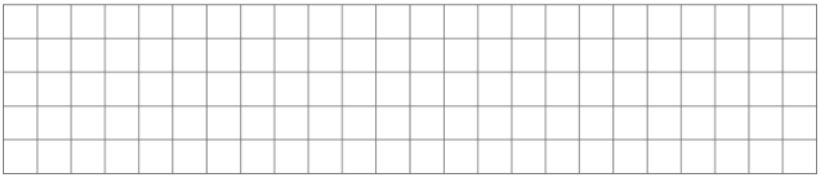


Figure 32.3.2

2. Draw a diagram of $x + 2$, and a separate diagram of $3x$, when x is 2.

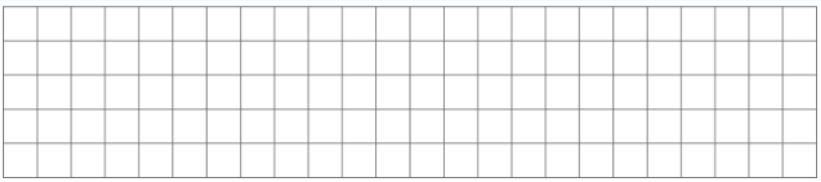


Figure 32.3.3

3. Draw a diagram of $x + 2$, and a separate diagram of $3x$, when x is 1.

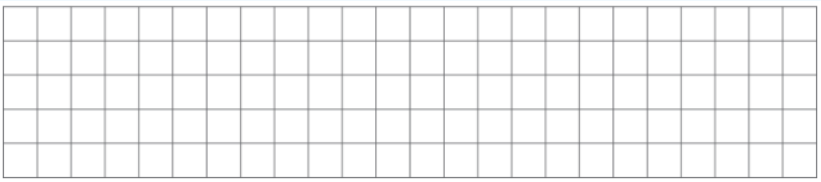


Figure 32.3.4

4. Draw a diagram of $x + 2$, and a separate diagram of $3x$, when x is 0.

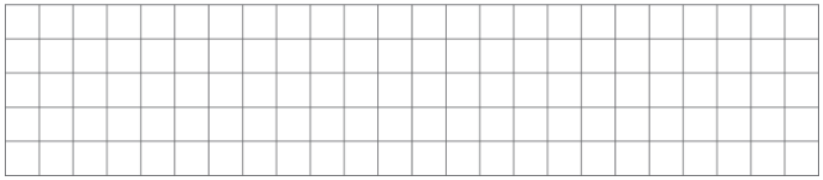


Figure 32.3.5

5. When are $x+2$ and $3x$ equal? When are they not equal? Use your diagrams to explain.
6. Draw a diagram of $x+3$, and a separate diagram of $3+x$.
7. When are $x+3$ and $3+x$ equal? When are they not equal? Use your diagrams to explain.

Exercise 32.3.3: Identifying Equivalent Expressions

Here is a list of expressions. Find any pairs of expressions that are equivalent. If you get stuck, try reasoning with diagrams.

$a+3$	$a \div \frac{1}{3}$	$\frac{1}{3}a$	$\frac{a}{3}$	a
$a+a+a$	$a \cdot 3$	$3a$	$1a$	$3+a$

Are you ready for more?

Below are four questions about equivalent expressions. For each one:

- Decide whether you think the expressions are equivalent.
- Test your guess by choosing numbers for x (and y, if needed).

1. Are $\frac{x \cdot x \cdot x \cdot x}{x}$ and $x \cdot x \cdot x$ equivalent expressions?
2. Are $\frac{x+x+x+x}{x}$ and $x+x+x$ equivalent expressions?
3. Are $2(x+y)$ and $2x+2y$ equivalent expressions?
4. Are $2xy$ and $2x \cdot 2y$ equivalent expressions?

Summary

We can use diagrams showing lengths of rectangles to see when expressions are equal. For example, the expressions $x+9$ and $4x$ are equal when x is 3, but are not equal for other values of x.

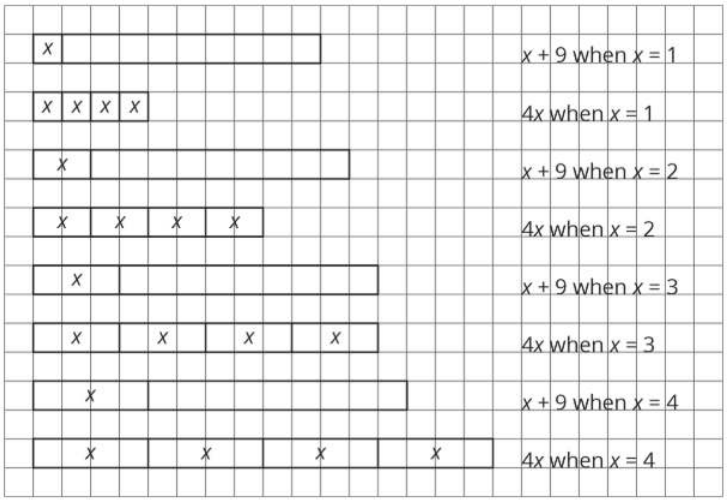


Figure 32.3.6: 8 tape diagrams on a grid with matching expressions. First diagram composed of 1 square unit labeled x and 9 square units combined which are blank, matched with x+9 when x=1. Second diagram composed of 4 square units each labeled x matched with 4x when x=1. Third diagram composed 2 combined square units labeled x and 9 combined square units blank, matched with x+9 when x=2. Fourth diagram composed of 2 combined square units labeled x created 4 total times, matched with 4x when x=2. Fifth diagram composed 3 combined square units labeled x and 9 combined square units blank, matched with x+9 when x=3. Sixth diagram composed 3 combined square units labeled x created 4 total times, matched with 4x when x=3. Seventh diagram composed of 4 combined square units labeled x and 9 combined square units blank, matched with x+9 when x=4. Eighth diagram composed of 4 combined square units labeled x created 4 total times, matched with 4x when x=4.

Sometimes two expressions are equal for only one particular value of their variable. Other times, they seem to be equal no matter what the value of the variable.

Expressions that are always equal for the same value of their variable are called **equivalent expressions**. However, it would be impossible to test every possible value of the variable. How can we know for sure that expressions are equivalent? We use the meaning of operations and properties of operations to know that expressions are equivalent. Here are some examples:

- $x+3$ is equivalent to $3+x$ because of the commutative property of addition.
- $4 \cdot y$ is equivalent to $y \cdot 4$ because of the commutative property of multiplication.
- $a+a+a+a+a$ is equivalent to $5 \cdot a$ because adding 5 copies of something is the same as multiplying it by 5.
- $b \div 3$ is equivalent to $b \cdot \frac{1}{3}$ because dividing by a number is the same as multiplying by its reciprocal.

In the coming lessons, we will see how another property, the distributive property, can show that expressions are equivalent.

Glossary Entries

Definition: Equivalent Expressions

Equivalent expressions are always equal to each other. If the expressions have variables, they are equal whenever the same value is used for the variable in each expression.

For example, $3x+4x$ is equivalent to $5x+2x$. No matter what value we use for x, these expressions are always equal. When x is 3, both expressions equal 21. When x is 10, both expressions equal 70.

Practice

Exercise 32.3.4

1. Draw a diagram of $x+3$ and a diagram of $2x$ when x is 1.

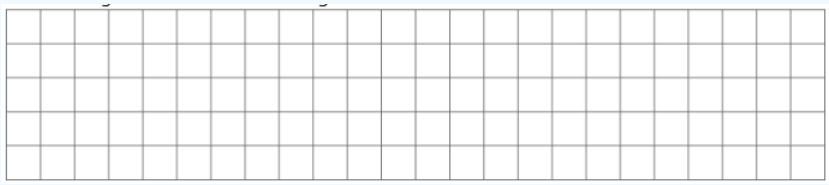


Figure 32.3.7

2. Draw a diagram of $x+3$ and of $2x$ when x is 2.

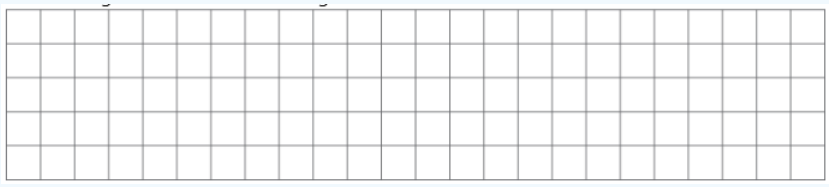


Figure 32.3.8

3. Draw a diagram of $x+3$ and of $2x$ when x is 3.

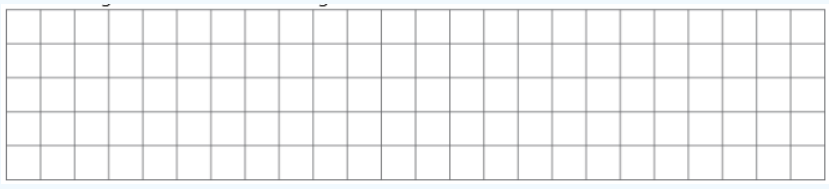


Figure 32.3.9

4. Draw a diagram of $x+3$ and of $2x$ when x is 4.

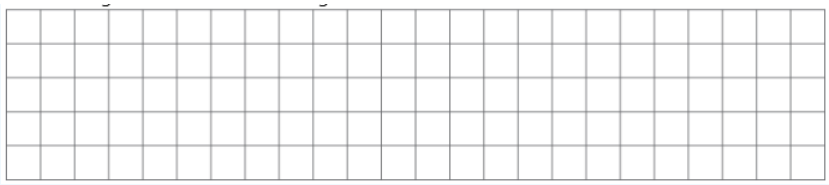


Figure 32.3.10

5. When are $x + 3$ and $2x$ equal? When are they not equal? Use your diagrams to explain.

Exercise 32.3.5

1. Do $4x$ and $15 + x$ have the same value when x is 5?
2. Are $4x$ and $15 + x$ equivalent expressions? Explain your reasoning.

Exercise 32.3.6

1. Check that $2b + b$ and $3b$ have the same value when b is 1, 2, and 3.
2. Do $2b + b$ and $3b$ have the same value for all values of b? Explain your reasoning.
3. Are $2b + b$ and $3b$ equivalent expressions?

Exercise 32.3.7

80% of x is equal to 100.

1. Write an equation that shows the relationship of 80%, x, and 100.
2. Use your equation to find x.

(From Unit 6.2.2)

Exercise 32.3.8

For each story problem, write an equation to represent the problem and then solve the equation. Be sure to explain the meaning of any variables you use.

1. Jada's dog was $5\frac{1}{2}$ inches tall when it was a puppy. Now her dog is $14\frac{1}{2}$ inches taller than that. How tall is Jada's dog now?
2. Lin picked $9\frac{3}{4}$ pounds of apples, which was 3 times the weight of the apples Andre picked. How many pounds of apples did Andre pick?

(From Unit 6.1.5)

Exercise 32.3.9

Find these products.

1. $(2.3) \cdot (1.4)$
2. $(1.72) \cdot (2.6)$
3. $(18.2) \cdot (0.2)$
4. $15 \cdot (1.2)$

(From Unit 5.3.4)

Exercise 32.3.10

Calculate $141.75 \div 2.5$ using a method of your choice. Show or explain your reasoning.

(From Unit 5.4.5)

32.3: Equal and Equivalent is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

32.4: The Distributive Property, Part 1

Lesson

Let's use the distributive property to make calculating easier.

Exercise 32.4.1: Number Talk: Ways to Multiply

Find each product mentally.

$5 \cdot 102$

$5 \cdot 98$

$5 \cdot 999$

Exercise 32.4.2: Ways to Represent Area of a Rectangle

1. Select **all** the expressions that represent the area of the large, outer rectangle in figure A. Explain your reasoning.
   - $6+3+2$
   - $6 \cdot 3+6 \cdot 2$
   - $6 \cdot 3+2$
   - $6 \cdot 5$
   - $6(3+2)$
   - $6 \cdot 3 \cdot 2$

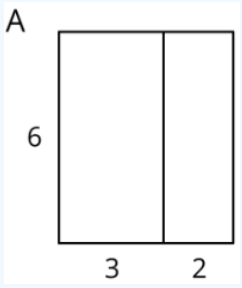


Figure 32.4.1: One rectangle diagram. Labeled A. Large outer rectangle partitioned into 2 smaller rectangles, vertical side 6. First rectangle bottom side length 3. Second rectangle bottom side length 2.

2. Select **all** the expressions that represent the area of the shaded rectangle on the left side of figure B. Explain your reasoning.
   - $4 \cdot 7+4 \cdot 2$
   - $4 \cdot 7 \cdot 2$
   - $4 \cdot 5$
   - $4 \cdot 7-4 \cdot 2$
   - $4(7-2)$
   - $4(7+2)$
   - $4 \cdot 2-4 \cdot 7$

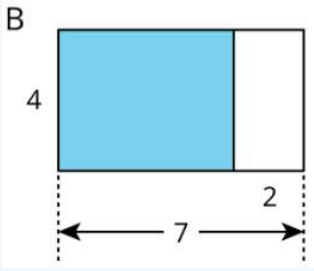


Figure 32.4.2: One rectangle diagram. Labeled B. Large rectangle with bottom side length 7, partitioned into 2 smaller rectangles. First rectangle vertical side 4, bottom side length blank, area shaded blue. Second rectangle vertical side 4, bottom side length 2.

Exercise 32.4.3: Distributive Practice

Complete the table. If you get stuck, skip an entry and come back to it, or consider drawing a diagram of two rectangles that share a side.

Table 32.4.1

column 1	column 2	column 3	column 4	value
$5 \cdot 98$	$5(100-2)$	$5 \cdot 100 - 5 \cdot 2$	$500 - 10$	490
$33 \cdot 12$	$33(10+2)$			
		$3 \cdot 10 - 3 \cdot 4$	$30 - 12$	
	$100(0.04+0.06)$			
		$8 \cdot \frac{1}{2} + 8 \cdot \frac{1}{4}$		
			$9+12$	
			$24-16$	

Are you ready for more?

1. Use the distributive property to write two expressions that equal 360. (There are many correct ways to do this.)
2. Is it possible to write an expression like $a(b+c)$ that equals 360 where a is a fraction? Either write such an expression, or explain why it is impossible.
3. Is it possible to write an expression like $a(b-c)$ that equals 360? Either write such an expression, or explain why it is impossible.
4. How many ways do you think there are to make 360 using the distributive property?

Summary

A **term** is a single number or variable, or variables and numbers multiplied together. Some examples of terms are 10, $8x$, ab, and $7yz$.

When we need to do mental calculations, we often come up with ways to make the calculation easier to do mentally.

Suppose we are grocery shopping and need to know how much it will cost to buy 5 cans of beans at 79 cents a can. We may calculate mentally in this way:

$$\begin{aligned} &5 \cdot 79 \\ &5 \cdot 70 + 5 \cdot 9 \\ &350 + 45 \\ &395 \end{aligned}$$

In general, when we multiply two terms (or factors), we can break up one of the factors into parts, multiply each part by the other factor, and then add the products. The result will be the same as the product of the two original factors. When we break up one of the factors and multiply the parts we are using the distributive property.

The distributive property also works with subtraction. Here is another way to find $5 \cdot 79$:

$$\begin{aligned} &5 \cdot 79 \\ &5 \cdot (80 - 1) \\ &400 - 5 \\ &395 \end{aligned}$$

Glossary Entries

Definition: Equivalent Expressions

Equivalent expressions are always equal to each other. If the expressions have variables, they are equal whenever the same value is used for the variable in each expression.

For example, $3x + 4x$ is equivalent to $5x + 2x$. No matter what value we use for x, these expressions are always equal. When x is 3, both expressions equal 21. When x is 10, both expressions equal 70.

Definition: Term

A term is a part of an expression. It can be a single number, a variable, or a number and a variable that are multiplied together. For example, the expression $5x + 18$ has two terms. The first term is $5x$ and the second term is 18.

Practice

Exercise 32.4.4

Select **all** the expressions that represent the area of the large, outer rectangle.

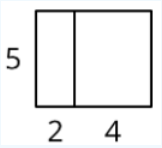


Figure 32.4.3: Area diagram. A large rectangle is partitioned vertically into two smaller rectangles. First rectangle, vertical side 5, bottom side length 2. Second rectangle, vertical side 5, bottom side length 4.

A. $5(2+4)$
B. $5 \cdot 2 + 4$
C. $5 \cdot 2 + 5 \cdot 4$
D. $5 \cdot 2 \cdot 4$
E. $5 + 2 + 4$
F. $5 \cdot 6$

Exercise 32.4.5

Draw and label diagrams that show these two methods for calculating $19 \cdot 50$.

- First find $10 \cdot 50$ and then add $9 \cdot 50$.
- First find $20 \cdot 50$ and then take away 50.

Exercise 32.4.6

Complete each calculation using the distributive property.

$98 \cdot 24 (100 - 2) \cdot 24 \ldots$

$21 \cdot 15 (20 + 1) \cdot 15 \ldots$

$0.51 \cdot 40 (0.5 + 0.01) \cdot 40 \ldots$

Exercise 32.4.7

A group of 8 friends go to the movies. A bag of popcorn costs $2.99. How much will it cost to get one bag of popcorn for each friend? Explain how you can calculate this amount mentally.

Exercise 32.4.8

1. On graph paper, draw diagrams of $a+a+a+a$ and $4a$ when a is 1, 2, and 3. What do you notice?
2. Do $a+a+a+a$ and $4a$ have the same value for any value of a? Explain how you know.

(From Unit 6.2.3)

Exercise 32.4.9

120% of x is equal to 78.

1. Write an equation that shows the relationship of 120%, x, and 78.
2. Use your equation to find x. Show your reasoning.

(From Unit 6.2.2)

Exercise 32.4.10

Kiran's aunt is 17 years older than Kiran.

1. How old will Kiran's aunt be when Kiran is:
   15 years old?
   30 years old?
   x years old?
2. How old will Kiran be when his aunt is 60 years old?

(From Unit 6.2.1)

32.4: The Distributive Property, Part 1 is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

32.5: The Distributive Property, Part 2

Lesson

Let's use rectangles to understand the distributive property with variables.

Exercise 32.5.1: Possible Areas

1. A rectangle has a width of 4 units and a length of m units. Write an expression for the area of this rectangle.
2. What is the area of the rectangle if m is:
   3 units?
   2.2 units?
   $\frac{1}{5}$ unit?
3. Could the area of this rectangle be 11 square units? Why or why not?

Exercise 32.5.2: Partitioned Rectangles When Lengths are Unknown

1. Here are two rectangles. The length and width of one rectangle are 8 and 5. The width of the other rectangle is 5, but its length is unknown so we labeled it x.
   Write an expression for the sum of the areas of the two rectangles.

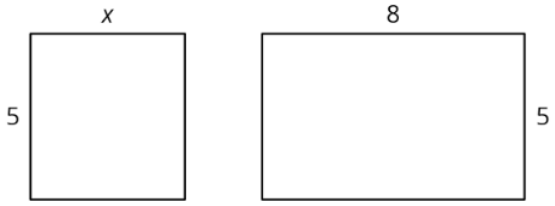


Figure 32.5.1

2. The two rectangles can be composed into one larger rectangle as shown.
   What are the width and length of the new, large rectangle?



Figure 32.5.2

3. Write an expression for the total area of the large rectangle as the product of its width and its length.

Exercise 32.5.3: Areas of Partitioned Rectangles

For each rectangle, write expressions for the length and width and two expressions for the total area. Record them in the table. Check your expressions in each row with your group and discuss any disagreements.

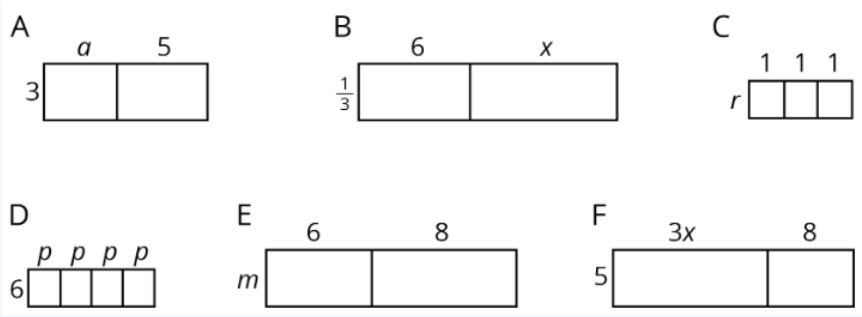


Figure 32.5.3: Six different sized rectangles labeled A, B, C, D, E, and F. Rectangle A is partitioned by a vertical line segment into two smaller rectangles. The vertical side is labeled 3 and the top horizontal side lengths are labeled a and 5. Rectangle B is partitioned by a vertical line segment into two smaller rectangles. The vertical side is labeled one third and the top horizontal side lengths are labeled 6 and x. Rectangle C is partitioned by 2 vertical line segments into three equally sized rectangles. The vertical side is labeled r and the top horizontal side lengths are each labeled 1. Rectangle D is partitioned by 3 vertical line segments into 4 equally sized rectangles. The vertical side is labeled 6, and the top horizontal side lengths are each labeled 4. Rectangle E is partitioned by a vertical line segment into two smaller rectangles. The vertical side is labeled m and the top horizontal side lengths are labeled 6 and 8. Rectangle F is partitioned by a vertical line segment into two smaller rectangles. The vertical side is labeled 5 and the top horizontal side lengths are labeled 3 x and 8.

Table 32.5.1

rectangle	width	length	area as a product of width times length	area as a sum of the areas of the smaller rectangles
A				
B				
C				
D				
E				
F				

Are you ready for more?

Here is an area diagram of a rectangle.

	y	z
w	A	24
x	18	72

Figure 32.5.4: Area diagram. A large rectangle is partitioned vertically and horizontally into four smaller rectangles. Starting with the top left rectangle and moving clockwise, vertical side w, top side y, area A. Top right rectangle, top side length z, area 24. Bottom right rectangle, area 72. Bottom left rectangle, vertical side x, area 18.

1. Find the lengths $w, x, y,$and z, and the area A. All values are whole numbers.
2. Can you find another set of lengths that will work? How many possibilities are there?

Summary

Here is a rectangle composed of two smaller rectangles A and B.

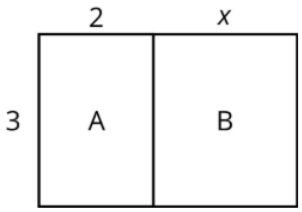


Figure 32.5.5: A rectangle is partitioned by a vertical line segment creating two smaller rectangles, A and B. Rectangle A has a vertical side length of 3 and horizontal side length of 2. Rectangle B has a horizontal side length of x.

Based on the drawing, we can make several observations about the area of the rectangle:

- One side length of the large rectangle is 3 and the other is $2+x$, so its area is $3(2+x)$.
- Since the large rectangle can be decomposed into two smaller rectangles, A and B, with no overlap, the area of the large rectangle is also the sum of the areas of rectangles A and B: $3(2)+3(x)$ or $6+3x$.
- Since both expressions represent the area of the large rectangle, they are equivalent to each other. $3(2+x)$ is equivalent to $6+3x$.

We can see that multiplying 3 by the sum $2+x$ is equivalent to multiplying 3 by 2 and then 3 by x and adding the two products. This relationship is an example of the *distributive property*.

$$3(2+x)=3\cdot 2+3\cdot x$$

Glossary Entries

Definition: Equivalent Expressions

Equivalent expressions are always equal to each other. If the expressions have variables, they are equal whenever the same value is used for the variable in each expression.

For example, $3x+4x$ is equivalent to $5x+2x$. No matter what value we use for x, these expressions are always equal. When x is 3, both expressions equal 21. When x is 10, both expressions equal 70.

Definition: Term

A term is a part of an expression. It can be a single number, a variable, or a number and a variable that are multiplied together. For example, the expression $5x+18$ has two terms. The first term is $5x$ and the second term is 18.

Practice

Exercise 32.5.4

Here is a rectangle.



Figure 32.5.6: Area diagram. A rectangle partitioned vertically into 3 smaller rectangles. First rectangle, vertical side, a, bottom side 2. Second rectangle, vertical side, a, bottom side, 3. Third rectangle, vertical side, a, bottom side, 4.

1. Explain why the area of the large rectangle is $2a+3a+4a$.
2. Explain why the area of the large rectangle is $(2+3+4)a$.

Exercise 32.5.5

Is the area of the shaded rectangle $6(2-m)$ or $6(m-2)$?

Explain how you know.

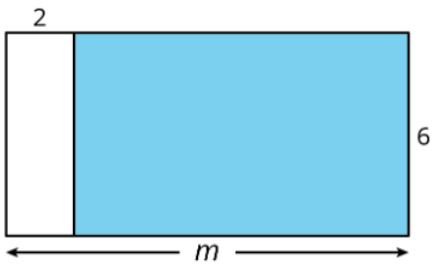


Figure 32.5.7: Area diagram of two attached rectangles one with a shaded area. The height of the rectangle is 6 and has a total width of m. Smaller attached rectangle shares the height of 6 and has a width of 2.

Exercise 32.5.6

Choose the expressions that do *not* represent the total area of the rectangle. Select **all** that apply.



Figure 32.5.8

A. $5t+4t$
B. $t+5+4$
C. $9t$
D. $4\cdot 5\cdot t$
E. $t(5+4)$

Exercise 32.5.7

Evaluate each expression mentally.

1. $35\cdot 91-35\cdot 89$
2. $22\cdot 87+22\cdot 13$
3. $\frac{9}{11}\cdot\frac{7}{10}-\frac{9}{11}\cdot\frac{3}{10}$

(From Unit 6.2.4)

Exercise 32.5.8

Select **all** the expressions that are equivalent to $4b$.

A. $b+b+b+b$
B. $b+4$
C. $2b+2b$
D. $b\cdot b\cdot b\cdot b$
E. $b\div\frac{1}{4}$

(From Unit 6.2.3)

Exercise 32.5.9

Solve each equation. Show your reasoning.

$111=14a$

$13.65=b+4.88$

https://math.libretexts.org/@go/page/40607

$c + \frac{1}{3} = 5\frac{1}{8}$

$\frac{2}{5}d = \frac{17}{4}$

$5.16 = 4e$

(From Unit 6.1.4)

Exercise 32.5.10

Andre ran $5\frac{1}{2}$ laps of a track in 8 minutes at a constant speed. It took Andre x minutes to run each lap. Select **all** the equations that represent this situation.

A. $\left(5\frac{1}{2}\right)x = 8$
B. $5\frac{1}{2} + x = 8$
C. $5\frac{1}{2} - x = 8$
D. $5\frac{1}{2} \div x = 8$
E. $x = 8 \div \left(5\frac{1}{2}\right)$
F. $x = \left(5\frac{1}{2}\right) \div 8$

(From Unit 6.1.2)

32.5: The Distributive Property, Part 2 is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

32.6: The Distributive Property, Part 3

Lesson

Let's practice writing equivalent expressions by using the distributive property.

Exercise 32.6.1: The Shaded Region

A rectangle with dimensions 6 cm and w cm is partitioned into two smaller rectangles.

Explain why each of these expressions represents the area, in cm^2, of the shaded region.

- $6w - 24$
- $6(w - 4)$

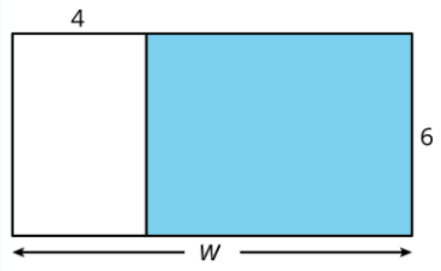


Figure 32.6.1: A rectangle with height labeled 6 and total width labeled w. Rectangle is partition into two smaller rectangles. First rectangle shares height of 6 and width of 4. Second smaller rectangle has an area shaded blue.

Exercise 32.6.2: Matching to Practice Distributive Property

Match each expression in column 1 to an equivalent expression in column 2. If you get stuck, consider drawing a diagram.

Column 1

1. $a(1+2+3)$
2. $2(12-4)$
3. $12a+3b$
4. $\frac{2}{3}(15a-18)$
5. $6a+10b$
6. $0.4(5-2.5a)$
7. $2a+3a$

Column 2

1. $3(4a+b)$
2. $12 \cdot 2 - 4 \cdot 2$
3. $2(3a+5b)$
4. $(2+3)a$
5. $a+2a+4a$
6. $10a-12$
7. $2-a$

Exercise 32.6.3: Writing Equivalent Expressions Using the Distributive Property

The distributive property can be used to write equivalent expressions. In each row, use the distributive property to write an equivalent expression. If you get stuck, consider drawing a diagram.

Table 32.6.1

product	sum or difference
$3(3+x)$	

product	sum or difference
	$4x-20$
$(9-5)x$	
	$4x+7x$
$3(2x+1)$	
	$10x-5$
	$x+2x+3x$
$\frac{1}{2}(x-6)$	
$y(3x+4z)$	
	$2xyz-3yz+4xz$

Are you ready for more?

This rectangle has been cut up into squares of varying sizes. Both small squares have side length 1 unit. The square in the middle has side length x units.

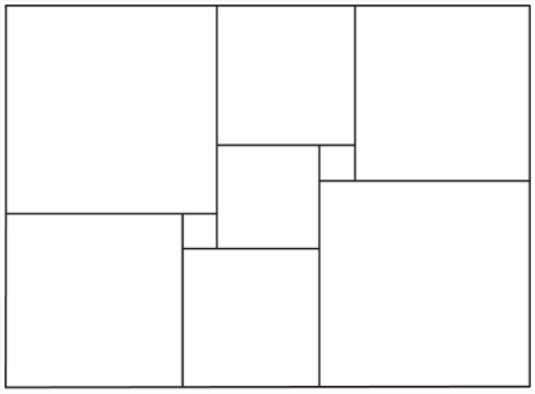


Figure 32.6.2

1. Suppose that x is 3. Find the area of each square in the diagram. Then find the area of the large rectangle.
2. Find the side lengths of the large rectangle assuming that x is 3. Find the area of the large rectangle by multiplying the length times the width. Check that this is the same area you found before.
3. Now suppose that we do not know the value of x. Write an expression for the side lengths of the large rectangle that involves x.

Summary

The distributive property can be used to write a sum as a product, or write a product as a sum. You can always draw a partitioned rectangle to help reason about it, but with enough practice, you should be able to apply the distributive property without making a drawing.

Here are some examples of expressions that are equivalent due to the distributive property.

$$\begin{aligned} 9+18 &= 9(1+2) \\ 2(3x+4) &= 6x+8 \\ 2n+3n+n &= n(2+3+1) \\ 11b-99a &= 11(b-9a) \\ k(c+d-e) &= kc+kd-ke \end{aligned}$$

Glossary Entries

Definition: Equivalent Expressions

Equivalent expressions are always equal to each other. If the expressions have variables, they are equal whenever the same value is used for the variable in each expression.

For example, $3x + 4x$ is equivalent to $5x + 2x$. No matter what value we use for x, these expressions are always equal. When x is 3, both expressions equal 21. When x is 10, both expressions equal 70.

Definition: Term

A term is a part of an expression. It can be a single number, a variable, or a number and a variable that are multiplied together. For example, the expression $5x + 18$ has two terms. The first term is $5x$ and the second term is 18.

Practice

Exercise 32.6.4

For each expression, use the distributive property to write an equivalent expression.

1. $4(x + 2)$
2. $(6 + 8) \cdot x$
3. $4(2x + 3)$
4. $6(x + y + z)$

Exercise 32.6.5

Priya rewrites the expression $8y - 24$ as $8(y - 3)$. Han rewrites $8y - 24$ as $2(4y - 12)$. Are Priya's and Han's expressions each equivalent to $8y - 24$? Explain your reasoning.

Exercise 32.6.6

Select **all** the expressions that are equivalent to $16x + 36$.

A. $16(x + 20)$
B. $x(16 + 36)$
C. $4(4x + 9)$
D. $2(8x + 18)$
E. $2(8x + 36)$

Exercise 32.6.7

The area of a rectangle is $30 + 12x$. List at least 3 possibilities for the length and width of the rectangle.

Exercise 32.6.8

Select **all** the expressions that are equivalent to $\frac{1}{2}z$.

A. $z + z$
B. $z \div 2$
C. $z \cdot z$
D. $\frac{1}{4}z + \frac{1}{4}z$
E. $2z$

(From Unit 6.2.3)

Exercise 32.6.9

1. What is the perimeter of a square with side length:
   3 cm?
   7 cm?
   s cm?
2. If the perimeter of a square is 360 cm, what is its side length?
3. What is the area of a square with side length:
   3 cm?
   7 cm?
   s cm?
4. If the area of a square is 121 cm^2, what is its side length?

(From Unit 6.2.1)

Exercise 32.6.10

Solve each equation.

$10 = 4a$

$5b = 17.5$

$1.036 = 10c$

$0.6d = 1.8$

$15 = 0.1e$

(From Unit 6.1.5)

32.6: The Distributive Property, Part 3 is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

Section 33: Expressions with Exponents

This page titled Section 33: Expressions with Exponents is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

33.1: Meaning of Exponents

Lesson

Let's see how exponents show repeated multiplication.

Exercise 33.1.1: Notice and Wonder: Dots and Lines

What do you notice? What do you wonder?

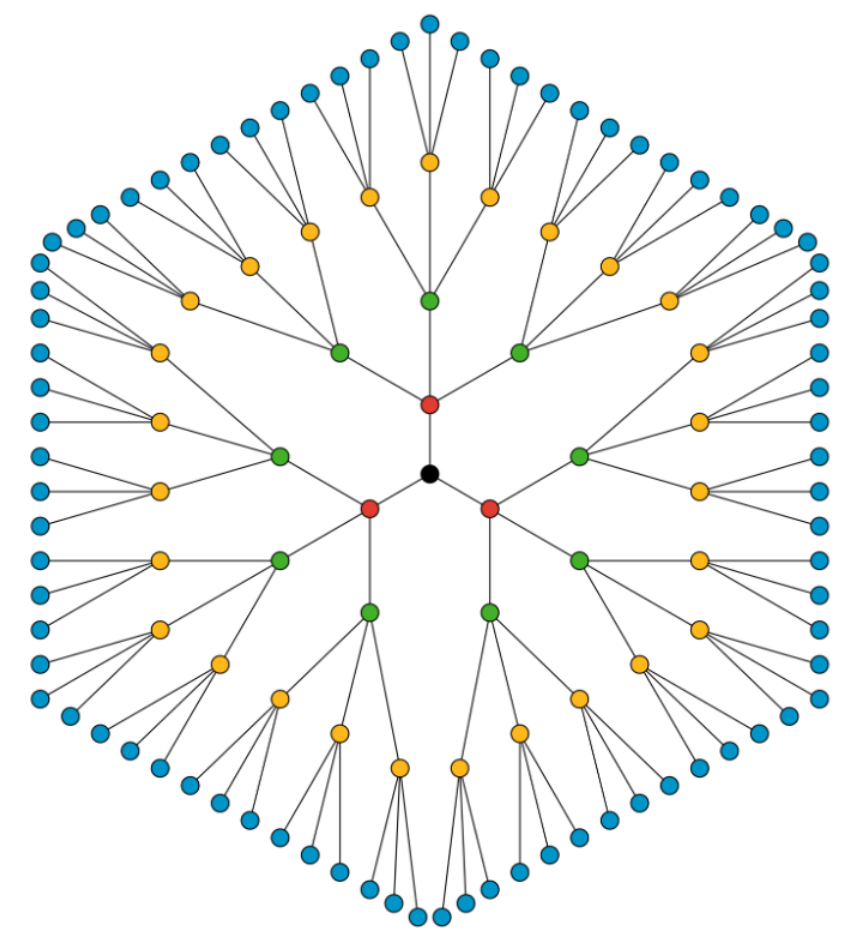


Figure 33.1.1: A figure of a series of dot branches. In the center is a black dot. Three branches extend from the black dot with one red dot at the end of each branch. There are three branches that extend from each red dot with one green dot at the end of each branch. There are three branches that extend from each green dot with one yellow dot at the end of each branch. There are three branches that extend from each yellow dot with one blue dot at the end of each branch.

Exercise 33.1.2: The Genie's Offer

You find a brass bottle that looks really old. When you rub some dirt off of the bottle, a genie appears! The genie offers you a reward. You must choose one:

- $50,000; or
- A magical $1 coin. The coin will turn into two coins on the first day. The two coins will turn into four coins on the second day. The four coins will double to 8 coins on the third day. The genie explains the doubling will continue for 28 days.

1. The number of coins on the third day will be $2 \cdot 2 \cdot 2$. Write an equivalent expression using exponents.
2. What do 2^5 and 2^6 represent in this situation? Evaluate 2^5 and 2^6 without a calculator.
3. How many days would it take for the number of magical coins to exceed $50,000?
4. Will the value of the magical coins exceed a million dollars within the 28 days? Explain or show your reasoning.

Explore the applet. (Why do you think it stops?)

Are you ready for more?

A scientist is growing a colony of bacteria in a petri dish. She knows that the bacteria are growing and that the number of bacteria doubles every hour.

When she leaves the lab at 5 p.m., there are 100 bacteria in the dish. When she comes back the next morning at 9 a.m., the dish is completely full of bacteria. At what time was the dish half full?

Exercise 33.1.3: Make 81

1. Here are some expressions. All but one of them equals 16. Find the one that is *not* equal to 16 and explain how you know.

$$2^3 \cdot 2 \qquad 4^2 \qquad \frac{2^5}{2} \qquad 8^2$$

2. Write three expressions containing exponents so that each expression equals 81.

Summary

When we write an expression like 2^n, we call n the exponent.

If n is a positive whole number, it tells how many factors of 2 we should multiply to find the value of the expression. For example, $2^1 = 2$, and $2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$.

There are different ways to say 2^5. We can say “two raised to the power of five” or “two to the fifth power” or just “two to the fifth.”

Practice

Exercise 33.1.4

Select **all** the expressions that are equivalent to 64.

A. 2^6
B. 2^8
C. 4^3
D. 8^2
E. 16^4
F. 32^2

Exercise 33.1.5

Select **all** the expressions that equal 3^4.

A. 7
B. 4^3
C. 12
D. 81
E. 64
F. 9^2

Exercise 33.1.6

4^5 is equal to 1,024. Evaluate each expression.

1. 4^6
2. 4^4
3. $4^3 \cdot 4^2$

Exercise 33.1.7

$6^3 = 216$. Using exponents, write three more expressions whose value is 216.

Exercise 33.1.8

Find two different ways to rewrite $3xy + 6yz$ using the distributive property.

(From Unit 6.2.6)

Exercise 33.1.9

Solve each equation.

$a - 2.01 = 5.5$

$b + 2.01 = 5.5$

$10c = 13.71$

$100d = 13.71$

(From Unit 6.1.5)

Exercise 33.1.10

Which expressions represent the total area of the large rectangle? Select **all** that apply.

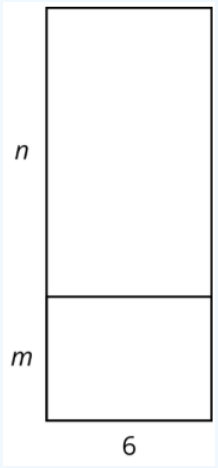


Figure 33.1.2

A. $6(m+n)$
B. $6n+m$
C. $6n+6m$
D. $6mn$
E. $(n+m)6$

(From Unit 6.2.5)

Exercise 33.1.11

Is each statement true or false? Explain your reasoning.

1. $\frac{45}{100} \cdot 72 = \frac{45}{72} \cdot 100$
2. 16% of 250 is equal to 250% of 16

(From Unit 3.4.7)

33.1: Meaning of Exponents is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

33.2: Expressions with Exponents

Lesson

Let's use the meaning of exponents to decide if equations are true.

Exercise 33.2.1: Which One Doesn't Belong: Twos

Which one doesn't belong?

$2 \cdot 2 \cdot 2 \cdot 2$

16

2^4

$4 \cdot 2$

Exercise 33.2.2: Is the Equation True?

Decide whether each equation is true or false, and explain how you know.

1. $2^4 = 2 \cdot 4$
2. $3+3+3+3+3 = 3^5$
3. $5^3 = 5 \cdot 5 \cdot 5$
4. $2^3 = 3^2$
5. $16^1 = 8^2$
6. $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = 4 \cdot \frac{1}{2}$
7. $\left(\frac{1}{2}\right)^4 = \frac{1}{8}$
8. $8^2 = 4^3$

Exercise 33.2.3: What's Your Reason?

In each list, find expressions that are equivalent to each other and explain to your partner why they are equivalent. Your partner listens to your explanation. If you disagree, explain your reasoning until you agree. Switch roles for each list. (There may be more than two equivalent expressions in each list.)

1. a. $5 \cdot 5$
   b. 2^5
   c. 5^2
   d. $2 \cdot 5$
2. a. 4^3
   b. 3^4
   c. $4 \cdot 4 \cdot 4$
   d. $4+4+4$
3. a. $6+6+6$
   b. 6^3
   c. 3^6
   d. $3 \cdot 6$
4. a. 11^5
   b. $11 \cdot 11 \cdot 11 \cdot 11 \cdot 11$
   c. $11 \cdot 5$
   d. 5^{11}
5. a. $\frac{1}{5} \cdot \frac{1}{5} \cdot \frac{1}{5}$
   b. $\left(\frac{1}{5}\right)^3$

c. $\frac{1}{15}$
d. $\frac{1}{125}$

6. a. $\left(\frac{5}{3}\right)^2$
b. $\left(\frac{3}{5}\right)^2$
c. $\frac{10}{6}$
d. $\frac{25}{9}$

Are you ready for more?

What is the last digit of $3^{1,000}$? Show or explain your reasoning.

Summary

When working with exponents, the bases don't have to always be whole numbers. They can also be other kinds of numbers, like fractions, decimals, and even variables. For example, we can use exponents in each of the following ways:

$\left(\frac{2}{3}\right)^4 = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3}$

$(1.7)^3 = (1.7) \cdot (1.7) \cdot (1.7)$

$x^5 = x \cdot x \cdot x \cdot x \cdot x$

Practice

Exercise 33.2.4

Select **all** expressions that are equal to $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$.

A. $3 \cdot 5$
B. 3^5
C. $3^4 \cdot 3$
D. $5 \cdot 3$
E. 5^3

Exercise 33.2.5

Noah starts with 0 and then adds the number 5 four times. Diego starts with 1 and then multiplies by the number 5 four times. For each expression, decide whether it is equal to Noah's result, Diego's result, or neither.

1. $4 \cdot 5$
2. $4 + 5$
3. 4^5
4. 5^4

Exercise 33.2.6

Decide whether each equation is true or false, and explain how you know.

1. $9 \cdot 9 \cdot 3 = 3^5$
2. $7 + 7 + 7 = 3 + 3 + 3 + 3 + 3 + 3 + 3$
3. $\frac{1}{7} \cdot \frac{1}{7} \cdot \frac{1}{7} = \frac{3}{7}$
4. $4^1 = 4 \cdot 1$
5. $6 + 6 + 6 = 6^3$

Exercise 33.2.7

1. What is the area of a square with side lengths of $\frac{3}{5}$ units?
2. What is the side length of a square with area $\frac{1}{16}$ square units?
3. What is the volume of a cube with edge lengths of $\frac{2}{3}$ units?
4. What is the edge length of a cube with volume $\frac{27}{64}$ cubic units?

Exercise 33.2.8

Select **all** the expressions that represent the area of the shaded rectangle.

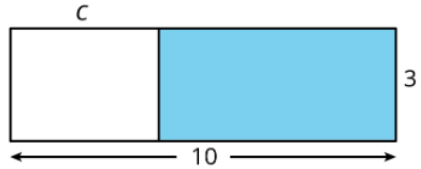


Figure 33.2.1: Area diagram. A large rectangle, bottom side length 10, is partitioned vertically into two smaller rectangles. First rectangle, vertical side 3, top side c. Second rectangle, vertical side 3, top side length unknown, area shaded blue.

A. $3(10-c)$
B. $3(c-10)$
C. $10(c-3)$
D. $10(3-c)$
E. $30-3c$
F. $30-10c$

(From Unit 6.2.5)

Exercise 33.2.9

A ticket at a movie theater costs $8.50. One night, the theater had $29,886 in ticket sales.

1. Estimate about how many tickets the theater sold. Explain your reasoning.
2. How many tickets did the theater sell? Explain your reasoning.

(From Unit 5.4.5)

Exercise 33.2.10

A fence is being built around a rectangular garden that is $8\frac{1}{2}$ feet by $6\frac{1}{3}$ feet. Fencing comes in panels. Each panel is $\frac{2}{3}$ of a foot wide. How many panels are needed? Explain or show your reasoning.

(From Unit 4.4.1)

33.2: Expressions with Exponents is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

33.3: Evaluating Expressions with Exponents

Lesson

Let's find the values of expressions with exponents.

Exercise 33.3.1: Revisiting the Cube

Based on the given information, what other measurements of the square and cube could we find?

Exercise 33.3.2: Calculating Surface Area

A cube has side length 10 inches. Jada says the surface area of the cube is 600 in^2, and Noah says the surface area of the cube is 3,600 in^2. Here is how each of them reasoned:

Jada's Method:

$6 \cdot 10^2$
$6 \cdot 100$
600

Noah's Method:

$6 \cdot 10^2$
60^2
$3,600$

Do you agree with either of them? Explain your reasoning.

Exercise 33.3.3: Row Game: Expression Explosion

Evaluate the expressions in one of the columns. Your partner will work on the other column. Check with your partner after you finish each row. Your answers in each row should be the same. If your answers aren't the same, work together to find the error.

Table 33.3.1

column A	column B
$5^2 + 4$	$2^2 + 25$
$2^4 \cdot 5$	$2^3 \cdot 10$
$3 \cdot r^2$	$12 \cdot 2^2$
$20 + 2^3$	$1 + 3^3$
$9 \cdot 2^1$	$3 \cdot 6^1$
$\frac{1}{9} \cdot \left(\frac{1}{2}\right)^3$	$\frac{1}{8} \cdot \left(\frac{1}{3}\right)^2$

Are you ready for more?

1. Consider this equation: $\square^2 + \square^2 = \square^2$. An example of 3 different whole numbers that could go in the boxes are 3, 4, and 5, since $3^2 + 4^2 = 5^2$. (That is, $9 + 16 = 25$.)
   Can you find a different set of 3 whole numbers that make the equation true?
2. How many sets of 3 different whole numbers can you find?
3. Can you find a set of 3 different whole numbers that make this equation true? $\square^3 + \square^3 = \square^3$
4. How about this one? $\square^4 + \square^4 = \square^4$

Once you have worked on this a little while, you can understand a problem that is famous in the history of math. (Alas, this space is too small to contain it.) If you are interested, consider doing some further research on *Fermat's Last Theorem*.

Summary

Exponents give us a new way to describe operations with numbers, so we need to understand how exponents get along with the other operations we know.

When we write $6 \cdot 4^2$, we want to make sure everyone agrees about how to evaluate this. Otherwise some people might multiply first and others compute the exponent first, and different people would get different values for the same expression!

Earlier we saw situations in which $6 \cdot 4^2$ represented the surface area of a cube with side lengths 4 units. When computing the surface area, we evaluate 4^2 first (or find the area of one face of the cube first) and then multiply the result by 6. In many other expressions that use exponents, the part with an exponent is intended to be evaluated first.

To make everyone agree about the value of expressions like $6 \cdot 4^2$, the convention is to *evaluate the part of the expression with the exponent first*. Here are a couple of examples:

$$\begin{aligned} 6 \cdot 4^2 \qquad & \qquad 45 + 5^2 \\ 6 \cdot 16 \qquad & \qquad 45 + 25 \\ 96 \qquad & \qquad 70 \end{aligned}$$

If we want to communicate that 6 and 4 should be multiplied first and then squared, then we can use parentheses to group parts together:

$$\begin{aligned} (6 \cdot 4)^2 \qquad & \qquad (45 + 5)^2 \\ 24^2 \qquad & \qquad 50^2 \\ 576 \qquad & \qquad 2,500 \end{aligned}$$

Practice

Exercise 33.3.4

Lin says, "I took the number 8, and then multiplied it by the square of 3." Select **all** the expressions that equal Lin's answer.

A. $8 \cdot 3^2$
B. $(8 \cdot 3)^2$
C. $8 \cdot 2^3$
D. $3^2 \cdot 8$
E. 24^2
F. 72

Exercise 33.3.5

Evaluate each expression.

1. $7 + 2^3$
2. $9 \cdot 3^1$
3. $20 - 2^4$
4. $2 \cdot 6^2$
5. $8 \cdot \left(\frac{1}{2}\right)^2$
6. $\frac{1}{3} \cdot 3^3$
7. $\left(\frac{1}{5} \cdot 5\right)^5$

Exercise 33.3.6

Andre says, “I multiplied 4 by 5, then cubed the result.” Select **all** the expressions that equal Andre’s answer.

A. $4 \cdot 5^3$
B. $(4 \cdot 5)^3$
C. $(4 \cdot 5)^2$
D. $5^3 \cdot 4$
E. 20^3
F. 500
G. $8,000$

Exercise 33.3.7

Han has 10 cubes, each 5 inches on a side.

1. Find the total volume of Han’s cubes. Express your answer as an expression using an exponent.
2. Find the total surface area of Han’s cubes. Express your answer as an expression using an exponent.

Exercise 33.3.8

Priya says that $\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{4}{3}$. Do you agree with Priya? Explain or show your reasoning.

(From Unit 6.3.2)

Exercise 33.3.9

Answer each question. Show your reasoning.

1. 125% of e is 30. What is e?
2. 35% of f is 14. What is f?

(From Unit 6.2.2)

Exercise 33.3.10

Which expressions are solutions to the equation $2.4y = 13.75$? Select **all** that apply.

A. $13.75 - 1.4$
B. $13.75 \cdot 2.4$
C. $13.75 \div 2.4$
D. $\frac{13.75}{2.4}$
E. $2.4 \div 13.75$

(From Unit 6.1.5)

Exercise 33.3.11

Jada explains how she finds $15 \cdot 23$:

“I know that ten 23s is 230, so five 23s will be half of 230, which is 115.
15 is 10 plus 5, so $15 \cdot 23$ is 230 plus 115, which is 345.”

1. Do you agree with Jada? Explain.
2. Draw a 15 by 23 rectangle. Partition the rectangle into two rectangles and label them to show Jada’s reasoning.

(From Unit 5.3.3)

33.3: Evaluating Expressions with Exponents is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

33.4: Equivalent Exponential Expressions

Lesson

Let's investigate expressions with variables and exponents.

Exercise 33.4.1: Up or Down?

Find the values of 3^x and $\left(\frac{1}{3}\right)^x$ for different values of x. What patterns do you notice?

Table 33.4.1

x	3^x	$\left(\frac{1}{3}\right)^x$
1		
2		
3		
4		

Exercise 33.4.2: What's the Value?

Evaluate each expression for the given value of x.

1. $3x^2$ when x is 10
2. $3x^2$ when x is $\frac{1}{9}$
3. $\frac{x^3}{4}$ when x is 4
4. $\frac{x^3}{4}$ when x is $\frac{1}{2}$
5. $9 + x^7$ when x is 1
6. $9 + x^7$ when x is $\frac{1}{2}$

Exercise 33.4.3: Exponent Experimentation

Find a solution to each equation in the list. (Numbers in the list may be a solution to more than one equation, and not all numbers in the list will be used.)

1. $64 = x^2$
2. $64 = x^3$
3. $2^x = 32$
4. $x = \left(\frac{2}{5}\right)^3$
5. $\frac{16}{9} = x^2$
6. $2 \cdot 2^5 = 2^x$
7. $2x = 2^4$
8. $4^3 = 8^x$

List:

$$\frac{8}{125} \quad \frac{6}{15} \quad \frac{5}{8} \quad \frac{8}{9} \quad 1\frac{4}{3} \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 8$$

Are you ready for more?

This fractal is called a Sierpinski Tetrahedron. A tetrahedron is a polyhedron that has four faces. (The plural of tetrahedron is tetrahedra.)

The small tetrahedra form four medium-sized tetrahedra: blue, red, yellow, and green. The medium-sized tetrahedra form one large tetrahedron.

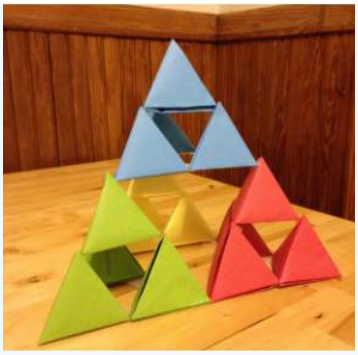


Figure 33.4.1: A large tetrahedron is formed by four medium-sized tetrahedra of different colors: blue, red, yellow, green. Each medium-sized tetrahedron is formed by four small tetrahedra.

1. How many small faces does this fractal have? Be sure to include faces you can't see. Try to find a way to figure this out so that you don't have to count every face.
2. How many small tetrahedra are in the bottom layer, touching the table?
3. To make an even bigger version of this fractal, you could take four fractals like the one pictured and put them together. Explain where you would attach the fractals to make a bigger tetrahedron.
4. How many small faces would this bigger fractal have? How many small tetrahedra would be in the bottom layer?
5. What other patterns can you find?

Summary

In this lesson, we saw expressions that used the letter x as a variable. We evaluated these expressions for different values of x.

- To evaluate the expression $2x^3$ when x is 5, we replace the letter x with 5 to get $2 \cdot 5^3$. This is equal to $2 \cdot 125$ or just 250. So the value of $2x^3$ is 250 when x is 5.
- To evaluate $\frac{x^2}{8}$ when x is 4, we replace the letter x with 4 to get $\frac{4^2}{8} = \frac{16}{8}$, which equals 2. So $\frac{x^2}{8}$ has a value of 2 when x is 4.

We also saw equations with the variable x and had to decide what value of x would make the equation true.

- Suppose we have an equation $10 \cdot 3^x = 90$ and a list of possible solutions: $1, 2, 3, 9, 11$ The only value of x that makes the equation true is 2 because $10 \cdot 3^2 = 10 \cdot 3 \cdot 3$, which equals 90. So 2 is the solution to the equation.

Practice

Exercise 33.4.4

Evaluate each expression if $x = 3$.

1. 2^x
2. x^2
3. 1^x
4. x^1
5. $\left(\frac{1}{2}\right)^x$

Exercise 33.4.5

Evaluate each expression for the given value of each variable.

1. $2 + x^3$, x is 3
2. x^2, x is $\frac{1}{2}$
3. $3x^2 + y$, x is 5 y is 3
4. $10y + x^2$, x is 6 y is 4

Exercise 33.4.6

Decide if the expressions have the same value. If not, determine which expression has the larger value.

1. 2^3 and 3^2
2. 1^{31} and 31^1
3. 4^2 and 2^4
4. $\left(\frac{1}{2}\right)^3$ and $\left(\frac{1}{3}\right)^2$

Exercise 33.4.7

Match each equation to its solution.

A. $7 + x^2 = 16$
B. $5 - x^2 = 1$
C. $2 \cdot 2^3 = 2^x$
D. $\frac{3^4}{3^x} = 27$

1. $x = 1$
2. $x = 2$
3. $x = 3$
4. $x = 4$

Exercise 33.4.8

An adult pass at the amusement park costs 1.6 times as much as a child's pass.

1. How many dollars does an adult pass cost if a child's pass costs:
   $5?
   $10?
   w dollars?
2. A child's pass costs $15. How many dollars does an adult pass cost?

(From Unit 6.2.1)

Exercise 33.4.9

Jada reads 5 pages every 20 minutes. At this rate, how many pages can she read in 1 hour?

- Use a double number line to find the answer.

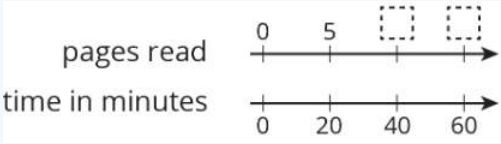


Figure 33.4.2: Double number line, 4 evenly spaced tick marks. Top line, pages read. Beginning at first tick mark, labels: 0, 5, box, box. Bottom line, time minutes. Beginning at first tick mark, labeled 0, 20, 40, 60.

- Use a table to find the answer.

Table 33.4.2

pages read	time in minutes
5	20

Which strategy do you think is better, and why?

(From Unit 2.4.4)

https://math.libretexts.org/@go/page/40612

33.4: Equivalent Exponential Expressions is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

Section 34: Relationship Between Quantities

This page titled Section 34: Relationship Between Quantities is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

34.1: Two Related Quantities, Part 1

Lesson

Let's use equations and graphs to describe relationships with ratios.

Exercise 34.1.1: Which One Would You Choose?

Which one would you choose? Be prepared to explain your reasoning.

- A 5-pound jug of honey for $15.35
- Three 1.5-pound jars of honey for $13.05

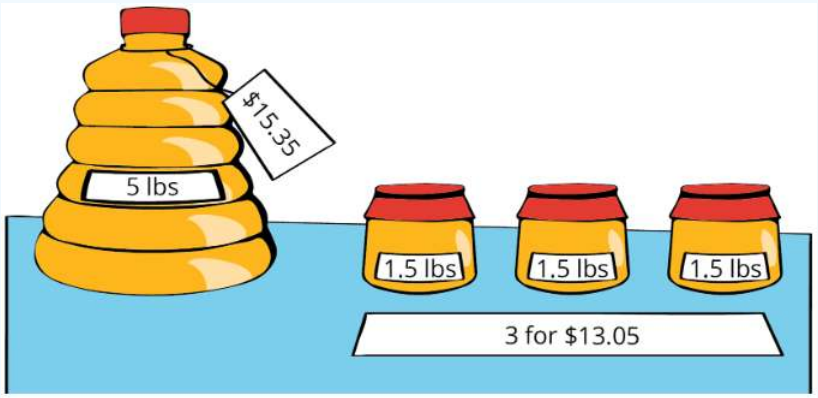


Figure 34.1.1

Exercise 34.1.2: Painting the Set

Lin needs to mix a specific shade of orange paint for the set of the school play. The color uses 3 parts yellow for every 2 parts red.

1. Complete the table to show different combinations of red and yellow paint that will make the shade of orange Lin needs.

Table 34.1.1

cups of red paint (r)	cups of yellow paint (y)	total cups of paint (t)
2	3	
6		
		20
	18	
14		
16		
		50
	42	

2. Lin notices that the number of cups of red paint is always $\frac{2}{5}$ of the total number of cups. She writes the equation $r = \frac{2}{5}t$ to describe the relationship. Which is the **independent variable**? Which is the **dependent variable**? Explain how you know.
3. Write an equation that describes the relationship between r and y where y is the independent variable.
4. Write an equation that describes the relationship between y and r where r is the independent variable.
5. Use the points in the table to create two graphs that show the relationship between r and y. Match each relationship to one of the equations you wrote.

Are you ready for more?

A fruit stand sells apples, peaches, and tomatoes. Today, they sold 4 apples for every 5 peaches. They sold 2 peaches for every 3 tomatoes. They sold 132 pieces of fruit in total. How many of each fruit did they sell?

Summary

Equations are very useful for describing sets of equivalent ratios. Here is an example.

A pie recipe calls for 3 green apples for every 5 red apples. We can create a table to show some equivalent ratios.

We can see from the table that r is always $\frac{5}{3}$ as large as g and that g is always $\frac{3}{5}$ as large as r.

green apples (g)	red apples (r)
3	5
6	10
9	15
12	20

Table$34.1.2$

We can write equations to describe the relationship between g and r.

- When we know the number of green apples and want to find the number of red apples, we can write:

$$r = \frac{5}{3}g$$

In this equation, if g changes, r is affected by the change, so we refer to g as the **independent variable** and r as the **dependent variable**.

We can use this equation with any value of g to find r. If 270 green apples are used, then $\frac{5}{3} \cdot (270)$ or 450 red apples are used.

- When we know the number of red apples and want to find the number of green apples, we can write:

$$g = \frac{3}{5}r$$

In this equation, if r changes, g is affected by the change, so we refer to r as the independent variable and g as the dependent variable.

We can use this equation with any value of r to find g. If 275 red apples are used, then $\frac{3}{5} \cdot (275)$ or 165 green apples are used.

We can also graph the two equations we wrote to get a visual picture of the relationship between the two quantities.

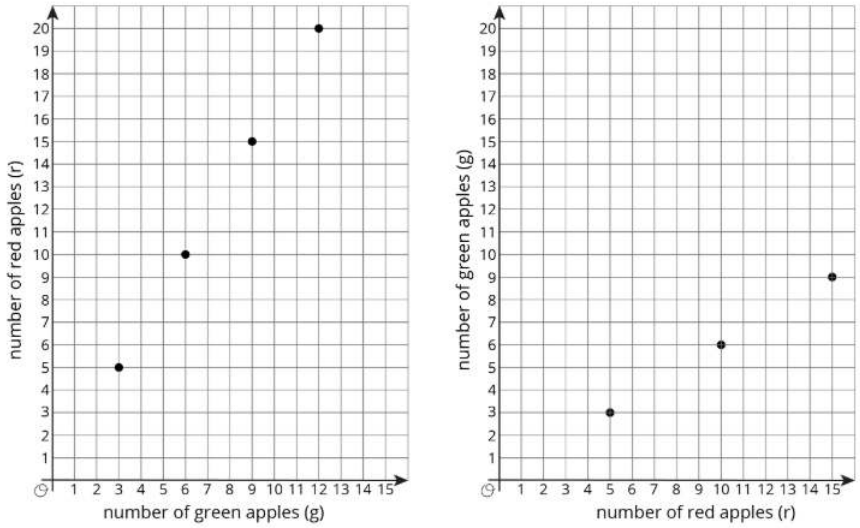


Figure 34.1.2: Two graphs that represent a ratio of two quantities. The graph on the left has a horizontal axis labeled number of green apples and the numbers 1 through 15 are indicated, The vertical axis is labeled number of red apples and the numbers 1 through 20 are indicated. The following four points are indicated on the graph: 3 comma 5, 6 comma 10, 9 comma 15, and 12 comma 20. The graph on the right has a horizontal axis labeled number of red apples and the numbers 1 through 15 are indicated. The vertical axis is labeled number of green apples and the numbers 1 through 20 are indicated. The following three points are indicated on the graph: 5 comma 3, 10 comma 6, and 15 comma 9.

Glossary Entries

Definition: Dependent Variable

The dependent variable is the result of a calculation.

For example, a boat travels at a constant speed of 25 miles per hour. The equation $d = 25t$ describes the relationship between the boat's distance and time. The dependent variable is the distance traveled, because d is the result of multiplying 25 by t.

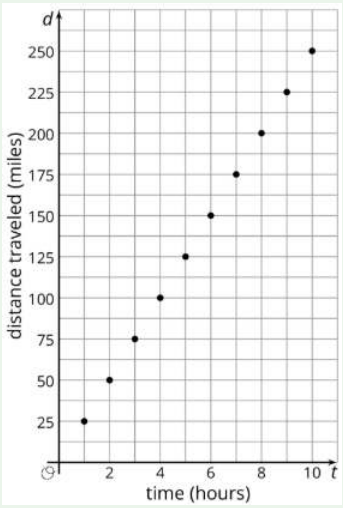


Figure 34.1.3: A graph of 10 points plotted in the coordinate plane with the origin labeled O. The horizontal t axis is labeled time in hours. The numbers 0 through 10, in increments of 2, are indicated, and there are vertical grid lines midway between. The vertical axis is labeled distance traveled in miles. The numbers 0 through 250, in increments of 25, are indicated, and there are horizontal grid lines midway between. The data are as follows: 1 comma 25. 2 comma 50. 3 comma 75. 4 comma 100. 5 comma 125. 6 comma 150. 7 comma 175. 8 comma 200. 9 comma 225. 10 comma 250.

Definition: Independent Variable

The independent variable is used to calculate the value of another variable.

For example, a boat travels at a constant speed of 25 miles per hour. The equation $d = 25t$ describes the relationship between the boat's distance and time. The independent variable is time, because t is multiplied by 25 to get d.

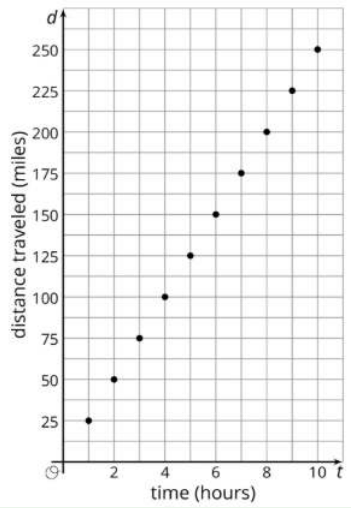


Figure 34.1.4: A graph of 10 points plotted in the coordinate plane with the origin labeled O. The horizontal t axis is labeled time in hours. The numbers 0 through 10, in increments of 2, are indicated, and there are vertical grid lines midway between. The vertical axis is labeled distance traveled in miles. The numbers 0 through 250, in increments of 25, are indicated, and there are horizontal grid lines midway between. The data are as follows: 1 comma 25. 2 comma 50. 3 comma 75. 4 comma 100. 5 comma 125. 6 comma 150. 7 comma 175. 8 comma 200. 9 comma 225. 10 comma 250.

Practice

Exercise 34.1.3

Here is a graph that shows some values for the number of cups of sugar, s, required to make x batches of brownies.

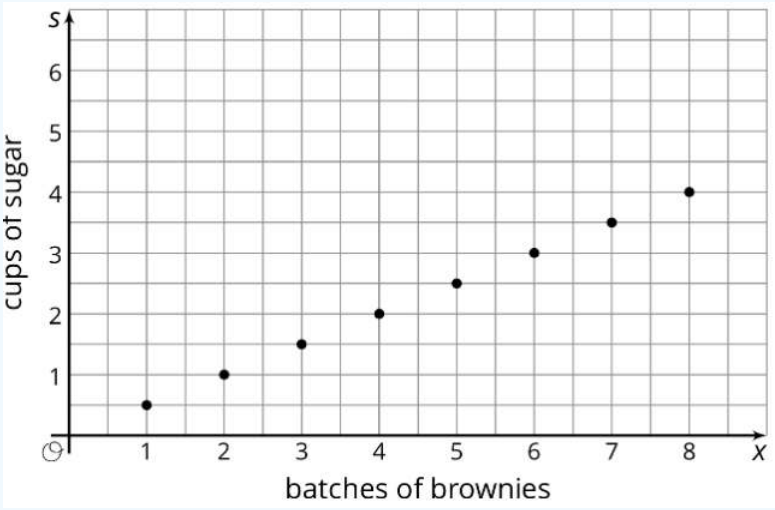


Figure 34.1.5: Eight points plotted on the coordinate plane with the origin labeled O. The x axis is labeled batches of brownies and the numbers 0 through 8 are indicated. The s axis is labeled cups of sugar and the numbers 0 through 6 are indicated. There are horizontal grid lines halfway between each integer. The data are as follows: 1 comma one half. 2 comma 1. 3 comma one and one half. 4 comma 2. 5 comma 2 and one half. 6 comma 3. 7 comma 3 and one half. 8 comma 4.

1. Complete the table so that the pair of numbers in each column represents the coordinates of a point on the graph.

Table 34.1.3

x	1	2	3	4	5	6	7
s							

2. What does the point $(8, 4)$ mean in terms of the amount of sugar and number of batches of brownies?
3. Write an equation that shows the amount of sugar in terms of the number of batches.

Exercise 34.1.4

Each serving of a certain fruit snack contains 90 calories.

1. Han wants to know how many calories he gets from the fruit snacks. Write an equation that shows the number of calories, c, in terms of the number of servings, n.
2. Tyler needs some extra calories each day during his sports season. He wants to know how many servings he can have each day if all the extra calories come from the fruit snack. Write an equation that shows the number of servings, n, in terms of the number of calories, c.

Exercise 34.1.5

Kiran shops for books during a 20% off sale.

1. What percent of the original price of a book does Kiran pay during the sale?
2. Complete the table to show how much Kiran pays for books during the sale.
3. Write an equation that relates the sale price, s, to the original price p.
4. On graph paper, create a graph showing the relationship between the sale price and the original price by plotting the points from the table.

Table 34.1.4

original price in dollars (p)	sale price in dollars (s)
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

Exercise 34.1.6

Evaluate each expression when x is 4 and y is 6.

1. $(6-x)^3+y$
2. $2+x^3$
3. 2^x-2y
4. $\left(\frac{1}{2}\right)^x$
5. 1^x+2^x
6. $\frac{2^x}{x^2}$

(From Unit 6.3.4)

Exercise 34.1.7

Find $(12.34) \cdot (0.7)$. Show your reasoning.

(From Unit 5.3.4)

Exercise 34.1.8

For each expression, write another division expression that has the same value and that can be used to help find the quotient. Then, find each quotient.

1. $302.1 \div 0.5$
2. $12.15 \div 0.02$
3. $1.375 \div 0.11$

(From Unit 5.4.5)

34.1: Two Related Quantities, Part 1 is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

34.2: Two Related Quantities, Part 2

Lesson

Let's use equations and graphs to describe stories with constant speed.

Exercise 34.2.1: Walking to the Library

Lin and Jada each walk at a steady rate from school to the library. Lin can walk 13 miles in 5 hours, and Jada can walk 25 miles in 10 hours. They each leave school at 3:00 and walk $3\frac{3}{4}$ miles to the library. What time do they each arrive?

Exercise 34.2.2: The Walk-a-thon

Diego, Elena, and Andre participated in a walk-a-thon to raise money for cancer research. They each walked at a constant rate, but their rates were different.

1. Complete the table to show how far each participant walked during the walk-a-thon.

Table 34.2.1

time in hours	miles walked by Diego	miles walked by Elena	miles walked by Andre
1			
2	6		
	12	11	
5			17.5

2. How fast was each participant walking in miles per hour?
3. How long did it take each participant to walk one mile?
4. Graph the progress of each person in the coordinate plane. Use a different color for each participant.
5. Diego says that $d = 3t$ represents his walk, where d is the distance walked in miles and t is the time in hours.
   a. Explain why $d = 3t$ relates the distance Diego walked to the time it took.
   b. Write two equations that relate distance and time: one for Elena and one for Andre.
6. Use the equations you wrote to predict how far each participant would walk, at their same rate, in 8 hours.
7. For Diego's equation and the equations you wrote, which is the dependent variable and which is the independent variable?

Are you ready for more?

