PRE-ALGEBRA I (ILLUSTRATIVE MATHEMATICS - GRADE 7)



Illustrative Mathematics

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Pre-Algebra I (Illustrative Mathematics - Grade 7)

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Glossary

Detailed Licensing

Licensing

*A detailed breakdown of this resource's licensing can be found in **Back Matter/Detailed Licensing**.*

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SECTION OVERVIEW

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1.1.2: Corresponding Parts and Scale Factors

Lesson

Let's describe features of scaled copies.

Exercise 1.1.2.1: Number Talk: Multiplying by a Unit Fraction

Find each product mentally.

$\frac{1}{4} \cdot 32$

$(7.2) \cdot \frac{1}{9}$

$\frac{1}{4} \cdot (5.6)$

Exercise 1.1.2.2: Corresponding Parts

One road sign for railroad crossings is a circle with a large X in the middle and two R's—with one on each side. Here is a picture with some points labeled and two copies of the picture. Drag and turn the moveable angle tool to compare the angles in the copies with the angles in the original.

1. Complete this table to show **corresponding parts** in the three pictures.

Table 1.1.2.1

original	Copy 1	Copy 2
point L		
segment LM		
	segment ED	
		point X
angle KLM		
		angle XYZ

2. Is either copy a scaled copy of the original road sign? Explain your reasoning.
3. Use the moveable angle tool to compare angle KLM with its corresponding angles in Copy 1 and Copy 2. What do you notice?
4. Use the moveable angle tool to compare angle NOP with its corresponding angles in Copy 1 and Copy 2. What do you notice?

Exercise 1.1.2.3: Scaled Triangles

Here is Triangle O, followed by a number of other triangles.

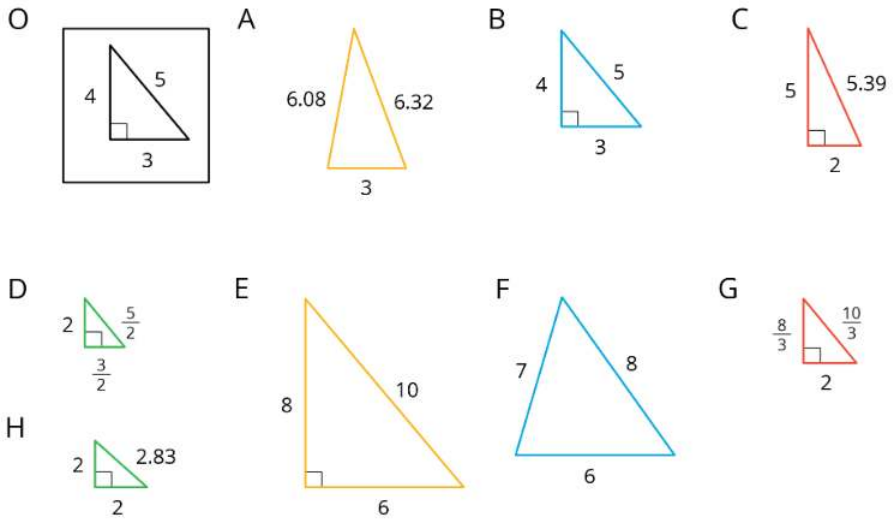


Figure 1.1.2.1

Your teacher will assign you two of the triangles to look at.

1. For each of your assigned triangles, is it a scaled copy of Triangle O? Be prepared to explain your reasoning.
2. As a group, identify *all* the scaled copies of Triangle O in the collection. Discuss your thinking. If you disagree, work to reach an agreement.
3. List all the triangles that are scaled copies in the table. Record the side lengths that correspond to the side lengths of Triangle O listed in each column.

Table 1.1.2.2

Triangle O	3	4	5

4. Explain or show how each copy has been scaled from the original (Triangle O).

Are you ready for more?

Choose one of the triangles that is not a scaled copy of Triangle O. Describe how you could change at least one side to make a scaled copy, while leaving at least one side unchanged.

Summary

A figure and its scaled copy have **corresponding parts**, or parts that are in the same position in relation to the rest of each figure. These parts could be points, segments, or angles. For example, Polygon 2 is a scaled copy of Polygon 1.

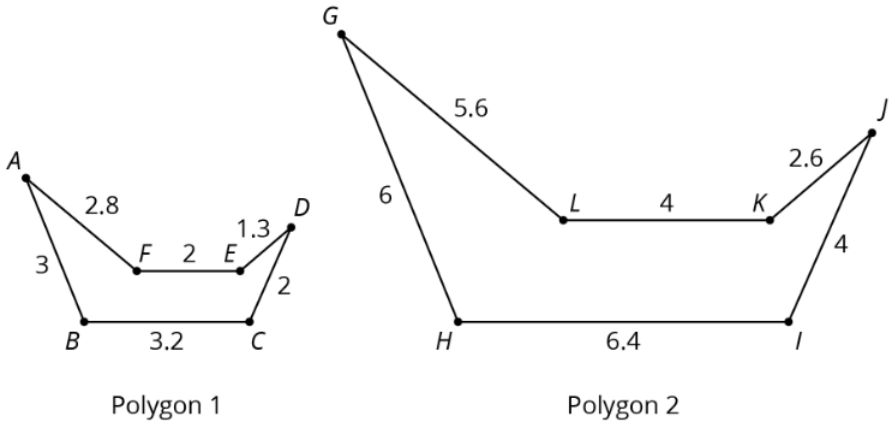


Figure 1.1.2.2

- Each point in Polygon 1 has a *corresponding point* in Polygon 2.
  For example, point B corresponds to point H and point C corresponds to point I.
- Each segment in Polygon 1 has a *corresponding segment* in Polygon 2.
  For example, segment AF corresponds to segment GL.
- Each angle in Polygon 1 also has a *corresponding angle* in Polygon 2.
  For example, angle DEF corresponds to angle JKL.

The **scale factor** between Polygon 1 and Polygon 2 is 2, because all of the lengths in Polygon 2 are 2 times the corresponding lengths in Polygon 1. The angle measures in Polygon 2 are the same as the corresponding angle measures in Polygon 1. For example, the measure of angle JKL is the same as the measure of angle DEF.

Glossary Entries

Definition: Corresponding

When part of an original figure matches up with part of a copy, we call them corresponding parts. These could be points, segments, angles, or distances.

For example, point B in the first triangle corresponds to point E in the second triangle. Segment AC corresponds to segment DF.

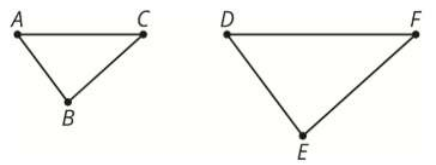


Figure 1.1.2.3

Definition: Scale Factor

To create a scaled copy, we multiply all the lengths in the original figure by the same number. This number is called the scale factor.

In this example, the scale factor is 1.5, because $4 \cdot (1.5) = 6$, $5 \cdot (1.5) = 7.5$, and $6 \cdot (1.5) = 9$.

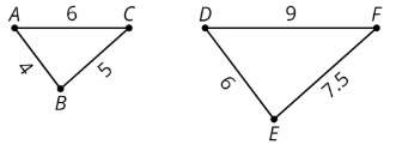


Figure 1.1.2.4

Definition: Scaled Copy

A scaled copy is a copy of an figure where every length in the original figure is multiplied by the same number.

For example, triangle DEF is a scaled copy of triangle ABC. Each side length on triangle ABC was multiplied by 1.5 to get the corresponding side length on triangle DEF.

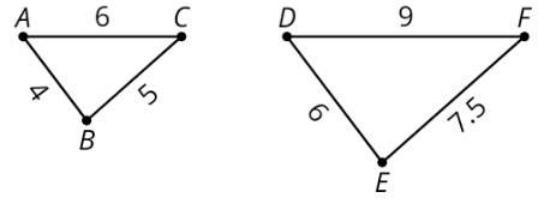


Figure 1.1.2.5

Practice

Exercise 1.1.2.4

The second H-shaped polygon is a scaled copy of the first.

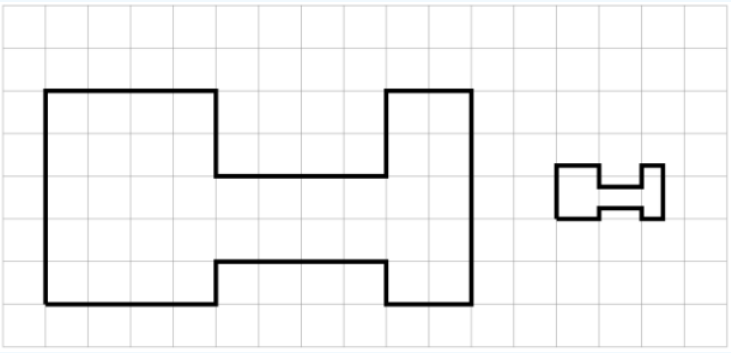


Figure 1.1.2.6

1. Show one pair of corresponding points and two pairs of corresponding sides in the original polygon and its copy. Consider using colored pencils to highlight corresponding parts or labeling some of the vertices.
2. What scale factor takes the original polygon to its smaller copy? Explain or show your reasoning.

Exercise 1.1.2.5

Figure B is a scaled copy of Figure A. Select **all** of the statements that must be true:

A. Figure B is larger than Figure A.
B. Figure B has the same number of edges as Figure A.
C. Figure B has the same perimeter as Figure A.
D. Figure B has the same number of angles as Figure A.
E. Figure B has angles with the same measures as Figure A.

Exercise 1.1.2.6

Polygon B is a scaled copy of Polygon A.

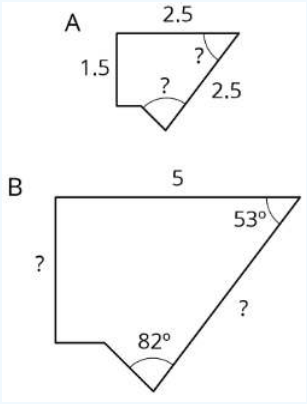


Figure 1.1.2.7: Pentagon A clockwise from the top has length 2.5, angle unknown, length 2.5, angle unknown, 2 unknown lengths, length 1.5. Pentagon B has top length 5, angle of 53 degrees, unknown length, angle of 82 degrees, and all the rest unknown.

1. What is the scale factor from Polygon A to Polygon B? Explain your reasoning.
2. Find the missing length of each side marked with ? in Polygon B.
3. Determine the measure of each angle marked with ? in Polygon A.

Exercise 1.1.2.7

Compare each equation with a number that makes it true.

1. $8 \cdot \_\_\_\_ = 40$
2. $8 + \_\_\_\_ = 40$
3. $21 \div \_\_\_\_ = 7$
4. $21 - \_\_\_\_ = 7$
5. $21 \cdot \_\_\_\_ = 7$

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1.1.3: Making Scaled Copies

Lesson

Let's draw scaled copies.

Exercise 1.1.3.1: More or Less?

For each problem, select the answer from the two choices.

1. The value of $25 \cdot (8.5)$ is:
   a. More than 205
   b. Less than 205
2. The value of $(9.93) \cdot (0.984)$ is:
   a. More than 10
   b. Less than 10
3. The value of $(0.24) \cdot (0.67)$ is:
   a. More than 0.2
   b. Less than 0.2

Exercise 1.1.3.2: Drawing Scaled Copies

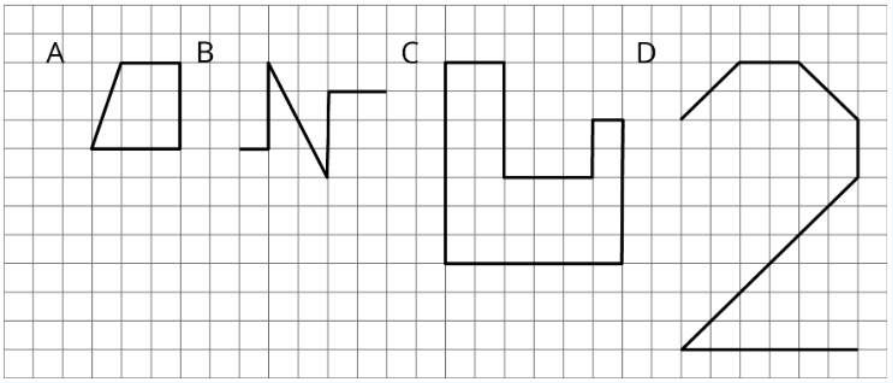


Figure 1.1.3.1

1. Draw a scaled copy of either Figure A or B using a scale factor of 3.
2. Draw a scaled copy of either Figure C or D using a scale factor of $\frac{1}{2}$.

Exercise 1.1.3.3: Which Operations? (Part 1)

Diego and Jada want to scale this polygon so the side that corresponds to 15 units in the original is 5 units in the scaled copy.

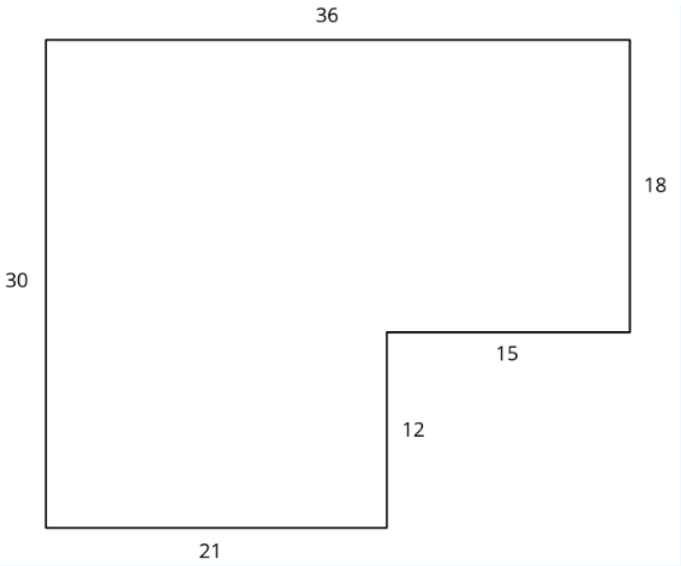


Figure 1.1.3.2

Diego and Jada each use a different operation to find the new side lengths. Here are their finished drawings.

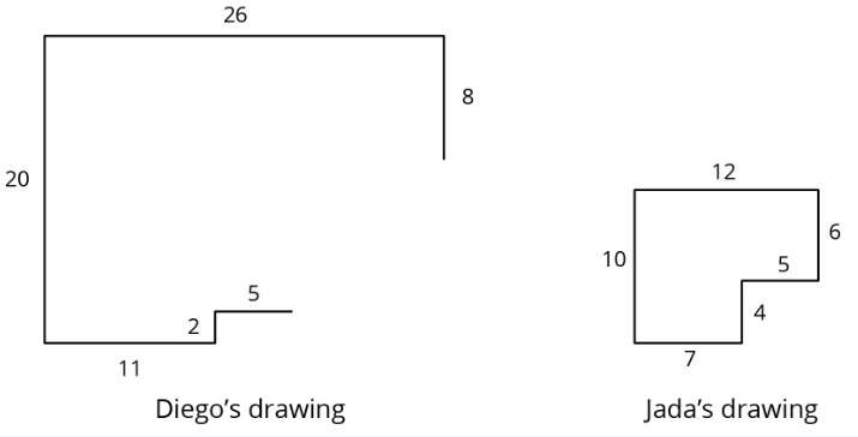


Figure 1.1.3.3

1. What operation do you think Diego used to calculate the lengths for his drawing?
2. What operation do you think Jada used to calculate the lengths for her drawing?
3. Did each method produce a scaled copy of the polygon? Explain your reasoning.

Exercise 1.1.3.4: Which Operations? (Part 2)

Andre wants to make a scaled copy of Jada's drawing so the side that corresponds to 4 units in Jada's polygon is 8 units in his scaled copy.

1. Andre says "I wonder if I should add 4 units to the lengths of all of the segments?" What would you say in response to Andre? Explain or show your reasoning.
2. Create the scaled copy that Andre wants. If you get stuck, consider using the edge of an index card or paper to measure the lengths needed to draw the copy.

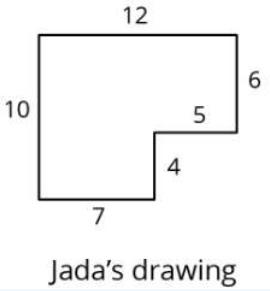


Figure 1.1.3.4

Are you ready for more?

The side lengths of Triangle B are all 5 more than the side lengths of Triangle A. Can Triangle B be a scaled copy of Triangle A? Explain your reasoning.

Summary

Creating a scaled copy involves *multiplying* the lengths in the original figure by a scale factor.

For example, to make a scaled copy of triangle ABC where the base is 8 units, we would use a scale factor of 4. This means multiplying all the side lengths by 4, so in triangle DEF, each side is 4 times as long as the corresponding side in triangle ABC.

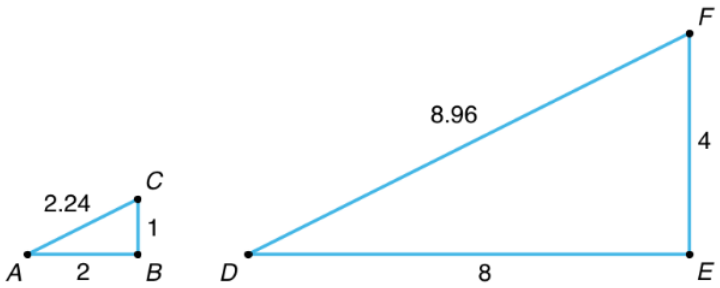


Figure 1.1.3.5

Glossary Entries

Definition: Corresponding

When part of an original figure matches up with part of a copy, we call them corresponding parts. These could be points, segments, angles, or distances.

For example, point B in the first triangle corresponds to point E in the second triangle. Segment AC corresponds to segment DF.

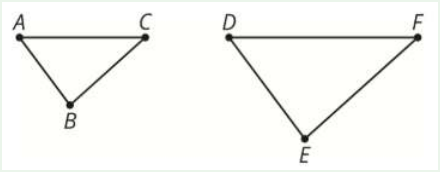


Figure 1.1.3.6

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To create a scaled copy, we multiply all the lengths in the original figure by the same number. This number is called the scale factor.

In this example, the scale factor is 1.5, because $4 \cdot (1.5) = 6$, $5 \cdot (1.5) = 7.5$, and $6 \cdot (1.5) = 9$.

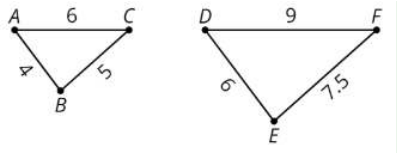


Figure 1.1.3.7

Definition: Scaled Copy

A scaled copy is a copy of an figure where every length in the original figure is multiplied by the same number.

For example, triangle DEF is a scaled copy of triangle ABC. Each side length on triangle ABC was multiplied by 1.5 to get the corresponding side length on triangle DEF.

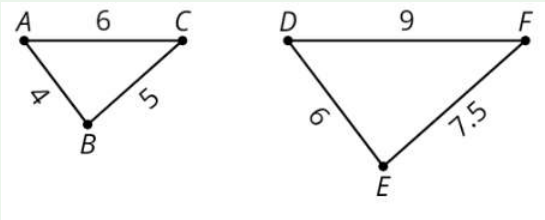


Figure 1.1.3.8

Practice

Exercise 1.1.3.5

Here are 3 polygons.

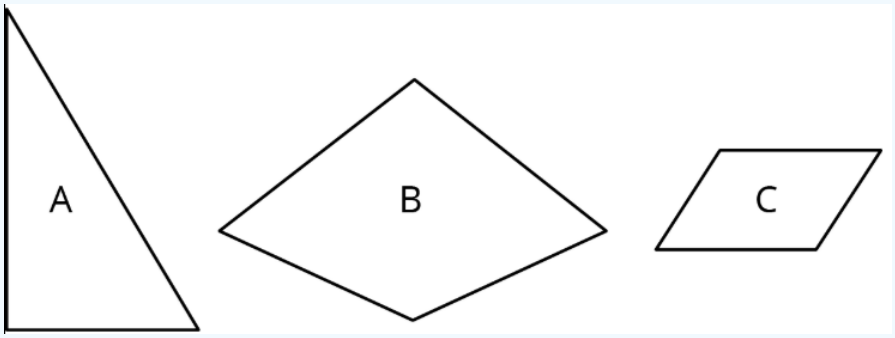


Figure 1.1.3.9

Draw a scaled copy of Polygon A using a scale factor of 2.

Draw a scaled copy of Polygon B using a scale factor of $\frac{1}{2}$.

Draw a scaled copy of Polygon C using a scale factor of $\frac{3}{2}$.

Exercise 1.1.3.6

Quadrilateral A has side lengths 6, 9, 9, and 12. Quadrilateral B is a scaled copy of Quadrilateral A, with its shortest side of length 2. What is the perimeter of Quadrilateral B?

Exercise 1.1.3.7

Here is a polygon on a grid.



Figure 1.1.3.10: An L shaped figure on a square grid. The polygon is composed of two shapes: a rectangle adjoined with a square so that the bottom horizontal base of the polygon is 2 units. The rectangle is 1 unit wide and 3 units in length. The square is 1 unit wide and 1 unit in length.

Draw a scaled copy of this polygon that has a perimeter of 30 units. What is the scale factor? Explain how you know.

Exercise 1.1.3.8

Priya and Tyler are discussing the figures shown below. Priya thinks that B, C, and D are scaled copies of A. Tyler says B and D are scaled copies of A. Do you agree with Priya, or do you agree with Tyler? Explain your reasoning.

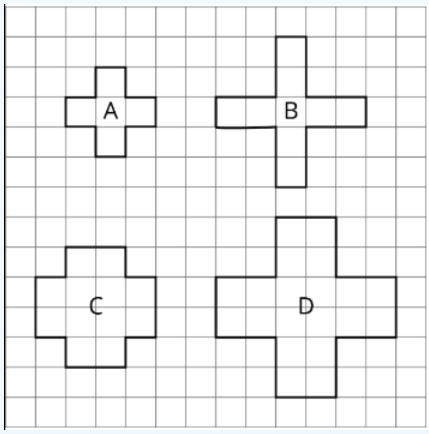


Figure 1.1.3.11: Four shapes that look like plus signs. Shape A is one square with a square appended to each side. Shape B is one square with a 1 by 2 rectangle appended to each side. Shape C is a 2 by 2 square with a 1 by 2 rectangle appended to each side. Shape D is a 2 by 2 square with a 2 by 2 square appended to each side.

(From Unit 1.1.1)

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1.1.4: Scaled Relationships

Lesson

Let's find relationships between scaled copies.

Exercise 1.1.4.1: Three Quadrilaterals (Part 1)

Each of these polygons is a scaled copy of the others.

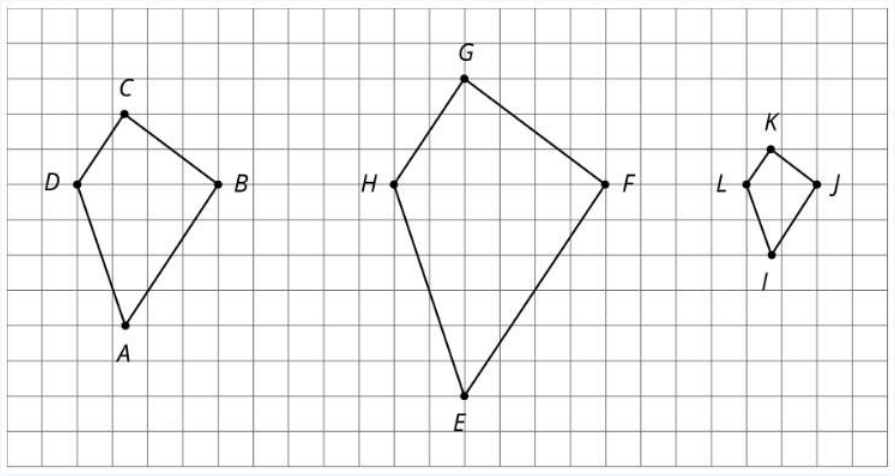


Figure 1.1.4.1

1. Name two pairs of corresponding angles. What can you say about the sizes of these angles?
2. Check your prediction by measuring at least one pair of corresponding angles using a protractor. Record your measurements to the nearest 5°.

Exercise 1.1.4.2: Three Quadrilaterals (Part 2)

Each of these polygons is a scaled copy of the others. You already checked their corresponding angles.

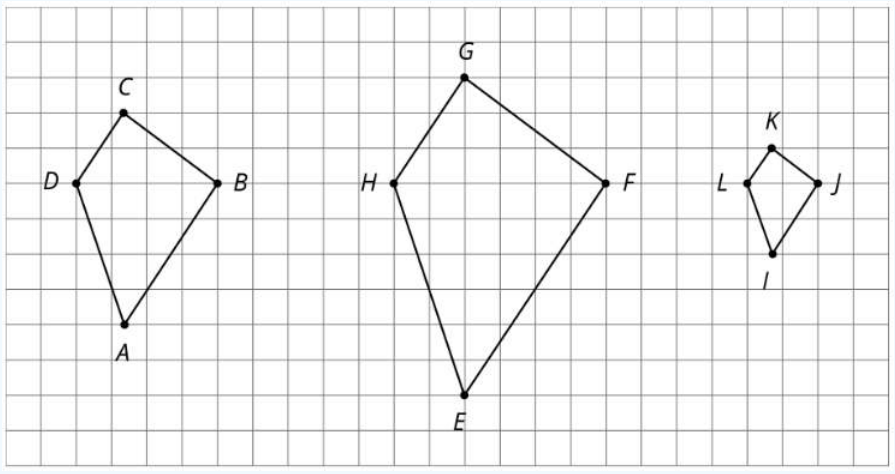


Figure 1.1.4.2

1. The side lengths of the polygons are hard to tell from the grid, but there are other *corresponding distances* that are easier to compare. Identify the distances in the other two polygons that correspond to DB and AC, and record them in the table.

Table 1.1.4.1

quadrilateral	distance that corresponds to DB	distance that corresponds to AC
$ABCD$	$DB = 4$	$AC = 6$

quadrilateral	distance that corresponds to DB	distance that corresponds to AC
$EFGH$		
$IJKL$		

2. Look at the values in the table. What do you notice?

Pause here so your teacher can review your work.

3. The larger figure is a scaled copy of the smaller figure.

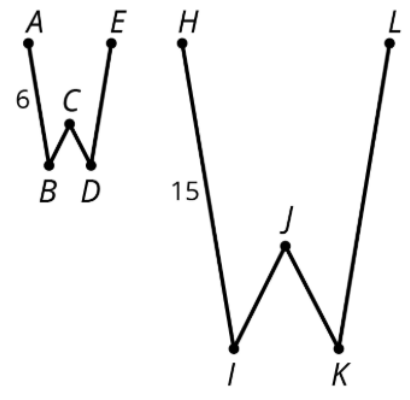


Figure 1.1.4.3

a. If $AE = 4$, how long is the corresponding distance in the second figure? Explain or show your reasoning.

b. If $IK = 5$, how long is the corresponding distance in the first figure? Explain or show your reasoning.

Exercise 1.1.4.3: Scaled or Not Scaled?

Here are two quadrilaterals.

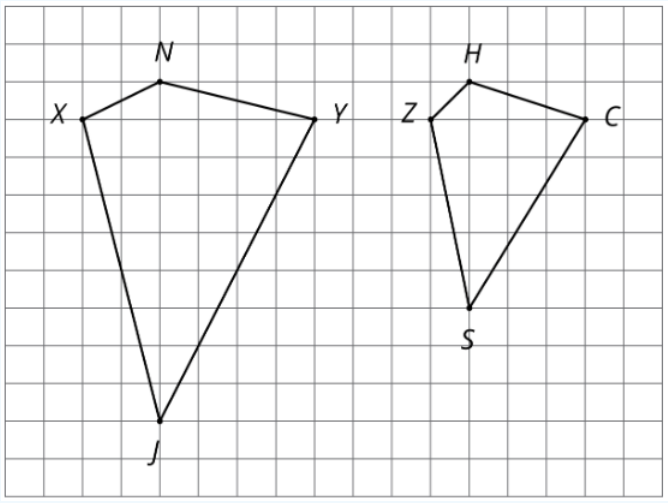


Figure 1.1.4.4

1. Mai says that Polygon $ZSCH$ is a scaled copy of Polygon $XJYN$, but Noah disagrees. Do you agree with either of them? Explain or show your reasoning.
2. Record the corresponding distances in the table. What do you notice?

Table 1.1.4.2

quadrilateral	horizontal distance	vertical distance
$XJYN$	$XY =$	$JN =$

	quadrilateral	horizontal distance	vertical distance
eral	$ZSCH$	$ZC =$	$SH =$

3. Measure at least three pairs of corresponding angles in $XJYN$ and $ZSCH$ using a protractor. Record your measurements to the nearest $5°$. What do you notice?
4. Do these results change your answer to the first question? Explain.
5. Here are two more quadrilaterals.

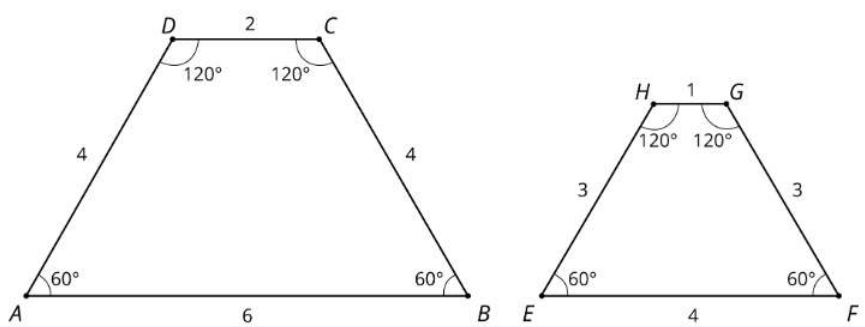


Figure 1.1.4.5: The angle measures, in degrees, for both trapezoids are: 60, 60, 120, 120. In A, B, C, D, the top length is 2, bottom length is 6, both sides lengths are 4. In E, F, G, H, the top length is 1, bottom length is 4 and both side lengths are 3.

Kiran says that Polygon $EFGH$ is a scaled copy of $ABCD$, but Lin disagrees. Do you agree with either of them? Explain or show your reasoning.

Are you ready for more?

All side lengths of quadrilateral $MNOP$ are 2, and all side lengths of quadrilateral $QRST$ are 3. Does $MNOP$ have to be a scaled copy of $QRST$? Explain your reasoning.

Exercise 1.1.4.4: Comparing Pictures of Birds

Here are two pictures of a bird. Find evidence that one picture is not a scaled copy of the other. Be prepared to explain your reasoning.

Summary

When a figure is a scaled copy of another figure, we know that:

- All distances in the copy can be found by multiplying the *corresponding distances* in the original figure by the same scale factor, whether or not the endpoints are connected by a segment.

For example, Polygon $STUVWX$ is a scaled copy of Polygon $ABCDEF$. The scale factor is 3. The distance from T to X is 6, which is three times the distance from B to F.

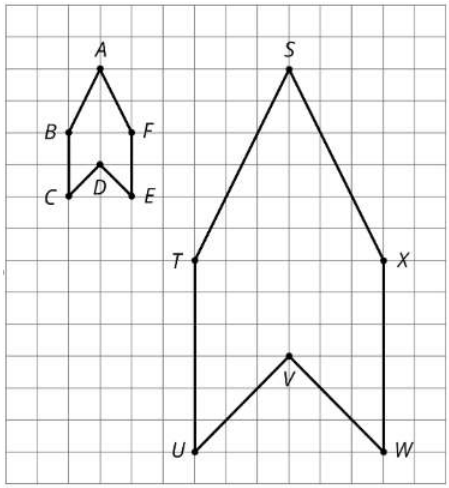


Figure 1.1.4.6

- All angles in the copy have the same measure as the corresponding angles in the original figure, as in these triangles.

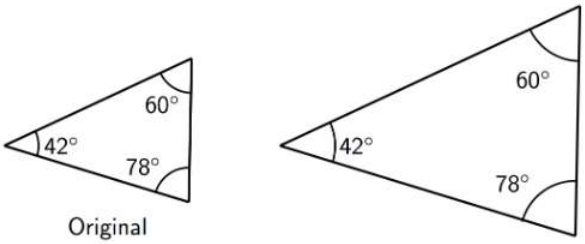


Figure 1.1.4.7

These observations can help explain why one figure is *not* a scaled copy of another.

For example, even though their corresponding angles have the same measure, the second rectangle is not a scaled copy of the first rectangle, because different pairs of corresponding lengths have different scale factors, $2 \cdot \frac{1}{2} = 1$ but $3 \cdot \frac{2}{3} = 2$.



Figure 1.1.4.8

Glossary Entries

Definition: Corresponding

When part of an original figure matches up with part of a copy, we call them corresponding parts. These could be points, segments, angles, or distances.

For example, point B in the first triangle corresponds to point E in the second triangle. Segment AC corresponds to segment DF.

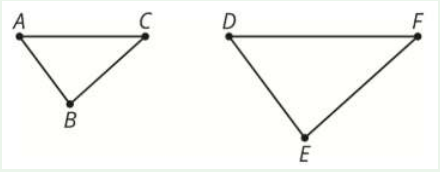


Figure 1.1.4.9

Definition: Scale Factor

To create a scaled copy, we multiply all the lengths in the original figure by the same number. This number is called the scale factor.

In this example, the scale factor is 1.5, because $4 \cdot (1.5) = 6$, $5 \cdot (1.5) = 7.5$, and $6 \cdot (1.5) = 9$.

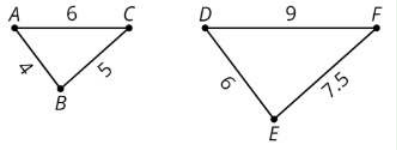


Figure 1.1.4.10

Definition: Scaled Copy

A scaled copy is a copy of an figure where every length in the original figure is multiplied by the same number.

For example, triangle DEF is a scaled copy of triangle ABC. Each side length on triangle ABC was multiplied by 1.5 to get the corresponding side length on triangle DEF.

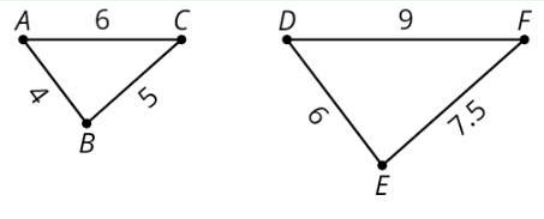


Figure 1.1.4.11

Practice

Exercise 1.1.4.5

Select **all** the statements that must be true for *any* scaled copy Q of Polygon P.

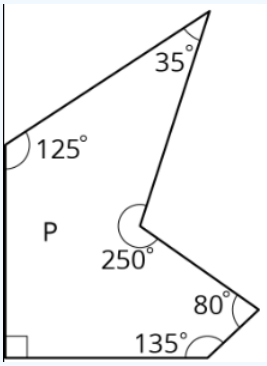


Figure 1.1.4.12

A. The side lengths are all whole numbers.
B. The angle measures are all whole numbers.
C. Q has exactly 1 right angle.
D. If the scale factor between P and Q is $\frac{1}{5}$, then each side length of P is multiplied by $\frac{1}{5}$ to get the corresponding side length of Q.
E. If the scale factor is 2, each angle in P is multiplied by 2 to get the corresponding angle in Q.
F. Q has 2 acute angles and 3 obtuse angles.

Exercise 1.1.4.6

Here is Quadrilateral $ABCD$.

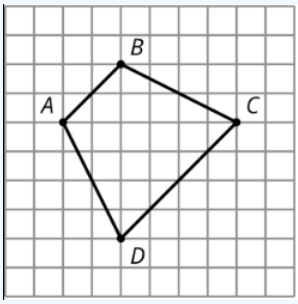


Figure 1.1.4.13: Quadrilateral ABCD is on a grid. Point a is 2 units right and 4 units down from the edge of the grid. Point B is 2 units right and 2 units up from point A. Point C is 6 units right from point A. Point D is 2 units right and 4 units down from point A.

Quadrilateral $PQRS$ is a scaled copy of Quadrilateral $ABCD$. Point P corresponds to A, Q to B, R to C, and S to D.

If the distance from P to R is 3 units, what is the distance from Q to S? Explain your reasoning.

Exercise 1.1.4.7

Figure 2 is a scaled copy of Figure 1.

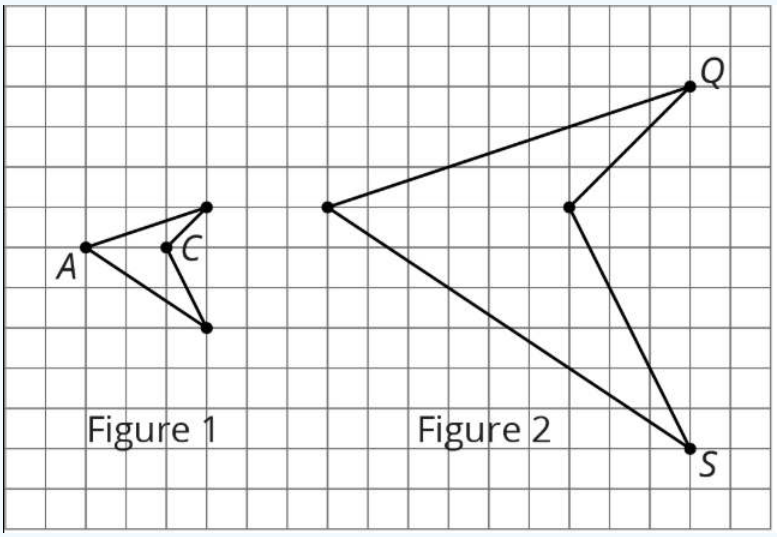


Figure 1.1.4.14

1. Identify the points in Figure 2 that correspond to the points A and C in Figure 1. Label them P and R. What is the distance between P and R?
2. Identify the points in Figure 1 that correspond to the points Q and S in Figure 2. Label them B and D. What is the distance between B and D?
3. What is the scale factor that takes Figure 1 to Figure 2?
4. G and H are two points on Figure 1, but they are not shown. The distance between G and H is 1. What is the distance between the corresponding points on Figure 2?

Exercise 1.1.4.8

To make 1 batch of lavender paint, the ratio of cups of pink paint to cups of blue paint is 6 to 5. Find two more ratios of cups of pink paint to cups of blue paint that are equivalent to this ratio.

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1.1.5: The Size of the Scale Factor

Lesson

Let's look at the effects of different scale factors.

Exercise 1.1.5.1: Number Talk: Missing Factor

Solve each equation mentally.

$16x = 176$

$16x = 8$

$16x = 1$

$\frac{1}{5}x = 1$

$\frac{2}{5}x = 1$

Exercise 1.1.5.2: Card Sort: Scaled Copies

Your teacher will give you a set of cards. On each card, Figure A is the original and Figure B is a scaled copy.

1. Sort the cards based on their scale factors. Be prepared to explain your reasoning.
2. Examine cards 10 and 13 more closely. What do you notice about the shapes and sizes of the figures? What do you notice about the scale factors?
3. Examine cards 8 and 12 more closely. What do you notice about the figures? What do you notice about the scale factors?

Are you ready for more?

Triangle B is a scaled copy of Triangle A with scale factor $\frac{1}{2}$.

1. How many times bigger are the side lengths of Triangle B when compared with Triangle A?
2. Imagine you scale Triangle B by a scale factor of $\frac{1}{2}$ to get Triangle C. How many times bigger will the side lengths of Triangle C be when compared with Triangle A?
3. Triangle B has been scaled once. Triangle C has been scaled twice. Imagine you scale triangle A n times to get Triangle N, always using a scale factor of $\frac{1}{2}$. How many times bigger will the side lengths of Triangle N be when compared with Triangle A?

Exercise 1.1.5.3: Scaling A Puzzle

Your teacher will give you 2 pieces of a 6-piece puzzle.

1. If you drew scaled copies of your puzzle pieces using a scale factor of $\frac{1}{2}$, would they be larger or smaller than the original pieces? How do you know?

2. Create a scaled copy of each puzzle piece on a blank square, with a scale factor of $\frac{1}{2}$.

3. When everyone in your group is finished, put all 6 of the original puzzle pieces together like this:

1	2	3
4	5	6

Figure 1.1.5.1

Next, put all 6 of your scaled copies together. Compare your scaled puzzle with the original puzzle. Which parts seem to be scaled correctly and which seem off? What might have caused those parts to be off?

4. Revise any of the scaled copies that may have been drawn incorrectly.

5. If you were to lose one of the pieces of the original puzzle, but still had the scaled copy, how could you recreate the lost piece?

Exercise 1.1.5.4: Missing Figure, Factor, or Copy

1. What is the scale factor from the original triangle to its copy? Explain or show your reasoning.

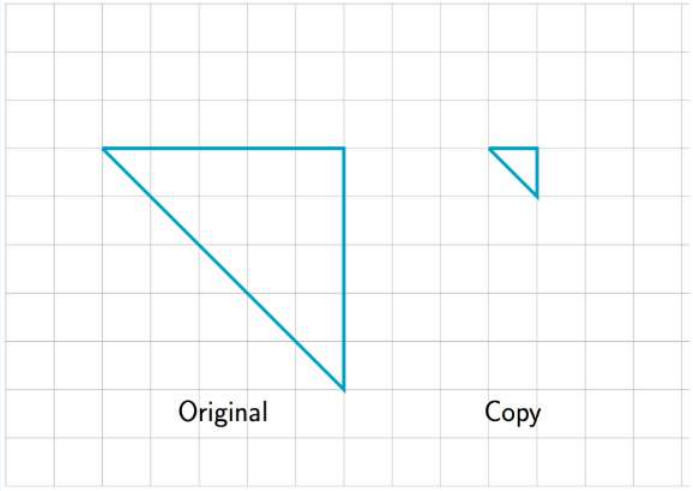


Figure 1.1.5.2

2. The scale factor from the original trapezoid to its copy is 2. Draw the scaled copy.

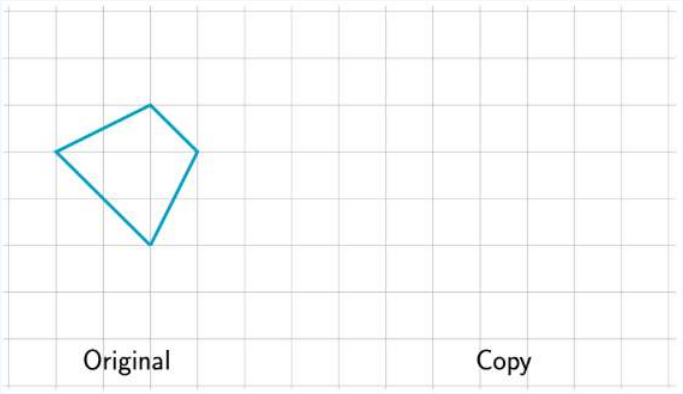


Figure 1.1.5.3

3. The scale factor from the original figure to its copy is $\frac{3}{2}$. Draw the original figure.

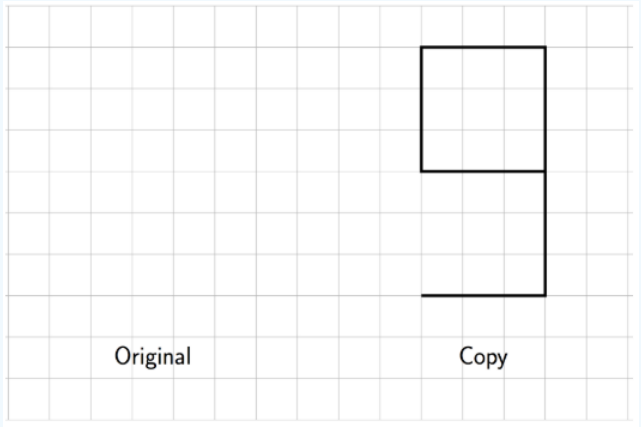


Figure 1.1.5.4

4. What is the scale factor from the original figure to the copy? Explain how you know.

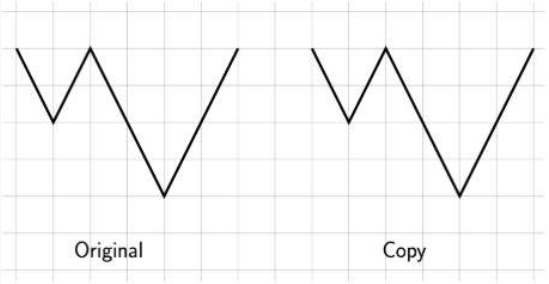


Figure 1.1.5.5

5. The scale factor from the original figure to its scaled copy is 3. Draw the scaled copy.

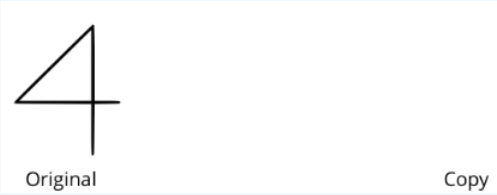


Figure 1.1.5.6: A triangle with the horizontal side extended to the right and the vertical side extended upward creating a figure that resembles the number 4. The figure is labeled original. On the right, the position labeled copy is blank.

Summary

The size of the scale factor affects the size of the copy. When a figure is scaled by a scale factor greater than 1, the copy is larger than the original. When the scale factor is less than 1, the copy is smaller. When the scale factor is exactly 1, the copy is the same size as the original.

Triangle DEF is a larger scaled copy of triangle ABC, because the scale factor from ABC to DEF is $\frac{3}{2}$. Triangle ABC is a smaller scaled copy of triangle DEF, because the scale factor from DEF to ABC is $\frac{2}{3}$.

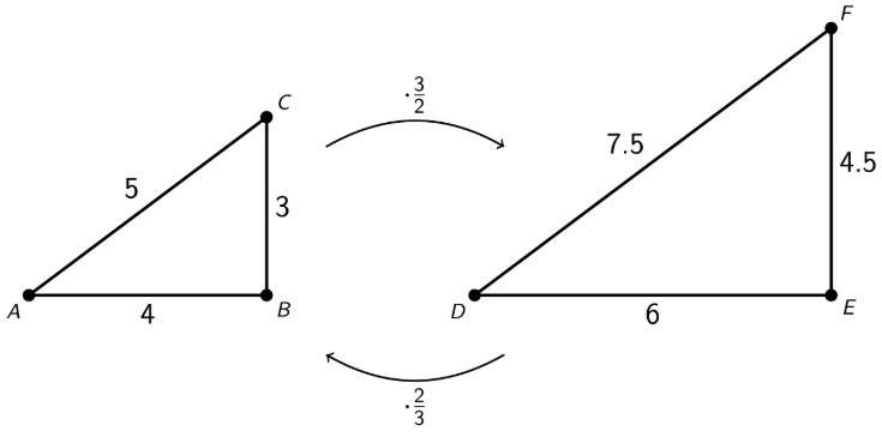


Figure 1.1.5.7: Two triangles; one labeled A B C with horizontal A B and the other D E F with horizontal D E. The length of A B is labeled 4. The length of B C is labeled 3. The length of C A is labeled 5. The length of D E is labeled 6. The length of E F is labeled 4.5. The length of F D is labeled 7.5. An arrow from triangle A B C pointing to triangle D E F is labeled, times 3 halves. An arrow from triangle D E F pointing to triangle A B C is labeled times 2 thirds.

This means that triangles ABC and DEF are scaled copies of each other. It also shows that scaling can be reversed using **reciprocal** scale factors, such as $\frac{2}{3}$ and $\frac{3}{2}$.

In other words, if we scale Figure A using a scale factor of 4 to create Figure B, we can scale Figure B using the reciprocal scale factor, $\frac{1}{4}$, to create Figure A.

Glossary Entries

Definition: Corresponding

When part of an original figure matches up with part of a copy, we call them corresponding parts. These could be points, segments, angles, or distances.

For example, point B in the first triangle corresponds to point E in the second triangle. Segment AC corresponds to segment DF.

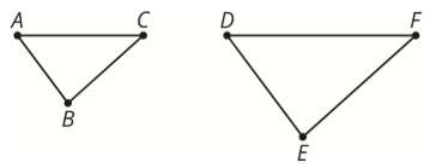


Figure 1.1.5.8

Definition: Reciprocal

Dividing 1 by a number gives the reciprocal of that number. For example, the reciprocal of 12 is $\frac{1}{12}$, and the reciprocal of $\frac{2}{5}$ is $\frac{5}{2}$.

Definition: Scale Factor

To create a scaled copy, we multiply all the lengths in the original figure by the same number. This number is called the scale factor.

In this example, the scale factor is 1.5, because $4 \cdot (1.5) = 6$, $5 \cdot (1.5) = 7.5$, and $6 \cdot (1.5) = 9$.

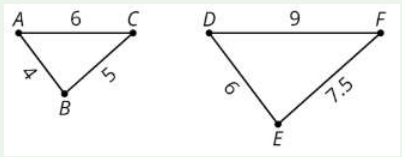


Figure 1.1.5.9

Definition: Scaled Copy

A scaled copy is a copy of an figure where every length in the original figure is multiplied by the same number.

For example, triangle DEF is a scaled copy of triangle ABC. Each side length on triangle ABC was multiplied by 1.5 to get the corresponding side length on triangle DEF.

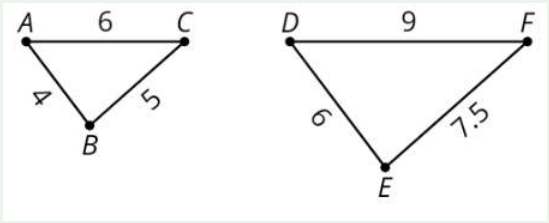


Figure 1.1.5.10

Practice

Exercise 1.1.5.5

Rectangles P, Q, R, and S are scaled copies of one another. For each pair, decide if the scale factor from one to the other is greater than 1, equal to 1, or less than 1.

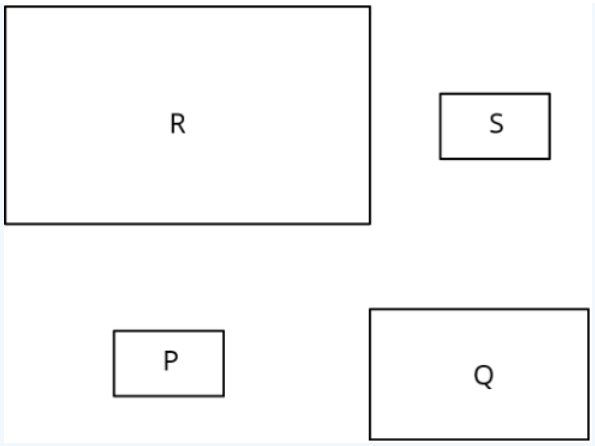


Figure 1.1.5.11: Four rectangles, labeled P, Q, R and S. Each rectangle is a scaled copy of one another. Ranked in order from least to greatest, the area of the rectangles are as follows: the area of P is equal to S, which are less than the area of Q, which is less than the area of R.

1. from P to Q
2. from P to R
3. from Q to S
4. from Q to R
5. from S to P
6. from R to P
7. from P to S

Exercise 1.1.5.6

Triangle S and Triangle L are scaled copies of one another.

1. What is the scale factor from S to L?
2. What is the scale factor from L to S?
3. Triangle M is also a scaled copy of S. The scale factor from S to M is $\frac{3}{2}$. What is the scale factor from M to S?

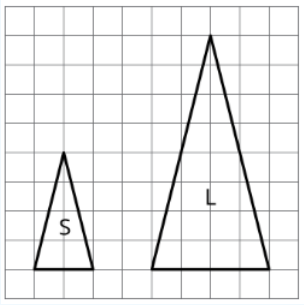


Figure 1.1.5.12

Exercise 1.1.5.7

Are two squares with the same side lengths scaled copies of one another? Explain your reasoning.

Exercise 1.1.5.8

Quadrilateral A has side lengths 2, 3, 5, and 6. Quadrilateral B has side lengths 4, 5, 8, and 10. Could one of the quadrilaterals be a scaled copy of the other? Explain.

(From Unit 1.1.2)

Exercise 1.1.5.9

Select **all** the ratios that are equivalent to the ratio $12 : 3$

A. $6 : 1$
B. $1 : 4$
C. $4 : 1$
D. $24 : 6$
E. $15 : 6$
F. $1,200 : 300$
G. $112 : 13$

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1.1.6: Scaling and Area

Lesson

Let's build scaled shapes and investigate their areas.

Exercise 1.1.6.1: Scaling a Pattern Block

Use the applets to explore the pattern blocks. Work with your group to build the scaled copies described in each question.



Figure 1.1.6.1

1. How many blue rhombus blocks does it take to build a scaled copy of Figure A:
   a. Where each side is twice as long?
   b. Where each side is 3 times as long?
   c. Where each side is 4 times as long?
2. How many green triangle blocks does it take to build a scaled copy of Figure B:
   a. Where each side is twice as long?
   b. Where each side is 3 times as long?
   c. Using a scale factor of 4?
3. How many red trapezoid blocks does it take to build a scaled copy of Figure C:
   a. Using a scale factor of 2?
   b. Using a scale factor of 3?
   c. Using a scale factor of 4?
4. Make a prediction: How many blocks would it take to build scaled copies of these shapes using a scale factor of 5? Using a scale factor of 6? Be prepared to explain your reasoning.

Exercise 1.1.6.2: Scaling More Pattern Blocks

Your teacher will assign your group one of these figures, each made with original-size blocks.

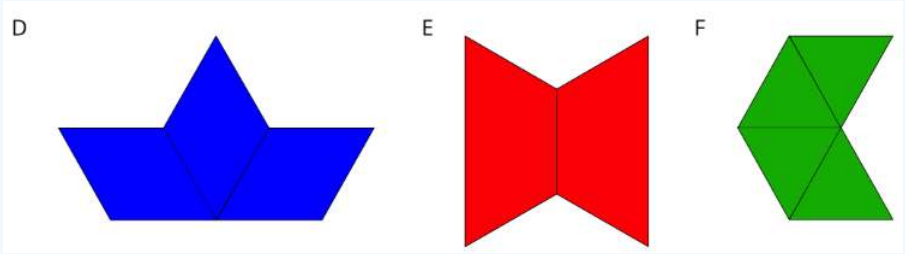


Figure 1.1.6.2: 3 shapes composed of pattern blocks. First shape, labeled figure D composed of 3 blue rhombuses. Second, labeled figure E composed of 2 red trapezoids. Third, labeled figure F composed of 4 green triangles.

1. In the applet, move the slider to see a scaled copy of your assigned shape, using a scale factor of 2. Use the original-size blocks to build a figure to match it. How many blocks did it take?
2. Your classmate thinks that the scaled copies in the previous problem will each take 4 blocks to build. Do you agree or disagree? Explain you reasoning.
3. Move the slider to see a scaled copy of your assigned shape using a scale factor of 3. Start building a figure with the original-size blocks to match it. Stop when you can tell for sure how many blocks it would take. Record your answer.
4. Predict: How many blocks would it take to build scaled copies using scale factors 4, 5, and 6? Explain or show your reasoning.
5. How is the pattern in this activity the same as the pattern you saw in the previous activity? How is it different?

Are you ready for more?

1. How many blocks do you think it would take to build a scaled copy of one yellow hexagon where each side is twice as long? Three times as long?
2. Figure out a way to build these scaled copies.
3. Do you see a pattern for the number of blocks used to build these scaled copies? Explain your reasoning.

Exercise 1.1.6.3: Area of Scaled Parallelograms and Triangles

1. Your teacher will give you a figure with measurements in centimeters. What is the **area** of your figure? How do you know?
2. Work with your partner to draw scaled copies of your figure, using each scale factor in the table. Complete the table with the measurements of your scaled copies.

Table 1.1.6.1

scale factor	base (cm)	height (cm)	area (cm^2)
1			
2			
3			
$\frac{1}{2}$			
$\frac{1}{3}$			

3. Compare your results with a group that worked with a different figure. What is the same about your answers? What is different?
4. If you drew scaled copies of your figure with the following scale factors, what would their areas be? Discuss your thinking. If you disagree, work to reach an agreement. Be prepared to explain your reasoning.

Table 1.1.6.2

scale factor	area (cm^2)
5	
$\frac{3}{5}$	

Summary

Scaling affects lengths and areas differently. When we make a scaled copy, all original lengths are multiplied by the scale factor. If we make a copy of a rectangle with side lengths 2 units and 4 units using a scale factor of 3, the side lengths of the copy will be 6 units and 12 units, because $2 \cdot 3 = 6$ and $4 \cdot 3 = 12$.

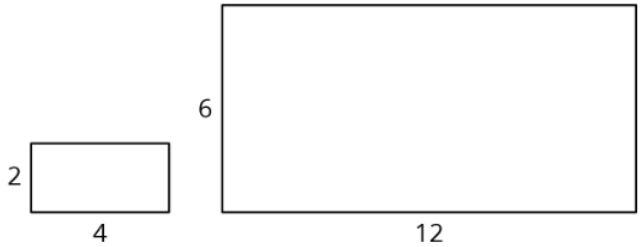


Figure 1.1.6.3

The area of the copy, however, changes by a factor of (scale factor)^2. If each side length of the copy is 3 times longer than the original side length, then the area of the copy will be 9 times the area of the original, because $3 \cdot 3$, or 3^2, equals 9.

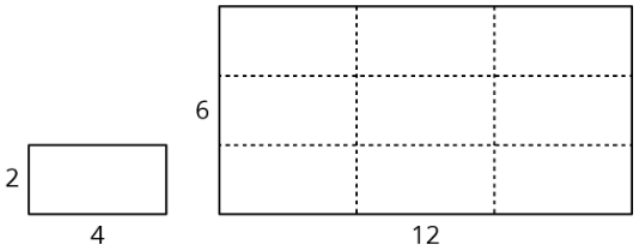


Figure 1.1.6.4: Two rectangles. The first rectangle has the vertical side labeled 2 and the horizontal side labeled 4. The second rectangle has the vertical side labeled 6 and the horizontal side labeled 12. Two horizontal dashed lines and 2 vertical dashed lines are drawn in the second rectangle dividing it into 9 identical smaller rectangles.

In this example, the area of the original rectangle is 8 units^2 and the area of the scaled copy is 72 units^2, because $9 \cdot 8 = 72$. We can see that the large rectangle is covered by 9 copies of the small rectangle, without gaps or overlaps. We can also verify this by multiplying the side lengths of the large rectangle: $6 \cdot 12 = 72$.

Lengths are one-dimensional, so in a scaled copy, they change by the scale factor. Area is two-dimensional, so it changes by the *square* of the scale factor. We can see this is true for a rectangle with length l and width w. If we scale the rectangle by a scale factor of s, we get a rectangle with length $s \cdot l$ and width $s \cdot w$. The area of the scaled rectangle is $A = (s \cdot l) \cdot (s \cdot w)$, so $A = (s^2) \cdot (l \cdot w)$. The fact that the area is multiplied by the square of the scale factor is true for scaled copies of other two-dimensional figures too, not just for rectangles.

Glossary Entries

Definition: Area

Area is the number of square units that cover a two-dimensional region, without any gaps or overlaps.

For example, the area of region A is 8 square units. The area of the shaded region of B is $\frac{1}{2}$ square unit.

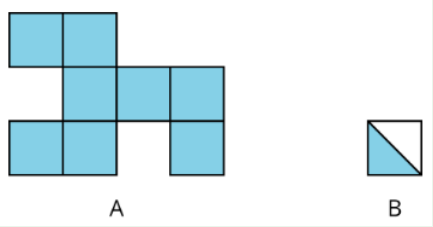


Figure 1.1.6.5

Definition: Corresponding

When part of an original figure matches up with part of a copy, we call them corresponding parts. These could be points, segments, angles, or distances.

For example, point B in the first triangle corresponds to point E in the second triangle. Segment AC corresponds to segment DF.

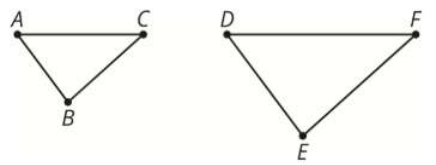


Figure 1.1.6.6

Definition: Reciprocal

Dividing 1 by a number gives the reciprocal of that number. For example, the reciprocal of 12 is $\frac{1}{12}$, and the reciprocal of $\frac{2}{5}$ is $\frac{5}{2}$.

Definition: Scale Factor

To create a scaled copy, we multiply all the lengths in the original figure by the same number. This number is called the scale factor.

In this example, the scale factor is 1.5, because $4 \cdot (1.5) = 6$, $5 \cdot (1.5) = 7.5$, and $6 \cdot (1.5) = 9$.

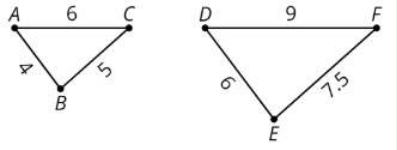


Figure 1.1.6.7

Definition: Scaled Copy

A scaled copy is a copy of an figure where every length in the original figure is multiplied by the same number.

For example, triangle DEF is a scaled copy of triangle ABC. Each side length on triangle ABC was multiplied by 1.5 to get the corresponding side length on triangle DEF.

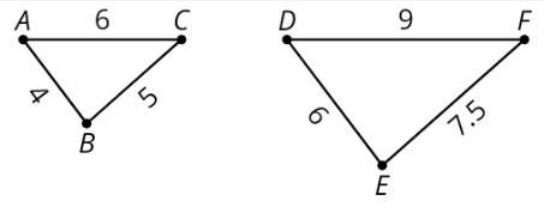


Figure 1.1.6.8

Practice

Exercise 1.1.6.4

On the grid, draw a scaled copy of Polygon Q using a scale factor of 2. Compare the perimeter and area of the new polygon to those of Q.

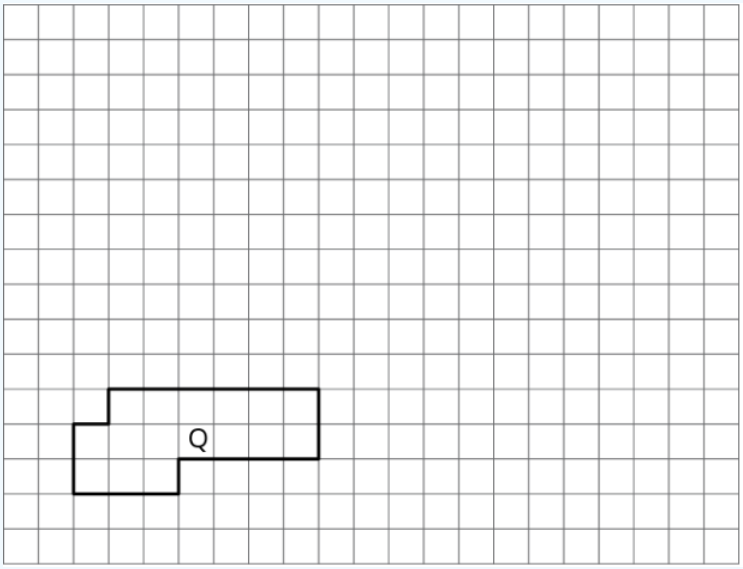


Figure 1.1.6.9: A polygon aligned to a square grid is labeled Q. The shape is composed of 3 rectangles. The left part of the shape is 2 squares tall and 1 square wide. The next part of the shape is 3 squares tall and 2 squares wide. The next part of the shape begins 1 square up from the bottom of the shape and is 2 squares tall by 4 squares wide.

Exercise 1.1.6.5

A right triangle has an area of 36 square units.

If you draw scaled copies of this triangle using the scale factors in the table, what will the areas of these scaled copies be? Explain or show your reasoning.

Table 1.1.6.3

scale factor	area (units^2)
1	36
2	
3	
5	
$\frac{1}{2}$	
$\frac{2}{3}$	

Exercise 1.1.6.6

Diego drew a scaled version of a Polygon P and labeled it Q.

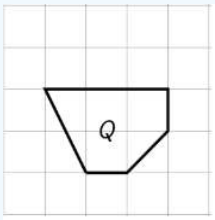


Figure 1.1.6.10: A polygon aligned to a square grid labeled Q. The polygon is made up of 4 shapes, joined together along their edges. On the left is a right triangle with a base of 1 unit and a height of 2 units. To the right of the triangle is a 2 unit by 1 unit rectangle aligned with its 2 unit side next to the 2 unit side of the triangle. To the right of this is a 1 unit square, on top of a one-half unit triangle.

If the area of Polygon P is 72 square units, what scale factor did Diego use to go from P to Q? Explain your reasoning.

Exercise 1.1.6.7

Here is an unlabeled polygon, along with its scaled copies Polygons A–D. For each copy, determine the scale factor. Explain how you know.

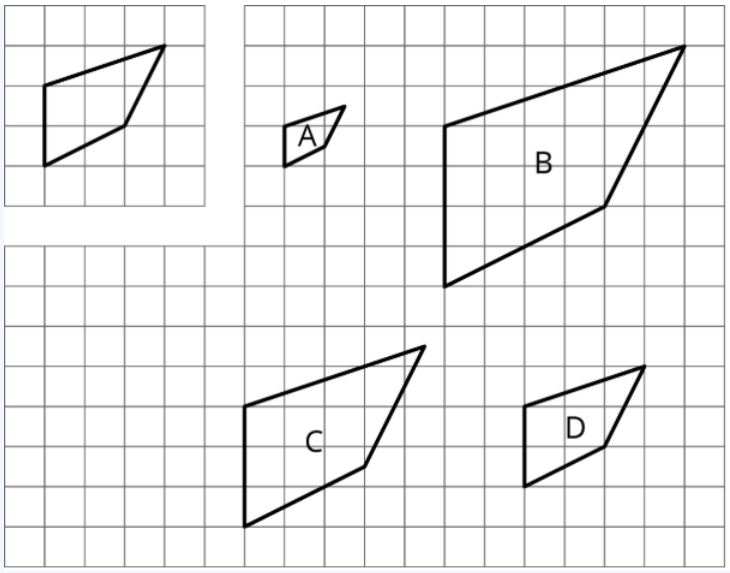


Figure 1.1.6.11: An unlabeled four sided polygon with four different scale copies labeled A, B, C and D on a square grid. The unlabeled polygon has 4 sides, with a vertical heightof 2 on the left side. The height of the vertical side in each polygon is as follows: Polygon A: 1 unit. Polygon B: 4 units. Polygon C: 3 units. Polygon D: 2 units.

(From Unit 1.1.2)

Exercise 1.1.6.8

Solve each equation mentally.

1. $\frac{1}{7} \cdot x = 1$
2. $x \cdot \frac{1}{11} = 1$
3. $1 \div \frac{1}{5} = x$

(From Unit 1.1.5)

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1.2: Scale Drawings

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1.2.1: Scale Drawings

Lesson

Let's explore scale drawings.

Exercise 1.2.1.1: What is a Scale Drawing?

Here are some drawings of a school bus, a quarter, and the subway lines around Boston, Massachusetts. The first three drawings are **scale drawings** of these objects.



Figure 1.2.1.1: "MBTA Boston subway map", by Citynoise. CC BY-SA. Wikimedia Commons. Source.

The next three drawings are *not* scale drawings of these objects.

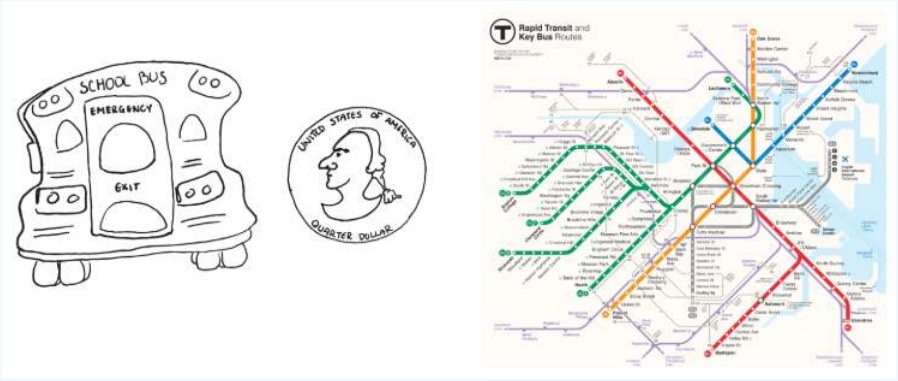


Figure 1.2.1.2: "2013 unofficial MBTA subway map by Michael Kvrivishvili" , by Michael Kvrivishvili. CC BY 2.0. Wikimedia Commons. Source.

Discuss with your partner what a scale drawing is.

Exercise 1.2.1.2: Sizing Up a Basketball Court

Your teacher will give you a scale drawing of a basketball court. The drawing does not have any measurements labeled, but it says that 1 centimeter represents 2 meters.

1. Measure the distances on the scale drawing that are labeled a–d to the nearest tenth of a centimeter. Record your results in the first row of the table.
2. The statement "1 cm represents 2 m" is the **scale** of the drawing. It can also be expressed as "1 cm to 2 m," or "1 cm for every 2 m." What do you think the scale tells us?
3. How long would each measurement from the first question be on an actual basketball court? Explain or show your reasoning.

Table 1.2.1.1

measurement	(a) length of court	(b) width of court	(c) hoop to hoop	(d) 3 point line to sideline

	measurement	(a) length of court	(b) width of court	(c) hoop to hoop	(d) 3 point line to sideline
ıent	scale drawing				
ıent	actual court				

4. On an actual basketball court, the bench area is typically 9 meters long.
   a. Without measuring, determine how long the bench area should be on the scale drawing.
   b. Check your answer by measuring the bench area on the scale drawing. Did your prediction match your measurement?

Exercise 1.2.1.3: Tall Structures

Here is a scale drawing of some of the world's tallest structures.

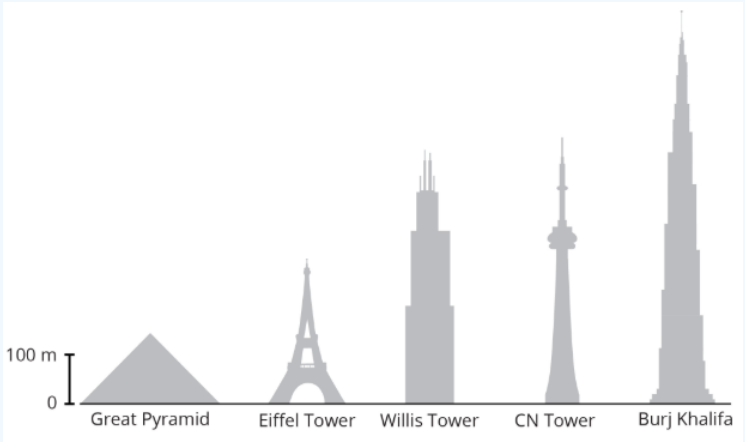


Figure 1.2.1.3

1. About how tall is the actual Willis Tower? About how tall is the actual Great Pyramid? Be prepared to explain your reasoning.
2. About how much taller is the Burj Khalifa than the Eiffel Tower? Explain or show your reasoning.
3. Measure the line segment that shows the scale to the nearest tenth of a centimeter. Express the scale of the drawing using numbers and words.

Are you ready for more?

The tallest mountain in the United States, Mount Denali in Alaska, is about 6,190 m tall. If this mountain were shown on the scale drawing, how would its height compare to the heights of the structures? Explain or show your reasoning.

Summary

Scale drawings are two-dimensional representations of actual objects or places. Floor plans and maps are some examples of scale drawings. On a scale drawing:

- Every part corresponds to something in the actual object.
- Lengths on the drawing are enlarged or reduced by the same scale factor.
- A **scale** tells us how actual measurements are represented on the drawing. For example, if a map has a scale of "1 inch to 5 miles" then a $\frac{1}{2}$-inch line segment on that map would represent an actual distance of 2.5 miles

Sometimes the scale is shown as a segment on the drawing itself. For example, here is a scale drawing of a stop sign with a line segment that represents 25 cm of actual length.

The width of the octagon in the drawing is about three times the length of this segment, so the actual width of the sign is about $3 \cdot 25$, or 75 cm.

Because a scale drawing is two-dimensional, some aspects of the three-dimensional object are not represented. For example, this scale drawing does not show the thickness of the stop sign.

A scale drawing may not show every detail of the actual object; however, the features that are shown correspond to the actual object and follow the specified scale.

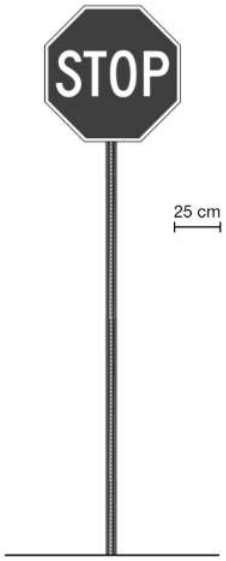


Figure 1.2.1.4

Glossary Entries

Definition: Scale

A scale tells how the measurements in a scale drawing represent the actual measurements of the object.

For example, the scale on this floor plan tells us that 1 inch on the drawing represents 8 feet in the actual room. This means that 2 inches would represent 16 feet, and $\frac{1}{2}$ inch would represent 4 feet.

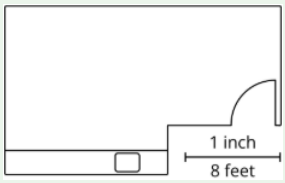


Figure 1.2.1.5

Definition: Scale Drawing

A scale drawing represents an actual place or object. All the measurements in the drawing correspond to the measurements of the actual object by the same scale.



Figure 1.2.1.6

Practice

Exercise 1.2.1.4

The Westland Lysander was an aircraft used by the Royal Air Force in the 1930s. Here are some scale drawings that show the top, side, and front views of the Lysander.

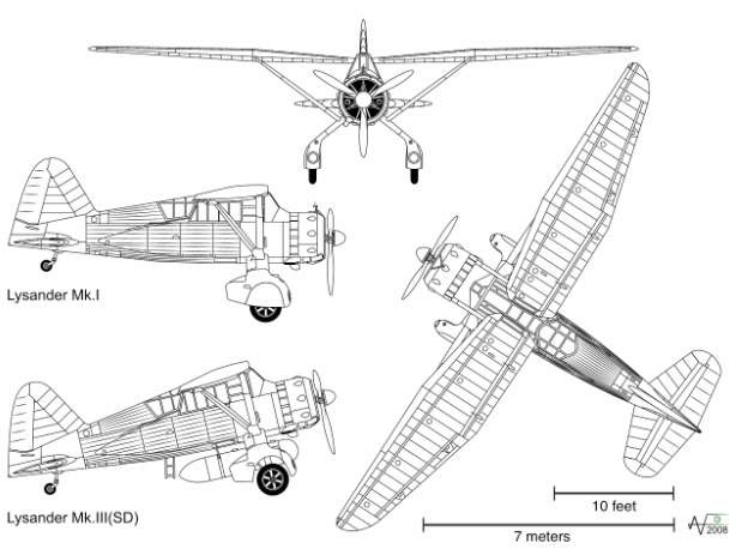


Figure 1.2.1.7: Westland Lysander, by Emoscopes 23:36, 28 April 2008 (UTC) (self made using XaraXtreme). CC BY 3.0. Wikimedia Commons. Source.

Use the scales and scale drawings to approximate the actual lengths of:

1. the wingspan of the plane, to the nearest foot
2. the height of the plane, to the nearest foot
3. the length of the Lysander Mk. I, to the nearest meter

Exercise 1.2.1.5

A blueprint for a building includes a rectangular room that measures 3 inches long and 5.5 inches wide. The scale for the blueprint says that 1 inch on the blueprint is equivalent to 10 feet in the actual building. What are the dimensions of this rectangular room in the actual building?

Exercise 1.2.1.6

Here is a scale map of Lafayette Square, a rectangular garden north of the White House.

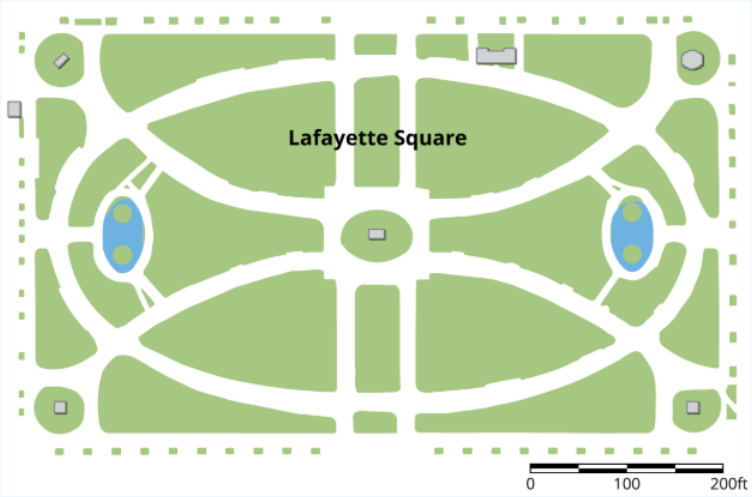


Figure 1.2.1.8: By American Fact Finder by the United States Census Bureau. Public Domain. American Fact Finder. Source.

1. The scale is shown in the lower right corner. Find the actual side lengths of Lafayette Square in feet.
2. Use an inch ruler to measure the line segment of the graphic scale. About how many feet does one inch represent on this map?

Exercise 1.2.1.7

Here is Triangle A. Lin created a scaled copy of Triangle A with an area of 72 square units.

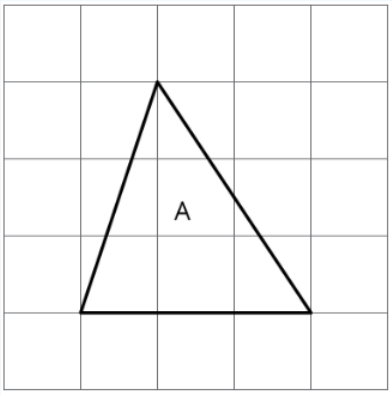


Figure 1.2.1.9

1. How many times larger is the area of of the scaled copy compared to that of Triangle A?
2. What scale factor did Lin apply to the Triangle A to create the copy?
3. What is the length of bottom side of the scaled copy?

(From Unit 1.1.6)

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1.2.2: Scale Drawings and Maps

Lesson

Let's use scale drawings to solve problems.

Exercise 1.2.2.1: A Train And a Car

Two cities are 243 miles apart.

- It takes a train 4 hours to travel between the two cities at a constant speed.
- A car travels between the two cities at a constant speed of 65 miles per hour.

Which is traveling faster, the car or the train? Be prepared to explain your reasoning.

Exercise 1.2.2.2: Driving on I-90

1. A driver is traveling at a constant speed on Interstate 90 outside of Chicago. If she traveled from Point A to Point B in 8 minutes, did she obey the speed limit of 55 miles per hour? Explain your reasoning.
2. A traffic helicopter flew directly from Point A to Point B in 8 minutes. Did the helicopter travel faster or slower than the driver? Explain or show your reasoning.



Figure 1.2.2.1

Use the Distance tool to measure the shortest distance between two points.

Exercise 1.2.2.3: Biking through Kansas

A cyclist rides at a constant speed of 15 miles per hour. At this speed, about how long would it take the cyclist to ride from Garden City to Dodge City, Kansas?



Figure 1.2.2.2: By skeeze. Public Domain. Pixabay. Source.



Figure 1.2.2.3: Map with a line, trending downward and right. Endpoints are Garden City and Dodge City. Scale with line segment labeled 4 miles. The distance between the endpoints is about 12 and one half of the scale line segments. By American Fact Finder by the United States Census Bureau. Public Domain. American Fact Finder. Source.

Are you ready for more?

Jada finds a map that says, “Note: This map is not to scale.” What do you think this means? Why is this information important?

Summary

Maps with scales are useful for making calculations involving speed, time, and distance. Here is a map of part of Alabama.



Figure 1.2.2.4: A map of a part of Alabama. Three points indicate three different cities on the map. The bottom point is labeled Montgomery, the middle point is labeled Centreville, and the top point is labeled Birmingham. A scale on the map shows 1 inch equals 20 miles. The distance between Birmingham and Montgomery is approximately 4 point 5 inches on the map. By United States Census Bureau. Public Domain. American Fact Finder. Source.

Suppose it takes a car 1 hour and 30 minutes to travel at constant speed from Birmingham to Montgomery. How fast is the car traveling?

To make an estimate, we need to know about how far it is from Birmingham to Montgomery. The scale of the map represents 20 miles, so we can estimate the distance between these cities is about 90 miles.

Since 90 miles in 1.5 hours is the same speed as 180 miles in 3 hours, the car is traveling about 60 miles per hour.

time (hours)	distance (miles)
1.5	90
3	180
1	60

Figure 1.2.2.5: Table with arrows. Time, hours. Data, from top to bottom, 1.5, 3, 1. Distance, miles. Data from top to bottom, 90, 180, 60. On the left and right of the table, two downward arrows labeled times 2 and times one third.

Suppose a car is traveling at a constant speed of 60 miles per hour from Montgomery to Centreville. How long will it take the car to make the trip? Using the scale, we can estimate that it is about 70 miles. Since 60 miles per hour is the same as 1 mile per minute, it will take the car about 70 minutes (or 1 hour and 10 minutes) to make this trip.

Glossary Entries

Definition: Scale

A scale tells how the measurements in a scale drawing represent the actual measurements of the object.

For example, the scale on this floor plan tells us that 1 inch on the drawing represents 8 feet in the actual room. This means that 2 inches would represent 16 feet, and $\frac{1}{2}$ inch would represent 4 feet.

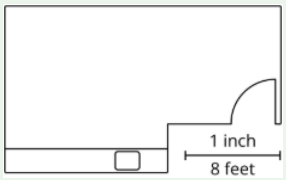


Figure 1.2.2.6

Definition: Scale Drawing

A scale drawing represents an actual place or object. All the measurements in the drawing correspond to the measurements of the actual object by the same scale.



Figure 1.2.2.7

Practice

Exercise 1.2.2.4

Here is a map that shows parts of Texas and Oklahoma.



Figure 1.2.2.8: By United States Census Bureau. Public Domain. American Fact Finder. Source.

1. About how far is it from Amarillo to Oklahoma City? Explain your reasoning.
2. Driving at a constant speed of 70 miles per hour, will it be possible to make this trip in 3 hours? Explain how you know.

Exercise 1.2.2.5

A local park is in the shape of a square. A map of the local park is made with the scale 1 inch to 200 feet.

1. If the park is shown as a square on the map, each side of which is one foot long, how long is each side of the square park?
2. If a straight path in the park is 900 feet long, how long would the path be when represented on the map?

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1.2.3: Creating Scale Drawings

Lesson

Let's create our own scale drawings.

Exercise 1.2.3.1: Number Talk: Which is Greater?

Without calculating, decide which quotient is larger.

$11 \div 23$ or $7 \div 13$

$0.63 \div 2$ or $0.55 \div 3$

$15 \div \frac{1}{3}$ or $15 \div \frac{1}{4}$

Exercise 1.2.3.2: Bedroom Floor Plan

Here is a rough sketch of Noah's bedroom (not a scale drawing).

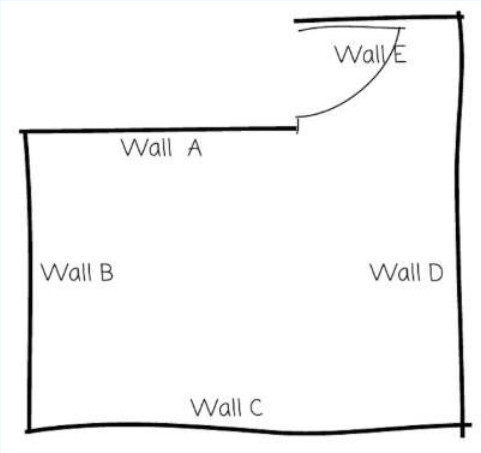


Figure 1.2.3.1

Noah wants to create a floor plan that is a scale drawing.

1. The actual length of Wall C is 4 m. To represent Wall C, Noah draws a segment 16 cm long. What scale is he using? Explain or show your reasoning.
2. Find another way to express the scale.
3. Discuss your thinking with your partner. How do your scales compare?
4. The actual lengths of Wall A, Wall B, and Wall D are 2.5 m, 2.75 m, and 3.75 m. Determine how long these walls will be on Noah's scale floor plan.
5. Use the Point tool



Figure 1.2.3.2

and the Segment tool

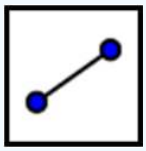


Figure 1.2.3.3

to draw the walls of Noah's scale floor plan in the applet.

Are you ready for more?

If Noah wanted to draw another floor plan on which Wall C was 20 cm, would 1 cm to 5 m be the right scale to use? Explain your reasoning.

Exercise 1.2.3.3: Two Maps of Utah

A rectangle around Utah is about 270 miles wide and about 350 miles tall. The upper right corner that is missing is about 110 miles wide and about 70 miles tall.

1. Make a scale drawing of Utah where 1 centimeter represents 50 miles.
   Make a scale drawing of Utah where 1 centimeter represents 75 miles.
2. How do the two drawings compare? How does the choice of scale influence the drawing?

Summary

If we want to create a scale drawing of a room's floor plan that has the scale "1 inch to 4 feet," we can divide the actual lengths in the room (in feet) by 4 to find the corresponding lengths (in inches) for our drawing.

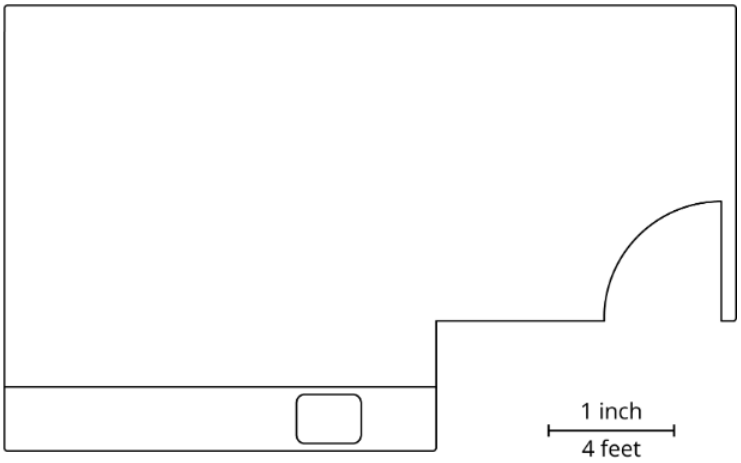


Figure 1.2.3.4

Suppose the longest wall is 15 feet long. We should draw a line 3.75 inches long to represent this wall, because $15 \div 4 = 3.75$.

There is more than one way to express this scale. These three scales are all equivalent, since they represent the same relationship between lengths on a drawing and actual lengths:

- 1 inch to 4 feet
- $\frac{1}{2}$ inch to 2 feet
- $\frac{1}{4}$ inch to 1 foot

Any of these scales can be used to find actual lengths and scaled lengths (lengths on a drawing). For instance, we can tell that, at this scale, an 8-foot long wall should be 2 inches long on the drawing because $\frac{1}{4} \cdot 8 = 2$.

The size of a scale drawing is influenced by the choice of scale. For example, here is another scale drawing of the same room using the scale 1 inch to 8 feet.

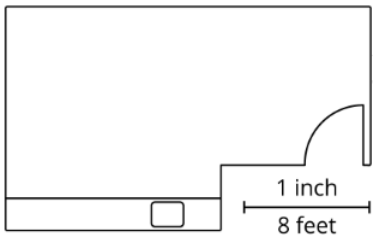


Figure 1.2.3.5

Notice this drawing is smaller than the previous one. Since one inch on this drawing represents twice as much actual distance, each side length only needs to be half as long as it was in the first scale drawing.

Glossary Entries

Definition: Scale

A scale tells how the measurements in a scale drawing represent the actual measurements of the object.

For example, the scale on this floor plan tells us that 1 inch on the drawing represents 8 feet in the actual room. This means that 2 inches would represent 16 feet, and $\frac{1}{2}$ inch would represent 4 feet.

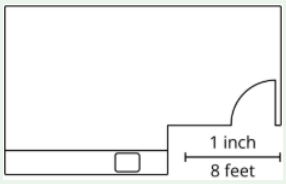


Figure 1.2.3.6

Definition: Scale Drawing

A scale drawing represents an actual place or object. All the measurements in the drawing correspond to the measurements of the actual object by the same scale.



Figure 1.2.3.7

Practice

Exercise 1.2.3.4

An image of a book shown on a website is 1.5 inches wide and 3 inches tall on a computer monitor. The actual book is 9 inches wide.

1. What scale is being used for the image?
2. How tall is the actual book?

Exercise 1.2.3.5

The flag of Colombia is a rectangle that is 6 ft long with three horizontal strips.



Figure 1.2.3.8: A rectangular figure that represents a flag. The rectangle is divided by two horizontal lines. The first horizontal line is drawn dividing the rectangle in half and the top half is shaded yellow. The second horizontal line is drawn halfway between the first horizontal line and the base of the rectangle. The middle section is shaded blue and the bottom section is shaded red. Flag of Colombia, by SKopp. Public Domain. Wikimedia Commons. Source.

- The top stripe is 2 ft tall and is yellow.
- The middle stripe is 1 ft tall and is blue.
- The bottom stripe is also 1 ft tall and is red.

1. Create a scale drawing of the Colombian flag with a scale of 1 cm to 2 ft.
2. Create a scale drawing of the Colombian flag with a scale of 2 cm to 1 ft.

Exercise 1.2.3.6

These triangles are scaled copies of each other.

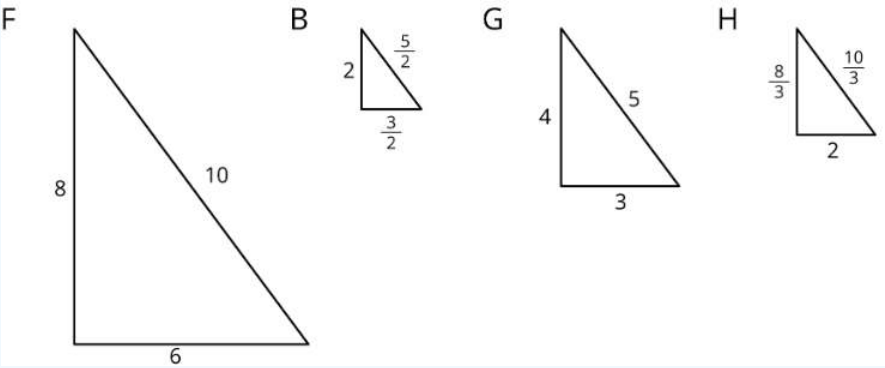


Figure 1.2.3.9: Four right triangles labeled F, B, G, and H. Triangle F has the vertical side labeled 8, the horizontal side labeled 6, and the longest side labeled 10. Triangle B has the vertical side labeled 2, the horizontal side labeled 3 halves, and the longest side labeled 5 halves. Triangle G has the vertical side labeled 4, the horizontal side labeled 3, and the longest side labeled 5. Triangle H has the vertical side labeled 8 thirds, the horizontal side labeled 2, and the longest side labeled 10 thirds.

For each pair of triangles listed, the area of the second triangle is how many times larger than the area of the first?

1. Triangle G and Triangle F
2. Triangle G and Triangle B
3. Triangle B and Triangle F
4. Triangle F and Triangle H
5. Triangle G and Triangle H
6. Triangle H and Triangle B

(From Unit 1.1.6)

Exercise 1.2.3.7

Here is an unlabeled rectangle, followed by other quadrilaterals that are labeled.

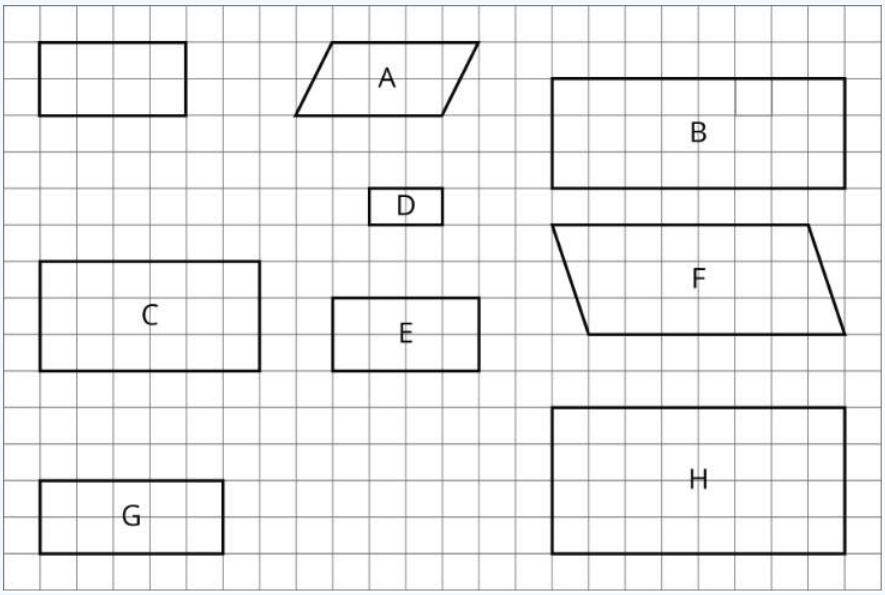


Figure B has side lengths 8, 3, 8, 3. C has side lengths 6, 3, 6, 3. D has 2, 1, 2, 1. E has 4, 2, 4, 2. F has 7, diagonal side down 3, right 1, 7, and diagonal side left 1, up 3. G has 5, 2, 5, 2. H has 8, 4, 8, 4.

1. Select **all** quadrilaterals that are scaled copies of the unlabeled rectangle. Explain how you know.
2. On graph paper, draw a different scaled version of the original rectangle.

(From Unit 1.1.3)

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1.2.4: Changing Scales in Scale Drawings

Lesson

Let's explore different scale drawings of the same actual thing.

Exercise 1.2.4.1: Appropriate Measurements

1. If a student uses a ruler like this to measure the length of their foot, which choices would be appropriate measurements? Select **all** that apply. Be prepared to explain your reasoning.

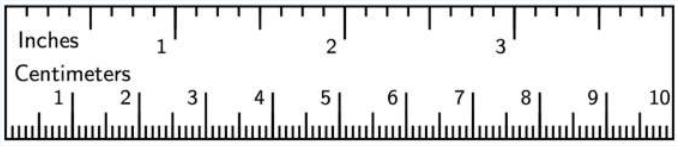


Figure 1.2.4.1

a. $9\frac{1}{4}$ inches
b. $9\frac{5}{64}$ inches
c. 23.47659centimeters
d. 23.5 centimeters
e. 23.48 centimeters

2. Here is a scale drawing of an average seventh-grade student's foot next to a scale drawing of a foot belonging to the person with the largest feet in the world. Estimate the length of the larger foot.

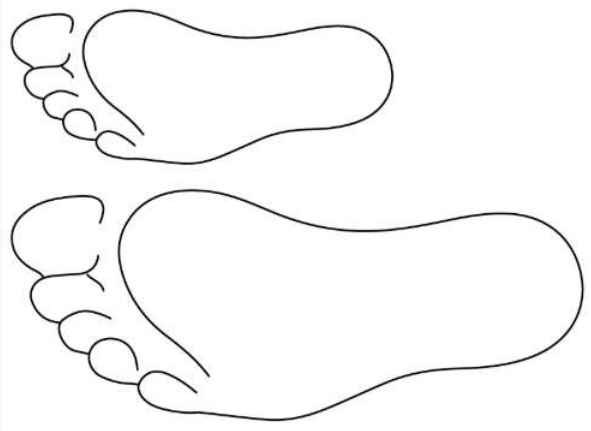


Figure 1.2.4.2

Exercise 1.2.4.2: Same Plot, Different Drawings

Here is a map showing a plot of land in the shape of a right triangle.



Figure 1.2.4.3: By American Fact Finder. Public Domain. American Fact Finder. Source.

1. Your teacher will assign you a scale to use. On centimeter graph paper, make a scale drawing of the plot of land. Make sure to write your scale on your drawing.
2. What is the area of the triangle you drew? Explain or show your reasoning.
3. How many square meters are represented by 1 square centimeter in your drawing?
4. After everyone in your group is finished, order the scale drawings from largest to smallest. What do you notice about the scales when your drawings are placed in this order?

Are you ready for more?

Noah and Elena each make a scale drawing of the same triangular plot of land, using the following scales. Make a prediction about the size of each drawing. How would they compare to the scale drawings made by your group?

1. Noah uses the scale 1 cm to 200 m.
2. Elena uses the scale 2 cm to 25 m.

Exercise 1.2.4.3: A New Drawing of the Playground

Here is a scale drawing of a playground.

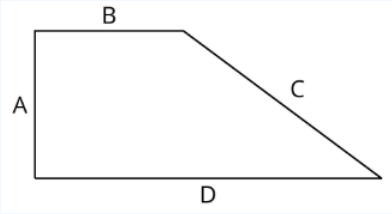


Figure 1.2.4.4

The scale is 1 centimeter to 30 meters.

1. Make another scale drawing of the same playground at a scale of 1 centimeter to 20 meters.
2. How do the two scale drawings compare?

Summary

Sometimes we have a scale drawing of something, and we want to create another scale drawing of it that uses a different scale. We can use the original scale drawing to find the size of the actual object. Then we can use the size of the actual object to figure out the size of our new scale drawing.

For example, here is a scale drawing of a park where the scale is 1 cm to 90 m.

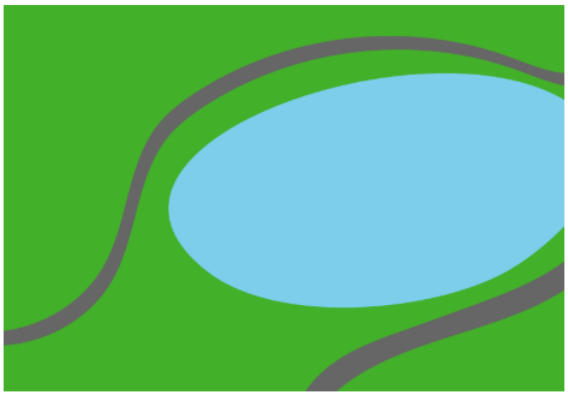


Figure 1.2.4.5

The rectangle is 10 cm by 4 cm, so the actual dimensions of the park are 900 m by 360 m, because $10 \cdot 90 = 900$ and $4 \cdot 90 = 360$.

Suppose we want to make another scale drawing of the park where the scale is 1 cm to 30 meters. This new scale drawing should be 30 cm by 12 cm, because $900 \div 30 = 30$ and $360 \div 30 = 12$.

Another way to find this answer is to think about how the two different scales are related to each other. In the first scale drawing, 1 cm represented 90 m. In the new drawing, we would need 3 cm to represent 90 m. That means each length in the new scale drawing should be 3 times as long as it was in the original drawing. The new scale drawing should be 30 cm by 12 cm, because $3 \cdot 10 = 30$ and $3 \cdot 4 = 12$.

Since the length and width are 3 times as long, the area of the new scale drawing will be 9 times as large as the area of the original scale drawing, because $3^2 = 9$.

Glossary Entries

Definition: Scale

A scale tells how the measurements in a scale drawing represent the actual measurements of the object.

For example, the scale on this floor plan tells us that 1 inch on the drawing represents 8 feet in the actual room. This means that 2 inches would represent 16 feet, and $\frac{1}{2}$ inch would represent 4 feet.

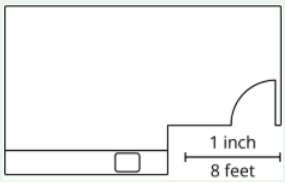


Figure 1.2.4.6

Definition: Scale Drawing

A scale drawing represents an actual place or object. All the measurements in the drawing correspond to the measurements of the actual object by the same scale.



Figure 1.2.4.7

Practice

Exercise 1.2.4.4

Here is a scale drawing of a swimming pool where 1 cm represents 1 m.



Figure 1.2.4.8

1. How long and how wide is the actual swimming pool?
2. Will a scale drawing where 1 cm represents 2 m be larger or smaller than this drawing?
3. Make a scale drawing of the swimming pool where 1 cm represents 2 m.

Exercise 1.2.4.5

A map of a park has a scale of 1 inch to 1,000 feet. Another map of the same park has a scale of 1 inch to 500 feet. Which map is larger? Explain or show your reasoning.

Exercise 1.2.4.6

On a map with a scale of 1 inch to 12 feet, the area of a restaurant is 60 in^2. Han says that the actual area of the restaurant is 720 ft^2. Do you agree or disagree? Explain your reasoning.

Exercise 1.2.4.7

If Quadrilateral Q is a scaled copy of Quadrilateral P created with a scale factor of 3, what is the perimeter of Q?

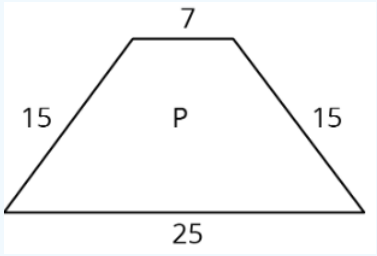


Figure 1.2.4.9: A trapezoid labeled P. The parallel sides form the top and bottom of the shape. The length of the top side is labeled 7 units and the length of the bottom side is labeled 25 units. The two legs that join the two parallel sides together are each labeled 15 units.

(From Unit 1.1.3)

Exercise 1.2.4.8

Triangle DEF is a scaled copy of triangle ABC. For each of the following parts of triangle ABC, identify the corresponding part of triangle DEF.

- angle ABC
- angle BCA
- segment AC
- segment BA

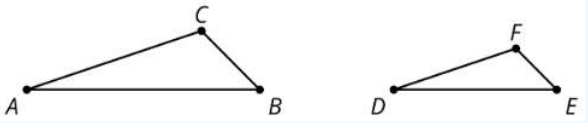


Figure 1.2.4.10: Two triangles labeled ABC and DEF. Triangle ABC has horizontal side AB with point B to the right of point A and point C is directly above side AB. Triangle DEF has horizontal side DE with point E to the right of point D and point F is directly above side DE.

(From Unit 1.1.2)

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1.2.5: Scales without Units

Lesson

Let's explore a different way to express scales.

Exercise 1.2.5.1: One to One Hundred

A map of a park says its scale is 1 to 100.

1. What do you think that means?
2. Give an example of how this scale could tell us about measurements in the park.

Exercise 1.2.5.2: Apollo Lunar Module

Your teacher will give you a drawing of the Apollo Lunar Module. It is drawn at a scale of 1 to 50.

1. The “legs” of the spacecraft are its landing gear. Use the drawing to estimate the actual length of each leg on the sides. Write your answer to the nearest 10 centimeters. Explain or show your reasoning.
2. Use the drawing to estimate the actual height of the Apollo Lunar Module to the nearest 10 centimeters. Explain or show your reasoning.
3. Neil Armstrong was 71 inches tall when he went to the surface of the Moon in the Apollo Lunar Module. How tall would he be in the drawing if he were drawn with his height to scale? Show your reasoning.
4. Sketch a stick figure to represent yourself standing next to the Apollo Lunar Module. Make sure the height of your stick figure is to scale. Show how you determined your height on the drawing.

Are you ready for more?

The table shows the distance between the Sun and 8 planets in our solar system.

1. If you wanted to create a scale model of the solar system that could fit somewhere in your school, what scale would you use?
2. The diameter of Earth is approximately 8,000 miles. What would the diameter of Earth be in your scale model?

Table 1.2.5.1

planet	average distance (millions of miles)
Mercury	35
Venus	67
Earth	93
Mars	142
Jupiter	484
Saturn	887
Uranus	1, 784
Neptune	2, 795

Exercise 1.2.5.3: Same Drawing, Different Scales

A rectangular parking lot is 120 feet long and 75 feet wide.

- Lin made a scale drawing of the parking lot at a scale of 1 inch to 15 feet. The drawing she produced is 8 inches by 5 inches.
- Diego made another scale drawing of the parking lot at a scale of 1 to 180. The drawing he produced is also 8 inches by 5 inches.

1. Explain or show how each scale would produce an 8 inch by 5 inch drawing.
2. Make another scale drawing of the same parking lot at a scale of 1 inch to 20 feet. Be prepared to explain your reasoning.
3. Express the scale of 1 inch to 20 feet as a scale without units. Explain your reasoning.

Summary

In some scale drawings, the scale specifies one unit for the distances on the drawing and a different unit for the actual distances represented. For example, a drawing could have a scale of 1 cm to 10 km.

In other scale drawings, the scale does not specify any units at all. For example, a map may simply say the scale is 1 to 1,000. In this case, the units for the scaled measurements and actual measurements can be any unit, so long as the same unit is being used for both. So if a map of a park has a scale 1 to 1,000, then 1 inch on the map represents 1,000 inches in the park, and 12 centimeters on the map represent 12,000 centimeters in the park. In other words, 1,000 is the scale factor that relates distances on the drawing to actual distances, and $\frac{1}{1000}$ is the scale factor that relates an actual distance to its corresponding distance on the drawing.

A scale with units can be expressed as a scale without units by converting one measurement in the scale into the same unit as the other (usually the unit used in the drawing). For example, these scales are equivalent:

- 1 inch to 200 feet
- 1 inch to 2,400 inches (because there are 12 inches in 1 foot, and $200 \cdot 12 = 2,400$)
- 1 to 2,400

This scale tells us that all actual distances are 2,400 times their corresponding distances on the drawing, and distances on the drawing are $\frac{1}{2400}$ times the actual distances they represent.

Glossary Entries

Definition: Scale

A scale tells how the measurements in a scale drawing represent the actual measurements of the object.

For example, the scale on this floor plan tells us that 1 inch on the drawing represents 8 feet in the actual room. This means that 2 inches would represent 16 feet, and $\frac{1}{2}$ inch would represent 4 feet.

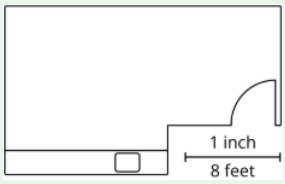


Figure 1.2.5.1

Definition: Scale Drawing

A scale drawing represents an actual place or object. All the measurements in the drawing correspond to the measurements of the actual object by the same scale.



Figure 1.2.5.2

Practice

Exercise 1.2.5.4

A scale drawing of a car is presented in the following three scales. Order the scale drawings from smallest to largest. Explain your reasoning. (There are about 1.1 yards in a meter, and 2.54 cm in an inch.)

1. 1 in to 1 ft
2. 1 in to 1 m
3. 1 in to 1 yd

Exercise 1.2.5.5

Which scales are equivalent to 1 inch to 1 foot? Select **all** that apply.

A. 1 to 12
B. $\frac{1}{12}$ to 1
C. 100 to 0.12
D. 5 to 60
E. 36 to 3
F. 9 to 108

Exercise 1.2.5.6

A model airplane is built at a scale of 1 to 72. If the model plane is 8 inches long, how many feet long is the actual airplane?

Exercise 1.2.5.7

Quadrilateral A has side lengths 3, 6, 6, and 9. Quadrilateral B is a scaled copy of A with a shortest side length equal to 2. Jada says, “Since the side lengths go down by 1 in this scaling, the perimeter goes down by 4 in total.” Do you agree with Jada? Explain your reasoning.

(From Unit 1.1.3)

Exercise 1.2.5.8

Polygon B is a scaled copy of Polygon A using a scale factor of 5. Polygon A’s area is what fraction of Polygon B’s area?

(From Unit 1.1.6)

Exercise 1.2.5.9

Figures R, S, and T are all scaled copies of one another. Figure S is a scaled copy of R using a scale factor of 3. Figure T is a scaled copy of S using a scale factor of 2. Find the scale factors for each of the following:

1. From T to S
2. From S to R
3. From R to T
4. From T to R

(From Unit 1.1.5)

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1.2.6: Units in Scale Drawings

Lesson

Let's use different scales to describe the same drawing.

Exercise 1.2.6.1: Centimeters in a Mile

There are 2.54 cm in an inch, 12 inches in a foot, and 5,280 feet in a mile. Which expression gives the number of centimeters in a mile? Explain your reasoning.

1. $\frac{2.54}{12 \cdot 5{,}280}$
2. $5{,}280 \cdot 12 \cdot (2.54)$
3. $\frac{1}{5{,}280 \cdot 12 \cdot (2.54)}$
4. $5{,}280 + 12 + 2.54$
5. $\frac{5{,}280 \cdot 12}{2.54}$

Exercise 1.2.6.2: Card Sort: Scales

Your teacher will give you some cards with a scale on each card.

1. Sort the cards into sets of equivalent scales. Be prepared to explain how you know that the scales in each set are equivalent. Each set should have at least two cards.
2. Trade places with another group and check each other's work. If you disagree about how the scales should be sorted, work to reach an agreement.
   Pause here so your teacher can review your work.
3. Next, record one of the sets with three equivalent scales and explain why they are equivalent.

Exercise 1.2.6.3: The World's Largest Flag

As of 2016, Tunisia holds the world record for the largest version of a national flag. It was almost as long as four soccer fields. The flag has a circle in the center, a crescent moon inside the circle, and a star inside the crescent moon.

1. Complete the table. Explain or show your reasoning.

Table 1.2.6.1

	flag length	flag height	height of crescent moon
actual	396 m		99 m
at 1 to 2,000 scale		13.2 cm	

2. Complete each scale with the value that makes it equivalent to the scale of 1 to 2,000. Explain or show your reasoning.
   a. 1 cm to _____ cm
   b. 1 cm to _____ m
   c. 1 cm to _____ km
   d. 2 m to _____ m
   e. 5 cm to _____ m
   f. _____ cm to 1,000 m
   g. _____ mm to 20 m
3. a. What is the area of the large flag?
   b. What is the area of the smaller flag?
   c. The area of the large flag is how many times the area of the smaller flag?

Exercise 1.2.6.4: Pondering Pools

Your teacher will give you a floor plan of a recreation center.

1. What is the scale of the floor plan if the actual side length of the square pool is 15 m? Express your answer both as a scale with units and without units.
2. Find the actual area of the large rectangular pool. Show your reasoning.
3. The kidney-shaped pool has an area of 3.2 cm^2 on the drawing. What is its actual area? Explain or show your reasoning.

Are you ready for more?

1. Square A is a scaled copy of Square B with scale factor 2. If the area of Square A is 10 $units^2$, what is the area of Square B?
2. Cube A is a scaled copy of Cube B with scale factor 2. If the volume of Cube A is 10 $units^3$, what is the volume of Cube B?
3. The four-dimensional Hypercube A is a scaled copy of Hypercube B with scale factor 2. If the "volume" of Hypercube A is 10 $units^4$, what do you think the "volume" of Hypercube B is?

Summary

Sometimes scales come with units, and sometimes they don't. For example, a map of Nebraska may have a scale of 1 mm to 1 km. This means that each millimeter of distance on the map represents 1 kilometer of distance in Nebraska. Notice that there are 1,000 millimeters in 1 meter and 1,000 meters in 1 kilometer. This means there are $1,000 \cdot 1,000$ or 1,000,000 millimeters in 1 kilometer. So, the same scale without units is 1 to 1,000,000, which means that each unit of distance on the map represents 1,000,000 units of distance in Nebraska. This is true for *any* choice of unit to express the scale of this map.

Sometimes when a scale comes with units, it is useful to rewrite it without units. For example, let's say we have a different map of Rhode Island, and we want to use the two maps to compare the size of Nebraska and Rhode Island. It is important to know if the maps are at the same scale. The scale of the map of Rhode Island is 1 inch to 10 miles. There are 5,280 feet in 1 mile, and 12 inches in 1 foot, so there are 63,360 inches in 1 mile (because $5,280 \cdot 12 = 63,360$). Therefore, there are 633,600 inches in 10 miles. The scale of the map of Rhode Island without units is 1 to 633,600. The two maps are not at the same scale, so we should not use these maps to compare the size of Nebraska to the size of Rhode Island.

Here is some information about equal lengths that you may find useful.

Customary Units

1 foot (ft) = 12 inches (in)
1 yard (yd) = 36 inches
1 yard = 3 feet
1 mile = 5,280 feet

Metric Units

1 meter (m) = 1,000 millimeters (mm)
1 meter = 100 centimeters
1 kilometer (km) = 1,000 meters

Equal Lengths in Different Systems

1 inch = 2.54 centimeters
1 foot $\approx$ 0.30 meter
1 mile $\approx$ 1.61 kilometers

1 centimeter $\approx$ 0.39 inch
1 meter $\approx$ 39.37 inches
1 kilometer $\approx$ 0.62 mile

Glossary Entries

Definition: Scale

A scale tells how the measurements in a scale drawing represent the actual measurements of the object.

For example, the scale on this floor plan tells us that 1 inch on the drawing represents 8 feet in the actual room. This means that 2 inches would represent 16 feet, and $\frac{1}{2}$ inch would represent 4 feet.

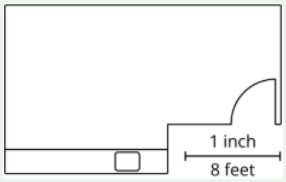


Figure 1.2.6.1

Definition: Scale Drawing

A scale drawing represents an actual place or object. All the measurements in the drawing correspond to the measurements of the actual object by the same scale.



Figure 1.2.6.2

Practice

Exercise 1.2.6.5

The Empire State Building in New York City is about 1,450 feet high (including the antenna at the top) and 400 feet wide. Andre wants to make a scale drawing of the front view of the Empire State Building on an $8\frac{1}{2}$-inch-by-11-inch piece of paper. Select a scale that you think is the most appropriate for the scale drawing. Explain your reasoning.

1. 1 inch to 1 foot
2. 1 inch to 100 feet
3. 1 inch to 1 mile
4. 1 centimeter to 1 meter
5. 1 centimeter to 50 meters
6. 1 centimeter to 1 kilometer

Exercise 1.2.6.6

Elena finds that the area of a house on a scale drawing is 25 square inches. The actual area of the house is 2,025 square feet. What is the scale of the drawing?

Exercise 1.2.6.7

Which of these scales are equivalent to 3 cm to 4 km? Select **all** that apply. Recall that 1 inch is 2.54 centimeters.

A. 0.75 cm to 1 km
B. 1 cm to 12 km
C. 6 mm to 2 km
D. 0.3 mm to 40 m
E. 1 inch to 7.62 km

Exercise 1.2.6.8

These two triangles are scaled copies of one another. The area of the smaller triangle is 9 square units. What is the area of the larger triangle? Explain or show how you know.

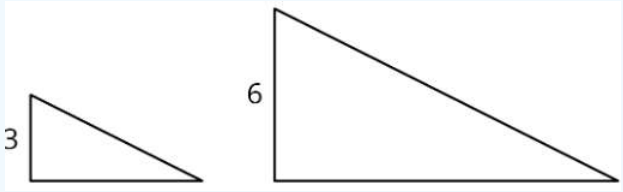


Figure 1.2.6.3

Exercise 1.2.6.9

Water costs $1.25 per bottle. At this rate, what is the cost of:

1. 10 bottles?
2. 20 bottles?
3. 50 bottles?

Exercise 1.2.6.10

The first row of the table shows the amount of dish detergent and water needed to make a soap solution.

1. Complete the table for 2, 3, and 4 batches.
2. How much water and detergent is needed for 8 batches? Explain your reasoning.

Table 1.2.6.2

number of batches	cups of water	cups of detergent
1	6	1
2		
3		
4		

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1.3: Let's Put It to Work

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1.3.1: Draw It to Scale

Lesson

Let's draw a floor plan.

Exercise 1.3.1.1: Which Measurements Matter?

Which measurements would you need in order to draw a scale floor plan of your classroom? List which parts of the classroom you would measure and include in the drawing. Be as specific as possible.