1. Two trains are traveling toward each other, on parallel tracks. Train A is moving at a constant speed of 70 miles per hour. Train B is moving at a constant speed of 50 miles per hour. The trains are initially 320 miles apart. How long will it take them to meet? One way to start thinking about this problem is to make a table. Add as many rows as you like.
2. How long will it take a train traveling at 120 miles per hour to go 320 miles?
3. Explain the connection between these two problems.

Table 34.2.2

	train A	train B
starting position	0 miles	320 miles
after 1 hour	70 miles	270 miles
after 2 hours		

Summary

Equations are very useful for solving problems with constant speeds. Here is an example.

A boat is traveling at a constant speed of 25 miles per hour.

1. How far can the boat travel in 3.25 hours?
2. How long does it take for the boat to travel 60 miles?

We can write equations to help us answer questions like these.

Let's use t to represent the time in hours and d to represent the distance in miles that the boat travels.

When we know the time and want to find the distance, we can write: $d = 25t$ In this equation, if t changes, d is affected by the change, so we t is the independent variable and d is the dependent variable.

This equation can help us find d when we have any value of t. In 3.25 hours, the boat can travel $25(3.25)$ or 81.25 miles.

When we know the distance and want to find the time, we can write: $t = \frac{d}{25}$ In this equation, if d changes, t is affected by the change, so we d is the independent variable and t is the dependent variable.

This equation can help us find t when for any value of d. To travel 60 miles, it will take $\frac{60}{25}$ or $2\frac{2}{5}$ hours.

These problems can also be solved using important ratio techniques such as a table of equivalent ratios. The equations are particularly valuable in this case because the answers are not round numbers or easy to quickly evaluate.

We can also graph the two equations we wrote to get a visual picture of the relationship between the two quantities:

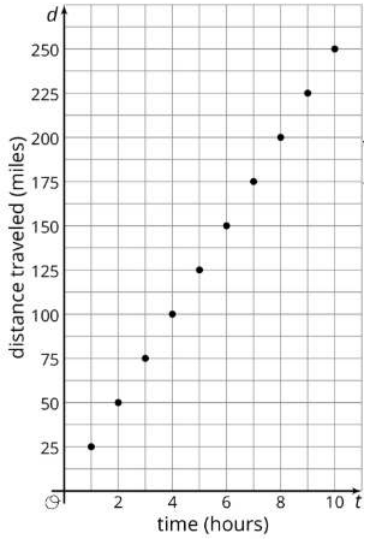


Figure 34.2.1: A graph of 10 points plotted in the coordinate plane with the origin labeled O. The horizontal t axis is labeled time in hours. The numbers 0 through 10, in increments of 2, are indicated, and there are vertical grid lines midway between. The vertical axis is labeled distance traveled in miles. The numbers 0 through 250, in increments of 25, are indicated, and there are horizontal grid lines midway between. The data are as follows: 1 comma 25. 2 comma 50. 3 comma 75. 4 comma 100. 5 comma 125. 6 comma 150. 7 comma 175. 8 comma 200. 9 comma 225. 10 comma 250.

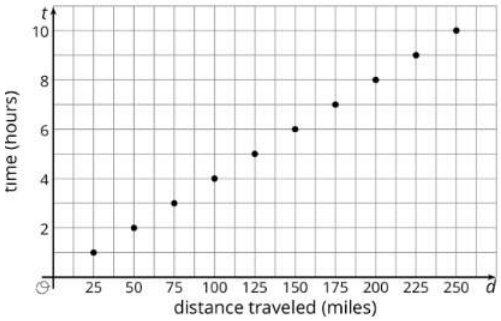


Figure 34.2.2: A graph of 10 points plotted on the coordinate plane with the origin labeled O. The horizontal d axis is labeled distance traveled in miles. The numbers 0 through 250, in increments of 25, are indicated, and there are vertical grid lines midway between. The vertical t axis is labeled time in hours. The numbers 0 through 10, in increments of 2, are indicated, and there are horizontal grid lines midway between. The data are as follows: 25 comma 1. 50 comma 2. 75 comma 3. 100 comma 4. 125 comma 5. 150 comma 6. 175 comma 7. 200 comma 8. 225 comma 9. 250 comma 10.

Glossary Entries

Definition: Coordinate Plane

The coordinate plane is a system for telling where points are. For example. point R is located at $(3, 2)$ on the coordinate plane, because it is three units to the right and two units up.

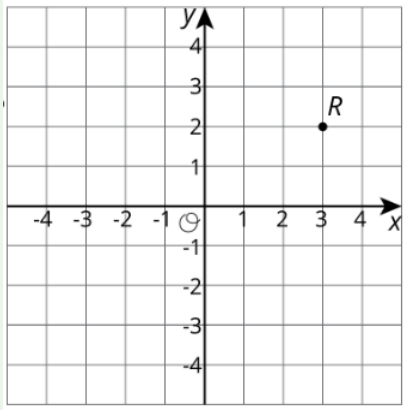


Figure 34.2.3

Definition: Dependent Variable

The dependent variable is the result of a calculation.

For example, a boat travels at a constant speed of 25 miles per hour. The equation $d = 25t$ describes the relationship between the boat's distance and time. The dependent variable is the distance traveled, because d is the result of multiplying 25 by t.

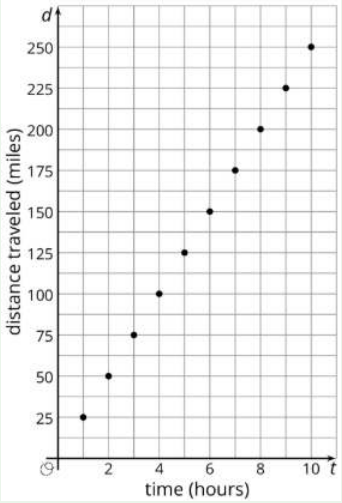


Figure 34.2.4: A graph of 10 points plotted in the coordinate plane with the origin labeled O. The horizontal t axis is labeled time in hours. The numbers 0 through 10, in increments of 2, are indicated, and there are vertical grid lines midway between. The vertical axis is labeled distance traveled in miles. The numbers 0 through 250, in increments of 25, are indicated, and there are horizontal grid lines midway between. The data are as follows: 1 comma 25. 2 comma 50. 3 comma 75. 4 comma 100. 5 comma 125. 6 comma 150. 7 comma 175. 8 comma 200. 9 comma 225. 10 comma 250.

Definition: Independent Variable

The independent variable is used to calculate the value of another variable.

For example, a boat travels at a constant speed of 25 miles per hour. The equation $d = 25t$ describes the relationship between the boat's distance and time. The independent variable is time, because t is multiplied by 25 to get d.

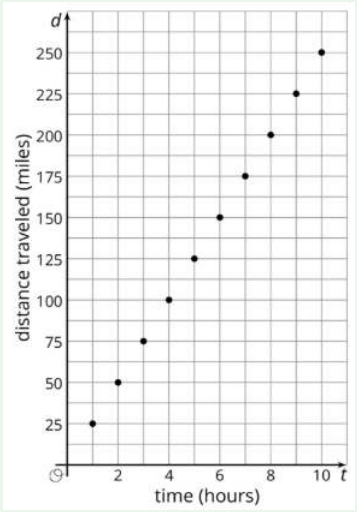


Figure 34.2.5: A graph of 10 points plotted in the coordinate plane with the origin labeled O. The horizontal t axis is labeled time in hours. The numbers 0 through 10, in increments of 2, are indicated, and there are vertical grid lines midway between. The vertical axis is labeled distance traveled in miles. The numbers 0 through 250, in increments of 25, are indicated, and there are horizontal grid lines midway between. The data are as follows: 1 comma 25. 2 comma 50. 3 comma 75. 4 comma 100. 5 comma 125. 6 comma 150. 7 comma 175. 8 comma 200. 9 comma 225. 10 comma 250.

Practice

Exercise 34.2.3

A car is traveling down a road at a constant speed of 50 miles per hour.

1. Complete the table with the amounts of time it takes the car to travel certain distances, or the distances traveled for certain amounts of time.

2. Write an equation that represents the distance traveled by the car, d, for an amount of time, t.
3. In your equation, which is the dependent variable and which is the independent variable?

Table 34.2.3

time (hours)	distance (miles)
2	
1.5	
t	
	50
	300
	d

Exercise 34.2.4

The graph represents the amount of time in hours it takes a ship to travel various distances in miles.

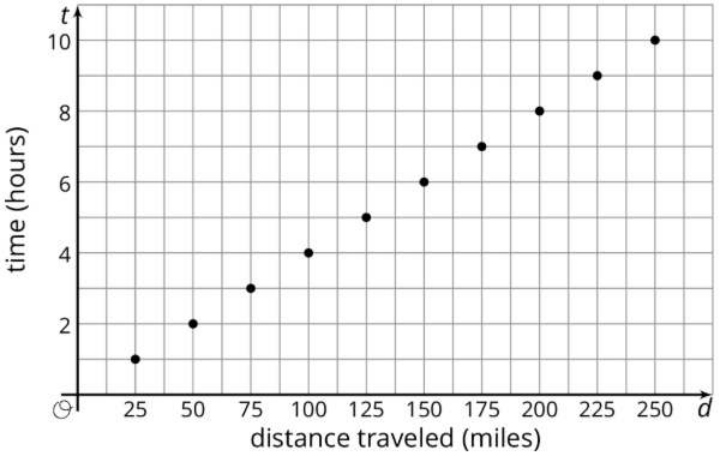


Figure 34.2.6: A graph of 10 points plotted on the coordinate plane with the origin labeled O. The horizontal d axis is labeled distance traveled in miles. The numbers 0 through 250, in increments of 25, are indicated, and there are vertical grid lines midway between. The vertical t axis is labeled time in hours. The numbers 0 through 10, in increments of 2, are indicated, and there are horizontal grid lines midway between. The data are as follows: 25 comma 1. 50 comma 2. 75 comma 3. 100 comma 4. 125 comma 5. 150 comma 6. 175 comma 7. 200 comma 8. 225 comma 9. 250 comma 10.

1. Write the coordinates of one of point on the graph. What does the point represent?
2. What is the speed of the ship in miles per hour?
3. Write an equation that relates the time, t, it takes to travel a given distance, d.

Exercise 34.2.5

Find a solution to each equation in the list that follows (not all numbers will be used):

1. $2^x = 8$
2. $2^x = 2$
3. $x^2 = 100$
4. $x^2 = \frac{1}{100}$
5. $x^1 = 7$
6. $2^x \cdot 2^3 = 2^7$
7. $\frac{2^x}{2^3} = 2^5$

List:

$\frac{1}{10}$ $\frac{1}{3}$ 1 2 3 4 5 7 8 10 16

(From Unit 6.3.4)

Exercise 34.2.6

Select **all** the expressions that are equivalent to $5x + 30x - 15x$.

A. $5(x + 6x - 3x)$
B. $(5 + 30 - 15) \cdot x$
C. $x(5 + 30x - 15x)$
D. $5x(1 + 6 - 3)$
E. $5(x + 30x - 15x)$

(From Unit 6.2.6)

Exercise 34.2.7

Evaluate each expression if x is 1, y is 2, and z is 3.

1. $7x^2 - z$
2. $(x + 4)^3 - y$
3. $y(x + 3^3)$
4. $(7 - y + z)^2$
5. $0.241x + x^3$

(From Unit 6.3.4)

34.2: Two Related Quantities, Part 2 is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

34.3: More Relationships

Lesson

Let's use graphs and equations to show relationships involving area, volume, and exponents.

Exercise 34.3.1: Which One Doesn't Belong: Graphs

Which one doesn’t belong?

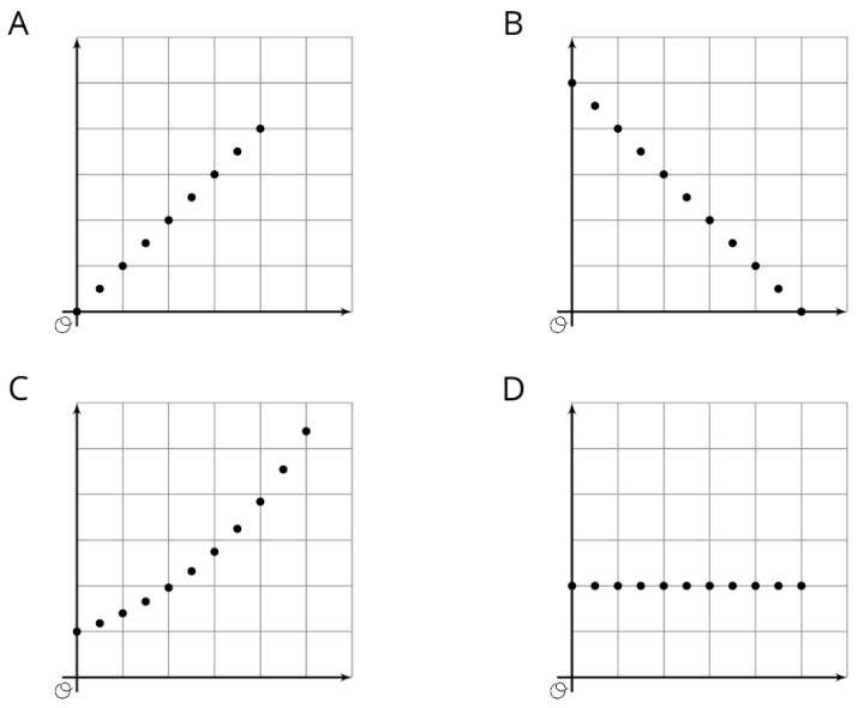


Figure 34.3.1: Four coordinate planes labeled A, B, C, and D, each with the origin labeled O. For graph A, nine points are graphed and the trend of the data points move linearly upward and to the right, where the first point begins at the origin. For graph B, 11 points are graphed and the trend of the data points move linearly downward and to the right, where the first point begins on the vertical axis, high above the origin. For graph C, 11 points are graphed and the trend of the data points move in a curve that moves upward and to the right, where the first point begins on the vertical axis, slightly above the origin. For graph D, 11 points are graphed and the trend of the data points move horizontally and to the right, where the first point begins on the vertical axis and slightly above the origin.

Exercise 34.3.2: Making a Banner

Mai is creating a rectangular banner to advertise the school play. The material for the banner is sold by the square foot. Mai has enough money to buy 36 square feet of material. She is trying to decide on the length and width of the banner.

1. If the length is 6 feet, what is the width?
2. If the length is 4 feet, what is the width?
3. If the length is 9 feet, what is the width?
4. To find different combinations of length and width that give an area of 36 square feet, Mai uses the equation $w = \frac{36}{l}$, where w is the width and l is the length. Compare your strategy and Mai's method for finding the width. How were they alike or different?
5. Use several combinations of length and width to create a graph that shows the relationship between the side lengths of various rectangles with area 36 square feet.
6. Explain how the graph describes the relationship between length and width for different rectangles with area 36.
7. Suppose Mai used the equation $l = \frac{36}{w}$ to find the length for different values of the width. Would the graph be different if she graphed length on the vertical axis and width on the horizontal axis? Explain how you know.

Exercise 34.3.3: Cereal Boxes

A cereal manufacturer needs to design a cereal box that has a volume of 225 cubic inches and a height that is no more than 15 inches.

1. The designers know that the volume of a rectangular prism can be calculated by multiplying the area of its base and its height. Complete the table with pairs of values that will make the volume 225 in^3.

Table 34.3.1

height (in)		5	9	12		$7\frac{1}{2}$
area of base (in ^{2})	75				15	

2. Describe how you found the missing values for the table.
3. Write an equation that shows how the area of the base, A, is affected by changes in the height, h, for different rectangular prisms with volume 225 in^3.
4. Plot the ordered pairs from the table on the graph to show the relationship between the area of the base and the height for different boxes box with volume 225 in^3.

Exercise 34.3.4: Multiplying Mosquitoes

A researcher who is studying mosquito populations collects the following data:

Table 34.3.2

day in the study (d)	**number of mosquitoes (n)**
1	2
2	4
3	8
4	16
5	32

1. The researcher said that, for these five days, the number of mosquitoes, n, can be found with the equation $n = 2^d$ where d is the day in the study. Explain why this equation matches the data.
2. Use the ordered pairs in the table to graph the relationship between number of mosquitoes and day in the study for these five days.
3. Describe the graph. Compare how the data, equation, and graph illustrate the relationship between the day in the study and the number of mosquitoes.
4. If the pattern continues, how many mosquitoes will there be on day 6?

Summary

Equations can represent relationships between geometric quantities. For instance:

- If s is the side length of a square, then the area A is related to s by $A = s^2$.
- Sometimes the relationships are more specific. For example, the perimeter P of a rectangle with length l and width w is $P = 2l + 2$. If we consider only rectangles with a length of 10, then the relationship between the perimeter and the width is $P = 20 + 2w$.

Here is another example of an equation with exponent expressing the relationship between quantities:

- A super ball is dropped from 10 feet. On each successive bounce, it only goes as high as on the previous bounce.

This means that on the first bounce, the ball will bounce 5 feet high, and then on the second bounce it will only go $2\frac{1}{2}$ feet high, and so on. We can represent this situation with an equation to find how high the super ball will bounce after any number of

bounces.

To find how high the super ball bounces on the nth bounce, we have to multiply 10 feet (the initial height) by $\frac{1}{2}$ and multiply by $\frac{1}{2}$ again for each bounce thereafter; we need to do this n times. So the height, h, of the ball on the nth bounce will be $h = 10\left(\frac{1}{2}\right)^n$. In this equation, the dependent variable, h, is affected by changes in the independent variable, n.

Equations and graphs can give us insight into different kinds of relationships between quantities and help us answer questions and solve problems.

Glossary Entries

Definition: Coordinate Plane

The coordinate plane is a system for telling where points are. For example. point R is located at $(3, 2)$ on the coordinate plane, because it is three units to the right and two units up.

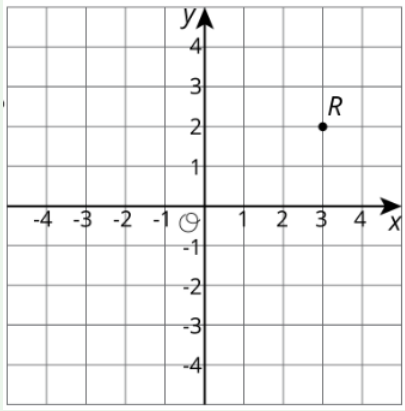


Figure 34.3.2

Definition: Dependent Variable

The dependent variable is the result of a calculation.

For example, a boat travels at a constant speed of 25 miles per hour. The equation $d = 25t$ describes the relationship between the boat's distance and time. The dependent variable is the distance traveled, because d is the result of multiplying 25 by t.

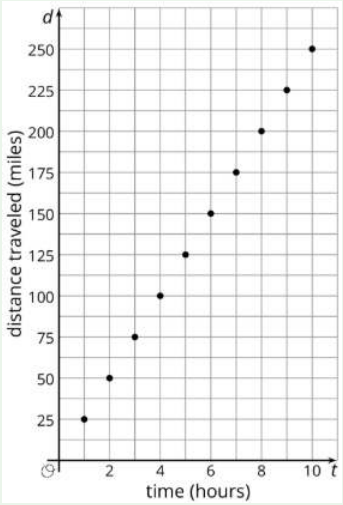


Figure 34.3.3: A graph of 10 points plotted in the coordinate plane with the origin labeled O. The horizontal t axis is labeled time in hours. The numbers 0 through 10, in increments of 2, are indicated, and there are vertical grid lines midway between. The vertical axis is labeled distance traveled in miles. The numbers 0 through 250, in increments of 25, are indicated, and there are horizontal grid lines midway between. The data are as follows: 1 comma 25. 2 comma 50. 3 comma 75. 4 comma 100. 5 comma 125. 6 comma 150. 7 comma 175. 8 comma 200. 9 comma 225. 10 comma 250.

Definition: Independent Variable

The independent variable is used to calculate the value of another variable.

For example, a boat travels at a constant speed of 25 miles per hour. The equation $d = 25t$ describes the relationship between the boat's distance and time. The independent variable is time, because t is multiplied by 25 to get d.

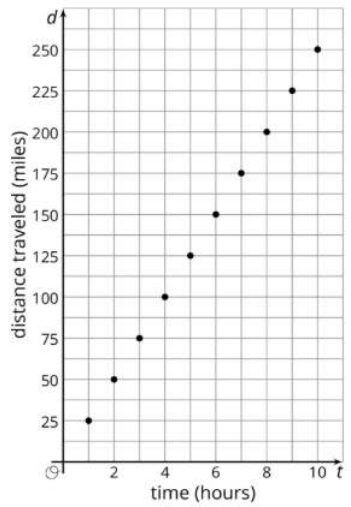


Figure 34.3.4: A graph of 10 points plotted in the coordinate plane with the origin labeled O. The horizontal t axis is labeled time in hours. The numbers 0 through 10, in increments of 2, are indicated, and there are vertical grid lines midway between. The vertical axis is labeled distance traveled in miles. The numbers 0 through 250, in increments of 25, are indicated, and there are horizontal grid lines midway between. The data are as follows: 1 comma 25. 2 comma 50. 3 comma 75. 4 comma 100. 5 comma 125. 6 comma 150. 7 comma 175. 8 comma 200. 9 comma 225. 10 comma 250.

Practice

Exercise 34.3.5

Elena is designing a logo in the shape of a parallelogram. She wants the logo to have an area of 12 square inches. She draws bases of different lengths and tries to compute the height for each.

1. Write an equation Elena can use to find the height, h, for each value of the base, b.
2. Use your equation to find the height of a parallelogram with base 1.5 inches.

Exercise 34.3.6

Han is planning to ride his bike 24 miles.

1. How long will it take if he rides at a rate of:
   3 miles per hour?
   4 miles per hour?
   6 miles per hour?
2. Write an equation that Han can use to find t, the time it will take to ride 24 miles, if his rate in miles per hour is represented by r.
3. On graph paper, draw a graph that shows t in terms of r for a 24-mile ride.

Exercise 34.3.7

The graph of the equation $V = 10s^3$ contains the points $(2, 80)$ and $(4, 640)$.

1. Create a story that is represented by this graph.
2. What do the points mean in the context of your story?

Exercise 34.3.8

You find a brass bottle that looks really old. When you rub some dirt off of the bottle, a genie appears! The genie offers you a reward. You must choose one:

$50,000 or a magical $1 coin.

The coin will turn into two coins on the first day. The two coins will turn into four coins on the second day. The four coins will double to 8 coins on the third day. The genie explains the doubling will continue for 28 days.

1. Write an equation that shows the number of coins, n, in terms of the day, d.
2. Create a table that shows the number of coins for each day for the first 15 days.
3. Create a graph for days 7 through 12 that shows how the number of coins grows with each day.

Exercise 34.3.9

At a market, 3.1 pounds of peaches cost $7.72. How much did the peaches cost per pound? Explain or show your reasoning. Round your answer to the nearest cent.

(From Unit 5.4.5)

Exercise 34.3.10

Andre set up a lemonade stand last weekend. It cost him $0.15 to make each cup of lemonade, and he sold each cup for $0.35.

1. If Andre collects $9.80, how many cups did he sell?
2. How much money did it cost Andre to make this amount of lemonade?
3. How much money did Andre make in profit?

(From Unit 5.4.5)

34.3: More Relationships is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

Section 35: Let's Put it to Work

This page titled Section 35: Let's Put it to Work is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

35.1: Tables, Equations, and Graphs, Oh My!

Lesson

Let's explore some equations from real-world situations.

Exercise 35.1.1: Matching Equations and Tables

Match each equation with a table that represents the same relationship. Be prepared to explain your reasoning.

$S-2=T$ $G=J+13$ $P=I-47.50$ $C+273.15=K$

$e=6s$ $m=8.96V$ $y=\frac{1}{12}x$ $t=\frac{d}{2.5}$

$g=28.35z$

Table 1:

Table 35.1.1

independent variable	dependent variable
20	8
58.85	23.54
804	321.6

Table 2:

Table 35.1.2

independent variable	dependent variable
5	18
36	49
75	88

Table 3:

Table 35.1.3

independent variable	dependent variable
2.5	22.4
20	179.2
75	672

Table 4:

Table 35.1.4

independent variable	dependent variable
20	$1\frac{2}{3}$
36	3
804	67

Table 5:

Table 35.1.5

independent variable	dependent variable
58.85	11.35
175.5	128
804	756.6

Table 6:

Table 35.1.6

independent variable	dependent variable
2.5	275.65
20	293.15
58.85	332

Table 7:

Table 35.1.7

independent variable	dependent variable
5	3
20	18
36	34

Table 8:

Table 35.1.8

independent variable	dependent variable
2.6	73.71
20	567
36	1, 020, 6

Table 9:

Table 35.1.9

independent variable	dependent variable
2.6	15.6
36	216
58.85	353.1

Exercise 35.1.2: Getting to Know an Equation

The equations in the previous activity represent situations.

- $S-2=T$ where S is the number of sides on a polygon and T is the number of triangles you can draw inside it (from one vertex to the others, without overlapping)
- $G=J+13$ where G is a day in the Gregorian calendar and J is the same day in the Julian calendar
- $P=I-47.50$ where I is the amount of income and P is the profit after \$47.50 in expenses
- $C+273.15=K$ where C is a temperature in degrees Celsius and K is the same temperature in Kelvin

- $e = 6s$ where e is the total edge length of a regular tetrahedron and s is the length of one side
- $m = 8.96V$ where V is the volume of a piece of copper and m is its mass
- $y = \frac{1}{12}x$ where x is the number of eggs and y is how many dozens that makes
- $t = \frac{d}{2.5}$ where t is the amount of time it takes in seconds to jog a distance of d meters at a constant speed of 2.5 meters per second
- $g = 28.35z$ where g is the mass in grams and z is the same amount in ounces

Your teacher will assign you one of these equations to examine more closely.

1. Rewrite your equation using words. Use words like product, sum, difference, quotient, and term.
2. In the previous activity, you matched equations and tables. Copy the values from the table that matched your assigned equation into the first 3 rows of this table. Make sure to label what each column represents.

Table 35.1.10

independent variable: _____	**dependent variable:** _____
60	
	300

3. Select one of the first 3 rows of the table and explain what those values mean in this situation.
4. Use your equation to find the values that complete the last 2 rows of the table. Explain your reasoning.
5. On graph paper, create a graph that represents this relationship. Make sure to label your axes.

Exercise 35.1.3: Sharing Your Equation with Others

Create a visual display of your assigned relationships that includes:

- your equation along with an explanation of each variable
- a verbal description of the relationship
- your table
- your graph

If you have time, research more about your relationship and add more details or illustrations to help explain the situation.

35.1: Tables, Equations, and Graphs, Oh My! is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

SECTION OVERVIEW

7: Rational Numbers

Section 36: Negative Numbers and Absolute Value

- 36.1: Positive and Negative Numbers
- 36.2: Points on the Number Line
- 36.3: Comparing Positive and Negative Numbers
- 36.4: Ordering Rational Numbers
- 36.5: Using Negative Numbers to Make Sense of Contexts
- 36.6: Absolute Value of Numbers
- 36.7: Comparing Numbers and Distance from Zero

Section 37: Inequalities

- 37.1: Writing and Graphing Inequalities
- 37.2: Solutions of Inequalities
- 37.3: Interpreting Inequalities

Section 38: The Coordinate Plane

- 38.1: Points on the Coordinate Plane
- 38.2: Constructing the Coordinate Plane
- 38.3: Interpreting Points on a Coordinate Plane
- 38.4: Distances on a Coordinate Plane
- 38.5: Shapes on the Coordinate Plane

Section 39: Common Factors and Common Multiples

- 39.1: Common Factors
- 39.2: Common Multiples
- 39.3: Using Common Multiples and Common Factors

Section 40: Let's Put it to Work

- 40.1: Drawing on the Coordinate Plane

This page titled 7: Rational Numbers is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

Section 36: Negative Numbers and Absolute Value

This page titled Section 36: Negative Numbers and Absolute Value is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

36.1: Positive and Negative Numbers

Lesson

Let's explore how we represent temperatures and elevations.

Exercise 36.1.1: Notice and Wonder: Memphis and Bangor



Figure 36.1.1

What do you notice? What do you wonder?

Exercise 36.1.2: Above and Below Zero

1. Here are three situations involving changes in temperature. Represent each change on the applet, and draw it on a number line. Then, answer the question.
   1. At noon, the temperature was 5 degrees Celsius. By late afternoon, it has risen 6 degrees Celsius. What was the temperature late in the afternoon?
   2. The temperature was 8 degrees Celsius at midnight. By dawn, it has dropped 12 degrees Celsius. What was the temperature at dawn?
   3. Water freezes at 0 degrees Celsius, but the freezing temperature can be lowered by adding salt to the water. A student discovered that adding half a cup of salt to a gallon of water lowers its freezing temperature by 7 degrees Celsius. What is the freezing temperature of the gallon of salt water?
2. Discuss with a partner:
   1. How did you name the resulting temperature in each situation? Did both of you refer to each resulting temperature by the same name or different names?
   2. What does it mean when the resulting temperature is above 0 on the number line? What does it mean when a temperature is below 0?
   3. Do numbers below 0 make sense outside of the context of temperature? If you think so, give some examples to show how they make sense. If you don't think so, give some examples to show otherwise.

Exercise 36.1.3: High Places, Low Places

1. Here is a table that shows elevations of various cities.

city	elevation (feet)
Harrisburg, PA	320
Bethell, IN	1, 211
Denver, CO	5, 280
Coachella, CA	−22
Death Valley, CA	−282
New York City, NY	33
Miami, FL	0

Table 36.1.1

a. On the list of cities, which city has the second highest elevation?
b. How would you describe the elevation of Coachella, CA, in relation to sea level?
c. How would you describe the elevation of Death Valley, CA, in relation to sea level?
d. If you are standing on a beach right next to the ocean, what is your elevation?
e. How would you describe the elevation of Miami, FL?
f. A city has a higher elevation than Coachella, CA. Select all numbers that could represent the city's elevation. Be prepared to explain your reasoning.
   i. -11 feet
   ii. -35 feet
   iii. 4 feet
   iv. -8 feet
   v. 0 feet

2. Here are two tables that show the elevations of highest points on land and lowest points in the ocean. Distances are measured from sea level. Drag the points marking the mountains and trenches to the vertical number line and answer the questions.

point	mountain	continent	elevation (meters)
C	Everest	Asia	8,848
H	Kilimanjaro	Africa	5,895
E	Denali	North America	6,168
A	Pikchu Pikchu	South America	5,664

Table 36.1.2

point	trench	ocean	elevation (meters)
F	Mariana Trench	Pacific	$-11,033$
B	Puerto Rico Trench	Atlantic	$-8,600$
D	Tonga Trench	Pacific	$-10,882$
G	Sunda Trench	Indian	$-7,725$

Table 36.1.3

a. Which point in the ocean is the lowest in the world? What is its elevation?
b. Which mountain is the highest in the world? What is its elevation?
c. If you plot the elevations of the mountains and trenches on a vertical number line, what would 0 represent? What would points above 0 represent? What about points below 0?
d. Which is farther from sea level: the deepest point in the ocean, or the top of the highest mountain in the world? Explain.

Are you ready for more?

A spider spins a web in the following way:

- It starts at sea level.
- It moves up one inch in the first minute.
- It moves down two inches in the second minute.
- It moves up three inches in the third minute.
- It moves down four inches in the fourth minute.

Assuming that the pattern continues, what will the spider's elevation be after an hour has passed?

Summary

Positive numbers are numbers that are greater than 0. **Negative numbers** are numbers that are less than zero. The meaning of a negative number in a context depends on the meaning of zero in that context.

For example, if we measure temperatures in degrees Celsius, then 0 degrees Celsius corresponds to the temperature at which water freezes.

In this context, positive temperatures are warmer than the freezing point and negative temperatures are colder than the freezing point. A temperature of -6 degrees Celsius means that it is 6 degrees away from 0 and it is less than 0. This thermometer shows a temperature of -6 degrees Celsius.

If the temperature rises a few degrees and gets very close to 0 degrees without reaching it, the temperature is still a negative number.

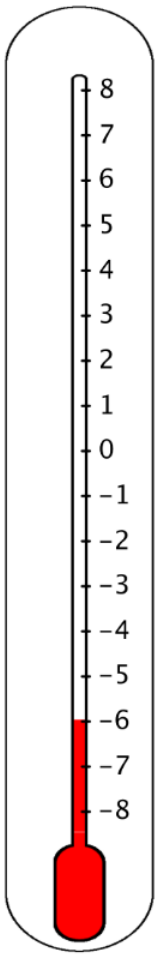


Figure 36.1.2

Another example is elevation, which is a distance above or below sea level. An elevation of 0 refers to the sea level. Positive elevations are higher than sea level, and negative elevations are lower than sea level.

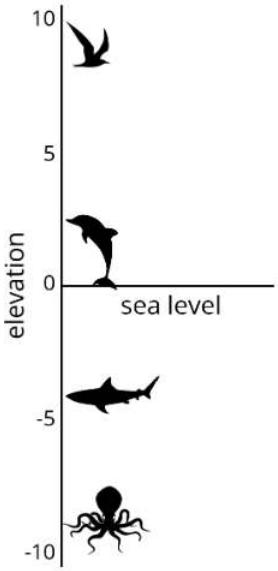


Figure 36.1.3

Glossary Entries

Definition: Negative Number

A negative number is a number that is less than zero. On a horizontal number line, negative numbers are usually shown to the left of 0.

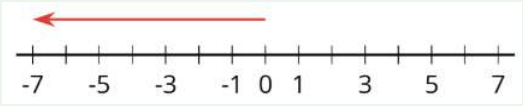


Figure 36.1.4

Definition: Positive Number

A positive number is a number that is greater than zero. On a horizontal number line, positive numbers are usually shown to the right of 0.

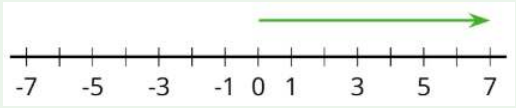


Figure 36.1.5

Practice

Exercise 36.1.4

1. Is a temperature of -11 degrees warmer or colder than a temperature of -15 degrees?
2. Is an elevation of -10 feet closer or farther from the surface of the ocean than an elevation of -8 feet?
3. It was 8 degrees at nightfall. The temperature dropped 10 degrees by midnight. What was the temperature at midnight?
4. A diver is 25 feet below sea level. After he swims up 15 feet toward the surface, what is his elevation?

Exercise 36.1.5

1. A whale is at the surface of the ocean to breathe. What is the whale's elevation?
2. The whale swims down 300 feet to feed. What is the whale's elevation now?
3. The whale swims down 150 more feet more. What is the whale's elevation now?
4. Plot each of the three elevations as a point on a vertical number line. Label each point with its numeric value.

Exercise 36.1.6

Explain how to calculate a number that is equal to $\frac{2.1}{1.5}$.

(From Unit 6.1.5)

Exercise 36.1.7

Write an equation to represent each situation and then solve the equation.

1. Andre drinks 15 ounces of water, which is $\frac{3}{5}$ of a bottle. How much does the bottle hold? Use x for the number of ounces of water the bottle holds.
2. A bottle holds 15 ounces of water. Jada drank 8.5 ounces of water. How many ounces of water are left in the bottle? Use y for the number of ounces of water left in the bottle.
3. A bottle holds z ounces of water. A second bottle holds 16 ounces, which is $\frac{8}{5}$ times as much water. How much does the first bottle hold?

(From Unit 6.1.4)

Exercise 36.1.8

A rectangle has an area of 24 square units and a side length of $2\frac{3}{4}$ units. Find the other side length of the rectangle. Show your reasoning.

(From Unit 4.4.2)

This page titled 36.1: Positive and Negative Numbers is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

36.2: Points on the Number Line

Lesson

Let's plot positive and negative numbers on the number line.

Exercise 36.2.1: A Point on the Number Line

Which of the following numbers could be B?

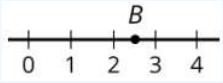


Figure 36.2.1

$2.5 \quad \frac{2}{5} \quad \frac{5}{2} \quad \frac{25}{10} \quad 2.49$

Exercise 36.2.2: What's the Temperature?

1. Here are five thermometers. The first four thermometers show temperatures in Celsius. Write the temperatures in the blanks.

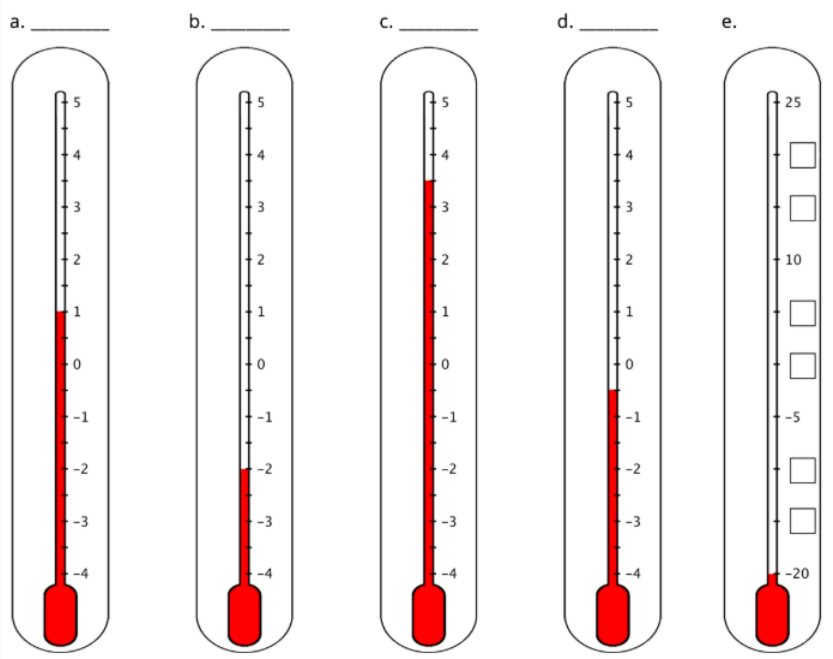


Figure 36.2.2: Five vertical thermometers, a through e. Thermometer a is shaded up to 1. Thermometer b is shaded up to 2. Thermometer c is shaded up to the halfway point between 3 and 4. Thermometer d is shaded to the halfway point between negative 1 and 0. Thermometer has tick marks labeled negative 20, blank, blank, negative 5, blank, blank, ten, blank, blank, 25.

The last thermometer is missing some numbers. Write them in the boxes.

2. Elena says that the thermometer shown here reads -2.5°C because the line of the liquid is above -2°C. Jada says that it is -1.5°C. Do you agree with either one of them? Explain your reasoning.

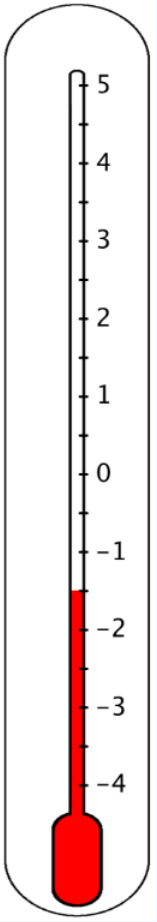


Figure 36.2.3

3. One morning, the temperature in Phoenix, Arizona, was 8°C and the temperature in Portland, Maine, was 12°C cooler. What was the temperature in Portland?

Exercise 36.2.3: Folded Number Lines

Your teacher will give you a sheet of tracing paper on which to draw a number line.

1. Follow the steps to make your own number line.
   - Use a straightedge or a ruler to draw a horizontal line. Mark the middle point of the line and label it 0.
   - To the right of 0, draw tick marks that are 1 centimeter apart. Label the tick marks 1, 2, 3. . . 10. This represents the positive side of your number line.
   - Fold your paper so that a vertical crease goes through 0 and the two sides of the number line match up perfectly.
   - Use the fold to help you trace the tick marks that you already drew onto the opposite side of the number line. Unfold and label the tick marks -1, -2, -3. . . -10. This represents the negative side of your number line.
2. Use your number line to answer these questions:
   a. Which number is the same distance away from zero as is the number 4?
   b. Which number is the same distance away from zero as is the number -7?
   c. Two numbers that are the same distance from zero on the number line are called **opposites**. Find another pair of opposites on the number line.
   d. Determine how far away the number 5 is from 0. Then, choose a positive number and a negative number that is each farther away from zero than is the number 5.
   e. Determine how far away the number -2 is from 0. Then, choose a positive number and a negative number that is each farther away from zero than is the number -2.
   Pause here so your teacher can review your work.

3. Here is a number line with some points labeled with letters. Determine the location of points P, X, and Y.

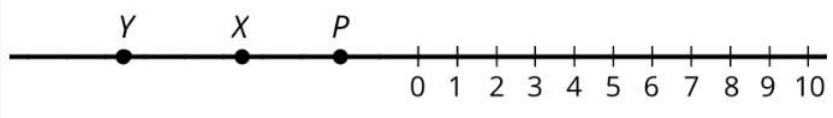


Figure 36.2.4

If you get stuck, trace the number line and points onto a sheet of tracing paper, fold it so that a vertical crease goes through 0, and use the folded number line to help you find the unknown values.

Are you ready for more?

At noon, the temperatures in Portland, Maine, and Phoenix, Arizona, had opposite values. The temperature in Portland was 18°C lower than in Phoenix. What was the temperature in each city? Explain your reasoning.

Summary

Here is a number line labeled with positive and negative numbers. The number 4 is positive, so its location is 4 units to the right of 0 on the number line. The number -1.1 is negative, so its location is 1.1 units to the left of 0 on the number line.

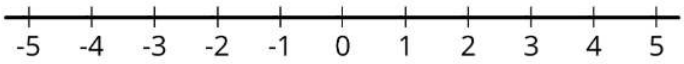


Figure 36.2.5

We say that the *opposite* of 8.3 is -8.3, and that the *opposite* of $\frac{-3}{2}$ is $\frac{3}{2}$. Any pair of numbers that are equally far from 0 are called **opposites**.

Points A and B are opposites because they are both 2.5 units away from 0, even though A is to the left of 0 and B is to the right of 0.

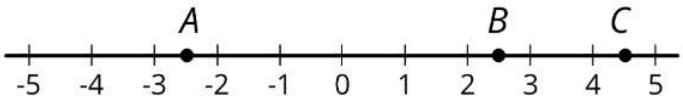


Figure 36.2.6

A positive number has a negative number for its opposite. A negative number has a positive number for its opposite. The opposite of 0 is itself.

You have worked with positive numbers for many years. All of the positive numbers you have seen—whole and non-whole numbers—can be thought of as fractions and can be located on a the number line.

To locate a non-whole number on a number line, we can divide the distance between two whole numbers into fractional parts and then count the number of parts. For example, 2.7 can be written as $2\frac{7}{10}$. The segment between 2 and 3 can be partitioned into 10 equal parts or 10 tenths. From 2, we can count 7 of the tenths to locate 2.7 on the number line.

All of the fractions and their opposites are what we call **rational numbers**. For example, $4, -1.1, 8.3, -8.3, \frac{-3}{2}$,and $\frac{3}{2}$ are all rational numbers.

Glossary Entries

Definition: Negative Number

A negative number is a number that is less than zero. On a horizontal number line, negative numbers are usually shown to the left of 0.

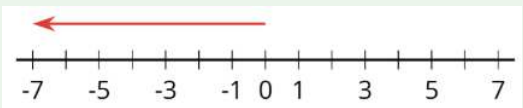


Figure 36.2.7

Definition: Opposite

Two numbers are opposites if they are the same distance from 0 and on different sides of the number line.

For example, 4 is the opposite of -4, and -4 is the opposite of 4. They are both the same distance from 0. One is negative, and the other is positive.

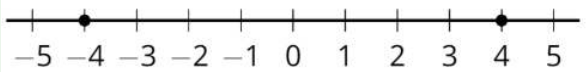


Figure 36.2.8

Definition: Positive Number

A positive number is a number that is greater than zero. On a horizontal number line, positive numbers are usually shown to the right of 0.

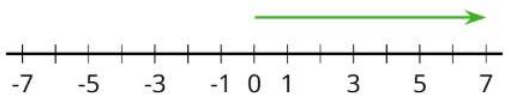


Figure 36.2.9

Definition: Rational Number

A rational number is a fraction or the opposite of a fraction.

For example, 8 and -8 are rational numbers because they can be written as $\frac{8}{1}$ and $-\frac{8}{1}$.

Also, 0.75 and -0.75 are rational numbers because they can be written as $\frac{75}{100}$ and $-\frac{75}{100}$.

Practice

Exercise 36.2.4

For each number, name its opposite.

1. -5
2. 28
3. -10.4

1. 0.875
2. 0
3. $-8,003$

Exercise 36.2.5

Plot the numbers -1.5, $\frac{3}{2}$, $-\frac{3}{2}$, and $-\frac{4}{3}$ on the number line. Label each point with its numeric value.

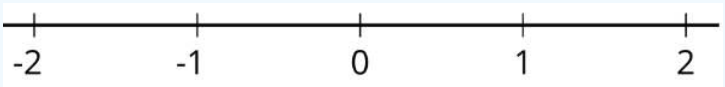


Figure 36.2.10

Exercise 36.2.6

Plot these points on a number line.

- -1.5
- the opposite of -2
- the opposite of 0.5
- -2

Exercise 36.2.7

1. Represent each of these temperatures in degrees Fahrenheit with a positive or negative number.
   - 5 degrees above zero
   - 3 degrees below zero
   - 6 degrees above zero
   - $2\frac{3}{4}$ degrees below zero
2. Order the temperatures above from the coldest to the warmest.

(From Unit 7.1.1)

Exercise 36.2.8

Solve each equation.

1. $8x = \frac{2}{3}$
2. $1\frac{1}{2} = 2x$
3. $5x = \frac{2}{7}$
4. $\frac{1}{4}x = 5$
5. $\frac{1}{5} = \frac{2}{3}x$

(From Unit 6.1.5)

Exercise 36.2.9

Write the solution to each equation as a fraction and as a decimal.

1. $2x = 3$
2. $5y = 3$
3. $0.3z = 0.009$

(From Unit 6.1.5)

Exercise 36.2.10

There are 15.24 centimeters in 6 inches.

1. How many centimeters are in 1 foot?
2. How many centimeters are in 1 yard?

(From Unit 3.2.3)

36.2: Points on the Number Line is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

36.3: Comparing Positive and Negative Numbers

Lesson

Let's compare numbers on the number line.

Exercise 36.3.1: Which One Doesn't Belong: Inequalities

Which inequality doesn't belong?

- $\frac{5}{4} < 2$
- $8.5 > 0.95$
- $8.5 < 7$
- $10.00 < 100$

Exercise 36.3.2: Comparing Temperatures

Here are the low temperatures, in degrees Celsius, for a week in Anchorage, Alaska.

Table 36.3.1

day	Mon	Tues	Weds	Thurs	Fri	Sat	Sun
temperature	5	-1	-5.5	-2	3	4	0

1. Plot the temperatures on a number line. Which day of the week had the lowest low temperature?
2. The lowest temperature ever recorded in the United States was -62 degrees Celsius, in Prospect Creek Camp, Alaska. The average temperature on Mars is about -55 degrees Celsius.
   a. Which is warmer, the coldest temperature recorded in the USA, or the average temperature on Mars? Explain how you know.
   b. Write an inequality to show your answer.
3. On a winter day the low temperature in Anchorage, Alaska, was -21 degrees Celsius and the low temperature in Minneapolis, Minnesota, was -14 degrees Celsius.
   Jada said, “I know that 14 is less than 21, so -14 is also less than -21. This means that it was colder in Minneapolis than in Anchorage.”
   Do you agree? Explain your reasoning.

Are you ready for more?

Another temperature scale frequently used in science is the *Kelvin scale*. In this scale, 0 is the lowest possible temperature of anything in the universe, and it is -273.15 degrees in the Celsius scale. Each 1K is the same as 1°C, so 10K is the same as -263.15°C.

1. Water boils at 100°C. What is this temperature in K?
2. Ammonia boils at -35.5°C. What is the boiling point of ammonia in K?
3. Explain why only positive numbers (and 0) are needed to record temperature in K.

Exercise 36.3.3: Rational Numbers on a Number Line

1. Plot the numbers -2, 4, -7, and 10 on the number line. Label each point with its numeric value.

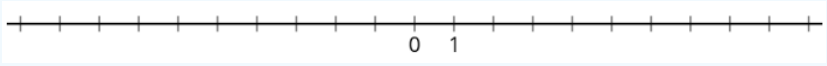


Figure 36.3.1

2. Decide whether each inequality statement is true or false. Be prepared to explain your reasoning.
   a. $-2 < 4$
   b. $-2 < -7$
   c. $4 > -7$

d. $-7 > 10$

Drag each point to its proper place on the number line. Use your observations to help answer the questions that follow.

3. Andre says that $\frac{1}{4}$ is less than $-\frac{3}{4}$ because, of the two numbers, $\frac{1}{4}$ is closer to 0. Do you agree? Explain your reasoning.
4. Answer each question. Be prepared to explain how you know.
   a. Which number is greater: $\frac{1}{4}$ or $\frac{5}{4}$?
   b. Which number is farther from 0: $\frac{1}{4}$ or $\frac{5}{4}$?
   c. Which number is greater: $-\frac{3}{4}$ or $\frac{5}{8}$?
   d. Which number is farther from 0: $-\frac{3}{4}$ or $\frac{5}{8}$?
   e. Is the number that is farther from 0 always the greater number? Explain your reasoning.

Summary

We use the words *greater than* and *less than* to compare numbers on the number line. For example, the numbers -2.7, 0.8, and -1.3, are shown on the number line.

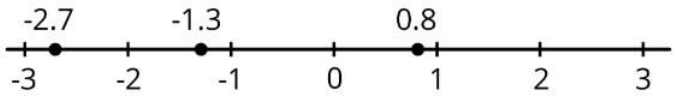


Figure 36.3.2

Because -2.7 is to the left of -1.3, we say that -2.7 is less than -1.3. We write:

$$-2.7 < -1.3$$

In general, any number that is to the left of a number n is less than n.

We can see that -1.3 is greater than -2.7 because -1.3 is to the right of -2.7. We write:

$$-1.3 > -2.7$$

In general, any number that is to the right of a number is greater than

We can also see that $0.8 > -1.3$ and $0.8 > -2.7$. In general, any positive number is greater than any negative number.

Glossary Entries

Definition: Negative Number

A negative number is a number that is less than zero. On a horizontal number line, negative numbers are usually shown to the left of 0.

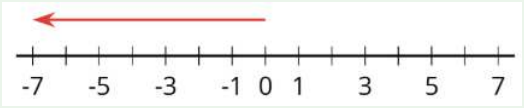


Figure 36.3.3

Definition: Opposite

Two numbers are opposites if they are the same distance from 0 and on different sides of the number line.

For example, 4 is the opposite of -4, and -4 is the opposite of 4. They are both the same distance from 0. One is negative, and the other is positive.

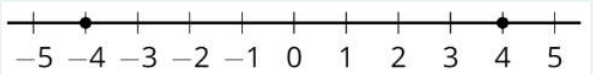


Figure 36.3.4

Definition: Positive Number

A positive number is a number that is greater than zero. On a horizontal number line, positive numbers are usually shown to the right of 0.

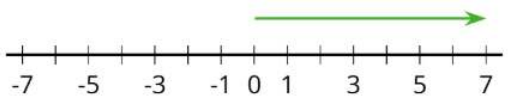


Figure 36.3.5

Definition: Rational Number

A rational number is a fraction or the opposite of a fraction.

For example, 8 and -8 are rational numbers because they can be written as $\frac{8}{1}$ and $-\frac{8}{1}$.

Also, 0.75 and -0.75 are rational numbers because they can be written as $\frac{75}{100}$ and $-\frac{75}{100}$.

Definition: Sign

The sign of any number other than 0 is either positive or negative.

For example, the sign of 6 is positive. The sign of -6 is negative. Zero does not have a sign, because it is not positive or negative.

Practice

Exercise 36.3.4

Decide whether each inequality statement is true or false. Explain your reasoning.

1. $-5 > 2$
2. $3 > -8$
3. $-12 > -15$
4. $-12.5 > -12$

Exercise 36.3.5

Here is a true statement: $-8.7 < -8.4$. Select **all** of the statements that are equivalent to $-8.7 < -8.4$.

A. -8.7 is further to the right on the number line than -8.4.
B. -8.7 is further to the left on the number line than -8.4.
C. -8.7 is less than -8.4.
D. -8.7 is greater than -8.4.
E. -8.4 is less than -8.7.
F. -8.4 is greater than -8.7.

Exercise 36.3.6

Plot each of the following numbers on the number line. Label each point with its numeric value. 0.4, -1.5, $-1\frac{7}{10}$, $-\frac{11}{10}$

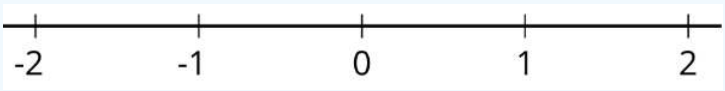


Figure 36.3.6

(From Unit 7.1.2)

Exercise 36.3.7

The table shows five states and the lowest point in each state.

Table 36.3.2

state	lowest elevation (feet)
California	-282
Colorado	3350
Louisiana	-8
New Mexico	2842
Wyoming	3099

Put the states in order by their lowest elevation, from least to greatest.

(From Unit 7.1.4)

Exercise 36.3.8

Each lap around the track is 400 meters.

1. How many meters does someone run if they run:
   2 laps?
   5 laps?
   x laps?
2. If Noah ran 14 laps, how many meters did he run?
3. If Noah ran 7,600 meters, how many laps did he run?

(From Unit 6.1.6)

Exercise 36.3.9

A stadium can seat 16,000 people at full capacity.

1. If there are 13,920 people in the stadium, what percentage of the capacity is filled? Explain or show your reasoning.
2. What percentage of the capacity is not filled?

(From Unit 3.4.7)

36.3: Comparing Positive and Negative Numbers is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

36.4: Ordering Rational Numbers

Lesson

Let's order rational numbers.

Exercise 36.4.1: How Do They Compare?

Use the symbols >, <, or = to compare each pair of numbers. Be prepared to explain your reasoning.

- $12 \_\_ 19$
- $15 \_\_ 1.5$
- $6.050 \_\_ 6.05$
- $\frac{19}{24} \_\_ \frac{19}{21}$
- $212 \_\_ 190$
- $9.02 \_\_ 9.2$
- $0.4 \_\_ \frac{9}{40}$
- $\frac{16}{17} \_\_ \frac{11}{12}$

Exercise 36.4.2: Ordering Rational Number Cards

Your teacher will give you a set of number cards. Order them from least to greatest.

Your teacher will give you a second set of number cards. Add these to the correct places in the ordered set.

Exercise 36.4.3: Comparing Points on A Line

1.

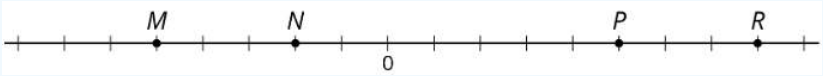


Figure 36.4.1

Use each of the following terms at least once to describe or compare the values of points M, N, P, R.

- greater than
- less than
- opposite of (or opposites)
- negative number

2. Tell what the value of each point would be if:

   a. P is $2\frac{1}{2}$
   b. N is -0.4
   c. R is 200
   d. M is -15

Are you ready for more?

The list of fractions between 0 and 1 with denominators between 1 and 3 looks like this:

$$\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}$$

We can put them in order like this: $\frac{0}{1} < \frac{1}{3} < \frac{1}{2} < \frac{2}{3} < \frac{1}{1}$

Now let's expand the list to include fractions with denominators of 4. We won't include $\frac{2}{4}$, because $\frac{1}{2}$ is already on the list.

$$\frac{0}{1} < \frac{1}{4} < \frac{1}{3} < \frac{1}{2} < \frac{2}{3} < \frac{3}{4} < \frac{1}{1}$$

1. Expand the list again to include fractions that have denominators of 5.
2. Expand the list you made to include fractions have have denominators of 6.

3. When you add a new fraction to the list, you put it in between two "neighbors." Go back and look at your work. Do you see a relationship between a new fraction and its two neighbors?

Summary

To order rational numbers from least to greatest, we list them in the order they appear on the number line from left to right. For example, we can see that the numbers

$$-2.7, -1.3, 0.8$$

are listed from least to greatest because of the order they appear on the number line.

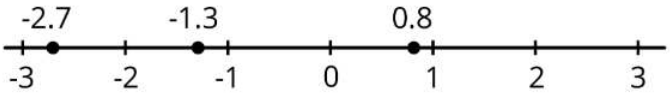


Figure 36.4.2

Glossary Entries

Definition: Negative Number

A negative number is a number that is less than zero. On a horizontal number line, negative numbers are usually shown to the left of 0.

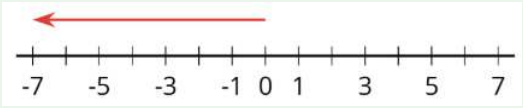


Figure 36.4.3

Definition: Opposite

Two numbers are opposites if they are the same distance from 0 and on different sides of the number line.

For example, 4 is the opposite of -4, and -4 is the opposite of 4. They are both the same distance from 0. One is negative, and the other is positive.

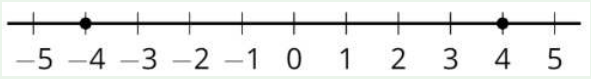


Figure 36.4.4

Definition: Positive Number

A positive number is a number that is greater than zero. On a horizontal number line, positive numbers are usually shown to the right of 0.

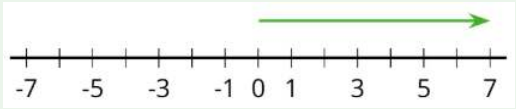


Figure 36.4.5

Definition: Rational Number

A rational number is a fraction or the opposite of a fraction.

For example, 8 and -8 are rational numbers because they can be written as $\frac{8}{1}$ and $-\frac{8}{1}$.

Also, 0.75 and -0.75 are rational numbers because they can be written as $\frac{75}{100}$ and $-\frac{75}{100}$.

Definition: Sign

The sign of any number other than 0 is either positive or negative.

For example, the sign of 6 is positive. The sign of -6 is negative. Zero does not have a sign, because it is not positive or negative.

Practice

Exercise 36.4.4

Select **all** of the numbers that are *greater than* -5.

A. 1.3
B. -6
C. -12
D. $\frac{1}{7}$
E. -1
F. -4

Exercise 36.4.5

Order these numbers from least to greatest: $\frac{1}{2}, 0, 1, -1\frac{1}{2}, -\frac{1}{2}, -1$

Exercise 36.4.6

Here are the boiling points of certain elements in degrees Celsius:

- Argon: -185.8
- Chlorine: -34
- Fluorine: -188.1
- Hydrogen: -252.87
- Krypton: -153.2

List the elements from least to greatest boiling points.

Exercise 36.4.7

Explain why zero is considered its own opposite.

(From Unit 7.1.2)

Exercise 36.4.8

Explain how to make these calculations mentally.

1. $99 + 54$
2. $244 - 99$
3. $99 \cdot 6$
4. $99 \cdot 15$

(From Unit 6.2.4)

Exercise 36.4.9

Find the quotients.

1. $\frac{1}{2} \div 2$
2. $2 \div 2$

3. $\frac{1}{2} \div \frac{1}{2}$
4. $\frac{38}{79} \div \frac{38}{79}$

(From Unit 4.3.2)

Exercise 36.4.10

Over several months, the weight of a baby measured in pounds doubles. Does its weight measured in kilograms also double? Explain.

(From Unit 3.2.3)

36.4: Ordering Rational Numbers is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

36.5: Using Negative Numbers to Make Sense of Contexts

Lesson

Let's make sense of negative amounts of money.

Exercise 36.5.1: Notice and Wonder: It Comes and Goes

Table 36.5.1

activity	amount
do my chores	30.00
babysit my cousin	45.00
buy my lunch	−10.80
get my allowance	15.00
buy a shirt	−18.69
pet my dog	0.00

What do you notice? What do you wonder?

Exercise 36.5.2: The Concession Stand

The manager of the concession stand keeps records of all of the supplies she buys and all of the items she sells. The table shows some of her records for Tuesday.

Table 36.5.2

item	quantity	value in dollars
doughnuts	−58	37.70
straws	3,000	−10.35
hot dogs	−39	48.75
pizza	13	−116.87
apples	−40	14.00
french fries	−88	132.00

1. Which items did she sell? Explain your reasoning.
2. How can we interpret -58 in this situation?
3. How can we interpret -10.35 in this situation?
4. On which item did she spend the most amount of money? Explain your reasoning.

Exercise 36.5.3: Drinks for Sale

A vending machine in an office building sells bottled beverages. The machine keeps track of all changes in the number of bottles from sales and from machine refills and maintenance. This record shows the changes for every 5-minute period over one hour.

1. What might a positive number mean in this context? What about a negative number?
2. What would a “0” in the second column mean in this context?
3. Which numbers—positive or negative—result in fewer bottles in the machine?

4. At what time was there the greatest change to the number of bottles in the machine? How did that change affect the number of remaining bottles in the machine?
5. At which time period, 8:05–8:09 or 8:25–8:29, was there a greater change to the number of bottles in the machine? Explain your reasoning.
6. The machine must be emptied to be serviced. If there are 40 bottles in the machine when it is to be serviced, what number will go in the second column in the table?

Table 36.5.3

time	number of bottles
8:00-8:04	-1
8:05-8:09	$+12$
8:10-8:14	-4
8:15-8:19	-1
8:20-8:24	-5
8:25-8:29	-12
8:30-8:34	-2
8:35-8:39	0
8:40-8:44	0
8:45-8:49	-6
8:50-8:54	$+24$
8:55-8:59	0
service	

Are you ready for more?

Priya, Mai, and Lin went to a cafe on a weekend. Their shared bill came to $25. Each student gave the server a $10 bill. The server took this $30 and brought back five $1 bills in change. Each student took $1 back, leaving the rest, $2, as a tip for the server.

As she walked away from the cafe, Lin thought, “Wait—this doesn’t make sense. Since I put in $10 and got $1 back, I wound up paying $9. So did Mai and Priya. Together, we paid $27. Then we left a $2 tip. That makes $29 total. And yet we originally gave the waiter $30. Where did the extra dollar go?”

Think about the situation and about Lin’s question. Do you agree that the numbers didn’t add up properly? Explain your reasoning.

Summary

Sometimes we represent changes in a quantity with positive and negative numbers. If the quantity increases, the change is positive. If it decreases, the change is negative.

- Suppose 5 gallons of water is put in a washing machine. We can represent the change in the number of gallons as +5. If 3 gallons is emptied from the machine, we can represent the change as -3.

It is especially common to represent money we receive with positive numbers and money we spend with negative numbers.

- Suppose Clare gets $30.00 for her birthday and spends $18.00 buying lunch for herself and a friend. To her, the value of the gift can be represented as +30.00 and the value of the lunch as -18.00.

Whether a number is considered positive or negative depends on a person’s perspective. If Clare’s grandmother gives her $20 for her birthday, Clare might see this as +20, because to her, the amount of money she has increased. But her grandmother might see it

as -20, because to her, the amount of money she has decreased.

In general, when using positive and negative numbers to represent changes, we have to be very clear about what it means when the change is positive and what it means when the change is negative.

Glossary Entries

Definition: Negative Number

A negative number is a number that is less than zero. On a horizontal number line, negative numbers are usually shown to the left of 0.

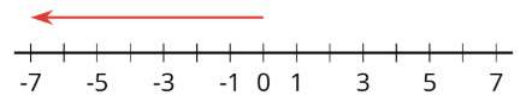


Figure 36.5.1

Definition: Opposite

Two numbers are opposites if they are the same distance from 0 and on different sides of the number line.

For example, 4 is the opposite of -4, and -4 is the opposite of 4. They are both the same distance from 0. One is negative, and the other is positive.

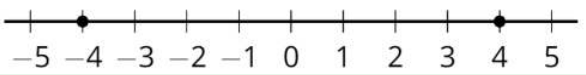


Figure 36.5.2

Definition: Positive Number

A positive number is a number that is greater than zero. On a horizontal number line, positive numbers are usually shown to the right of 0.

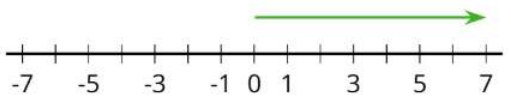


Figure 36.5.3

Definition: Rational Number

A rational number is a fraction or the opposite of a fraction.

For example, 8 and -8 are rational numbers because they can be written as $\frac{8}{1}$ and $-\frac{8}{1}$.

Also, 0.75 and -0.75 are rational numbers because they can be written as $\frac{75}{100}$ and $-\frac{75}{100}$.

Definition: Sign

The sign of any number other than 0 is either positive or negative.

For example, the sign of 6 is positive. The sign of -6 is negative. Zero does not have a sign, because it is not positive or negative.

Practice

Exercise 36.5.4

Write a positive or negative number to represent each change in the high temperature.

1. Tuesday's high temperature was 4 degrees less than Monday's high temperature.
2. Wednesday's high temperature was 3.5 degrees less than Tuesday's high temperature.
3. Thursday's high temperature was 6.5 degrees more than Wednesday's high temperature.

4. Friday's high temperature was 2 degrees less than Thursday's high temperature.

Exercise 36.5.5

Decide which of the following quantities can be represented by a positive number and which can be represented by a negative number. Give an example of a quantity with the opposite sign in the same situation.

1. Tyler's puppy gained 5 pounds.
2. The aquarium leaked 2 gallons of water.
3. Andre received a gift of $10.
4. Kiran gave a gift of $10.
5. A climber descended 550 feet.

Exercise 36.5.6

Make up a situation where a quantity is changing.

1. Explain what it means to have a negative change.
2. Explain what it means to have a positive change.
3. Give an example of each.

Exercise 36.5.7

1. On the number line, label the points that are 4 units away from 0.

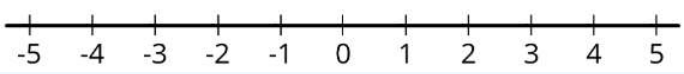


Figure 36.5.4

2. If you fold the number line so that a vertical crease goes through 0, the points you label would match up. Explain why this happens.
3. On the number line, label the points that are units from 0. What is the distance between these points?

(From Unit 7.1.2)

Exercise 36.5.8

Evaluate each expression.

- $2^3 \cdot 3$
- $\frac{4^2}{2}$
- 3^1
- $6^2 \div 4$
- $2^3 - 2$
- $10^2 + 5^2$

(From Unit 6.3.1)

36.5: Using Negative Numbers to Make Sense of Contexts is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

36.6: Absolute Value of Numbers

Lesson

Let's explore distances from zero more closely.

Exercise 36.6.1: Number Talk: Closer to Zero

For each pair of expressions, decide mentally which one has a value that is closer to 0.

$\frac{9}{11}$ or $\frac{15}{11}$

$\frac{1}{5}$ or $\frac{1}{9}$

1.25 or $\frac{5}{4}$

0.01 or 0.001

Exercise 36.6.2: Jumping Flea

Move the bug to a starting point, choose a jump distance, and press the jump button. You may need to zoom in or out if your bug jumps off the screen.

1. A bug is jumping around on a number line.
   a. If the bug starts at 1 and jumps 4 units to the right, where does it end up? How far away from 0 is this?
   b. If the bug starts at 1 and jumps 4 units to the left, where does it end up? How far away from 0 is this?
   c. If the bug starts at 0 and jumps 3 units away, where might it land?
   d. If the bug jumps 7 units and lands at 0, where could it have started?
   e. The **absolute value** of a number is the distance it is from 0. The bug is currently to the left of 0 and the absolute value of its location is 4. Where on the number line is it?
   f. If the bug is to the left of 0 and the absolute value of its location is 5, where on the number line is it?
   g. If the bug is to the right of 0 and the absolute value of its location is 2.5, where on the number line is it?
2. We use the notation $|-2|$ to say "the absolute value of -2," which means "the distance of -2 from 0 on the number line."
   a. What does $|-7|$ mean and what is its value?
   b. What does $|-1.8|$ mean and what is its value?

Exercise 36.6.3: Absolute Elevation and Temperature

1. A part of the city of New Orleans is 6 feet below sea level. We can use "-6 feet" to describe its elevation, and "$|-6|$ feet" to describe its vertical distance from sea level. In the context of elevation, what would each of the following numbers describe?
   a. 25 feet
   b. $|25|$ feet
   c. -8 feet
   d. $|-8|$ feet
2. The elevation of a city is different from sea level by 10 feet. Name the two elevations that the city could have.
3. We write "-5°C" to describe a temperature that is 5 degrees Celsius below freezing point and "6°C" for a temperature that is 5 degrees above freezing. In this context, what do each of the following numbers describe?
   a. 1°C
   b. -4°C
   c. $|12|^\circ\text{C}$
   d. $|-7|^\circ\text{C}$
4. a. Which temperature is colder: -6°C or 3°C?
   b. Which temperature is closer to freezing temperature: -6°C or 3°C?
   c. Which temperature has a smaller absolute value? Explain how you know.

5. Are you ready for more?

Summary

We compare numbers by comparing their positions on the number line: the one farther to the right is greater; the one farther to the left is less.

Sometimes we wish to compare which one is closer to or farther from 0. For example, we may want to know how far away the temperature is from the freezing point of $0^\circ C$, regardless of whether it is above or below freezing.

The **absolute value** of a number tells us its distance from 0.

The absolute value of -4 is 4, because -4 is 4 units to the left of 0. The absolute value of 4 is also 4, because 4 is 4 units to the right of 0. Opposites always have the same absolute value because they both have the same distance from 0.

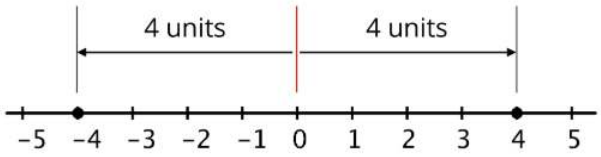


Figure 36.6.1

The distance from 0 to itself is 0, so the absolute value of 0 is 0. Zero is the *only* number whose distance to 0 is 0. For all other absolute values, there are always two numbers—one positive and one negative—that have that distance from 0.

To say "the absolute value of 4," we write: $|4|$

To say that "the absolute value of -8 is 8," we write: $|-8| = |8|$

Glossary Entries

Definition: Absolute Value

The absolute value of a number is its distance from 0 on the number line.