Exercise 1.3.1.2: Creating a Floor Plan (Part 1)

1. On a blank sheet of paper, make a *rough sketch* of a floor plan of the classroom. Include parts of the room that the class has decided to include or that you would like to include. Accuracy is not important for this rough sketch, but be careful not to omit important features like a door.
2. Trade sketches with a partner and check each other's work. Specifically, check if any parts are missing or incorrectly placed. Return their work and revise your sketch as needed.
3. Discuss with your group a plan for measuring. Work to reach an agreement on:
   - Which classroom features must be measured and which are optional.
   - The units to be used.
   - How to record and organize the measurements (on the sketch, in a list, in a table, etc.).
   - How to share the measuring and recording work (or the role each group member will play).
4. Gather your tools, take your measurements, and record them as planned. Be sure to double-check your measurements.
5. Make your own copy of all the measurements that your group has gathered, if you haven't already done so. You will need them for the next activity.

Exercise 1.3.1.3: Creating a Floor Plan (Part 2)

Your teacher will give you several paper options for your scale floor plan.

1. Determine an appropriate scale for your drawing based on your measurements and your paper choice. Your floor plan should fit on the paper and not end up too small.
2. Use the scale and the measurements your group has taken to draw a scale floor plan of the classroom. Make sure to:
   - Show the scale of your drawing.
   - Label the key parts of your drawing (the walls, main openings, etc.) with their actual measurements.
   - Show your thinking and organize it so it can be followed by others.

Are you ready for more?

1. If the flooring material in your classroom is to be replaced with 10-inch by 10-inch tiles, how many tiles would it take to cover the entire room? Use your scale drawing to approximate the number of tiles needed.
2. How would using 20-inch by 20-inch tiles (instead of 10-inch by 10-inch tiles) change the number of tiles needed? Explain your reasoning.

Exercise 1.3.1.4: Creating a Floor Plan (part 3)

1. Trade floor plans with another student who used the same paper size as you. Discuss your observations and thinking.
2. Trade floor plans with another student who used a different paper size than you. Discuss your observations and thinking.
3. Based on your discussions, record ideas for how your floor plan could be improved.

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1.1.1: What are Scaled Copies?

Lesson

Let's explore scaled copies.

Exercise 1.1.1.1: Printing Portraits

Here is a portrait of a student. Move the slider under each image, A–E, to see it change.

1. How is each one the same as or different from the original portrait of the student?
2. Some of the sliders make **scaled copies** of the original portrait. Which ones do you think are scaled copies? Explain your reasoning.
3. What do you think “scaled copy” means?

Exercise 1.1.1.2: Scaling F

Here is an original drawing of the letter F and some other drawings.

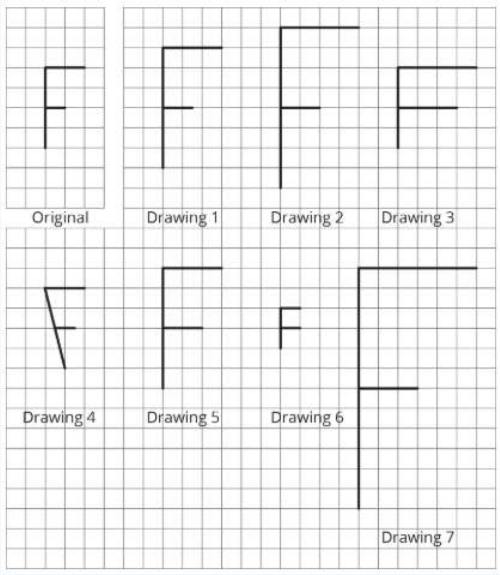


Figure 1.1.1.1

1. Identify **all** the drawings that are scaled copies of the original letter F drawing. Explain how you know.
2. Examine all the scaled copies more closely, specifically, the lengths of each part of the letter F. How do they compare to the original? What do you notice?
3. On the grid, draw a different scaled copy of the original letter F.

Exercise 1.1.1.3: Pairs of Scaled Polygons

Your teacher will give you a set of cards that have polygons drawn on a grid. Mix up the cards and place them all face up.

1. Take turns with your partner to match a pair of polygons that are scaled copies of one another.
   a. For each match you find, explain to your partner how you know it’s a match.
   b. For each match your partner finds, listen carefully to their explanation, and if you disagree, explain your thinking.
2. When you agree on all of the matches, check your answers with the answer key. If there are any errors, discuss why and revise your matches.
3. Select one pair of polygons to examine further. Use the grid below to produce both polygons. Explain or show how you know that one polygon is a scaled copy of the other.

Are you ready for more?

Is it possible to draw a polygon that is a scaled copy of both Polygon A and Polygon B? Either draw such a polygon, or explain how you know this is impossible.

Summary

What is a **scaled copy** of a figure? Let's look at some examples.

The second and third drawings are both scaled copies of the original Y.

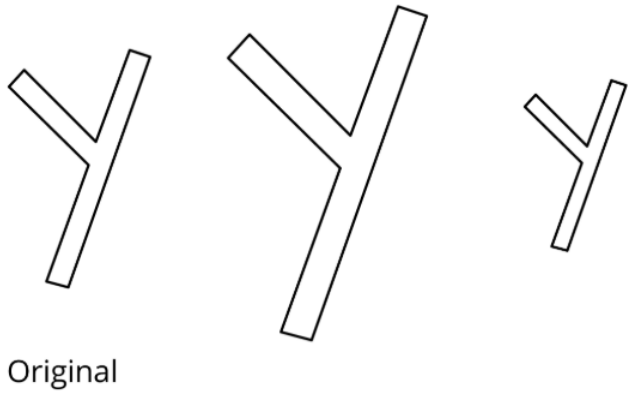


Figure 1.1.1.2

However, here, the second and third drawings are *not* scaled copies of the original W.

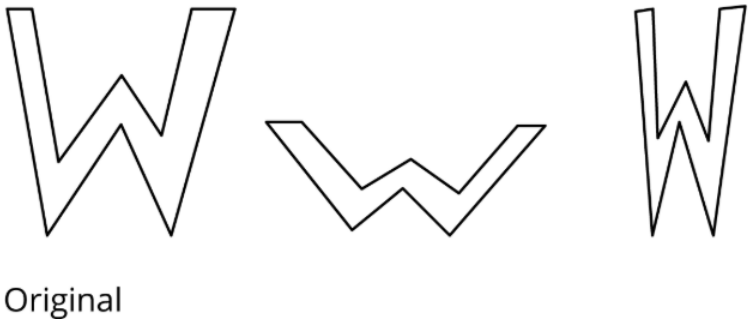


Figure 1.1.1.3

The second drawing is spread out (wider and shorter). The third drawing is squished in (narrower, but the same height).

We will learn more about what it means for one figure to be a scaled copy of another in upcoming lessons.

Glossary Entries

Definition: Scaled Copy

A scaled copy is a copy of an figure where every length in the original figure is multiplied by the same number.

For example, triangle DEF is a scaled copy of triangle ABC. Each side length on triangle ABC was multiplied by 1.5 to get the corresponding side length on triangle DEF.

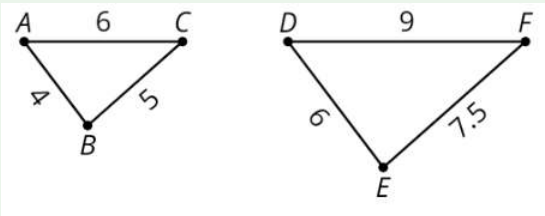


Figure 1.1.1.4

Practice

Exercise 1.1.1.4

Here is a figure that looks like the letter A, along with several other figures. Which figures are scaled copies of the original A? Explain how you know.

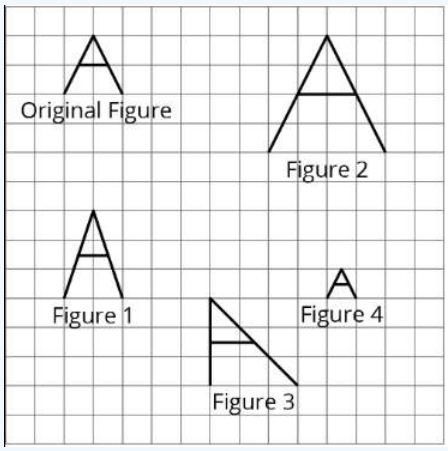


Figure 1.1.1.5

Exercise 1.1.1.5

Tyler says that Figure B is a scaled copy of Figure A because all of the peaks are half as tall.

Do you agree with Tyler? Explain your reasoning.

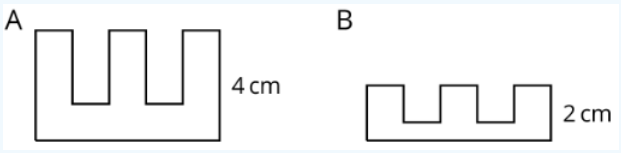


Figure 1.1.1.6: Two 12 sided figures in the shape of a W are labeled A and B. Both figures are composed of vertical and horizontal line segments that meet at a right angle. In figure A, the height is labeled 4 centimeters and in figure B, the height is labeled 2 centimeters. The width of both figures, appear to be the same, and is not labeled.

Exercise 1.1.1.6

Here is a picture of the Rose Bowl Stadium in Pasadena, CA.



Figure 1.1.1.7: Rose Bowl aerial, by NASA (http://worldwind.arc.nasa.gov/). Public Domain. Wikimedia Commons. Source.

Here are some copies of the picture. Select **all** the pictures that are scaled copies of the original picture.

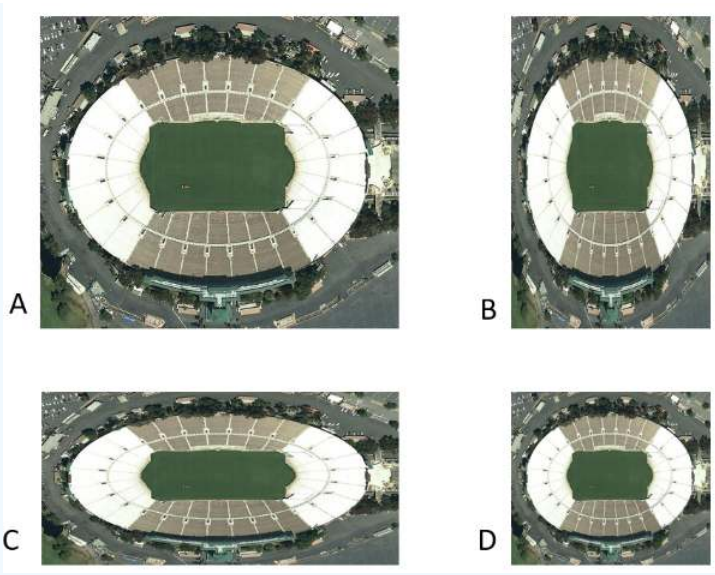


Figure 1.1.1.8

Exercise 1.1.1.7

Complete each equation with a number that makes it true.

1. $5 \cdot \_\_\_\_ = 15$
2. $4 \cdot \_\_\_\_ = 32$
3. $6 \cdot \_\_\_\_ = 9$
4. $12 \cdot \_\_\_\_ = 3$

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CHAPTER OVERVIEW

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2.1: Representing Proportional Relationships with Tables

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2.1.1: One of These Things is Not Like the Others

Lesson

Let's remember what equivalent ratios are.

Exercise 2.1.1.1: Remembering Double Number Lines

1. Complete the double number line diagram with the missing numbers.

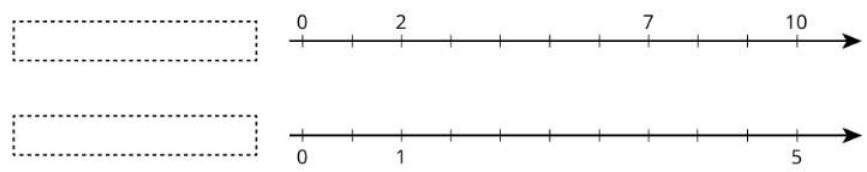


Figure 2.1.1.1: A double number line with 11 evenly spaced tick marks. For the top number line, a dashed box represents the title and the number 0 is on the first tick mark, 2 on the third, 7 on the eighth, and 10 on the eleventh. For the bottom number line, a dashed box represents the title and the number 0 is on the first tick mark, 1 on the third, and 5 on the eleventh.

2. What could each of the number lines represent? Invent a situation and label the diagram.

3. Make sure your labels include appropriate units of measure.

Exercise 2.1.1.2: Mystery Mixtures

Your teacher will show you three mixtures. Two taste the same, and one is different.

1. Which mixture tastes different? Describe how it is different.
2. Here are the recipes that were used to make the three mixtures:
   - 1 cup of water with teaspoons of powdered drink mix
   - 2 cups of water with teaspoon of powdered drink mix
   - 1 cup of water with teaspoon of powdered drink mix

Which of these recipes is for the stronger tasting mixture? Explain how you know.

Are you ready for more?

Salt and sugar give two distinctly different tastes, one salty and the other sweet. In a mixture of salt and sugar, it is possible for the mixture to be salty, sweet or both. Will any of these mixtures taste exactly the same?

- Mixture A: 2 cups water, 4 teaspoons salt, 0.25 cup sugar
- Mixture B: 1.5 cups water, 3 teaspoons salt, 0.2 cup sugar
- Mixture C: 1 cup water, 2 teaspoons salt, 0.125 cup sugar

Exercise 2.1.1.3: Crescent Moons

Here are four different crescent moon shapes.

1. What do Moons A, B, and C all have in common that Moon D doesn't?
2. Use numbers to describe how Moons A, B, and C are differnt from Moon D.
3. Use a table or a double number line to show how Moons A, B, and C are different from Moon D.

Are you ready for more?

Can you make one moon cover another by changing its size? What does that tell you about its dimensions?

GeoGebra Applet tbmsMsJZ

Summary

When two different situations can be described by **equivalent ratios**, that means they are alike in some important way.

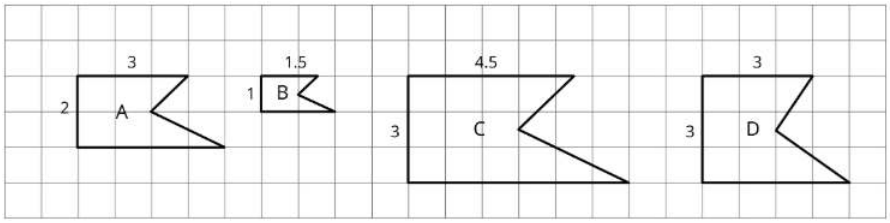
An example is a recipe. If two people make something to eat or drink, the taste will only be the same as long as the ratios of the ingredients are equivalent. For example, all of the mixtures of water and drink mix in this table taste the same, because the ratios of cups of water to scoops of drink mix are all equivalent ratios.

Table 2.1.1.1

water (cups)	drink mix (scoops)
3	1
12	4
1.5	0.5

If a mixture were not equivalent to these, for example, if the ratio of cups of water to scoops of drink mix were $6 : 4$, then the mixture would taste different.

Notice that the ratios of pairs of corresponding side lengths are equivalent in figures A, B, and C. For example, the ratios of the length of the top side to the length of the left side for figures A, B, and C are equivalent ratios. Figures A, B, and C are *scaled copies* of each other; this is the important way in which they are alike.



Figures A, B, C, and D drawn on a grid. A has left side 2 and top side 3. B has left side 1 and top side 1.5. C has left side 3 and top side 4.5. D has left side 3 and top side 3.

If a figure has corresponding sides that are not in a ratio equivalent to these, like figure D, then it's not a scaled copy. In this unit, you will study relationships like these that can be described by a set of equivalent ratios.

Glossary Entries

Definition: Equivalent Ratios

Two ratios are equivalent if you can multiply each of the numbers in the first ratio by the same factor to get the numbers in the second ratio. For example, $8 : 6$ is equivalent to $4 : 3$, because $8 \cdot \frac{1}{2} = 4$ and $6 \cdot \frac{1}{2} = 3$.

A recipe for lemonade says to use 8 cups of water and 6 lemons. If we use 4 cups of water and 3 lemons, it will make half as much lemonade. Both recipes taste the same, because and are equivalent ratios.

Table 2.1.1.2

cups of water	number of lemons
8	6
4	3

Practice

Exercise 2.1.1.4

Which one of these shapes is not like the others? Explain what makes it different by representing each width and height pair with a ratio.

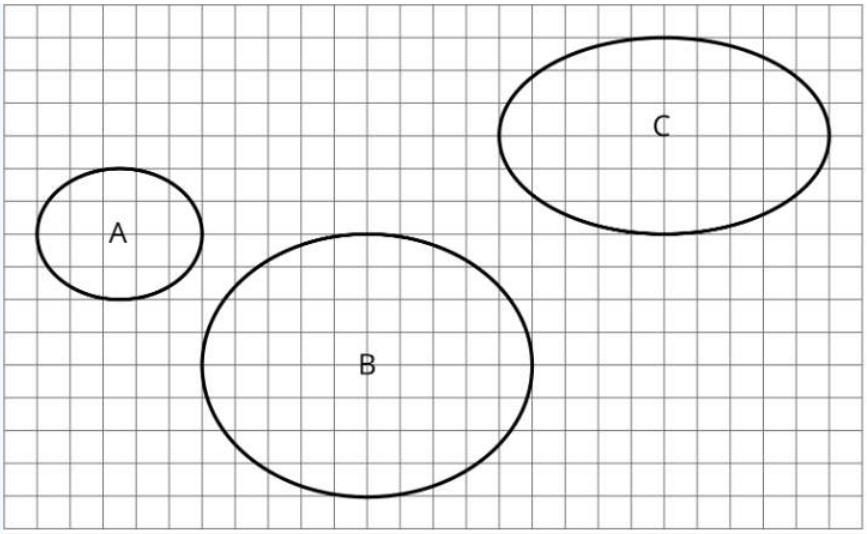


Figure 2.1.1.3: Three ovals, A, B and C on a grid. At each oval’s widest point, oval A has width 5 units and height 4 units. Oval B has width 10 units and height 8 units. Oval C has width 10 units and height 6 units.

Exercise 2.1.1.5

In one version of a trail mix, there are 3 cups of peanuts mixed with 2 cups of raisins. In another version of trail mix, there are 4.5 cups of peanuts mixed with 3 cups of raisins. Are the ratios equivalent for the two mixes? Explain your reasoning.

Exercise 2.1.1.6

For each object, choose an appropriate scale for a drawing that fits on a regular sheet of paper. Not all of the scales on the list will be used.

Objects

1. A person
2. A football field (120 yards by $53\frac{1}{3}$ yards)
3. The state of Washington (about 240 miles by 360 miles)
4. The floor plan of a house
5. A rectangular farm (6 miles by 2 mile)

Scales

- 1 in : 1 ft
- 1 cm : 1 m
- 1: 1000
- 1 ft: 1 mile
- 1: 100,000
- 1 mm: 1 km
- 1: 10,000,000

(From Unit 1.2.6)

Exercise 2.1.1.7

Which scale is equivalent to 1 cm to 1 km?

A. 1 to 1000
B. 10,000 to 1
C. 1 to 100,000
D. 100,000 to 1

E. 1 to 1,000,000

(From Unit 1.2.5)

Exercise 2.1.1.8

1. Find 3 different ratios that are equivalent to $7 : 3$.
2. Explain why these ratios are equivalent.

(From Unit 1.2.4)

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2.1.2: Introducing Proportional Relationships with Tables

Lesson

Let's solve problems involving proportional relationships using tables.

Exercise 2.1.2.1: NOtice and Wonder: Paper Towels by The Case

Here is a table that shows how many rolls of paper towels a store receives when they order different numbers of cases.

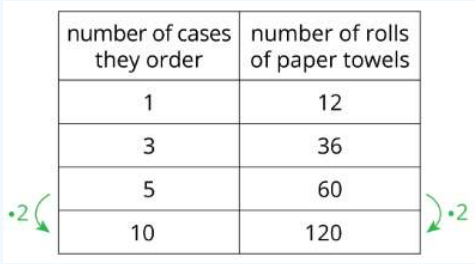


Figure 2.1.2.1: Table with 2 columns and 4 rows of data. The columns are: number of cases they order and number of rolls of paper towels. The table has the ordered pairs (1 comma 12), (2 comma 36), (5 comma 60) and (10 comma 120). There is an arrow pointing from row 3 to row 4 on each side with times 2 next to the arrows.

What do you notice about the table? What do you wonder?

Exercise 2.1.2.2: Feeding a Crowd

1. A recipe says that 2 cups of dry rice will serve 6 people. Complete the table as you answer the questions. Be prepared to explain your reasoning.
   a. How many people will 10 cups of rice serve?
   b. How many cups of rice are needed to serve 45 people?

Table 2.1.2.1

cups of rice	number of people
2	6
3	9
10	
	45

2. A recipe says that 6 spring rolls will serve 3 people. Complete the table.

Table 2.1.2.2

number of spring rolls	number of people
6	3
30	
40	
	28

Exercise 2.1.2.3: Making Bread Dough

A bakery uses 8 tablespoons of honey for every 10 cups of flour to make bread dough. Some days they bake bigger batches and some days they bake smaller batches, but they always use the same ratio of honey to flour. Complete the table as you answer the questions. Be prepared to explain your reasoning.

1. How many cups of flour do they use with 20 tablespoons of honey?
2. How many cups of flour do they use with 13 tablespoons of honey?
3. How many tablespoons of honey do they use with 20 cups of flour?
4. What is the **proportional relationship** represented by this table?

Table 2.1.2.3

honey (tbsp)	flour (c)
8	10
20	
13	
	20

Exercise 2.1.2.4: Quarters and Dimes

4 quarters are equal in value to 10 dimes.

1. How many dimes equal the value of 6 quarters?
2. How many dimes equal the value of 14 quarters?
3. What value belongs next to the 1 in the table? What does it mean in this context?

Table 2.1.2.4

number of quarters	number of dimes
1	
4	20
6	
14	

Are you ready for more?

Pennies made before 1982 are 95% copper and weigh about 3.11 grams each. (Pennies made after that date are primarily made of zinc). Some people claim that the value of the copper in one of these pennies is greater than the face value of the penny. Find out how much copper is worth right now, and decide if this claim is true.

Summary

If the ratios between two corresponding quantities are always equivalent, the relationship between the quantities is called a **proportional relationship**.

This table shows different amounts of milk and chocolate syrup. The ingredients in each row, when mixed together, would make a different total amount of chocolate milk, but these mixtures would all taste the same.

Table 2.1.2.5

tablespoons of chocolate syrup	cups of milk
4	1
6	$1\frac{1}{2}$
8	2
$\frac{1}{2}$	$\frac{1}{8}$
12	3

tablespoons of chocolate syrup	cups of milk
1	$\frac{1}{4}$

Notice that each row in the table shows a ratio of tablespoons of chocolate syrup to cups of milk that is equivalent to $4 : 1$.

About the relationship between these quantities, we could say:

- The relationship between amount of chocolate syrup and amount of milk is proportional.
- The relationship between the amount of chocolate syrup and the amount of milk is a proportional relationship.
- The table represents a proportional relationship between the amount of chocolate syrup and amount of milk.
- The amount of milk is proportional to the amount of chocolate syrup.

We could multiply any value in the chocolate syrup column by $\frac{1}{4}$ to get the value in the milk column. We might call $\frac{1}{4}$ a *unit rate*, because $\frac{1}{4}$ cups of milk are needed for 1 tablespoon of chocolate syrup. We also say that $\frac{1}{4}$ is the **constant of proportionality** for this relationship. It tells us how many cups of milk we would need to mix with 1 tablespoon of chocolate syrup.

Glossary Entries

Definition: Constant of Proportionality

In a proportional relationship, the values for one quantity are each multiplied by the same number to get the values for the other quantity. This number is called the constant of proportionality.

In this example, the constant of proportionality is 3, because $2 \cdot 3 = 6$, $3 \cdot 3 = 9$, and $5 \cdot 3 = 15$. This means that there are 3 apples for every 1 orange in the fruit salad.

Table 2.1.2.6

number of oranges	number of apples
2	6
3	9
5	15

Definition: Equivalent Ratios

Two ratios are equivalent if you can multiply each of the numbers in the first ratio by the same factor to get the numbers in the second ratio. For example, $8 : 6$ is equivalent to $4 : 3$, because $8 \cdot \frac{1}{2} = 4$ and $6 \cdot \frac{1}{2} = 3$.

A recipe for lemonade says to use 8 cups of water and 6 lemons. If we use 4 cups of water and 3 lemons, it will make half as much lemonade. Both recipes taste the same, because and are equivalent ratios.

Table 2.1.2.7

cups of water	number of lemons
8	6
4	3

Definition: Proportional Relationship

In a proportional relationship, the values for one quantity are each multiplied by the same number to get the values for the other quantity.

For example, in this table every value of p is equal to 4 times the value of s on the same row.

We can write this relationship as $p = 4s$. This equation shows that s is proportional to p.

Table 2.1.2.8

s	p
2	8
3	12
5	20
10	40

Practice

Exercise 2.1.2.5

When Han makes chocolate milk, he mixes 2 cups of milk with 3 tablespoons of chocolate syrup. Here is a table that shows how to make batches of different sizes. Use the information in the table to complete the statements. Some terms are used more than once.

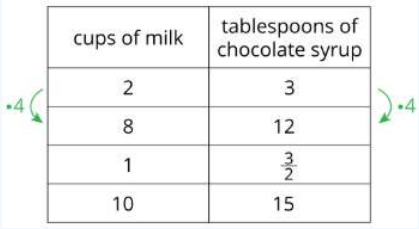


Figure 2.1.2.2: Table with 2 columns and 4 rows of data. The columns are: cups of milk and tablespoons of chocolate syrup. The table has the ordered pairs (2 comma 3), (8 comma 12), (1 comma the fraction 3 over 2) and (10 comma 15). There is an arrow pointing from row 1 to row 2 on each side with times 4 next to the arrows.

1. The table shows a proportional relationship between _____ and _____.
2. The scale factor shown is _____.
3. The constant of proportionality for this relationship is_____.
4. The units for the constant of proportionality are _____ per _____.

Bank of Terms: tablespoons of chocolate syrup, 4, cups of milk, cup of milk, $\frac{3}{2}$

Exercise 2.1.2.6

A certain shade of pink is created by adding 3 cups of red paint to 7 cups of white paint.

1. How many cups of red paint should be added to 1 cup of white paint?

Table 2.1.2.9

cups of white paint	cups of red paint
1	
7	3

2. What is the constant of proportionality?

Exercise 2.1.2.7

A map of a rectangular park has a length of 4 inches and a width of 6 inches. It uses a scale of 1 inch for every 30 miles.

1. What is the actual area of the park? Show how you know.
2. The map needs to be reproduced at a different scale so that it has an area of 6 square inches and can fit in a brochure. At what scale should the map be reproduced so that it fits on the brochure? Show your reasoning.

(From Unit 1.2.6)

Exercise 2.1.2.8

Noah drew a scaled copy of Polygon P and labeled it Polygon Q.

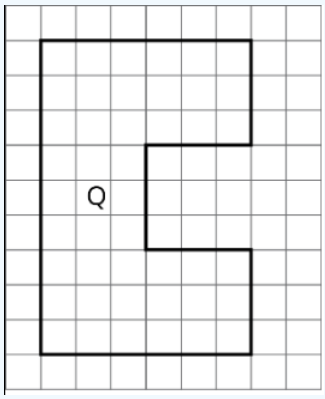


Figure 2.1.2.3: Polygon Q on a grid. Polygon Q has 8 sides. Starting at the bottom left corner, the first side is 9 units up, the second side is 6 units right, the third side is 3 units down, the fourth side is 3 units left, the fifth side is 3 units down, the sixth side is 3 units right, the seventh side is 3 units down, and the eighth side is 6 units left.

If the area of Polygon P is 5 square units, what scale factor did Noah apply to Polygon P to create Polygon Q? Explain or show how you know.

(From Unit 1.1.6)

Exercise 2.1.2.9

Select **all** the ratios that are equivalent to each other.

A. $4 : 7$
B. $8 : 15$
C. $16 : 28$
D. $2 : 3$
E. $20 : 35$

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2.1.3: More about Constant of Proportionality

Lesson

Let's solve more problems involving proportional relationships using tables.

Exercise 2.1.3.1: Equal Measures

Use the numbers and units from the list to find as many equivalent measurements as you can. For example, you might write "30 minutes is $\frac{1}{2}$ hour."

You can use the numbers and units more than once.

1	$\frac{1}{2}$	0.3	centimeter
12	40	24	meter
0.4	0.01	$\frac{1}{10}$	hour
60	$3\frac{1}{3}$	6	feet
50	30	2	minute
		$\frac{2}{5}$	inch

Exercise 2.1.3.2: Centimeters and Millimeters

There is a proportional relationship between any length measured in centimeters and the same length measured in millimeters.

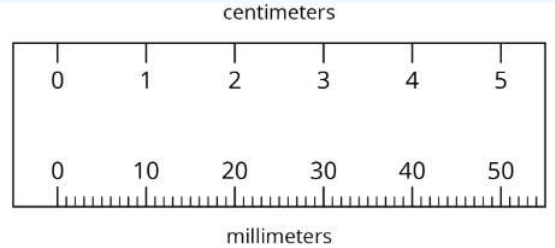


Figure 2.1.3.1

There are two ways of thinking about this proportional relationship.

1. If you know the length of something in centimeters, you can calculate its length in millimeters.
   a. Complete the table.
   b. What is the constant of proportionality?

Table 2.1.3.1

length (cm)	length (mm)
9	
12.5	
50	
88.49	

2. If you know the length of something in millimeters, you can calculate its length in centimeters.
   a. Complete the table.
   b. What is the constant of proportionality?

Table 2.1.3.2

length (mm)	length (cm)
70	

length (mm)	length (cm)
245	
4	
699.1	

3. How are these two constants of proportionality related to each other?
4. Complete each sentence:
   a. To convert from centimeters to millimeters, you can multiply by _____.
   b. To convert from millimeters to centimeters, you can divide by _____ *or* multiply by _____.

Are you ready for more?

1. How many square millimeters are there in a square centimeter?
2. How do you convert square centimeters to square millimeters? How do you convert the other way?

Exercise 2.1.3.3: Pittsburgh to Phoenix

On its way from New York to San Diego, a plane flew over Pittsburgh, Saint Louis, Albuquerque, and Phoenix traveling at a constant speed.

Complete the table as you answer the questions. Be prepared to explain your reasoning.

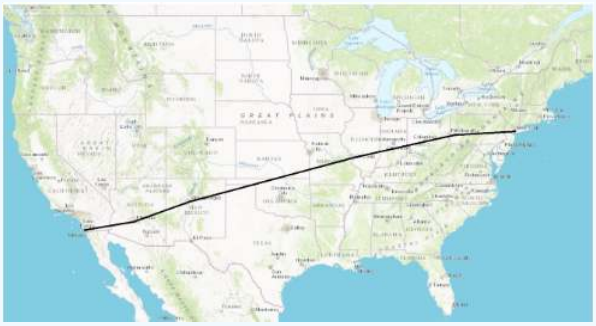


Figure 2.1.3.2: A United States map with 5 segments representing the distance a plane flew. The segments are New York to Pittsburgh, Pittsburgh to Saint Louis, Saint Louis to Albuquerque, Albuquerque to Phoenix and Phoenix to San Diego. By United States Census Bureau. Public Domain. American Fact Finder. Source.

Table 2.1.3.3

segment	time	distance	speed
Pittsburgh to Saint Louis	1 hour	550 miles	
Saint Louis to Albuquerque	1 hour 42 minutes		
Albuquerque to Phoenix		330 miles	

1. What is the distance between Saint Louis and Albuquerque?
2. How many minutes did it take to fly between Albuquerque and Phoenix?
3. What is the proportional relationship represented by this table?
4. Diego says the constant of proportionality is 550. Andre says the constant of proportionality is $9\frac{1}{6}$. Do you agree with either of them? Explain your reasoning.

Summary

When something is traveling at a constant speed, there is a proportional relationship between the time it takes and the distance traveled. The table shows the distance traveled and elapsed time for a bug crawling on a sidewalk.

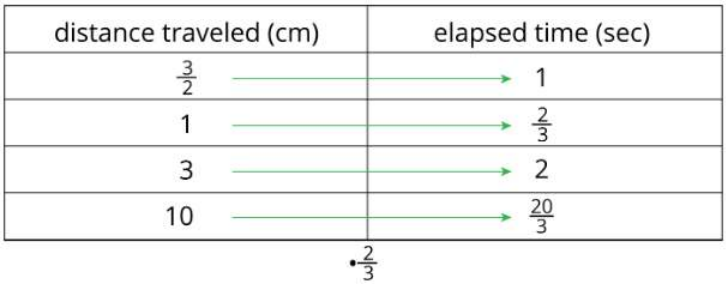


Figure 2.1.3.3: Table with 2 columns and 4 rows of data. The columns are: distance traveled (cm) and elapsed time (sec). The table has the ordered pairs (the fraction 3 over 2 comma 1), (1 comma the fraction 2 over 3), (3 comma 2) and (10 comma the fraction 20 over 3). Each pair has an arrow pointing from the value in the first column to the value in the second column. Times the fraction 2 over 3 is below the table.

We can multiply any number in the first column by $\frac{2}{3}$ to get the corresponding number in the second column. We can say that the elapsed time is proportional to the distance traveled, and the constant of proportionality is $\frac{2}{3}$. This means that the bug's *pace* is $\frac{2}{3}$ seconds per centimeter.

This table represents the same situation, except the columns are switched.

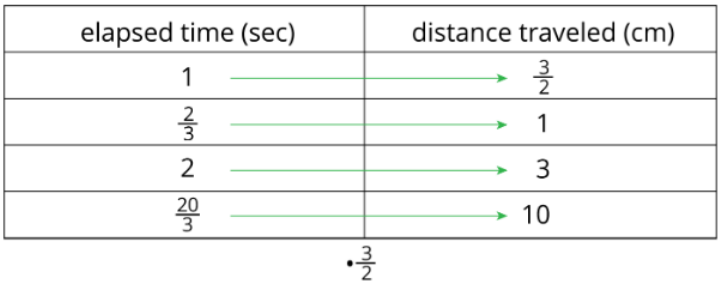


Figure 2.1.3.4: Table with 2 columns and 4 rows of data. The columns are: elapsed time (sec) and distance traveled (cm). The table has the ordered pairs (1 comma the fraction 3 over 2), (the fraction 2 over 3 comma 1), (2 comma 3) and (the fraction 20 over 3 comma 10). Each pair has an arrow pointing from the value in the first column to the value in the second column. Times the fraction three over 2 is below the table.

We can multiply any number in the first column by $\frac{3}{2}$ to get the corresponding number in the second column. We can say that the distance traveled is proportional to the elapsed time, and the constant of proportionality is $\frac{3}{2}$. This means that the bug's *speed* is $\frac{3}{2}$ centimeters per second.

Notice that $\frac{3}{2}$ is the reciprocal of $\frac{2}{3}$. When two quantities are in a proportional relationship, there are two constants of proportionality, and they are always reciprocals of each other. When we represent a proportional relationship with a table, we say the quantity in the second column is proportional to the quantity in the first column, and the corresponding constant of proportionality is the number we multiply values in the first column to get the values in the second.

Glossary Entries

Definition: Constant of Proportionality

In a proportional relationship, the values for one quantity are each multiplied by the same number to get the values for the other quantity. This number is called the constant of proportionality.

In this example, the constant of proportionality is 3, because $2 \cdot 3 = 6$, $3 \cdot 3 = 9$, and $5 \cdot 3 = 15$. This means that there are 3 apples for every 1 orange in the fruit salad.

Table 2.1.3.4

number of oranges	number of apples
2	6
3	9
5	15

Definition: Equivalent Ratios

Two ratios are equivalent if you can multiply each of the numbers in the first ratio by the same factor to get the numbers in the second ratio. For example, $8 : 6$ is equivalent to $4 : 3$, because $8 \cdot \frac{1}{2} = 4$ and $6 \cdot \frac{1}{2} = 3$.

A recipe for lemonade says to use 8 cups of water and 6 lemons. If we use 4 cups of water and 3 lemons, it will make half as much lemonade. Both recipes taste the same, because and are equivalent ratios.

Table 2.1.3.5

cups of water	number of lemons
8	6
4	3

Definition: Proportional Relationship

In a proportional relationship, the values for one quantity are each multiplied by the same number to get the values for the other quantity.

For example, in this table every value of p is equal to 4 times the value of s on the same row.

We can write this relationship as $p = 4s$. This equation shows that s is proportional to p.

Table 2.1.3.6

s	p
2	8
3	12
5	20
10	40

Practice

Exercise 2.1.3.4

Noah is running a portion of a marathon at a constant speed of 6 miles per hour.

Complete the table to predict how long it would take him to run different distances at that speed, and how far he would run in different time intervals.

Table 2.1.3.7

time in hours	miles traveled at 6 miles per hour
1	
$\frac{1}{2}$	
$1\frac{1}{3}$	
	$1\frac{1}{2}$
	9
	$4\frac{1}{2}$

Exercise 2.1.3.5

One kilometer is 1000 meters.

1. Complete the tables. What is the interpretation of the constant of proportionality in each case?

Table 2.1.3.8

meters	kilometers
1,000	1
250	
12	
1	

The constant of proportionality tells us that:

Table 2.1.3.9

kilometers	meters
1	1,000
5	
20	
0.3	

The constant of proportionality tells us that:

2. What is the relationship between two constants of proportionality?

Exercise 2.1.3.6

Jada and Lin are comparing inches and feet. Jada says that the constant of proportionality is 12. Lin says it is $\frac{1}{12}$. Do you agree with either of them? Explain your reasoning.

Exercise 2.1.3.7

The area of the Mojave desert is 25,000 square miles. A scale drawing of the Mojave desert has an area of 10 square inches. What is the scale of the map?

(From Unit 1.2.6)

Exercise 2.1.3.8

Which of these scales is equivalent to the scale 1 cm to 5 km? Select **all** that apply.

A. 3 cm to 15 km
B. 1 mm to 150 km
C. 5 cm to 1 km
D. 5 mm to 2.5 km
E. 1 mm to 500 m

(From Unit 1.2.5)

Exercise 2.1.3.9

Which one of these pictures is not like the others? Explain what makes it different using ratios.

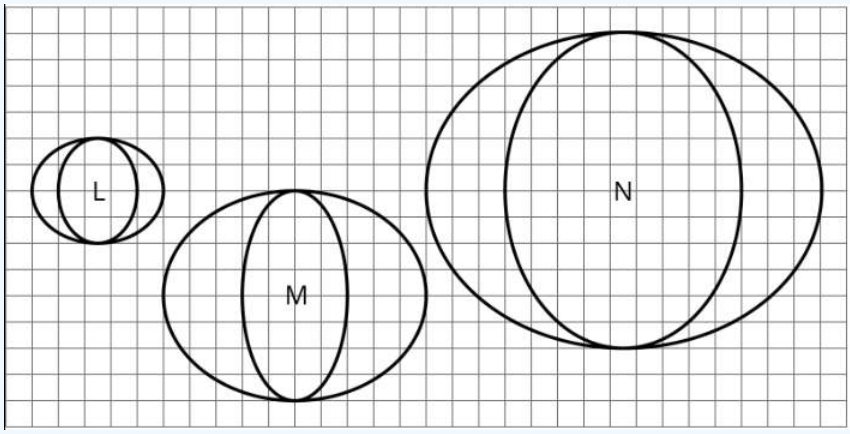


Figure 2.1.3.5

(From Unit 2.1.1)

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2.2: Representing Proportional Relationships with Equations

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2.2.1: Proportional Relationships and Equations

Lesson

Let's write equations describing proportional relationships.

Exercise 2.2.1.1: NUmber Talk: Division

Find each quotient mentally.

$645 \div 100$

$645 \div 50$

$48.6 \div 30$

$48.6 \div x$

Exercise 2.2.1.2: Feeding a Crowd, Revisited

1. A recipe says that 2 cups of dry rice will serve 6 people. Complete the table as you answer the questions. Be prepared to explain your reasoning.

   a. How many people will 1 cup of rice serve?
   b. How many people will 3 cups of rice serve? 12 cups? 43 cups?
   c. How many people will x cups of rice serve?

Table 2.2.1.1

cups of dry rice	number of people
1	
2	6
3	
12	
43	
x	

2. A recipe says that 6 spring rolls will serve 3 people. Complete the table as you answer the questions. Be prepared to explain your reasoning.

   a. How many people will 1 spring roll serve?
   b. How many people will 10 spring rolls serve? 16 spring rolls? 25 spring rolls?
   c. How many people will n spring rolls serve?

Table 2.2.1.2

number of spring rolls	number of people
1	
6	3
10	
16	
25	
n	

3. How was completing this table different from the previous table? How as it the same?

Exercise 2.2.1.3: Denver to Chicago

A plane flew at a constant speed between Denver and Chicago. It took the plane 1.5 hours to fly 915 miles.



Figure 2.2.1.1: By United States Census Bureau. Public Domain. American Fact Finder. Source.

1. Complete the table.

Table 2.2.1.3

time (hours)	distance (miles)	speed (miles per hour)
1		
1.5	915	
2		
2.5		
t		

2. How far does the plane fly in one hour?
3. How far would the plane fly in t hours at this speed?
4. If d represents the distance that the plane flies at this speed for t hours, write an equation that relates t and d.
5. How far would the plane fly in 3 hours at this speed? in 3.5 hours? Explain or show your reasoning.

Are you ready for more?

A rocky planet orbits Proxima Centauri, a star that is about 1.3 parsecs from Earth. This planet is the closest planet outside of our solar system.

1. How long does it take light from Proxima Centauri to reach Earth? (A parsec is about 3.26 light years. A light year is the distance light travels in one year.)
2. There are two twins. One twin leaves on a spaceship to explore the planet near Proxima Centauri traveling at 90% of the speed of light, while the other twin stays home on Earth. How much does the twin on Earth age while the other twin travels to Proxima Centauri? (Do you think the answer would be the same for the other twin? Consider researching “The Twin Paradox” to learn more.)

Exercise 2.2.1.4: Revisiting a Bread Dough

A bakery uses 8 tablespoons of honey for every 10 cups of flour to make bread dough. Some days they bake bigger batches and some days they bake smaller batches, but they always use the same ratio of honey to flour.

1. Complete the table.
2. If f is the cups of flour needed for h tablespoons of honey, write an equation that relates f and h.
3. How much flour is needed for 15 tablespoons of honey? 17 tablespoons? Explain or show your reasoning.

Table 2.2.1.4

honey (tbsp)	flour (c)

honey (tbsp)	flour (c)
1	
8	10
16	
20	
h	

Summary

The table shows the amount of red paint and blue paint needed to make a certain shade of purple paint, called Venusian Sunset.

Note that "parts" can be *any* unit for volume. If we mix 3 cups of red with 12 cups of blue, you will get the same shade as if we mix 3 teaspoons of red with 12 teaspoons of blue.

Table 2.2.1.5

red paint (parts)	blue paint (parts)
3	12
1	4
7	28
$\frac{1}{4}$	1
r	$4r$

The last row in the table says that if we know the amount of red paint needed, r, we can always multiply it by 4 to find the amount of blue paint needed, b, to mix with it to make Venusian Sunset. We can say this more succinctly with the equation $b = 4r$. So the amount of blue paint is proportional to the amount of red paint and the constant of proportionality is 4.

We can also look at this relationship the other way around.

If we know the amount of blue paint needed, b, we can always multiply it by $\frac{1}{4}$ to find the amount of red paint needed, r, to mix with it to make Venusian Sunset. So $r = \frac{1}{4}b$. The amount of blue paint is proportional to the amount of red paint and the constant of proportionality $\frac{1}{4}$.

Table 2.2.1.6

blue paint (parts)	red paint (parts)
12	3
4	1
28	7
1	$\frac{1}{4}$
b	$\frac{1}{4}b$

In general, when y is proportional to x, we can always multiply x by the same number k—the constant of proportionality—to get y. We can write this much more succinctly with the equation $y = kx$.

Practice

Exercise 2.2.1.5

A certain ceiling is made up of tiles. Every square meter of ceiling requires 10.75 tiles. Fill in the table with the missing values.

Table 2.2.1.7

square meters of ceiling	number of tiles
1	
10	
	100
a	

Exercise 2.2.1.6

On a flight from New York to London, an airplane travels at a constant speed. An equation relating the distance traveled in miles, d, to the number of hours flying, t, is $t = \frac{1}{500}d$. How long will it take the airplane to travel 800 miles?

Exercise 2.2.1.7

Each table represents a proportional relationship. For each, find the constant of proportionality, and write an equation that represents the relationship.

Table 2.2.1.8

s	P
2	8
3	12
5	20
10	40

Constant of proportionality:

Equation: $P =$

Table 2.2.1.9

d	C
2	6.28
3	9.42
5	15.7
10	31.4

Constant of proportionality:

Equation: $C =$

Exercise 2.2.1.8

A map of Colorado says that the scale is 1 inch to 20 miles or 1 to 1,267,200. Are these two ways of reporting the scale the same? Explain your reasoning.

(From Unit 1.2.5)

Exercise 2.2.1.9

Here is a polygon on a grid.

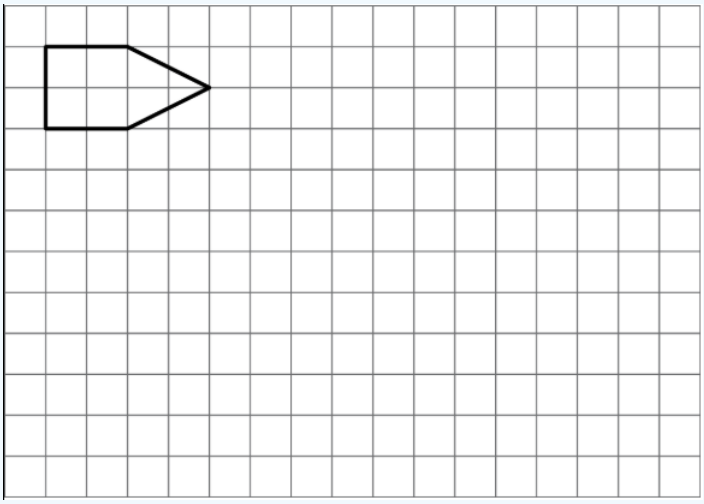


Figure 2.2.1.2

1. Draw a scaled copy of the polygon using a scale factor 3. Label the copy A.
2. Draw a scaled copy of the polygon with a scale factor $\frac{1}{2}$. Label it B.
3. Is Polygon A a scaled copy of Polygon B? If so, what is the scale factor that takes B to A?

(From Unit 1.1.3)

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2.2.2: Two Equations for Each Relationship

Lesson

Let's investigate the equations that represent proportional relationships.

Exercise 2.2.2.1: Missing Figures

Here are the second and fourth figures in a pattern.

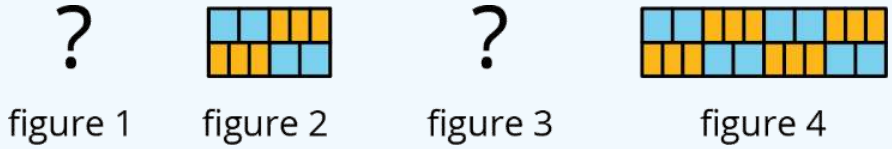


Figure 4 has 3 yellow, then 2 blue, then 3 yellow and 2 blue tiles on the bottom row and 2 blue, 3 yellow, 2 blue and 3 yellows on the top row.

1. What do you think the first and third figures in the pattern look like?
2. Describe the 10th figure in the pattern.

Exercise 2.2.2.2: Meters and Centimeters

There are 100 centimeters (cm) in every meter (m).

Table 2.2.2.1

length (m)	length (cm)
1	100
0.94	
1.67	
57.24	
x	

Table 2.2.2.2

length (cm)	length (m)
100	1
250	
78.2	
123.9	
y	

1. Complete each of the tables.
2. For each table, find the constant of proportionality.
3. What is the relationship between these constants of proportionality?
4. For each table, write an equation for the proportional relationship. Let x represent a length measured in meters and y represent the same length measured in centimeters.

Are you ready for more?

1. How many cubic centimeters are there in a cubic meter?
2. How do you convert cubic centimeters to cubic meters?
3. How do you convert the other way?

Exercise 2.2.2.3: Filling a Water Cooler

It took Priya 5 minutes to fill a cooler with 8 gallons of water from a faucet that was flowing at a steady rate. Let w be the number of gallons of water in the cooler after t minutes.

1. Which of the following equations represent the relationship between w and t? Select **all** that apply.
   a. $w = 1.6t$
   b. $w = 0.625t$
   c. $t = 1.6w$
   d. $t = 0.625w$
2. What does 1.6 tell you about the situation?
3. What does 0.625 tell you about the situation?
4. Priya changed the rate at which water flowed through the faucet. Write an equation that represents the relationship of w and t when it takes 3 minutes to fill the cooler with 1 gallon of water.
5. Was the cooler filling faster before or after Priya changed the rate of water flow? Explain how you know.

Exercise 2.2.2.4: Feeding Shrimp

At an aquarium, a shrimp is fed $\frac{1}{5}$ gram of food each feeding and is fed 3 times each day.

1. How much food does a shrimp get fed in one day?
2. Complete the table to show how many grams of food the shrimp is fed over different numbers of days.

number of days	food in grams
1	
7	
30	

Table 2.2.2.3



Figure 2.2.2.1

3. What is the constant of proportionality? What does it tell us about the situation?
4. If we switched the columns in the table, what would be the constant of proportionality? Explain your reasoning.
5. Use d for number of days and f for amount of food in grams that a shrimp eats to write *two* equations that represent the relationship between d and f.
6. If a tank has 10 shrimp in it, how much food is added to the tank each day?
7. If the aquarium manager has 300 grams of shrimp food for this tank of 10 shrimp, how many days will it last? Explain or show your reasoning.

Summary

If Kiran rode his bike at a constant 10 miles per hour, his distance in miles, d, is proportional to the number of hours, t, that he rode. We can write the equation $d = 10t$. With this equation, it is easy to find the distance Kiran rode when we know how long it took because we can just multiply the time by 10.

We can rewrite the equation:

$$d = 10t$$
$$\left(\frac{1}{10}\right) d = t$$
$$t = \left(\frac{1}{10}\right) d$$

This version of the equation tells us that the amount of time he rode is proportional to the distance he traveled, and the constant of proportionality is $\frac{1}{10}$. That form is easier to use when we know his distance and want to find how long it took because we can just multiply the distance by $\frac{1}{10}$.

When two quantities x and y are in a proportional relationship, we can write the equation $y = kx$ and say, "y is proportional to x." In this case, the number k is the corresponding constant of proportionality. We can also write the equation $x = \frac{1}{k}y$ and say, "x is proportional to y." In this case, the number $\frac{1}{k}$ is the corresponding constant of proportionality. Each one can be useful depending on the information we have and the quantity we are trying to figure out.

Practice

Exercise 2.2.2.5

The table represents the relationship between a length measured in meters and the same length measured in kilometers.

Table 2.2.2.4

meters	kilometers
1, 000	1
3, 500	
500	
75	
1	
x	

1. Complete the table.
2. Write an equation for converting the number of meters to kilometers. Use x for number of meters and y for number of kilometers.

Exercise 2.2.2.6

Concrete building blocks weigh 28 pounds each. Using b for the number of concrete blocks and w for the weight, write two equations that relate the two variables. One equation should begin with $w =$ and the other should begin with $b =$.

Exercise 2.2.2.7

A store sells rope by the meter. The equation $p = 0.8L$ represents the price p (in dollars) of a piece of nylon rope that is L meters long.

1. How much does the nylon rope cost per meter?
2. How long is a piece of nylon rope that costs \$1.00?

Exercise 2.2.2.8

The table represents a proportional relationship. Find the constant of proportionality and write an equation to represent the relationship.

Table 2.2.2.5

a	y
2	$\frac{2}{3}$
3	1
10	$\frac{10}{3}$
12	4

Constant of proportionality: _____

Equation: $y =$

(From Unit 2.2.1)

Exercise 2.2.2.9

On a map of Chicago, 1 cm represents 100 m. Select **all** statements that express the same scale.

A. 5 cm on the map represents 50 m in Chicago.
B. 1 mm on the map represents 10 m in Chicago.
C. 1 km in Chicago is represented by 10 cm the map.
D. 100 cm in Chicago is represented by 1 m on the map.

(From Unit 1.2.2)

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2.2.3: Using Equations to Solve Problems

Lesson

Let's use equations to solve problems involving proportional relationships.

Exercise 2.2.3.1: Number Talk: Quotients with Decimal Points

Without calculating, order the quotients of these expressions from least to greatest.

$42.6 \div 0.07$

$42.6 \div 70$

$42.6 \div 0.7$

$426 \div 70$

Place the decimal point in the appropriate location in the quotient: $42.6 \div 7 = 608571$

Use this answer to find the quotient of *one* of the previous expressions.

Exercise 2.2.3.2: Concert Ticket Sales

A performer expects to sell 5,000 tickets for an upcoming concert. They want to make a total of $311,000 in sales from these tickets.

1. Assuming that all tickets have the same price, what is the price for one ticket?
2. How much will they make if they sell 7,000 tickets?
3. How much will they make if they sell 10,000 tickets? 50,000? 120,000? a million? x tickets?
4. If they make $404,300, how many tickets have they sold?
5. How many tickets will they have to sell to make $5,000,000?

Exercise 2.2.3.3: Recycling

Aluminum cans can be recycled instead of being thrown in the garbage. The weight of 10 aluminum cans is 0.16 kilograms. The aluminum in 10 cans that are recycled has a value of $0.14.

1. If a family threw away 2.4 kg of aluminum in a month, how many cans did they throw away? Explain or show your reasoning.
2. What would be the recycled value of those same cans? Explain or show your reasoning.
3. Write an equation to represent the number of cans c given their weight w.
4. Write an equation to represent the recycled value r of c cans.
5. Write an equation to represent the recycled value r of w kilograms of aluminum.

Are you ready for more?

The EPA estimated that in 2013, the average amount of garbage produced in the United States was 4.4 pounds per person per day. At that rate, how long would it take your family to produce a ton of garbage? (A ton is 2,000 pounds.)

Summary

Remember that if there is a proportional relationship between two quantities, their relationship can be represented by an equation of the form $y = kx$. Sometimes writing an equation is the easiest way to solve a problem.

For example, we know that Denali, the highest mountain peak in North America, is 20,300 feet above sea level. How many miles is that? There are 5,280 feet in 1 mile. This relationship can be represented by the equation

$f = 5,280m$

where f represents a distance measured in feet and m represents the same distance measured miles. Since we know Denali is 20,310 feet above sea level, we can write

$20,310 = 5,280m$

So $m = \frac{20,310}{5,280}$, which is approximately 3.85 miles.

Practice

Exercise 2.2.3.4

A car is traveling down a highway at a constant speed, described by the equation $d = 65t$, where d represents the distance, in miles, that the car travels at this speed in t hours.

1. What does the 65 tell us in this situation?
2. How many miles does the car travel in 1.5 hours?
3. How long does it take the car to travel 26 miles at this speed?

Exercise 2.2.3.5

Elena has some bottles of water that each holds 17 fluid ounces.

1. Write an equation that relates the number of bottles of water (b) to the total volume of water (w) in fluid ounces.
2. How much water is in 51 bottles?
3. How many bottles does it take to hold 51 fluid ounces of water?

Exercise 2.2.3.6

There are about 1.61 kilometers in 1 mile. Let x represent a distance measured in kilometers and y represent the same distance measured in miles. Write two equations that relate a distance measured in kilometers and the same distance measured in miles.

(From Unit 2.2.2)

Exercise 2.2.3.7

In Canadian coins, 16 quarters is equal in value to 2 toonies.

Table 2.2.3.1

number of quarters	number of toonies
1	
16	2
20	
24	

1. Complete the table.
2. What does the value next to 1 mean in this situation?

(From Unit 2.1.2)

Exercise 2.2.3.8

Each table represents a proportional relationship. For each table:

1. Fill in the missing parts of the table.
2. Draw a circle around the constant of proportionality.

Table 2.2.3.2

x	y

x	y
2	10
	15
7	
1	

Table 2.2.3.3

a	b
12	3
20	
	10
1	

Table 2.2.3.4

m	n
5	3
10	
	18
1	

(From Unit 2.1.2)

Exercise 2.2.3.9

Describe some things you could notice in two polygons that would help you decide that they were not scaled copies.

(From Unit 1.1.4)

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2.3: Comparing Proportional and Nonproportional Relationships

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2.3.1: Comparing Relationships with Tables

Lesson

Let's explore how proportional relationships are different from other relationships.

Exercise 2.3.1.1: Adjusting a Recipe

A lemonade recipe calls for the juice of 5 lemons, 2 cups of water, and 2 tablespoons of honey.

Invent four new versions of this lemonade recipe:

1. One that would make more lemonade but taste the same as the original recipe.
2. One that would make less lemonade but taste the same as the original recipe.
3. One that would have a stronger lemon taste than the original recipe.
4. One that would have a weaker lemon taste than the original recipe.

Exercise 2.3.1.2: Visiting the State Park

Entrance to a state park costs $6 per vehicle, plus $2 per person in the vehicle.

1. How much would it cost for a car with 2 people to enter the park? 4 people? 10 people? Record your answers in the table.

Table 2.3.1.1

number of people in vehicle	total entrance cost in dollars
2	
4	
10	

2. For each row in the table, if each person in the vehicle splits the entrance cost equally, how much will each person pay?
3. How might you determine the entrance cost for a bus with 50 people?
4. Is the relationship between the number of people and the total entrance cost a proportional relationship? Explain how you know.

Are you ready for more?

What equation could you use to find the total entrance cost for a vehicle with any number of people?

Exercise 2.3.1.3: Running Laps

Han and Clare were running laps around the track. The coach recorded their times at the end of laps 2, 4, 6, and 8.

Han's run:

Table 2.3.1.2

distance (laps)	time (minutes)	minutes per lap
2	4	
4	9	
6	15	
8	23	

Clare's run:

Table 2.3.1.3

distance (laps)	time (minutes)	minutes per lap
2	5	
4	10	
6	15	
8	20	

1. Is Han running at a constant pace? Is Clare? How do you know?
2. Write an equation for the relationship between distance and time for anyone who is running at a constant pace.

Summary

Here are the prices for some smoothies at two different smoothie shops:

Smoothie Shop A

Table 2.3.1.4

smoothie size (oz)	price ($)	dollars per ounce
8	6	0.75
12	9	0.75
16	12	0.75
s	$0.75s$	0.75

Smoothie Shop B

Table 2.3.1.5

smoothie size (oz)	prize ($)	dollars per ounce
8	6	0.75
12	8	0.67
16	10	0.625
s	???	???

For Smoothie Shop A, smoothies cost \$0.75 per ounce no matter which size we buy. There could be a proportional relationship between smoothie size and the price of the smoothie. An equation representing this relationship is $p = 0.75s$ where s represents size in ounces and p represents price in dollars. (The relationship could still not be proportional, if there were a different size on the menu that did not have the same price per ounce.)

For Smoothie Shop B, the cost per ounce is different for each size. Here the relationship between smoothie size and price is definitely *not* proportional.

In general, two quantities in a proportional relationship will always have the same quotient. When we see some values for two related quantities in a table and we get the same quotient when we divide them, that means they might be in a proportional relationship—but if we can't see all of the possible pairs, we can't be completely sure. However, if we know the relationship can be represented by an equation is of the form $y = kx$, then we are sure it is proportional.

Practice

Exercise 2.3.1.4

Decide whether each table could represent a proportional relationship. If the relationship could be proportional, what would the constant of proportionality be?

1. How loud a sound is depending on how far away you are.

Table 2.3.1.6

distance to listener (ft)	sound level (dB)
5	85
10	79
20	73
40	67

2. The cost of fountain drinks at Hot Dog Hut.

Table 2.3.1.7

volume (fluid ounces)	cost ($)
16	$1.49
20	$1.59
30	$1.89

Exercise 2.3.1.5

A taxi service charges \$1.00 for the first $\frac{1}{10}$ mile then \$0.10 for each additional $\frac{1}{10}$ mile after that.

Fill in the table with the missing information then determine if this relationship between distance traveled and price of the trip is a proportional relationship.

Table 2.3.1.8

distance traveled (mi)	price (dollars)
$\frac{9}{10}$	
2	
$3\frac{1}{10}$	
10	

Exercise 2.3.1.6

A rabbit and turtle are in a race. Is the relationship between distance traveled and time proportional for either one? If so, write an equation that represents the relationship.

Turtle's run:

Table 2.3.1.9

distance (meters)	time (minutes)
108	2
405	7.5
540	10

distance (meters)	time (minutes)
1, 768.5	32.75

Rabbit's run:

Table 2.3.1.10

distance (meters)	time (minutes)
800	1
900	5
1, 107.5	20
1, 524	32.5

Exercise 2.3.1.7

For each table, answer: What is the constant of proportionality?

Table 2.3.1.11

a	b
2	14
5	35
9	63
$\frac{1}{3}$	$\frac{7}{3}$

Table 2.3.1.12

a	b
3	60
5	600
8	960
12	1440

Table 2.3.1.13

a	b
75	3
200	8
1525	61
10	0.4

Table 2.3.1.14

a	b
4	10
6	15

a	b
22	55
3	$7\frac{1}{2}$

(From Unit 2.1.2)

Exercise 2.3.1.8

Kiran and Mai are standing at one corner of a rectangular field of grass looking at the diagonally opposite corner. Kiran says that if the the field were twice as long and twice as wide, then it would be twice the distance to the far corner. Mai says that it would be more than twice as far, since the diagonal is even longer than the side lengths. Do you agree with either of them?

(From Unit 1.1.4)

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2.3.2: Comparing Relationships with Equations

Lesson

Let's develop methods for deciding if a relationship is proportional.

Exercise 2.3.2.1: Notice and Wonder: Patterns with Rectangles

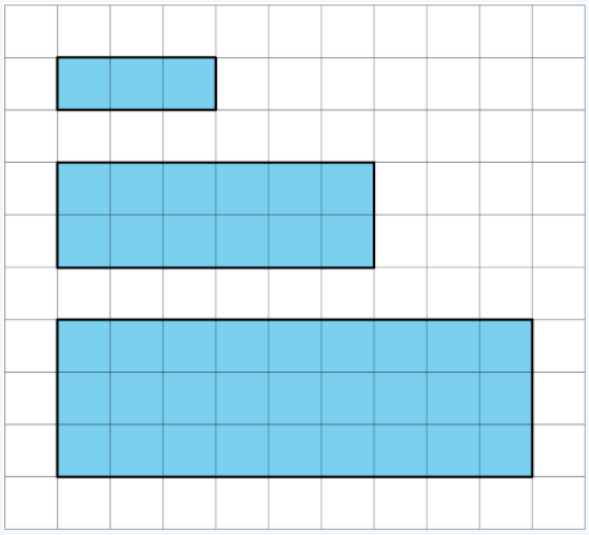


Figure 2.3.2.1

Do you see a pattern? What predictions can you make about future rectangles in the set if your pattern continues?

Exercise 2.3.2.2: More Conversions

The other day you worked with converting meters, centimeters, and millimeters. Here are some more unit conversions.

1. Use the equation $F = \frac{9}{5}C + 32$, where F represents degrees Fahrenheit and C represents degrees Celsius, to complete the table.

Table 2.3.2.1

temperature (°C)	temperature (°F)
20	
4	
175	

2. Use the equation $c = 2.54n$, where c represents the length in centimeters and n represents the length in inches, to complete the table.

Table 2.3.2.2

length (in)	length (cm)
10	
8	
$3\frac{1}{2}$	

3. Are these proportional relationships? Explain why or why not.

Exercise 2.3.2.3: Total Edge Length, Surface Area, and Volume

Here are some cubes with different side lengths. Complete each table. Be prepared to explain your reasoning.

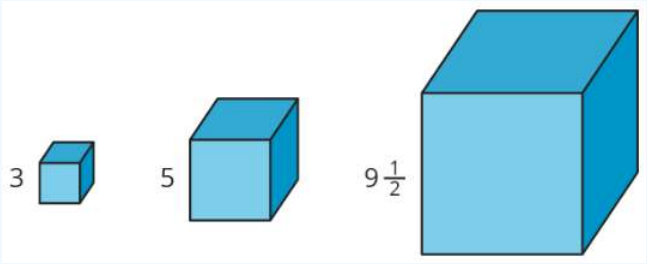


Figure 2.3.2.2

1. How long is the total edge length of each cube?

Table 2.3.2.3

side length	total edge length
3	
5	
$9\frac{1}{2}$	
s	

2. What is the surface area of each cube?

Table 2.3.2.4

side length	surface area
3	
5	
$9\frac{1}{2}$	
s	

3. What is the volume of each cube?

Table 2.3.2.5

side length	volume
3	
5	
$9\frac{1}{2}$	
s	

4. Which of these relationships is proportional? Explain how you know.
5. Write equations for the total edge length E, total surface area A, and volume V of a cube with side length s.

Are you ready for more?

1. A rectangular solid has a square base with side length l, height 8, and volume V. Is the relationship between l and V a proportional relationship?
2. A different rectangular solid has length l, width 10, height 5, and volume V. Is the relationship between l and V a proportional relationship?
3. Why is the relationship between the side length and the volume proportional in one situation and not the other?

Exercise 2.3.2.4: All Kinds of Equations

Here are six different equations.

$y = 4 + x$ $\quad y = 4x$ $\quad y = \frac{4}{x}$

$y = \frac{x}{4}$ $\quad y = 4^x$ $\quad y = x^4$

$y = 4 + x$

x	y	$\frac{y}{x}$
2		
3		
4		
5		

$y = 4x$

x	y	$\frac{y}{x}$
2		
3		
4		
5		

$y = \frac{4}{x}$

x	y	$\frac{y}{x}$
2		
3		
4		
5		

$y = \frac{x}{4}$

x	y	$\frac{y}{x}$
2		
3		
4		
5		

$y = 4^x$

x	y	$\frac{y}{x}$
2		
3		
4		
5		

$y = x^4$

x	y	$\frac{y}{x}$
2		
3		
4		
5		

Figure 2.3.2.3

1. Predict which of these equations represent a proportional relationship.
2. Complete each table using the equation that represents the relationship.
3. Do these results change your answer to the first question? Explain your reasoning.
4. What do the equations of the proportional relationships have in common?

Summary

If two quantities are in a proportional relationship, then their quotient is always the same. This table represents different values of a and b, two quantities that are in a proportional relationship.

Table 2.3.2.6

a	b	$\frac{b}{a}$
20	100	5
3	15	5
11	55	5
1	5	5

Notice that the quotient of b and a is always 5. To write this as an equation, we could say $\frac{b}{a} = 5$. If this is true, then $b = 5a$. (This doesn't work if $a = 0$, but it works otherwise.)

If quantity y is proportional to quantity x, we will always see this pattern: $\frac{y}{x}$ will always have the same value. This value is the constant of proportionality, which we often refer to as k. We can represent this relationship with the equation $\frac{y}{x} = k$ (as long as x is not 0) or $y = kx$.

Note that if an equation cannot be written in this form, then it does not represent a proportional relationship.

Practice

Exercise 2.3.2.5

The relationship between a distance in yards (y) and the same distance in miles (m) is described by the equation $y = 1760m$.

1. Find measurements in yards and miles for distances by completing the table.

Table 2.3.2.7

distance measured in miles	distance measured in yards
1	
5	
	3, 520
	17, 600

2. Is there a proportional relationship between a measurement in yards and a measurement in miles for the same distance? Explain why or why not.

Exercise 2.3.2.6

Decide whether or not each equation represents a proportional relationship.

1. The remaining length (L) of 120-inch rope after x inches have been cut off: $120 - x = L$
2. The total cost (t) after 8% sales tax is added to an item's price (p): $1.08p = t$
3. The number of marbles each sister gets (x) when m marbles are shared equally among four sisters: $x = \frac{m}{4}$
4. The volume (V) of a rectangular prism whose height is 12 cm and base is a square with side lengths s cm: $V = 12s^2$

Exercise 2.3.2.7

1. Use the equation $y = \frac{5}{2}x$ to complete the table.

Is y proportional to x and y? Explain why or why not.

Table 2.3.2.8

x	y
2	
3	
6	

2. Use the equation $y = 3.2x + 5$ to complete the table.

Is y proportional to x and y? Explain why or why not.

Table 2.3.2.9

x	y
1	
2	
4	

Exercise 2.3.2.8

To transmit information on the internet, large files are broken into packets of smaller sizes. Each packet has 1,500 bytes of information. An equation relating packets to bytes of information is given by $b = 1,500p$ where p represents the number of packets and b represents the number of bytes of information.

1. How many packets would be needed to transmit 30,000 bytes of information?
2. How much information could be transmitted in 30,000 packets?
3. Each byte contains 8 bits of information. Write an equation to represent the relationship between the number of packets and the number of bits.

(From Unit 2.2.3)

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2.3.3: Solving Problems about Proportional Relationships

Lesson

Let's solve problems about proportional relationships.

Exercise 2.3.3.1: What Do You Want to Know?

Consider the problem: A person is running a distance race at a constant rate. What time will they finish the race?

What information would you need to be able to solve the problem?

Exercise 2.3.3.2: Info Gap: Biking and Rain

Your teacher will give you either a *problem card* or a *data card*. Do not show or read your card to your partner.

If your teacher gives you the *problem card*:

1. Silently read your card and think about what information you need to be able to answer the question.
2. Ask your partner for the specific information that you need.
3. Explain how you are using the information to solve the problem.
   Continue to ask questions until you have enough information to solve the problem.
4. Share the *problem card* and solve the problem independently.
5. Read the *data card* and discuss your reasoning.

If your teacher gives you the *data card*:

1. Silently read your card.
2. Ask your partner "*What specific information do you need?*" and wait for them to *ask* for information.
   If your partner asks for information that is not on the card, do not do the calculations for them. Tell them you don't have that information.
3. Before sharing the information, ask "*Why do you need that information?*" Listen to your partner's reasoning and ask clarifying questions.
4. Read the *problem card* and solve the problem independently.
5. Share the *data card* and discuss your reasoning.

Pause here so your teacher can review your work. Ask your teacher for a new set of cards and repeat the activity, trading roles with your partner.

Exercise 2.3.3.3: Moderating Comments

A company is hiring people to read through all the comments posted on their website to make sure they are appropriate. Four people applied for the job and were given one day to show how quickly they could check comments.

- Person 1 worked for 210 minutes and checked a total of 50,000 comments.
- Person 2 worked for 200 minutes and checked 1,325 comments every 5 minutes.
- Person 3 worked for 120 minutes, at a rate represented by $c = 331t$,
  where is the number of comments checked and is the time in minutes.
- Person 4 worked for 150 minutes, at a rate represented by $t = \left(\frac{3}{800}\right) c$.

1. Order the people from greatest to least in terms of total number of comments checked.
2. Order the people from greatest to least in terms of how fast they checked the comments.

Are you ready for more?

1. Write equations for each job applicant that allow you to easily decide who is working the fastest.
2. Make a table that allows you to easily compare how many comments the four job applicants can check.

Summary

Whenever we have a situation involving constant rates, we are likely to have a proportional relationship between quantities of interest.

- When a bird is flying at a constant speed, then there is a proportional relationship between the flying time and distance flown.
- If water is filling a tub at a constant rate, then there is a proportional relationship between the amount of water in the tub and the time the tub has been filling up.
- If an aardvark is eating termites at a constant rate, then there is proportional relationship between the number of termites the aardvark has eaten and the time since it started eating.

Sometimes we are presented with a situation, and it is not so clear whether a proportional relationship is a good model. How can we decide if a proportional relationship is a good representation of a particular situation?

- If you aren't sure where to start, look at the quotients of corresponding values. If they are not always the same, then the relationship is definitely not a proportional relationship.
- If you can see that there is a single value that we always multiply one quantity by to get the other quantity, it is definitely a proportional relationship.

After establishing that it is a proportional relationship, setting up an equation is often the most efficient way to solve problems related to the situation.

Practice

Exercise 2.3.3.4

For each situation, explain whether you think the relationship is proportional or not. Explain your reasoning.

1. The weight of a stack of standard 8.5x11 copier paper vs. number of sheets of paper.
2. The weight of a stack of different-sized books vs. the number of books in the stack.



Figure 2.3.3.1: Paper ream , by Bluesnap. Public Domain. Pixabay. Source.



Figure 2.3.3.2: Stack of books, by Hermann. Public Domain. Pixabay. Source.

Exercise 2.3.3.5

Every package of a certain toy also includes 2 batteries.

1. Are the number of toys and number of batteries in a proportional relationship? If so, what are the two constants of proportionality? If not, explain your reasoning.
2. Use t for the number of toys and b for the number of batteries to write two equations relating the two variables.

$$b = \qquad\qquad t =$$

Exercise 2.3.3.6

Lin and her brother were born on the same date in different years. Lin was 5 years old when her brother was 2.

1. Find their ages in different years by filling in the table.

Table 2.3.3.1

Lin's age	Her brother's age
5	2
6	
15	
	25

2. Is there a proportional relationship between Lin's age and her brother's age? Explain your reasoning.

Exercise 2.3.3.7

A student argues that $y = \frac{x}{9}$ does not represent a proportional relationship between x and y because we need to multiply one variable by the same constant to get the other one and not divide it by a constant. Do you agree or disagree with this student?

(From Unit 2.3.2)

Exercise 2.3.3.8

Quadrilateral A has side lengths 3, 4, 5, and 6. Quadrilateral B is a scaled copy of Quadrilateral A with a scale factor of 2. Select **all** of the following that are side lengths of Quadrilateral B.

A. 5
B. 6
C. 7
D. 8
E. 9

(From Unit 1.1.3)

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2.4: Representing Proportional Relationships with Graphs

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2.4.1: Introducing Graphs of Proportional Relationships

Lesson

Let's see how graphs of proportional relationships differ from graphs of other relationships.

Exercise 2.4.1.1: Notice These Points

1. Plot the points $(0, 10), (1, 8), (2, 6), (3, 4), (4, 2)$.
2. What do you notice about the graph?

Exercise 2.4.1.2: T-shirts for Sale

Some T-shirts cost $8 each.

Table 2.4.1.1

x	y
1	8
2	16
3	24
4	32
5	40
6	48

1. Use the table to answer these questions.
   a. What does x represent?
   b. What does y represent?
   c. Is there a proportional relationship between x and y?
2. Plot the pairs in the table on the coordinate plane.
3. What do you notice about the graph?

Exercise 2.4.1.3: Matching Tables and Graphs

Your teacher will give you papers showing tables and graphs.

1. Examine the graphs closely. What is the same and what is different about the graphs?
2. Sort the graphs into categories of your choosing. Label each category. Be prepared to explain why you sorted the graphs the way you did.
3. Take turns with a partner to match a table with a graph.
   a. For each match you find, explain to your partner how you know it is a match.
   b. For each match your partner finds, listen carefully to their explanation. If you disagree, work to reach an agreement. Pause here so your teacher can review your work.
4. Trade places with another group. How are their categories the same as your group's categories? How are they different?
5. Return to your original place. Discuss any changes you may wish to make to your categories based on what the other group did.
6. Which of the relationships are proportional?
7. What have you noticed about the graphs of proportional relationships? Do you think this will hold true for *all* graphs of proportional relationships?

Are you ready for more?

1. All the graphs in this activity show points where both coordinates are positive. Would it make sense for any of them to have one or more coordinates that are negative?
2. The equation of a proportional relationship is of the form $y = kx$, where k is a positive number, and the graph is a line through $(0, 0)$. What would the graph look like if k were a negative number?

Summary

One way to represent a proportional relationship is with a graph. Here is a graph that represents different amounts that fit the situation, "Blueberries cost $6 per pound."

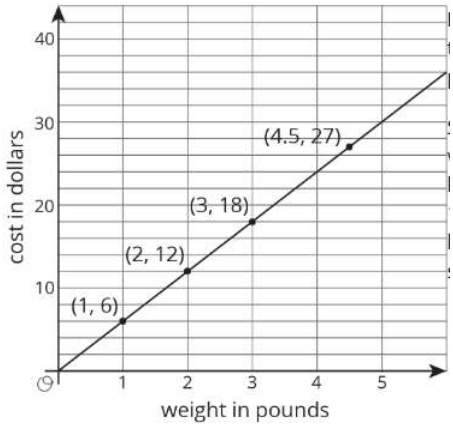


Figure 2.4.1.1

Different points on the graph tell us, for example, that 2 pounds of blueberries cost $12, and 4.5 pounds of blueberries cost $27.

Sometimes it makes sense to connect the points with a line, and sometimes it doesn't. We could buy, for example, 4.5 pounds of blueberries or 1.875 pounds of blueberries, so all the points in between the whole numbers make sense in the situation, so any point on the line is meaningful.

If the graph represented the cost for different *numbers of sandwiches* (instead of pounds of blueberries), it might not make sense to connect the points with a line, because it is often not possible to buy 4.5 sandwiches or 1.875 sandwiches. Even if only points make sense in the situation, though, sometimes we connect them with a line anyway to make the relationship easier to see.

Graphs that represent proportional relationships all have a few things in common:

- Points that satisfy the relationship lie on a straight line.
- The line that they lie on passes through the **origin**, $(0, 0)$.

Here are some graphs that do *not* represent proportional relationships:

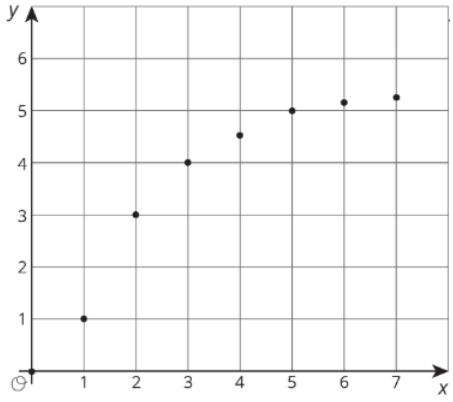


Figure 2.4.1.2

These points do not lie on a line.

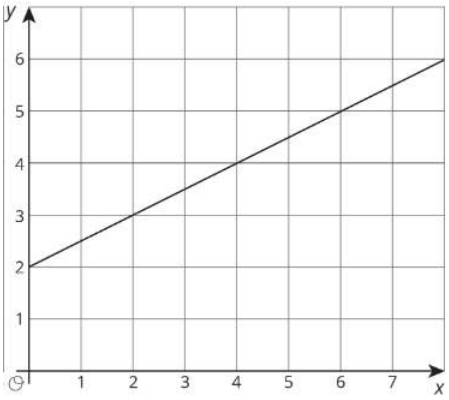


Figure 2.4.1.3

This is a line, but it doesn't go through the origin.

Glossary Entries

Definition: Coordinate Plane

The coordinate plane is a system for telling where points are. For example. point R is located at $(3, 2)$ on the coordinate plane, because it is three units to the right and two units up.

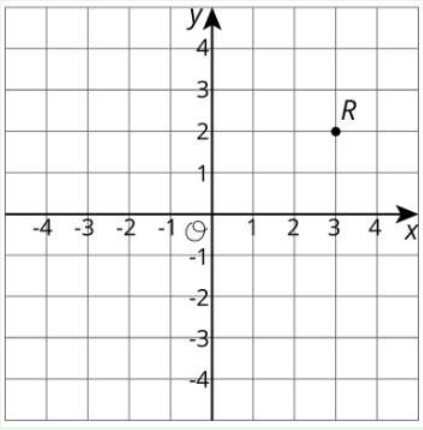


Figure 2.4.1.4

Definition: Origin

The origin is the point $(0, 0)$ in the coordinate plane. This is where the horizontal axis and the vertical axis cross.

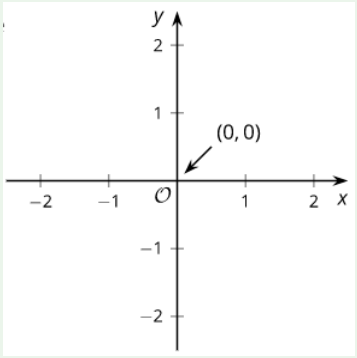


Figure 2.4.1.5

Practice

Exercise 2.4.1.4

Which graphs could represent a proportional relationship?

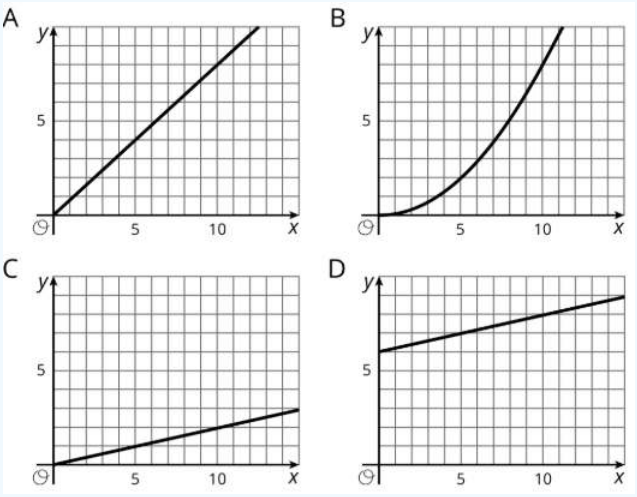


Figure 2.4.1.6

A. A
B. B
C. C
D. D

Exercise 2.4.1.5

A lemonade recipe calls for $\frac{1}{4}$ cup of lemon juice for every cup of water.

1. Use the table to answer these questions.
   a. What does x represent?
   b. What does y represent?
   c. Is there a proportional relationship between x and y?
2. Plot the pairs in the table in a coordinate plane.

Table 2.4.1.2

x	y
1	$\frac{1}{4}$
2	$\frac{1}{2}$
3	$\frac{3}{4}$
4	1

Exercise 2.4.1.6

Select **all** the pieces of information that would tell you x and y have a proportional relationship. Let y represent the distance in meters between a rock and a turtle's current position and x represent the time in minutes the turtle has been moving.

A. $y = 3x$
B. After 4 minutes, the turtle has walked 12 feet away from the rock.
C. The turtle walks for a bit, then stops for a minute before walking again.
D. The turtle walks away from the rock at a constant rate.

(From Unit 2.3.3)

Exercise 2.4.1.7

Decide whether each table could represent a proportional relationship. If the relationship could be proportional, what would be the constant of proportionality?

1. The sizes you can print a photo.

Table 2.4.1.3

width of photo (inches)	height of photo (inches)
2	3
4	6
5	7
8	10

2. The distance from which a lighthouse is visible.

Table 2.4.1.4

height of a lighthouse (feet)	distance it can be seen (miles)
20	6
45	9
70	11
95	13

(From Unit 2.3.1)

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2.4.2: Interpreting Graphs of Proportional Relationships

Lesson

Let's read stories from the graphs of proportional relationships.

Exercise 2.4.2.1: What Could the Graph Represent?

Here is a graph that represents a proportional relationship.

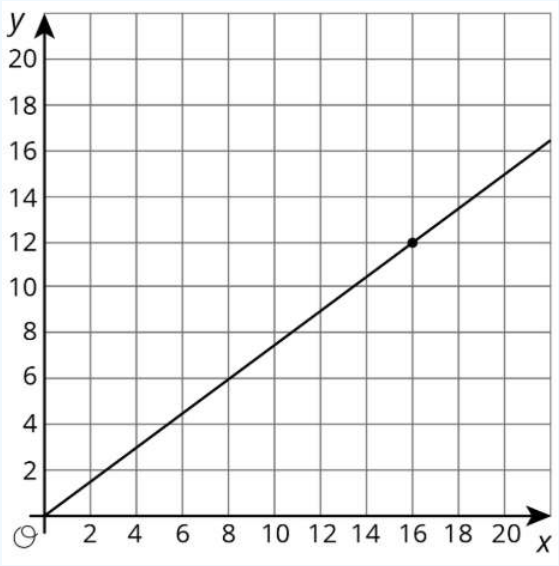


Figure 2.4.2.1

1. Invent a situation that could be represented by this graph.
2. Label the axes with the quantities in your situation.
3. Give the graph a title.
4. There is a point on the graph. What are its coordinates? What does it represent in your situation?

Exercise 2.4.2.2: Tyler's Walk

Tyler was at the amusement park. He walked at a steady pace from the ticket booth to the bumper cars.

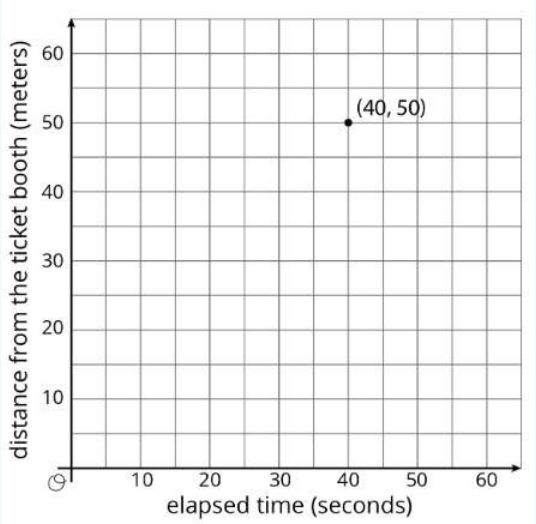


Figure 2.4.2.2

1. The point on the graph shows his arrival at the bumper cars. What do the coordinates of the point tell us about the situation?

2. The table representing Tyler's walk shows other values of time and distance. Complete the table. Next, plot the pairs of values on the grid.
3. What does the point $(0, 0)$ mean in this situation?
4. How far away from the ticket booth was Tyler after 1 second? Label the point on the graph that shows this information with its coordinates.
5. What is the constant of proportionality for the relationship between time and distance? What does it tell you about Tyler's walk? Where do you see it in the graph?

Table 2.4.2.1

time (seconds)	distance (meters)
0	0
20	25
30	37.5
40	50
1	

Are you ready for more?

If Tyler wanted to get to the bumper cars in half the time, how would the graph representing his walk change? How would the table change? What about the constant of proportionality?

Exercise 2.4.2.3: Seagulls Eat What?

4 seagulls ate 10 pounds of garbage. Assume this information describes a proportional relationship.

1. Plot a point that shows the number of seagulls and the amount of garbage they ate.
2. Use a straight edge to draw a line through this point and $(0, 0)$.
3. Plot the point $(1, k)$ on the line. What is the value of k? What does the value of k tell you about this context?

Summary

For the relationship represented in this table, y is proportional to x. We can see in the table that $\frac{5}{4}$ is the constant of proportionality because it's the y value when x is 1.

The equation $y = \frac{5}{4}x$ also represents this relationship.

Table 2.4.2.2

x	y
4	5
5	$\frac{25}{4}$
8	10
1	$\frac{5}{4}$

Here is the graph of this relationship.

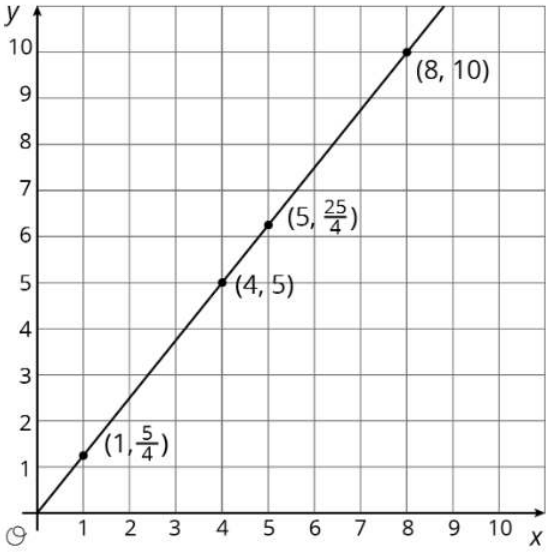


Figure 2.4.2.3

If y represents the distance in feet that a snail crawls in x minutes, then the point $(4, 5)$ tells us that the snail can crawl 5 feet in 4 minutes.

If y represents the cups of yogurt and x represents the teaspoons of cinnamon in a recipe for fruit dip, then the point $(4, 5)$ tells us that you can mix 4 teaspoons of cinnamon with 5 cups of yogurt to make this fruit dip.

We can find the constant of proportionality by looking at the graph, because $\frac{5}{4}$ is the y-coordinate of the point on the graph where the x-coordinate is 1. This could mean the snail is traveling $\frac{5}{4}$ feet per minute or that the recipe calls for $1\frac{1}{4}$ cups of yogurt for every teaspoon of cinnamon.

In general, when y is proportional to x, the corresponding constant of proportionality is the y-value when $x = 1$.

Glossary Entries

Definition: Coordinate Plane

The coordinate plane is a system for telling where points are. For example. point R is located at $(3, 2)$ on the coordinate plane, because it is three units to the right and two units up.

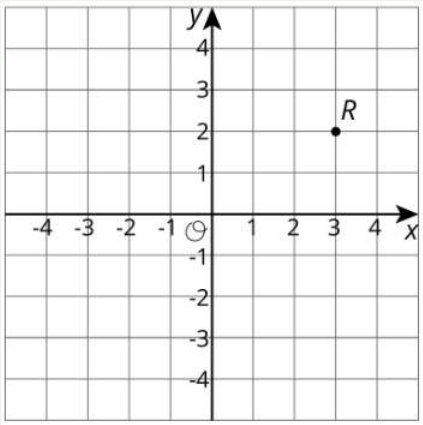


Figure 2.4.2.4

Definition: Origin

The origin is the point $(0, 0)$ in the coordinate plane. This is where the horizontal axis and the vertical axis cross.

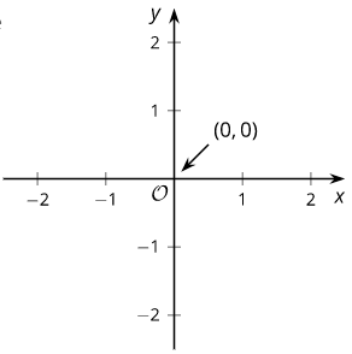


Figure 2.4.2.5

Practice

Exercise 2.4.2.4

There is a proportional relationship between the number of months a person has had a streaming movie subscription and the total amount of money they have paid for the subscription. The cost for 6 months is \$47.94. The point $(6, 47.94)$is shown on the graph below.

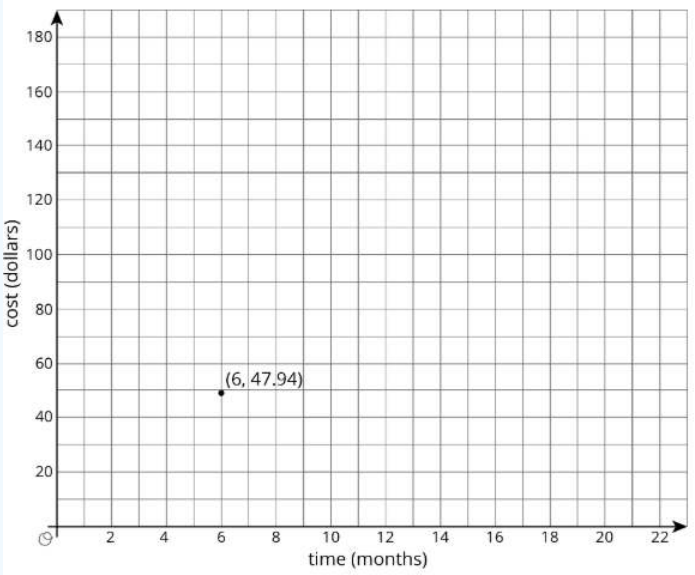


Figure 2.4.2.6

1. What is the constant of proportionality in this relationship?
2. What does the constant of proportionality tell us about the situation?
3. Add at least three more points to the graph and label them with their coordinates.
4. Write an equation that represents the relationship between C, the total cost of the subscription, and m, the number of months.

Exercise 2.4.2.5

The graph shows the amounts of almonds, in grams, for different amounts of oats, in cups, in a granola mix. Label the point $(1, k)$ on the graph, find the value of k, and explain its meaning.

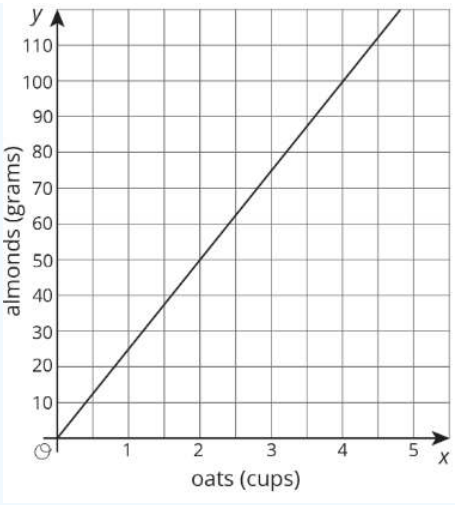


Figure 2.4.2.7

Exercise 2.4.2.6

To make a friendship bracelet, some long strings are lined up then taking one string and tying it in a knot with each of the other strings to create a row of knots. A new string is chosen and knotted with the all the other strings to create a second row. This process is repeated until there are enough rows to make a bracelet to fit around your friend's wrist.

Are the number of knots proportional to the number of rows? Explain your reasoning.

(From Unit 2.3.3)

Exercise 2.4.2.7

What information do you need to know to write an equation relating two quantities that have a proportional relationship?

(From Unit 2.3.3)

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2.4.3: Using Graphs to Compare Relationships

Lesson

Let's graph more than one relationship on the same grid.

Exercise 2.4.3.1: Number Talk: Fraction Multiplication and Division

Find each product or quotient mentally.

$\frac{2}{3} \cdot \frac{1}{2}$

$\frac{4}{3} \cdot \frac{1}{4}$

$4 \div \frac{1}{5}$

$\frac{9}{6} \div \frac{1}{2}$

Exercise 2.4.3.2: Race to the Bumper Cars

Diego, Lin, and Mai went from the ticket booth to the bumper cars.

1. Use each description to complete the table representing that person's journey.

- Diego left the ticket booth at the same time as Tyler. Diego jogged ahead at a steady pace and reached the bumper cars in 30 seconds.
- Lin left the ticket booth at the same time as Tyler. She ran at a steady pace and arrived at the bumper cars in 20 seconds.
- Mai left the booth 10 seconds later than Tyler. Her steady jog enabled her to catch up with Tyler just as he arrived at the bumper cars.

Table 2.4.3.1

Diego's time (seconds)	Diego's distance (meters)
0	
15	
30	50
1	

Table 2.4.3.2

Lim's time (seconds)	Lin's distance (meters)
	0
	25
20	50
1	

Table 2.4.3.3

Mai's time (seconds)	Mai's distance (meters)
	0
	25
40	50
1	

2. Using a different color for each person, draw a graph of all four people's journeys (including Tyler's from the other day).

- Drag the names to the correct lines to label them.
- If you choose to, you can use the Paint Brush tool to change the color of each line. Select the tool, click on a color in the palette below the graph, and then click on a line. Click on the Move tool (the arrow) before changing to a new paint brush color.
- You can hide any points you create with the checkbox below the graph.

3. Which person is moving the most quickly? How is that reflected in the graph?

Are you ready for more?

Write equations to represent each person's relationship between time and distance.

Exercise 2.4.3.3: Space Rocks and the Price of Rope

1. Meteoroid Perseid 245 and Asteroid x travel through the solar system. Explore the applet to learn about the distance they had each traveled after a given time.

Is Asteroid x traveling faster or slower than Perseid 245? Explain how you know.

2. The graph shows the price of different lengths of two types of rope.

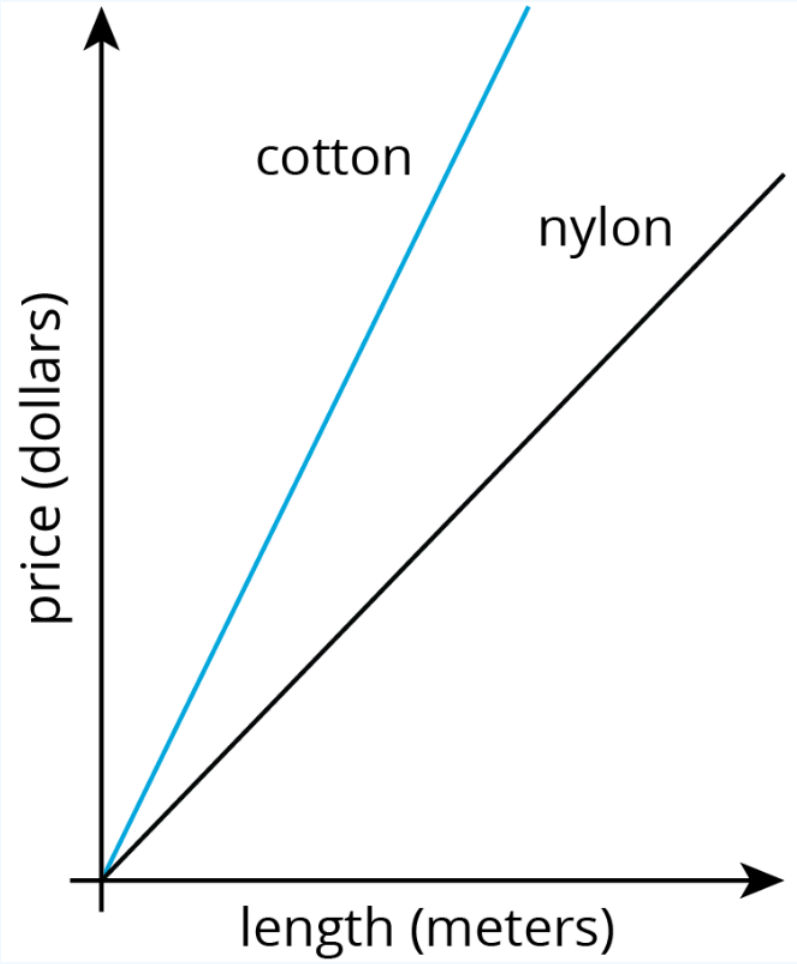


Figure 2.4.3.1: Two lines on an x y plane. The vertical axis is labeled price, dollars. The horizontal axis is labeled length, meters. A line, slanting up and to the right is labeled cotton. It goes through the origin. Another line, below the cotton line, is slanting up and to the right, through the origin, is labeled nylon.

If you buy $1.00 of each kind of rope, which one will be longer? Explain how you know.

Summary

Here is a graph that shows the price of blueberries at two different stores. Which store has a better price?

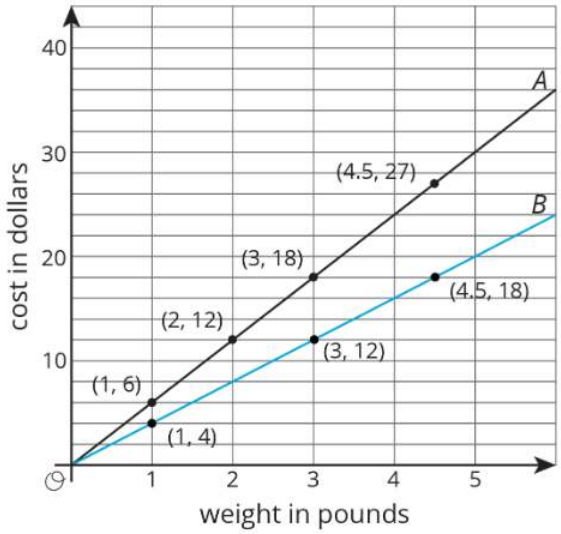


Figure 2.4.3.2

We can compare points that have the same x value or the same y value. For example, the points $(2, 12)$ and $(3, 12)$ tell us that at store B you can get more pounds of blueberries for the same price.

The points $(3, 12)$ and $(3, 18)$ tell us that at store A you have to pay more for the same quantity of blueberries. This means store B has the better price.

We can also use the graphs to compare the constants of proportionality. The line representing store B goes through the point $(1, 4)$, so the constant of proportionality is 4. This tells us that at store B the blueberries cost \$4 per pound. This is cheaper than the \$6 per pound unit price at store A.

Glossary Entries

Definition: Coordinate Plane

The coordinate plane is a system for telling where points are. For example. point R is located at $(3, 2)$ on the coordinate plane, because it is three units to the right and two units up.

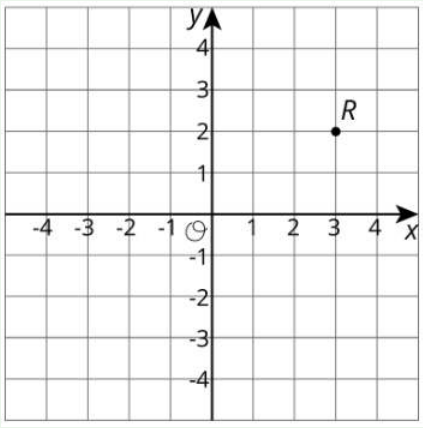


Figure 2.4.3.3

Definition: Origin

The origin is the point $(0, 0)$ in the coordinate plane. This is where the horizontal axis and the vertical axis cross.

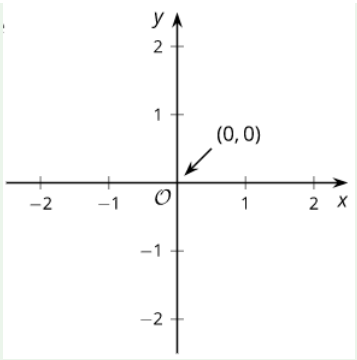


Figure 2.4.3.4

Practice

Exercise 2.4.3.4

The graphs below show some data from a coffee shop menu. One of the graphs shows cost (in dollars) vs. drink volume (in ounces), and one of the graphs shows calories vs. drink volume (in ounces).

_____ vs volume

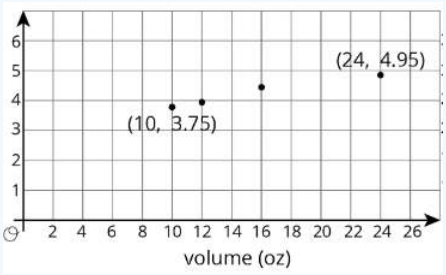


Figure 2.4.3.5

_____ vs volume

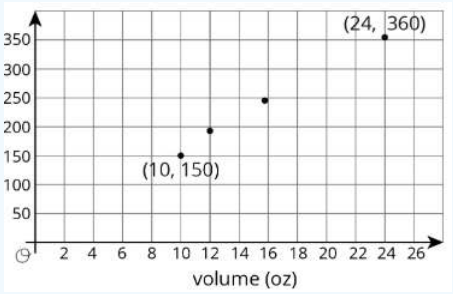


Figure 2.4.3.6

1. Which graph is which? Give them the correct titles.
2. Which quantities appear to be in a proportional relationship? Explain how you know.
3. For the proportional relationship, find the constant of proportionality. What does that number mean?

Exercise 2.4.3.5

Lin and Andre biked home from school at a steady pace. Lin biked 1.5 km and it took her 5 minutes. Andre biked 2 km and it took him 8 minutes.

1. Draw a graph with two lines that represent the bike rides of Lin and Andre.
2. For each line, highlight the point with coordinates $(1, k)$ and find k.
3. Who was biking faster?

Exercise 2.4.3.6

Match each equation to its graph.

1. $y = 2x$
2. $y = \frac{4}{5}x$
3. $y = \frac{1}{4}x$
4. $y = \frac{2}{3}x$
5. $y = \frac{4}{3}x$
6. $y = \frac{3}{2}x$

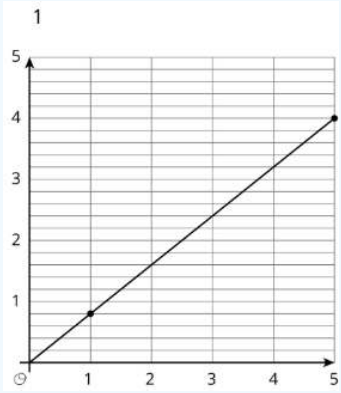


Figure 2.4.3.7

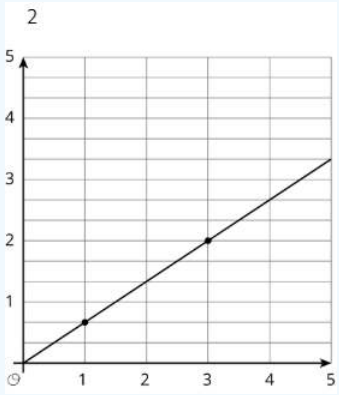


Figure 2.4.3.8

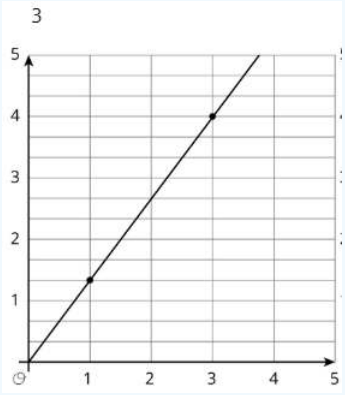


Figure 2.4.3.9

4

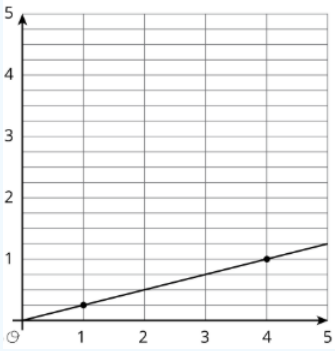


Figure 2.4.3.10

5

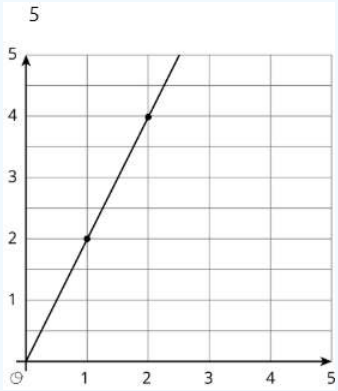


Figure 2.4.3.11

6

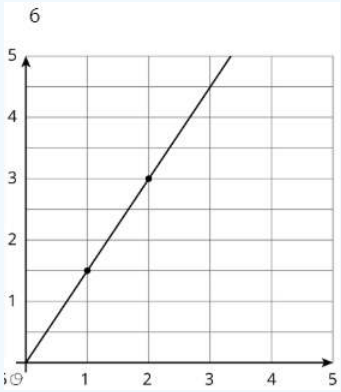


Figure 2.4.3.12

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2.4.4: Two Graphs for Each Relationship

Lesson

Let's use tables, equations, and graphs to answer questions about proportional relationships.

Exercise 2.4.4.1: True or False: Fractions and Decimals

Decice whether each equation is true or false. Be prepared to explain your reasoning.

1. $\frac{3}{2} \cdot 16 = 3 \cdot 8$
2. $\frac{3}{4} \div \frac{1}{2} = \frac{6}{4} \div \frac{1}{4}$
3. $(2.8) \cdot (13) = (0.7) \cdot (52)$

Exercise 2.4.4.2: Tables, Graphs, and Equations

Explore the graph. Start by dragging the gray bar on the left across the screen until you can see both the table and the graph. Notice the values in the table and the coordinates of the labeled point. Grab the point and move it around.

- What stays the same and what changes in the table? in the equation? on the graph?
- Choose one row in the table and write it here. To what does this row correspond on the graph?

Table 2.4.4.1

x	y

Grab and drag the point until you see the equation $y = \frac{3}{2}x$.

- Do not move the point. Choose three rows from the table, other than the origin. Record x and y, and compute $\frac{y}{x}$.

Table 2.4.4.2

x	y	$\frac{y}{x}$

- What do you notice? What does this have to do with the equation of the line?
- Do not move the point. Check the box to view the coordinates $(1, ?)$. What are the coordinates of this point? What does this correspond to in the table? What does this correspond to in the equation?
- Drag the point to a different location. Record the equation of the line, the coordinates of three points, and the value of $\frac{y}{x}$.

Equation of the line: _____

Table 2.4.4.3

x	y	$\frac{y}{x}$

- Based on your observations, summarize any connections you see between the table, characteristics of the graph, and the equation.

Are you ready for more?

The graph of an equation of the form $y = kx$, where k is a positive number, is a line through $(0, 0)$ and the point $(1, k)$.

1. Name at least one line through $(0, 0)$ that cannot be represented by an equation like this.
2. If you could draw the graphs of *all* of the equations of this form in the same coordinate plane, what would it look like?

Exercise 2.4.4.3: Hot Dog Eating Contest

Andre and Jada were in a hot dog eating contest. Andre ate 10 hot dogs in 3 minutes. Jada ate 12 hot dogs in 5 minutes.

1. The points shown on the *first* set of axes display information about Andre's and Jada's consumption. Which point indicates Andre's consumption? Which indicates Jada's consumption? Label them.
2. Draw two lines: one through the origin and Andre's point, and one through the origin and Jada's point. Write an equation for each line. Use to represent time in minutes, and to represent number of hot dogs.
3. For each equation, what does the constant of proportionality tell you?
4. The points shown on the second set of axes display information about Andre's and Jada's consumption. Which point indicates Andre's consumption? Which indicates Jada's consumption? Label them.
5. Draw lines from the origin through each of the two points. Write an equation for each line. What does the constant of proportionality tell you in each case?

Summary

Imagine that a faucet is leaking at a constant rate and that every 2 minutes, 10 milliliters of water leaks from the faucet. There is a proportional relationship between the volume of water and elapsed time.

- We could say that the elapsed time is proportional to the volume of water. The corresponding constant of proportionality tells us that the faucet is leaking at a rate of $\frac{1}{5}$ of a minute per milliliter.
- We could say that the volume of water is proportional to the elapsed time. The corresponding constant of proportionality tells us that the faucet is leaking at a rate of 5 milliliters per minute.

Let's use v to represent volume in milliliters and t to represent time in minutes. Here are graphs and equations that represent both ways of thinking about this relationship:

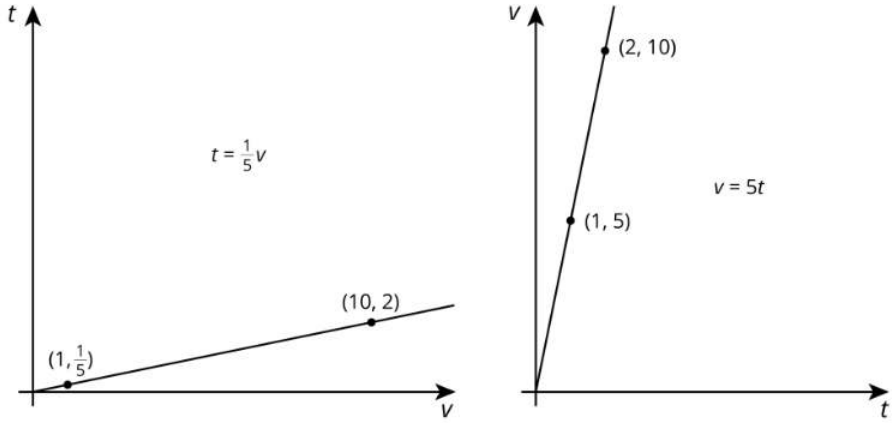


Figure 2.4.4.1: Two line graphs: The first line graph has a horizontal v-axis and a vertical t-axis. The line begins at the origin and moves upwards and to the right passing through the points with coordinates (1, 1/5) and (10, 2). The graph is labeled with the equation t=1/5v. The second line graph has a horizontal t-axis and vertical v-axis. The line begins at the origin and moves upwards and to the right passing through the points with coordinates (1,5) and (2,10). The graph is labeled with the equation v =5t.

Even though the relationship between time and volume is the same, we are making a different choice in each case about which variable to view as the independent variable. The graph on the left has v as the independent variable, and the graph on the right has t as the independent variable.

Glossary Entries

Definition: Coordinate Plane

The coordinate plane is a system for telling where points are. For example. point R is located at $(3, 2)$ on the coordinate plane, because it is three units to the right and two units up.

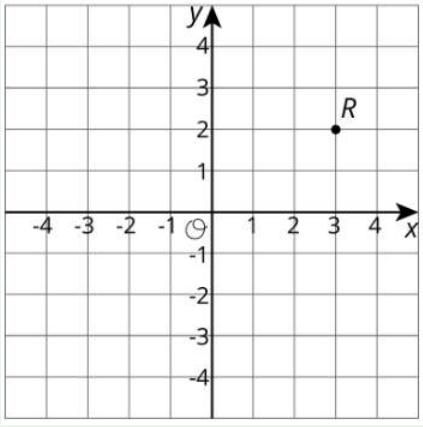


Figure 2.4.4.2

Definition: Origin

The origin is the point $(0, 0)$ in the coordinate plane. This is where the horizontal axis and the vertical axis cross.

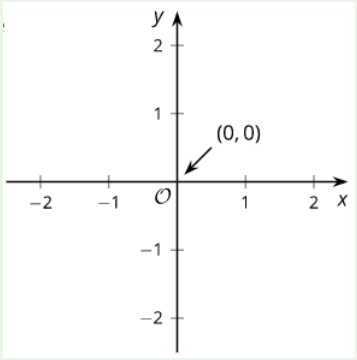


Figure 2.4.4.3

Practice

Exercise 2.4.4.4

At the supermarket you can fill your own honey bear container. A customer buys 12 oz of honey for $5.40.

1. How much does honey cost per ounce?
2. How much honey can you buy per dollar?
3. Write two different equations that represent this situation. Use h for ounces of honey and c for cost in dollars.

- Choose one of your equations, and sketch its graph. Be sure to label the axes.

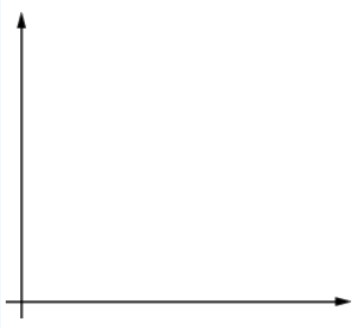


Figure 2.4.4.4

Exercise 2.4.4.5

The point $(3, \frac{6}{5})$ lies on the graph representing a proportional relationship. Which of the following points also lie on the same graph? Select **all** that apply.

A. $(1, 0.4)$
B. $1.5, \frac{6}{10})$
C. $\frac{6}{5}, 3)$
D. $(4, \frac{11}{5})$
E. $(15, 6)$

Exercise 2.4.4.6

A trail mix recipe asks for 4 cups of raisins for every 6 cups of peanuts. There is proportional relationship between the amount of raisins, r (cups), and the amount of peanuts, p (cups), in this recipe.

1. Write the equation for the relationship that has constant of proportionality greater than 1. Graph the relationship.
2. Write the equation for the relationship that has constant of proportionality less than 1. Graph the relationship.

Exercise 2.4.4.7

Here is a graph that represents a proportional relationship.

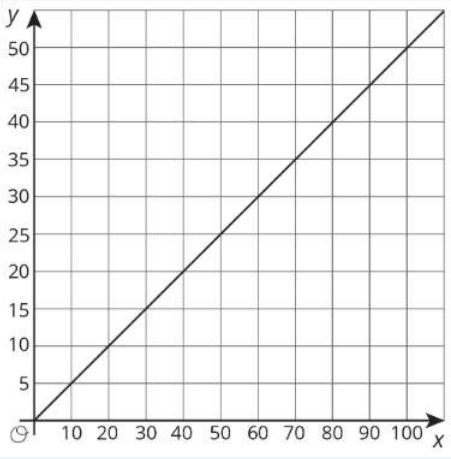


Figure 2.4.4.5

1. Come up with a situation that could be represented by this graph.
2. Label the axes with the quantities in your situation.
3. Give the graph a title.
4. Choose a point on the graph. What do the coordinates represent in your situation?

(From Unit 2.4.2)

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2.5: Let's Put it to Work

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2.5.1: Four Representations

Lesson

Let's contrast relationships that are and are not proportional in four different ways.

Exercise 2.5.1.1: Which is the Bluest?

1. Which group of blocks is the bluest?

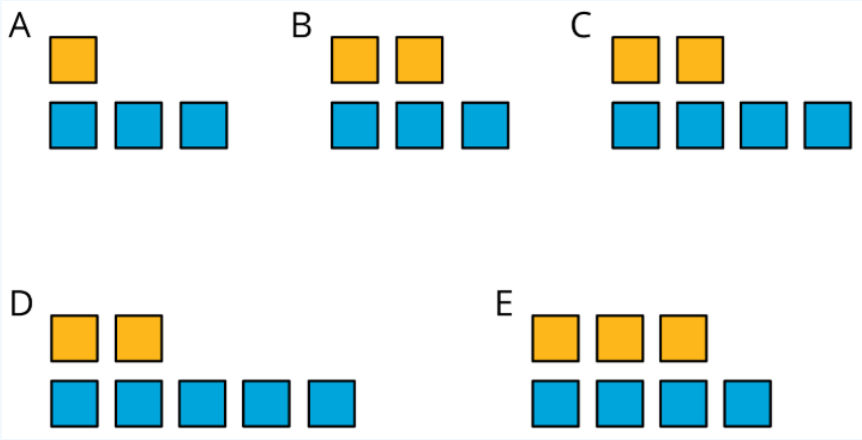


Figure E has 4 blue blocks on bottom row and 3 yellow blocks on top row.

2. Order the groups of blocks from least blue to bluest.

Exercise 2.5.1.2: One Scenario, Four Representations

1. Select two things from different lists. Make up a situation where there is a *proportional relationship* between quantities that involve these things.
   - creatures
     - starfish
     - centipedes
     - earthworms
     - dinosaurs
   - length
     - centimeters
     - cubits
     - kilometers
     - parsecs
   - time
     - nanoseconds
     - minutes
     - years
     - millennia
   - volume
     - milliliters
     - gallons
     - bushels
     - cubic miles
   - body parts
     - legs

        - eyes
        - neurons
        - digits
    - area
        - square microns
        - acres
        - hides
        - square light-years
    - weight
        - nanograms
        - ounces
        - deben
        - metric tonnes
    - substance
        - helium
        - oobleck
        - pitch
        - glue
2. Select two other things from the lists, and make up a situation where there is a relationship between quantities that involve these things, but the relationship is *not* proportional.
3. Your teacher will give you two copies of the "One Scenario, Four Representations" sheet. For each of your situations, describe the relationships in detail. If you get stuck, consider asking your teacher for a copy of the sample response.
    a. Write one or more sentences describing the relationship between the things you chose.
    b. Make a table with titles in each column and at least 6 pairs of numbers relating the two things.
    c. Graph the situation and label the axes.
    d. Write an equation showing the relationship and explain in your own words what each number and letter in your equation means.
    e. Explain how you know whether each relationship is proportional or not proportional. Give as many reasons as you can.

Exercise 2.5.1.3: Make a Poster

Create a visual display of your two situations that includes all the information from the previous activity.

Summary

The constant of proportionality for a proportional relationship can often be easily identified in a graph, a table, and an equation that represents it. Here is an example of all three representations for the same relationship. The constant of proportionality is circled:

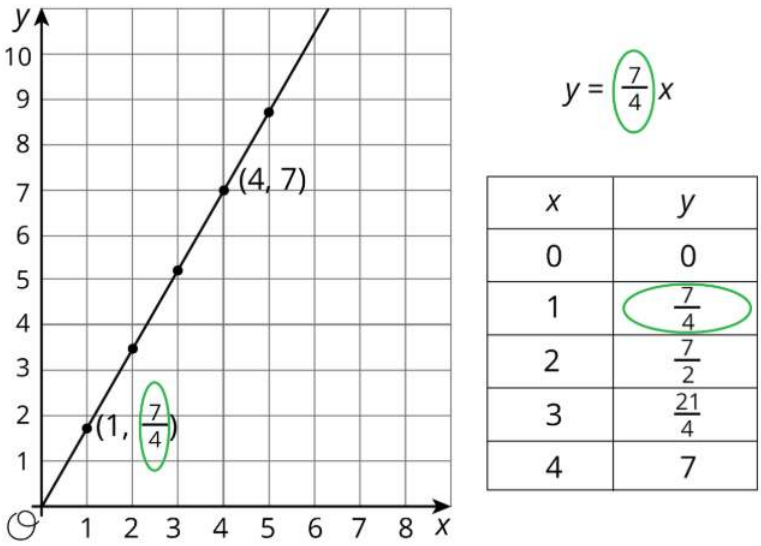


Figure 2.5.1.2

On the other hand, some relationships are not proportional. If the graph of a relationship is not a straight line through the origin, if the equation cannot be expressed in the form $y = kx$, or if the table does not have a constant of proportionality that you can multiply by any number in the first column to get the associated number in the second column, then the relationship between the quantities is not a proportional relationship.

Practice

Exercise 2.5.1.4

The equation $c = 2.95g$ shows how much it costs to buy gas at a gas station on a certain day. In the equation, c represents the cost in dollars, and g represents how many gallons of gas were purchased.

1. Write down at least four (gallons of gas, cost) pairs that fit this relationship.
2. Create a graph of the relationship.
3. What does 2.95 represent in this situation?
4. Jada's mom remarks, "You can get about a third of a gallon of gas for a dollar." Is she correct? How did she come up with that?

Exercise 2.5.1.5

There is a proportional relationship between a volume measured in cups and the same volume measured in tablespoons. 3 cups is equivalent to 48 tablespoons, as shown in the graph.

1. Plot and label at least two more points that represent the relationship.
2. Use a straightedge to draw a line that represents this proportional relationship.
3. For which value y is $(1, y)$ on the line you just drew?
4. What is the constant of proportionality for this relationship?
5. Write an equation representing this relationship. Use c for cups and t for tablespoons.

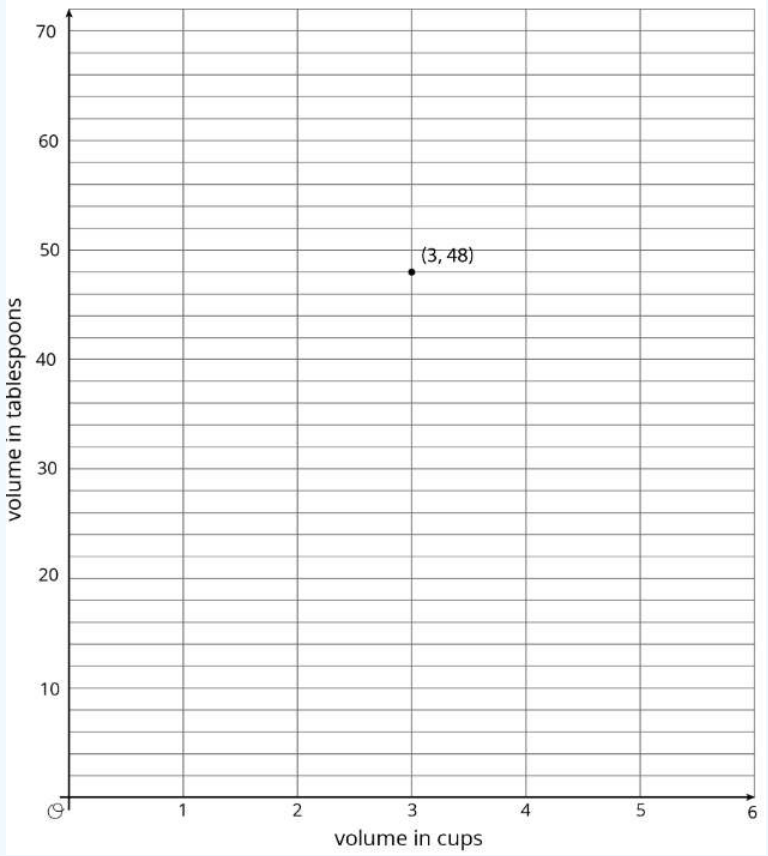


Figure 2.5.1.3

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2.5.2: Using Water Efficiently

Lesson

Let's investigate saving water.

Exercise 2.5.2.1: Comparing Baths and Showers

Some people say that it uses more water to take a bath than a shower. Others disagree.

1. What information would you collect in order to answer the question?
2. Estimate some reasonable values for the things you suggest.

Exercise 2.5.2.2: Saving Water: Bath or Shower?

1. Describe a method for comparing the water usage for a bath and a shower.
2. Find out values for the measurements needed to use the method you described. You may ask your teacher or research them yourself.
3. Under what conditions does a bath use more water? Under what conditions does a shower use more water?

Exercise 2.5.2.3: Representing Water Usage

1. Continue considering the problem from the previous activity. Name two quantities that are in a proportional relationship. Explain how you know they are in a proportional relationship.
2. What are two constants of proportionality for the proportional relationship? What do they tell us about the situation?
3. On graph paper, create a graph that shows how the two quantities are related. Make sure to label the axes.
4. Write two equations that relate the quantities in your graph. Make sure to record what each variable represents.

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CHAPTER OVERVIEW

3: Measuring Circles

3.1: Circumference of a Circle

3.1.1: How Well Can You Measure?

3.1.2: Exploring Circles

3.1.3: Exploring Circumference

3.1.4: Applying Circumference

3.1.5: Circumference and Wheels

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3.1: Circumference of a Circle

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3.1.1: How Well Can You Measure?

Lesson

Let's see how accurately we can measure.

Exercise 3.1.1.1: Estimating a Percentage

A student got 16 out of 21 questions correct on a quiz. Use mental estimation to answer these questions.

1. Did the student answer less than or more than 80% of the questions correctly?
2. Did the student answer less than or more than 75% of the questions correctly?

Exercise 3.1.1.2: Perimeter of a Square

Your teacher will give you a picture of 9 different squares and will assign your group 3 of these squares to examine more closely.

1. For each of your assigned squares, measure the length of the diagonal and the perimeter of the square in centimeters. Check your measurements with your group. After you come to an agreement, record your measurements in the table.

Table 3.1.1.1

	diagonal (cm)	perimeter (cm)
square A		
square B		
square C		
square D		
square E		
square F		
square G		
square H		
square I		

2. Plot the diagonal and perimeter values from the table on the coordinate plane.
3. What do you notice about the points on the graph?
   Pause here so your teacher can review your work.
4. Record measurements of the other squares to complete your data.

Exercise 3.1.1.3: Area of a Square

1. In the table, record the length of the diagonal for each of your assigned squares from the previous activity. Next, calculate the area of each of your squares.

Table 3.1.1.2

	diagonal (cm)	area (cm^2)
square A		
square B		
square C		
square D		

	diagonal (cm)	area (cm^2)
square E		
square F		
square G		
square H		
square I		

Pause here so your teacher can review your work. Be prepared to share your values with the class.

2. Examine the class graph of these values. What do you notice?

3. How is the relationship between the diagonal and area of a square the same as the relationship between the diagonal and perimeter of a square from the previous activity? How is it different?

Are you ready for more?

Here is a rough map of a neighborhood.

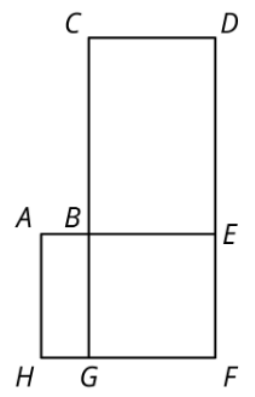


Figure 3.1.1.1

There are 4 mail routes during the week.

- On Monday, the mail truck follows the route A-B-E-F-G-H-A, which is 14 miles long.
- On Tuesday, the mail truck follows the route B-C-D-E-F-G-B, which is 22 miles long.
- On Wednesday, the truck follows the route A-B-C-D-E-F-G-H-A, which is 24 miles long.
- On Thursday, the mail truck follows the route B-E-F-G-B.

How long is the route on Thursdays?

Summary

When we measure the values for two related quantities, plotting the measurements in the coordinate plane can help us decide if it makes sense to model them with a proportional relationship. If the points are close to a line through $(0, 0)$, then a proportional relationship is a good model. For example, here is a graph of the values for the height, measured in millimeters, of different numbers of pennies placed in a stack.

Because the points are close to a line through $(0, 0)$, the height of the stack of pennies appears to be proportional to the number of pennies in a stack. This makes sense because we can see that the heights of the pennies only vary a little bit.

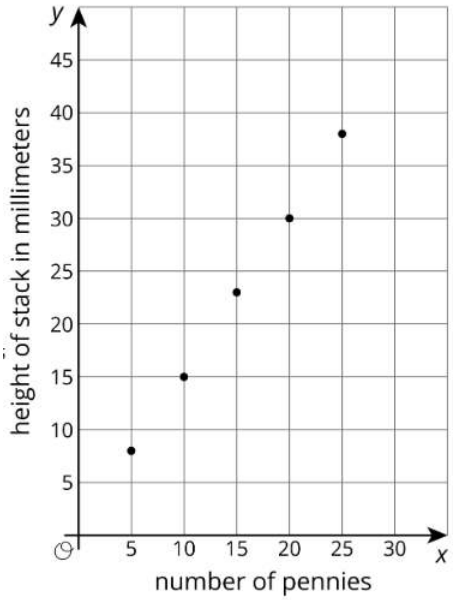


Figure 3.1.1.2

An additional way to investigate whether or not a relationship is proportional is by making a table. Here is some data for the weight of different numbers of pennies in grams, along with the corresponding number of grams per penny.

Table 3.1.1.3

number of pennies	grams	grams per penny
1	3.1	3.1
2	5.6	2.8
5	13.1	2.6
10	25.6	2.6

Though we might expect this relationship to be proportional, the quotients are not very close to one another. In fact, the metal in pennies changed in 1982, and older pennies are heavier. This explains why the weight per penny for different numbers of pennies are so different!

Practice

Exercise 3.1.1.4

Estimate the side length of a square that has a 9 cm long diagonal.

Exercise 3.1.1.5

Select **all** quantities that are proportional to the diagonal length of a square.

A. Area of the square
B. Perimeter of the square
C. Side length of the square

Exercise 3.1.1.6

Diego made a graph of two quantities that he measured and said, “The points all lie on a line except one, which is a little bit above the line. This means that the quantities can’t be proportional.” Do you agree with Diego? Explain.

Exercise 3.1.1.7

The graph shows that while it was being filled, the amount of water in gallons in a swimming pool was approximately proportional to the time that has passed in minutes.

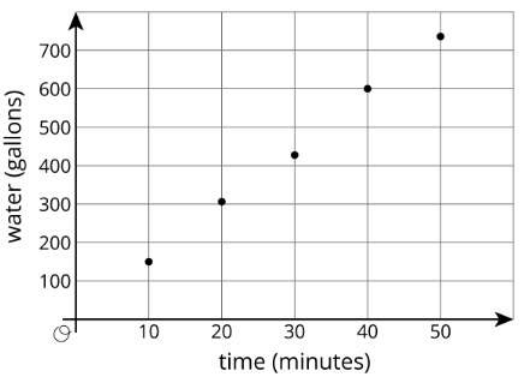


Figure 3.1.1.3

1. About how much water was in the pool after 25 minutes?
2. Approximately when were there 500 gallons of water in the pool?
3. Estimate the constant of proportionality for the gallons of water per minute going into the pool.

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3.1.2: Exploring Circles

Lesson

Let's explore circles.

Exercise 3.1.2.1: How Do You Figure?

Here are two figures.

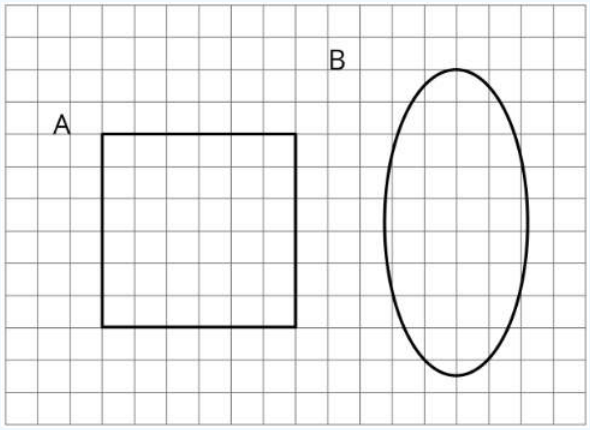


Figure B is a curved figure with horizontal distance of approximately 4 units and vertical distance of approximately 9.5 units.

Figure C looks more like Figure A than Figure B. Sketch what Figure C might look like. Explain your reasoning.

Exercise 3.1.2.2: Sorting Round Objects

Your teacher will give you some pictures of different objects.

1. How could you sort these pictures into two groups? Be prepared to share your reasoning.
2. Work with your partner to sort the pictures into the categories that your class has agreed on. Pause here so your teacher can review your work.
3. What are some characteristics that all **circles** have in common?
4. Put the circular objects in order from smallest to largest.
5. Select one of the pictures of a circular object. What are some ways you could measure the actual size of your circle?

Are you ready for more?

On January 3rd, Earth is 147,500,000 kilometers away from the Sun. On July 4th, Earth is 152,500,000 kilometers away from the Sun. The Sun has a radius of about 865,000 kilometers.

Could Earth's orbit be a circle with some point in the Sun as its center? Explain your reasoning.

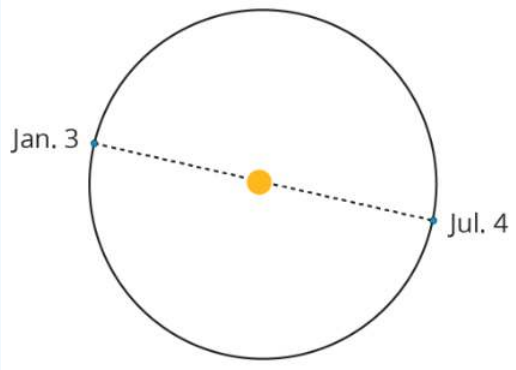


Figure 3.1.2.2: Circle with a center point in orange. One point on the circle, Jan 3 and the other point, Jul 4. These points are directly across from each other and go through the center point.

Exercise 3.1.2.3: Measuring Circles

Priya, Han, and Mai each measured one of the circular objects from earlier.

- Priya says that the bike wheel is 24 inches.
- Han says that the yo-yo trick is 24 inches.
- Mai says that the glow necklace is 24 inches.

1. Do you think that all these circles are the same size?
2. What part of the circle did each person measure? Explain your reasoning.

Exercise 3.1.2.4: Drawing Circles

Spend some time familiarizing yourself with the tools that are available in this applet.

Draw and label each circle.

1. Circle A, with a **diameter** of 6 cm.
2. Circle B, with a **radius** of 5 cm. Pause here so your teacher can review your work.
3. Circle C, with a radius that is equal to Circle A's diameter.
4. Circle D, with a diameter that is equal to Circle B's radius.
5. Use a compass to recreate one of these designs.

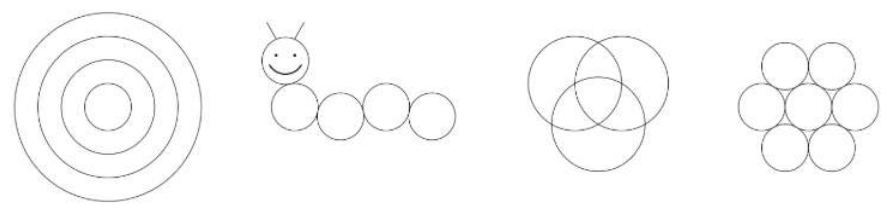


Figure 3.1.2.3: 4 images. Image 1, circles within circles. Image 2, caterpillar made of 5 circles. Image 3, 3 interlocking circles. Image 4, 7 circles in the shape of a flower.

Summary

A **circle** consists of all of the points that are the same distance away from a particular point called the *center* of the circle.

A segment that connects the center with any point on the circle is called a **radius**. For example, segments QG, QH, QI, and QJ are all radii of circle 2. (We say one radius and two radii.) The length of any radius is always the same for a given circle. For this reason, people also refer to this distance as the *radius* of the circle.

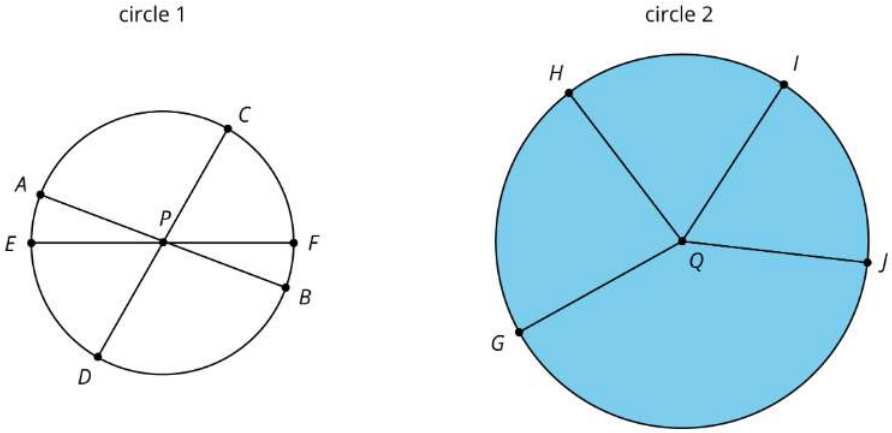


Figure 3.1.2.4: Two circles labeled circle 1 and circle 2. Circle 1 has a center at P and the points A, C, F, B, D, and E lie on the circle. Diameters A B, C D, and E F are drawn. Circle 2 has a center at Q and the points G, H, I and J lie on the circle. Line segments Q G, Q H, Q I, and Q J are drawn.

A segment that connects two opposite points on a circle (passing through the circle's center) is called a **diameter**. For example, segments AB, CD, and EF are all diameters of circle 1. All diameters in a given circle have the same length because they are

composed of two radii. For this reason, people also refer to the length of such a segment as the *diameter* of the circle.

The **circumference** of a circle is the distance around it. If a circle was made of a piece of string and we cut it and straightened it out, the circumference would be the length of that string. A circle always encloses a circular region. The region enclosed by circle 2 is shaded, but the region enclosed by circle 1 is not. When we refer to the area of a circle, we mean the area of the enclosed circular region.

Glossary Entries

Definition: Circle

A circle is made out of all the points that are the same distance from a given point.

For example, every point on this circle is 5 cm away from point A, which is the center of the circle.

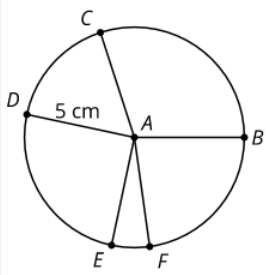


Figure 3.1.2.5

Definition: Circumference

The circumference of a circle is the distance around the circle. If you imagine the circle as a piece of string, it is the length of the string. If the circle has radius r then the circumference is $2\pi r$.

The circumference of a circle of radius 3 is $2 \cdot \pi \cdot 3$, which is 6π, or about 18.85.

Definition: Diameter

A diameter is a line segment that goes from one edge of a circle to the other and passes through the center. A diameter can go in any direction. Every diameter of the circle is the same length. We also use the word *diameter* to mean the length of this segment.

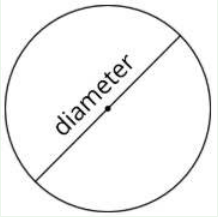


Figure 3.1.2.6

Definition: Radius

A radius is a line segment that goes from the center to the edge of a circle. A radius can go in any direction. Every radius of the circle is the same length. We also use the word *radius* to mean the length of this segment.

For example, r is the radius of this circle with center O.

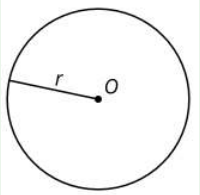


Figure 3.1.2.7

Practice

Exercise 3.1.2.5

Use a geometric tool to draw a circle. Draw and measure a radius and a diameter of the circle.

Exercise 3.1.2.6

Here is a circle with center H and some line segments and curves joining points on the circle.

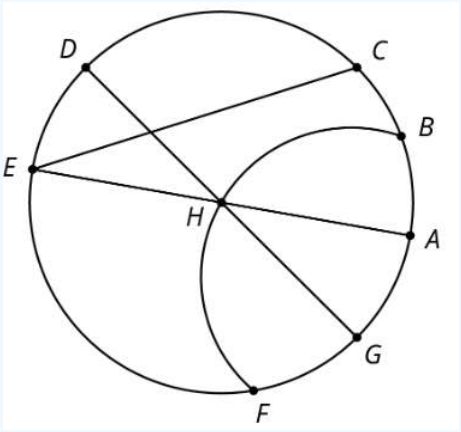


Figure 3.1.2.8: A circle, with a center labeled H. Around the edge of the circle are points labeled A, B, C, D, E, F, and G. A line extends from point A through center H to point E. A curved extends from point B to center H. A line extends from point C to point E. A line extends from point D through center H to point G. A curved line extends from point F to center H.

Identify examples of the following. Explain your reasoning.

1. Diameter
2. Radius

Exercise 3.1.2.7

Lin measured the diameter of a circle in two different directions. Measuring vertically, she got 3.5 cm, and measuring horizontally, she got 3.6 cm. Explain some possible reasons why these measurements differ.

Exercise 3.1.2.8

A small, test batch of lemonade used $\frac{1}{4}$ cup of sugar added to 1 cup of water and $\frac{1}{4}$ cup of lemon juice. After confirming it tasted good, a larger batch is going to be made with the same ratios using 10 cups of water. How much sugar should be added so that the large batch tastes the same as the test batch?

(From Unit 2.1.1)

Exercise 3.1.2.9

The graph of a proportional relationship contains the point with coordinates $(3, 12)$. What is the constant of proportionality of the relationship?

(From Unit 2.4.4)

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3.1.3: Exploring Circumference

Lesson

Let's explore the circumference of circles.

Exercise 3.1.3.1: Which is Greater?

Clare wonders if the height of the toilet paper tube or the distance around the tube is greater. What information would she need in order to solve the problem? How could she find this out?

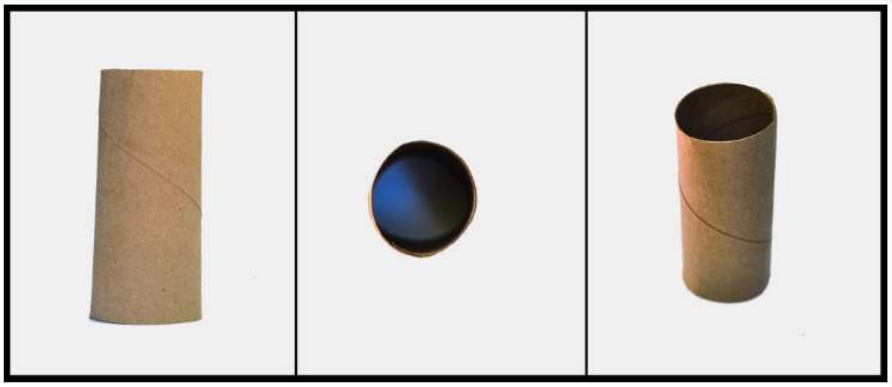


Figure 3.1.3.1: Three different views of a circular shaped toilet paper tube. The first view is of the vertical height of the tube. The second view is of the circular base of the tube. The third view is of both the base and the height of the tube.

Exercise 3.1.3.2: Measuring Circumference and Diameter

Your teacher will give you several circular objects.

1. Explore the applet to find the diameter and the circumference of three circular objects to the nearest tenth of a unit. Record your measurements in the table.
2. Plot the diameter and circumference values from the table on the coordinate plane. What do you notice?
3. Plot the points from two other groups on the same coordinate plane. Do you see the same pattern that you noticed earlier?

Exercise 3.1.3.3: Calculating Circumference and Diameter

Here are five circles. One measurement for each circle is given in the table.

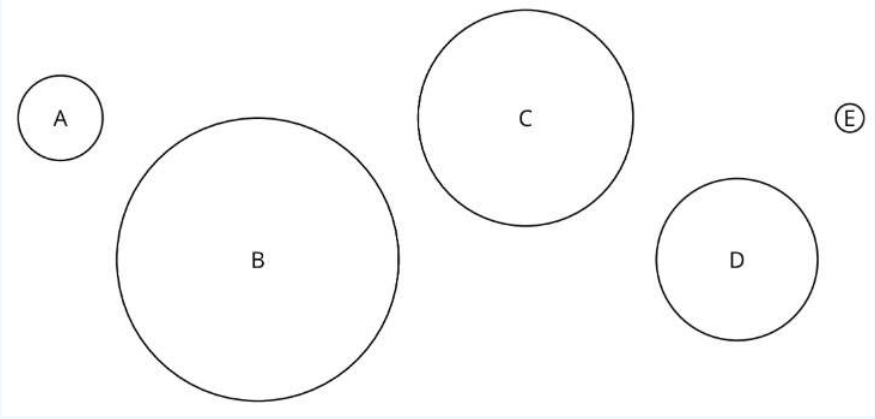


Figure 3.1.3.2

Use the constant of proportionality estimated in the previous activity to complete the table.

Table 3.1.3.1

	diameter (cm)	circumference (cm)

	diameter (cm)	circumference (cm)
circle A	3	
circle B	10	
circle C		24
circle D		18
circle E	1	

Are you ready for more?

The circumference of Earth is approximately 40,000 km. If you made a circle of wire around the globe, that is only 10 meters (0.01 km) longer than the circumference of the globe, could a flea, a mouse, or even a person creep under it?

Summary

There is a proportional relationship between the diameter and circumference of any circle. That means that if we write C for circumference and d for diameter, we know that $C = kd$, where k is the constant of proportionality.

The exact value for the constant of proportionality is called π. Some frequently used approximations for π are $\frac{22}{7}$, 3.14, and 3.14159, but none of these is exactly π.

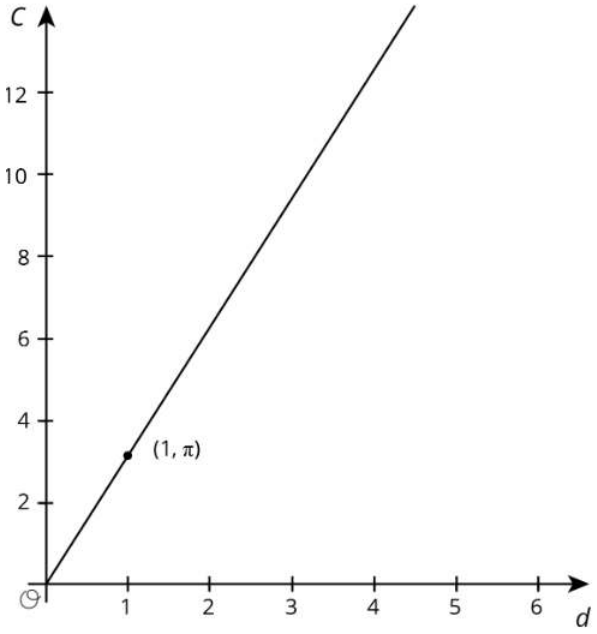


Figure 3.1.3.3

We can use this to estimate the circumference if we know the diameter, and vice versa. For example, using 3.1 as an approximation for π, if a circle has a diameter of 4 cm, then the circumference is about $(3.1) \cdot 4 = 12.4$ or 12.4 cm.

The relationship between the circumference and the diameter can be written as

$C = \pi d$

Glossary Entries

Definition: Circle

A circle is made out of all the points that are the same distance from a given point.

For example, every point on this circle is 5 cm away from point A, which is the center of the circle.

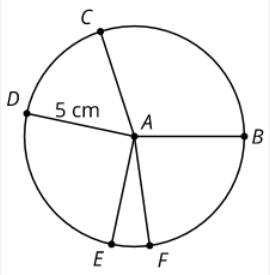


Figure 3.1.3.4

Definition: Circumference

The circumference of a circle is the distance around the circle. If you imagine the circle as a piece of string, it is the length of the string. If the circle has radius r then the circumference is $2\pi r$.

The circumference of a circle of radius 3 is $2 \cdot \pi \cdot 3$, which is 6π, or about 18.85.

Definition: Diameter

A diameter is a line segment that goes from one edge of a circle to the other and passes through the center. A diameter can go in any direction. Every diameter of the circle is the same length. We also use the word *diameter* to mean the length of this segment.

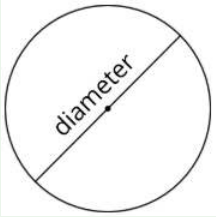


Figure 3.1.3.5

Definition: pi (π)

There is a proportional relationship between the diameter and circumference of any circle. The constant of proportionality is pi. The symbol for pi is π.

We can represent this relationship with the equation $C = \pi d$, where C represents the circumference and d represents the diameter.

Some approximations for π are $\frac{22}{7}$, 3.14, and 3.14159.

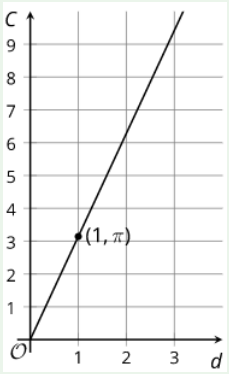


Figure 3.1.3.6

Definition: Radius

A radius is a line segment that goes from the center to the edge of a circle. A radius can go in any direction. Every radius of the circle is the same length. We also use the word *radius* to mean the length of this segment.

For example, r is the radius of this circle with center O.

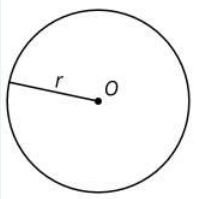


Figure 3.1.3.7

Practice

Exercise 3.1.3.4

Diego measured the diameter and circumference of several circular objects and recorded his measurements in the table.

Table 3.1.3.2

object	diameter (cm)	circumference (cm)
half dollar coin	3	10
flying disc	23	28
jar lid	8	25
flower pot	15	48

One of his measurements is inaccurate. Which measurement is it? Explain how you know.

Exercise 3.1.3.5

Complete the table. Use one of the approximate values for π discussed in class (for example 3.14, $\frac{22}{7}$, 3.1416). Explain or show your reasoning.

Table 3.1.3.3

object	diameter	circumference
hula hoop	35 in	
circular pond		556 ft
magnifying glass	5.2 cm	
car tire		71.6 in

Exercise 3.1.3.6

A is the center of the circle, and the length of is 15 centimeters.

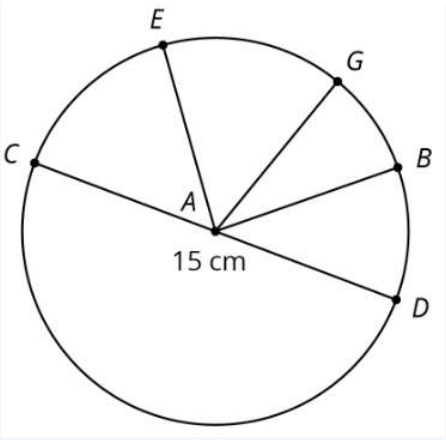


Figure 3.1.3.8

1. Name a segment that is a radius. How long is it?
2. Name a segment that is a diameter. How long is it?

(From Unit 3.1.2)

Exercise 3.1.3.7

1. Consider the equation $y = 1.5x + 2$. Find four pairs of x and y values that make the equation true. Plot the points (x, y) on the coordinate plane.

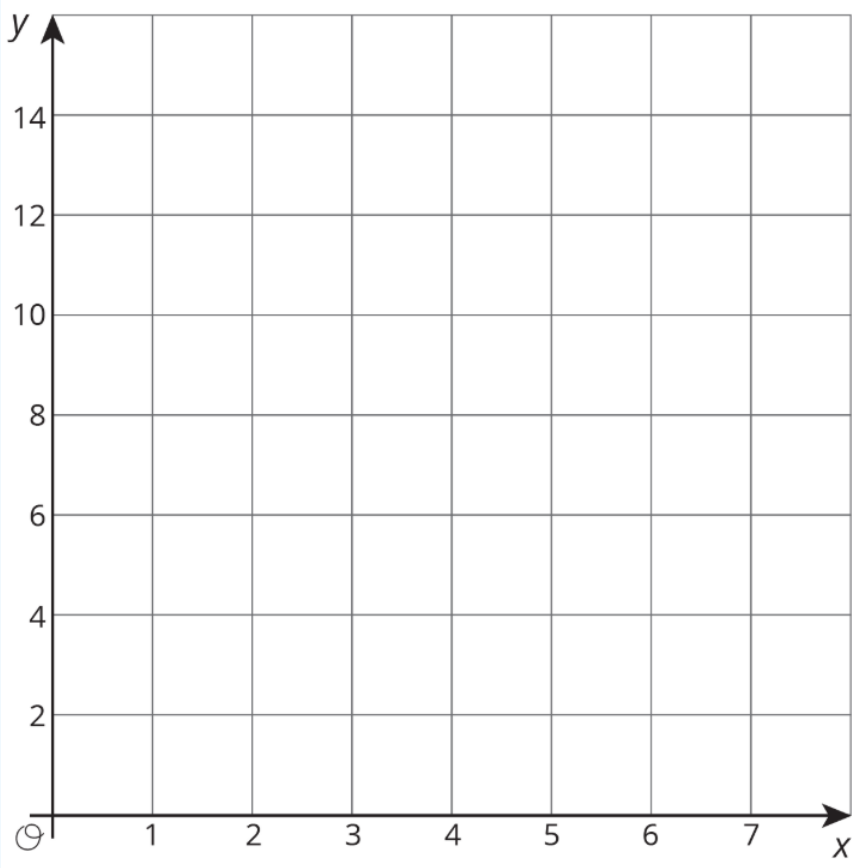


Figure 3.1.3.9

2. Based on the graph, can this be a proportional relationship? Why or why not?

(From Unit 2.4.1)

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3.1.4: Applying Circumference

Lesson

Let's use π to solve problems.

Exercise 3.1.4.1: What Do We Know? What Can we Estimate?

Here are some pictures of circular objects, with measurement tools shown. The measurement tool on each picture reads as follows:

- Wagon wheel: 3 feet
- Plane propeller: 24 inches
- Sliced Orange: 20 centimeters

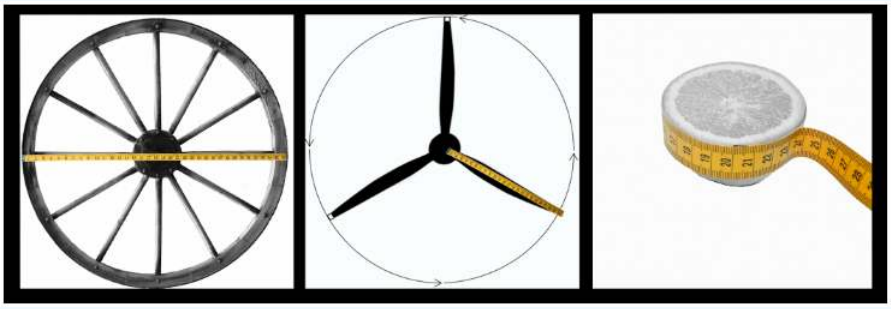


Figure 3.1.4.1: A picture of three different circular objects. The leftmost object is a wagon wheel with a measuring tool starting from one point on the wheel, goes through the wheel center to a point on the other side of the wheel. The center object is a plane propeller with three identical propeller blades. A measuring tool starts from the center of the propeller and goes to the end of the blade. The third object is of a sliced orange. A measuring tool goes around the entire circular region of the orange.

1. For each picture, which measurement is shown?
2. Based on this information, what measurement(s) could you estimate for each picture?

Exercise 3.1.4.2: Using π

In the previous activity, we looked at pictures of circular objects. One measurement for each object is listed in the table.

Your teacher will assign an approximation for π for you to use in this activity.

1. Complete the table.

Table 3.1.4.1

object	radius	diameter	circumference
wagon wheel		3 ft	
airplane propeller	24 in		
orange slice			20 cm

2. A bug was sitting on the tip of the propeller blade when the propeller started to rotate. The bug held on for 5 rotations before flying away. How far did the bug travel before it flew off?

- If you choose to, you can change the settings in the applet and enter your calculation in the box at the bottom to check your work.
- Just for fun, use the slider marked "turn" to watch the bug's motion.

Exercise 3.1.4.3: Around the Running Track

The field inside a running track is made up of a rectangle that is 84.39 m long and 73 m wide, together with a half-circle at each end.

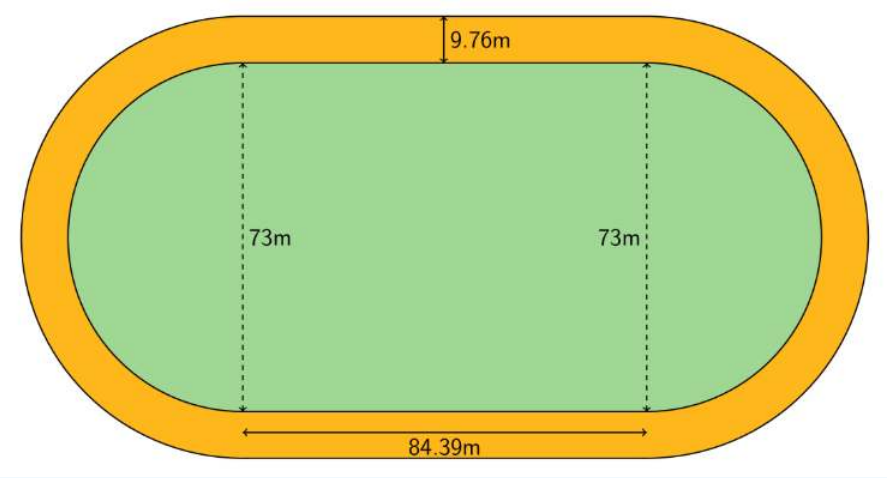


Figure 3.1.4.2: A picture of a field inside a running track. The field inside the track is composed of a rectangle, indicated by two dashed vertical lines labeled 73 meters and a horizontal length labeled 84 point 3 9 meters. There is a semi circle on each vertical side of the rectangle. The running track goes completely around the field and has a width of 9 point 7 6 meters.

1. What is the distance around the inside of the track? Explain or show your reasoning.
2. The track is 9.76 m wide all the way around. What is the distance around the outside of the track? Explain or show your reasoning.

Are you ready for more?

This size running track is usually called a 400-meter track. However, if a person ran as close to the “inside” as possible on the track, they would run less than 400 meters in one lap. How far away from the inside border would someone have to run to make one lap equal exactly 400 meters?

Exercise 3.1.4.4: Measuring a Picture Frame

Kiran bent some wire around a rectangle to make a picture frame. The rectangle is 8 inches by 10 inches.

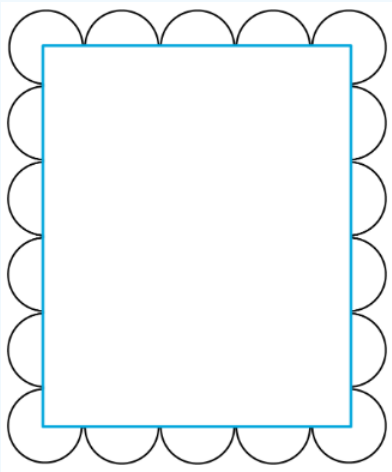


Figure 3.1.4.3: A rectangular figure that represents a picture frame. On each vertex is a three-quarters-circle where each vertex is the center of the three-quarters-circle. There are 3 identical half-circles on each horizontal length and 4 identical half-circles on each vertical width.

1. Find the perimeter of the wire picture frame. Explain or show your reasoning.
2. If the wire picture frame were stretched out to make one complete circle, what would its radius be?

Summary

The circumference of a circle, C, is π times the diameter, d. The diameter is twice the radius, r. So if we know any one of these measurements for a particular circle, we can find the others. We can write the relationships between these different measures using equations:

$d = 2r \qquad C = \pi d \qquad C = 2\pi r$

If the diameter of a car tire is 60 cm, that means the radius is 30 cm and the circumference is $60 \cdot \pi$ or about 188 cm.

If the radius of a clock is 5 in, that means the diameter is 10 in, and the circumference is $10 \cdot \pi$ or about 31 in.

If a ring has a circumference of 44 mm, that means the diameter is $44 \div \pi$, which is about 14 mm, and the radius is about 7 mm.

Glossary Entries

Definition: Circle

A circle is made out of all the points that are the same distance from a given point.

For example, every point on this circle is 5 cm away from point A, which is the center of the circle.

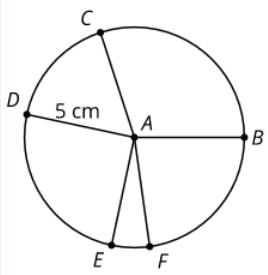


Figure 3.1.4.4

Definition: Circumference

The circumference of a circle is the distance around the circle. If you imagine the circle as a piece of string, it is the length of the string. If the circle has radius r then the circumference is $2\pi r$.

The circumference of a circle of radius 3 is $2 \cdot \pi \cdot 3$, which is 6π, or about 18.85.

Definition: Diameter

A diameter is a line segment that goes from one edge of a circle to the other and passes through the center. A diameter can go in any direction. Every diameter of the circle is the same length. We also use the word *diameter* to mean the length of this segment.

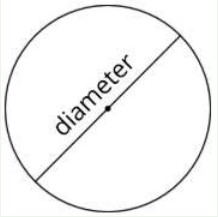


Figure 3.1.4.5

Definition: pi (π)

There is a proportional relationship between the diameter and circumference of any circle. The constant of proportionality is pi. The symbol for pi is π.

We can represent this relationship with the equation $C = \pi d$, where C represents the circumference and d represents the diameter.

Some approximations for π are $\frac{22}{7}$, 3.14, and 3.14159.

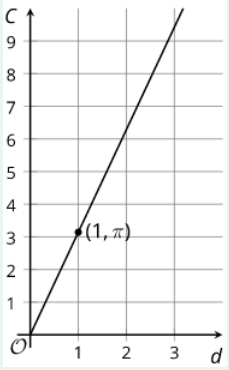


Figure 3.1.4.6

Definition: Radius

A radius is a line segment that goes from the center to the edge of a circle. A radius can go in any direction. Every radius of the circle is the same length. We also use the word *radius* to mean the length of this segment.

For example, r is the radius of this circle with center O.

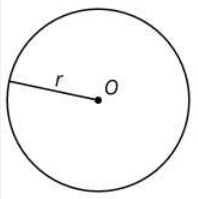


Figure 3.1.4.7

Practice

Exercise 3.1.4.5

Here is a picture of a Ferris wheel. It has a diameter of 80 meters.


Figure 3.1.4.8: "Steiger Ferris Wheel 1102009 1", by Zonk43. Public Domain. Wikimedia Commons. Source.

1. On the picture, draw and label a diameter.
2. How far does a rider travel in one complete rotation around the Ferris wheel?

Exercise 3.1.4.6

Identify each measurement as the diameter, radius, or circumference of the circular object. Then, estimate the other two measurements for the circle.

1. The length of the minute hand on a clock is 5 in.

2. The distance across a sink drain is 3.8 cm.
3. The tires on a mining truck are 14 ft tall.
4. The fence around a circular pool is 75 ft long.
5. The distance from the tip of a slice of pizza to the crust is 7 in.
6. Breaking a cookie in half creates a straight side 10 cm long.
7. The length of the metal rim around a glass lens is 190 mm.
8. From the center to the edge of a DVD measures 60 mm.

Exercise 3.1.4.7

A half circle is joined to an equilateral triangle with side lengths of 12 units. What is the perimeter of the resulting shape?

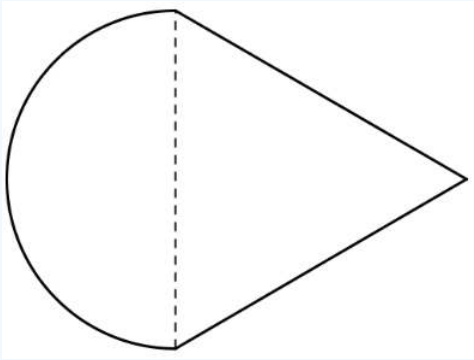


Figure 3.1.4.9: The figure shows a half circle and a triangle. The triangle is drawn adjacent and to the right of the half circle such that the dashed vertical base of the triangle is the diameter of the half circle.

Exercise 3.1.4.8

Circle A has a diameter of 1 foot. Circle B has a circumference of 1 meter. Which circle is bigger? Explain your reasoning. (1 inch = 2.54 centimeters)

Exercise 3.1.4.9

The circumference of Tyler's bike tire is 72 inches. What is the diameter of the tire?

(From Unit 3.1.3)

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3.1.5: Circumference and Wheels

Lesson

Let's explore how far different wheels roll.

Exercise 3.1.5.1: A Rope and a Wheel

Han says that you can wrap a 5-foot rope around a wheel with a 2-foot diameter because $\frac{5}{2}$ is less than pi. Do you agree with Han? Explain your reasoning.

Exercise 3.1.5.2: Rolling, Rolling, Rolling

Your teacher will give you a circular object.

1. Follow these instructions to create the drawing:
   - On a separate piece of paper, use a ruler to draw a line all the way across the page.
   - Roll your object along the line and mark where it completes one rotation.
   - Use your object to draw tick marks along the line that are spaced as far apart as the diameter of your object.
2. What do you notice?
3. Use your ruler to measure each distance. Record these values in the first row of the table:
   a. the diameter of your object
   b. how far your object rolled in one complete rotation
   c. the quotient of how far your object rolled divided by the diameter of your object

Table 3.1.5.1

object	diameter	distance traveled in one rotation	distance ÷ diameter

4. If you wanted to trace two complete rotations of your object, how long of a line would you need?
5. Share your results with your group and record their measurements in the table.
6. If each person in your group rolled their object along the entire length of the classroom, which object would complete the most rotations? Explain or show your reasoning.

Exercise 3.1.5.3: Rotations and Distance

1. A car wheel has a diameter of 20.8 inches.
   a. About how far does the car wheel travel in 1 rotation? 5 rotations? 30 rotations?
   b. Write an equation relating the distance the car travels in inches, c, to the number of wheel rotations, x.
   c. About how many rotations does the car wheel make when the car travels 1 mile? Explain or show your reasoning.
2. A bike wheel has a radius of 13 inches.
   a. About how far does the bike wheel travel in 1 rotation? 5 rotations? 30 rotations?
   b. Write an equation relating the distance the bike travels in inches, b, to the number of wheel rotations, x.
   c. About how many rotations does the bike wheel make when the bike travels 1 mile? Explain or show your reasoning.

Are you ready for more?

Here are some photos of a spring toy.

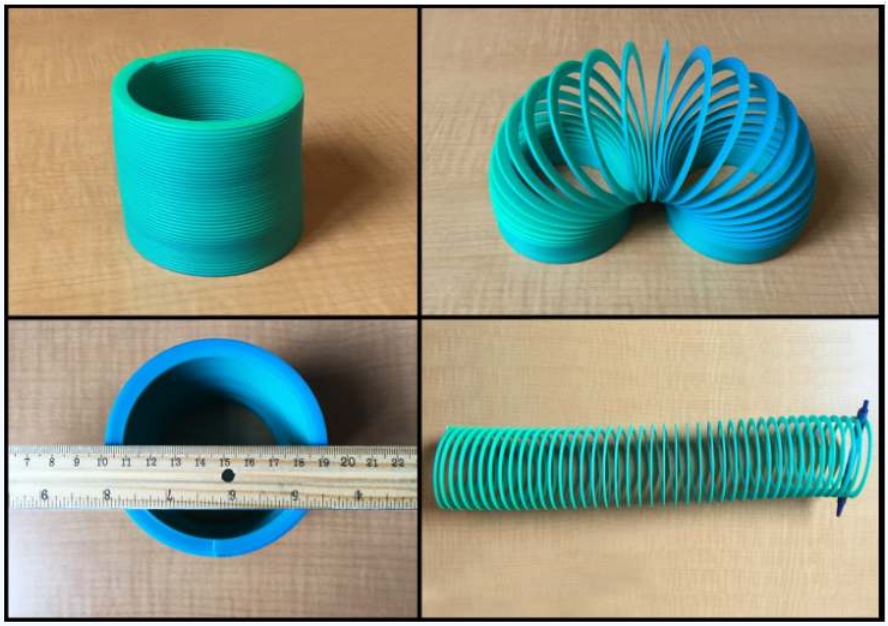


Figure 3.1.5.1: 4 images of a slinky. Image top left is a compressed slinky. Image top right is slinky with both ends on the floor. Image bottom left is measuring the diameter. Image bottom right is the slinky stretched out.

If you could stretch out the spring completely straight, how long would it be? Explain or show your reasoning.

Exercise 3.1.5.4: Rotations and Speed

The circumference of a car wheel is about 65 inches.

1. If the car wheel rotates once per second, how far does the car travel in one minute?
2. If the car wheel rotates once per second, about how many miles does the car travel in one hour?
3. If the car wheel rotates 5 times per second, about how many miles does the car travel in one hour?
4. If the car is traveling 65 miles per hour, about how many times per second does the wheel rotate?

Summary

The circumference of a circle is the distance around the circle. This is also how far the circle rolls on flat ground in one rotation. For example, a bicycle wheel with a diameter of 24 inches has a circumference of 24π inches and will roll 24π inches (or 2π feet) in one complete rotation. There is an equation relating the number of rotations of the wheel to the distance it has traveled. To see why, let's look at a table showing how far the bike travels when the wheel makes different numbers of rotations.

Table 3.1.5.2

number of rotations	distance traveled (feet)
1	2π
2	4π
3	6π
10	20π
50	100π
x	?

In the table, we see that the relationship between the distance traveled and the number of wheel rotations is a proportional relationship. The constant of proportionality is 2π.

To find the missing value in the last row of the table, note that each rotation of the wheel contributes 2π feet of distance traveled. So after x rotations the bike will travel $2\pi x$ feet. If d is the distance, in feet, traveled when this wheel makes x rotations, we have the relationship: $d = 2\pi x$.

Glossary Entries

Definition: Circle

A circle is made out of all the points that are the same distance from a given point.

For example, every point on this circle is 5 cm away from point A, which is the center of the circle.

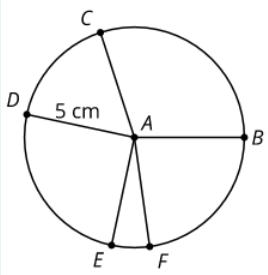


Figure 3.1.5.2

Definition: Circumference

The circumference of a circle is the distance around the circle. If you imagine the circle as a piece of string, it is the length of the string. If the circle has radius r then the circumference is $2\pi r$.

The circumference of a circle of radius 3 is $2 \cdot \pi \cdot 3$, which is 6π, or about 18.85.

Definition: Diameter

A diameter is a line segment that goes from one edge of a circle to the other and passes through the center. A diameter can go in any direction. Every diameter of the circle is the same length. We also use the word *diameter* to mean the length of this segment.

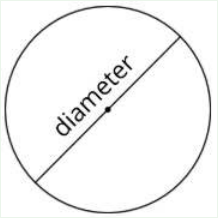


Figure 3.1.5.3

Definition: pi (π)

There is a proportional relationship between the diameter and circumference of any circle. The constant of proportionality is pi. The symbol for pi is π.

We can represent this relationship with the equation $C = \pi d$, where C represents the circumference and d represents the diameter.

Some approximations for π are $\frac{22}{7}$, 3.14, and 3.14159.

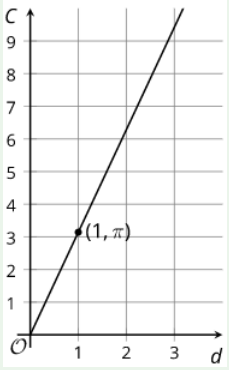


Figure 3.1.5.4

Definition: Radius

A radius is a line segment that goes from the center to the edge of a circle. A radius can go in any direction. Every radius of the circle is the same length. We also use the word *radius* to mean the length of this segment.

For example, r is the radius of this circle with center O.

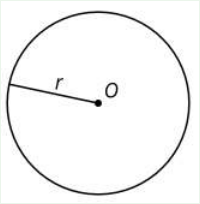


Figure 3.1.5.5

Practice

Exercise 3.1.5.5

The diameter of a bike wheel is 27 inches. If the wheel makes 15 complete rotations, how far does the bike travel?

Exercise 3.1.5.6

The wheels on Kiran's bike are 64 inches in circumference. How many times do the wheels rotate if Kiran rides 300 yards?

Exercise 3.1.5.7

The numbers are measurements of radius, diameter, and circumference of circles A and B. Circle A is smaller than circle B. Which number belongs to which quantity?

$2.6, 6, 7.6, 15.2, 15.7, 47.7$

(From Unit 3.1.4)

Exercise 3.1.5.8

Circle A has circumference $2\frac{2}{3}$ m. Circle B has a diameter that is $1\frac{1}{2}$ times as long as Circle A's diameter. What is the circumference of Circle B?

(From Unit 3.1.3)

Exercise 3.1.5.9

The length of segment AE is 5 centimeters.

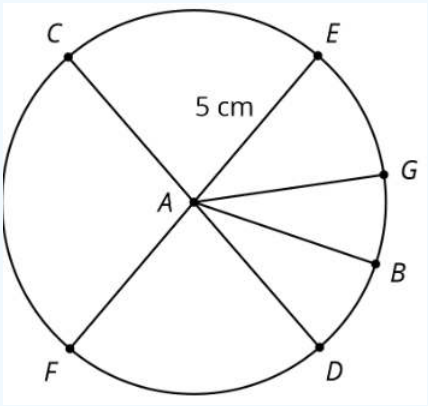


Figure 3.1.5.6

1. What is the length of segment CD?
2. What is the length of segment AB?
3. Name a segment that has the same length as segment AB?

(From Unit 3.1.2)

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3.2: Area of a Circle

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3.2.1: Estimating Areas

Lesson

Let's estimate the areas of weird shapes.

Exercise 3.2.1.1: Mental Calculations

Find a strategy to make each calculatoin mentally:

$599 + 87$

$254 - 88$

$99 \cdot 75$

Exercise 3.2.1.2: House Floorplan

Here is a floor plan of a house. Approximate lengths of the walls are given. What is the approximate area of the home, including the balcony? Explain or show your reasoning.

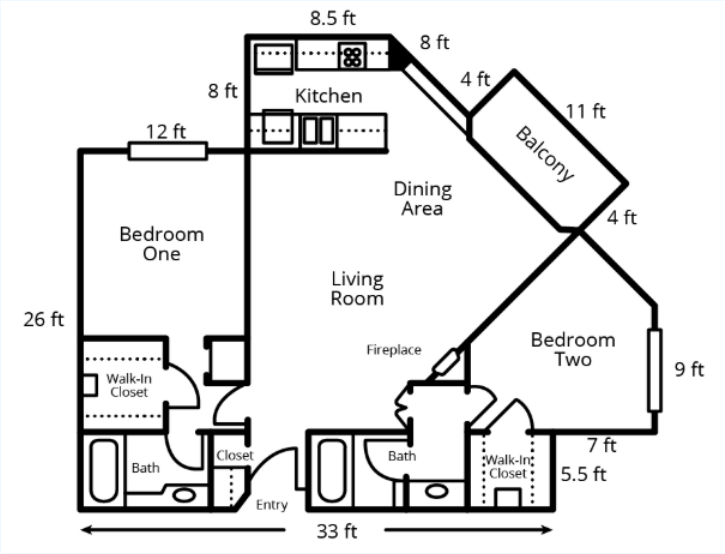


Figure 3.2.1.1: Floor plan of a house. Bottom, from left to right, horizontal segment = 33 feet, vertical segment, upward = 5 and 5 tenths feet, horizontal segment = 7 feet. Right side, from bottom to top, vertical segment = 9 feet, slanted segment, up and to the left, not labeled, slanted segment, up and to the right = 4 feet, slanted segment, up and to the left = 11 feet, slanted segment, down and to the left = 4 feet, slanted segment, up and to the left = 8 feet. Top, from right to left, horizontal segment = 8.5 feet, vertical segment, down = 8 feet, horizontal segment = 12 feet. Left side, vertical segment = 26 feet.

Exercise 3.2.1.3: Area of Nevada

Estimate the area of Nevada in square miles. Explain or show your reasoning.



Figure 3.2.1.2

Are you ready for more?

The two triangles are equilateral, and the three pink regions are identical. The blue equilateral triangle has the same area as the three pink regions taken together. What is the ratio of the sides of the two equilateral triangles?

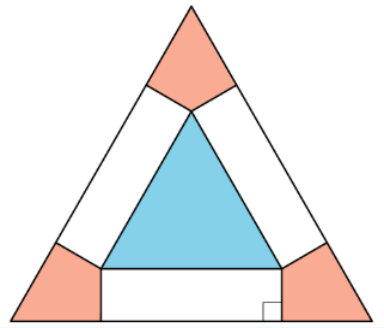


Figure 3.2.1.3

Summary

We can find the area of some complex polygons by surrounding them with a simple polygon like a rectangle. For example, this octagon is contained in a rectangle.

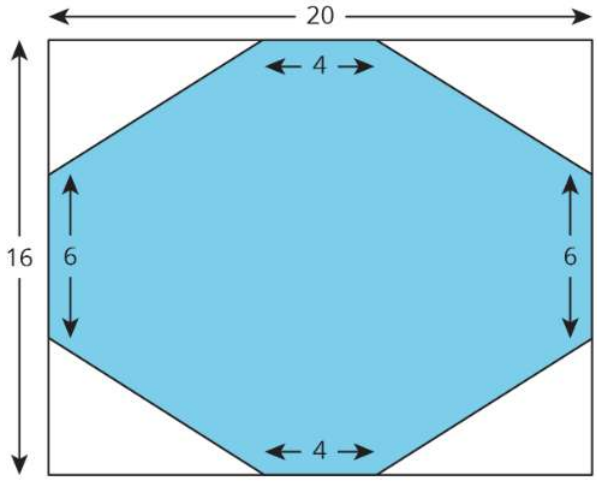


Figure 3.2.1.4

The rectangle is 20 units long and 16 units wide, so its area is 320 square units. To get the area of the octagon, we need to subtract the areas of the four right triangles in the corners. These triangles are each 8 units long and 5 units wide, so they each have an area of 20 square units. The area of the octagon is $320 - (4 \cdot 20)$ or 240 square units.

We can estimate the area of irregular shapes by approximating them with a polygon and finding the area of the polygon. For example, here is a satellite picture of Lake Tahoe with some one-dimensional measurements around the lake.

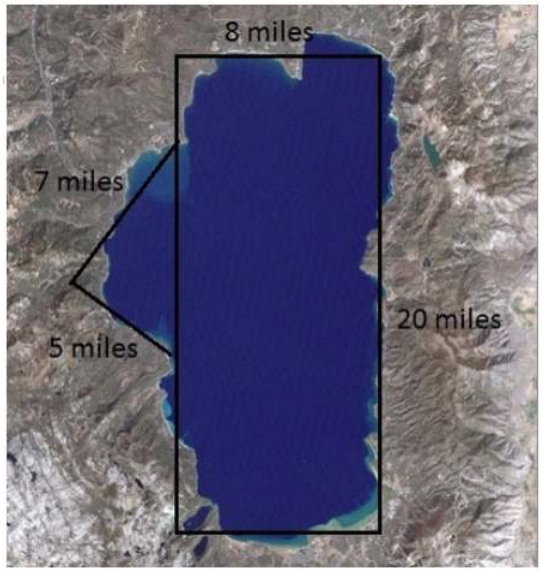


Figure 3.2.1.5: By US Geological Survey. Public Domain. US Geological Survey. Source.

The area of the rectangle is 160 square miles, and the area of the triangle is 17.5 square miles for a total of 177.5 square miles. We recognize that this is an approximation, and not likely the exact area of the lake.

Practice

Exercise 3.2.1.4

Find the area of the polygon.

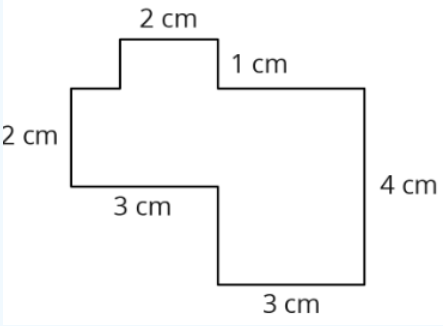


Figure 3.2.1.6

Exercise 3.2.1.5

1. Draw polygons on the map that could be used to approximate the area of Virginia.



Figure 3.2.1.7: Map of Virginia. Bottom is approximately straight horizontal segment. Left side, slanted segment up and to the right. Slope of left slide starts less steep, steepness increases significantly about halfway to the top. Right side, slanted segment down and to the right. Slope is approximately consistent. By United States Census Bureau. Public Domain. American Fact Finder. Source.

2. Which measurements would you need to know in order to calculate an approximation of the area of Virginia? Label the sides of the polygons whose measurements you would need. (Note: You aren't being asked to calculate anything.)

Exercise 3.2.1.6

Jada's bike wheels have a diameter of 20 inches. How far does she travel if the wheels rotate 37 times?

(From Unit 3.1.5)

Exercise 3.2.1.7

The radius of Earth is approximately 6,400 km. The equator is the circle around Earth dividing it into the northern and southern hemispheres. (The center of the earth is also the center of the equator.) What is the length of the equator?

(From Unit 3.1.4)

Exercise 3.2.1.8

Here are several recipes for sparkling lemonade. For each recipe, describe how many tablespoons of lemonade mix it takes per cup of sparkling water.

1. 4 tablespoons of lemonade mix and 12 cups of sparkling water
2. 4 tablespoons of lemonade mix and 6 cups of sparkling water
3. 3 tablespoons of lemonade mix and 5 cups of sparkling water
4. $\frac{1}{2}$ tablespoon of lemonade mix and $\frac{3}{4}$ cups of sparkling water

(From Unit 2.1.1)

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3.2.2: Exploring the Area of a Circle

Lesson

Let's investigate the areas of circles.

Exercise 3.2.2.1: Estimating Areas

Your teacher will show you some figures. Decide which figure has the largest area. Be prepared to explain your reasoning.

Exercise 3.2.2.2: Estimating Areas of Circles

Your teacher will give your group two circles of different sizes.

1. Set the diameter of your assigned circle and use the applet to help estimate the area of the circle.
   Note: to create a polygon, select the Polygon tool, and click on each vertex. End by clicking the first vertex again. For example, to draw triangle ABC, click on $A - B - C - A$.

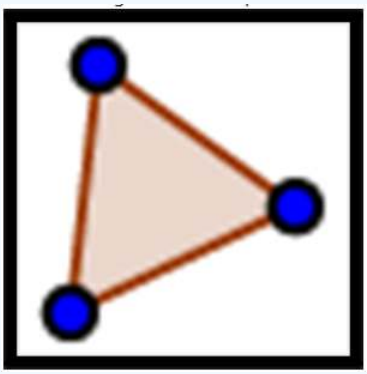


Figure 3.2.2.1

2. Record the diameter in column D and the corresponding area in column A for your circles and others from your classmates.
3. In a previous lesson, you graphed the relationship between the diameter and circumference of a circle. How is this graph the same? How is it different?

Are you ready for more?

How many circles of radius 1 unit can you fit inside each of the following so that they do not overlap?

1. a circle of radius 2 units?
2. a circle of radius 3 units?
3. a circle of radius 4 units?

If you get stuck, consider using coins or other circular objects.

Exercise 3.2.2.3: Covering a Circle

Here is a square whose side length is the same as the radius of the circle.

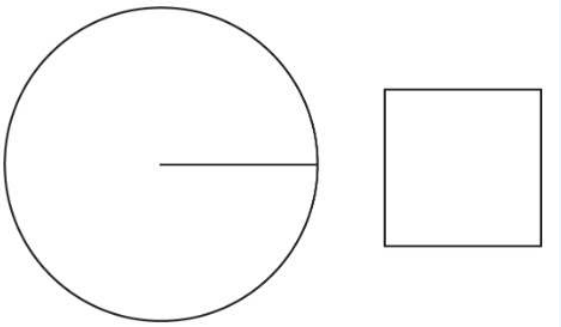


Figure 3.2.2.2

How many of these squares do you think it would take to cover the circle exactly?

Summary

The circumference C of a circle is proportional to the diameter d, and we can write this relationship as $C = \pi d$. The circumference is also proportional to the radius of the circle, and the constant of proportionality is $2 \cdot \pi$ because the diameter is twice as long as the radius. However, the **area of a circle** is *not* proportional to the diameter (or the radius).

The area of a circle with radius r is a little more than 3 times the area of a square with side r so the area of a circle of radius r is approximately $3r^2$. We saw earlier that the circumference of a circle of radius r is $2\pi r$. If we write C for the circumference of a circle, this proportional relationship can be written $C = 2\pi r$.

The area A of a circle with radius r is approximately $3r^2$. Unlike the circumference, the area is not proportional to the radius because $3r^2$ cannot be written in the form kr for a number k. We will investigate and refine the relationship between the area and the radius of a circle in future lessons.

Glossary Entries

Definition: Area of a Circle

If the radius of a circle is r units, then the area of the circle is πr^2 square units.

For example, a circle has radius 3 inches. Its area is $\pi 3^2$ square inches, or 9π square inches, which is approximately 28.3 square inches.

Practice

Exercise 3.2.2.4

The x-axis of each graph has the diameter of a circle in meters. Label the y-axis on each graph with the appropriate measurement of a circle:
radius (m), circumference (m), or area (m^2).

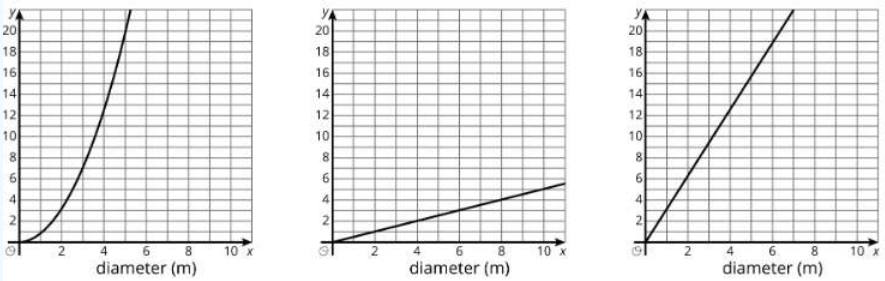


Figure 3.2.2.3

Exercise 3.2.2.5

Circle A has area 500 in^2. The diameter of circle B is three times the diameter of circle A. Estimate the area of circle B.

Exercise 3.2.2.6

Lin's bike travels 100 meters when her wheels rotate 55 times. What is the circumference of her wheels?

(From Unit 3.1.5)

Exercise 3.2.2.7

Priya drew a circle whose circumference is 25 cm. Clare drew a circle whose diameter is 3 times the diameter of Priya's circle. What is the circumference of Clare's circle?

(From Unit 3.1.3)

Exercise 3.2.2.8

1. Here is a picture of two squares and a circle. Use the picture to explain why the area of this circle is more than 2 square units but less than 4 square units.

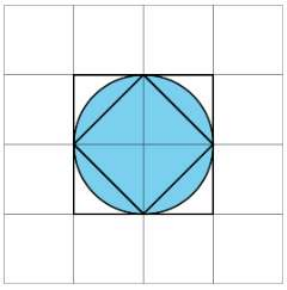


Figure 3.2.2.4: Two squares and circle on a grid of 4 units by 4 units. In the center of the grid is a square that measures two units high by two units wide. Within the square is a circle, with a diameter of 2 units. Within the circle is another square, rotated so that the each of the four vertices of the square meet with a point of the circle and the midpoint of the larger square.

2. Here is another picture of two squares and a circle. Use the picture to explain why the area of this circle is more than 18 square units and less than 36 square units.

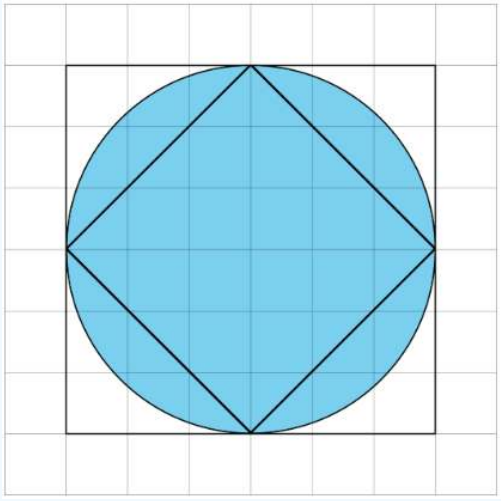


Figure 3.2.2.5: A square with a circle inside of it and a square inside of the circle. The shapes are drawn on a grid. The outer square has side lengths = 6 units. The Circle has diameter = 6 units. The inner square is turned such that its four corners correspond with the midpoints of each side of the outer square.

Exercise 3.2.2.9

Point A is the center of the circle, and the length of CD is 15 centimeters. Find the circumference of this circle.

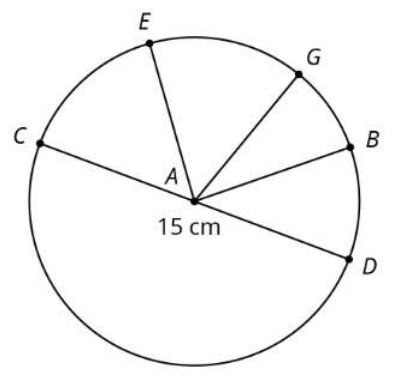


Figure 3.2.2.6

(From Unit 3.1.3)

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3.2.3: Relating Area to Circumference

Lesson

Let's rearrange circles to calculate their areas.

Exercise 3.2.3.1: Irrigating a Field

A circular field is set into a square with an 800 m side length. Estimate the field's area.

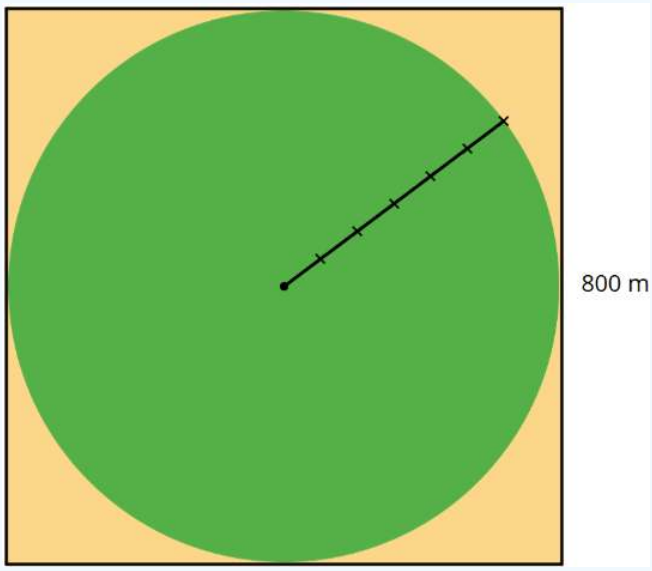


Figure 3.2.3.1: A square where the vertical distance is labeled 800 meters. The largest possible circle is drawn inside the square with a line segment is that begins from the center of the circle to a point on the edge of the circle.

- About $5,000\text{m}^2$
- About $50,000\text{m}^2$
- About $500,000\text{m}^2$
- About $5,000,000\text{m}^2$
- About $50,000,000\text{m}^2$

Exercise 3.2.3.2: Making a Polygon out of a Circle

Your teacher will give you a circular object, a marker, and two pieces of paper of different colors.

Follow these instructions to create a visual display:

1. Using a thick marker, trace your circle in two separate places on the same piece of paper.
2. Cut out both circles, cutting around the marker line.
3. Fold and cut one of the circles into fourths.
4. Arrange the fourths so that straight sides are next to each other, but the curved edges are alternately on top and on bottom. Pause here so your teacher can review your work.
5. Fold and cut the fourths in half to make eighths. Arrange the eighths next to each other, like you did with the fourths.
6. If your pieces are still large enough, repeat the previous step to make sixteenths.
7. Glue the remaining circle and the new shape onto a piece of paper that is a different color.

After you finish gluing your shapes, answer the following questions.

1. How do the areas of the two shapes compare?
2. What polygon does the shape made of the circle pieces most resemble?
3. How could you find the area of this polygon?

Exercise 3.2.3.3: Making Another Polygon out of a Circle

Imagine a circle made of rings that can bend but not stretch.



Figure 3.2.3.2

Watch the animation.

1. What polygon does the new shape resemble?
2. How does the area of the polygon compare to the area of the circle?
3. How can you find the area of the polygon?
4. Show, in detailed steps, how you could find the polygon's area in terms of the circle's measurements. Show your thinking. Organize it so it can be followed by others.
5. After you finish, trade papers with a partner and check each other's work. If you disagree, work to reach an agreement. Discuss:
   - Do you agree or disagree with each step?
   - Is there a way they can make the explanation clearer?
6. Return your partner's work, and revise your explanation based on the feedback you received.

Exercise 3.2.3.4: Tiling a Table

Elena wants to tile the top of a circular table. The diameter of the table top is 28 inches. What is its area?

Are you ready for more?

A box contains 20 square tiles that are 2 inches on each side. How many boxes of tiles will Elena need to tile the table?

Summary

If C is a circle's circumference and r is its radius, then $C = 2\pi r$. The area of a circle can be found by taking the product of half the circumference and the radius.

If A is the area of the circle, this gives the equation:

$$A = \frac{1}{2}(2\pi r) \cdot r$$

This equation can be rewritten as:

$$A = \pi r^2$$

(Remember that when we have we can write and we can say " **squared**.")

This means that if we know the radius, we can find the area. For example, if a circle has radius 10 cm, then the area is about $(3.14) \cdot 100$ which is 314 cm^2.

If we know the diameter, we can figure out the radius, and then we can find the area. For example, if a circle has a diameter of 30 ft, then the radius is 15 ft, and the area is about $(3.14) \cdot 225$ which is approximately 707 ft^2.

Glossary Entries

Definition: Area of a Circle

If the radius of a circle is r units, then the area of the circle is πr^2 square units.

For example, a circle has radius 3 inches. Its area is $\pi 3^2$ square inches, or 9π square inches, which is approximately 28.3 square inches.

Definition: Squared

We use the word *squared* to mean "to the second power." This is because a square with side length s has an area of $s \cdot s$, or s^2.

Practice

Exercise 3.2.3.5

The picture shows a circle divided into 8 equal wedges which are rearranged.

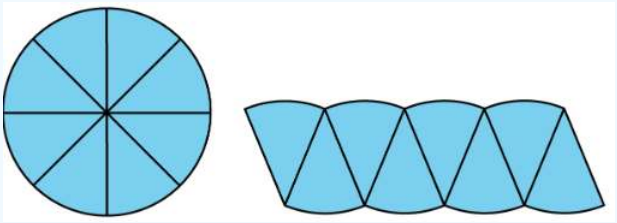


Figure 3.2.3.3: Two shapes. The first shape is a circle divided into 8 identical wedges. The second shape is the same 8 wedges arranged horizontally, alternating between wedges with their arc ends on top and wedges with their arc ends on the bottom, making a shape somewhat like a rectangle.

The radius of the circle is r and its circumference is $2\pi r$. How does the picture help to explain why the area of the circle is πr^2?

Exercise 3.2.3.6

A circle's circumference is approximately 76 cm. Estimate the radius, diameter, and area of the circle.

Exercise 3.2.3.7

Jada paints a circular table that has a diameter of 37 inches. What is the area of the table?

Exercise 3.2.3.8

The Carousel on the National Mall has 4 rings of horses. Kiran is riding on the inner ring, which has a radius of 9 feet. Mai is riding on the outer ring, which is 8 feet farther out from the center than the inner ring is.

1. In one rotation of the carousel, how much farther does Mai travel than Kiran?
2. One rotation of the carousel takes 12 seconds. How much faster does Mai travel than Kiran?

(From Unit 3.1.4)

Exercise 3.2.3.9

Here are the diameters of four coins:

Table 3.2.3.1

coin	penny	nickel	dime	quarter
diameter	1.9 cm	2.1 cm	1.8 cm	2.4 cm

1. A coin rolls a distance of 33 cm in 5 rotations. Which coin is it?
2. A quarter makes 8 rotations. How far did it roll?
3. A dime rolls 41.8 cm. How many rotations did it make?

(From Unit 3.1.5)

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3.2.4: Applying Area of Circles

Lesson

Lets's find the areas of shapes made up of circles.

Exercise 3.2.4.1: Still Irrigating the Field

The area of this field is about 500,000 m^2. What is the field's area to the nearest square meter? Assume that the side lengths of the square are exactly 800 m.

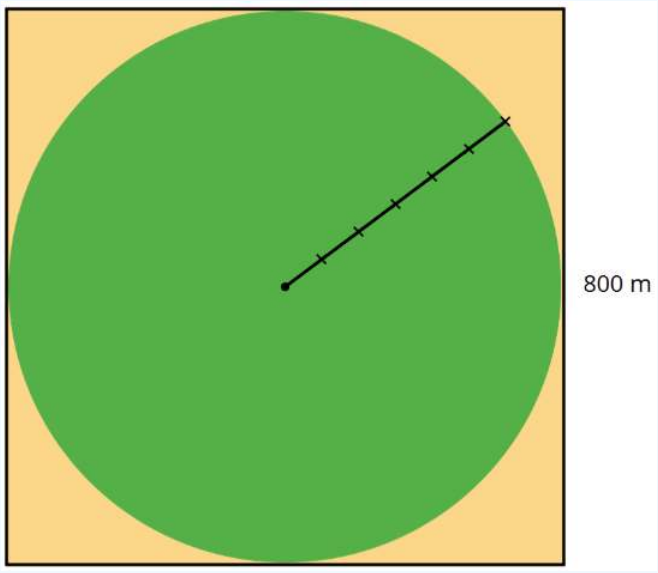


Figure 3.2.4.1: A square where the vertical distance is labeled 800 meters. The largest possible circle is drawn inside the square with a line segment is that begins from the center of the circle to a point on the edge of the circle.

- $502,400 m^2$
- $502,640 m^2$
- $502,655 m^2$
- $502,656 m^2$
- $502,857 m^2$

Exercise 3.2.4.2: Comparing Areas Made of Circles

1. Each square has a side length of 12 units. Compare the areas of the shaded regions in the 3 figures. Which figure has the largest shaded region? Explain or show your reasoning.

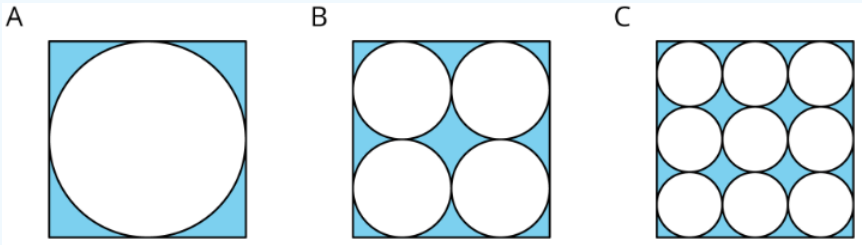


Figure C is a square with nine identical circles inside of it that do not overlap. The circles are arranged in three rows of three circles, are tangent to each other and to the sides of the square. The area outside of the circles is shaded blue.

2. Each square in Figures D and E has a side length of 1 unit. Compare the area of the two figures. Which figure has more area? How much more? Explain or show your reasoning.

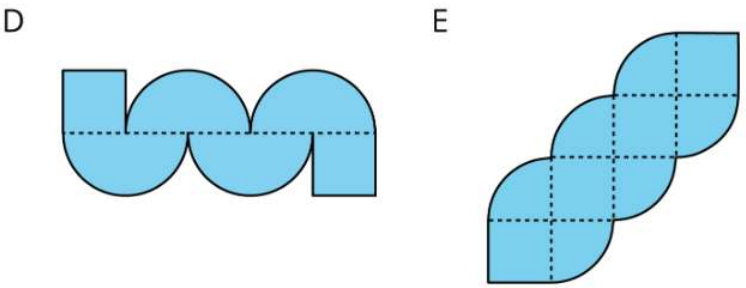


Figure D is composed of one square and two semi circles on top, and two bottom halves of circles and one square on the bottom. The top and bottom are divided by a horizontal dashed line. Fiqure E is composed of 4 squares and 6 quarter circles. The 4 square are arranged diagonally, where the top right vertex of the first square is the same point as the bottom left vertex of the second square, the top right vertex of the second square is the same point as the bottom left vertex of the third square, and the top right vertex of the third square is the same point as the bottom left vertex of the fourth square. The corner circles align in between the sides of each square.

Are you ready for more?

Which figure has a longer perimeter, Figure D or Figure E? How much longer?

Exercise 3.2.4.3: The Running Track Revisited

The field inside a running track is made up of a rectangle 84.39 m long and 73 m wide, together with a half-circle at each end. The running lanes are 9.76 m wide all the way around.

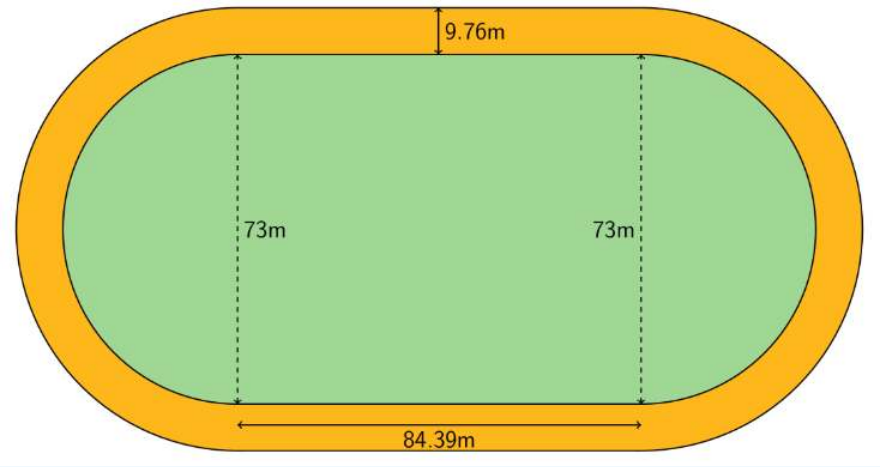


Figure 3.2.4.4: A picture of a field inside a running track. The field inside the track is composed of a rectangle, indicated by two dashed vertical lines labeled 73 meters and a horizontal length labeled 84 point 3 9 meters. There is a semi circle on each vertical side of the rectangle. The running track goes completely around the field and has a width of 9 point 7 6 meters.

What is the area of the running track that goes around the field? Explain or show your reasoning.

Summary

The relationship between A, the area of a circle, and r, its radius, is $A = \pi r^2$. We can use this to find the area of a circle if we know the radius. For example, if a circle has a radius of 10 cm, then the area is $\pi \cdot 10^2$ or 100π cm^2. We can also use the formula to find the radius of a circle if we know the area. For example, if a circle has an area of 49π m^2 then its radius is 7 m and its diameter is 14 m.

Sometimes instead of leaving π in expressions for the area, a numerical approximation can be helpful. For the examples above, a circle of radius 10 cm has area about 314 cm^2. In a similar way, a circle with area 154 m^2 has radius about 7 m.

We can also figure out the area of a fraction of a circle. For example, the figure shows a circle divided into 3 pieces of equal area. The shaded part has an area of $\frac{1}{3}\pi r^2$.

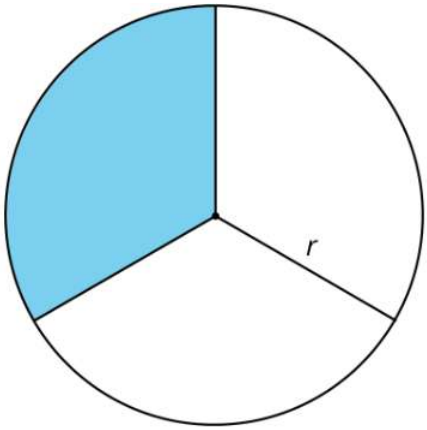


Figure 3.2.4.5: A circle divided into three equal sections. From the center of the circle three line segments extend to a point on the circle. The line segment extending downward and to the right is labeled r. The upper left region of the circle is shaded.

Glossary Entries

Definition: Area of a Circle

If the radius of a circle is r units, then the area of the circle is πr^2 square units.

For example, a circle has radius 3 inches. Its area is $\pi 3^2$ square inches, or 9π square inches, which is approximately 28.3 square inches.

Definition: Squared

We use the word *squared* to mean "to the second power." This is because a square with side length s has an area of $s \cdot s$, or s^2.

Practice

Exercise 3.2.4.4

A circle with a 12-inch diameter is folded in half and then folded in half again. What is the area of the resulting shape?

Exercise 3.2.4.5

Find the area of the shaded region. Express your answer in terms of π.

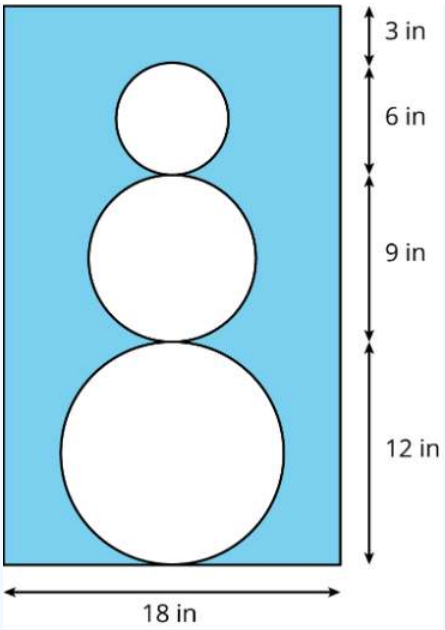


Figure 3.2.4.6: A shaded rectangle with three unshaded circles stacked one above the other inside the rectangle. The horizontal length of the rectangle is 18 inches. On the right side of the rectangle, the length of the vertical side is subdivided into 4 different lengths using arrows. The first arrow indicates that the distance from the top of the rectangle to the top of the first circle is 3 inches. The second arrow indicates that the distance from the top to the bottom of the first circle is 6 inches. The third arrow indicates that the distance from the top to the bottom of the second circle is 9 inches. The fourth arrow indicates that the distance between the top to the bottom of the third circle is 12 inches.

Exercise 3.2.4.6

The face of a clock has a circumference of 63 in. What is the area of the face of the clock?

(From Unit 3.2.3)

Exercise 3.2.4.7

Which of these pairs of quantities are proportional to each other? For the quantities that are proportional, what is the constant of proportionality?

1. Radius and diameter of a circle
2. Radius and circumference of a circle
3. Radius and area of a circle
4. Diameter and circumference of a circle
5. Diameter and area of a circle

(From Unit 3.2.2)

Exercise 3.2.4.8

Find the area of this shape in two different ways.

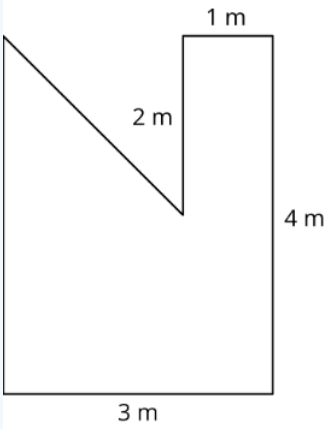


Figure 3.2.4.7

(From Unit 3.2.1)

Exercise 3.2.4.9

Elena and Jada both read at a constant rate, but Elena reads more slowly. For every 4 pages that Elena can read, Jada can read 5.

1. Complete the table.

Table 3.2.4.1

pages read by Elena	pages read by Jada
4	5
1	
9	
e	
	15
	j

2. Here is an equation for the table: $j = 1.25e$. What does the 1.25 mean?
3. Write an equation for this relationship that starts $e = \ldots$

(From Unit 2.2.2)

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3.3: Let's Put it to Work

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3.3.1: Distinguishing Circumference and Area

Lesson

Let's contrast circumference and area.

Exercise 3.3.1.1: Filling the Plate

About how many cheese puffs can fit on the plate in a single layer? Be prepared to explain your reasoning.



Figure 3.3.1.1: A picture of a tray with covered with one layer of circular cheese puffs. On top of the cheese puffs is a circular plate with 1 cheese puff placed in the center of the plate. The diameter of the plate is approximately the same as the width of the tray. Approximately 14 cheese puffs are used to measure the width of the tray. Cheese Balls, by Andrew Stadel . CC BY-SA. Estimation 180. Source.

Exercise 3.3.1.2: Cart Sort: Circle Problems

Your teacher will give you cards with questions about circles.

1. Sort the cards into two groups based on whether you would use the circumference or the area of the circle to answer the question. Pause here so your teacher can review your work.
2. Your teacher will assign you a card to examine more closely. What additional information would you need in order to answer the question on your card?
3. Estimate measurements for the circle on your card.
4. Use your estimates to calculate the answer to the question.

Exercise 3.3.1.3: Visual Display of Circle Problem

In the previous activity you estimated the answer to a question about circles.

Create a visual display that includes:

- The question you were answering
- A diagram of a circle labeled with your estimated measurements
- Your thinking, organized so that others can follow it
- Your answer, expressed in terms of π and also expressed as a decimal approximation

Exercise 3.3.1.4: Analyzing Circle Claims

Here are two students' answers for each question. Do you agree with either of them? Explain or show your reasoning.

1. How many feet are traveled by a person riding once around the merry-go-round?



Figure 3.3.1.2

- Clare says, "The radius of the merry-go-round is about 4 feet, so the distance around the edge is about 8π feet."
- Andre says, "The diameter of the merry-go-round is about 4 feet, so the distance around the edge is about 4π feet."

2. How much room is there to spread frosting on the cookie?



Figure 3.3.1.3

- Clare says "The radius of the cookie is about 3 centimeters, so the space for frosting is about 6π cm^2."
- Andre says "The diameter of the cookie is about 3 inches, so the space for frosting is about 2.25π in^2."

3. How far does the unicycle move when the wheel makes 5 full rotations?



Figure 3.3.1.4

- Clare says, "The diameter of the unicycle wheel is about 0.5 meters. In 5 complete rotations it will go about $\frac{5}{2}\pi$ m^2."
- Andre says, "I agree with Clare's estimate of the diameter, but that means the unicycle will go about $\frac{5}{4}\pi$ m."

Are you ready for more?

A goat (point G) is tied with a 6-foot rope to the corner of a shed. The floor of the shed is a square whose sides are each 3 feet long. The shed is closed and the goat can't go inside. The space all around the shed is flat, grassy, and the goat can't reach any other structures or objects. What is the area over which the goat can roam?

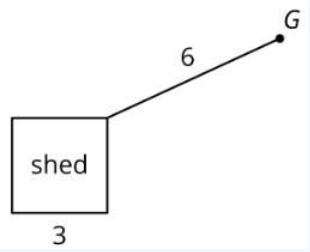


Figure 3.3.1.5: Square, side length = 3 units. Square labeled shed. Line segment from top right corner of square, running up and to the right diagonally, ends at point labeled G. Line segment labeled 6.

Summary

Sometimes we need to find the circumference of a circle, and sometimes we need to find the area. Here are some examples of quantities related to the circumference of a circle:

- The length of a circular path.
- The distance a wheel will travel after one complete rotation.
- The length of a piece of rope coiled in a circle.

Here are some examples of quantities related to the area of a circle:

- The amount of land that is cultivated on a circular field.
- The amount of frosting needed to cover the top of a round cake.
- The number of tiles needed to cover a round table.

In both cases, the radius (or diameter) of the circle is all that is needed to make the calculation. The circumference of a circle with radius r is $2\pi r$ while its area is πr^2. The circumference is measured in linear units (such as cm, in, km) while the area is measured in square units (such as cm^2, in^2, km^2).

Practice

Exercise 3.3.1.5

For each problem, decide whether the circumference of the circle or the area of the circle is most useful for finding a solution. Explain your reasoning.

1. A car's wheels spin at 1000 revolutions per minute. The diameter of the wheels is 23 inches. You want to know how fast the car is travelling.
2. A circular kitchen table has a diameter of 60 inches. You want to know how much fabric is needed to cover the table top.
3. A circular puzzle is 20 inches in diameter. All of the pieces are about the same size. You want to know about how many pieces there are in the puzzle.
4. You want to know about how long it takes to walk around a circular pond.

Exercise 3.3.1.6

The city of Paris, France is completely contained within an almost circular road that goes around the edge. Use the map with its scale to:

1. Estimate the circumference of Paris.
2. Estimate the area of Paris.

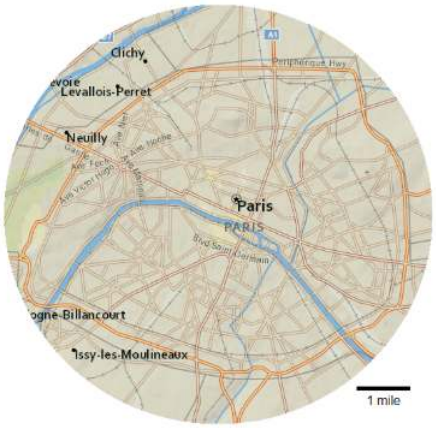


Figure 3.3.1.6: Map of the city of Paris, France. Paris is completely contained within an almost circular road. Map scale indicates 1 mile = a line segment that is approximately one sixth of the diameter of the circle. By United States Census Bureau. Public Domain. American Fact Finder. Source.

Exercise 3.3.1.7

Here is a diagram of a softball field:

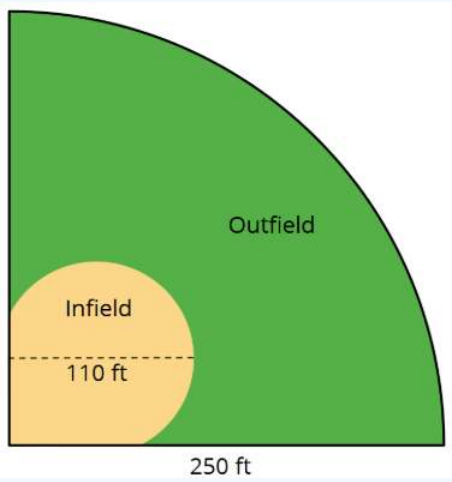


Figure 3.3.1.7: A quarter-circle that represents the diagram of a softball field. A small circular-like region inside the quarter-circle is labeled Infield and a dashed line goes through the center of the circular region connecting to two points on its edge is labeled 110 ft. The area outside of the circular region is labeled Outfield and the horizontal base of the quarter-circle is labeled 250 ft.

1. About how long is the fence around the field?
2. About how big is the outfield?

Exercise 3.3.1.8

While in math class, Priya and Kiran come up with two ways of thinking about the proportional relationship shown in the table.

Table 3.3.1.1

x	y
2	?
5	1750

Both students agree that they can solve the equation $5k = 1750$ to find the constant of proportionality.

- Priya says, “I can solve this equation by dividing 1750 by 5.”
- Kiran says, “I can solve this equation by multiplying 1750 by $\frac{1}{5}$.”

1. What value of k would each student get using their own method?
2. How are Priya and Kiran's approaches related?
3. Explain how each student might approach solving the equation $\frac{2}{3}k = 50$.

(From Unit 2.2.2)

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3.3.2: Stained-Glass Windows

Lesson

Let's use circumference and area to design stained-glass windows.

Exercise 3.3.2.1: Cost of a Stained-Glass Window

The students in art class are designing a stained-glass window to hang in the school entryway. The window will be 3 feet tall and 4 feet wide. Here is their design.

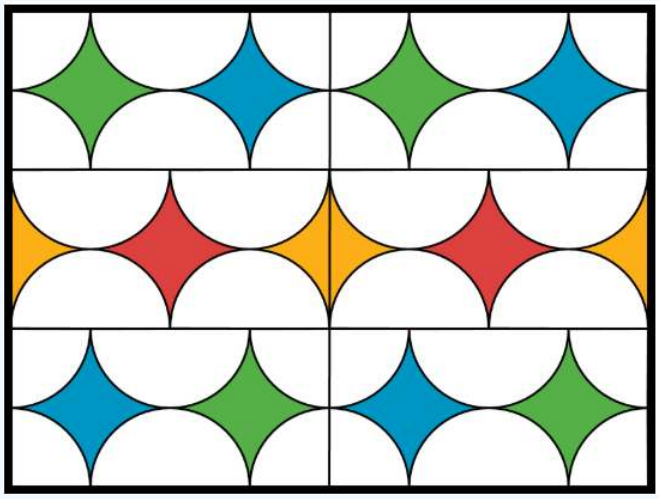


Figure 3.3.2.1: A stained glass window composed of six rectangular panels of equal size. The six panels are arranged two across and three down to form a three foot by four foot rectangle. Four panels are composed of two half circles and four quarter circles of clear glass that are tangent to each other. The two radii of the quarter circles are on both the horizontal and vertical sides of the rectangle and the diameter of the semi circles are on the horizontal side of the rectangle. The remaining regions in the panel are composed of colored glass. Two panels are composed of four half circles of clear glass that are tangent to each other. The diameters of the semi circles are on the horizontal sides of the rectangular panel and are half the length of the horizontal side. The remaining regions in the panel are composed of colored glass.

They have raised $100 for the project. The colored glass costs $5 per square foot and the clear glass costs $2 per square foot. The material they need to join the pieces of glass together costs 10 cents per foot and the frame around the window costs $4 per foot.

Do they have enough money to cover the cost of making the window?

Exercise 3.3.2.2: A Bigger Window

A local community member sees the school's stained-glass window and really likes the design. They ask the students to create a larger copy of the window using a scale factor of 3. Would $450 be enough to buy the materials for the larger window? Explain or show your reasoning.

Exercise 3.3.2.3: Invent Your Own Design

Draw a stained-glass window design that could be made for less than $450. Show your thinking. Organize your work so it can be followed by others.

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CHAPTER OVERVIEW

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4.1.4: Half as Much Again

4.1.5: Say It with Decimals

4.2: Percent Increase and Decrease

4.2.1: Increasing and Decreasing

4.2.2: One Hundred Percent

4.2.3: Percent Increase and Decrease with Equations

4.2.4: More and Less than 1 Percent

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4.4: Let's Put it to Work

4.4.1: Posing Percentage Problems

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4.1: Proportional Relationships with Fractions

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4.1.1: Lots of Flags

Lesson

Let's explore the U.S. flag.

Exercise 4.1.1.1: Scaled or Not?

1. Which of the geometric objects are scaled versions of each other?

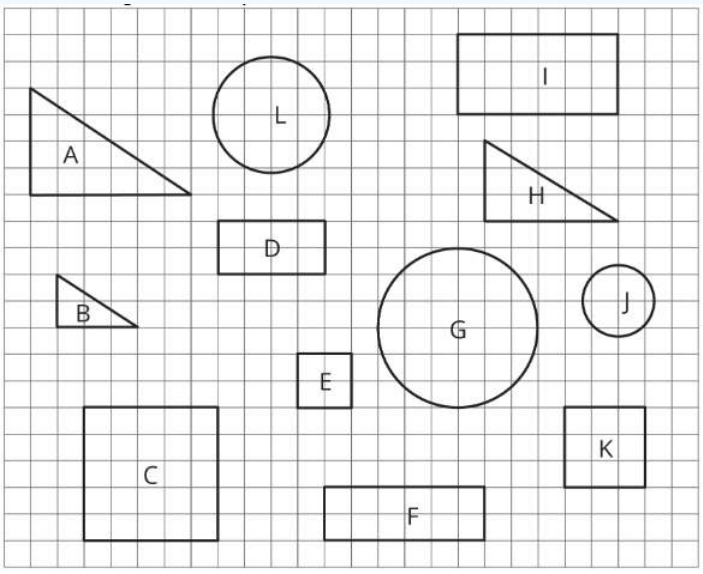


Figure 4.1.1.1

2. Pick two of the objects that are scaled copies and find the scale factor.

Exercise 4.1.1.2: Flags are Many Sizes

One standard size for the United States flag is 19 feet by 10 feet. On a flag of this size, the union (the blue rectangle in the top-left corner) is $7\frac{5}{8}$ feet by $5\frac{3}{8}$ feet.

There are many places that display flags of different sizes.

- Many classrooms display a U.S. flag.
- Flags are often displayed on stamps.
- There was a flag on the space shuttle.
- Astronauts on the Apollo missions had a flag on a shoulder patch.



Figure 4.1.1.2: By papapishu. Public Domain. OpenClipArt.org. Source.

1. Choose one of the four options and decide on a size that would be appropriate for this flag. Find the size of the union.
2. Share your answer with another group that used a different option. What do your dimensions have in common?

Exercise 4.1.1.3: What Percentage is the Union?

On a U.S. flag that is 19 feet by 10 feet, the union is $7\frac{5}{8}$ feet by $5\frac{3}{8}$ feet. For each question, first estimate the answer and then compute the actual percentage.

1. What **percentage** of the flag is taken up by the union?
2. What percentage of the flag is red? Be prepared to share your reasoning.

Are you ready for more?

The largest U.S. flag in the world is 225 feet by 505 feet.

1. Is the ratio of the length to the width equivalent to $1 : 1.9$, the ratio for official government flags?
2. If a square yard of the flag weighs about 3.8 ounces, how much does the entire flag weigh in pounds?

Summary

Imagine you have a painting that is 15 feet wide and 5 feet high. To sketch a scaled copy of the painting, the ratio of the width and height of a scaled copy must be equivalent to $15 : 5$. What is the height of a scaled copy that is 2 feet across?

Table 4.1.1.1

width	height
15	5
2	h

We know that the height is $\frac{1}{3}$ the width, so $h = \frac{1}{3} \cdot 2$ or $\frac{2}{3}$.

Sometimes ratios include fractions and decimals. We will be working with these kinds of ratios in the next few lessons.

Glossary Entries

Definition: Percentage

A percentage is a rate per 100.

For example, a fish tank can hold 36 liters. Right now there is 27 liters of water in the tank. The percentage of the tank that is full is 75%.

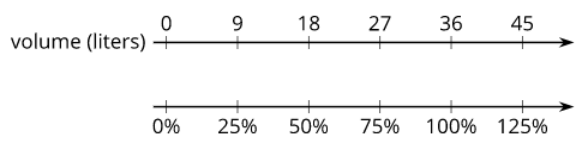


Figure 4.1.1.3

Practice

Exercise 4.1.1.4

A rectangle has a height to width ratio of $3 : 4.5$. Give two examples of dimensions for rectangles that could be scaled versions of this rectangle.

Exercise 4.1.1.5

One rectangle measures 2 units by 7 units. A second rectangle measures 11 units by 37 units. Are these two figures scaled versions of each other? If so, find the scale factor. If not, briefly explain why.

Exercise 4.1.1.6

Ants have 6 legs. Elena and Andre write equations showing the proportional relationship between the number of ants, a, to the number of ant legs l. Elena writes $a = 6 \cdot l$ and Andre writes $l = \frac{1}{6} \cdot a$. Do you agree with either of the equations? Explain your reasoning.

(From Unit 2.2.2)

Exercise 4.1.1.7

On the grid, draw a scaled copy of quadrilateral ABCD with a scale factor $\frac{2}{3}$.

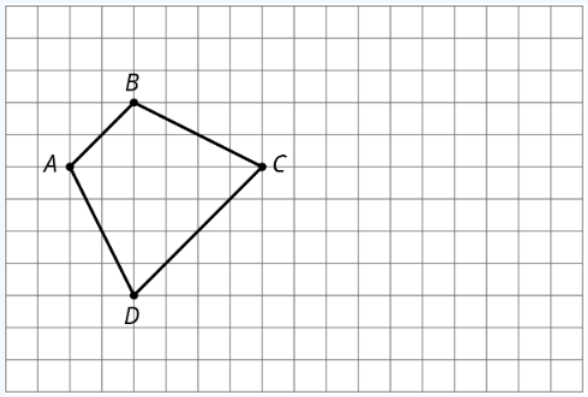


Figure 4.1.1.4

(From Unit 1.1.4)

Exercise 4.1.1.8

Solve each equation mentally.

1. $\frac{5}{2} \cdot x = 1$
2. $x \cdot \frac{7}{3} = 1$
3. $1 \div \frac{11}{2} = x$

(From Unit 1.1.5)

Exercise 4.1.1.9

Lin has a scale model of a modern train. The model is created at a scale of 1 to 48.

1. The height of the model train is 102 millimeters. What is the actual height of the train in meters? Explain your reasoning.
2. On the scale model, the distance between the wheels on the left and the wheels on the right is $1\frac{1}{4}$ inches. The state of Wyoming has old railroad tracks that are 4.5 feet apart. Can the modern train travel on those tracks? Explain your reasoning.

(From Unit 1.2.5)

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4.1.2: Ratios and Rates With Fractions

Lesson

Let's calculate some rates with fractions.

Exercise 4.1.2.1: Number Talk: Division

Find each quotient mentally.

$5 \div \frac{1}{3}$

$2 \div \frac{1}{3}$

$\frac{1}{2} \div \frac{1}{3}$

$2\frac{1}{2} \div \frac{1}{3}$

Exercise 4.1.2.2: A Train is Traveling at...

A train is traveling at a constant speed and goes 7.5 kilometers in 6 minutes. At that rate:

1. How far does the train go in 1 minute?
2. How far does the train go in 100 minutes?



Figure 4.1.2.1: By hpgruesen. Public Domain. Pixabay. Source.

Exercise 4.1.2.3: Comparing Running Speeds

Lin ran $2\frac{3}{4}$ miles in $\frac{2}{5}$ of an hour. Noah ran $8\frac{2}{3}$ miles in $\frac{4}{3}$ of an hour.

1. Pick one of the questions that was displayed, but don't tell anyone which question you picked. Find the answer to the question.
2. When you and your partner are both done, share the answer you got (do not share the question) and ask your partner to guess which question you answered. If your partner can't guess, explain the process you used to answer the question.
3. Switch with your partner and take a turn guessing the question that your partner answered.

Are you ready for more?

Nothing can go faster than the speed of light, which is 299,792,458 meters per second. Which of these are possible?

1. Traveling a billion meters in 5 seconds.
2. Traveling a meter in 2.5 nanoseconds. (A nanosecond is a billionth of a second.)
3. Traveling a parsec in a year. (A parsec is about 3.26 light years and a light year is the distance light can travel in a year.)

Exercise 4.1.2.4: Scaling the Mona Lisa

In real life, the Mona Lisa measures $2\frac{1}{2}$ feet by $1\frac{3}{4}$ feet. A company that makes office supplies wants to print a scaled copy of the Mona Lisa on the cover of a notebook that measures 11 inches by 9 inches.

The applet is here to help you experiment with the situation. (It won't solve the problems for you.) Use the sliders to scale the image and drag the red circle to place it on the book. Measure the side lengths with the Distance or Length tool.

1. What size should they use for the scaled copy of the Mona Lisa on the notebook cover?

2. What is the scale factor from the real painting to its copy on the notebook cover?
3. Discuss your thinking with your partner. Did you use the same scale factor? If not, is one more reasonable than the other?

Summary

There are 12 inches in a foot, so we can say that for every 1 foot, there are 12 inches, or the ratio of feet to inches is $1 : 12$. We can find the **unit rates** by dividing the numbers in the ratio:

$1 \div 12 = \frac{1}{2}$ so there is $\frac{1}{12}$ foot per inch.

$12 \div 1 = 12$ so there are 12 inches per foot.

The numbers in a ratio can be fractions, and we calculate the unit rates the same way: by dividing the numbers in the ratio. For example, if someone runs $\frac{3}{4}$ mile in $\frac{11}{2}$ minutes, the ratio of minutes to miles is $\frac{11}{2} : \frac{3}{4}$.

$\frac{11}{2} \div \frac{3}{4} = \frac{22}{3}$, so the person's pace is $\frac{22}{3}$ minutes per mile.

$\frac{3}{4} \div \frac{11}{2} = \frac{3}{22}$, so the person's speed is $\frac{3}{22}$ miles per minute.

Glossary Entries

Definition: Percentage

A percentage is a rate per 100.

For example, a fish tank can hold 36 liters. Right now there is 27 liters of water in the tank. The percentage of the tank that is full is 75%.

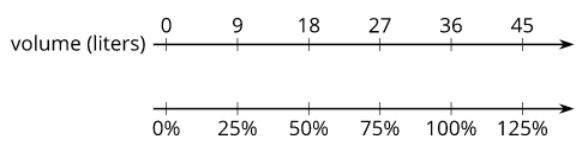


Figure 4.1.2.2

Definition: Unit Rate

A unit rate is a rate per 1.

For example, 12 people share 2 pies equally. One unit rate is 6 people per pie, because $12 \div 2 = 6$. The other unit rate is $\frac{1}{6}$ of a pie per person, because $2 \div 12 = \frac{1}{6}$.

Practice

Exercise 4.1.2.5

A cyclist rode 3.75 miles in 0.3 hours.

1. How fast was she going in miles per hour?
2. At that rate, how long will it take her to go 4.5 miles?

Exercise 4.1.2.6

A recipe for sparkling grape juice calls for $1\frac{1}{2}$ quarts of sparkling water and $\frac{3}{4}$ quart of grape juice.

1. How much sparkling water would you need to mix with 9 quarts of grape juice?
2. How much grape juice would you need to mix with $\frac{15}{4}$ quarts of sparkling water?
3. How much of each ingredient would you need to make 100 quarts of punch?

Exercise 4.1.2.7

At a deli counter,

- Someone bought $1\frac{3}{4}$ pounds of ham for $14.50.
- Someone bought $2\frac{1}{2}$ pounds of turkey for $26.25.
- Someone bought $\frac{3}{8}$ pounds of roast beef for $5.50.

Which meat is the least expensive per pound? Which meat is the most expensive per pound? Explain how you know.

Exercise 4.1.2.8

1. Draw a scaled copy of the circle using a scale factor of 2.

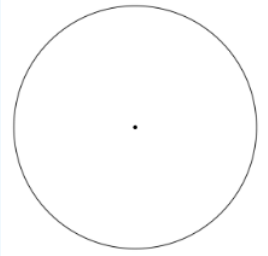


Figure 4.1.2.3

2. How does the circumference of the scaled copy compare to the circumference of the original circle?

3. How does the area of the scaled copy compare to the area of the original circle?

(From Unit 3.3.1)

Exercise 4.1.2.9

Jada has a scale map of Kansas that fits on a page in her book. The page is 5 inches by 8 inches. Kansas is about 210 miles by 410 miles. Select **all** scales that could be a scale of the map. (There are 2.54 centimeters in an inch.)

A. 1 in to 1 mi
B. 1 cm to 1 km
C. 1 in to 10 mi
D. 1 ft to 100 mi
E. 1 cm to 200 km
F. 1 in to 100 mi
G. 1 cm to 1000 km

(From Unit 1.2.5)

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4.1.3: Revisiting Proportional Relationships

Lesson

Let's use constants of proportionality to solve more problems.

Exercise 4.1.3.1: Recipe Ratios

A recipe calls for $\frac{1}{2}$ cup sugar and 1 cup flour. Complete the table to show how much sugar and flour to use in different numbers of batches of the recipe.

Table 4.1.3.1

sugar (cups)	flour (cups)
$\frac{1}{2}$	1
$\frac{3}{4}$	
	$1\frac{3}{4}$
1	
	$2\frac{1}{2}$

Exercise 4.1.3.2: The Price of Rope

Two students are solving the same problem: At a hardware store, they can cut a length of rope off of a big roll, so you can buy any length you like. The cost for 6 feet of rope is $7.50. How much would you pay for 50 feet of rope, at this rate?

1. Kiran knows he can solve the problem this way.

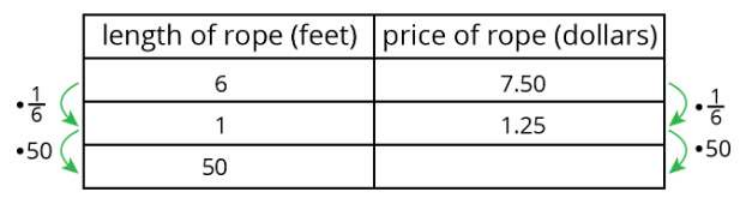


Figure 4.1.3.1

What would be Kiran's answer?

2. Kiran wants to know if there is a more efficient way of solving the problem. Priya says she can solve the problem with only 2 rows in the table.