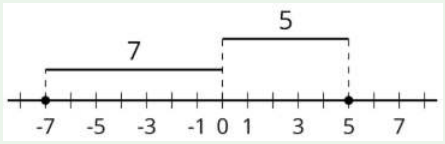


Figure 36.6.2

The absolute value of -7 is 7, because it is 7 units away from 0. The absolute value of 5 is 5, because it is 5 units away from 0.

Definition: Negative Number

A negative number is a number that is less than zero. On a horizontal number line, negative numbers are usually shown to the left of 0.

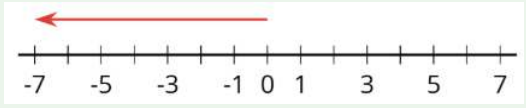


Figure 36.6.3

Definition: Opposite

Two numbers are opposites if they are the same distance from 0 and on different sides of the number line.

For example, 4 is the opposite of -4, and -4 is the opposite of 4. They are both the same distance from 0. One is negative, and the other is positive.

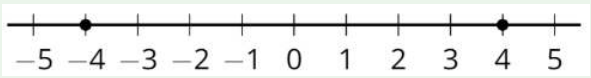


Figure 36.6.4

Definition: Positive Number

A positive number is a number that is greater than zero. On a horizontal number line, positive numbers are usually shown to the right of 0.

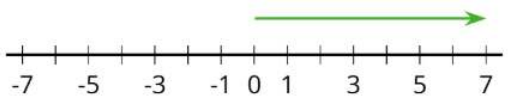


Figure 36.6.5

Definition: Rational Number

A rational number is a fraction or the opposite of a fraction.

For example, 8 and -8 are rational numbers because they can be written as $\frac{8}{1}$ and $-\frac{8}{1}$.

Also, 0.75 and -0.75 are rational numbers because they can be written as $\frac{75}{100}$ and $-\frac{75}{100}$.

Definition: Sign

The sign of any number other than 0 is either positive or negative.

For example, the sign of 6 is positive. The sign of -6 is negative. Zero does not have a sign, because it is not positive or negative.

Practice

Exercise 36.6.4

On the number line, plot and label all numbers with an absolute value of $\frac{3}{2}$.

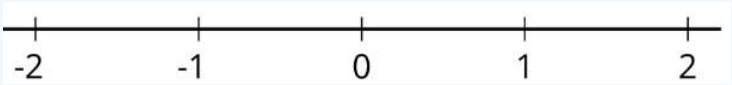


Figure 36.6.6

Exercise 36.6.5

The temperature at dawn is 6°C away from 0. Select **all** the temperatures that are possible.

A. -12°C
B. -6°C
C. 0°C
D. 6°C
E. 12°C

Exercise 36.6.6

Put these numbers in order, from least to greatest.

$$|-2.7| \qquad 0 \qquad 1.3 \qquad |-1| \qquad 2$$

Exercise 36.6.7

Lin's family needs to travel 325 miles to reach her grandmother's house.

1. At 26 miles, what percentage of the trip's distance have they completed?
2. How far have they traveled when they have completed 72% of the trip's distance?
3. At 377 miles, what percentage of the trip's distance have they completed?

(From Unit 5.4.3)

Exercise 36.6.8

Elena donates some money to charity whenever she earns money as a babysitter. The table shows how much money, d, she donates for different amounts of money, m, that she earns.

Table 36.6.1

d	4.44	1.80	3.12	3.60	2.16
m	37	15	26	30	18

1. What percent of her income does Elena donate to charity? Explain or show your work.
2. Which quantity, m or d, would be the better choice for the dependent variable in an equation describing the relationship between m and d? Explain your reasoning.
3. Use your choice from the second question to write an equation that relates m and d.

(From Unit 6.4.1)

Exercise 36.6.9

How many times larger is the first number in the pair than the second?

1. 3^4 is _____ times larger than 3^3.
2. 5^3 is _____ times larger than 5^2.
3. 7^{10} is _____ times larger than 7^8.
4. 17^6 is _____ times larger than 17^4.
5. 5^{10} is _____ times larger than 5^4.

(From Unit 6.3.1)

36.6: Absolute Value of Numbers is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

36.7: Comparing Numbers and Distance from Zero

Lesson

Let's use absolute value and negative numbers to think about elevation.

Exercise 36.7.1: Opposites

1. a is a rational number. Choose a value for a and plot it on the number line.

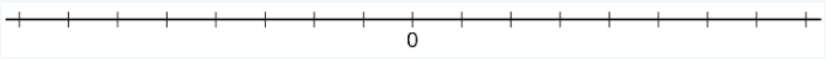


Figure 36.7.1

1. a. Based on where you plotted a, plot $-a$ on the same number line.
   b. What is the value of $-a$ that you plotted?
2. Noah said, "If a is a rational number, $-a$ will always be a negative number." Do you agree with Noah? Explain your reasoning.

Exercise 36.7.2: Submarine

A submarine is at an elevation of -100 feet (100 feet below sea level). Let's compare the elevations of these four people to that of the submarine:

- Clare's elevation is greater than the elevation of the submarine. Clare is farther from sea level than the submarine.
- Andre's elevation is less than the elevation of the submarine. Andre is farther away from sea level than the submarine.
- Han's elevation is greater than the elevation of the submarine. Han is closer to sea level than is the submarine.
- Lin's elevation is the same distance away from sea level as the submarine's.

1. Complete the table as follows.
   a. Write a possible elevation for each person.
   b. Use $<$, $>$, or $=$ to compare the elevation of that person to that of the submarine.
   c. Use absolute value to tell how far away the person is from sea level (elevation 0).
   As an example, the first row has been filled with a possible elevation for Clare.

Table 36.7.1

	possible elevation	compare to submarine	distance from sea level
Clare	150 feet	$150 > -100$	$\lvert 150 \rvert$ or 150 feet
Andre			
Han			
Lin			

2. Priya says her elevation is less than the submarine's and she is closer to sea level. Is this possible? Explain your reasoning.

Exercise 36.7.3: Info Gap: Points on the Number Line

Your teacher will give you either a *problem card* or a *data card*. Do not show or read your card to your partner.

If your teacher gives you the *problem card*:

1. Silently read your card and think about what information you need to be able to answer the question.
2. Ask your partner for the specific information that you need.
3. Explain how you are using the information to solve the problem.
   Continue to ask questions until you have enough information to solve the problem.
4. Share the *problem card* and solve the problem independently.
5. Read the *data card* and discuss your reasoning.

If your teacher gives you the *data card*:

1. Silently read your card.
2. Ask your partner "*What specific information do you need?*" and wait for them to *ask* for information.
   If your partner asks for information that is not on the card, do not do the calculations for them. Tell them you don't have that information.
3. Before sharing the information, ask "*Why do you need that information?*" Listen to your partner's reasoning and ask clarifying questions.
4. Read the *problem card* and solve the problem independently.
5. Share the *data card* and discuss your reasoning.

Exercise 36.7.4: Inequality Mix and Match

Here are some numbers and inequality symbols. Work with your partner to write true comparison statements.

-0.7	$-\frac{3}{5}$	1	4	$\lvert -8\rvert$	$<$
$-\frac{6}{3}$	-2.5	2.5	8	$\lvert 0.7\rvert$	$=$
-4	0	$\frac{7}{2}$	$\lvert 3\rvert$	$\lvert -\frac{5}{2}\rvert$	$>$

One partner should select two numbers and one comparison symbol and use them to write a true statement using symbols. The other partner should write a sentence in words with the same meaning, using the following phrases:

- is equal to
- is the absolute value of
- is greater than
- is less than

For example, one partner could write $4 < 8$ and the other would write, "4 is less than 8." Switch roles until each partner has three true mathematical statements and three sentences written down.

Are you ready for more?

For each question, choose a value for each variable to make the whole statement true. (When the word *and* is used in math, both parts have to be true for the whole statement to be true.) Can you do it if one variable is negative and one is positive? Can you do it if both values are negative?

1. $x < y$ and $\lvert x\rvert < y$.
2. $a < b$ and $\lvert a\rvert < \lvert b\rvert$.
3. $c < d$ and $\lvert c\rvert > d$.
4. $t < u$ and $\lvert t\rvert > \lvert u\rvert$.

Summary

We can use elevation to help us compare two rational numbers or two absolute values.

- Suppose an anchor has an elevation of -10 meters and a house has an elevation of 12 meters. To describe the anchor having a lower elevation than the house, we can write $-10 < 12$ and say "-10 is less than 12."
- The anchor is closer to sea level than the house is to sea level (or elevation of 0). To describe this, we can write $\lvert -10\rvert < \lvert 12\rvert$ and say "the distance between -10 and 0 is less than the distance between 12 and 0."

We can use similar descriptions to compare rational numbers and their absolute values outside of the context of elevation.

- To compare the distance of -47.5 and 5.2 from 0, we can say: $\lvert -47.5\rvert$ is 47.5 units away from 0, and $\lvert 5.2\rvert$ is 5.2 units away from 0, so $\lvert -47.5\rvert > \lvert 5.2\rvert$.
- $\lvert -18\rvert > 4$ means that the absolute value of -18 is greater than 4. This is true because 18 is greater than 4.

Glossary Entries

Definition: Absolute Value

The absolute value of a number is its distance from 0 on the number line.

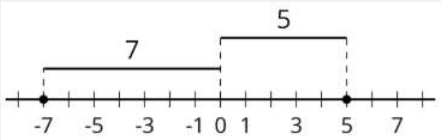


Figure 36.7.2

The absolute value of -7 is 7, because it is 7 units away from 0. The absolute value of 5 is 5, because it is 5 units away from 0.

Definition: Negative Number

A negative number is a number that is less than zero. On a horizontal number line, negative numbers are usually shown to the left of 0.

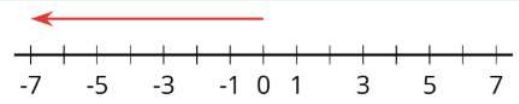


Figure 36.7.3

Definition: Opposite

Two numbers are opposites if they are the same distance from 0 and on different sides of the number line.

For example, 4 is the opposite of -4, and -4 is the opposite of 4. They are both the same distance from 0. One is negative, and the other is positive.

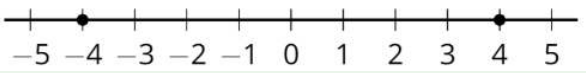


Figure 36.7.4

Definition: Positive Number

A positive number is a number that is greater than zero. On a horizontal number line, positive numbers are usually shown to the right of 0.

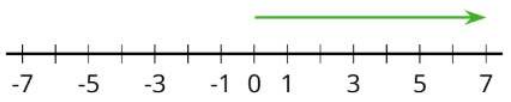


Figure 36.7.5

Definition: Rational Number

A rational number is a fraction or the opposite of a fraction.

For example, 8 and -8 are rational numbers because they can be written as $\frac{8}{1}$ and $-\frac{8}{1}$.

Also, 0.75 and -0.75 are rational numbers because they can be written as $\frac{75}{100}$ and $-\frac{75}{100}$.

Definition: Sign

The sign of any number other than 0 is either positive or negative.

For example, the sign of 6 is positive. The sign of -6 is negative. Zero does not have a sign, because it is not positive or negative.

Practice

Exercise 36.7.5

In the context of elevation, what would $|-7|$ feet mean?

Exercise 36.7.6

Match the the statements written in English with the mathematical statements.

A. The number -4 is a distance of 4 units away from 0 on the number line.
B. The number -63 is more than 4 units away from 0 on the number line.
C. The number 4 is greater than the number -4.
D. The numbers 4 and -4 are the same distance away from 0 on the number line.
E. The number -63 is less than the number 4.
F. The number -63 is further away from 0 than the number 4 on the number line.

1. $|-63| > 4$
2. $-63 < 4$
3. $|-63| > |4|$
4. $|-4| = 4$
5. $4 > -4$
6. $|4| = |-4|$

Exercise 36.7.7

Compare each pair of expressions using >, <, or =.

- $-32 \_\_ 15$
- $|-32| \_\_ |15|$
- $5 \_\_ -5$
- $|5| \_\_ |-5|$
- $2 \_\_ -17$
- $2 \_\_ |-17|$
- $|-27| \_\_ |-45|$
- $|-27| \_\_ -45$

Exercise 36.7.8

Mai received and spent money in the following ways last month. For each example, write a signed number to represent the change in money from her perspective.

1. Her grandmother gave her $25 in a birthday card.
2. She earned $14 dollars babysitting.
3. She spent $10 on a ticket to the concert.
4. She donated $3 to a local charity.
5. She got $2 interest on money that was in her savings account.

(From Unit 7.1.5)

Exercise 36.7.9

Here are the lowest temperatures recorded in the last 2 centuries for some US cities.

- Death Valley, CA was -45°F in January of 1937.
- Danbury, CT was -37°F in February of 1943.
- Monticello, FL was -2°F in February of 1899.

- East Saint Louis, IL was -36°F in January of 1999.
- Greenville, GA was -17°F in January of 1940.

1. Which of these states has the lowest record temperature?
2. Which state has a lower record temperature, FL or GA?
3. Which state has a lower record temperature, CT or IL?
4. How many more degrees colder is the record temperature for GA than for FL?

(From Unit 7.1.1)

Exercise 36.7.10

Find the quotients.

1. $0.024 \div 0.015$
2. $0.24 \div 0.015$
3. $0.024 \div 0.15$
4. $24 \div 15$

(From Unit 5.4.5)

36.7: Comparing Numbers and Distance from Zero is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

Section 37: Inequalities

This page titled Section 37: Inequalities is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

37.1: Writing and Graphing Inequalities

Lesson

Let's write inequalities.

Exercise 37.1.1: Estimate Heights of People

1. Here is a picture of a man.



Figure 37.1.1

   a. Name a number, in feet, that is clearly too high for this man's height.
   b. Name a number, in feet, that is clearly too low for his height.
   c. Make an estimate of his height.

Pause here for a class discussion.

2. Here is a picture of the same man standing next to a child.



Figure 37.1.2

If the man's actual height is 5 feet 10 inches, what can you say about the height of the child in this picture?

Be prepared to explain your reasoning.

Exercise 37.1.2: Stories about 9

1. Drag the green and red open points over the blue points. Match each story with a numberline and a description.
2. Think about how you would represent each set of numbers with an inequality statement, talk about it with your partner, and be prepared to discuss your ideas with the class. Then, record your final matching decisions here.
   a. A fishing boat can hold fewer than 9 people. How many people (x) can it hold?

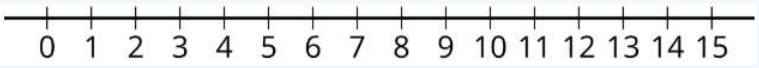
- Description or list:
- Number line: 

Figure 37.1.3
- Inequality:

b. Lin needs more than 9 ounces of butter to make cookies for her party. How many ounces of butter (x) would be enough?

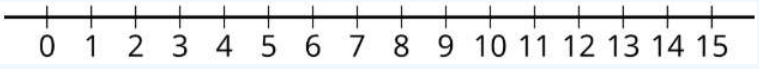
- Description or list:
- Number line: 

Figure 37.1.4
- Inequality:

c. A magician will perform her magic tricks only if there are at least 9 people in the audience. For how many people (x) will she perform her magic tricks?

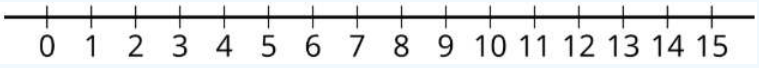
- Description or list:
- Number line: 

Figure 37.1.5
- Inequality:

d. A food scale can measure up to 9 kilograms of weight. What weights (x) can the scale measure?

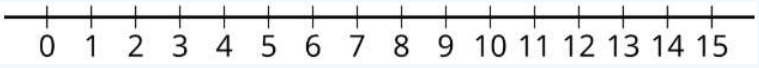
- Description or list:
- Number line: 

Figure 37.1.6
- Inequality:

Exercise 37.1.3: How High and How Loc Can it Be?

Here is a picture of a person and a basketball hoop. Based on the picture, what do you think are reasonable estimates for the maximum and minimum heights of the basketball hoop?

1. Complete the first blank in each sentence with an estimate, and the second blank with “taller” or “shorter.”


Figure 37.1.7

a. I estimate the *minimum* height of the basketball hoop to be _____ feet; this means the hoop cannot be _____ than this height.

b. I estimate the *maximum* height of the basketball hoop to be _____ feet; this means the hoop cannot be _____ than this height.

2. Write two inequalities—one to show your estimate for the *minimum* height of the basketball hoop, and another for the *maximum* height. Use an inequality symbol and the variable to represent the unknown height.
3. Plot each estimate for minimum or maximum value on a number line.

- Minimum:

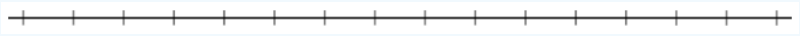


Figure 37.1.8

- Maximum:



Figure 37.1.9

4. Suppose a classmate estimated the value of h to be 19 feet. Does this estimate agree with your inequality for the maximum height? Does it agree with your inequality for the minimum height? Explain or show how you know.
5. Ask a partner for an estimate of h. Record the estimate and check if it agrees with your inequalities for maximum and minimum heights.

Are you ready for more?

1. Find 3 different numbers that a could be if $|a| < 5$. Plot these points on the number line. Then plot as many other possibilities for a as you can.

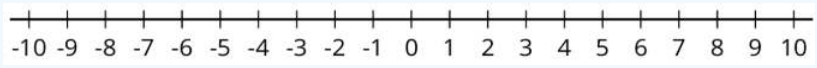


Figure 37.1.10

2. Find 3 different numbers that b could be if $|b| > 3$. Plot these points on the number line. Then plot as many other possibilities for b as you can.

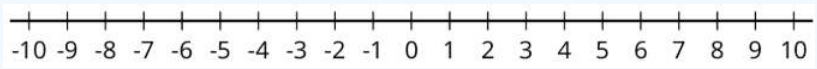


Figure 37.1.11

Summary

An inequality tells us that one value is *less than* or *greater than* another value.

Suppose we knew the temperature is *less than* 3°F, but we don't know exactly what it is. To represent what we know about the temperature t in $^\circ\text{F}$ we can write the inequality: $t < 3$

The temperature can also be graphed on a number line. Any point to the left of 3 is a possible value for t. The open circle at 3 means that t cannot be *equal* to 3, because the temperature is *less than* 3.

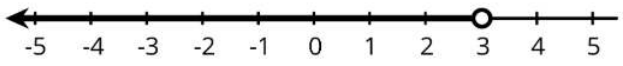


Figure 37.1.12

Here is another example. Suppose a young traveler has to be at least 16 years old to fly on an airplane without an accompanying adult.

If a represents the age of the traveler, any number greater than 16 is a possible value for a, and 16 itself is also a possible value of a. We can show this on a number line by drawing a closed circle at 16 to show that it meets the requirement (a 16-year-old person can travel alone). From there, we draw a line that points to the right.



Figure 37.1.13

We can also write an inequality and equation to show possible values for a:

$$a > 16$$

$$a = 16$$

Practice

Exercise 37.1.4

At the book sale, all books cost less than $5.

1. What is the most expensive a book could be?
2. Write an inequality to represent costs of books at the sale.
3. Draw a number line to represent the inequality.

Exercise 37.1.5

Kiran started his homework *before* 7:00 p.m. and finished his homework *after* 8:00 p.m. Let h represent the number of hours Kiran worked on his homework.

Decide if each statement it is definitely true, definitely not true, or possibly true. Explain your reasoning.

1. $h > 1$
2. $h > 2$
3. $h < 1$
4. $h < 2$

Exercise 37.1.6

Consider a rectangular prism with length 4 and width and height d.

1. Find an expression for the volume of the prism in terms of d.
2. Compute the volume of the prism when $d = 1$, when $d = 2$, and when $d = \frac{1}{2}$.

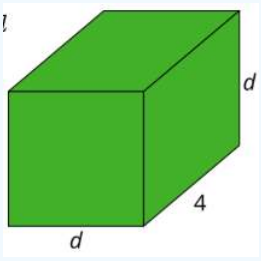


Figure 37.1.14

(From Unit 6.3.3)

Exercise 37.1.7

Match the statements written in English with the mathematical statements. All of these statements are true.

A. The number -15 is further away from 0 than the number -12 on the number line.
B. The number -12 is a distance of 12 units away from 0 on the number line.
C. The distance between -12 and 0 on the number line is greater than -15.
D. The numbers 12 and -12 are the same distance away from 0 on the number line.
E. The number -15 is less than the number -12.
F. The number 12 is greater than the number -12.

1. $|-12| > -15$
2. $-15 < -12$
3. $|-15| > |-12|$
4. $|-12| = 12$
5. $12 > -12$

6. $|12| = |-12|$

Exercise 37.1.8

Here are five sums. Use the distributive property to write each sum as a product with two factors.

1. $2a + 7a$
2. $5z - 10$
3. $c - 2cd$
4. $r + r + r + r$
5. $2x - \frac{1}{2}$

(From Unit 6.2.6)

37.1: Writing and Graphing Inequalities is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

37.2: Solutions of Inequalities

Lesson

Let's think about the solutions to inequalities.

Exercise 37.2.1: Unknowns on a Number Line

The number line shows several points, each labeled with a letter.

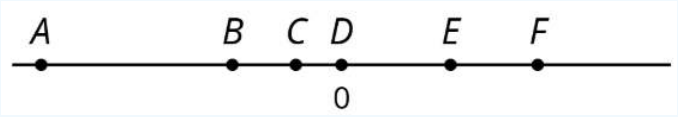


Figure 37.2.1

1. Fill in each blank with a letter so that the inequality statements are true.
   a. _____ > _____
   b. _____ < _____
2. Jada says that she found three different ways to complete the first question correctly. Do you think this is possible? Explain your reasoning.
3. List a possible value for each letter on the number line based on its location.

Exercise 37.2.2: Amusement Park Rides

Priya finds these height requirements for some of the rides at an amusement park.

Table 37.2.1

To ride the . . .	you must be . . .
High Bounce	between 55 and 72 inches tall
Climn-A-Thon	under 60 inches tall
Twirl-O-Coaster	58 inches minimum

1. Write an inequality for each of the the three height requirements. Use h for the unknown height. Then, represent each height requirement on a number line.
   - High Bounce

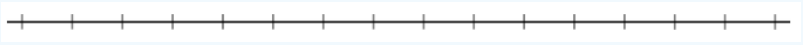
     

     Figure 37.2.2
   - Climb-A-Thon

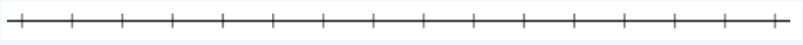
     

     Figure 37.2.3
   - Twirl-O-Coaster

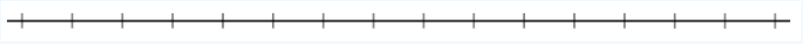
     

     Figure 37.2.4

     Pause here for additional instructions from your teacher.
2. Han's cousin is 55 inches tall. Han doesn't think she is tall enough to ride the High Bounce, but Kiran believes that she is tall enough. Do you agree with Han or Kiran? Be prepared to explain your reasoning.
3. Priya can ride the Climb-A-Thon, but she cannot ride the High Bounce or the Twirl-O-Coaster. Which, if any, of the following could be Priya's height? Be prepared to explain your reasoning.
   - 59 inches
   - 53 inches
   - 56 inches

4. Jada is 56 inches tall. Which rides can she go on?
5. Kiran is 60 inches tall. Which rides can he go on?
6. The inequalities $h < 75$ and $h > 64$ represent the height restrictions, in inches, of another ride. Write three values that are **solutions** to both of these inequalities.

Are you ready for more?

1. Represent the height restrictions for all three rides on a single number line, using a different color for each ride.

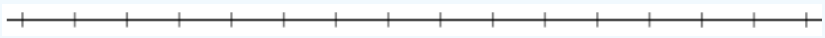


Figure 37.2.5

2. Which part of the number line is shaded with all 3 colors?
3. Name one possible height a person could be in order to go on all three rides.

Exercise 37.2.3: What Number Am I?

Your teacher will give your group two sets of cards—one set shows inequalities and the other shows numbers. Place the inequality cards face up where everyone can see them. Shuffle the number cards and stack them face down.

To play:

- One person in your group is the detective. The other people will give clues.
- Pick one number card from the stack and show it to everyone except the detective.
- The people giving clues each choose an inequality that will help the detective identify the unknown number.
- The detective studies the inequalities and makes three guesses.
  - If the detective does not guess the right number, each person chooses another inequality to help.
  - When the detective does guess the right number, a new person becomes the detective.
- Repeat the game until everyone has had a turn being the detective.

Summary

Let's say a movie ticket costs less than \$10. If c represents the cost of a movie ticket, we can use to express what we know about the cost of a ticket.

Any value of c that makes the inequality true is called a **solution to the inequality**.

For example, 5 is a solution to the inequality $c < 10$ because $5 < 10$ (or "5 is less than 10") is a true statement, but 12 is not a solution because $12 < 10$ ("12 is less than 10") is *not* a true statement.

If a situation involves more than one boundary or limit, we will need more than one inequality to express it.

For example, if we knew that it rained for *more* than 10 minutes but *less* than 30 minutes, we can describe the number of minutes that it rained (r) with the following inequalities and number lines.

$r > 10$

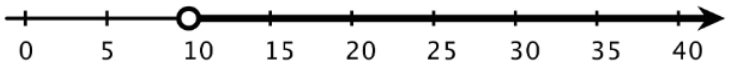


Figure 37.2.6

$r < 30$

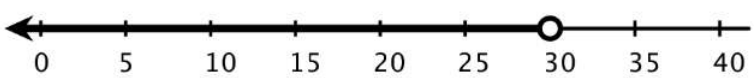


Figure 37.2.7

Any number of minutes greater than 10 is a solution to $r > 10$, and any number less than 30 is a solution to $r < 30$. But to meet the condition of "more than 10 but less than 30," the solutions are limited to the numbers between 10 and 30 minutes, *not* including 10 and 30.

We can show the solutions visually by graphing the two inequalities on one number line.

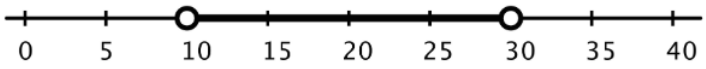


Figure 37.2.8

Glossary Entries

Definition: Solution to an Inequality

A solution to an inequality is a number that can be used in place of the variable to make the inequality true.

For example, 5 is a solution to the inequality $c < 10$, because it is true that $5 < 10$. Some other solutions to this inequality are 9.9, 0, and -4.

Practice

Exercise 37.2.4

1. Select **all** numbers that are solutions to the inequality $k > 5$.

$$4 \qquad 5 \qquad 6 \qquad 5.2 \qquad 5.01 \qquad 0.5$$

2. Draw a number line to represent this inequality.

Exercise 37.2.5

A sign on the road says: "Speed limit, 60 miles per hour."

1. Let s be the speed of a car. Write an inequality that matches the information on the sign.
2. Draw a number line to represent the solutions to the inequality.
3. Could 60 be a value of s? Explain your reasoning.

Exercise 37.2.6

One day in Boston, MA, the high temperature was 60 degrees Fahrenheit, and the low temperature was 52 degrees.

1. Write one or more inequalities to describe the temperatures T that are between the high and low temperature on that day.
2. Show the possible temperatures on a number line.

Exercise 37.2.7

Select **all** the true statements.

A. $-5 < |-5|$
B. $|-6| < -5$
C. $|-6| < 3$
D. $4 < |-7|$
E. $|-7| < |-8|$

(From Unit 7.1.7)

Exercise 37.2.8

Match each equation to its solution.

1. $x^4 = 81$
2. $x^2 = 100$
3. $x^3 = 64$
4. $x^5 = 32$

- 2

- 3
- 4
- 10

(From Unit 6.3.4)

Exercise 37.2.9

1. The price of a cell phone is usually $250. Elena's mom buys one of these cell phones for $150. What percentage of the usual price did she pay?
2. Elena's dad buys another type of cell phone that also usually sells for $250. He pays 75% of the usual price. How much did he pay?

(From Unit 3.4.5)

37.2: Solutions of Inequalities is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

37.3: Interpreting Inequalities

Lesson

Let's examine what inequalities can tell us.

Exercise 37.3.1: True or False: Fractions and Decimals

Is each equation true or false? Be prepared to explain your reasoning.

1. $3(12+5) = (3 \cdot 12) \cdot (3 \cdot 5)$
2. $\frac{1}{3} \cdot \frac{3}{4} = \frac{3}{4} \cdot \frac{2}{6}$
3. $2 \cdot (1.5) \cdot 12 = 4 \cdot (0.75) \cdot 6$

Exercise 37.3.2: Basketball Game

Noah scored n points in a basketball game.

1. What does $15 < n$ mean in the context of the basketball game?
2. What does $n < 25$ mean in the context of the basketball game?
3. Draw two number lines to represent the solutions to the two inequalities.
4. Name a possible value for n that is a solution to both inequalities.
5. Name a possible value for n that is a solution to $15 < n$, but not a solution to $n < 25$.
6. Can -8 be a solution to n in this context? Explain your reasoning.

Exercise 37.3.3: Unbalanced Hangers

1. Here is a diagram of an unbalanced hanger.

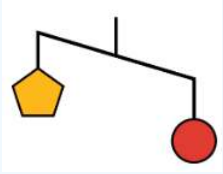


Figure 37.3.1

   a. Jada says that the weight of one circle is greater than the weight of one pentagon. Write an inequality to represent her statement. Let p be the weight of one pentagon and c be the weight of one circle.
   b. A circle weighs 12 ounces. Use this information to write another inequality to represent the relationship of the weights. Then, describe what this inequality means in this context.

2. Here is another diagram of an unbalanced hanger.



Figure 37.3.2

   a. Write an inequality to represent the relationship of the weights. Let p be the weight of one pentagon and s be the weight of one square.
   b. One pentagon weighs 8 ounces. Use this information to write another inequality to represent the relationship of the weights. Then, describe what this inequality means in this context.
   c. Graph the solutions to this inequality on a number line.

3. Based on your work so far, can you tell the relationship between the weight of a square and the weight of a circle? If so, write an inequality to represent that relationship. If not, explain your reasoning.
4. This is another diagram of an unbalanced hanger.

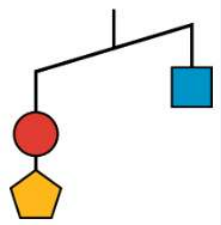


Figure 37.3.3

Andre writes the following inequality: $c+p<s$. Do you agree with his inequality? Explain your reasoning.

5. Jada looks at another diagram of an unbalanced hangar and writes: $s+c>2t$, where t represents the weight of one triangle. Draw a sketch of the diagram.

Are you ready for more?

Here is a picture of a balanced hanger. It shows that the total weight of the three triangles is the same as the total weight of the four squares.

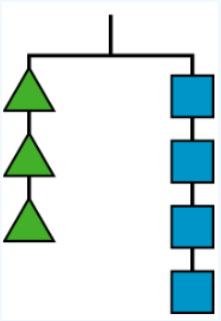


Figure 37.3.4

1. What does this tell you about the weight of one square when compared to one triangle? Explain how you know.
2. Write an equation or an inequality to describe the relationship between the weight of a square and that of a triangle. Let s be the weight of a square and t be the weight of a triangle.

Summary

When we find the solutions to an inequality, we should think about its context carefully. A number may be a solution to an inequality outside of a context, but may not make sense when considered in context.

- Suppose a basketball player scored more than 11 points in a game, and we represent the number of points she scored, s, with the inequality $s>11$. By looking only at $s>11$, we can say that numbers such as $12, 14\frac{1}{2}$, and 130.25 are all solutions to the inequality because they each make the inequality true.

$$12>11 \qquad 14\tfrac{1}{2}>11 \qquad 130.25>11$$

In a basketball game, however, it is only possible to score a whole number of points, so fractional and decimal scores are not possible. It is also highly unlikely that one person would score more than 130 points in a single game.

In other words, the context of an inequality may limit its solutions.

Here is another example:

- The solutions to $r<30$ can include numbers such as $27\frac{3}{4}, 18.5, 0$, and -7. But if r represents the number of minutes of rain yesterday (and it did rain), then our solutions are limited to positive numbers. Zero or negative number of minutes would not make sense in this context.

To show the upper and lower boundaries, we can write two inequalities:

$$0<r \qquad r<30$$

Inequalities can also represent comparison of two unknown numbers.

- Let's say we knew that a puppy weighs more than a kitten, but we did not know the weight of either animal. We can represent the weight of the puppy, in pounds, with p and the weight of the kitten, in pounds, with k, and write this inequality: $p > k$

Glossary Entries

Definition: Solution to an Inequality

A solution to an inequality is a number that can be used in place of the variable to make the inequality true.

For example, 5 is a solution to the inequality $c < 10$, because it is true that $5 < 10$. Some other solutions to this inequality are 9.9, 0, and -4.

Practice

Exercise 37.3.4

There is a closed carton of eggs in Mai's refrigerator. The carton contains e eggs and it can hold 12 eggs.

1. What does the inequality $e < 12$ mean in this context?
2. What does the inequality $e > 0$ mean in this context?
3. What are some possible values of e that will make both $e < 12$ and $e > 0$ true?

Exercise 37.3.5

Here is a diagram of an unbalanced hanger.

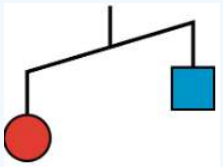


Figure 37.3.5

1. Write an inequality to represent the relationship of the weights. Use s to represent the weight of the square in grams and c to represent the weight of the circle in grams.
2. One red circle weighs 12 grams. Write an inequality to represent the weight of one blue square.
3. Could 0 be a value of s? Explain your reasoning.

Exercise 37.3.6

1. Jada is taller than Diego. Diego is 54 inches tall (4 feet, 6 inches). Write an inequality that compares Jada's height in inches, j, to Diego's height.
2. Jada is shorter than Elena. Elena is 5 feet tall. Write an inequality that compares Jada's height in inches, j, to Elena's height.

(From Unit 7.2.1)

Exercise 37.3.7

Tyler has more than \$10. Elena has more money than Tyler. Mai has more money than Elena. Let t be the amount of money that Tyler has, let e be the amount of money that Elena has, and let m be the amount of money that Mai has. Select **all** statements that are true:

A. $t < j$
B. $m > 10$
C. $e > 10$
D. $t > 10$
E. $e > m$
F. $t < e$

Exercise 37.3.8

Which is greater, $\frac{-9}{20}$ or -0.5? Explain how you know. If you get stuck, consider plotting the numbers on a number line.

(From Unit 7.1.3)

Exercise 37.3.9

Select **all** the expressions that are equivalent to $\left(\frac{1}{2}\right)^3$.

A. $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$
B. $\frac{1}{2^3}$
C. $\left(\frac{1}{3}\right)^2$
D. $\frac{1}{6}$
E. $\frac{1}{8}$

(From Unit 6.3.2)

37.3: Interpreting Inequalities is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

Section 38: The Coordinate Plane

This page titled Section 38: The Coordinate Plane is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

38.1: Points on the Coordinate Plane

Lesson

Let's explore and extend the coordinate plane.

Exercise 38.1.1: Guess My Line

1. Choose a horizontal or a vertical line on the grid. Draw 4 points on the line and label each point with its coordinates.

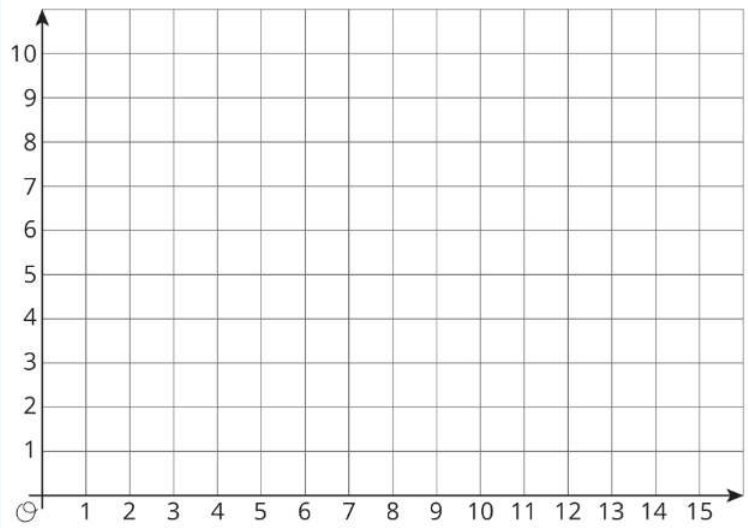


Figure 38.1.1

2. Tell your partner whether your line is horizontal or vertical, and have your partner guess the locations of your points by naming coordinates.
   If a guess is correct, put an X through the point. If your partner guessed a point that is on your line but not the point that you plotted, say, “That point is on my line, but is not one of my points.”
   Take turns guessing each other’s points, 3 guesses per turn.

Exercise 38.1.2: The Coordinate Plane

1.



Figure 38.1.2

The colored points on the coordinate plane are like targets. Hit each point by entering its coordinates as an ordered pair in the Input Bar, like this:

2. What do you notice about the locations and ordered pairs of B, C, and D? How are they different from those for point A?
3. Plot a point at $(-2, 5)$. Label it E. Plot another point at $(3, -4.5)$. Label it F.
4. The coordinate plane is divided into four quadrants, I, II, III, and IV, as shown here.

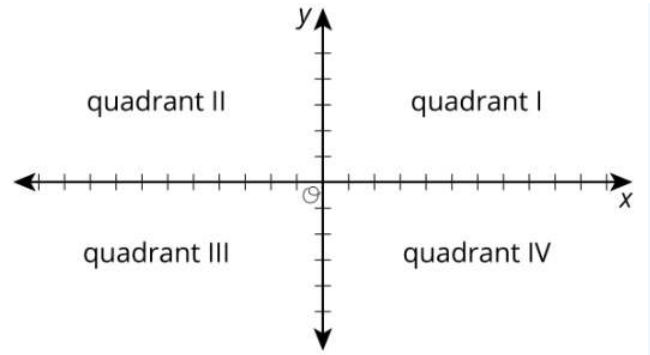


Figure 38.1.3

- $G = (5, 2)$
- $H = (-1, -5)$
- $I = (7, -4)$

a. In which quadrant is G located? H? I?
b. A point has a positive y-coordinate. In which quadrant could it be?

Exercise 38.1.3: Coordinated Archery

Here is an image of an archery target on a coordinate plane. The scores for landing an arrow in the colored regions are:

- Yellow: 10 points
- Red: 8 points
- Blue: 6 points
- Green: 4 points
- White: 2 points

Name the coordinates for a possible landing point to score:

1. 6 points
2. 10 points
3. 2 points
4. No points
5. 4 points
6. 8 points

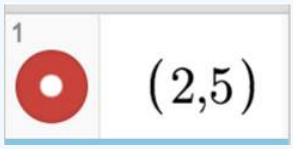


Figure 38.1.4

Type the coordinates for each point in a separate line, using parentheses. Like this:

Are you ready for more?

Pretend you are stuck in a coordinate plane. You can only take vertical and horizontal steps that are one unit long.

1. How many ways are there to get from the point $(-3, 2)$ to $(-1, -1)$ if you will only step down and to the right?
2. How many ways are there to get from the point $(-1, -2)$ to $(4, 0)$ if you can only step up and to the right?
3. Make up some more problems like this and see what patterns you notice.

Summary

Just as the number line can be extended to the left to include negative numbers, the x- and y-axis of a coordinate plane can also be extended to include negative values.

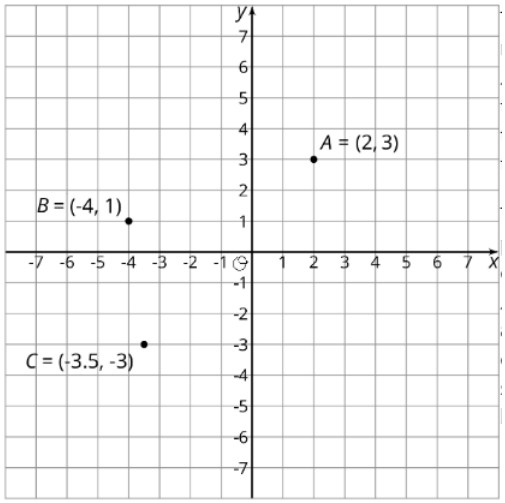


Figure 38.1.5

The ordered pair (x, y) can have negative x- and y-values. For $B = (-4, 1)$, the x-value of -4 tells us that the point is 4 units to the left of the y-axis. The y-value of 1 tells us that the point is one unit above the x-axis.

The same reasoning applies to the points A and C. The x- and y-coordinates for point A are positive, so A is to the right of the y-axis and above the x-axis. The x- and y-coordinates for point C are negative, so C is to the left of the y-axis and below the x-axis.

Glossary Entries

Definition: Quadrant

The coordinate plane is divided into 4 regions called quadrants. The quadrants are numbered using Roman numerals, starting in the top right corner.

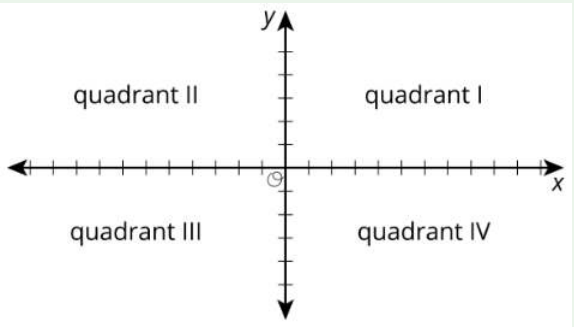


Figure 38.1.6

Practice

Exercise 38.1.4

1. Graph these points in the coordinate plane: $(-2, 3), (2, 3), (-2, -3), (2, -3)$

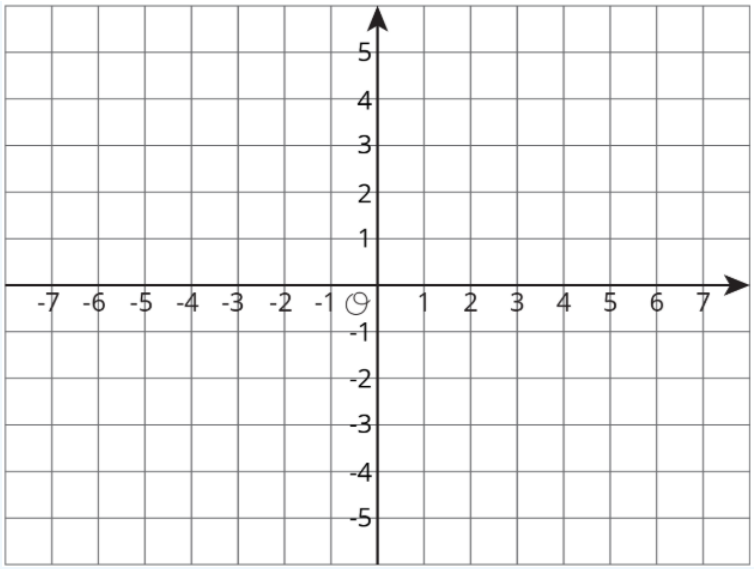


Figure 38.1.7

2. Connect all of the points. Describe the figure.

Exercise 38.1.5

Write the coordinates of each point.

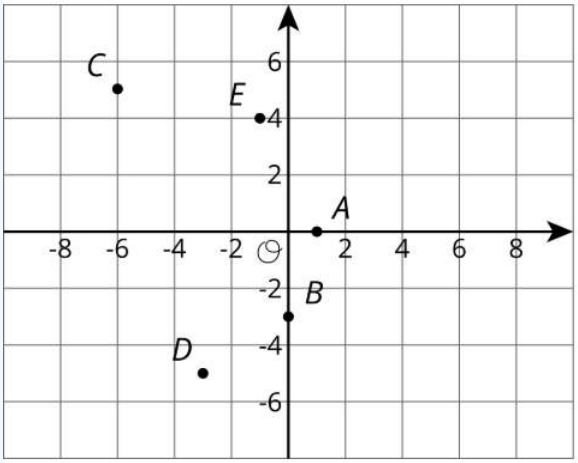


Figure 38.1.8: Coordinate plane, o at origin, horizontal axis -8 to 8 by twos, vertical axis negative 6 to 6 by twos. Point A on the horizontal axis, halfway between 0 and 2. Point B, on the vertical axis, halfway between negative 2 and negative 4. Point C above negative 6 on the horizontal axis, halfway between 4 and 6. Point D halfway between negative 4 and negative 2 on the horizontal axis, halfway between negative 4 and negative 6 on the vertical axis. Point E between negative 2 and zero on the horizontal axis, next to 4 on the vertical axis.

Exercise 38.1.6

These three points form a horizontal line: $(-3.5, 4)$, $(0, 4)$,and $(6.2, 4)$. Name two additional points that fall on this line.

Exercise 38.1.7

One night, it is 24°C warmer in Tucson than it was in Minneapolis. If the temperatures in Tucson and Minneapolis are opposites, what is the temperature in Tucson?

A. -24°C
B. -12°C
C. 12°C

D. 24°C

(From Unit 7.1.2)

Exercise 38.1.8

Lin ran 29 meters in 10 seconds. She ran at a constant speed.

1. How far did Lin run every second?
2. At this rate, how far can she run in 1 minute?

(From Unit 2.3.4)

Exercise 38.1.9

Noah is helping his band sell boxes of chocolate to fund a field trip. Each box contains 20 bars and each bar sells for $1.50.

1. Complete the table for values of m.

Table 38.1.1

boxes sold (b)	money collected (m)
1	
2	
3	
4	
5	
6	
7	
8	

2. Write an equation for the amount of money, m, that will be collected if b boxes of chocolate bars are sold. Which is the independent variable and which is the dependent variable in your equation?
3. Write an equation for the number of boxes, b, that were sold if m dollars were collected. Which is the independent variable and which is the dependent variable in your equation?

(From Unit 6.4.1)

38.1: Points on the Coordinate Plane is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

38.2: Constructing the Coordinate Plane

Lesson

Let's investigate different ways of creating a coordinate plane.

Exercise 38.2.1: English Writer

The following data were collected over one December afternoon in England.

Table 38.2.1

time after noon (hours)	temperature (°C)
0	5
1	3
2	4
3	2
4	1
5	−2
6	−3
7	−4
8	−4

A

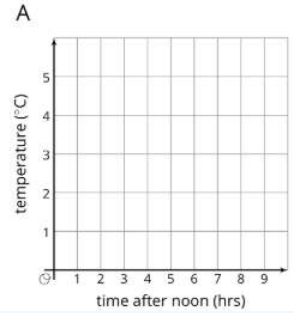


Figure 38.2.1

B

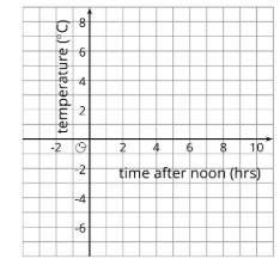


Figure 38.2.2

C

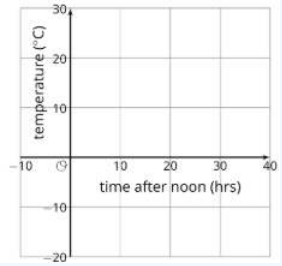


Figure 38.2.3

1. Which set of axes would you choose to represent these data? Explain your reasoning.
2. Explain why the other two sets of axes did not seem as appropriate as the one you chose.

Exercise 38.2.2: Axes Drawing Decisions

1. Here are three sets of coordinates. For each set, draw and label an appropriate pair of axes and plot the points.

   a. $(1, 2), (3, -4), (-5, -2), (0, 2.5)$

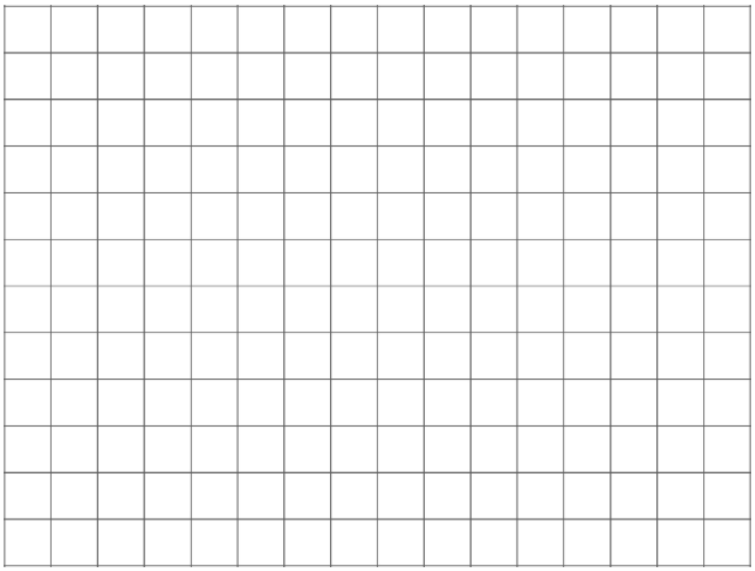


Figure 38.2.4

   b. $(50, 50), (0, 0), (-10, -30), (-35, 40)$

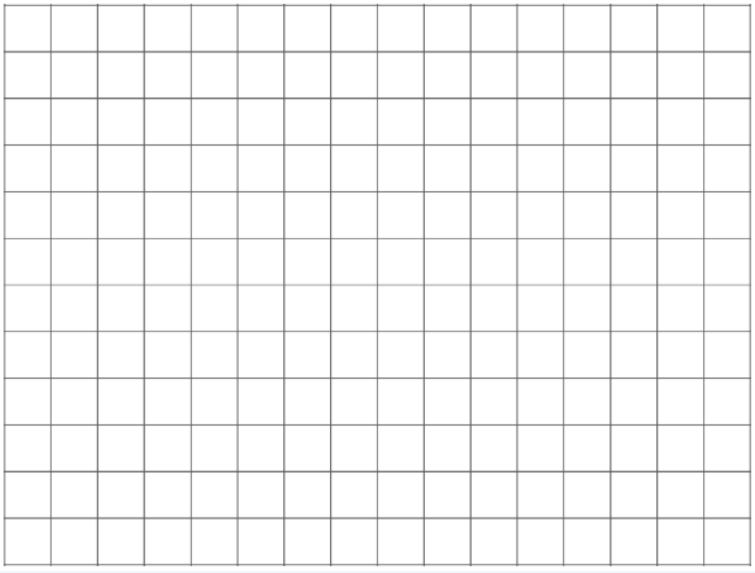


Figure 38.2.5

c. $\left(\frac{1}{4}, \frac{3}{4}\right), \left(\frac{-5}{4}, \frac{1}{2}\right), \left(-1\frac{1}{4}, \frac{-3}{4}\right), \left(\frac{1}{4}, \frac{-1}{2}\right)$

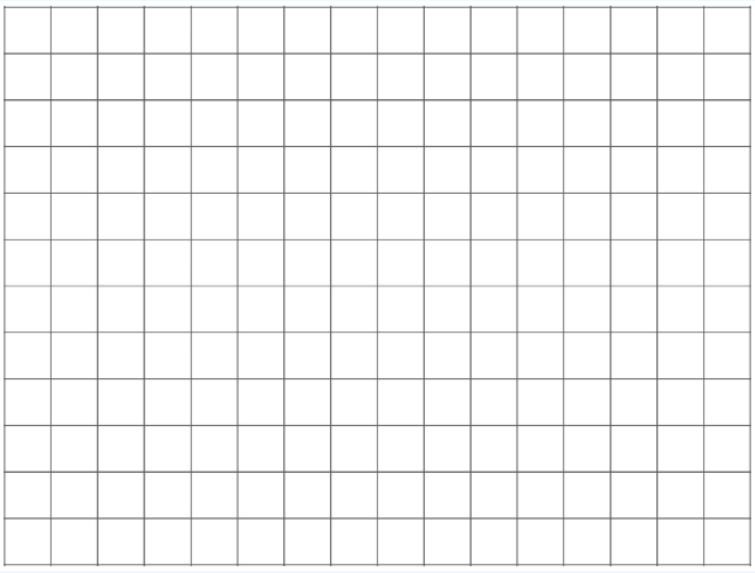


Figure 38.2.6

2. Discuss with a partner:

- How are the axes and labels of your three drawings different?
- How did the coordinates affect the way you drew the axes and label the numbers?

Exercise 38.2.3: Positively A-maze-ing

Here is a maze on a coordinate plane. The black point in the center is (0, 0). The side of each grid square is 2 units long.

1. Enter the above maze at the location marked with a green segment. Draw line segments to show your way through and out of the maze. Label each turning point with a letter. Then, list all the letters and write their coordinates.
2. Choose any 2 turning points that share the same line segment. What is the same about their coordinates? Explain why they share that feature.

Are you ready for more?

To get from the point $(2, 1)$ to $(-4, 3)$ you can go two units up and six units to the left, for a total distance of eight units. This is called the "taxicab distance," because a taxi driver would have to drive eight blocks to get between those two points on a map.

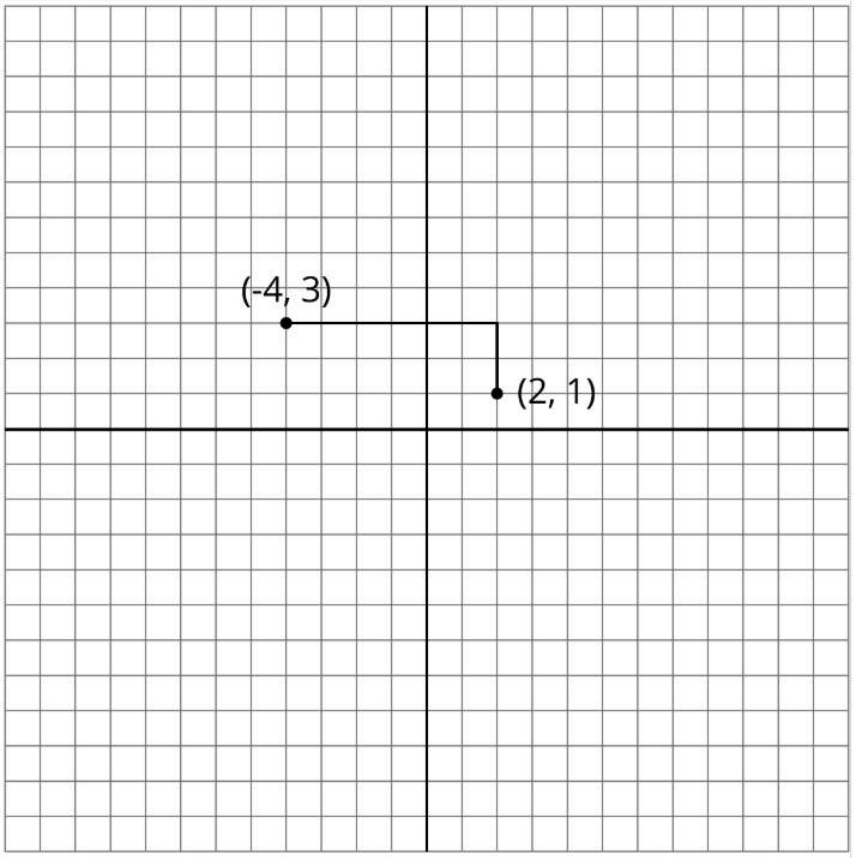


Figure 38.2.7

Find as many points as you can that have a taxicab distance of eight units away from $(2, 1)$. What shape do these points make?

Summary

The coordinate plane can be used to show information involving pairs of numbers.

When using the coordinate plane, we should pay close attention to what each axis represents and what scale each uses.

Suppose we want to plot the following data about the temperatures in Minneapolis one evening.

Table 38.2.2

time (hours from midnight)	temperature (degrees C)
-4	3
-1	-2
0	-4
3	-8

We can decide that the x-axis represents number of hours in relation to midnight and the y-axis represents temperatures in degrees Celsius.

- In this case, x-values less than 0 represent hours before midnight, and and x-values greater than 0 represent hours after midnight.
- On the y-axis, the values represents temperatures above and below the the freezing point of 0 degrees Celsius.

The data involve whole numbers, so it is appropriate that the each square on the grid represents a whole number.

- On the left of the origin, the x-axis needs to go as far as -4 or less (farther to the left). On the right, it needs to go to 3 or greater.
- Below the origin, the y-axis has to go as far as -8 or lower. Above the origin, it needs to go to 3 or higher.

Here is a graph of the data with the axes labeled appropriately.

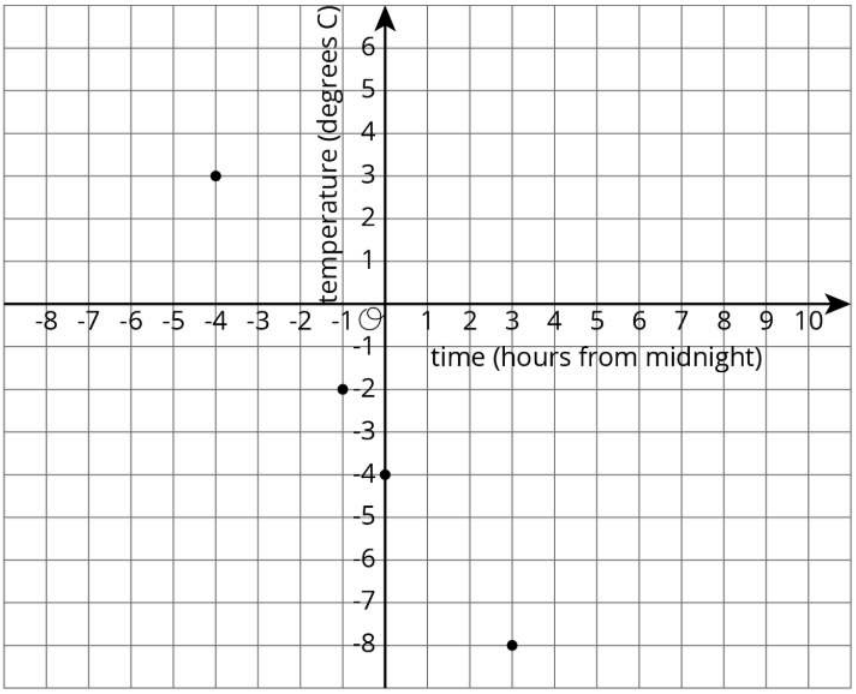


Figure 38.2.8: Coordinate plane, origin O, horizontal axis, time in hours after midnight, negative 8 to 10 by ones, vertical axis, temperature in degrees Celsius, negative 8 to 6 by ones. Points at (negative 4 comma 30), (negative 1 comma negative 2), (zero comma negative 4), (3 comma negative 8).

On this coordinate plane, a point at $(0, 0)$ would mean a temperature of 0 degrees Celsius at midnight. The point at $(-4, 3)$ means a temperature of 3 degrees Celsius at 4 hours before midnight (or 8 p.m.).

Glossary Entries

Definition: Quadrant

The coordinate plane is divided into 4 regions called quadrants. The quadrants are numbered using Roman numerals, starting in the top right corner.

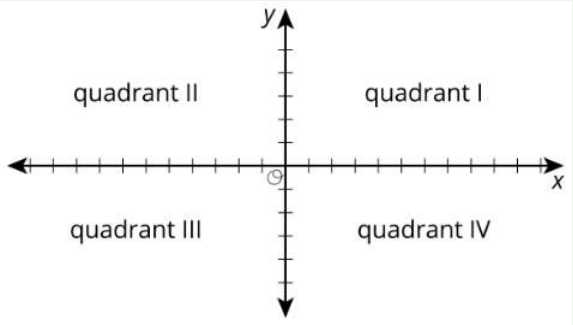


Figure 38.2.9

Practice

Exercise 38.2.4

Draw and label an appropriate pair of axes and plot the points.

$\left(\frac{1}{5}, \frac{4}{5}\right)$

$\left(-\frac{3}{5}, \frac{2}{5}\right)$

$\left(-1\frac{1}{5}, -\frac{4}{5}\right)$

$\left(\frac{1}{5}, -\frac{3}{5}\right)$

Exercise 38.2.5

Diego was asked to plot these points: $(-50, 0), (150, 100), (200, -100), (350, 50), (-250, 0)$. What interval could he use for each axis? Explain your reasoning.

Exercise 38.2.6

1. Name 4 points that would form a square with the origin at its center.
2. Graph these points to check if they form a square.

Exercise 38.2.7

Which of the following changes would you represent using a negative number? Explain what a positive number would represent in that situation.

1. A loss of 4 points
2. A gain of 50 yards
3. A loss of $10
4. An elevation above sea level

(From Unit 7.1.5)

Exercise 38.2.8

Jada is buying notebooks for school. The cost of each notebook is $1.75.

1. Write an equation that shows the cost of Jada's notebooks, c, in terms of the number of notebooks, n, that she buys.
2. Which of the following could be points on the graph of your equation?

$(1.75, 1)$ $(2, 3.50)$ $(5, 8.75)$ $(17.50, 10)$ $(9, 15.35)$

(From Unit 6.4.1)

Exercise 38.2.9

A corn field has an area of 28.6 acres. It requires about 15,000,000 gallons of water. About how many gallons of water per acre is that?

A. $5,000$
B. $50,000$
C. $500,000$
D. $5,000,000$

(From Unit 5.4.5)

38.2: Constructing the Coordinate Plane is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

38.3: Interpreting Points on a Coordinate Plane

Lesson

Let's examine what points on the coordinate plane can tell us.

Exercise 38.3.1: Unlabeled Points

Label each point on the coordinate plane with the appropriate letter and ordered pair.

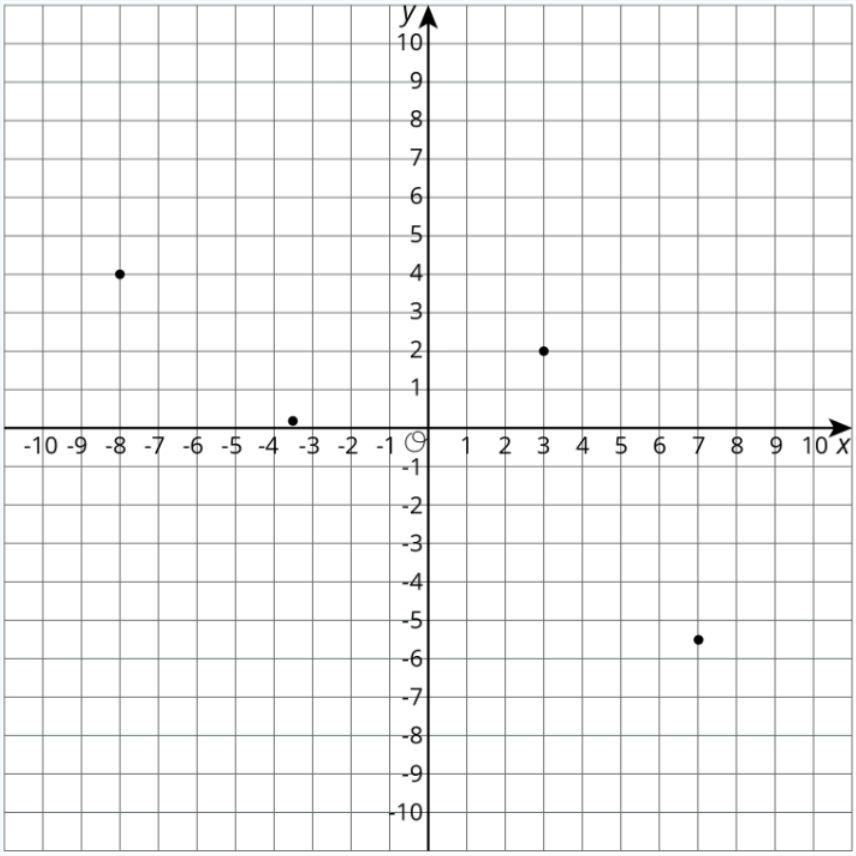


Figure 38.3.1: Coordinate plane, horizontal axis, negative 10 to 10 by ones, vertical axis, negative 10 to 10 by ones. One point is 8 units left and 4 units up from the origin. A point is 3 point 5 left and point 2 up from the origin. A point is three units right and 2 units up from the origin. A point is 7 units right and between 5 and 6 units below the origin.

$$A = (7, -5.5) \qquad B = (-8, 4) \qquad C = (3, 2) \qquad D = (-3.5, 0.2)$$

Exercise 38.3.2: Account Balance

The graph shows the balance in a bank account over a period of 14 days. The axis labeled b represents account balance in dollars. The axis labeled d represents the day.

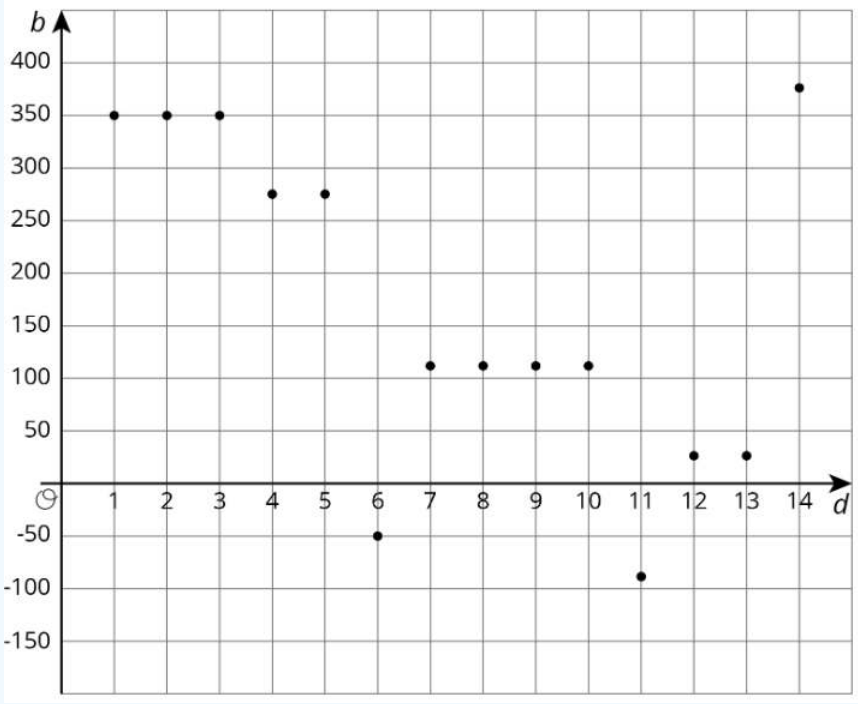


Figure 38.3.2: Coordinate plane, horizontal axis, d, 0 to 14 by ones, vertical axis, b, negative 150 to 400 by fifties. Points at (1 comma 350), (2 comma 350), (3 comma 350), (4 comma 275), (5 comma 275), (6 comma negative 50), (7 comma 110), (8 comma 110), (9 comma 110), (10 comma 110), (11 comma negative 90), (12 comma 25), (13 comma 25).

1. Estimate the greatest account balance. On which day did it occur?
2. Estimate the least account balance. On which day did it occur?
3. What does the point $(6, -50)$ tell you about the account balance?
4. How can we interpret $|-50|$ in the context?

Exercise 38.3.3: High and Low Temperatures

The coordinate plane shows the high and low temperatures in Nome, Alaska over a period of 8 days. The axis labeled T represents temperatures in degrees Fahrenheit. The axis labeled d represents the day.

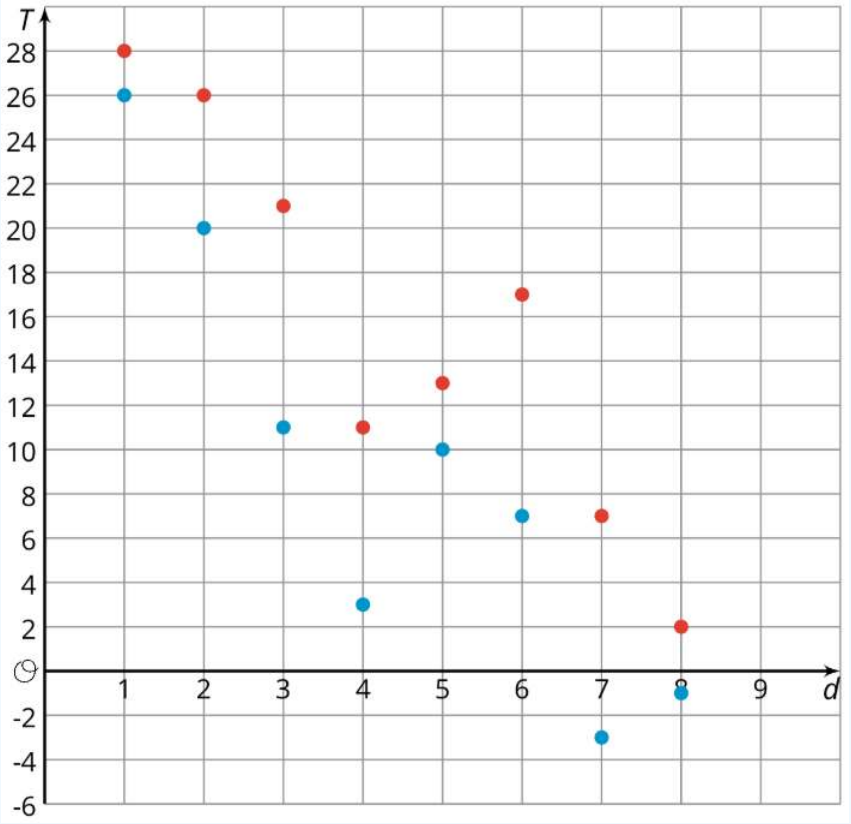


Figure 38.3.3: Coordinate plane, origin O, horizontal axis, d, 0 to 9 by ones, vertical axis, T, negative 6 to 28 by twos. Red points at (one comma 28), (2 comma 26), (3 comma 21), (4 comma 11), (5 comma 13), (6 comma 17), (7 comma 7), (8 comma 2). Blue points at (1 comma 28), (2 comma 20), (3 comma 11), (4 comma 3), (5 comma 10), (6 comma 7), (7 comma negative 3), (8 comma negative 1).

1. a. What was the warmest high temperature?
   b. Write an inequality to describe the high temperatures, H, over the 8-day period.
2. a. What was the coldest low temperature?
   b. Write an inequality to describe the low temperatures, L, over the 8-day period.
3. a. On which day(s) did the *largest* difference between the high and low temperatures occur? Write down this difference.
   b. On which day(s) did the *smallest* difference between the high and low temperatures occur? Write down this difference.

Are you ready for more?

Before doing this problem, do the problem about taxicab distance in an earlier lesson.

The point $(0, 3)$ is 4 taxicab units away from $(-4, 3)$ and 4 taxicab units away from $(2, 1)$.