Table 4.1.3.2

length of rope (feet)	price of rope (dollars)
6	7.50
50	

What do you think Priya's method is?

Exercise 4.1.3.3: Swimming, Manufacturing, and Painting

1. Tyler swims at a constant speed, 5 meters every 4 seconds. How long does it take him to swim 114 meters?

Table 4.1.3.3

distance (meters)	time (seconds)

distance (meters)	time (seconds)
5	4
114	

2. A factory produces 3 bottles of sparkling water for every 8 bottles of plain water. How many bottles of sparkling water does the company produce when it produces 600 bottles of plain water?

Table 4.1.3.4

number of bottles of sparkling water	number of bottles of plain water

3. A certain shade of light blue paint is made by mixing $1\frac{1}{2}$ quarts of blue paint with 5 quarts of white paint. How much white paint would you need to mix with 4 quarts of blue paint?
4. For each of the previous three situations, write an equation to represent the proportional relationship.

Are you ready for more?

Different nerve signals travel at different speeds.

- Pressure and touch signals travel about 250 feet per second.
- Dull pain signals travel about 2 feet per second.

1. How long does it take you to feel an ant crawling on your foot?
2. How much longer does it take to feel a dull ache in your foot?

Exercise 4.1.3.4: Finishing the Race and More Orange Juice

1. Lin runs $2\frac{3}{4}$ miles in $\frac{2}{5}$ of an hour. Tyler runs $8\frac{2}{3}$ miles in $\frac{4}{3}$ of an hour. How long does it take each of them to run 10 miles at that rate?
2. Priya mixes $2\frac{1}{2}$ cups of water with $\frac{1}{3}$ cup of orange juice concentrate. Diego mixes $1\frac{2}{3}$ cups of water with $\frac{1}{4}$ cup orange juice concentrate. How much concentrate should each of them mix with 100 cups of water to make juice that tastes the same as their original recipe? Explain or show your reasoning.

Summary

If we identify two quantities in a problem and one is proportional to the other, then we can calculate the constant of proportionality and use it to answer other questions about the situation. For example, Andre runs at a constant speed, 5 meters every 2 seconds. How long does it take him to run 91 meters at this rate?

In this problem there are two quantities, time (in seconds) and distance (in meters). Since Andre is running at a constant speed, time is proportional to distance. We can make a table with distance and time as column headers and fill in the given information.

Table 4.1.3.5

distance (meters)	time (seconds)
5	2
91	

To find the value in the right column, we multiply the value in the left column by $\frac{2}{5}$ because $\frac{2}{5} \cdot 5 = 2$. This means that it takes Andre $\frac{2}{5}$ seconds to run one meter.

At this rate, it would take Andre $\frac{2}{5} \cdot 91 = \frac{182}{5}$, or 36.4 seconds to walk 91 meters. In general, if t is the time it takes to walk d meters at that pace, then $t = \frac{2}{5}d$.

Glossary Entries

Definition: Percentage

A percentage is a rate per 100.

For example, a fish tank can hold 36 liters. Right now there is 27 liters of water in the tank. The percentage of the tank that is full is 75%.

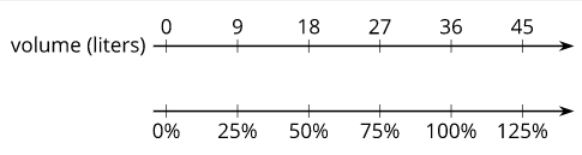


Figure 4.1.3.2

Definition: Unit Rate

A unit rate is a rate per 1.

For example, 12 people share 2 pies equally. One unit rate is 6 people per pie, because $12 \div 2 = 6$. The other unit rate is $\frac{1}{6}$ of a pie per person, because $2 \div 12 = \frac{1}{6}$.

Practice

Exercise 4.1.3.5

It takes an ant farm 3 days to consume $\frac{1}{2}$ of an apple. At that rate, in how many days will the ant farm consume 3 apples?

Exercise 4.1.3.6

To make a shade of paint called jasper green, mix 4 quarts of green paint with $\frac{2}{3}$ cups of black paint. How much green paint should be mixed with 4 cups of black paint to make jasper green?

Exercise 4.1.3.7

An airplane is flying from New York City to Los Angeles. The distance it travels in miles, d, is related to the time in seconds, t, by the equation $d = 0.15t$.

1. How fast is it flying? Be sure to include the units.
2. How far will it travel in 30 seconds?
3. How long will it take to go 12.75 miles?

Exercise 4.1.3.8

A grocer can buy strawberries for $1.38 per pound.

1. Write an equation relating c, the cost, and p, the pounds of strawberries.
2. A strawberry order cost $241.50. How many pounds did the grocer order?

Exercise 4.1.3.9

Crater Lake in Oregon is shaped like a circle with a diameter of about 5.5 miles.

1. How far is it around the perimeter of Crater Lake?
2. What is the area of the surface of Crater Lake?

(From Unit 3.3.1)

Exercise 4.1.3.10

A 50-centimeter piece of wire is bent into a circle. What is the area of this circle?

(From Unit 3.2.3)

Exercise 4.1.3.11

Suppose Quadrilaterals A and B are both squares. Are A and B necessarily scaled copies of one another? Explain.

(From Unit 1.1.2)

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4.1.4: Half as Much Again

Lesson

Let's use fractions to describe increases and decreases.

Exercise 4.1.4.1: Notice and Wonder: TApe Diagrams

What do you notice? What do you wonder?

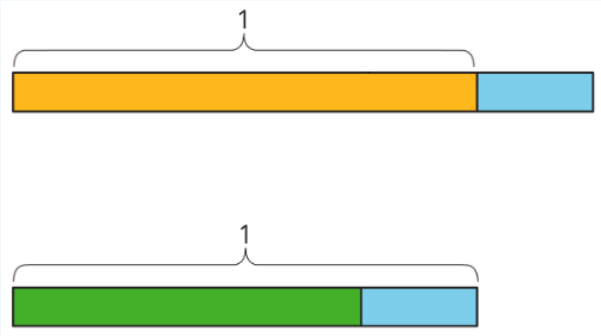


Figure 4.1.4.1: 2 tape diagrams. First diagram. One longer orange section, labeled 1. One smaller blue section. Second diagram. One longer green section and one smaller blue section, all labeled 1.

Exercise 4.1.4.2: Walking Half as Much Again

1. Complete the table to show the total distance walked in each case.
   a. Jada's pet turtle walked 10 feet, and then half that length again.
   b. Jada's baby brother walked 3 feet, and then half that length again.
   c. Jada's hamster walked 4.5 feet, and then half that length again.
   d. Jada's robot walked 1 foot, and then half that length again.
   e. A person walked x feet and then half that length again.

Table 4.1.4.1

initial distance	total distance
10	
3	
4.5	
1	
x	

2. Explain how you computed the total distance in each case.
3. Two students each wrote an equation to represent the relationship between the initial distance walked (x) and the total distance walked (y).
   - Mai wrote $y = x + \frac{1}{2}x$.
   - Kiran wrote $y = \frac{3}{2}x$.

Do you agree with either of them? Explain your reasoning.

Are you ready for more?

Zeno jumped 8 meters. Then he jumped half as far again (4 meters). Then he jumped half as far again (2 meters). So after 3 jumps, he was $8 + 4 + 2 = 14$ meters from his starting place.

1. Zeno kept jumping half as far again. How far would he be after 4 jumps? 5 jumps? 6 jumps?

2. Before he started jumping, Zeno put a mark on the floor that was exactly 16 meters from his starting place. How close can Zeno get to the mark if he keeps jumping half as far again?
3. If you enjoyed thinking about this problem, consider researching Zeno's Paradox.

Exercise 4.1.4.3: More and Less

1. Match each situation with a diagram. A diagram may not have a match.

A

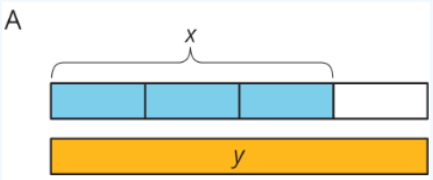


Figure 4.1.4.2

B

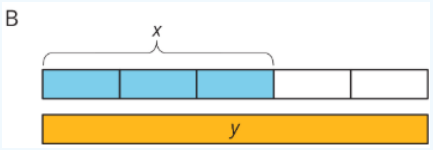


Figure 4.1.4.3

C

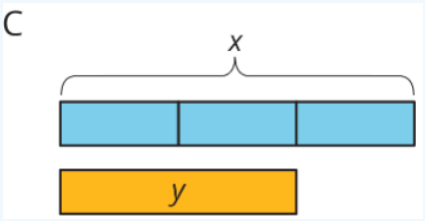


Figure 4.1.4.4

D

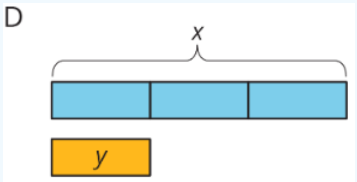


Figure 4.1.4.5

- Han ate x ounces of blueberries. Mai ate $\frac{1}{3}$ less than that.
- Mai biked x miles. Han biked $\frac{2}{3}$ more than that.
- Han bought x pounds of apples. Mai bought $\frac{2}{3}$ of that.

2. For each diagram, write an equation that represents the relationship between x and y.
   a. Diagram A:
   b. Diagram B:
   c. Diagram C:
   d. Diagram D:
3. Write a story for one of the diagrams that doesn't have a match.

Exercise 4.1.4.4: Card Sort: Representations of Proportional Relationships

Your teacher will give you a set of cards that have proportional relationships represented three different ways: as descriptions, equations, and tables. Mix up the cards and place them all face-up.

1. Take turns with a partner to match a description with an equation and a table.
   a. For each match you find, explain to your partner how you know it's a match.
   b. For each match your partner finds, listen carefully to their explanation, and if you disagree, explain your thinking.

2. When you agree on all of the matches, check your answers with the answer key. If there are any errors, discuss why and revise your matches.

Summary

Using the distributive property provides a shortcut for calculating the final amount in situations that involve adding or subtracting a fraction of the original amount.

For example, one day Clare runs 4 miles. The next day, she plans to run that same distance plus half as much again. How far does she plan to run the next day?

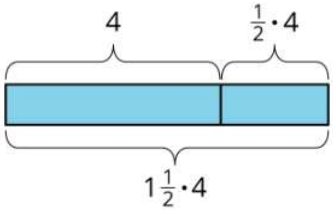


Figure 4.1.4.6

Tomorrow she will run 4 miles plus $\frac{1}{2}$ of 4 miles. We can use the distributive property to find this in one step: $1 \cdot 4 + \frac{1}{2} \cdot 4 = \left(1 + \frac{1}{2}\right) \cdot 4$

Clare plans to run $1\frac{1}{2} \cdot 4$, or 6 miles.

This works when we decrease by a fraction, too. If Tyler spent x dollars on a new shirt, and Noah spent $\frac{1}{3}$ less than Tyler, then Noah spent $\frac{2}{3}x$ dollars since $x - \frac{1}{3}x = \frac{2}{3}x$.

Glossary Entries

Definition: Percentage

A percentage is a rate per 100.

For example, a fish tank can hold 36 liters. Right now there is 27 liters of water in the tank. The percentage of the tank that is full is 75%.

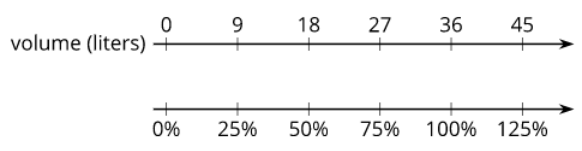


Figure 4.1.4.7

Definition: Tape Diagram

A tape diagram is a group of rectangles put together to represent a relationship between quantities.

For example, this tape diagram shows a ratio of 30 gallons of yellow paint to 50 gallons of blue paint.

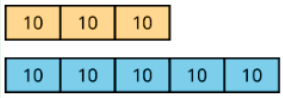


Figure 4.1.4.8

If each rectangle were labeled 5, instead of 10, then the same picture could represent the equivalent ratio of 15 gallons of yellow paint to 25 gallons of blue paint.

Definition: Unit Rate

A unit rate is a rate per 1.

For example, 12 people share 2 pies equally. One unit rate is 6 people per pie, because $12 \div 2 = 6$. The other unit rate is $\frac{1}{6}$ of a pie per person, because $2 \div 12 = \frac{1}{6}$.

Practice

Exercise 4.1.4.5

Match each situation with a diagram.

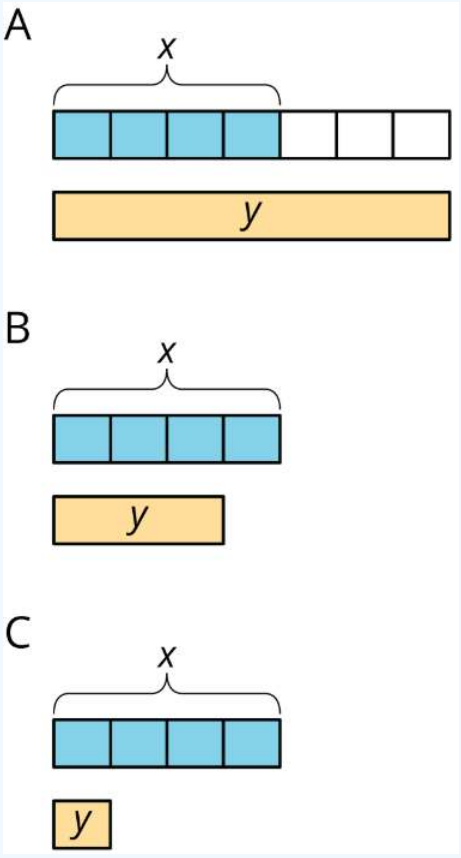


Figure 4.1.4.9: Tape diagram A. 8 equal sections, 4 blue labeled x and 3 white. One longer orange section labeled y. Tape diagram B, 4 equal blue sections labeled x. One orange section labeled y that is the same size as 3 blue sections. Tape diagram C, 4 equal blue sections labeled x. One orange section labeled y that is the same size as one blue section.

A. Diagram A
B. Diagram B
C. Diagram C

1. Diego drank x ounces of juice. Lin drank $\frac{1}{4}$ less than that.
2. Lin ran x miles. Diego ran $\frac{3}{4}$ more than that.
3. Diego bought x pounds of almonds. Lin bought $\frac{1}{4}$ of that.

Exercise 4.1.4.6

Elena walked 12 miles. Then she walked $\frac{1}{4}$ that distance. How far did she walk all together? Select **all** that apply.

A. $12 + \frac{1}{4}$
B. $12 \cdot \frac{1}{4}$
C. $12 + \frac{1}{4} \cdot 12$
D. $12\left(1 + \frac{1}{4}\right)$
E. $12 \cdot \frac{3}{4}$

F. $12 \cdot \frac{5}{4}$

Exercise 4.1.4.7

Write a story that can be represented by the equation $y = x + \frac{1}{4}x$.

Exercise 4.1.4.8

Select **all** ratios that are equivalent to $4 : 5$.

A. $2 : 2.5$
B. $2 : 3$
C. $3 : 3.75$
D. $7 : 8$
E. $8 : 10$
F. $14 : 27.5$

(From Unit 4.1.1)

Exercise 4.1.4.9

Jada is making circular birthday invitations for her friends. The diameter of the circle is 12 cm. She bought 180 cm of ribbon to glue around the edge of each invitation. How many invitations can she make?

(From Unit 3.3.1)

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4.1.5: Say It with Decimals

Lesson

Let's use decimals to describe increases and decreases.

Exercise 4.1.5.1: Notice and Wonder: Fractions to Decimals

A calculator gives the following decimal representations for some unit fractions:

$\frac{1}{2} = 0.5$ $\quad$ $\frac{1}{7} = 0.142857143$

$\frac{1}{3} = 0.333333$ $\quad$ $\frac{1}{8} = 0.125$

$\frac{1}{4} = 0.25$ $\quad$ $\frac{1}{9} = 0.1111111$

$\frac{1}{5} = 0.2$ $\quad$ $\frac{1}{10} = 0.1$

$\frac{1}{6} = 0.1666667$ $\quad$ $\frac{1}{11} = 0.0909091$

What do you notice? What do you wonder?

Exercise 4.1.5.2: Repeating Decimals

1. Use **long division** to express each fraction as a decimal.
   $\frac{9}{25}$ $\quad$ $\frac{11}{30}$ $\quad$ $\frac{4}{11}$
2. What is similar about your answers to the previous question? What is different?
3. Use the decimal representations to decide which of these fractions has the greatest value. Explain your reasoning.

Are you ready for more?

One common approximation for π is $\frac{22}{7}$. Express this fraction as a decimal. How does this approximation compare to 3.14?

Exercise 4.1.5.3: More and Less With Decimals

1. Match each diagram with a description and an equation.

Diagrams:

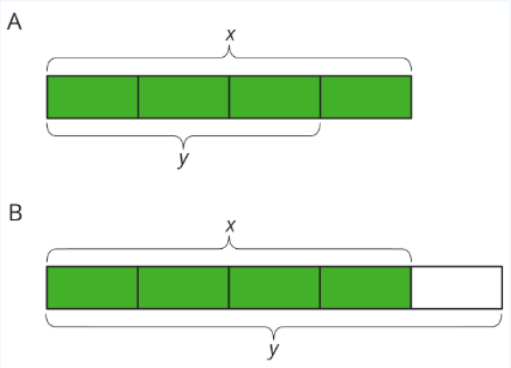


Figure 4.1.5.1: Tape diagram A. 4 green equal sections, all labeled x and 3 labeled y. Tape diagram B. 5 equal sections. 4 green and labeled x. 1 white section. The entire tape diagram labeled y.

Descriptions:

An increase by $\frac{1}{4}$

An increase by $\frac{1}{3}$

An increase by $\frac{2}{3}$

A decrease by $\frac{1}{5}$

A decrease by $\frac{1}{4}$

Equations:

$y = 1.\overline{6}x$

$y = 1.\overline{3}x$

$y = 0.75x$

$y = 0.4x$

$y = 1.25x$

2. Draw a diagram for one of the unmatched equations.

Exercise 4.1.5.4: Card Sort: More Representations

Your teacher will give you a set of cards that have proportional relationships represented 2 different ways: as descriptions and equations. Mix up the cards and place them all face-up.

Take turns with a partner to match a description with an equation.

1. For each match you find, explain to your partner how you know it's a match.
2. For each match your partner finds, listen carefully to their explanation, and if you disagree, explain your thinking.
3. When you have agreed on all of the matches, check your answers with the answer key. If there are any errors, discuss why and revise your matches.

Summary

Long division gives us a way of finding decimal representations for fractions.

For example, to find a decimal representation for $\frac{9}{8}$, we can divide 9 by 8.

$$\begin{array}{r} 1.125 \\ 8\overline{)9.000} \\ \underline{8} \\ 10 \\ \underline{8} \\ 20 \\ \underline{16} \\ 40 \\ \underline{40} \\ 0 \end{array}$$

Figure 4.1.5.2

So $\frac{9}{8} = 1.125$.

Sometimes it is easier to work with the decimal representation of a number, and sometimes it is easier to work with its fraction representation. It is important to be able to work with both. For example, consider the following pair of problems:

- Priya earned x dollars doing chores, and Kiran earned $\frac{6}{5}$ as much as Priya. How much did Kiran earn?
- Priya earned x dollars doing chores, and Kiran earned 1.2 times as much as Priya. How much did Kiran earn?

Since $\frac{6}{5} = 1.2$, these are both exactly the same problem, and the answer is $\frac{6}{5}x$ or $1.2x$. When we work with percentages in later lessons, the decimal representation will come in especially handy.

Glossary Entries

Definition: Long Division

Long division is a way to show the steps for dividing numbers in decimal form. It finds the quotient one digit at a time, from left to right.

For example, here is the long division for $57 \div 4$.

$$\begin{array}{r} 14.25 \\ 4\overline{)57.00} \\ -4 \\ \hline 17 \\ -16 \\ \hline 10 \\ -8 \\ \hline 20 \\ -20 \\ \hline 0 \end{array}$$

Figure 4.1.5.3

Definition: Percentage

A percentage is a rate per 100.

For example, a fish tank can hold 36 liters. Right now there is 27 liters of water in the tank. The percentage of the tank that is full is 75%.

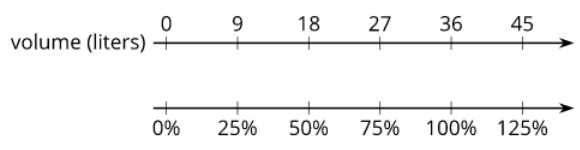


Figure 4.1.5.4

Definition: Repeating Decimal

A repeating decimal has digits that keep going in the same pattern over and over. The repeating digits are marked with a line above them.

For example, the decimal representation for $\frac{1}{3}$ is $0.\overline{3}$, which means 0.3333333 . . . The decimal representation for $\frac{25}{22}$ is $1.1\overline{36}$ which means 1.136363636 . . .

Definition: Tape Diagram

A tape diagram is a group of rectangles put together to represent a relationship between quantities.

For example, this tape diagram shows a ratio of 30 gallons of yellow paint to 50 gallons of blue paint.

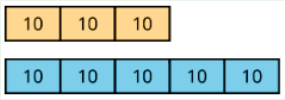


Figure 4.1.5.5

If each rectangle were labeled 5, instead of 10, then the same picture could represent the equivalent ratio of 15 gallons of yellow paint to 25 gallons of blue paint.

Definition: Unit Rate

A unit rate is a rate per 1.

For example, 12 people share 2 pies equally. One unit rate is 6 people per pie, because $12 \div 2 = 6$. The other unit rate is $\frac{1}{6}$ of a pie per person, because $2 \div 12 = \frac{1}{6}$.

Practice

Exercise 4.1.5.5

1. Match each diagram with a description and an equation.

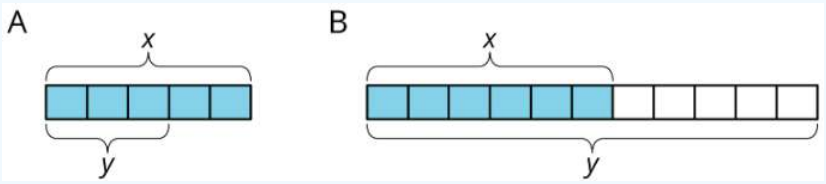


Figure 4.1.5.6

Descriptions:

An increase by $\frac{2}{3}$

An increase by $\frac{5}{6}$

A decrease by $\frac{2}{5}$

A decrease by $\frac{5}{11}$

Equations:

$y = 1.8\overline{3}x$

$y = 1.\overline{6}x$

$y = 0.6x$

$y = 0.4x$

2. Draw a diagram for one of the unmatched equations.

Exercise 4.1.5.6

At the beginning of the month, there were 80 ounces of peanut butter in the pantry. Since then, the family ate 0.3 of the peanut butter. How many ounces of peanut butter are in the pantry now?

A. $0.7 \cdot 80$
B. $0.3 \cdot 80$
C. $80 - 0.3$
D. $(1 + 0.3) \cdot 80$

Exercise 4.1.5.7

1. On a hot day, a football team drank an entire 50-gallon cooler of water and half as much again. How much water did they drink?
2. Jada has 12 library books checked out and Han has $\frac{1}{3}$ less than that. How many books does Han have checked out?

(From Unit 4.1.4)

Exercise 4.1.5.8

If x represents a positive number, select **all** expressions whose value is greater than x.

A. $\left(1 - \frac{1}{4}\right) x$
B. $\left(1 + \frac{1}{4}\right) x$
C. $\frac{7}{8}x$
D. $\frac{9}{8}x$

(From Unit 4.1.4)

Exercise 4.1.5.9

A person's resting heart rate is typically between 60 and 100 beats per minute. Noah looks at his watch, and counts 8 heartbeats in 10 seconds.

1. Is his heart rate typical? Explain how you know.
2. Write an equation for h, the number of times Noah's heart beats (at this rate) in m minutes.

(From Unit 2.2.3)

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4.2: Percent Increase and Decrease

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4.2.1: Increasing and Decreasing

Lesson

Let's use percentages to describe increases and decreases.

Exercise 4.2.1.1: Improving Their Game

Here are the scores from 3 different sports teams from their last 2 games.

Table 4.2.1.1

sports team	total points in game 1	total points in game 2
football team	22	30
basketball team	100	108
baseball team	4	12

1. What do you notice about the teams' scores? What do you wonder?
2. Which team improved the most? Explain your reasoning.

Exercise 4.2.1.2: More Cereal and a Discounted Shirt

1. A cereal box says that now it contains 20% more. Originally, it came with 18.5 ounces of cereal. How much cereal does the box come with now?



Figure 4.2.1.1

2. The price of a shirt is $18.50, but you have a coupon that lowers the price by 20%. What is the price of the shirt after using the coupon?



Figure 4.2.1.2

Exercise 4.2.1.3: Using Tape Diagrams

1. Match each situation to a diagram. Be prepared to explain your reasoning.

    a. Compared with last year’s strawberry harvest, this year’s strawberry harvest is a 25% increase.
    b. This year’s blueberry harvest is 75% of last year’s.
    c. Compared with last year, this year’s peach harvest decreased 25%.

d. This year's plum harvest is 125% of last year's plum harvest.

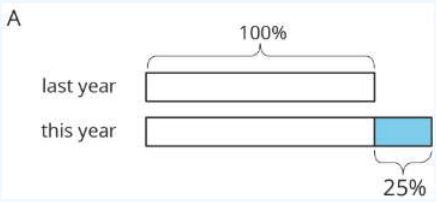


Figure 4.2.1.3

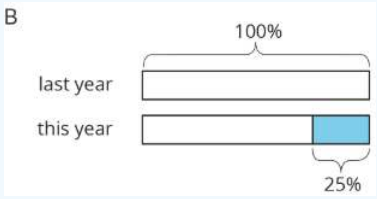


Figure 4.2.1.4

2. Draw a diagram to represent these situations.

   a. The number of ducks living at the pond increased by 40%.
   b. The number of mosquitoes decreased by 80%.

Are you ready for more?

What could it mean to say there is a 100% decrease in a quantity? Give an example of a quantity where this makes sense.

Exercise 4.2.1.4: Agree or Disagree: Percentages

Do you agree or disagree with each statement? Explain your reasoning.

1. Employee A gets a pay raise of 50%. Employee B gets a pay raise of 45%. So Employee A gets the bigger pay raise.
2. Shirts are on sale for 20% off. You buy two of them. As you pay, the cashier says, "20% off of each shirt means 40% off of the total price."

Summary

Imagine that it takes Andre $\frac{3}{4}$ more than the time it takes Jada to get to school. Then we know that Andre's time is $1\frac{3}{4}$ or 1.75 times Jada's time. We can also describe this in terms of percentages:

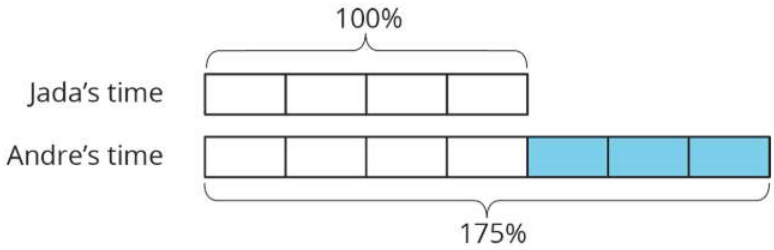


Figure 4.2.1.5

We say that Andre's time is 75% more than Jada's time. We can also see that Andre's time is 175% of Jada's time. In general, the terms **percent increase** and **percent decrease** describe an increase or decrease in a quantity as a percentage of the starting amount.

For example, if there were 500 grams of cereal in the original package, then "20% more" means that 20% of 500 grams has been added to the initial amount, $500 + (0.2) \cdot 500 = 600$, so there are 600 grams of cereal in the new package.



Figure 4.2.1.6

We can see that the new amount is 120% of the initial amount because

$500 + (0.2) \cdot 500 = (1 + 0.2)500$

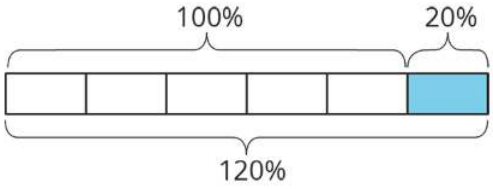


Figure 4.2.1.7: Tape diagram. 6 equal sections, 1 shaded blue. 120 percent labels the entire tape. 100 percent labels 5 sections. 20 percent labels the one blue shaded section.

Glossary Entries

Definition: Percentage Decrease

A percentage decrease tells how much a quantity went down, expressed as a percentage of the starting amount.

For example, a store had 64 hats in stock on Friday. They had 48 hats left on Saturday. The amount went down by 16.

This was a 25% decrease, because 16 is 25% of 64.

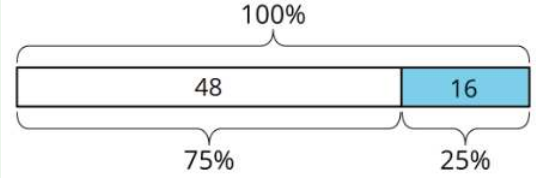


Figure 4.2.1.8

Definition: Percentage Increase

A percentage increase tell how much a quantity went up, expressed as a percentage of the starting amount.

For example, Elena had $50 in the bank on Monday. She had $56 on Tuesday. The amount went up by $6.

This was a 12% increase, because 6 is 12% of 50.

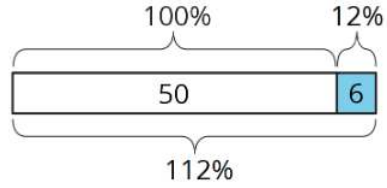


Figure 4.2.1.9

Practice

Exercise 4.2.1.5

For each diagram, decide if y is an increase or a decrease relative to x. Then determine the percent increase or decrease.

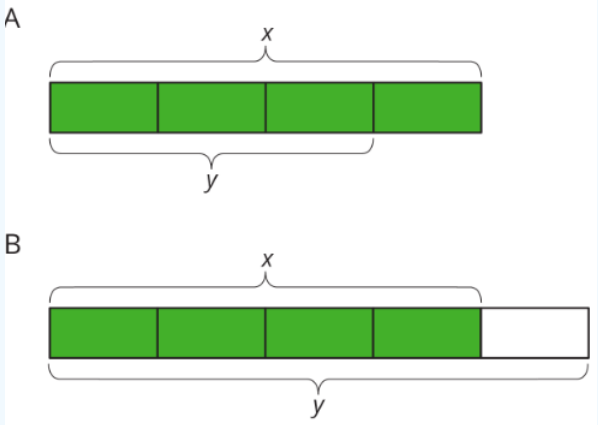


Figure 4.2.1.10: Tape diagram A. 4 green equal sections, all labeled x and 3 labeled y. Tape diagram B. 5 equal sections. 4 green and labeled x. 1 white section. The entire tape diagram labeled y.

Exercise 4.2.1.6

Draw diagrams to represent the following situations.

1. The amount of flour that the bakery used this month was a 50% increase relative to last month.
2. The amount of milk that the bakery used this month was a 75% decrease relative to last month.

Exercise 4.2.1.7

Write each percent increase or decrease as a percentage of the initial amount. The first one is done for you.

1. This year, there was 40% more snow than last year.
   The amount of snow this year is 140% of the amount of snow last year.
2. This year, there were 25% fewer sunny days than last year.
3. Compared to last month, there was a 50% increase in the number of houses sold this month.
4. The runner's time to complete the marathon was a 10% less than the time to complete the last marathon.

Exercise 4.2.1.8

The graph shows the relationship between the diameter and the circumference of a circle with the point $(1, \pi$ shown. Find 3 more points that are on the line.

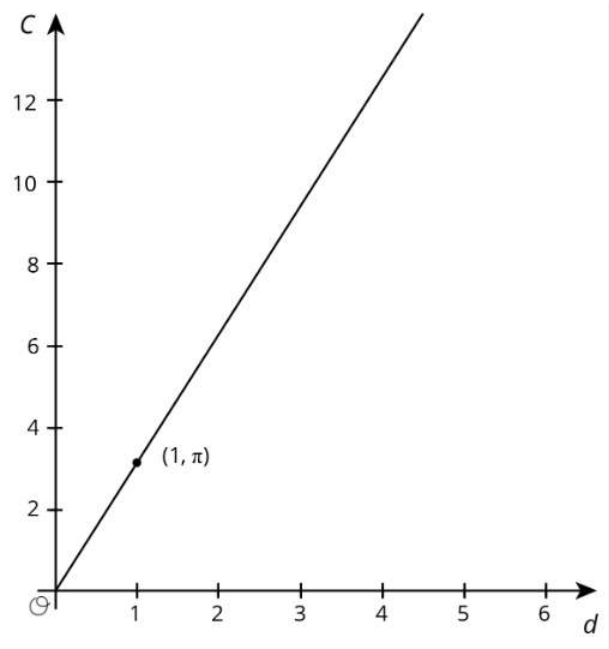


Figure 4.2.1.11

(From Unit 3.1.3)

Exercise 4.2.1.9

Priya bought x grams of flour. Clare bought $\frac{3}{8}$ more than that. Select **all** equations that represent the relationship between the amount of flour that Priya bought, x, and the amount of flour that Clare bought, y.

A. $y = \frac{3}{8}x$
B. $y = \frac{5}{8}x$
C. $y = x + \frac{3}{8}x$
D. $y = x - \frac{3}{8}x$
E. $y = \frac{11}{8}x$

(From Unit 4.1.4)

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4.2.2: One Hundred Percent

Lesson

Let's solve more problems about percent increase and percent decrease.

Exercise 4.2.2.1: Notice and Wonder: Double Number Line

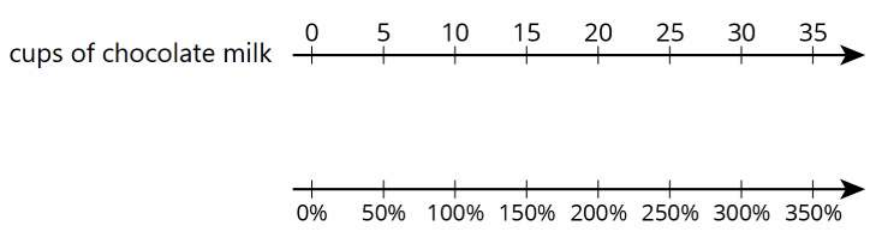


Figure 4.2.2.1

What do you notice? What do you wonder?

Exercise 4.2.2.2: Double Number Lines

For each problem, complete the double number line diagram to show the percentages that correspond to the original amount and to the new amount.

1. The gas tank in dad's car holds 12 gallons. The gas tank in mom's truck holds 50% more than that. How much gas does the truck's tank hold?

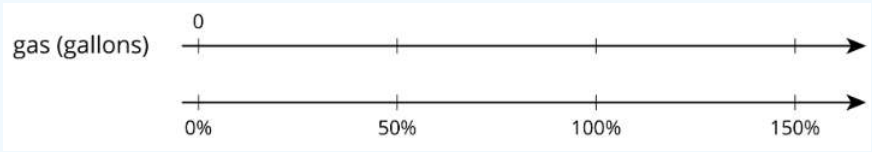


Figure 4.2.2.2

2. At a movie theater, the size of popcorn bags decreased 20%. If the old bags held 15 cups of popcorn, how much do the new bags hold?

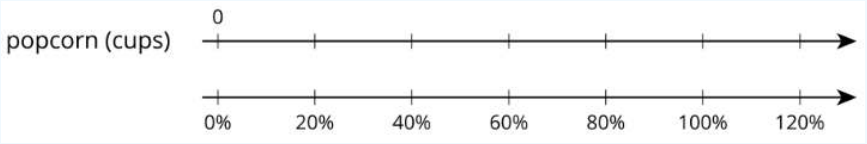


Figure 4.2.2.3: A double number line for popcorn in cups with 7 evenly spaced tick marks. On the top number line the number zero is on the first tick mark and the remaining tick marks are not labeled. The bottom number line, starting with the first tick mark, zero percent, 20 percent, 40 percent, 60 percent, 80 percent, 100 percent, and 120 percent are labeled.

3. A school had 1,200 students last year and only 1,080 students this year. What was the percentage decrease in the number of students?

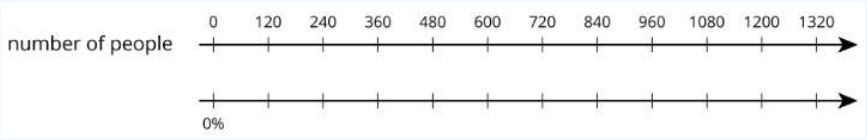


Figure 4.2.2.4: Double number line, top line number of people, evenly spaced tick marks labeled 0, 120, 240, 360, 480, 600, 720, 840, 960, 1080, 1200. Bottom line, corresponding tick marks, first labeled 0% and the rest blank.

4. One week gas was $1.25 per gallon. The next week gas was $1.50 per gallon. By what percentage did the price increase?

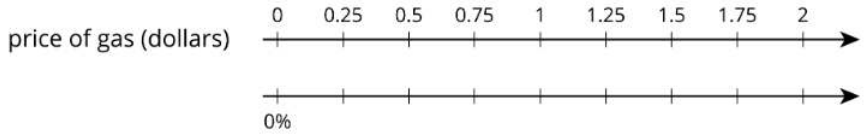


Figure 4.2.2.5: A double number line for price of gas in dollars with 9 evenly spaced tick marks. On the top number line starting with the first tick mark zero, zero point 2 5, zero point 5, zero point 7 5, one, one point 2 5, one point 5, one point 7 5 and 2 are labeled. On the bottom number line, zero percent is on the first tick mark and the remaining tick marks are not labeled.

5. After a 25% discount, the price of a T-shirt was $12. What was the price before the discount?

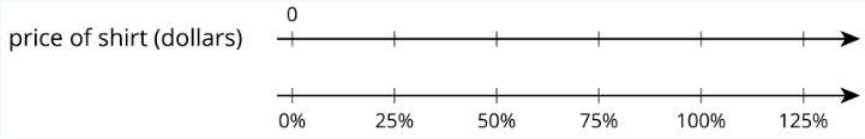


Figure 4.2.2.6: A double number line for price of shirts in dollars with 6 evenly spaced tick marks. On the top number line, the number zero is on the first tick mark and the remaining tick marks are not labeled. On the bottom number line, starting with the first tick mark, zero percent, 25 percent, 50 percent, 75 percent, 100 percent and 125 percent are labeled.

6. Compared to last year, the population of Boom Town has increased 25%.The population is now 6,600. What was the population last year?

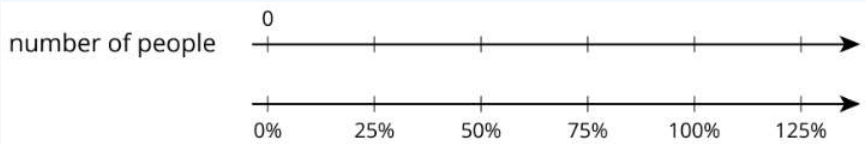


Figure 4.2.2.7: A double number line for number of people with 6 evenly spaced tick marks. For the top number line, the number 0 is on the first tick mark and the remaining tick marks are blank. For the bottom number line, starting with the first tick mark, the percentages 0%, 25%, 50%, 75%, 100%, and 125% are labeled.

Exercise 4.2.2.3: Representing More Juice

Two students are working on the same problem:

A juice box has 20% more juice in its new packaging. The original packaging held 12 fluid ounces. How much juice does the new packaging hold?

- Here is how Priya set up her double number line.

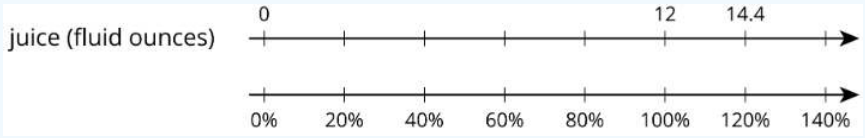


Figure 4.2.2.8: A double number line for juice in fluid ounces with 8 evenly spaced tick marks. The top number line, the number 0 is on the first tick mark, 12 on the sixth, and 14 point 4 on the seventh. The bottom number line, starting with the first tick mark, zero percent, 20 percent, 40 percent, 60 percent, 80 percent, 100 percent, 120 percent and 140 percent are labeled.

- Here is how Clare set up her double number line.

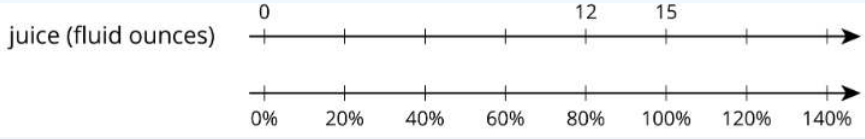


Figure 4.2.2.9: A double number line for juice in fluid ounces with 8 evenly spaced tick marks. The top number line has the number zero on the first tick mark, 12 on the fifth, and 15 on the sixth. The bottom number line, starting with the first tick mark, zero percent, 20 percent, 40 percent, 60 percent, 80 percent, 100 percent, 120 percent and 140 percent are labeled.

Do you agree with either of them? Explain or show your reasoning.

Are you ready for more?

Clare's diagram could represent a percent decrease. Describe a situation that could be represented with Clare's diagram.

Exercise 4.2.2.4: Protecting the Green Sea Turtle

Green sea turtles live most of their lives in the ocean, but come ashore to lay their eggs. Some beaches where turtles often come ashore have been made into protected sanctuaries so the eggs will not be disturbed.



Figure 4.2.2.10: By Dominique Feldwick-Davis . Public Domain. Pexels. Source.

1. One sanctuary had 180 green sea turtles come ashore to lay eggs last year. This year, the number of turtles increased by 10%. How many turtles came ashore to lay eggs in the sanctuary this year?
2. At another sanctuary, the number of nesting turtles decreased by 10%. This year there were 234 nesting turtles. How many nesting turtles were at this sanctuary last year?

Summary

We can use a double number line diagram to show information about percent increase and percent decrease:

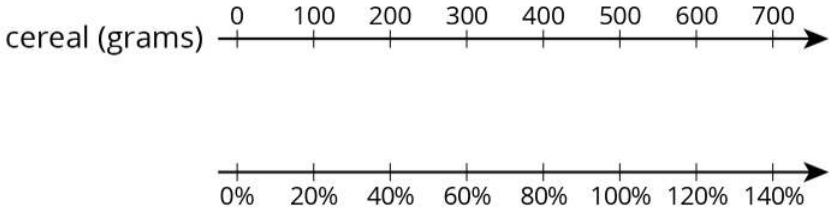


Figure 4.2.2.11: Double number line, evenly spaced tick marks. Top line, cereal, grams. Beginning at first tick mark, labels: 0, 100, 200, 300, 400, 500, 600, 700. Bottom line, beginning at first tick mark, labels: 0 percent, 20 percent, 40 percent, 60 percent, 80 percent, 100 percent, 120 percent, 140 percent.

The initial amount of cereal is 500 grams, which is lined up with 100% in the diagram. We can find a 20% *increase* to 500 by adding 20% of 500:

$$\begin{aligned} 500 + (0.2) \cdot 500 &= (1.20) \cdot 500 \\ &= 600 \end{aligned}$$

In the diagram, we can see that 600 corresponds to 120%.

If the initial amount of 500 grams is *decreased* by 40%, we can find how much cereal there is by subtracting 40% of the 500 grams:

$$\begin{aligned} 500 - (0.4) \cdot 500 &= (0.6) \cdot 500 \\ &= 300 \end{aligned}$$

So a 40% decrease is the same as 60% of the initial amount. In the diagram, we can see that 300 is lined up with 60%.

To solve percentage problems, we need to be clear about what corresponds to 100%. For example, suppose there are 20 students in a class, and we know this is an increase of 25% from last year. In this case, the number of students in the class *last* year corresponds to 100%. So the initial amount (100%) is unknown and the final amount (125%) is 20 students.

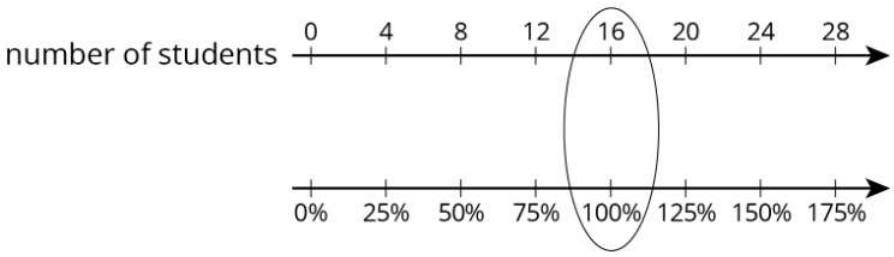


Figure 4.2.2.12: Double number line, evenly spaced tick marks. Top line, number of students. Beginning at first tick mark, labels: 0, 4, 8, 12, 16, 20, 24, 28. Bottom line, beginning at first tick mark, labels: 0 percent, 25 percent, 50 percent, 75 percent, 100 percent, 125 percent, 150 percent, 175 percent. 16 and 100 percent are circled.

Looking at the double number line, if 20 students is a 25% increase from the previous year, then there were 16 students in the class last year.

Glossary Entries

Definition: Percentage Decrease

A percentage decrease tells how much a quantity went down, expressed as a percentage of the starting amount.

For example, a store had 64 hats in stock on Friday. They had 48 hats left on Saturday. The amount went down by 16.

This was a 25% decrease, because 16 is 25% of 64.

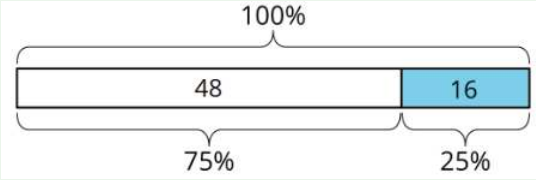


Figure 4.2.2.13

Definition: Percentage Increase

A percentage increase tell how much a quantity went up, expressed as a percentage of the starting amount.

For example, Elena had $50 in the bank on Monday. She had $56 on Tuesday. The amount went up by $6.

This was a 12% increase, because 6 is 12% of 50.

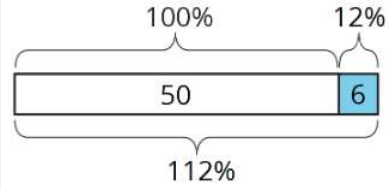


Figure 4.2.2.14

Practice

Exercise 4.2.2.5

A bakery used 25% more butter this month than last month. If the bakery used 240 kilograms of butter last month, how much did it use this month?

Exercise 4.2.2.6

Last week, the price of oranges at the farmer's market was $1.75 per pound. This week, the price has decreased by 20%. What is the price of oranges this week?

Exercise 4.2.2.7

Noah thinks the answers to these two questions will be the same. Do you agree with him? Explain your reasoning.

- This year, a herd of bison had a 10% increase in population. If there were 550 bison in the herd last year, how many are in the herd this year?
- This year, another herd of bison had a 10% decrease in population. If there are 550 bison in the herd this year, how many bison were there last year?

Exercise 4.2.2.8

Elena walked 12 miles. Then she walked 0.25 that distance. How far did she walk all together? Select **all** that apply.

A. $12 + 0.25 \cdot 12$
B. $12(1 + 0.25)$
C. $12 \cdot 1.25$
D. $12 \cdot 0.25$
E. $12 + 0.25$

(From Unit 4.1.5)

Exercise 4.2.2.9

A circle's circumference is 600 m. What is a good approximation of the circle's area?

A. $300\,\text{m}^2$
B. $3,000\,\text{m}^2$
C. $30,000\text{m}^2$
D. $300,000\text{m}^2$

(From Unit 3.2.3)

Exercise 4.2.2.10

The equation $d = 3t$ represents the relationship between the distance (d) in inches that a snail is from a certain rock and the time (t) in minutes.

1. What does the number 3 represent?
2. How many minutes does it take the snail to get 9 inches from the rock?
3. How far will the snail be from the rock after 9 minutes?

(From Unit 2.2.3)

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4.2.3: Percent Increase and Decrease with Equations

Lesson

Let's use equations to represent increases and decreases.

Exercise 4.2.3.1: Number Talk: From 100 to 106

How do you get from one number to the next using multiplication or division?

From 100 to 106

From 100 to 90

From 90 to 100

From 106 to 100

Exercise 4.2.3.2: Interest and Depreciation

1. Money in a particular savings account increases by about 6% after a year. How much money will be in the account after one year if the initial amount is \$100? \$50? \$200? \$125? x dollars? If you get stuck, consider using diagrams or a table to organize your work.
2. The value of a new car decreases by about 15% in the first year. How much will a car be worth after one year if its initial value was \$1,000? \$5,000? \$5,020? x dollars? If you get stuck, consider using diagrams or a table to organize your work.



Figure 4.2.3.1



Figure 4.2.3.2

Exercise 4.2.3.3: Matching Equations

Match an equation to each of these situations. Be prepared to share your reasoning.

1. The water level in a reservoir is now 52 meters. If this was a 23% increase, what was the initial depth?
2. The snow is now 52 inches deep. If this was a 77% decrease, what was the initial depth?

$0.23x = 52$

$0.77x = 52$

$1.23x = 52$

$1.77x = 52$

Are you ready for more?

An astronaut was exploring the moon of a distant planet, and found some glowing goo at the bottom of a very deep crater. She brought a 10-gram sample of the goo to her laboratory. She found that when the goo was exposed to light, the total amount of goo increased by 100% every hour.

1. How much goo will she have after 1 hour? After 2 hours? After 3 hours? After n hours?
2. When she put the goo in the dark, it shrank by 75% every hour. How many hours will it take for the goo that was exposed to light for n hours to return to the original size?

Exercise 4.2.3.4: Representing Percent Increase and Decrease: EQuations

1. The gas tank in dad's car holds 12 gallons. The gas tank in mom's truck holds 50% more than that. How much gas does the truck's tank hold? Explain why this situation can be represented by the equation $(1.5) \cdot 12 = t$. Make sure that you explain what t represents.
2. Write an equation to represent each of the following situations.
   a. A movie theater decreased the size of its popcorn bags by 20%. If the old bags held 15 cups of popcorn, how much do the new bags hold?
   b. After a 25% discount, the price of a T-shirt was $12. What was the price before the discount?
   c. Compared to last year, the population of Boom Town has increased by 25%.The population is now 6,600. What was the population last year?

Summary

We can use equations to express percent increase and percent decrease. For example, if y is 15% more than x,

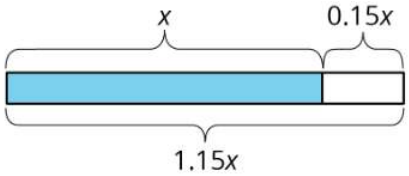


Figure 4.2.3.3

we can represent this using any of these equations:

$$y = x + 0.15x \qquad y = (1 + 0.15)x \qquad y = 1.15x$$

So if someone makes an investment of x dollars, and its value increases by 15% to $1250, then we can write and solve the equation $1.15x = 1250$to find the value of the initial investment.

Here is another example: if a is 7% less than b,

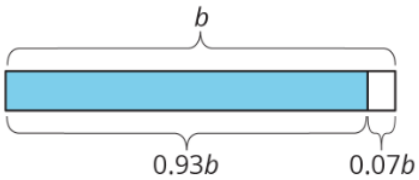


Figure 4.2.3.4

we can represent this using any of these equations:

$$a = b - 0.07b \qquad a = (1 - 0.07)b \qquad a = 0.93b$$

So if the amount of water in a tank decreased 7% from its starting value of b to its ending value of 348 gallons, then you can write $0.93b = 348$.

Often, an equation is the most efficient way to solve a problem involving percent increase or percent decrease.

Glossary Entries

Definition: Percentage Decrease

A percentage decrease tells how much a quantity went down, expressed as a percentage of the starting amount.

For example, a store had 64 hats in stock on Friday. They had 48 hats left on Saturday. The amount went down by 16.

This was a 25% decrease, because 16 is 25% of 64.

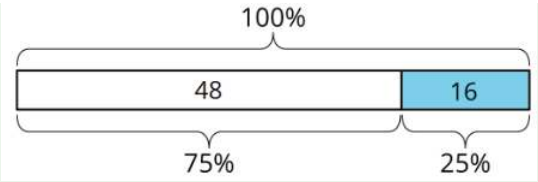


Figure 4.2.3.5

Definition: Percentage Increase

A percentage increase tell how much a quantity went up, expressed as a percentage of the starting amount.

For example, Elena had $50 in the bank on Monday. She had $56 on Tuesday. The amount went up by $6.

This was a 12% increase, because 6 is 12% of 50.

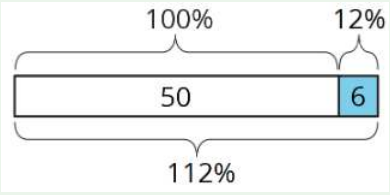


Figure 4.2.3.6

Practice

Exercise 4.2.3.5

A pair of designer sneakers was purchased for $120. Since they were purchased, their price has increased by 15%. What is the new price?

Exercise 4.2.3.6

Elena's aunt bought her a $150 savings bond when she was born. When Elena is 20 years old, the bond will have earned 105% in interest. How much will the bond be worth when Elena is 20 years old?

Exercise 4.2.3.7

In a video game, Clare scored 50% more points than Tyler. If c is the number of points that Clare scored and t is the number of points that Tyler scored, which equations are correct? Select **all** that apply.

A. $c = 1.5t$
B. $c = t + 0.5$
C. $c = t + 0.5t$
D. $c = t + 50$
E. $c = (1 + 0.5)t$

Exercise 4.2.3.8

Draw a diagram to represent each situation:

1. The number of miles driven this month was a 30% decrease of the number of miles driven last month.
2. The amount of paper that the copy shop used this month was a 25% increase of the amount of paper they used last month.

(From Unit 4.2.1)

Exercise 4.2.3.9

Which decimal is the best estimate of the fraction $\frac{29}{40}$?

A. 0.5
B. 0.6
C. 0.7
D. 0.8

(From Unit 4.1.5)

Exercise 4.2.3.10

Could 7.2 inches and 28 inches be the diameter and circumference of the same circle? Explain why or why not.

(From Unit 3.1.3)

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4.2.4: More and Less than 1 Percent

Lesson

Let's explore percentages smaller than 1%.

Exercise 4.2.4.1: Number Talk: What Percentage?

Determine the percentage mentally.

10 is what percentage of 50?

5 is what percentage of 50?

1 is what percentage of 50?

17 is what percentage of 50?

Exercise 4.2.4.2: Waiting Tables

During one waiter's shift, he delivered appetizers, entrées, and desserts. What percentage of the dishes were desserts? appetizers? entrées? What do your percentages add up to?

Exercise 4.2.4.3: Fractions of a Percent

1. Find each percentage of 60. What do you notice about your answers?

$$30\% \text{ of } 60 \qquad 3\% \text{ of } 60 \qquad 0.3\% \text{ of } 60 \qquad 0.03\% \text{ of } 60$$

2. 20% of 5,000 is 1,000 and 21% of 5,000 is 1,050. Find each percentage of 5,000 and be prepared to explain your reasoning. If you get stuck, consider using the double number line diagram.

a. 1% of 5,000
b. 0.1% of 5,000
c. 20.1% of 5,000
d. 20.4% of 5,000

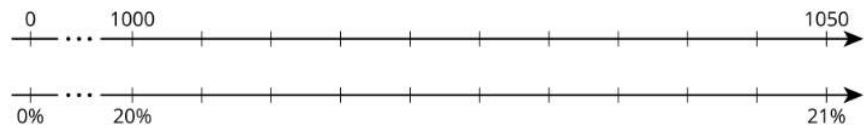


Figure 4.2.4.1: A double number line with 12 tick marks. The first tick mark is followed by a break and then 11 evenly spaced tick marks. For the top number line, the number 0 is on the first tick mark, 1000 on the second, and 1050 on the twelfth. For the bottom number line, the percentage 0% is on the first tick mark, 20% on the second, and 21% on the twelfth.

3. 15% of 80 is 12 and 16% of 80 is 12.8. Find each percentage of 80 and be prepared to explain your reasoning.

a. 15.1% of 80
b. 15.7% of 80

Are you ready for more?

To make Sierpinski's triangle,

- Start with an equilateral triangle. This is step 1.
- Connect the midpoints of every side, and remove the middle triangle, leaving three smaller triangles. This is step 2.
- Do the same to each of the remaining triangles. This is step 3.
- Keep repeating this process.

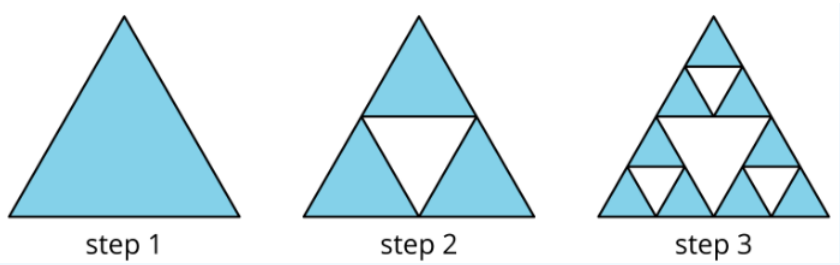


Figure 4.2.4.2: Step 1, blue triangle. Step 2, triangle broken into 4 equal sized triangles, 1 white and 3 blue. Step 3, triangle broken into 12 small triangles, 9 blue and 3 white and one larger white triangle in center.

1. What percentage of the area of the original triangle is left after step 2? Step 3? Step 10?
2. At which step does the percentage first fall below 1%?

Exercise 4.2.4.4: Population Growth

1. The population of City A was approximately 243,000 people, and it increased by 8% in one year. What was the new population?
2. The population of city B was approximately 7,150,000, and it increased by 0.8% in one year. What was the new population?

Summary

A percentage, such as 30%, is a rate per 100. To find 30% of a quantity, we multiply it by $30 \div 100$, or 0.3.

The same method works for percentages that are not whole numbers, like 7.8% or 2.5%. In the square, 2.5% of the area is shaded.

To find 2.5% of a quantity, we multiply it by $2.5 \div 100$, or 0.025. For example, to calculate 2.5% interest on a bank balance of \$80, we multiply $(0.025) \cdot 80 = 2$, so the interest is \$2.

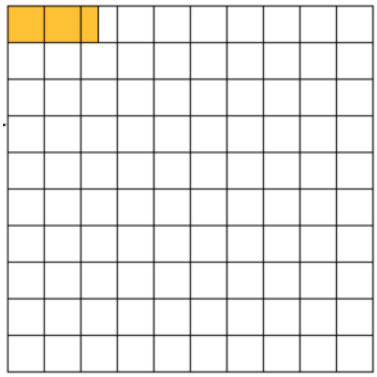


Figure 4.2.4.3

We can sometimes find percentages like 2.5% mentally by using convenient whole number percents. For example, 25% of 80 is one fourth of 80, which is 20. Since 2.5 is one tenth of 25, we know that 2.5% of 80 is one tenth of 20, which is 2.

Glossary Entries

Definition: Percentage Decrease

A percentage decrease tells how much a quantity went down, expressed as a percentage of the starting amount.

For example, a store had 64 hats in stock on Friday. They had 48 hats left on Saturday. The amount went down by 16.

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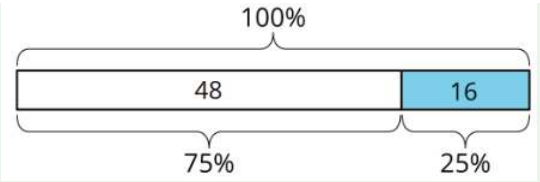


Figure 4.2.4.4

Definition: Percentage Increase

A percentage increase tell how much a quantity went up, expressed as a percentage of the starting amount.

For example, Elena had $50 in the bank on Monday. She had $56 on Tuesday. The amount went up by $6.

This was a 12% increase, because 6 is 12% of 50.

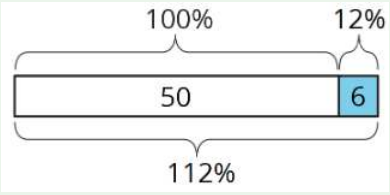


Figure 4.2.4.5

Practice

Exercise 4.2.4.5

The student government snack shop sold 32 items this week. For each snack type, what percentage of all snacks sold were of that type?

Table 4.2.4.1

snack type	number of items sold
fruit cup	8
veggie sticks	6
chips	14
water	4

Exercise 4.2.4.6

Select **all** the options that have the same value as $3\frac{1}{2}\%$ of 20.

A. 3.5% of 20
B. $3\frac{1}{2} \cdot 20$
C. $(0.35) \cdot 20$
D. $(0.035) \cdot 20$
E. 7% of 10

Exercise 4.2.4.7

22% of 65 is 14.3. What is 22.6% of 65? Explain your reasoning.

Exercise 4.2.4.8

A bakery used 30% more sugar this month than last month. If the bakery used 560 kilograms of sugar last month, how much did it use this month?

(From Unit 4.2.2)

Exercise 4.2.4.9

Match each situation to a diagram. The diagrams can be used more than once.

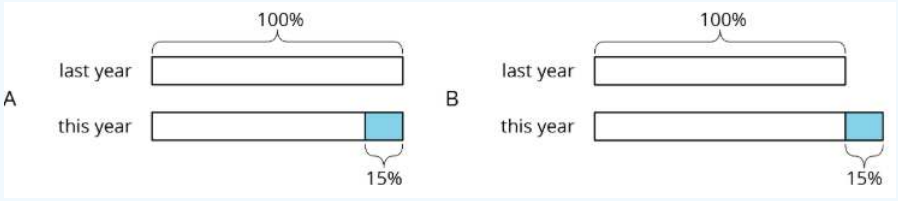


Figure 4.2.4.6: Tape diagram A. Both tapes same size. Last year, entire tape labeled 100 percent. This year, small blue portion of tape labeled 15 percent. Tape diagram B. Last year, entire tape labeled 100 percent. This year, small blue portion of tape labeled 15 percent and extends farther than last year tape.

A. The amount of apples this year decreased by 15% compared with last year's amount.
B. The amount of pears this year is 85% of last year's amount.
C. The amount of cherries this year increased by 15% compared with last year's amount.
D. The amount of oranges this year is 115% of last year's amount.

1. Diagram A
2. Diagram B

(From Unit 4.2.1)

Exercise 4.2.4.10

A certain type of car has room for 4 passengers.

1. Write an equation relating the number of cars (n) to the number of passengers (p).
2. How many passengers could fit in 78 cars?
3. How many cars would be needed to fit 78 passengers?

(From Unit 2.2.3)

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4.3: Applying Percentages

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4.3.1: Tax and Tip

Lesson

Let's learn about sales tax and tips.

Exercise 4.3.1.1: Notice and Wonder: The Price of SUnglasses

You are on vacation and want to buy a pair of sunglasses for $10 or less. You find a pair with a price tag of $10. The cashier says the total cost will be $10.45.



Figure 4.3.1.1

What do you notice? What do you wonder?

Exercise 4.3.1.2: Shopping in Two Different Cities

Different cities have different sales tax rates. Here are the sales tax charges on the same items in two different cities. Complete the tables.

City 1

Table 4.3.1.1

item	price (dollars)	sales tax (dollars)	total cost (dollars)
paper towels	8.00	0.48	8.48
lamp	25.00	\(1.50	
pack of gum	1.00		
laundry soap	12.00		
	x		

City 2

Table 4.3.1.2

item	price (dollars)	sales tax (dollars)	total cost (dollars)

item	price (dollars)	sales tax (dollars)	total cost (dollars)
paper towels	8.00	0.64	8.64
lamp	25.00	2.00	
pack of gum	1.00		
laundry soap	12.00		
	x		

Exercise 4.3.1.3: Shopping in a Third City

Here is the sales tax on the same items in City 3.

Table 4.3.1.3

item	price (dollars)	sales tax (dollars)
paper towels	8.00	0.58
lamp	25.00	1.83
pack of gum	1.00	0.07
laundry soap	12.00	

1. What is the tax rate in this city?
2. For the sales tax on the laundry soap, Kiran says it should be $0.84. Lin says it should be $0.87. Do you agree with either of them? Explain your reasoning.

Exercise 4.3.1.4: Dining at a Restaurant

1. Jada has a meal in a restaurant. She adds up the prices listed on the menu for everything they ordered and gets a subtotal of $42.00.

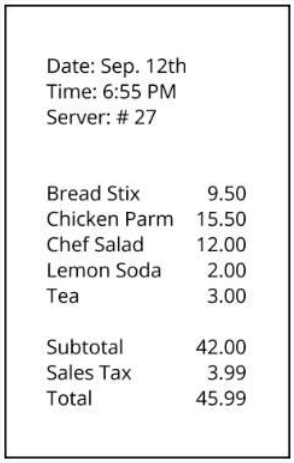


Figure 4.3.1.2

   a. When the check comes, it says they also need to pay $3.99 in sales tax. What percentage of the subtotal is the sales tax?
   b. After tax, the total is $45.99. What percentage of the subtotal is the total?
   c. They actually pay $52.99. The additional $7 is a tip for the server. What percentage of the subtotal is the tip?

2. The tax rate at this restaurant is 9.5%.



Figure 4.3.1.3

Another person's subtotal is $24.95. How much will their sales tax be?

Some other person's sales tax is $1.61. How much was their subtotal?

Are you ready for more?

Elena's cousins went to a restaurant. The part of the entire cost of the meal that was tax and tip together was 25% of the cost of the food alone. What could the tax rate and tip rate be?

Summary

Many places have *sales tax*. A sales tax is an amount of money that a government agency collects on the sale of certain items. For example, a state might charge a tax on all cars purchased in the state. Often the tax rate is given as a percentage of the cost. For example, a state's tax rate on car sales might be 2%, which means that for every car sold in that state, the buyer has to pay a tax that is 2% of the sales price of the car.

Fractional percentages often arise when a state or city charges a sales tax on a purchase. For example, the sales tax in Arizona is 7.5%. This means that when someone buys something, they have to add 0.075 times the amount on the price tag to determine the total cost of the item.

For example, if the price tag on a T-shirt in Arizona says $11.50, then the sales tax is $(0.075) \cdot 11.5 = 0.8625$, which rounds to 86 cents. The customer pays $11.50 + 0.86$, or $12.36 for the shirt.

The total cost to the customer is the item price plus the sales tax. We can think of this as a percent increase. For example, in Arizona, the total cost to a customer is 107.5% of the price listed on the tag.

A *tip* is an amount of money that a person gives someone who provides a service. It is customary in many restaurants to give a tip to the server that is between 10% and 20% of the cost of the meal. If a person plans to leave a 15% tip on a meal, then the total cost will be 115% of the cost of the meal.

Practice

Exercise 4.3.1.5

In a city in Ohio, the sales tax rate is 7.25%. Complete the table to show the sales tax and the total price including tax for each item.

Table 4.3.1.4

item	price before tax ($)	sales tax ($)	price including tax ($)
pillow	8.00		
blanket	22.00		

item	price before tax ($)	sales tax ($)	price including tax ($)
trash can	14.50		

Exercise 4.3.1.6

The sales tax rate in New Mexico is 5.125%. Select **all** the equations that represent the sales tax, t, you would pay in New Mexico for an item of cost c?

A. $t = 5.125c$
B. $t = 0.5125c$
C. $t = 0.05125c$
D. $t = c \div 0.05125$
E. $t = \frac{5.125}{100}c$

Exercise 4.3.1.7

Here are the prices of some items and the amount of sales tax charged on each in Nevada.

Table 4.3.1.5

cost of item ($)	sales tax ($)
10	0.46
50	2.30
5	0.23

1. What is the sales tax rate in Nevada?
2. Write an expression for the amount of sales tax charged, in dollars, on an item that costs c dollars.

Exercise 4.3.1.8

Find each amount:

1. 3.8% of 25
2. 0.2% of 50
3. 180.5% of 99

(From Unit 4.2.4)

Exercise 4.3.1.9

On Monday, the high was 60 degrees Fahrenheit. On Tuesday, the high was 18% more. How much did the high increase from Monday to Tuesday?

(From Unit 4.2.3)

Exercise 4.3.1.10

Complete the table. Explain or show your reasoning.

Table 4.3.1.6

object	radius	circumference
ceiling fan	2.8 ft	
water bottle cap	13 mm	

object	radius	circumference
bowl		56.5 cm
drum		75.4 in

(From Unit 3.1.4)

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4.3.2: Percentage Contexts

Lesson

Let's learn about more situations that involve percentages.

Exercise 4.3.2.1: Leaving a Tip

Which of these expressions represent a 15% tip on a $20 meal? Which represent the total bill?

$15 \cdot 20$

$20 + 0.15 \cdot 20$

$1.15 \cdot 20$

$\frac{15}{100} \cdot 20$

Exercise 4.3.2.2: A Car Dealership

A car dealership pays a wholesale price of $12,000 to purchase a vehicle.

1. The car dealership wants to make a 32% profit.
    a. By how much will they mark up the price of the vehicle?
    b. After the markup, what is the retail price of the vehicle?



Figure 4.3.2.1: Cars , by Pexels. Public Domain. Pixabay. Source.

2. During a special sales event, the dealership offers a 10% discount off of the retail price. After the discount, how much will a customer pay for this vehicle?

Are you ready for more?

This car dealership pays the salesperson a bonus for selling the car equal to 6.5% of the sale price. How much commission did the salesperson lose when they decided to offer a 10% discount on the price of the car?

Exercise 4.3.2.3: Commission at a Gym

1. For each gym membership sold, the gym keeps $42 and the employee who sold it gets $8. What is the commission the employee earned as a percentage of the total cost of the gym membership?
2. If an employee sells a family pass for $135, what is the amount of the commission they get to keep?

Exercise 4.3.2.4: Card Sort: Percentage Situations

Your teacher will give you a set of cards. Take turns with your partner matching a situation with a descriptor. For each match, explain your reasoning to your partner. If you disagree, work to reach an agreement.

Summary

There are many everyday situations where a percentage of an amount of money is added to or subtracted from that amount, in order to be paid to some other person or organization:

Table 4.3.2.1

	goes to	how it works

	goes to	how it works
sales tax	the government	added to the price of the item
gratuity (tip)	the server	added to the cost of the meal
interest	the lender (or account holder)	added to the balance of the loan, credit card, or bank account
markup	the seller	added to the price of an item so the seller can make a profit
markdown (discount)	the customer	subtracted from the price of an item to encourage the customer to buy it
commission	the salesperson	subtracted from the payment that is collected

For example,

- If a restaurant bill is \$34 and the customer pays \$40, they left \$6 dollars as a tip for the server. That is 18% of \$34, so they left an 18% tip. From the customer's perspective, we can think of this as an 18% increase of the restaurant bill.
- If a realtor helps a family sell their home for \$200,000 and earns a 3% commission, then the realtor makes \$6,000, because $(0.03) \cdot 200,000 = 6,000$ and the family gets \$194,000, because $200,000 - 6,000 = 194,000$ From the family's perspective, we can think of this as a 3% decrease on the sale price of the home.

Practice

Exercise 4.3.2.5

A car dealership pays \$8,350 for a car. They mark up the price by 17.4% to get the retail price. What is the retail price of the car at this dealership?

Exercise 4.3.2.6

A store has a 20% off sale on pants. With this discount, the price of one pair of pants before tax is \$15.20. What was the original price of the pants?

A. \$3.04
B. \$12.16
C. \$18.24
D. \$19.00

Exercise 4.3.2.7

Lin is shopping for a couch with her dad and hears him ask the salesperson, “How much is your commission?” The salesperson says that her commission is $5\frac{1}{2}\%$ of the selling price.

1. How much commission will the salesperson earn by selling a couch for \$495?
2. How much money will the store get from the sale of the couch?

Exercise 4.3.2.8

A college student takes out a \$7,500 loan from a bank. What will the balance of the loan be after one year (assuming the student has not made any payments yet):

1. if the bank charges 3.8% interest each year?
2. if the bank charges 5.3% interest each year?

(From Unit 4.2.4)

Exercise 4.3.2.9

Match the situations with the equations.

1. Mai slept for x hours, and Kiran slept for $\frac{1}{10}$ less than that.
2. Kiran practiced the piano for x hours, and Mai practiced for $\frac{2}{5}$ less than that.
3. Mai drank x oz of juice and Kiran drank $\frac{4}{3}$ more than that.
4. Kiran spent x dollars and Mai spent $\frac{1}{4}$ less than that.
5. Mai ate x grams of almonds and Kiran ate 1.5 times more than that.
6. Kiran collected x pounds of recycling and Mai collected $\frac{3}{10}$ less than that.
7. Mai walked x kilometers and Kiran walked $\frac{3}{8}$ more than that.
8. Kiran completed x puzzles and Mai completed $\frac{3}{5}$ more than that.

$y = 2.33x$

$y = 1.375x$

$y = 0.6x$

$y = 0.9x$

$y = 0.75x$

$y = 1.6x$

$y = 0.7x$

$y = 2.5x$

(From Unit 4.1.5)

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4.3.3: Finding the Percentage

Lesson

Let's find unknown percentages.

Exercise 4.3.3.1: Tax, Tip, and Discount

What percentage of the car price is the tax?

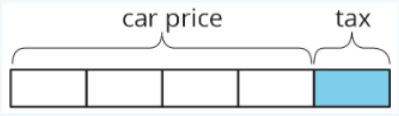


Figure 4.3.3.1

What percentage of the food cost is the tip?

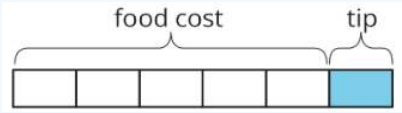


Figure 4.3.3.2

What percentage of the shirt cost is the discount?

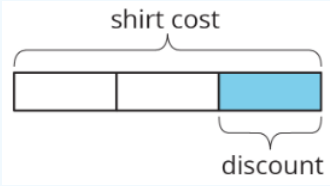


Figure 4.3.3.3

Exercise 4.3.3.2: What is the Percentage?

1. A salesperson sold a car for $18,250 and their commission is $693.50. What percentage of the sale price is their commission?
2. The bill for a meal was $33.75. The customer left $40.00. What percentage of the bill was the tip?
3. The original price of a bicycle was $375. Now it is on sale for $295. What percentage of the original price was the markdown?

Are you ready for more?

To make a Koch snowflake,

- Start with an equilateral triangle. This is step 1.
- Divide each side into 3 equal pieces. Construct a smaller equilateral triangle on the middle third. This is step 2.
- Do the same to each of the newly created sides. This is step 3.
- Keep repeating this process.

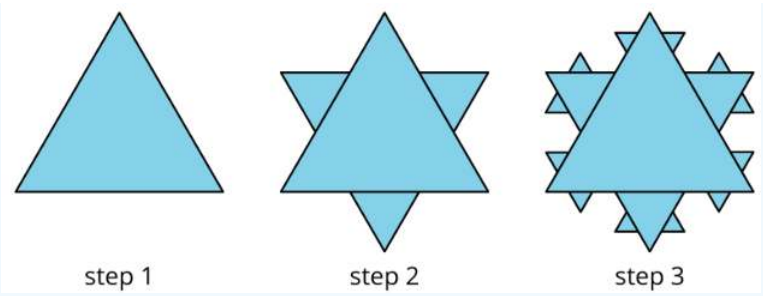


Figure 4.3.3.4: Three figures. First, an equilateral triangle. Second, the same equilateral triangle, but each lateral edge is divided into thirds and the middle third forms the base of an equilateral triangle. Third, the second image, but each lateral edge is divided into thirds and the middle third forms the base of an equilateral triangle.

By what percentage does the perimeter increase at step 2? Step 3? Step 10?

Exercise 4.3.3.3: Info Gap: Sporting Goods

Your teacher will give you either a *problem card* or a *data card*. Do not show or read your card to your partner.

If your teacher gives you the *problem card*:

1. Silently read your card and think about what information you need to be able to answer the question.
2. Ask your partner for the specific information that you need.
3. Explain how you are using the information to solve the problem.Continue to ask questions until you have enough information to solve the problem.
4. Share the *problem card* and solve the problem independently.
5. Read the *data card* and discuss your reasoning.

If your teacher gives you the *data card*:

1. Silently read your card.
2. Ask your partner *"What specific information do you need?"* and wait for them to *ask* for information.
   If your partner asks for information that is not on the card, do not do the calculations for them. Tell them you don't have that information.
3. Before sharing the information, ask "*Why do you need that information?*" Listen to your partner's reasoning and ask clarifying questions.
4. Read the *problem card* and solve the problem independently.
5. Share the *data card* and discuss your reasoning.

Pause here so your teacher can review your work. Ask your teacher for a new set of cards and repeat the activity, trading roles with your partner.

Summary

To find a 30% increase over 50, we can find 130% of 50.

$$1.3 \cdot 50 = 65$$

To find a 30% decrease from 50, we can find 70% of 50.

$$0.7 \cdot 50 = 35$$

If we know the initial amount and the final amount, we can also find the percent increase or percent decrease. For example, a plant was 12 inches tall and grew to be 15 inches tall. What percent increase is this? Here are two ways to solve this problem:

The plant grew 3 inches, because $15 - 12 = 3$. We can divide this growth by the original height, $3 \div 12 = 0.25$. So the height of the plant increased by 25%.

The plant's new height is 125% of the original height, because $15 \div 12 = 1.25$. This means the height increased by 25%, because $125 - 100 = 25$.

Here are two ways to solve the problem: A rope was 2.4 meters long. Someone cut it down to 1.9 meters. What percent decrease is this?

The rope is now $2.4 - 1.9$, or 0.5 meters shorter. We can divide this decrease by the original length, $0.5 \div 2.4 = 0.208\overline{3}$. So the length of the rope decreased by approximately 20.8%.

The rope's new length is about 79.2% of the original length, because $1.9 \div 2.4 = 0.791\overline{6}$. The length decreased by approximately 20.8%, because $100 - 79.2 = 20.8$.

Practice

Exercise 4.3.3.4

A music store marks up the instruments it sells by 30%.

1. If the store bought a guitar for $45, what will be its store price?
2. If the price tag on a trumpet says $104, how much did the store pay for it?
3. If the store paid $75 for a clarinet and sold it for $100, did the store mark up the price by 30%?

Exercise 4.3.3.5

A family eats at a restaurant. The bill is $42. The family leaves a tip and spends $49.77.

1. How much was the tip in dollars?
2. How much was the tip as a percentage of the bill?

Exercise 4.3.3.6

The price of gold is often reported per ounce. At the end of 2005, this price was $513. At the end of 2015, it was $1060. By what percentage did the price per ounce of gold increase?

Exercise 4.3.3.7

A phone keeps track of the number of steps taken and the distance traveled. Based on the information in the table, is there a proportional relationship between the two quantities? Explain your reasoning.

Table 4.3.3.1

number of steps	distance in kilometers
950	1
2,852	3
4,845	5.1

(From Unit 2.3.1)

Exercise 4.3.3.8

Noah picked 3 kg of cherries. Mai picked half as many cherries as Noah. How many total kg of cherries did Mai and Noah pick?

A. $3 + \frac{1}{2}$
B. $3 - \frac{1}{2}$
C. $\left(1 + \frac{1}{2}\right) \cdot 3$
D. $1 + \frac{1}{2} \cdot 3$

(From Unit 4.1.4)

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4.3.4: Measurement Error

Lesson

Let's use percentages to describe how accurately we can measure.

Exercise 4.3.4.1: MEasuring to the Nearest

Your teacher will give you two rulers and three line segments labeled A, B, and C.

1. Use the centimeter ruler to measure each line segment to the nearest centimeter. Record these lengths in the first column of the table.
2. Use the millimeter ruler to measure each line segment to the nearest tenth of a centimeter. Record these lengths in the second column of the table.

Table 4.3.4.1

line segment	length (cm) as measured with the first ruler	length (cm) as measured with the second ruler
A		
B		
C		

Exercise 4.3.4.2: Measuring a Soccer Field

A soccer field is 120 yards long. Han measures the length of the field using a 30-foot-long tape measure and gets a measurement of 358 feet, 10 inches.

1. What is the amount of the error?
2. Express the error as a percentage of the actual length of the field.

Exercise 4.3.4.3: Measuring Your Classroom

Your teacher will tell you which three items to measure. Keep using the paper rulers from the earlier activity.

1.Between you and your partner, decide who will use which ruler.

2. Measure the three items assigned by your teacher and record your measurements in the first column of the appropriate table.

Using the cm ruler:

Table 4.3.4.2

item	measured length (cm)	actual length (cm)	difference	percentage

Using the mm ruler:

Table 4.3.4.3

item	measured length (cm)	actual length (cm)	difference	percentage

3. After you finish measuring the items, share your data with your partner. Next, ask your teacher for the actual lengths.

4. Calculate the difference between your measurements and the actual lengths in both tables.

5. For each difference, what percentage of the actual length is this amount? Record your answers in the last column of the tables.

Are you ready for more?

Before there were standard units of measurement, people often measured things using their hands or feet.

1. Measure the length of your foot to the nearest centimeter with your shoe on.
2. How many foot-lengths long is your classroom? Try to determine this as precisely as possible by carefully placing your heel next to your toe as you pace off the room.
3. Use this information to estimate the length of your classroom in centimeters.
4. Use a tape measure to measure the length of your classroom. What is the difference between the two measurements? Which one do you think is more accurate?

Summary

When we are measuring a length using a ruler or measuring tape, we can get a measurement that is different from the actual length. This could be because we positioned the ruler incorrectly, or it could be because the ruler is not very precise. There is always at least a small difference between the actual length and a measured length, even if it is a microscopic difference!

Here are two rulers with different markings.

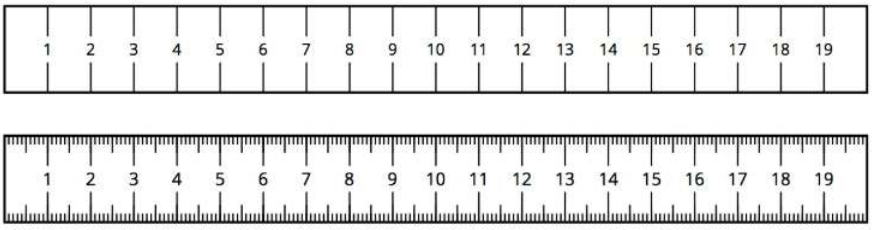


Figure 4.3.4.1: Two rulers, of equal length, each measuring 19 centimeters long. The top ruler has centimeter markings only. The bottom ruler has centimeter markings with 10 milimeter markings in between each centimeter marking.

The second ruler is marked in millimeters, so it is easier to get a measurement to the nearest tenth of a centimeter with this ruler than with the first. For example, a line that is actually 6.2 cm long might be measured to be 6 cm long by the first ruler, because we measure to the nearest centimeter.

The **measurement error** is the positive difference between the measurement and the actual value. Measurement error is often expressed as a percentage of the actual value. We always use a positive number to express measurement error and, when appropriate, use words to describe whether the measurement is greater than or less than the actual value.

For example, if we get 6 cm when we measure a line that is actually 6.2 cm long, then the measurement error is 0.2 cm, or about 3.2%, because $0.2 \div 6.2 \approx 0.032$.

Glossary Entries

Definition: Measurement Error

Measurement error is the positive difference between a measured amount and the actual amount.

For example, Diego measures a line segment and gets 5.3 cm. The actual length of the segment is really 5.32 cm. The measurement error is 0.02 cm, because $5.32 - 5.3 = 0.02$.

Practice

Exercise 4.3.4.4

The depth of a lake is 15.8 m.

1. Jada accurately measured the depth of the lake to the nearest meter. What measurement did Jada get?
2. By how many meters does the measured depth differ from the actual depth?
3. Express the measurement error as a percentage of the actual depth.

Exercise 4.3.4.5

A watermelon weighs 8,475 grams. A scale measured the weight with an error of 12% under the actual weight. What was the measured weight?

Exercise 4.3.4.6

Noah's oven thermometer gives a reading that is 2% greater than the actual temperature.

1. If the actual temperature is 325°F, what will the thermometer reading be?
2. If the thermometer reading is 76°F, what is the actual temperature?

Exercise 4.3.4.7

At the beginning of the month, there were 80 ounces of peanut butter in the pantry. Now, there is $\frac{1}{3}$ less than that. How many ounces of peanut butter are in the pantry now?

A. $\frac{2}{3} \cdot 80$
B. $\frac{1}{3} \cdot 80$
C. $80 - \frac{1}{3}$
D. $\left(1 + \frac{1}{3}\right) \cdot 80$

(From Unit 4.1.4)

Exercise 4.3.4.8

1. Fill in the table for side length and area of different squares.

Table 4.3.4.4

side length (cm)	area (cm^2)
3	
100	
25	
s	

2. Is the relationship between the side length of a square and the area of a square proportional?

(From Unit 3.2.2)

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4.3.5: Percent Error

Lesson

Let's use percentages to describe other situations that involve error.

Exercise 4.3.5.1: Number Talk: Estimating a Percentage of a Number

Estimate.

25% of 15.8

9% of 38

1.2% of 127

0.53% of 6

0.06% of 202

Exercise 4.3.5.2: Plants, Bicycles, and Crowds

1. Instructions to care for a plant say to water it with $\frac{3}{4}$ cup of water every day. The plant has been getting 25% too much water. How much water has the plant been getting?
2. The pressure on a bicycle tire is 63 psi. This is 5% higher than what the manual says is the correct pressure. What is the correct pressure?
3. The crowd at a sporting event is estimated to be 3,000 people. The exact attendance is 2,486 people. What is the **percent error**?

Are you ready for more?

A micrometer is an instrument that can measure lengths to the nearest micron (a micron is a millionth of a meter). Would this instrument be useful for measuring any of the following things? If so, what would the largest percent error be?

1. The thickness of an eyelash, which is typically about 0.1 millimeters.
2. The diameter of a red blood cell, which is typically about 8 microns.
3. The diameter of a hydrogen atom, which is about 100 picometers (a picometer is a trillionth of a meter).

Exercise 4.3.5.3: Measuring in the Heat

A metal measuring tape expands when the temperature goes above $50^\circ F$. For every degree Fahrenheit above 50, its length increases by 0.00064%.

1. The temperature is 100 degrees Fahrenheit. How much longer is a 30-foot measuring tape than its correct length?
2. What is the percent error?

Summary

Percent error can be used to describe any situation where there is a correct value and an incorrect value, and we want to describe the relative difference between them. For example, if a milk carton is supposed to contain 16 fluid ounces and it only contains 15 fluid ounces:

- the measurement error is 1 oz, and
- the percent error is 6.25% because $1 \div 16 = 0.0625$.

We can also use percent error when talking about estimates. For example, a teacher estimates there are about 600 students at their school. If there are actually 625 students, then the percent error for this estimate was 4%, because $625 - 600 = 25$ and $25 \div 625 = 0.04$.

Glossary Entries

Definition: Measurement Error

Measurement error is the positive difference between a measured amount and the actual amount.

For example, Diego measures a line segment and gets 5.3 cm. The actual length of the segment is really 5.32 cm. The measurement error is 0.02 cm, because $5.32 - 5.3 = 0.02$.

Definition: Percent Error

Percent error is a way to describe error, expressed as a percentage of the actual amount.

For example, a box is supposed to have 150 folders in it. Clare counts only 147 folders in the box. This is an error of 3 folders. The percent error is 2%, because 3 is 2% of 150.

Practice

Exercise 4.3.5.4

A student estimated that it would take 3 hours to write a book report, but it actually took her 5 hours. What is the percent error for her estimate?

Exercise 4.3.5.5

A radar gun measured the speed of a baseball at 103 miles per hour. If the baseball was actually going 102.8 miles per hour, what was the percent error in this measurement?

Exercise 4.3.5.6

It took 48 minutes to drive downtown. An app estimated it would be less than that. If the error was 20%, what was the app's estimate?

Exercise 4.3.5.7

A farmer estimated that there were 25 gallons of water left in a tank. If this is an underestimate by 16%, how much water was actually in the tank?

Exercise 4.3.5.8

For each story, write an equation that describes the relationship between the two quantities.

1. Diego collected x kg of recycling. Lin collected $\frac{2}{5}$ more than that.
2. Lin biked x km. Diego biked $\frac{3}{10}$ less than that.
3. Diego read for x minutes. Lin read $\frac{4}{7}$ of that.

(From Unit 4.1.4)

Exercise 4.3.5.9

For each diagram, decide if y is an increase or a decrease of x. Then determine the percentage.

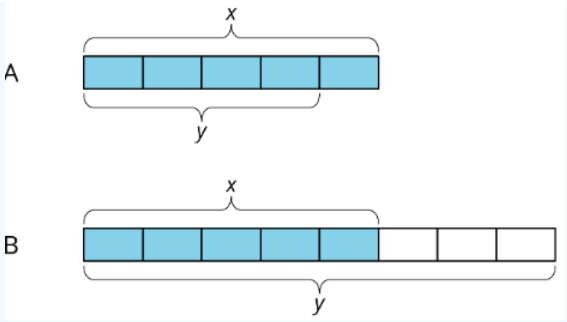


Figure 4.3.5.1

(From Unit 4.3.3)

Exercise 4.3.5.10

Lin is making a window covering for a window that has the shape of a half circle on top of a square of side length 3 feet. How much fabric does she need?

(From Unit 3.3.1)

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4.3.6: Error Intervals

Lesson

Let's solve more problems about percent error.

Exercise 4.3.6.1: A Lot of Iron Ore

An industrial scale is guaranteed by the manufacturer to have a percent error of no more than 1%. What is a possible reading on the scale if you put 500 kilograms of iron ore on it?

Exercise 4.3.6.2: Saw Mill

1. A saw mill cuts boards that are 16 ft long. After they are cut, the boards are inspected and rejected if the length has a percent error of 1.5% or more.
   a. List some board lengths that should be accepted.
   b. List some board lengths that should be rejected.
2. The saw mill also cuts boards that are 10, 12, and 14 feet long. An inspector rejects a board that was 2.3 inches too long. What was the intended length of the board?

Exercise 4.3.6.3: Info Gap: Quality Control

Your teacher will give you either a *problem card* or a *data card*. Do not show or read your card to your partner.

If your teacher gives you the *problem card*:

1. Silently read your card and think about what information you need to be able to answer the question.
2. Ask your partner for the specific information that you need.
3. Explain how you are using the information to solve the problem.
   Continue to ask questions until you have enough information to solve the problem.
4. Share the *problem card* and solve the problem independently.
5. Read the *data card* and discuss your reasoning.

If your teacher gives you the *data card*:

1. Silently read your card.
2. Ask your partner "*What specific information do you need?*" and wait for them to *ask* for information.
   If your partner asks for information that is not on the card, do not do the calculations for them. Tell them you don't have that information.
3. Before sharing the information, ask "*Why do you need that information?*" Listen to your partner's reasoning and ask clarifying questions.
4. Read the *problem card* and solve the problem independently.
5. Share the *data card* and discuss your reasoning.

Pause here so your teacher can review your work. Ask your teacher for a new set of cards and repeat the activity, trading roles with your partner.

Summary

Percent error is often used to express a range of possible values. For example, if a box of cereal is guaranteed to have 750 grams of cereal, with a margin of error of less than 5%, what are possible values for the actual number of grams of cereal in the box? The error could be as large as $(0.05) \cdot 750 = 37.5$ and could be either above or below than the correct amount.

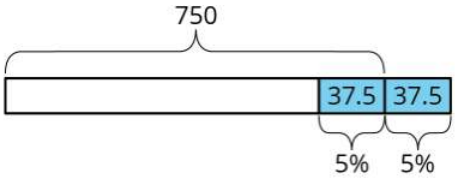


Figure 4.3.6.1: Tape diagram. One longer white section and 2 smaller blue sections both labeled 37 point 5 and 5 percent. Long white section and one blue section are labeled 750.

Therefore, the box can have anywhere between 712.5 and 787.5 grams of cereal in it, but it should not have 700 grams or 800 grams, because both of those are more than 37.5 grams away from 750 grams.

Glossary Entries

Definition: Measurement Error

Measurement error is the positive difference between a measured amount and the actual amount.

For example, Diego measures a line segment and gets 5.3 cm. The actual length of the segment is really 5.32 cm. The measurement error is 0.02 cm, because $5.32 - 5.3 = 0.02$.

Definition: Percent Error

Percent error is a way to describe error, expressed as a percentage of the actual amount.

For example, a box is supposed to have 150 folders in it. Clare counts only 147 folders in the box. This is an error of 3 folders. The percent error is 2%, because 3 is 2% of 150.

Practice

Exercise 4.3.6.4

Jada measured the height of a plant in a science experiment and finds that, to the nearest $\frac{1}{4}$ of an inch, it is $4\frac{3}{4}$ inches.

1. What is the largest the actual height of the plant could be?
2. What is the smallest the actual height of the plant could be?
3. How large could the percent error in Jada's measurement be?

Exercise 4.3.6.5

The reading on a car's speedometer has 1.6% maximum error. The speed limit on a road is 65 miles per hour.

1. The speedometer reads 64 miles per hour. Is it possible that the car is going over the speed limit?
2. The speedometer reads 66 miles per hour. Is the car definitely going over the speed limit?

Exercise 4.3.6.6

Water is running into a bathtub at a constant rate. After 2 minutes, the tub is filled with 2.5 gallons of water. Write two equations for this proportional relationship. Use w for the amount of water (gallons) and t for time (minutes). In each case, what does the constant of proportionality tell you about the situation?

(From Unit 2.2.2)

Exercise 4.3.6.7

Noah picked 3 kg of cherries. Jada picked half as many cherries as Noah. How many total kg of cherries did Jada and Noah pick?

A. $3 + 0.5$
B. $3 - 0.5$
C. $(1 + 0.5) \cdot 3$

D. $1 + 0.5 \cdot 3$

(From Unit 4.1.5)

Exercise 4.3.6.8

Here is a shape with some measurements in cm.

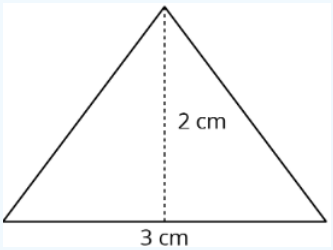


Figure 4.3.6.2

1. Complete the table showing the area of different scaled copies of the triangle.

Table 4.3.6.1

scale factor	area (cm^2)
1	
2	
5	
s	

2. Is the relationship between the scale factor and the area of the scaled copy proportionally?

(From Unit 3.2.2)

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4.4: Let's Put it to Work

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4.4.1: Posing Percentage Problems

Lesson

Let's explore how percentages are used in the news.

Exercise 4.4.1.1: Sorting the News

Your teacher will give you a variety of news clippings that include percentages.

1. Sort the clippings into two piles: those that are about increases and those that are about decreases.
2. Were there any clippings that you had trouble deciding which pile they should go in?

Exercise 4.4.1.2: Investigating

In the previous activity, you sorted news clippings into two piles.

1. For each pile, choose one example. Draw a diagram that shows how percentages are being used to describe the situation.
   a. Increase Example:
   b. Decrease Example:
2. For each example, write *two* questions that you can answer with the given information. Next, find the answers. Explain or show your reasoning.

Exercise 4.4.1.3: Displaying the News

1. Choose the example that you find the most interesting. Create a visual display that includes:
   - a title that describes the situation
   - the news clipping
   - your diagram of the situation
   - the two questions you asked about the situation
   - the answers to each of your questions
   - an explanation of how you calculated each answer

   Pause here so your teacher can review your work.
2. Examine each display. Write one comment and one question for the group.
3. Next, read the comments and questions your classmates wrote for your group. Revise your display using the feedback from your classmates.

Summary

Statements about percentage increase or decrease need to specify what the whole is to be mathematically meaningful. Sometimes advertisements, media, etc. leave the whole ambiguous in order to make somewhat misleading claims. We should be careful to think critically about what mathematical claim is being made.

For example, if a disinfectant claims to "kill 99% of all bacteria," does it mean that

- It kills 99% of the number of bacteria on a surface?
- Or is it 99% of the types of bacteria commonly found inside the house?
- Or 99% of the total mass or volume of bacteria?
- Does it even matter if the remaining 1% are the most harmful bacteria?

Resolving questions of this type is an important step in making informed decisions.

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CHAPTER OVERVIEW

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  - 5.1.1: Interpreting Negative Numbers
- 5.2: Adding and Subtracting Rational Numbers
  - 5.2.1: Changing Temperatures
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  - 5.2.3: Money and Debts
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  - 5.6.1: The Stock Market

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5.1: Interpreting Negative Numbers

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5.1.1: Interpreting Negative Numbers

Lesson

Let's review what we know about signed numbers.

Exercise 5.1.1.1: Using the Thermometer

Here is a weather thermometer. Three of the numbers have been left off.

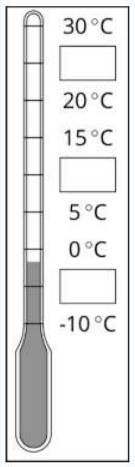


Figure 5.1.1.1: A vertical thermometer measured in degrees Celsius. There are 9 evenly spaced tick marks and starting from the bottom of the thermometer, negative 10 is on the first tick mark, zero on the third, 5 on the fourth, 15 on the sixth, 20 on the seventh, and 30 on the ninth. The second, fifth, and eighth tick marks each are labeled with a box. The thermometer is shaded starting from the bottom of the thermometer to halfway between the second and third tick marks.

1. What numbers go in the boxes?
2. What temperature does the thermometer show?

Exercise 5.1.1.2: Fractions of a Degree

1. What temperature is shown on each thermometer?
2. Which thermometer shows the highest temperature?
3. Which thermometer shows the lowest temperature?
4. Suppose the temperature outside is $-4^\circ \mathrm{C}$. Is that colder or warmer than the coldest temperature shown? How do you know?

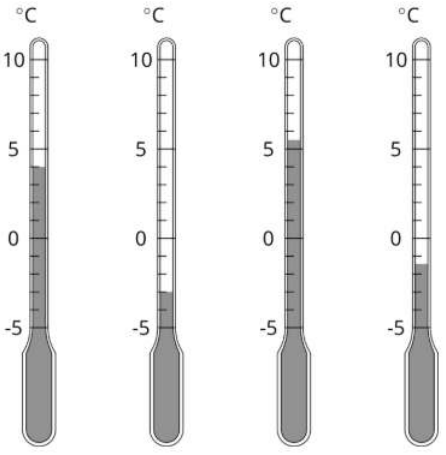


Figure 5.1.1.2: Four vertical thermometers measured in degrees Celsius. There are 16 evenly spaced tick marks and starting from the bottom of the thermometer, negative 5 is on the first tick mark, zero on the sixth, 5 on the eleventh, and 10 on the sixteenth. The first thermometer is shaded starting from the bottom of the thermometer to the tenth tickmark. The second thermometer is shaded starting from the bottom of the thermometer to the third tickmark. The third thermometer is shaded starting from the bottom of the thermometer to between the eleventh and twelfth tickmark. The fourth thermometer is shaded starting from the bottom of the thermometer to between the fourth and fifth tickmark.

Exercise 5.1.1.3: Seagulls Soar, SHarks Swim

Here is a picture of some sea animals. The number line on the left shows the vertical position of each animal above or below sea level, in meters.

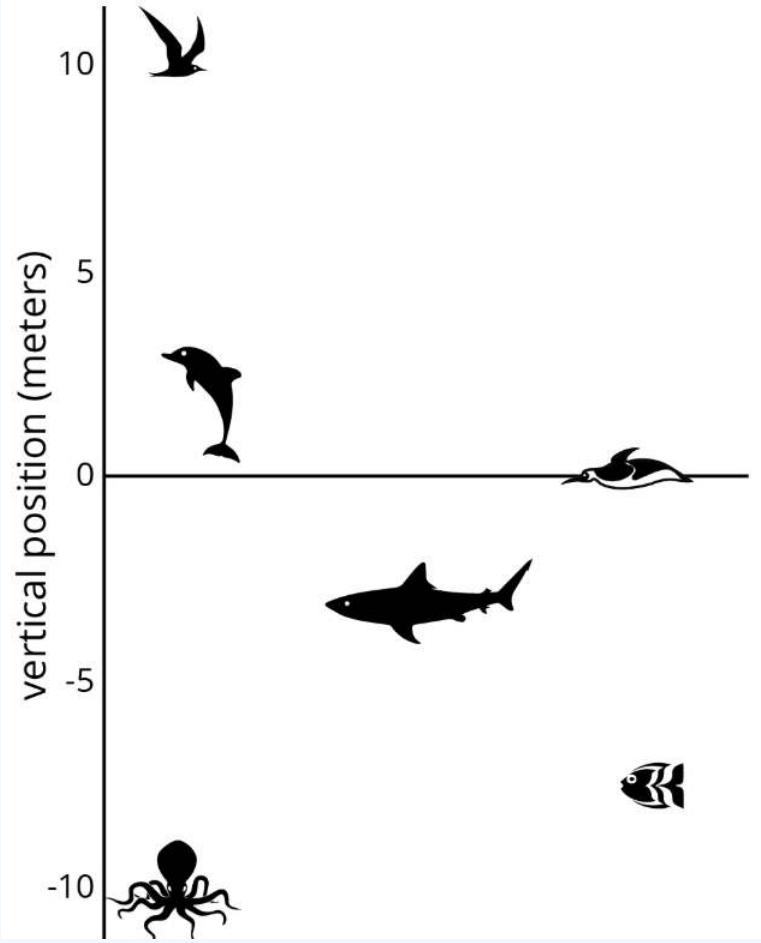


Figure 5.1.1.3: A graph. Vertical axis is labeled vertical position, in meters. 5 tick marks. Scale from negative 15 to 15, by 5s. Horizontal axis has no label or scale. Seagull is at 10. Dolphin is at 3. Penguin is at 0. Shark is at negative 3. Fish is at negative 7. Octopus is at negative 10.

1. How far above or below sea level is each animal? Measure to their eye level.
2. A mobula ray is 3 meters above the surface of the ocean. How does its vertical position compare to the height or depth of:
   The jumping dolphin?
   The flying seagull?
   The octopus?
3. An albatross is 5 meters above the surface of the ocean. How does its vertical position compare to the height or depth of:
   The jumping dolphin?
   The flying seagull?
   The octopus?
4. A clownfish is 2 meters below the surface of the ocean. How does its vertical position compare to the height or depth of:
   The jumping dolphin?
   The flying seagull?
   The octopus?
5. The vertical distance of a new dolphin from the dolphin in the picture is 3 meters. What is its distance from the surface of the ocean?

Are you ready for more?

The north pole is in the middle of the ocean. A person at sea level at the north pole would be 3,949 miles from the center of Earth. The sea floor below the north pole is at an elevation of approximately -2.7 miles. The elevation of the south pole is about 1.7 miles. How far is a person standing on the south pole from a submarine at the sea floor below the north pole?

Exercise 5.1.1.4: Card Sort: Rational Numbers

1. Your teacher will give your group a set of cards. Order the cards from least to greatest.
2. Pause here so your teacher can review your work. Then, your teacher will give you a second set of cards.
3. Add the new set of cards to the first set so that all of the cards are ordered from least to greatest.

Summary

We can use **positive numbers** and **negative numbers** to represent temperature and elevation.

When numbers represent temperatures, positive numbers indicate temperatures that are warmer than zero and negative numbers indicate temperatures that are colder than zero. This thermometer shows a temperature of -1 degree Celsius, which we write -1°C.

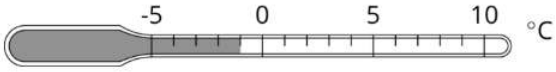


Figure 5.1.1.4: A thermometer positioned horizontally and with the bottom of the thermometer on the left, is labeled degrees Celsius. There are 4 evenly spaced large tick marks. Starting on the left, the large tick marks are labeled negative 5, 0, 5, and 10. Between each large tick mark, there are 4 evenly spaced small tick marks. The thermometer is shaded, starting from the left, up to negative 1.

When numbers represent elevations, positive numbers indicate positions above sea level and negative numbers indicate positions below sea level.

We can see the order of signed numbers on a number line.

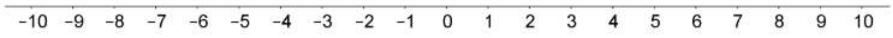


Figure 5.1.1.5

A number is always less than numbers to its right. So $-7 < -3$.

We use **absolute value** to describe how far a number is from 0. The numbers 15 and -15 are both 15 units from 0, so $|15| = 15$ and $|-15| = 15$. We call 15 and -15 *opposites*. They are on opposite sides of 0 on the number line, but the same distance from 0.

Glossary Entries

Definition: Absolute Value

The absolute value of a number is its distance from 0 on the number line.

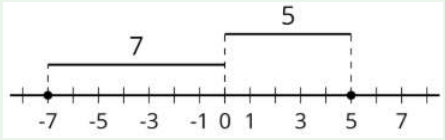


Figure 5.1.1.6

The absolute value of -7 is 7, because it is 7 units away from 0. The absolute value of 5 is 5, because it is 5 units away from 0.

Definition: Negative Number

A negative number is a number that is less than zero. On a horizontal number line, negative numbers are usually shown to the left of 0.

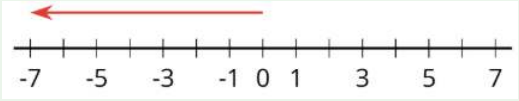


Figure 5.1.1.7

Definition: Positive Number

A positive number is a number that is greater than zero. On a horizontal number line, positive numbers are usually shown to the right of 0.

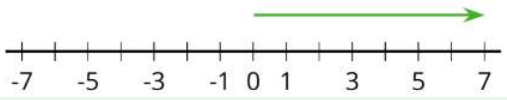


Figure 5.1.1.8

Practice

Exercise 5.1.1.5

It was -5°C in Copenhagen and -12°C in Oslo. Which city was colder?

Exercise 5.1.1.6

1. A fish is 12 meters below the surface of the ocean. What is its elevation?
2. A sea bird is 28 meters above the surface of the ocean. What is its elevation?
3. If the bird is directly above the fish, how far apart are they?

Exercise 5.1.1.7

Compare using >, =, or <.

1. 3 _____ -3
2. 12 _____ 24
3. -12 _____ -24
4. 5 _____ $-(-5)$
5. 7.2 _____ 7
6. -7.2 _____ -7
7. -1.5 _____ $\frac{-3}{2}$
8. $\frac{-4}{5}$ _____ $\frac{-5}{4}$
9. $\frac{-3}{5}$ _____ $\frac{-6}{10}$
10. $\frac{-2}{3}$ _____ $\frac{1}{3}$

Exercise 5.1.1.8

Han wants to buy a $30 ticket to a game, but the pre-order tickets are sold out. He knows there will be more tickets sold the day of the game, with a markup of 200%. How much should Han expect to pay for the ticket if he buys it the day of the game?

(From Unit 4.2.2)

Exercise 5.1.1.9

A type of green paint is made by mixing 2 cups of yellow with 3.5 cups of blue.

1. Find a mixture that will make the same shade of green but a smaller amount.
2. Find a mixture that will make the same shade of green but a larger amount.
3. Find a mixture that will make a different shade of green that is bluer.
4. Find a mixture that will make a different shade of green that is more yellow.

(From Unit 2.1.1)

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5.2: Adding and Subtracting Rational Numbers

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5.2.1: Changing Temperatures

Lesson

Let's add signed numbers.

Exercise 5.2.1.1: Which One Doesn't Belong: Arrows

Which pair of arrows doesn't belong?

1.

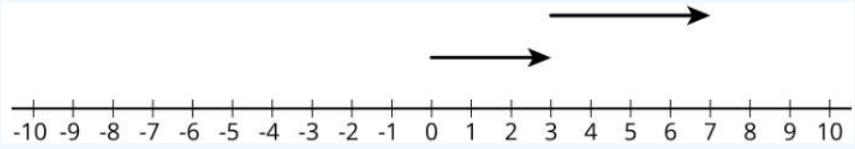


Figure 5.2.1.1: Number line. 21 evenly spaced tick marks. Scale negative 10 to 10, by 1s. Two arrows. One arrow points to the right from 0 to 3. One arrow points to the right from 3 to 7.

2.

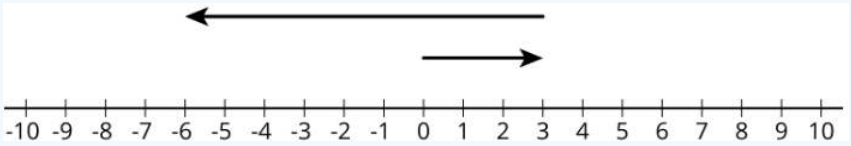


Figure 5.2.1.2: Number line. 21 evenly spaced tick marks. Scale negative 10 to 10, by 1s. Two arrows. One arrow points to the right from 0 to 3. One arrow points to the left from 3 to negative 6.

3.

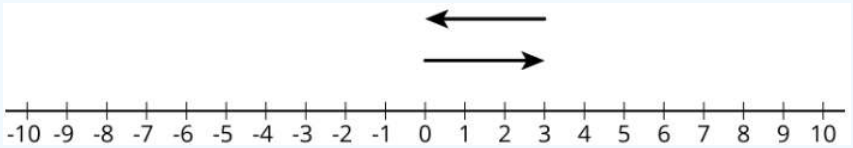


Figure 5.2.1.3: Number line. 21 evenly spaced tick marks. Scale negative 10 to 10, by 1s. Two arrows. One arrow points to the right from 0 to 3. One arrow points to the left from 3 to 0.

4.

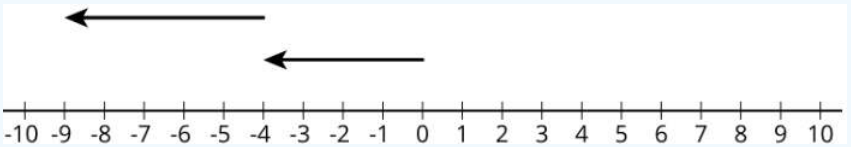


Figure 5.2.1.4: Number line. 21 evenly spaced tick marks. Scale negative 10 to 10, by 1s. Two arrows pointing to the left, one from 0 to negative 4 and another from negative 4 to negative 9.

Exercise 5.2.1.2: Warmer and Colder

1. Complete the table and draw a number line diagram for each situation.

Table 5.2.1.1

	start (°C)	change (°C)	final (°C)	addition equation
a	$+40$	10 degrees warmer	$+50$	$40+10=50$
b	$+40$	5 degrees colder		
c	$+40$	30 degrees colder		
d	$+40$	40 degrees colder		
e	$+40$	50 degrees colder		

a.

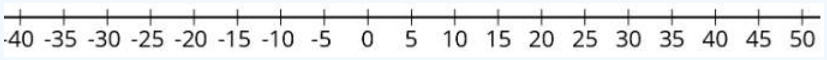


Figure 5.2.1.5

b.

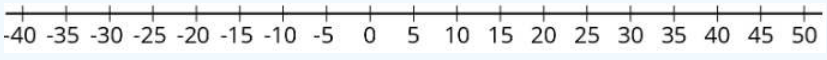


Figure 5.2.1.6

c.

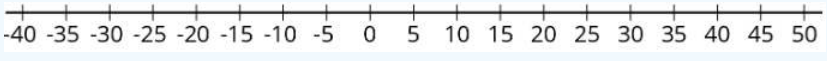


Figure 5.2.1.7

d.

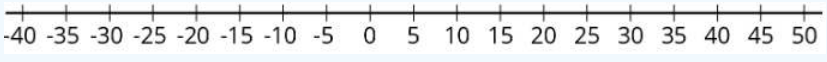


Figure 5.2.1.8

e.

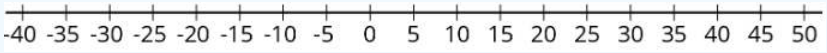


Figure 5.2.1.9

2. Complete the table and draw a number line diagram for each situation.

Table 5.2.1.2

	start (°C)	change (°C)	final (°C)	addition equation
a	−20	30 degrees warmer		
b	−20	35 degrees warmer		
c	−20	15 degrees warmer		
d	−20	15 degrees colder		

a.

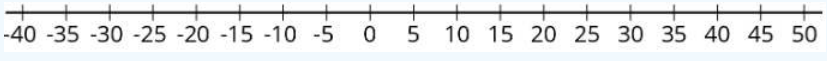


Figure 5.2.1.10

b.

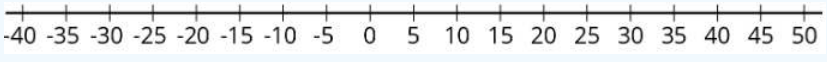


Figure 5.2.1.11

c.

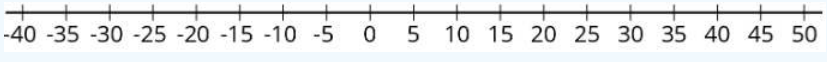


Figure 5.2.1.12

d.

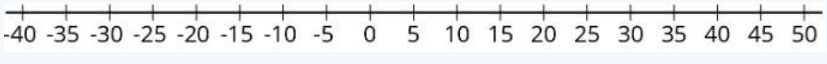


Figure 5.2.1.13

Are you ready for more?

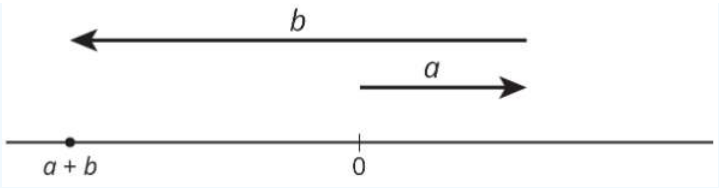


Figure 5.2.1.14: Number line. 2 tick marks. First tick mark is labeled a plus b. Second tick mark, to the right of the first tick mark, is labeled 0. Arrow a starts at zero and extends to the right. Arrow b is directly above arrow a, points to the left and begins where arrow a ends, and ends at the point labeled a plus b.

For the numbers a and b represented in the figure, which expression is equal to $|a+b|$?

$$|a|+|b| \qquad |a|-|b| \qquad |b|-|a|$$

Exercise 5.2.1.3: Winter Temperatures

One winter day, the temperature in Houston is 8° Celsius. Find the temperatures in these other cities. Explain or show your reasoning.

1. In Orlando, it is 10° warmer than it is in Houston.
2. In Salt Lake City, it is 8° colder than it is in Houston.
3. In Minneapolis, it is 20° colder than it is in Houston.
4. In Fairbanks, it is 10° colder than it is in Minneapolis.
5. Use the thermometer applet to verify your answers and explore your own scenarios.

Summary

If it is 42° outside and the temperature increases by 7°, then we can add the initial temperature and the change in temperature to find the final temperature.

$$42+7=49$$

If the temperature decreases by 7°, we can either subtract $42-7$ to find the final temperature, or we can think of the change as -7°. Again, we can add to find the final temperature.

$$42+(-7)=35$$

In general, we can represent a change in temperature with a positive number if it increases and a negative number if it decreases. Then we can find the final temperature by adding the initial temperature and the change. If it is 3° and the temperature decreases by 7°, then we can add to find the final temperature.

$$3+(-7)=-4$$

We can represent signed numbers with arrows on a number line. We can represent positive numbers with arrows that start at 0 and points to the right. For example, this arrow represents +10 because it is 10 units long and it points to the right.

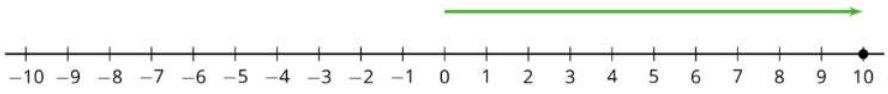


Figure 5.2.1.15

We can represent negative numbers with arrows that start at 0 and point to the left. For example, this arrow represents -4 because it is 4 units long and it points to the left.



Figure 5.2.1.16

To represent addition, we put the arrows “tip to tail.” So this diagram represents $3+5$:

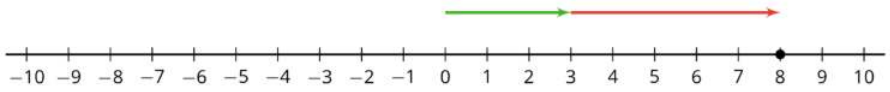


Figure 5.2.1.17: A number line with the numbers negative 10 through 10 indicated. An arrow starts at 0, points to the right, and ends at 3. A second arrow starts at 3, points to the right, and ends at 8. there is a solid dot indicated at 8.

And this represents $3+(-5)$:

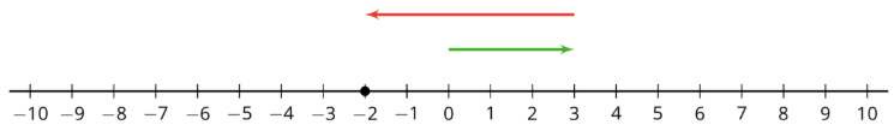


Figure 5.2.1.18: A number line with the numbers negative 10 through 10 indicated. An arrow starts at 0, points to the right, and ends at three. A second arrow starts at 3, points to the left, and ends at negative 2. There is a solid dot indicated at negative

Practice

Exercise 5.2.1.4

1. The temperature is -2°C. If the temperature rises by 15°C, what is the new temperature?
2. At midnight the temperature is -6°C. At midday the temperature is 9°C. By how much did the temperature rise?

Exercise 5.2.1.5

Draw a diagram to represent each of these situations. Then write an addition expression that represents the final temperature.

1. The temperature was 80°F and then fell 20°F.
2. The temperature was -13°F and then rose 9°F.
3. The temperature was -5°F and then fell 8°F.

Exercise 5.2.1.6

Complete each statement with a number that makes the statement true.

1. _____ $< 7^\circ\text{C}$
2. _____ $< -3^\circ\text{C}$
3. $-0.8^\circ\text{C} <$ _____ $< -0.1^\circ\text{C}$
4. _____ $> -2^\circ\text{C}$

(From Unit 5.1.1)

Exercise 5.2.1.7

Decide whether each table could represent a proportional relationship. If the relationship could be proportional, what would be the constant of proportionality?

1. The number of wheels on a group of buses.

Table 5.2.1.3

number of buses	number of wheels	wheels per bus
5	30	
8	48	
10	60	
15	90	

2. The number of wheels on a train.

Table 5.2.1.4

number of train cars	number of wheels	wheels per train car
20	184	
30	264	
40	344	
50	424	

(From Unit 2.3.1)

Exercise 5.2.1.8

Noah was assigned to make 64 cookies for the bake sale. He made 125% of that number. 90% of the cookies he made were sold. How many of Noah's cookies were left after the bake sale?

(From Unit 4.2.2)

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5.2.2: Changing Elevation

Lesson

Let's solve problems about adding signed numbers.

Exercise 5.2.2.1: That's the Opposite

1. Draw arrows on a number line to represents these situations:

   a. The temperature was -5 degrees. Then the temperature rose 5 degrees.

Figure 5.2.2.1

   b. A climber was 30 feet above sea level. Then she descended 30 feet.

Figure 5.2.2.2

2. What's the opposite?

   a. Running 150 feet east.
   b. Jumping down 10 steps.
   c. Pouring 8 gallons into a fish tank.

Exercise 5.2.2.2: Cliffs and Caves

Explore the applet and then answer the questions.

1. A mountaineer is climbing on a cliff. She is 200 feet above the ground. If she climbs up, this will be a positive change. If she climbs down, this will be a negative change.



Figure 5.2.2.3: By makamuki0. Public Domain. Pixabay. Source.

a. Complete the table.

Table 5.2.2.1

	starting elevation (feet)	change (feet)	final elevation (feet)
A	$+200$	75 up	
B	$+200$	75 down	
C	$+200$	200 down	
D	$+200$		$+25$

b. Write an addition equation and draw a number line diagram for B. Include the starting elevation, change, and final elevation in your diagram.

2. A spelunker is down in a cave next to the cliff. If she climbs down deeper into the cave, this will be a negative change. If she climbs up, whether inside the cave or out of the cave and up the cliff, this will be a positive change.

a. Complete the table.

Table 5.2.2.2

	starting elevation (feet)	change (feet)	final elevation (feet)
A	-20	15 down	
B	-20	10 up	
C	-20	20 up	

	starting elevation (feet)	change (feet)	final elevation (feet)
D	-20	25 up	
E	-20		-50

b. Write an addition equation and draw a number line diagram for C and D. Include the starting elevation, change, and final elevation in your diagram.

3. What does the expression $-45+60$ tell us about the spelunker? What does the value of the expression tell us?

Exercise 5.2.2.3: Adding Rational Numbers

Find the sums.

1. $-35+(30+5)$
2. $-0.15+(-0.85)+12.5$
3. $\frac{1}{2}+\left(-\frac{3}{4}\right)$

Are you ready for more?

Find the sum without a calculator.

$$10+21+32+43+54+(-54)+(-43)+(-32)+(-21)+(-10)$$

Exercise 5.2.2.4: School Supply Number Line

Your teacher will give you a long strip of paper.

Follow these instructions to create a number line.

1. Fold the paper in half along its length and along its width.
2. Unfold the paper and draw a line along each crease.
3. Label the line in the middle of the paper 0. Label the right end of the paper $+$ and the left end of the paper $-$.
4. Select two objects of different lengths, for example a pen and a gluestick. The length of the longer object is a and the length of the shorter object is b.
5. Use the objects to measure and label each of the following points on your number line.

a	$2b$	$-b$
b	$a+b$	$a+-b$
$2a$	$-a$	$b+-a$

6. Complete each statement using <,>, or =. Use your number line to explain your reasoning.
   a. a _____ b
   b. $-a$ _____ $-b$
   c. $a+-a$ _____ $b+-b$
   d. $a+-b$ _____ $b+-a$
   e. $a+-b$ _____ $-a+b$

Summary

The opposite of a number is the same distance from 0 but on the other side of 0.

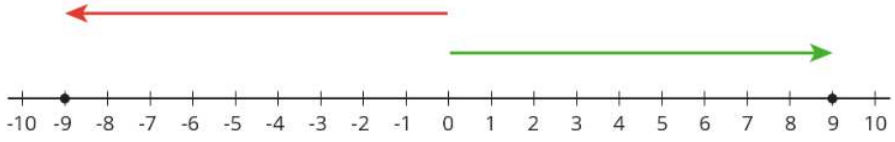


Figure 5.2.2.4: A number line with the numbers negative 10 through 10 indicated. An arrow starts at 0, points to the left, and ends at negative 9. There is a solid dot indicated at negative 9. A second arrow starts at 0, points to the right, and ends at 9. There is a solid dot indicated at 9.

The opposite of -9 is 9. When we add opposites, we always get 0. This diagram shows that $9 + -9 = 0$.

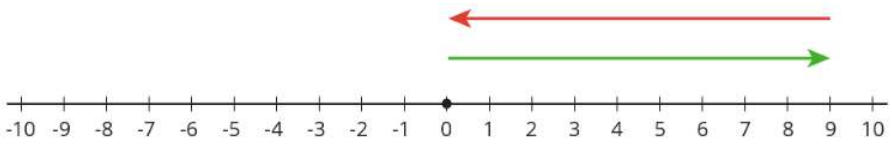


Figure 5.2.2.5: A number line with the numbers negative 10 through 10 indicated. An arrow starts at 0, points to the right, and ends at 9. A second arrow starts at 9, points to the left, and ends at 0. there is a solid dot indicated at 0.

When we add two numbers with the same sign, the arrows that represent them point in the same direction. When we put the arrows tip to tail, we see the sum has the same sign.

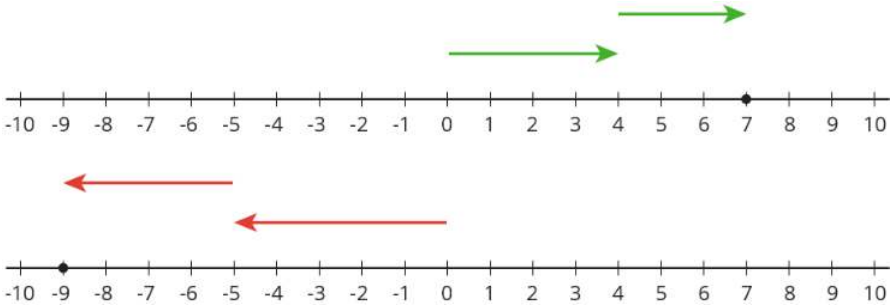


Figure 5.2.2.6: Two identical number lines with the numbers negative 10 through 10 indicated. On the top number line, an arrow starts at 0, points to the right, and ends at 4. A second arrow starts at 4, points to the right, and ends at 7. There is a solid dot indicated at 7. On the bottom number line, an arrow starts at 0, points to the left, and ends at negative 5. A second arrow starts at negative 5, points to the left, and ends at negative 9. There is a solid dot indicated at negative 9.

To find the sum, we add the magnitudes and give it the correct sign. For example, $(-5) + (-4) = -(5 + 4)$.

On the other hand, when we add two numbers with different signs, we subtract their magnitudes (because the arrows point in the opposite direction) and give it the sign of the number with the larger magnitude. For example, $(-5) + 12 = +(12 - 5)$.

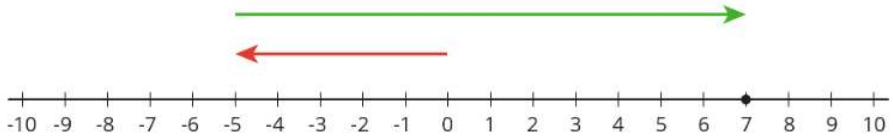


Figure 5.2.2.7: A number line with the numbers negative 10 through 10 indicated. An arrow starts at 0, points to the left, and ends at negative five. A second arrow starts at negative five, points to the right, and ends at 7. There is a solid dot indicated at 7.

Practice

Exercise 5.2.2.5

What is the final elevation if

1. A bird starts at 20 m and changes 16 m?
2. A butterfly starts at 20 m and changes -16 m?
3. A diver starts at 5 m and changes -16 m?
4. A whale starts at -9 m and changes 11 m?
5. A fish starts at -9 meters and changes -11 meters?

Exercise 5.2.2.6

One of the particles in an atom is called an electron. It has a charge of -1. Another particle in an atom is a proton. It has charge of +1. The charge of an atom is the sum of the charges of the electrons and the protons. A carbon atom has an overall charge of 0, because it has 6 electrons and 6 protons and $-6 + 6 = 0$. Find the overall charge for the rest of the elements on the list.

Table 5.2.2.3

	charge from electrons	charge from protons	overall charge

	charge from electrons	charge from protons	overall charge
carbon	-6	$+6$	0
neon	-10	$+10$	
oxide	-10	$+8$	
copper	-27	$+29$	
tin	-50	$+50$	

Exercise 5.2.2.7

Add.

1. $14.7+28.9$
2. $-9.2+4.4$
3. $-81.4+(-12)$
4. $51.8+(-0.8)$

Exercise 5.2.2.8

Last week, the price, in dollars, of a gallon of gasoline was g. This week, the price of gasoline per gallon increased by 5%. Which expressions represent this week's price, in dollars, of a gallon of gasoline? Select **all** that apply.

A. $g+0.05$
B. $g+0.05g$
C. $1.05g$
D. $0.05g$
E. $(1+0.05)g$

(From Unit 4.2.3)

Exercise 5.2.2.9

Decide whether each table could represent a proportional relationship. If the relationship could be proportional, what would be the constant of proportionality?

1. Annie's Attic is giving away $5 off coupons.

Table 5.2.2.4

original price	sale price
$15	$10
$25	$20
$35	$30

2. Bettie's Boutique is having a 20% off sale.

Table 5.2.2.5

original price	sale price
$15	$12
$25	$20
$35	$28

(From Unit 2.3.1)

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5.2.3: Money and Debts

Lesson

Let's apply what we know about signed numbers to money.

Exercise 5.2.3.1: Concert Tickets

Priya wants to buy three tickets for a concert. She has earned $135 and each ticket costs $50. She borrows the rest of the money she needs from a bank and buys the tickets.

1. How can you represent the amount of money that Priya has after buying the tickets?
2. How much more money will Priya need to earn to pay back the money she borrowed from the bank?
3. How much money will she have after she pays back the money she borrowed from the bank?

Exercise 5.2.3.2: Cafeteria Food Debt

At the beginning of the month Kiran had $24 in his school cafeteria account. Use a variable to represent the unknown quantity in each transaction below and write an equation to represent it. Then, represent each transaction on a number line. What is the unknown quantity in each case?

1. In the first week he spent $16 on lunches. How much was in his account then?
2. Then he deposited some more money and his account balance was $28. How much did he deposit?
3. Then he spent $34 on lunches the next week. How much was in his account then?
4. Then he deposited enough money to pay off his debt to the cafeteria. How much did he deposit?
5. Explain why it makes sense to use a negative number to represent Kiran's account balance when he owes money.

Exercise 5.2.3.3: Bank Statement

Here is a bank statement.



Responsible Bank
210 2nd Street
Anytown, MH 06930

Checking Account Statement
Page: 1 of 1

Andre Person
1729 Euclid Ave
Anytown, MH 06930

Statement Period	Account No.
2017-10-01 to 2017-11-01	1120635978

Date	Description	Withdrawals	Deposits	Balance
2017-10-03	Previous Balance			39.87
2017-10-05	Check Number 256	28.50		11.37
2017-10-06	ATM Deposit - Cash		45.00	56.37
2017-10-10	Wire Transfer	37.91		18.46
2017-10-17	Point of Sale - Grocery Store	16.43		2.03
2017-10-25	Funds Transfer from Savings		50.00	52.03
2017-10-28	Check Number 257	42.00		10.03
2017-10-29	Online Payment - Phone Services	72.50		-62.47

Figure 5.2.3.1

1. If we put withdrawals and deposits in the same column, how can they be represented?
2. Andre withdraws $40 to buy a music player. What is his new balance?
3. If Andre deposits $100 in this account, will he still be in debt? How do you know?

Are you ready for more?

The *national debt* of a country is the total amount of money the government of that country owes. Imagine everyone in the United States was asked to help pay off the national debt. How much would each person have to pay?

Summary

Banks use positive numbers to represent money that gets put into an account and negative numbers to represent money that gets taken out of an account. When you put money into an account, it is called a **deposit**. When you take money out of an account, it is called a **withdrawal**.

People also use negative numbers to represent debt. If you take out more money from your account than you put in, then you owe the bank money, and your account balance will be a negative number to represent that debt. For example, if you have $200 in your bank account, and then you write a check for $300, you will owe the bank $100 and your account balance will be -$100.

Table 5.2.3.1

starting balance	deposits and withdrawals	new balance
0	50	$0+50$
50	150	$50+150$
200	-300	$200+(-300)$
-100		

In general, you can find a new account balance by adding the value of the deposit or withdrawal to it. You can also tell quickly how much money is needed to repay a debt using the fact that to get to zero from a negative value you need to add its opposite.

Glossary Entries

Definition: Deposit

When you put money into an account, it is called a *deposit*.

For example, a person added $60 to their bank account. Before the deposit, they had $435. After the deposit, they had $495, because $435+60=495$.

Definition: Withdrawal

When you take money out of an account, it is called a withdrawal.

For example, a person removed $25 from their bank account. Before the withdrawal, they had $350. After the withdrawal, they had $325, because $350-25=325$.

Practice

Exercise 5.2.3.4

The table shows five transactions and the resulting account balance in a bank account, except some numbers are missing. Fill in the missing numbers.

Table 5.2.3.2

	transaction amount	account balance
transaction 1	200	200
transaction 2	-147	53
transaction 3	90	
transaction 4	-229	
transaction 5		0

Exercise 5.2.3.5

1. Clare has $54 in her bank account. A store credits her account with a $10 refund. How much does she now have in the bank?
2. Mai's bank account is overdrawn by $60, which means her balance is -$60. She gets $85 for her birthday and deposits it into her account. How much does she now have in the bank?
3. Tyler is overdrawn at the bank by $180. He gets $70 for his birthday and deposits it. What is his account balance now?
4. Andre has $37 in his bank account and writes a check for $87. After the check has been cashed, what will the bank balance show?

Exercise 5.2.3.6

Last week, it rained g inches. This week, the amount of rain decreased by 5%. Which expressions represent the amount of rain that fell this week? Select **all** that apply.

A. $g-0.05$
B. $g-0.05g$
C. $0.95g$
D. $0.05g$
E. $(1-0.05)g$

(From Unit 4.2.3)

Exercise 5.2.3.7

Decide whether or not each equation represents a proportional relationship.

1. Volume measured in cups (c) vs. the same volume measured in ounces (z): $c = \frac{1}{8}z$
2. Area of a square (A) vs. the side length of the square (s): $A = s^2$
3. Perimeter of an equilateral triangle (P) vs. the side length of the triangle (s): $3s = P$
4. Length (L) vs. width (w) for a rectangle whose area is 60 square units: $L = \frac{60}{w}$

(From Unit 2.3.2)

Exercise 5.2.3.8

Add.

1. $5\frac{3}{4} + \left(-\frac{1}{4}\right)$
2. $-\frac{2}{3} + \frac{1}{6}$
3. $-\frac{8}{5} + \left(-\frac{3}{4}\right)$

(From Unit 5.2.2)

Exercise 5.2.3.9

In each diagram, x represents a different value.

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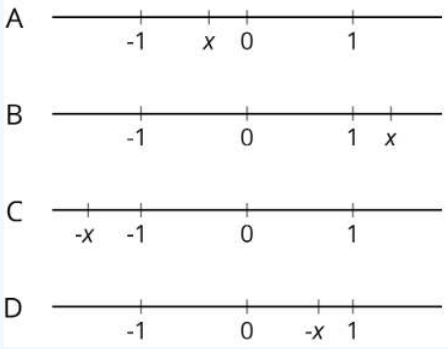


Figure 5.2.3.2

For each diagram,

1. What is something that is *definitely* true about the value of x?
2. What is something that *could be* true about the value of x?

(From Unit 5.1.1)

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5.2.4: Representing Subtraction

Lesson

Let's subtract signed numbers.

Exercise 5.2.4.1: Equivalent Equations

Consider the equation $2+3=5$. Here are some more equations, using the same numbers, that express the same relationship in a different way:

$$3+2=5 \qquad 5-3=2 \qquad 5-2=3$$

For each equation, write two more equations, using the same numbers, that express the same relationship in a different way.

1. $9+(-1)=8$
2. $-11+x=7$

Exercise 5.2.4.2: Subtraction with Number Lines

1. Here is an unfinished number line diagram that represents a sum of 8.

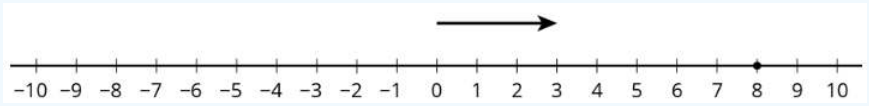


Figure 5.2.4.1: A number line, 21 evenly spaced tick marks. Tick marks begin at negative 10 and count up by 1 to the final tick mark at 10. Above the number line is an arrow that begins at 0 and stops at 3. There is a dot on the tick mark labeled 8.

   a. How long should the arrow be?
   b. For an equation that goes with this diagram, Mai writes $3+?=8$.
      Tyler writes $8-3=?$. Do you agree with either of them?
   c. What is the unknown number? How do you know?

2. Here are two more unfinished diagrams that represent sums.

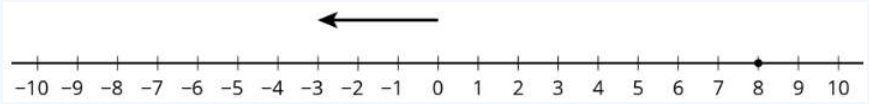


Figure 5.2.4.2: Number line. 21 evenly spaced tick marks. Scale negative 10 to 10, by 1s. An arrow, pointing to the left, goes from 0 to negative 3.

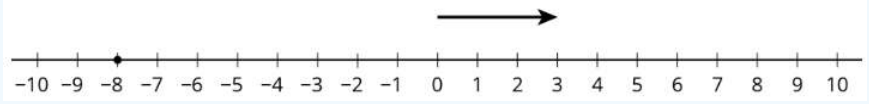


Figure 5.2.4.3: Number line. 21 evenly spaced tick marks. Scale negative 10 to 10, by 1s. There is a point at negative 8. An arrow, pointing to the right, goes from 0 to 3.

For each diagram:

   a. What equation would Mai write if she used the same reasoning as before?
   b. What equation would Tyler write if he used the same reasoning as before?
   c. How long should the other arrow be?
   d. What number would complete each equation? Be prepared to explain your reasoning.

3. Draw a number line diagram for $(-8)-(3)=?$ What is the unknown number? How do you know?

Exercise 5.2.4.3: We can Add Instead

1. Match each diagram to one of these expressions:

$$3+7 \qquad 3-7 \qquad 3+(-7) \qquad 3-(-7)$$

   a.

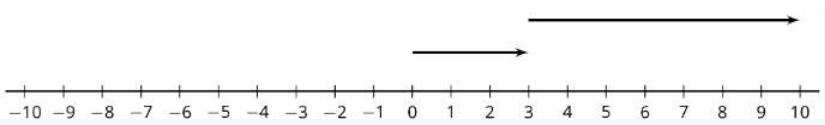


Figure 5.2.4.4

b.

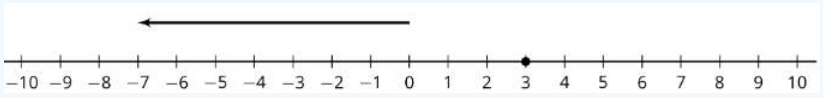


Figure 5.2.4.5

c.

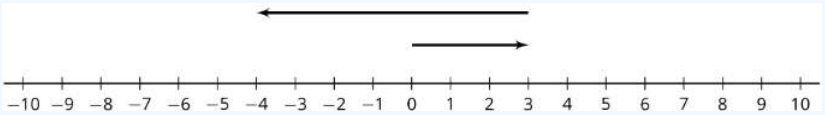


Figure 5.2.4.6

d.

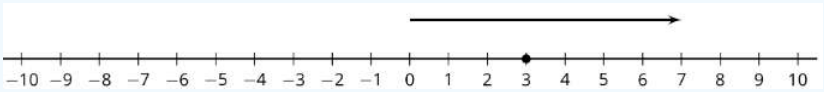


Figure 5.2.4.7

2. Which expressions in the first question have the same value? What do you notice?

3. Complete each of these tables. What do you notice?

Table 5.2.4.1

expression	value
$8+(-8)$	
$8-8$	
$8+(-5)$	
$8-5$	
$8+(-12)$	
$8-12$	

Table 5.2.4.2

expression	value
$-5+5$	
$-5-(-5)$	
$-5+9$	
$-5-(-9)$	
$-5+2$	
$-5-(-2)$	

Are you ready for more?

It is possible to make a new number system using *only* the numbers 0, 1, 2, and 3. We will write the symbols for adding and subtracting in this system like this: $2 \oplus 1 = 3$ and $2 \ominus 1 = 1$. The table shows some of the sums.

Table 5.2.4.3

$\oplus$	**0**	**1**	**2**	**3**
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3				

1. In this system, $1 \oplus 2 = 3$ and $2 \oplus 3 = 1$. How can you see that in the table?
2. What do you think $3 \oplus 1$ should be?
3. What about $3 \oplus 3$?
4. What do you think $3 \ominus 1$ should be?
5. What about $2 \ominus 3$?
6. Can you think of any uses for this number system?

Summary

The equation $7 - 5 = ?$ is equivalent to $? + 5 = 7$. The diagram illustrates the second equation.



Figure 5.2.4.8: A number line with the numbers negative 10 through 10, indicated. An arrow starts at 0, points to the right, ends at 2, and is labeled with a question mark. A second arrow starts at 2, points to the right, ends at 7, and is labeled plus 5. There is a solid dot indicated at 7.

Notice that the value of $7 + (-5)$ is 2.

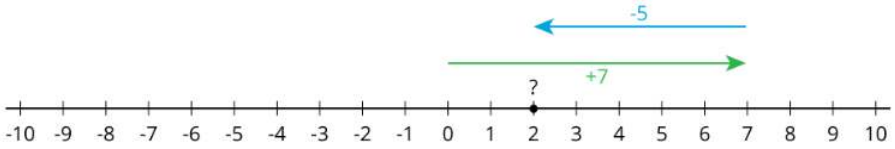


Figure 5.2.4.9: A number line with the numbers negative 10 through 10, indicated. An arrow starts at 0, points to the right, ends at 7, and is labeled plus 7. A second arrow starts at 7, points to the left, ends at 2, and is labeled minus 5. There is a solid dot and a question mark labeled at 2.

We can solve the equation $? + 5 = 7$ by adding -5 to both sides. This shows that $7 - 5 = 7 + (-5)$

Likewise, $3 - 5 = ?$ is equivalent to $? + 5 = 3$.

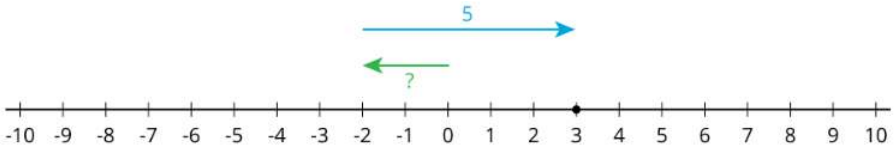


Figure 5.2.4.10: A number line with the numbers negative 10 through 10, indicated. An arrow starts at 0, points to the left, ends at negative 2, and is labeled with a question mark. A second arrow starts at negative 2, points to the right, ends at 3, and is labeled 5. There is a solid dot indicated at 3.

Notice that the value of $3 + (-5)$ is -2.

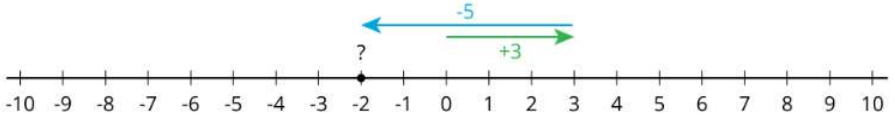


Figure 5.2.4.11: A number line with the numbers negative 10 through 10 indicated. An arrow starts at 0, points to the right, ends at 3, and is labeled plus 3. A second arrow starts at 3, points to the left, ends at negative two, and is labeled minus 5. There is a solid dot and a question mark labeled at 2.

We can solve the equation $? + 5 = 3$ by adding -5 to both sides. This shows that $3 - 5 = 3 + (-5)$

In general:

$$a - b = a + (-b)$$

If $a - b = x$, then $x + b = a$. We can add $-b$ to both sides of this second equation to get that $x = a + (-b)$

Glossary Entries

Definition: Deposit

When you put money into an account, it is called a *deposit*.

For example, a person added \$60 to their bank account. Before the deposit, they had \$435. After the deposit, they had \$495, because $435 + 60 = 495$.

Definition: Withdrawal

When you take money out of an account, it is called a withdrawal.

For example, a person removed \$25 from their bank account. Before the withdrawal, they had \$350. After the withdrawal, they had \$325, because $350 - 25 = 325$.

Practice

Exercise 5.2.4.4

Write each subtraction equation as an addition equation.

1. $a - 9 = 6$
2. $p - 20 = -30$
3. $z - (-12) = 15$
4. $x - (-7) = -10$

Exercise 5.2.4.5

Find each difference. If you get stuck, consider drawing a number line diagram.

1. $9 - 4$
2. $4 - 9$
3. $9 - (-4)$
4. $-9 - (-4)$
5. $-9 - 4$
6. $4 - (-9)$
7. $-4 - (-9)$
8. $-4 - 9$

Exercise 5.2.4.6

A restaurant bill is \$59 and you pay \$72. What percentage gratuity did you pay?

(From Unit 4.3.1)

Exercise 5.2.4.7

Find the solution to each equation mentally.

1. $30 + a = 40$
2. $500 + b = 200$
3. $-1 + c = -2$
4. $d + 3,567 = 0$

Exercise 5.2.4.8

One kilogram is 2.2 pounds. Complete the tables. What is the interpretation of the constant of proportionality in each case?

Table 5.2.4.4

pounds	kilograms
2.2	1
11	
5.5	
1	

_____ kilogram per pound

Table 5.2.4.5

kilograms	pounds
1	2.2
7	
30	
0.5	

_____ pounds per kilogram

(From Unit 2.1.3)

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5.2.5: Subtracting Rational Numbers

Lesson

Let's bring addition and subtraction together.

Exercise 5.2.5.1: Number Talk: Missing Addend

Solve each equation mentally. Rewrite each addition equation as a subtraction equation.

$247 + c = 458$

$c + 43.87 = 58.92$

$\frac{15}{8} + c = \frac{51}{8}$

Exercise 5.2.5.2: Expressions with Altitude

A mountaineer is changing elevations. Write an expression that represents the difference between the final elevation and beginning elevation. Then write the value of the change. The first one is done for you.



Figure 5.2.5.1: By Orca. Public Domain. Pixabay. Source.

Table 5.2.5.1

beginning elevation (feet)	final elevation (feet)	difference between final and beginning	change
$+400$	$+900$	$900 - 400$	$+500$
$+400$	$+50$		
$+400$	-120		
-200	$+610$		
-200	-50		
-200	-500		
-200	0		

Are you ready for more?

Fill in the table so that every row and every column sums to 0. Can you find another way to solve this puzzle?

Table 5.2.5.2

	-12	0		5
0			-18	25
25		-18	5	-12

-12				-18
	-18	25	-12	

Table 5.2.5.3

	-12	0		5
0			-18	25
25		-18	5	-12
-12				-18
	-18	25	-12	

Exercise 5.2.5.3: Does the Order Matter?

1. Find the value of each subtraction expression.

Table 5.2.5.4

A	B
$3-2$	$2-3$
$5-(-9)$	$(-9)-5$
$(-11)-2$	$2-(-11)$
$(-6)-(-3)$	$(-3)-(-6)$
$(-1.2)-(-3.6)$	$(-3.6)-(-1.2)$
$(-2\frac{1}{2})-(-3\frac{1}{2})$	$(-3\frac{1}{2})-(-2\frac{1}{2})$

2. What do you notice about the expressions in Column A compared to Column B?

3. What do you notice about their values?

Summary

When we talk about the difference of two numbers, we mean, “subtract them.” Usually, we subtract them in the order they are named. For example, the difference of $+8$ and -6 is $8-(-6)$.

The difference of two numbers tells you how far apart they are on the number line. 8 and -6 are 14 units apart, because $8-(-6)=14$:

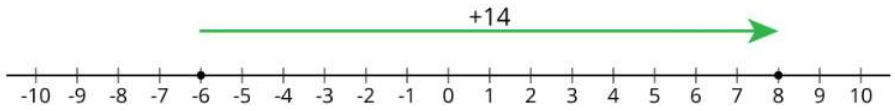


Figure 5.2.5.2: A number line with the numbers negative 10 through 10 indicated. Two solid dots are on the number line located at, negative 6 and 8. An arrow starts at negative 6, points to the right, ends at 8, and is labeled positive 14.

Notice that if you subtract them in the opposite order, you get the opposite number:

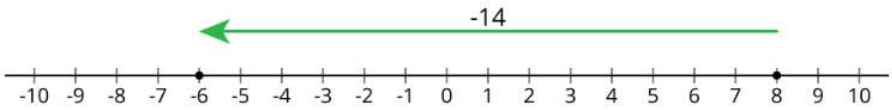


Figure 5.2.5.3: A number line with the numbers negative 10 through 10 indicated. Two solid dots are on the number line located at, negative 6 and 8. An arrow starts at 8, points to the left, ends at negative 6, and is labeled negative 14.

$(-6)-8=-14$

In general, the distance between two numbers a and b on the number line is $|a-b|$. Note that the *distance* between two numbers is always positive, no matter the order. But the *difference* can be positive or negative, depending on the order.

Glossary Entries

Definition: Deposit

When you put money into an account, it is called a *deposit*.

For example, a person added \$60 to their bank account. Before the deposit, they had \$435. After the deposit, they had \$495, because $435+60=495$

Definition: Withdrawal

When you take money out of an account, it is called a withdrawal.

For example, a person removed \$25 from their bank account. Before the withdrawal, they had \$350. After the withdrawal, they had \$325, because $350-25=325$.

Practice

Exercise 5.2.5.4

Write a sentence to answer each question:

1. How much warmer is 82 than 40?
2. How much warmer is 82 than -40?

Exercise 5.2.5.5

1. What is the difference in height between 30 m up a cliff and 87 m up a cliff? What is the distance between these positions?
2. What is the difference in height between an albatross flying at 100 m above the surface of the ocean and a shark swimming 30 m below the surface? What is the distance between them if the shark is right below the albatross?

Exercise 5.2.5.6

A company produces screens of different sizes. Based on the table, could there be a relationship between the number of pixels and the area of the screen? If so, write an equation representing the relationship. If not, explain your reasoning.

Table 5.2.5.5

square inches of screen	number of pixels
6	31, 104
72	373, 248
105	544, 320
300	1, 555, 200

(From Unit 2.3.2)

Exercise 5.2.5.7

Find each difference.

1. $(-5)-6$
2. $35-(-8)$

1. $\frac{2}{5}-\frac{3}{5}$

2. $-4\frac{3}{8} - \left(-1\frac{1}{4}\right)$

Exercise 5.2.5.8

A family goes to a restaurant. When the bill comes, this is printed at the bottom of it:

> Gratuity Guide For Your Convenience:
> 15% would be $4.89
> 18% would be $5.87
> 20% would be $6.52

How much was the price of the meal? Explain your reasoning.

(From Unit 4.3.1)

Exercise 5.2.5.9

Which is a scaled copy of Polygon A? Identify a pair of corresponding sides and a pair of corresponding angles. Compare the areas of the scaled copies.

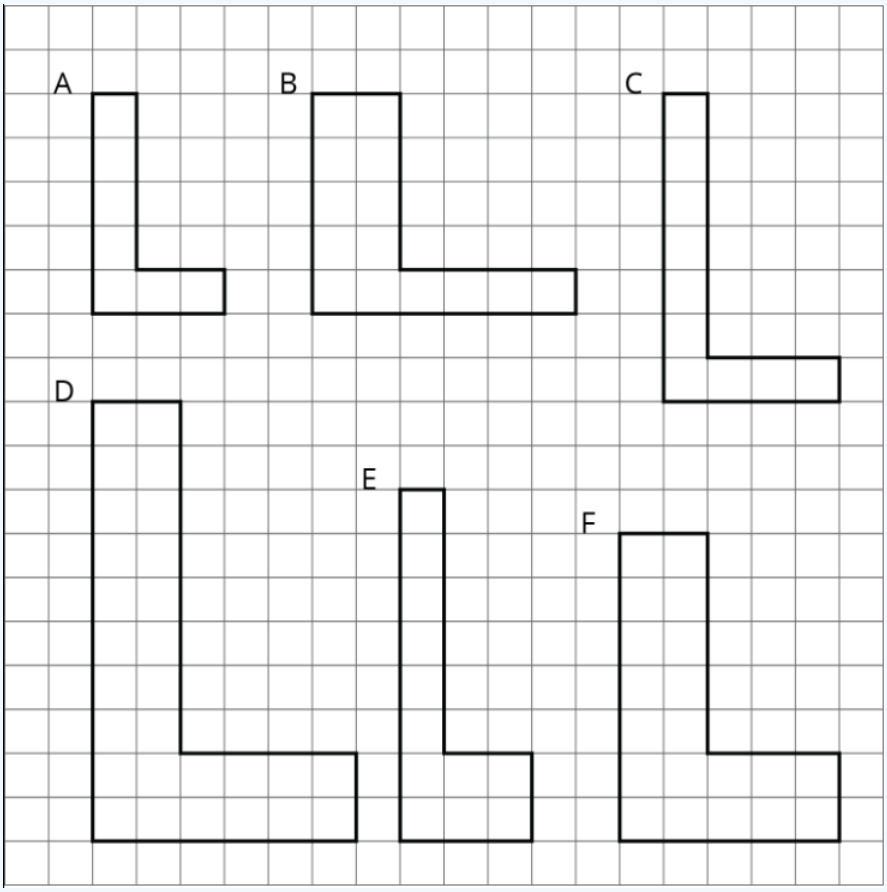


Figure 5.2.5.4

(From Unit 1.1.2)

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5.2.6: Adding and Subtracting to Solve Problems

Lesson

Let's apply what we know about signed numbers to different situations.

Exercise 5.2.6.1: Positive or Negative?

Without computing:

1. Is the solution to $-2.7 + x = -3.5$ positive or negative?
2. Select **all** the expressions that are solutions to $-2.7 + x = -3.5$.
   a. $-3.5 + 2.7$
   b. $3.5 - 2.7$
   c. $-3.5 - (-2.7)$
   d. $-3.5 - 2.7$

Exercise 5.2.6.2: Phone Inventory

A store tracks the number of cell phones it has in stock and how many phones it sells.

The table shows the inventory for one phone model at the beginning of each day last week. The inventory changes when they sell phones or get shipments of phones into the store.

Table 5.2.6.1

	inventory	change
Monday	18	-2
Tuesday	16	-5
Wednesday	11	-7
Thursday	4	-6
Friday	-2	20

1. What do you think it means when the change is positive? Negative?
2. What do you think it means when the inventory is positive? Negative?
3. Based on the information in the table, what do you think the inventory will be at on Saturday morning? Explain your reasoning.
4. What is the difference between the greatest inventory and the least inventory?

Exercise 5.2.6.3: Solar Power

Han's family got a solar panel. Each month they get a credit to their account for the electricity that is generated by the solar panel. The credit they receive varies based on how sunny it is.



Figure 5.2.6.1: Solar panels on a roof, by Pujanak (Own work). Public Domain. Wikimedia Commons. Source.

Table 5.2.6.2

Current charges: $83.56 Solar Credit: -$6.75 Amount due: $76.81	In January they used $83.56 worth of electricity and generated $6.75 worth of electricity.

1. In July they were traveling away from home and only used $19.24 worth of electricity. Their solar panel generated $22.75 worth of electricity. What was their amount due in July?
2. The table shows the value of the electricity they used and the value of the electricity they generated each week for a month. What amount is due for this month?

Table 5.2.6.3

	used ($)	generated ($)
week 1	13.45	−6.33
week 2	21.78	−8.94
week 3	18.12	−7.70
week 4	24.05	−5.36

3. What is the difference between the value of the electricity generated in week 1 and week 2? Between week 2 and week 3?

Are you ready for more?

While most rooms in any building are all at the same level of air pressure, hospitals make use of "positive pressure rooms" and "negative pressure rooms." What do you think it means to have negative pressure in this setting? What could be some uses of these rooms?

Exercise 5.2.6.4: Differences and Distances

Plot these points on the coordinate grid: $A = (5, 4),\ B = (5, -2),\ C = (-3, -2),\ D = (-3, 4)$

1. What shape is made if you connect the dots in order?
2. What are the side lengths of figure $ABCD$?
3. What is the difference between the x-coordinates of B and C?
4. What is the difference between the x-coordinates of C and B?
5. How do the differences of the coordinates relate to the distances between the points?

Summary

Sometimes we use positive and negative numbers to represent quantities in context. Here are some contexts we have studied that can be represented with positive and negative numbers:

- temperature
- elevation
- inventory
- an account balance
- electricity flowing in and flowing out

In these situations, using positive and negative numbers, and operations on positive and negative numbers, helps us understand and analyze them. To solve problems in these situations, we just have to understand what it means when the quantity is positive, when it is negative, and what it means to add and subtract them.

When two points in the coordinate plane lie on a horizontal line, you can find the distance between them by subtracting their x-coordinates.

When two points in the coordinate plane lie on a vertical line, you can find the distance between them by subtracting their y-coordinates.

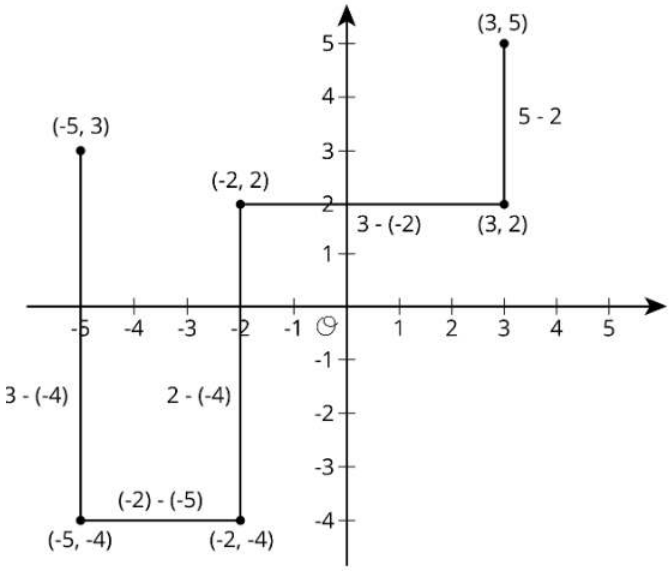


Figure 5.2.6.2

Remember: the *distance* between two numbers is independent of the order, but the *difference* depends on the order.

Glossary Entries

Definition: Deposit

When you put money into an account, it is called a *deposit*.

For example, a person added \$60 to their bank account. Before the deposit, they had \$435. After the deposit, they had \$495, because $435 + 60 = 495$

Definition: Withdrawal

When you take money out of an account, it is called a withdrawal.

For example, a person removed \$25 from their bank account. Before the withdrawal, they had \$350. After the withdrawal, they had \$325, because $350 - 25 = 325$.

Practice

Exercise 5.2.6.5

The table shows four transactions and the resulting account balance in a bank account, except some numbers are missing. Fill in the missing numbers.

Table 5.2.6.4

	transaction amount	account balance
transaction 1	360	360
transaction 2	-22.50	337.50
transaction 3		182.35
transaction 4		-41.40

Exercise 5.2.6.6

The *departure from the average* is the difference between the actual amount of rain and the average amount of rain for a given month. The historical average for rainfall in Albuquerque, NM for June, July, and August is shown in the table.

Table 5.2.6.5

June	July	August
0.67	1.5	1.57

1. Last June only 0.17 inches of rain fell all month. What is the difference between the average rainfall and the actual rainfall for last June?
2. The departure from the average rainfall last July was -0.36 inches. How much rain fell last July?
3. How much rain would have to fall in August so that the total amount of rain equals the average rainfall for these three months? What would the departure from the average be in August in that situation?

Exercise 5.2.6.7

1. How much higher is 500 than 400 m?
2. How much higher is 500 than -400 m?
3. What is the change in elevation from 8,500 m to 3,400 m?
4. What is the change in elevation between 8,500 m and -300 m?
5. How much higher is -200 m than 450 m?

(From Unit 5.2.5)

Exercise 5.2.6.8

Tyler orders a meal that costs $15.

1. If the tax rate is 6.6%, how much will the sales tax be on Tyler's meal?
2. Tyler also wants to leave a tip for the server. How much do you think he should pay in all? Explain your reasoning.

(From Unit 4.3.1)

Exercise 5.2.6.9

In a video game, a character is healed at a constant rate as long as they are standing in a certain circle. Complete the table.

Table 5.2.6.6

time in circle (seconds)	health gained (points)
4	100
10	
3	
	1, 000

(From Unit 2.1.3)

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5.3: Multiplying and Dividing Rational Numbers

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5.3.1: Position, Speed, and Direction

Lesson

Let's use signed numbers to represent movement.

Exercise 5.3.1.1: Distance, Rate, Time

1. An airplane moves at a constant speed of 120 miles per hour for 3 hours. How far does it go?
2. A train moves at constant speed and travels 6 miles in 4 minutes. What is its speed in miles per minute?
3. A car moves at a constant speed of 50 miles per hour. How long does it take the car to go 200 miles?

Exercise 5.3.1.2: Going Left, Going Right

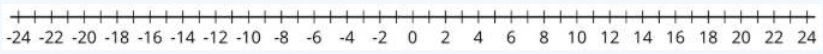


Figure 5.3.1.1

1. After each move, record your location in the table. Then write an expression to represent the ending position that uses the starting position, the speed, and the time. The first row is done for you.

Table 5.3.1.1

starting position	direction	speed (units per second)	time (seconds)	ending position (units)	expression
0	right	5	3	$+15$	$0+5\cdot 3$
0	left	4	6		
0	right	2	8		
0	right	6	2		
0	left	1.1	5		

2. How can you see the *direction* of movement in the expression?
3. Using a starting position p, a speed s, and a time t, write two expressions for an ending position. One expression should show the result of moving right, and one expression should show the result of moving left.

Exercise 5.3.1.3: Velocity

A traffic safety engineer was studying travel patterns along a highway. She set up a camera and recorded the speed and direction of cars and trucks that passed by the camera. Positions to the east of the camera are positive, and to the west are negative.

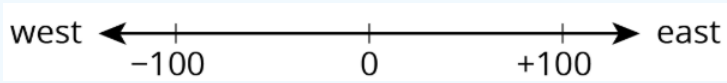


Figure 5.3.1.2: Number line. To the left is the word west. To the right is the word east. Three evenly spaced tick marks labeled, from left to right, negative 100, 0, plus 100.

Vehicles that are traveling towards the east have a positive velocity, and vehicles that are traveling towards the west have a negative velocity.

1. Complete the table with the position of each vehicle if the vehicle is traveling at a constant speed for the indicated time period. Then write an equation.

Table 5.3.1.2

velocity (meters per second)	time after passing the camera (seconds)	ending position (meters)	equation describing the position
$+25$	$+10$	$+250$	$25\cdot 10=250$

velocity (meters per second)	time after passing the camera (seconds)	ending position (meters)	equation describing the position
-20	$+30$		
$+32$	$+40$		
-35	$+20$		
$+28$	0		

2. If a car is traveling east when it passes the camera, will its position be positive or negative 60 seconds after it passes the camera? If we multiply two positive numbers, is the result positive or negative?
3. If a car is traveling west when it passes the camera, will its position be positive or negative 60 seconds after it passes the camera? If we multiply a negative and a positive number, is the result positive or negative?

Are you ready for more?

In many contexts we can interpret negative rates as "rates in the opposite direction." For example, a car that is traveling -35 miles per hour is traveling in the opposite direction of a car that is traveling 40 miles per hour.

1. What could it mean if we say that water is flowing at a rate of -5 gallons per minute?
2. Make up another situation with a negative rate, and explain what it could mean.

Summary

We can use signed numbers to represent the position of an object along a line. We pick a point to be the reference point, and call it zero. Positions to the right of zero are positive. Positions to the left of zero are negative.

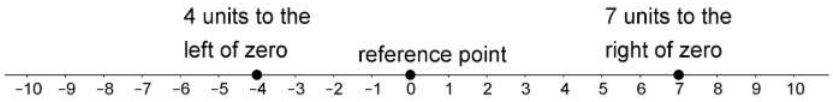


Figure 5.3.1.3: A number line with the numbers negative 10 through 10 indicated. A point is indicated at zero and is labeled reference point. Another point is indicated at negative 4 and is labeled 4 units to the left of zero. A third point is indicated at 7 and is labeled 7 units to the right of zero.

When we combine speed with direction indicated by the sign of the number, it is called *velocity*. For example, if you are moving 5 meters per second to the right, then your velocity is +5 meters per second. If you are moving 5 meters per second to the left, then your velocity is -5 meters per second.

If you start at zero and move 5 meters per second for 10 seconds, you will be $5 \cdot 10 = 50$ meters to the right of zero. In other words, $5 \cdot 10 = 50$.

If you start at zero and move -5 meters per second for 10 seconds, you will be $5 \cdot 10 = 50$ meters to the *left* of zero. In other words,

$$-5 \cdot 10 = -50$$

In general, a negative number times a positive number is a negative number.

Practice

Exercise 5.3.1.4

A number line can represent positions that are north and south of a truck stop on a highway. Decide whether you want positive positions to be north or south of the truck stop. Then plot the following positions on a number line.

1. The truck stop
2. 5 miles north of the truck stop
3. 3.5 miles south of the truck stop

Exercise 5.3.1.5

1. How could you distinguish between traveling west at 5 miles per hour and traveling east at 5 miles per hour without using the words “east” and “west”?
2. Four people are cycling. They each start at the same point. (0 represents their starting point.) Plot their finish points after five seconds of cycling on a number line.
   - Lin cycles at 5 meters per second
   - Diego cycles at -4 meters per second
   - Elena cycles at 3 meters per second
   - Noah cycles at -6 meters per second

Exercise 5.3.1.6

Find the value of each expression.

1. $16.2 + -8.4$
2. $\frac{2}{5} - \frac{3}{5}$
3. $-9.2 + -7$
4. $-4\frac{3}{8} - (-1\frac{1}{4})$

(From Unit 5.2.5)

Exercise 5.3.1.7

For each equation, write two more equations using the same numbers that express the same relationship in a different way.

1. $3 + 2 = 5$
2. $7.1 + 3.4 = 10.5$
3. $15 - 8 = 7$
4. $\frac{3}{2} + \frac{9}{5} = \frac{33}{10}$

(From Unit 5.2.4)

Exercise 5.3.1.8

A shopper bought a watermelon, a pack of napkins, and some paper plates. In his state, there is no tax on food. The tax rate on non-food items is 5%. The total for the three items he bought was $8.25 before tax, and he paid $0.19 in tax. How much did the watermelon cost?

(From Unit 4.3.1)

Exercise 5.3.1.9

Which graphs could not represent a proportional relationship? Explain how you decided.

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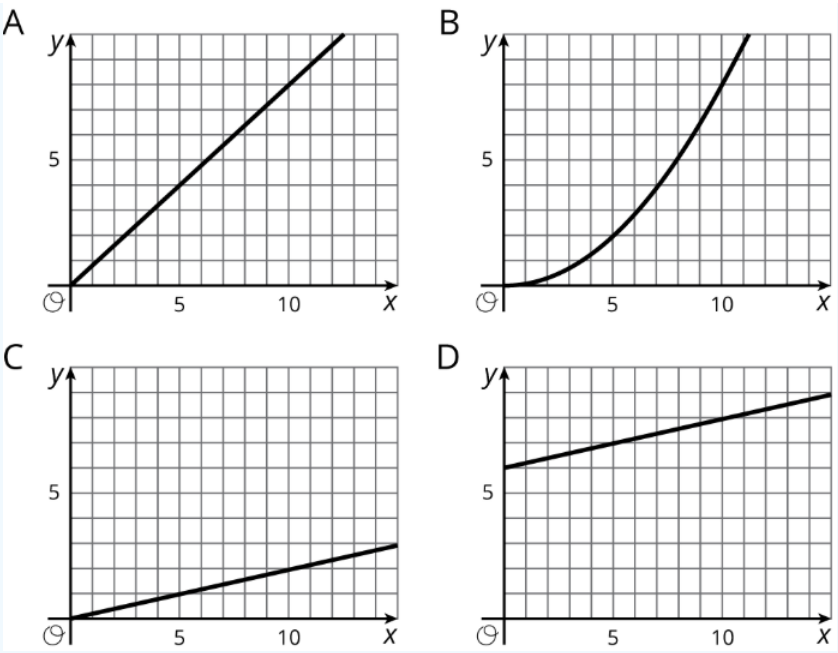


Figure 5.3.1.4

(From Unit 2.4.1)

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5.3.2: Multiplying Rational Numbers

Lesson

Let's multiply signed numbers.

Exercise 5.3.2.1: Before and After



Figure 5.3.2.1

Where was the girl:

1. 5 seconds *after* this picture was taken? Mark her approximate location on the picture.
2. 5 seconds *before* this picture was taken? Mark her approximate location on the picture.

Exercise 5.3.2.2: Backwards in Time

A traffic safety engineer was studying travel patterns along a highway. She set up a camera and recorded the speed and direction of cars and trucks that passed by the camera. Positions to the east of the camera are positive, and to the west are negative.

1. Here are some positions and times for one car:

position (feet)	-180	-120	-60	0	60	120
time (seconds)	-3	-2	-1	0	1	2

Table 5.3.2.1

   a. In what direction is this car traveling?
   b. What is its velocity?

2. a. What does it mean when the time is zero?
   b. What could it mean to have a negative time?

3. Here are the positions and times for a different car whose velocity is -50 feet per second:

position (feet)				0	-50	-100
time (seconds)	-3	-2	-1	0	1	2

Table 5.3.2.2

   a. Complete the table with the rest of the positions.
   b. In what direction is this car traveling? Explain how you know.

4. Complete the table for several different cars passing the camera.

Table 5.3.2.3

	velocity (meters per second)	**time after passing the camera (seconds)**	**ending position (meters)**	**equation**
car C	$+25$	$+10$	$+250$	$25 \cdot 10 = 250$
car D	-20	$+30$		

	velocity (meters per second)	time after passing the camera (seconds)	ending position (meters)	equation
car E	$+32$	-40		
car F	-35	-20		
car G	-15	-8		

5. a. If a car is traveling east when it passes the camera, will its position be positive or negative 60 seconds *before* it passes the camera?
   b. If we multiply a positive number and a negative number, is the result positive or negative?
6. a. If a car is traveling west when it passes the camera, will its position be positive or negative 60 seconds *before* it passes the camera?
   b. If we multiply two negative numbers, is the result positive or negative?

Exercise 5.3.2.3: Cruising

Around noon, a car was traveling -32 meters per second down a highway. At exactly noon (when time was 0), the position of the car was 0 meters.

1. Complete the table.

Table 5.3.2.4

time (s)	-10	-7	-4	-1	2	5	8	11
position (m)								

2. Graph the relationship between the time and the car's position.
   a. What was the position of the car at -3 seconds?
   b. What was the position of the car at 6.5 seconds?

Are you ready for more?

Find the value of these expressions without using a calculator.

$(-1)^2$ $(-1)^3$ $(-1)^4$ $(-1)^{99}$

Exercise 5.3.2.4: Rational Numbers Multiplication Grid

Look at the patterns along the rows and columns and continue those patterns to complete the table. When you have filled in all the boxes you can see, click on the "More Boxes" button.

What does this tell you about multiplication by a negative?

Summary

We can use signed numbers to represent time relative to a chosen point in time. We can think of this as starting a stopwatch. The positive times are after the watch starts, and negative times are times before the watch starts.

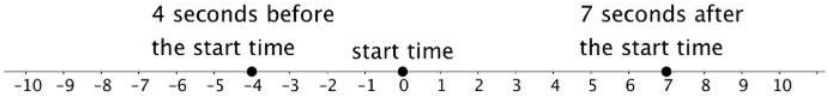


Figure 5.3.2.2: Three points are labeled on a number line. The numbers negative 10 through 10, in increments of 1, are indicated. A dot at negative 4 is labeled 4 seconds before the start time. A dot at 0 is labeled start time. A dot at 7 is labeled 7 seconds after the start time.

If a car is at position 0 and is moving in a positive direction, then for times after that (positive times), it will have a positive position. A positive times a positive is positive.

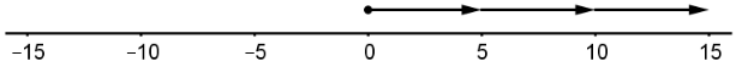


Figure 5.3.2.3: Number line. 7 evenly spaced tick marks. Scale negative 15 to 15, by 5s. Three equal sized arrows pointing to the right, one from 0 to 5, 5 to 10, and 10 to 15.

If a car is at position 0 and is moving in a negative direction, then for times after that (positive times), it will have a negative position. A negative times a positive is negative.

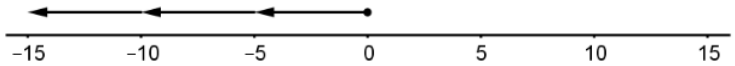


Figure 5.3.2.4: A blank horizontal number line from negative 15 to 15 by 5s. Above the number line with three arrows pointing left and a dot are plotted. The first arrow points from 0 to negative 5. The second arrow points from negative 5 to negative 10. The third arrow points from negative 10 to negative 15. A dot is above 0.

If a car is at position 0 and is moving in a positive direction, then for times *before* that (negative times), it must have had a negative position. A positive times a negative is negative.

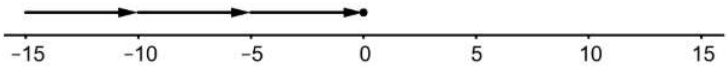


Figure 5.3.2.5: A blank horizontal number line from negative 15 to 15 by 5s. Above the number line with three arrows pointing right and a dot are plotted. The first arrow points from negative 15 to negative 10. The second arrow points from negative 10 to negative 5. The third arrow points from negative 5 to 0. A dot is above 0.

If a car is at position 0 and is moving in a negative direction, then for times *before* that (negative times), it must have had a positive position. A negative times a negative is positive.

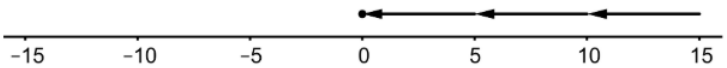


Figure 5.3.2.6: A blank horizontal number line from negative 15 to 15 by 5s. Above the number line with three arrows pointing left and a dot are plotted. The first arrow points from 15 to 10. The second arrow points from 10 to 5. The third arrow points from 5 to 0. A dot is above 0.

Here is another way of seeing this:

We can think of $3 \cdot 5$ as $5+5+5$, which has a value of 15.

We can think of $3 \cdot (-5)$ as $-5+-5+-5$, which has a value of -15.

We know we can multiply positive numbers in any order: $3 \cdot 5 = 5 \cdot 3$

If we can multiply signed numbers in any order, then $(-5) \cdot 3$ would also equal -15.

Now let's think about multiplying two negatives.

We can find $-5 \cdot (3+-3)$ in two ways:

- Applying the distributive property:
  $-5 \cdot 3 + -5 \cdot (-3)$
- Adding the numbers in parentheses:
  $-5 \cdot (0) = 0$

This means that these expressions must be equal.

$$-5 \cdot 3 + -5 \cdot (-3) = 0$$

Multiplying the first two numbers gives

$$-15 + -5 \cdot (-3) = 0$$

Which means that

$$-5 \cdot (-3) = 15$$

There was nothing special about these particular numbers. This always works!

- A positive times a positive is always positive.
- A negative times a positive or a positive times a negative is always negative.
- A negative times a negative is always positive.

Practice

Exercise 5.3.2.5

Fill in the missing numbers in these equations

1. $-2 \cdot (-4.5) = ?$
2. $(-8.7) \cdot (-10) = ?$
3. $(-7) \cdot ? = 14$
4. $? \cdot (-10) = 90$

Exercise 5.3.2.6

A weather station on the top of a mountain reports that the temperature is currently $0°\text{C}$ and has been falling at a constant rate of $3°\text{C}$ per hour. If it continues to fall at this rate, find each indicated temperature. Explain or show your reasoning.

1. What will the temperature be in 2 hours?
2. What will the temperature be in 5 hours?
3. What will the temperature be in half an hour?
4. What was the temperature 1 hour ago?
5. What was the temperature 3 hours ago?
6. What was the temperature 4.5 hours ago?

Exercise 5.3.2.7

Find the value of each expression.

1. $\frac{1}{4} \cdot (-12)$
2. $-\frac{1}{3} \cdot 39$
3. $(-\frac{4}{5}) \cdot (-75)$
4. $-\frac{2}{5} \cdot (-\frac{3}{4})$
5. $\frac{8}{3} \cdot -42$

Exercise 5.3.2.8

To make a specific hair dye, a hair stylist uses a ratio of $1\frac{1}{8}$ oz of red tone, $\frac{3}{4}$ oz of gray tone, and $\frac{5}{8}$ oz of brown tone.

1. If the stylist needs to make 20 oz of dye, how much of each dye color is needed?
2. If the stylist needs to make 100 oz of dye, how much of each dye color is needed?

(From Unit 4.1.2)

Exercise 5.3.2.9

1. Here are the vertices of rectangle $FROG$: $(-2, 5),\ (-2, 1),\ (6, 5),\ (6, 1)$
   Find the perimeter of this rectangle. If you get stuck, try plotting the points on a coordinate plane.
2. Find the area of the rectangle $FROG$.
3. Here are the coordinates of rectangle $PLAY$: $(-11, 20),\ (-11, -3),\ (-1, 20),\ (-1, -3)$ Find the perimeter and area of this rectangle. See if you can figure out its side lengths without plotting the points.

(From Unit 5.2.6)

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5.3.3: Multiply!

Lesson

Let's get more practice multiplying signed numbers.

Exercise 5.3.3.1: Which One Doesn't Belong: Expressions

Which expression doesn't belong?

$7.9x$	$-7.9+x$
$7.9\cdot(-10)$	-79

Exercise 5.3.3.2: Card Sort: Matching Expressions

Your teacher will give you cards with multiplication expressions on them. Match the expressions that are equal to each other. There will be 3 cards in each group.

Exercise 5.3.3.3: Row Game: Multiplying Rational Numbers

Evaluate the expressions in one of the columns. Your partner will work on the other column. Check in with your partner after you finish each row. Your answers in each row should be the same. If your answers aren't the same, work together to find the error.

Table 5.3.3.1

column A	column B
$790\div 10$	$(7.9)\cdot 10$
$-\frac{6}{7}\cdot 7$	$(0.1)\cdot -60$
$(2.1)\cdot -2$	$(-8.4)\cdot \frac{1}{2}$
$(2.5)\cdot(-3.25)$	$-\frac{5}{2}\cdot\frac{13}{4}$
$-10\cdot(3.2)\cdot(-7.3)$	$5\cdot(-1.6)\cdot(-29.2)$

Are you ready for more?

A sequence of rational numbers is made by starting with 1, and from then on, each term is one more than the reciprocal of the previous term. Evaluate the first few expressions in the sequence. Can you find any patterns? Find the 10th term in this sequence.

$$1 \qquad 1+\frac{1}{1} \qquad 1+\frac{1}{1+1} \qquad 1+\frac{1}{1+\frac{1}{1+1}} \qquad 1+\frac{1}{1+\frac{1}{1+\frac{1}{1+1}}} \qquad \ldots$$

Summary

- A positive times a positive is always positive.
  For example, $\frac{3}{5}\cdot\frac{7}{8}=\frac{21}{40}$.
- A negative times a negative is also positive.
  For example, $-\frac{3}{5}\cdot -\frac{7}{8}=\frac{21}{40}$.
- A negative times a positive or a positive times a negative is always negative.
  For example, $\frac{3}{5}\cdot -\frac{7}{8}=-\frac{3}{5}\cdot\frac{7}{8}=-\frac{21}{40}$.
- A negative times a negative times a negative is also negative.
  For example, $-3\cdot -4\cdot -5=-60$.

Practice

Exercise 5.3.3.4

Evaluate each expression:

1. $-12 \cdot \frac{1}{3}$
2. $-12 \cdot -\frac{1}{3}$
3. $12 \cdot \left(-\frac{5}{4}\right)$
4. $-12 \cdot \left(-\frac{5}{4}\right)$

Exercise 5.3.3.5

Evaluate each expression:

1. $-1 \cdot 2 \cdot 3$
2. $-1 \cdot (-2) \cdot 3$
3. $-1 \cdot (-2) \cdot (-3)$

Exercise 5.3.3.6

Order each set of numbers from least to greatest.

1. $4, 8, -2, -6, 0$
2. $-5, -5.2, 5.5, -5\frac{1}{2}, -\frac{5}{2}$

(From Unit 5.1.1)

Exercise 5.3.3.7

$30 + -30 = 0$

1. Write another sum of two numbers that equals 0.
2. Write a sum of three numbers that equals 0.
3. Write a sum of four numbers that equals 0, none of which are opposites.

(From Unit 5.2.2)

Exercise 5.3.3.8

A submarine is searching for underwater features. It is accompanied by a small aircraft and an underwater robotic vehicle.

At one time the aircraft is 200 m above the surface, the submarine is 55 m below the surface, and the underwater robotic vehicle is 227 m below the surface.

1. What is the difference in height between the submarine and the aircraft?
2. What is the distance between the underwater robotic vehicle and the submarine?

(From Unit 5.2.5)

Exercise 5.3.3.9

1. Clare is cycling at a speed of 12 miles per hour. If she starts at a position chosen as zero, what will her position be after 45 minutes?
2. Han is cycling at a speed of -8 miles per hour; if he starts at the same zero point, what will his position be after 45 minutes?
3. What will the distance between them be after 45 minutes?

(From Unit 5.3.1)

Exercise 5.3.3.10

Fill in the missing numbers in these equations

1. $(-7)\cdot ? = -14$
2. $? \cdot 3 = -15$
3. $? \cdot 4 = 32$
4. $-49 \cdot 3 = ?$

(From Unit 5.3.2)

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5.3.4: Dividing Rational Numbers

Lesson

Let's divide signed numbers.

Exercise 5.3.4.1: Tell Me Your Sign

Consider the equation: $-27x = -35$

Without computing:

1. Is the **solution** to this equation positive or negative?
2. Are either of these two number solutions to the equation?

$$\frac{35}{27} \qquad -\frac{35}{27}$$

Exercise 5.3.4.2: Multiplication and Division

1. Find the missing values in the equations
   a. $-3 \cdot 4 = ?$
   b. $-3 \cdot ? = 12$
   c. $3 \cdot ? = 12$
   d. $? \cdot -4 = 12$
   e. $? \cdot 4 = -12$
2. Rewrite the unknown factor problems as division problems.
3. Complete the sentences. Be prepared to explain your reasoning.
   a. The sign of a positive number divided by a positive number is always:
   b. The sign of a positive number divided by a negative number is always:
   c. The sign of a negative number divided by a positive number is always:
   d. The sign of a negative number divided by a negative number is always:
4. Han and Clare walk towards each other at a constant rate, meet up, and then continue past each other in opposite directions. We will call the position where they meet up 0 feet and the time when they meet up 0 seconds.
   - Where is each person 10 seconds before they meet up?
   - When is each person at the position -10 feet from the meeting place?

   a. Where is each person 10 seconds before they meet up?
   b. When is each person at the position -10 feet from the meeting place?

Are you ready for more?

It is possible to make a new number system using *only* the numbers 0, 1, 2, and 3. We will write the symbols for multiplying in this system like this: $1 \otimes 2 = 2$. The table shows some of the products.

Table 5.3.4.1

$\otimes$	**0**	**1**	**2**	**3**
0	0	0	0	0
1		1	2	3
2			0	2
3				

1. In this system, $1 \otimes 3 = 3$ and $2 \otimes 3 = 2$. How can you see that in the table?
2. What do you think $2 \otimes 1$ is?
3. What about $3 \otimes 3$?

4. What do you think the solution to $3 \otimes n = 2$ is?
5. What about $2 \otimes n = 3$?

Exercise 5.3.4.3: Drilling Down

A water well drilling rig has dug to a height of -60 feet after one full day of continuous use.

1. Assuming the rig drilled at a constant rate, what was the height of the drill after 15 hours?
2. If the rig has been running constantly and is currently at a height of -147.5 feet, for how long has the rig been running?



Figure 5.3.4.1: "US Navy 090226-N-9584H-018 Construction Electrician Construction man Greg Langdon, assigned to Naval Mobile Construction Battalion (NMCB) 1, installs a new section of drill steel during a water well drilling operation", by U.S. Navy photo by Mass Communication Specialist Seaman Ernesto Hernandez Fonte. Public Domain. Source.

1. Use the coordinate grid to show the drill's progress.
2. At this rate, how many hours will it take until the drill reaches -250 feet?

Summary

Any division problem is actually a multiplication problem:

- $6 \div 2 = 3$ because $2 \cdot 3 = 6$
- $6 \div -2 = -3$ because $-2 \cdot -3 = 6$
- $-6 \div 2 = -3$ because $2 \cdot -3 = -6$
- $-6 \div 2 = 3$ because $-2 \cdot 3 = -6$

Because we know how to multiply signed numbers, that means we know how to divide them.

- The sign of a positive number divided by a negative number is always negative.
- The sign of a negative number divided by a positive number is always negative.
- The sign of a negative number divided by a negative number is always positive.

A number that can be used in place of the variable that makes the equation true is called a **solution** to the equation. For example, for the equation $x \div -2 = 5$, the solution is -10, because it is true that $-10 \div -2 = 5$.

Glossary Entries

Definition: Solution to an Equation

A solution to an equation is a number that can be used in place of the variable to make the equation true.

For example, 7 is the solution to the equation $m + 1 = 8$, because it is true that $7 + 1 = 8$. The solution to $m + 1 = 8$ is not 9, because $9 + 1 \neq 8$.

Practice

Exercise 5.3.4.4

Find the quotients:

1. $24 \div -6$
2. $-15 \div 0.3$
3. $-4 \div -20$

Exercise 5.3.4.5

Find the quotients.

1. $\frac{2}{5} \div \frac{3}{4}$
2. $\frac{9}{4} \div \frac{-3}{4}$
3. $\frac{-5}{7} \div \frac{-1}{3}$
4. $\frac{-5}{3} \div \frac{1}{6}$

Exercise 5.3.4.6

Is the solution positve or negative?

1. $2 \cdot x = 6$
2. $-2 \cdot x = 6.1$
3. $2.9 \cdot x = -6.04$
4. $-2.473 \cdot x = -6.859$

Exercise 5.3.4.7

Find the solution mentally.

1. $3 \cdot -4 = a$
2. $b = \cdot(-3) = -12$
3. $-12 \cdot c = 12$
4. $d \cdot 24 = -12$

Exercise 5.3.4.8

In order to make a specific shade of green paint, a painter mixes $1\frac{1}{2}$ quarts of blue paint, 2 cups of green paint, and $\frac{1}{2}$ gallon of white paint. How much of each color is needed to make 100 cups of this shade of green paint?

(From Unit 4.1.2)

Exercise 5.3.4.9

Here is a list of the highest and lowest elevation on each continent.

Table 5.3.4.2

	highest point (m)	lowest point (m)
Europe	4, 810	−28
Asia	8, 848	−427
Africa	5, 895	−155
Australia	4, 884	−15

	highest point (m)	lowest point (m)
North America	$6,198$	-86
South America	$6,960$	-105
Antartica	$4,892$	-50

1. Which continent has the largest difference in elevation? The smallest?
2. Make a display (dot plot, box plot, or histogram) of the data set and explain why you chose that type of display to represent this data set.

(From Unit 5.2.2)

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5.3.5: Negative Rates

Lesson

Let's apply what we know about signed numbers.

Exercise 5.3.5.1: Graper per Minute

1. If you eat 5 grapes per minute for 8 minutes, how many grapes will you eat?
2. If you hear 9 new songs per day for 3 days, how many new songs will you hear?
3. If you run 15 laps per practice, how many practices will it take you to run 30 laps?

Exercise 5.3.5.2: Water Level in the Aquarium

1. A large aquarium should contain 10,000 liters of water when it is filled correctly. It will overflow if it gets up to 12,000 liters. The fish will get sick if it gets down to 4,000 liters. The aquarium has an automatic system to help keep the correct water level. If the water level is too low, a faucet fills it. If the water level is too high, a drain opens.
   One day, the system stops working correctly. The faucet starts to fill the aquarium at a rate of 30 liters per minute, and the drain opens at the same time, draining the water at a rate of 20 liters per minute.
   a. Is the water level rising or falling? How do you know?
   b. How long will it take until the tank starts overflowing or the fish get sick?
2. A different aquarium should contain 15,000 liters of water when filled correctly. It will overflow if it gets to 17,600 liters. One day there is an accident and the tank cracks in 4 places. Water flows out of each crack at a rate of $\frac{1}{2}$ liter per hour. An emergency pump can re-fill the tank at a rate of 2 liters per minute. How many minutes must the pump run to replace the water lost each hour?

Exercise 5.3.5.3: Up and Down with the Piccards

1. Challenger Deep is the deepest known point in the ocean, at 35,814 feet below sea level. In 1960, Jacques Piccard and Don Walsh rode down in the Trieste and became the first people to visit the Challenger Deep.



Figure 5.3.5.1: Bathyscaphe Trieste, by U.S. Navy photo by Mass Communication Specialist Seaman Ernesto Hernandez Fonte. Public Domain. Wikimedia Commons. Source.

   a. If sea level is represented by 0 feet, explain how you can represent the depth of a submarine descending from sea level to the bottom of Challenger Deep.
   b. Trieste's descent was a change in depth of -3 feet per second. We can use the relationship $y = -3x$ to model this, where y is the depth (in feet) and x is the time (in seconds). Using this model, how much time would the Trieste take to reach the bottom?
   c. It took the Trieste 3 hours to ascend back to sea level. This can be modeled by a different relationship $y = kx$. What is the value of k in this situation?

2. The design of the Trieste was based on the design of a hot air balloon built by Auguste Piccard, Jacques's father. In 1932, Auguste rode in his hot-air balloon up to a record-breaking height.

   a. Auguste's ascent took 7 hours and went up 51,683 feet. Write a relationship $y = kx$ to represent his ascent from his starting location.
   b. Auguste's descent took 3 hours and went down 52,940 feet. Write another relationship to represent his descent.
   c. Did Auguste Piccard end up at a greater or lesser altitude than his starting point? How much higher or lower?

Are you ready for more?

During which part of either trip was a Piccard changing vertical position the fastest? Explain your reasoning.

- Jacques's descent
- Jacques's ascent
- Auguste's ascent
- Auguste's descent

Summary

We saw earlier that we can represent speed with direction using signed numbers. Speed with direction is called *velocity*. Positive velocities always represent movement in the opposite direction from negative velocities.

We can do this with vertical movement: moving up can be represented with positive numbers, and moving down with negative numbers. The magnitude tells you how fast, and the sign tells you which direction. (We could actually do it the other way around if we wanted to, but usually we make up positive and down negative.)

Glossary Entries

Definition: Solution to an Equation

A solution to an equation is a number that can be used in place of the variable to make the equation true.

For example, 7 is the solution to the equation $m+1=8$, because it is true that $7+1=8$. The solution to $m+1=8$ is not 9, because $9+1 \neq 8$.

Practice

Exercise 5.3.5.4

Describe a situation where each of the following quantities might be useful.

1. -20 gallons per hour
2. -10 feet per minute
3. -0.1 kilograms per second

Exercise 5.3.5.5

A submarine is only allowed to change its depth by rising toward the surface in 60-meter stages. It starts off at -340 meters.

1. At what depth is it after:
   a. 1 stage
   b. 2 stages
   c. 4 stages
2. How many stages will it take to return to the surface?

Exercise 5.3.5.6

Some boats were traveling up and down a river. A satellite recorded the movements of several boats.

1. A motor boat traveled -3.4 miles per hour for 0.75 hours. How far did it go?
2. A tugboat traveled -1.5 miles in 0.3 hours. What was its velocity?
3. What do you think that negative distances and velocities could mean in this situation?

Exercise 5.3.5.7

1. A cookie recipe uses 3 cups of flour to make 15 cookies. How many cookies can you make with this recipe with 4 cups of flour? (Assume you have enough of the other ingredients.)
2. A teacher uses 36 centimeters of tape to hang up 9 student projects. At that rate, how much tape would the teacher need to hang up 10 student projects?

(From Unit 4.1.3)

Exercise 5.3.5.8

Evaluate each expression. When the answer is not a whole number, write your answer as a fraction.

1. $-4 \cdot -6$
2. $-24 \cdot \frac{-7}{6}$
3. $4 \div -6$
4. $\frac{4}{3} \div -24$

(From Unit 5.3.4)

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5.4: Four Operations with Rational Numbers

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5.4.1: Expressions with Rational Numbers

Lesson

Let's develop our signed number sense.

Exercise 5.4.1.1: True or False: Rational Numbers

Decide if each statement is true or false. Be prepared to explain your reasoning.

1. $(-38.76)(-15.6)$is negative
2. $10,000 - 99,999 < 0$
3. $\left(\frac{3}{4}\right)\left(-\frac{4}{3}\right) = 0$
4. $(30)(-80) - 50 = 50 - (30)(-80)$

Exercise 5.4.1.2: Card Sort: The Same but Different

Your teacher will give you a set of cards. Group them into pairs of expressions that have the same value.

Exercise 5.4.1.3: Near and Far From Zero

Table 5.4.1.1

a	b	$-a$	$-4b$	$-a+b$	$a \div -b$	a^2	b^3
$-\frac{1}{2}$	6						
$\frac{1}{2}$	-6						
-6	$-\frac{1}{2}$						

1. For each set of values for a and b, evaluate the given expressions and record your answers in the table.
2. When $a = -\frac{1}{2}$ and $b = 6$, which expression:
   a. has the largest value?
   b. has the smallest value?
   c. is the closest to zero?
3. When $a = \frac{1}{2}$ and $b = -6$, which expression:
   a. has the largest value?
   b. has the smallest value?
   c. is the closest to zero?
4. When $a = -6$ and $b = -\frac{1}{2}$, which expression:
   a. has the largest value?
   b. has the smallest value?
   c. is the closest to zero?

Are you ready for more?

Are there any values could you use for a and b that would make all of these expressions have the same value? Explain your reasoning.

Exercise 5.4.1.4: Seagulls and Sharks Again

A seagull has a vertical position a, and a shark has a vertical position b.

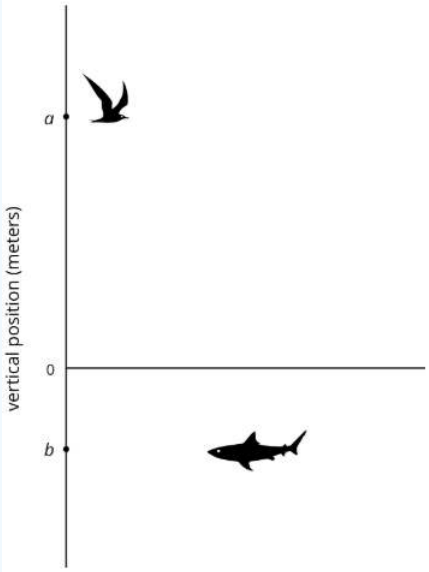


Figure 5.4.1.1: A vertical number line with the number zero indicated is labeled vertical position in meters. On the number line, a is high above zero, b is slightly below zero and a horizontal line extends to the right at zero. A seagull has the vertical position of a and a shark has the vertical position of b.

In the applet, you may choose to start by clicking on the open circles on the seagull and shark to drag them to new vertical positions. Once you have them in place, drag each of the other animals to the vertical axis to show its position, determined by the expression next to it.

1. A dragonfly at d, where $d = -b$
2. A jellyfish at j, where $j = 2b$
3. An eagle at e, where $4e = a$
4. A clownfish at c, where $c = \frac{-a}{2}$
5. A vulture at v, where $v = a + b$
6. A goose at g, where $g = a - b$

Summary

We can represent sums, differences, products, and quotients of **rational numbers**, and combinations of these, with numerical and algebraic expressions.

Sums:	Differences:	Products:	Quotients:
$\frac{1}{2} = -9$	$\frac{1}{2} - -9$	$\left(\frac{1}{2}\right)(-9)$	$\frac{1}{2} \div -9$
$-8.5 + x$	$-8.5 - x$	$-8.5x$	$\frac{-8.5}{x}$

We can write the product of two numbers in different ways.

- By putting a little dot between the factors, like this: $-8.5 \cdot x$.
- By putting the factors next to each other without any symbol between them at all, like this: $-8.5x$.

We can write the quotient of two numbers in different ways as well.

- By writing the division symbol between the numbers, like this: $-8.5 \div x$.
- By writing a fraction bar between the numbers like this: $\frac{-8.5}{x}$.

When we have an algebraic expression like $\frac{-8.5}{x}$ and are given a value for the variable, we can find the value of the expression. For example, if x is 2, then the value of the expression is -4.25, because $-8.5 \div 2 = -4.25$.

Glossary Entries

Definition: Rational Number

A rational number is a fraction or the opposite of a fraction.

For example, 8 and -8 are rational numbers because they can be written as $\frac{8}{1}$ and $-\frac{8}{1}$.

Also, 0.75 and -0.75 are rational numbers because they can be written as $\frac{75}{100}$ and $-\frac{75}{100}$.

Practice

Exercise 5.4.1.5

The value of x is $\frac{-1}{4}$. Order these expressions from least to greatest:

$$x \qquad 1-x \qquad x-1 \qquad -1 \div x$$

Exercise 5.4.1.6

Here are four expressions that have the value $\frac{-1}{2}$:

$$\frac{-1}{4}+\left(\frac{-1}{4}\right) \qquad \frac{1}{2}-1 \qquad -2 \cdot \frac{1}{4} \qquad -1 \div 2$$

Write five expressions: a sum, a difference, a product, a quotient, and one that involves at least two operations that have the value $\frac{-3}{4}$.

Exercise 5.4.1.7

Find the value of each expression.

1. $-22+5$
2. $-22-(-5)$
3. $(-22)(-5)$
4. $-22 \div 5$

Exercise 5.4.1.8

The price of an ice cream cone is $3.25, but it costs $3.51 with tax. What is the sales tax rate?

(From Unit 4.3.1)

Exercise 5.4.1.9

Two students are both working on the same problem: A box of laundry soap has 25% more soap in its new box. The new box holds 2 kg. How much soap did the old box hold?

- Here is how Jada set up her double number line.

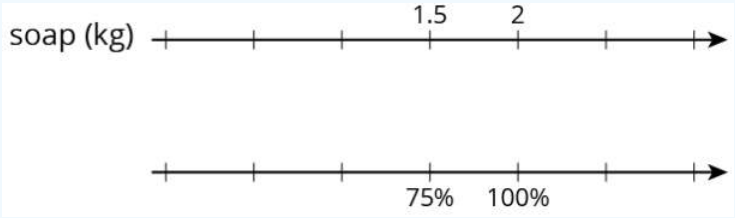


Figure 5.4.1.2

- Here is how Lin set up her double number line.

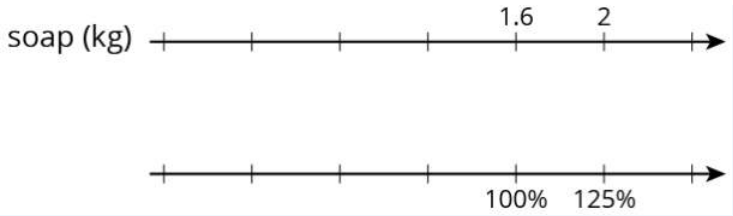


Figure 5.4.1.3

Do you agree with either of them? Explain or show your reasoning.

(From Unit 4.2.2)

Exercise 5.4.1.10

1. A coffee maker's directions say to use 2 tablespoons of ground coffee for every 6 ounces of water. How much coffee should you use for 33 ounces of water?
2. A runner is running a 10 km race. It takes her 17.5 minutes to reach the 2.5 km mark. At that rate, how long will it take her to run the whole race?

(From Unit 4.1.3)

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5.4.2: Solving Problems with Rational Numbers

Lesson

Let's use all four operations with signed numbers to solve problems.

Exercise 5.4.2.1: Which One Doesn't Belong? Equations

Which equation doesn't belong?

$\frac{1}{2}x = -50$ $\quad$ $x + 90 = -100$

$-60t = 30$ $\quad$ $-0.01 = -0.001x$

Exercise 5.4.2.2: Draining and Filling a Tank

A tank of water is being drained. Due to a problem, the sensor does not start working until some time into the draining process. The sensor starts its recording at time zero when there are 770 liters in the tank.

1. Given that the drain empties the tank at a constant rate of 14 liters per minute, complete the table:

Table 5.4.2.1

time after sensor starts (minutes)	change in water (liters)	expression	water in the tank (liters)
0	0	$770 + (0)(-14)$	770
1	-14	$770 + (1)(-14)$	756
5	-70		
10			

2. Later, someone wants to use the data to find out how long the tank had been draining before the sensor started. Complete this table:

Table 5.4.2.2

time after sensor starts (minutes)	change in water (liters)	expression	water in the tank (liters)
1	-14	$770 + (1)(-14)$	756
0	0	$770 + (0)(-14)$	770
-1	14	$770 + (-1)(-14)$	784
-2	28		
-3			
-4			
-5			

3. If the sensor started working 15 minutes into the tank draining, how much was in the tank to begin with?

Exercise 5.4.2.3: Buying and Selling Power

A utility company charges \$0.12 per kilowatt-hour for energy a customer uses. They give a credit of \$0.025 for every kilowatt-hour of electricity a customer with a solar panel generates that they don't use themselves.

A customer has a charge of \$82.04 and a credit of -\$4.10 on this month's bill.

1. What is the amount due this month?
2. How many kilowatt-hours did they use?
3. How many kilowatt-hours did they generate that they didn't use themselves?

Are you ready for more?

1. Find the value of the expression without a calculator.

$$(2)(-30)+(-3)(-20)+(-6)(-10)-(2)(3)(10)$$

2. Write an expression that uses addition, subtraction, multiplication, and division and only negative numbers that has the same value.

Summary

We can apply the rules for arithmetic with rational numbers to solve problems.

In general: $a-b=a+-b$

If $a-b=x$, then $x+b=a$. We can add $-b$ to both sides of this second equation to get that $x=a+-b$

Remember: the *distance* between two numbers is independent of the order, but the *difference* depends on the order.

And when multiplying or dividing:

- The sign of a positive number multiplied or divided by a negative number is always negative.
- The sign of a negative number multiplied or divided by a positive number is always negative.
- The sign of a negative number multiplied or divided by a negative number is always positive.

Glossary Entries

Definition: Rational Number

A rational number is a fraction or the opposite of a fraction.

For example, 8 and -8 are rational numbers because they can be written as $\frac{8}{1}$ and $-\frac{8}{1}$.

Also, 0.75 and -0.75 are rational numbers because they can be written as $\frac{75}{100}$ and $-\frac{75}{100}$.

Practice

Exercise 5.4.2.4

A bank charges a service fee of $7.50 per month for a checking account.

A bank account has $85.00. If no money is deposited or withdrawn except the service charge, how many months until the account balance is negative?

Exercise 5.4.2.5

The table shows transactions in a checking account.

Table 5.4.2.3

January	February	March	April
-38.50	250.00	-14.00	-86.80
126.30	-135.20	99.90	-570.00
429.40	35.50	-82.70	100.00
-265.00	-62.30	-1.50	(-280.10\)

1. Find the total of the transactions for each month.
2. Find the mean total for the four months.

Exercise 5.4.2.6

A large aquarium of water is being filled with a hose. Due to a problem, the sensor does not start working until some time into the filling process. The sensor starts its recording at the time zero minutes. The sensor initially detects the tank has 225 liters of water in it.

1. The hose fills the aquarium at a constant rate of 15 liters per minute. What will the sensor read at the time 5 minutes?
2. Later, someone wants to use the data to find the amount of water at times before the sensor started. What should the sensor have read at the time -7 minutes?

Exercise 5.4.2.7

A furniture store pays a wholesale price for a mattress. Then, the store marks up the retail price to 150% of the wholesale price. Later, they put the mattress on sale for 50% off of the retail price. A customer just bought the mattress on sale and paid $1,200.

1. What was the retail price of the mattress, before the discount?
2. What was the wholesale price, before the markup?

(From Unit 4.3.2)

Exercise 5.4.2.8

1. A restaurant bill is $21. You leave a 15% tip. How much do you pay including the tip?
2. Which of the following represents the amount a customer pays including the tip of 15% if the bill was b dollars? Select **all** that apply.
   - $15b$
   - $b+0.15b$
   - $1.15b$
   - $1.015b$
   - $b+\frac{15}{100}b$
   - $b+0.15$
   - $0.15b$

(From Unit 4.3.1)

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5.5: Solving Equations When There Are Negative Numbers

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5.5.1: Solving Equations with Rational Numbers

Lesson

Let's solve equations that include negative values.

Exercise 5.5.1.1: Number Talk: Opposites and Reciprocals

The **variables** a through h all represent *different* numbers. Mentally find numbers that make each equation true.

$\frac{3}{5} \cdot \frac{5}{3} = a$

$7 \cdot b = 1$

$c \cdot d = 1$

$-6 + 6 = e$

$11 + f = 0$

$g + h = 0$

Exercise 5.5.1.2: Match Solutions

Match each equation to a value that makes it true by dragging the answer to the corresponding equation. Be prepared to explain your reasoning.

Exercise 5.5.1.3: Trip to the Mountains

The Hiking Club is on a trip to hike up a mountain.

1. The members increased their elevation 290 feet during their hike this morning. Now they are at an elevation of 450 feet.
   a. Explain how to find their elevation before the hike.
   b. Han says the equation $e + 290 = 450$ describes the situation. What does the variable e represent?
   c. Han says that he can rewrite his equation as $e = 450 + -290$ to solve for e. Compare Han's strategy to your strategy for finding the beginning elevation.
2. The temperature fell 4 degrees in the last hour. Now it is 21 degrees. Write and solve an equation to find the temperature it was 1 hour ago.
3. There are 3 times as many students participating in the hiking trip this year than last year. There are 42 students on the trip this year.
   a. Explain how to find the number of students that came on the hiking trip last year.
   b. Mai says the equation $3s = 42$ describes the situation. What does the variable s represent?
   c. Mai says that she can rewrite her equation as $s = \frac{1}{3} \cdot 42$ to solve for s. Compare Mai's strategy to your strategy for finding the number of students on last year's trip.
4. The cost of the hiking trip this year is $\frac{2}{3}$ of the cost of last year's trip. This year's trip cost $32. Write and solve an equation to find the cost of last year's trip.

Are you ready for more?

A number line is shown below. The numbers 0 and 1 are marked on the line, as are two other rational numbers a and b.

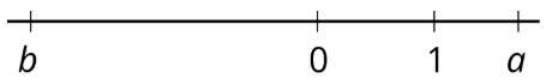


Figure 5.5.1.1: A number line with 4 tick marks. The number 0 is indicated. Rational number a is to the right of 0 and 1 is slightly more than halfway between 0 and a. Rational number b is to the far left of 0, where b is a greater distance from 0 than a is from 0.

Decide which of the following numbers are positive and which are negative.

$$a-1 \qquad a-2 \qquad -b \qquad a+b \qquad a-b \qquad ab+1$$

Exercise 5.5.1.4: Card Sort: Matching Inverses

Your teacher will give you a set of cards with numbers on them.

1. Match numbers with their additive inverses.
2. Next, match numbers with their multiplicative inverses.
3. What do you notice about the numbers and their inverses?

Summary

To solve the equation $x+8=-5$, we can add the opposite of 8, or -8, to each side:

$$\begin{aligned} x+8 &= -5 \\ (x+8)+-8 &= (-5)+-8 \\ x &= -13 \end{aligned}$$

Because adding the opposite of a number is the same as subtracting that number, we can also think of it as subtracting 8 from each side.

We can use the same approach for this equation:

$$\begin{aligned} -12 &= t+-\frac{2}{9} \\ (-12)+\frac{2}{9} &= \left(t+-\frac{2}{9}\right)+\frac{2}{9} \\ -11\frac{7}{9} &= t \end{aligned}$$

To solve the equation $8x=-5$, we can multiply each side by the reciprocal of 8, or $\frac{1}{8}$:

$$\begin{aligned} 8x &= -5 \\ \frac{1}{8}(8x) &= \frac{1}{8}(-5) \\ x &= -\frac{5}{8} \end{aligned}$$

Because multiplying by the reciprocal of a number is the same as dividing by that number, we can also think of it as dividing by 8.

We can use the same approach for this equation:

$$\begin{aligned} -12 &= -\frac{2}{9}t \\ -\frac{9}{2}(-12) &= -\frac{9}{2}\left(-\frac{2}{9}t\right) \\ 54 &= t \end{aligned}$$

Glossary Entries

Definition: Variable

A variable is a letter that represents a number. You can choose different numbers for the value of the variable.

For example, in the expression $10-x$, the variable is x. If the value of x is 3, then $10-x=7$, because $10-3=7$. If the value of x is 6, then $10-x=4$, because $10-6=4$.

Practice

Exercise 5.5.1.5

Solve:

1. $\frac{2}{5}t=6$
2. $-4.5=a-8$

3. $\frac{1}{2} + p = -3$
4. $12 = x \cdot 3$
5. $-12 = -3y$

Exercise 5.5.1.6

Match each equation to a step that will help solve the equation.

A. $5x = 0.4$
B. $\frac{x}{5} = 8$
C. $3 = \frac{-x}{5}$
D. $7 = -5x$

1. Multiply each side by 5.
2. Multiply each side by -5.
3. Multiply each side by $\frac{1}{5}$.
4. Multiply each side by $\frac{-1}{5}$.

Exercise 5.5.1.7

Evaluate each expression if x is $\frac{2}{5}$, y is -4, and z is -0.2.

1. $x + y$
2. $2x - z$
3. $x + y + z$
4. $y \cdot x$

(From Unit 5.4.1)

Exercise 5.5.1.8

1. Write an equation where a number is added to a variable, and a solution is -8.
2. Write an equation where a number is multiplied by a variable, and a solution is $\frac{-4}{5}$.

Exercise 5.5.1.9

The markings on the number line are evenly spaced. Label the other markings on the number line.

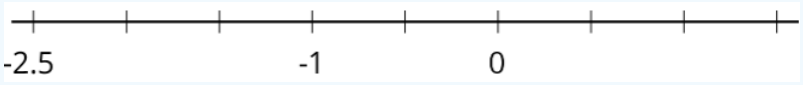


Figure 5.5.1.2: A blank number line with 9 evenly spaced tick marks. Starting on the left, the first tick mark is labeled negative 2 point 5, the fourth tick mark is labeled negative 1, and the sixth tick mark is labeled 0.

(From Unit 5.3.1)

Exercise 5.5.1.10

In 2012, James Cameron descended to the bottom of Challenger Deep in the Marianas Trench; the deepest point in the ocean. The vessel he rode in was called DeepSea Challenger.

Challenger Deep is 35,814 feet deep at its lowest point

1. DeepSea Challenger's descent was a change in depth of (-4) feet per second. We can use the equation $y = -4x$ to model this relationship, where y is the depth and x is the time in seconds that have passed. How many seconds does this model suggest it would take for DeepSea Challenger to reach the bottom?
2. To end the mission DeepSea Challenger made a one-hour ascent to the surface. How many seconds is this?
3. The ascent can be modeled by a different proportional relationship $y = kx$. What is the value of k in this case?

(From Unit 5.3.5)

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5.5.2: Representing Contexts with Equations

Lesson

Let's write equations that represent situations.

Exercise 5.5.2.1: Don't Solve It

Is the solution positive or negative?

$(-8.7)(1.4) = a$

$-8.7b = 1.4$

$-8.7 + c = -1.4$

$-8.7 - d = -1.4$

Exercise 5.5.2.2: Warmer or Colder than Before?

For each situation,

- Find *two* equations that could represent the situation from the bank of equations. (Some equations will not be used.)
- Explain what the variable v represents in the situation.
- Determine the value of the variable that makes the equation true, and explain your reasoning.

Bank of equations:

Table 5.5.2.1

$-3v = 9$	$v = -16 + 6$	$v = \frac{1}{3} \cdot (-6)$	$v + 12 = 4$	$-4 \cdot 3 = v$
$v = 4 + (-12)$	$v = -16 - (6)$	$v = 9 + 3$	$-4 \cdot -3 = v$	$-3v = -6$
$-6 + v = -16$	$-4 = \frac{1}{3}v$	$v = -\frac{1}{3} \cdot 9$	$v = -\frac{1}{3} \cdot (-6)$	$v = 4 + 12$

1. Between 6 a.m. and noon, the temperature rose 12 degrees Fahrenheit to 4 degrees Fahrenheit.
2. At midnight the temperature was -6 degrees. By 4 a.m. the temperature had fallen to -16 degrees.
3. The temperature is 0 degrees at midnight and dropping 3 degrees per hour. The temperature is -6 degrees at a certain time.
4. The temperature is 0 degrees at midnight and dropping 3 degrees per hour. The temperature is 9 degrees at a certain time.
5. The temperature at 9 p.m. is one third the temperature at midnight.

You may use this applet if you find it helpful.

Exercise 5.5.2.3: Animals Changing Altitudes

1. Match each situation with a diagram.
   a. A penguin is standing 3 feet above sea level and then dives down 10 feet. What is its depth?
   b. A dolphin is swimming 3 feet below sea level and then jumps up 10 feet. What is its height at the top of the jump?
   c. A sea turtle is swimming 3 feet below sea level and then dives down 10 feet. What is its depth?
   d. An eagle is flying 10 feet above sea level and then dives down to 3 feet above sea level. What was its change in altitude?
   e. A pelican is flying 10 feet above sea level and then dives down reaching 3 feet below sea level. What was its change in altitude?
   f. A shark is swimming 10 feet below sea level and then swims up reaching 3 feet below sea level. What was its change in depth?
2. Next, write an equation to represent each animal's situation and answer the question. Be prepared to explain your reasoning.

Diagrams

A

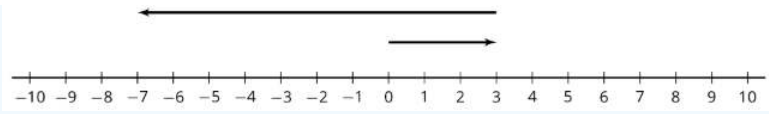


Figure 5.5.2.1

B

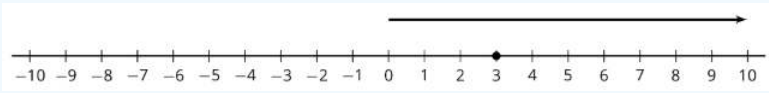


Figure 5.5.2.2

C

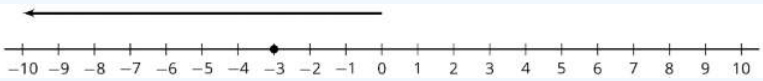


Figure 5.5.2.3

D

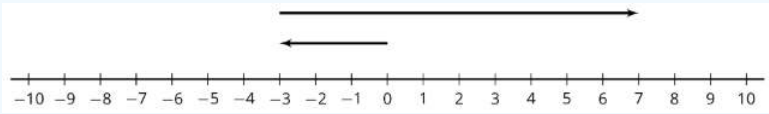


Figure 5.5.2.4

E

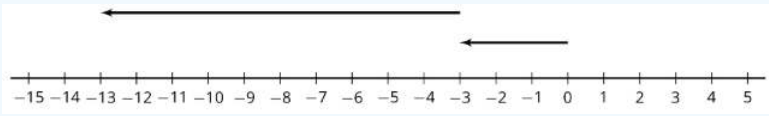


Figure 5.5.2.5

F

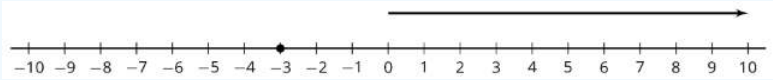


Figure 5.5.2.6

Exercise 5.5.2.4: Equations Tell a Story

Your teacher will assign your group *one* of these situations. Create a visual display about your situation that includes:

- An equation that represents your situation
- What your variable and each term in the equation represent
- How the operations in the equation represent the relationships in the story
- How you use inverses to solve for the unknown quantity
- The solution to your equation

1. As a $7\frac{1}{4}$ inch candle burns down, its height decreases $\frac{3}{4}$ inch each hour. How many hours does it take for the candle to burn completely?
2. On Monday $\frac{1}{9}$ of the enrolled students in a school were absent. There were 4,512 students present. How many students are enrolled at the school?
3. A hiker begins at sea level and descends 25 feet every minute. How long will it take to get to an elevation of -750 feet?
4. Jada practices the violin for the same amount of time every day. On Tuesday she practices for 35 minutes. How much does Jada practice in a week?
5. The temperature has been dropping $2\frac{1}{2}$ degrees every hour and the current temperature is $-15^{\circ}\mathrm{F}$. How many hours ago was the temperature $0^{\circ}\mathrm{F}$?
6. The population of a school increased by 12%, and now the population is 476. What was the population before the increase?
7. During a 5% off sale, Diego pays $74.10 for a new hockey stick. What was the original price?

8. A store buys sweaters for $8 and sells them for $26. How many sweaters does the store need to sell to make a profit of $990?

Are you ready for more?

Diego and Elena are 2 miles apart and begin walking towards each other. Diego walks at a rate of 3.7 miles per hour and Elena walks 4.3 miles per hour. While they are walking, Elena's dog runs back and forth between the two of them, at a rate of 6 miles per hour. Assuming the dog does not lose any time in turning around, how far has the dog run by the time Diego and Elena reach each other?

Summary

We can use variables and equations involving signed numbers to represent a story or answer questions about a situation.

For example, if the temperature is -3°C and then falls to -17°C, we can let x represent the temperature change and write the equation:

$-3 + x = -17$

We can solve the equation by adding 3 to each side. Since $-17 + 3 = -14$, the change is -14°C.

Here is another example: if a starfish is descending by $\frac{3}{2}$ feet every hour then we can solve $-\frac{3}{2}h = -6$ to find out how many hours h it takes the starfish to go down 6 feet.

We can solve this equation by multiplying each side by $-\frac{2}{3}$. Since $-\frac{2}{3} \cdot -6 = 4$, we know it will take the starfish 4 hours to descend 6 feet.

Glossary Entries

Definition: Variable

A variable is a letter that represents a number. You can choose different numbers for the value of the variable.

For example, in the expression $10 - x$, the variable is x. If the value of x is 3, then $10 - x = 7$, because $10 - 3 = 7$. If the value of x is 6, then $10 - x = 4$, because $10 - 6 = 4$.

Practice

Exercise 5.5.2.5

Match each situation to one of the equations.

A. A whale was diving at a rate of 2 meters per second. How long will it take for the whale to get from the surface of the ocean to an elevation of -12 meters at that rate?
B. A swimmer dove below the surface of the ocean. After 2 minutes, she was 12 meters below the surface. At what rate was she diving?
C. The temperature was -12 degrees Celsius and rose to 2 degrees Celsius. What was the change in temperature?
D. The temperature was 2 degrees Celsius and fell to -12 degrees Celsius. What was the change in temperature?

1. $-12 + x = 2$
2. $2 + x = -12$
3. $-2x = -12$
4. $2x = -12$

Exercise 5.5.2.6

Starting at noon, the temperature dropped steadily at a rate of 0.8 degrees Celsius every hour.

For each of these situations, write and solve an equation and describe what your variable represents.

1. How many hours did it take for the temperature to decrease by 4.4 degrees Celsius?
2. If the temperature after the 4.4 degree drop was -2.5 degrees Celsius, what was the temperature at noon?

Exercise 5.5.2.7

Kiran mixes $\frac{3}{4}$ cups of raisins, 1 cup peanuts, and $\frac{1}{2}$ cups of chocolate chips to make trail mix. How much of each ingredient would he need to make 10 cups of trail mix? Explain your reasoning.

(From Unit 4.1.3)

Exercise 5.5.2.8

Find the value of each expression.

1. $12 + -10$
2. $-5 - 6$
3. $-42 + 17$
4. $35 - -8$
5. $-4\frac{1}{2} + 3$

(From Unit 5.1.5)

Exercise 5.5.2.9

The markings on the number line are evenly spaced. Label the other markings on the number line.

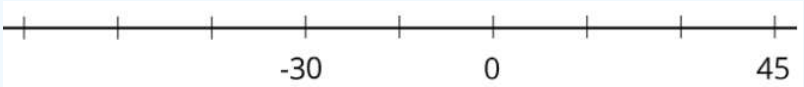


Figure 5.5.2.7

(From Unit 5.3.1)

Exercise 5.5.2.10

Kiran drinks 6.4 oz of milk each morning. How many days does it take him to finish a 32 oz container of milk?

1. Write and solve an equation for the situation.
2. What does the variable represent?

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5.6: Let's Put It to Work

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5.6.1: The Stock Market

Lesson

Let's learn about the stock market.

Exercise 5.6.1.1: Revisiting Interest and Depreciation

1. Lin deposited $300 in a savings account that has a 2% interest rate per year. How much is in her account after 1 year? After 2 years?
2. Diego wants to sell his bicycle. It cost $150 when he bought it but has depreciated by 15%. How much should he sell it for?

Exercise 5.6.1.2: Gains and Losses

1. Here is some information from the stock market in September 2016. Complete the table.

Table 5.6.1.1

company	value on day 1 (dollars)	value on day 2 (dollars)	change in value (dollars)	change in value as a percentage of day 1 value
Mobile Tech Company	107.95	111.77	3.82	3.54%
Electrical Appliance Company		114.03	2.43	2.18%
Oil Corporation	26.14	25.14		−3.83%
Department Store Company	7.38	7.17		
Jewelry Company		70.30		2.27%

2. Which company's change in dollars had the largest magnitude?
3. Which company's change in percentage had the larges magnitude?

Exercise 5.6.1.3: What is a Stock Portfolio?

A person who wants to make money by investing in the stock market usually buys a portfolio, or a collection of different stocks. That way, if one of the stocks decreases in value, they won't lose all of their money at once.

1. Here is an example of someone's stock portfolio. Complete the table to show the total value of each investment.

Table 5.6.1.2

name	price (dollars)	number of shares	total value (dollars)
Technology Company	107.75	98	
Airline Company	133.54	27	
Film Company	95.95	135	
Sports Clothing Company	58.96	100	

2. Here is the same portfolio the next year. Complete the table to show the new total value of each investment.

Table 5.6.1.3

company	old price (dollars)	price change	new price (dollars)	number of shares	total value (dollars)

company	old price (dollars)	price change	new price (dollars)	number of shares	total value (dollars)
Technology Company	107.75	$+2.43\%$		98	
Airline Company	133.54	-7.67%		27	
Film Company	95.95		87.58	135	
Sports Clothing Company	58.96	-5.56%		100	

3. Did the entire portfolio increase or decrease in value over the year?

Exercise 5.6.1.4: Your Own Stock Portfolio

Your teacher will give you a list of stocks.

1. Select a combination of stocks with a total value close to, but no more than, $100.
2. Using the new list, how did the total value of your selected stocks change?

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CHAPTER OVERVIEW

6: Expressions, Equations, and Inequalities

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6.1: Representing Situations of the Form px+q=r and p(x+q)=r

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6.1.1: Relationships between Quantities

Lesson

Let's try to solve some new kinds of problems.

Exercise 6.1.1.1: Pricing Theater Popcorn

A movie theater sells popcorn in bags of different sizes. The table shows the volume of popcorn and the price of the bag.

Complete one column of the table with prices where popcorn is priced at a constant rate. That is, the amount of popcorn is proportional to the price of the bag. Then complete the other column with realistic example prices where the amount of popcorn and price of the bag are not in proportion.

Table 6.1.1.1

volume of popcorn (ounces)	price of bag, proportional ($)	price of bag, not proportional ($)
10	6	6
20		
35		
48		

Exercise 6.1.1.2: Entrance Fees

A state park charges an entrance fee based on the number of people in a vehicle. A car containing 2 people is charged $14, a car containing 4 people is charged $20, and a van containing 8 people is charged $32.

1. How much do you think a bus containing 30 people would be charged?
2. If a bus is charged $122, how many people do you think it contains?
3. What rule do you think the state park uses to decide the entrance fee for a vehicle?

Exercise 6.1.1.3: Making Toast

A toaster has 4 slots for bread. Once the toaster is warmed up, it takes 35 seconds to make 4 slices of toast, 70 seconds to make 8 slices, and 105 seconds to make 12 slices.

1. How long do you think it will take to make 20 slices?
2. If someone makes as many slices of toast as possible in 4 minutes and 40 seconds, how many slices do think they can make?

Are you ready for more?

What is the smallest number that has a remainder of 1, 2, and 3 when divided by 2, 3, and 4, respectively? Are there more numbers that have this property?

Summary

In much of our previous work that involved relationships between two quantities, we were often able to describe amounts as being so much more than another, or so many times as much as another. We wrote equations like $x + 3 = 8$ and $4x = 20$ and solved for unknown amounts.

In this unit, we will see situations where relationships between amounts involve more operations. For example, a pizza store might charge the amounts shown in the table for delivering pies.

Table 6.1.1.2

number of pies	total cost in dollars

number of pies	total cost in dollars
1	13
2	23
3	33
5	53

We can see that each additional pie adds \$10 to the total cost, and that each total includes a \$3 additional cost, maybe representing a delivery fee. In this situation, 8 pies will cost $8 \cdot 10 + 3$ and a total cost of \$63 means 6 pies were ordered.

In this unit, we will see many situations like this one, and will learn how to use diagrams and equations to answer questions about unknown amounts.

Practice

Exercise 6.1.1.4

Lin and Tyler are drawing circles. Tyler's circle has twice the diameter of Lin's circle. Tyler thinks that his circle will have twice the area of Lin's circle as well. Do you agree with Tyler?

(From Unit 3.2.2)

Exercise 6.1.1.5

Jada and Priya are trying to solve the equation $\frac{2}{3} + x = 4$.

- Jada says, "I think we should multiply each side by $\frac{3}{2}$ because that is the reciprocal of $\frac{2}{3}$."
- Priya says, "I think we should add $-\frac{2}{3}$ to each side because that is the opposite of $\frac{2}{3}$."

1. Which person's strategy should they use? Why?
2. Write an equation that can be solved using the other person's strategy.

(From Unit 5.5.1)

Exercise 6.1.1.6

What are the missing operations?

1. $48?(-8) = (-6)$
2. $(-40)?8 = (-5)$
3. $12?(-2) = 14$
4. $18?(-12) = 6$
5. $18?(-20) = -2$
6. $22?(-0.5) = -11$

(From Unit 5.4.1)

Exercise 6.1.1.7

In football, the team that has the ball has four chances to gain at least ten yards. If they don't gain at least ten yards, the other team gets the ball. Positive numbers represent a gain and negative numbers represent a loss. Select **all** of the sequences of four plays that result in the team getting to keep the ball.

A. $8, -3, 4, 21$
B. $30, -7, -8, -12$
C. $2, 16, -5, -3$
D. $5, -2, 20, -1$

E. $20, -3, -13, 2$

(From Unit 5.4.2)

Exercise 6.1.1.8

A sandwich store charges $20 to have 3 turkey subs delivered and $26 to have 4 delivered.

1. Is the relationship between number of turkey subs delivered and amount charged proportional? Explain how you know.
2. How much does the store charge for 1 additional turkey sub?
3. Describe a rule for determining how much the store charges based on the number of turkey subs delivered.

Exercise 6.1.1.9

Which question cannot be answered by the solution to the equation $3x = 27$?

A. Elena read three times as many pages as Noah. She read 27 pages. How many pages did Noah read?
B. Lin has 27 stickers. She gives 3 stickers to each of her friends. With how many friends did Lin share her stickers?
C. Diego paid $27 to have 3 pizzas delivered and $35 to have 4 pizzas delivered. What is the price of one pizza?
D. The coach splits a team of 27 students into 3 groups to practice skills. How many students are in each group?

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6.1.2: Reasoning about Contexts with Tape Diagrams

Lesson

Let's use tape diagrams to make sense of different kinds of stories.

Exercise 6.1.2.1: Notice and Wonder: Remembering Tape Diagrams

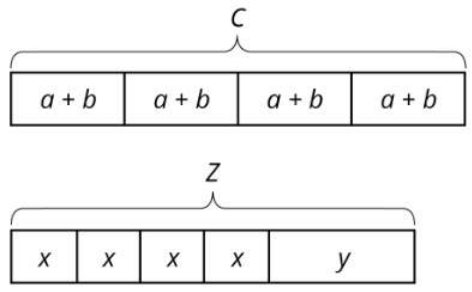


Figure 6.1.2.1

1. What do you notice? What do you wonder?
2. What are some possible values for a, b, and c in the first diagram?

For x, y, and z in the second diagram? How did you decide on those values?

Exercise 6.1.2.2: Every Picture Tells a Story

Here are three stories with a diagram that represents it. With your group, decide who will go first. That person explains why the diagram represents the story. Work together to find any unknown amounts in the story. Then, switch roles for the second diagram and switch again for the third.

1. Mai made 50 flyers for five volunteers in her club to hang up around school. She gave 5 flyers to the first volunteer, 18 flyers to the second volunteer, and divided the remaining flyers equally among the three remaining volunteers.

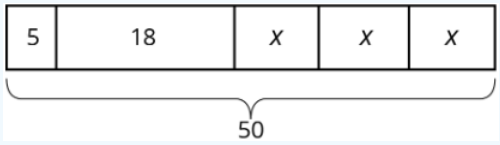


Figure 6.1.2.2

2. To thank her five volunteers, Mai gave each of them the same number of stickers. Then she gave them each two more stickers. Altogether, she gave them a total of 30 stickers.

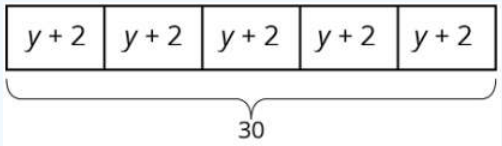


Figure 6.1.2.3

3. Mai distributed another group of flyers equally among the five volunteers. Then she remembered that she needed some flyers to give to teachers, so she took 2 flyers from each volunteer. Then, the volunteers had a total of 40 flyers to hang up.

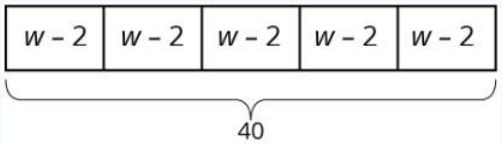


Figure 6.1.2.4

Exercise 6.1.2.3: Every story Needs a Picture

Here are three more stories. Draw a tape diagram to represent each story. Then describe how you would find any unknown amounts in the stories.

1. Noah and his sister are making gift bags for a birthday party. Noah puts 3 pencil erasers in each bag. His sister puts x stickers in each bag. After filling 4 bags, they have used a total of 44 items.
2. Noah's family also wants to blow up a total of 60 balloons for the party. Yesterday they blew up 24 balloons. Today they want to split the remaining balloons equally between four family members.
3. Noah's family bought some fruit bars to put in the gift bags. They bought one box each of four flavors: apple, strawberry, blueberry, and peach. The boxes all had the same number of bars. Noah wanted to taste the flavors and ate one bar from each box. There were 28 bars left for the gift bags.

Are you ready for more?

Design a tiling that uses a repeating pattern consisting of 2 kinds of shapes (e.g., 1 hexagon with 3 triangles forming a triangle). How many times did you repeat the pattern in your picture? How many individual shapes did you use?

Summary

Tape diagrams are useful for representing how quantities are related and can help us answer questions about a situation.

Suppose a school receives 46 copies of a popular book. The library takes 26 copies and the remainder are split evenly among 4 teachers. How many books does each teacher receive? This situation involves 4 equal parts and one other part. We can represent the situation with a rectangle labeled 26 (books given to the library) along with 4 equal-sized parts (books split among 4 teachers). We label the total, 46, to show how many the rectangle represents in all. We use a letter to show the unknown amount, which represents the number of books each teacher receives. Using the same letter, x, means that the same number is represented four times.

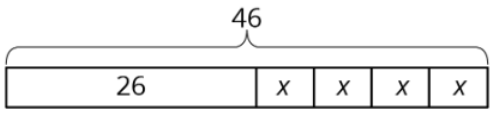


Figure 6.1.2.5

Some situations have parts that are all equal, but each part has been increased from an original amount:

A company manufactures a special type of sensor, and packs them in boxes of 4 for shipment. Then a new design increases the weight of each sensor by 9 grams. The new package of 4 sensors weighs 76 grams. How much did each sensor weigh originally?

We can describe this situation with a rectangle representing a total of 76 split into 4 equal parts. Each part shows that the new weight, $x+9$, is 9 more than the original weight, x.