1. Find as many other points as you can that are 4 taxicab units away from *both* $(-4, 3)$ and $(2, 1)$.
2. Are there any points that are 3 taxicab units away from both points?

Summary

Points on the coordinate plane can give us information about a context or a situation. One of those contexts is about money.

To open a bank account, we have to put money into the account. The account balance is the amount of money in the account at any given time. If we put in $350 when opening the account, then the account balance will be 350.

Sometimes we may have no money in the account and need to borrow money from the bank. In that situation, the account balance would have a negative value. If we borrow $200, then the account balance is -200.

A coordinate grid can be used to display both the balance and the day or time for any balance. This allows to see how the balance changes over time or to compare the balances of different days.

Similarly, if we plot on the coordinate plane data such as temperature over time, we can see how temperature changes over time or compare temperatures of different times.

Glossary Entries

Definition: Quadrant

The coordinate plane is divided into 4 regions called quadrants. The quadrants are numbered using Roman numerals, starting in the top right corner.

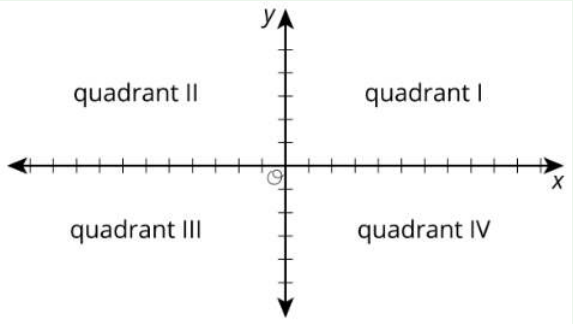


Figure 38.3.4

Practice

Exercise 38.3.4

The elevation of a submarine is shown in the table. Draw and label coordinate axes with an appropriate scale and plot the points.

Table 38.3.1

time after noon (hours)	elevation (meters)
0	-567
1	-892
2	$-1,606$
3	$-1,289$
4	-990
5	-702
6	-365

Exercise 38.3.5

The inequalities $h > 42$ and $h < 60$ represent the height requirements for an amusement park ride, where h represents a person's height in inches.

Write a sentence or draw a sign that describes these rules as clearly as possible.

(From Unit 7.2.1)

Exercise 38.3.6

The x-axis represents the number of hours before or after noon, and the y-axis represents the temperature in degrees Celsius.

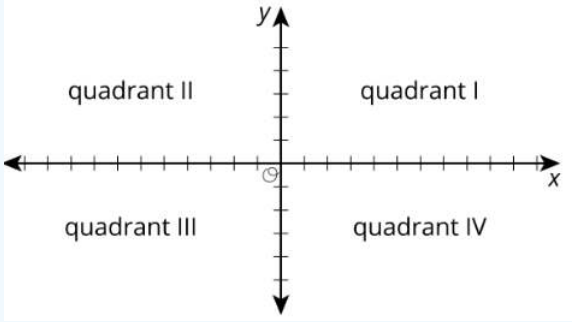


Figure 38.3.5

1. At 9 a.m., it was below freezing. In what quadrant would this point be plotted?
2. At 11 a.m., it was 10°C. In what quadrant would this point be plotted?
3. Choose another time and temperature. Then tell the quadrant where the point should be plotted.
4. What does the point $(0, 0)$ represent in this context?

Exercise 38.3.7

Solve each equation.

$3a = 12$ $\quad$ $b + 3.3 = 8.9$ $\quad$ $1 = \frac{1}{4}c$

$5\frac{1}{2} = d + \frac{1}{4}$ $\quad$ $2e = 6.4$

(From Unit 6.1.4)

38.3: Interpreting Points on a Coordinate Plane is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

38.4: Distances on a Coordinate Plane

Lesson

Let's explore distance on the coordinate plane.

Exercise 38.4.1: Coordinate Patterns

Plot points in your assigned quadrant and label them with their coordinates.

Exercise 38.4.2: Signs of Numbers in Coordinates

1. Write the coordinates of each point.
   - $A =$
   - $B =$
   - $C =$
   - $D =$
   - $E =$
2. Answer these questions for each pair of points.
   - How are the coordinates the same? How are they different?
   - How far away are they from the y-axis? To the left or to the right of it?
   - How far away are they from the x-axis? Above or below it?

   a. A and B
   b. B and D
   c. A and D

   Pause here for a class discussion.
3. Point F has the same coordinates as point C, except its y-coordinate has the opposite sign.
   a. Plot point F on the coordinate plane and label it with its coordinates.
   b. How far away are F and C from the x-axis?
   c. What is the distance between F and C?
4. Point G has the same coordinates as point E, except its x-coordinate has the opposite sign.
   a. Plot point G on the coordinate plane and label it with its coordinates.
   b. How far away are G and E from the y-axis?
   c. What is the distance between and ?
5. Point H has the same coordinates as point B, except *both* of its coordinates have the opposite signs. In which quadrant is point H?

Exercise 38.4.3: Finding Distances on a Coordinate Plane

1. Label each point with its coordinates.
2. Find the distance between each of the following pairs of points.
   a. Point B and C
   b. Point D and B
   c. Point D and E
3. Which of the points are 5 units from $(-1.5, -3)$?
4. Which of the points are 2 units from $(0.5, -4.5)$?
5. Plot a point that is both 2.5 units from A and 9 units from E. Label that point F and write down its coordinates.

Are you ready for more?

Priya says, “There are exactly four points that are 3 units away from $(-5, 0)$.” Lin says, “I think there are a whole bunch of points that are 3 units away from $(-5, 0)$.”

Do you agree with either of them? Explain your reasoning.

Summary

The points $A = (5, 2), B = (-5, 2), C = (-5, -2),$ and $D = (5, -2)$ are shown in the plane. Notice that they all have almost the same coordinates, except the signs are different. They are all the same distance from each axis but are in different quadrants.

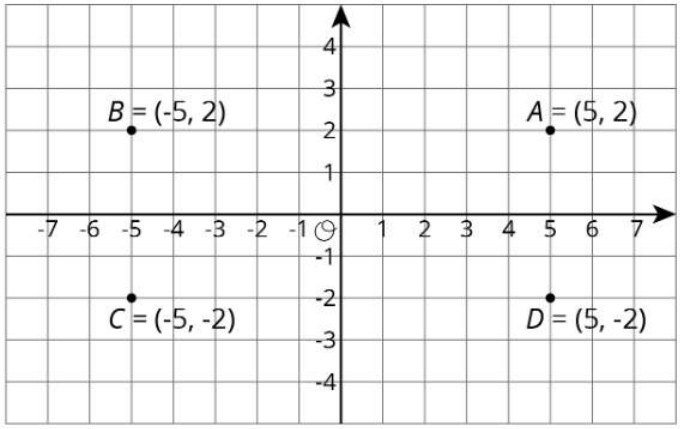


Figure 38.4.1

Notice that the vertical distance between points A and D is 4 units, because point A is 2 units above the horizontal axis and point D is 2 units below the horizontal axis. The horizontal distance between points A and B is 10 units, because point B is 5 units to the left of the vertical axis and point A is 5 units to the right of the vertical axis.

We can always tell which quadrant a point is located in by the signs of its coordinates.

Table 38.4.1

x	y	quadrant
positive	positive	I
negative	positive	II
negative	negative	III
positive	negative	IV

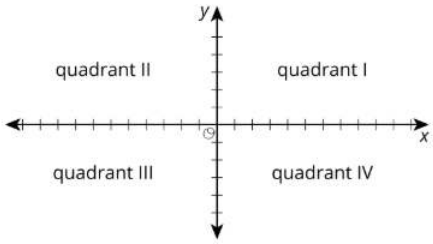


Figure 38.4.2

In general:

- If two points have x-coordinates that are opposites (like 5 and -5), they are the same distance away from the vertical axis, but one is to the left and the other to the right.
- If two points have y-coordinates that are opposites (like 2 and -2), they are the same distance away from the horizontal axis, but one is above and the other below.

When two points have the same value for the first or second coordinate, we can find the distance between them by subtracting the coordinates that are different. For example, consider $(1, 3)$ and $(5, 3)$:

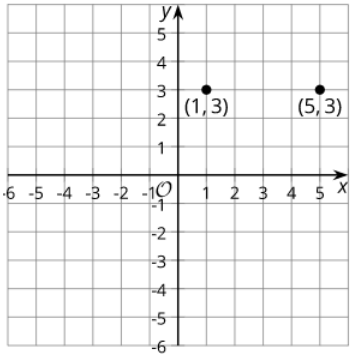


Figure 38.4.3

They have the same y-coordinate. If we subtract the x-coordinates, we get $5 - 1 = 4$. These points are 4 units apart.

Glossary Entries

Definition: Quadrant

The coordinate plane is divided into 4 regions called quadrants. The quadrants are numbered using Roman numerals, starting in the top right corner.

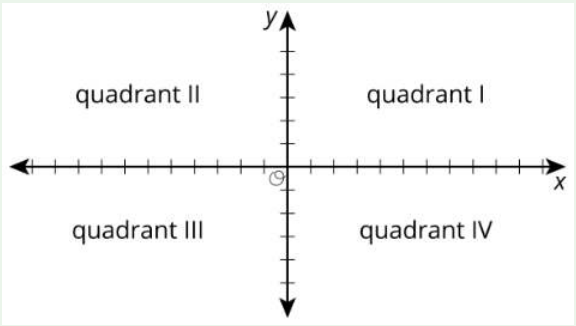


Figure 38.4.4

Practice

Exercise 38.4.4

Here are 4 points on a coordinate plane.

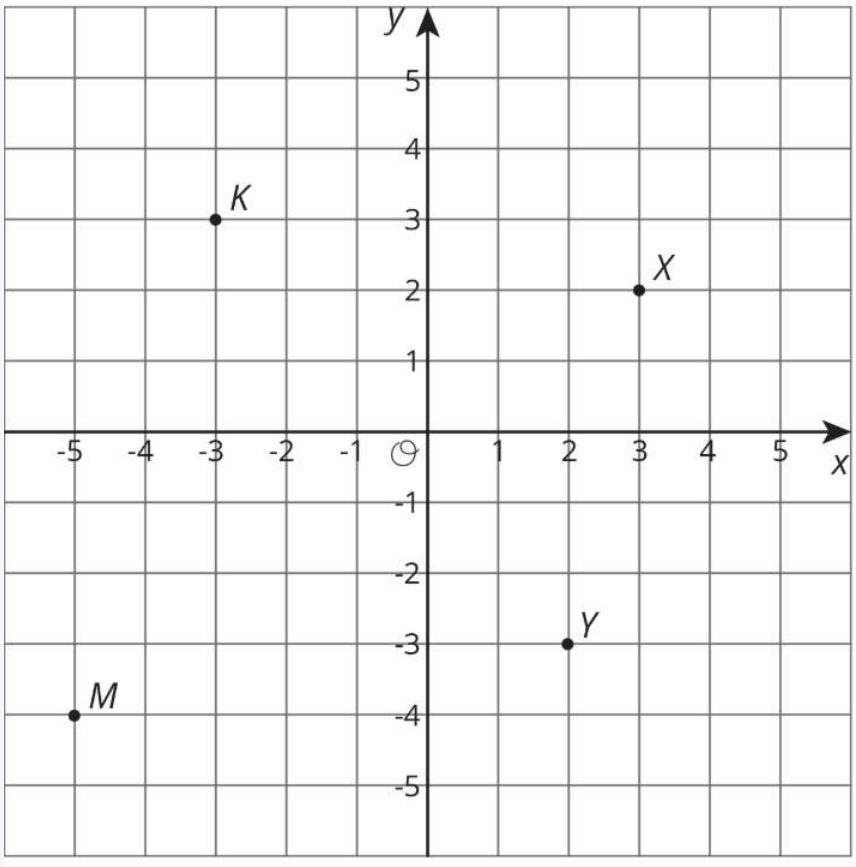


Figure 38.4.5

1. Label each point with its coordinates.
2. Plot a point that is 3 units from point K. Label it P.
3. Plot a point that is 2 units from point M. Label it W.

Exercise 38.4.5

Each set of points are connected to form a line segment. What is the length of each?

1. $A = (3, 5)$ and $B = (3, 6)$
2. $C = (-2, -3)$ and $D = (-2, -6)$
3. $E = (-3, 1)$ and $F = (-3, -1)$

Exercise 38.4.6

On the coordinate plane, plot four points that are each 3 units away from point $P = (-2, -1)$. Write the coordinates of each point.

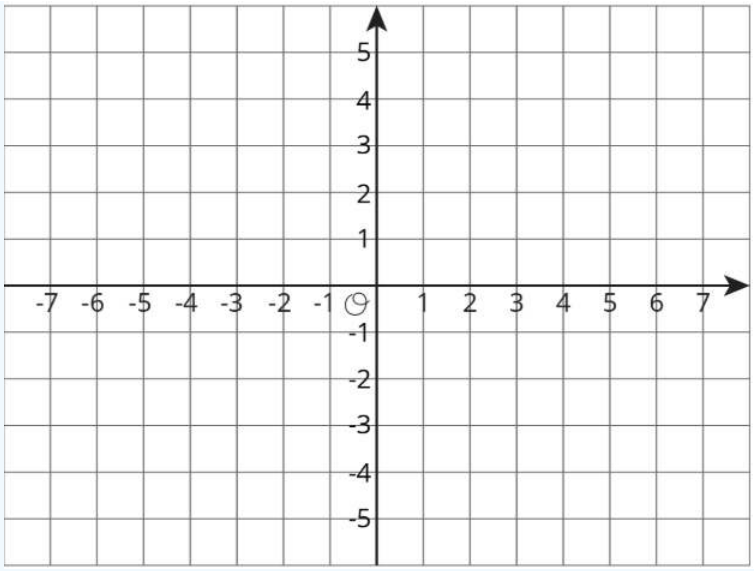


Figure 38.4.6

Exercise 38.4.7

Noah's recipe for sparkling orange juice uses 4 liters of orange juice and 5 liters of soda water.

1. Noah prepares large batches of sparkling orange juice for school parties. He usually knows the total number of liters, t, that he needs to prepare. Write an equation that shows how Noah can find s, the number of liters of soda water, if he knows t.
2. Sometimes the school purchases a certain number, j, of liters of orange juice and Noah needs to figure out how much sparkling orange juice he can make. Write an equation that Noah can use to find t if he knows j.

(From Unit 6.4.1)

Exercise 38.4.8

For a suitcase to be checked on a flight (instead of carried by hand), it can weigh at most 50 pounds. Andre's suitcase weighs 23 kilograms. Can Andre check his suitcase? Explain or show your reasoning. (Note: 10 kilograms $\approx$ 22 pounds)

(From Unit 3.2.3)

38.4: Distances on a Coordinate Plane is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

38.5: Shapes on the Coordinate Plane

Lesson

Let's use the coordinate plane to solve problems and puzzles.

Exercise 38.5.1: Figuring Out The Coordinate Plane

1. Draw a figure in the coordinate plane with at least three of following properties:
   - 6 vertices
   - 1 pair of parallel sides
   - At least 1 right angle
   - 2 sides the same length
2. Is your figure a polygon? Explain how you know.

Exercise 38.5.2: Plotting Polygons

Here are the coordinates for four polygons. Move the slider to choose the polygon you want to plot. Move the points, in order, to their locations on the coordinate plane. Sketch each one before changing the slider.

1. Polygon 1: $(-7, 4), (-8, 5), (-8, 6), (-7, 7), (-5, 7), (-5, 5), (-7, 4)$
2. Polygon 2: $(4, 3), (3, 3), (2, 2), (2, 1), (3, 0), (4, 0), (5, 1), (5, 2), (4, 3)$
3. Polygon 3: $(-8, -5), (-8, -8), (-5, -8), (-5, -5), (-8, -5)$
4. Polygon 4: $(-5, 1), (-3, -3), (-1, -2), (0, 3), (-3, 3), (-5, 1)$

Are you ready for more?

Find the area of Polygon D in this activity.

Exercise 38.5.3: Four Quadrants of A-Maze-ing

1. The following diagram shows Andre's route through a maze. He started from the lower right entrance.

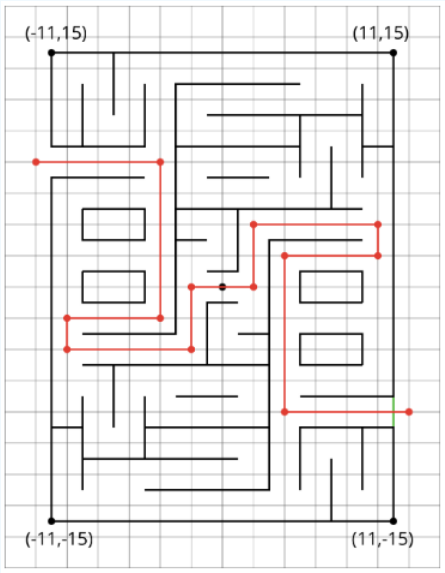


Figure 38.5.1

   a. What are the coordinates of the first two and the last two points of his route?
   b. How far did he walk from his starting point to his ending point? Show how you know.

2. Jada went into the maze and stopped at $(-7, 2)$.

   a. Plot that point and other points that would lead her out of the maze (through the exit on the upper left side).

b. How far from $(-7, 2)$ must she walk to exit the maze? Show how you know.

Summary

We can use coordinates to find lengths of segments in the coordinate plane.

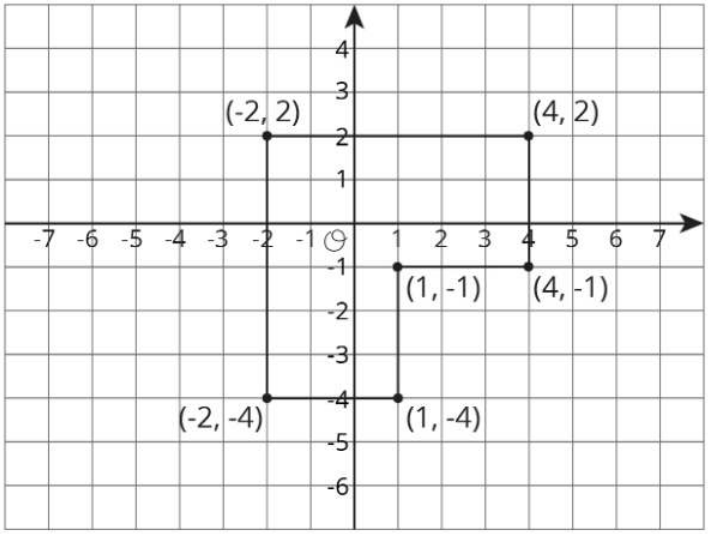


Figure 38.5.2: Coordinate plane, origin O, axes labeled by ones. Segments connecting, in order, (negative 2 comma 2), (4 comma 2), (4 comma negative 1), (1 comma negative 1), (1 comma negative 4), (negative 2 comma negative 4), (negative 2 comma 2).

For example, we can find the perimeter of this polygon by finding the sum of its side lengths. Starting from $(-2, 2)$ and moving clockwise, we can see that the lengths of the segments are 6, 3, 3, 3, 3, and 6 units. The perimeter is therefore 24 units.

In general:

- If two points have the same x-coordinate, they will be on the same vertical line, and we can find the distance between them.
- If two points have the same y-coordinate, they will be on the same horizontal line, and we can find the distance between them.

Glossary Entries

Definition: Quadrant

The coordinate plane is divided into 4 regions called quadrants. The quadrants are numbered using Roman numerals, starting in the top right corner.

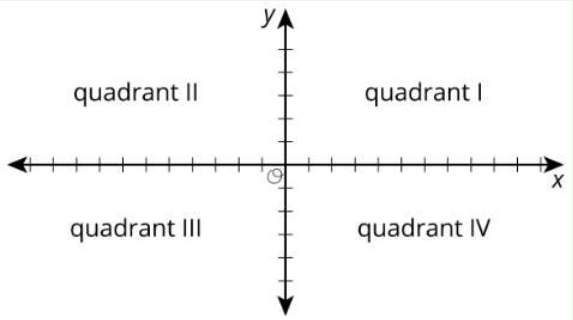


Figure 38.5.3

Practice

Exercise 38.5.4

The coordinates of a rectangle are $(3, 0), (3, -5), (-4, 0)$ and $(-4, -5)$

1. What is the length and width of this rectangle?
2. What is the perimeter of the rectangle?

3. What is the area of the rectangle?

Exercise 38.5.5

Draw a square with one vertex on the point $(-3, 5)$ and a perimeter of 20 units. Write the coordinates of each other vertex.

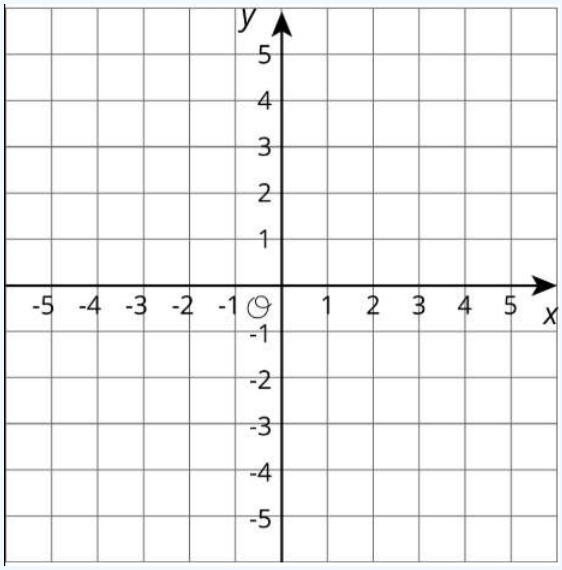


Figure 38.5.4

Exercise 38.5.6

1. Plot and connect the following points to form a polygon.

$$(-3, 2), (2, 2), (2, -4), (-1, -4), (-1, -2), (-3, -2), (-3, 2)$$

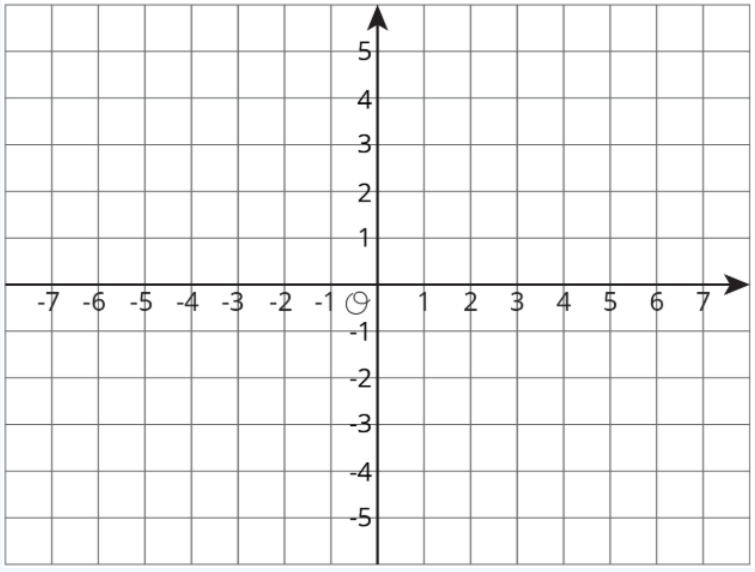


Figure 38.5.5

2. Find the perimeter of the polygon.

Exercise 38.5.7

For each situation, select **all** the equations that represent it. Choose one equation and solve it.

1. Jada's cat weighs 3.45 kg. Andre's cat weighs 1.2 kg more than Jada's cat. How much does Andre's cat weigh?

$$x = 3.45 + 1.2 \qquad x = 3.45 - 1.2 \qquad x + 1.2 = 3.45 \qquad x - 1.2 = 3.45$$

2. Apples cost $1.60 per pound at the farmer's market. They cost 1.5 times as much at the grocery store. How much do the apples cost per pound at the grocery store?

$$y = (1.5) \cdot (1.60) \qquad y = 1.60 \div 1.5 \qquad (1.5)y = 1.60 \qquad \frac{y}{1.5} = 1.60$$

(From Unit 6.1.4)

38.5: Shapes on the Coordinate Plane is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

Section 39: Common Factors and Common Multiples

This page titled Section 39: Common Factors and Common Multiples is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

39.1: Common Factors

Lesson

Let's use factors to solve problems.

Exercise 39.1.1: Figures Made of Squares

How are the pairs of figures alike? How are they different?

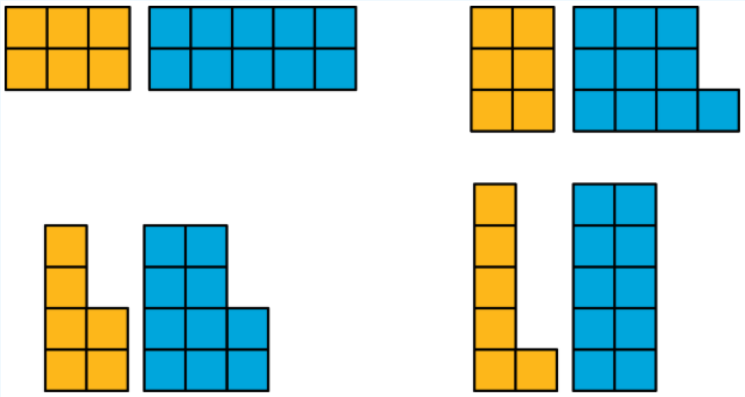


Figure 39.1.1

Exercise 39.1.2: Diego's Bake Sale

Diego is preparing brownies and cookies for a bake sale. He would like to make equal-size bags for selling all of the 48 brownies and 64 cookies that he has. Organize your answer to each question so that it can be followed by others.

1. How can Diego package all the 48 brownies so that each bag has the same number of them? How many bags can he make, and how many brownies will be in each bag? Find all the possible ways to package the brownies.
2. How can Diego package all the 64 cookies so that each bag has the same number of them? How many bags can he make, and how many cookies will be in each bag? Find all the possible ways to package the cookies.
3. How can Diego package all the 48 brownies and 64 cookies so that each bag has the same combination of items? How many bags can he make, and how many of each will be in each bag? Find all the possible ways to package both items.
4. What is the largest number of combination bags that Diego can make with no left over? Explain to your partner how you know that it is the largest possible number of bags.

Exercise 39.1.3: Greatest Common Factor

1. The **greatest common factor** of 30 and 18 is 6. What do you think the term “greatest common factor” means?
2. Find all of the **factors** of 21 and 6. Then, identify the greatest common factor of 21 and 6.
3. Find all of the factors of 28 and 12. Then, identify the greatest common factor of 28 and 12.
4. A rectangular bulletin board is 12 inches tall and 27 inches wide. Elena plans to cover it with squares of colored paper that are all the same size. The paper squares come in different sizes; all of them have whole-number inches for their side lengths.
   a. What is the side length of the largest square that Elena could use to cover the bulletin board completely without gaps and overlaps? Explain or show your reasoning.
   b. How is the solution to this problem related to greatest common factor?

Are you ready for more?

A school has 1,000 lockers, all lined up in a hallway. Each locker is closed. Then . . .

- One student goes down the hall and opens each locker.
- A second student goes down the hall and closes every second locker: lockers 2, 4, 6, and so on.

- A third student goes down the hall and changes every third locker. If a locker is open, he closes it. If a locker is closed, he opens it.
- A fourth student goes down the hall and changes every fourth locker.

This process continues up to the thousandth student! At the end of the process, which lockers will be open? (Hint: you may want to try this problem with a smaller number of lockers first.)

Summary

A factor of a whole number n is a whole number that divides n evenly without a remainder. For example, 1, 2, 3, 4, 6, and 12 are all factors of 12 because each of them divides 12 evenly and without a remainder.

A **common factor** of two whole numbers is a factor that they have in common. For example, 1, 3, 5, and 15 are factors of 45; they are also factors of 60. We call 1, 3, 5, and 15 common factors of 45 and 60.

The **greatest common factor** (sometimes written as GCF) of two whole numbers is the greatest of all of the common factors. For example, 15 is the greatest common factor for 45 and 60.

One way to find the greatest common factor of two whole numbers is to list all of the factors for each, and then look for the greatest factor they have in common. Let's try to find the greatest common factor of 18 and 24. First, we list all the factors of each number.

- Factors of 18: **1**, **2**, **3**, **6**, 9,18
- Factors of 24: **1**, **2**, **3**, 4, **6**, 8, 12, 24

The common factors are 1, 2, 3, and 6. Of these, 6 is the greatest one, so 6 is the greatest common factor of 18 and 24.

Glossary Entries

Definition: Common Factor

A common factor of two numbers is a number that divides evenly into both numbers. For example, 5 is a common factor of 15 and 20, because $15 \div 5 = 3$ and $20 \div 5 = 4$. Both of the quotients, 3 and 4, are whole numbers.

- The factors of 15 are 1, 3, *5*, and 15.
- The factors of 20 are 1, 2, 4, *5*, 10, and 20.

Definition: Greatest Common Factor

The greatest common factor of two numbers is the largest number that divides evenly into both numbers. Sometimes we call this the GCF. For example, 15 is the greatest common factor of 45 and 60.

- The factors of 45 are 1, 3, 5, 9, *15*, and 45.
- The factors of 60 are 1, 2, 3, 4, 5, 6, 10, 12, *15*, 20, 30, and 60.

Practice

Exercise 39.1.4

A teacher is making gift bags. Each bag is to be filled with pencils and stickers. The teacher has 24 pencils and 36 stickers to use. Each bag will have the same number of each item, with no items left over. For example, she could make 2 bags with 12 pencils and 18 stickers each.

What are the other possibilities? Explain or show your reasoning.

Exercise 39.1.5

1. List all the factors of 42.
2. What is the greatest common factor of 42 and 15?
3. What is the greatest common factor of 42 and 50?

Exercise 39.1.6

A school chorus has 90 sixth-grade students and 75 seventh-grade students. The music director wants to make groups of performers, with the same combination of sixth- and seventh-grade students in each group. She wants to form as many groups as possible.

1. What is the largest number of groups that could be formed? Explain or show your reasoning.
2. If that many groups are formed, how many students of each grade level would be in each group?

Exercise 39.1.7

Here are some bank transactions from a bank account last week. Which transactions represent negative values?

Monday: $650 paycheck deposited

Tuesday: $40 withdrawal from the ATM at the gas pump

Wednesday: $20 credit for returned merchandise

Thursday: $125 deducted for cell phone charges

Friday: $45 check written to pay for book order

Saturday: $80 withdrawal for weekend spending money

Sunday: $10 cash-back reward deposited from a credit card company

(From Unit 7.3.3)

Exercise 39.1.8

Find the quotients.

1. $\frac{1}{7} \div \frac{1}{8}$
2. $\frac{12}{5} \div \frac{6}{5}$
3. $\frac{1}{10} \div 10$
4. $\frac{9}{10} \div \frac{10}{9}$

(From Unit 4.3.2)

Exercise 39.1.9

An elephant can travel at a constant speed of 25 miles per hour, while a giraffe can travel at a constant speed of 16 miles in $\frac{1}{2}$ hour.

1. Which animal runs faster? Explain your reasoning.
2. How far can each animal run in 3 hours?

(From Unit 2.3.4)

39.1: Common Factors is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

39.2: Common Multiples

Lesson

Let's use multiples to solve problems.

Exercise 39.2.1: Notice and Wonder: Multiples

Circle all the multiples of 4 in this list.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26

Circle all the multiples of 6 in this list.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26

What do you notice? What do you wonder?

Exercise 39.2.2: The Florist's Order

A florist can order roses in bunches of 12 and lilies in bunches of 8. Last month she ordered the same number of roses and lilies.

1. If she ordered no more than 100 of each kind of flower, how many bunches of each could she have ordered? Find all the possible combinations.
2. What is the smallest number of bunches of roses that she could have ordered? What about the smallest number of bunches of lilies? Explain your reasoning.

Exercise 39.2.3: Least Common Multiple

The **least common multiple** of 6 and 8 is 24.

1. What do you think the term “least common multiple” means?
2. Find all of the **multiples** of 10 and 8 that are less than 100. Find the least common multiple of 10 and 8.
3. Find all of the multiples of 7 and 9 that are less than 100. Find the least common multiple of 7 and 9.

Are you ready for more?

1. What is the least common multiple of 10 and 20?
2. What is the least common multiple of 4 and 24?
3. In the previous two questions, one number is a multiple of the other. What do you notice about their least common multiple? Do you think this will always happen when one number is a multiple of the other? Explain your reasoning.

Exercise 39.2.4: Prizes on Grand Opening Day

Lin’s uncle is opening a bakery. On the bakery’s grand opening day, he plans to give away prizes to the first 50 customers that enter the shop. Every fifth customer will get a free bagel. Every ninth customer will get a free blueberry muffin. Every 12th customer will get a free slice of carrot cake.

1. Diego is waiting in line and is the 23rd customer. He thinks that he should get farther back in line in order to get a prize. Is he right? If so, how far back should he go to get at least one prize? Explain your reasoning.
2. Jada is the 36th customer.
   a. Will she get a prize? If so, what prize will she get?
   b. Is it possible for her to get more than one prize? How do you know? Explain your reasoning.
3. How many prizes total will Lin’s uncle give away? Explain your reasoning.

Summary

A multiple of a whole number is a product of that number with another whole number. For example, 20 is a multiple of 4 because $20 = 5 \cdot 4$.

A **common multiple** for two whole numbers is a number that is a multiple of both numbers. For example, 20 is a multiple of 2 and a multiple of 5, so 20 is a common multiple of 2 and 5.

The **least common multiple** (sometimes written as LCM) of two whole numbers is the smallest multiple they have in common. For example, 30 is the least common multiple of 6 and 10.

One way to find the least common multiple of two numbers is to list multiples of each in order until we find the smallest multiple they have in common. Let's find the least common multiple for 4 and 10. First, we list some multiples of each number.

- Multiples of 4: 4, 8, 12, 16, **20**, 24, 28, 32, 36, **40**, 44 . . .
- Multiples of 10: 10, **20**, 30, **40**, 50, . . .

20 and 40 are both common multiples of 4 and 10 (as are 60, 80, . . .), but 20 is the smallest number that is on *both* lists, so 20 is the least common multiple.

Glossary Entries

Definition: Common Factor

A common factor of two numbers is a number that divides evenly into both numbers. For example, 5 is a common factor of 15 and 20, because $15 \div 5 = 3$ and $20 \div 5 = 4$. Both of the quotients, 3 and 4, are whole numbers.

- The factors of 15 are 1, 3, *5*, and 15.
- The factors of 20 are 1, 2, 4, *5*, 10, and 20.

Definition: Common Multiple

A common multiple of two numbers is a product you can get by multiplying each of the two numbers by some whole number. For example, 30 is a common multiple of 3 and 5, because $3 \cdot 10 = 30$ and $5 \cdot 6 = 30$. Both of the factors, 10 and 6, are whole numbers.

- The multiples of 3 are 3, 6, 9, 12, *15*, 18, 21, 24, 27, *30*, 33 . . .
- The multiples of 5 are 5, 10, *15*, 20, 25, *30*, 35, 40 . . .

The common multiples of 3 and 5 are 15, 30, 45, 60 . . .

Definition: Greatest Common Factor

The greatest common factor of two numbers is the largest number that divides evenly into both numbers. Sometimes we call this the GCF. For example, 15 is the greatest common factor of 45 and 60.

- The factors of 45 are 1, 3, 5, 9, *15*, and 45.
- The factors of 60 are 1, 2, 3, 4, 5, 6, 10, 12, *15*, 20, 30, and 60.

Definition: Least Common Multiple

The least common multiple of two numbers is the smallest product you can get by multiplying each of the two numbers by some whole number. Sometimes we call this the LCM. For example, 30 is the least common multiple of 6 and 10.

- The multiples of 6 are 6, 12, 18, 24, *30*, 36, 42, 48, 54, 60 . . .
- The multiples of 10 are 10, 20, *30*, 40, 50, 60, 70, 80 . . .

Practice

Exercise 39.2.5

1. A green light blinks every 4 seconds and a yellow light blinks every 5 seconds. When will both lights blink at the same time?
2. A red light blinks every 12 seconds and a blue light blinks every 9 seconds. When will both lights blink at the same time?
3. Explain how to determine when 2 lights blink together.

Exercise 39.2.6

1. List all multiples of 10 up to 100.
2. List all multiples of 15 up to 100.
3. What is the least common multiple of 10 and 15?

Exercise 39.2.7

Cups are sold in packages of 8. Napkins are sold in packages of 12.

1. What is the fewest number of packages of cups and the fewest number of packages of napkins that can be purchased so there will be the same number of cups as napkins?
2. How many sets of cups and napkins will there be?

Exercise 39.2.8

1. Plot and connect these points to form a polygon.

$(-5, 3), \quad (3, 3), \quad (1, -2), \quad (-3, -2)$

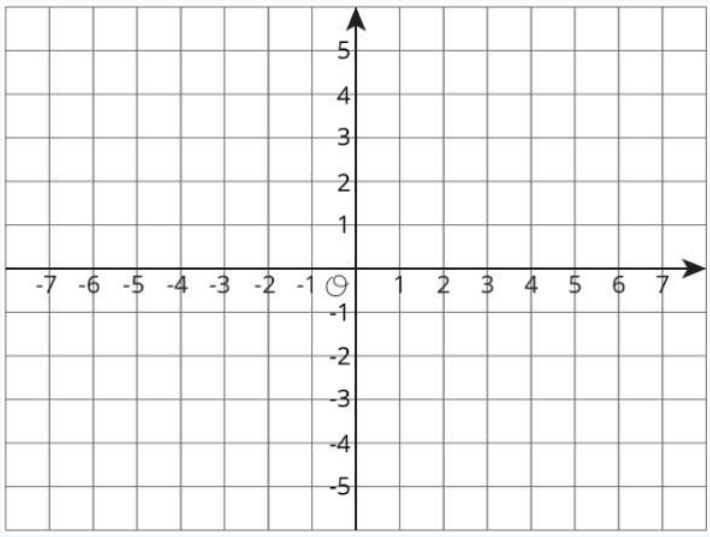


Figure 39.2.1

2. Find the lengths of the two horizontal sides of the polygon.

(From Unit 7.3.5)

Exercise 39.2.9

Rectangle ABCD is drawn on a coordinate plane. $A = (-6, 9)$ and $B = (5, 9)$. What could be the locations of points C and D?

(From Unit 7.3.4)

Exercise 39.2.10

A school wants to raise $2,500 to support its music program.

1. If it has met 20% of its goal so far, how much money has it raised?
2. If it raises 175% of its goal, how much money will the music program receive? Show your reasoning.

(From Unit 3.4.5)

39.2: Common Multiples is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

39.3: Using Common Multiples and Common Factors

Lesson

Let's use common factors and common multiple to solve problems.

Exercise 39.3.1: Keeping a Steady Beat

Your teacher will give you instructions for playing a rhythm game. As you play the game, think about these questions:

- When will the two sounds happen at the same time?
- How does this game relate to common factors or common multiples?

Exercise 39.3.2: Factors and Multiples

Work with your partner to solve the following problems.

1. **Party.** Elena is buying cups and plates for her party. Cups are sold in packs of 8 and plates are sold in packs of 6. She wants to have the same number of plates and cups.
   a. Find a number of plates and cups that meets her requirement.
   b. How many packs of each supply will she need to buy to get that number?
   c. Name two other quantities of plates and cups she could get to meet her requirement.
2. **Tiles**. A restaurant owner is replacing the restaurant's bathroom floor with square tiles. The tiles will be laid side-by-side to cover the entire bathroom with no gaps, and none of the tiles can be cut. The floor is a rectangle that measures 24 feet by 18 feet.
   a. What is the largest possible tile size she could use? Write the side length in feet. Explain how you know it's the largest possible tile.
   b. How many of these largest size tiles are needed?
   c. Name more tile sizes that are whole number of feet that she could use to cover the bathroom floor. Write the side lengths (in feet) of the square tiles.
3. **Stickers**. To celebrate the first day of spring, Lin is putting stickers on some of the 100 lockers along one side of her middle school's hallway. She puts a skateboard sticker on every 4th locker (starting with locker 4), and a kite sticker on every 5th locker (starting with locker 5).
   a. Name three lockers that will get both stickers.
   b. After Lin makes her way down the hall, will the 30th locker have no stickers, 1 sticker, or 2 stickers? Explain how you know.
4. **Kits.** The school nurse is assembling first-aid kits for the teachers. She has 75 bandages and 90 throat lozenges. All the kits must have the same number of each supply, and all supplies must be used.
   a. What is the largest number of kits the nurse can make?
   b. How many bandages and lozenges will be in each kit?
5. What kind of mathematical work was involved in each of the previous problems? Put a checkmark to show what the questions were about.

Table 39.3.1

problem	finding multiples	finding least common multiple	finding factors	finding greatest common factor
Party				
Tiles				
Stickers				
Kits				

Are you ready for more?

You probably know how to draw a five-pointed star without lifting your pencil. One way to do this is to start with five dots arranged in a circle, then connect every second dot.

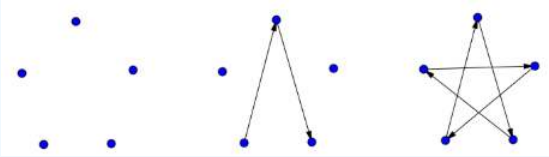


Figure 39.3.1

If you try the same thing with six dots arranged in a circle, you will have to lift your pencil. Once you make the first triangle, you'll have to find an empty dot and start the process over. Your six-pointed star has two pieces that are each drawn without lifting the pencil.

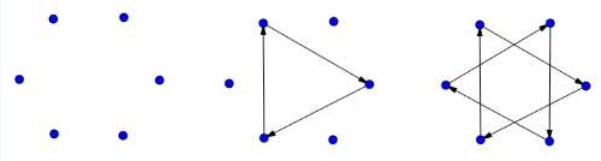


Figure 39.3.2

With twelve dots arranged in a circle, we can make some twelve-pointed stars.

1. Start with one dot and connect every second dot, as if you were drawing a five-pointed star. Can you draw the twelve-pointed star without lifting your pencil? If not, how many pieces does the twelve-pointed star have?

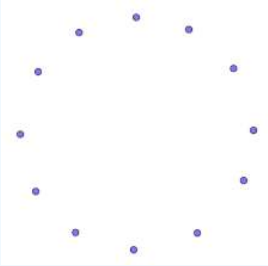


Figure 39.3.3

2. This time, connect every third dot. Can you draw this twelve-pointed star without lifting your pencil? If not, how many pieces do you get?

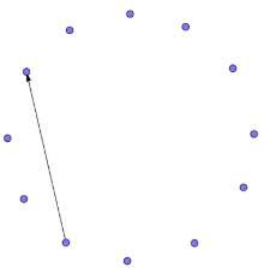


Figure 39.3.4

3. What do you think will happen if you connect every fourth dot? Try it. How many pieces do you get?

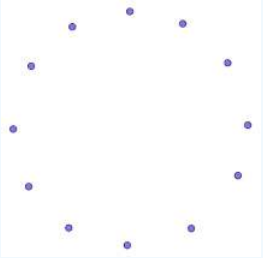


Figure 39.3.5

4. Do you think there is any way to draw a twelve-pointed star without lifting your pencil? Try it out.

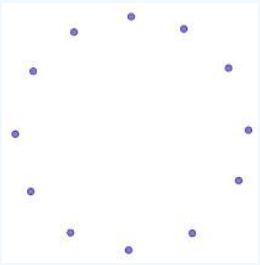


Figure 39.3.6

5. Now investigate eight-pointed stars, nine-pointed stars, and ten-pointed stars. What patterns do you notice?

Exercise 39.3.3: More Factors and Multiples

Here are five more problems. Read and discuss each one with your group. *Without solving*, predict whether each problem involves finding common multiples or finding common factors. Circle one or more options to show your prediction.

1. **Soccer.** Diego and Andre are both in a summer soccer league. During the month of August, Diego has a game every 3rd day, starting August 3rd, and Andre has a game every 4th day, starting August 4th.

   - common multiples
   - least common multiple
   - common factors
   - greatest common factor

   a. What is the first date that both boys will have a game?
   b. How many of their games fall on the same date?

2. **Performances.** During a performing arts festival, students from elementary and middle schools will be grouped together for various performances. There are 32 elementary students and 40 middle-school students. The arts director wants identical groups for the performances, with students from both schools in each group. Each student can be a part of only one group.

   - common multiples
   - least common multiple
   - common factors
   - greatest common factor

   a. Name all possible groupings.
   b. What is the largest number of groups that can be formed? How many elementary school students and how many middle school students will be in each group?

3. **Lights.** There is a string of holiday lights with red, gold, and blue lights. The red lights are set to blink every 12 seconds, the gold lights are set to blink every 8 seconds, and the blue lights are set to blink every 6 seconds. The lights are on an automatic timer that starts each day at 7:00 p.m. and stops at midnight.

   - common multiples
   - least common multiple

- common factors
- greatest common factor

a. After how much time with all 3 lights blink at the exact same time?
b. How many times total will this happen in one day?

4. **Banners.** Noah has two pieces of cloth. He is making square banners for students to hold during the opening day game. One piece of cloth is 72 inches wide. The other is 90 inches wide. He wants to use all the cloth, and each square banner must be of equal width and as wide as possible.

   - common multiples
   - least common multiple
   - common factors
   - greatest common factor

   a. How wide should he cut the banners?
   b. How many banners can he cut?

5. **Dancers.** At Elena's dance recital her performance begins with a line of 48 dancers that perform in the dark with a black light that illuminates white clothing. All 48 dancers enter the stage in a straight line. Every 3rd dancer wears a white headband, every 5th dancer wears a white belt, and every 9th dancer wears a set of white gloves.

   - common multiples
   - least common multiple
   - common factors
   - greatest common factor

   a. If Elena is the 30th dancer, what accessories will she wear?
   b. Will any of the dancers wear all 3 accessories? If so, which one(s)?
   c. How many of each accessory will the dance teacher need to order?

6. Your teacher will assign your group a problem. Work with your group to solve the problem. Show your reasoning. Pause here so your teacher can review your work.
7. Work with your group to create a visual display that includes a diagram, an equation, and a math vocabulary word that would help to explain your mathematical thinking while solving the problem.
8. Prepare a short presentation in which all group members are involved. Your presentation should include: the problem (read aloud), your group's prediction of what mathematical concept the problem involved, and an explanation of each step of the solving process.

Exercise 39.3.4: Factors and Multiples Bingo

Your teacher will explain the directions for a bingo game. Here are some things to keep in mind:

- Share one bingo board and some bingo chips with a partner.
- To play the game, your teacher will read statements aloud. You may help one another decide what numbers fit each statement, but speak only in a whisper. If the teacher hears anything above a whisper, you are out.
- The first person to call bingo needs to call out each number and identify the statement that it corresponds to. If there is an error in identifying statements, that player is out and the round continues.

Good luck, and have fun!

Summary

If a problem requires dividing two whole numbers by the same whole number, solving it involves looking for a common factor. If it requires finding the *largest* number that can divide into the two whole numbers, we are looking for the *greatest common factor*.

Suppose we have 12 bagels and 18 muffins and want to make bags so each bag has the same combination of bagels and muffins. The common factors of 12 and 18 tell us possible number of bags that can be made.

The common factors of 18 are 1, 2, 3, and 6. For these numbers of bags, here are the number of bagels and muffins per bag.

- 1 bag: 12 bagels and 18 muffins
- 2 bags: 6 bagels and 9 muffins
- 3 bags: 4 bagels and 6 muffins
- 6 bags: 2 bagels and 3 muffins

We can see that the largest number of bags that can be made, 6, is the greatest common factor.

If a problem requires finding a number that is a multiple of two given numbers, solving it involves looking for a common multiple. If it requires finding the *first* instance the two numbers share a multiple, we are looking for the *least common multiple*.

Suppose forks are sold in boxes of 9 and spoons are sold in boxes of 15, and we want to buy an equal number of each. The multiples of 9 tell us how many forks we could buy, and the multiples of 15 tell us how many spoons we could buy, as shown here.

- Forks: 9, 18, 27, 36, 45, 54, 63, 72, 90. . .
- Spoons: 15, 30, 45, 60, 75, 90. . .

If we want as many forks as spoons, our options are 45, 90, 135, and so on, but the smallest number of utensils we could buy is 45, the least common multiple. This means buying 5 boxes of forks ($5 \cdot 9 = 45$) and 3 boxes of spoons ($3 \cdot 15 = 45$).

Glossary Entries

Definition: Common Factor

A common factor of two numbers is a number that divides evenly into both numbers. For example, 5 is a common factor of 15 and 20, because $15 \div 5 = 3$ and $20 \div 5 = 4$. Both of the quotients, 3 and 4, are whole numbers.

- The factors of 15 are 1, 3, *5*, and 15.
- The factors of 20 are 1, 2, 4, *5*, 10, and 20.

Definition: Common Multiple

A common multiple of two numbers is a product you can get by multiplying each of the two numbers by some whole number. For example, 30 is a common multiple of 3 and 5, because $3 \cdot 10 = 30$ and $5 \cdot 6 = 30$. Both of the factors, 10 and 6, are whole numbers.

- The multiples of 3 are 3, 6, 9, 12, *15*, 18, 21, 24, 27, *30*, 33 . . .
- The multiples of 5 are 5, 10, *15*, 20, 25, *30*, 35, 40 . . .

The common multiples of 3 and 5 are 15, 30, 45, 60 . . .

Definition: Greatest Common Factor

The greatest common factor of two numbers is the largest number that divides evenly into both numbers. Sometimes we call this the GCF. For example, 15 is the greatest common factor of 45 and 60.

- The factors of 45 are 1, 3, 5, 9, *15*, and 45.
- The factors of 60 are 1, 2, 3, 4, 5, 6, 10, 12, *15*, 20, 30, and 60.

Definition: Least Common Multiple

The least common multiple of two numbers is the smallest product you can get by multiplying each of the two numbers by some whole number. Sometimes we call this the LCM. For example, 30 is the least common multiple of 6 and 10.

- The multiples of 6 are 6, 12, 18, 24, *30*, 36, 42, 48, 54, 60 . . .
- The multiples of 10 are 10, 20, *30*, 40, 50, 60, 70, 80 . . .

Practice

Exercise 39.3.5

Mai, Clare, and Noah are making signs to advertise the school dance. It takes Mai 6 minutes to complete a sign, it takes Clare 8 minutes to complete a sign, and it takes Noah 5 minutes to complete a sign. They keep working at the same rate for a half hour.

1. Will Mai and Clare complete a sign at the same time? Explain your reasoning.
2. Will Mai and Noah complete a sign at the same time? Explain your reasoning.
3. Will Clare and Noah complete a sign at the same time? Explain your reasoning.
4. Will all three students complete a sign at the same time? Explain your reasoning.

Exercise 39.3.6

Diego has 48 chocolate chip cookies, 64 vanilla cookies, and 100 raisin cookies for a bake sale. He wants to make bags that have all three cookie flavors and the same number of each flavor per bag.

1. How many bags can he make without having any cookies left over?
2. Find the another solution to this problem.

(From Unit 7.4.1)

Exercise 39.3.7

1. Find the product of 12 and 8.
2. Find the greatest common factor of 12 and 8.
3. Find the least common multiple of 12 and 8.
4. Find the product of the greatest common factor and the least common multiple of 12 and 8.
5. What do you notice about the answers to question 1 and question 4?
6. Choose 2 other numbers and repeat the previous steps. Do you get the same results?

Exercise 39.3.8

1. Given the point $(5.5, -7)$, name a second point so that the two points form a vertical segment. What is the length of the segment?
2. Given the point $(3, 3.5)$, name a second point so that the two points form a horizontal segment. What is the length of the segment?

(From Unit 7.3.1)

Exercise 39.3.9

Find the value of each expression mentally.

1. $\frac{1}{2} \cdot 37 - \frac{1}{2} \cdot 7$
2. $3.5 \cdot 40 + 3.5 \cdot 60$
3. $999 \cdot 5$

(From Unit 6.2.4)

39.3: Using Common Multiples and Common Factors is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

Section 40: Let's Put it to Work

This page titled Section 40: Let's Put it to Work is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

40.1: Drawing on the Coordinate Plane

Lesson

Let's draw on the coordinate plane.

Exercise 40.1.1: Cat Pictures

Recreate this image in the applet by entering ordered pairs.

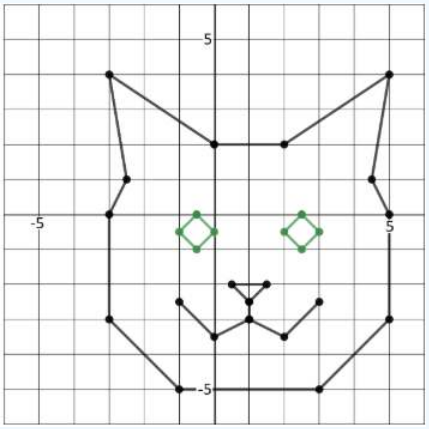


Figure 40.1.1: A cat drawn on a coordinate grid. Horizontal axis goes from negative 6 to 6, by 1s. The vertical axis goes from negative 6 to 6, by 1s. The outline of the cat is marked by the points negative 3 comma 4, negative 2 point 5 comma 1, negative 3 comma 0, negative 3 comma negative 3, negative 1 comma negative 5, 3 comma negative 5, 5 comma negative 3, 5 comma 0, 4 point 5 comma 1, 5 comma 4, 2 comma 2, 0 comma 2. An eye is marked by the points negative 0 point 5 comma 0, negative 1 comma negative point 5, negative point 5 comma negative 1, and 0 comma negative 0 point 5. Another eye is marked by the point 2 point 5 comma 0, 3 comma negative 0 point 5, 2 point 5 comma negative 1, 2 comma negative 0 point 5. The outline of the nose and mouth is marked by 0 point 5 comma negative 2, 1 point 5 comma negative 2, 1 comma negative 2 point 5, 1 comma negative 3, 0 comma negative 3, negative 1 comma 2 point 5, 2 comma negative 3 point 5, 3 comma negative 2 point 5.

If you have time, consider adding more details to your image such as whiskers, the inside of the ears, a bow, or a body.

Are you ready for more?

Add these statements to the list of things being graphed.

$x > 6$

$y > 5$

$x < -4$

$y < -6$

Describe the result. Why do you think that happened?

Exercise 40.1.2: Design Your Own Image

Create an image of your own design by listing the ordered pairs in the applet. You could draw a different animal, a vehicle, a building, or something else. Make sure your image includes at least 4 points in each quadrant of the coordinate plane.

40.1: Drawing on the Coordinate Plane is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

SECTION OVERVIEW

8: Data Sets and Distributions

Section 41: Data, Variability, and Statistical Questions

41.1: Got Data?
41.2: Statistical Questions

Section 42: Dot Plots and Histograms

42.1: Representing Data Graphically
42.2: Dot Plots
42.3: Using Dot Plots to Answer Statistical Questions
42.4: Interpreting Histograms
42.5: Using Histograms to Answer Statistical Questions
42.6: Describing Distributions on Histograms

Section 43: Measures of Center and Variability

43.1: Mean
43.2: Finding and Interpreting the Mean as the Balance Point
43.3: Variability and MAD
43.4: Using Mean and MAD to Make Comparisons

Section 44: Median and IQR

44.1: Median
44.2: Comparing Mean and Median
44.3: Quartiles and Interquartile Range
44.4: Box Plots
44.4.1: Using Box Plots

Section 45: Let's Put it to Work

45.1: Using Data to Solve Problems

This page titled 8: Data Sets and Distributions is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

CHAPTER OVERVIEW

Section 41: Data, Variability, and Statistical Questions

41.1: Got Data?
41.2: Statistical Questions

This page titled Section 41: Data, Variability, and Statistical Questions is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

41.1: Got Data?

Lesson

Let's explore different kinds of data.

Exercise 41.1.1: Dots of Data

Here is a **dot plot** for a data set.

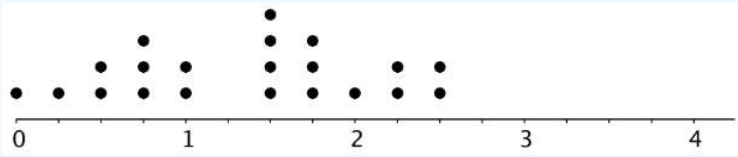


Figure 41.1.1

1. Determine if each of the following would be an appropriate label to represent the data in the dot plot? Be prepared to explain your reasoning.
   a. Number of children per class.
   b. Distance between home and school, in miles.
   c. Hours spent watching TV each day.
   d. Weight of elephants, in pounds.
   e. Points received on a homework assignment.
2. Think of another label that can be used with the dot plot.
   a. Write it below the scale of the dot plot. Be sure to include the unit of measurement.
   b. In your scenario, what does one dot represent?
   c. In your scenario, what would a data point of 0 mean? What would a data point of $3\frac{1}{4}$ mean?

Exercise 41.1.2: Surveying the Class

Here are some survey questions. Your teacher will explain which questions can be used to learn more about the students in your class and how the responses will be collected. The data that your class collects will be used in upcoming activities.

1. How long does it usually take you to travel to school? Answer to the nearest minute.
2. How do you travel to school on most days? Choose one.
   - Public transport
   - Other
   - Walk
   - Bike
   - Scooter or skateboard
   - Car
   - School bus
3. How tall are you without your shoes on? Answer to the nearest centimeter.
4. What is the length of your right foot without your shoe on? Answer to the nearest centimeter.
5. What is your arm span? Stretch your arms open, and measure the distance from the tip of your right hand's middle finger to the tip of your left hand's middle finger, across your back. Answer to the nearest centimeter.
6. How important are the following issues to you? Rate each on a scale from 0 (not important) to 10 (very important).
   a. Reducing pollution
   b. Recycling
   c. Conserving water
7. Do you have any siblings? _____ Yes _____ No
8. How many hours of sleep per night do you usually get when you have school the next day? Answer to the nearest half hour.

9. How many hours of sleep per night do you usually get when you do not have school the next day? Answer to the nearest half hour.
10. Other than traveling from school, what do you do right after school on most days?
    - Have a snack
    - Do homework
    - Read a book
    - Talk on the phone
    - Practice a sport
    - Do chores
    - Use the computer
    - Participate in an extracurricular activity
11. If you could meet one of these celebrities, who would you choose?
    - A city or state leader
    - A champion athlete
    - A movie star
    - A musical artist
    - A best-selling author
12. Estimate how much time per week you usually spend on each of these activities. Answer to the nearest quarter of an hour.
    a. Playing sports or doing outdoor activities
    b. Using a screen for fun (watching TV, playing computer games, etc.)
    c. Doing homework
    d. Reading

Exercise 41.1.3: Numerical and Categorical Data

The list of survey questions in the activity earlier can help you complete these exercises.

1. The first survey question about travel *time* produces **numerical data**. Identify two other questions that produce numerical data. For each, describe what was measured and its unit of measurement.
    a. Question #: _____
    What was measured:
    Unit of measurement:
    b. Question #: _____
    What was measured:
    Unit of measurement:
2. The second survey question about travel *method* produces **categorical data**. Identify two other questions that produce categorical data. For each, describe what characteristic or feature was being studied.
    a. Question #: _____
    Characteristic being studied:
    b. Question #: _____
    Characteristic being studied:
3. Think about the responses to these survey questions. Do they produce numerical or categorical data? Be prepared to explain how you know.
    a. How many pets do you have?
    b. How many years have you lived in this state?
    c. What is your favorite band?
    d. What kind of music do you like best?
    e. What is the area code of your school's phone number?
    f. Where were you born?
    g. How much does your backpack weigh?

4. Name two characteristics you could investigate to learn more about your classmates. Make sure one would give categorical data and the other would give numerical data.

Are you ready for more?

Priya and Han collected data on the birth months of students in their class. Here are the lists of their records for the same group of students.

This list shows Priya's records.

Jan Apr Jan Feb Oct May June July Aug Aug Sep Jan Feb Mar Apr
Nov Nov Dec Feb Mar

This list shows Han's records.

1 4 1 2 10 5 6 7 8 8 9 1 2 3 4 11 11 12 2 3

1. How are their records alike? How are they different?
2. What kind of data—categorical or numerical—do you think the variable "birth month" produces? Explain how you know.

Summary

The table contains data about 10 dogs.

Table 41.1.1

dog name	weight (kg)	breed
Duke	36	German shepherd
Coco	6	pug
Pierre	7	pug
Ginger	35	German shepherd
Lucky	10	beagle
Daisy	10	beagle
Buster	35	German shepherd
Pepper	7	pug
Rocky	7	beagle
Lady	32	German shepherd

- The weights of the dogs are an example of **numerical data**, which is data that are numbers, quantities, or measurements. The weights of the dogs are measurements in kilograms.
- The dog breeds are an example of **categorical data**, which is data containing values that can be sorted into categories. In this case, there are three categories for dog breeds: pug, beagle, and German shepherd.

Some data with numbers are categorical because the numbers are *not* quantities or measurements. For example, telephone area codes are categorical data, because the numbers are labels rather than quantities or measurements.

Numerical data can be represented with a **dot plot** (sometimes called a line plot). Here is a dot plot that shows the weights of the dogs.

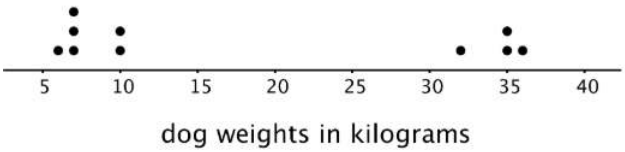


Figure 41.1.2: A dot plot for dog weights in kilograms. The numbers 5 through 40, in increments of 5, are indicated. The data are as follows: 6 kilograms, 1 dot. 7 kilograms, 3 dots. 10 kilograms, 2 dots. 32 kilograms, 1 dot. 35 kilograms, 2 dots. 36 kilograms, 1 dot.

We can collect and study both kinds of data by doing surveys or taking measurements. When we do, it is important to think about what feature we are studying (for example, breeds of dogs or weights of dogs) and what units of measurement are used.

Glossary Entries

Definition: Categorical Data

A set of categorical data has values that are words instead of numbers.

For example, Han asks 5 friends to name their favorite color. Their answers are: blue, blue, green, blue, orange.

Definition: Dot Plot

A dot plot is a way to represent data on a number line. Each time a value appears in the data set, we put another dot above that number on the number line.

For example, in this dot plot there are three dots above the 9. This means that three different plants had a height of 9 cm.

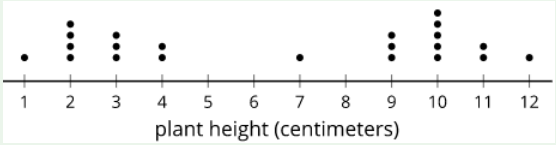


Figure 41.1.3

Definition: Numerical Data

A set of numerical data has values that are numbers.

For example, Han lists the ages of people in his family: 7, 10, 12, 36, 40, 67.

Practice

Exercise 41.1.4

Tyler asked 10 students at his school how much time in minutes it takes them to get from home to school. Determine if each of these dot plots could represent the data Tyler collected. Explain your reasoning for each dot plot.

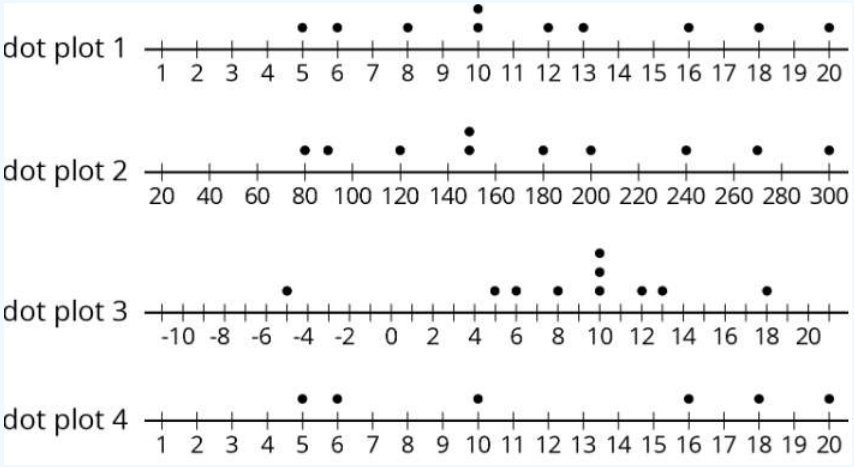


Figure 41.1.4

Exercise 41.1.5

Here is a list of questions. For each question, decide if the responses will produce numerical data or categorical data and give two possible responses.

1. What is your favorite breakfast food?
2. How did you get to school this morning?

3. How many different teachers do you have?
4. What is the last thing you ate or drank?
5. How many minutes did it take you to get ready this morning—from waking up to leaving for school?

Exercise 41.1.6

1. Write two questions that you could ask the students in your class that would result in categorical data. For each question, explain how you know that responses to it would produce categorical data.
2. Write two questions that you could ask the students in your class that would result in numerical data. For each question, explain how you know that responses to it would produce numerical data.

Exercise 41.1.7

Triangle DEF has vertices $D = (-4, -4), E = (-2, -4),$ and $F = (-3, -1)$.

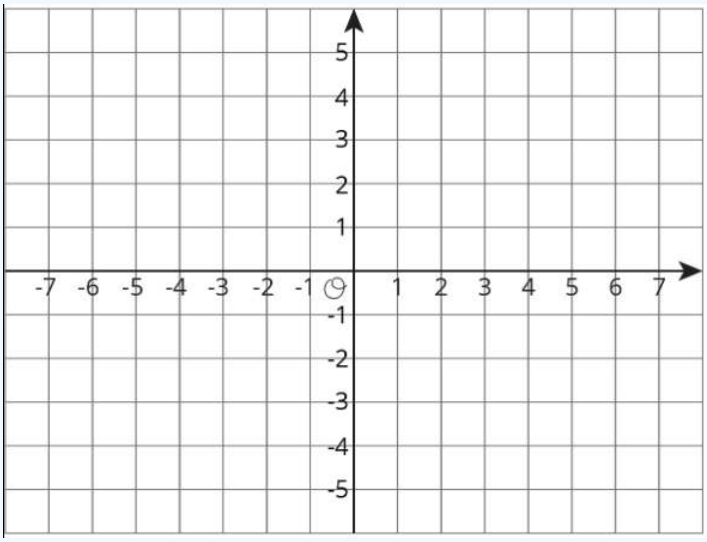


Figure 41.1.5

1. Plot the triangle in the coordinate plane and label the vertices.
2. Name the coordinates of 3 points that are inside the triangle.
3. What is the area of the triangle? Show your reasoning.

(From Unit 7.3.5)

This page titled 41.1: Got Data? is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

41.2: Statistical Questions

Lesson

Let's look more closely at data and the questions they can help to answer.

Exercise 41.2.1: Pencils on A Plot

1. Measure your pencil to the nearest $\frac{1}{4}$-inch. Then, plot your measurement on the class dot plot.
2. What is the difference between the longest and shortest pencil lengths in the class?
3. What is the most common pencil length?
4. Find the difference in lengths between the most common length and the shortest pencil.

Exercise 41.2.2: What's in the Data?

Ten sixth-grade students at a school were each asked five survey questions. Their answers to each question are shown here.

data set A	0	1	1	3	0	0	0	2	1	1
data set B	12	12	12	12	12	12	12	12	12	12
data set C	6	5	7	6	4	5	3	4	6	8
data set D	6	6	6	6	6	6	6	6	6	6
data set E	3	7	9	11	6	4	2	16	6	10

1. Here are the five survey questions. Match each question to a data set that could represent the students' answers. Explain your reasoning.
   - Question 1: Flip a coin 10 times. How many heads did you get?
   - Question 2: How many books did you read in the last year?
   - Question 3: What grade are you in?
   - Question 4: How many dogs and cats do you have?
   - Question 5: How many inches are in 1 foot?
2. How are survey questions 3 and 5 different from the other questions?

Exercise 41.2.3: What Makes a Statistical Question?

These three questions are examples of **statistical questions**:

- What is the most common color of the cars in the school parking lot?
- What percentage of students in the school have a cell phone?
- Which kind of literature—fiction or nonfiction—is more popular among students in the school?

These three questions are not examples of **statistical questions**:

- What color is the principal's car?
- Does Elena have a cell phone?
- What kind of literature—fiction or nonfiction—does Diego prefer?

1. Study the examples and non-examples. Discuss with your partner:
   a. How are the three statistical questions alike? What do they have in common?
   b. How are the three non-statistical questions alike? What do they have in common?
   c. How can you find answers to the statistical questions? How about answers to non-statistical questions?
   d. What makes a question a statistical question?
   Pause here for a class discussion.
2. Read each question. Think about the data you might collect to answer it and whether you expect to see **variability** in the data. Complete each blank with "Yes" or "No."
   a. How many cups of water do my classmates drink each day?
      - Is variability expected in the data? _____

- Is the question statistical? _____

b. Where in town does our math teacher live?
   - Is variability expected in the data? _____
   - Is the question statistical? _____

c. How many minutes does it take students in my class to get ready for school in the morning?
   - Is variability expected in the data? _____
   - Is the question statistical? _____

d. How many minutes of recess do sixth-grade students have each day?
   - Is variability expected in the data? _____
   - Is the question statistical? _____

e. Do all students in my class know what month it is?
   - Is variability expected in the data? _____
   - Is the question statistical? _____

Exercise 41.2.4: Sifting for Statistical Questions

1. Your teacher will give you and your partner a set of cards with questions. Sort them into three piles: Statistical Questions, Not Statistical Questions, and Unsure.
2. Compare your sorting decisions with another group of students. Start by discussing the two piles that your group sorted into the Statistical Questions and Not Statistical Questions piles. Then, review the cards in the Unsure pile. Discuss the questions until both groups reach an agreement and have no cards left in the Unsure pile. If you get stuck, think about whether the question could be answered by collecting data and if there would be variability in that data.
3. Record the letter names of the questions in each pile.
   - Statistical questions:
   - Non-statistical questions:

Are you ready for more?

Tyler and Han are discussing the question, “Which sixth-grade student lives the farthest from school?”

- Tyler says, “I don’t think the question is a statistical question. There is only one person who lives the farthest from school, so there would not be variability in the data we collect.”
- Han says: “I think it is a statistical question. To answer the question, we wouldn’t actually be asking everyone, 'Which student lives the farthest from school?' We would have to ask each student how far away from school they live, and we can expect their responses to have variability.”

Do you agree with either one of them? Explain your reasoning.

Summary

We often collect data to answer questions about something. The data we collect may show **variability**, which means the data values are not all the same.

Some data sets have more variability than others. Here are two sets of figures.

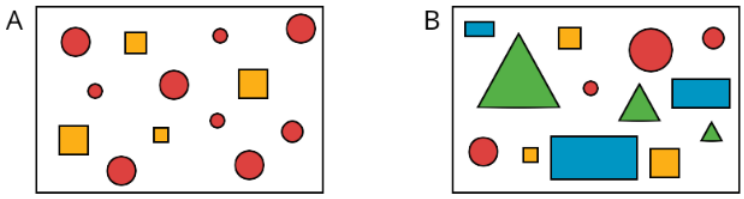


Figure B is a box containing 3 different sized triangles, 3 different sized rectangles, 4 different sized circles, and 3 different sized squares.

Set A has more figures with the same shape, color, or size. Set B shows more figures with different shapes, colors, or sizes, so set B has greater variability than set A.

Both numerical and categorical data can show variability. Numerical sets can contain different numbers, and categorical sets can contain different categories or types.

When a question can only be answered by using data and we expect that data to have variability, we call it a **statistical question**. Here are some examples.

- Who is the most popular musical artist at your school?
- When do students in your class typically eat dinner?
- Which classroom in your school has the most books?

To answer the question about books, we may need to count all of the books in each classroom of a school. The data we collect would likely show variability because we would expect each classroom to have a different number of books.

In contrast, the question "How many books are in your classroom?" is *not* a statistical question. If we collect data to answer the question (for example, by asking everyone in the class to count books), the data can be expected to show the same value. Likewise, if we ask all of the students at a school where they go to school, that question is not a statistical question because the responses will all be the same.

Glossary Entries

Definition: Categorical Data

A set of categorical data has values that are words instead of numbers.

For example, Han asks 5 friends to name their favorite color. Their answers are: blue, blue, green, blue, orange.

Definition: Dot Plot

A dot plot is a way to represent data on a number line. Each time a value appears in the data set, we put another dot above that number on the number line.

For example, in this dot plot there are three dots above the 9. This means that three different plants had a height of 9 cm.

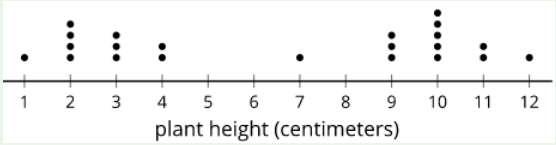


Figure 41.2.2

Definition: Numerical Data

A set of numerical data has values that are numbers.

For example, Han lists the ages of people in his family: 7, 10, 12, 36, 40, 67.

Definition: Statistical Question

A statistical question can be answered by collecting data that has variability. Here are some examples of statistical questions:

- Who is the most popular musical artist at your school?
- When do students in your class typically eat dinner?
- Which classroom in your school has the most books?

Definition: Variability

Variability means having different values.

For example, data set B has more variability than data set A. Data set B has many different values, while data set A has more of the same values.

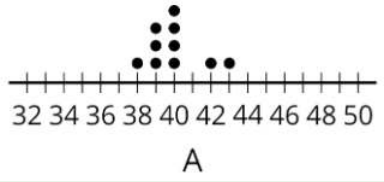


Figure 41.2.3

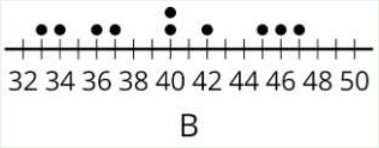


Figure 41.2.4

Practice

Exercise 41.2.5

Sixth-grade students were asked, “What grade are you in?” Explain why this is *not* a statistical question.

Exercise 41.2.6

Lin and her friends went out for ice cream after school. The following questions came up during their trip. Select **all** the questions that are statistical questions.

A. How far are we from the ice cream shop?
B. What is the most popular ice cream flavor this week?
C. What does a group of 4 people typically spend on ice cream at this shop?
D. Do kids usually prefer to get a cup or a cone?
E. How many toppings are there to choose from?

Exercise 41.2.7

Here is a list of questions about the students and teachers at a school. Select **all** the questions that are statistical questions.

A. What is the most popular lunch choice?
B. What school do these students attend?
C. How many math teachers are in the school?
D. What is a common age for the teachers at the school?
E. About how many hours of sleep do students generally get on a school night?
F. How do students usually travel from home to school?

Exercise 41.2.8

Here is a list of statistical questions. What data would you collect and analyze to answer each question? For numerical data, include the unit of measurement that you would use.

1. What is a typical height of female athletes on a team in the most recent international sporting event?
2. Are most adults in the school football fans?
3. How long do drivers generally need to wait at a red light in Washington, DC?

Exercise 41.2.9

Describe the scale you would use on the coordinate plane to plot each set of points. What value would you assign to each unit of the grid?

1. $(1, -6), (-7, -8), (-3, 7), (0, 9)$

2. $(-20, -30), (-40, 10), (20, -10), (5, -20)$
3. $(\frac{-1}{3}, -1), (\frac{2}{3}, -1\frac{1}{3}), (\frac{-4}{3}, \frac{2}{3}), (\frac{1}{6}, 0)$

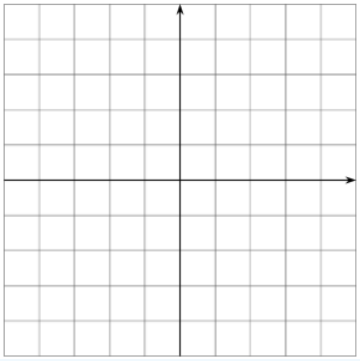


Figure 41.2.5

(From Unit 7.3.3)

Exercise 41.2.10

Noah's water bottle contains more than 1 quart of water but less than $1\frac{1}{2}$ quarts. Let w be the amount of water in Noah's bottle, in quarts. Select **all** the true statements.

A. w could be $\frac{3}{4}$.
B. w could be 1.
C. $w > 1$
D. w could be $\frac{4}{3}$.
E. w could be $\frac{5}{4}$.
F. w could be $\frac{5}{3}$.
G. $w > 1.5$

(From Unit 7.2.2)

Exercise 41.2.11

Order these numbers from least to greatest:

$$|-17| \qquad |-18| \qquad -18 \qquad |19| \qquad 20$$

(From Unit 7.1.7)

41.2: Statistical Questions is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

Section 42: Dot Plots and Histograms

This page titled Section 42: Dot Plots and Histograms is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

42.1: Representing Data Graphically

Lesson

Let's represent data with dot plots and bar graphs.

Exercise 42.1.1: Curious about Caps

Clare collects bottle caps and keeps them in plastic containers.



Figure 42.1.1

Write one statistical question that someone could ask Clare about her collection. Be prepared to explain your reasoning.

Exercise 42.1.2: Estimating Caps

1. Write down the statistical question your class is trying to answer.
2. Look at the dot plot that shows the data from your class. Write down one thing you notice and one thing you wonder about the dot plot.
3. Use the dot plot to answer the statistical question. Be prepared to explain your reasoning.

Exercise 42.1.3: Been There, Done That!

Priya wants to know if basketball players on a men's team and a women's team have had prior experience in international competitions. She gathered data on the number of times the players were on a team before 2016.

men's team

3 0 0 0 0 1 0 0 0 0 0 0

women's team

2 3 3 1 0 2 0 1 1 0 3 1

1. Did Priya collect categorical or numerical data?
2. Organize the information on the two basketball teams into these tables.

Men's Basketball Team Players

Table 42.1.1

number of prior competitions	frequency (number)
0	
1	
2	
3	
4	

Women's Basketball Team Players

Table 42.1.2

number of prior competitions	frequency (number)
0	
1	
2	
3	
4	

3. Make a dot plot for each table.

Men's Basketball Team Players

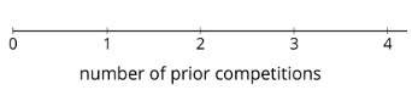


Figure 42.1.2

Women's Basketball Team Players

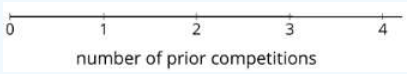


Figure 42.1.3

4. Study your dot plots. What do they tell you about the competition participation of:
   a. the players on the men's basketball team?
   b. the players on the women's basketball team?
5. Explain why a dot plot is an appropriate representation for Priya's data.

Are you ready for more?

Combine the data for the players on the men's and women's teams and represent it as a single dot plot. What can you say about the repeat participation of the basketball players?

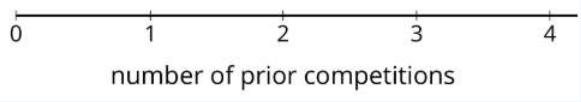


Figure 42.1.4

Exercise 42.1.4: Favorite Summer Sports

Kiran wants to know which three summer sports are most popular in his class. He surveyed his classmates on their favorite summer sport. Here are their responses.

swimming	gymnastics	track and field	volleyball
swimming	swimming	diving	track and field
gymnastics	basketball	basketball	volleyball
track and field	track and field	volleyball	gymnastics
diving	gymnastics	volleyball	rowing
track and field	track and field	soccer	swimming
gymnastics	track and field	swimming	rowing
diving	soccer		

1. Did Kiran collect categorical or numerical data?
2. Organize the responses in a table to help him find which summer sports are most popular in his class.

Table 42.1.3

sport	frequency

3. Represent the information in the table as a bar graph.

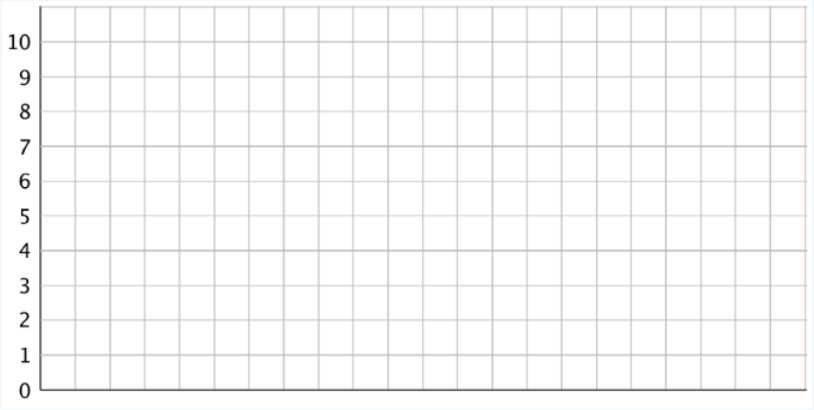


Figure 42.1.5

4. a. How can you use the bar graph to find how many classmates Kiran surveyed?
   b. Which three summer sports are most popular in Kiran's class?
   c. Use your bar graph to describe at least one observation about Kiran's classmates' preferred summer sports.
5. Could a dot plot be used to represent Kiran's data? Explain your reasoning.

Summary

When we analyze data, we are often interested in the **distribution**, which is information that shows all the data values and how often they occur.

In a previous lesson, we saw data about 10 dogs. We can see the distribution of the dog weights in a table such as this one.

Table 42.1.4

weight in kilograms	frequency
6	1
7	3
10	2
32	1
35	2
36	1

The term **frequency** refers to the number of times a data value occurs. In this case, we see that there are three dogs that weigh 7 kilograms, so "3" is the frequency for the value "7 kilograms."

Recall that dot plots are often used to to represent numerical data. Like a frequency table, a dot plot also shows the distribution of a data set. This dot plot, which you saw in an earlier lesson, shows the distribution of dog weights.

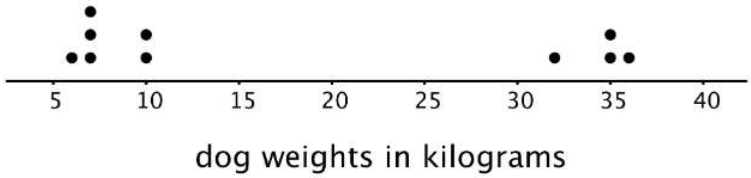


Figure 42.1.6: A dot plot for dog weights in kilograms. The numbers 5 through 40, in increments of 5, are indicated. The data are as follows: 6 kilograms, 1 dot. 7 kilograms, 3 dots. 10 kilograms, 2 dots. 32 kilograms, 1 dot. 35 kilograms, 2 dots. 36 kilograms, 1 dot.

A dot plot uses a horizontal number line. We show the frequency of a value by the number of dots drawn above that value. Here, the two dots above the number 35 tell us that there are two dogs weighing 35 kilograms.

The distribution of categorical data can also be shown in a table. This table shows the distribution of dog breeds.

Table 42.1.5

breed	frequency
pug	9
beagle	9
German shepherd	12

We often represent the distribution of categorical data using a bar graph.

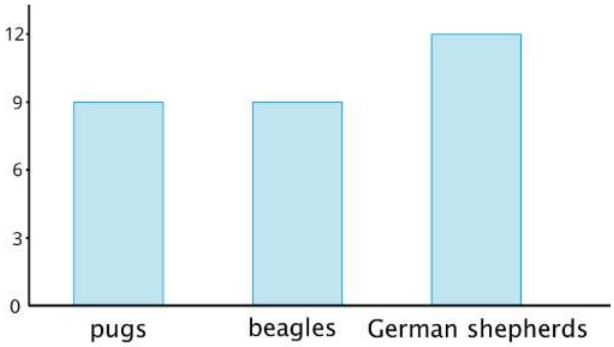


Figure 42.1.7: A bar graph. The categories pugs, beagles, and German shepherds are labeled on the horizontal axis. The numbers 0 through 4 are indicated on the vertical axis. The data represented by the bars are as follows: pugs, 3. beagles, 3. German shepherds, 4.

A bar graph also uses a horizontal line. Above it we draw a rectangle (or “bar”) to represent each category in the data set. The height of a bar tells us the frequency of the category. There are 12 German shepherds in the data set, so the bar for this category is 12 units tall. Below the line we write the labels for the categories.

In a dot plot, a data value is placed according to its position on the number line. A weight of 10 kilograms must be shown as a dot above 10 on the number line.

In a bar graph, however, the categories can be listed in any order. The bar that shows the frequency of pugs can be placed anywhere along the horizontal line.

Glossary Entries

Definition: Distribution

The distribution tells how many times each value occurs in a data set. For example, in the data set blue, blue, green, blue, orange, the distribution is 3 blues, 1 green, and 1 orange.

Here is a dot plot that shows the distribution for the data set 6, 10, 7, 35, 7, 36, 32, 10, 7, 35.

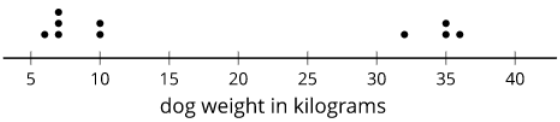


Figure 42.1.8

Definition: Frequency

The frequency of a data value is how many times it occurs in the data set.

For example, there were 20 dogs in a park. The table shows the frequency of each color.

Table 42.1.6

color	frequency
white	4
brown	7
black	3
multi-color	6

Practice

Exercise 42.1.5

A teacher drew a line segment that was 20 inches long on the blackboard. She asked each of her students to estimate the length of the segment and used their estimates to draw this dot plot.

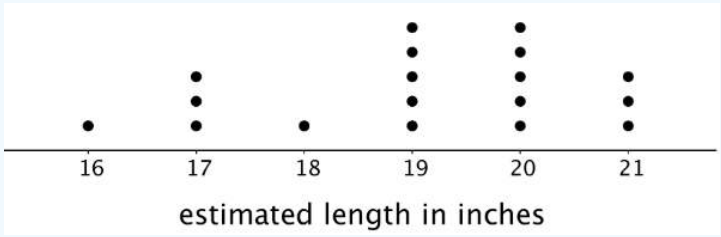


Figure 42.1.9

1. How many students were in the class?
2. Were students generally accurate in their estimates of the length of the line? Explain your reasoning.

Exercise 42.1.6

Here are descriptions of data sets. Select **all** descriptions of data sets that could be graphed as dot plots.

A. Class size for the classes at an elementary school
B. Colors of cars in a parking lot
C. Favorite sport of each student in a sixth-grade class
D. Birth weights for the babies born during October at a hospital
E. Number of goals scored in each of 20 games played by a school soccer team

Exercise 42.1.7

Priya recorded the number of attempts it took each of 12 of her classmates to successfully throw a ball into a basket. Make a dot plot of Priya's data.

1 2 1 3 1 4 4 3 1 2 5 2

Exercise 42.1.8

Solve each equation.

1. $9v = 1$
2. $1.37w = 0$
3. $1 = \frac{7}{10}x$
4. $12.1 = 12.1 + y$
5. $\frac{3}{5} + z = 1$

(From Unit 6.1.4)

Exercise 42.1.9

Find the quotients.

1. $\frac{2}{5} \div 2$
2. $\frac{2}{5} \div 5$
3. $2 \div \frac{2}{5}$
4. $5 \div \frac{2}{5}$

(From Unit 4.3.2)

Exercise 42.1.10

Find the area of each triangle.

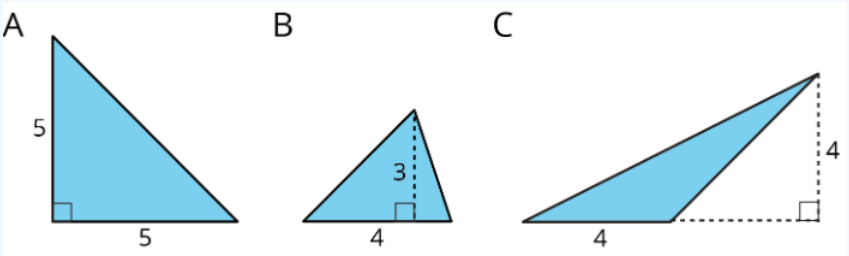


Figure 42.1.10: Three triangles, labeled A, B, C. Triangle A, right, base = 5, height = 5. Triangle B, acute, base = 4, height = 3. Triangle C, obtuse, base 4, height= 4.

(From Unit 1.3.3)

42.1: Representing Data Graphically is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

42.2: Dot Plots

Lesson

Let's investigate what dot plots and bar graphs can tell us.

Exercise 42.2.1: Pizza Toppings (Part 1)

Fifteen customers in a pizza shop were asked, “How many toppings did you add to your cheese pizza?” Here are their responses:

1 2 1 3 0 1 1 2 0 3 0 0 1 2 2

1. Could you use a dot plot to represent the data? Explain your reasoning.
2. Complete the table.

Table 42.2.1

number of toppings	frequency (number)
0	
1	
2	
3	

Exercise 42.2.2: Pizza Toppings (Part 2)

1. Use the tables from the warm-up to display the number of toppings as a dot plot. Label your drawing clearly.

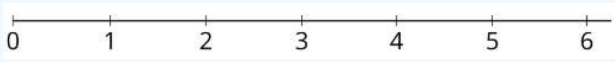


Figure 42.2.1

2. Use your dot plot to study the distribution for number of toppings. What do you notice about the number of toppings that this group of customers ordered? Write 2–3 sentences summarizing your observations.

Are you ready for more?

Think of a statistical question that can be answered with the data about the number of toppings ordered, as displayed on the dot plot. Then answer this question.

Exercise 42.2.3: Homework Time

Twenty-five sixth-grade students answered the question: “How many hours do you generally spend on homework each week?”

This dot plot shows the number of hours per week that these 25 students reported spending on homework.



Figure 42.2.2: Studying, by English106. CC BY 2.0. Flickr. Source.

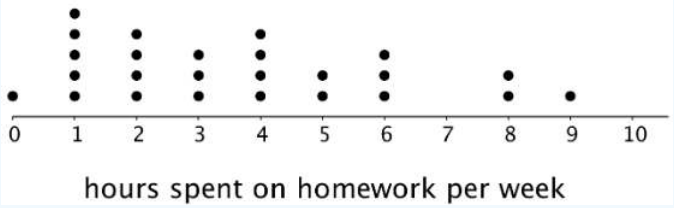


Figure 42.2.3

Use the dot plot to answer the following questions. For each, show or explain your reasoning.

1. What percentage of the students reported spending 1 hour on homework each week?
2. What percentage of the students reported spending 4 or fewer hours on homework each week?
3. Would 6 hours per week be a good description of the number of hours this group of students spends on homework per week? What about 1 hour per week? Explain your reasoning.
4. What value do you think would be a good description of the homework time of the students in this group? Explain your reasoning.
5. Someone said, "In general, these students spend roughly the same number of hours doing homework." Do you agree? Explain your reasoning.

Summary

We often collect and analyze data because we are interested in learning what is "typical," or what is common and can be expected in a group.

Sometimes it is easy to tell what a typical member of the group is. For example, we can say that a typical shape in this set is a large circle.

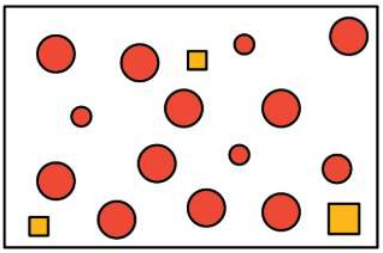


Figure 42.2.4

Just looking at the members of a group doesn't always tell us what is typical, however. For example, if we are interested in the side length typical of squares in this set, it isn't easy to do so just by studying the set visually.

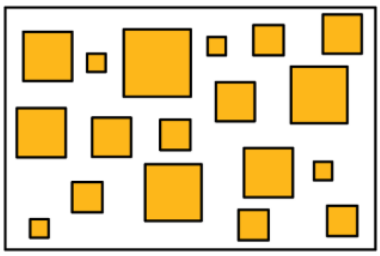


Figure 42.2.5

In a situation like this, it is helpful to gather the side lengths of the squares in the set and look at their distribution, as shown in this dot plot.

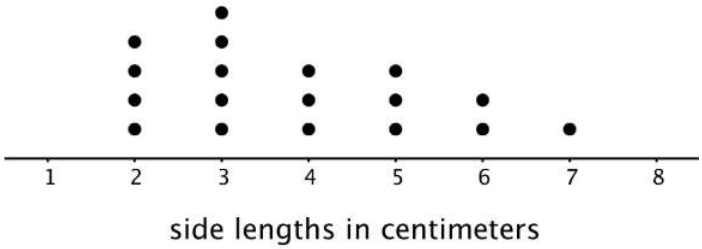


Figure 42.2.6: A dot plot for side lengths in centimeters. The numbers 1 through 8 are indicated. The data are as follows: 2 centimeters, 4 dots. 3 centimeters, 5 dots. 4 centimeters, 3 dots. 5 centimeters, 3 dots. 6 centimeters, 2 dots. 7 centimeters, 1 dot.

We can see that many of the data points are between 2 and 4, so we could say that side lengths between 2 and 4 centimeters or close to these lengths are typical of squares in this set.

Glossary Entries

Definition: Distribution

The distribution tells how many times each value occurs in a data set. For example, in the data set blue, blue, green, blue, orange, the distribution is 3 blues, 1 green, and 1 orange.

Here is a dot plot that shows the distribution for the data set 6, 10, 7, 35, 7, 36, 32, 10, 7, 35.

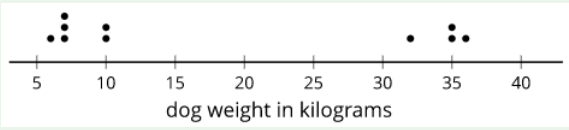


Figure 42.2.7

Definition: Frequency

The frequency of a data value is how many times it occurs in the data set.

For example, there were 20 dogs in a park. The table shows the frequency of each color.

Table 42.2.2

color	frequency
white	4
brown	7
black	3
multi-color	6

Practice

Exercise 42.2.4

Clare recorded the amounts of time spent doing homework, in hours per week, by students in sixth, eighth, and tenth grades. She made a dot plot of the data for each grade and provided the following summary.

- Students in sixth grade tend to spend less time on homework than students in eighth and tenth grades.
- The homework times for the tenth-grade students are more alike than the homework times for the eighth-grade students.

Use Clare's summary to match each dot plot to the correct grade (sixth, eighth, or tenth).

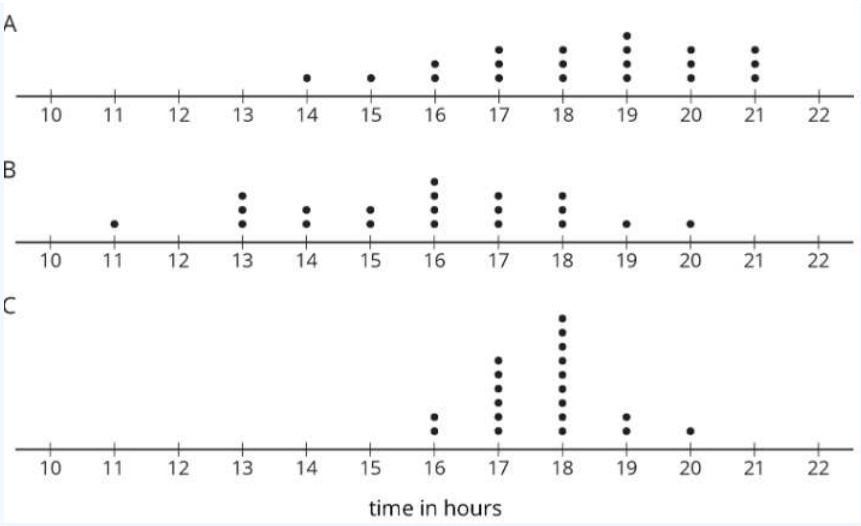


Figure 42.2.8: Three dot plots, time in hours, 10 to 22 by ones. Dot plot A, starting at 14, the number of dots above each increment is 1, 1, 2, 3, 3, 4, 3, 3. Dot plot B, starting at 11, the number of dots above each increment is 1, 0, 3, 2, 2, 4, 3, 3, 1, 1. Dot plot C, starting with 16, the number of dots above each increment is 2, 6, 9, 2, 1.

Exercise 42.2.5

Mai played 10 basketball games. She recorded the number of points she scored and made a dot plot. Mai said that she scored between 8 and 14 points in most of the 10 games, but one game was exceptional. During that game she scored more than double her typical score of 9 points. Use the number line to make a dot plot that fits the description Mai gave.

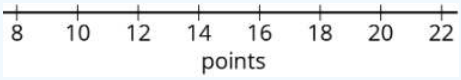


Figure 42.2.9

Exercise 42.2.6

A movie theater is showing three different movies. The dot plots represent the ages of the people who were at the Saturday afternoon showing of each of these movies.

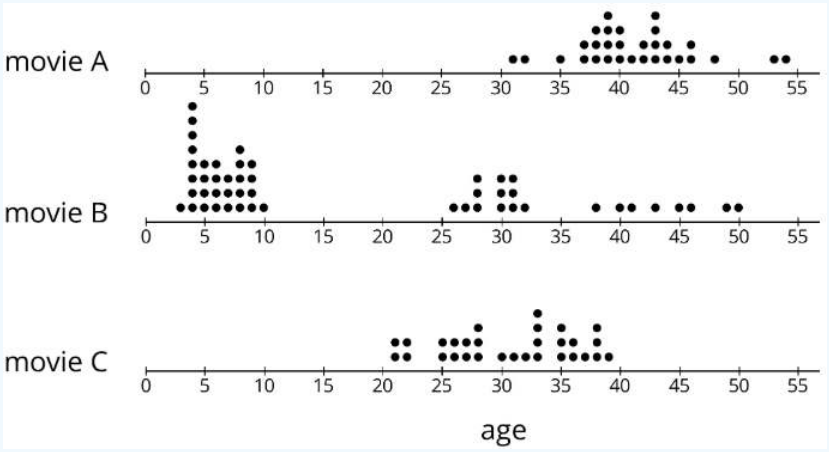


Figure 42.2.10: Three dot plots from 0 to 55 by 1's. Age. Plots labeled movie A, movie B and movie C. Movie A, from left to right , beginning at 31, number of dots above each increment is: 1, 1 , 0, 0, 1, 0, 2, 3, 4, 3, 1, 2, 4, 2, 1, 2, 0, 1, 0, 0, 0, 0, 0, 1,1, 0. last dot above 54. Movie B, from left to right , beginning at 3, number of dots above each increment is: 1, 8, 4, 4 , 3, 5, 4, 1. next dot is at 26, from 26, number of dots above each increment is: 1, 1, 3, 0, 3, 3, 1, 0, 0, 0, 0, 1, 0, 1, 1, 0, 1, 0,1, 1, 0, 0, 1, 1. last dot above 50. Movie C, from left to right , beginning at 21, number of dots above each increment is: 2, 2, 0, 0, 2,2,2,3,0,1,1,1,4,0,3,2,1,3,1. last dot above 39.

1. One of these movies was an animated movie rated G for general audiences. Do you think it was movie A, B, or C? Explain your reasoning.

2. Which movie has a dot plot with ages that that center at about 30 years?
3. What is a typical age for the people who were at Movie A?

Exercise 42.2.7

Find the value of each expression.

1. $3.727 + 1.384$
2. $3.727 - 1.384$
3. $5.01 \cdot 4.8$
4. $5.01 \div 4.8$

(From Unit 5.4.5)

42.2: Dot Plots is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

42.3: Using Dot Plots to Answer Statistical Questions

Lesson

Let's use dot plots to describe distributions and answer questions.

Exercise 42.3.1: Packs on Backs

This dot plot shows the weights of backpacks, in kilograms, of 50 sixth-grade students at a school in New Zealand.

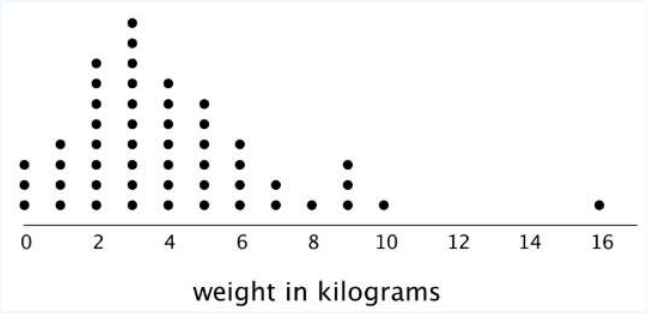


Figure 42.3.1

1. The dot plot shows several dots at 0 kilograms. What could a value of 0 mean in this context?
2. Clare and Tyler studied the dot plot.
   - Clare said, “I think we can use 3 kilograms to describe a typical backpack weight of the group because it represents 20%—or the largest portion—of the data.”
   - Tyler disagreed and said, “I think 3 kilograms is too low to describe a typical weight. Half of the dots are for backpacks that are heavier than 3 kilograms, so I would use a larger value.”

Do you agree with either of them? Explain your reasoning.

Exercise 42.3.2: On the Phone

Twenty-five sixth-grade students were asked to estimate how many hours a week they spend talking on the phone. This dot plot represents their reported number of hours of phone usage per week.

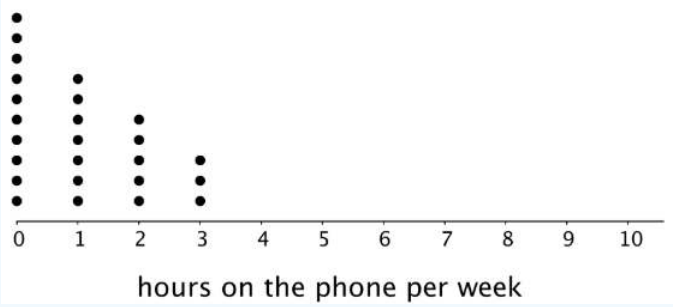


Figure 42.3.2

1. a. How many of the students reported not talking on the phone during the week? Explain how you know.
   b. What percentage of the students reported not talking on the phone?
2. a. What is the largest number of hours a student spent talking on the phone per week?
   b. What percentage of the group reported talking on the phone for this amount of time?
3. a. How many hours would you say that these students typically spend talking on the phone?
   b. How many minutes per day would that be?
4. a. How would you describe the **spread** of the data? Would you consider these students’ amounts of time on the phone to be alike or different? Explain your reasoning.
   b. Here is the dot plot from an earlier activity. It shows the number of hours per week the same group of 25 sixth-grade students reported spending on homework.

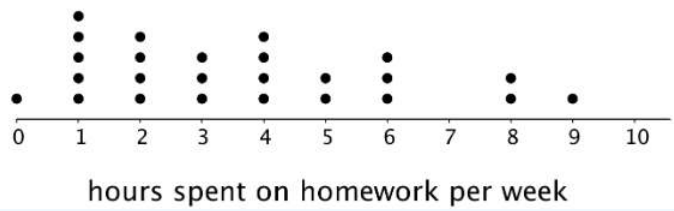


Figure 42.3.3

Overall, are these students more alike in the amount of time they spend talking on the phone or in the amount of time they spend on homework? Explain your reasoning.

5. Suppose someone claimed that these sixth-grade students spend too much time on the phone. Do you agree? Use your analysis of the dot plot to support your answer.

Exercise 42.3.3: Click-Clack

1. A keyboarding teacher wondered: "Do typing speeds of students improve after taking a keyboarding course?" Explain why her question is a statistical question.
2. The teacher recorded the number of words that her students could type per minute at the beginning of a course and again at the end. The two dot plots show the two data sets.

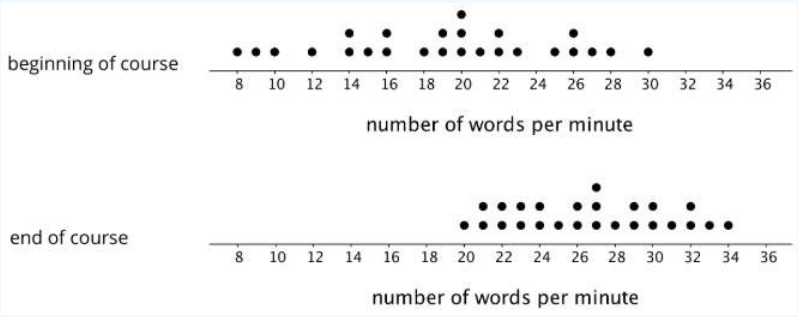


Figure 42.3.4: Two dot plots. Both dot plots, 8 to 36 by 1's. Number of words per minute. Top dot plot, beginning of course. Beginning at 8, number of dots above each increment is 1, 1, 1, 0, 1, 0, 2, 1, 2, 0, 1, 2, 3, 1, 2, 1, 0, 1, 2, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0. Bottom dot plot, end of course. Beginning at 8, number of dots above each increment is 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 2, 2, 2, 2, 1, 2, 3, 1, 2, 2, 1, 2, 1, 1, 0, 0.

Based on the dot plots, do you agree with each of the following statements about this group of students? Be prepared to explain your reasoning.

   a. Overall, the students' typing speed did not improve. They typed at the same speed at the end of the course as they did at the beginning.
   b. 20 words per minute is a good estimate for how fast, in general, the students typed at the beginning of the course.
   c. 20 words per minute is a good description of the **center** of the data set at the end of the course.
   d. There was more variability in the typing speeds at the beginning of the course than at the end, so the students' typing speeds were more alike at the end.

3. Overall, how fast would you say that the students typed after completing the course? What would you consider the center of the end-of-course data?

Are you ready for more?

Use one of these suggestions (or make up your own). Research to create a dot plot with at least 10 values. Then, describe the center and spread of the distribution.

- Points scored by your favorite sports team in its last 10 games
- Length of your 10 favorite movies (in minutes)
- Ages of your favorite 10 celebrities

Summary

One way to describe what is typical or characteristic for a data set is by looking at the **center** and **spread** of its distribution.

Let's compare the distribution of cat weights and dog weights shown on these dot plots.

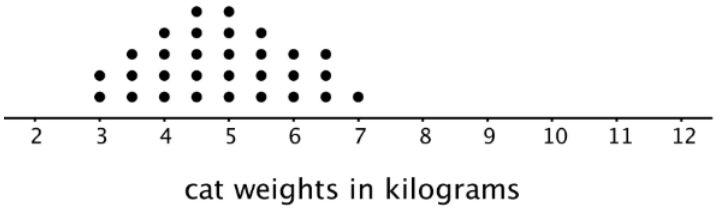


Figure 42.3.5: A dot plot for cat weights in kilograms. The numbers 2 through 12 are indicated. The data are as follows: 3 kilograms, 2 dots. 3.5 kilograms, 3 dots. 4 kilograms, 4 dots. 4.5 kilograms, 5 dots. 5 kilograms, 5 dots. 5.5 kilograms, 4 dots. 6 kilograms, 3 dots. 6.5 kilograms, 3 dots. 7 kilograms, 1 dot.

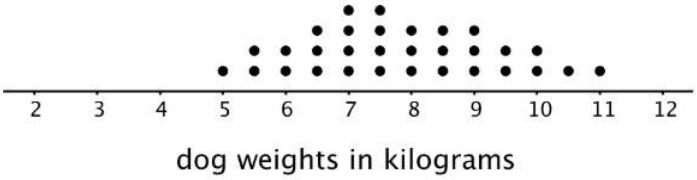


Figure 42.3.6: A dot plot for dog weights in kilograms. The numbers 2 through 12 are indicated. The data are as follows: 5 kilograms, 1 dot. 5.5 kilograms, 2 dots. 6 kilograms, 2 dots. 6.5 kilograms, 3 dots. 7 kilograms, 4 dots. 7.5 kilograms, 4 dots. 8 kilograms, 3 dots. 8.5 kilograms, 3 dots. 9 kilograms, 3 dots. 9.5 kilograms, 2 dots. 10 kilograms, 2 dots. 10.5 kilograms, 1 dot. 11 kilograms, 1 dot.

The collection of points for the cat data is further to the left on the number line than the dog data. Based on the dot plots, we may describe the center of the distribution for cat weights to be between 4 and 5 kilograms and the center for dog weights to be between 7 and 8 kilograms.

We often say that values at or near the center of a distribution are typical for that group. This means that a weight of 4–5 kilograms is typical for a cat in the data set, and weight of 7–8 kilograms is typical for a dog.

We also see that the dog weights are more spread out than the cat weights. The difference between the heaviest and lightest cats is only 4 kilograms, but the difference between the heaviest and lightest dogs is 6 kilograms.

A distribution with greater spread tells us that the data have greater variability. In this case, we could say that the cats are more similar in their weights than the dogs.

In future lessons, we will discuss how to measure the center and spread of a distribution.

Glossary Entries

Definition: Center

The center of a set of numerical data is a value in the middle of the distribution. It represents a typical value for the data set.

For example, the center of this distribution of cat weights is between 4.5 and 5 kilograms.

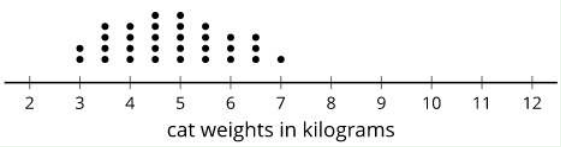


Figure 42.3.7

Definition: Distribution

The distribution tells how many times each value occurs in a data set. For example, in the data set blue, blue, green, blue, orange, the distribution is 3 blues, 1 green, and 1 orange.

Here is a dot plot that shows the distribution for the data set 6, 10, 7, 35, 7, 36, 32, 10, 7, 35.

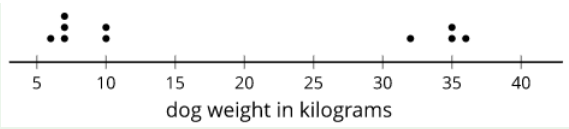


Figure 42.3.8

Definition: Frequency

The frequency of a data value is how many times it occurs in the data set.

For example, there were 20 dogs in a park. The table shows the frequency of each color.

Table 42.3.1

color	frequency
white	4
brown	7
black	3
multi-color	6

Definition: Spread

The spread of a set of numerical data tells how far apart the values are.

For example, the dot plots show that the travel times for students in South Africa are more spread out than for New Zealand.

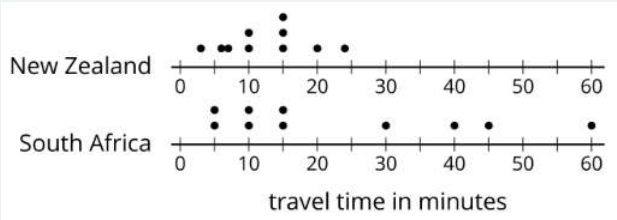


Figure 42.3.9

Practice

Exercise 42.3.4

Three sets of data about ten sixth-grade students were used to make three dot plots. The person who made these dot plots forgot to label them. Match each dot plot with the appropriate label.

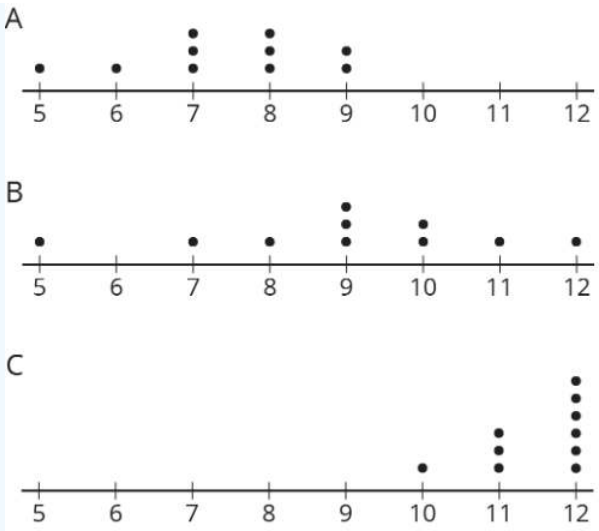


Figure 42.3.10: Three dot plots are labeled A, B, and C. All dot plots have the numbers 5 through 12 indicated. Dot plot A has the following data: 5, 1 dot 6, 1 dot 7, 3 dots 8, 3 dots 9, 2 dots 10, 0 dots 11, 0 dots 12, 0 dots. Dot plot B has the following data: 5, 1 dot 6, 0 dots 7, 1 dot 8, 1 dot 9, 3 dots 10, 2 dots 11, 1 dot 12, 1 dot. Dot plot C has the following data: 5, 0 dots 6, 0 dots 7, 0 dots 8, 0 dots 9, 0 dots 10, 1 dot 11, 3 dots 12, 6 dots

A. Dot Plot A
B. Dot Plot B
C. Dot Plot C

1. Ages in years
2. Number of hours of sleep on nights before school days
3. Numbers of hours of sleep on nights before non-school days

Exercise 42.3.5

The dot plots show the time it takes to get to school for ten sixth-grade students from the United States, Canada, Australia, New Zealand, and South Africa.

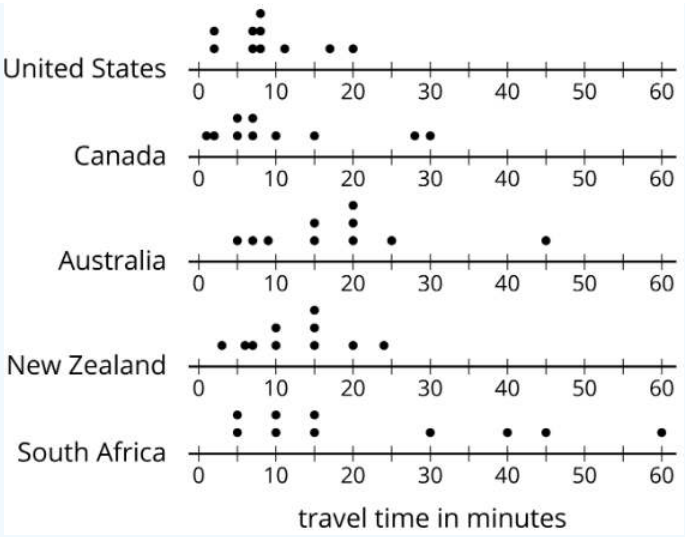


Figure 42.3.11: Five dot plots for travel time in minutes labeled United States, Canada, Australia, New Zealand, and South Africa. Each dot plot has the numbers 0 through 60, in increments of 10. There are also tick marks midway between. The approximate data for United States are as follows: 2 minutes, 2 dots; 7 minutes, 2 dots; 8 minutes, 3 dots; 11 minutes, 1 dot; 17 minutes, 1 dot; 20 minutes, 1 dot. The approximate data for Canada are as follows: 1 minute, 1 dot; 2 minutes, 1 dot; 5 minutes, 2 dots; 7 minutes, 2 dots; 10 minutes, 1 dot; 15 minutes, 1 dot; 28 minutes, 1 dot; 30 minutes, 1 dot. The approximate data for Australia are as follows: 5 minutes, 1 dot; 7 minutes, 1 dot; 9 minutes, 1 dot; 15 minutes, 2 dots; 20 minutes, 3 dots; 25 minutes, 1 dot; 45 minutes, 1 dot. The approximate data for New Zealand are as follows: 3 minutes, 1 dot; 6 minutes, 1 dot; 7 minutes, 1 dot; 10 minutes, 2 dots; 15 minutes, 3 dots; 20 minutes, 1 dot; 24 minutes, 1 dot. The approximate data for South Africa are as follows: 5 minutes, 2 dots; 10 minutes, 2 dots; 15 minutes, 2 dots; 30 minutes, 1 dot; 40 minutes, 1 dot; 45 minutes, 1 dot; 60 minutes, 1 dot.

1. List the countries in order of *typical travel times*, from shortest to longest.
2. List the countries in order of *variability in travel times*, from the least variability to the greatest.

Exercise 42.3.6

Twenty-five students were asked to rate—on a scale of 0 to 10—how important it is to reduce pollution. A rating of 0 means "not at all important" and a rating of 10 means "very important." Here is a dot plot of their responses.

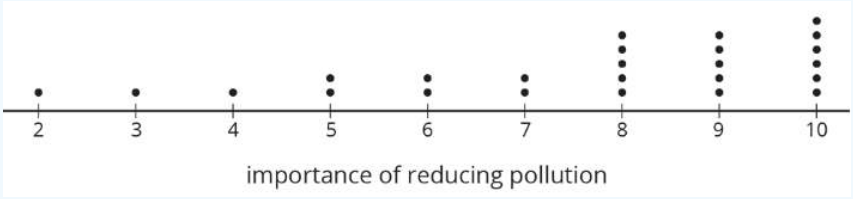


Figure 42.3.12

Explain why a rating of 6 is *not* a good description of the center of this data set.

Exercise 42.3.7

Tyler wants to buy some cherries at the farmer's market. He has $10 and cherries cost $4 per pound.

1. If c is the number of pounds of cherries that Tyler can buy, write one or more inequalities or equations describing c.
2. Can 2 be a value of c? Can 3 be a value of c? What about -1? Explain your reasoning.
3. If m is the amount of money, in dollars, Tyler can spend, write one or more inequalities or equations describing m.
4. Can 8 be a value of m? Can 2 be a value of m? What about 10.5? Explain your reasoning.

(From Unit 7.2.3)

42.3: Using Dot Plots to Answer Statistical Questions is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

42.4: Interpreting Histograms

Lesson

Let's explore how histograms represent data sets.

Exercise 42.4.1: Dog Show (Part 1)

Here is a dot plot showing the weights, in pounds, of 40 dogs at a dog show.

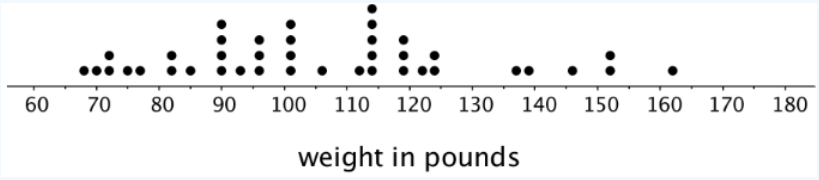


Figure 42.4.1: A dot plot, weight in pounds, labeled 60 to 180 by tens. Starting at 68 with intervals of 2, the number of dots above each increment is 1, 1, 2, 0, 1, 1, 0, 2, 1, 0, 0, 4, 1, 3, 0, 0, 4, 0, 0, 1, 0, 0, 1, 5, 0, 3, 0, 1, 2, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 1, 0, 0, 2, 0, 0, 0, 0, 1.

1. Write two statistical questions that can be answered using the dot plot.
2. What would you consider a typical weight for a dog at this dog show? Explain your reasoning.

Exercise 42.4.2: Dog Show (Part 2)

Here is a **histogram** that shows some dog weights in pounds.

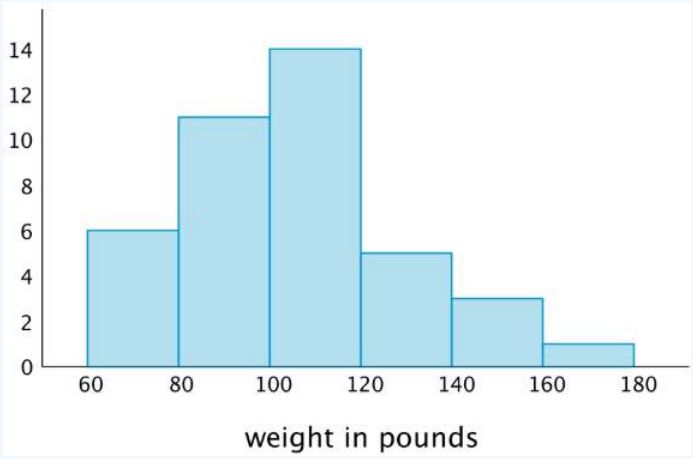


Figure 42.4.2

Each bar includes the left-end value but not the right-end value. For example, the first bar includes dogs that weigh 60 pounds and 68 pounds but not 80 pounds.

1. Use the histogram to answer the following questions.
   a. How many dogs weigh at least 100 pounds?
   b. How many dogs weigh exactly 70 pounds?
   c. How many dogs weigh at least 120 and less than 160 pounds?
   d. How much does the heaviest dog at the show weigh?
   e. What would you consider a typical weight for a dog at this dog show? Explain your reasoning.
2. Discuss with a partner:
   - If you used the dot plot to answer the same five questions you just answered, how would your answers be different?
   - How are the histogram and the dot plot alike? How are they different?

Exercise 42.4.3: Population of States

Every ten years, the United States conducts a census, which is an effort to count the entire population. The dot plot shows the population data from the 2010 census for each of the fifty states and the District of Columbia (DC).

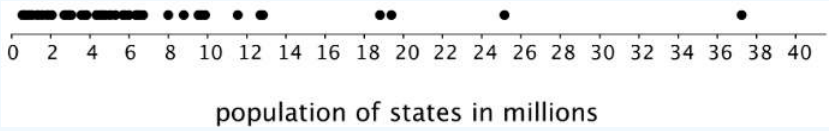


Figure 42.4.3: A dot plot, population of states in millions, labeled 0 to 40 by twos. A cluster of approximately 30 dots have a population between point six and six point 4. There is one dot above each of the tick marks for 8, 8 point 7, 9, 9 point five, 11, 12, 12 point 1, 18 point 8, 19 point 2, 25, 37 point 2.

1. Here are some statistical questions about the population of the fifty states and DC. How difficult would it be to answer the questions using the *dot plot*?
   In the middle column, rate each question with an E (easy to answer), H (hard to answer), or I (impossible to answer). Be prepared to explain your reasoning.

Table 42.4.1

statistical question	using the dot plot	using the histogram
a. How many states have populations greater than 15 million?		
b. Which states have populations greater than 15 million?		
c. How many states have populations less than 5 million?		
d. What is a typical state population?		
e. Are there more states with fewer than 5 million people or more states with between 5 and 10 million people?		
f. How would you describe the distribution of state populations?		

2. Here are the population data for all states and the District of Columbia from the 2010 census. Use the information to complete the table.

Alabama	4.78	Illinois	12.83	Montana	0.99	Rhode Island	1.05
Alaska	0.71	Indiana	6.48	Nebraska	1.83	South Carolina	4.63
Arizona	6.39	Iowa	3.05	Nevada	2.70	South Dakota	0.81
Arkansas	2.92	Kansas	2.85	New Hampshire	1.32	Tennessee	6.35
California	37.25	Kentucky	4.34	New Jersey	8.79	Texas	25.15
Colorado	5.03	Louisiana	4.53	New Mexico	2.06	Utah	2.76
Connecticut	3.57	Maine	1.33	New York	19.38	Vermont	0.63
Delaware	0.90	Maryland	5.77	North Carolina	9.54	Virginia	8.00
District of Columbia	0.60	Massachusetts	6.55	North Dakota	0.67	Washington	6.72
Florida	18.80	Michigan	9.88	Ohio	11.54	West Virginia	1.85
Georgia	9.69	Minnesota	5.30	Oklahoma	3.75	Wisconsin	5.69
Hawaii	1.36	Mississippi	2.97	Oregon	3.83	Wyoming	0.56
Idaho	1.57	Missouri	5.99	Pennsylvania	12.70		

Figure 42.4.4: A table of population of states in millions. Alabama, 4 point 7 8, Alaska, zero point 7 1, Arizona, 6 point 3 9, Arkansas, 2 point 9 2, California, 37 point 2 5, Colorado, 5 point 0 3, Connecticut, 3 point 5 7, Delaware, 0 point 9 0, District of Columbia, 0 point 6 0, Florida, 18 point 8 0, Georgia, 9 point 6 9, Hawaii, 1 point 3 6, Idaho, 1 point 5 7, Illinois, 12 point 8 3, Indiana, 6 point 4 8, Iowa, 3 point 0 5, Kansas, 2 point 8 5, Kentucky, 4 point 3 4, Louisiana, 4 point 5 3, Maine, 1 point 3 3, Maryland, 5 point 7 7, Massachusetts, 6 point 5 5, Michigan, 9 point 8 8, Minnesota, 5 point 3 0, Mississippi, 2 point 9 7, Missouri, 5 point 9 9, Montana, 0 point 9 9, Nebraska, 1 point 8 3, Nevada, 2 point 7 0, New Hampshire, 1 point 3 2, New Jersey, 8 point 7 9, New Mexico, 2 point 0 6, New York, 19 point 3 8, North Carolina, 9 point 5 4. North Dakota, 0 point 6 7, Ohio, 11 point 5 4, Oklahoma, 3 point 7 5, Oregon, 3 point 8 3, Pennsylvania, 12 point 7 0, Rhode Island, 1 point 0 5, South Carolina, 4 point 6 3, South Dakota, 0 point 8 1, Tennessee, 6 point 3 5, Texas, 25 point 1 5, Utah, 2 point 7 6, Vermont, 0 point 6 3, Virginia, 8 point 0 0, Washington, 6 point 7 2, West Virginia, 1 point 8 5, Wisconsin 5 point 6 9, Wyoming, 0 point 5 6.

Table 42.4.2

population (millions)	frequency
0-5	
5-10	
10-15	
15-20	
20-25	
25-30	
30-35	
35-40	

3. Use the grid and the information in your table to create a histogram.

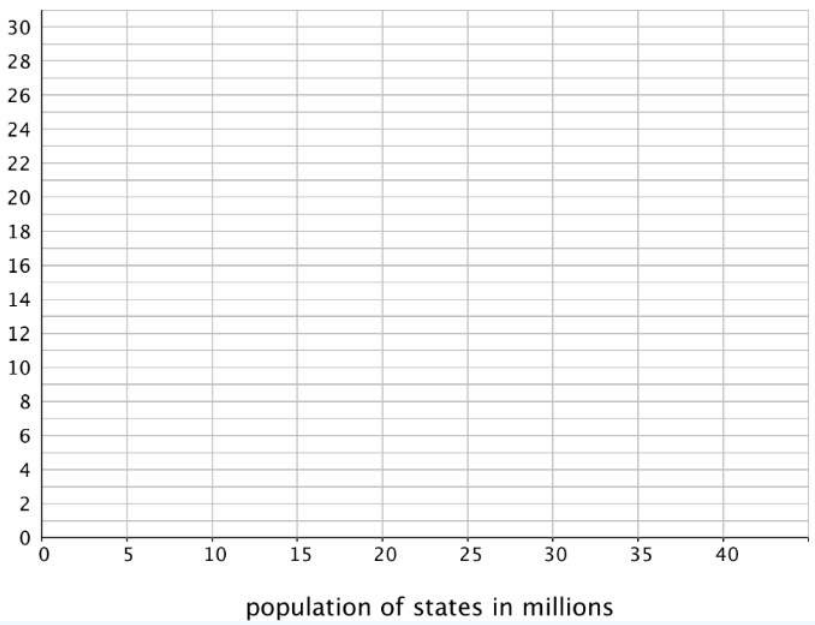


Figure 42.4.5

4. Return to the statistical questions at the beginning of the activity. Which ones are now easier to answer?
   In the last column of the table, rate each question with an E (easy), H (hard), and I (impossible) based on how difficult it is to answer them. Be prepared to explain your reasoning.

Are you ready for more?

Think of two more statistical questions that can be answered using the data about populations of states. Then, decide whether each question can be answered using the dot plot, the histogram, or both.

Summary

In addition to using dot plots, we can also represent distributions of numerical data using **histograms**.

Here is a dot plot that shows the weights, in kilograms, of 30 dogs, followed by a histogram that shows the same distribution.

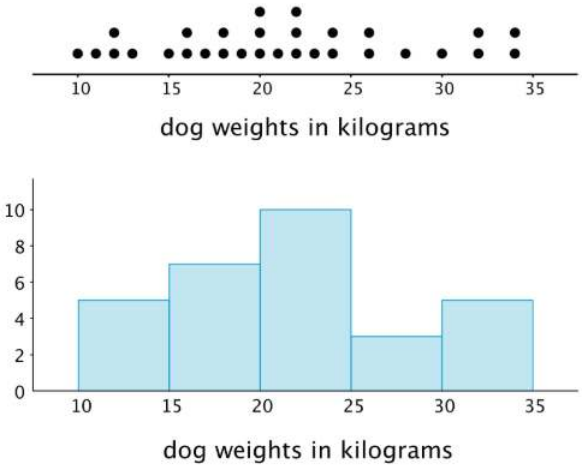


Figure 42.4.6: A dotplot and histogram for dog weights in kilograms. For the dot plot, the numbers 10 through 35, in increments of 5, are indicated. The 30 data values are as follows: 10 kilograms, 1 dot. 11 kilograms, 1 dot. 12 kilograms, 2 dots. 13 kilograms, 1 dot. 15 kilograms, 1 dot. 16 kilograms, 2 dots. 17 kilograms, 1 dot. 18 kilograms, 2 dots. 19 kilograms, 1 dot. 20 kilograms, 3 dots. 21 kilograms, 1 dot. 22 kilograms, 3 dots. 23 kilograms, 1 dot. 24 kilograms, 2 dots. 26 kilograms, 2 dots. 28 kilograms, 1 dot. 30 kilograms, 1 dot. 32 kilograms, 2 dots. 34 kilograms, 2 dots. For the histogram, the horizontal axis is labeled dog weights in kilograms and the numbers 10 through 35, in increments of 5, are indicated. On the vertical axis the numbers 0 through 10, in increments of 2, are indicated. The data represented by the bars are as follows: Weight from 10 up to 15, 5. Weight from 15 up to 20, 7. Weight from 20 up to 25, 10. Weight from 25 up to 30, 3. Weight from 30 up to 35, 5.

In a histogram, data values are placed in groups or “bins” of a certain size, and each group is represented with a bar. The height of the bar tells us the frequency for that group.

For example, the height of the tallest bar is 10, and the bar represents weights from 20 to less than 25 kilograms, so there are 10 dogs whose weights fall in that group. Similarly, there are 3 dogs that weigh anywhere from 25 to less than 30 kilograms.

Notice that the histogram and the dot plot have a similar shape. The dot plot has the advantage of showing all of the data values, but the histogram is easier to draw and to interpret when there are a lot of values or when the values are all different.

Here is a dot plot showing the weight distribution of 40 dogs. The weights were measured to the nearest 0.1 kilogram instead of the nearest kilogram.

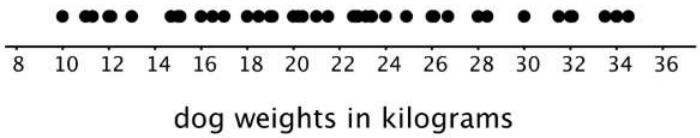


Figure 42.4.7: A dot plot for dog weights in kilograms. The numbers 8 through 36, in increments of 2, are indicated. There is 1 dot on each of the following values: 10 kilograms,, 11 kilograms, 11.3 kilograms, 12 kilograms, 12.1 kilograms, 13 kilograms, 14.7 kilograms, 15 kilograms, 15.1 kilograms, 16 kilograms, 16.5 kilograms, 17 kilograms, 18 kilograms, 18.5 kilograms, 19 kilograms, 19.1 kilograms, 20 kilograms, 20.2 kilograms, 20.4 kilograms, 21 kilograms, 21.5 kilograms, 22.6 kilograms, 22.7 kilograms, 22.8 kilograms, 23.2 kilograms, 23.4 kilograms, 24 kilograms, 24.9 kilograms, 26 kilograms, 26.1 kilograms, 26.7 kilograms, 28 kilograms, 28.4 kilograms, 20 kilograms, 31.5 kilograms, 32 kilograms, 32.1 kilograms, 33.5 kilograms, 34 kilograms, 34.4 kilograms.

Here is a histogram showing the same distribution.

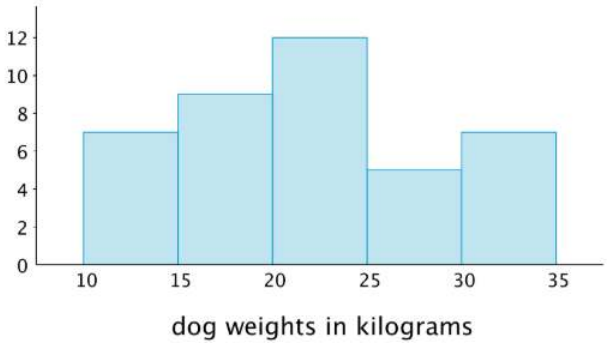


Figure 42.4.8

In this case, it is difficult to make sense of the distribution from the dot plot because the dots are so close together and all in one line. The histogram of the same data set does a much better job showing the distribution of weights, even though we can’t see the individual data values.

Glossary Entries

Definition: Center

The center of a set of numerical data is a value in the middle of the distribution. It represents a typical value for the data set.

For example, the center of this distribution of cat weights is between 4.5 and 5 kilograms.

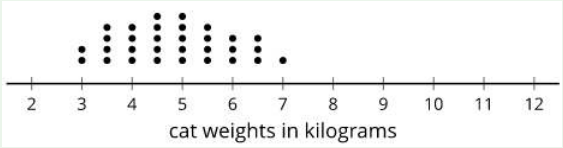


Figure 42.4.9

Definition: Distribution

The distribution tells how many times each value occurs in a data set. For example, in the data set blue, blue, green, blue, orange, the distribution is 3 blues, 1 green, and 1 orange.

Here is a dot plot that shows the distribution for the data set 6, 10, 7, 35, 7, 36, 32, 10, 7, 35.

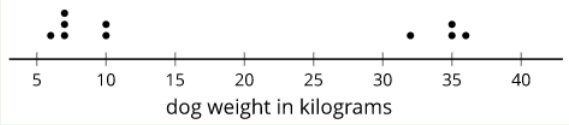


Figure 42.4.10

Definition: Frequency

The frequency of a data value is how many times it occurs in the data set.

For example, there were 20 dogs in a park. The table shows the frequency of each color.

Table 42.4.3

color	frequency
white	4
brown	7
black	3
multi-color	6

Definition: Histogram

A histogram is a way to represent data on a number line. Data values are grouped by ranges. The height of the bar shows how many data values are in that group.

This histogram shows there were 10 people who earned 2 or 3 tickets. We can't tell how many of them earned 2 tickets or how many earned 3. Each bar includes the left-end value but not the right-end value. (There were 5 people who earned 0 or 1 tickets and 13 people who earned 6 or 7 tickets.)

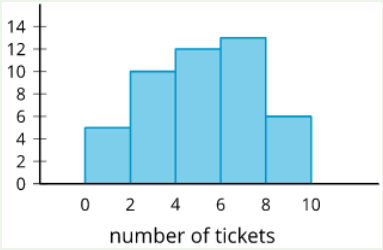


Figure 42.4.11

Definition: Spread

The spread of a set of numerical data tells how far apart the values are.

For example, the dot plots show that the travel times for students in South Africa are more spread out than for New Zealand.

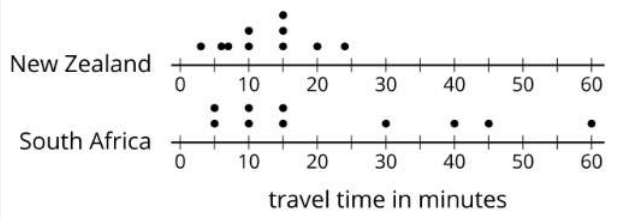


Figure 42.4.12

Practice

Exercise 42.4.4

Match histograms A through E to dot plots 1 through 5 so that each match represents the same data set.

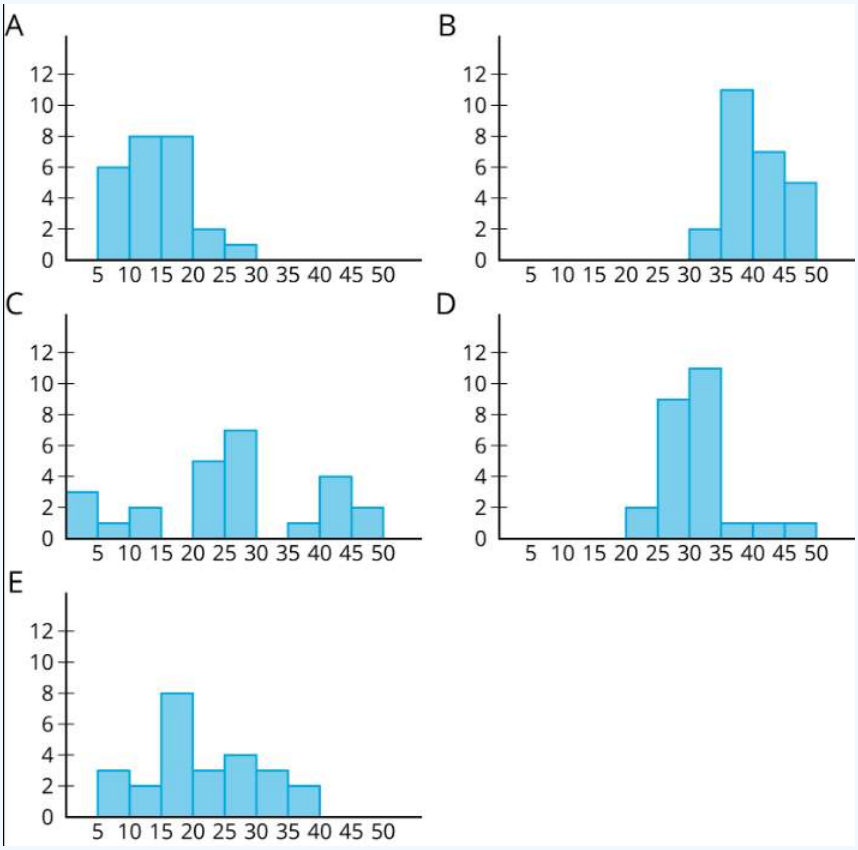


Figure 42.4.13: Five histograms, 0 to 20 by fives. Histogram A, beginning at 5 up to but not including 10, height of bar at each interval is 6, 8, 8, 2, 1. Histogram B, beginning at 30 up to but not including 35, height of bar at each interval is 2, 11, 7, 5. Histogram C, beginning at 0 up to but no including 5, height of bar at each interval is 3, 1, 2, 0, 5, 7, 0, 1, 4, 2. Histogram D, beginning at 20 up to but not including 25, height of bar at each interval is 2, 9, 11, 1, 1, 1. Histogram E, beginning at 5 up to but not including 10, height of bar at each interval is 3, 2, 8, 3, 4, 3, 2.

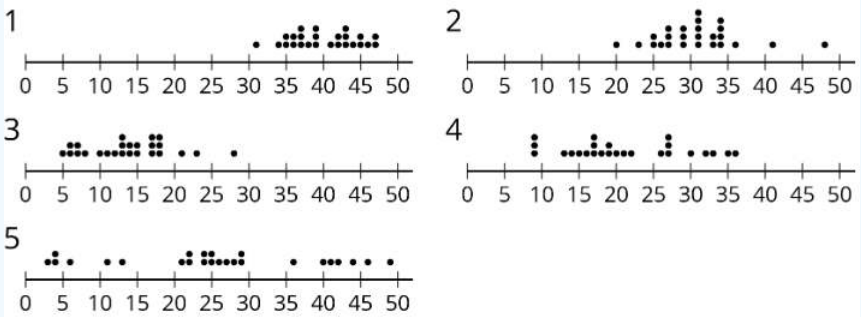


Figure 42.4.14: There are five dot plots labeled 1, 2, 3, 4, and 5. Each dot plot has the numbers 0 through 50, in increments of 5, indicated. The data for dot plot 1 are: 31, 1 dot; 34, 1 dot; 35, 2 dots; 36, 2 dots; 37, 3 dots; 38, 1 dot; 39, 3 dots; 41, 1 dot; 42, 2 dots; 43, 3 dots; 44, 1 dot; 45, 2 dots; 46, 1 dot; 47, 2 dots. The data for dot plot 2 are: 20, 1 dot; 23, 1 dot; 25, 2 dots; 26, 1 dot; 27, 3 dots; 29, 3 dots; 31, 5 dots; 33, 2 dots; 34,4 dots; 36, 1 dot; 41, 1 dot; 48, 1 dot. The data for dot plot 3 are: 5, 1 dot; 6, 2 dots; 7 two dots; 8, 1 dot; 10, 1 dot; 11, 1 dot; 12, 1 dot; 13, 3 dots; 14, 2 dots; 15, 2 dots; 17, 3 dots; 18, 3 dots; 21, 1 dot; 23, 1 dot; 30, 1 dot. The data for dot plot 4 are: 9, 3 dots; 13, 1 dot; 14, 1 dot; 15, 1 dot; 16, 1 dot; 17, 3 dots; 18, 1 dot; 19, 2 dots; 20, 1 dot; 21, 1 dot; 22, 1 dot; 26, 1 dot; 27, 3 dots; 30, 1 dot; 32, 1 dot; 33, 1 dot; 35, 1 dot; 36, 1 dot. The data for dot plot 5 are: 3, 1 dot; 4, 2 dots; 6, 1 dot; 11, 1 dot; 13, 1 dot; 21, 1 dot; 22, 2 dots; 24, 2 dots; 25, 2 dots; 26, 1 dot; 27, 1 dot; 28, 1 dot; 29, 2 dots; 36, 1 dot; 40, 1 dot; 42, 1 dot, 44, 1 dot; 46, 1 dot; 49, 1 dot.