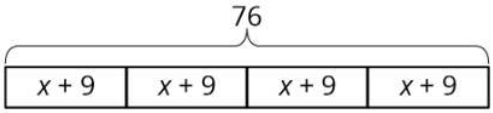


Figure 6.1.2.6

Practice

Exercise 6.1.2.4

The table shows the number of apples and the total weight of the apples.

Table 6.1.2.1

number of apples	weight of apples (grams)
2	511
5	1200
8	2016

Estimate the weight of 6 apples.

(From Unit 3.1.1)

Exercise 6.1.2.5

Select **all** stories that the tape diagram can represent.

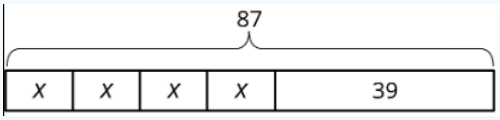


Figure 6.1.2.7

A. There are 87 children and 39 adults at a show. The seating in the theater is split into 4 equal sections.
B. There are 87 first graders in after-care. After 39 students are picked up, the teacher put the remaining students into 4 groups for an activity.
C. Lin buys a pack of 87 pencils. She gives 39 to her teacher and shared the remaining pencils between herself and 3 friends.
D. Andre buys 4 packs of paper clips with 39 paper clips in each. Then he gives 87 paper clips to his teacher.
E. Diego's family spends $87 on 4 tickets to the fair and a $39 dinner.

Exercise 6.1.2.6

Andre wants to save $40 to buy a gift for his dad. Andre's neighbor will pay him weekly to mow the lawn, but Andre always gives a $2 donation to the food bank in weeks when he earns money. Andre calculates that it will take him 5 weeks to earn the money for his dad's gift. He draws a tape diagram to represent the situation.

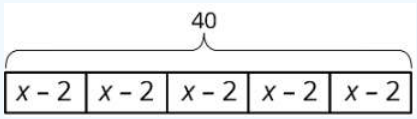


Figure 6.1.2.8

1. Explain how the parts of the tape diagram represent the story.
2. How much does Andre's neighbor pay him each week to mow the lawn?

Exercise 6.1.2.7

Without evaluating each expression, determine which value is the greatest. Explain how you know.

1. $7\frac{5}{6} - 9\frac{3}{4}$
2. $(-7\frac{5}{6}) + (-9\frac{3}{4})$
3. $(-7\frac{5}{6}) \cdot 9\frac{3}{4}$
4. $(-7\frac{5}{6}) \div (-9\frac{3}{4})$

(From Unit 5.4.1)

Exercise 6.1.2.8

Solve each equation.

1. $(8.5) \cdot (-3) = a$
2. $(-7) + b = (-11)$
3. $c - (-3) = 15$
4. $d \cdot (-4) = 32$

(From Unit 5.5.1)

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6.1.3: Reasoning about Equations with Tape Diagrams

Lesson

Let's see how equations can describe tape diagrams.

Exercise 6.1.3.1: Find Equivalent Expressions

Select **all** the expressions that are **equivalent** to $7(2-3n)$. Explain how you know each expression you select is equivalent.

1. $9-10n$
2. $14-3n$
3. $14-21n$
4. $(2-3n)\cdot 7$
5. $7\cdot 2\cdot(-3n)$

Exercise 6.1.3.2: Matching Equations to Tape Diagrams

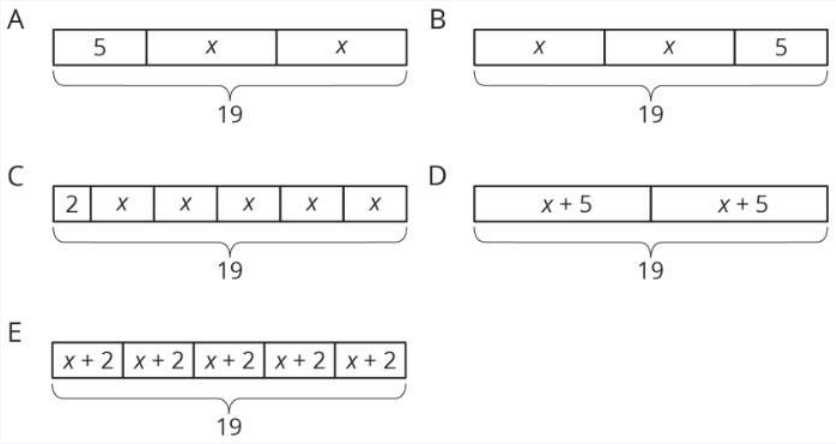


Figure 6.1.3.1

1. Match each equation to one of the tape diagrams. Be prepared to explain how the equation matches the diagram.
2. Sort the equations into categories of your choosing. Explain the criteria for each category.

- $2x+5=19$
- $2+5x=19$
- $2(x+5)=19$
- $5(x+2)=19$
- $19=5+2x$
- $(x+5)\cdot 2=19$
- $19=(x+2)\cdot 5$
- $19\div 2=x+5$
- $19-2=5x$

Exercise 6.1.3.3: Drawing Tape Diagrams to Represent Equations

1. Draw a tape diagram to match each equation.
   a. $114=3x+18$
   b. $114=3(y+18)$
2. Use any method to find values for x and y that make the equations true.

Are you ready for more?

To make a Koch snowflake:

- Start with an equilateral triangle that has side lengths of 1. This is step 1.
- Replace the middle third of each line segment with a small equilateral triangle with the middle third of the segment forming the base. This is step 2.
- Do the same to each of the line segments. This is step 3.
- Keep repeating this process.

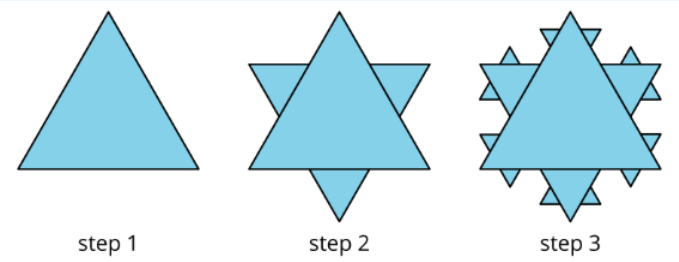


Figure 6.1.3.2: Three figures. First, an equilateral triangle. Second, the same equilateral triangle, but each lateral edge is divided into thirds and the middle third forms the base of an equilateral triangle. Third, the second image, but each lateral edge is divided into thirds and the middle third forms the base of an equilateral triangle.

1. What is the perimeter after step 2? Step 3?
2. What happens to the perimeter, or the length of line traced along the outside of the figure, as the process continues?

Summary

We have seen how tape diagrams represent relationships between quantities. Because of the meaning and properties of addition and multiplication, more than one equation can often be used to represent a single tape diagram.

Let's take a look at two tape diagrams.

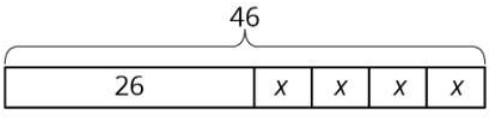


Figure 6.1.3.3

We can describe this diagram with several different equations. Here are some of them:

- $26+4x=46$, because the parts add up to the whole.
- $4x+26=46$, because addition is commutative.
- $46=4x+26$, because if two quantities are equal, it doesn't matter how we arrange them around the equal sign.
- $4x=46-26$, because one part (the part made up of $4x$'s) is the difference between the whole and the other part.

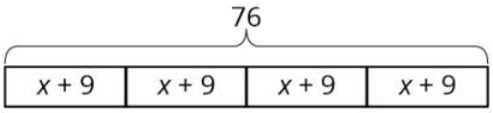


Figure 6.1.3.4

For this diagram:

- $4(x+9)=76$, because multiplication means having multiple groups of the same size.
- $(x+9)\cdot 4=76$, because multiplication is commutative.
- $76\div 4=x+9$, because division tells us the size of each equal part.

Glossary Entries

Definition: Equivalent Expressions

Equivalent expressions are always equal to each other. If the expressions have variables, they are equal whenever the same value is used for the variable in each expression.

For example, $3x+4x$ is equivalent to $5x+2x$. No matter what value we use for x, these expressions are always equal. When x is 3, both expressions equal 21. When x is 10, both expressions equal 70.

Practice

Exercise 6.1.3.4

Solve each equation mentally.

1. $2x = 10$
2. $-3x = 21$
3. $\frac{1}{3}x = 6$
4. $-\frac{1}{2}x = -7$

(From Unit 5.5.1)

Exercise 6.1.3.5

Complete the magic squares so that the sum of each row, each column, and each diagonal in a grid are all equal.

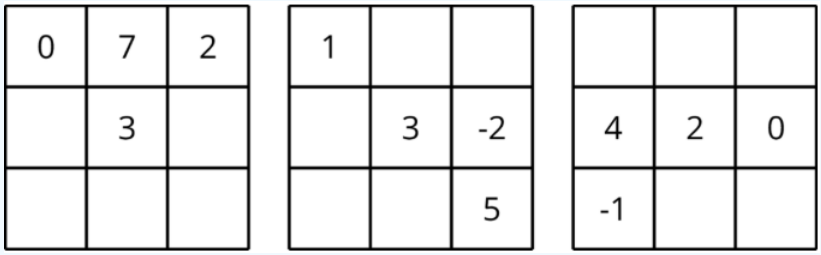


Figure 6.1.3.5: Three square grids, each with 3 columns and 3 rows. First grid contains 0, 7, 2, blank, 3, blank, blank, blank, blank. Second grid contains 1, blank, blank, blank, 3, negative 2, blank, blank, five. Third grid contains blank, blank, blank, 4, 2, 0, negative 1, blank, blank.

(From Unit 5.2.2)

Exercise 6.1.3.6

Draw a tape diagram to match each equation.

1. $5(x+1) = 20$
2. $5x + 1 = 20$

Exercise 6.1.3.7

Select **all** the equations that match the tape diagram.

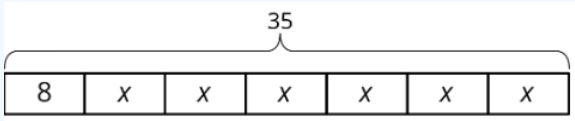


Figure 6.1.3.6

A. $35 = 8 + x + x + x + x + x + x$
B. $35 = 8 + 6x$
C. $6 + 8x = 35$
D. $6x + 8 = 35$
E. $6x + 8x = 35x$
F. $35 - 8 = 6x$

Exercise 6.1.3.8

Each car is traveling at a constant speed. Find the number of miles each car travels in 1 hour at the given rate.

1. 135 miles in 3 hours

2. 22 miles in $\frac{1}{2}$ hour
3. 7.5 miles in $\frac{1}{4}$ hour
4. $\frac{100}{3}$ miles in $\frac{2}{3}$ hour
5. $97\frac{1}{2}$ miles in $\frac{3}{2}$ hour

(From Unit 4.1.2)

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6.1.4: Reasoning about Equations and Tape Diagrams (Part 1)

Lesson

Let's see how tape diagrams can help us answer questions about unknown amounts in stories.

Exercise 6.1.4.1: Algebra Talk: Seeing Structure

Find a solution to each equation without writing anything down.

$x + 1 = 5$

$2(x + 1) = 10$

$3(x + 1) = 15$

$500 = 100(x + 1)$

Exercise 6.1.4.2: Situations and Diagrams

Draw a tape diagram to represent each situation. For some of the situations, you need to decide what to represent with a variable.

1. Diego has 7 packs of markers. Each pack has x markers in it. After Lin gives him 9 more markers, he has a total of 30 markers.
2. Elena is cutting a 30-foot piece of ribbon for a craft project. She cuts off 7 feet, and then cuts the remaining piece into 9 equal lengths of x feet each.
3. A construction manager weighs a bundle of 9 identical bricks and a 7-pound concrete block. The bundle weighs 30 pounds.
4. A skating rink charges a group rate of $9 plus a fee to rent each pair of skates. A family rents 7 pairs of skates and pays a total of $30.
5. Andre bakes 9 pans of brownies. He donates 7 pans to the school bake sale and keeps the rest to divide equally among his class of 30 students.

Exercise 6.1.4.3: Situations, Diagrams, and Equations

Each situation in the previous activity is represented by one of the equations.

- $7x + 9 = 30$
- $30 = 9x + 7$
- $30x + 7 = 9$

1. Match each situation to an equation.
2. Find the solution to each equation. Use your diagrams to help you reason.
3. What does each solution tell you about its situation?

Are you ready for more?

While in New York City, is it a better deal for a group of friends to take a taxi or the subway to get from the Empire State Building to the Metropolitan Museum of Art? Explain your reasoning.

Summary

Many situations can be represented by equations. Writing an equation to represent a situation can help us express how quantities in the situation are related to each other, and can help us reason about unknown quantities whose value we want to know. Here are three situations:

1. An architect is drafting plans for a new supermarket. There will be a space 144 inches long for rows of nested shopping carts. The first cart is 34 inches long and each nested cart adds another 10 inches. The architect want to know how many shopping carts will fit in each row.

2. A bakery buys a large bag of sugar that has 34 cups. They use 10 cups to make some cookies. Then they use the rest of the bag to make 144 giant muffins. Their customers want to know how much sugar is in each muffin.
3. Kiran is trying to save $144 to buy a new guitar. He has $34 and is going to save $10 a week from money he earns mowing lawns. He wants to know how many weeks it will take him to have enough money to buy the guitar.

We see the same three numbers in the situations: 10, 34, and 144. How could we represent each situation with an equation?

In the first situation, there is one shopping cart with length 34 and then an unknown number of carts with length 10. Similarly, Kiran has 34 dollars saved and then will save 10 each week for an unknown number of weeks. Both situations have one part of 34 and then equal parts of size 10 that all add together to 144. Their equation is $34 + 10x = 144$.

Since it takes 11 groups of 10 to get from 34 to 144, the value of x in these two situations is $(144 - 34) \div 10$ or 11. There will be 11 shopping carts in each row, and it will take Kiran 11 weeks to raise the money for the guitar.

In the bakery situation, there is one part of 10 and then 144 equal parts of unknown size that all add together to 34. The equation is $10 + 144x = 34$. Since 24 is needed to get from 10 to 34, the value of x is $(34 - 10) \div 144$ or $\frac{1}{6}$. There is $\frac{1}{6}$ cup of sugar in each giant muffin.

Glossary Entries

Definition: Equivalent Expressions

Equivalent expressions are always equal to each other. If the expressions have variables, they are equal whenever the same value is used for the variable in each expression.

For example, $3x + 4x$ is equivalent to $5x + 2x$. No matter what value we use for x, these expressions are always equal. When x is 3, both expressions equal 21. When x is 10, both expressions equal 70.

Practice

Exercise 6.1.4.4

Draw a square with side length 7 cm.

1. Predict the perimeter and the length of the diagonal of the square.
2. Measure the perimeter and the length of the diagonal of the square.
3. Describe how close the predictions and measurements are.

(From Unit 3.1.1)

Exercise 6.1.4.5

Find the products.

1. $(100) \cdot (-0.09)$
2. $(-7) \cdot (-1.1)$
3. $(-7.3) \cdot (5)$
4. $(-0.2) \cdot (-0.3)$

(From Unit 5.3.2)

Exercise 6.1.4.6

Here are three stories:

- A family buys 6 tickets to a show. They also pay a $3 parking fee. They spend $27 to see the show.
- Diego has 27 ounces of juice. He pours equal amounts for each of his 3 friends and has 6 ounces left for himself.
- Jada works for 6 hours preparing for the art fair. She spends 3 hours on a sculpture and then paints 27 picture frames.

Here are three equations.

- $3x + 6 = 27$

- $6x + 3 = 27$
- $27x + 3 = 6$

1. Decide which equation represents each story. What does x represent in each equation?
2. Find the solution to each equation. Explain or show your reasoning.
3. What does each solution tell you about its situation?

Exercise 6.1.4.7

Here is a diagram and its corresponding equation. Find the solution to the equation and explain your reasoning.

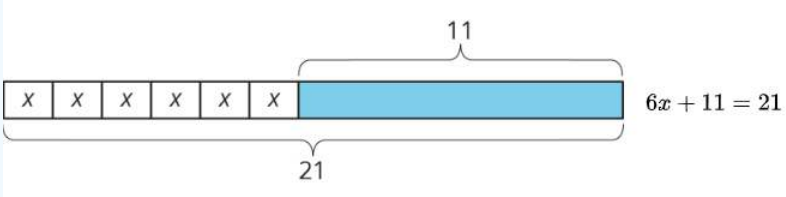


Figure 6.1.4.1

Exercise 6.1.4.8

1. Plot these points on the coordinate plane:

$$A = (3, 2),\ B = (7.5, 2),\ C = (7.5, -2.5),\ D = (3, -2)$$

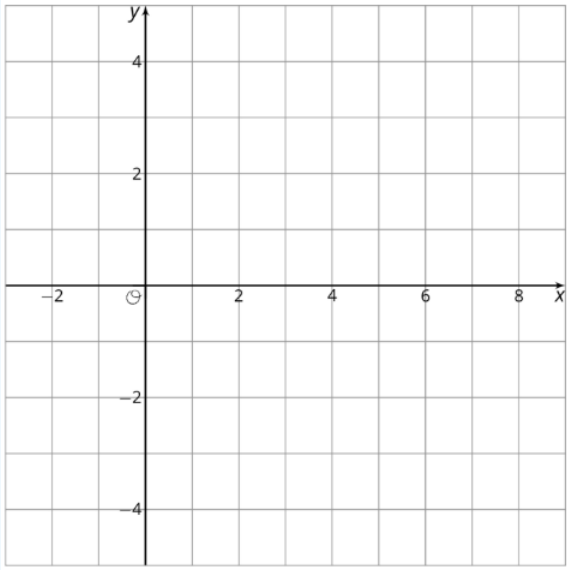


Figure 6.1.4.2

2. What is the vertical difference between D and A?

3. Write an expression that represents the vertical distance between B and C?

(From Unit 5.2.6)

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6.1.5: Reasoning about Equations and Tape Diagrams (Part 2)

Lesson

Let's use tape diagrams to help answer questions about situations where the equation has parentheses.

Exercise 6.1.5.1: Algebra Talk: Seeing Structure

Solve each equation mentally.

$x - 1 = 5$

$2(x - 1) = 10$

$3(x - 1) = 15$

$500 = 100(x - 1)$

Exercise 6.1.5.2: More Situations and Diagrams

Draw a tape diagram to represent each situation. For some of the situations, you need to decide what to represent with a variable.

1. Each of 5 gift bags contains x pencils. Tyler adds 3 more pencils to each bag. Altogether, the gift bags contain 20 pencils.
2. Noah drew an equilateral triangle with sides of length 5 inches. He wants to increase the length of each side by x inches so the triangle is still equilateral and has a perimeter of 20 inches.
3. An art class charges each student $3 to attend plus a fee for supplies. Today, $20 was collected for the 5 students attending the class.
4. Elena ran 20 miles this week, which was three times as far as Clare ran this week. Clare ran 5 more miles this week than she did last week.

Exercise 6.1.5.3: More Situations, Diagrams, and Equations

Each situation in the previous activity is represented by one of the equations.

- $(x + 3) \cdot 5 = 20$
- $3(x + 5) = 20$

1. Match each situation to an equation.
2. Find the solution to each equation. Use your diagrams to help you reason.
3. What does each solution tell you about its situation?

Are you ready for more?

Han, his sister, his dad, and his grandmother step onto a crowded bus with only 3 open seats for a 42-minute ride. They decide Han's grandmother should sit for the entire ride. Han, his sister, and his dad take turns sitting in the remaining two seats, and Han's dad sits 1.5 times as long as both Han and his sister. How many minutes did each one spend sitting?

Summary

Equations with parentheses can represent a variety of situations.

1. Lin volunteers at a hospital and is preparing toy baskets for children who are patients. She adds 2 items to each basket, after which the supervisor's list shows that 140 toys have been packed into a group of 10 baskets. Lin wants to know how many toys were in each basket before she added the items.
2. A large store has the same number of workers on each of 2 teams to handle different shifts. They decide to add 10 workers to each team, bringing the total number of workers to 140. An executive at the company that runs this chain of stores wants to know how many employees were in each team before the increase.

Each bag in the first story has an unknown number of toys, x, that is increased by 2. Then ten groups of $x + 2$ give a total of 140 toys. An equation representing this situation is $10(x + 2) = 140$. Since 10 times a number is 140, that number is 14, which is the

total number of items in each bag. Before Lin added the 2 items there were $14-2$ or 12 toys in each bag.

The executive in the second story knows that the size of each team of employees has been increased by 10. There are now 2 teams of $y+10$ each. An equation representing this situation is $2(y+10)=140$. Since 2 times an amount is 140, that amount is 70, which is the new size of each team. The value of y is $70-10$ or 60. There were 60 employees on each team before the increase.

Glossary Entries

Definition: Equivalent Expressions

Equivalent expressions are always equal to each other. If the expressions have variables, they are equal whenever the same value is used for the variable in each expression.

For example, $3x+4x$ is equivalent to $5x+2x$. No matter what value we use for x, these expressions are always equal. When x is 3, both expressions equal 21. When x is 10, both expressions equal 70.

Practice

Exercise 6.1.5.4

Here are some prices customers paid for different items at a farmer's market. Find the cost for 1 pound of each item.

1. \$5 for 4 pounds of apples
2. \$3.50 for $\frac{1}{2}$ pound of cheese
3. \$8.25 for $1\frac{1}{2}$ pounds of coffee beans
4. \$6.75 for $\frac{3}{4}$ pounds of fudge
5. \$5.50 for a $6\frac{1}{4}$ pound pumpkin

(From Unit 4.1.2)

Exercise 6.1.5.5

Find the products.

1. $\frac{2}{3}\cdot\left(\frac{-4}{5}\right)$
2. $\left(\frac{-5}{7}\right)\cdot\left(\frac{7}{5}\right)$
3. $\left(\frac{-2}{39}\right)\cdot 39$
4. $\left(\frac{2}{5}\right)\cdot\left(\frac{-3}{4}\right)$

(From Unit 5.3.2)

Exercise 6.1.5.6

Here are two stories:

- A family buys 6 tickets to a show. They also *each* spend \$3 on a snack. They spend \$24 on the show.
- Diego has 24 ounces of juice. He pours equal amounts for each of his 3 friends, and then adds 6 more ounces for each.

Here are two equations:

- $3(x+6)=24$
- $6(x+3)=24$

1. Which equation represents which story?
2. What does x represent in each equation?
3. Find the solution to each equation. Explain or show your reasoning.
4. What does each solution tell you about its situation?

Exercise 6.1.5.7

Here is a diagram and its corresponding equation. Find the solution to the equation and explain your reasoning.

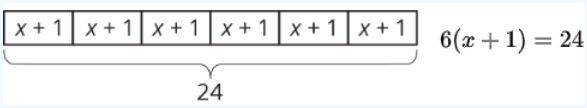


Figure 6.1.5.1

Exercise 6.1.5.8

Below is a set of data about temperatures. The *range* of a set of data is the distance between the lowest and highest value in the set. What is the range of these temperatures?

$$9^\circ\text{C},\ -3^\circ\text{C},\ 22^\circ\text{C},\ -5^\circ\text{C},\ 11^\circ\text{C},\ 15^\circ\text{C}$$

(From Unit 5.2.6)

Exercise 6.1.5.9

A store is having a 25% off sale on all shirts. Show two different ways to calculate the sale price for a shirt that normally costs $24.

(From Unit 4.3.2)

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6.1.6: Distinguishing between Two Types of Situations

Lesson

Let's think about equations with and without parentheses and the kinds of situations they describe.

Exercise 6.1.6.1: Which One Doesn't Belong: Seeing Structure

Which equation doesn't belong?

$4(x+3)=9$

$4 \cdot x+12=9$

$4+3x=9$

$9=12+4x$

Exercise 6.1.6.2: Card Sort: Categories of Equations

Your teacher will give you a set of cards that show equations. Sort the cards into 2 categories of your choosing. Be prepared to explain the meaning of your categories. Then, sort the cards into 2 categories in a different way. Be prepared to explain the meaning of your new categories.

Exercise 6.1.6.3: Even More Situations, Diagrams, and Equations

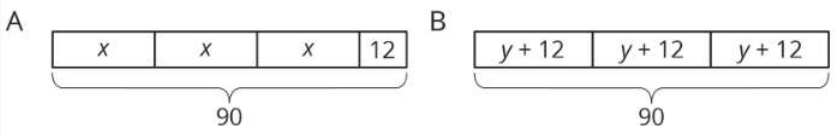


Figure 6.1.6.1

Story 1: Lin had 90 flyers to hang up around the school. She gave 12 flyers to each of three volunteers. Then she took the remaining flyers and divided them up equally between the three volunteers.

Story 2: Lin had 90 flyers to hang up around the school. After giving the same number of flyers to each of three volunteers, she had 12 left to hang up by herself.

1. Which diagram goes with which story? Be prepared to explain your reasoning.
2. In each diagram, what part of the story does the variable represent?
3. Write an equation corresponding to each story. If you get stuck, use the diagram.
4. Find the value of the variable in the story.

Are you ready for more?

A tutor is starting a business. In the first year, they start with 5 clients and charge $10 per week for an hour of tutoring with each client. For each year following, they double the number of clients and the number of hours each week. Each new client will be charged 150% of the charges of the clients from the previous year.

1. Organize the weekly earnings for each year in a table.
2. Assuming a full-time week is 40 hours per week, how many years will it take to reach full time and how many new clients will be taken on that year?
3. After reaching full time, what is the tutor's annual salary if they take 2 weeks of vacation?
4. Is there another business model you'd recommend for the tutor? Explain your reasoning.

Summary

In this unit, we encounter two main types of situations that can be represented with an equation. Here is an example of each type:

1. After adding 8 students to each of 6 same-sized teams, there were 72 students altogether.

2. After adding an 8-pound box of tennis rackets to a crate with 6 identical boxes of ping pong paddles, the crate weighed 72 pounds.

The first situation has all equal parts, since additions are made to *each* team. An equation that represents this situation is $6(x+8)=72$, where x represents the original number of students on each team. Eight students were added to each group, there are 6 groups, and there are a total of 72 students.

In the second situation, there are 6 equal parts added to one other part. An equation that represents this situation is $6x+8=72$, where x represents the weight of a box of ping pong paddles, there are 6 boxes of ping pong paddles, there is an additional box that weighs 8 pounds, and the crate weighs 72 pounds altogether.

In the first situation, there were 6 equal groups, and 8 students added to each group. $6(x+8)=72$.

In the second situation, there were 6 equal groups, but 8 more pounds in addition to that. $6x+8=72$.

Glossary Entries

Definition: Equivalent Expressions

Equivalent expressions are always equal to each other. If the expressions have variables, they are equal whenever the same value is used for the variable in each expression.

For example, $3x+4x$ is equivalent to $5x+2x$. No matter what value we use for x, these expressions are always equal. When x is 3, both expressions equal 21. When x is 10, both expressions equal 70.

Practice

Exercise 6.1.6.4

A school ordered 3 large boxes of board markers. After giving 15 markers to each of 3 teachers, there were 90 markers left. The diagram represents the situation. How many markers were originally in each box?

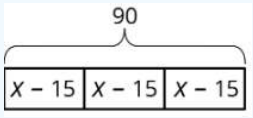


Figure 6.1.6.2

(From Unit 6.1.2)

Exercise 6.1.6.5

The diagram can be represented by the equation $25=2+6x$. Explain where you can see the 6 in the diagram.

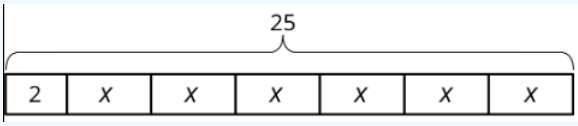


Figure 6.1.6.3

(From Unit 6.1.3)

Exercise 6.1.6.6

Match each equation to a story. (Two of the stories match the same equation.)

1. $3(x+5)=17$
2. $3x+5=17$
3. $5(x+3)=17$
4. $5x+3=17$

1. Jada's teacher fills a travel bag with 5 copies of a textbook. The weight of the bag and books is 17 pounds. The empty travel bag weighs 3 pounds. How much does each book weigh?

2. A piece of scenery for the school play is in the shape of a 5-foot-long rectangle. The designer decides to increase the length. There will be 3 identical rectangles with a total length of 17 feet. By how much did the designer increase the length of each rectangle?
3. Elena spends $17 and buys a $3 book and a bookmark for each of her 5 cousins. How much does each bookmark cost?
4. Noah packs up bags at the food pantry to deliver to families. He packs 5 bags that weigh a total of 17 pounds. Each bag contains 3 pounds of groceries and a packet of papers with health-related information. How much does each packet of papers weigh?
5. Andre has 3 times as many pencils as Noah and 5 pens. He has 17 pens and pencils all together. How many pencils does Noah have?

Exercise 6.1.6.7

Elena walked 20 minutes more than Lin. Jada walked twice as long as Elena. Jada walked for 90 minutes. The equation $2(x+20)=90$ describes this situation. Match each expression with the statement in the story with the expression it represents.

A. x
B. $x+20$
C. $2(x+20)$
D. 90

1. The number of minutes that Jada walked
2. The number of minutes that Elena walked
3. The number of minutes that Lin walked

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6.2: Solving Equations of the Form px+q=r and p(x+q)=r and Problems That Lead to Those Equations

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6.2.1: Reasoning about Solving Equations (Part 1)

Lesson

Let's see how a balanced hanger is like an equation and moving its weight is like solving the equation.

Exercise 6.2.1.1: Hanger Diagrams

In the two diagrams, all the triangles weigh the same and all the squares weigh the same.

For each diagram, come up with . . .

1. One thing that *must* be true
2. One thing that *could* be true
3. One thing that *cannot possibly* be true

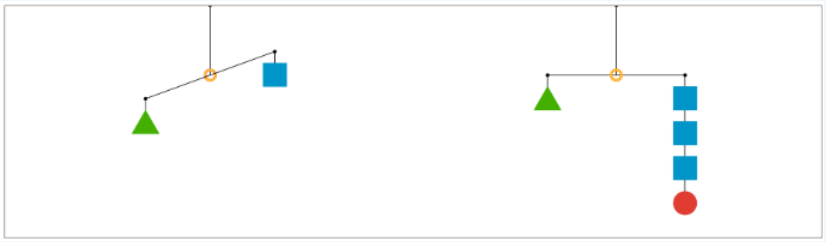


Figure 6.2.1.1: Two hanger diagrams. First hanger, unbalanced with left side lower, left side, green triangle, right side, blue square. Second hanger, balanced, left side, green triangle, right side, three blue squares, one red circle.

Exercise 6.2.1.2: Hanger and Equation Matching

On each balanced hanger, figures with the same letter have the same weight.

1. Match each hanger to an equation. Complete the equation by writing x, y, z, or w in the empty box.
   - $2\square + 3 = 5$
   - $3\square + 2 = 3$
   - $6 = 2\square + 3$
   - $7 = 3\square + 1$
2. Find the solution to each equation. Use the hanger to explain what the solution means.

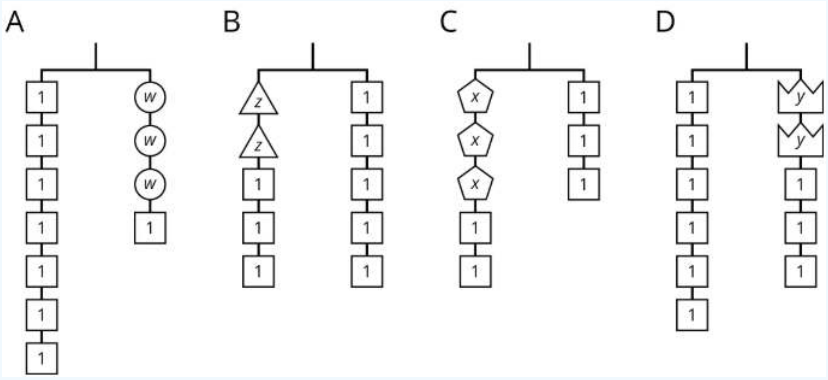


Figure 6.2.1.2: Four balanced hanger diagrams, A, B, C, D. Diagram A, left side, 7 squares, 1, right side, three circles, w, 1 square, 1. Diagram B, left side, 2 triangles, z, 3 squares, 1, right side, 5 squares, 1. Diagram C, left side, 3 pentagons, x, 2 squares, 1, right side, 3 squares, 1. Diagram D, left side, 6 squares, right side, 1, 2 crowns, y, 3 squares, 1.

Exercise 6.2.1.3: Use Hangers to Understand Equation Solving

Here are some balanced hangers where each piece is labeled with its weight. For each diagram:

1. Write an equation.
2. Explain how to figure out the weight of a piece labeled with a letter by reasoning about the diagram.

3. Explain how to figure out the weight of a piece labeled with a letter by reasoning about the equation.

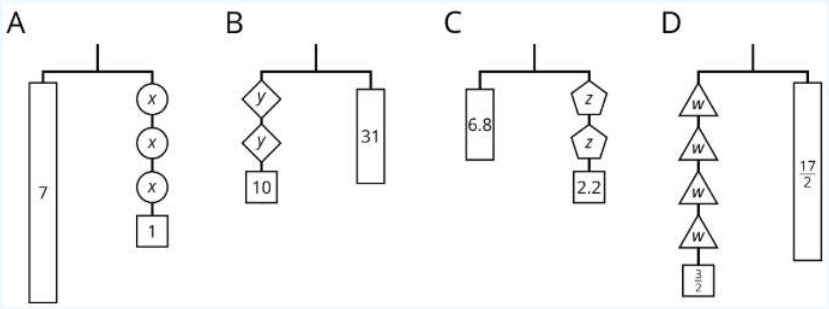


Figure 6.2.1.3: Four balanced hanger diagrams, A, B, C, D. Diagram A, left side, rectangle, 7, right side, 3 circles, x, square, 1. Diagram B, left side, 2 diamonds, y, square, 10, right side, rectangle, 31. Diagram C, left side, rectangle 6.8, right side 2 pentagons, z, square 2.2. Diagram D, left side, 4 triangles, w, square, 3 over 2, right side, rectangle, 17 over 2.

Summary

In this lesson, we worked with two ways to show that two amounts are equal: a balanced hanger and an equation. We can use a balanced hanger to think about steps to finding an unknown amount in an associated equation.

The hanger shows a total weight of 7 units on one side that is balanced with 3 equal, unknown weights and a 1-unit weight on the other. An equation that represents the relationship is $7 = 3x + 1$.

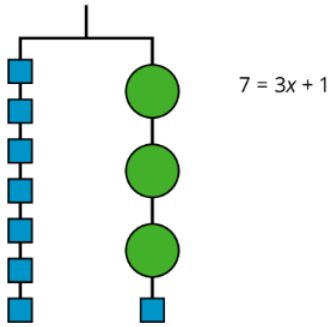


Figure 6.2.1.4

We can remove a weight of 1 unit from each side and the hanger will stay balanced. This is the same as subtracting 1 from each side of the equation.

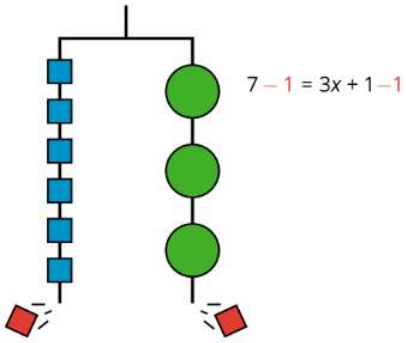


Figure 6.2.1.5: Balanced hanger, left side, 6 blue squares and one red squared being removed. Right side, 3 green squares, and one red square being removed. To the side, an equation 7 minus 1 = 3 + 1 minus 1, with each minus 1 written in red.

An equation for the new balanced hanger is $6 = 3x$.

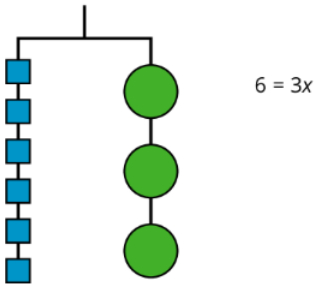


Figure 6.2.1.6

So the hanger will balance with $\frac{1}{3}$ of the weight on each side: $\frac{1}{3} \cdot 6 = \frac{1}{3} \cdot 3x$.

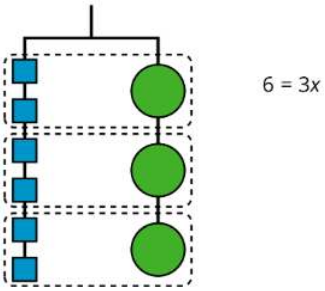


Figure 6.2.1.7: Balanced hanger, left side 6 blue squares, right side, three green circles. A dotted line is drawn around each of 3 groups, each groups consists of two blue square s from the left side and one green circle from the right side. To the side, an equation says 6=3x.

The two sides of the hanger balance with these weights: 6 1-unit weights on one side and 3 weights of unknown size on the other side.

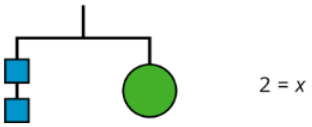


Figure 6.2.1.8

Here is a concise way to write the steps above:

$$\begin{aligned} 7 &= 3x + 1 \\ 6 &= 3x \qquad \text{after subtracting 1 from each side} \\ 2 &= x \qquad \text{after multiplying each side by } \tfrac{1}{3} \end{aligned}$$

Practice

Exercise 6.2.1.4

There is a proportional relationship between the volume of a sample of helium in liters and the mass of that sample in grams. If the mass of a sample is 5 grams, its volume is 28 liters. (5, 28) is shown on the graph below.

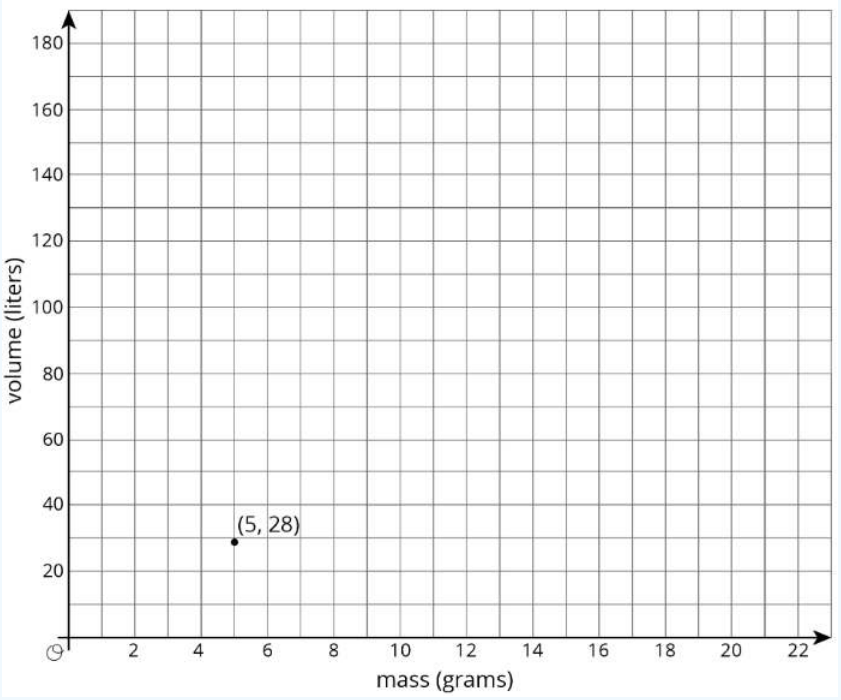


Figure 6.2.1.9

1. What is the constant of proportionality in this relationship?
2. In this situation, what is the meaning of the number you found in part a?
3. Add at least three more points to the graph above, and label with their coordinates.
4. Write an equation that shows the relationship between the mass of a sample of helium and its volume. Use m for mass and v for volume.

(From Unit 2.4.2)

Exercise 6.2.1.5

Explain how the parts of the balanced hanger compare to the parts of the equation.

$$7 = 2x + 3$$

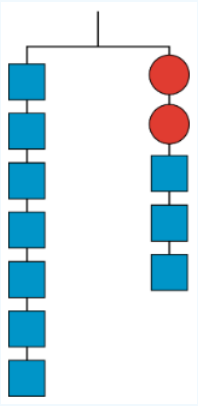


Figure 6.2.1.10

Exercise 6.2.1.6

For the hanger below:

1. Write an equation to represent the hanger.
2. Draw more hangers to show each step you would take to find x. Explain your reasoning.

3. Write an equation to describe each hanger you drew. Describe how each equation matches its hanger.

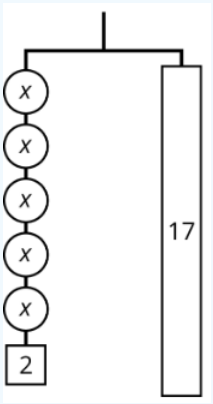


Figure 6.2.1.11

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6.2.2: Reasoning about Solving Equations (Part 2)

Lesson

Let's use hangers to understand two different ways of solving equations with parentheses.

Exercise 6.2.2.1: Equivalent to $2(x + 3)$

Select **all** the expressions equivalent to $2(x+3)$.

1. $2 \cdot (x+3)$
2. $(x+3)2$
3. $2 \cdot x + 2 \cdot 3$
4. $2 \cdot x + 3$
5. $(2 \cdot x) + 3$
6. $(2+x)3$

Exercise 6.2.2.2: Either Or

1. Explain why either of these equations could represent this hanger:

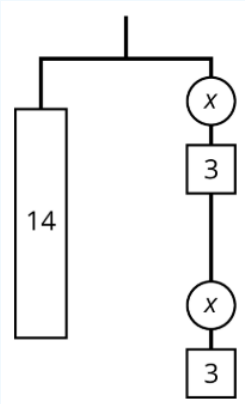


Figure 6.2.2.1

$14 = 2(x+3)$ or $14 = 2x + 6$

2. Find the weight of one circle. Be prepared to explain your reasoning.

Exercise 6.2.2.3: Use Hangers to Understand Equation Solving, Again

Here are some balanced hangers. Each piece is labeled with its weight.

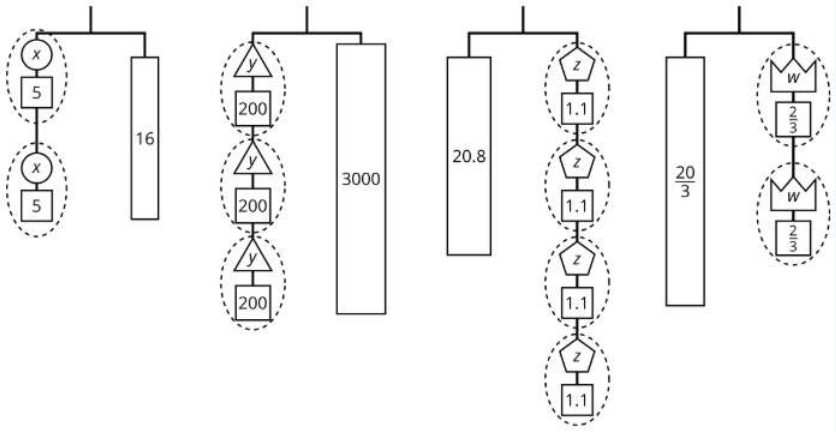


Figure 6.2.2.2: Four balanced hanger diagram. First diagram, left side, two groups, each group has circle x, square 5, right side, rectangle, 16. Second diagram, left side, two groups, each group has triangle y, square 200, right side, rectangle, 3000. Third diagram, left side, rectangle 20 point 8, right side 4 groups, each group has pentagon z, square 1 point 1. Fourth diagram, left side, rectangle, 20 over 3, left side, two groups, each group has crown w and square 2 over 3.

1. Assign one of these equations to each hanger:

$$2(x+5)=16 \qquad 3(y+200)=3,000$$
$$20.8=4(z+1.1) \qquad \tfrac{20}{3}=2(w+\tfrac{2}{3})$$

2. Explain how to figure out the weight of a piece labeled with a letter by reasoning about the diagram.
3. Explain how to figure out the weight of a piece labeled with a letter by reasoning about the equation.

Summary

The balanced hanger shows 3 equal, unknown weights and 3 2-unit weights on the left and an 18-unit weight on the right.

There are 3 unknown weights plus 6 units of weight on the left. We could represent this balanced hanger with an equation and solve the equation the same way we did before.

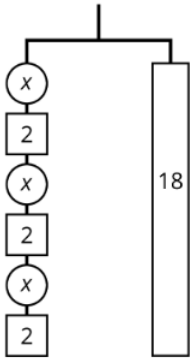


Figure 6.2.2.3

$$\begin{aligned} 3x+6 &= 18 \\ 3x &= 12 \\ x &= 4 \end{aligned}$$

Since there are 3 groups of $x+2$ on the left, we could represent this hanger with a different equation: $3(x+2)=18$.

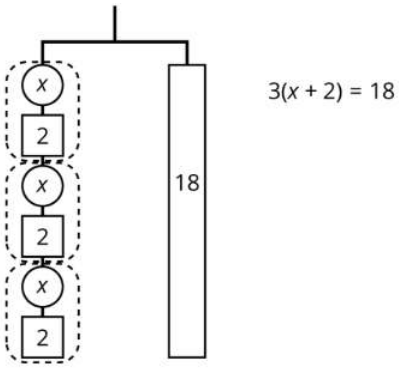


Figure 6.2.2.4

The two sides of the hanger balance with these weights: 3 groups of $x+2$ on one side, and 18, or 3 groups of 6, on the other side.

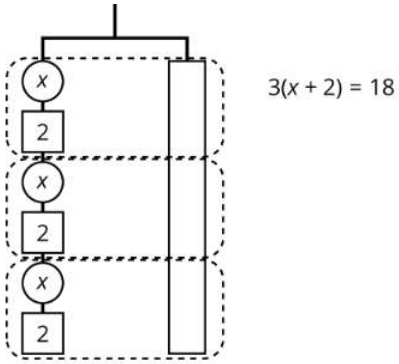


Figure 6.2.2.5: Balanced hanger, left side, circle labeled x, square labeled 2, circle labeled x, square labeled 2, circle labeled x, square labeled 2, right side, rectangle not labeled. A dotted line is drawn around three groups, each group contains one circle and one square from the left side and a third of the rectangle on the right side. To the side, an equation says 3 (x + 2) = 18.

The two sides of the hanger will balance with $\frac{1}{3}$ of the weight on each side: $\frac{1}{3} \cdot 3(x+2) = \frac{1}{3} \cdot 18$.

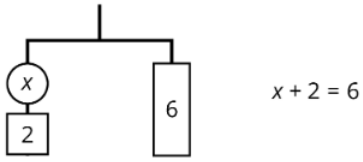


Figure 6.2.2.6

We can remove 2 units of weight from each side, and the hanger will stay balanced. This is the same as subtracting 2 from each side of the equation.

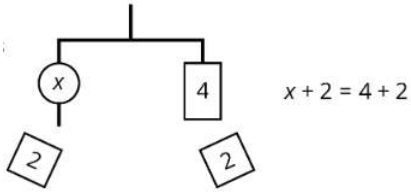


Figure 6.2.2.7: Balanced hanger. Left side, circle labeled x and square labeled 2, the square appears to be loose from the hanger. Right side, rectangle labeled 4 and square labeled 2, the square appears to be loose from the hanger. To the side, and equation says x + 2 = 4 + 2.

An equation for the new balanced hanger is $x = 4$. This gives the solution to the original equation.

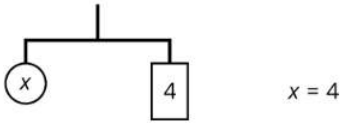


Figure 6.2.2.8

Here is a concise way to write the steps above:

$$
\begin{aligned}
&3(x+2)=18 && \\
&x+2=6 && \text{after multiplying each side by } \tfrac{1}{3} \\
&x=4 && \text{after subtracting 2 from each side}
\end{aligned}
$$

Practice

Exercise 6.2.2.4

Here is a hanger:

1. Write an equation to represent the hanger.
2. Solve the equation by reasoning about the equation or the hanger. Explain your reasoning.

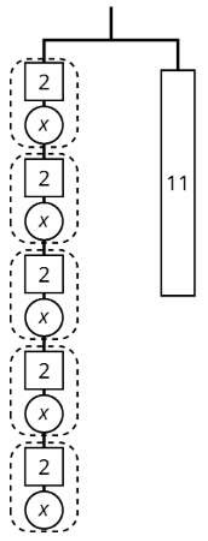


Figure 6.2.2.9

Exercise 6.2.2.5

Explain how each part of the equation $9=3(x+2)$ is represented in the hanger.

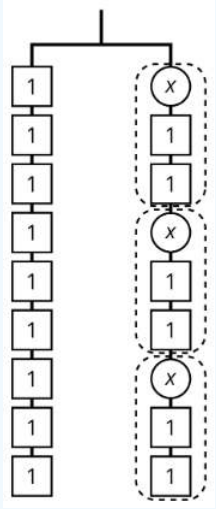


Figure 6.2.2.10

- x
- 9
- 3

- $x+2$
- $3(x+2)$
- the equal sign

Exercise 6.2.2.6

Select the word from the following list that best describes each situation.

A. You deposit money in a savings account, and every year the amount of money in the account increases by 2.5%.
B. For every car sold, a car salesman is paid 6% of the car's price.
C. Someone who eats at a restaurant pays an extra 20% of the food price. This extra money is kept by the person who served the food.
D. An antique furniture store pays $200 for a chair, adds 50% of that amount, and sells the chair for $300.
E. The normal price of a mattress is $600, but it is on sale for 10% off.
F. For any item you purchase in Texas, you pay an additional 6.25% of the item's price to the state government.

1. Tax
2. Commission
3. Discount
4. Markup
5. Tip or gratuity
6. Interest

(From Unit 4.3.2)

Exercise 6.2.2.7

Clare drew this diagram to match the equation $2x+16=50$, but she got the wrong solution as a result of using this diagram.

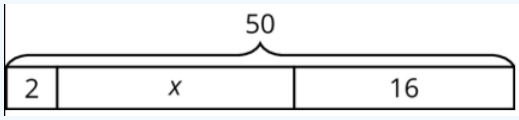


Figure 6.2.2.11

1. What value of x can be found using the diagram?
2. Show how to fix Clare's diagram to correctly match the equation.
3. Use th new diagram to find a correct value for x.
4. Explain how the mistake Clare made when she drew her diagram.

(From Unit 6.1.3)

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6.2.3: Dealing with Negative Numbers

Lesson

Let's show that doing the same to each side works for negative numbers too.

Exercise 6.2.3.1: Which One Doesn't Belong: Rational Number Arithmetic

Which equation doesn't belong?

$15 = -5 \cdot -3$ $\quad$ $4 - -2 = 6$

$2 + -5 = -3$ $\quad$ $-3 \cdot -4 = -12$

Exercise 6.2.3.2: Old and New Ways to Solve

Solve each equation. Be prepared to explain your reasoning.

1. $x + 6 = 4$
2. $x - -4 = -6$
3. $2(x - 1) = -200$
4. $2x + -3 = -23$

Exercise 6.2.3.3: Keeping It True

Here are some equations that all have the same solution.

$$\begin{aligned} x &= -6 \\ x - 3 &= -9 \\ -9 &= x - 3 \\ 900 &= -100(x - 3) \\ 900 &= (x - 3) \cdot (-100) \\ 900 &= -100x + 300 \end{aligned}$$

1. Explain how you know that each equation has the same solution as the previous equation. Pause for discussion before moving to the next question.
2. Keep your work secret from your partner. Start with the equation $-5 = x$. Do the same thing to each side at least three times to create an equation that has the same solution as the starting equation. Write the equation you ended up with on a slip of paper, and trade equations with your partner.
3. See if you can figure out what steps they used to transform $-5 = x$ into their equation. When you think you know, check with them to see if you are right.

Summary

When we want to find a solution to an equation, sometimes we just think about what value in place of the variable would make the equation true. Sometimes we perform the same operation on each side (for example, subtract the same amount from each side). The balanced hangers helped us to understand that doing the same to each side of an equation keeps the equation true.

Since negative numbers are just numbers, then doing the same thing to each side of an equation works for negative numbers as well. Here are some examples of equations that have negative numbers and steps you could take to solve them.

Example:

$$\begin{aligned} 2(x - 5) &= -6 \\ \frac{1}{2} \cdot 2(x - 5) &= \frac{1}{2} \cdot (-6) && \text{multiply each side by } \frac{1}{2} \\ x - 5 &= -3 \\ x - 5 + 5 &= -3 + 5 && \text{add 5 to each side} \\ x &= 2 \end{aligned}$$

Example:

$$\begin{aligned} -2x + -5 &= 6 \\ -2x + -5 - -5 &= 6 - -5 \qquad \text{subtract } -5 \text{ from each side} \\ -2x &= 11 \\ -2x \div -2 &= 11 \div 2 \qquad \text{divide each side by } -2 \\ x &= -\frac{11}{2} \end{aligned}$$

Doing the same thing to each side maintains equality even if it is not helpful to solving for the unknown amount. For example, we could take the equation $-3x + 7 = -8$ and add -2 to each side:

$$\begin{aligned} -3x + 7 &= -8 \\ -3x + 7 + -2 &= -8 + -2 \qquad \text{add } -2 \text{ to each side} \\ -3x + 5 &= -10 \end{aligned}$$

If $-2x + 7 = -8$ is true then $-3x + 5 = -10$ is also true, but we are no closer to a solution than we were before adding -2. We can use moves that maintain equality to make new equations that all have the same solution. Helpful combinations of moves will eventually lead to an equation like $x = 5$, which gives the solution to the original equation (and every equation we wrote in the process of solving).

Practice

Exercise 6.2.3.4

Solve each equation.

1. $4x = -28$
2. $x - -6 = -2$
3. $-x + 4 = -9$
4. $-3x + 7 = 1$
5. $25x + -11 = -86$

Exercise 6.2.3.5

Here is an equation $2x + 9 = -15$. Write three different equations that have the same solution as $2x + 9 = -15$. Show or explain how you found them.

Exercise 6.2.3.6

Select **all** the equations that match the diagram.

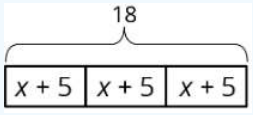


Figure 6.2.3.1

A. $x + 5 = 18$
B. $18 \div 3 = x + 5$
C. $3(x + 5) = 18$
D. $x + 5 = \frac{1}{3} \cdot 18$
E. $3x + 5 = 18$

(From Unit 6.1.3)

Exercise 6.2.3.7

There are 88 seats in a theater. The seating in the theater is split into 4 identical sections. Each section has 14 red seats and some blue seats.

1. Draw a tape diagram to represent the situation.
2. What unknown amounts can be found by by using the diagram or reasoning about the situation?

(From Unit 6.1.2)

Exercise 6.2.3.8

Match each story to an equation.

A. A stack of nested paper cups is 8 inches tall. The first cup is 4 inches tall and each of the rest of the cups in the stack adds $\frac{1}{4}$ inch to the height of the stack.
B. A baker uses 4 cups of flour. She uses $\frac{1}{4}$ cup to flour the counters and the rest to make 8 identical muffins.
C. Elena has an 8-foot piece of ribbon. She cuts off a piece that is $\frac{1}{4}$ of a foot long and cuts the remainder into four pieces of equal length.

1. $\frac{1}{4} + 4x = 8$
2. $4 + \frac{1}{4}x = 8$
3. $8x + \frac{1}{4} = 4$

(From Unit 6.1.4)

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6.2.4: Different Options for Solving One Equation

Lesson

Let's think about which way is easier when we solve equations with parentheses.

Exercise 6.2.4.1: Algebra Talk: Solve Each Equation

$100(x-3)=1,000$

$500(x-3)=5,000$

$0.03(x-3)=0.3$

$0.72(x+2)=7.2$

Exercise 6.2.4.2: Analyzing Solution Methods

Three students each attempted to solve the equation $2(x-9)=10$, but got different solutions. Here are their methods. Do you agree with any of their methods, and why?

Noah's method:

$$\begin{aligned} 2(x-9) &= 10 \\ 2(x-9)+9 &= 10+9 && \text{add 9 to each side} \\ 2x &= 19 \\ 2x \div 2 &= 19 \div 2 && \text{divide each side by 2} \\ x &= \frac{19}{2} \end{aligned}$$

Elena's method:

$$\begin{aligned} 2(x-9) &= 10 \\ 2x-18 &= 10 && \text{apply the distributive property} \\ 2x-18-18 &= 10-18 && \text{subtract 18 from each side} \\ 2x &= -8 \\ 2x \div 2 &= -8 \div 2 && \text{divide each side by 2} \\ x &= -4 \end{aligned}$$

Andre's method:

$$\begin{aligned} 2(x-9) &= 10 \\ 2x-18 &= 10 && \text{apply the distributive property} \\ 2x-18+18 &= 10+18 && \text{add 18 to each side} \\ 2x &= 28 \\ 2x \div 2 &= 28 \div 2 && \text{divide each side by 2} \\ x &= 14 \end{aligned}$$

Exercise 6.2.4.3: Solution Pathways

For each equation, try to solve the equation using each method (dividing each side first, or applying the distributive property first). Some equations are easier to solve by one method than the other. When that is the case, stop doing the harder method and write down the reason you stopped.

1. $2,000(x-0.03)=6,000$
2. $2(x+1.25)=3.5$
3. $\frac{1}{4}(4+x)=\frac{4}{3}$
4. $-10(x-1.7)=-3$
5. $5.4=0.3(x+8)$

Summary

Equations can be solved in many ways. In this lesson, we focused on equations with a specific structure, and two specific ways to solve them.

Suppose we are trying to solve the equation $\frac{4}{5}(x+27)=16$. Two useful approaches are:

- divide each side by $\frac{4}{5}$
- apply the distributive property

In order to decide which approach is better, we can look at the numbers and think about which would be easier to compute. We notice that $\frac{4}{5}\cdot 27$ will be hard, because 27 isn't divisible by 5. But $16\div\frac{4}{5}$ gives us $16\cdot\frac{5}{4}$, and 16 is divisible by 4. Dividing each side by $\frac{4}{5}$ gives:

$$\begin{aligned}\frac{4}{5}(x+27)&=16\\ \frac{5}{4}\cdot\frac{4}{5}(x+27)&=16\cdot\frac{5}{4}\\ x+27&=20\\ x&=-7\end{aligned}$$

Sometimes the calculations are simpler if we first use the distributive property. Let's look at the equation $100(x+0.06)=21$. If we first divide each side by 100, we get $\frac{21}{100}$ or 0.21 on the right side of the equation. But if we use the distributive property first, we get an equation that only contains whole numbers.

$$\begin{aligned}100(x+0.06)&=21\\ 100x+6&=21\\ 100x&=15\\ x&=\frac{15}{100}\end{aligned}$$

Practice

Exercise 6.2.4.4

Andre wants to buy a backpack. The normal price of the backpack is \$40. He notices that a store that sells the backpack is having a 30% off sale. What is the sale price of the backpack?

(From Unit 4.3.2)

Exercise 6.2.4.5

On the first math exam, 16 students received an A grade. On the second math exam, 12 students received an A grade. What percentage decrease is that?

(From Unit 4.3.3)

Exercise 6.2.4.6

Solve each equation.

1. $2(x-3)=14$
2. $-5(x-1)=40$
3. $12(x+10)=24$
4. $\frac{1}{6}(x+6)=11$
5. $\frac{5}{7}(x-9)=25$

Exercise 6.2.4.7

Select **all** expressions that represent a correct solution to the equation $6(x+4)=20$.

A. $(20-4)\div 6$
B. $\frac{1}{6}(20-4)$
C. $20-6-4$
D. $20\div 6-4$
E. $\frac{1}{6}(20-24)$
F. $(20-24)\div 6$

Exercise 6.2.4.8

Lin and Noah are solving the equation $7(x+2)=91$.

Lin starts by using the distributive property. Noah starts by dividing each side by 7.

1. Show what Lin's and Noah's full solution methods might look like.
2. What is the same and what is different about their methods?

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6.2.5: Using Equations to Solve Problems

Lesson

Let's use tape diagrams, equations, and reasoning to solve problems.

Exercise 6.2.5.1: Remember Tape Diagrams

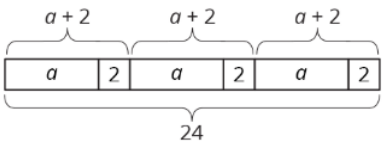


Figure 6.2.5.1

1. Write a story that could be represented by this tape diagram.
2. Write an equation that could be represented by this tape diagram.

Exercise 6.2.5.2: At the Fair

1. Tyler is making invitations to the fair. He has already made some of the invitations, and he wants to finish the rest of them within a week. He is trying to spread out the remaining work, to make the same number of invitations each day. Tyler draws a diagram to represent the situation.

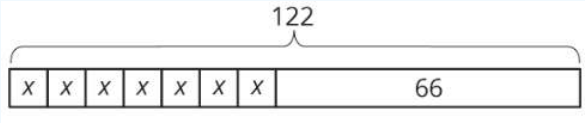


Figure 6.2.5.2

a. Explain how each part of the situation is represented in Tyler's diagram:
   How many total invitations Tyler is trying to make.
   How many invitations he has made already.
   How many days he has to finish the invitations.
b. How many invitations should Tyler make each day to finish his goal within a week? Explain or show your reasoning.
c. Use Tyler's diagram to write an equation that represents the situation. Explain how each part of the situation is represented in your equation.
d. Show how to solve your equation.

2. Noah and his sister are making prize bags for a game at the fair. Noah is putting 7 pencil erasers in each bag. His sister is putting in some number of stickers. After filling 3 of the bags, they have used a total of 57 items.

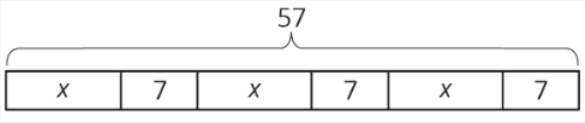


Figure 6.2.5.3

a. Explain how the diagram represents the situation.
b. Noah writes the equation $3(x+7)=57$ to represent the situation. Do you agree with him? Explain your reasoning.
c. How many stickers is Noah's sister putting in each prize bag? Explain or show your reasoning.

3. A family of 6 is going to the fair. They have a coupon for $1.50 off each ticket. If they pay $46.50 for all their tickets, how much does a ticket cost without the coupon? Explain or show your reasoning. If you get stuck, consider drawing a diagram or writing an equation.

Exercise 6.2.5.3: Running Around

Priya, Han, and Elena, are members of the running club at school.

1. Priya was busy studying this week and ran 7 fewer miles than last week. She ran 9 times as far as Elena ran this week. Elena only had time to run 4 miles this week.
   a. How many miles did Priya run last week?
   b. Elena wrote the equation $\frac{1}{9}(x-7)=4$ to describe the situation. She solved the equation by multiplying each side by 9 and then adding 7 to each side. How does her solution compare to the way you found Priya's miles?
2. One day last week, 6 teachers joined $\frac{5}{7}$ of the members of the running club in an after-school run. Priya counted a total of 31 people running that day. How many members does the running club have?
3. Priya and Han plan a fundraiser for the running club. They begin with a balance of -80 because of expenses. In the first hour of the fundraiser they collect equal donations from 9 family members, which brings their balance to -44. How much did each parent give?
4. The running club uses the money they raised to pay for a trip to a canyon. At one point during a run in the canyon, the students are at an elevation of 128 feet. After descending at a rate of 50 feet per minute, they reach an elevation of -472 feet. How long did the descent take?

Are you ready for more?

A musician performed at three local fairs. At the first he doubled his money and spent $30. At the second he tripled his money and spent $54. At the third, he quadrupled his money and spent $72. In the end he had $48 left. How much did he have before performing at the fairs?

Summary

Many problems can be solved by writing and solving an equation. Here is an example:

Clare ran 4 miles on Monday. Then for the next six days, she ran the same distance each day. She ran a total of 22 miles during the week. How many miles did she run on each of the 6 days?

One way to solve the problem is to represent the situation with an equation, $4+6x=22$, where x represents the distance, in miles, she ran on each of the 6 days. Solving the equation gives the solution to this problem.

$$\begin{aligned} 4+6x &= 22 \\ 6x &= 18 \\ x &= 3 \end{aligned}$$

Practice

Exercise 6.2.5.4

Find the value of each variable.

1. $a \cdot 3 = -30$
2. $-9 \cdot b = 45$
3. $-89 \cdot 12 = c$
4. $d \cdot 88 = -88,000$

(From Unit 5.3.2)

Exercise 6.2.5.5

Match each equation to its solution and to the story it describes.

Equations:

1. $5x-7=3$
2. $7=3(5+x)$
3. $3x+5=-7$

4. $\frac{1}{3}(x+7)=5$

Solutions:

1. -4
2. $\frac{-8}{3}$
3. 2
4. 8

Stories:

- The temperature is -7. Since midnight the temperature tripled and then rose 5 degrees. What was temperature at midnight?
- Jada has 7 pink roses and some white roses. She gives all of them away: 5 roses to each of her 3 favorite teachers. How many white roses did she give away?
- A musical instrument company reduced the time it takes for a worker to build a guitar. Before the reduction it took 5 hours. Now in 7 hours they can build 3 guitars. By how much did they reduce the time it takes to build each guitar?
- A club puts its members into 5 groups for an activity. After 7 students have to leave early, there are only 3 students left to finish the activity. How many students were in each group?

Exercise 6.2.5.6

The baby giraffe weighed 132 pounds at birth. He gained weight at a steady rate for the first 7 months until his weight reached 538 pounds. How much did he gain each month?

Exercise 6.2.5.7

Six teams are out on the field playing soccer. The teams all have the same number of players. The head coach asks for 2 players from each team to come help him move some equipment. Now there are 78 players on the field. Write and solve an equation whose solution is the number of players on each team.

Exercise 6.2.5.8

A small town had a population of 960 people last year. The population grew to 1200 people this year. By what percentage did the population grow?

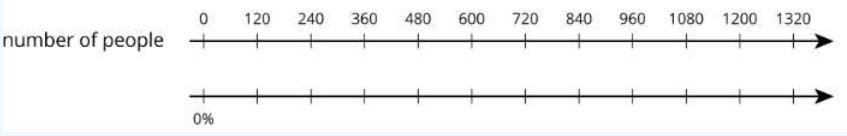


Figure 6.2.5.4: Double number line, top line number of people, evenly spaced tick marks labeled 0, 120, 240, 360, 480, 600, 720, 840, 960, 1080, 1200. Bottom line, corresponding tick marks, first labeled 0% and the rest blank.

(From Unit 4.2.2)

Exercise 6.2.5.9

The gas tank of a truck holds 30 gallons. The gas tank of a passenger car holds 50% less. How many gallons does it hold?

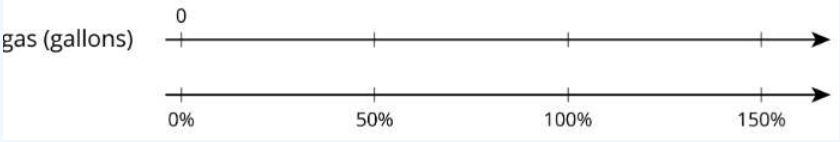


Figure 6.2.5.5

(From Unit 4.2.2)

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6.2.6: Solving Problems about Percent Increase or Decrease

Lesson

Let's use tape diagrams, equations, and reasoning to solve problems with negatives and percents.

Exercise 6.2.6.1: 20% Off

An item costs x dollars and then a 20% discount is applied. Select **all** the expressions that could represent the price of the item after the discount.

1. $\frac{20}{100}x$
2. $x - \frac{20}{100}x$
3. $(1 - 0.20)x$
4. $\frac{100-20}{100}x$
5. $0.80x$
6. $(100 - 20)x$

Exercise 6.2.6.2: Walking More Each Day

1. Mai started a new exercise program. On the second day, she walked 5 minutes more than on the first day. On the third day, she increased her walking time from day 2 by 20% and walked for 42 minutes. Mai drew a diagram to show her progress.

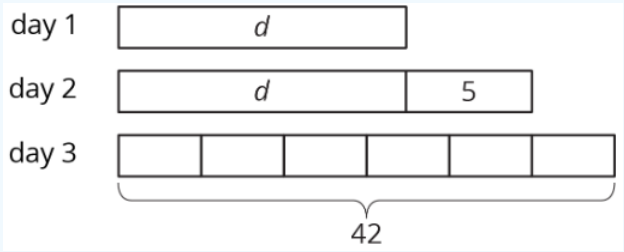


Figure 6.2.6.1: Three tape diagrams, day 1, day 2, day 3. Diagram Day 1, 1 part, labeled d. Diagram Day 2, 2 parts, first part, d, same length as d above, second part labeled 5. Diagram day 3, has 6 equal parts with no labels. The first five parts together are the length of the Day 3 diagram above, the total is 42.

Explain how the diagram represents the situation.

2. Noah said the equation $1.20(d+5) = 42$ also represents the situation. Do you agree with Noah? Explain your reasoning.

3. Find the number of minutes Mai walked on the first day. Did you use the diagram, the equation, or another strategy? Explain or show your reasoning.

4. Mai has been walking indoors because of cold temperatures. On Day 4 at noon, Mai hears a report that the temperature is only 9 degrees Fahrenheit. She remembers the morning news reporting that the temperature had doubled since midnight and was expected to rise 15 degrees by noon. Mai is pretty sure she can draw a diagram to represent this situation but isn't sure if the equation is $9 = 15 + 2t$ or $2(t + 15) = 9$. What would you tell Mai about the diagram and the equation and how they might be useful to find the temperature, t, at midnight?

Exercise 6.2.6.3: A Sale on Shoes

1. A store is having a sale where all shoes are discounted by 20%. Diego has a coupon for $3 off of the regular price for one pair of shoes. The store first applies the coupon and then takes 20% off of the reduced price. If Diego pays $18.40 for a pair of shoes, what was their original price before the sale and without the coupon?
2. Before the sale, the store had 100 pairs of flip flops in stock. After selling some, they notice that $\frac{3}{5}$ of the flip flops they have left are blue. If the store has 39 pairs of blue flip flops, how many pairs of flip flops (any color) have they sold?
3. When the store had sold $\frac{2}{9}$ of the boots that were on display, they brought out another 34 pairs from the stock room. If that gave them 174 pairs of boots out, how many pairs were on display originally?

4. On the morning of the sale, the store donated 50 pairs of shoes to a homeless shelter. Then they sold 64% of their remaining inventory during the sale. If the store had 288 pairs after the donation and the sale, how many pairs of shoes did they have at the start?

Are you ready for more?

A coffee shop offers a special: 33% extra free or 33% off the regular price. Which offer is a better deal? Explain your reasoning.

Summary

We can solve problems where there is a percent increase or decrease by using what we know about equations. For example, a camping store increases the price of a tent by 25%. A customer then uses a $10 coupon for the tent and pays $152.50. We can draw a diagram that shows first the 25% increase and then the $10 coupon.

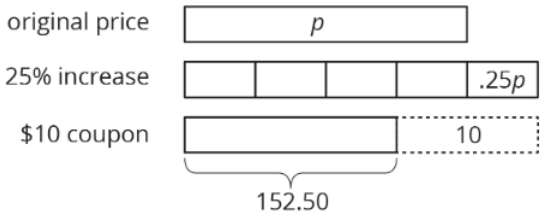


Figure 6.2.6.2: Three tape diagrams of unequal length. Top diagram, original price, one part labeled p. Middle diagram, 25% increase, 4 equal parts which total to the same length as p above, with another equal part on the end labeled point 25 p. Third diagram, 10 dollar coupon, first part the same length as three parts above, labeled 152 point 50, second part, dotted outline, same length as two parts above, labeled 10.

The price after the 25% increase is $p + .25p$ or $1.25p$. An equation that represents the situation could be $1.25p - 10 = 152.50$. To find the original price before the increase and discount, we can add 10 to each side and divide each side by 1.25, resulting in $p = 130$. The original price of the tent was $130.

Practice

Exercise 6.2.6.4

A backpack normally costs $25 but it is on sale for $21. What percentage is the discount?

(From Unit 4.3.3)

Exercise 6.2.6.5

Find each product

1. $\frac{2}{5} \cdot (-10)$
2. $-8 \cdot \left(\frac{-3}{2}\right)$
3. $\frac{10}{6} \cdot 0.6$
4. $\left(\frac{-100}{37}\right) \cdot (-0.37)$

(From Unit 5.3.2)

Exercise 6.2.6.6

Select **all** expressions that show x increased by 35%.

A. $1.35x$
B. $\frac{35}{100}x$
C. $x + \frac{35}{100}x$
D. $(1 + 0.35)x$
E. $\frac{100+35}{100}x$

F. $(100+35)x$

Exercise 6.2.6.7

Complete each sentence with the word *discount*, *deposit*, or *withdrawal*.

1. Clare took $20 out of her bank account. She made a _____.
2. Kiran used a coupon when he bought a pair of shoes. He got a _____.
3. Priya put $20 into her bank account. She made a _____.
4. Lin paid less than usual for a pack of gum because it was on sale. She got a _____.

(From Unit 4.3.2)

Exercise 6.2.6.8

Here are two stories:

- The initial freshman class at a college is 10% smaller than last year's class. But then during the first week of classes, 20 more students enroll. There are then 830 students in the freshman class.
- A store reduces the price of a computer by $20. Then during a 10% off sale, a customer pays $830.

Here are two equations:

- $0.9x+20=830$
- $0.9(x-20)=830$

1. Decide which equation represents each story.
2. Explain why one equation has parentheses and the other doesn't.
3. Solve each equation, and explain what the solution means in the situation.

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6.3: Inequalities

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6.3.1: Reintroducing Inequalities

Lesson

Let's work with inequalities.

Exercise 6.3.1.1: Greater Than One

The number line shows values of x that make the inequality $x > 1$ true.

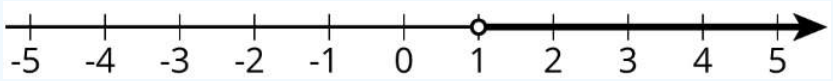


Figure 6.3.1.1

1. Select **all** the values of x from this list that make the inequality $x > 1$ true.
   a. 3
   b. -3
   c. 1
   d. 700
   e. 1.05
2. Name two more values of x that are solutions to the inequality.

Exercise 6.3.1.2: The Roller Coaster

A sign next to a roller coaster at an amusement park says, “You must be at least 60 inches tall to ride.” Noah is happy to know that he is tall enough to ride.



Figure 6.3.1.2: Baishamen Park - amusement park - roller coaster, adult-01, by Anna Frodesiak (Own work). Public Domain. Wikimedia Commons. Source.

1. Noah is x inches tall. Which of the following can be true: $x > 60$, $x = 60$, or $x < 60$? Explain how you know.
2. Noah’s friend is 2 inches shorter than Noah. Can you tell if Noah’s friend is tall enough to go on the ride? Explain or show your reasoning.
3. List one possible height for Noah that means that his friend is tall enough to go on the ride, and another that means that his friend is too short for the ride.
4. On the number line below, show all the possible heights that Noah’s friend could be.

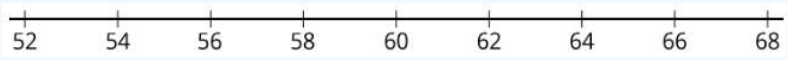


Figure 6.3.1.3

5. Noah's friend is y inches tall. Use y and any of the symbols $<$, $=$, $>$ to express this height.

Exercise 6.3.1.3: Is the Inequality True or False?

The table shows four inequalities and four possible values for x. Decide whether each value makes each inequality true, and complete the table with “true” or “false.” Discuss your thinking with your partner. If you disagree, work to reach an agreement.

Table 6.3.1.1

x	0	100	-100	25
$x \leq 25$				
$100 < 4x$				
$-3x > -75$				
$100 \geq 35 - x$				

Are you ready for more?

Find an example of in inequality used in the real world and describe it using a number line.

Summary

We use inequalities to describe a range of numbers. In many places, you are allowed to get a driver's license when you are at least 16 years old. When checking if someone is old enough to get a license, we want to know if their age is at least 16. If h is the age of a person, then we can check if they are allowed to get a driver's license by checking if their age makes the inequality $h > 16$ (they are older than 16) or the equation $h = 16$ (they are 16) true. The symbol $\geq$, pronounced "greater than or equal to," combines these two cases and we can just check if $h \geq 16$ (their age is greater than or equal to 16). The inequality $h \geq 16$ can be represented on a number line:

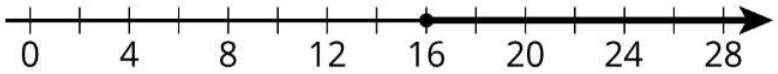


Figure 6.3.1.4

Practice

Exercise 6.3.1.4

For each inequality, find two values for x that make the inequality true and two values that make it false.

1. $x + 3 > 70$
2. $x + 3 < 70$
3. $-5x < 2$
4. $5x < 2$

Exercise 6.3.1.5

Here is an inequality: $-3x > 18$.

1. List some values for x that would make this inequality true.
2. How are the solutions to the inequality $-3x \geq 18$ different from the solutions to $-3x > 18$? Explain your reasoning.

Exercise 6.3.1.6

Here are the prices for cheese pizza at a certain pizzeria:

Table 6.3.1.2

pizza store	price in dollars
small	11.160
medium	
large	16.25

1. You had a coupon that made the price of a large pizza $13.00. For what percent off was the coupon?

2. Your friend purchased a medium pizza for $10.31 with a 30% off coupon. What is the price of a medium pizza without a coupon?
3. Your friend has a 15% off coupon and $10. What is the largest pizza that your friend can afford, and how much money will be left over after the purchase?

(From Unit 4.3.3)

Exercise 6.3.1.7

Select **all** the stories that can be represented by the diagram.

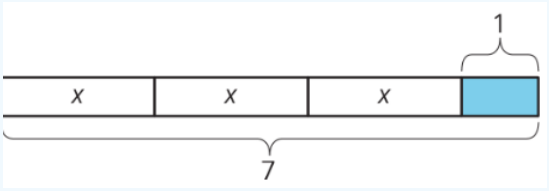


Figure 6.3.1.5

A. Andre studies 7 hours this week for end-of-year exams. He spends 1 hour on English and an equal number of hours each on math, science, and history.
B. Lin spends $3 on 7 markers and a $1 pen.
C. Diego spends $1 on 7 stickers and 3 marbles.
D. Noah shares 7 grapes with 3 friends. He eats 1 and gives each friend the same number of grapes.
E. Elena spends $7 on 3 notebooks and a $1 pen.

(From Unit 6.1.4)

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6.3.2: Finding Solutions to Inequalities in Context

Lesson

Let's solve more complicated inequalities.

Exercise 6.3.2.1: Solutions to Equations and Solutions to Inequalities

1. Solve $-x = 10$
2. Find 2 solutions to $-x > 10$
3. Solve $2x = -20$
4. Find 2 solutions to $2x > -20$

Exercise 6.3.2.2: Earning Money For Soccer Stuff

1. Andre has a summer job selling magazine subscriptions. He earns $25 per week plus $3 for every subscription he sells. Andre hopes to make at least enough money this week to buy a new pair of soccer cleats.
   a. Let n represent the number of magazine subscriptions Andre sells this week. Write an expression for the amount of money he makes this week.
   b. The least expensive pair of cleats Andre wants costs $68. Write and solve an equation to find out how many magazine subscriptions Andre needs to sell to buy the cleats.
   c. If Andre sold 16 magazine subscriptions this week, would he reach his goal? Explain your reasoning.
   d. What are some other numbers of magazine subscriptions Andre could have sold and still reached his goal?
   e. Write an *inequality* expressing that Andre wants to make at least $68.
   f. Write an inequality to describe the number of subscriptions Andre must sell to reach his goal.
2. Diego has budgeted $35 from his summer job earnings to buy shorts and socks for soccer. He needs 5 pairs of socks and a pair of shorts. The socks cost different amounts in different stores. The shorts he wants cost $19.95.
   a. Let x represent the price of one pair of socks. Write an expression for the total cost of the socks and shorts.
   b. Write and solve an equation that says that Diego spent exactly $35 on the socks and shorts.
   c. List some other possible prices for the socks that would still allow Diego to stay within his budget.
   d. Write an inequality to represent the amount Diego can spend on a single pair of socks.

Exercise 6.3.2.3: Granola Bars and Savings

1. Kiran has $100 saved in a bank account. (The account doesn't earn interest.) He asked Clare to help him figure out how much he could take out each month if he needs to have at least $25 in the account a year from now.
   a. Clare wrote the inequality $-12x + 100 \geq 25$, where x represents the amount Kiran takes out each month. What does $-12x$ represent?
   b. Find some values of x that would work for Kiran.
   c. We could express *all* the values that would work using either $x \leq$ __ or $x \geq$ __. Which one should we use?
   d. Write the answer to Kiran's question using mathematical notation.
2. A teacher wants to buy 9 boxes of granola bars for a school trip. Each box usually costs $7, but many grocery stores are having a sale on granola bars this week. Different stores are selling boxes of granola bars at different discounts.
   a. If x represents the dollar amount of the discount, then the amount the teacher will pay can be expressed as $9(7 - x)$. In this expression, what does the quantity $7 - x$ represent?
   b. The teacher has $36 to spend on the granola bars. The equation $9(7 - x) = 36$ represents a situation where she spends all $36. Solve this equation.
   c. What does the solution mean in this situation?
   d. The teacher does not have to spend all $36. Write an inequality relating 36 and $9(7 - x)$ representing this situation.
   e. The solution to this inequality must either look like $x \geq 3$ or $x \leq 3$. Which do you think it is? Explain your reasoning.

Are you ready for more?

Jada and Diego baked a large batch of cookies.

- They selected $\frac{1}{4}$ of the cookies to give to their teachers.
- Next, they threw away one burnt cookie.
- They delivered $\frac{2}{5}$ of the remaining cookies to a local nursing home.
- Next, they gave 3 cookies to some neighborhood kids.
- They wrapped up $\frac{2}{3}$ of the remaining cookies to save for their friends.

After all this, they had 15 cookies left. How many cookies did they bake?

Summary

Suppose Elena has $5 and sells pens for $1.50 each. Her goal is to save $20. We could solve the equation $1.5x + 5 = 20$ to find the number of pens, x, that Elena needs to sell in order to save *exactly* $20. Adding -5 to both sides of the equation gives us $1.5x = 15$, and then dividing both sides by 1.5 gives the solution $x = 10$ pens.

What if Elena wants to have some money left over? The inequality $1.5x + 5 > 20$ tells us that the amount of money Elena makes needs to be *greater* than $20. The solution to the previous equation will help us understand what the solutions to the inequality will be. We know that if she sells 10 pens, she will make $20. Since each pen gives her more money, she needs to sell *more* than 10 pens to make more than $20. So the **solution to the inequality** is $x > 10$.

Glossary Entries

Definition: Solution to an Inequality

A solution to an inequality is a number that can be used in place of the variable to make the inequality true.

For example, 5 is a solution to the inequality $c < 10$, because it is true that $5 < 10$. Some other solutions to this inequality are 9.9, 0, and -4.

Practice

Exercise 6.3.2.4

The solution to $5 - 3x > 35$ is either $x > -10$ or $-10 > x$. Which solution is correct? Explain how you know.

Exercise 6.3.2.5

The school band director determined from past experience that if they charge t dollars for a ticket to the concert, they can expect attendance of $1000 - 50t$. The director used this model to figure out that the ticket price needs to be $8 or greater in order for at least 600 to attend. Do you agree with this claim? Why or why not?

Exercise 6.3.2.6

Which inequality is true when the value of x is -3?

A. $-x - 6 < -3.5$
B. $-x - 6 > 3.5$
C. $-x - 6 > -3.5$
D. $x - 6 > -3.5$

(From Unit 6.3.1)

Exercise 6.3.2.7

Draw the solution set for each of the following inequalities.

1. $x \leq 5$

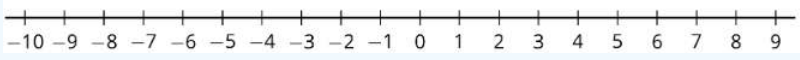


Figure 6.3.2.1

2. $x < \frac{5}{2}$

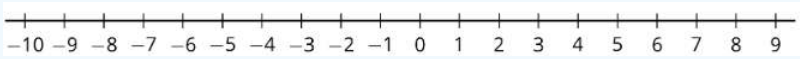


Figure 6.3.2.2

(From Unit 6.3.1)

Exercise 6.3.2.8

Write three different equations that match the tape diagram.

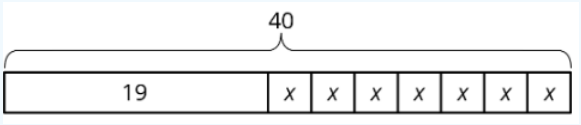


Figure 6.3.2.3

(From Unit 6.1.3)

Exercise 6.3.2.9

A baker wants to reduce the amount of sugar in his cake recipes. He decides to reduce the amount used in 1 cake by $\frac{1}{2}$ cup. He then uses $4\frac{1}{2}$ cups of sugar to bake 6 cakes.

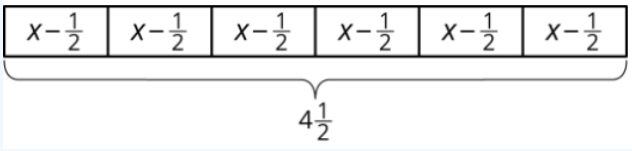


Figure 6.3.2.4

1. Describe how the tape diagram represents the story.
2. How much sugar was originally in each cake recipe?

(From Unit 6.1.2)

Exercise 6.3.2.10

One year ago, Clare was 4 feet 6 inches tall. Now Clare is 4 feet 10 inches tall. By what percentage did Clare's height increase in the last year?

(From Unit 4.3.3)

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6.3.3: Efficiently Solving Inequalities

Lesson

Let's solve more complicated inequalities.

Exercise 6.3.3.1: Lots of Negatives

Here is an inequality: $-x \geq -4$.

1. Predict what you think the solutions on the number line will look like.
2. Select **all** the values that are solutions to $-x \geq -4$:
   a. 3
   b. -3
   c. 4
   d. -4
   e. 4.001
   f. -4.001
3. Graph the solutions to the inequality on the number line:

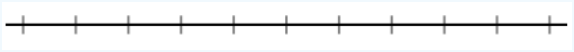


Figure 6.3.3.1

Exercise 6.3.3.2: Inequalities with Tables

1. Let's investigate the inequality $x - 3 > -2$.

Table 6.3.3.1

x	-4	-3	-2	-1	0	1	2	3	4
$x-3$	-7		-5				-1		1

   a. Complete the table.
   b. For which values of x is it true that $x - 3 = -2$?
   c. For which values of x is it true that $x - 3 > -2$?
   d. Graph the solutions to $x - 3 > -2$ on the number line:

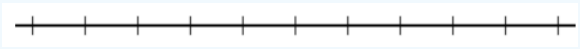


Figure 6.3.3.2

2. Here is an inequality: $2x < 6$.
   a. Predict which values of x will make the inequality $2x < 6$ true.
   b. Complete the table. Does it match your prediction?

Table 6.3.3.2

x	-4	-3	-2	-1	0	1	2	3	4
$2x$									

   c. Graph the solutions to $2x < 6$ on the number line:

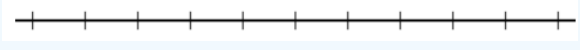


Figure 6.3.3.3

3. Here is an inequality: $-2x < 6$.
   a. Predict which values of x will make the inequality $-2x < 6$ true.
   b. Complete the table. Does it match your prediction?

Table 6.3.3.4

x	-4	-3	-2	-1	0	1	2	3	4
$-2x$									

c. Graph the solutions to $-2x < 6$ on the number line:

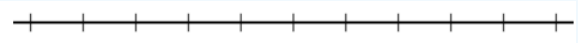


Figure 6.3.3.4

d. How are the solutions to $2x < 6$ different from the solutions to $-2x < 6$?

Exercise 6.3.3.3: Which Side are the Solutions?

1. Let's investigate $-4x + 5 \geq 25$.

   a. Solve $-4x + 5 = 25$.
   b. Is $-4x + 5 \geq 25$ true when x is 0? What about when x is 7? What about when x is -7?
   c. Graph the solutions to $-4x + 5 \geq 25$ on the number line.

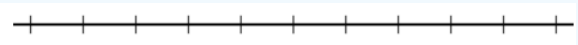


Figure 6.3.3.5

2. Let's investigate $\frac{4}{3}x + 3 < \frac{23}{3}$.

   a. Solve $\frac{4}{3}x + 3 = \frac{23}{3}$.
   b. Is $\frac{4}{3}x + 3 < \frac{23}{3}$ true when x is 0?
   c. Graph the solutions to $\frac{4}{3}x + 3 < \frac{23}{3}$ on the number line.

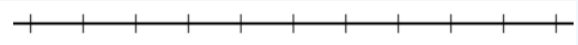


Figure 6.3.3.6

3. Solve the inequality $3(x + 4) > 17.4$ and graph the solutions on the number line.

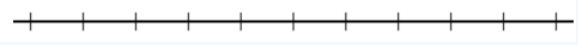


Figure 6.3.3.7

4. Solve the inequality $-3\left(x - \frac{4}{3}\right) \leq 6$ and graph the solutions on the number line.

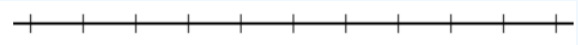


Figure 6.3.3.8

Are you ready for more?

Write at least three different inequalities whose solution is $x > -10$. Find one with x on the left side that uses a $<$.

Summary

Here is an inequality: $3(10 - 2x) < 18$. The solution to this inequality is all the values you could use in place of x to make the inequality true.

In order to solve this, we can first solve the related equation $3(10 - 2x) = 18$ to get the solution $x = 2$. That means 2 is the boundary between values of that make the inequality true and values that make the inequality false.

To solve the inequality, we can check numbers greater than 2 and less than 2 and see which ones make the inequality true.

Let's check a number that is greater than 2: $x = 5$. Replacing x with 5 in the inequality, we get $3(10 - 2 \cdot 5) < 18$ or just $0 < 18$. This is true, so $x = 5$ is a solution. This means that all values greater than 2 make the inequality true. We can write the solutions as $x > 2$ and also represent the solutions on a number line:

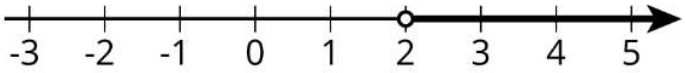


Figure 6.3.3.9

Notice that 2 itself is not a solution because it's the value of x that makes $3(10-2x)$ *equal* to 18, and so it does not make $3(10-2x)<18$ true.

For confirmation that we found the correct solution, we can also test a value that is less than 2. If we test $x=0$, we get $3(10-2\cdot 0)<18$ or just $30<18$. This is false, so $x=0$ and all values of x that are less than 2 are not solutions.

Glossary Entries

Definition: Solution to an Inequality

A solution to an inequality is a number that can be used in place of the variable to make the inequality true.

For example, 5 is a solution to the inequality $c<10$, because it is true that $5<10$. Some other solutions to this inequality are 9.9, 0, and -4.

Practice

Exercise 6.3.3.4

1. Consider the inequality $-1\leq \frac{x}{2}$.
   a. Predict which values of x will make the inequality true.
   b. Complete the table to check your prediction.

Table 6.3.3.5

x	-4	-3	-2	-1	0	1	2	3	4
$\frac{x}{2}$									

2. Consider the inequality $1\leq \frac{-x}{2}$.
   a. Predict which values of x will make it true.
   b. Complete the table to check your prediction.

Table 6.3.3.6

x	-4	-3	-2	-1	0	1	2	3	4
$-\frac{x}{2}$									

Exercise 6.3.3.5

Diego is solving the inequality $100-3x\geq -50$. He solves the equation $100-3x=-50$ and gets $x=50$. What is the solution to the inequality?

A. $x<50$
B. $x\leq 50$
C. $x>50$
D. $x\geq 50$

Exercise 6.3.3.6

Solve the inequality $-5(x-1)>-40$, and graph the solution on a number line.

Exercise 6.3.3.7

Select **all** values of x that make the inequality $-x+6 \geq 10$ true.

A. -3.9
B. 4
C. -4.01
D. -4
E. 4.01
F. 3.9
G. 0
H. -7

(From Unit 6.3.1)

Exercise 6.3.3.8

Draw the solution set for each of the following inequalities.

1. $x > 7$

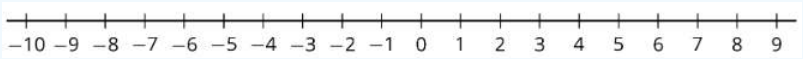


Figure 6.3.3.10

2. $x \geq -4.2$

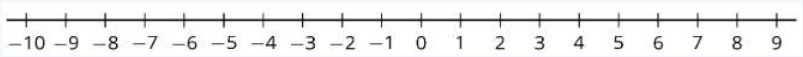


Figure 6.3.3.11

(From Unit 6.3.1)

Exercise 6.3.3.9

The price of a pair of earrings is $22 but Priya buys them on sale for $13.20.

1. By how much was the price discounted?
2. What was the percentage of the discount?

(From Unit 4.3.3)

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6.3.4: Interpreting Inequalities

Lesson

Let's write inequalities.

Exercise 6.3.4.1: Solve Some Inequalities!

For each inequality, find the value or values of x that make it true.

1. $8x + 21 \leq 56$
2. $56 < 7(7 - x)$

Exercise 6.3.4.2: Club Activities Matching

Choose the inequality that best matches each given situation. Explain your reasoning.

1. The Garden Club is planting fruit trees in their school's garden. There is one large tree that needs 5 pounds of fertilizer. The rest are newly planted trees that need $\frac{1}{2}$ pound fertilizer each.
   a. $25x + 5 \leq \frac{1}{2}$
   b. $\frac{1}{2}x + 5 \leq 25$
   c. $\frac{1}{2}x + 25 \leq 5$
   d. $5x + \frac{1}{2} \leq 25$
2. The Chemistry Club is experimenting with different mixtures of water with a certain chemical (sodium polyacrylate) to make fake snow.
   To make each mixture, the students start with some amount of water, and then add $\frac{1}{7}$ of that amount of the chemical, and then 9 more grams of the chemical. The chemical is expensive, so there can't be more than a certain number of grams of the chemical in any one mixture.
   a. $\frac{1}{7}x + 9 \leq 26.25$
   b. $9x + \frac{1}{7} \leq 26.25$
   c. $26.25x + 9 \leq \frac{1}{7}$
   d. $\frac{1}{7}x + 26.25 \leq 9$
3. The Hiking Club is on a hike down a cliff. They begin at an elevation of 12 feet and descend at the rate of 3 feet per minute.
   a. $37x - 3 \geq 12$
   b. $3x - 37 \geq 12$
   c. $12 - 3x \geq -37$
   d. $12x - 37 \geq -3$
4. The Science Club is researching boiling points. They learn that at high altitudes, water boils at lower temperatures. At sea level, water boils at 212°F. With each increase of 500 feet in elevation, the boiling point of water is lowered by about 1°F.
   a. $212 - \frac{1}{500}e < 195$
   b. $\frac{1}{500}e - 195 < 212$
   c. $195 - 212e < \frac{1}{500}$
   d. $212 - 195e < \frac{1}{500}$

Exercise 6.3.4.3: Club Activities Display

Your teacher will assign your group *one* of the situations from the last task. Create a visual display about your situation. In your display:

- Explain what the variable and each part of the inequality represent
- Write a question that can be answered by the solution to the inequality
- Show how you solved the inequality
- Explain what the solution means in terms of the situation

Are you ready for more?

$\{3, 4, 5, 6\}$is a set of four consecutive integers whose sum is 18.

1. How many sets of three consecutive integers are there whose sum is between 51 and 60? Can you be sure you've found them all? Explain or show your reasoning.
2. How many sets of four consecutive integers are there whose sum is between 59 and 82? Can you be sure you've found them all? Explain or show your reasoning.

Summary

We can represent and solve many real-world problems with inequalities. Writing the inequalities is very similar to writing equations to represent a situation. The expressions that make up the inequalities are the same as the ones we have seen in earlier lessons for equations. For inequalities, we also have to think about how expressions compare to each other, which one is bigger, and which one is smaller. Can they also be equal?

For example, a school fundraiser has a minimum target of $500. Faculty have donated $100 and there are 12 student clubs that are participating with different activities. How much money should each club raise to meet the fundraising goal? If n is the amount of money that each club raises, then the solution to $100 + 12n = 500$ is the minimum amount each club has to raise to meet the goal. It is more realistic, though, to use the inequality $100 + 12n \geq 500$ since the more money we raise, the more successful the fundraiser will be. There are many solutions because there are many different amounts of money the clubs could raise that would get us above our minimum goal of $500.

Glossary Entries

Definition: Solution to an Inequality

A solution to an inequality is a number that can be used in place of the variable to make the inequality true.

For example, 5 is a solution to the inequality $c < 10$, because it is true that $5 < 10$. Some other solutions to this inequality are 9.9, 0, and -4.

Practice

Exercise 6.3.4.4

Priya looks at the inequality $12 - x > 5$ and says "I subtract a number from 12 and want a result that is bigger than 5. That means that the solutions should be values of that are smaller than something."

Do you agree with Priya? Explain your reasoning and include solutions to the inequality in your explanation.

Exercise 6.3.4.5

When a store had sold $\frac{2}{5}$ of the shirts that were on display, they brought out another 30 from the stockroom. The store likes to keep at least 150 shirts on display. The manager wrote the inequality $\frac{3}{5}x + 30 \geq 150$ to describe the situation.

1. Explain what $\frac{3}{5}$ means in the inequality.
2. Solve the inequality.
3. Explain what the solution means in the situation.

Exercise 6.3.4.6

You know x is a number less than 4. Select **all** the inequalities that *must* be true.

A. $x < 2$
B. $x + 6 < 10$
C. $5x < 20$
D. $x - 2 > 2$
E. $x < 8$

(From Unit 6.3.1)

Exercise 6.3.4.7

Here is an unbalanced hanger.

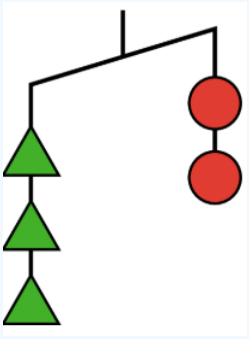


Figure 6.3.4.1

1. If you knew each circle weighed 6 grams, what would that tell you about the weight of each triangle? Explain your reasoning.
2. If you knew each triangle weighed 3 grams, what would that tell you about the weight of each circle? Explain your reasoning.

(From Unit 6.3.1)

Exercise 6.3.4.8

Match each sentence with the inequality that could represent the situation.

A. Han got \$2 from Clare, but still has less than \$20.
B. Mai spent \$2 and has less than \$20.
C. If Tyler had twice the amount of money he has, he would have less than \$20.
D. If Priya had half the money she has, she would have less than \$20.

1. $x - 2 < 20$
2. $2x < 20$
3. $x + 2 < 20$
4. $\frac{1}{2}x < 20$

(From Unit 6.3.1)

Exercise 6.3.4.9

At a skateboard shop:

1. The price tag on a shirt says \$12.58. Sales tax is 7.5% of the price. How much will you pay for the shirt?
2. The store buys a helmet for \$19.00 and sells it for \$31.50. What percentage was the markup?
3. The shop pays workers \$14.25 per hour plus 5.5% commission. If someone works 18 hours and sells \$250 worth of merchandise, what is the total amount of their paycheck for this pay period? Explain or show your reasoning.

(From Unit 4.3.3)

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6.3.5: Modeling with Inequalities

Lesson

Let's look at solutions to inequalities.

Exercise 6.3.5.1: Possible Values

The stage manager of the school musical is trying to figure out how many sandwiches he can order with the \$83 he collected from the cast and crew. Sandwiches cost \$5.99 each, so he lets x represent the number of sandwiches he will order and writes $5.99x \leq 83$. He solves this to 2 decimal places, getting $x \leq 13.86$.

Which of these are valid statements about this situation? (Select **all** that apply.)

1. He can call the sandwich shop and order exactly 13.86 sandwiches.
2. He can round up and order 14 sandwiches.
3. He can order 12 sandwiches.
4. He can order 9.5 sandwiches.
5. He can order 2 sandwiches.
6. He can order -4 sandwiches.

Exercise 6.3.5.2: Elevator

A mover is loading an elevator with many identical 48-pound boxes.
The mover weighs 185 pounds. The elevator can carry at most 2000 pounds.

1. Write an inequality that says that the mover will not overload the elevator on a particular ride. Check your inequality with your partner.
2. Solve your inequality and explain what the solution means.
3. Graph the solution to your inequality on a number line.
4. If the mover asked, "How many boxes can I load on this elevator at a time?" what would you tell them?

Exercise 6.3.5.3: Info Gap: Giving Advice

Your teacher will give you either a *problem card* or a *data card*. Do not show or read your card to your partner.

If your teacher gives you the *problem card*:

1. Silently read your card and think about what information you need to be able to answer the question.
2. Ask your partner for the specific information that you need.
3. Explain how you are using the information to solve the problem.
   Continue to ask questions until you have enough information to solve the problem.
4. Share the *problem card* and solve the problem independently.
5. Read the *data card* and discuss your reasoning.

If your teacher gives you the *data card*:

1. Silently read your card.
2. Ask your partner "*What specific information do you need?*" and wait for them to *ask* for information.
   If your partner asks for information that is not on the card, do not do the calculations for them. Tell them you don't have that information.
3. Before sharing the information, ask "*Why do you need that information?*"
   Listen to your partner's reasoning and ask clarifying questions.
4. Read the *problem card* and solve the problem independently.
5. Share the *data card* and discuss your reasoning.

Pause here so your teacher can review your work. Ask your teacher for a new set of cards and repeat the activity, trading roles with your partner.

Are you ready for more?

In a day care group, nine babies are five months old and 12 babies are seven months old. How many full months from now will the average age of the 21 babies first surpass 20 months old?

Summary

We can represent and solve many real-world problems with inequalities. Whenever we write an inequality, it is important to decide what quantity we are representing with a variable. After we make that decision, we can connect the quantities in the situation to write an expression, and finally, the whole inequality.

As we are solving the inequality or equation to answer a question, it is important to keep the meaning of each quantity in mind. This helps us to decide if the final answer makes sense in the context of the situation.

For example: Han has 50 centimeters of wire and wants to make a square picture frame with a loop to hang it that uses 3 centimeters for the loop. This situation can be represented by $3+4s=50$, where s is the length of each side (if we want to use all the wire). We can also use $3+4s \leq 50$ if we want to allow for solutions that don't use all the wire. In this case, any positive number that is less or equal to 11.75 cm is a solution to the inequality. Each solution represents a possible side length for the picture frame since Han can bend the wire at any point. In other situations, the variable may represent a quantity that increases by whole numbers, such as with numbers of magazines, loads of laundry, or students. In those cases, only whole-number solutions make sense.

Glossary Entries

Definition: Solution to an Inequality

A solution to an inequality is a number that can be used in place of the variable to make the inequality true.

For example, 5 is a solution to the inequality $c < 10$, because it is true that $5 < 10$. Some other solutions to this inequality are 9.9, 0, and -4.

Practice

Exercise 6.3.5.4

28 students travel on a field trip. They bring a van that can seat 12 students. Elena and Kiran's teacher asks other adults to drive cars that seat 3 children each to transport the rest of the students.

Elena wonders if she should use the inequality $12+3n > 28$ or $12+3n \geq 28$ to figure out how many cars are needed. Kiran doesn't think it matters in this case. Do you agree with Kiran? Explain your reasoning.

Exercise 6.3.5.5

1. In the cafeteria, there is one large 10-seat table and many smaller 4-seat tables. There are enough tables to fit 200 students. Write an inequality whose solution is the possible number of 4-seat tables in the cafeteria.
2. 5 barrels catch rainwater in the schoolyard. Four barrels are the same size, and the fifth barrel holds 10 liters of water. Combined, the 5 barrels can hold at least 200 liters of water. Write an inequality whose solution is the possible size of each of the 4 barrels.
3. How are these two problems similar? How are they different?

Exercise 6.3.5.6

Solve each equation.

1. $5(n-4)=-60$
2. $-3t+-8=25$
3. $7p-8=-22$
4. $\frac{2}{5}(j+40)=-4$
5. $4(w+1)=-6$

(From Unit 6.2.3)

Exercise 6.3.5.7

Select **all** the inequalities that have the same graph as $x < 4$.

A. $x < 2$
B. $x + 6 < 10$
C. $5x < 20$
D. $x - 2 > 2$
E. $x < 8$

(From Unit 6.3.1)

Exercise 6.3.5.8

A 200 pound person weighs 33 pounds on the Moon.

1. How much did the person's weight decrease?
2. By what percentage did the person's weight decrease?

(From Unit 4.3.3)

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6.4: Writing Equivalent Expressions

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6.4.1: Subtraction in Equivalent Expressions

Lesson

Let's find ways to work with subtraction in expressions.

Exercise 6.4.1.1: Number Talk: Additive Inverses

Find each sum or difference mentally.

$-30 + -10$

$-10 + -30$

$-30 - 10$

$10 - -30$

Exercise 6.4.1.2: A Helpful Observation

Lin and Kiran are trying to calculate $7\frac{3}{4} + 3\frac{5}{6} - 1\frac{3}{4}$. Here is their conversation:

Lin: "I plan to first add $7\frac{3}{4}$ and $3\frac{5}{6}$, so I will have to start by finding equivalent fractions with a common denominator."

Kiran: "It would be a lot easier if we could start by working with the $1\frac{3}{4}$ and $7\frac{3}{4}$. Can we rewrite it like $7\frac{3}{4} + 1\frac{3}{4} - 3\frac{5}{6}$?"

Lin: "You can't switch the order of numbers in a subtraction problem like you can with addition; $2 - 3$ is not equal to $3 - 2$."

Kiran: "That's true, but do you remember what we learned about rewriting subtraction expressions using addition? $2 - 3$ is equal to $2 + (-3)$."

1. Write an expression that is equivalent to that uses addition instead of subtraction.
2. If you wrote the **terms** of your new expression in a different order, would it still be equivalent? Explain your reasoning.

Exercise 6.4.1.3: Organizing Work

1. Write two expressions for the area of the big rectangle.

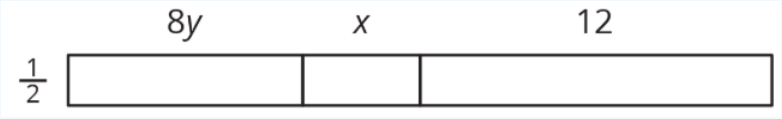


Figure 6.4.1.1

2. Use the distributive property to write an expression that is equivalent to $\frac{1}{2}(8y + -x + -12)$. The boxes can help you organize your work.

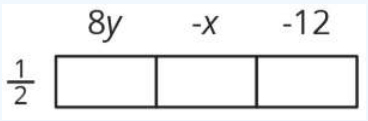


Figure 6.4.1.2

3. Use the distributive property to write an expression that is equivalent to $\frac{1}{2}(8y - x - 12)$.

Are you ready for more?

Here is a calendar for April 2017.

April 2017						
Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
						1
2	3	4	5	6	7	8
9	10	11	12	13	14	15
16	17	18	19	20	21	22
23	24	25	26	27	28	29
30						

Figure 6.4.1.3

Let's choose a date: the 10th. Look at the numbers above, below, and to either side of the 10th: 3, 17, 9, 11.

1. Average these four numbers. What do you notice?
2. Choose a different date that is in a location where it has a date above, below, and to either side. Average these four numbers. What do you notice?
3. Explain why the same thing will happen for any date in a location where it has a date above, below, and to either side.

Summary

Working with subtraction and signed numbers can sometimes get tricky. We can apply what we know about the relationship between addition and subtraction—that subtracting a number gives the same result as adding its opposite—to our work with expressions. Then, we can make use of the properties of addition that allow us to add and group in any order. This can make calculations simpler. For example:

$$\frac{5}{8}-\frac{2}{3}-\frac{1}{8}$$
$$\frac{5}{8}+-\frac{2}{3}+-\frac{1}{8}$$
$$\frac{5}{8}+-\frac{1}{8}+-\frac{2}{3}$$
$$\frac{4}{8}+-\frac{2}{3}$$

We can also organize the work of multiplying signed numbers in expressions. The product $\frac{3}{2}(6y-2x-8)$ can be found by drawing a rectangle with the first factor, $\frac{3}{2}$, on one side, and the three terms inside the parentheses on the other side:

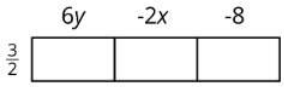


Figure 6.4.1.4

Multiply $\frac{3}{2}$ by each term across the top and perform the multiplications:

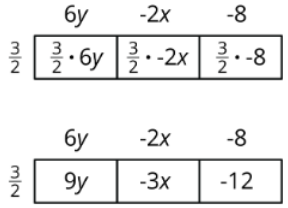


Figure 6.4.1.5: Two area diagrams. First diagram, one row, 3 columns. To the left of the row, 3 over 2. Starting with the first column, 6 y, negative 2 x, negative 8. Inside the boxes, 3 over 2 times 6 y, 3 over 2 times negative 2 x, 3 over 2 times negative 8. Second diagram, 1 row, 3 columns. To the left of the row, 3 over 2. Starting with the first column, 6 y, negative 2 x, negative 8. Inside the boxes, 9 y, negative 3 x, negative 12.

Reassemble the parts to get the expanded version of the original expression:

$$\frac{3}{2}(6y-2x-8)=9y-3x-12$$

Glossary Entries

Definition: Term

A term is a part of an expression. It can be a single number, a variable, or a number and a variable that are multiplied together. For example, the expression $5x + 18$ has two terms. The first term is $5x$ and the second term is 18.

Practice

Exercise 6.4.1.4

For each expression, write an equivalent expression that uses only addition.

1. $20 - 9 + 8 - 7$
2. $4x - 7y - 5x + 6$
3. $-3x - 8y - 4 - \frac{8}{7}z$

Exercise 6.4.1.5

Use the distributive property to write an expression that is equivalent to each expression. If you get stuck, consider drawing boxes to help organize your work.

1. $9(4x - 3y - \frac{2}{3})$
2. $-2(-6x + 3y - 1)$
3. $\frac{1}{5}(20y - 4x - 13)$
4. $8(-x - \frac{1}{2})$
5. $-8(-x - \frac{3}{4}y + \frac{7}{2})$

Exercise 6.4.1.6

Kiran wrote the expression $x - 10$ for this number puzzle: "Pick a number, add -2, and multiply by 5."

Lin thinks Kiran made a mistake.

1. How can she convince Kiran he made a mistake?
2. What would be a correct expression for this number puzzle?

Exercise 6.4.1.7

The output from a coal power plant is shown in the table:

Table 6.4.1.1

energy in megawatts	number of days
1, 200	2.4
1, 800	3.6
4, 000	8
10, 000	20

Similarly, the output from a solar power plant is shown in the table:

Table 6.4.1.2

energy in megawatts	number of days
100	1
650	4
1, 200	7

energy in megawatts	number of days
1, 750	10

Based on the tables, is the energy output in proportion to the number of days for either plant? If so, write an equation showing the relationship. If not, explain your reasoning.

(From Unit 2.3.1)

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6.4.2: Expanding and Factoring

Lesson

Let's use the distributive property to write expressions in different ways.

Exercise 6.4.2.1: Number Talk: Parentheses

Find the value of each expression mentally.

$2+3\cdot 4$

$(2+3)(4)$

$2-3\cdot 4$

$2-(3+4)$

Exercise 6.4.2.2: Factoring and Expanding with Negative Numbers

In each row, write the equivalent expression. If you get stuck, use a diagram to organize your work. The first row is provided as an example. Diagrams are provided for the first three rows.

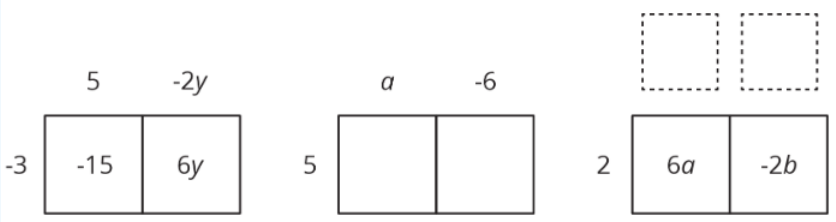


Figure 6.4.2.1: Three area diagrams, each 1 row, 2 columns. First diagram, row labeled negative 3, columns labeled 5 and negative 2y. Second diagram, row labeled 5, columns labeled a and negative 6. Third diagram, row labeled 2, boxes labeled 6 a and negative 2 b, blanks above the columns indicated missing labels.

Table 6.4.2.1

factored	expanded
$-3(5-2y)$	$-15+6y$
$5(a-6)$	
	$6a-2b$
$-4(2w-5z)$	
$-2(2x-3y)$	
	$20x-10y+15z$
$k(4-17)$	
	$10a-13a$
$-2x(3y-z)$	
	$ab-bc-3bd$
$-x(3y-z+4w)$	

Are you ready for more?

Expand to create an equivalent expression that uses the fewest number of terms: $((((x+1)\frac{1}{2})+1)\frac{1}{2})+1$. If we wrote a new expression following the same pattern so that there were 20 sets of parentheses, how could it be expanded into an equivalent expression that uses the fewest number of terms?

Summary

We can use properties of operations in different ways to rewrite expressions and create equivalent expressions. We have already seen that we can use the distributive property to **expand** an expression, for example $3(x+5)=3x+15$. We can also use the distributive property in the other direction and **factor** an expression, for example $8x+12=4(2x+3)$.

We can organize the work of using distributive property to rewrite the expression $12x-8$. In this case we know the product and need to find the factors.

The terms of the product go inside:

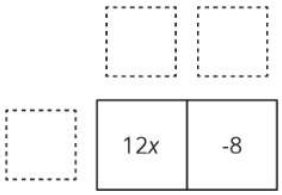


Figure 6.4.2.2

We look at the expressions and think about a factor they have in common. $12x$ and -8 each have a factor of 4. We place the common factor on one side of the large rectangle:

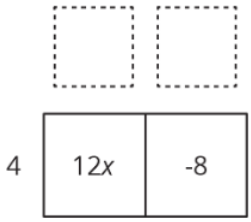


Figure 6.4.2.3

Now we think: "4 times *what* is $12x$?" "4 times *what* is -8?" and write the other factors on the other side of the rectangle:

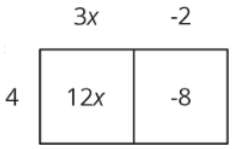


Figure 6.4.2.4

So, $12x-8$ is equivalent to $4(3x-2)$.

Glossary Entries

Definition: Expand

To expand an expression, we use the distributive property to rewrite a product as a sum. The new expression is equivalent to the original expression.

For example, we can expand the expression $5(4x+7)$ to get the equivalent expression $20x+25$.

Definition: Factor (an expression)

To factor an expression, we use the distributive property to rewrite a sum as a product. The new expression is equivalent to the original expression.

For example, we can factor the expression $20x+35$ to get the equivalent expression $5(4x+7)$.

Definition: Term

A term is a part of an expression. It can be a single number, a variable, or a number and a variable that are multiplied together. For example, the expression $5x+18$ has two terms. The first term is $5x$ and the second term is 18.

Practice

Exercise 6.4.2.3

1. Expand to write an equivalent expression: $\frac{-1}{4}(-8x+12y)$
2. Factor to write an equivalent expression: $36a-16$

Exercise 6.4.2.4

Lin missed math class on the day they worked on expanding and factoring. Kiran is helping Lin catch up.

1. Lin understands that expanding is using the distributive property, but she doesn't understand what factoring is or why it works. How can Kiran explain factoring to Lin?
2. Lin asks Kiran how the diagrams with boxes help with factoring. What should Kiran tell Lin about the boxes?
3. Lin asks Kiran to help her factor the expression $-4xy-12xz+20xw$. How can Kiran use this example to Lin understand factoring?

Exercise 6.4.2.5

Complete the equation with numbers that makes the expression on the right side of the equal sign equivalent to the expression on the left side.

$$75a+25b=\_\_(\_\_a+b)$$

Exercise 6.4.2.6

Elena makes her favorite shade of purple paint by mixing 3 cups of blue paint, $1\frac{1}{2}$ cups of red paint, and $\frac{1}{2}$ of a cup of white paint. Elena has $\frac{2}{3}$ of a cup of white paint.

1. Assuming she has enough red paint and blue paint, how much purple paint can Elena make?
2. How much blue paint and red paint will Elena need to use with the $\frac{2}{3}$ of a cup of white paint?

(From Unit 4.1.3)

Exercise 6.4.2.7

Solve each equation.

1. $\frac{-1}{8}d-4=\frac{-3}{8}$
2. $\frac{-1}{4}m+5=16$
3. $10b+-45=-43$
4. $-8(y-1.25)=4$
5. $3.2(s+10)=32$

(From Unit 6.2.3)

Exercise 6.4.2.8

Select **all** the inequalities that have the same solutions as $-4x<20$.

A. $-x<5$
B. $4x>-20$
C. $\$4x$
D. $\$x$
E. $x>5$
F. $x>-5$

(From Unit 6.3.1)

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6.4.3: Combining Like Terms (Part 1)

Lesson

Let's see how we can tell that expressions are equivalent.

Exercise 6.4.3.1: Why is it True?

Explain why each statement is true.

1. $5+2+3=5+(2+3)$
2. $9a$ is equivalent to $11a-2a$.
3. $7a+4-2a$ is equivalent to $7a+-2a+4$.
4. $8a-(8a-8)$ is equivalent to 8.

Exercise 6.4.3.2: A's and B's

Diego and Jada are both trying to write an expression with fewer terms that is equivalent to $7a+5b-3a+4b$

- Jada thinks $10a+1b$ is equivalent to the original expression.
- Diego thinks $4a+9b$ is equivalent to the original expression.

1. We can show expressions are equivalent by writing out all the variables. Explain why the expression on each row (after the first row) is equivalent to the expression on the row before it.
   $7a+5b-3a+4b$
   $(a+a+a+a+a+a+a)+(b+b+b+b+b)-(a+a+a)+(b+b+b+b)$
   $(a+a+a+a)+(a+a+a)+(b+b+b+b+b)-(a+a+a)+(b+b+b+b)$
   $(a+a+a+a)+(b+b+b+b+b)+(a+a+a)-(a+a+a)+(b+b+b+b)$
   $(a+a+a+a)+(b+b+b+b+b)+(b+b+b+b)$
   $(a+a+a+a)+(b+b+b+b+b+b+b+b+b)$
   $4a+9b$
2. Here is another way we can rewrite the expressions. Explain why the expression on each row (after the first row) is equivalent to the expression on the row before it.
   $7a+5b-3a+4b$
   $7a+5b+(-3a)+4b$
   $ya+(-3a)+5b+4b$
   $(7+-3)a+(5+4)b$
   $4a+9b$

Are you ready for more?

Follow the instructions for a number puzzle:

- Take the number formed by the first 3 digits of your phone number and multiply it by 40
- Add 1 to the result
- Multiply by 500
- Add the number formed by the last 4 digits of your phone number, and then add it again
- Subtract 500
- Multiply by $\frac{1}{2}$

1. What is the final number?
2. How does this number puzzle work?
3. Can you invent a new number puzzle that gives a surprising result?

Exercise 6.4.3.3: Making Sides Equal

Replace each ? with an expression that will make the left side of the equation equivalent to the right side.

Set A

1. $6x+?=10x$
2. $6x+?=2x$
3. $6x+?=-10x$
4. $6x+?=0$
5. $6x+?=10$

Check your results with your partner and resolve any disagreements. Next move on to Set B.

Set B

1. $6x-?=2x$
2. $6x-?=10x$
3. $6x-?=x$
4. $6x-?=6$
5. $6x-?=4x-10$

Summary

There are many ways to write equivalent expressions that may look very different from each other. We have several tools to find out if two expressions are equivalent.

- Two expressions are definitely not equivalent if they have different values when we substitute the same number for the variable. For example, $2(-3+x)+8$ and $2x+5$ are not equivalent because when x is 1, the first expression equals 4 and the second expression equals 7.
- If two expressions are equal for many different values we substitute for the variable, then the expressions *may* be equivalent, but we don't know for sure. It is impossible to compare the two expressions for all values. To know for sure, we use properties of operations. For example, $2(-3+x)+8$ is equivalent to $2x+2$ because:

$$\begin{aligned} &2(-3+x)+8 && \\ &-6+2x+8 && \text{by the distributive property} \\ &2x+-6+8 && \text{by the commutative property} \\ &2x+(-6+8) && \text{by the associative property} \\ &2x+2 && \end{aligned}$$

Glossary Entries

Definition: Expand

To expand an expression, we use the distributive property to rewrite a product as a sum. The new expression is equivalent to the original expression.

For example, we can expand the expression $5(4x+7)$ to get the equivalent expression $20x+25$.

Definition: Factor (an expression)

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Definition: Term

A term is a part of an expression. It can be a single number, a variable, or a number and a variable that are multiplied together. For example, the expression $5x+18$ has two terms. The first term is $5x$ and the second term is 18.

Practice

Exercise 6.4.3.4

Andre says that $10x+6$ and $5x+11$ are equivalent because they both equal 16 when x is 1. Do you agree with Andre? Explain your reasoning.

Exercise 6.4.3.5

Select **all** expressions that can be subtracted from $9x$ to result in the expression $3x+5$.

A. $-5+6x$
B. $5-6x$
C. $6x+5$
D. $6x-5$
E. $-6x+5$

Exercise 6.4.3.6

Select **all** the statements that are true for any value of x.

A. $7x+(2x+7)=9x+7$
B. $7x+(2x-1)=9x+1$
C. $\frac{1}{2}x+(3-\frac{1}{2}x)=3$
D. $5x-(8-6x)=-x-8$
E. $0.4x-(0.2x+8)=0.2x-8$
F. $6x-(2x-4)=4x+4$

Exercise 6.4.3.7

For each situation, would you describe it with $x<25$, $x>25$, $x\leq 25$, or $x\geq 25$?

1. The library is having a party for any student who read at least 25 books over the summer. Priya read x books and was invited to the party.
2. Kiran read x books over the summer but was not invited to the party.
3.

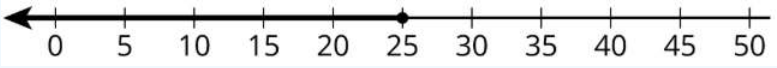


Figure 6.4.3.1

4.

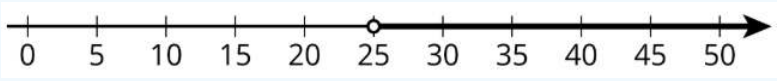


Figure 6.4.3.2

(From Unit 6.3.1)

Exercise 6.4.3.8

Consider the problem: A water bucket is being filled with water from a water faucet at a constant rate. When will the bucket be full? What information would you need to be able to solve the problem?

(From Unit 2.3.3)

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6.4.4: Combining Like Terms (Part 2)

Lesson

Let's see how to use properties correctly to write equivalent expressions.

Exercise 6.4.4.1: True or False?

Select **all** the statements that are true. Be prepared to explain your reasoning.

1. $4-2(3+7)=4-2\cdot 3-2\cdot 7$
2. $4-2(3+7)=4-2\cdot 3+-2\cdot 7$
3. $4-2(3+7)=4-2\cdot 3+2\cdot 7$
4. $4-2(3+7)=4-(2\cdot 3+2\cdot 7)$

Exercise 6.4.4.2: Seeing it Differently

Some students are trying to write an expression with fewer terms that is equivalent to

$8-3(4-9x)$

Noah says, “I worked the problem from left to right and ended up with $20-45x$."

$8-3(4-9x)$

$5(4-9x)$

$20-45x$

Lin says, "I started inside the parentheses and ended up with $23x$."

$8-3(4-9x)$

$8-3(-5x)$

$8+15x$

$23x$

Jada says, “I used the distributive property and ended up with $27x-4$.”

$8-3(4-9x)$

$8-(12-27x)$

$8-12-(-27x)$

$27x-4$

Andre says, “I also used the distributive property, but I ended up with $-4-27x$."

$8-3(4-9x)$

$8-12-27x$

$-4-27x$

1. Do you agree with any of them? Explain your reasoning.
2. For each strategy that you disagree with, find and describe the errors.

Are you ready for more?

1. Jada’s neighbor said, “My age is the difference between twice my age in 4 years and twice my age 4 years ago.” How old is Jada’s neighbor?
2. Another neighbor said, “My age is the difference between twice my age in 5 years and and twice my age 5 years ago.” How old is this neighbor?

3. A third neighbor had the same claim for 17 years from now and 17 years ago, and a fourth for 21 years. Determine those neighbors' ages.

Exercise 6.4.4.3: Grouping Differently

Diego was taking a math quiz. There was a question on the quiz that had the expression $8x - 9 - 12x + 5$. Diego's teacher told the class there was a typo and the expression was supposed to have one set of parentheses in it.

1. Where could you put parentheses in $8x - 9 - 12x + 5$ to make a new expression that is still equivalent to the original expression? How do you know that your new expression is equivalent?
2. Where could you put parentheses in $8x - 9 - 12x + 5$ to make a new expression that is not equivalent to the original expression? List as many different answers as you can.

Summary

Combining like terms allows us to write expressions more simply with fewer terms. But it can sometimes be tricky with long expressions, parentheses, and negatives. It is helpful to think about some common errors that we can be aware of and try to avoid:

- $6x - x$ is not equivalent to 6. While it might be tempting to think that subtracting x makes the x disappear, the expression is really saying take $1x$ away from $6x$'s, and the distributive property tells us that $6x - x$ is equivalent to $(6 - 1)x$.
- $7 - 2x$ is not equivalent to $5x$. The expression $7 - 2x$ tells us to double an unknown amount and subtract it from 7. This is not always the same as taking 5 copies of the unknown.
- $7 - 4(x + 2)$ is not equivalent to $3(x + 2)$. The expression tells us to subtract 4 copies of an amount from 7, not to take $(7 - 4)$ copies of the amount.

If we think about the meaning and properties of operations when we take steps to rewrite expressions, we can be sure we are getting equivalent expressions and are not changing their value in the process.

Glossary Entries

Definition: Expand

To expand an expression, we use the distributive property to rewrite a product as a sum. The new expression is equivalent to the original expression.

For example, we can expand the expression $5(4x + 7)$ to get the equivalent expression $20x + 35$.

Definition: Factor (an expression)

To factor an expression, we use the distributive property to rewrite a sum as a product. The new expression is equivalent to the original expression.

For example, we can factor the expression $20x + 35$ to get the equivalent expression $5(4x + 7)$.

Definition: Term

A term is a part of an expression. It can be a single number, a variable, or a number and a variable that are multiplied together. For example, the expression $5x + 18$ has two terms. The first term is $5x$ and the second term is 18.

Practice

Exercise 6.4.4.4

- Noah says that $9x - 2x + 4x$ is equivalent to $3x$, because the subtraction sign tells us to subtract everything that comes after $9x$.
- Elena says that $9x - 2x + 4x$ is equivalent to $11x$, because the subtraction only applies to $2x$.

Do you agree with either of them? Explain your reasoning.

Exercise 6.4.4.5

Identify the error in generating an expression equivalent to $4+2x-\frac{1}{2}(10-4x)$. Then correct the error.

$4+2x+\frac{-1}{2}(10+-4x)$

$4+2x+-5+2x$

$4+2x-5+2x$

-1

Exercise 6.4.4.6

Select **all** expressions that are equivalent to $5x-15-20x+10$.

A. $5x-(15+20x)+10$
B. $5x+-15+-20x+10$
C. $5(x-3-4x+2)$
D. $-5(-x+3+4x+-2)$
E. $-15x-5$
F. $-5(3x+1)$
G. $-15(x-\frac{1}{3})$

Exercise 6.4.4.7

The school marching band has a budget of up to $750 to cover 15 new uniforms and competition fees that total $300. How much can they spend for one uniform?

1. Write an inequality to represent this situation.
2. Solve the inequality and describe what it means in the situation.

(From Unit 6.3.2)

Exercise 6.4.4.8

Solve the inequality that represents each story. Then interpret what the solution means in the story.

1. For every $9 that Elena earns, she gives x dollars to charity. This happens 7 times this month. Elena wants to be sure she keeps at least $42 from this month's earnings. $7(9-x) \geq 42$
2. Lin buys a candle that is 9 inches tall and burns down x inches per minute. She wants to let the candle burn for 7 minutes until it is less than 6 inches tall. $9-7x < 6$

(From Unit 6.3.4)

Exercise 6.4.4.9

A certain shade of blue paint is made by mixing $1\frac{1}{2}$ quarts of blue paint with 5 quarts of white paint. If you need a total of 16.25 gallons of this shade of blue paint, how much of each color should you mix?

(From Unit 4.1.3)

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6.4.5: Combining Like Terms (Part 3)

Lesson

Let's see how we can combine terms in an expression to write it with less terms.

Exercise 6.4.5.1: Are They Equal?

Select **all** expressions that are equal to $8-12-(6+4)$.

1. $8-6-12+4$
2. $8-12-6-4$
3. $8-12+(6+4)$
4. $8-12-6+4$
5. $8-4-12-6$

Exercise 6.4.5.2: X's and Y's

Match each expression in column A with an equivalent expression from column B. Be prepared to explain your reasoning.

A

1. $(9x+5y)+(3x+7y)$
2. $(9x+5y)-(3x+7y)$
3. $(9x+5y)-(3x-7y)$
4. $9x-7y+3x+5y$
5. $9x-7y+3x-5y$
6. $9x7y-3x-5y$

B

1. $12(x+y)$
2. $12(x-y)$
3. $6(x-2y)$
4. $9x+5y+3x-7y$
5. $9x+5y-3x+7y$
6. $9x-3x+5y-7y$

Exercise 6.4.5.3: Seeing Structure and Factoring

Write each expression with fewer terms. Show or explain your reasoning.

1. $3\cdot 15+4\cdot 15-5\cdot 15$
2. $3x+4x-5x$
3. $3(x-2)+4(x-2)-5(x-2)$
4. $3(\frac{5}{2}x+6\frac{1}{2})+4(\frac{5}{2}x+5\frac{1}{2})-5(\frac{5}{2}x+6\frac{1}{2})$

Summary

Combining like terms is a useful strategy that we will see again and again in our future work with mathematical expressions. It is helpful to review the things we have learned about this important concept.

- Combining like terms is an application of the distributive property. For example:

$$2x+9x$$
$$(2+9)\cdot x$$
$$11x$$

- It often also involves the commutative and associative properties to change the order or grouping of addition. For example:

$$2a+3b+4a+5b$$
$$2a+4a+3b+5b$$
$$(2a+4a)+(3b+5b)$$
$$6a+8b$$

- We can't change order or grouping when subtracting; so in order to apply the commutative or associative properties to expressions with subtraction, we need to rewrite subtraction as addition. For example:

$$2a-3b-4a-5b$$
$$2a+-3b+-4a+-5b$$
$$2a+-4a+-3b+-5b$$
$$-2a+-8b$$
$$-2a-8b$$

- Since combining like terms uses properties of operations, it results in expressions that are equivalent.
- The like terms that are combined do not have to be a single number or variable; they may be longer expressions as well. Terms can be combined in any sum where there is a common factor in all the terms. For example, each term in the expression $5(x+3)-0.5(x+3)+2(x+3)$ has a factor of $(x+3)$. We can rewrite the expression with fewer terms by using the distributive property:

$$5(x+3)-0.5(x+3)+2(x+3)$$
$$(5-0.5+2)(x+3)$$
$$6.5(x+3)$$

Glossary Entries

Definition: Expand

To expand an expression, we use the distributive property to rewrite a product as a sum. The new expression is equivalent to the original expression.

For example, we can expand the expression $5(4x+7)$ to get the equivalent expression $20x+35$.

Definition: Factor (an expression)

To factor an expression, we use the distributive property to rewrite a sum as a product. The new expression is equivalent to the original expression.

For example, we can factor the expression $20x+35$ to get the equivalent expression $5(4x+7)$.

Definition: Term

A term is a part of an expression. It can be a single number, a variable, or a number and a variable that are multiplied together. For example, the expression $5x+18$ has two terms. The first term is $5x$ and the second term is 18.

Practice

Exercise 6.4.5.4

Jada says, "I can tell that $\frac{-2}{3}(x+5)+4(x=5)-\frac{10}{3}(x+5)$ equals 0 just by looking at it." Is Jada correct? Explain how you know.

Exercise 6.4.5.5

In each row, decide whether the expression in column A is equivalent to the expression in column B. If they are not equivalent, show how to change one expression to make them equivalent.

A

1. $3x-2x+0.5x$
2. $3(x+4)-2(x+4)$
3. $6(x+4)-2(x+5)$
4. $3(x+4)-2(x+4)+0.5(x+4)$
5. $20(\frac{2}{5}x+\frac{3}{4}y-\frac{1}{2})$

B

1. $1.5x$
2. $x+3$
3. $2(2x+7)$
4. $2(2x+7)$
5. 1.5
6. $\frac{1}{2}(16x+30y-20)$

Exercise 6.4.5.6

For each situation, write an expression for the new balance using as few terms as possible.

1. A checking account has a balance of -$126.89. A customer makes two deposits, one $3\frac{1}{2}$ times the other, and then withdraws $25.
2. A checking account has a balance of $350. A customer makes two withdrawals, one $50 more than the other. Then he makes a deposit of $75.

(From Unit 6.4.3)

Exercise 6.4.5.7

Tyler is using the distributive property on the expression $9-4(5x-6)$. Here is his work:

$9-4(5x-6)$

$9+(-4)(5x+-6)$

$9+-20x+-6$

$3-20x$

Mai thinks Tyler's answer is incorrect. She says, "If expressions are equivalent then they are equal for any value of the variable. Why don't you try to substitute the same value for x in all the equations and see where they are not equal?"

1. Find the step where Tyler made an error.
2. Explain what he did wrong.
3. Correct Tyler's work.

(From Unit 6.4.4)

Exercise 6.4.5.8

1. If $(11+x)$ is positive, but $(4+x)$ is negative, what is one number that x could be?
2. If $(-3+y)$ is positive, but $(-9+y)$ is negative, what is one number that y could be?
3. If $(-5+z)$ is positive, but $(-6+z)$ is negative, what is one number that z could be?

(From Unit 6.3.1)

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6.5: Let's Put it to Work

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6.5.1: Applications of Expressions

Lesson

Let's use expressions to solve problems.

Exercise 6.5.1.1: Algebra Talk: Equivalent to $0.75t - 21$

Decide whether each expression is equivalent to $0.75t - 21$. Be prepared to explain how you know.

$\frac{3}{4}t - 21$

$\frac{3}{4}(t - 21)$

$0.75(t - 28)$

$t - 0.25t - 21$

Exercise 6.5.1.2: Two Ways to Calculate

Usually when you want to calculate something, there is more than one way to do it. For one or more of these situations, show how the two different ways of calculating are equivalent to each other.

1. Estimating the temperature in Fahrenheit when you know the temperature in Celsius
   a. Double the temperature in Celsius, then add 30.
   b. Add 15 to the temperature in Celsius, then double the result.
2. Calculating a 15% tip on a restaurant bill
   a. Take 10% of the bill amount, take 5% of the bill amount, and add those two values together.
   b. Multiply the bill amount by 3, divide the result by 2, and then take $\frac{1}{10}$ of that result.
3. Changing a distance in miles to a distance in kilometers
   a. Take the number of miles, double it, then decrease the result by 20%.
   b. Divide the number of miles by 5, then multiply the result by 8.

Exercise 6.5.1.3: Which Way?

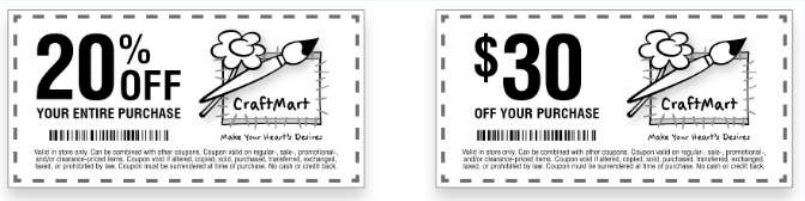


Figure 6.5.1.1

You have two coupons to the same store: one for 20% off and one for $30 off. The cashier will let you use them both, and will let you decide in which order to use them.

- Mai says that it doesn't matter in which order you use them. You will get the same discount either way.
- Jada says that you should apply the 20% off coupon first, and then the $30 off coupon.
- Han says that you should apply the $30 off coupon first, and then the 20% off coupon.
- Kiran says that it depends on how much you are spending.

Do you agree with any of them? Explain your reasoning.

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CHAPTER OVERVIEW

7: Angles, Triangles, and Prisms

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7.1.1: Relationships of Angles

7.1.2: Adjacent Angles

7.1.3: Nonadjacent Angles

7.1.4: Solving for Unknown Angles

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7.4.1: Building Prisms

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7.1: Angle Relationships

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7.1.1: Relationships of Angles

Lesson

Let's examine some special angles.

Exercise 7.1.1.1: Visualizing Angles

Use the applet to answer the questions.

1. Which angle is bigger, a or b?
2. Identify an obtuse angle in the diagram.

Exercise 7.1.1.2: Pattern Block Angles

1. Look at the different pattern blocks inside the applet. Each block contains either 1 or 2 angles with different degree measures. Which blocks have only 1 unique angle? Which have 2?

2. If you place three copies of the hexagon together so that one vertex from each hexagon touches the same point, as shown, they fit together without any gaps or overlaps. Use this to figure out the degree measure of the angle inside the hexagon pattern block.

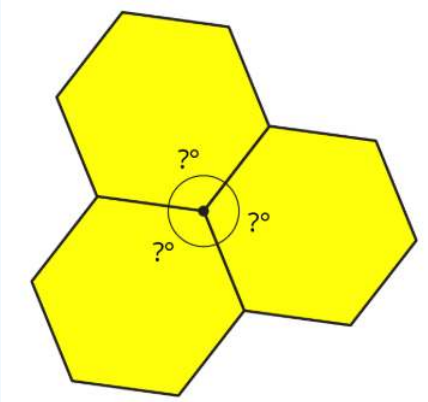


Figure 7.1.1.1

3. Figure out the degree measure of all of the other angles inside the pattern blocks. (Hint: turn on the grid to help align the pieces.) Be prepared to explain your reasoning.

Are you ready for more?

We saw that it is possible to fit three copies of a regular hexagon snugly around a point.

Each interior angle of a regular pentagon measures 108°. Is it possible to fit copies of a regular pentagon snugly around a point? If yes, how many copies does it take? If not, why not?

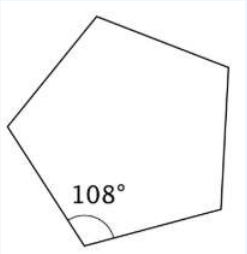


Figure 7.1.1.2

Exercise 7.1.1.3: More Pattern Block Angles

1. Use pattern blocks to determine the measure of each of these angles.

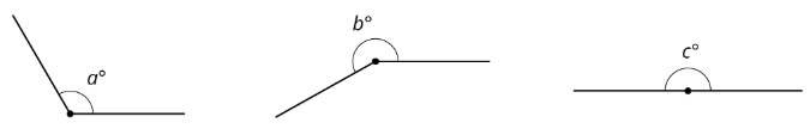


Figure 7.1.1.3

2. If an angle has a measure of 180° then the two legs form a straight line. An angle that forms a straight line is called a straight angle. Find as many different combinations of pattern blocks as you can that make a straight angle.

Use the applet if you choose. (Hint: turn on the grid to help align the pieces.)

Exercise 7.1.1.4: Measuring Like This or That

Tyler and Priya were both measuring angle TUS.

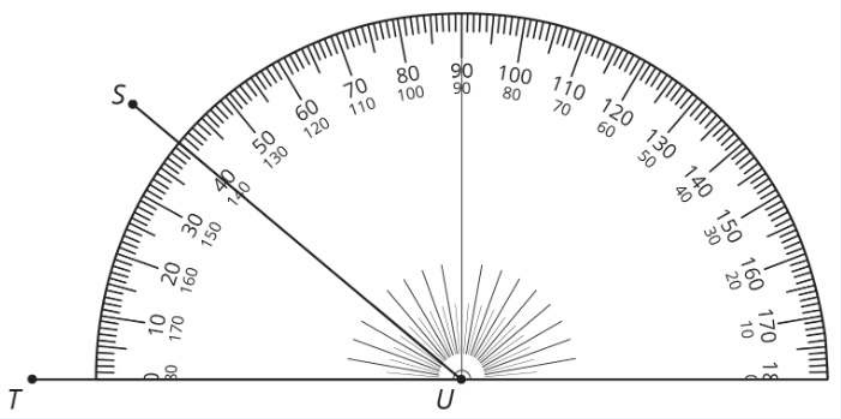


Figure 7.1.1.4

Priya thinks the angle measures 40 degrees. Tyler thinks the angle measures 140 degrees. Do you agree with either of them? Explain your reasoning.

Summary

When two lines intersect and form four equal angles, we call each one a **right angle**. A right angle measures 90°. You can think of a right angle as a quarter turn in one direction or the other.

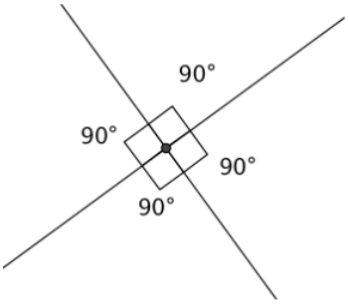


Figure 7.1.1.5

An angle in which the two sides form a straight line is called a **straight angle**. A straight angle measures 180°. A straight angle can be made by putting right angles together. You can think of a straight angle as a half turn, so that you are facing in the opposite direction after you are done.

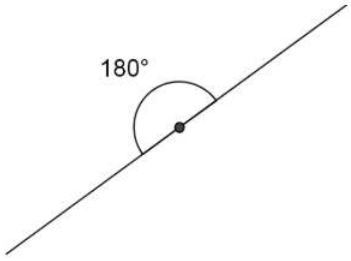


Figure 7.1.1.6

If you put two straight angles together, you get an angle that is 360°. You can think of this angle as turning all the way around so that you are facing the same direction as when you started the turn.

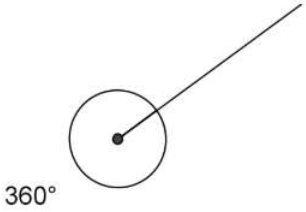


Figure 7.1.1.7

When two angles share a side and a vertex, and they don't overlap, we call them **adjacent angles**.

Glossary Entries

Definition: Adjacent Angles

Adjacent angles share a side and a vertex.

In this diagram, angle ABC is adjacent to angle DBC.

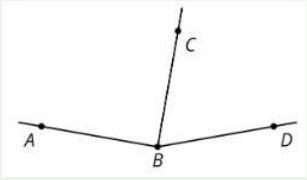


Figure 7.1.1.8

Definition: Right Angle

A right angle is half of a straight angle. It measures 90 degrees.

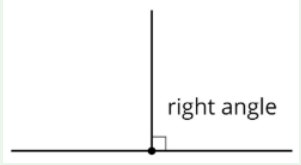


Figure 7.1.1.9

Definition: Straight Angle

A straight angle is an angle that forms a straight line. It measures 180 degrees.

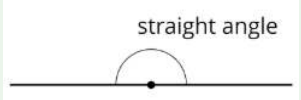


Figure 7.1.1.10

Practice

Exercise 7.1.1.5

Here are questions about two types of angles.

1. Draw a right angle. How do you know it's a right angle? What is its measure in degrees?
2. Draw a straight angle. How do you know it's a straight angle? What is its measure in degrees?

Exercise 7.1.1.6

An equilateral triangle's angles each have a measure of 60 degrees.

1. Can you put copies of an equilateral triangle together to form a straight angle? Explain or show your reasoning.
2. Can you put copies of an equilateral triangle together to form a right angle? Explain or show your reasoning.

Exercise 7.1.1.7

Here is a square and some regular octagons.

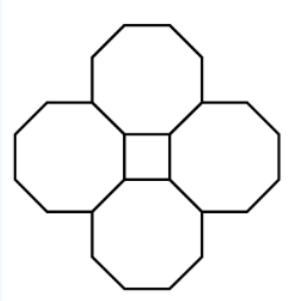


Figure 7.1.1.11

In this pattern, all of the angles inside the octagons have the same measure. The shape in the center is a square. Find the measure of one of the angles inside one of the octagons.

Exercise 7.1.1.8

The height of the water in a tank decreases by 3.5 cm each day. When the tank is full, the water is 10 m deep. The water tank needs to be refilled when the water height drops below 4 m.

1. Write a question that could be answered by solving the equation $10 - 0.035d = 4$.
2. Is 100 a solution of $10 - 0.035d > 4$? Write a question that solving this problem could answer.

(From Unit 6.3.5)

Exercise 7.1.1.9

Use the distributive property to write an expression that is equivalent to each given expression.

1. $-3(2x - 4)$
2. $0.1(-90 + 50a)$
3. $-7(-x - 9)$
4. $\frac{4}{5}(10y + -x + -15)$

(From Unit 6.4.1)

Exercise 7.1.1.10

Lin's puppy is gaining weight at a rate of 0.125 pounds per day. Describe the weight gain in days per pound.

(From Unit 2.1.3)

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7.1.2: Adjacent Angles

Lesson

Let's look at some special pairs of angles.

Exercise 7.1.2.1: Estimating Angle Measures

Estimate the degree measure of each indicated angle.

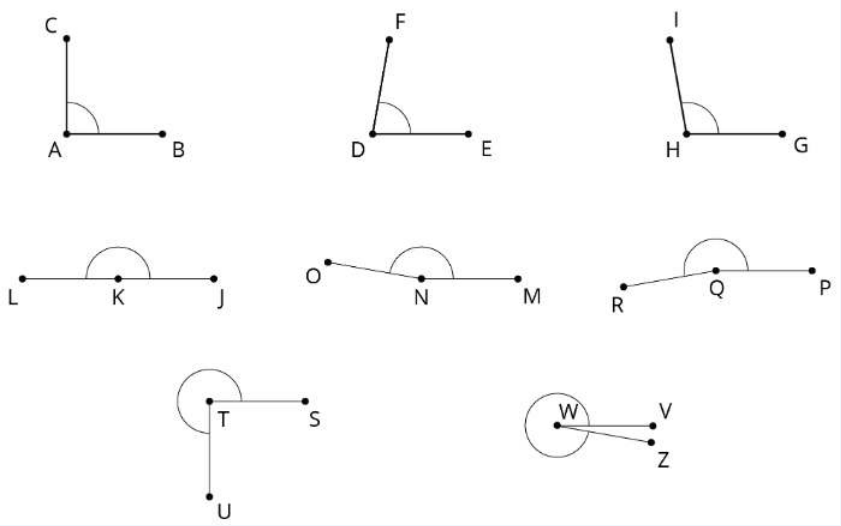


Figure 7.1.2.1

Exercise 7.1.2.2: Cutting Rectangles

Your teacher will give you two small, rectangular papers.

1. On one of the papers, draw a small half-circle in the middle of one side.

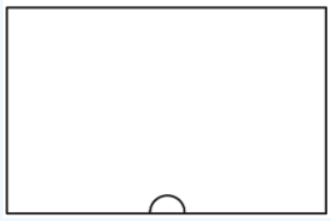


Figure 7.1.2.2

2. Cut a straight line, starting from the center of the half-circle, all the way across the paper to make 2 separate pieces. (Your cut does not need to be perpendicular to the side of the paper.)

3. On each of these two pieces, measure the angle that is marked by part of a circle. Label the angle measure on the piece.

4. What do you notice about these angle measures?

5. Clare measured 70 degrees on one of her pieces. Predict the angle measure of her other piece.

6. On the other rectangular paper, draw a small quarter-circle in one of the corners.



Figure 7.1.2.3

7. Repeat the previous steps to cut, measure, and label the two angles marked by part of a circle.
8. What do you notice about these angle measures?
9. Priya measured 53 degrees on one of her pieces. Predict the angle measure of her other piece.

Exercise 7.1.2.3: Is It a Complement or Supplement?

1. Use the protractor in the picture to find the measure of angles BCA and BCD.

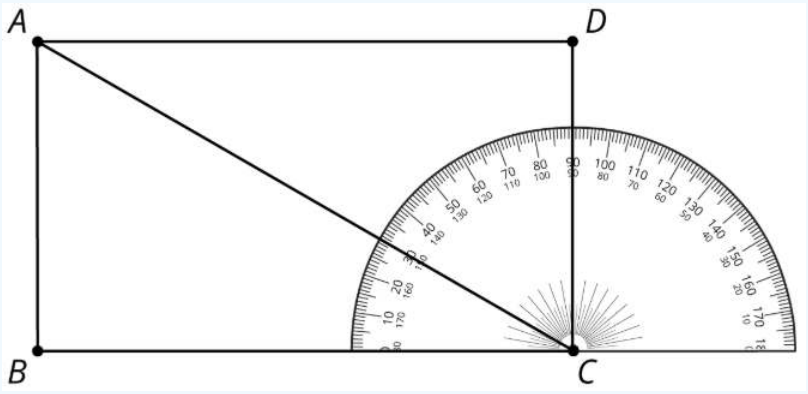


Figure 7.1.2.4: A protractor is shown. Point C is on the center of the protractor. Segment B, A lies along the 0 or 180 line, segment C D lies along the 90 degree line. Segment A, C goes through the 30 or 130 line. Points A and B are connected, and points A and D are connected.

2. Explain how to find the measure of angle ACD without repositioning the protractor.
3. Use the protractor in the picture to find the measure of angles LOK and LOM.

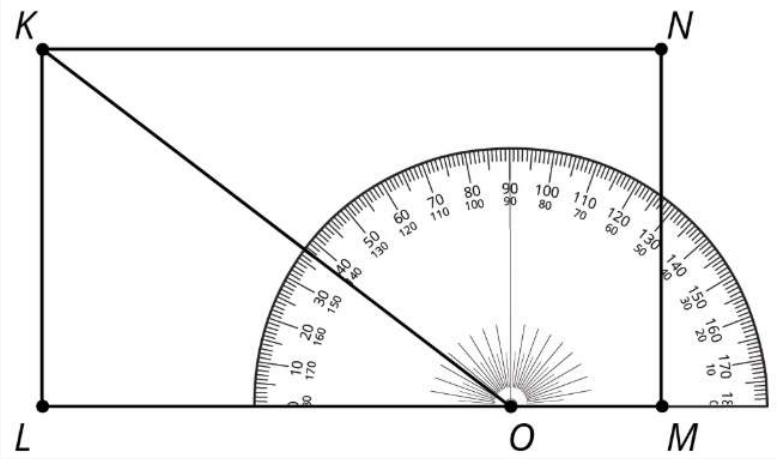


Figure 7.1.2.5

4. Explain how to find the measure of angle KOM without repositioning the protractor.
5. Angle BAC s a right angle. Find the measure of angle CAD.

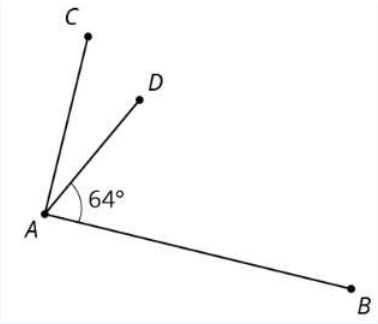


Figure 7.1.2.6

6. Point O is on line RS. Find the measure of angle SOP.

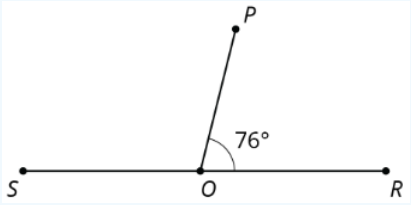


Figure 7.1.2.7

Are you ready for more?

Clare started with a rectangular piece of paper. She folded up one corner, and then folded up the other corner, as shown in the photos.



Figure 7.1.2.8

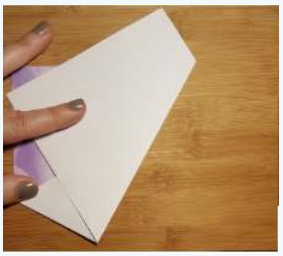


Figure 7.1.2.9



Figure 7.1.2.10

1. Try this yourself with any rectangular paper. Fold the left corner up at any angle, and then fold the right corner up so that the edges of the paper meet.
2. Clare thought that the angle at the bottom looked like a 90 degree angle. Does yours also look like it is 90 degrees?
3. Can you explain why the bottom angle *always has to be* 90 degrees? Hint: the third photo shows Clare's paper, unfolded. The crease marks have dashed lines, and the line where the two paper edges met have a solid line. Mark these on your own paper as well.

Summary

If two angle measures add up to 90°, then we say the angles are **complementary**. Here are three examples of pairs of complementary angles.

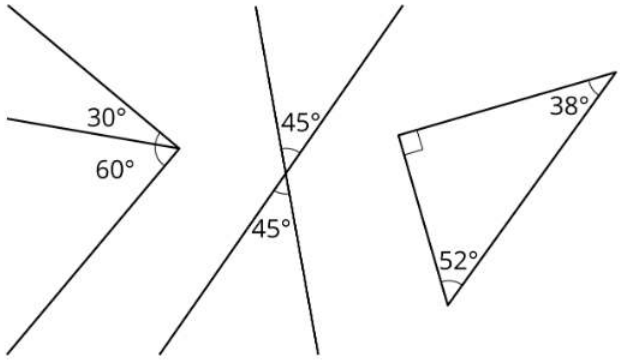


Figure 7.1.2.11

If two angle measures add up to 180°, then we say the angles are **supplementary**. Here are three examples of pairs of supplementary angles.

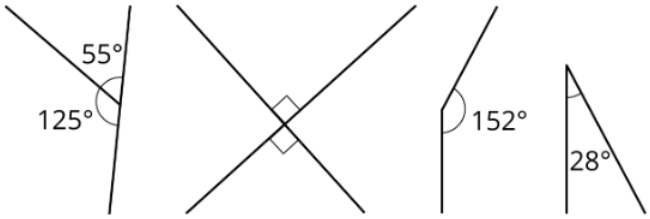


Figure 7.1.2.12

Glossary Entries

Definition: Adjacent Angles

Adjacent angles share a side and a vertex.

In this diagram, angle ABC is adjacent to angle DBC.

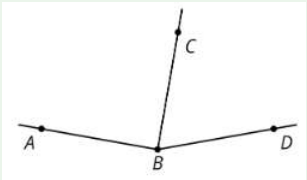


Figure 7.1.2.13

Definition: Complementary

Complementary angles have measures that add up to 90 degrees.

For example, a 15° angle and a 75° angle are complementary.

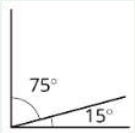


Figure 7.1.2.14

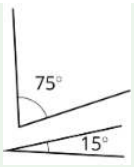


Figure 7.1.2.15

Definition: Right Angle

A right angle is half of a straight angle. It measures 90 degrees.

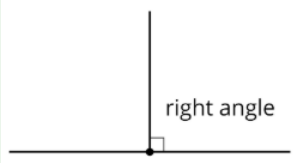


Figure 7.1.2.16

Definition: Straight Angle

A straight angle is an angle that forms a straight line. It measures 180 degrees.

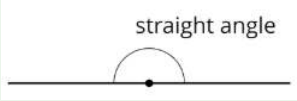


Figure 7.1.2.17

Definition: Supplementary

Supplementary angles have measures that add up to 180 degrees.

For example, a 15° angle and a 165° angle are supplementary.



Figure 7.1.2.18

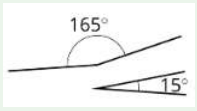


Figure 7.1.2.19

Practice

Exercise 7.1.2.4

Angles A and C are supplementary. Find the measure of angle C.

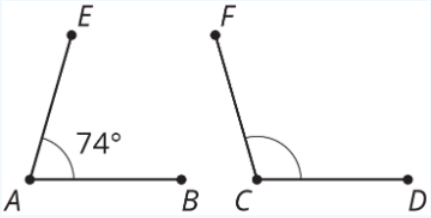


Figure 7.1.2.20

Exercise 7.1.2.5

1. List two pairs of angles in square $CDFG$ that are complementary.
2. Name three angles that sum to $180°$.

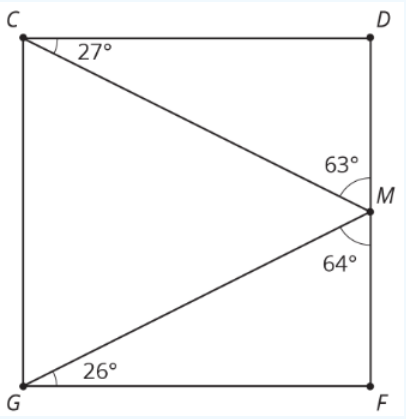


Figure 7.1.2.21

Exercise 7.1.2.6

Complete the equation with a number that makes the expression on the right side of the equal sign equivalent to the expression on the left side.

$$5x - 2.5 + 6x - 3 = \_\_(2x - 1)$$

(From Unit 6.4.5)

Exercise 7.1.2.7

Match each table with the equation that represents the same proportional relationship.

A:

Table 7.1.2.1

x	y
2	8
3	12
4	16
5	20

1. $y = 1.5x$
2. $y = 1.25x$
3. $y = 4x$

B:

Table 7.1.2.2

x	y
3	4.5
6	9
7	10.5
10	15

C:

Table 7.1.2.3

x	y
2	$\frac{5}{2}$
4	5
6	$\frac{15}{2}$
12	15

(From Unit 2.2.1)

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7.1.3: Nonadjacent Angles

Lesson

Let's look at angles that are not right next to one another.

Exercise 7.1.3.1: Finding Related Statements

Given a and b are numbers, and $a+b=180$, which statements also must be true?

$a=180-b$ $a-180=b$ $360=2a+2b$ $a=90$ and $b=90$

Exercise 7.1.3.2: Polygon Angles

Use any useful tools in the geometry toolkit to identify any pairs of angles in these figures that are complementary or supplementary.

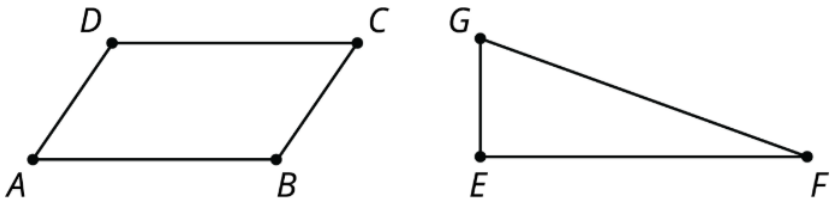


Figure 7.1.3.1

Exercise 7.1.3.3: Vertical Angles

Use a straightedge to draw two intersecting lines. Use a protractor to measure all four angles whose vertex is located at the intersection.

Compare your drawing and measurements to the people in your group. Make a conjecture about the relationships between angle measures at an intersection.

Exercise 7.1.3.4: Row Game: Angles

Find the measure of the angles in one column. Your partner will work on the other column. Check in with your partner after you finish each row. Your answers in each row should be the same. If your answers aren't the same, work together to find the error and correct it.

column A

P is on line m. Find the value of a.

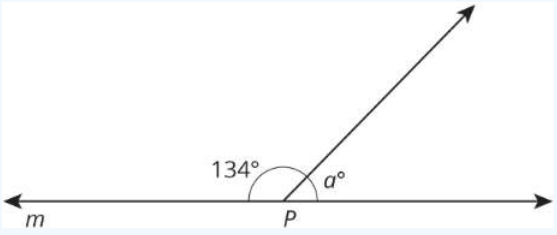


Figure 7.1.3.2

Find the value of a.

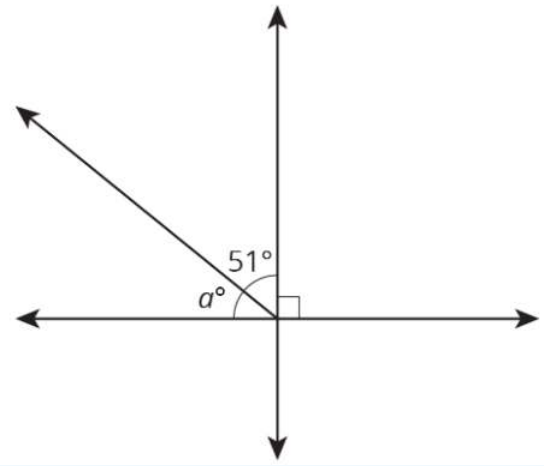


Figure 7.1.3.3

column B

Find the value of b.

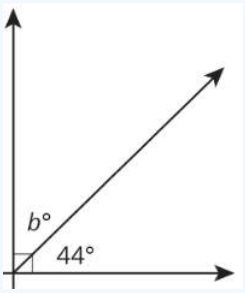


Figure 7.1.3.4

In right triangle LMN, angles L and M are complementary. Find the measure of angle L.

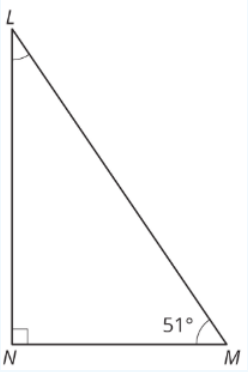


Figure 7.1.3.5

column A

Angle C and angle E are supplementary. Find the measure of angle E.

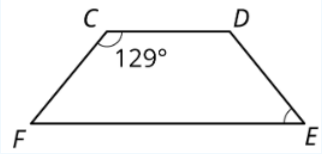


Figure 7.1.3.6

Find the value of c.

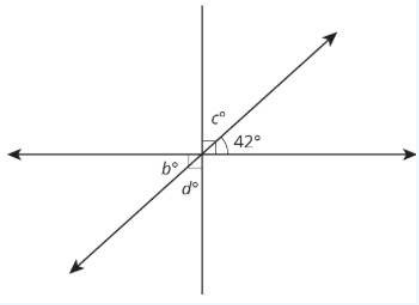


Figure 7.1.3.7

Two angles are complementary. One angle measures 37 degrees. Find the measure of the other angle.

column B

X is on line WY. Find the value of b.

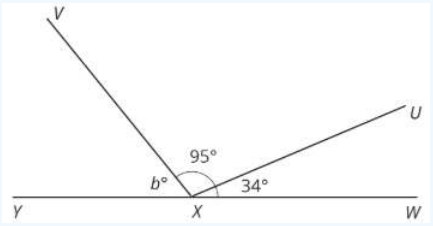


Figure 7.1.3.8

B is on line FW. Find the measure of angle CBW.

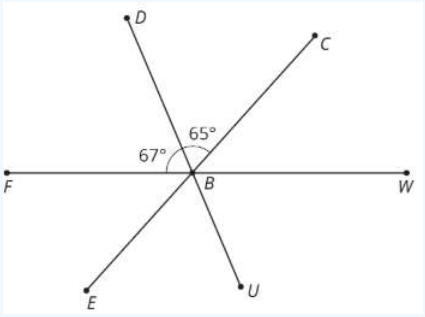


Figure 7.1.3.9

Two angles are supplementary. One angle measures 127 degrees. Find the measure of the other angle.

Summary

When two lines cross, they form two pairs of **vertical angles**. Vertical angles are across the intersection point from each other.

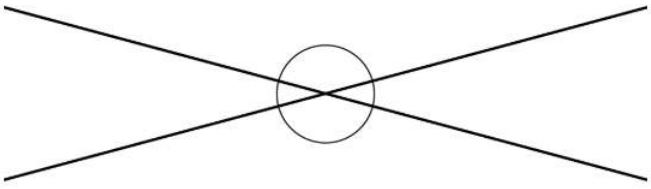


Figure 7.1.3.10

Vertical angles always have equal measure. We can see this because they are always supplementary with the same angle. For example:



Figure 7.1.3.11

This is always true!

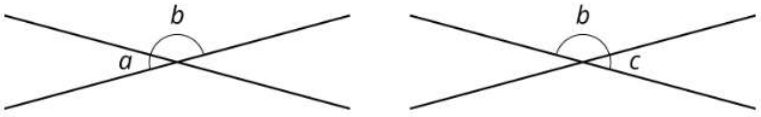


Figure 7.1.3.12

$a+b=180$ so $a=180-b$.

$c+b=180$ so $c=180-b$.

That means $a=c$.

Glossary Entries

Definition: Adjacent Angles

Adjacent angles share a side and a vertex.

In this diagram, angle ABC is adjacent to angle DBC.

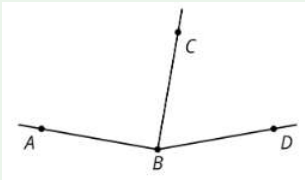


Figure 7.1.3.13

Definition: Complementary

Complementary angles have measures that add up to 90 degrees.

For example, a 15° angle and a 75° angle are complementary.

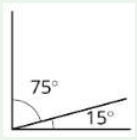


Figure 7.1.3.14

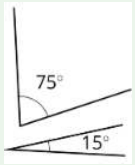


Figure 7.1.3.15

Definition: Right Angle

A right angle is half of a straight angle. It measures 90 degrees.

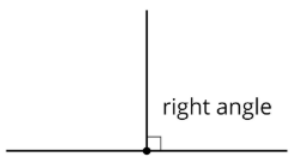


Figure 7.1.3.16

Definition: Straight Angle

A straight angle is an angle that forms a straight line. It measures 180 degrees.

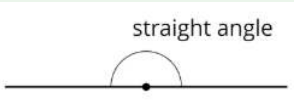


Figure 7.1.3.17

Definition: Supplementary

Supplementary angles have measures that add up to 180 degrees.

For example, a 15° angle and a 165° angle are supplementary.

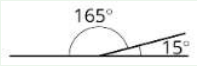


Figure 7.1.3.18

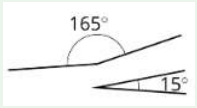


Figure 7.1.3.19

Definition: Vertical Angles

Vertical angles are opposite angles that share the same vertex. They are formed by a pair of intersecting lines. Their angle measures are equal.

For example, angles AEC and DEB are vertical angles. If angle AEC measures 120°, then angle DEB must also measure 120°.

Angles AED and BEC are another pair of vertical angles.

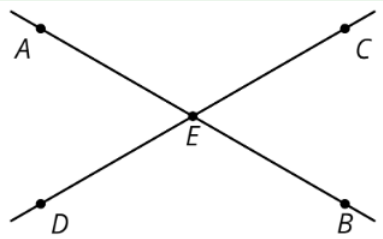


Figure 7.1.3.20

Practice

Exercise 7.1.3.5

Two lines intersect. Find the value of b and c.

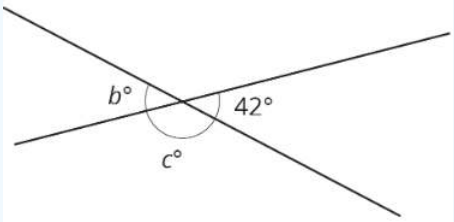


Figure 7.1.3.21

Exercise 7.1.3.6

In this figure, angles R and S are complementary. Find the measure of angle S.

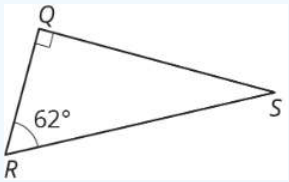


Figure 7.1.3.22

Exercise 7.1.3.7

If two angles are both vertical and supplementary, can we determine the angles? Is it possible to be both vertical and complementary? If so, can you determine the angles? Explain how you know.

Exercise 7.1.3.8

Match each expression in the first list with an equivalent expression from the second list.

A. $5(x+1)-2x+11$
B. $2x+2+x+5$
C. $\frac{-3}{8}x-9+\frac{5}{8}x+1$
D. $2.06x-5.53+4.98-9.02$
E. $99x+44$

1. $\frac{1}{4}x-8$
2. $\frac{1}{2}(6x+14)$
3. $11(9x+4)$
4. $3x+16$
5. $2.06x+(-5.53)+4.98+(-9.02)$

(From Unit 6.4.5)

Exercise 7.1.3.9

Factor each expression.

1. $15a-13a$
2. $-6x-18y$
3. $36abc+54ad$

(From Unit 6.4.2)

Exercise 7.1.3.10

The directors of a dance show expect many students to participate but don't yet know how many students will come. The directors need 7 students to work on the technical crew. The rest of the students work on dance routines in groups of 9. For the show to work, they need at least 6 full groups working on dance routines.

1. Write and solve an inequality to represent this situation, and graph the solution on a number line.
2. Write a sentence to the directors about the number of students they need.

(From Unit 6.3.5)

Exercise 7.1.3.11

A small dog gets fed $\frac{3}{4}$ cup of dog food twice a day. Using d for the number of days and f for the amount of food in cups, write an equation relating the variables. Use the equation to find how many days a large bag of dog food will last if it contains 210 cups of food.

(From Unit 2.2.2)

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7.1.4: Solving for Unknown Angles

Lesson

Let's figure out some missing angles.

Exercise 7.1.4.1: True or False: Length Relationships

Here are some line segments.



Figure 7.1.4.1

Decide if each of these equations is true or false. Be prepared to explain your reasoning.

$CD + BC = BD$

$AB + BD = CD + AD$

$AC - AB = AB$

$BD - CD = AC - AB$

Exercise 7.1.4.2: Info Gap: ANgle Finding

Your teacher will give you either a *problem card* or a *data card*. Do not show or read your card to your partner.

If your teacher gives you the *problem card*:

1. Silently read your card and think about what information you need to be able to answer the question.
2. Ask your partner for the specific information that you need.
3. Explain how you are using the information to solve the problem.
   Continue to ask questions until you have enough information to solve the problem.
4. Share the *problem card* and solve the problem independently.
5. Read the *data card* and discuss your reasoning.

If your teacher gives you the *data card*:

1. Silently read your card.
2. Ask your partner "*What specific information do you need?*" and wait for them to *ask* for information.
   If your partner asks for information that is not on the card, do not do the calculations for them. Tell them you don't have that information.
3. Before sharing the information, ask "*Why do you need that information?*" Listen to your partner's reasoning and ask clarifying questions.
4. Read the *problem card* and solve the problem independently.
5. Share the *data card* and discuss your reasoning.

Pause here so your teacher can review your work. Ask your teacher for a new set of cards and repeat the activity, trading roles with your partner.

Exercise 7.1.4.3: What's the Match?

Match each figure to an equation that represents what is seen in the figure. For each match, explain how you know they are a match.

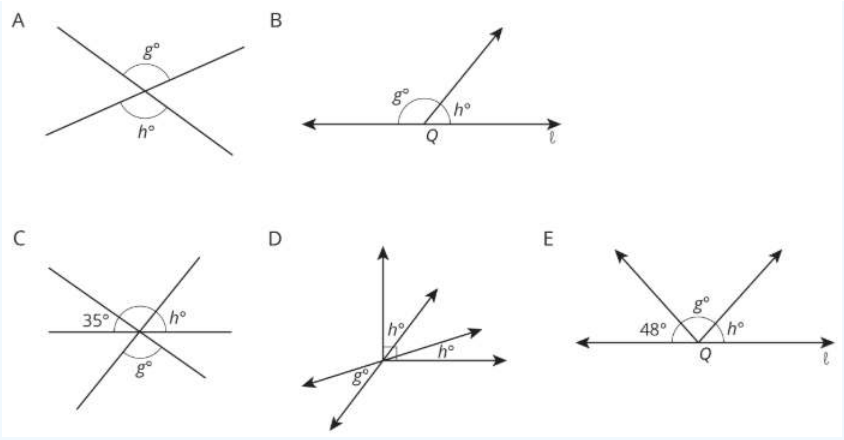


Figure 7.1.4.2

1. $g+h=180$
2. $g=h$
3. $2h+g=90$
4. $g+h+48=180$
5. $g+h+35=180$

Are you ready for more?

1. What is the angle between the hour and minute hands of a clock at 3:00?
2. You might think that the angle between the hour and minute hands at 2:20 is 60 degrees, but it is not! The hour hand has moved beyond the 2. Calculate the angle between the clock hands at 2:20.
3. Find a time where the hour and minute hand are 40 degrees apart. (Assume that the time has a whole number of minutes.) Is there just one answer?

Summary

We can write equations that represent relationships between angles.

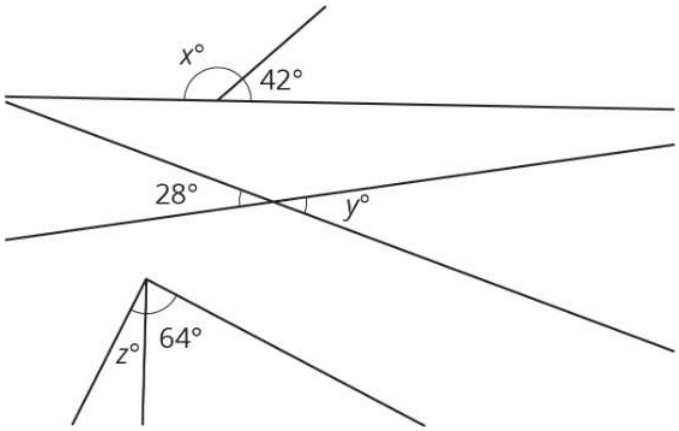


Figure 7.1.4.3

- The first pair of angles are supplementary, so $x+42=180$.
- The second pair of angles are vertical angles, so $y=28$.
- Assuming the third pair of angles form a right angle, they are complementary, so $z+64=90$.

Glossary Entries

Definition: Adjacent Angles

Adjacent angles share a side and a vertex.

In this diagram, angle ABC is adjacent to angle DBC.

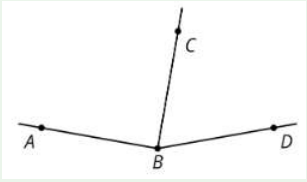


Figure 7.1.4.4

Definition: Complementary

Complementary angles have measures that add up to 90 degrees.

For example, a 15° angle and a 75° angle are complementary.

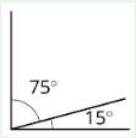


Figure 7.1.4.5

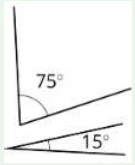


Figure 7.1.4.6

Definition: Right Angle

A right angle is half of a straight angle. It measures 90 degrees.

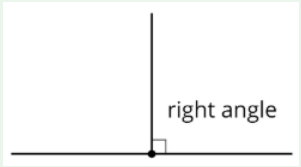


Figure 7.1.4.7

Definition: Straight Angle

A straight angle is an angle that forms a straight line. It measures 180 degrees.

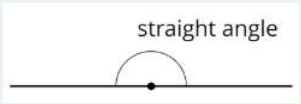


Figure 7.1.4.8

Definition: Supplementary

Supplementary angles have measures that add up to 180 degrees.

For example, a 15° angle and a 165° angle are supplementary.



Figure 7.1.4.9

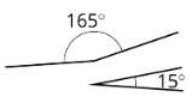


Figure 7.1.4.10

Definition: Vertical Angles

Vertical angles are opposite angles that share the same vertex. They are formed by a pair of intersecting lines. Their angle measures are equal.

For example, angles AEC and DEB are vertical angles. If angle AEC measures $120°$, then angle DEB must also measure $120°$.

Angles AED and BEC are another pair of vertical angles.

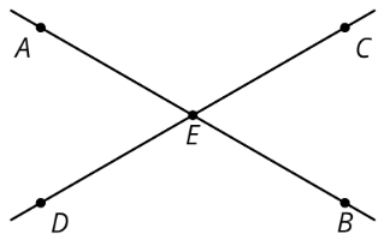


Figure 7.1.4.11

Practice

Exercise 7.1.4.4

M is a point on line segment KL. NM is a line segment. Select **all** the equations that represent the relationship between the measures of the angles in the figure.

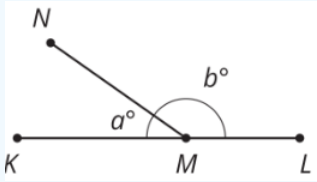


Figure 7.1.4.12

A. $a = b$
B. $a + b = 90$
C. $b = 90 - a$
D. $a + b = 180$
E. $180 - a = b$
F. $180 = b - a$

Exercise 7.1.4.5

Which equation represents the relationship between the angles in the figure?

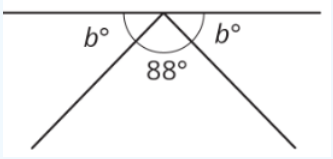


Figure 7.1.4.13

A. $88 + b = 90$
B. $88 + b = 180$

C. $2b+88=90$
D. $2b+88=180$

Exercise 7.1.4.6

Segments AB, EF, and CD intersect at point C, and angle ACD is a right angle. Find the value of g.

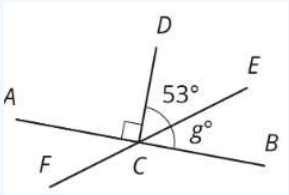


Figure 7.1.4.14

Exercise 7.1.4.7

Select **all** the expressions that are the result of decreasing x by 80%.

A. $\frac{20}{100}x$
B. $x-\frac{80}{100}x$
C. $\frac{100-20}{100}x$
D. $0.80x$
E. $(1-0.8)x$

(From Unit 6.2.6)

Exercise 7.1.4.8

Andre is solving the equation $4(x+\frac{3}{2})=7$. He says, "I can subtract $\frac{3}{2}$ from each side to get $4x=\frac{11}{2}$ and then divide by 4 to get $x=\frac{11}{8}$." Kiran says, "I think you made a mistake."

1. How can Kiran know for sure that Andre's solution is incorrect?
2. Describe Andre's error and explain how to correct his work.

(From Unit 6.2.2)

Exercise 7.1.4.9

Solve each equation.

$\frac{1}{7}a+\frac{3}{4}=\frac{9}{8}$ $\quad$ $\frac{2}{3}+\frac{1}{5}b=\frac{5}{6}$ $\quad$ $\frac{3}{2}=\frac{4}{3}c+\frac{2}{3}$

$0.3d+7.9=9.1$ $\quad$ $11.03=8.78+0.02e$

(From Unit 6.2.1)

Exercise 7.1.4.10

A train travels at a constant speed for a long distance. Write the two constants of proportionality for the relationship between distance traveled and elapsed time. Explain what each of them means.

Table 7.1.4.1

time elapsed (hr)	distance (mi)
1.2	54
3	135
4	180

(From Unit 2.2.2)

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7.1.5: Using Equations to Solve for Unknown Angles

Lesson

Let's figure out missing angles using equations.

Exercise 7.1.5.1: Is this Enough?

Tyler thinks that this figure has enough information to figure out the values of a and b.

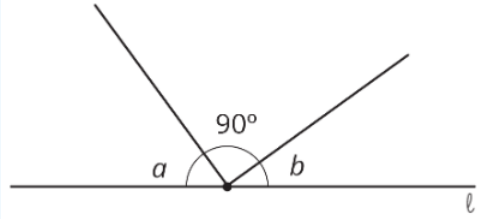


Figure 7.1.5.1

Do you agree? Explain your reasoning.

Exercise 7.1.5.2: What Does It Look Like?

Elena and Diego each wrote equations to represent these diagrams. For each diagram, decide which equation you agree with, and solve it. You can assume that angles that look like right angles are indeed right angles.

1. Elena: $x = 35$

Diego: $x + 35 = 180$

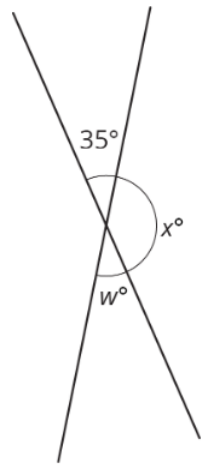


Figure 7.1.5.2

2. Elena: $35 + w + 41 = 180$

Diego: $w + 35 = 180$

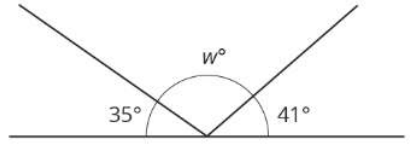


Figure 7.1.5.3

3. Elena: $w + 35 = 90$

Diego: $2w + 35 = 90$

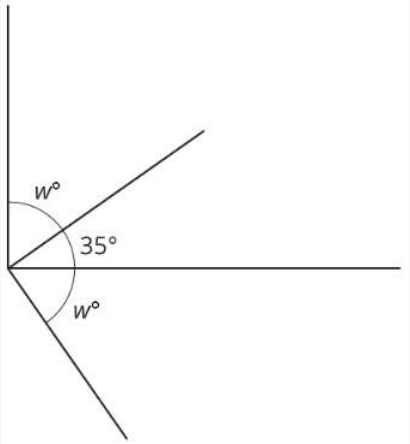


Figure 7.1.5.4

4. Elena: $2w + 35 = 90$

Diego: $w + 35 = 90$

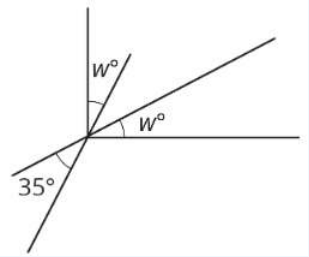


Figure 7.1.5.5

5. Elena: $w + 148 = 180$

Diego: $x + 90 = 148$

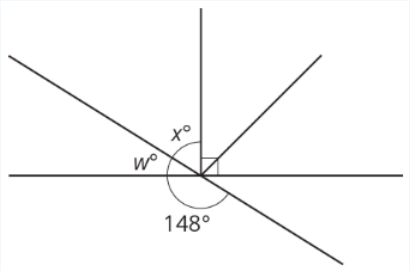


Figure 7.1.5.6

Exercise 7.1.5.3: Calculate the Measure

Find the unknown angle measures. Show your thinking. Organize it so it can be followed by others.

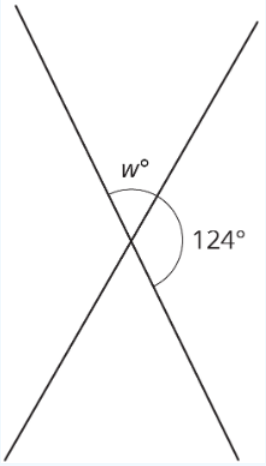


Figure 7.1.5.7

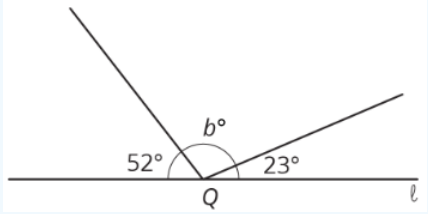


Figure 7.1.5.8

Lines l and m are perpendicular.

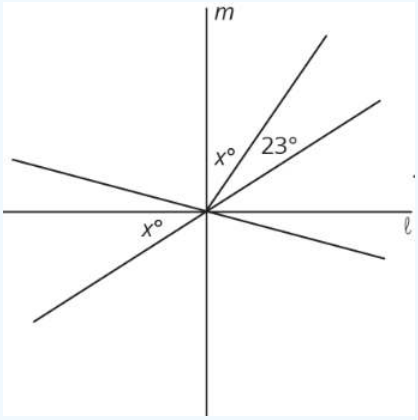


Figure 7.1.5.9

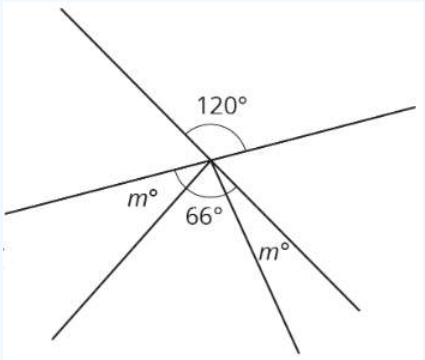


Figure 7.1.5.10

Are you ready for more?

The diagram contains three squares. Three additional segments have been drawn that connect corners of the squares. We want to find the exact value of $a+b+c$.

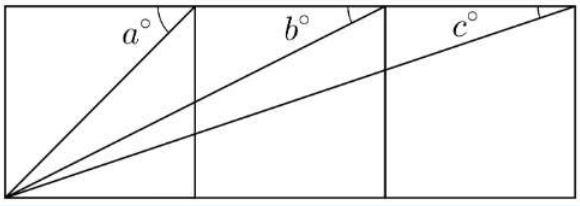


Figure 7.1.5.11

1. Use a protractor to measure the three angles. Use your measurements to conjecture about the value of $a+b+c$.
2. Find the exact value of $a+b+c$ by reasoning about the diagram.

Summary

To find an unknown angle measure, sometimes it is helpful to write and solve an equation that represents the situation. For example, suppose we want to know the value of x in this diagram.

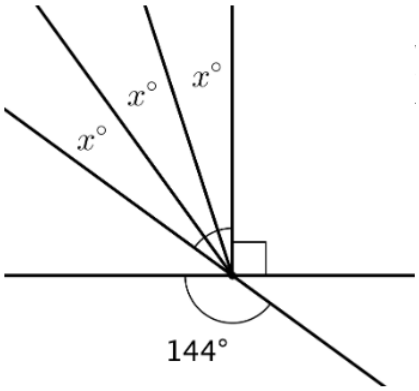


Figure 7.1.5.12

Using what we know about vertical angles, we can write the equation $3x+90=144$ to represent this situation. Then we can solve the equation.

$$\begin{aligned} 3x+90 &= 144 \\ 3x+90-90 &= 144-90 \\ 3x &= 54 \\ 3x\cdot\frac{1}{3} &= 54\cdot\frac{1}{3} \\ x &= 18 \end{aligned}$$

Glossary Entries

Definition: Adjacent Angles

Adjacent angles share a side and a vertex.

In this diagram, angle ABC is adjacent to angle DBC.

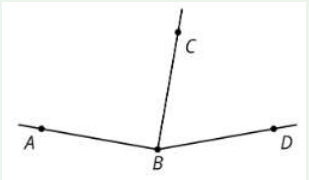


Figure 7.1.5.13

Definition: Complementary

Complementary angles have measures that add up to 90 degrees.

For example, a 15° angle and a 75° angle are complementary.

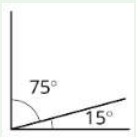


Figure 7.1.5.14

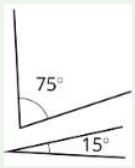


Figure 7.1.5.15

Definition: Right Angle

A right angle is half of a straight angle. It measures 90 degrees.

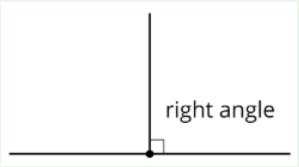


Figure 7.1.5.16

Definition: Straight Angle

A straight angle is an angle that forms a straight line. It measures 180 degrees.

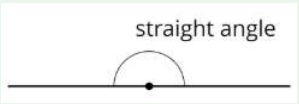


Figure 7.1.5.17

Definition: Supplementary

Supplementary angles have measures that add up to 180 degrees.

For example, a 15° angle and a 165° angle are supplementary.

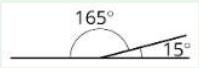


Figure 7.1.5.18

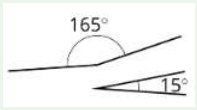


Figure 7.1.5.19

Definition: Vertical Angles

Vertical angles are opposite angles that share the same vertex. They are formed by a pair of intersecting lines. Their angle measures are equal.

For example, angles AEC and DEB are vertical angles. If angle AEC measures 120°, then angle DEB must also measure 120°.

Angles AED and BEC are another pair of vertical angles.

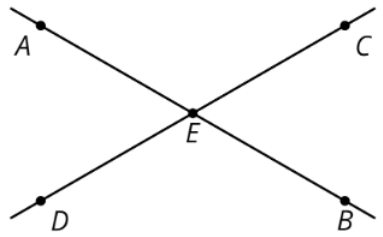


Figure 7.1.5.20

Practice

Exercise 7.1.5.4

Segments AB, DC, and EC intersect at point C. Angle DCE measures 148°. Find the value of x.

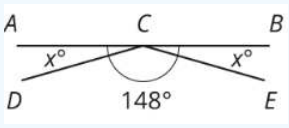


Figure 7.1.5.21

Exercise 7.1.5.5

Line l is perpendicular to line m. Find the value of x and w.

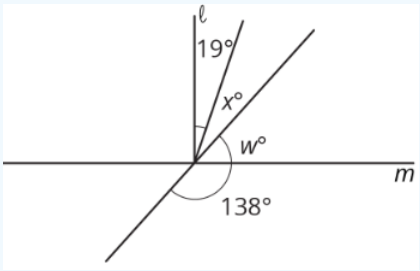


Figure 7.1.5.22

Exercise 7.1.5.6

If you knew that two angles were complementary and were given the measure of one of those angles, would you be able to find the measure of the other angle? Explain your reasoning.

Exercise 7.1.5.7

For each inequality, decide whether the solution is represented by $x < 4.5$ or $x > 4.5$.

1. $-24 > -6(x - 0.5)$
2. $-8x + 6 > -30$
3. $-2(x + 3.2) < -15.4$

(From Unit 6.3.3)

Exercise 7.1.5.8

A runner ran $\frac{2}{3}$ of a 5 kilometer race in 21 minutes. They ran the entire race at a constant speed.

1. How long did it take to run the entire race?
2. How many minutes did it take to run 1 kilometer?

(From Unit 4.1.2)

Exercise 7.1.5.9

Jada, Elena, and Lin walked a total of 37 miles last week. Jada walked 4 more miles than Elena, and Lin walked 2 more miles than Jada. The diagram represents this situation:

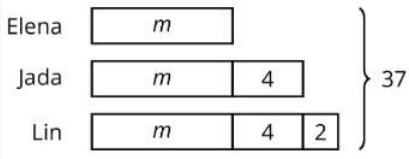


Figure 7.1.5.23

Find the number of miles that they each walked. Explain or show your reasoning.

(From Unit 6.2.6)

Exercise 7.1.5.10

Select **all** the expressions that are equivalent to $-36x+54y-90$.

A. $-9(4x-6y-10)$
B. $-18(2x-3y+5)$
C. $-6(6x+9y-15)$
D. $18(-2x+3y-5)$
E. $-2(18x-27y+45)$
F. $2(-18x+54y-90)$

(From Unit 6.4.2)

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7.2: Drawing Polygons with Given Conditions

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7.2.1: Building Polygons (Part 1)

Lesson

Let's build shapes.

Exercise 7.2.1.1: True or False: Signed Numbers

Decide whether each equation is true or false. Be prepared to explain your reasoning.

$4 \cdot (-6) = (-6) + (-6) + (-6) + (-6)$

$-8 \cdot 4 = (-8 \cdot 3) + 4$

$6 \cdot (-7) = 7 \cdot (-7) + 7$

$-10 - 6 = -10 - (-6)$

Exercise 7.2.1.2: What Can You build?

1. Use the segments in the applet to build several polygons, including at least one triangle and one quadrilateral.
2. After you finish building several polygons, select one triangle and one quadrilateral that you have made.
   a. Measure all the angles in the two shapes you selected. Note: select points in order counterclockwise, like a protractor.
   b. Using these measurements along with the side lengths as marked, draw your triangle and quadrilateral as accurately as possible on separate paper.

Exercise 7.2.1.3: Building Diego's and Jada's Shapes

1. Diego built a quadrilateral using side lengths of 4 in, 5 in, 6 in, and 9 in.
   a. Build such a shape.
   b. Is your shape an identical copy of Diego's shape? Explain your reasoning.
2. Jada built a triangle using side lengths of 4 in, 5 in, and 8 in.
   a. Build such a shape.
   b. Is your shape an identical copy of Jada's shape? Explain your reasoning.

Exercise 7.2.1.4: Building Han's Shape

Han built a polygon using side lengths of 3 in, 4 in, and 9 in.

1. Build such a shape.
2. What do you notice?

Summary

Sometimes we are given a polygon and asked to find the lengths of the sides. What options do you have if you need to build a polygon with some side lengths? Sometimes, we can make lots of different figures. For example, if you have side lengths 5, 7, 11, and 14, here are some of the many, many quadrilaterals we can make with those side lengths:

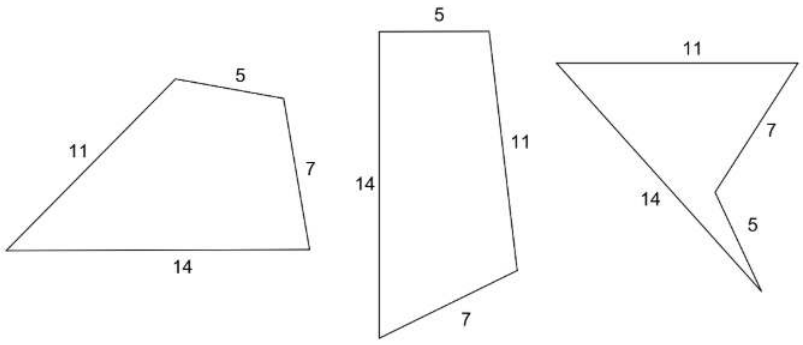


Figure 7.2.1.1

Sometimes, it is not possible to make a figure with certain side lengths. For example, 18, 1, 1, 1 (try it!).

We will continue to investigate the figures that can be made with given measures.

Practice

Exercise 7.2.1.5

A rectangle has side lengths of 6 units and 3 units. Could you make a quadrilateral that is not identical using the same four side lengths? If so, describe it.

Exercise 7.2.1.6

Come up with an example of three side lengths that can not possibly make a triangle, and explain how you know.

Exercise 7.2.1.7

Find x, y, and z.

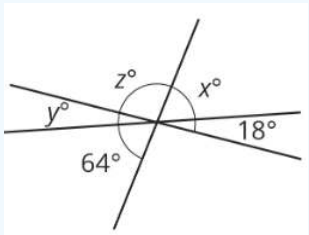


Figure 7.2.1.2

(From Unit 7.1.3)

Exercise 7.2.1.8

How many right angles need to be put together to make:

1. 360 degrees?
2. 180 degrees?
3. 270 degrees?
4. A straight angle?

(From Unit 7.1.1)

Exercise 7.2.1.9

Solve each equation.

$\frac{1}{7}(x+\frac{3}{4})=\frac{1}{8}$ $\frac{9}{2}=\frac{3}{4}(z+\frac{2}{3})$

$1.5=0.6(w+0.4)$ $0.08(7.97+v)=0.832$

(From Unit 6.2.2)

Exercise 7.2.1.10

1. You can buy 4 bottles of water from a vending machine for $7. At this rate, how many bottles of water can you buy for $28? If you get stuck, consider creating a table.
2. It costs $20 to buy 5 sandwiches from a vending machine. At this rate, what is the cost for 8 sandwiches? If you get stuck, consider creating a table.

(From Unit 4.1.3)

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7.2.2: Building Polygons (Part 2)

Lesson

Let's build more triangles.

Exercise 7.2.2.1: Where is Lin?

At a park, the slide is 5 meters east of the swings. Lin is standing 3 meters away from the slide.

1. Draw a diagram of the situation including a place where Lin could be.
2. How far away from the swings is Lin in your diagram?
3. Where are some other places Lin could be?

Exercise 7.2.2.2: How Long is the Third Side?

Use the applet to answer the questions.

1. Build as many different triangles as you can that have one side length of 5 inches and one of 4 inches. Record the side lengths of each triangle you build.
2. Are there any other lengths that could be used for the third side of the triangle but aren't values of the sliders?
3. Are there any lengths that are values of the sliders but could not be used as the third side of the triangle?

Are you ready for more?

Assuming you had access to strips of any length, and you used the 9-inch and 5-inch strips as the first two sides, complete the sentences:

1. The third side can't be _____ inches or longer.
2. The third side can't be _____ inches or shorter.

Exercise 7.2.2.3: Swinging the Sides Around

We'll explore a method for drawing a triangle that has three specific side lengths. Use the applet to answer the questions.

1. Follow these instructions to mark the possible endpoints of one side:
   a. For now, ignore segment AC, the 3-inch side length on the left side

   Figure 7.2.2.1

   b. Let segment BD be the 3-unit side length on the right side. Right-click on point D, check Trace On. Rotate the point, drawing all the places where a 3-inch side could end.
2. What shape have you drawn while moving BD around? Why? Which tool in your geometry toolkit can do something similar?
3. Use your drawing to create two unique triangles, each with a base of length 4 inches and a side of length 3 inches. Use a different color to draw each triangle.
4. Repeat the previous instructions, letting segment AC be the 3-unit side length.
5. Using a third color, draw a point where the two traces intersect. Using this third color, draw a triangle with side lengths of 4 inches, 3 inches, and 3 inches.

Summary

If we want to build a polygon with two given side lengths that share a vertex, we can think of them as being connected by a hinge that can be opened or closed:

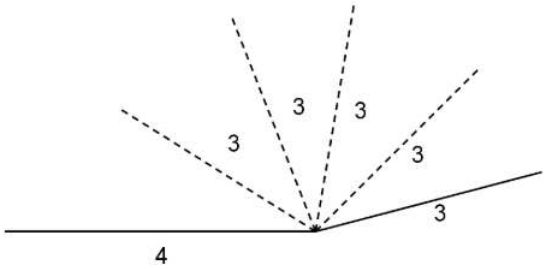


Figure 7.2.2.2

All of the possible positions of the endpoint of the moving side form a circle:

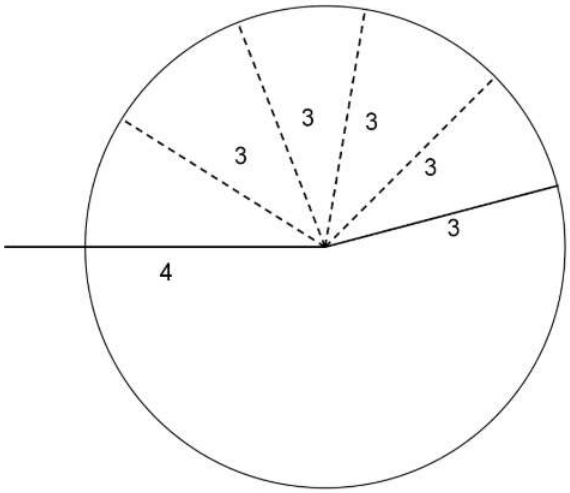


Figure 7.2.2.3

You may have noticed that sometimes it is not possible to build a polygon given a set of lengths. For example, if we have one really, really long segment and a bunch of short segments, we may not be able to connect them all up. Here's what happens if you try to make a triangle with side lengths 21, 4, and 2:



Figure 7.2.2.4

The short sides don't seem like they can meet up because they are too far away from each other.

If we draw circles of radius 4 and 2 on the endpoints of the side of length 21 to represent positions for the shorter sides, we can see that there are no places for the short sides that would allow them to meet up and form a triangle.

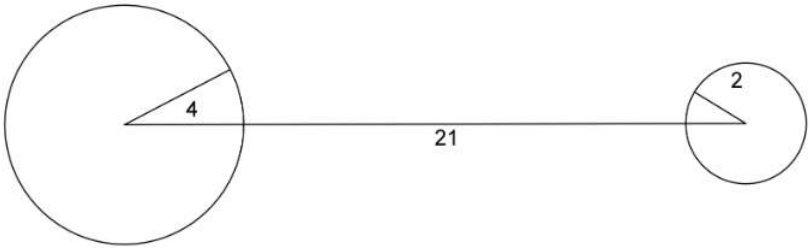


Figure 7.2.2.5

In general, the longest side length must be less than the sum of the other two side lengths. If not, we can't make a triangle!

If we *can* make a triangle with three given side lengths, it turns out that the measures of the corresponding angles will *always* be the same. For example, if two triangles have side lengths 3, 4, and 5, they will have the same corresponding angle measures.

Practice

Exercise 7.2.2.4

In the diagram, the length of segment AB is 10 units and the radius of the circle centered at A is 4 units. Use this to create two unique triangles, each with a side of length 10 and a side of length 4. Label the sides that have length 10 and 4.

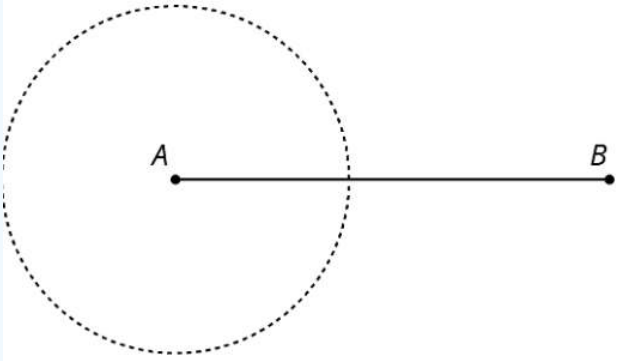


Figure 7.2.2.6

Exercise 7.2.2.5

Select **all** the sets of three side lengths that will make a triangle.

A. $3, 4, 8$
B. $7, 6, 12$
C. $5, 11, 13$
D. $4, 6, 12$
E. $4, 6, 10$

Exercise 7.2.2.6

Based on signal strength, a person knows their lost phone is exactly 47 feet from the nearest cell tower. The person is currently standing 23 feet from the same cell tower. What is the closest the phone could be to the person? What is the furthest their phone could be from them?

Exercise 7.2.2.7

Each row contains the degree measures of two complementary angles. Complete the table.

Table 7.2.2.1

measure of an angle	measure of its complement
80°	
25°	
54°	
x	

(From Unit 7.1.2)

Exercise 7.2.2.8

Here are two patterns made using identical rhombuses. Without using a protractor, determine the value of a and b. Explain or show your reasoning.

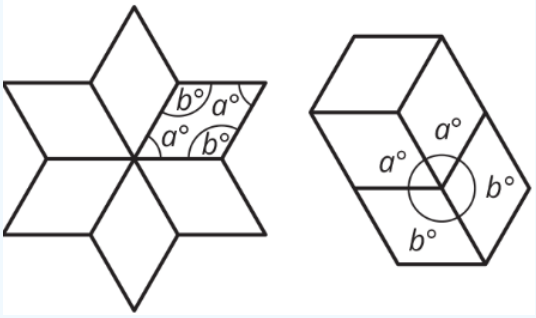


Figure 7.2.2.7

(From Unit 7.1.1)

Exercise 7.2.2.9

Mai's family is traveling in a car at a constant speed of 65 miles per hour.

1. At that speed, how long will it take them to travel 200 miles?
2. How far do they travel in 25 minutes?

(From Unit 4.1.3)

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7.2.3: Triangles with 3 Common Measures

Lesson

Let's contrast triangles.

Exercise 7.2.3.1: 3 Sides; 3 Angles

Examine each set of triangles. What do you notice? What is the same about the triangles in the set? What is different?

Set 1:

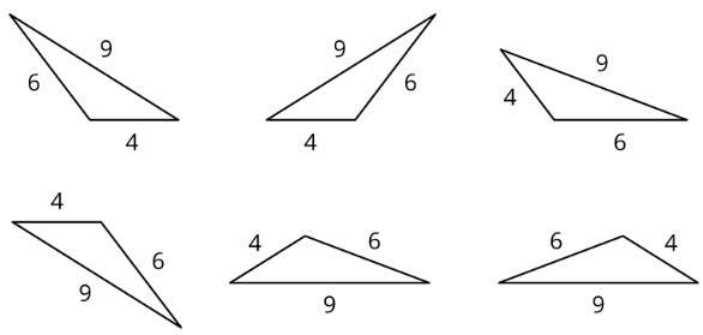


Figure 7.2.3.1

Set 2:

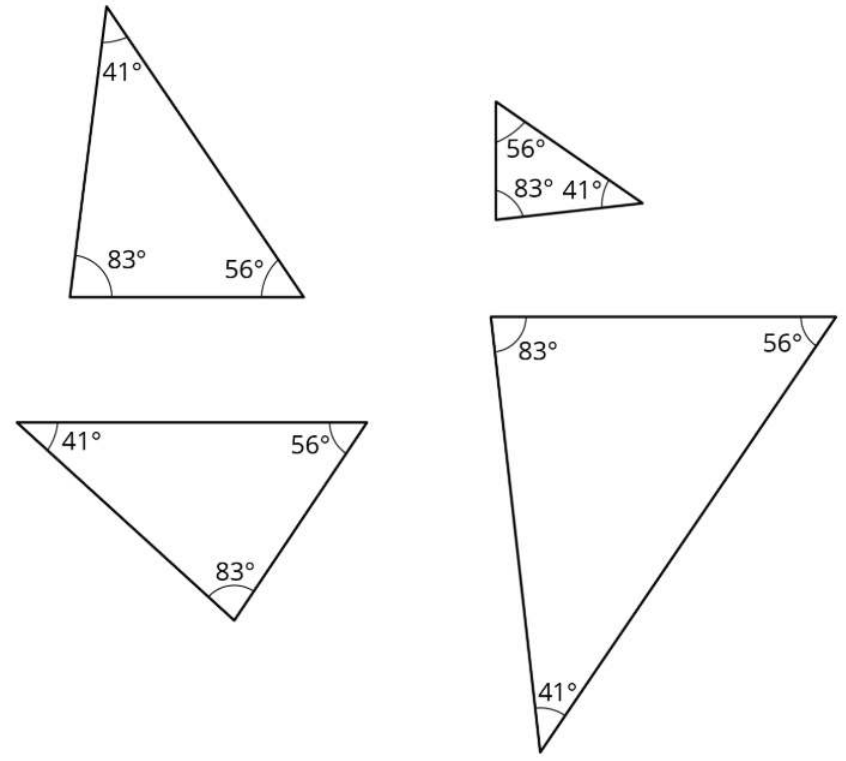


Figure 7.2.3.2

Exercise 7.2.3.2: 2 Sides and 1 Angle

Examine this set of triangles.

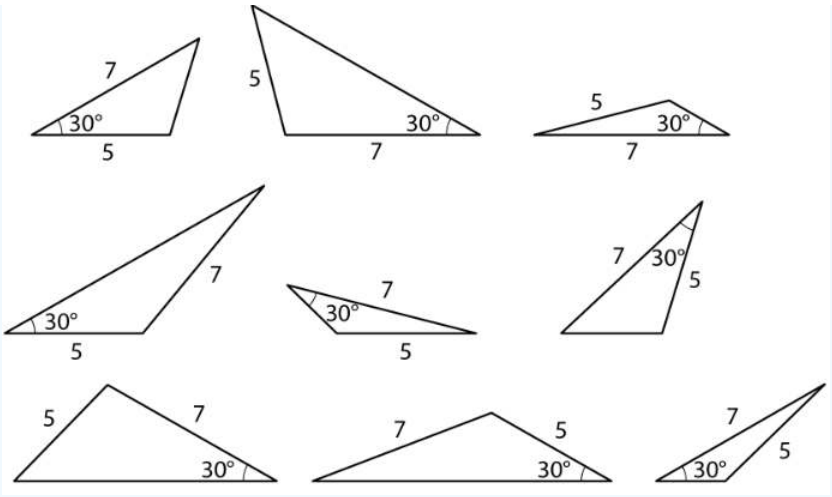


Figure 7.2.3.3

1. What is the same about the triangles in the set? What is different?
2. How many different triangles are there? Explain or show your reasoning.

Exercise 7.2.3.3: 2 Angles and 1 Side

Examine this set of triangles.

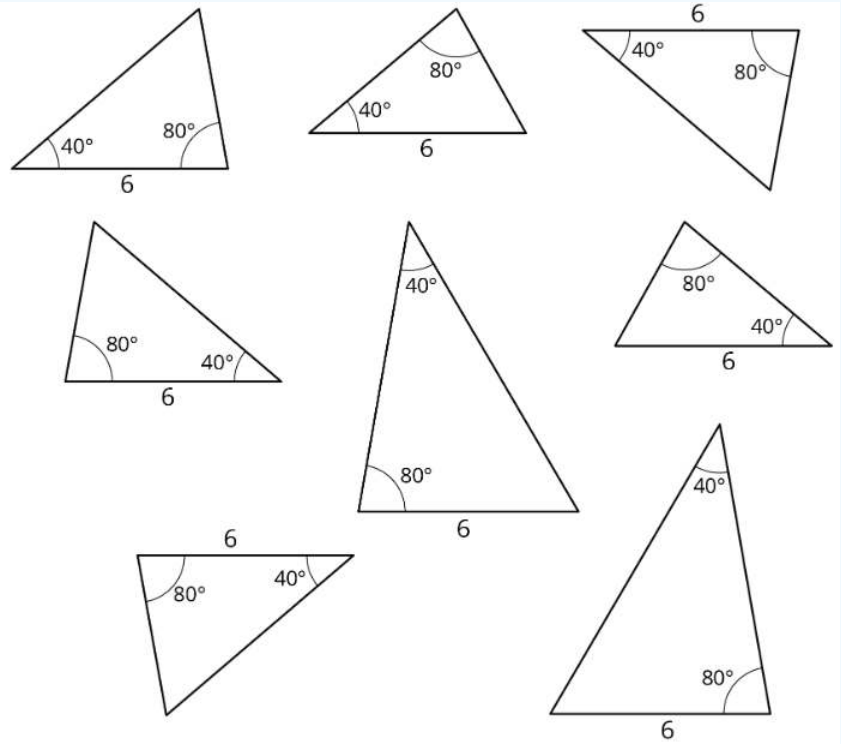


Figure 7.2.3.4

1. What is the same about the triangles in the set? What is different?
2. How many different triangles are there? Explain or show your reasoning.

Summary

Both of these quadrilaterals have a right angle and side lengths 4 and 5.

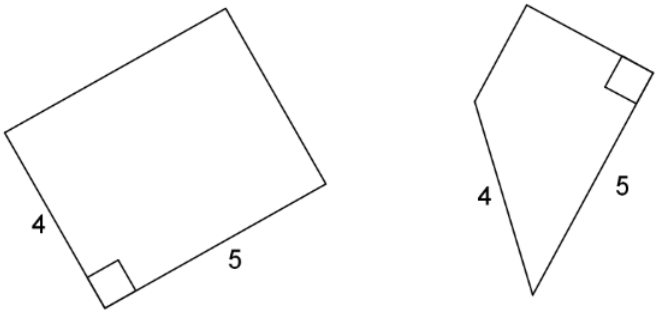


Figure 7.2.3.5: Two quadrilaterals each with two given side lengths labeled 4 and 5, and a right angle. On the left, the quadrilaterial is a rectangle with the right angle between adjacent side lengths 4 and 5. On the right, the quadrilateral is a trapezoid with the bottom base labeled 5 and one leg labeled 4. There is a right angle between the bottom base and the leg not labeled.

However, in one case, the right angle is *between* the two given side lengths; in the other, it is not.

If we create two triangles with three equal measures, but these measures are not next to each other in the same order, that usually means the triangles are different. Here is an example:

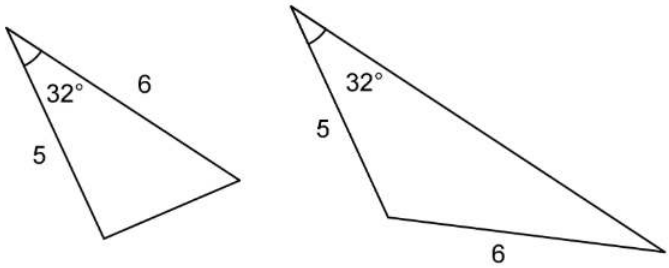


Figure 7.2.3.6: Two triangles. The triangle on the left has the angle labeled 32 degrees between the adjacent side lengths 5 and 6. The triangle on the right has the angle labeled 32 degrees between the side length labeled 5 and the third side of the triangle that is not labeled.

Practice

Exercise 7.2.3.4

Are these two triangles identical? Explain how you know.

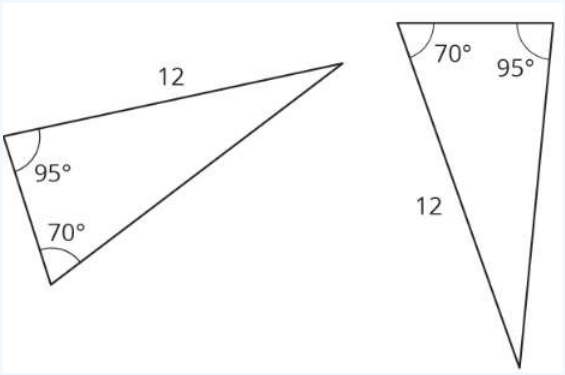


Figure 7.2.3.7

Exercise 7.2.3.5

Are these triangles identical? Explain your reasoning.

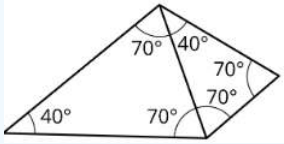


Figure 7.2.3.8: Two triangles shown that share one side. Starting from the bottom left angle, going clockwise, the triangle on the left has the angle measurements of 40 degrees, 70 degrees, and 70 degrees. The triangle on the right, starting at the top angle and going clockwise, has the angle measurements of 40 degrees, 70 degrees, and 70 degrees.

Exercise 7.2.3.6

Tyler claims that if two triangles each have a side length of 11 units and a side length of 8 units, and also an angle measuring 100°, they must be identical to each other. Do you agree? Explain your reasoning.

Exercise 7.2.3.7

The markings on the number line are equally spaced. Label the other markings on the number line.

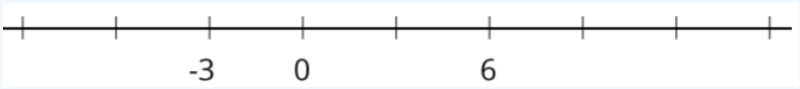


Figure 7.2.3.9

(From Unit 5.3.1)

Exercise 7.2.3.8

A passenger on a ship dropped his camera into the ocean. If it is descending at a rate of -4.2 meters per second, how long until it hits the bottom of the ocean, which is at -1,875 meters?

(From Unit 5.3.2)

Exercise 7.2.3.9

Apples cost $1.99 per pound.

1. How much do $3\frac{1}{4}$ pounds of apples cost?
2. How much do x pounds of apples cost?
3. Clare spent $5.17 on apples. How many pounds of apples did Clare buy?

(From Unit 4.1.3)

Exercise 7.2.3.10

Diego has a glue stick with a diameter of 0.7 inches. He sets it down 3.5 inches away from the edge of the table, but it rolls onto the floor. How many rotations did the glue stick make before it fell off of the table?

(From Unit 3.1.5)

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7.2.4: Drawing Triangles (Part 1)

Lesson

Let's see how many different triangles we can draw with certain measurements.

Exercise 7.2.4.1: Which One Doesn't Belong: Triangles

Which one doesn't belong?

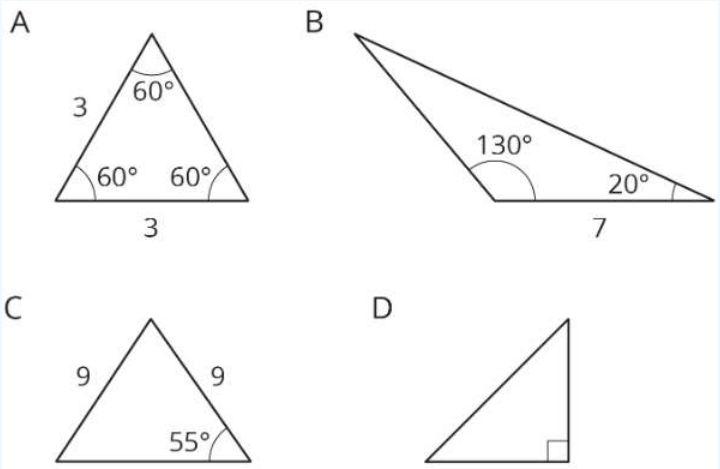


Figure 7.2.4.1: Four triangles, A, B, C, D. Triangle A has 3 60 degree angles and 2 sides with length 3. Triangle B has angles labeled 130 degrees, 20 degrees, and the side between them length 7. Triangle C has 2 sides length 9, angle 55 degrees. Triangle D has one angle marked 90 degrees.

Exercise 7.2.4.2: Does Your Triangle Match Theirs?

Three students have each drawn a triangle. For each description:

1. Drag the vertices to create a triangle with the given measurements.
2. Make note of the different side lengths and angle measures in your triangle.
3. Decide whether the triangle you made must be an identical copy of the triangle that the student drew. Explain your reasoning.

Jada's triangle has one angle measuring 75°.

Andre's triangle has one angle measuring 75° and one angle measuring 45°.

Lin's triangle has one angle measuring 75°, one angle measuring 45°, and one side measuring 5 cm.

Exercise 7.2.4.3: How Many Can You Draw?

1. Draw as many different triangles as you can with each of these sets of measurements:
   a. Two angles measure 60°, and one side measures 4 cm.
   b. Two angles measure 90°, and one side measures 4 cm.
   c. One angle measures 60°, one angle measures 90°, and one side measures 4 cm.
2. Which sets of measurements determine one unique triangle? Explain or show your reasoning.

Are you ready for more?

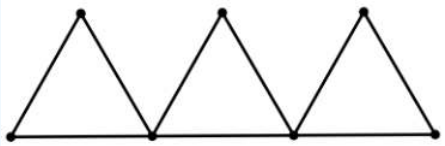


Figure 7.2.4.2

In the diagram, 9 toothpicks are used to make three equilateral triangles. Figure out a way to move only 3 of the toothpicks so that the diagram has exactly 5 equilateral triangles.

Summary

Sometimes, we are given two different angle measures and a side length, and it is impossible to draw a triangle. For example, there is no triangle with side length 2 and angle measures 120° and 100°:

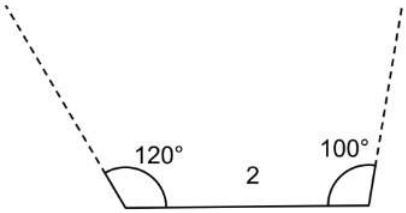


Figure 7.2.4.3: In the figure a horizontal line segment is drawn and labeled 2. On the left end of the line segment, a dashed line is drawn upward and to the left. The angle formed between the dashed line and the horizontal line is labeled 120 degrees. On the right end of the horizontal line, a dashed line is drawn upward and to the right. The angle formed between the dashed line and horizontal line is labeled 100 degrees.

Sometimes, we are given two different angle measures and a side length between them, and we *can* draw a unique triangle. For example, if we draw a triangle with a side length of 4 between angles 90° and 60°, there is only one way they can meet up and complete to a triangle:

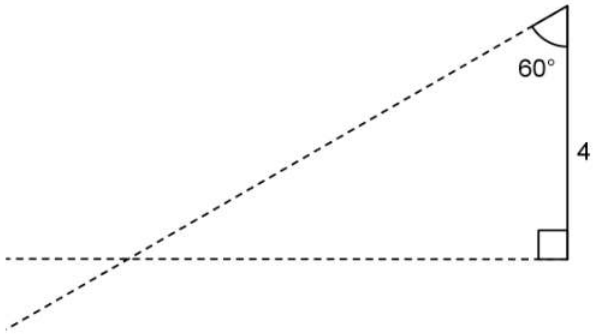


Figure 7.2.4.4

Any triangle drawn with these three conditions will be identical to the one above, with the same side lengths and same angle measures.

Practice

Exercise 7.2.4.4

Use a protractor to try to draw each triangle. Which of these three triangles is impossible to draw?

1. A triangle where one angle measures 20° and another angle measures 45°
2. A triangle where one angle measures 120° and another angle measures 50°
3. A triangle where one angle measures 90° and another angle measures 100°

Exercise 7.2.4.5

A triangle has an angle measuring 90°, an angle measuring 20°, and a side that is 6 units long. The 6-unit side is in between the 90° and 20° angles.

1. Sketch this triangle and label your sketch with the given measures.
2. How many unique triangles can you draw like this?

Exercise 7.2.4.6

1. Find a value for x that makes $-x$ less than $2x$.
2. Find a value for x that makes $-x$ greater than $2x$.

(From Unit 5.4.1)

Exercise 7.2.4.7

One of the particles in atoms is called an electron. It has a charge of -1. Another particle in atoms is a proton. It has charge of +1.

The overall charge of an atom is the sum of the charges of the electrons and the protons. Here is a list of common elements.

Table 7.2.4.1

	charge from electrons	charge from protons	overall charge
carbon	-6	$+6$	0
aluminum	-10	$+13$	
phosphide	-18	$+15$	
iodide	-54	$+53$	
tin	-50	$+50$	

Find the overall charge for the rest of the atoms on the list.

(From Unit 5.2.2)

Exercise 7.2.4.8

A factory produces 3 bottles of sparkling water for every 7 bottles of plain water. If those are the only two products they produce, what percentage of their production is sparkling water? What percentage is plain?

(From Unit 4.1.3)

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7.2.5: Drawing Triangles (Part 2)

Lesson

Let's draw some more triangles.

Exercise 7.2.5.1: Using a Compass to Estimate Length

1. Draw a 40° angle.
2. Use a compass to make sure both sides of your angle have a length of 5 centimeters.
3. If you connect the ends of the sides you drew to make a triangle, is the third side longer or shorter than 5 centimeters? How can you use a compass to explain your answer?

Exercise 7.2.5.2: Revisiting How Many Can You Draw?

Use the applet to draw triangles.

1. Draw as many different triangles as you can with each of these sets of measurements:
   a. One angle measures 40°, one side measures 4 cm and one side measures 5 cm.
   b. Two sides measure 6 cm and one angle measures 100°.
2. Did either of these sets of measurements determine one unique triangle? How do you know?

Exercise 7.2.5.3: Three Angles

Use the applet to draw triangles. Sides can overlap.

1. Draw as many different triangles as you can with each of these sets of measurements:
   a. One angle measures 50°, one measures 60°, and one measures 70°.
   b. One angle measures 50°, one measures 60°, and one measures 100°.
2. Did either of these sets of measurements determine one unique triangle? How do you know?

Are you ready for more?

Using only the point, segment, and compass tools provided, create an equilateral triangle. You are only successful if the triangle *remains* equilateral while dragging its vertices around.

GeoGebra Applet DsB2VFYv

Summary

A triangle has six measures: three side lengths and three angle measures.

If we are given three measures, then sometimes, there is no triangle that can be made. For example, there is no triangle with side lengths 1, 2, 5, and there is no triangle with all three angles measuring 150°.

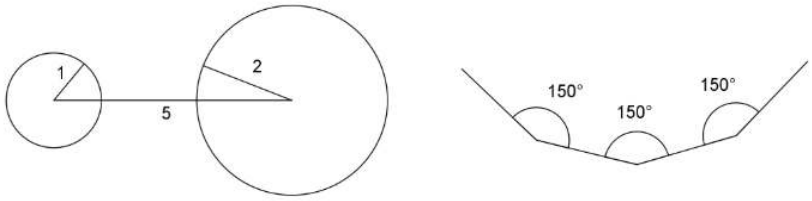


Figure 7.2.5.1

Sometimes, only one triangle can be made. By this we mean that any triangle we make will be the same, having the same six measures. For example, if a triangle can be made with three given side lengths, then the corresponding angles will have the same measures. Another example is shown here: an angle measuring 45° between two side lengths of 6 and 8 units. With this information, one unique triangle can be made.

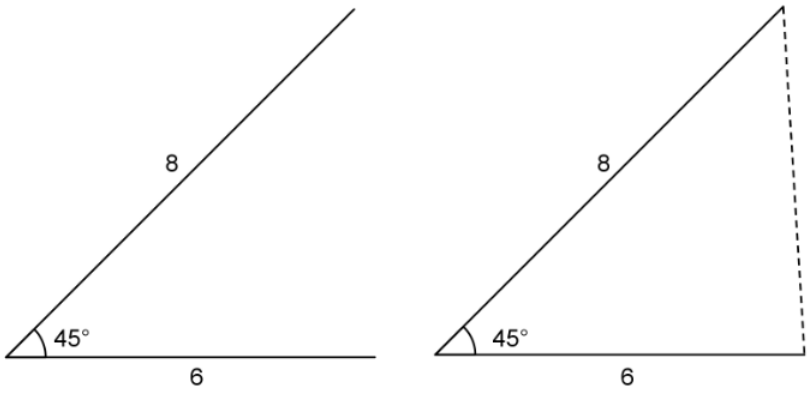


Figure 7.2.5.2

Sometimes, two or more different triangles can be made with three given measures. For example, here are two different triangles that can be made with an angle measuring 45° and side lengths 6 and 8. Notice the angle is not between the given sides.

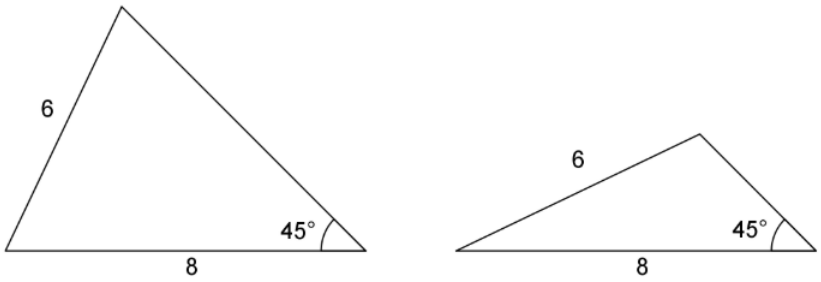


Figure 7.2.5.3

Three pieces of information about a triangle's side lengths and angle measures may determine no triangles, one unique triangle, or more than one triangle. It depends on the information.

Practice

Exercise 7.2.5.4

Three pieces of information about a triangle's side lengths and angle measures may determine no triangles, one unique triangle, or more than one triangle. It depends on the information.

Exercise 7.2.5.5

A triangle has one side that is 5 units long and an adjacent angle that measures 25°. The two other angles in the triangle measure 90° and 65°. Complete the two diagrams to create two *different* triangles with these measurements.

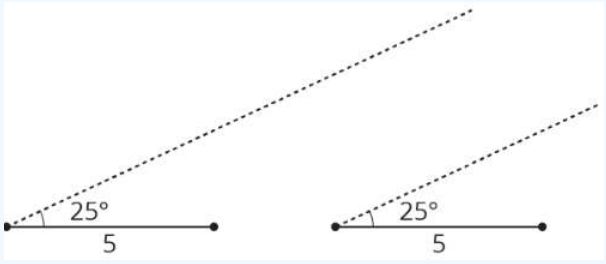


Figure 7.2.5.4

Exercise 7.2.5.6

Is it possible to make a triangle that has angles measuring 90 degrees, 30 degrees, and 100 degrees? If so, draw an example. If not, explain your reasoning.

Exercise 7.2.5.7

Segments CD, AB, and FG intersect at point E. Angle FEC is a right angle. Identify any pairs of angles that are complementary.

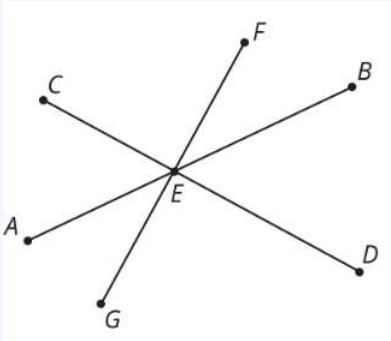


Figure 7.2.5.5

(From Unit 7.1.2)

Exercise 7.2.5.8

Match each equation to a step that will help solve the equation for x.

A. $3x = -4$
B. $-4.5 = x - 3$
C. $3 = \frac{-x}{3}$
D. $\frac{1}{3} = -3x$
E. $x - \frac{1}{3} = 0.4$
F. $3 + x = 8$
G. $\frac{x}{3} = 15$
H. $7 = \frac{1}{3} + x$

1. Add $\frac{1}{3}$ to each side.
2. Add $\frac{-1}{3}$ to each side.
3. Add 3 to each side.
4. Add -3 to each side.
5. Multiply each side by 3.
6. Multiply each side by -3.
7. Multiply each side by $\frac{1}{3}$.
8. Multiply each side by $\frac{-1}{3}$.

(From Unit 5.5.1)

Exercise 7.2.5.9

1. If you deposit $300 in an account with a 6% interest rate, how much will be in your account after 1 year?
2. If you leave this money in the account, how much will be in your account after 2 years?

(From Unit 4.2.3)

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7.3: Solid Geometry

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7.3.1: Slicing Solids

Lesson

Let's see what shapes you get when you slice a three-dimensional object.

Exercise 7.3.1.1: Prisms, Pyramids, and Polyhedra

Describe each shape as precisely as you can. Click on the applet and drag the mouse to show the object turning in 3D.

Exercise 7.3.1.2: What's the Cross Section

Here are a rectangular prism and a pyramid with the same base and same height. Drag the large red point up and down to move the plane through the solids.

1. If we slice each solid parallel to its base halfway up, what shape cross sections would we get? What is the same about the cross sections? What is different?
2. If we slice each solid parallel to its base near the top, what shape cross sections would we get? What is the same about the cross sections? What is different?

Are you ready for more?

Describe the cross sections that would result from slicing each solid perpendicular to its base.

Exercise 7.3.1.3: Card Sort: Cross Sections

Your teacher will give you a set of cards. Sort the images into groups that make sense to you. Be prepared to explain your reasoning.

Exercise 7.3.1.4: Drawing Cross Sections

Use the applet to draw each cross section and describe it in words. Start each one by dragging the gray bar on the left side across the screen until you see the whole 3D image on the left and the controls on the right.

1. Here is an applet with a rectangular prism, 4 units by 2 units by 3 units.
   a. A plane cuts the prism parallel to the bottom and top faces.
   b. The plane moves up and cuts the prism at a different height.
   c. A vertical plane cuts the prism diagonally.
2. A square pyramid has a base that is 4 units by 4 units. Its height is also 4 units.
   a. A plane cuts the pyramid parallel to the base.
   b. A vertical plane cuts the prism.
3. A cube has an edge of length 4.
   a. A plane cuts off the corner of the cube.
   b. The plane moves farther from the corner and makes a cut through the middle of the cube.

Summary

When we slice a three-dimensional object, we expose new faces that are two dimensional. The two-dimensional face is a **cross section**. Many different cross sections are possible when slicing the same three-dimensional object.

Here are two peppers. One is sliced horizontally, and the other is sliced vertically, producing different cross sections.



Figure 7.3.1.1

The imprints of the slices represent the two-dimensional faces created by each slice.

It takes practice imagining what the cross section of a three-dimensional object will be for different slices. It helps to experiment and see for yourself what happens!

Glossary Entries

Definition: Base (of a prism or pyramid)

The word *base* can also refer to a face of a polyhedron.

A prism has two identical bases that are parallel. A pyramid has one base.

A prism or pyramid is named for the shape of its base.

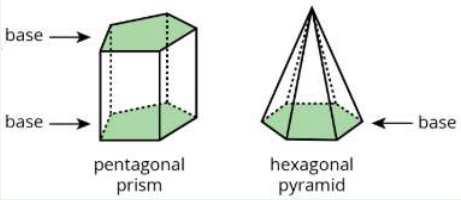


Figure 7.3.1.2: The figure on the left is labeled pentagonal prism. There are two identical pentagons on the top and bottom. Each vertex of a pentagon is connected by a vertical segment to the corresponding vertex of the other pentagons. The pentagons are each shaded, with the word base pointing to each. The figure on the right is labeled hexagonal pyramid. There is a hexagon on the bottom shaded green. From a point above the hexagon extend 6 segments, each connected to a vertex of the hexagon.

Definition: Cross Section

A cross section is the new face you see when you slice through a three-dimensional figure.

For example, if you slice a rectangular pyramid parallel to the base, you get a smaller rectangle as the cross section.

Definition: Prism

A prism is a type of polyhedron that has two bases that are identical copies of each other. The bases are connected by rectangles or parallelograms.