Exercise 42.4.5

$(-2, 3)$ is one vertex of a square on a coordinate plane. Name three points that could be the other vertices.

(From Unit 7.3.2)

Exercise 42.4.6

Here is a histogram that summarizes the lengths, in feet, of a group of adult female sharks. Select **all** the statements that are true, according to the histogram.

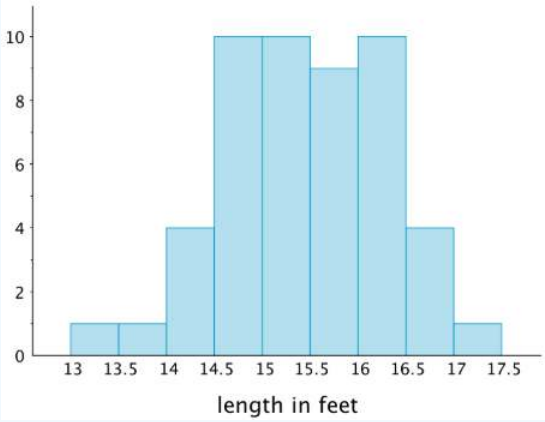


Figure 42.4.15

A. A total of 9 sharks were measured.
B. A total of 50 sharks were measured.
C. The longest shark that was measured was 10 feet long.
D. Most of the sharks that were measured were over 16 feet long.
E. Two of the sharks that were measured were less than 14 feet long.

Exercise 42.4.7

This table shows the times, in minutes, it took 40 sixth-grade students to run 1 mile.

Table 42.4.4

time (minutes)	frequency
4 to less than 6	1
6 to less than 8	5
8 to less than 10	13
10 to less than 12	12
12 to less than 14	7
14 to less than 16	2

Draw a histogram for the information in the table.

42.4: Interpreting Histograms is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

42.5: Using Histograms to Answer Statistical Questions

Lesson

Let's draw histograms and use them to answer questions.

Exercise 42.5.1: Which One Doesn't Belong: Questions

Here are four questions about the population of Alaska. Which question does not belong? Be prepared to explain your reasoning.

1. In general, at what age do Alaska residents retire?
2. At what age can Alaskans vote?
3. What is the age difference between the youngest and oldest Alaska residents with a full-time job?
4. Which age group is the largest part of the population: 18 years or younger, 19–25 years, 25–34 years, 35–44 years, 45–54 years, 55–64 years, or 65 years or older?

Exercise 42.5.2: Measuring Earthworms

An earthworm farmer set up several containers of a certain species of earthworms so that he could learn about their lengths. The lengths of the earthworms provide information about their ages. The farmer measured the lengths of 25 earthworms in one of the containers. Each length was measured in millimeters.



Figure 42.5.1: By JKpics. Public Domain. Pixabay. Source.

1. Using a ruler, draw a line segment for each length:
   - 20 millimeters
   - 40 millimeters
   - 60 millimeters
   - 80 millimeters
   - 100 millimeters
2. Here are the lengths, in millimeters, of the 25 earthworms.

   6 11 18 19 20 23 23 25 25 26 27 27 28 29 32 33 41 42 48 52 54 59 60 77 93

Complete the table for the lengths of the 25 earthworms.

Table 42.5.1

length	frequency
0 millimeters to less than 20 millimeters	
20 millimeters to less than 40 millimeters	
40 millimeters to less than 60 millimeters	
60 millimeters to less than 80 millimeters	
80 millimeters to less than 100 millimeters	

3. Use the grid and the information in the table to draw a histogram for the worm length data. Be sure to label the axes of your histogram.

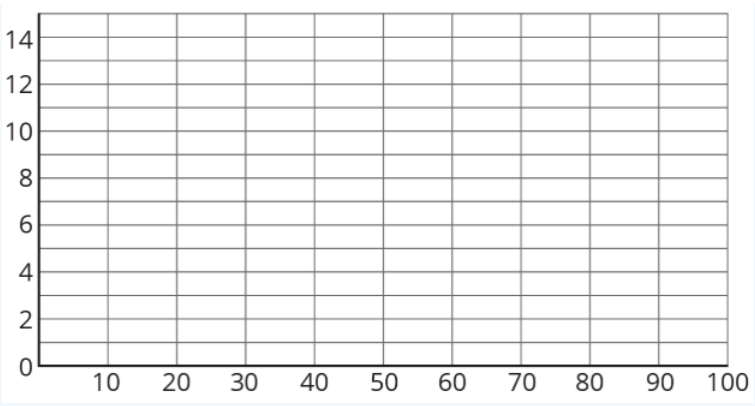


Figure 42.5.2

4. Based on the histogram, what is a typical length for these 25 earthworms? Explain how you know.
5. Write 1–2 sentences to describe the spread of the data. Do most of the worms have a length that is close to your estimate of a typical length, or are they very different in length?

Are you ready for more?

Here is another histogram for the earthworm measurement data. In this histogram, the measurements are in different groupings.

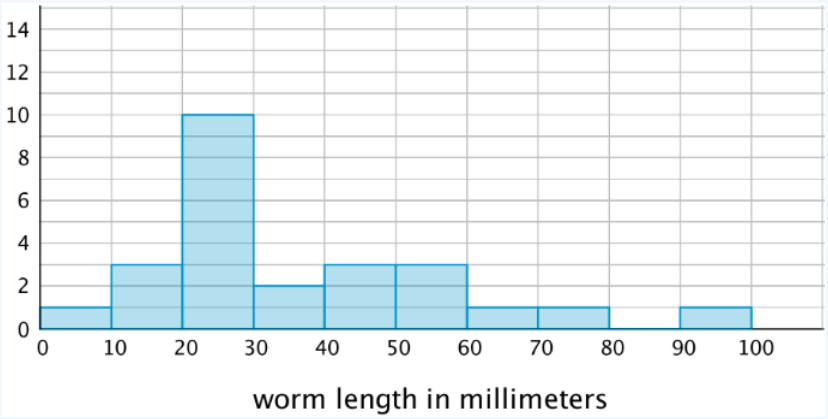


Figure 42.5.3

1. Based on this histogram, what is your estimate of a typical length for the 25 earthworms?
2. Compare this histogram with the one you drew. How are the distributions of data summarized in the two histograms the same? How are they different?
3. Compare your estimates of a typical earthworm length for the two histograms. Did you reach different conclusions about a typical earthworm length from the two histograms?

Exercise 42.5.3: Tall and Taller Players

Professional basketball players tend to be taller than professional baseball players.

Here are two histograms that show height distributions of 50 male professional baseball players and 50 male professional basketball players.

1. Decide which histogram shows the heights of baseball players and which shows the heights of basketball players. Be prepared to explain your reasoning.

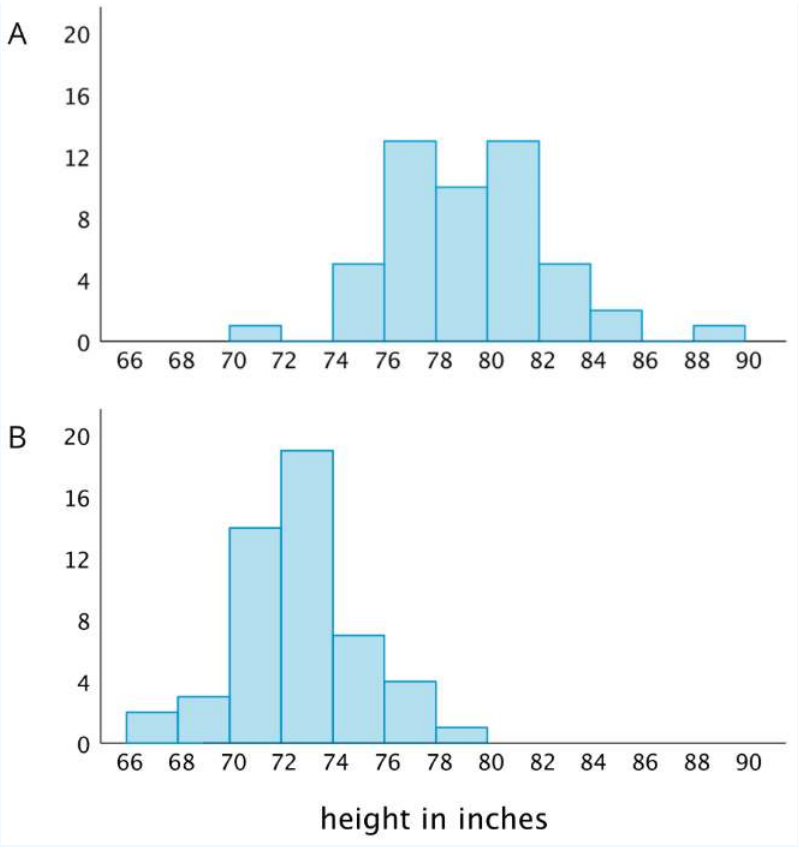


Figure 42.5.4: Two histograms, height in inches, 66 through 90 by twos. Histogram A, beginning with 70 up to but not including 72, height of bar for each interval is 1, 0, 4, 12, 10, 12, 4, 2, 0, 1. Histogram B, beginning with 66 up to but not including 68, height of bar for each interval is 2, 3, 14, 19, 7, 5, 1.

2. Write 2–3 sentences that describe the distribution of the heights of the basketball players. Comment on the center and spread of the data.
3. Write 2–3 sentences that describe the distribution of the heights of the baseball players. Comment on the center and spread of the data.

Summary

Here are the weights, in kilograms, of 30 dogs.

10 11 12 12 13 15 16 16 17 18 18 19 20 20 20 21 22 22 22 23 24 24 26 26
28 30 32 32 34 34

Before we draw a histogram, let's consider a couple of questions.

- What are the smallest and largest values in our data set? This gives us an idea of the distance on the number line that our histogram will cover. In this case, the minimum is 10 and the maximum is 34, so our number line needs to extend from 10 to 35 at the very least.

(Remember the convention we use to mark off the number line for a histogram: we include the left boundary of a bar but exclude the right boundary. If 34 is the right boundary of the last bar, it won't be included in that bar, so the number line needs to go a little greater than the maximum value.)

- What group size or bin size seems reasonable here? We could organize the weights into bins of 2 kilograms (10, 12, 14, . . .), 5 kilograms, (10, 15, 20, 25, . . .), 10 kilograms (10, 20, 30, . . .), or any other size. The smaller the bins, the more bars we will have, and vice versa.

Let's use bins of 5 kilograms for the dog weights. The boundaries of our bins will be: 10, 15, 20, 25, 30, 35. We stop at 35 because it is greater than the maximum.

Next, we find the frequency for the values in each group. It is helpful to organize the values in a table.

Table 42.5.2

weights in kilograms	frequency
10 to less than 15	5
15 to less than 20	7
20 to less than 25	10
25 to less than 30	3
30 to less than 35	5

Now we can draw the histogram.

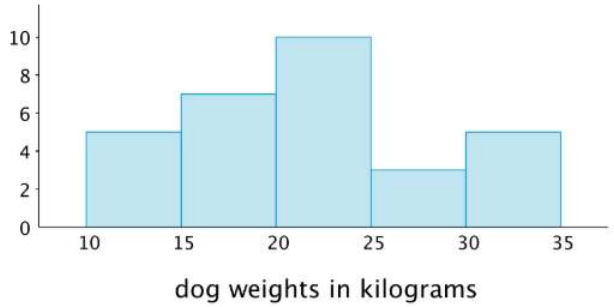


Figure 42.5.5: A histogram: The horizontal axis is labeled dog weights in kilograms and the numbers 10 through 35, in increments of 5, are indicated. On the vertical axis the numbers 0 through 10, in increments of 2, are indicated. The data represented by the bars are as follows: Weight from 10 up to 15, 5. Weight from 15 up to 20, 7. Weight from 20 up to 25, 10. Weight from 25 up to 30, 3. Weight from 30 up to 35, 5.

The histogram allows us to learn more about the dog weight distribution and describe its center and spread.

Glossary Entries

Definition: Center

The center of a set of numerical data is a value in the middle of the distribution. It represents a typical value for the data set.

For example, the center of this distribution of cat weights is between 4.5 and 5 kilograms.

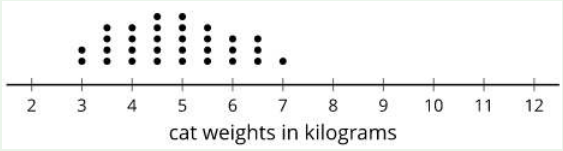


Figure 42.5.6

Definition: Distribution

The distribution tells how many times each value occurs in a data set. For example, in the data set blue, blue, green, blue, orange, the distribution is 3 blues, 1 green, and 1 orange.

Here is a dot plot that shows the distribution for the data set 6, 10, 7, 35, 7, 36, 32, 10, 7, 35.

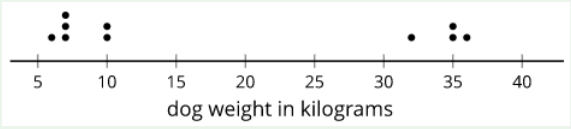


Figure 42.5.7

Definition: Frequency

The frequency of a data value is how many times it occurs in the data set.

For example, there were 20 dogs in a park. The table shows the frequency of each color.

Table 42.5.3

color	frequency
white	4
brown	7
black	3
multi-color	6

Definition: Histogram

A histogram is a way to represent data on a number line. Data values are grouped by ranges. The height of the bar shows how many data values are in that group.

This histogram shows there were 10 people who earned 2 or 3 tickets. We can't tell how many of them earned 2 tickets or how many earned 3. Each bar includes the left-end value but not the right-end value. (There were 5 people who earned 0 or 1 tickets and 13 people who earned 6 or 7 tickets.)

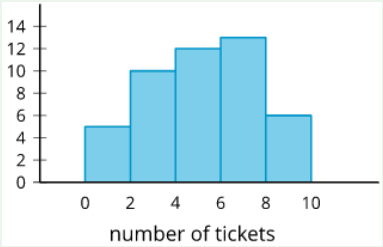


Figure 42.5.8

Definition: Spread

The spread of a set of numerical data tells how far apart the values are.

For example, the dot plots show that the travel times for students in South Africa are more spread out than for New Zealand.

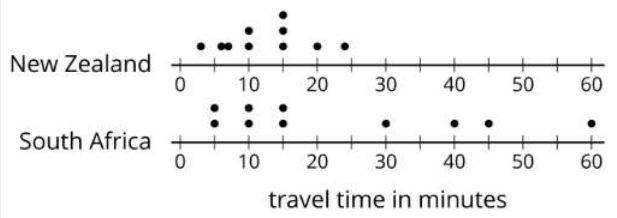


Figure 42.5.9

Practice

Exercise 42.5.4

These two histograms show the number of text messages sent in one week by two groups of 100 students. The first histogram summarizes data from sixth-grade students. The second histogram summarizes data from seventh-grade students.

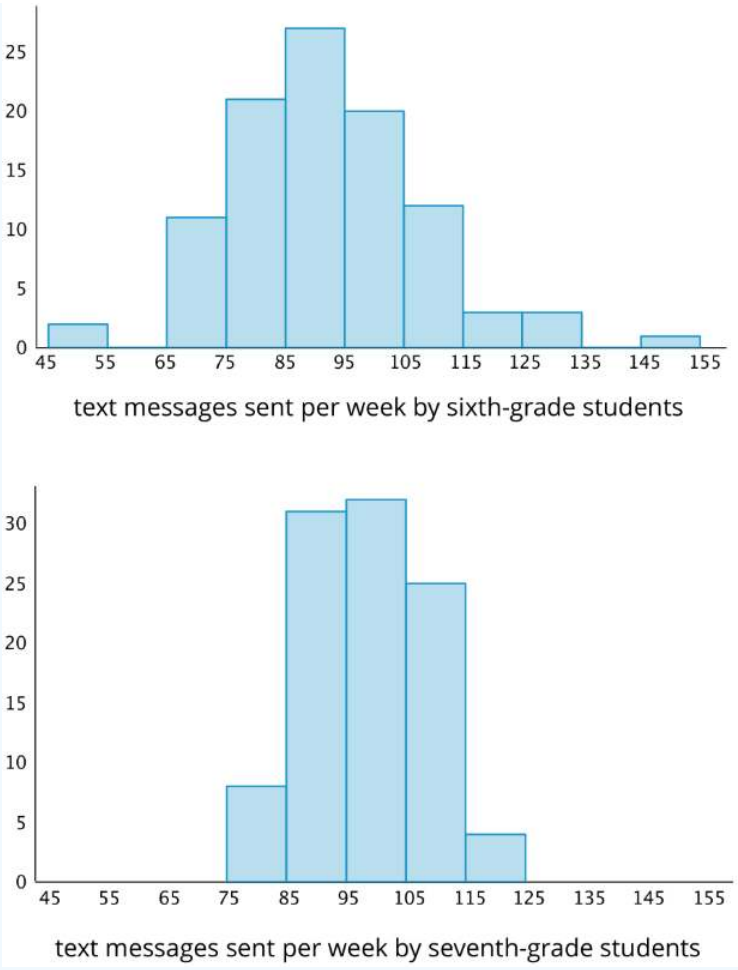


Figure 42.5.10: Two histograms, text messages sent per week by sixth grade students and seventh grade students, 45, to 155 by tens. First graph, sixth grade students, beginning at 45 up to but not including 44, height of bar at each interval is 2, 0 11, 21, 27, 24, 12, 3, 3, 0, 1. Second graph, seventh grade students, beginning at 75 up to but not including 85, height of bar at each interval is 7, 30, 31, 24, 3.

1. Do the two data sets have approximately the same center? If so, explain where the center is located. If not, which one has the greater center?
2. Which data set has greater spread? Explain your reasoning.
3. Overall, which group of students—sixth- or seventh-grade—sent more text messages?

Exercise 42.5.5

Forty sixth-grade students ran 1 mile. Here is a histogram that summarizes their times, in minutes. The center of the distribution is approximately 10 minutes.

On the blank axes, draw a second histogram that has:

- a distribution of times for a different group of 40 sixth-grade students.
- a center at 10 minutes.
- less variability than the distribution shown in the first histogram.

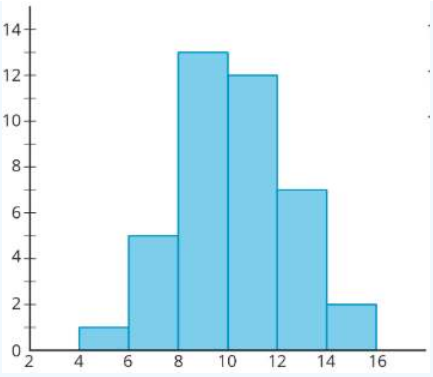


Figure 42.5.11

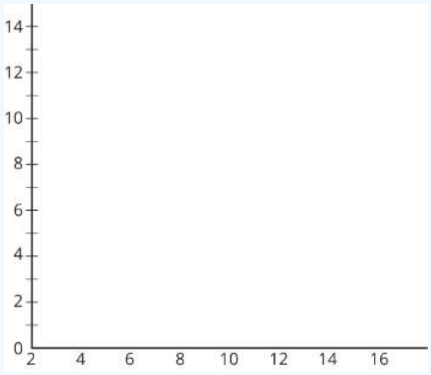


Figure 42.5.12

Exercise 42.5.6

Jada has d dimes. She has more than 30 cents but less than a dollar.

1. Write two inequalities that represent how many dimes Jada has.
2. Can d be 10?
3. How many possible solutions make both inequalities true? If possible, describe or list the solutions.

(From Unit 7.2.2)

Exercise 42.5.7

Order these numbers from greatest to least: $-4, \frac{1}{4}, 0, 4, -3\frac{1}{2}, \frac{7}{4}, -\frac{5}{4}$

(From Unit 7.1.4)

42.5: Using Histograms to Answer Statistical Questions is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

42.6: Describing Distributions on Histograms

Lesson

Let's describe distributions displayed in histograms.

Exercise 42.6.1: Which One Doesn't Belong: Histograms

Which histogram does not belong? Be prepared to explain your reasoning.

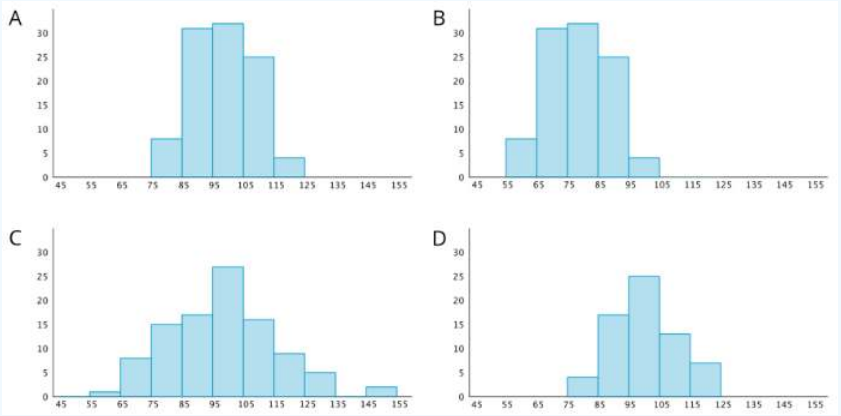


Figure 42.6.1: Four histograms, horizontal axis 45 to 155 by tens. Histogram A, beginning at 75 up to but not including 85, height of bar at each interval is 7, 29, 30, 25, 3. Histogram B, beginning at 55 up to but not including 65, height of bar at each interval is 7, 29, 30, 25, 3. Histogram C, beginning at 55 up to but not including 65, height of bar at each interval is 1, 6, 15, 17, 26, 16, 6, 4, 0, 2. Histogram D, beginning 75 up to but not including 85, height of bar at each interval is 4, 19, 25, 12, 7.

Exercise 42.6.2: Sorting Histograms

1. Your teacher will give your group a set of histogram cards. Sort them into two piles—one for histograms that are approximately symmetrical, and another for those that are not.
2. Discuss your sorting decisions with another group. Do both groups agree which cards should go in each pile? If not, discuss the reasons behind the differences and see if you can reach agreement. Record your final decisions.
   - Histograms that are approximately symmetrical:
   - Histograms that are not approximately symmetrical:
3. Histograms are also described by how many major peaks they have. Histogram A is an example of a distribution with a single peak that is not symmetrical.
   Which other histograms have this feature?
4. Some histograms have a gap, a space between two bars where there are no data points. For example, if some students in a class have 7 or more siblings, but the rest of the students have 0, 1, or 2 siblings, the histogram for this data set would show gaps between the bars because no students have 3, 4, 5, or 6 siblings.
   Which histograms do you think show one or more gaps?
5. Sometimes there are a few data points in a data set that are far from the center. Histogram A is an example of a distribution with this feature.
   Which other histograms have this feature?

Exercise 42.6.3: Getting to School

Your teacher will provide you with some data that your class collected the other day.

1. Use the data to draw a histogram that shows your class's travel times.

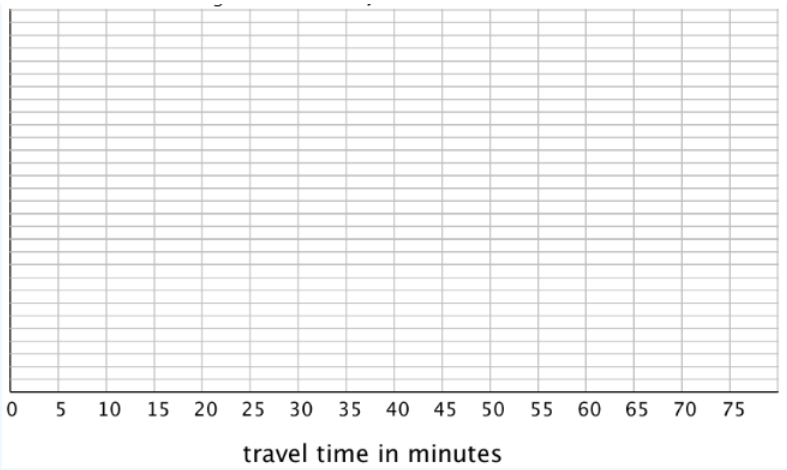


Figure 42.6.2

2. Describe the distribution of travel times. Comment on the center and spread of the data, as well as the shape and features.
3. Use the data on methods of travel to draw a bar graph. Include labels for the horizontal axis.



Figure 42.6.3

4. Describe what you learned about your class's methods of transportation to school. Comment on any patterns you noticed.
5. Compare the histogram and the bar graph that you drew. How are they the same? How are they different?

Are you ready for more?

Use one of these suggestions (or make up your own). Research data to create a histogram. Then, describe the distribution.

- Heights of 30 athletes from multiple sports
- Heights of 30 athletes from the same sport
- High temperatures for each day of the last month in a city you would like to visit
- Prices for all the menu items at a local restaurant

Summary

We can describe the shape and features of the distribution shown on a histogram. Here are two distributions with very different shapes and features.

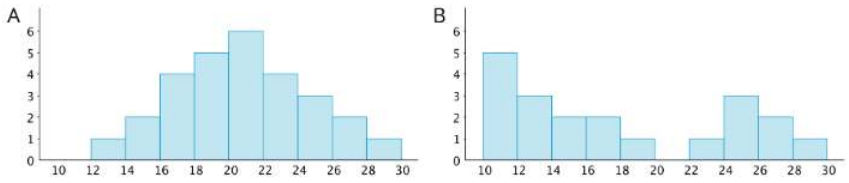


Figure 42.6.4: Two histograms, labeled A and B where the horizontal axis has the numbers 10 through 30, in increments of 2, indicated and on the vertical axis, the numbers 0 through 6 are indicated. The data represented by the bars on histogram A are: 10 up to 12, 0; 12 up to 14, 1; 14 up to 16, 2, 16 up to 18, 4; 18 up to 20, 5; 20 up to 22, 6; 22 up to 24, 4, 24 up to 26, 3; 26 up to 28, 2; 28 up to 30, 1. The data represented by the bars on histogram B are: 10 up to 12, 5; 12 up to 14, 3; 14 up to 16, 2; 16 up to 18, 2; 18 up to 20, 1; 20 up to 22, 0; 22 up to 24, 1; 24 up to 26, 3; 26 up to 28, 2; 28 up to 30, 1.

- Histogram A is very symmetrical and has a peak near 21. Histogram B is not symmetrical and has two peaks, one near 11 and one near 25.
- Histogram B has two clusters. A cluster forms when many data points are near a particular value (or a neighborhood of values) on a number line.
- Histogram B also has a gap between 20 and 22. A gap shows a location with no data values.

Here is a bar graph showing the breeds of 30 dogs and a histogram for their weights.

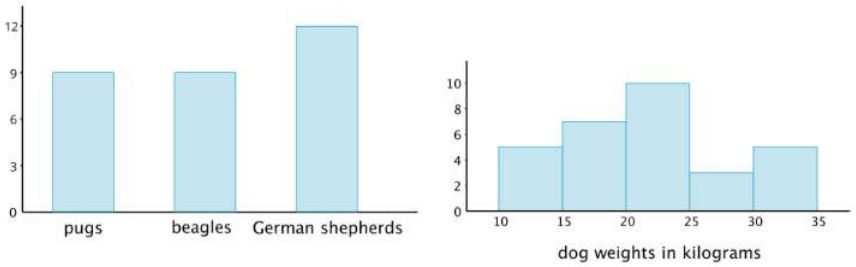


Figure 42.6.5: A bar graph and a histogram. The bar graph has three categories on the horizontal axis labeled pugs, beagles, and German shepherds. The vertical axis has the numbers 0 through 12, in increments of 3, indicated. The bar labeled pugs has a height of 9. The bar labeled beagles has a height of 9 and the bar labeled German shepherds has a height of 12. The histogram has a horizontal axis labeled dog weights in kilograms. The vertical axis has the numbers 0 through 10, in increments of 2, indicated. The data represented by the bars are as follows: 10 up to 15 kilograms, 5; 15 up to 20 kilograms, 7; 20 up to 25 kilograms, 10; 25 up to 30 kilograms, 3; 30 up to 35 kilograms, 5.

Bar graphs and histograms may seem alike, but they are very different.

- Bar graphs represent categorical data. Histograms represent numerical data.
- Bar graphs have spaces between the bars. Histograms show a space between bars *only* when no data values fall between the bars.
- Bars in a bar graph can be in any order. Histograms must be in numerical order.
- In a bar graph, the number of bars depends on the number of categories. In a histogram, we choose how many bars to use.

Glossary Entries

Definition: Center

The center of a set of numerical data is a value in the middle of the distribution. It represents a typical value for the data set.

For example, the center of this distribution of cat weights is between 4.5 and 5 kilograms.

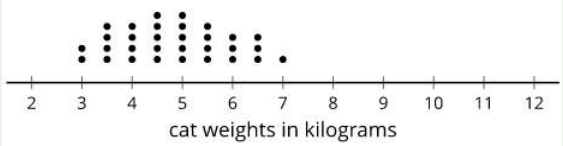


Figure 42.6.6

Definition: Distribution

The distribution tells how many times each value occurs in a data set. For example, in the data set blue, blue, green, blue, orange, the distribution is 3 blues, 1 green, and 1 orange.

Here is a dot plot that shows the distribution for the data set 6, 10, 7, 35, 7, 36, 32, 10, 7, 35.

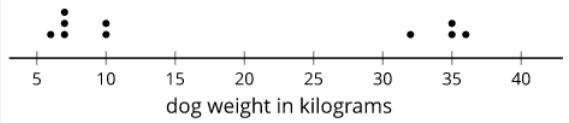


Figure 42.6.7

Definition: Frequency

The frequency of a data value is how many times it occurs in the data set.

For example, there were 20 dogs in a park. The table shows the frequency of each color.

Table 42.6.1

color	frequency
white	4
brown	7
black	3
multi-color	6

Definition: Histogram

A histogram is a way to represent data on a number line. Data values are grouped by ranges. The height of the bar shows how many data values are in that group.

This histogram shows there were 10 people who earned 2 or 3 tickets. We can't tell how many of them earned 2 tickets or how many earned 3. Each bar includes the left-end value but not the right-end value. (There were 5 people who earned 0 or 1 tickets and 13 people who earned 6 or 7 tickets.)

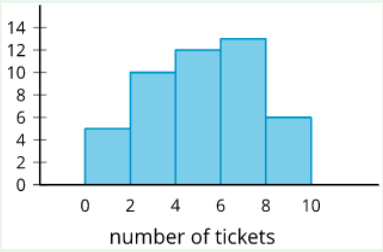


Figure 42.6.8

Definition: Spread

The spread of a set of numerical data tells how far apart the values are.

For example, the dot plots show that the travel times for students in South Africa are more spread out than for New Zealand.

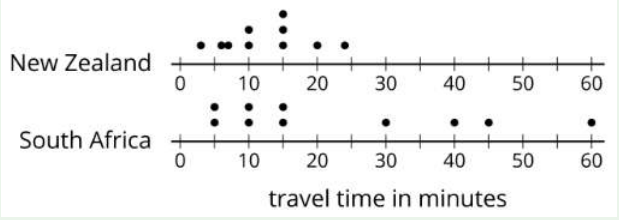


Figure 42.6.9

Practice

Exercise 42.6.4

The histogram summarizes the data on the body lengths of 143 wild bears. Describe the distribution of body lengths. Be sure to comment on shape, center, and spread.

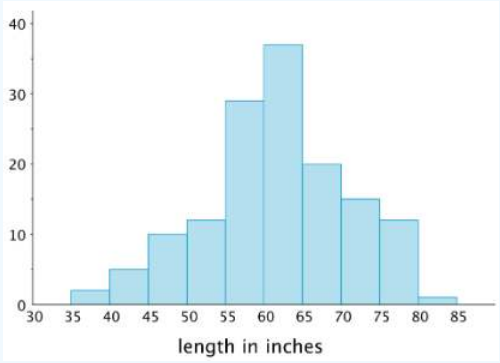


Figure 42.6.10

Exercise 42.6.5

Which data set is more likely to produce a histogram with a symmetric distribution? Explain your reasoning.

- Data on the number of seconds on a track of music in a pop album.
- Data on the number of seconds spent talking on the phone yesterday by everyone in the school.

Exercise 42.6.6

Evaluate the expression $4x^3$ for each value of x.

1. 1
2. 2
3. $\frac{1}{2}$

(From Unit 6.3.4)

Exercise 42.6.7

Decide if each data set might produce one or more gaps when represented by a histogram. For each data set that you think might produce gaps, briefly describe or give an example of how the values in the data set might do so.

1. The ages of students in a sixth-grade class.
2. The ages of people in an elementary school.
3. The ages of people eating in a family restaurant.
4. The ages of people who watch football.
5. The ages of runners in a marathon.

Exercise 42.6.8

Jada drank 12 ounces of water from her bottle. This is 60% of the water the bottle holds.

1. Write an equation to represent this situation. Explain the meaning of any variables you use.
2. How much water does the bottle hold?

(From Unit 6.2.2)

42.6: Describing Distributions on Histograms is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

Section 43: Measures of Center and Variability

This page titled Section 43: Measures of Center and Variability is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

43.1: Mean

Lesson

Let's explore the mean of a data set and what it tells us.

Exercise 43.1.1: Close to Four

Use the digits 0–9 to write an expression with a value as close as possible to 4. Each digit can be used only one time in the expression.

$(\square + \square + \square + \square) \div 4$

Figure 43.1.1

Exercise 43.1.2: Spread Out and Share

1. The kittens in a room at an animal shelter are arranged in five crates, as shown.

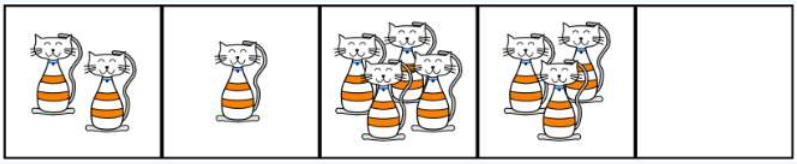


Figure 43.1.2: By Clker-Free-Vector-Images. Public Domain. Pixabay. Source.

   a. The manager of the shelter wants the kittens distributed equally among the crates. How might that be done? How many kittens will end up in each crate?
   b. The number of kittens in each crate after they are equally distributed is called the **mean** number of kittens per crate, or the **average** number of kittens per crate.
   Explain how the expression $10 \div 5$ is related to the average.
   c. Another room in the shelter has 6 crates. No two crates contain the same number of kittens, and there is an average of 3 kittens per crate.
   Draw or describe at least two different arrangements of kittens that match this description. You may choose to use the applet to help.

2. Five servers were scheduled to work the number of hours shown. They decided to share the workload, so each one would work equal hours.
   Server A: 3
   Server B: 6
   Server C: 11
   Server D: 7
   Server E: 4

   a. On the grid on the left, draw 5 bars whose heights represent the hours worked by servers A, B, C, D, and E.

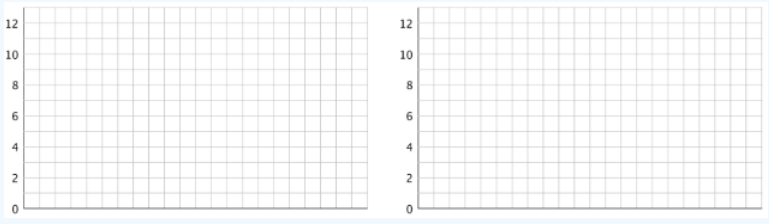


Figure 43.1.3

   b. Think about how you would rearrange the hours so that each server gets a fair share. Then, on the grid on the right, draw a new graph to represent the rearranged hours. Be prepared to explain your reasoning.
   c. Based on your second drawing, what is the average or mean number of hours that the servers will work?
   d. Explain why we can also find the mean by finding the value of $31 \div 5$.
   e. Which server will see the biggest change to work hours? Which server will see the least change?

Are you ready for more?

Server F, working 7 hours, offers to join the group of five servers, sharing their workload. If server F joins, will the mean number of hours worked increase or decrease? Explain how you know.

Exercise 43.1.3: Getting to School

For the past 12 school days, Mai has recorded how long her bus rides to school take in minutes. The times she recorded are shown in the table.

9	8	6	9	10	7	6	12	9	8	10	8

1. Find the mean for Mai's data. Show your reasoning.
2. In this situation, what does the mean tell us about Mai's trip to school?
3. For 5 days, Tyler has recorded how long his walks to school take in minutes. The mean for his data is 11 minutes. Without calculating, predict if each of the data sets shown could be Tyler's. Explain your reasoning.
   - data set A: 11, 8, 7, 9, 8
   - data set B: 12, 7, 13, 9, 14
   - data set C: 11, 20, 6, 9, 10
   - data set D: 8, 10, 9, 11, 11
4. Determine which data set is Tyler's. Explain how you know.

Summary

Sometimes a general description of a distribution does not give enough information, and a more precise way to talk about center or spread would be more useful. The **mean**, or **average**, is a number we can use to summarize a distribution.

We can think about the mean in terms of "fair share" or "leveling out." That is, a mean can be thought of as a number that each member of a group would have if all the data values were combined and distributed equally among the members.

For example, suppose there are 5 bottles which have the following amounts of water: 1 liter, 4 liters, 2 liters, 3 liters, and 0 liters.

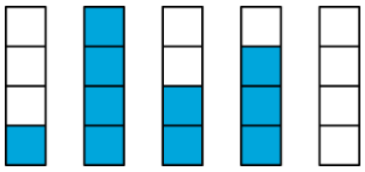


Figure 43.1.4: There are 5 identical tape diagrams that are each partitioned into 4 equal parts. The first diagram has 1 part shaded. The second diagram has 4 parts shaded. The third diagram has 2 parts shaded. The fourth diagram has 3 parts shaded. The fifth diagram has no parts shaded.

To find the mean, first we add up all of the values. We can think of as putting all of the water together: $1+4+2+3+0=10$.



Figure 43.1.5

To find the "fair share," we divide the 10 liters equally into the 5 containers: $10 \div 5 = 2$.

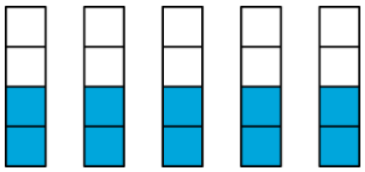


Figure 43.1.6

Suppose the quiz scores of a student are 70, 90, 86, and 94. We can find the mean (or average) score by finding the sum of the scores $(70+90+86+94=340)$ and dividing the sum by four $(340 \div 4 = 85)$. We can then say that the student scored, on average, 85 points on the quizzes.

In general, to find the mean of a data set with n values, we add all of the values and divide the sum by n.

Glossary Entries

Definition: Average

The average is another name for the mean of a data set.

For the data set 3, 5, 6, 8, 11, 12, the average is 7.5.

$3+5+6+8+11+12=45$

$45\div 6=7.5$

Definition: Mean

The mean is one way to measure the center of a data set. We can think of it as a balance point. For example, for the data set 7, 9, 12, 13, 14, the mean is 11.

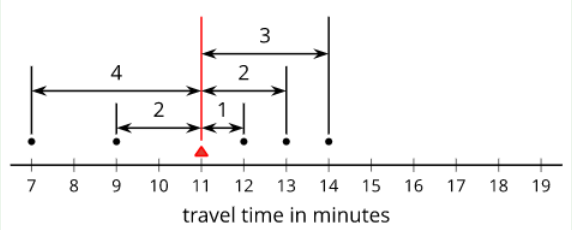


Figure 43.1.7

To find the mean, add up all the numbers in the data set. Then, divide by how many numbers there are. $7+9+12+13+14=55$ and $55\div 5=11$.

Practice

Exercise 43.1.4

A preschool teacher is rearranging four boxes of playing blocks so that each box contains an equal number of blocks. Currently Box 1 has 32 blocks, Box 2 has 18, Box 3 has 41, and Box 4 has 9.

Select **all** the ways he could make each box have the same number of blocks.

A. Remove all the blocks and make four equal piles of 25, then put each pile in one of the boxes.
B. Remove 7 blocks from Box 1 and place them in Box 2.
C. Remove 21 blocks from Box 3 and place them in Box 4.
D. Remove 7 blocks from Box 1 and place them in Box 2, and remove 21 blocks from Box 3 and place them in Box 4.
E. Remove 7 blocks from Box 1 and place them in Box 2, and remove 16 blocks from Box 3 and place them in Box 4.

Exercise 43.1.5

In a round of mini-golf, Clare records the number of strokes it takes to hit the ball into the hole of each green.

2 3 1 4 5 2 3 4 3

She said that, if she redistributed the strokes on different greens, she could tell that her average number of strokes per hole is 3. Explain how Clare is correct.

Exercise 43.1.6

Three sixth-grade classes raised $25.50, $49.75, and $37.25 for their classroom libraries. They agreed to share the money raised equally. What is each class's equal share? Explain or show your reasoning.

Exercise 43.1.7

In her English class, Mai's teacher gives 4 quizzes each worth 5 points. After 3 quizzes, she has the scores 4, 3, and 4. What does she need to get on the last quiz to have a mean score of 4? Explain or show your reasoning.

Exercise 43.1.8

An earthworm farmer examined two containers of a certain species of earthworms so that he could learn about their lengths. He measured 25 earthworms in each container and recorded their lengths in millimeters.

Here are histograms of the lengths for each container.

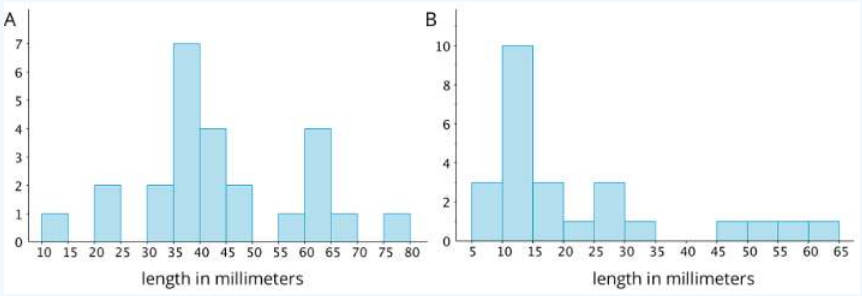


Figure 43.1.8: Two histograms, length in millimeters. Histogram A from 10 to 80 by fives, Beginning at 10 up to but not including 15, height of bar at each interval is 1, 0, 2, 0, 2, 7, 4, 2, 0, 1, 4, 1, 0, 1. Histogram B from 5 to 65 by fives. Beginning at 5 up to but not including 10, height of bar at each interval is 3, 10, 3, 1, 3, 1, 0, 0, 1, 1, 1, 1.

1. Which container tends to have longer worms than the other container?
2. For which container would 15 millimeters be a reasonable description of a typical length of the worms in the container?
3. If length is related to age, which container had the most young worms?

(From Unit 8.2.5)

Exercise 43.1.9

Diego thinks that $x = 3$ is a solution to the equation $x^2 = 16$. Do you agree? Explain or show your reasoning.

(From Unit 6.3.4)

43.1: Mean is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

43.2: Finding and Interpreting the Mean as the Balance Point

Lesson

Let's look at another way to understand the mean of a data set.

Exercise 43.2.1: Which One Doesn't Belong: Division

Which expression does not belong? Be prepared to explain your reasoning.

$$\frac{8+8+4+4}{4} \qquad \frac{10+10+4}{4} \qquad \frac{9+9+5+5}{4} \qquad \frac{6+6+6+6+6}{5}$$

Exercise 43.2.2: Travel Times (Part 1)

Here is the data set from an earlier lesson showing how long it takes for Diego to walk to school, in minutes, over 5 days. The mean number of minutes is 11.

12 7 13 9 14

1. Represent Diego's data on a dot plot. Mark the location of the mean with a triangle.
2. The mean can also be seen as a **measure of center** that balances the points in a data set. If we find the distance between every point and the mean, add the distances on each side of the mean, and compare the two sums, we can see this balancing.
   a. Record the distance between each point and 11 and its location relative to 11.

Table 43.2.1

time in minutes	distance from 11	left of 11 or right of 11?
12		
7		
13		
9		
14		

   b. Sum of distances left of 11:_____ Sum of distances right of 11:_____
   What do you notice about the two sums?
3. Can another point that is *not* the mean produce similar sums of distances?
   Let's investigate whether 10 can produce similar sums as those of 11.
   a. Complete the table with the distance of each data point from 10.

Table 43.2.2

time in minutes	distance from 10	left of 10 or right of 10?
12		
7		
13		
9		
14		

   b. Sum of distances left of 10:_____ Sum of distances right of 10:_____
   What do you notice about the two sums?
4. Based on your work so far, explain why the mean can be considered a balance point for the data set.

Exercise 43.2.3: Travel Times (Part 2)

1. Here are dot plots showing how long Diego's trips to school took in minutes—which you studied earlier—and how long Andre's trips to school took in minutes. The dot plots include the means for each data set, marked by triangles.

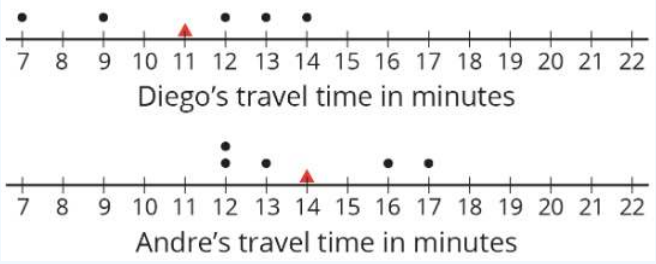


Figure 43.2.1: Two dot plots, travel time in minutes, 7 to 22 by ones. First plot, Deigo's, has one dot each at 7, 9, 12, 13, 14 and a triangle at 11. Second plot, Andre's, has 2 dots at 12, one dot each at 13, 16, 17, and a triangle at 1

   a. Which of the two data sets has a larger mean? In this context, what does a larger mean tell us?
   b. Which of the two data sets has larger sums of distances to the left and right of the mean? What do these sums tell us about the variation in Diego's and Andre's travel times?

2. Here is a dot plot showing lengths of Lin's trips to school.

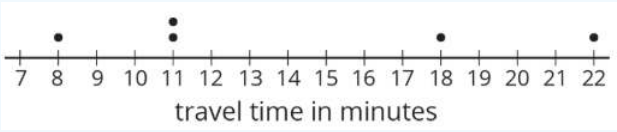


Figure 43.2.2

   a. Calculate the mean of Lin's travel times.
   b. Complete the table with the distance between each point and the mean as well whether the point is to the left or right of the mean.

Table 43.2.3

time in minutes	distance from the mean	left or right of the mean?
22		
18		
11		
8		
11		

   c. Find the sum of distances to the left of the mean and the sum of distances to the right of the mean.
   d. Use your work to compare Lin's travel times to Andre's. What can you say about their average travel times? What about the variability in their travel times?

Summary

The mean is often used as a **measure of center** of a distribution. This is because the mean of a distribution can be seen as the "balance point" for the distribution. Why is this a good way to think about the mean? Let's look at a very simple set of data on the number of cookies that each of eight friends baked:

19 20 20 21 21 22 22 23

Here is a dot plot showing the data set.

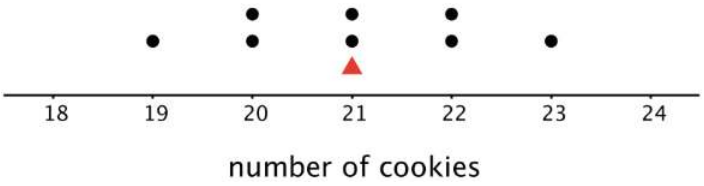


Figure 43.2.3: Dot plot for “number of cookies”. The numbers 18 through 24 are indicated. There is a triangle indicated at 21 cookies. The data are as follows: 18 cookies, 0 dots. 19 cookies, 1 dot. 20 cookies, 2 dots. 21 cookies, 2 dots. 22 cookies, 2 dots. 23 cookies, 1 dot. 24 cookies, 0 dots.

The distribution shown is completely symmetrical. The mean number of cookies is 21, because $(19+20+20+21+21+22+22+23)\div 8=21$. If we mark the location of the mean on the dot plot, we can see that the data points could balance at 21.

In this plot, each point on either side of the mean has a mirror image. For example, the two points at 20 and the two at 22 are the same distance from 21, but each pair is located on either side of 21. We can think of them as balancing each other around 21.

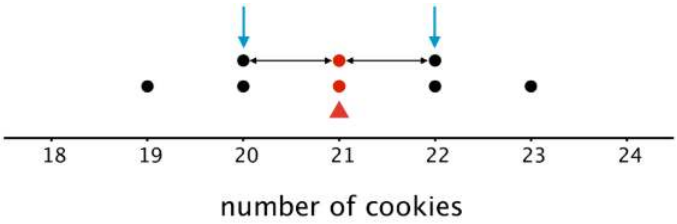


Figure 43.2.4: A dot plot for "number of cookies". The numbers 18 through 24 are indicated. There is a triangle indicated at 21 cookies. There are arrows pointing to the dots at 20 and 22. There are lines indicating the distance between 20 and 21 and the distance between 21 and 22. The data are as follows: 18 cookies, 0 dots. 19 cookies, 1 dot. 20 cookies, 2 dots. 21 cookies, 2 dots. 22 cookies, 2 dots. 23 cookies, 1 dot. 24 cookies, 0 dots.

Similarly, the points at 19 and 23 are the same distance from 21 but are on either side of it. They, too, can be seen as balancing each other around 21.

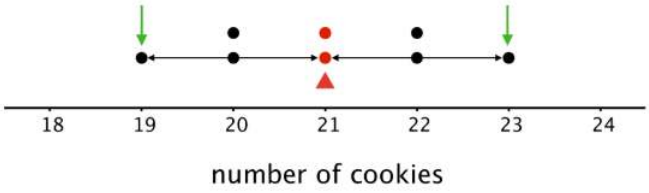


Figure 43.2.5: A dot plot for "number of cookies". The numbers 18 through 24 are indicated. There is a triangle indicated at 21 cookies. There are arrows pointing to the dots at 19 and 23. There are lines indicating the distance between 19 and 21 and the distance between 21 and 23. The data are as follows: 18 cookies, 0 dots. 19 cookies, 1 dot. 20 cookies, 2 dots. 21 cookies, 2 dots. 22 cookies, 2 dots. 23 cookies, 1 dot. 24 cookies, 0 dots.

We can say that the distribution of the cookies has a center at 21 because that is its balance point, and that the eight friends, on average, baked 21 cookies.

Even when a distribution is not completely symmetrical, the distances of values below the mean, on the whole, balance the distances of values above the mean.

Glossary Entries

Definition: Average

The average is another name for the mean of a data set.

For the data set 3, 5, 6, 8, 11, 12, the average is 7.5.

$3+5+6+8+11+12=45$

$45\div 6=7.5$

Definition: Mean

The mean is one way to measure the center of a data set. We can think of it as a balance point. For example, for the data set 7, 9, 12, 13, 14, the mean is 11.

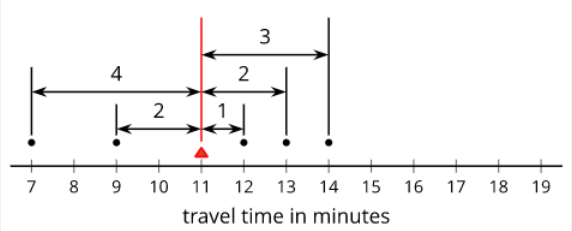


Figure 43.2.6

To find the mean, add up all the numbers in the data set. Then, divide by how many numbers there are. $7+9+12+13+14=55$ and $55\div 5=11$.

Definition: Measure of Center

A measure of center is a value that seems typical for a data distribution.

Mean and median are both measures of center.

Practice

Exercise 43.2.4

On school days, Kiran walks to school. Here are the lengths of time, in minutes, for Kiran's walks on 5 school days:

16 11 18 12 13

1. Create a dot plot for Kiran's data.
2. Without calculating, decide if 15 minutes would be a good estimate of the mean. If you think it is a good estimate, explain your reasoning. If not, give a better estimate and explain your reasoning.
3. Calculate the mean for Kiran's data.
4. In the table, record the distance of each data point from the mean and its location relative to the mean.

Table 43.2.4

time in minutes	distance from the mean	left or right of the mean?
16		
11		
18		
12		
13		

5. Calculate the sum of all distances to the left of the mean, then calculate the sum of distances to the right of the mean. Explain how these sums show that the mean is a balance point for the values in the data set.

Exercise 43.2.5

Noah scored 20 points in a game. Mai's score was 30 points. The mean score for Noah, Mai, and Clare was 40 points. What was Clare's score? Explain or show your reasoning.

Exercise 43.2.6

Compare the numbers using >, <, or =.

1. -2____3
2. $|-12|$____$|15|$
3. 3____-4
4. $|15|$____$|-12|$
5. 7____-11
6. -4____$|5|$

(From Unit 7.1.7)

Exercise 43.2.7

1. Plot $\frac{2}{3}$ and $\frac{3}{4}$ on a number line.
2. Is $\frac{2}{3} < \frac{3}{4}$, or is $\frac{3}{4} < \frac{2}{3}$? Explain how you know.

(From Unit 7.1.3)

Exercise 43.2.8

Select **all** the expressions that represent the total area of the large rectangle.

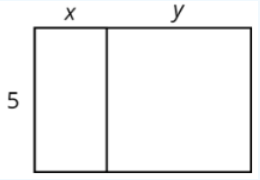


Figure 43.2.7: A large rectangle is partitioned by a vertical line segment into 2 smaller rectangles. The left rectangle has a vertical side length of 5 and horizontal side length of x. The right rectangle has horizontal side length of y.

A. $5(x+y)$
B. $5+xy$
C. $5x+5y$
D. $2(5+x+y)$
E. $5xy$

(From Unit 6.2.5)

43.2: Finding and Interpreting the Mean as the Balance Point is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

43.3: Variability and MAD

Lesson

Let's study distances between data points and the mean and see what they tell us.

Exercise 43.3.1: Shooting Hoops (Part 1)

Elena, Jada, and Lin enjoy playing basketball during recess. Lately, they have been practicing free throws. They record the number of baskets they make out of 10 attempts. Here are their data sets for 12 school days.

Elena

4	5	1	6	9	7	2	8	3	3	5	7

Jada

2	4	5	4	6	6	4	7	3	4	8	7

Lin

3	6	6	4	5	5	3	5	4	6	6	7

1. Calculate the mean number of baskets each player made, and compare the means. What do you notice?
2. What do the means tell us in this context?

Exercise 43.3.2: Shooting Hoops (Part 2)

Here are the dot plots showing the number of baskets Elena, Jada, and Lin each made over 12 school days.

1. On each dot plot, mark the location of the mean with a triangle (Δ). Then, contrast the dot plot distributions. Write 2–3 sentences to describe the shape and spread of each distribution.

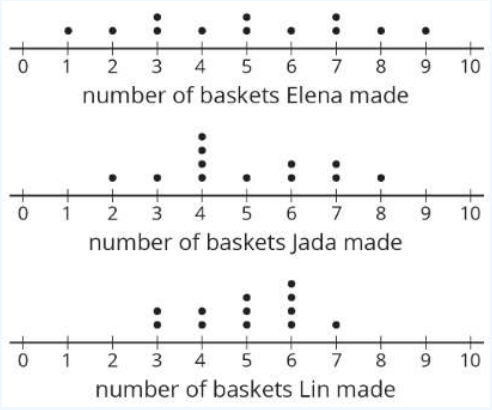


Figure 43.3.1: Three dot plots, number of baskets made, 0 to 10 by ones. First plot, Elena, beginning at 1 number of dots above each increment is 1, 1, 2, 1, 2, 1, 2, 1, 1. Second plot, Jada, beginning at 2 the number of dots above each increment is 1, 1, 4, 1, 2, 2, 1. Third plot, Lin, beginning at 3 the number of dots above each increment is 2, 2, 3, 4, 1.

2. Discuss the following questions with your group. Explain your reasoning.
   a. Would you say that all three students play equally well?
   b. Would you say that all three students play equally consistently?
   c. If you could choose one player to be on your basketball team based on their records, who would you choose?

Exercise 43.3.3: Shooting Hoops (Part 3)

The tables show Elena, Jada, and Lin's basketball data from an earlier activity. Recall that the mean of Elena's data, as well as that of Jada and Lin's data, was 5.

1. Record the distance between each of Elena's scores and the mean.

Table 43.3.1

Elena	4	5	1	6	9	7	2	8	3	3	5	7
distance from 5	1			1								

Now find *the average of the distances* in the table. Show your reasoning and round your answer to the nearest tenth.

This value is the **mean absolute deviation (MAD)** of Elena's data.

Elena's MAD: _____

2. Find the mean absolute deviation of Jada's data. Round it to the nearest tenth.

Table 43.3.2

Jada	2	4	5	4	6	6	4	7	3	4	8	7
distance from 5												

Jada's MAD: _____

3. Find the mean absolute deviation of Lin's data. Round it to the nearest tenth.

Table 43.3.3

Lin	3	6	6	4	5	5	3	5	4	6	6	7
distance from 5												

Lin's MAD: _____

4. Compare the MADs and dot plots of the three students' data. Do you see a relationship between each student's MAD and the distribution on her dot plot? Explain your reasoning.

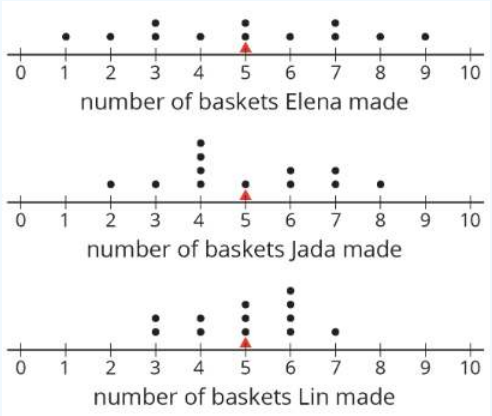


Figure 43.3.2: Three dot plots, number of baskets made, 0 to 10 by ones. First plot, Elena, beginning at 1 number of dots above each increment is 1, 1, 2, 1, 2, 1, 2, 1, 1, triangle at 5. Second plot, Jada, beginning at 2 the number of dots above each increment is 1, 1, 4, 1, 2, 2, 1, triangle at 5. Third plot, Lin, beginning at 3 the number of dots above each increment is 2, 2, 3, 4, 1, triangle at 5.

Are you ready for more?

Invent another data set that also has a mean of 5 but has a MAD greater than 2. Remember, the values in the data set must be whole numbers from 0 to 10.

Exercise 43.3.4: Game of 22

Your teacher will give your group a deck of cards. Shuffle the cards, and put the deck face down on the playing surface.

- To play: Draw 3 cards and add up the values. An ace is a 1. A jack, queen, and king are each worth 10. Cards 2–10 are each worth their face value. If your sum is anything other than 22 (either above or below 22), say: "My sum deviated from 22 by ____ ," or "My sum was off from 22 by ____ ."
- To keep score: Record each sum and each distance from 22 in the table. After five rounds, calculate the average of the distances. The player with the lowest average distance from 22 wins the game.

Table 43.3.4

player A	round 1	round 2	round 3	round 4	round 5
sum of cards					
distance from 22					

Average distance from 22: _____

Table 43.3.5

player B	round 1	round 2	round 3	round 4	round 5
sum of cards					
distance from 22					

Average distance from 22: _____

Table 43.3.6

player C	round 1	round 2	round 3	round 4	round 5
sum of cards					
distance from 22					

Average distance from 22: _____

Whose average distance from 22 is the smallest? Who won the game?

Summary

We use the mean of a data set as a measure of center of its distribution, but two data sets with the same mean could have very different distributions.

This dot plot shows the weights, in grams, of 22 cookies.

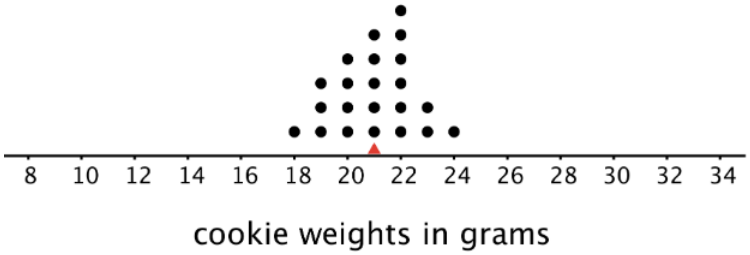


Figure 43.3.3: A dot plot for "cookie weights in grams." The numbers 8 through 34, in increments of 2, are indicated. A triangle is indicated at 21 grams. The data are as follows: 18 grams, 1 dot; 19 grams, 3 dots; 20 grams, 4 dots; 21 grams, 5 dots; 22 grams, 6 dots; 23 grams, 2 dots, 24 grams, 1 dot.

The mean weight is 21 grams. All the weights are within 3 grams of the mean, and most of them are even closer. These cookies are all fairly close in weight.

This dot plot shows the weights, in grams, of a different set of 30 cookies.

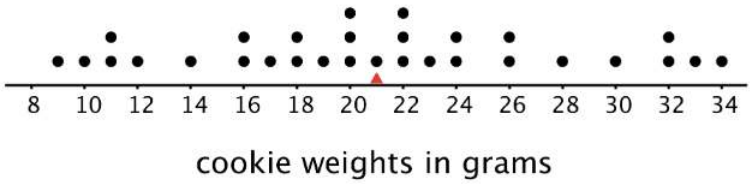


Figure 43.3.4: A dot plot for “cookie weights in grams.” The numbers 8 through 34, in increments of 2, are indicated. A triangle is indicated at 21 grams. The data are as follows: 9 grams, 1 dot; 10 grams, 1 dot; 11 grams, 2 dots; 12 grams, 1 dot; 14 grams, 1 dot; 16 grams, 2 dots; 17 grams, 1 dot; 18 grams, 2 dots; 19 grams, 1 dot; 20 grams, 3 dots; 21 grams, 1 dot; 22 grams, 3 dots; 23 grams, 1 dot; 24 grams, 2 dots; 26 grams, 2 dots; 28 grams, 1 dot; 30 grams, 1 dot; 32 grams, 2 dots; 33 grams, 1 dot; 34 grams, 1 dot.

The mean weight for this set of cookies is also 21 grams, but some cookies are half that weight and others are one-and-a-half times that weight. There is a lot more variability in the weight.

There is a number we can use to describe how far away, or how spread out, data points generally are from the mean. This *measure of spread* is called the **mean absolute deviation (MAD)**.

Here the MAD tells us how far cookie weights typically are from 21 grams. To find the MAD, we find the distance between each data value and the mean, and then calculate the mean of those distances.

For instance, the point that represents 18 grams is 3 units away from the mean of 21 grams. We can find the distance between each point and the mean of 21 grams and organize the distances into a table, as shown.

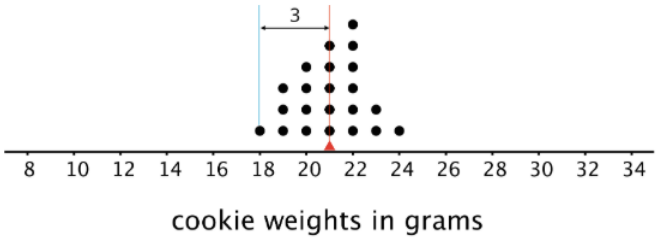


Figure 43.3.5: A dot plot for “cookie weights in grams.” The numbers 8 through 34, in increments of 2, are indicated. A triangle is indicated at 21 grams. There are two perpendicular lines drawn, one at 18 grams and the other at 21 grams. A horizontal line between the two lines is labeled 3. The data are as follows: 18 grams, 1 dot; 19 grams, 3 dots; 20 grams, 4 dots; 21 grams, 5 dots; 22 grams, 6 dots; 23 grams, 2 dots, 24 grams, 1 dot.

Table 43.3.7

weight in grams	18	19	19	19	20	20	20	20	21	21	21	21	21	22	22	22	22	22	22	23	23	**24**
distance from mean	3	2	2	2	1	1	1	1	0	0	0	0	0	1	1	1	1	1	1	2	2	**3**

The values in the first row of the table are the cookie weights for the first set of cookies. Their mean, 21 grams, is the *mean of the cookie weights*.

The values in the second row of the table are the *distances* between the values in the first row and 21. The mean of these distances is the *MAD of the cookie weights*.

What can we learn from the averages of these distances once they are calculated?

- In the first set of cookies, the distances are all between 0 and 3. The MAD is 1.2 grams, which tells us that the cookie weights are typically within 1.2 grams of 21 grams. We could say that a typical cookie weighs between 19.8 and 22.2 grams.

- In the second set of cookies, the distances are all between 0 and 13. The MAD is 5.6 grams, which tells us that the cookie weights are typically within 5.6 grams of 21 grams. We could say a typical cookie weighs between 15.4 and 26.6 grams.

The MAD is also called a *measure of the variability* of the distribution. In these examples, it is easy to see that a higher MAD suggests a distribution that is more spread out, showing more variability.

Glossary Entries

Definition: Average

The average is another name for the mean of a data set.

For the data set 3, 5, 6, 8, 11, 12, the average is 7.5.

$3+5+6+8+11+12=45$

$45\div 6=7.5$

Definition: Mean

The mean is one way to measure the center of a data set. We can think of it as a balance point. For example, for the data set 7, 9, 12, 13, 14, the mean is 11.

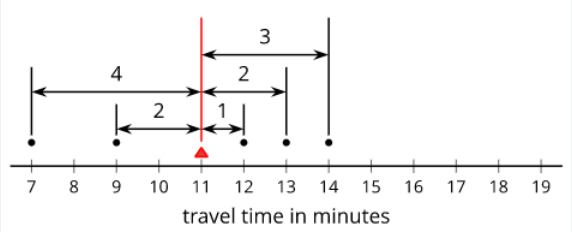


Figure 43.3.6

To find the mean, add up all the numbers in the data set. Then, divide by how many numbers there are. $7+9+12+13+14=55$ and $55\div 5=11$.

Definition: Mean Absolute Deviation (MAD)

The mean absolute deviation is one way to measure how spread out a data set is. Sometimes we call this the MAD. For example, for the data set 7, 9, 12, 13, 14, the MAD is 2.4. This tells us that these travel times are typically 2.4 minutes away from the mean, which is 11.

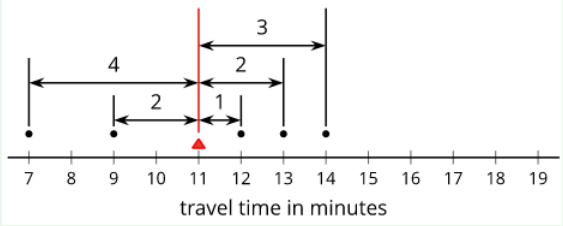


Figure 43.3.7

To find the MAD, add up the distance between each data point and the mean. Then, divide by how many numbers there are.

$4+2+1+2+3=12$ and $12\div 5=2.4$

Definition: Measure of Center

A measure of center is a value that seems typical for a data distribution.

Mean and median are both measures of center.

Practice

Exercise 43.3.5

Han recorded the number of pages that he read each day for five days. The dot plot shows his data.

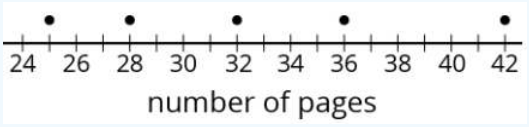


Figure 43.3.8

1. Is 30 pages a good estimate of the mean number of pages that Han read each day? Explain your reasoning.
2. Find the mean number of pages that Han read during the five days. Draw a triangle to mark the mean on the dot plot.
3. Use the dot plot and the mean to complete the table.

Table 43.3.8

number of pages	distance from mean	left or right of mean
25		left
28		
32		
36		
42		

4. Calculate the mean absolute deviation (MAD) of the data. Explain or show your reasoning.

Exercise 43.3.6

Ten sixth-grade students recorded the amounts of time each took to travel to school. The dot plot shows their travel times.

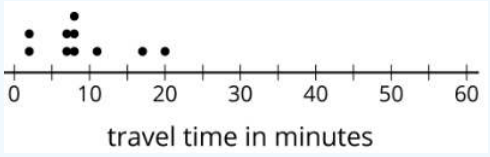


Figure 43.3.9

The mean travel time for these students is approximately 9 minutes. The MAD is approximately 4.2 minutes.

1. Which number of minutes—9 or 4.2—is a typical amount of time for the ten sixth-grade students to travel to school? Explain your reasoning.
2. Based on the mean and MAD, Jada believes that travel times between 5 and 13 minutes are common for this group. Do you agree? Explain your reasoning.
3. A different group of ten sixth-grade students also recorded their travel times to school. Their mean travel time was also 9 minutes, but the MAD was about 7 minutes. What could the dot plot of this second data set be? Describe or draw how it might look.

Exercise 43.3.7

In an archery competition, scores for each round are calculated by averaging the distance of 3 arrows from the center of the target.

An archer has a mean distance of 1.6 inches and a MAD distance of 1.3 inches in the first round. In the second round, the archer's arrows are farther from the center but are more consistent. What values for the mean and MAD would fit this description for the second round? Explain your reasoning.

43.3: Variability and MAD is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

43.4: Using Mean and MAD to Make Comparisons

Lesson

Let's use mean and MAD to describe and compare distributions.

Exercise 43.4.1: Number Talk: Decimal Division

Find the value of each expression mentally.

$42 \div 12$

$2.4 \div 12$

$44.4 \div 12$

$46.8 \div 12$

Exercise 43.4.2: Which Player Would You Choose?

1. Andre and Noah joined Elena, Jada, and Lin in recording their basketball scores. They all recorded their scores in the same way: the number of baskets made out of 10 attempts. Each collected 12 data points.
   - Andre's mean number of baskets was 5.25, and his MAD was 2.6.
   - Noah's mean number of baskets was also 5.25, but his MAD was 1.

Here are two dot plots that represent the two data sets. The triangle indicates the location of the mean.

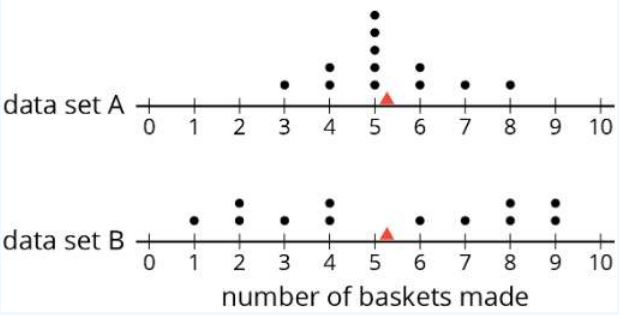


Figure 43.4.1: Two dot plots, number of baskets made, 0 to 10 by ones. Data set A, beginning at 3, number of dots above each increment is 1, 2, 5, 2, 1, 1, triangle approximately 5 point 2. Data set B, beginning at 1, number of dots above each increment is 1, 2, 1, 2, 0, 1, 1, 2, 2, triangle approximately 5 point 2.

   a. Without calculating, decide which dot plot represents Andre's data and which represents Noah's. Explain how you know.
   b. If you were the captain of a basketball team and could use one more player on your team, would you choose Andre or Noah? Explain your reasoning.

2. An eighth-grade student decided to join Andre and Noah and kept track of his scores. His data set is shown here. The mean number of baskets he made is 6.

Table 43.4.1

eighth-grade student	6	5	4	7	6	5	7	8	5	6	5	8
distance from 6												

   a. Calculate the MAD. Show your reasoning.
   b. Draw a dot plot to represent his data and mark the location of the mean with a triangle (Δ).

c. Compare the eighth-grade student's mean and MAD to Noah's mean and MAD. What do you notice?
d. Compare their dot plots. What do you notice about the distributions?
e. What can you say about the two players' shooting accuracy and consistency?

Are you ready for more?

Invent a data set with a mean of 7 and a MAD of 1.

Exercise 43.4.3: Swimmers Over the Years

In 1984, the mean age of swimmers on the U.S. women's swimming team was 18.2 years and the MAD was 2.2 years. In 2016, the mean age of the swimmers was 22.8 years, and the MAD was 3 years.

1. How has the average age of the women on the U.S. swimming team changed from 1984 to 2016? Explain your reasoning.
2. Are the swimmers on the 1984 team closer in age to one another than the swimmers on the 2016 team are to one another? Explain your reasoning.
3. Here are dot plots showing the ages of the women on the U.S. swimming team in 1984 and in 2016. Use them to make two other comments about how the women's swimming team has changed over the years.

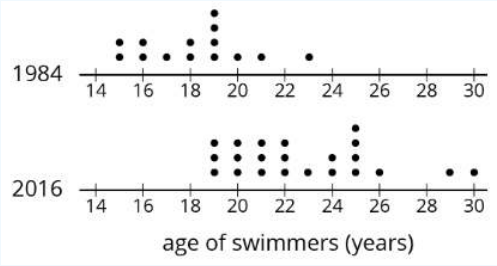


Figure 43.4.2: Two dot plots, age of swimmers in years, 14 to 30 by twos. First plot labeled 1984, beginning with 15 in increments of 1, the number of dots above each increment is 2, 2, 1, 2, 4, 1, 1, 0, 1. Second plot labeled 2016, beginning with 19 increments of 1, the number of dots above each increment is 3, 3, 3, 3, 1, 2, 4, 1, 0, 0, 1, 1.

Summary

Sometimes two distributions have different means but the same MAD.

Pugs and beagles are two different dog breeds. The dot plot shows two sets of weight data—one for pugs and the other for beagles.

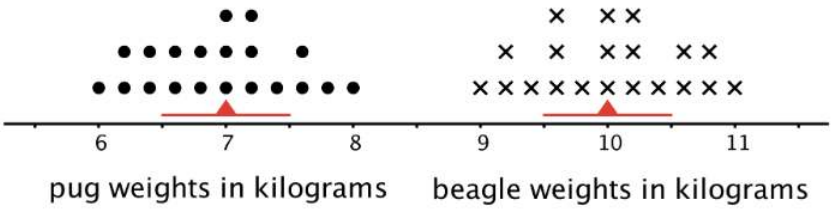


Figure 43.4.3: A dot plot for two sets of data: "pug weights in kilograms" and "beagle weights in kilograms". The numbers 6 through 11 are indicated and there are tick marks midway between each indicated number. There are also two triangles indicated. The first triangle is at 7 kilograms with a horizontal line drawn below the triangle that begins at 6.5 and ends at 7.5 kilograms. The second triangle is indicated at 10 kilograms with a horizontal line drawn below the triangle that begins at 9.5 and ends at 10.5 kilograms. The data for "pug weights in kilograms" are as follows: 6 kilograms, 1 dot. 6.2 kilograms, 2 dots. 6.4 kilograms, 2 dots. 6.6 kilograms, 2 dots. 6.8 kilograms, 2 dots. 7 kilograms, 3 dots. 7.2 kilograms, 3 dots. 7.4 kilograms, 1 dot. 7.6 kilograms, 2 dots. 7.8 kilograms, 1 dot. 8 kilograms, 1 dot. The data for "beagle weights in kilograms" are as follows: 9 kilograms, 1 x. 9.2 kilograms, 2 x's. 9.4 kilograms, 1 x. 9.6 kilograms, 3 x's. 9.8 kilograms, 1 x. 10 kilograms, 3 x's. 10.2 kilograms, 3 x's. 10.4 kilograms, 1 x. 10.6 kilograms, 2 x's. 10.8 kilograms, 2 x's. 11 kilograms, 1 x.

- The mean weight for pugs is 7 kilograms, and the MAD is 0.5 kilogram.
- The mean weight for beagles is 10 kilograms, and the MAD is 0.5 kilogram.

We can say that, in general, the beagles are heavier than the pugs. A typical weight for the beagles in this group is about 3 kilograms heavier than a typical weight for the pugs.

The variability of pug weights, however, is about the same as the variability of beagle weights. In other words, the weights of pugs and the weights of beagles are equally spread out.

Glossary Entries

Definition: Average

The average is another name for the mean of a data set.

For the data set 3, 5, 6, 8, 11, 12, the average is 7.5.

$3+5+6+8+11+12=45$

$45 \div 6=7.5$

Definition: Mean

The mean is one way to measure the center of a data set. We can think of it as a balance point. For example, for the data set 7, 9, 12, 13, 14, the mean is 11.

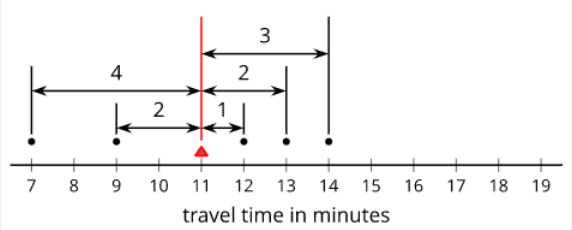


Figure 43.4.4

To find the mean, add up all the numbers in the data set. Then, divide by how many numbers there are. $7+9+12+13+14=55$ and $55 \div 5=11$.

Definition: Mean Absolute Deviation (MAD)

The mean absolute deviation is one way to measure how spread out a data set is. Sometimes we call this the MAD. For example, for the data set 7, 9, 12, 13, 14, the MAD is 2.4. This tells us that these travel times are typically 2.4 minutes away from the mean, which is 11.

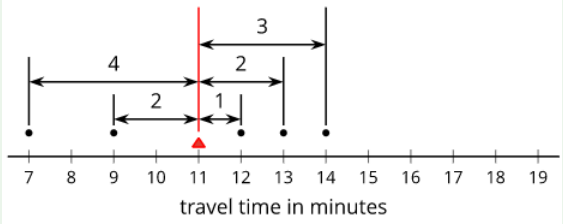


Figure 43.4.5

To find the MAD, add up the distance between each data point and the mean. Then, divide by how many numbers there are.

$4+2+1+2+3=12$ and $12 \div 5=2.4$

Definition: Measure of Center

A measure of center is a value that seems typical for a data distribution.

Mean and median are both measures of center.

Practice

Exercise 43.4.4

The dot plots show the amounts of time that ten U.S. students and ten Australian students took to get to school.

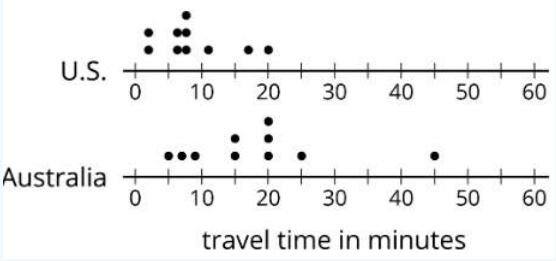


Figure 43.4.6: Two dot plots, travel time in minutes, 0 to 60 by tens. First plot, U S, data indicates 2 minutes, 2 dots. 6 minutes, 2 dots. 7 minutes, 3 dots. 11 minutes, 1 dot. 17 minutes, 1 dot. 20 minutes, 1 dot. Second plot, Australia, data indicates 5 minutes, 1 dot. 7 minutes, 1 dot. 9 minutes, 1 dot. 15 minutes, 2 dots. 20 minutes, 3 dots. 25 minutes, 1 dot. 45 minutes, 1 dot.

Which statement is true about the MAD of the Australian data set?

A. It is significantly less than the MAD of the U.S. data set.
B. It is exactly equal to the MAD of the U.S. data set.
C. It is approximately equal to the MAD of the U.S. data set.
D. It is significantly greater than the MAD of the U.S. data set.

Exercise 43.4.5

The dot plots show the amounts of time that ten South African students and ten Australian students took to get to school. Without calculating, answer the questions.

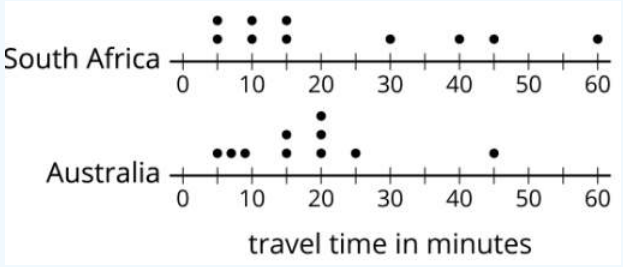


Figure 43.4.7: Two dot plots, travel time in minutes, 0 to 60 by tens. First plot, South Africa, data indicates 5 minutes, 2 dots. 10 minutes, 2 dots. 15 minutes, 2 dots. 30 minutes, 1 dot. 40 minutes, 1 dot. 45 minutes, 1 dot. 60 minutes, 1 dot. Second plot, Australia, data indicates 5 minutes, 1 dot. 7 minutes, 1 dot. 9 minutes, 1 dot. 15 minutes, 2 dots. 20 minutes, 3 dots. 25 minutes, 1 dot. 45 minutes, 1 dot.

1. Which data set has the smaller mean? Explain your reasoning.
2. Which data set has the smaller MAD? Explain your reasoning.
3. What does a smaller mean tell us in this context?
4. What does a smaller MAD tell us in this context?

Exercise 43.4.6

Two high school basketball teams have identical records of 15 wins and 2 losses. Sunnyside High School's mean score is 50 points and its MAD is 4 points. Shadyside High School's mean score is 60 points and its MAD is 15 points.

Lin read the records of each team's score. She likes the team that had nearly the same score for every game it played. Which team do you think Lin likes? Explain your reasoning.

Exercise 43.4.7

Jada thinks the perimeter of this rectangle can be represented with the expression $a+a+b+b$. Andre thinks it can be represented with $2a+2b$. Do you agree with either of them? Explain your reasoning.

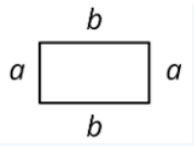


Figure 43.4.8

(From Unit 6.2.3)

Exercise 43.4.8

Draw a number line.

1. Plot and label three numbers between -2 and -8 (not including -2 and -8).
2. Use the numbers you plotted and the symbols $<$ and $>$ to write three inequality statements.

(From Unit 7.1.3)

Exercise 43.4.9

Adult elephant seals generally weigh about 5,500 pounds. If you weighed 5 elephant seals, would you expect each seal to weigh exactly 5,500 pounds? Explain your reasoning.

(From Unit 8.1.2)

43.4: Using Mean and MAD to Make Comparisons is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

Section 44: Median and IQR

This page titled Section 44: Median and IQR is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

44.1: Median

Lesson

Let's explore the median of a data set and what it tells us.

Exercise 44.1.1: The Plot of the Story

1. Here are two dot plots and two stories. Match each story with a dot plot that could represent it. Be prepared to explain your reasoning.

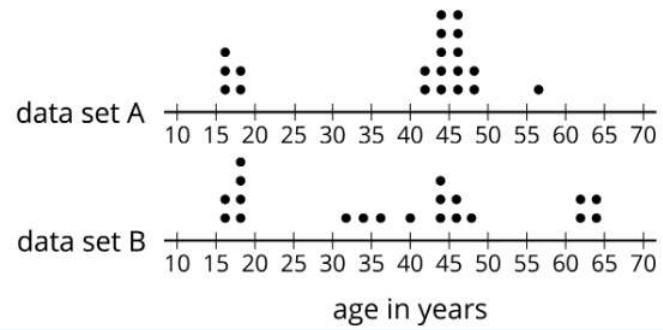


Figure 44.1.1: Two dot plots from 10 to 75 by 5's. Age in years. Labeled data set A, data set B. Top dot plot labeled data set A. Beginning at 10, 5 dots between 15 and 20. 7 dots between 40 and 45. 7 dots between 45 and 50. 1 dot between 55 and 60. Bottom dot plot labeled data set B. Beginning at 10, 6 dots between 15 and 20. 2 dots between 30 and 35. 1 dot between 35 and 40. 1 dot on 40. 3 dots between 40 and 45. 3 dots between 45 and 50. 4 dots between 60 and 65.

   - Twenty people—high school students, teachers, and invited guests—attended a rehearsal for a high school musical. The mean age was 38.5 years and the MAD was 16.5 years.
   - High school soccer team practice is usually watched by supporters of the players. One evening, twenty people watched the team practice. The mean age was 38.5 years and the MAD was 12.7 years.

2. Another evening, twenty people watched the soccer team practice. The mean age was similar to that from the first evening, but the MAD was greater (about 20 years).
   Make a dot plot that could illustrate the distribution of ages in this story.

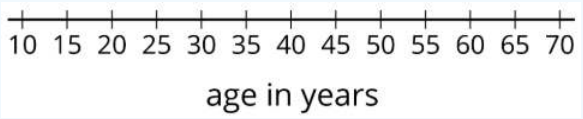


Figure 44.1.2

Exercise 44.1.2: Siblings in the House

Here is data that shows the numbers of siblings of ten students in Tyler's class.

1 0 2 1 7 0 2 0 1 10

1. Represent the data shown with a dot plot.
2. Without making any calculations, estimate the center of the data based on your dot plot. What is a typical number of siblings for these sixth-grade students? Mark the location of that number on your dot plot.
3. Find the mean. Show your reasoning.
4. a. How does the mean compare to the value that you marked on the dot plot as a typical number of siblings? (Is it a little larger, a lot larger, exactly the same, a little smaller, or a lot smaller than your estimate?)
   b. Do you think the mean summarizes the data set well? Explain your reasoning.

Are you ready for more?

Invent a data set with a mean that is significantly lower than what you would consider a typical value for the data set.

Exercise 44.1.3: Finding the Middle

1. Your teacher will give you an index card. Write your first and last names on the card. Then record the total number of letters in your name. After that, pause for additional instructions from your teacher.
2. Here is the data set on numbers of siblings from an earlier activity.

$$1 \quad 0 \quad 2 \quad 1 \quad 7 \quad 0 \quad 2 \quad 0 \quad 1 \quad 10$$

   a. Sort the data from least to greatest, and then find the **median**.
   b. In this situation, do you think the median is a good measure of a typical number of siblings for this group? Explain your reasoning.

3. Here is the dot plot showing the travel time, in minutes, of Elena's bus rides to school.

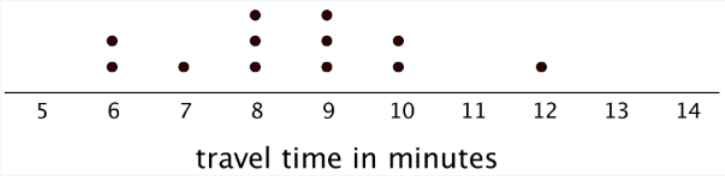


Figure 44.1.3

   a. Find the median travel time. Be prepared to explain your reasoning.
   b. What does the median tell us in this context?

Summary

The **median** is another measure of center of a distribution. It is the middle value in a data set when values are listed in order. Half of the values in a data set are less than or equal to the median,and half of the values are greater than or equal to the median.

To find the median, we order the data values from least to greatest and find the number in the middle.

Suppose we have 5 dogs whose weights, in pounds, are shown in the table. The median weight for this group of dogs is 32 pounds because three dogs weigh less than or equal to 32 pounds and three dogs weigh greater than or equal to 32 pounds.

$$20 \quad 25 \quad 32 \quad 40 \quad 55$$

Now suppose we have 6 cats whose weights, in pounds, are as shown in the table. Notice that there are *two* values in the middle: 7 and 8.

$$4 \quad 6 \quad 7 \quad 8 \quad 10 \quad 10$$

The median weight must be between 7 and 8 pounds, because half of the cats weigh less or equal to 7 pounds and half of the cats weigh greater than or equal to 8 pounds.

In general, when we have an even number of values, we take the number exactly in between the two middle values. In this case, the median cat weight is 7.5 pounds because $(7+8) \div 2 = 7.5$.

Glossary Entries

Definition: Median

The median is one way to measure the center of a data set. It is the middle number when the data set is listed in order.

For the data set 7, 9, 12, 13, 14, the median is 12.

For the data set 3, 5, 6, 8, 11, 12, there are two numbers in the middle. The median is the average of these two numbers. $6+8=14$ and $14 \div 2 = 7$.

Practice

https://math.libretexts.org/@go/page/40885

Exercise 44.1.4

Here is data that shows a student's scores for 10 rounds of a video game.

130 150 120 170 130 120 160 160 190 140

What is the median score?

A. 125
B. 145
C. 147
D. 150

Exercise 44.1.5

When he sorts the class's scores on the last test, the teacher notices that exactly 12 students scored better than Clare and exactly 12 students scored worse than Clare. Does this mean that Clare's score on the test is the median? Explain your reasoning.

Exercise 44.1.6

The medians of the following dot plots are 6, 12, 13, and 15, but not in that order. Match each dot plot with its median.

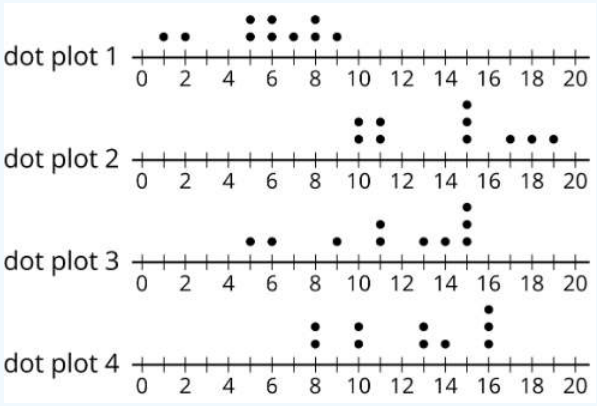


Figure 44.1.4: Four dot plots, 0 to 20 by twos. Dot plot 1, beginning with 1 in increments of 1, the number of dots above each increment is 1, 1, 0, 0, 2, 2, 1, 2, 1. Dot plot 2, beginning with 10 in increments of 1, the number of dots above each increment are 2, 2, 0, 0, 0, 3, 0, 1, 1, 1. Dot plot 3, beginning with 5 in increments of 1, the number of dots above each increment are 1, 1, 0, 0, 1, 0, 2, 0, 1, 1, 3. Dot plot 4, beginning with 8 in increments of 1, the number of dots above each increment is 2, 0, 2, 0, 0, 2, 1, 0, 3.

Exercise 44.1.7

Invent a data set with five numbers that has a mean of 10 and a median of 12.

Exercise 44.1.8

Ten sixth-grade students reported the hours of sleep they get on nights before a school day. Their responses are recorded in the dot plot.

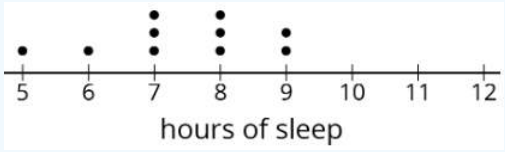


Figure 44.1.5

Looking at the dot plot, Lin estimated the mean number of hours of sleep to be 8.5 hours. Noah's estimate was 7.5 hours. Diego's estimate was 6.5 hours.

Which estimate do you think is best? Explain how you know.

(From Unit 8.3.2)

Exercise 44.1.9

In one study of wild bears, researchers measured the weights, in pounds, of 143 wild bears that ranged in age from newborn to 15 years old. The data were used to make this histogram.

1. What can you say about the heaviest bear in this group?
2. What is a typical weight for the bears in this group?
3. Do more than half of the bears in this group weigh less than 250 pounds?
4. If weight is related to age, with older bears tending to have greater body weights, would you say that there were more old bears or more young bears in the group? Explain your reasoning.

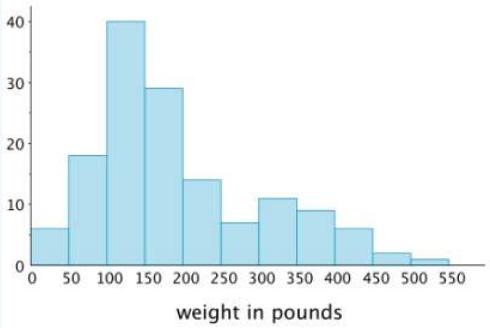


Figure 44.1.6

(From Unit 8.2.6)

44.1: Median is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

44.2: Comparing Mean and Median

Lesson

Let's compare the mean and median of data sets.

Exercise 44.2.1: Heights of Presidents

Here are two dot plots. The first dot plot shows the heights of the first 22 U.S. presidents. The second dot plot shows the heights of the next 22 presidents.

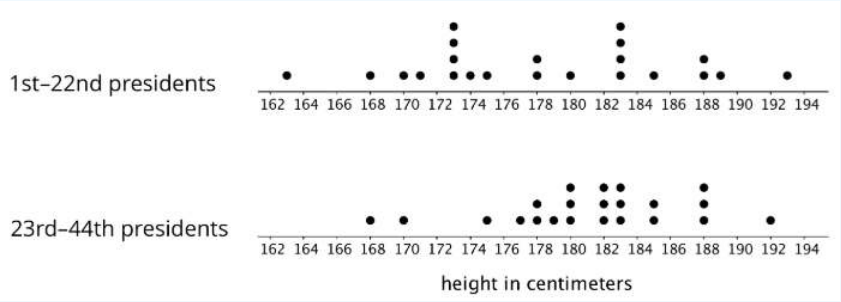


Figure 44.2.1: Two dot plots from 162 to 194 by 2's. Height in centimeters. Top dot plot labeled 1st-22nd presidents. Bottom dot plot labeled 23rd-44th presidents. Top dot plot beginning at 162, number of dots above each increment is 0, 1, 0, 0, 0, 1, 0, 1, 1, 0, 4, 1, 1, 0, 0, 2, 0, 1, 0, 0, 4, 0, 1, 0, 0, 2, 1, 0, 0, 0 , 1, 0. Bottom dot plot, number of dots above each increment is 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 1, 2, 1, 3, 0, 3, 3, 0, 2, 0, 0, 3, 0, 0, 0, 1, 0, 0.

Based on the two dot plots, decide if you agree or disagree with each of the following statements. Be prepared to explain your reasoning.

1. The median height of the first 22 presidents is 178 centimeters.
2. The mean height of the first 22 presidents is about 183 centimeters.
3. A typical height for a president in the second group is about 182 centimeters.
4. U.S. presidents have become taller over time.
5. The heights of the first 22 presidents are more alike than the heights of the second 22 presidents.
6. The MAD of the second data set is greater than the MAD of the first set.

Exercise 44.2.2: The Tallest and the Smallest in the World

Your teacher will provide the height data for your class. Use the data to complete the following questions.

1. Find the mean height of your class in centimeters.
2. Find the median height in centimeters. Show your reasoning.
3. Suppose that the world's tallest adult, who is 251 centimeters tall, joined your class.
   a. Discuss the following questions with your group and explain your reasoning.
      - How would the mean height of the class change?
      - How would the median height change?
   b. Find the new mean.
   c. Find the new median.
   d. Which measure of center—the mean or the median—changed more when this new person joined the class? Explain why the value of one measure changed more than the other.
4. The world's smallest adult is 63 centimeters tall. Suppose that the world's tallest and smallest adults both joined your class.
   a. Discuss the following questions with your group and explain your reasoning.
      - How would the mean height of the class change from the original mean?
      - How would the median height change from the original median?
   b. Find the new mean.
   c. Find the new median.

d. How did the measures of center—the mean and the median—change when these two people joined the class? Explain why the values of the mean and median changed the way they did.

Exercise 44.2.3: Mean or Median?

1. Your teacher will give you six cards. Each has either a dot plot or a histogram. Sort the cards into *two* piles based on the distributions shown. Be prepared to explain your reasoning.
2. Discuss your sorting decisions with another group. Did you have the same cards in each pile? If so, did you use the same sorting categories? If not, how are your categories different?
   Pause here for a class discussion.
3. Use the information on the cards to answer the following questions.
   a. Card A: What is a typical age of the dogs being treated at the animal clinic?
   b. Card B: What is a typical number of people in the Irish households?
   c. Card C: What is a typical travel time for the New Zealand students?
   d. Card D: Would 15 years old be a good description of a typical age of the people who attended the birthday party?
   e. Card E: Is 15 minutes or 24 minutes a better description of a typical time it takes the students in South Africa to get to school?
   f. Card F: Would 21.3 years old be a good description of a typical age of the people who went on a field trip to Washington, D.C.?
4. How did you decide which measure of center to use for the dot plots on Cards A–C? What about for those on Cards D–F?

Are you ready for more?

Most teachers use the mean to calculate a student's final grade, based on that student's scores on tests, quizzes, homework, projects, and other graded assignments.

Diego thinks that the median might be a better way to measure how well a student did in a course. Do you agree with Diego? Explain your reasoning.

Summary

Both the mean and the median are ways of measuring the center of a distribution. They tell us slightly different things, however.

The dot plot shows the weights of 30 cookies. The mean weight is 21 grams (marked with a triangle). The median weight is 20.5 grams (marked with a diamond).



Figure 44.2.2

The mean tells us that if the weights of all cookies were distributed so that each one weighed the same, that weight would be 21 grams. We could also think of 21 grams as a balance point for the weights of all of the cookies in the set.

The median tells us that half of the cookies weigh more than 20.5 grams and half weigh less than 20.5 grams. In this case, both the mean and the median could describe a typical cookie weight because they are fairly close to each other and to most of the data points.

Here is a different set of 30 cookies. It has the same mean weight as the first set, but the median weight is 23 grams.

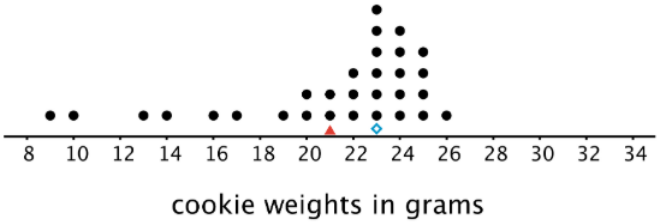


Figure 44.2.3

In this case, the median is closer to where most of the data points are clustered and is therefore a better measure of center for this distribution. That is, it is a better description of a typical cookie weight. The mean weight is influenced (in this case, pulled down) by a handful of much smaller cookies, so it is farther away from most data points.

In general, when a distribution is symmetrical or approximately symmetrical, the mean and median values are close. But when a distribution is not roughly symmetrical, the two values tend to be farther apart.

Glossary Entries

Definition: Median

The median is one way to measure the center of a data set. It is the middle number when the data set is listed in order.

For the data set 7, 9, 12, 13, 14, the median is 12.

For the data set 3, 5, 6, 8, 11, 12, there are two numbers in the middle. The median is the average of these two numbers. $6+8=14$ and $14\div 2=7$.

Practice

Exercise 44.2.4

Here is a dot plot that shows the ages of teachers at a school.

Which of these statements is true of the data set shown in the dot plot?

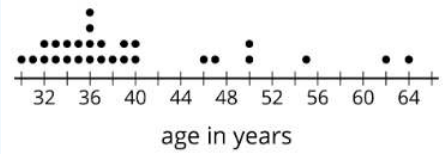


Figure 44.2.4

A. The mean is less than the median.
B. The mean is approximately equal to the median.
C. The mean is greater than the median.
D. The mean cannot be determined.

Exercise 44.2.5

Priya asked each of five friends to attempt to throw a ball in a trash can until they succeeded. She recorded the number of unsuccessful attempts made by each friend as: 1, 8, 6, 2, 4. Priya made a mistake: The 8 in the data set should have been 18.

How would changing the 8 to 18 affect the mean and median of the data set?

A. The mean would decrease and the median would not change.
B. The mean would increase and the median would not change.
C. The mean would decrease and the median would increase.
D. The mean would increase and the median would increase.

Exercise 44.2.6

In his history class, Han's homework scores are:

100 100 100 100 95 100 90 100 0

The history teacher uses the mean to calculate the grade for homework. Write an argument for Han to explain why median would be a better measure to use for his homework grades.

Exercise 44.2.7

The dot plots show how much time, in minutes, students in a class took to complete each of five different tasks. Select **all** the dot plots of tasks for which the mean time is approximately equal to the median time.

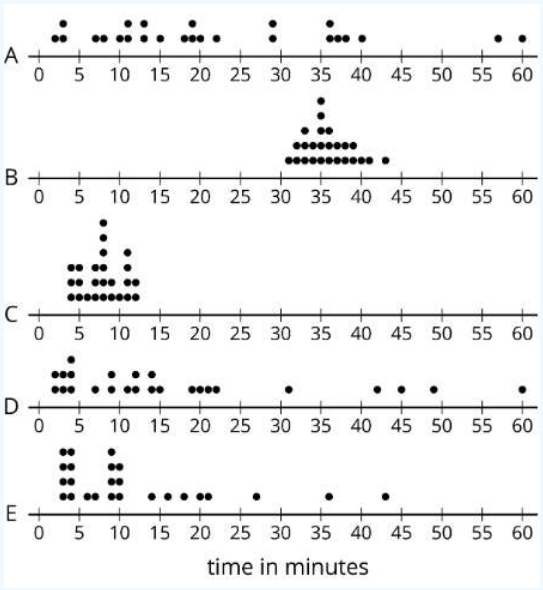


Figure 44.2.5: Five dot plots labeled A, B, C, D, and E. On each dot plot, the numbers 0 through 60, in increments of 5, are indicated. The data for dot plot A are as follows: 2, 1 dot. 3, 2 dots. 7, 1 dot. 8, 1 dot. 10, 1 dot. 11, 2 dots. 13, 2 dots. 15, 1 dot. 18, 1 dot. 19, 2 dots. 20, 1 dot. 22, 1 dot. 29, 2 dots. 36, 2 dots. 37, 1 dot. 38, 1 dot. 40, 1 dot. 57, 1 dot. 60, 1 dot. The data for dot plot B are as follows: 31, 1 dot. 32, 2 dots. 33, 3 dots. 34, 2 dots. 35, 5 dots. 36, 3 dots. 37, 2 dots. 38, 2 dots. 39, 2 dots. 40, 1 dot. 41, 1 dot. 44, 1 dot. The data for dot plot C are as follows: 4, 3 dots. 5, 3 dots. 6, 1 dot. 7, 3 dots. 8, 6 dots. 9, 2 dots. 10, 1 dot. 11, 4 dots. 12, 2 dots. The data for dot plot D are as follows: 2, 2 dots. 3, 2 dots. 4, 3 dots. 7, 1 dot. 9, 2 dots. 11, 1 dot. 12, 2 dots. 14, 2 dots. 15, 1 dot. 19, 1 dot. 20, 1 dot. 21, 1 dot. 22, 1 dot. 31, 1 dot. 43, 1 dot. 45, 1 dot. 49, 1 dot. 60, 1 dot. The data for dot plot E are as follows: 3, 4 dots. 4, 4 dots. 6, 1 dot. 7, 1 dot. 9, 4 dots. 10, 3 dots. 14, 1 dot. 16, 1 dot. 18, 1 dot. 20, 1 dot. 27, 1 dot. 36, 1 dot. 44, 1 dot.

Exercise 44.2.8

Zookeepers recorded the ages, weights, genders, and heights of the 10 pandas at their zoo. Write two statistical questions that could be answered using these data sets.

(From Unit 8.1.2)

Exercise 44.2.9

Here is a set of coordinates. Draw and label an appropriate pair of axes and plot the points. $A = (1, 0), B = (0, 0.5), C = (4, 3.5), D = (1.5, 0.5)$

(From Unit 7.3.2)

44.2: Comparing Mean and Median is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

44.3: Quartiles and Interquartile Range

Lesson

Let's look at other measures for describing distributions.

Exercise 44.3.1: Notice and Wonder: Two Parties

Here are dot plots that show the ages of people at two different parties. The mean of each distribution is marked with a triangle.

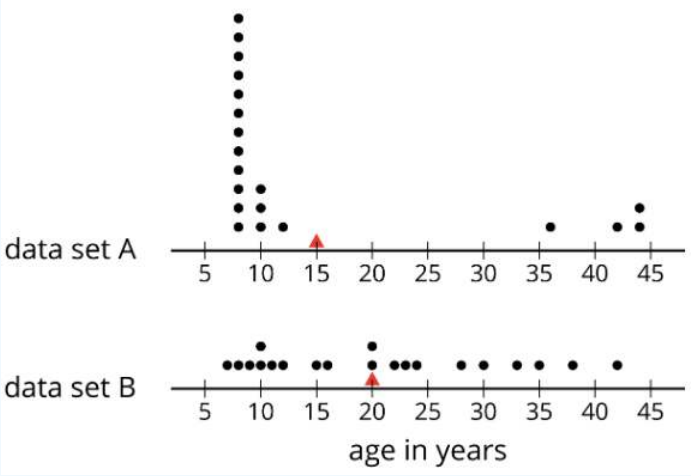


Figure 44.3.1: Two dot plots from 5 to 45 by 5's. Age in years. Top dot plot labeled data set A. Bottom dot plot labeled data set B.There is a red triangle indicated at 15 on "data set A" and at 20 on "data set B". The data for "data set A" are as follows: 8 years, 12 dots. 10 years, 3 dots. 12 years, 1 dot. 15 years, red triangle. 36 years, 1 dot. 42 years, 1 dot. 44 years, 2 dots. The data for "data set B" are as follows: 7 years, 1 dot. 8 years, 1 dot. 9 years, 1 dot. 10 years, 2 dots. 15 years, 1 dot. 16 years, 1 dot. 20 years, 2 dots and 1 red triangle. 22 years, 1 dot. 23 years, 1 dot. 24 years, 1 dot. 28 years, 1 dot. 30 years, 1 dot. 33 years, 1 dot. 35 years, 1 dot. 38 years, 1 dot. 42 years, 1 dot.

What do you notice and what do you wonder about the distributions in the two dot plots?

Exercise 44.3.2: The Five-Number Summary

Here are the ages of the people at one party, listed from least to greatest.

7 8 9 10 10 11 12 15 16 20 20 22 23 24 28 30 33 35 38 42

1. a. Find the median of the data set and label it "50th percentile." This splits the data into an upper half and a lower half.
   b. Find the middle value of the *lower* half of the data, without including the median. Label this value "25th percentile."
   c. Find the middle value of the *upper* half of the data, without including the median. Label this value "75th percentile."
2. You have split the data set into four pieces. Each of the three values that split the data is called a **quartile**.
   - We call the 25th percentile the *first quartile*. Write "Q1" next to that number.
   - The median can be called the *second quartile*. Write "Q2" next to that number.
   - We call the 75th percentile the *third quartile*. Write "Q3" next to that number.
3. Label the lowest value in the set "minimum" and the greatest value "maximum."
4. The values you have identified make up the *five-number summary* for the data set. Record them here.
   minimum: _____ Q1: _____ Q2: _____ Q3: _____ maximum: _____
5. The median of this data set is 20. This tells us that half of the people at the party were 20 years old or younger, and the other half were 20 or older. What do each of these other values tell us about the ages of the people at the party?
   a. the third quartile
   b. the minimum
   c. the maximum

Are you ready for more?

There was another party where 21 people attended. Here is the five-number summary of their ages.

minimum: 5 Q1: 6 Q2: 27 Q3: 32 maximum: 60

1. Do you think this party had more children or fewer children than the earlier one? Explain your reasoning.
2. Were there more children or adults at this party? Explain your reasoning.

Exercise 44.3.3: Range and Interquartile Range

1. Here is a dot plot that shows the lengths of Elena's bus rides to school, over 12 days.

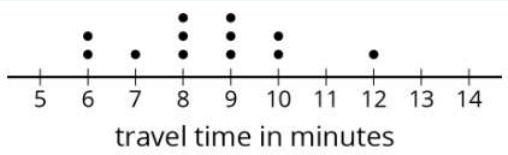


Figure 44.3.2

Write the five-number summary for this data set. Show your reasoning.

2. The **range** is one way to describe the *spread* of values in a data set. It is the difference between the maximum and minimum. What is the range of Elena's travel times?
3. Another way to describe the spread of values in a data set is the **interquartile range (IQR)**. It is the difference between the upper quartile and the lower quartile.
    a. What is the interquartile range (IQR) of Elena's travel times?
    b. What fraction of the data values are between the lower and upper quartiles?
4. Here are two more dot plots.

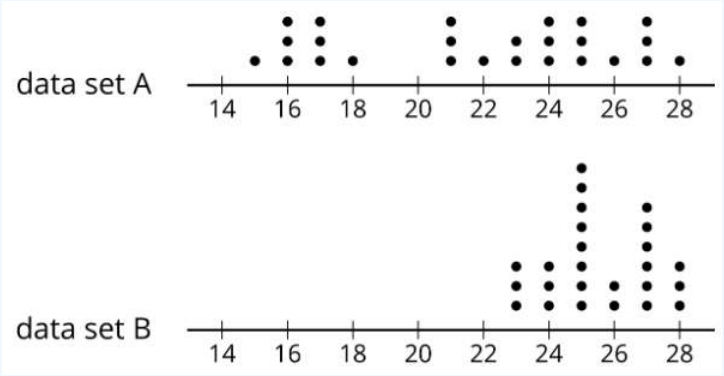


Figure 44.3.3: Two dot plots from 14 to 28 by 2's. Top dot plot labeled data set A. Bottom dot plot labeled data set B. Top dot plot beginning at 14, number of dots above each increment of 1 is 0, 1, 3, 3, 1, 0, 0, 3, 1, 2, 3, 4, 1, 3, 1. Bottom dot plot beginning at 14, number of dots above each increment of 1 is 0, 0, 0, 0, 0, 0, 0, 0, 0, 3, 3, 8, 2, 6, 3.

Without doing any calculations, predict:

    a. Which data set has the smaller range?
    b. Which data set has the smaller IQR?

5. Check your predictions by calculating the range and IQR for the data in each dot plot.

Summary

Earlier we learned that the mean is a measure of the center of a distribution and the MAD is a measure of the variability (or spread) that goes with the mean. There is also a measure of spread that goes with the median. It is called the interquartile range (IQR).

Finding the IQR involves splitting a data set into fourths. Each of the three values that splits the data into fourths is called a **quartile**.

- The median, or second quartile (Q2), splits the data into two halves.
- The first quartile (Q1) is the middle value of the lower half of the data.

- The third quartile (Q3) is the middle value of the upper half of the data.

For example, here is a data set with 11 values.

Table 44.3.1

12	19	20	21	22	33	34	35	40	40	49
		Q1			Q2			Q3		

- The median is 33.
- The first quartile is 20. It is the median of the numbers less than 33.
- The third quartile 40. It is the median of the numbers greater than 33.

The difference between the maximum and minimum values of a data set is the **range**. The difference between Q3 and Q1 is the **interquartile range (IQR)**. Because the distance between Q1 and Q3 includes the middle two-fourths of the distribution, the values between those two quartiles are sometimes called the *middle half of the data*.

The bigger the IQR, the more spread out the middle half of the data values are. The smaller the IQR, the closer together the middle half of the data values are. This is why we can use the IQR as a measure of spread.

A *five-number summary* can be used to summarize a distribution. It includes the minimum, first quartile, median, third quartile, and maximum of the data set. For the previous example, the five-number summary is 12, 20, 33, 40, and 49. These numbers are marked with diamonds on the dot plot.

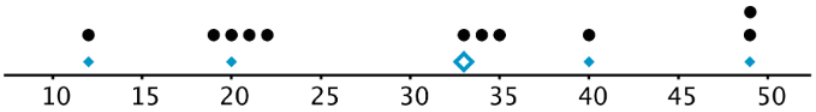


Figure 44.3.4: A dot plot. The numbers 10 through 50, in increments of 5, are indicated. There are diamonds indicated at 12, 20, 33, 40 and 49. The data are as follows: 12, 1 dot; 19, 1 dot; 20, 1 dot; 21, 1 dot; 22, 1 dot; 33, 1 dot; 34, 1 dot; 35, 1 dot; 40, 1 dot; 49, 2 dots.

Different data sets can have the same five-number summary. For instance, here is another data set with the same minimum, maximum, and quartiles as the previous example.

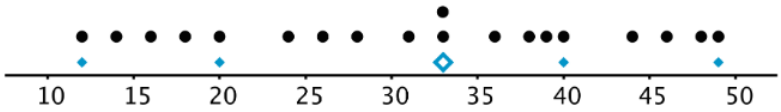


Figure 44.3.5: A dot plot. The numbers 10 through 50, in increments of 5, are indicated. There are diamonds indicated at 12, 20, 33, 40, and 49. The data are as follows: 12, 1 dot; 14, 1 dot; 16, 1 dot; 18, 1 dot; 20, 1 dot; 24, 1 dot; 26, 1 dot; 28, 1 dot; 31, 1 dot; 33, 2 dots; 36, 1 dot; 38, 1 dot; 39, 1 dot; 40, 1 dot; 44, 1 dot; 46, 1 dot; 48, 1 dot; 49, 1 dot.

Glossary Entries

Definition: Interquartile Range (IQR)

The interquartile range is one way to measure how spread out a data set is. We sometimes call this the IQR. To find the interquartile range we subtract the first quartile from the third quartile.

For example, the IQR of this data set is 20 because $50 - 30 = 20$.

Table 44.3.2

22	29	30	31	32	43	44	45	50	50	59
		Q1			Q2			Q3		

Definition: Median

The median is one way to measure the center of a data set. It is the middle number when the data set is listed in order.

For the data set 7, 9, 12, 13, 14, the median is 12.

For the data set 3, 5, 6, 8, 11, 12, there are two numbers in the middle. The median is the average of these two numbers. $6+8=14$ and $14\div 2=7$.

Definition: Quartile

Quartiles are the numbers that divide a data set into four sections that each have the same number of values.

For example, in this data set the first quartile is 30. The second quartile is the same thing as the median, which is 43. The third quartile is 50.

Table 44.3.3

22	29	30	31	32	43	44	45	50	50	59
		Q1			Q2			Q3		

Definition: Range

The range is the distance between the smallest and largest values in a data set. For example, for the data set 3, 5, 6, 8, 11, 12, the range is 9, because $12-3=9$.

Practice

Exercise 44.3.4

Suppose that there are 20 numbers in a data set and that they are all different.

1. How many of the values in this data set are between the first quartile and the third quartile?
2. How many of the values in this data set are between the first quartile and the median?

Exercise 44.3.5

In a word game, 1 letter is worth 1 point. This dot plot shows the scores for 20 common words.

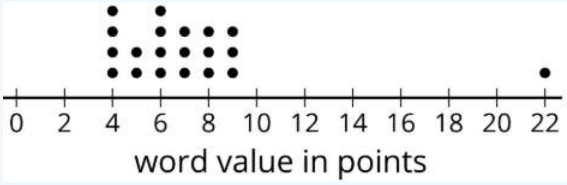


Figure 44.3.6

1. What is the median score?
2. What is the first quartile (Q1)?
3. What is the third quartile (Q3)?
4. What is the interquartile range (IQR)?

Exercise 44.3.6

Mai and Priya each played 10 games of bowling and recorded the scores. Mai's median score was 120, and her IQR was 5. Priya's median score was 118, and her IQR was 15. Whose scores probably had less variability? Explain how you know.

Exercise 44.3.7

Here are five dot plots that show the amounts of time that ten sixth-grade students in five countries took to get to school. Match each dot plot with the appropriate median and IQR.

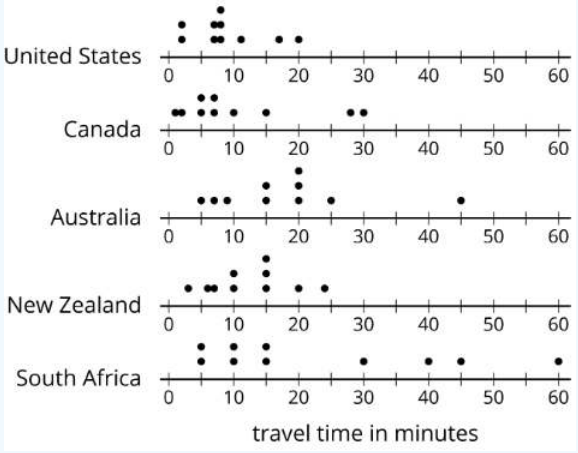


Figure 44.3.7: Five dot plots for "travel time in minutes" labeled “United States”, “Canada”, “Australia”, “New Zealand”, and “South Africa”. Each dot plot has the numbers 0 through 60, in increments of 10. There are also tick marks midway between. The approximate data for "United States" are as follows: 2 minutes, 2 dots; 7 minutes, 2 dots; 8 minutes, 3 dots; 11 minutes, 1 dot; 17 minutes, 1 dot; 20 minutes, 1 dot. The approximate data for "Canada" are as follows: 1 minute, 1 dot; 2 minutes, 1 dot; 5 minutes, 2 dots; 7 minutes, 2 dots; 10 minutes, 1 dot; 15 minutes, 1 dot; 28 minutes, 1 dot; 30 minutes, 1 dot. The approximate data for "Australia" are as follows: 5 minutes, 1 dot; 7 minutes, 1 dot; 9 minutes, 1 dot; 15 minutes, 2 dots; 20 minutes, 3 dots; 25 minutes, 1 dot; 45 minutes, 1 dot. The approximate data for "New Zealand" are as follows: 3 minutes, 1 dot; 6 minutes, 1 dot; 7 minutes, 1 dot; 10 minutes, 2 dots; 15 minutes, 3 dots; 20 minutes, 1 dot; 24 minutes, 1 dot. The approximate data for "South Africa" are as follows: 5 minutes, 2 dots; 10 minutes, 2 dots; 15 minutes, 2 dots; 30 minutes, 1 dot; 40 minutes, 1 dot; 45 minutes, 1 dot; 60 minutes, 1 dot.

1. Median: 17.5, IQR: 11
2. Median: 15, IQR: 30
3. Median: 8, IQR: 4
4. Median: 7, IQR: 10
5. Median: 12.5, IQR: 8

Exercise 44.3.8

Draw and label an appropriate pair of axes and plot the points. $A = (10, 50), B = (30, 25), C = (0, 30), D = (20, 35)$

(From Unit 7.3.2)

Exercise 44.3.9

There are 20 pennies in a jar. If 16% of the coins in the jar are pennies, how many coins are there in the jar?

(From Unit 6.2.2)

44.3: Quartiles and Interquartile Range is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

44.4: Box Plots

Lesson

Let's explore how box plots can help us summarize distributions.

Exercise 44.4.1: Notice and Wonder: Puppy Weights

Here are the birth weights, in ounces, of all the puppies born at a kennel in the past month.

13 14 15 15 16 16 16 16 17 17 17 17 17 17 17 18
18 18 18 18 18 18 18 19 20

What do you notice and wonder about the distribution of the puppy weights?

Exercise 44.4.2: Human Box Plot

Your teacher will give you the data on the lengths of names of students in your class. Write the five-number summary by finding the data set's minimum, Q1, Q2, Q3, and the maximum.

Pause for additional instructions from your teacher.

Exercise 44.4.3: Studying Blinks

Twenty people participated in a study about blinking. The number of times each person blinked while watching a video for one minute was recorded. The data values are shown here, in order from smallest to largest.

3 6 8 11 11 13 14 14 14 14 16 18 20 20 20 22 24
32 36 51

1.

a. Use the grid and axis to make a dot plot of this data set.

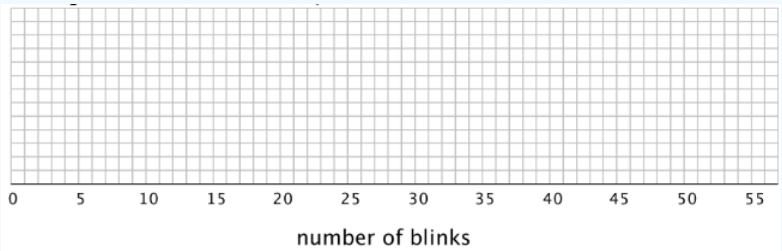


Figure 44.4.1

b. Find the median (Q2) and mark its location on the dot plot.
c. Find the first quartile (Q1) and the third quartile (Q3). Mark their locations on the dot plot.
d. What are the minimum and maximum values?

2. A **box plot** can be used to represent the five-number summary graphically. Let's draw a box plot for the number-of-blinks data. On the grid, *above* the dot plot:

a. Draw a box that extends from the first quartile (Q1) to the third quartile (Q3). Label the quartiles.
b. At the median (Q2), draw a vertical line from the top of the box to the bottom of the box. Label the median.
c. From the left side of the box (Q1), draw a horizontal line (a whisker) that extends to the minimum of the data set. On the right side of the box (Q3), draw a similar line that extends to the maximum of the data set.

3. You have now created a box plot to represent the number of blinks data. What fraction of the data values are represented by each of these elements of the box plot?

a. The left whisker
b. The box
c. The right whisker

Are you ready for more?

Suppose there were some errors in the data set: the smallest value should have been 6 instead of 3, and the largest value should have been 41 instead of 51. Determine if any part of the five-number summary would change. If you think so, describe how it would change. If not, explain how you know.

Summary

A **box plot** represents the five-number summary of a data set.

It shows the first quartile (Q1) and the third quartile (Q3) as the left and right sides of a rectangle or a box. The median (Q2) is shown as a vertical segment inside the box. On the left side, a horizontal line segment—a “whisker”—extends from Q1 to the minimum value. On the right, a whisker extends from Q3 to the maximum value.

The rectangle in the middle represents the middle half of the data. Its width is the IQR. The whiskers represent the bottom quarter and top quarter of the data set.

Earlier we saw dot plots representing the weights of pugs and beagles. The box plots for these data sets are shown above the corresponding dot plots.

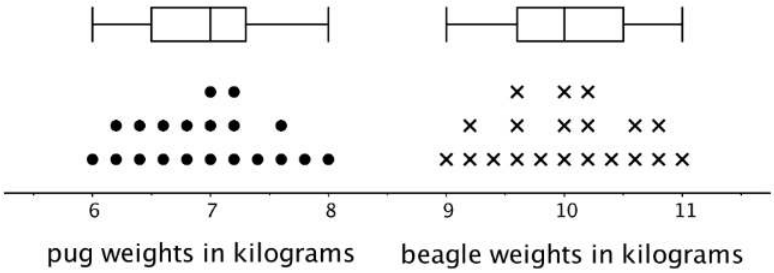


Figure 44.4.2: Box plots and dot plots for two sets of data: "pug weights in kilograms” and "beagle weights in kilograms". The numbers 6 through 11 are indicated and there are tick marks midway between each indicated number. Each box plot is above it's corresponding dot plot. The approximate data for the box plot for "pug weights in kilograms" are as follows: Minimum value, 6. Maximum value, 8. Q1, 6.5. Q2, 7. Q3, 7.3. The approximate data for the dot plot for "pug weights in kilograms" are as follows: 6 kilograms, 1 dot. 6.2 kilograms, 2 dots. 6.4 kilograms, 2 dots. 6.6 kilograms, 2 dots. 6.8 kilograms, 2 dots. 7 kilograms, 3 dots. 7.2 kilograms, 3 dots. 7.4 kilograms, 1 dot. 7.6 kilograms, 2 dots. 7.8 kilograms, 1 dot. 8 kilograms, 1 dot. The approximate data for the box plot for "beagle weights in kilograms" are as follows: Minimum value, 9. Maximum value, 11. Q1, 9.6. Q2, 10. Q3, 10.5. The approximate data for the dot plot for "beagle weights in kilograms" are as follows: 9 kilograms, 1 x. 9.2 kilograms, 2 x's. 9.4 kilograms, 1 x. 9.6 kilograms, 3 x's. 9.8 kilograms, 1 x. 10 kilograms, 3 x's. 10.2 kilograms, 3 x's. 10.4 kilograms, 1 x. 10.6 kilograms, 2 x's. 10.8 kilograms, 2 x's. 11 kilograms, 1 x.

We can tell from the box plots that, in general, the pugs in the group are lighter than the beagles: the median weight of pugs is 7 kilograms and the median weight of beagles is 10 kilograms. Because the two box plots are on the same scale and the rectangles have similar widths, we can also tell that the IQRs for the two breeds are very similar. This suggests that the variability in the beagle weights is very similar to the variability in the pug weights.

Glossary Entries

Definition: Box Plot

A box plot is a way to represent data on a number line. The data is divided into four sections. The sides of the box represent the first and third quartiles. A line inside the box represents the median. Lines outside the box connect to the minimum and maximum values.

For example, this box plot shows a data set with a minimum of 2 and a maximum of 15. The median is 6, the first quartile is 5, and the third quartile is 10.

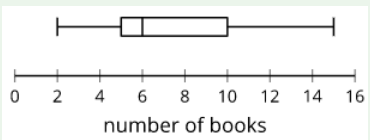


Figure 44.4.3

Definition: Interquartile Range (IQR)

The interquartile range is one way to measure how spread out a data set is. We sometimes call this the IQR. To find the interquartile range we subtract the first quartile from the third quartile.

For example, the IQR of this data set is 20 because $50 - 30 = 20$.

Table 44.4.1

22	29	30	31	32	43	44	45	50	50	59
		Q1			Q2			Q3		

Definition: Median

The median is one way to measure the center of a data set. It is the middle number when the data set is listed in order.

For the data set 7, 9, 12, 13, 14, the median is 12.

For the data set 3, 5, 6, 8, 11, 12, there are two numbers in the middle. The median is the average of these two numbers. $6 + 8 = 14$ and $14 \div 2 = 7$.

Definition: Quartile

Quartiles are the numbers that divide a data set into four sections that each have the same number of values.

For example, in this data set the first quartile is 30. The second quartile is the same thing as the median, which is 43. The third quartile is 50.

Table 44.4.2

22	29	30	31	32	43	44	45	50	50	59
		Q1			Q2			Q3		

Definition: Range

The range is the distance between the smallest and largest values in a data set. For example, for the data set 3, 5, 6, 8, 11, 12, the range is 9, because $12 - 3 = 9$.

Practice

Exercise 44.4.4

Each student in a class recorded how many books they read during the summer. Here is a box plot that summarizes their data.

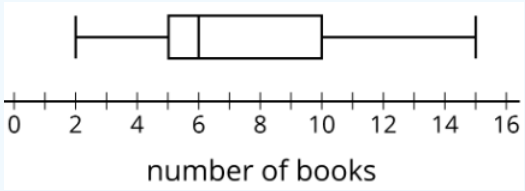


Figure 44.4.4

1. What is the greatest number of books read by a student in this group?
2. What is the median number of books read by the students?
3. What is the interquartile range (IQR)?

Exercise 44.4.5

Use this five-number summary to draw a box plot. All values are in seconds.

- Minimum: 40
- First quartile (Q1): 45
- Median: 48
- Third quartile (Q3): 50
- Maximum: 60

Exercise 44.4.6

The data shows the number of hours per week that each of 13 seventh-grade students spent doing homework. Create a box plot to summarize the data.

3 10 12 4 7 9 5 5 11 11 5 12 11

Exercise 44.4.7

The box plot displays the data on the response times of 100 mice to seeing a flash of light. How many mice are represented by the rectangle between 0.5 and 1 second?

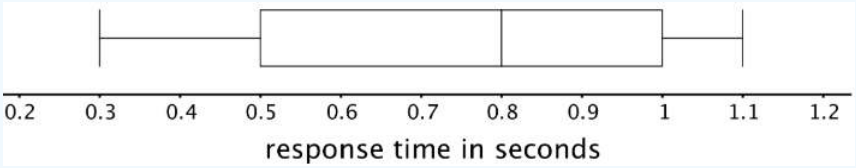


Figure 44.4.5

Exercise 44.4.8

Here is a dot plot that represents a data set. Explain why the mean of the data set is greater than its median.

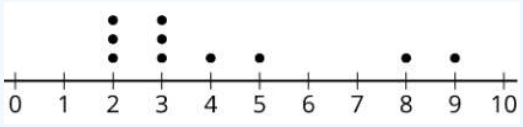


Figure 44.4.6

(From Unit 8.4.2)

Exercise 44.4.9

Jada earns money from babysitting, walking her neighbor's dogs, and running errands for her aunt. Every four weeks, she combines her earnings and divides them into three equal parts—one for spending, one for saving, and one for donating to a charity. Jada donated $26.00 of her earnings from the past four weeks to charity.

How much could she have earned from each job? Make two lists of how much she could have earned from the three jobs during the past four weeks.

(From Unit 8.3.1)

44.4: Box Plots is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

44.4.1: Using Box Plots

Lesson

Let's use box plots to make comparisons.

Exercise 44.4.1.1: Hours of Slumber

Ten sixth-grade students were asked how much sleep, in hours, they usually get on a school night. Here is the five-number summary of their responses.

- Minimum: 5 hours
- First quartile: 7 hours
- Median: 7.5 hours
- Third quartile: 8 hours
- Maximum: 9 hours

1. On the grid, draw a box plot for this five-number summary.



Figure 44.4.1.1

2. What questions could be answered by looking at this box plot?

Exercise 44.4.1.2: Info Gap: Sea Turtles

Your teacher will give you either a Problem Card or a Data Card about sea turtles that nest on the Outer Banks of North Carolina. Do not show or read your card to your partner.



Figure 44.4.1.2: Hawksbill Sea Turtle Carey de Concha (5840602412), by U.S. Fish and Wildlife Service Southeast Region. Public Domain. Wikimedia Commons. Source.

If your teacher gives you the *problem card*:

1. Silently read your card and think about what information you need to be able to answer the question.
2. Ask your partner for the specific information that you need.
3. Explain how you are using the information to solve the problem.
   Continue to ask questions until you have enough information to solve the problem.
4. Share the *problem card* and solve the problem independently.
5. Read the *data card* and discuss your reasoning.

If your teacher gives you the *data card*:

1. Silently read your card.
2. Ask your partner *"What specific information do you need?"* and wait for them to *ask* for information.
   If your partner asks for information that is not on the card, do not do the calculations for them. Tell them you don't have that information.

3. Before sharing the information, ask "*Why do you need that information?*" Listen to your partner's reasoning and ask clarifying questions.
4. Read the *problem card* and solve the problem independently.
5. Share the *data card* and discuss your reasoning.

Pause here so your teacher can review your work. Ask your teacher for a new set of cards and repeat the activity, trading roles with your partner.

Exercise 44.4.1.3: Paper Planes

Andre, Lin, and Noah each designed and built a paper airplane. They launched each plane several times and recorded the distance of each flight in yards.

Andre

25 26 27 27 27 28 28 28 29 30 30

Lin

20 20 21 24 26 28 28 29 29 30 32

Noah

13 14 15 18 19 20 21 23 23 24 25

Work with your group to summarize the data sets with numbers and box plots.

1. Write the five-number summary for the data for each airplane. Then, calculate the interquartile range for each data set.

Table 44.4.1.1

min	Q1	median	Q3	max	IQR

2. Draw three box plots, one for each paper airplane. Label the box plots clearly.

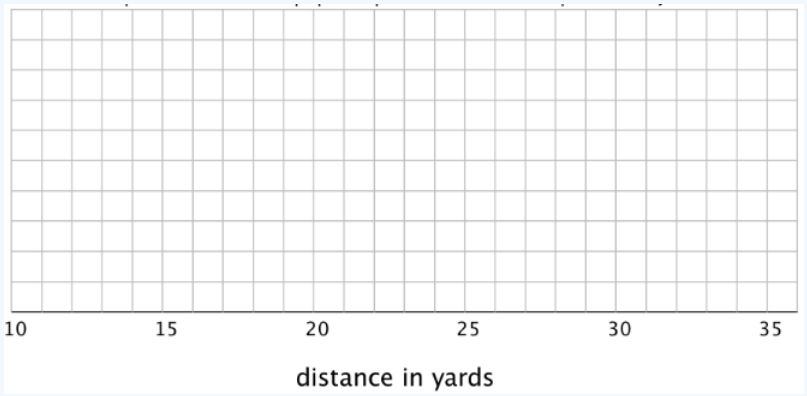


Figure 44.4.1.3

3. How are the results for Andre and Lin's planes the same? How are they different?
4. How are the results for Lin and Noah's planes the same? How are they different?

Are you ready for more?

Priya joined in the paper-plane experiments. She launched her plane eleven times and recorded the lengths of each flight. She found that her maximum and minimum were equal to Lin's. Her IQR was equal to Andre's.

Draw a box plot that could represent Priya's data.

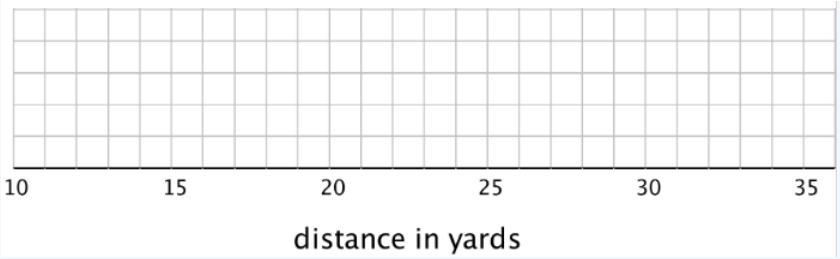


Figure 44.4.1.4

With the information given, can you estimate the median for Priya's data? Explain your reasoning.

Summary

Box plots are useful for comparing different groups. Here are two sets of plots that show the weights of some berries and some grapes.

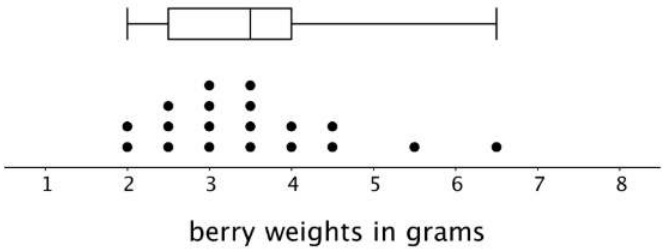


Figure 44.4.1.5: A box plot and dot plot for "berry weights in grams." The numbers 1 through 8 are indicated. The box plot is above the dot plot. The five-number summary for the box plot are as follows: Minimum value, 2. Maximum value, 6.5. Q1, 2.5. Q2, 3.5. Q3, 4. The data for the dot plot are as follows: 2 grams, 2 dots. 2.5 grams, 3 dots. 3 grams, 4 dots. 3.5 grams, 4 dots. 4 grams, 2 dots. 4.5 grams, 2 dots. 5.5 grams, 1 dot. 6.5 grams, 1 dot.

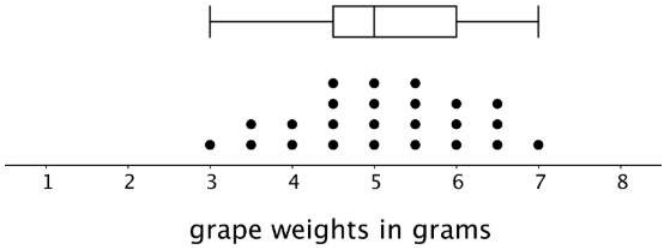


Figure 44.4.1.6: A box plot and dot plot for "grape weights in grams." The numbers 1 through 8 are indicated. The box plot is above the dot plot. The five-number summary for the box plot is as follows: Minimum value, 3. Maximum value, 9. Q1, 4.5. Q2, 5. Q3, 6. The data for the dot plot are as follows: 3 grams, 1 dot. 3.5 grams, 2 dots. 4 grams, 2 dots. 4.5 grams, 4 dots. 5 grams, 4 dots. 5.5 grams, 4 dots. 6 grams, 3 dots. 6.5 grams, 3 dots. 7 grams, 1 dot.

Notice that the median berry weight is 3.5 grams and the median grape weight is 5 grams. In both cases, the IQR is 1.5 grams. Because the grapes in this group have a higher median weight than the berries, we can say a grape in the group is typically heavier than a berry. Because both groups have the same IQR, we can say that they have a similar variability in their weights.

These box plots represent the length data for a collection ladybugs and a collection of beetles.

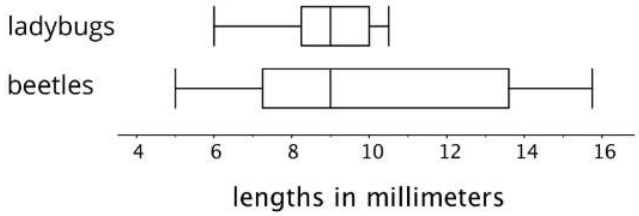


Figure 44.4.1.7: Two sets of box plots for "lengths in millimeters". The numbers 4 through 16 are indicated in increments of 2. There are tick marks midway between the indicated numbers. The top box plot is for "ladybugs". The five-number summary is as follows: Minimum value, 6. Maximum value, 10.5. Q1, 8.5. Q2, 9. Q3, 10. The bottom box plot is for "beetles". The five-number summary is as follows: Minimum value, 5. Maximum value, 15.5. Q1, 7.5. Q2, 9. Q3, 13.5.

The medians of the two collections are the same, but the IQR of the ladybugs is much smaller. This tells us that a typical ladybug length is similar to a typical beetle length, but the ladybugs are more alike in their length than the beetles are in their length.

Glossary Entries

Definition: Box Plot

A box plot is a way to represent data on a number line. The data is divided into four sections. The sides of the box represent the first and third quartiles. A line inside the box represents the median. Lines outside the box connect to the minimum and maximum values.

For example, this box plot shows a data set with a minimum of 2 and a maximum of 15. The median is 6, the first quartile is 5, and the third quartile is 10.

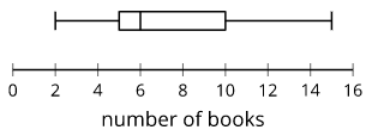


Figure 44.4.1.8

Definition: Interquartile Range (IQR)

The interquartile range is one way to measure how spread out a data set is. We sometimes call this the IQR. To find the interquartile range we subtract the first quartile from the third quartile.

For example, the IQR of this data set is 20 because $50 - 30 = 20$.

Table 44.4.1.1

22	29	30	31	32	43	44	45	50	50	59
		Q1			Q2			Q3		

Definition: Median

The median is one way to measure the center of a data set. It is the middle number when the data set is listed in order.

For the data set 7, 9, 12, 13, 14, the median is 12.

For the data set 3, 5, 6, 8, 11, 12, there are two numbers in the middle. The median is the average of these two numbers. $6 + 8 = 14$ and $14 \div 2 = 7$.

Definition: Quartile

Quartiles are the numbers that divide a data set into four sections that each have the same number of values.

For example, in this data set the first quartile is 30. The second quartile is the same thing as the median, which is 43. The third quartile is 50.

Table 44.4.1.2

22	29	30	31	32	43	44	45	50	50	59
		Q1			Q2			Q3		

Definition: Range

The range is the distance between the smallest and largest values in a data set. For example, for the data set 3, 5, 6, 8, 11, 12, the range is 9, because $12 - 3 = 9$.

Practice

Exercise 44.4.1.4

Here are box plots that summarize the heights of 20 professional male athletes in basketball, football, hockey, and baseball.

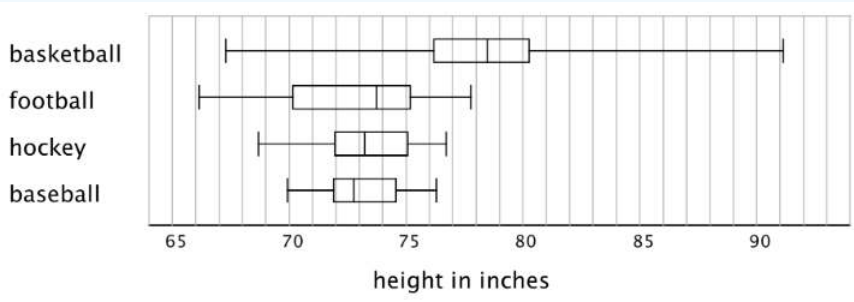


Figure 44.4.1.9: Four box plots from 65 to 90 by 5's. Height in inches. Labeled basketball, football, hockey, baseball. Top box plot labeled basketball. Whisker from 67 point 3 to 76 point 1.Box from 76 point 1 to 80 point 3 with vertical line at 78 point 5. Whisker from 80 point 3 to 91 point 1. Next box plot labeled football. Whisker from 66 point 1 to 70 point 1. Box from 70 point 1 to 75 point 1 with vertical line at 73 point 8. Whisker from 75 point 1 to 77 point 9. Next box plot labeled hockey. Whisker from 68 point 8 to 72. Box from 72 to 75 with vertical line at 73 point 1. Whisker from 75 to 76 point 8. Bottom box plot labeled baseball. Whisker from 70 to 72. Box from 72 to 74 point 5 with vertical line at 72 point 8. Whisker from 74 point 5 to 76 point 2.

1. In which two sports are the players' height distributions most alike? Explain your reasoning.
2. Which sport shows the greatest variability in players' heights? Which sport shows the least variability?

Exercise 44.4.1.5

Here is a box plot that summarizes data for the time, in minutes, that a fire department took to respond to 100 emergency calls.

Select **all** the statements that are true, according to the dot plot.

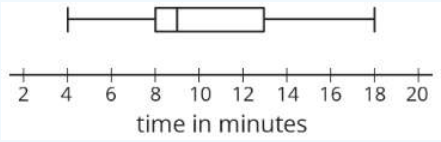


Figure 44.4.1.10

A. Most of the response times were under 13 minutes.
B. Fewer than 30 of the response times were over 13 minutes.
C. More than half of the response times were 11 minutes or greater.
D. There were more response times that were greater than 13 minutes than those that were less than 9 minutes.
E. About 75% of the response times were 13 minutes or less.

Exercise 44.4.1.6

Pineapples were packed in three large crates. For each crate, the weight of every pineapple in the crate was recorded. Here are three box plots that summarize the weights in each crate.

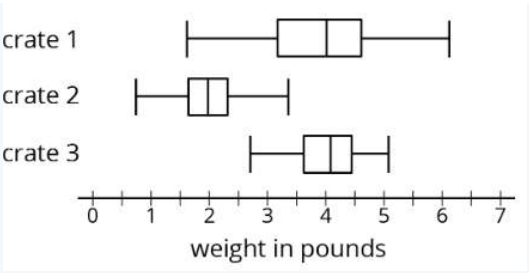


Figure 44.4.1.11: Three box plots, weight in pounds, 1 to 7 by ones. Crate 1, whisker from 1 point 5 to 3 point 2 5, box from 3 point 2 5 to 4 point 6 with a vertical line at 4, whisker from 4 point 6 to 6. Crate 2, whisker from point 7 5 to 1 point 2 5, box from 1 point 2 5 to 2 point 2 5 with a vertical line at 2, whisker from 2 point 2 5 to 3 point 4. Crate 3, whisker from 2 point 7 to 3 point 6, box from 3 point 6 to 4 point 4 with a vertical line at 4 point 1, whisker from 4 point 4 to 5 point 1.

Select **all** of the statements that are true, according to the box plots.

A. The weights of the pineapples in Crate 1 were the most variable.
B. The heaviest pineapple was in Crate 1.
C. The lightest pineapple was in Crate 1.
D. Crate 3 had the greatest median weight and the greatest IQR.
E. More than half the pineapples in Crate 1 and Crate 3 were heavier than the heaviest pineapple in Crate 2.

Exercise 44.4.1.7

Two TV shows each asked 100 viewers for their ages. For one show, the mean age of the viewers was 35 years and the MAD was 20 years. For the other show, the mean age of the viewers was 30 years and the MAD was 5 years.

A sixth-grade student says he watches one of the shows. Which show do you think he watches? Explain your reasoning.

(From Unit 8.3.4)

44.4.1: Using Box Plots is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

Section 45: Let's Put it to Work

This page titled Section 45: Let's Put it to Work is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

45.1: Using Data to Solve Problems

Lesson

Let's compare data sets using visual displays.

Exercise 45.1.1: Wild Bears

In one study on wild bears, researchers measured the head lengths and head widths, in inches, of 143 wild bears. The ages of the bears ranged from newborns (0 years) to 15 years. The box plots summarize the data from the study.

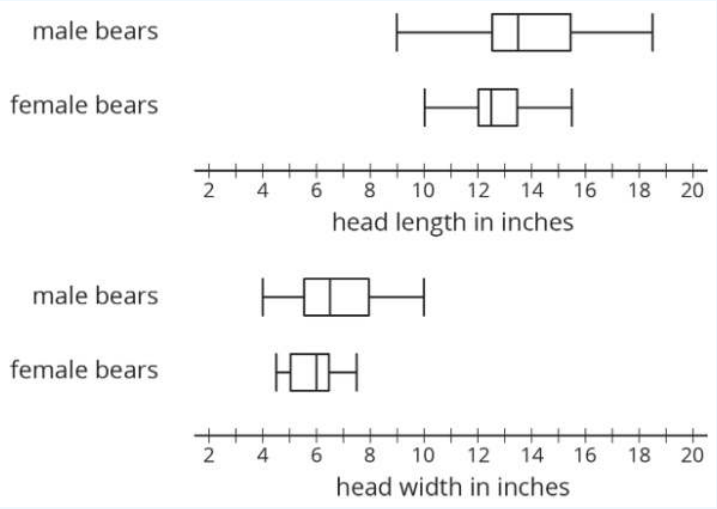


Figure 45.1.1: Two sets of two box plots from 2 to 10 by 2's. Top set labeled head length in inches. Bottom set labeled head width in inches. Top set, top box plot labeled male bears. Whisker from 9 to 12 point 5. Box from 12 point 5 to 15 point 5 with vertical line at 13 point 5. Whisker from 15 point 5 to 18 point 5. Bottom box plot labeled female bears. Whisker from 10 to 12. Box from 12 to 13 point 5 with vertical line at 12 point 5. Whisker from 13 point 5 to 15 point 5. Bottom set, top box plot labeled male bears. Whisker from 4 to 5 point 5. Box from 5 point 5 to 8 with vertical line 6 point 5. Whisker from 8 to 10. Bottom box plot whisker from 4 point 5 to 5. Box from 5 to 6 point 5 with vertical line at 6. Whisker from 6 point 5 to 7 point 5.

1. Write four statistical questions that could be answered using the box plots: two questions about the head length and two questions about the head width.
2. Trade questions with your partner.
   a. Decide if each question is a statistical question.
   b. Use the box plots to answer each question.

Exercise 45.1.2: Math Homework (Part 1)

Over a two-week period, Mai recorded the number of math homework problems she had each school day.

2 15 20 0 5 25 1 0 10 12

1. Calculate the following. Show your reasoning.
   a. The mean number of math homework problems.
   b. The mean absolute deviation (MAD).
2. Interpret the mean and MAD. What do they tell you about the number of homework problems Mai had over these two weeks?
3. Find or calculate the following values and show your reasoning.
   a. The median, quartile, maximum, and minimum of the same data on Mai's math homework problems.
   b. The interquartile range (IQR).
4. Which pair of measures of center and variability—mean and MAD, or median and IQR—do you think summarize the distribution of Mai's math homework assignments better? Explain your reasoning.

You may use the applet below to help if you choose to. Begin by dragging the left edge across the screen until you see only one column in the spreadsheet. Enter the values needed to calculate the IQR and the mean when prompted.

Exercise 45.1.3: Math Homework (Part 2)

Jada wanted to know whether a dot plot, a histogram, or a box plot would best summarize the center, variability, and other aspects of her homework data.

2 15 20 0 5 25 1 0 10 12

1. Use the axis to make a dot plot to represent the data. Mark the position of the mean, which you calculated earlier, on the dot plot using a triangle (Δ). From the triangle, draw a horizontal line segment to the left and right sides to represent the MAD.

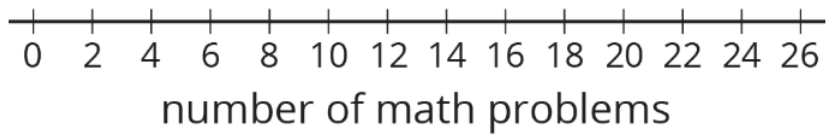


Figure 45.1.2

2. Use the five-number summary from the previous task and the grid to draw a box plot that represents Jada's homework data.

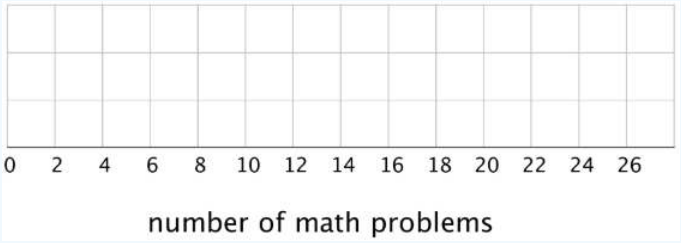


Figure 45.1.3

3. Work with your group to draw three histograms to represent Jada's homework data. The width of the bars in each histogram should represent a different number of homework problems, as specified.

   a. The width of one bar represents 10 problems.

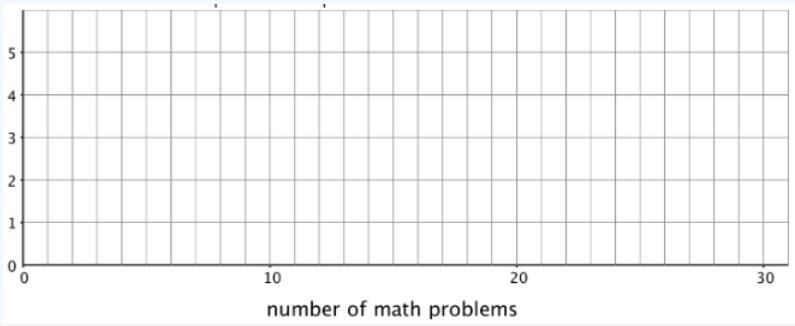


Figure 45.1.4

   b. The width of one bar represents 5 problems.

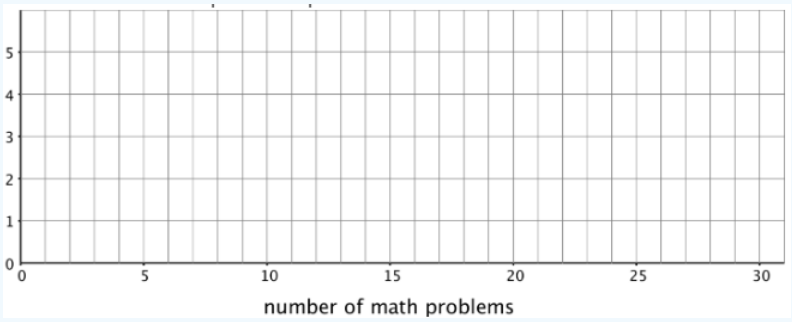


Figure 45.1.5

c. The width of one bar represents 2 problems.

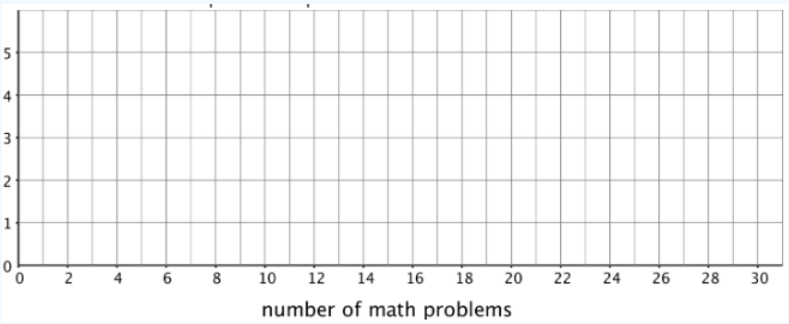


Figure 45.1.6

4. Which of the five representations should Jada use to summarize her data? Should she use a dot plot, box plot, or one of the histograms? Explain your reasoning.

You can use the applet to make each type of graph if you choose to. Begin by dragging the gray bar from the top of the applet down until you see all of the command boxes.

Exercise 45.1.4: Will the Yellow Perch Survive?

Scientists studying the yellow perch, a species of fish, believe that the length of a fish is related to its age. This means that the longer the fish, the older it is. Adult yellow perch vary in size, but they are usually between 10 and 25 centimeters.

Scientists at the Great Lakes Water Institute caught, measured, and released yellow perch at several locations in Lake Michigan. The following summary is based on a sample of yellow perch from one of these locations.

Table 45.1.1

length of fish in centimeters	number of fish
0 to less than 5	5
5 to less than 10	7
10 to less than 15	14
15 to less than 20	20
20 to less than 25	24
25 to less than 30	30

1. Use the data to make a histogram that shows the lengths of the captured yellow perch. Each bar should contain the lengths shown in each row in the table.

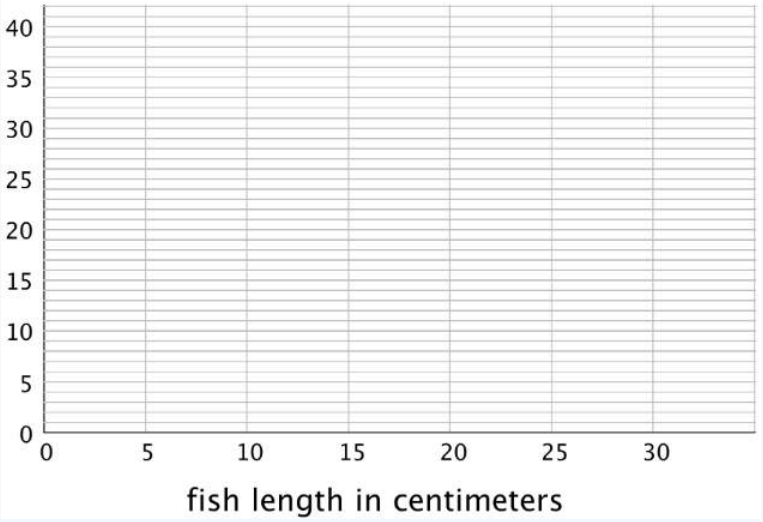


Figure 45.1.7

2. How many fish were measured? How do you know?
3. Use the histogram to answer the following questions.
   a. How would you describe the shape of the distribution?
   b. Estimate the median length for this sample. Describe how you made this estimate.
   c. Predict whether the mean length of this sample is greater than, less than, or nearly equal to the median length for this sample of fish? Explain your prediction.
   d. Would you use the mean or the median to describe a typical length of the fish being studied? Explain your reasoning.
4. Based on your work so far:
   a. Would you describe a typical age for the yellow perch in this sample as: “young,” “adult,” or “old”? Explain your reasoning.
   b. Some researchers are concerned about the survival of the yellow perch. Do you think the lengths (or the ages) of the fish in this sample are something to worry about? Explain your reasoning.

Summary

The dot plot shows the distribution of 30 cookie weights in grams.

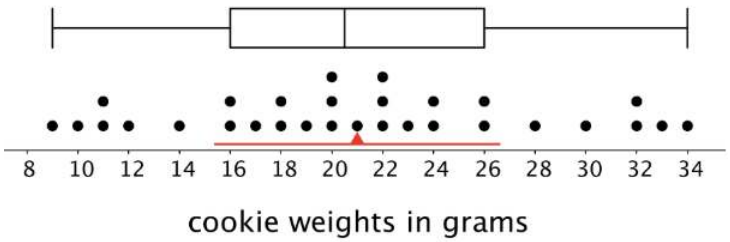


Figure 45.1.8: A box plot and a histogram for "cookie weights in grams". The numbers 8 through 34, in increments of two, are indicated. The box plot is above the dot plot. The five-number summary for the box plot is as follows: Minimum value, 9. Maximum value, 34. Q1, 16. Q2, 20.5. Q3, 26. For the dot plot a triangle is indicated at 21 grams. A horizontal line is drawn below the triangle and begins at 15.4 and ends at 26.6. The data for the dot plot are as follows: 9 grams, 1 dot. 10 grams, 1 dot. 11 grams, 2 dots. 12 grams, 1 dot. 14 grams, 1 dot. 16 grams, 2 dots. 17 grams, 1 dot. 18 grams, 2 dots. 19 grams, 1 dot. 20 grams, 3 dots. 21 grams, 1 dot. 22 grams, 3 dots. 23 grams, 1 dot. 24 grams, 2 dots. 26 grams, 2 dots. 28 grams, 1 dot. 30 grams, 1 dot. 32 grams, 2 dots. 33 grams, 1 dot. 34 grams, 1 dot.

The mean cookie weight, marked by the triangle, is 21 grams. This tells us that if the weights of all of the cookies were redistributed so they all had the same weight, each cookie would weigh 21 grams. The MAD is 5.6 grams, which suggests that a cookie typically weighs between 15.4 grams and 26.6 grams.

The box plot for the same data set is shown above the dot plot. The median shows that half of the weights are greater than or equal to 20.5 grams, and half are less than or equal to 20.5 grams. The box shows that the IQR is 10 and that the middle half of the cookies weigh between 16 and 26 grams.

In this case, the median weight is very close to the mean weight, and the IQR is about twice the MAD. This tells us that the two pairs of measures of center and spread are very similar.

Now let's look at another example of 30 different cookies.

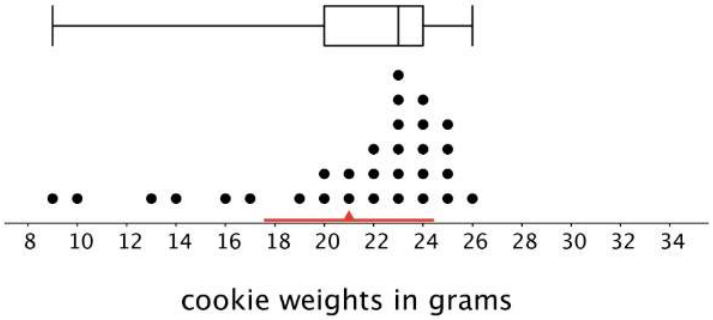


Figure 45.1.9: A box plot and a dot plot for "cookie weights in grams". The numbers 8 through 34, in increments of two, are indicated. For the box plot: The minimum value is 9 and the maximum value is 26. Q1 has a value of 20, Q2 has a value of 23, and Q3 has a value of 24. For the dot plot: A triangle is indicated at 21 grams. A line indicates the distance from 17.6 to 24.4 grams. The data is as follows: 9 grams, 1 dot. 10 grams, 1 dot. 13 grams, 1 dot. 14 grams, 1 dot. 16 grams, 1 dot. 17 grams, 1 dot. 19 grams, 1 dot. 20 grams, 2 dots. 21 grams, 2 dots. 22 grams, 3 dots. 23 grams, 6 dots. 24 grams, 5 dots. 25 grams, 4 dots. 26 grams, 1 dot.

Here the mean is 21 grams, and the MAD is 3.4 grams. This suggests that a cookie typically weighs between 17.6 and 24.4 grams. The median cookie weight is 23 grams, and the box plot shows that the middle half of the data are between 20 and 24 grams. These two pairs of measures paint very different pictures of the variability of the cookie weights.

The median (23 grams) is closer to the middle of the big cluster of values. If we were to ignore the smaller cookies, the median and IQR would give a more accurate picture of how much a cookie typically weighs.

When a distribution is not symmetrical, the median and IQR are often better measures of center and spread than the mean and MAD. However the decision on which pair of measures to use depends on what we want to know about the group we are investigating.

45.1: Using Data to Solve Problems is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

SECTION OVERVIEW

9: Putting it All Together

Section 46: Making Connections

46.1: Fermi Problems
46.2: If Our Class Were the World
46.3: Rectangle Madness

Section 47: Voting

47.1: More than Two Choices
47.2: Picking Representatives

Section 48: Let's Put it to Work

48.1: How Do We Choose?

This page titled 9: Putting it All Together is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

Section 46: Making Connections

This page titled Section 46: Making Connections is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

46.1: Fermi Problems

Lesson

Let's make some estimates.

Exercise 46.1.1: Ant Trek

How long would it take an ant to run from Los Angeles to New York City?

Exercise 46.1.2: Stacks and Stacks of Cereal Boxes

Imagine a warehouse that has a rectangular floor and that contains all of the boxes of breakfast cereal bought in the United States in one year.

If the warehouse is 10 feet tall, what could the side lengths of the floor be?

Exercise 46.1.3: Covering the Washington Monument

How many tiles would it take to cover the Washington Monument?

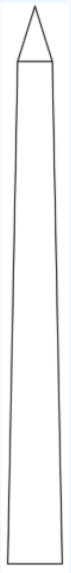


Figure 46.1.1

This page titled 46.1: Fermi Problems is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

46.2: If Our Class Were the World

Lesson

Let's use math to better understand our world.

Exercise 46.2.1: All 7.4 Billion of Us

There are 7.4 billion people in the world. If the whole world were represented by a 30-person class:

- 14 people would eat rice as their main food.
- 12 people would be under the age of 20.
- 5 people would be from Africa.

1. How many people in the class would *not* eat rice as their main food?
2. What percentage of the people in the class would be under the age of 20?
3. Based on the number of people in the class representing people from Africa, how many people live in Africa?

Exercise 46.2.2: About the People in the World

With the members of your group, write a list of questions about the people in the world. Your questions should begin with "How many people in the world. . ." Then, choose several questions on the list that you find most interesting.

Exercise 46.2.3: If Our Class Were the World

Suppose your class represents all the people in the world.

Choose several characteristics about the world's population that you have investigated. Find the number of students in *your* class that would have the same characteristics.

Create a visual display that includes a diagram that represents this information. Give your display the title "If Our Class Were the World."

This page titled 46.2: If Our Class Were the World is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

46.3: Rectangle Madness

Lesson

Let's cut up rectangles.

Exercise 46.3.1: Squares in Rectangles

1. Rectangle $ABCD$ is not a square. Rectangle $ABEF$ is a square.

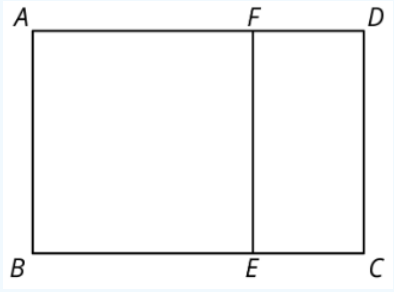


Figure 46.3.1

   a. Suppose segment AF were 5 units long and segment FD were 2 units long. How long would segment AD be?
   b. Suppose segment BC were 10 units long and segment BE were 6 units long. How long would segment EC be?
   c. Suppose segment AF were 12 units long and segment FD were 5 units long. How long would segment FE be?
   d. Suppose segment AD were 9 units long and segment AB were 5 units long. How long would segment F be?

2. Rectangle $JKXW$ has been decomposed into squares.

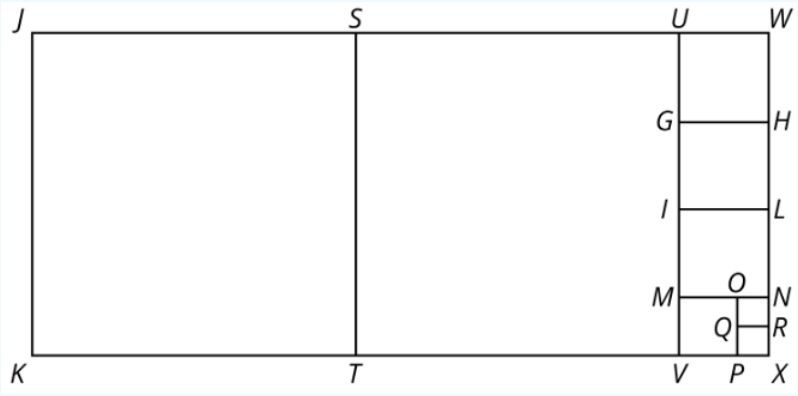


Figure 46.3.2: Tiling of rectangle KXWJ with 2 large squares, KJST and TSUV; 3 medium squares, GUWH, KGHL, and MILN; 1 small square, VMOP; and 2 tiny squares, QONR and PQRX. TSUV and KJST are stacked horizontally. TSUV is on the right of KJST. GUWH, KGHL, and MILN are stack vertically. All 3 are on the right of TSUV. VMOP is below MILN. QONR and PQRX are stacked vertically. Both are on the right of VMOP.

Segment JK is 33 units long and segment JW is 75 units long. Find the areas of all of the squares in the diagram.

3. Rectangle $ABCD$ is 16 units by 5 units.

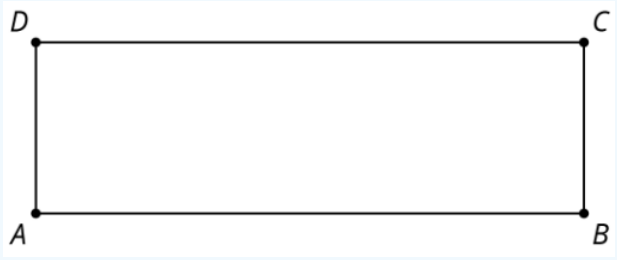


Figure 46.3.3

   a. In the diagram, draw a line segment that decomposes $ABCD$ into two regions: a square that is the largest possible and a new rectangle.

b. Draw another line segment that decomposes the *new* rectangle into two regions: a square that is the largest possible and another new rectangle.
c. Keep going until rectangle $ABCD$ is entirely decomposed into squares.
d. List the side lengths of all the squares in your diagram.

Are you ready for more?

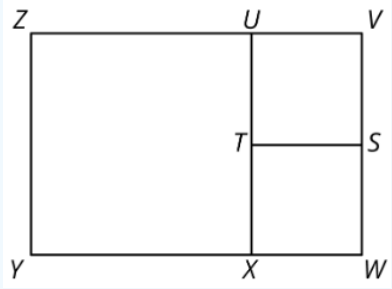


Figure 46.3.4: Rectangle VWYZ is decomposed into 3 squares. YZUX is a large square. UVTS and XTSW are two equal sized squares that are stacked vertically. Both are on the right of YZUX.

1. The diagram shows that rectangle $VWYZ$ has been decomposed into three squares. What could the side lengths of this rectangle be?
2. How many different side lengths can you find for rectangle $VWYZ$?
3. What are some rules for possible side lengths of rectangle $VWYZ$?

Exercise 46.3.2: More Rectangles, More Squares

1. Draw a rectangle that is 21 units by 6 units.
   a. In your rectangle, draw a line segment that decomposes the rectangle into a new rectangle and a square that is as large as possible. Continue until the diagram shows that your original rectangle has been entirely decomposed into squares.
   b. How many squares of each size are in your diagram?
   c. What is the side length of the smallest square?
2. Draw a rectangle that is 28 units by 12 units.
   a. In your rectangle, draw a line segment that decomposes the rectangle into a new rectangle and a square that is as large as possible. Continue until the diagram shows that your original rectangle has been decomposed into squares.
   b. How many squares of each size are in your diagram?
   c. What is the side length of the smallest square?
3. Write each of these fractions as a mixed number with the smallest possible numerator and denominator:
   a. $\frac{16}{5}$
   b. $\frac{21}{6}$
   c. $\frac{28}{12}$
4. What do the fraction problems have to do with the previous rectangle decomposition problems?

Exercise 46.3.3: Finding Equivalent Fractions

1. Accurately draw a rectangle that is 9 units by 4 units.
   a. In your rectangle, draw a line segment that decomposes the rectangle into a new rectangle and a square that is as large as possible. Continue until your original rectangle has been entirely decomposed into squares.
   b. How many squares of each size are there?
   c. What are the side lengths of the last square you drew?
   d. Write $\frac{9}{4}$ as a mixed number.
2. Accurately draw a rectangle that is 27 units by 12 units.
   a. In your rectangle, draw a line segment that decomposes the rectangle into a new rectangle and a square that is as large as possible. Continue until your original rectangle has been entirely decomposed into squares.
   b. How many squares of each size are there?

c. What are the side lengths of the last square you drew?
d. Write $\frac{27}{12}$ as a mixed number.
e. Compare the diagram you drew for this problem and the one for the previous problem. How are they the same? How are they different?

3. What is the greatest common factor of 9 and 4? What is the greatest common factor of 27 and 12? What does this have to do with your diagrams of decomposed rectangles?

Are you ready for more?

We have seen some examples of rectangle tilings. A *tiling* means a way to completely cover a shape with other shapes, without any gaps or overlaps. For example, here is a tiling of rectangle $KXWJ$ with 2 large squares, 3 medium squares, 1 small square, and 2 tiny squares.

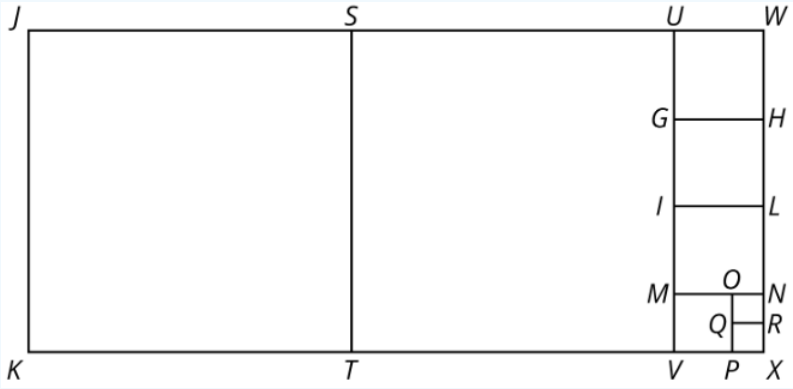


Figure 46.3.5: Tiling of rectangle KXWJ with 2 large squares, KJST and TSUV; 3 medium squares, GUWH, KGHL, and MILN; 1 small square, VMOP; and 2 tiny squares, QONR and PQRX. TSUV and KJST are stacked horizontally. TSUV is on the right of KJST. GUWH, KGHL, and MILN are stack vertically. All 3 are on the right of TSUV. VMOP is below MILN. QONR and PQRX are stacked vertically. Both are on the right of VMOP.

Some of the squares used to tile this rectangle have the same size.

Might it be possible to tile a rectangle with squares where the squares are *all different sizes*?

If you think it is possible, find such a rectangle and such a tiling. If you think it is not possible, explain why it is not possible.

Exercise 46.3.4: It's All About Fractions

1. Accurately draw a 37-by-16 rectangle. (Use graph paper, if possible.)
   a. In your rectangle, draw a line segment that decomposes the rectangle into a new rectangle and a square that is as large as possible. Continue until your original rectangle has been entirely decomposed into squares.
   b. How many squares of each size are there?
   c. What are the dimensions of the last square you drew?
   d. What does this have to do with $2 + \frac{1}{3+\frac{1}{5}}$?
2. Consider a 52-by-15 rectangle.
   a. In your rectangle, draw a line segment that decomposes the rectangle into a new rectangle and a square that is as large as possible. Continue until your original rectangle has been entirely decomposed into squares.
   b. Write a fraction equal to this expression: $3 + \frac{1}{2+\frac{1}{7}}$.
   c. Notice some connections between the rectangle and the fraction.
   d. What is the greatest common factor of 52 and 15?
3. Consider a 98-by-21 rectangle.
   a. In your rectangle, draw a line segment that decomposes the rectangle into a new rectangle and a square that is as large as possible. Continue until your original rectangle has been entirely decomposed into squares.
   b. Write a fraction equal to this expression: $4 + \frac{1}{1+\frac{7}{14}}$.
   c. Notice some connections between the rectangle and the fraction.

d. What is the greatest common factor of 98 and 21?

4. Consider a 121-by-38 rectangle.

a. Use the decomposition-into-squares process to write a continued fraction for $\frac{121}{38}$. Verify that it works.

b. What is the greatest common factor of 121 and 38?

This page titled 46.3: Rectangle Madness is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

Section 47: Voting

This page titled Section 47: Voting is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

47.1: More than Two Choices

Lesson

Let's explore different ways to determine a winner.

Exercise 47.1.1: Field Day

Students in a sixth-grade class were asked, “What activity would you most like to do for field day?” The results are shown in the table.

Table 47.1.1

activity	number of votes
softball game	16
scavenger hunt	10
dancing talent show	8
marshmallow throw	4
no preference	(2\)

1. What percentage of the class voted for softball?
2. What percentage did not vote for softball as their first choice?

Exercise 47.1.2: School Lunches (Part 1)

Suppose students at our school are voting for the lunch menu over the course of one week. The following is a list of options provided by the caterer.

Menu 1: Meat Lovers

- Meat loaf
- Hot dogs
- Pork cutlets
- Beef stew
- Liver and onions

Menu 2: Vegetarian

- Vegetable soup and peanut butter sandwich
- Hummus, pita, and veggie sticks
- Veggie burgers and fries
- Chef’s salad
- Cheese pizza every day
- Double desserts every day

Menu 3: Something for Everyone

- Chicken nuggets
- Burgers and fries
- Pizza
- Tacos
- Leftover day (all the week’s leftovers made into a casserole)
- Bonus side dish: pea jello (green gelatin with canned peas)

Menu 4: Concession Stand

- Choice of hamburger or hot dog, with fries, every day

To vote, draw one of the following symbols next to each menu option to show your first, second, third, and last choices. If you use the slips of paper from your teacher, use only the column that says “symbol.”



Figure 47.1.1

1. Meat Lovers _____
2. Vegetarian _____
3. Something for Everyone _____
4. Concession Stand _____

Here are two voting systems that can be used to determine the winner.

- Voting System #1. *Plurality*: The option with the most first-choice votes (stars) wins.
- Voting System #2. *Runoff*: If no choice received a majority of the votes, leave out the choice that received the fewest first-choice votes (stars). Then have another vote.

If your first vote is still a choice, vote for that. If not, vote for your second choice that you wrote down.

If there is still no majority, leave out the choice that got the fewest votes, and then vote again. Vote for your first choice if it’s still in, and if not, vote for your second choice. If your second choice is also out, vote for your third choice.

1. How many people in our class are voting? How many votes does it take to win a majority?
2. How many votes did the top option receive? Was this a majority of the votes?
3. People tend to be more satisfied with election results if their top choices win. For how many, and what percentage, of people was the winning option:
   a. their first choice?
   b. their second choice?
   c. their third choice?
   d. their last choice?
4. After the second round of voting, did any choice get a majority? If so, is it the same choice that got a plurality in Voting System #1?
5. Which choice won?
6. How satisfied were the voters by the election results? For how many, and what percentage, of people was the winning option:
   a. their first choice?
   b. their second choice?
   c. their third choice?
   d. their last choice?
7. Compare the satisfaction results for the plurality voting rule and the runoff rule. Did one produce satisfactory results for more people than the other?

Exercise 47.1.3: School Lunch (Part 2)

Let’s analyze a different election.

In another class, there are four clubs. Everyone in each club agrees to vote for the lunch menu exactly the same way, as shown in this table.

	Barbecue Club (21 members)	Garden Club (13 members)	Sports Boosters (7 members)	Film Club (9 members)
A. Meat Lovers				
B. Vegetarian				
C. Something for Everyone				
D. Concession Stand				

Figure 47.1.2: A 5 column table with 4 rows of data. The heading for the first column is blank. The heading for the second column is labeled Barbecue Club with 21 members. The third column is labeled Garden Club with 13 members.The fourth column is labeled Sports Boosters with 7 members. The fifth column is labeled Film Club with 9 members. The data are as follows: Row 1: A, Meat Lovers; Barbecue Club, Star; Garden Club, X; Sports Boosters, X; Film Club, X. Row 2: B, Vegetarian; Barbecue Club, Smiley Face; Garden Club, Star; Sports Boosters, Square Face; Film Club, Square Face. Row 3: C, Something for Everyone; Barbecue Club, Square Face; Garden Club, Square Face; Sports Boosters, Smiley Face; Film Club, Star. Row 4: D, Concession Stand; Barbecue Club, X; Garden Club, Smiley Face; Sports Boosters, Star; Film Club, Smiley Face.

1. Figure out which option won the election by answering these questions.
   a. On the first vote, when everyone voted for their first choice, how many votes did each option get? Did any choice get a majority?
   b. Which option is removed from the next vote?
   c. On the second vote, how many votes did each of the remaining three menu options get? Did any option get a majority?
   d. Which menu option is removed from the next vote?
   e. On the third vote, how many votes did each of the remaining two options get? Which option won?
2. Estimate how satisfied all the voters were.
   a. For how many people was the winner their first choice?
   b. For how many people was the winner their second choice?
   c. For how many people was the winner their third choice?
   d. For how many people was the winner their last choice?
3. Compare the satisfaction results for the plurality voting rule and the runoff rule. Did one produce satisfactory results for more people than the other?

Exercise 47.1.4: Just Vote Once

Your class just voted using the *instant runoff* system. Use the class data for following questions.

1. For our class, which choice received the most points?
2. Does this result agree with that from the runoff election in an earlier activity?
3. For the other class, which choice received the most points?
4. Does this result agree with that from the runoff election in an earlier activity?
5. The runoff method uses information about people's first, second, third, and last choices when it is not clear that there is a winner from everyone's first choices. How does the instant runoff method include the same information?
6. After comparing the results for the three voting rules (plurality, runoff, instant runoff) and the satisfaction surveys, which method do you think is fairest? Explain.

Are you ready for more?

Numbering your choices 0 through 3 might not really describe your opinions. For example, what if you really liked A and C a lot, and you really hated B and D? You might want to give A and C both a 3, and B and D both a 0.

1. Design a numbering system where the size of the number accurately shows how much you like a choice. Some ideas:
   - The same 0 to 3 scale, but you can choose more than one of each number, or even decimals between 0 and 3.

- A scale of 1 to 10, with 10 for the best and 1 for the worst.

2. Try out your system with the people in your group, using the same school lunch options for the election.
3. Do you think your system gives a more fair way to make choices? Explain your reasoning.

Exercise 47.1.5: Weekend Choices

Clare, Han, Mai, Tyler, and Noah are deciding what to do on the weekend. Their options are cooking, hiking, and bowling. Here are the points for their instant runoff vote. Each first choice gets 2 points, the second choice gets 1 point, and the last choice gets 0 points.

Table 47.1.2

	cooking	hiking	bowling
Clare	2	1	0
Han	2	1	0
Mai	2	1	0
Tyler	0	2	1
Noah	0	2	1

1. Which activity won using the instant runoff method? Show your calculations and use expressions or equations.
2. Which activity would have won if there was just a vote for their top choice, with a majority or plurality winning?
3. Which activity would have won if there was a runoff election?
4. Explain why this happened.

47.1: More than Two Choices is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

47.2: Picking Representatives

Lesson

Let's think about fair representation.

Exercise 47.2.1: Computers for Kids

A program gives computers to families with school-aged children. They have a certain number of computers to distribute fairly between several families. How many computers should each family get?

1. One month the program has 8 computers. The families have these numbers of school-aged children: 4, 2, 6, 2, 2.
   a. How many children are there in all?
   b. Counting all the children in all the families, how many children would use each computer? This is the number of children per computer. Call this number A.
   c. Fill in the third column of the table. Decide how many computers to give to each family if we use A as the basis for distributing the computers.

Table 47.2.1

family	number of children	number of computers, using A
Baum	4	
Chu	2	
Davila	6	
Eno	2	
Farouz	2	

   d. Check that 8 computers have been given out in all.
2. The next month they again have 8 computers. There are different families with these numbers of children: 3, 1, 2, 5, 1, 8.
   a. How many children are there in all?
   b. Counting all the children in all the families, how many children would use each computer? This is the number of children per computer. Call this number B.
   c. Does it make sense that B is not a whole number? Why?
   d. Fill in the third column of the table. Decide how many computers to give to each family if we use B as the basis for distributing the computers.

Table 47.2.2

family	number of children	number of computers, using B	number of computers, your way	children per computer, your way
Gray	3			
Hernandez	1			
Ito	2			
Jones	5			
Krantz	1			
Lo	8			

   e. Check that 8 computers have been given out in all.
   f. Does it make sense that the number of computers for one family is not a whole number? Explain your reasoning.
   g. Find and describe a way to distribute computers to the families so that each family gets a whole number of computers. Fill in the fourth column of the table.

h. Compute the number of children per computer in each family and fill in the last column of the table.
i. Do you think your way of distributing the computers is fair? Explain your reasoning.

Exercise 47.2.2: School Mascot (Part 1)

A school is deciding on a school mascot. They have narrowed the choices down to the Banana Slugs or the Sea Lions.

The principal decided that each class gets one vote. Each class held an election, and the winning choice was the one vote for the whole class. The table shows how three classes voted.

Table 47.2.3

	banana slugs	sea lions	class vote
class A	9	3	banana slug
class B	14	10	
class C	6	30	



Figure 47.2.1: Pawsox Mascot, by Paul Keleher. CC BY 2.0. Wikimedia Commons. Source.

1. Which mascot won, according to the principal's plan? What percentage of the votes did the winner get under this plan?
2. Which mascot received the most student votes in all? What percentage of the votes did this mascot receive?
3. The students thought this plan was not very fair. They suggested that bigger classes should have more votes to send to the principal. Make up a proposal for the principal where there are as few votes as possible, but the votes proportionally represent the number of students in each class.
4. Decide how to assign the votes for the results in the class. (Do they all go to the winner? Or should the loser still get some votes?)
5. In your system, which mascot is the winner?
6. In your system, how many representative votes are there? How many students does each vote represent?

Exercise 47.2.3: Advising the School Board

1. In a very small school district, there are four schools, D, E, F, and G. The district wants a total of 10 advisors for the students. Each school should have at least one advisor.

school	number of students	number of advisors, using A
D	48	
E	12	
F	24	
G	36	

Table 47.2.4

a. How many students are in this district in all?
b. If the advisors could represent students at different schools, how many students per advisor should there be? Call this number A.
c. Using A students per advisor, how many advisors should each school have? Complete the table with this information for schools D, E, F, and G.

2. Another district has four schools; some are large, others are small. The district wants 10 advisors in all. Each school should have at least one advisor.

school	number of students	number of advisors, using B	number of advisors, your way	students per advisor, your way
Dr. King School	500			
O'Connor School	200			
Science Magnet School	140			
Trombone Academy	10			

Table 47.2.5

a. How many students are in this district in all?
b. If the advisors didn't have to represent students at the same school, how many students per advisor should there be? Call this number B.
c. Using B students per advisor, how many advisors should each school have? Give your quotients to the tenths place. Fill in the first "number of advisors" column of the table. Does it make sense to have a tenth of an advisor?
d. Decide on a consistent way to assign advisors to schools so that there are only whole numbers of advisors for each school, and there is a total of 10 advisors among the schools. Fill in the "your way" column of the table.
e. How many students per advisor are there at each school? Fill in the last row of the table.
f. Do you think this is a fair way to assign advisors? Explain your reasoning.

Exercise 47.2.4: School Mascot (Part 2)

The whole town gets interested in choosing a mascot. The mayor of the town decides to choose representatives to vote.

There are 50 blocks in the town, and the people on each block tend to have the same opinion about which mascot is best. Green blocks like sea lions, and gold blocks like banana slugs. The mayor decides to have 5 representatives, each representing a district of 10 blocks.

Here is a map of the town, with preferences shown.



Figure 47.2.2: A figure that represents a map of a town composed of 50 green and gold squares that are arranged in 5 rows with 10 squares in each row. The top 2 rows each contain 10 gold squares and the bottom 3 rows each contain 10 green squares.

1. Suppose there were an election with each block getting one vote. How many votes would be for banana slugs? For sea lions? What percentage of the vote would be for banana slugs?
2. Suppose the districts are shown in the next map. What did the people in each district prefer? What did their representative vote? Which mascot would win the election?



Figure 47.2.3: A figure that represents a map of a town composed of 50 green and gold squares that are arranged in 5 rows with 10 squares in each row. The top 2 rows each contain 10 gold squares and are labeled 1 and 2. The bottom 3 rows each contain 10 green squares and the rows are labeled 3, 4, and 5.

Complete the table with this election's results.

Table 47.2.6

district	number of blocks for banana slugs	number of blocks for sea lions	percentage of blocks for banana slugs	representative's vote
1	10	0		banana slugs
2				
3				
4				
5				

3. Suppose, instead, that the districts are shown in the new map below. What did the people in each district prefer? What did their representative vote? Which mascot would win the election?

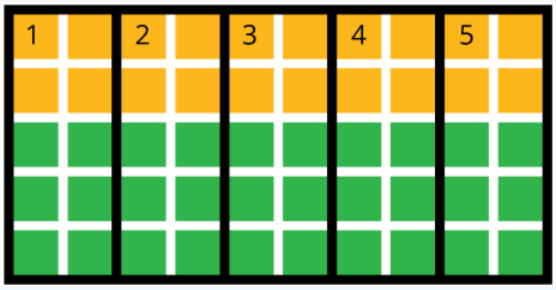


Figure 47.2.4: 50 gold and green squares are arranged in 5 rows with 10 squares in each row. The top 2 rows are gold and the bottom 3 rows are green. The grid is divided vertically into 5 equal rectangles labeled 1, 2, 3, 4, and 5. Each rectangle contains 4 gold squares at the top and 6 green squares directly underneath.

Complete the table with this election's results.

Table 47.2.7

district	number of blocks for banana slugs	number of blocks for sea lions	percentage of blocks for banana slugs	representative's vote
1				
2				
3				
4				
5				

4. Suppose the districts are designed in yet another way, as shown in the next map. What did the people in each district prefer? What did their representative vote? Which mascot would win the election?

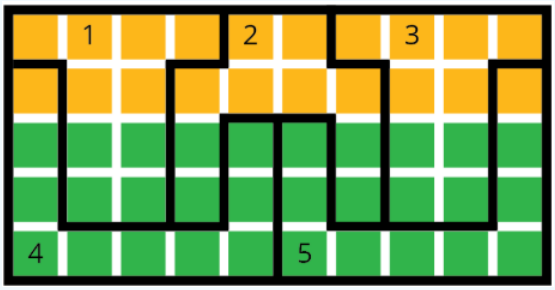


Figure 47.2.5: A 10 by 5 grid of squares with specific area boundaries numbered 1 through 5 indicated. The top two rows are gold and the bottom 3 rows are green. Each numbered area contains a combination of 10 gold and green squares. Area 1: Starting on the first row, region 1 has the first 4 gold squares. Under the second gold square in the first row is a row of 2 gold squares. Directly under the 2 gold squares in row 2 are 2 green squares. Directly under the 2 green squares are another 2 green squares. Area 2: Starting on the first row, region 2 has the fifth and sixth gold squares. Under the 2 gold squares and 1 place to the left are 4 gold squares side by side. Under the 4 gold squares are 1 green square, 2 spaces, and another green square. Under that row is an identical row with 1 green square, 2 spaces, and another green square. Area 3: Starting on the top row, region 3 has the last 4 gold squares in the first row. Under the second gold square in the first row is a row of 2 gold squares. Under the 2 gold squares in row 2 are 2 green squares. Under the 2 green squares are another two green squares. Area 3 is identical to area 1. Area 4: Starting in row 2, region 4 has starts with a gold square. Directly below in row 3 is 1 green square, then 3 spaces, and 1 green square. Row 4 is identical to row 3. Row 5 has 5 green squares side by side. Area 5: Starting in row 2, region 5 has the 10th gold square. In row 3 has the 6th green square, then 3 spaces, and 1 green square. Row 4 is identical to row 3. Row 5 has 5 green squares side by side.10 squares over is 1 yellow. In the next row down, 6 squares over is 1 green, 3 spaces, and 1 green square. Under the green square in the previous row is 1 green, 3 spaces, and 1 green square. Under the previous green square on the left, are 5 green squares.

Complete the table with this election's results.

Table 47.2.8

district	number of blocks for banana slugs	number of blocks for sea lions	percentage of blocks for banana slugs	representative's vote
1				
2				
3				
4				
5				

5. Write a headline for the local newspaper for each of the ways of splitting the town into districts.
6. Which systems on the three maps of districts do you think are more fair? Are any totally unfair?

Exercise 47.2.5: Fair and Unfair Districts

1. Smallville's map is shown, with opinions shown by block in green and gold. Decompose the map to create three connected, equal-area districts in two ways:

    a. Design three districts where *green* will win at least two of the three districts. Record results in Table 1.



Figure 47.2.6: A figure that represents a district composed of 30 green and gold squares that are arranged in 3 rows and 10 columns. The squares are arranged in the following order: Row 1: 8 green, 1 gold, 1 green. Row 2: 2 gold, 4 green, 1 gold, 1 green, 1 gold, 1 green. Row 3: 3 gold, 1 green, 1 gold, 2 green, 3 gold.

Table 1:

Table 47.2.9

district	number of blocks for green	number of blocks for gold	percentage of blocks for green	representative's vote
1				
2				
3				

b. Design three districts where *gold* will win at least two of the three districts. Record results in Table 2.



Figure 47.2.7: A figure that represents a district composed of 30 green and gold squares that are arranged in 3 rows and 10 columns. The squares are arranged in the following order: Row 1: 8 green, 1 gold, 1 green. Row 2: 2 gold, 4 green, 1 gold, 1 green, 1 gold, 1 green. Row 3: 3 gold, 1 green, 1 gold, 2 green, 3 gold.

Table 2:

Table 47.2.10

district	number of blocks for green	number of blocks for gold	percentage of blocks for green	representative's vote
1				
2				
3				

2. Squaretown's map is shown, with opinions by block shown in green and gold. Decompose the map to create five connected, equal-area districts in two ways:

a. Design five districts where *green* will win at least three of the five districts. Record the results in Table 3.

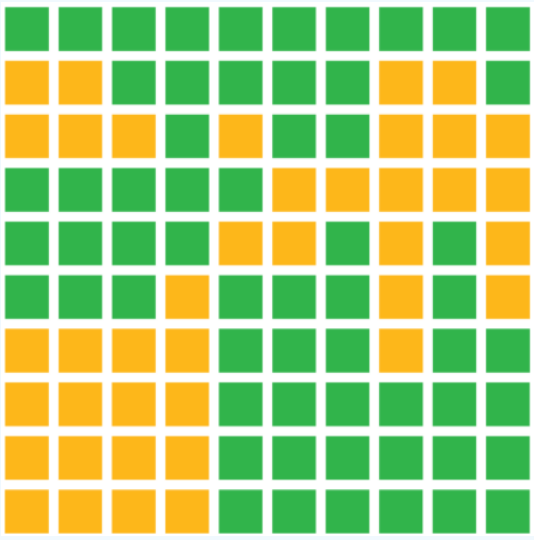


Figure 47.2.8: A figure that represents a district composed of 100 green and gold squares that are arranged in 10 rows and 10 columns. The squares are arranged in the following order: Row 1: 10 green. Row 2: 2 gold, 5 green, 2 gold, 1 green. Row 3: 3 gold, 1 green, 1 gold, 2 green, 3 gold. Row 4: 5 green, 5 gold. Row 5: 4 green, 2 gold, 1 green, 1 gold, 1 green, 1 gold. Row 6: 3 green, 1 gold, 3 green, 1 gold, 1 green, 1 gold. Row 7: 4 gold, 3 green, 1 gold, 2 green. Row 8: 4 gold, 6 green. Row 9: 4 gold, 6 green. Row 10: 4 gold, 6 green.

Table 3:

Table 47.2.11

district	number of blocks for green	number of blocks for gold	percentage of blocks for green	representative's vote
1				
2				
3				
4				
5				

b. Design five districts where *gold* will win at least three of the five districts. Record the results in Table 4.

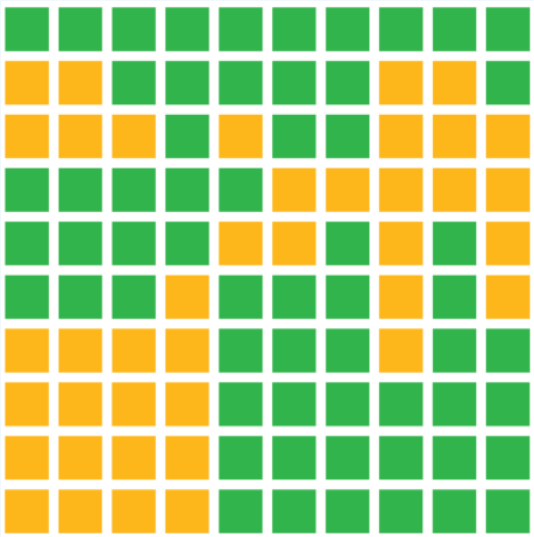


Figure 47.2.9: A figure that represents a district composed of 100 green and gold squares that are arranged in 10 rows and 10 columns. The squares are arranged in the following order: Row 1: 10 green. Row 2: 2 gold, 5 green, 2 gold, 1 green. Row 3: 3 gold, 1 green, 1 gold, 2 green, 3 gold. Row 4: 5 green, 5 gold. Row 5: 4 green, 2 gold, 1 green, 1 gold, 1 green, 1 gold. Row 6: 3 green, 1 gold, 3 green, 1 gold, 1 green, 1 gold. Row 7: 4 gold, 3 green, 1 gold, 2 green. Row 8: 4 gold, 6 green. Row 9: 4 gold, 6 green. Row 10: 4 gold, 6 green.

Table 4:

Table 47.2.12

district	number of blocks for green	number of blocks for gold	percentage of blocks for green	representative's vote
1				
2				
3				
4				
5				

3. Mountain Valley's map is shown, with opinions by block shown in green and gold. (This is a town in a narrow valley in the mountains.) Can you decompose the map to create three connected, equal-area districts in the two ways described here?

a. Design three districts where *green* will win at least 2 of the 3 districts. Record the results in Table 5.



Figure 47.2.10: A figure that represents a district composed of 18 green and gold squares that are arranged in the following order. Top row: 2 gold squares and 1 green square each side by side. Second Row: Starting under the green square in row 1, 1 gold square and 3 green squares each side by side. Third Row: Starting under the gold square in row 2, 1 gold square, 2 spaces, 2 green squares side by side, 2 spaces, and 3 gold squares each side by side. Fourth Row: Starting under the first green square in row 3, 3 green squares each side by side, then 1 gold square, and then 1 green square.

Table 5:

Table 47.2.13

district	number of blocks for green	number of blocks for gold	percentage of blocks for green	representative's vote
1				
2				
3				

b. Design three districts where *gold* will win at least 2 of the 3 districts. Record the results in Table 6.



Figure 47.2.11: A figure that represents a district composed of 18 green and gold squares that are arranged in the following order. Top row: 2 gold squares and 1 green square each side by side. Second Row: Starting under the green square in row 1, 1 gold square and 3 green squares each side by side. Third Row: Starting under the gold square in row 2, 1 gold square, 2 spaces, 2 green squares side by side, 2 spaces, and 3 gold squares each side by side. Fourth Row: Starting under the first green square in row 3, 3 green squares each side by side, then 1 gold square, and then 1 green square.

Table 6:

Table 47.2.14

district	number of blocks for green	number of blocks for gold	percentage of blocks for green	representative's vote
1				
2				
3				

47.2: Picking Representatives is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

Section 48: Let's Put it to Work

This page was auto-generated because a user created a sub-page to this page.

Section 48: Let's Put it to Work is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by LibreTexts.

48.1: How Do We Choose?

Lesson

Let's vote and choose a winner!

Exercise 48.1.1: Which Was "Yessier"?

Two sixth-grade classes, A and B, voted on whether to give the answers to their math problems in poetry. The “yes” choice was more popular in both classes.

Table 48.1.1

	yes	no
class A	24	16
class B	18	9

Was one class more in favor of math poetry, or were they equally in favor? Find three or more ways to answer the question.

Exercise 48.1.2: Which Class Voted Purpler?

The school will be painted over the summer. Students get to vote on whether to change the color to purple (a “yes” vote), or keep it a beige color (a “no” vote).

The principal of the school decided to analyze voting results by class. The table shows some results.

In both classes, a majority voted for changing the paint color to purple. Which class was more in favor of changing?

Table 48.1.2

	yes	no
class A	26	14
class B	31	19

Exercise 48.1.3: Supermajorities

1. Another school is also voting on whether to change their school’s color to purple. Their rules require a $\frac{2}{3}$ supermajority to change the colors. A total of 240 people voted, and 153 voted to change to purple. Were there enough votes to make the change?
2. This school also is thinking of changing their mascot to an armadillo. To change mascots, a 55% supermajority is needed. How many of the 240 students need to vote “yes” for the mascot to change?
3. At this school, which requires more votes to pass: a change of mascot or a change of color?

Exercise 48.1.4: Best Restaurant

A town’s newspaper held a contest to decide the best restaurant in town. Only people who subscribe to the newspaper can vote. 25% of the people in town subscribe to the newspaper. 20% of the subscribers voted. 80% of the people who voted liked Darnell’s BBQ Pit best.

Darnell put a big sign in his restaurant’s window that said, “80% say Darnell’s is the best!”

Do you think Darnell’s sign is making an accurate statement? Support your answer with:

- Some calculations
- An explanation in words

- A diagram that accurately represents the people in town, the newspaper subscribers, the voters, and the people who liked Darnell's best

This page titled 48.1: How Do We Choose? is shared under a CC BY 4.0 license and was authored, remixed, and/or curated by Illustrative Mathematics via source content that was edited to the style and standards of the LibreTexts platform.

Index

D

dire

Glossary

Sample Word 1 | Sample Definition 1

Detailed Licensing

Overview

Title: Basic Math (Grade 6)

Webpages: 214

All licenses found:

- CC BY 4.0: 99.5% (213 pages)
- Undeclared: 0.5% (1 page)

By Page

- Basic Math (Grade 6) - *CC BY 4.0*
  - Front Matter - *CC BY 4.0*
    - TitlePage - *CC BY 4.0*
    - InfoPage - *CC BY 4.0*
    - Table of Contents - *Undeclared*
    - Licensing - *CC BY 4.0*
  - 1: Area and Surface Area - *CC BY 4.0*
    - Section 1: Lessons Reasoning to Find Area - *CC BY 4.0*
      - 1.1: Tiling the Plane - *CC BY 4.0*
      - 1.2: Finding Area by Decomposing and Rearranging - *CC BY 4.0*
      - 1.3: Reasoning to Find Area - *CC BY 4.0*
    - Section 2: Parallelograms - *CC BY 4.0*
      - 2.1: Parallelograms - *CC BY 4.0*
      - 2.2: Bases and Heights of Parallelograms - *CC BY 4.0*
      - 2.3: Area of Parallelograms - *CC BY 4.0*
    - Section 3: Triangles - *CC BY 4.0*
      - 3.1: From Parallelograms to Triangles - *CC BY 4.0*
      - 3.2: Area of Triangles - *CC BY 4.0*
      - 3.3: Formula for the Area of a Triangle - *CC BY 4.0*
      - 3.4: Bases and Heights of Triangles - *CC BY 4.0*
    - Section 4: Polygons - *CC BY 4.0*
      - 4.1: Polygons - *CC BY 4.0*
    - Section 5: Surface Area - *CC BY 4.0*
      - 5.1: Polyhedra - *CC BY 4.0*
      - 5.2: Nets and Surface Area - *CC BY 4.0*
      - 5.3: More Nets, More Surface Area - *CC BY 4.0*
      - 5.4: Distinguishing Between Surface Area and Volume - *CC BY 4.0*
    - Section 6: Squares and Cubes - *CC BY 4.0*
      - 6.1: Squares and Cubes - *CC BY 4.0*
      - 6.2: Surface Area of a Cube - *CC BY 4.0*
    - Section 7: Surface Area - *CC BY 4.0*
      - 7.1: What is Surface Area? - *CC BY 4.0*
    - Section 8: Let's Put it to Work - *CC BY 4.0*
      - 8.1: Designing a Tent - *CC BY 4.0*
  - 2: Introducing Ratios - *CC BY 4.0*
    - Section 9: What are Ratios? - *CC BY 4.0*
      - 9.1: Introducing Ratios and Ratio Language - *CC BY 4.0*
      - 9.2: Representing Ratios with Diagrams - *CC BY 4.0*
    - Section 10: Equivalent Ratios - *CC BY 4.0*
      - 10.1: Recipes - *CC BY 4.0*
      - 10.2: Color Mixtures - *CC BY 4.0*
      - 10.3: Defining Equivalent Ratios - *CC BY 4.0*
    - Section 11: Representing Equivalent Ratios - *CC BY 4.0*
      - 11.1: Introducing Double Number Line Diagrams - *CC BY 4.0*
      - 11.2: Creating Double Number Line Diagrams - *CC BY 4.0*
      - 11.3: Constant Speed - *CC BY 4.0*
      - 11.4: Comparing Situations by Examining Ratios - *CC BY 4.0*
    - Section 12: Solving Ratio and Rate Problems - *CC BY 4.0*
      - 12.1: Representing Ratios with Tables - *CC BY 4.0*
      - 12.2: Navigating a Table of Equivalent Ratios - *CC BY 4.0*
      - 12.3: Tables and Double Number Line Diagrams - *CC BY 4.0*
      - 12.4: Solving Equivalent Ratio Problems - *CC BY 4.0*
    - Section 13: Part-Part-Whole Ratios - *CC BY 4.0*
      - 13.1: Part-Part-Whole Ratios - *CC BY 4.0*
      - 13.2: Solving More Ratio Problems - *CC BY 4.0*
    - Section 14: Let's Put it to Work - *CC BY 4.0*
      - 14.1: A Fermi Problem - *CC BY 4.0*

- 14.2: How Much for One? - *CC BY 4.0*

- 3: Unit Rates and Percentages - *CC BY 4.0*
  - Section 15: Unit of Measurement - *CC BY 4.0*
    - 15.1: The Burj Khalifa - *CC BY 4.0*
  - Section 16: Unit Conversion - *CC BY 4.0*
    - 16.1: Anchoring Units of Measurement - *CC BY 4.0*
    - 16.2: Measuring with Different-Sized Units - *CC BY 4.0*
    - 16.3: Converting Units - *CC BY 4.0*
  - Section 17: Rates - *CC BY 4.0*
    - 17.1: Comparing Speeds and Prices - *CC BY 4.0*
    - 17.2: Interpreting Rates - *CC BY 4.0*
    - 17.3: Equivalent Ratios Have the Same Unit Rates - *CC BY 4.0*
    - 17.4: More about Constant Speed - *CC BY 4.0*
    - 17.5: Solving Rate Problems - *CC BY 4.0*
  - Section 18: Percentages - *CC BY 4.0*
    - 18.1: Percentages and Double Number Lines - *CC BY 4.0*
    - 18.2: Percentages and Tape Diagrams - *CC BY 4.0*
    - 18.3: Benchmark Percentages - *CC BY 4.0*
    - 18.4: Solving Percentage Problems - *CC BY 4.0*
    - 18.5: Finding This Percent of That - *CC BY 4.0*
    - 18.6: Finding the Percentage - *CC BY 4.0*
  - Section 19: Let's Put it to Work - *CC BY 4.0*
    - 19.1: Painting a Room - *CC BY 4.0*
  - Section 20: New Page - *CC BY 4.0*
    - 20.1: What Are Percentages? - *CC BY 4.0*
- 4: Dividing Fractions - *CC BY 4.0*
  - Section 21: Making Sense of Division - *CC BY 4.0*
    - 21.1: Size of Divisor and Size of Quotient - *CC BY 4.0*
    - 21.2: Meanings of Division - *CC BY 4.0*
    - 21.3: Interpreting Division Situations - *CC BY 4.0*
  - Section 22: Meanings of Fraction Division - *CC BY 4.0*
    - 22.1: How Many Groups? (Part 1) - *CC BY 4.0*
    - 22.2: How Many Groups? (Part 2) - *CC BY 4.0*
    - 22.3: Using Diagrams to Find the Number of Groups - *CC BY 4.0*
    - 22.4: What Fraction of a Group? - *CC BY 4.0*
    - 22.5: How Much in Each Group? (Part 1) - *CC BY 4.0*
    - 22.6: How Much in Each Group? (Part 2) - *CC BY 4.0*
  - Section 23: Algorithm for Fraction Division - *CC BY 4.0*
    - 23.1: Dividing by Unit and Non-Unit Fractions - *CC BY 4.0*
    - 23.2: Using an Algorithm to Divide Fractions - *CC BY 4.0*
  - Section 24: Fractions in Lengths, Areas, and Volumes - *CC BY 4.0*
    - 24.1: Fractional Lengths - *CC BY 4.0*
    - 24.2: Rectangles with Fractional Side Lengths - *CC BY 4.0*
    - 24.3: Fractional Lengths in Triangles and Prisms - *CC BY 4.0*
    - 24.4: Volume of Prisms - *CC BY 4.0*
  - Section 25: Let's Put It to Work - *CC BY 4.0*
    - 25.1: Solving Problems Involving Fractions - *CC BY 4.0*
    - 25.2: Fitting Boxes into Boxes - *CC BY 4.0*
- 5: Arithmetic in Base Ten - *CC BY 4.0*
  - Section 26: Warming Up to Decimals - *CC BY 4.0*
    - 26.1: Using Decimals in a Shopping Context - *CC BY 4.0*
  - Section 27: Adding and Subtracting Decimals - *CC BY 4.0*
    - 27.1: Using Diagrams to Represent Addition and Subtraction - *CC BY 4.0*
    - 27.2: Adding and Subtracting Decimals with Few Non-Zero Digits - *CC BY 4.0*
    - 27.3: Adding and Subtracting Decimals with Many Non-Zero Digits - *CC BY 4.0*
  - Section 28: Multiplying Decimals - *CC BY 4.0*
    - 28.1: Decimal Points in Products - *CC BY 4.0*
    - 28.2: Methods for Multiplying Decimals - *CC BY 4.0*
    - 28.3: Using Diagrams to Represent Multiplication - *CC BY 4.0*
    - 28.4: Calculating Products of Decimals - *CC BY 4.0*
  - Section 29: Dividing Decimals - *CC BY 4.0*
    - 29.1: Using the Partial Quotients Method - *CC BY 4.0*
    - 29.2: Using Long Division - *CC BY 4.0*
    - 29.3: Dividing Numbers that Result in Decimals - *CC BY 4.0*
    - 29.4: Dividing Decimals by Whole Numbers - *CC BY 4.0*
    - 29.5: Dividing Decimals by Decimals - *CC BY 4.0*
  - Section 30: Let's Put It to Work - *CC BY 4.0*
    - 30.1: Using Operations on Decimals to Solve Problems - *CC BY 4.0*
    - 30.2: Making and Measuring Boxes - *CC BY 4.0*

- 6: Expressions and Equations - *CC BY 4.0*
  - Section 31: Equations in One Variable - *CC BY 4.0*
    - 31.1: Tape Diagrams and Equations - *CC BY 4.0*
    - 31.2: Truth and Equations - *CC BY 4.0*
    - 31.3: Staying in Balance - *CC BY 4.0*
    - 31.4: Practice Solving Equations and Representing Situations with Equations - *CC BY 4.0*
    - 31.5: A New Way to Interpret a over b - *CC BY 4.0*
  - Section 32: Equal and Equivalent - *CC BY 4.0*
    - 32.1: Write Expressions Where Letters Stand for Numbers - *CC BY 4.0*
    - 32.2: Revisit Percentages - *CC BY 4.0*
    - 32.3: Equal and Equivalent - *CC BY 4.0*
    - 32.4: The Distributive Property, Part 1 - *CC BY 4.0*
    - 32.5: The Distributive Property, Part 2 - *CC BY 4.0*
    - 32.6: The Distributive Property, Part 3 - *CC BY 4.0*
  - Section 33: Expressions with Exponents - *CC BY 4.0*
    - 33.1: Meaning of Exponents - *CC BY 4.0*
    - 33.2: Expressions with Exponents - *CC BY 4.0*
    - 33.3: Evaluating Expressions with Exponents - *CC BY 4.0*
    - 33.4: Equivalent Exponential Expressions - *CC BY 4.0*
  - Section 34: Relationship Between Quantities - *CC BY 4.0*
    - 34.1: Two Related Quantities, Part 1 - *CC BY 4.0*
    - 34.2: Two Related Quantities, Part 2 - *CC BY 4.0*
    - 34.3: More Relationships - *CC BY 4.0*
  - Section 35: Let's Put it to Work - *CC BY 4.0*
    - 35.1: Tables, Equations, and Graphs, Oh My! - *CC BY 4.0*
- 7: Rational Numbers - *CC BY 4.0*
  - Section 36: Negative Numbers and Absolute Value - *CC BY 4.0*
    - 36.1: Positive and Negative Numbers - *CC BY 4.0*
    - 36.2: Points on the Number Line - *CC BY 4.0*
    - 36.3: Comparing Positive and Negative Numbers - *CC BY 4.0*
    - 36.4: Ordering Rational Numbers - *CC BY 4.0*
    - 36.5: Using Negative Numbers to Make Sense of Contexts - *CC BY 4.0*
    - 36.6: Absolute Value of Numbers - *CC BY 4.0*
    - 36.7: Comparing Numbers and Distance from Zero - *CC BY 4.0*
  - Section 37: Inequalities - *CC BY 4.0*
    - 37.1: Writing and Graphing Inequalities - *CC BY 4.0*
    - 37.2: Solutions of Inequalities - *CC BY 4.0*
    - 37.3: Interpreting Inequalities - *CC BY 4.0*
  - Section 38: The Coordinate Plane - *CC BY 4.0*
    - 38.1: Points on the Coordinate Plane - *CC BY 4.0*
    - 38.2: Constructing the Coordinate Plane - *CC BY 4.0*
    - 38.3: Interpreting Points on a Coordinate Plane - *CC BY 4.0*
    - 38.4: Distances on a Coordinate Plane - *CC BY 4.0*
    - 38.5: Shapes on the Coordinate Plane - *CC BY 4.0*
  - Section 39: Common Factors and Common Multiples - *CC BY 4.0*
    - 39.1: Common Factors - *CC BY 4.0*
    - 39.2: Common Multiples - *CC BY 4.0*
    - 39.3: Using Common Multiples and Common Factors - *CC BY 4.0*
  - Section 40: Let's Put it to Work - *CC BY 4.0*
    - 40.1: Drawing on the Coordinate Plane - *CC BY 4.0*
- 8: Data Sets and Distributions - *CC BY 4.0*
  - Section 41: Data, Variability, and Statistical Questions - *CC BY 4.0*
    - 41.1: Got Data? - *CC BY 4.0*
    - 41.2: Statistical Questions - *CC BY 4.0*
  - Section 42: Dot Plots and Histograms - *CC BY 4.0*
    - 42.1: Representing Data Graphically - *CC BY 4.0*
    - 42.2: Dot Plots - *CC BY 4.0*
    - 42.3: Using Dot Plots to Answer Statistical Questions - *CC BY 4.0*
    - 42.4: Interpreting Histograms - *CC BY 4.0*
    - 42.5: Using Histograms to Answer Statistical Questions - *CC BY 4.0*
    - 42.6: Describing Distributions on Histograms - *CC BY 4.0*
  - Section 43: Measures of Center and Variability - *CC BY 4.0*
    - 43.1: Mean - *CC BY 4.0*
    - 43.2: Finding and Interpreting the Mean as the Balance Point - *CC BY 4.0*
    - 43.3: Variability and MAD - *CC BY 4.0*
    - 43.4: Using Mean and MAD to Make Comparisons - *CC BY 4.0*
  - Section 44: Median and IQR - *CC BY 4.0*
    - 44.1: Median - *CC BY 4.0*
    - 44.2: Comparing Mean and Median - *CC BY 4.0*
    - 44.3: Quartiles and Interquartile Range - *CC BY 4.0*

- 44.4: Box Plots - *CC BY 4.0*
  - 44.4.1: Using Box Plots - *CC BY 4.0*
- Section 45: Let's Put it to Work - *CC BY 4.0*
  - 45.1: Using Data to Solve Problems - *CC BY 4.0*

9: Putting it All Together - *CC BY 4.0*

- Section 46: Making Connections - *CC BY 4.0*
  - 46.1: Fermi Problems - *CC BY 4.0*
  - 46.2: If Our Class Were the World - *CC BY 4.0*
  - 46.3: Rectangle Madness - *CC BY 4.0*
- Section 47: Voting - *CC BY 4.0*
  - 47.1: More than Two Choices - *CC BY 4.0*
  - 47.2: Picking Representatives - *CC BY 4.0*
- Section 48: Let's Put it to Work - *CC BY 4.0*
  - 48.1: How Do We Choose? - *CC BY 4.0*

Back Matter - *CC BY 4.0*

- Index - *CC BY 4.0*
- Glossary - *CC BY 4.0*
- Detailed Licensing - *CC BY 4.0*