Here are some drawings of prisms.

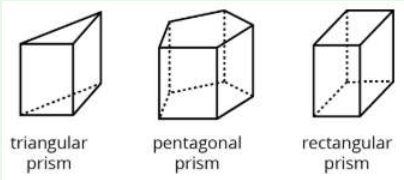


Figure 7.3.1.3

Definition: Pyramid

A pyramid is a type of polyhedron that has one base. All the other faces are triangles, and they all meet at a single vertex.

Here are some drawings of pyramids.

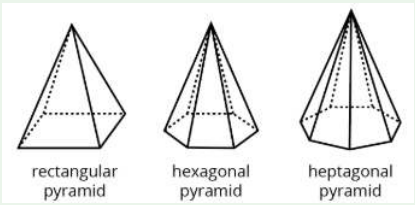


Figure 7.3.1.4

Practice

Exercise 7.3.1.5

A cube is cut into two pieces by a single slice that passes through points A, B, and C. What shape is the cross section?

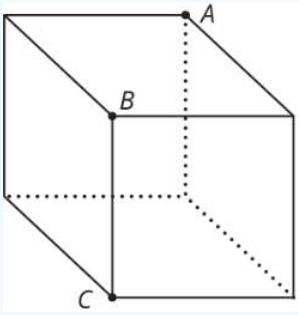


Figure 7.3.1.5

Exercise 7.3.1.6

Describe how to slice the three-dimensional figure to result in each cross section.

Three-dimensional figure:

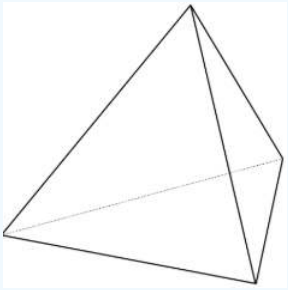


Figure 7.3.1.6

Cross sections:

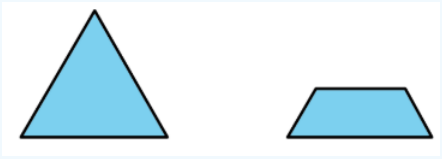


Figure 7.3.1.7

Exercise 7.3.1.7

Here are two three-dimensional figures.

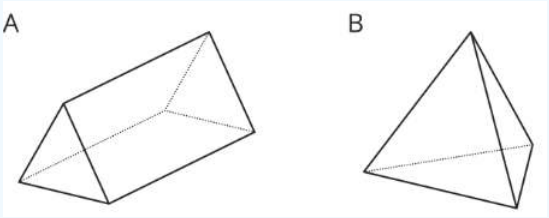


Figure 7.3.1.8

Describe a way to slice one of the figures so that the cross section is a rectangle.

https://math.libretexts.org/@go/page/38738

Exercise 7.3.1.8

Each row contains the degree measures of two supplementary angles. Complete the table.

Table 7.3.1.1

measure of an angle	measure of its supplement
80°	
25°	
119°	
x	

(From Unit 7.1.2)

Exercise 7.3.1.9

Two months ago, the price, in dollars, of a cell phone was c.

1. Last month, the price of the phone increased by 10%. Write an expression for the price of the phone last month.
2. This month, the price of the phone decreased by 10%. Write an expression for the price of the phone this month.
3. Is the price of the phone this month the same as it was two months ago? Explain your reasoning.

(From Unit 4.2.3)

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7.3.2: Volume of Right Prisms

Lesson

Let's look at volumes of prisms.

Exercise 7.3.2.1: Three Prisms with the Same Volume

Rectangles A, B, and C represent bases of three prisms.

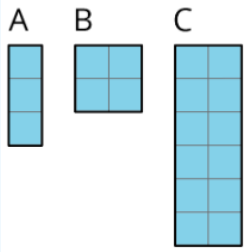


Figure 7.3.2.1

1. If each prism has the same height, which one will have the greatest **volume**, and which will have the least? Explain your reasoning.
2. If each prism has the same volume, which one will have the tallest height, and which will have the shortest? Explain your reasoning.

Exercise 7.3.2.2: Finding Volume with Cubes

This applet has 64 snap cubes, all sitting in the same spot on the screen, like a hidden stack of blocks. You will always know where the stack is because it sits on a gray square. You can keep dragging blocks out of the pile by their red points until you have enough to build what you want.

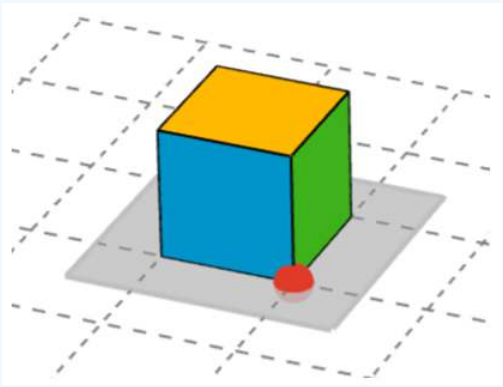


Figure 7.3.2.2

Click on the red points to change from left/right movement to up/down movement.

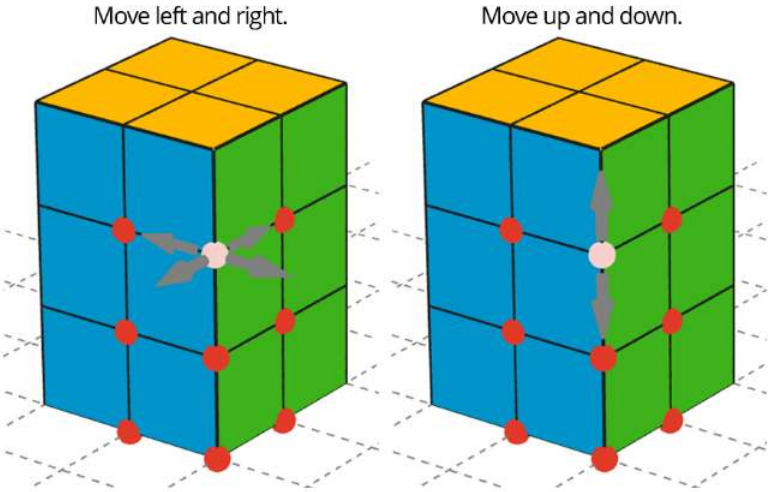


Figure 7.3.2.3

There is also a shape on the grid. It marks the footprint of the shapes you will be building.

1. Using the face of a snap cube as your area unit, what is the area of the shape? Explain or show your reasoning.
2. Use snap cubes to build the shape from the paper. Add another layer of cubes on top of the shape you have built. Describe this three-dimensional object.
3. What is the volume of your object? Explain your reasoning.
4. Right now, your object has a height of 2. What would the volume be
    a. if it had a height of 5?
    b. if it had a height of 8.5?

Exercise 7.3.2.3: Can You Find the Volume?

The applet has a set of three-dimensional figures.

1. For each figure, determine whether the shape is a prism.
2. For each prism:
    a. Find the area of the base of the prism.
    b. Find the height of the prism.
    c. Calculate the volume of the prism.

Table 7.3.2.1

Is it a prism?	area of prism base (cm^2)	height (cm)	volume (cm^3)

- Begin by grabbing the gray bar on the left and dragging it to the right until you see the slider.
- Choose a figure using the slider.
- Rotate the view using the Rotate 3D Graphics tool marked by two intersecting, curved arrows.
- Note that each polyhedron has only one label per unique face. Where no measurements are shown, the faces are identical copies.
- Use the distance tool, marked with the "cm," to click on any segment and find the height or length.
- Troubleshooting tip: the cursor must be on the 3D Graphics window for the full toolbar to appear.

Are you ready for more?

Imagine a large, solid cube made out of 64 white snap cubes. Someone spray paints all 6 faces of the large cube blue. After the paint dries, they disassemble the large cube into a pile of 64 snap cubes.

1. How many of those 64 snap cubes have exactly 2 faces that are blue?

2. What are the other possible numbers of blue faces the cubes can have? How many of each are there?
3. Try this problem again with some larger-sized cubes that use more than 64 snap cubes to build. What patterns do you notice?

Exercise 7.3.2.4: What's the Prism's Height?

There are 4 different prisms that all have the same volume. Here is what the base of each prism looks like.

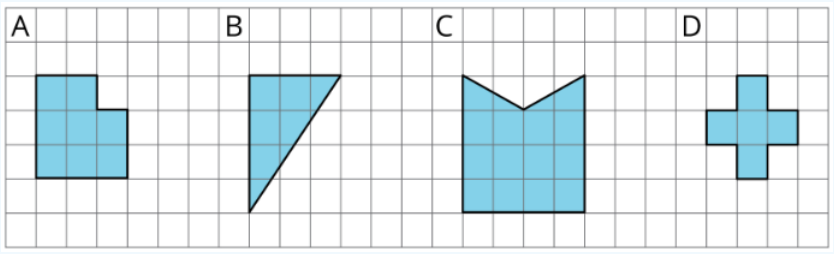


Figure 7.3.2.4: Four polygons on a grid. Polygon A, a 3 by 3 block with a 1 by 1 block missing from the corner. Polygon B, a right triangle 3 blocks by 4 blocks. Polygon C, a 4 by 4 rectangle with a 4 by 1 triangle missing from the top. Polygon D, 5 blocks arranged like a plus sign.

1. Order the prisms from shortest to tallest. Explain your reasoning.
2. If the volume of each prism is 60 units^3, what would be the height of each prism?
3. For a volume other than 60 units^3, what could be the height of each prism?
4. Discuss your thinking with your partner. If you disagree, work to reach an agreement.

Summary

Any cross section of a prism that is parallel to the base will be identical to the base. This means we can slice prisms up to help find their volume. For example, if we have a rectangular prism that is 3 units tall and has a base that is 4 units by 5 units, we can think of this as 3 layers, where each layer has $4 \cdot 5$ cubic units.

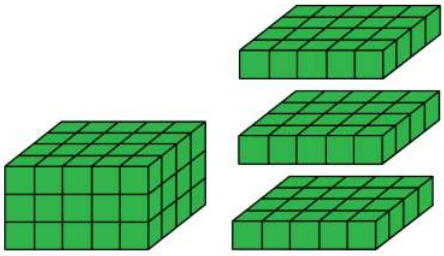


Figure 7.3.2.5

That means the volume of the original rectangular prism is $3(4 \cdot 5)$ cubic units.

This works with any prism! If we have a prism with height 3 cm that has a base of area 20 cm^2, then the volume is $3 \cdot 20$ cm^3 regardless of the shape of the base. In general, the volume of a prism with height h and area B is

$V = B \cdot h$

For example, these two prisms both have a volume of 100 cm^3.

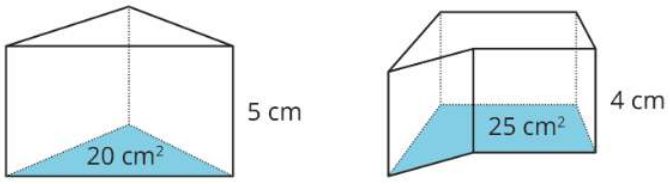


Figure 7.3.2.6

Definition: Base (of a prism or pyramid)

The word *base* can also refer to a face of a polyhedron.

A prism has two identical bases that are parallel. A pyramid has one base.

A prism or pyramid is named for the shape of its base.

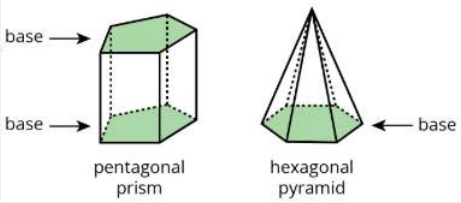


Figure 7.3.2.7: The figure on the left is labeled pentagonal prism. There are two identical pentagons on the top and bottom. Each vertex of a pentagon is connected by a vertical segment to the corresponding vertex of the other pentagons. The pentagons are each shaded, with the word base pointing to each. The figure on the right is labeled hexagonal pyramid. There is a hexagon on the bottom shaded green. From a point above the hexagon extend 6 segments, each connected to a vertex of the hexagon.

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A cross section is the new face you see when you slice through a three-dimensional figure.

For example, if you slice a rectangular pyramid parallel to the base, you get a smaller rectangle as the cross section.

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A prism is a type of polyhedron that has two bases that are identical copies of each other. The bases are connected by rectangles or parallelograms.

Here are some drawings of prisms.

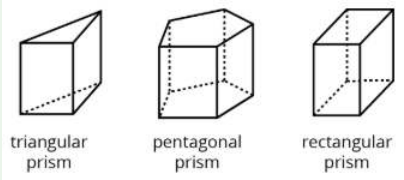


Figure 7.3.2.8

Definition: Pyramid

A pyramid is a type of polyhedron that has one base. All the other faces are triangles, and they all meet at a single vertex.

Here are some drawings of pyramids.

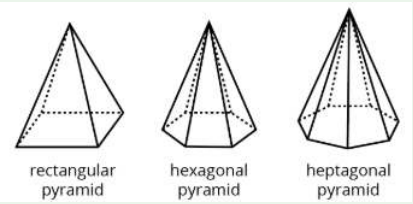


Figure 7.3.2.9

Definition: Volume

Volume is the number of cubic units that fill a three-dimensional region, without any gaps or overlaps.

For example, the volume of this rectangular prism is 60 units^3, because it is composed of 3 layers that are each 20 units^3.

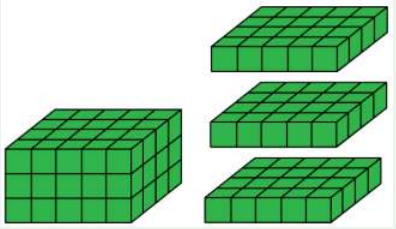


Figure 7.3.2.10

Practice

Exercise 7.3.2.5

1. Select **all** the prisms.
2. For each prism, shade one of its bases.

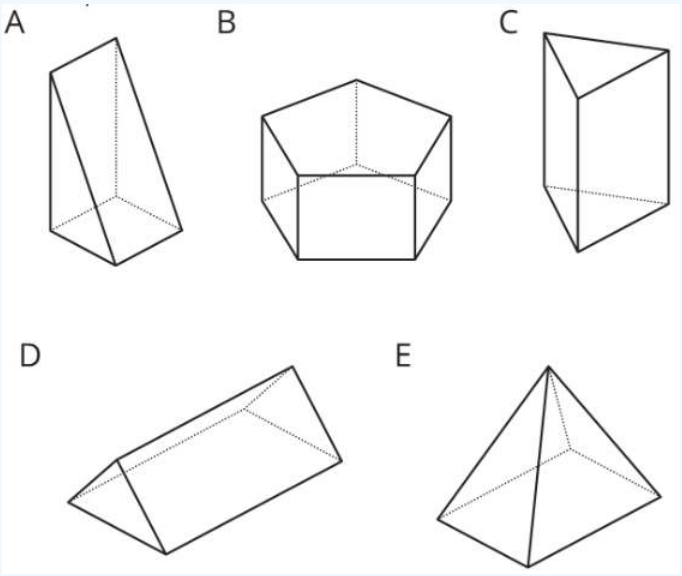


Figure 7.3.2.11

Exercise 7.3.2.6

The volume of both of these trapezoidal prisms is 24 cubic units. Their heights are 6 and 8 units, as labeled. What is the area of a trapezoidal base of each prism?

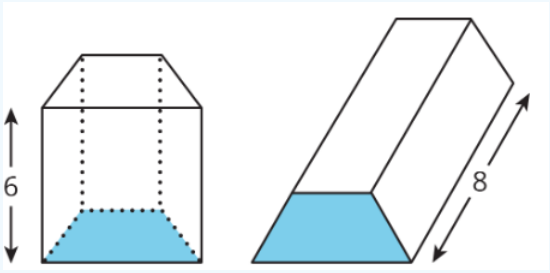


Figure 7.3.2.12

Exercise 7.3.2.7

Two angles are complementary. One has a measure of 19 degrees. What is the measure of the other?

(From Unit 7.1.2)

Exercise 7.3.2.8

Two angles are supplementary. One has a measure that is twice as large as the other. Find the two angle measures.

(From Unit 7.1.2)

Exercise 7.3.2.9

Match each expression in the first list with an equivalent expression from the second list.

A. $7(x+2)-x+3$
B. $6x+3+4x+5$
C. $\frac{-2}{5}x-7+\frac{3}{5}x-3$
D. $8x-5+4-9$
E. $24x+36$

1. $\frac{1}{5}x-10$
2. $6x+17$
3. $2(5x+4)$
4. $12(2x+3)$
5. $8x+(-5)+4+(-9)$

(From Unit 6.4.5)

Exercise 7.3.2.10

Clare paid 50% more for her notebook than Priya paid for hers. Priya paid x for her notebook and Clare paid y dollars for hers. Write an equation that represents the relationship between y and x.

(From Unit 4.2.3)

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7.3.3: Decomposing Bases for Area

Lesson

Let's look at how some people use volume.

Exercise 7.3.3.1: Are These Prisms?

1. Which of these solids are prisms? Explain how you know.

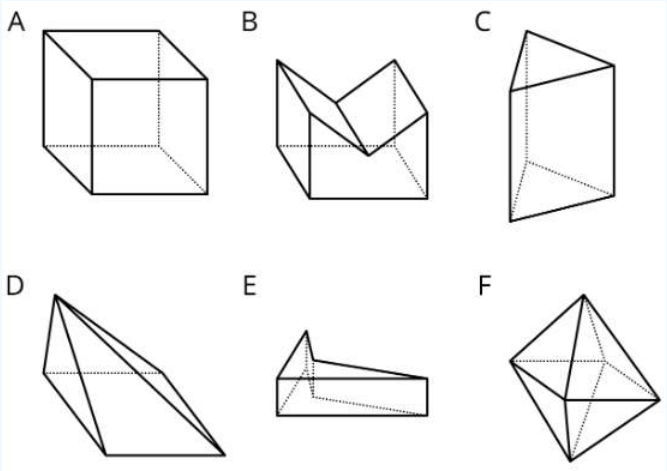


Figure 7.3.3.1

2. For each of the prisms, what does the base look like?

   a. Shade one base in the picture.
   b. Draw a cross section of the prism parallel to the base.

Exercise 7.3.3.2: A Box of Chocolates

A box of chocolates is a prism with a base in the shape of a heart and a height of 2 inches. Here are the measurements of the base.

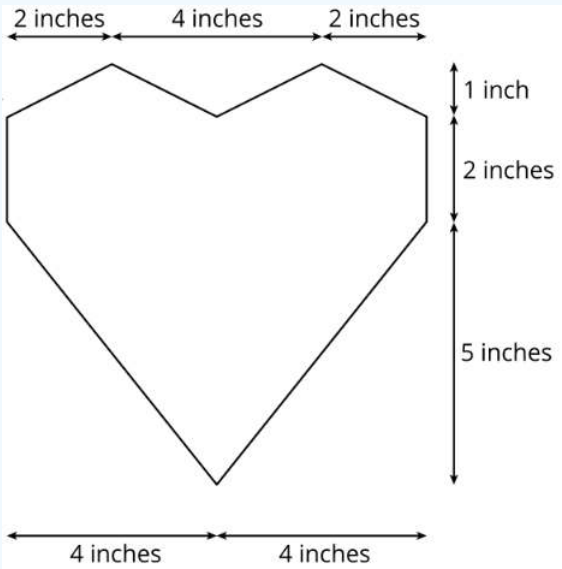


Figure 7.3.3.2

To calculate the volume of the box, three different students have each drawn line segments showing how they plan on finding the area of the heart-shaped base.

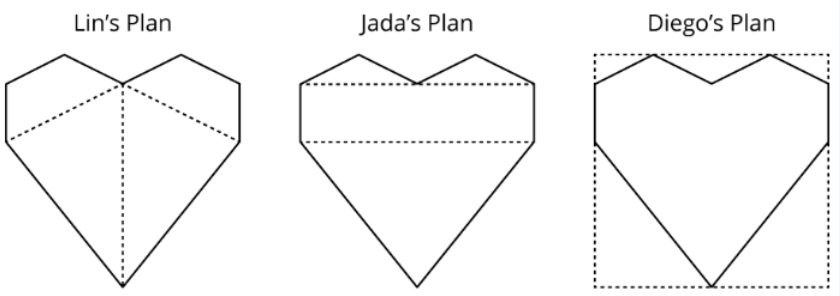


Figure 7.3.3.3

1. For each student's plan, describe the shapes the student must find the area of and the operations they must use to calculate the total area.
2. Although all three methods could work, one of them requires measurements that are not provided. Which one is it?
3. Between you and your partner, decide which of you will use which of the remaining two methods.
4. Using the quadrilaterals and triangles drawn in your selected plan, find the area of the base.
5. Trade with a partner and check each other's work. If you disagree, work to reach an agreement.
6. Return their work. Calculate the volume of the box of chocolates.

Are you ready for more?

The box has 30 pieces of chocolate in it, each with a volume of 1 in^3. If all the chocolates melt into a solid layer across the bottom of the box, what will be the height of the layer?

Exercise 7.3.3.3: Another Prism

A house-shaped prism is created by attaching a triangular prism on top of a rectangular prism.

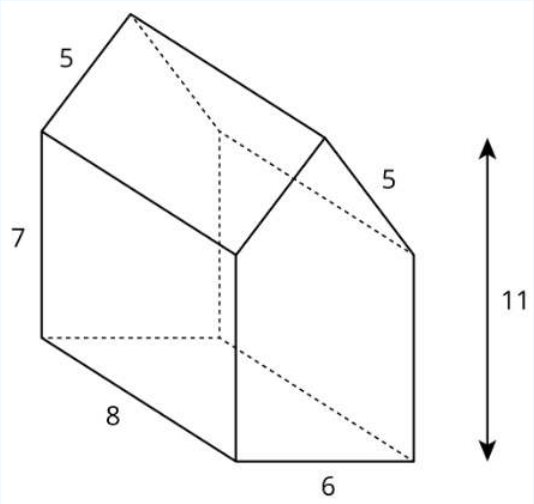


Figure 7.3.3.4

1. Draw the base of this prism and label its dimensions.
2. What is the area of the base? Explain or show your reasoning.
3. What is the volume of the prism?

Summary

To find the area of any polygon, you can decompose it into rectangles and triangles. There are always many ways to decompose a polygon.

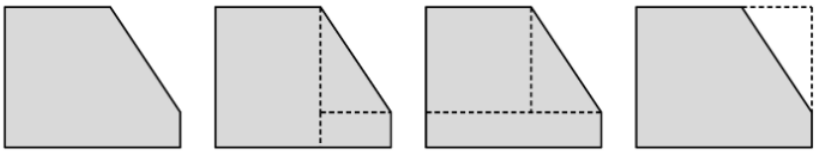


Figure 7.3.3.5

Sometimes it is easier to enclose a polygon in a rectangle and subtract the area of the extra pieces.

To find the volume of a prism with a polygon for a base, you find the area of the base, B, and multiply by the height, h.

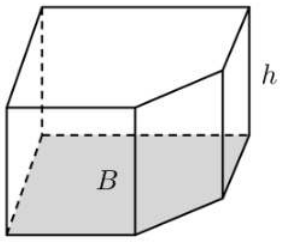


Figure 7.3.3.6

$V = Bh$

Glossary Entries

Definition: Base (of a prism or pyramid)

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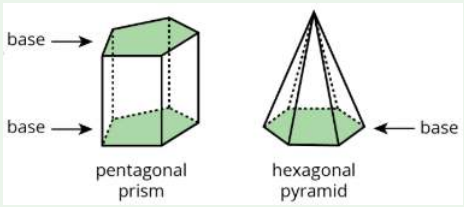


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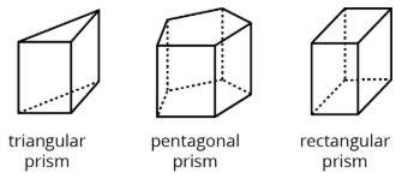


Figure 7.3.3.8

Definition: Pyramid

A pyramid is a type of polyhedron that has one base. All the other faces are triangles, and they all meet at a single vertex.

Here are some drawings of pyramids.

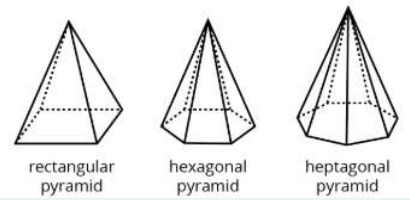


Figure 7.3.3.9

Definition: Volume

Volume is the number of cubic units that fill a three-dimensional region, without any gaps or overlaps.

For example, the volume of this rectangular prism is 60 units^3, because it is composed of 3 layers that are each 20 units^3.

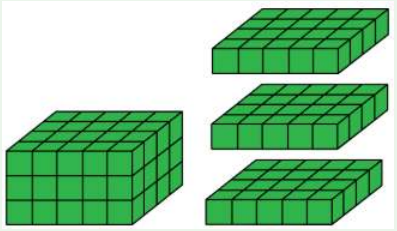


Figure 7.3.3.10

Practice

Exercise 7.3.3.4

You find a crystal in the shape of a prism. Find the volume of the crystal.

The point B is directly underneath point E, and the following lengths are known:

- From A to B: 2 mm
- From B to C: 3 mm
- From A to F: 6 mm
- From B to E: 10 mm
- From C to D: 7 mm
- From A to G: 4 mm

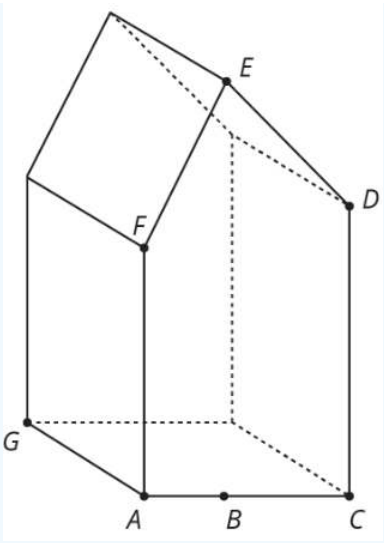


Figure 7.3.3.11

Exercise 7.3.3.5

A rectangular prism with dimensions 5 inches by 13 inches by 10 inches was cut to leave a piece as shown in the image. What is the volume of this piece? What is the volume of the other piece not pictured?

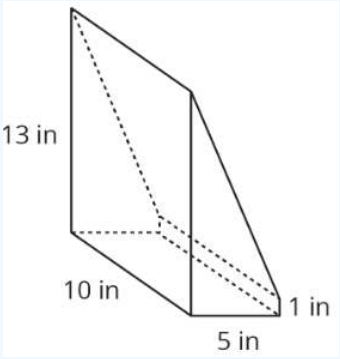


Figure 7.3.3.12

Exercise 7.3.3.6

A triangle has one side that is 7 cm long and another side that is 3 cm long.

1. Sketch this triangle and label your sketch with the given measures. (If you are stuck, try using a compass or cutting some straws to these two lengths.)
2. Draw one more triangle with these measures that is not identical to your first triangle.
3. Explain how you can tell they are not identical.

(From Unit 7.2.4)

Exercise 7.3.3.7

Select **all** equations that represent a relationship between angles in the figure.

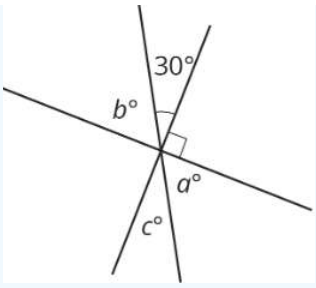


Figure 7.3.3.13

A. $90 - 30 = b$
B. $30 + b = a + c$
C. $a + c + 30 + b = 180$
D. $a = 30$
E. $a = c = 30$
F. $90 + a + c = 180$

(From Unit 7.1.4)

Exercise 7.3.3.8

A mixture of punch contains 1 quart of lemonade, 2 cups of grape juice, 4 tablespoons of honey, and $\frac{1}{2}$ gallon of sparkling water. Find the percentage of the punch mixture that comes from each ingredient. Round your answers to the nearest tenth of a percent. (Hint: 1 cup = 16 tablespoons)

(From Unit 4.2.4)

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7.3.4: Surface Area of Right Prisms

Lesson

Let's look at the surface area of prisms.

Exercise 7.3.4.1: Multifaceted

Your teacher will show you a prism.

1. What are some things you could measure about the object?
2. What units would you use for these measurements?

Exercise 7.3.4.2: So Many Faces

Here is a picture of your teacher's prism:

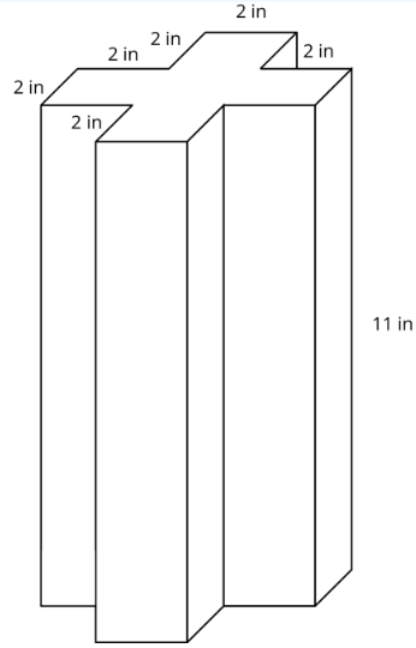


Figure 7.3.4.1

Three students are trying to calculate the **surface area** of this prism.

- Noah says, “This is going to be a lot of work. We have to find the areas of 14 different faces and add them up.”
- Elena says, “It’s not so bad. All 12 rectangles are identical copies, so we can find the area for one of them, multiply that by 12 and then add on the areas of the 2 bases.”
- Andre says, “Wait, I see another way! Imagine unfolding the prism into a net. We can use 1 large rectangle instead of 12 smaller ones.”

1. Do you agree with any of them? Explain your reasoning.
2. How big is the “1 large rectangle” Andre is talking about? Explain or show your reasoning. If you get stuck, consider drawing a net for the prism.
3. Will Noah’s method always work for finding the surface area of any prism? Elena’s method? Andre’s method? Be prepared to explain your reasoning.
4. Which method do you prefer? Why?

Exercise 7.3.4.3: Revisiting a Pentagonal Prism

1. Between you and your partner, choose who will use each of these two methods to find the surface area of the prism.
   - Adding the areas of all the faces
   - Using the perimeter of the base.
2. Use your chosen method to calculate the surface area of the prism. Show your thinking. Organize it so it can be followed by others.

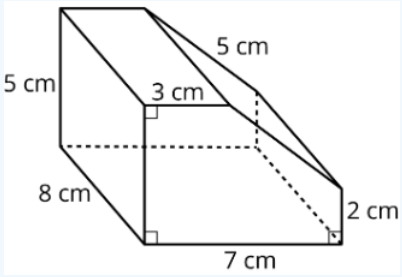


Figure 7.3.4.2

3. Trade papers with your partner, and check their work. Discuss your thinking. If you disagree, work to reach an agreement.

Are you ready for more?



Figure 7.3.4.3



Figure 7.3.4.4

In a deck of cards, each card measures 6 cm by 9 cm.

1. When stacked, the deck is 2 cm tall, as shown in the first photo. Find the volume of this deck of cards.
2. Then the cards are fanned out, as shown in the second picture. The distance from the rightmost point on the bottom card to the rightmost point on the top card is now 7 cm instead of 2 cm. Find the volume of the new stack.

Summary

To find the surface area of a three-dimensional figure whose faces are made up of polygons, we can find the area of each face, and add them up!

Sometimes there are ways to simplify our work. For example, all of the faces of a cube with side length s are the same. We can find the area of one face, and multiply by 6. Since the area of one face of a cube is s^2, the surface area of a cube is $6s^2$.

We can use this technique to make it faster to find the surface area of any figure that has faces that are the same.

For prisms, there is another way. We can treat the prism as having three parts: two identical bases, and one long rectangle that has been taped along the edges of the bases. The rectangle has the same height as the prism, and its width is the perimeter of the base. To find the surface area, add the area of this rectangle to the areas of the two bases.

Glossary Entries

Definition: Base (of a prism or pyramid)

The word *base* can also refer to a face of a polyhedron.

A prism has two identical bases that are parallel. A pyramid has one base.

A prism or pyramid is named for the shape of its base.

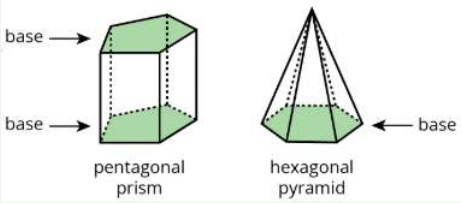


Figure 7.3.4.5: The figure on the left is labeled pentagonal prism. There are two identical pentagons on the top and bottom. Each vertex of a pentagon is connected by a vertical segment to the corresponding vertex of the other pentagons. The pentagons are each shaded, with the word base pointing to each. The figure on the right is labeled hexagonal pyramid. There is a hexagon on the bottom shaded green. From a point above the hexagon extend 6 segments, each connected to a vertex of the hexagon.

Definition: Cross Section

A cross section is the new face you see when you slice through a three-dimensional figure.

For example, if you slice a rectangular pyramid parallel to the base, you get a smaller rectangle as the cross section.

Definition: Prism

A prism is a type of polyhedron that has two bases that are identical copies of each other. The bases are connected by rectangles or parallelograms.

Here are some drawings of prisms.

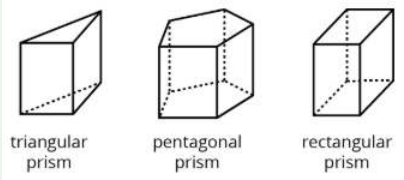


Figure 7.3.4.6

Definition: Pyramid

A pyramid is a type of polyhedron that has one base. All the other faces are triangles, and they all meet at a single vertex.

Here are some drawings of pyramids.

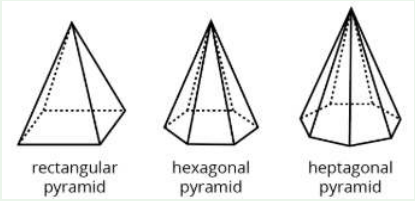


Figure 7.3.4.7

Definition: Surface Area

The surface area of a polyhedron is the number of square units that covers all the faces of the polyhedron, without any gaps or overlaps.

For example, if the faces of a cube each have an area of 9 cm^2, then the surface area of the cube is $6 \cdot 9$, or 54 cm^2.

Definition: Volume

Volume is the number of cubic units that fill a three-dimensional region, without any gaps or overlaps.

For example, the volume of this rectangular prism is 60 $units^3$, because it is composed of 3 layers that are each 20 $units^3$.

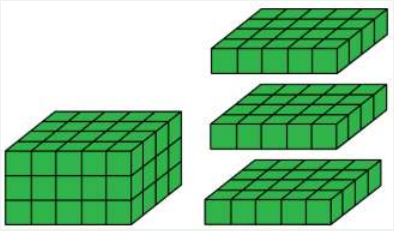


Figure 7.3.4.8

Practice

Exercise 7.3.4.4

Edge lengths are given in units. Find the surface area of each prism in square units.

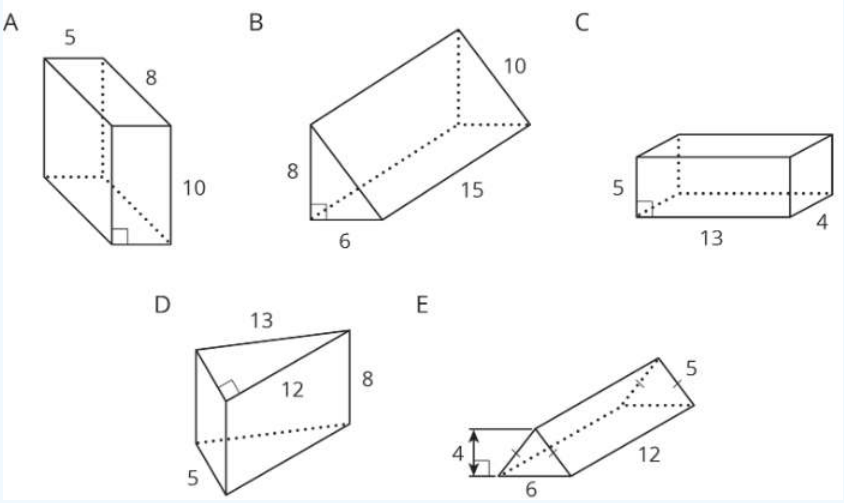


Figure 7.3.4.9: Five prisms. Prism A, rectangular prism with dimensions 5, 8, 10. Prism B, base is a right triangle with sides 6, 8, 10, prism height 15. Prism C, rectangular prism with dimensions 5, 13, 4. Prism D, base is a right triangle with sides 5, 12, and 13, prism height is 8. Prism E, base is a triangle with sides 6, 5, 5 and height perpendicular to the 6 side is 4, prism height 12.

Exercise 7.3.4.5

Priya says, “No matter which way you slice this rectangular prism, the cross section will be a rectangle.” Mai says, “I’m not so sure.” Describe a slice that Mai might be thinking of.

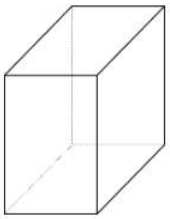


Figure 7.3.4.10

(From Unit 7.3.1)

Exercise 7.3.4.6

B is the intersection of line AC and line ED. Find the measure of each of the angles.

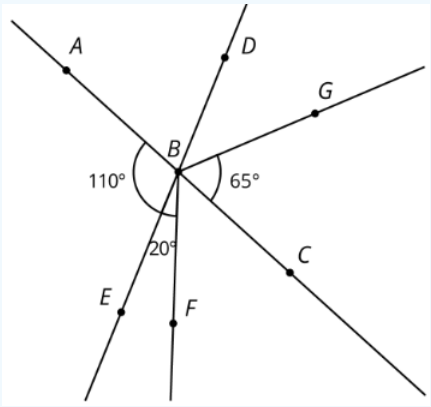


Figure 7.3.4.11: Two lines and two rays that intersect at point B creating 6 angles. Line AC is slanted downward and to the right and line ED is slanted upward and to the right. Ray BG is drawn between line segment BD and line segment BC. Angle GBC that is formed measures 65 degrees. Ray BF is drawn between line segment BC and line segment BE. Angle FBE that is formed measures 20 degrees. Angle ABE is labeled 110 degrees.

1. Angle ABF
2. Angle ABD
3. Angle EBC
4. Angle FBC
5. Angle DBG

(From Unit 7.1.5)

Exercise 7.3.4.7

Write each expression with fewer terms.

1. $12m - 4m$
2. $12m - 5k + m$
3. $9m + k - (3m - 2k)$

(From Unit 6.4.3)

Exercise 7.3.4.8

1. Find 44% of 625 using the facts that 40% of 625 is 250 and 4% of 625 is 25.
2. What is 4.4% of 625?
3. What is 0.44% of 625?

(From Unit 4.2.4)

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7.3.5: Distinguishing Volume and Surface Area

Lesson

Let's work with surface area and volume in context.

Exercise 7.3.5.1: THe Science Fair

Mai's science teacher told her that when there is more ice touching the water in a glass, the ice melts faster. She wants to test this statement so she designs her science fair project to determine if crushed ice or ice cubes will melt faster in a drink.

She begins with two cups of warm water. In one cup, she puts a cube of ice. In a second cup, she puts crushed ice with the same volume as the cube. What is your hypothesis? Will the ice cube or crushed ice melt faster, or will they melt at the same rate? Explain your reasoning.

Exercise 7.3.5.2: Revisiting the Box of Chocolates

The other day, you calculated the volume of this heart-shaped box of chocolates.

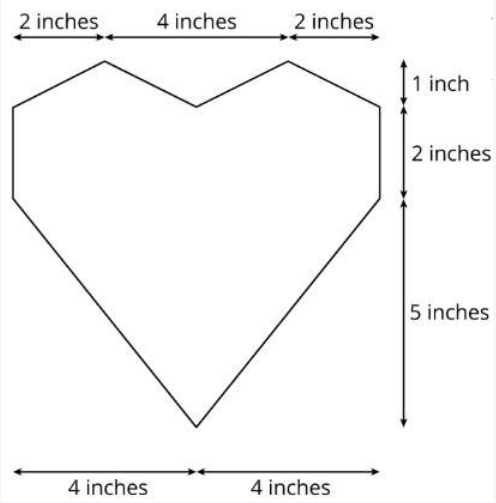


Figure 7.3.5.1

The depth of the box is 2 inches. How much cardboard is needed to create the box?

Exercise 7.3.5.3: Card Sort: Surface Area or Volume

Your teacher will give you cards with different figures and questions on them.

1. Sort the cards into two groups based on whether it would make more sense to think about the surface area or the volume of the figure when answering the question. Pause here so your teacher can review your work.
2. Your teacher will assign you a card to examine more closely. What additional information would you need to be able to answer the question on your card?
3. Estimate reasonable measurements for the figure on your card.
4. Use your estimated measurements to calculate the answer to the question.

Are you ready for more?

A cake is shaped like a square prism. The top is 20 centimeters on each side, and the cake is 10 centimeters tall. It has frosting on the sides and on the top, and a single candle on the top at the exact center of the square. You have a knife and a 20-centimeter ruler.

1. Find a way to cut the cake into 4 fair portions, so that all 4 portions have the same amount of cake and frosting.
2. Find another way to cut the cake into 4 fair portions.
3. Find a way to cut the cake into 5 fair portions.

Exercise 7.3.5.4: A Wheelbarrow of Concrete

A wheelbarrow is being used to carry wet concrete. Here are its dimensions.

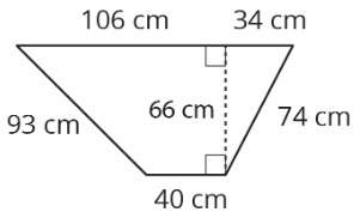


Figure 7.3.5.2

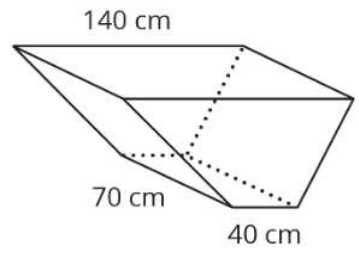


Figure 7.3.5.3

1. What volume of concrete would it take to fill the tray?
2. After dumping the wet concrete, you notice that a thin film is left on the inside of the tray. What is the area of the concrete coating the tray? (Remember, there is no top.)

Summary

Sometimes we need to find the volume of a prism, and sometimes we need to find the surface area.

Here are some examples of quantities related to volume:

- How much water a container can hold
- How much material it took to build a solid object

Volume is measured in cubic units, like in^3 or m^3.

Here are some examples of quantities related to surface area:

- How much fabric is needed to cover a surface
- How much of an object needs to be painted

Surface area is measured in square units, like in^2 or m^2.

Glossary Entries

Definition: Base (of a prism or pyramid)

The word *base* can also refer to a face of a polyhedron.

A prism has two identical bases that are parallel. A pyramid has one base.

A prism or pyramid is named for the shape of its base.

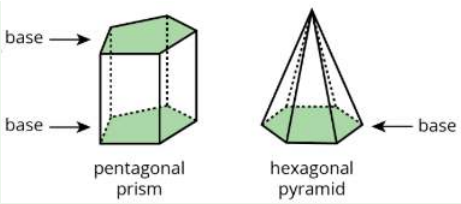


Figure 7.3.5.4: The figure on the left is labeled pentagonal prism. There are two identical pentagons on the top and bottom. Each vertex of a pentagon is connected by a vertical segment to the corresponding vertex of the other pentagons. The pentagons are each shaded, with the word base pointing to each. The figure on the right is labeled hexagonal pyramid. There is a hexagon on the bottom shaded green. From a point above the hexagon extend 6 segments, each connected to a vertex of the hexagon.

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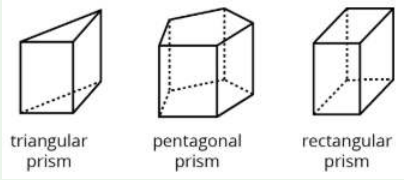


Figure 7.3.5.5

Definition: Pyramid

A pyramid is a type of polyhedron that has one base. All the other faces are triangles, and they all meet at a single vertex.

Here are some drawings of pyramids.

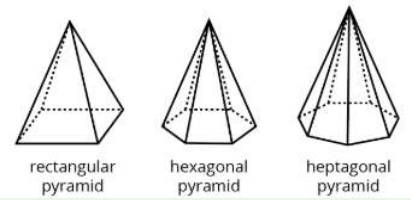


Figure 7.3.5.6

Definition: Surface Area

The surface area of a polyhedron is the number of square units that covers all the faces of the polyhedron, without any gaps or overlaps.

For example, if the faces of a cube each have an area of 9 cm^2, then the surface area of the cube is $6 \cdot 9$, or 54 cm^2.

Definition: Volume

Volume is the number of cubic units that fill a three-dimensional region, without any gaps or overlaps.

For example, the volume of this rectangular prism is 60 $units^3$, because it is composed of 3 layers that are each 20 $units^3$.

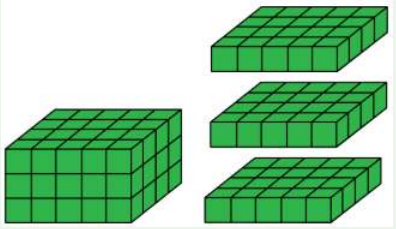


Figure 7.3.5.7

Practice

Exercise 7.3.5.5

Here is the base of a prism.

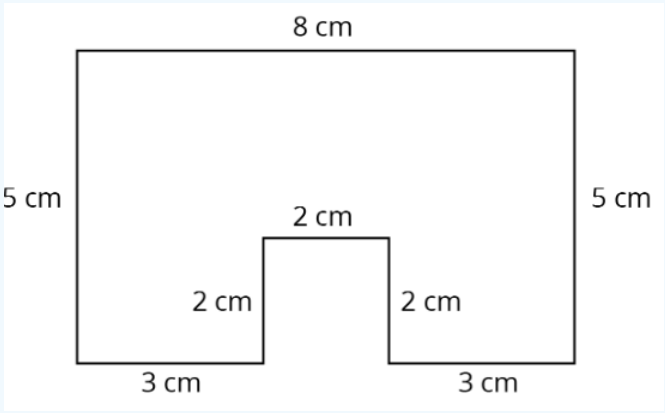


Figure 7.3.5.8

1. If the height of the prism is 5 cm, what is its surface area? What is its volume?
2. If the height of the prism is 10 cm, what is its surface area? What is its volume?
3. When the height doubled, what was the percent increase for the surface area? For the volume?

Exercise 7.3.5.6

Select **all** the situations where knowing the volume of an object would be more useful than knowing its surface area.

A. Determining the amount of paint needed to paint a barn.
B. Determining the monetary value of a piece of gold jewelry.
C. Filling an aquarium with buckets of water.
D. Deciding how much wrapping paper a gift will need.
E. Packing a box with watermelons for shipping.
F. Charging a company for ad space on your race car.
G. Measuring the amount of gasoline left in the tank of a tractor.

Exercise 7.3.5.7

Han draws a triangle with a 50° angle, a 40° angle, and a side of length 4 cm as shown. Can you draw a different triangle with the same conditions?

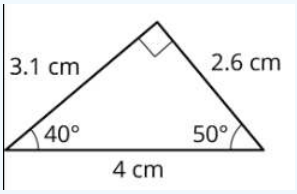


Figure 7.3.5.9

(From Unit 7.2.4)

Exercise 7.3.5.8

Angle H is half as large as angle J. Angle J is one fourth as large as angle K. Angle K has measure 240 degrees. What is the measure of angle H?

(From Unit 7.1.3)

Exercise 7.3.5.9

The Colorado state flag consists of three horizontal stripes of equal height. The side lengths of the flag are in the ratio $2:3$. The diameter of the gold-colored disk is equal to the height of the center stripe. What percentage of the flag is gold?



Figure 7.3.5.10

(From Unit 4.2.4)

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7.3.6: Applying Volume and Surface Area

Lesson

Let's explore things that are proportional to volume or surface area.

Exercise 7.3.6.1: You Decide

For each situation, decide if it requires Noah to calculate surface area or volume. Explain your reasoning.

1. Noah is planning to paint the bird house he built. He is unsure if he has enough paint.
2. Noah is planning to use a box with a trapezoid base to hold modeling clay. He is unsure if the clay will all fit in the box.

Exercise 7.3.6.2: Foam Play Structure

At a daycare, Kiran sees children climbing on this foam play structure.

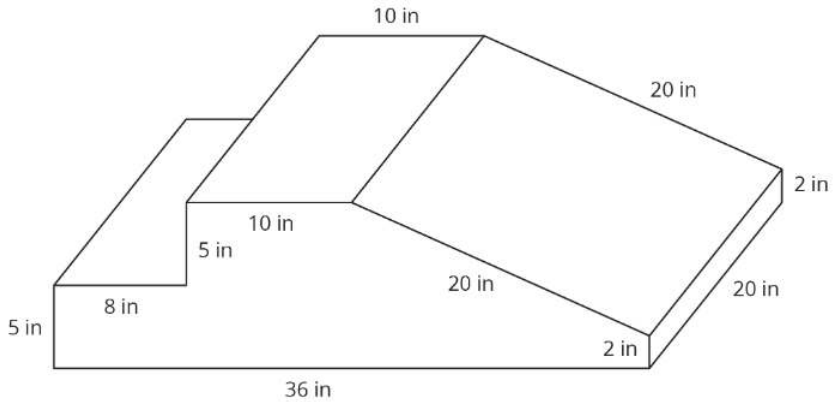


Figure 7.3.6.1

Kiran is thinking about building a structure like this for his younger cousins to play on.

1. The entire structure is made out of soft foam so the children don't hurt themselves. How much foam would Kiran need to build this play structure?
2. The entire structure is covered with vinyl so it is easy to wipe clean. How much vinyl would Kiran need to build this play structure?
3. The foam costs 0.8¢ per in^3. Here is a table that lists the costs for different amounts of vinyl. What is the total cost for all the foam and vinyl needed to build this play structure?

Table 7.3.6.1

vinyl (in^2)	cost ($)
75	0.45
125	0.75

Are you ready for more?

When he examines the play structure more closely, Kiran realizes it is really two separate pieces that are next to each other.

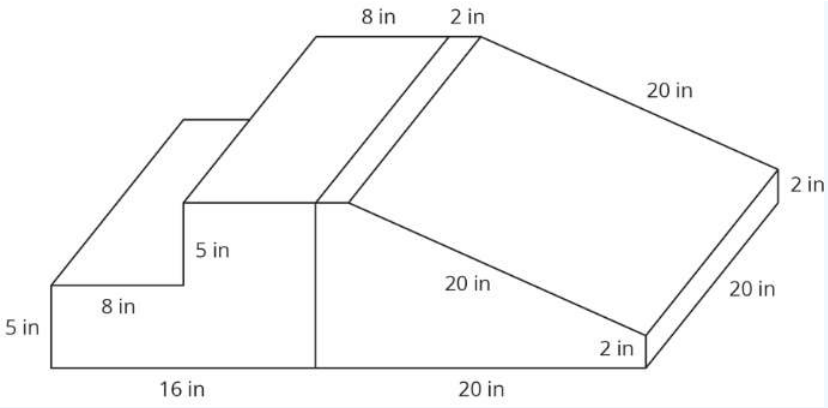


Figure 7.3.6.2

1. How does this affect the amount of foam in the play structure?
2. How does this affect the amount of vinyl covering the play structure?

Exercise 7.3.6.3: Filling the Sandbox

The daycare has two sandboxes that are both prisms with regular hexagons as their bases. The smaller sandbox has a base area of 1,146 in^2 and is filled 10 inches deep with sand.

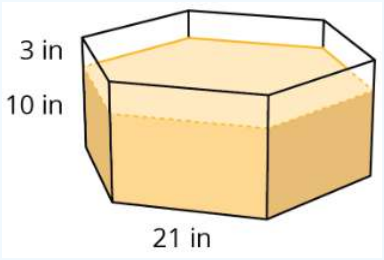


Figure 7.3.6.3

1. It took 14 bags of sand to fill the small sandbox to this depth. What volume of sand comes in one bag? (Round to the nearest whole cubic inch.)
2. The daycare manager wants to add 3 more inches to the depth of the sand in the small sandbox. How many bags of sand will they need to buy?
3. The daycare manager also wants to add 3 more inches to the depth of the sand in the large sandbox. The base of the large sandbox is a scaled copy of the base of the small sandbox, with a scale factor of 1.5. How many bags of sand will they need to buy for the large sandbox?
4. A lawn and garden store is selling 6 bags of sand for $19.50. How much will they spend to buy all the new sand for both sandboxes?

Summary

Suppose we wanted to make a concrete bench like the one shown in this picture. If we know that the finished bench has a volume of 10 ft^3 and a surface area of 44 ft^2 we can use this information to solve problems about the bench.



Figure 7.3.6.4: "Concrete seats, Balfour Street pocket park", by Didiunsw (Own work). CC BY-SA 4.0. Wikimedia Commons. Source.

For example,

- How much does the bench weigh?
- How long does it take to wipe the whole bench clean?
- How much will the materials cost to build the bench and to paint it?

To figure out how much the bench weighs, we can use its volume, 10 ft^3. Concrete weighs about 150 pounds per cubic foot, so this bench weighs about 1,500 pounds, because $10 \cdot 150 = 1,500$.

To figure out how long it takes to wipe the bench clean, we can use its surface area, 44 ft^2. If it takes a person about 2 seconds per square foot to wipe a surface clean, then it would take about 88 seconds to clean this bench, because $44 \cdot 2 = 88$. It may take a little less than 88 seconds, since the surfaces where the bench is touching the ground do not need to be wiped.

Would you use the volume or the surface area of the bench to calculate the cost of the concrete needed to build this bench? And for the cost of the paint?

Glossary Entries

Definition: Base (of a prism or pyramid)

The word *base* can also refer to a face of a polyhedron.

A prism has two identical bases that are parallel. A pyramid has one base.

A prism or pyramid is named for the shape of its base.

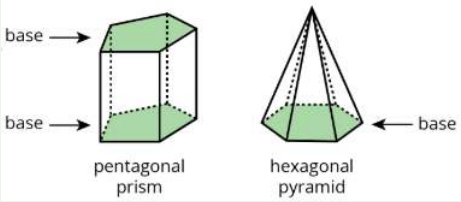


Figure 7.3.6.5: The figure on the left is labeled pentagonal prism. There are two identical pentagons on the top and bottom. Each vertex of a pentagon is connected by a vertical segment to the corresponding vertex of the other pentagons. The pentagons are each shaded, with the word base pointing to each. The figure on the right is labeled hexagonal pyramid. There is a hexagon on the bottom shaded green. From a point above the hexagon extend 6 segments, each connected to a vertex of the hexagon.

Definition: Cross Section

A cross section is the new face you see when you slice through a three-dimensional figure.

For example, if you slice a rectangular pyramid parallel to the base, you get a smaller rectangle as the cross section.

Definition: Prism

A prism is a type of polyhedron that has two bases that are identical copies of each other. The bases are connected by rectangles or parallelograms.

Here are some drawings of prisms.

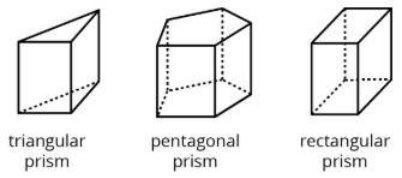


Figure 7.3.6.6

Definition: Pyramid

A pyramid is a type of polyhedron that has one base. All the other faces are triangles, and they all meet at a single vertex.

Here are some drawings of pyramids.

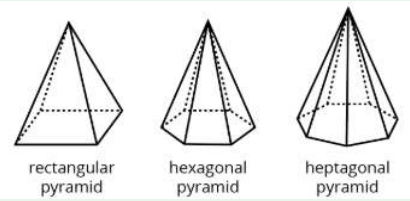


Figure 7.3.6.7

Definition: Surface Area

The surface area of a polyhedron is the number of square units that covers all the faces of the polyhedron, without any gaps or overlaps.

For example, if the faces of a cube each have an area of 9 cm^2, then the surface area of the cube is $6 \cdot 9$, or 54 cm^2.

Definition: Volume

Volume is the number of cubic units that fill a three-dimensional region, without any gaps or overlaps.

For example, the volume of this rectangular prism is 60 $units^3$, because it is composed of 3 layers that are each 20 $units^3$.

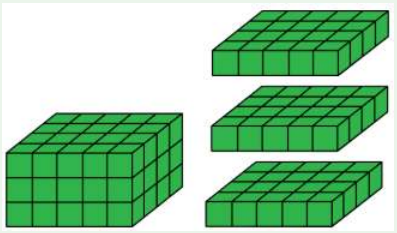


Figure 7.3.6.8

Practice

Exercise 7.3.6.4

A landscape architect is designing a pool that has this top view:

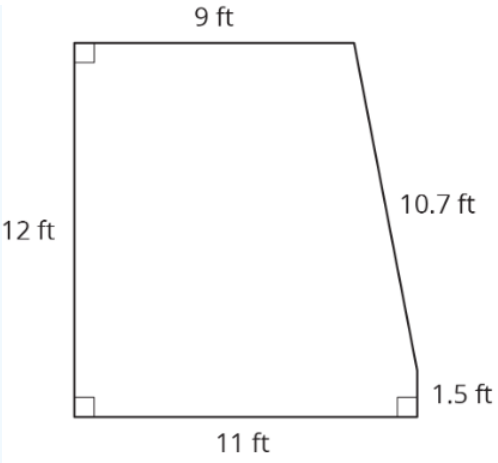


Figure 7.3.6.9

1. How much water will be needed to fill this pool 4 feet deep?
2. Before filling up the pool, it gets lined with a plastic liner. How much liner is needed for this pool?
3. Here are the prices for different amounts of plastic liner. How much will all the plastic liner for the pool cost?

Table 7.3.6.2

plastic liner (ft^2)	cost ($)
25	3.75
50	7.50
75	11.25

Exercise 7.3.6.5

Shade in a base of the trapezoidal prism. (The base is not the same as the bottom.)

1. Find the area of the base you shaded.
2. Find the volume of this trapezoidal prism.

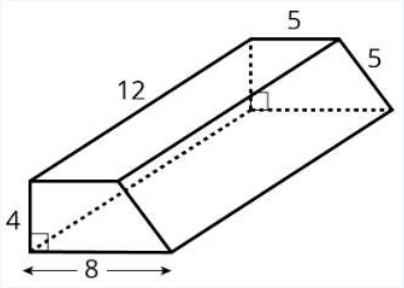


Figure 7.3.6.10

(From Unit 7.3.3)

Exercise 7.3.6.6

For each diagram, decide if y is an increase or a decrease of x. Then determine the percentage that x increased or decreased to result in y.

A

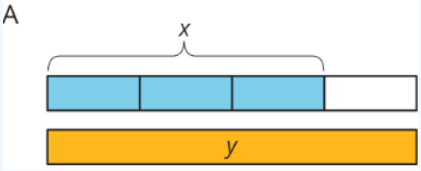


Figure 7.3.6.11

B

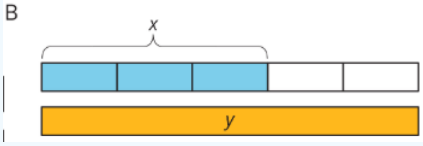


Figure 7.3.6.12

C

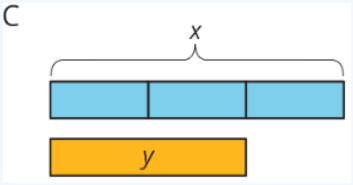


Figure 7.3.6.13

D

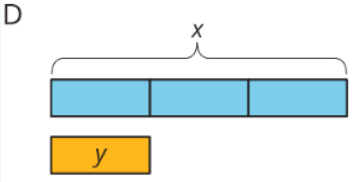


Figure 7.3.6.14

(From Unit 4.2.4)

Exercise 7.3.6.7

Noah is visiting his aunt in Texas. He wants to buy a belt buckle whose price is $25. He knows that the sales tax in Texas is 6.25%.

1. How much will the tax be on the belt buckle?
2. How much will Noah spend for the belt buckle including the tax?
3. Write an equation that represents the total cost, c, of an item whose price is p.

(From Unit 4.3.1)

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7.4: Let's Put It to work

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7.4.1: Building Prisms

Lesson

Let's build a triangular prism from scratch.

Exercise 7.4.1.1: Nets

Here are some nets for various prisms.

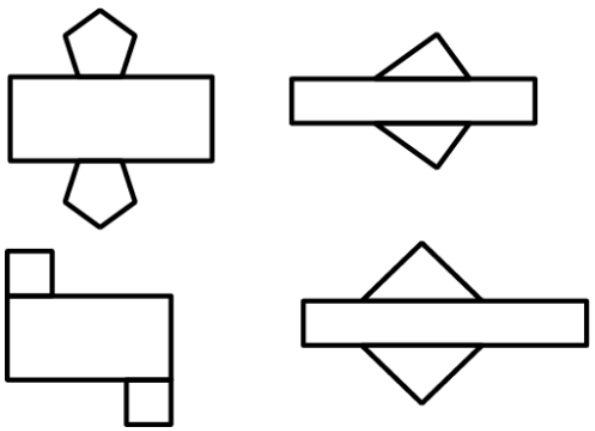


Figure 7.4.1.1

1. What would each net look like when folded?
2. What do you notice about the nets?

Exercise 7.4.1.2: Making the Base

The base of a triangular prism has one side that is 7 cm long, one side that is 5.5 cm long, and one angle that measures 45°.

1. Draw as many different triangles as you can with these given measurements.
2. Select one of the triangles you have drawn. Measure and calculate to approximate its area. Explain or show your reasoning.

Exercise 7.4.1.3: Making the Prism

Your teacher will give you an incomplete net. Follow these instructions to complete the net and assemble the triangular prism:

1. Draw an identical copy of the triangle you selected in the previous activity along the top of the rectangle, with one vertex on point A.
2. Draw another copy of your triangle, flipped upside down, along the bottom of the rectangle, with one vertex on point C.
3. Determine how long the rectangle needs to be to wrap all the way around your triangular bases. Pause here so your teacher can review your work.
4. Cut out and assemble your net.

After you finish assembling your triangular prism, answer these questions. Explain or show your reasoning.

1. What is the volume of your prism?
2. What is the surface area of your prism?
3. Stand your prism up so it is sitting on its triangular base.
    a. If you were to cut your prism in half horizontally, what shape would the cross section be?
    b. If you were to cut your prism in half vertically, what shape would the cross section be?

Exercise 7.4.1.4: Combining Prisms

1. Compare your prism with your partner's prism. What is the same? What is different?
2. Find a way you can put your prism and your partner's prism together to make one new, larger prism. Describe your new prism.
3. Draw the base of your new prism and label the lengths of the sides.
4. As you answer these questions about your new prism, look for ways you can use your calculations from the previous activity to help you. Explain or show your reasoning.
   a. What is the area of its base?
   b. What is its height?
   c. What is its volume?
   d. What is its surface area?

Are you ready for more?

How many identical copies of your prism would it take you to put together a new larger prism in which every dimension was twice as long?

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CHAPTER OVERVIEW

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8.1: Probabilities of Single Step Events

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8.1.1: Mystery Bags

Lesson

Let's make predictions based on what we know.

Exercise 8.1.1.1: Going Fishing

Andre and his dad have been fishing for 2 hours. In that time, they have caught 9 bluegills and 1 yellow perch.

The next time Andre gets a bite, what kind of fish do you think it will be? Explain your reasoning.

Exercise 8.1.1.2: Playing the Block Game

Your teacher will give your group a bag of colored blocks.

1. Follow these instructions to play one round of the game:
   a. Everyone in the group records the color written on the bag in the first column of the table.
   b. Without looking in the bag, one person takes out one of the blocks and shows it to the group.
   c. If they get a block that is the same color as the bag, they earn:
      - 1 point during round 1
      - 2 points during round 2
      - 3 points during round 3
   d. Next, they put the block back into the bag, shake the bag to mix up the blocks, and pass the bag to the next person in the group.
   e. Repeat these steps until everyone in your group has had 4 turns.
2. At the end of the round, record each person's score in the table.

Table 8.1.1.1

	What color bag?	person 1's score	person 2's score	person 3's score	person 4's score
round 1					
round 2					
round 3					

3. Pause here so your teacher can give you a new bag of blocks for the next round.
4. Repeat the previous steps to play rounds 2 and 3 of the game.
5. After you finish playing all 3 rounds, calculate the total score for each person in your group.

Are you ready for more?

Tyler's class played the block game using purple, orange, and yellow bags of blocks.

- During round 1, Tyler's group picked 4 purple blocks and 12 blocks of other colors.
- During round 2, Tyler's group picked 11 orange blocks and 5 blocks of other colors.
- During round 3, Tyler forgot to record how many yellow blocks his group picked.

For a final round, Tyler's group can pick one block from any of the three bags. Tyler's group decides that picking from the orange bag would give them the best chance of winning, and that picking from the purple bag would give them the worst chance of winning. What results from the yellow bag could have lead Tyler's group to this conclusion? Explain your reasoning.

Summary

One of the main ways that humans learn is by repeating experiments and observing the results. Babies learn that dropping their cup makes it hit the floor with a loud noise by repeating this action over and over. Scientists learn about nature by observing the results of repeated experiments again and again. With enough data about the results of experiments, we can begin to predict what may

happen if the experiment is repeated in the future. For example, a baseball player who has gotten a hit 33 out of 100 times at bat might be expected to get a hit about 33% of his times at bat in the future as well.

In some cases, we can predict the chances of things happening based on our knowledge of the situation. For example, a coin should land heads up about 50% of the time due to the symmetry of the coin.

In other cases, there are too many unknowns to predict the chances of things happening. For example, the chances of rain tomorrow are based on similar weather conditions we have observed in the past. In these situations, we can experiment, using past results to estimate chances.

Practice

Exercise 8.1.1.3

Lin is interested in how many of her classmates watch her favorite TV show, so she starts asking around at lunch. She gets the following responses:

yes yes yes no no no no no no no yes no no no

If she asks one more person randomly in the cafeteria, do you think they will say “yes” or “no”? Explain your reasoning.

Exercise 8.1.1.4

An engineer tests the strength of a new material by seeing how much weight it can hold before breaking. Previous tests have held these weights in pounds:

$$1,200 \quad 1,400 \quad 1,300 \quad 1,500 \quad 950 \quad 1,600 \quad 1,100$$

Do you think that this material will be able to hold more than 1,000 pounds in the next test? Explain your reasoning.

Exercise 8.1.1.5

A company tests two new products to make sure they last for more than a year.

- Product 1 had 950 out of 1,000 test items last for more than a year.
- Product 2 had 150 out of 200 last for more than a year.

If you had to choose one of these two products to use for more than a year, which one is more likely to last? Explain your reasoning.

Exercise 8.1.1.6

Put these numbers in order from least to greatest.

$$\frac{1}{2} \quad \frac{1}{3} \quad \frac{2}{5} \quad 0.6 \quad 0.3$$

Exercise 8.1.1.7

A small staircase is made so that the horizontal piece of each step is 10 inches long and 25 inches wide. Each step is 5 inches above the previous one. What is the surface area of this staircase?

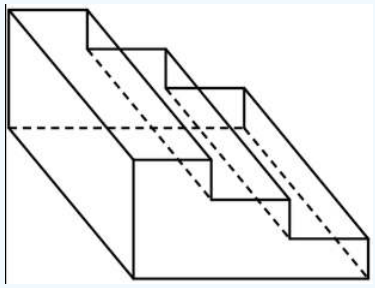


Figure 8.1.1.1

(From Unit 7.3.5)

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8.1.2: Chance Experiments

Lesson

Let's investigate chance.

Exercise 8.1.2.1: Which is More Likely?

Which is more likely to happen?

- When reaching into a dark closet and pulling out one shoe from a pile of 20 pairs of shoes, you pull out a left shoe.
- When listening to a playlist—which has 5 songs on it—in shuffle mode, the first song on the playlist plays first.

Exercise 8.1.2.2: How Likely Is it?

1. Label each event with one of these options:

impossible, unlikely, equally likely as not, likely, certain

a. You will win grand prize in a raffle if you purchased 2 out of the 100 tickets.
b. You will wait less than 10 minutes before ordering at a fast food restaurant.
c. You will get an even number when you roll a standard number cube.
d. A four-year-old child is over 6 feet tall.
e. No one in your class will be late to class next week.
f. The next baby born at a hospital will be a boy.
g. It will snow at our school on July 1.
h. The Sun will set today before 11:00 p.m.
i. Spinning this spinner will result in green.
j. Spinning this spinner will result in red.

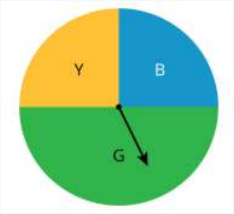


Figure 8.1.2.1

2. Discuss your answers to the previous question with your partner. If you disagree, work to reach an agreement.

3. Invent another situation for each label, for a total of 5 more events.

Exercise 8.1.2.3: Take a Chance

This applet displays a random number from 1 to 6, like a number cube. With a partner, you will play a game of chance.

- In the first round, one of you will score on an even roll and one of you will score on an odd roll. You decide that first.
- In the second round, the winner of round 1 will score on numbers $1-4$, and the other player will score on numbers $5-6$.
- Each round is 10 rolls. Be sure to turn on "History" after your first roll and wait for it to update before rolling again.

1. When each player had three numbers, dud one of them usually win?
2. When one player had four numbers, did you expect them to win? Explain your reasoning.

Are you ready for more?

On a game show, there are 3 closed doors. One door has a prize behind it. The contestant chooses one of the doors. The host of the game show, who knows where the prize is located, opens one of the *other* doors which does not have the prize. The contestant can choose to stay with their first choice or switch to the remaining closed door.

1. Do you think it matters if the contestant switches doors or stays?

2. Practice playing the game with your partner and record your results. Whoever is the host starts each round by secretly deciding which door has the prize.
   a. Play 20 rounds where the contestant always stays with their first choice.
   b. Play 20 more rounds where the contestant always switches doors.
3. Did the results from playing the game change your answer to the first question? Explain.

Exercise 8.1.2.4: Card Sort: Likelihood

1. Your teacher will give you some cards that describe events. Order the events from least likely to most likely.
2. After ordering the first set of cards, pause here so your teacher can review your work. Then, your teacher will give you a second set of cards.
3. Add the new set of cards to the first set so that all of the cards are ordered from least likely to most likely.

Summary

A **chance experiment** is something that happens where the outcome is unknown. For example, if we flip a coin, we don't know if the result will be a head or a tail. An **outcome** of a chance experiment is something that can happen when you do a chance experiment. For example, when you flip a coin, one possible outcome is that you will get a head. An **event** is a set of one or more outcomes.

We can describe events using these phrases:

- Impossible
- Unlikely
- Equally likely as not
- Likely
- Certain

For example, if you flip a coin:

- It is *impossible* that the coin will turn into a bottle of ketchup.
- It is *unlikely* the coin will land on its edge.
- It is *equally likely as not* that you will get a tail.
- It is *likely* that you will get a head or a tail.
- It is *certain* that the coin will land somewhere.

The *probability* of an event is a measure of the likelihood that an event will occur. We will learn more about probabilities in the lessons to come.

Glossary Entries

Definition: Chance Experiment

A chance experiment is something you can do over and over again, and you don't know what will happen each time.

For example, each time you spin the spinner, it could land on red, yellow, blue, or green.

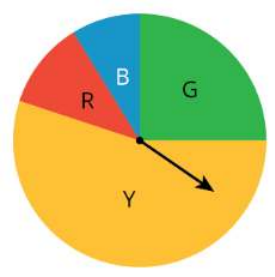


Figure 8.1.2.2

Definition: Event

An event is a set of one or more outcomes in a chance experiment. For example, if we roll a number cube, there are six possible outcomes.

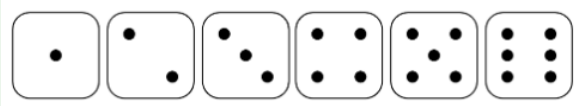


Figure 8.1.2.3

Examples of events are “rolling a number less than 3,” “rolling an even number,” or “rolling a 5.”

Definition: Outcome

An outcome of a chance experiment is one of the things that can happen when you do the experiment. For example, the possible outcomes of tossing a coin are heads and tails.

Practice

Exercise 8.1.2.5

The likelihood that Han makes a free throw in basketball is 60%. The likelihood that he makes a 3-point shot is 0.345. Which event is more likely, Han making a free throw or making a 3-point shot? Explain your reasoning.

Exercise 8.1.2.6

Different events have the following likelihoods. Sort them from least to greatest:

$$60\% \quad 8 \text{ out of } 10 \quad 0.37 \quad 20\% \quad \frac{5}{6}$$

Exercise 8.1.2.7

There are 25 prime numbers between 1 and 100. There are 46 prime numbers between 1 and 200. Which situation is more likely? Explain your reasoning.

- A computer produces a random number between 1 and 100 that is prime.
- A computer produces a random number between 1 and 200 that is prime.

Exercise 8.1.2.8

It takes $4\frac{3}{8}$ cups of cheese, $\frac{7}{8}$ cups of olives, and $2\frac{5}{8}$ cups of sausage to make a signature pizza. How much of each ingredient is needed to make 10 pizzas? Explain or show your reasoning.

(From Unit 4.1.2)

Exercise 8.1.2.9

Here is a diagram of a birdhouse Elena is planning to build. (It is a simplified diagram, since in reality, the sides will have a thickness.) About how many square inches of wood does she need to build this birdhouse?

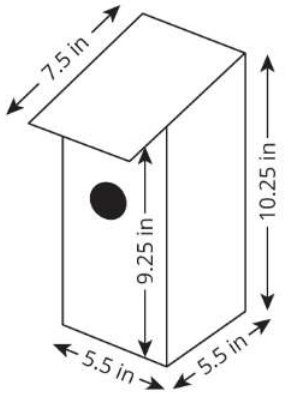


Figure 8.1.2.4

(From Unit 7.3.6)

Exercise 8.1.2.10

Select **all** the situations where knowing the surface area of an object would be more useful than knowing its volume.

A. Placing an order for tiles to replace the roof of a house.
B. Estimating how long it will take to clean the windows of a greenhouse.
C. Deciding whether leftover soup will fit in a container.
D. Estimating how long it will take to fill a swimming pool with a garden hose.
E. Calculating how much paper is needed to manufacture candy bar wrappers.
F. Buying fabric to sew a couch cover.
G. Deciding whether one muffin pan is enough to bake a muffin recipe.

(From Unit 7.3.5)

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8.1.3: What are Probabilities?

Lesson

Let's find out what's possible.

Exercise 8.1.3.1: Which Game Would You CHoose?

Which game would you choose to play? Explain your reasoning.

Game 1: You flip a coin and win if it lands showing heads.

Game 2: You roll a standard number cube and win if it lands showing a number that is divisible by 3.

Exercise 8.1.3.2: What's Possible?

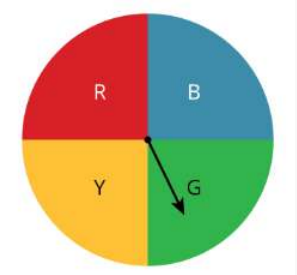


Figure 8.1.3.1

1. For each situation, list the **sample space** and tell how many outcomes there are.
   a. Han rolls a standard number cube once.
   b. Clare spins this spinner once.
   c. Kiran selects a letter at **random** from the word "MATH."
   d. Mai selects a letter at random from the alphabet.
   e. Noah picks a card at random from a stack that has cards numbered 5 through 20.
2. Next, compare the likelihood of these outcomes. Be prepared to explain your reasoning.
   a. Is Clare more likely to have the spinner stop on the red or blue section?
   b. Is Kiran or Mai more likely to get the letter T?
   c. Is Han or Noah more likely to get a number that is greater than 5?
3. Suppose you have a spinner that is evenly divided showing all the days of the week. You also have a bag of papers that list the months of the year. Are you more likely to spin the current day of the week or pull out the paper with the current month?

Are you ready for more?

Are there any outcomes for two people in this activity that have the same likelihood? Explain or show your reasoning.

Exercise 8.1.3.3: What's in the Bag?

Your teacher will give your group a bag of paper slips with something printed on them. Repeat these steps until everyone in your group has had a turn.

- As a group, guess what is printed on the papers in the bag and record your guess in the table.
- Without looking in the bag, one person takes out one of the papers and shows it to the group.
- Everyone in the group records what is printed on the paper.
- The person who took out the paper puts it back into the bag, shakes the bag to mix up the papers, and passes the bag to the next person in the group.

Table 8.1.3.1

	Guess the sample space.	What is printed on the paper?

	Guess the sample space.	What is printed on the paper?
person 1		
person 2		
person 3		
person 4		

1. How was guessing the sample space the fourth time different from the first?
2. What could you do to get a better guess of the sample space?
3. Look at all the papers in the bag. Were any of your guesses correct?
4. Are all of the possible outcomes equally likely? Explain.
5. Use the sample space to determine the **probability** that a fifth person would get the same outcome as person 1.

Summary

The **probability** of an event is a measure of the likelihood that the event will occur. Probabilities are expressed using numbers from 0 to 1.

- If the probability is 0, that means the event is impossible. For example, when you flip a coin, the probability that it will turn into a bottle of ketchup is 0. The closer the probability of some event is to 0, the less likely it is.
- If the probability is 1, that means the event is certain. For example, when you flip a coin, the probability that it will land somewhere is 1. The closer the probability of some event is to 1, the more likely it is.

If we list all of the possible outcomes for a chance experiment, we get the **sample space** for that experiment. For example, the sample space for rolling a standard number cube includes six outcomes: 1, 2, 3, 4, 5, and 6. The probability that the number cube will land showing the number 4 is $\frac{1}{6}$. In general, if all outcomes in an experiment are equally likely and there are n possible outcomes, then the probability of a single outcome is $\frac{1}{n}$.

Sometimes we have a set of possible outcomes and we want one of them to be selected at **random**. That means that we want to select an outcome in a way that each of the outcomes is *equally likely*. For example, if two people both want to read the same book, we could flip a coin to see who gets to read the book first.

Glossary Entries

Definition: Chance Experiment

A chance experiment is something you can do over and over again, and you don't know what will happen each time.

For example, each time you spin the spinner, it could land on red, yellow, blue, or green.

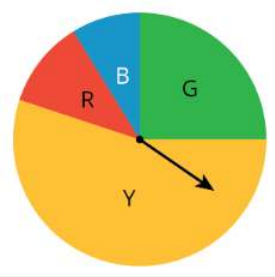


Figure 8.1.3.2

Definition: Event

An event is a set of one or more outcomes in a chance experiment. For example, if we roll a number cube, there are six possible outcomes.

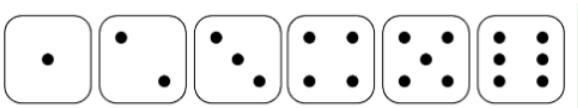


Figure 8.1.3.3

Examples of events are “rolling a number less than 3,” “rolling an even number,” or “rolling a 5.”

Definition: Outcome

An outcome of a chance experiment is one of the things that can happen when you do the experiment. For example, the possible outcomes of tossing a coin are heads and tails.

Definition: Probability

The probability of an event is a number that tells how likely it is to happen. A probability of 1 means the event will always happen. A probability of 0 means the event will never happen.

For example, the probability of selecting a moon block at random from this bag is $\frac{4}{5}$.



Figure 8.1.3.4

Definition: Random

Outcomes of a chance experiment are random if they are all equally likely to happen.

Definition: Sample Space

The sample space is the list of every possible outcome for a chance experiment.

For example, the sample space for tossing two coins is:

Table 8.1.3.2

heads-heads	tails-heads
heads-tails	tails-tails

Practice

Exercise 8.1.3.4

List the *sample space* for each chance experiment.

1. Flipping a coin
2. Selecting a random season of the year
3. Selecting a random day of the week

Exercise 8.1.3.5

A computer randomly selects a letter from the alphabet.

1. How many different outcomes are in the sample space?
2. What is the probability the computer produces the first letter of your first name?

Exercise 8.1.3.6

What is the probability of selecting a random month of the year and getting a month that starts with the letter "J?" If you get stuck, consider listing the sample space.

Exercise 8.1.3.7

E represents an object's weight on Earth and M represents that same object's weight on the Moon. The equation $M = \frac{1}{6}E$ represents the relationship between these quantities.

1. What does the $\frac{1}{6}$ represent in this situation?
2. Give an example of what a person might weigh on Earth and on the Moon.

(From Unit 2.2.1)

Exercise 8.1.3.8

Here is a diagram of the base of a bird feeder which is in the shape of a pentagonal prism. Each small square on the grid is 1 square inch.

The distance between the two bases is 8 inches. What will be the volume of the completed bird feeder?

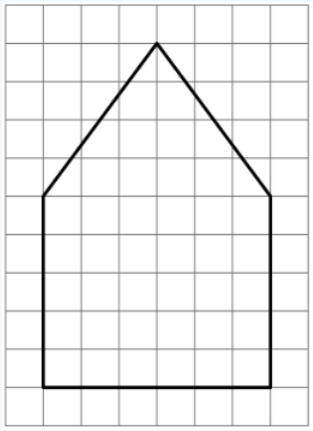


Figure 8.1.3.5

(From Unit 7.3.3)

Exercise 8.1.3.9

Find the surface area of the triangular prism.

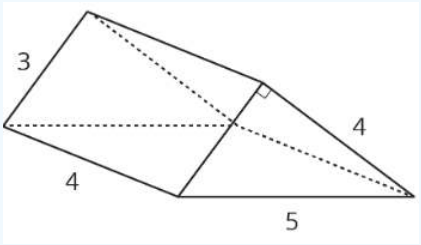


Figure 8.1.3.6

(From Unit 7.3.4)

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8.1.4: Estimating Probabilities Through Repeated Experiments

Lesson

Let's do some experimenting.

Exercise 8.1.4.1: Decimals on the Number Line

1. Locate and label these numbers on the number line.
   a. 0.5
   b. 0.75
   c. 0.33
   d. 0.67
   e. 0.25
2. Choose one of the numbers from the previous question. Describe a game in which that number represents your probability of winning.

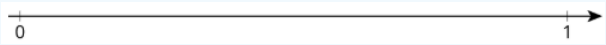


Figure 8.1.4.1

Exercise 8.1.4.2: In the Long Run

Mai plays a game in which she only wins if she rolls a 1 or a 2 with a standard number cube.

1. List the outcomes in the sample space for rolling the number cube.
2. What is the probability Mai will win the game? Explain your reasoning.
3. If Mai is given the option to flip a coin and win if it comes up heads, is that a better option for her to win?
4. Begin by dragging the gray bar below the toolbar down the screen until you see the table in the top window and the graph in the bottom window. This applet displays a random number from 1 to 6, like a number cube. Mai won with the numbers 1 and 2, but you can choose any two numbers from 1 to 6. Record them in the boxes in the center of the applet.
   Click the Roll button for 10 rolls and answer the questions.
5. What appears to be happening with the points on the graph?
6. a. After 10 rolls, what fraction of the total rolls were a win?
   b. How close is this fraction to the probability that Mai will win?
7. Roll the number cube 10 more times. Record your results in the table and on the graph from earlier.
8. a. After 20 rolls, what fraction of the total rolls were a win?
   b. How close is this fraction to the probability that Mai will win?

Exercise 8.1.4.3: Due For A Win

1. For each situation, do you think the result is surprising or not? Is it possible? Be prepared to explain your reasoning.
   a. You flip the coin once, and it lands heads up.
   b. You flip the coin twice, and it lands heads up both times.
   c. You flip the coin 100 times, and it lands heads up all 100 times.
2. If you flip the coin 100 times, how many times would you expect the coin to land heads up? Explain your reasoning.
3. If you flip the coin 100 times, what are some other results that would not be surprising?
4. You've flipped the coin 3 times, and it has come up heads once. The cumulative fraction of heads is currently $\frac{1}{3}$. If you flip the coin one more time, will it land heads up to make the cumulative fraction $\frac{2}{4}$?

Summary

A probabilityfor an event represents the proportion of the time we expect that event to occur in the long run. For example, the probability of a coin landing heads up after a flip is $\frac{1}{2}$, which means that if we flip a coin many times, we expect that it will land heads up about half of the time.

Even though the probability tells us what we should expect if we flip a coin many times, that doesn't mean we are more likely to get heads if we just got three tails in a row. The chances of getting heads are the same every time we flip the coin, no matter what the outcome was for past flips.

Glossary Entries

Definition: Chance Experiment

A chance experiment is something you can do over and over again, and you don’t know what will happen each time.

For example, each time you spin the spinner, it could land on red, yellow, blue, or green.

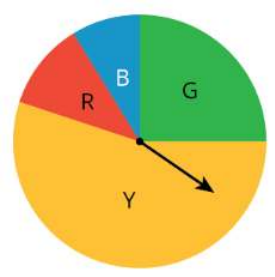


Figure 8.1.4.2

Definition: Event

An event is a set of one or more outcomes in a chance experiment. For example, if we roll a number cube, there are six possible outcomes.

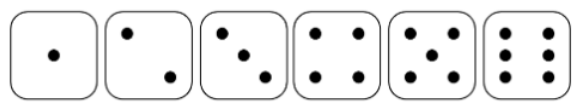


Figure 8.1.4.3

Examples of events are “rolling a number less than 3,” “rolling an even number,” or “rolling a 5.”

Definition: Outcome

An outcome of a chance experiment is one of the things that can happen when you do the experiment. For example, the possible outcomes of tossing a coin are heads and tails.

Definition: Probability

The probability of an event is a number that tells how likely it is to happen. A probability of 1 means the event will always happen. A probability of 0 means the event will never happen.

For example, the probability of selecting a moon block at random from this bag is $\frac{4}{5}$.



Figure 8.1.4.4

Definition: Random

Outcomes of a chance experiment are random if they are all equally likely to happen.

Definition: Sample Space

The sample space is the list of every possible outcome for a chance experiment.

For example, the sample space for tossing two coins is:

Table 8.1.4.2

heads-heads	tails-heads
heads-tails	tails-tails

Practice

Exercise 8.1.4.4

A carnival game has 160 rubber ducks floating in a pool. The person playing the game takes out one duck and looks at it.

- If there's a red mark on the bottom of the duck, the person wins a small prize.
- If there's a blue mark on the bottom of the duck, the person wins a large prize.
- Many ducks do not have a mark.

After 50 people have played the game, only 3 of them have won a small prize, and none of them have won a large prize.

Estimate the number of the 160 ducks that you think have red marks on the bottom. Then estimate the number of ducks you think have blue marks. Explain your reasoning.

Exercise 8.1.4.5

Lin wants to know if flipping a quarter really does have a probability of $\frac{1}{2}$ of landing heads up, so she flips a quarter 10 times. It lands heads up 3 times and tails up 7 times. Has she proven that the probability is not $\frac{1}{2}$? Explain your reasoning.

Exercise 8.1.4.6

A spinner has four equal sections, with one letter from the word "MATH" in each section.

1. You spin the spinner 20 times. About how many times do you expect it will land on A?
2. You spin the spinner 80 times. About how many times do you expect it will land on something other than A? Explain your reasoning.

Exercise 8.1.4.7

A spinner is spun 40 times for a game. Here is a graph showing the fraction of games that are wins under some conditions.

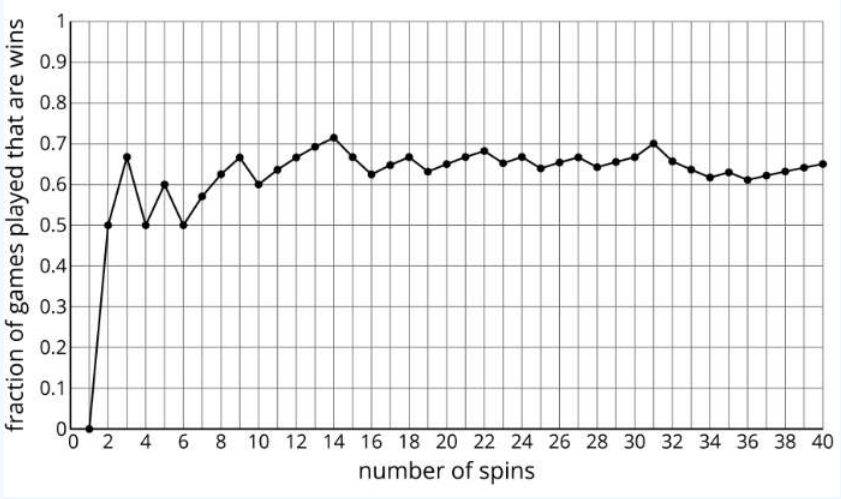


Figure 8.1.4.5

Estimate the probability of a spin winning this game based on the graph.

Exercise 8.1.4.8

Which event is more likely: rolling a standard number cube and getting an even number, or flipping a coin and having it land heads up?

(From Unit 8.1.2)

Exercise 8.1.4.9

Noah will select a letter at random from the word “FLUTE.” Lin will select a letter at random from the word “CLARINET.”

Which person is more likely to pick the letter “E?” Explain your reasoning.

(From Unit 8.1.3)

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8.1.5: More Estimating Probabilities

Lesson

Let's estimate some probabilities.

Exercise 8.1.5.1: Is it Likely?

1. If the weather forecast calls for a 20% chance of light rain tomorrow, would you say that it is likely to rain tomorrow?
2. If the probability of a tornado today is $\frac{1}{10}$, would you say that there will likely be a tornado today?
3. If the probability of snow this week is 0.85, would you say that it is likely to snow this week?

Exercise 8.1.5.2: Making My Head Spin

Work with your group to decide which person will use each spinner. Make sure each person selects a different spinner.

Answer the first set of questions on your own.

Spinner A

Spinner B

Spinner C

Spinner D

1. Spin your spinner 10 times and record your outcomes.
2. Did you get all of the different possible outcomes in your 10 spins?
3. What fraction of your 10 spins landed on 3?
   Work with your group to answer the next set of questions.
4. Share your outcomes with your group and record their outcomes.
   a. Outcomes for spinner A:
   b. Outcomes for spinner B:
   c. Outcomes for spinner C:
   d. Outcomes for spinner D:
5. Do any of the spinners have the same sample space? If so, do they have the same probabilities for each number to result?
6. For each spinner, what is the probability that it ends on the number 3? Explain or show your reasoning.
7. For each spinner, what is the probability that it lands on something other than the number 3? Explain or show your reasoning.
8. Noah put spinner D on top of his closed binder and spun it 10 times. It never landed on the number 1. How might you explain why this happened?
9. Han put spinner C on the floor and spun it 10 times. It never landed on the number 3, so he says that the probability of getting a 3 is 0. How might you explain why this happened?

Are you ready for more?

Design a spinner that has a $\frac{2}{3}$ probability of landing on the number 3. Explain how you could precisely draw this spinner.

Exercise 8.1.5.3: How much Green?

Your teacher will give you a bag of blocks that are different colors. Do not look into the bag or take out more than 1 block at a time. Repeat these steps until everyone in your group has had 4 turns.

- Take one block out of the bag and record whether or not it is green.
- Put the block back into the bag, and shake the bag to mix up the blocks.
- Pass the bag to the next person in the group.

1. What do you think is the probability of taking out a green block from this bag? Explain or show your reasoning.
2. How could you get a better estimate without opening the bag?

Summary

Suppose a bag contains 5 blocks. If we select a block at random from the bag, then the probability of getting any one of the blocks is $\frac{1}{5}$.

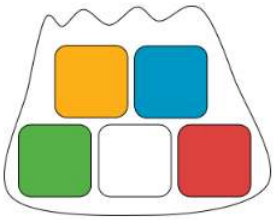


Figure 8.1.5.1

Now suppose a bag contains 5 blocks. Some of the blocks have a star, and some have a moon. If we select a block from the bag, then we will either get a star block or a moon block. The probability of getting a star block depends on how many there are in the bag.

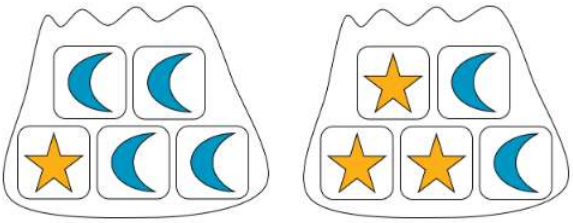


Figure 8.1.5.2

In this example, the probability of selecting a star block at random from the first bag is $\frac{1}{5}$, because it contains only 1 star block. (The probability of getting a moon block is $\frac{4}{5}$.) The probability of selecting a star block at random from the second bag is $\frac{3}{5}$, because it contains 3 star blocks. (The probability of getting a moon block from this bag is $\frac{2}{5}$.)

This shows that two experiments can have the same sample space, but different probabilities for each outcome.

Glossary Entries

Definition: Chance Experiment

A chance experiment is something you can do over and over again, and you don't know what will happen each time.

For example, each time you spin the spinner, it could land on red, yellow, blue, or green.

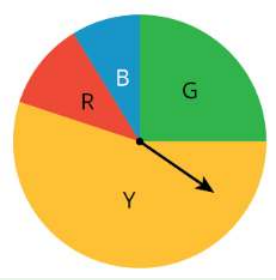


Figure 8.1.5.3

Definition: Event

An event is a set of one or more outcomes in a chance experiment. For example, if we roll a number cube, there are six possible outcomes.

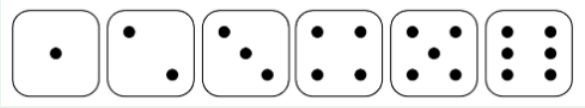


Figure 8.1.5.4

Examples of events are “rolling a number less than 3,” “rolling an even number,” or “rolling a 5.”

Definition: Outcome

An outcome of a chance experiment is one of the things that can happen when you do the experiment. For example, the possible outcomes of tossing a coin are heads and tails.

Definition: Probability

The probability of an event is a number that tells how likely it is to happen. A probability of 1 means the event will always happen. A probability of 0 means the event will never happen.

For example, the probability of selecting a moon block at random from this bag is $\frac{4}{5}$.



Figure 8.1.5.5

Definition: Random

Outcomes of a chance experiment are random if they are all equally likely to happen.

Definition: Sample Space

The sample space is the list of every possible outcome for a chance experiment.

For example, the sample space for tossing two coins is:

Table 8.1.5.2

heads-heads	tails-heads
heads-tails	tails-tails

Practice

Exercise 8.1.5.4

What is the same about these two experiments? What is different?

- Selecting a letter at random from the word “ALABAMA”
- Selecting a letter at random from the word “LAMB”

Exercise 8.1.5.5

Andre picks a block out of a bag 60 times and notes that 43 of them were green.

1. What should Andre estimate for the probability of picking out a green block from this bag?
2. Mai looks in the bag and sees that there are 6 blocks in the bag. Should Andre change his estimate based on this information? If so, what should the new estimate be? If not, explain your reasoning.

Exercise 8.1.5.6

Han has a number cube that he suspects is not so standard.

- Han rolls the cube 100 times, and it lands on a six 40 times.
- Kiran rolls the cube 50 times, and it lands on a six 21 times.
- Lin rolls the cube 30 times, and it lands on a six 11 times.

Based on these results, is there evidence to help prove that this cube is not a standard number cube? Explain your reasoning.

Exercise 8.1.5.7

A textbook has 428 pages numbered in order starting with 1. You flip to a random page in the book in a way that it is equally likely to stop at any of the pages.

1. What is the sample space for this experiment?

1. What is the probability that you turn to page 45?
2. What is the probability that you turn to an even numbered page?
3. If you repeat this experiment 50 times, about how many times do you expect you will turn to an even numbered page?

(From Unit 8.1.3)

Exercise 8.1.5.8

A rectangular prism is cut along a diagonal on each face to create two triangular prisms. The distance between A and B is 5 inches.

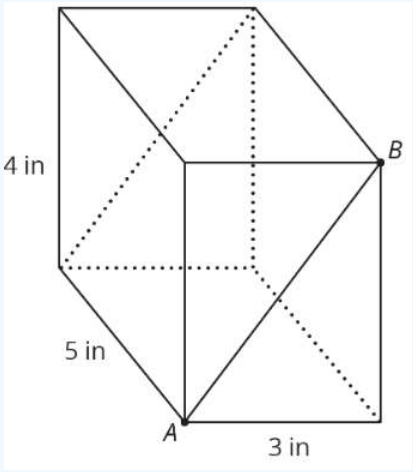


Figure 8.1.5.6: A rectangular prism with the horizontal side labeled 3 inches, a side length labeled 5 inches and a vertical height of 4 inches. Two vertices on the face of the prism are labeled A and B. Vertex A is located at the bottom left of the face and vertex B is located at the top right of the face. A diagonal is drawn connecting the two vertices.

What is the surface area of the original rectangular prism? What is the total surface area of the two triangular prisms together?

(From Unit 7.3.5)

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8.1.6: Estimating Probabilities Using Simulation

Lesson

Let's simulate real-world situations.

Exercise 8.1.6.1: Which One Doesn't Belong: Spinners

Which spinner doesn't belong?

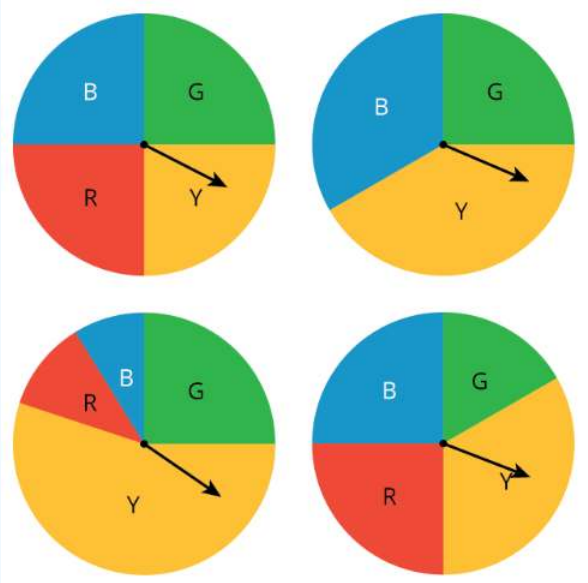


Figure 8.1.6.1

Exercise 8.1.6.2: Deigo's Walk

Your teacher will give your group the supplies for one of the three different simulations. Follow these instructions to simulate 15 days of Diego's walk. The first 3 days have been done for you.

- Simulate one day:
  - If your group gets a bag of papers, reach into the bag, and select one paper without looking inside.
  - If your group gets a spinner, spin the spinner, and see where it stops.
  - If your group gets two number cubes, roll both cubes, and add the numbers that land face up. A sum of 2–8 means Diego has to wait.
- Record in the table whether or not Diego had to wait more than 1 minute.
- Calculate the total number of days and the cumulative fraction of days that Diego has had to wait so far.
- On the graph, plot the number of days and the fraction that Diego has had to wait. Connect each point by a line.
- If your group has the bag of papers, put the paper back into the bag, and shake the bag to mix up the papers.
- Pass the supplies to the next person in the group.

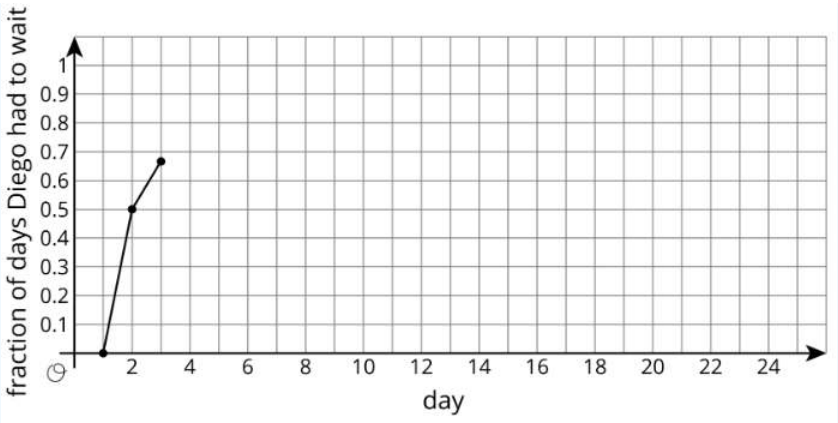


Figure 8.1.6.2: A graph of two connected line segments on a coordinate grid with the origin marked O. The horizontal axis is labeled day, with the numbers 0 through 24, in increments of 2, indicated. There are vertical grid lines midway between. The vertical axis is labeled fraction of days Diego had to wait with the numbers 0 point 1 through 1, in increments of 0 point 1, indicated. The first line segment begins at the point with coordinates 1 comma 0 and moves upward and to the right, ending at the point with coordinates 2 comma 0 point 5. The second line segment begins where the first line ends, and moves slightly upward and to the right, ending at the point with coordinates 3 comma 0 point 6 7.

Table 8.1.6.1

day	Does Diego have to wait more than 1 minute?	total number of days Diego had to wait	fraction of days Diego had to wait
1	no	0	$\frac{0}{1} = 0.00$
2	yes	1	$\frac{1}{2} = 0.50$
3	yes	2	$\frac{2}{3} \approx 0.67$
4			
5			
6			
7			
8			
9			
10			
11			
12			
13			
14			
15			

1. Based on the data you have collected, do you think the fraction of days Diego has to wait after the 16th day will be closer to 0.9 or 0.7? Explain or show your reasoning.
2. Continue the simulation for 10 more days. Record your results in this table and on the graph from earlier.

Table 8.1.6.2

day	Does Diego have to wait more than 1 minute?	total number of days Diego had to wait	fraction of days Diego had to wait
16			

day	Does Diego have to wait more than 1 minute?	total number of days Diego had to wait	fraction of days Diego had to wait
17			
18			
19			
20			
21			
22			
23			
24			
25			

3. What do you notice about the graph?
4. Based on the graph, estimate the probability that Diego will have to wait more than 1 minute to cross the crosswalk.

Are you ready for more?

Let's look at why the values tend to not change much after doing the simulation many times.

1. After doing the simulation 4 times, a group finds that Diego had to wait 3 times. What is an estimate for the probability Diego has to wait based on these results?
   a. If this group does the simulation 1 more time, what are the two possible outcomes for the fifth simulation?
   b. For each possibility, estimate the probability Diego has to wait.
   c. What are the differences between the possible estimates after 5 simulations and the estimate after 4 simulations?
2. After doing the simulation 20 times, this group finds that Diego had to wait 15 times. What is an estimate for the probability Diego has to wait based on these results?
   a. If this group does the simulation 1 more time, what are the two possible outcomes for the twenty-first simulation?
   b. For each possibility, estimate the probability Diego has to wait.
   c. What are the differences between the possible estimates after 21 simulations and the estimate after 20 simulations?
3. Use these results to explain why a single result after many simulations does not affect the estimate as much as a single result after only a few simulations.

Exercise 8.1.6.3: Designing Experiments

For each situation, describe a chance experiment that would fairly represent it.

1. Six people are going out to lunch together. One of them will be selected at random to choose which restaurant to go to. Who gets to choose?
2. After a robot stands up, it is equally likely to step forward with its left foot or its right foot. Which foot will it use for its first step?
3. In a computer game, there are three tunnels. Each time the level loads, the computer randomly selects one of the tunnels to lead to the castle. Which tunnel is it?
4. Your school is taking 4 buses of students on a field trip. Will you be assigned to the same bus that your math teacher is riding on?

Summary

Sometimes it is easier to estimate a probability by doing a **simulation**. A simulation is an experiment that approximates a situation in the real world. Simulations are useful when it is hard or time-consuming to gather enough information to estimate the probability of some event.

For example, imagine Andre has to transfer from one bus to another on the way to his music lesson. Most of the time he makes the transfer just fine, but sometimes the first bus is late and he misses the second bus. We could set up a simulation with slips of paper in a bag. Each paper is marked with a time when the first bus arrives at the transfer point. We select slips at random from the bag. After many trials, we calculate the fraction of the times that he missed the bus to estimate the probability that he will miss the bus on a given day.

Glossary Entries

Definition: Chance Experiment

A chance experiment is something you can do over and over again, and you don't know what will happen each time.

For example, each time you spin the spinner, it could land on red, yellow, blue, or green.

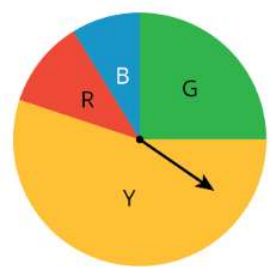


Figure 8.1.6.3

Definition: Event

An event is a set of one or more outcomes in a chance experiment. For example, if we roll a number cube, there are six possible outcomes.

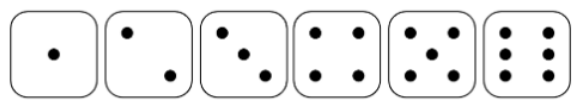


Figure 8.1.6.4

Examples of events are "rolling a number less than 3," "rolling an even number," or "rolling a 5."

Definition: Outcome

An outcome of a chance experiment is one of the things that can happen when you do the experiment. For example, the possible outcomes of tossing a coin are heads and tails.

Definition: Probability

The probability of an event is a number that tells how likely it is to happen. A probability of 1 means the event will always happen. A probability of 0 means the event will never happen.

For example, the probability of selecting a moon block at random from this bag is $\frac{4}{5}$.



Figure 8.1.6.5

Definition: Random

Outcomes of a chance experiment are random if they are all equally likely to happen.

Definition: Sample Space

The sample space is the list of every possible outcome for a chance experiment.

For example, the sample space for tossing two coins is:

Table 8.1.6.3

heads-heads	tails-heads
heads-tails	tails-tails

Definition: Simulation

A simulation is an experiment that is used to estimate the probability of a real-world event.

For example, suppose the weather forecast says there is a 25% chance of rain. We can simulate this situation with a spinner with four equal sections. If the spinner stops on red, it represents rain. If the spinner stops on any other color, it represents no rain.

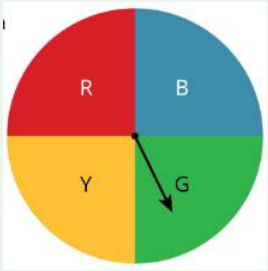


Figure 8.1.6.6: A circular spinner divided into 4 equal parts. The top left section is red and labeled R. The top right section is blue and labeled B. The bottom right section is green and labeled G. The bottom left section is yellow and labeled Y. The spinner dial points to the section labeled G.

Practice

Exercise 8.1.6.4

The weather forecast says there is a 75% chance it will rain later today.

1. Draw a spinner you could use to simulate this probability.
2. Describe another way you could simulate this probability.

Exercise 8.1.6.5

An experiment will produce one of ten different outcomes with equal probability for each. Why would using a standard number cube to simulate the experiment be a bad choice?

Exercise 8.1.6.6

An ice cream shop offers 40 different flavors. To simulate the most commonly chosen flavor, you could write the name of each flavor on a piece of paper and put it in a bag. Draw from the bag 100 times, and see which flavor is chosen the most. This simulation is not a good way to figure out the most-commonly chosen flavor. Explain why.

Exercise 8.1.6.7

Each set of three numbers represents the lengths, in units, of the sides of a triangle. Which set can *not* be used to make a triangle?

A. $7, 6, 14$
B. $4, 4, 4$

C. $6, 6, 2$
D. $7, 8, 13$

(From Unit 7.2.2)

Exercise 8.1.6.8

There is a proportional relationship between a volume measured in cups and the same volume measured in tablespoons. 48 tablespoons is equivalent to 3 cups, as shown in the graph.

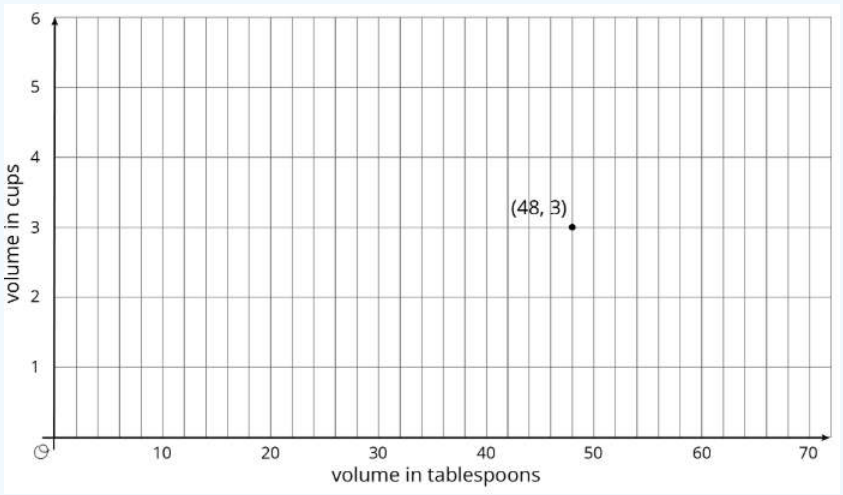


Figure 8.1.6.7: A point plotted in the coordinate plane with the origin labeled O. The horizontal axis is labeled volume in tablespoons and the numbers 0 through 70, in increments of 10, are indicated. There are four evenly spaced gridlines between each indicated number. The vertical axis is labeled volume in cups and the numbers 0 through 6 are indicated. The indicated point has coordinates 48 comma 3.

1. Plot and label some more points that represent the relationship.
2. Use a straightedge to draw a line that represents this proportional relationship.
3. For which value y is $(1, y)$ on the line you just drew?
4. What is the constant of proportionality for this relationship?
5. Write an equation representing this relationship. Use c for cups and t for tablespoons.

(From Unit 2.5.1)

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8.2: Probabilites of Multi-step Events

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8.2.1: Simulating Multi-step Experiments

Lesson

Let's simulate more complicated events.

Exercise 8.2.1.1: Notice and Wonder: Ski Business

What do you notice? What do you wonder?



Figure 8.2.1.1



Figure 8.2.1.2

Exercise 8.2.1.2: Alpine Zoom

Alpine Zoom is a ski business. To make money over spring break, they need it to snow at least 4 out of the 10 days. The weather forecast says there is a $\frac{1}{3}$ chance it will snow each day during the break.

Use the applet to simulate the weather for 10 days of break to see if Alpine Zoom will make money.

1. Describe a chance experiment that you could use to simulate whether it will snow on the first day of break.
2. How could this chance experiment be used to determine whether Alpine Zoom will make money?
   - In each trial, spin the spinner 10 times, and then record the number of 1's that appeared in the row.
   - The applet reports if the Alpine Zoom will make money or not in the last column.
   - Click Next to get the spin button back to start the next simulation.
3. Based on your simulations, estimate the probability that Alpine Zoom will make money.

Exercise 8.2.1.3: Kiran's Game

Kiran invents a game that uses a board with alternating black and white squares. A playing piece starts on a white square and must advance 4 squares to the other side of the board within 5 turns to win the game.



Figure 8.2.1.3: A game board with one row of alternating white and black squares. There are five squares in total and the board begins and ends with a white square. A playing piece is located on the first white square.

For each turn, the player draws a block from a bag containing 2 black blocks and 2 white blocks. If the block color matches the color of the next square on the board, the playing piece moves onto it. If it does not match, the playing piece stays on its current square.

1. Take turns playing the game until each person in your group has played the game twice.
2. Use the results from all the games your group played to estimate the probability of winning Kiran's game.
3. Do you think your estimate of the probability of winning is a good estimate? How could it be improved?

Are you ready for more?

How would each of these changes, on its own, affect the probability of winning the game?

1. Change the rules so that the playing piece must move 7 spaces within 8 moves.
2. Change the board so that all the spaces are black.
3. Change the blocks in the bag to 3 black blocks and 1 white block.

Exercise 8.2.1.4: Simulation Nation

Match each situation to a **simulation**.

Situations:

1. In a small lake, 25% of the fish are female. You capture a fish, record whether it is male or female, and toss the fish back into the lake. If you repeat this process 5 times, what is the probability that at least 3 of the 5 fish are female?
2. Elena makes about 80% of her free throws. Based on her past successes with free throws, what is the probability that she will make exactly 4 out of 5 free throws in her next basketball game?
3. On a game show, a contestant must pick one of three doors. In the first round, the winning door has a vacation. In the second round, the winning door has a car. What is the probability of winning a vacation and a car?
4. Your choir is singing in 4 concerts. You and one of your classmates both learned the solo. Before each concert, there is an equal chance the choir director will select you or the other student to sing the solo. What is the probability that you will be selected to sing the solo in exactly 3 of the 4 concerts?

Simulations:

1. Toss a standard number cube 2 times and record the outcomes. Repeat this process many times and find the proportion of the simulations in which a 1 or 2 appeared both times to estimate the probability.
2. Make a spinner with four equal sections labeled 1, 2, 3, and 4. Spin the spinner 5 times and record the outcomes. Repeat this process many times and find the proportion of the simulations in which a 4 appears 3 or more times to estimate the probability.
3. Toss a fair coin 4 times and record the outcomes. Repeat this process many times, and find the proportion of the simulations in which exactly 3 heads appear to estimate the probability.
4. Place 8 blue chips and 2 red chips in a bag. Shake the bag, select a chip, record its color, and then return the chip to the bag. Repeat the process 4 more times to obtain a simulated outcome. Then repeat this process many times and find the proportion of the simulations in which exactly 4 blues are selected to estimate the probability.

Summary

The more complex a situation is, the harder it can be to estimate the probability of a particular event happening. Well-designed simulations are a way to estimate a probability in a complex situation, especially when it would be difficult or impossible to determine the probability from reasoning alone.

To design a good simulation, we need to know something about the situation. For example, if we want to estimate the probability that it will rain every day for the next three days, we could look up the weather forecast for the next three days. Here is a table showing a weather forecast:

Table 8.2.1.1

	today (Tuesday)	Wednesday	Thursday	Friday
probability of rain	0.2	0.4	0.5	0.9

We can set up a simulation to estimate the probability of rain each day with three bags.

- In the first bag, we put 4 slips of paper that say “rain” and 6 that say “no rain.”
- In the second bag, we put 5 slips of paper that say “rain” and 5 that say “no rain.”
- In the third bag, we put 9 slips of paper that say “rain” and 1 that says “no rain.”

Then we can select one slip of paper from each bag and record whether or not there was rain on all three days. If we repeat this experiment many times, we can estimate the probability that there will be rain on all three days by dividing the number of times all three slips said “rain” by the total number of times we performed the simulation.

Practice

Exercise 8.2.1.5

Priya’s cat is pregnant with a litter of 5 kittens. Each kitten has a 30% chance of being chocolate brown. Priya wants to know the probability that at least two of the kittens will be chocolate brown.

To simulate this, Priya put 3 white cubes and 7 green cubes in a bag. For each trial, Priya pulled out and returned a cube 5 times. Priya conducted 12 trials.

Here is a table with the results.

Table 8.2.1.2

trial number	outcome
1	ggggg
2	gggwg
3	wgwgw
4	gwggg
5	gggwg
6	wwggg
7	gwggg
8	ggwgw
9	wwwgg
10	ggggw
11	wggwg
12	gggwg

1. How many successful trials were there? Describe how you determined if a trial was a success.
2. Based on this simulation, estimate the probability that *exactly* two kittens will be chocolate brown.
3. Based on this simulation, estimate the probability that *at least* two kittens will be chocolate brown.
4. Write and answer another question Priya could answer using this simulation.
5. How could Priya increase the accuracy of the simulation?

Exercise 8.2.1.6

A team has a 75% chance to win each of the 3 games they will play this week. Clare simulates the week of games by putting 4 pieces of paper in a bag, 3 labeled “win” and 1 labeled “lose.” She draws a paper, writes down the result, then replaces the paper and repeats the process two more times. Clare gets the result: win, win, lose. What can Clare do to estimate the probability the team will win at least 2 games?

Exercise 8.2.1.7

1. List the sample space for selecting a letter a random from the word “PINEAPPLE.”
2. A letter is randomly selected from the word “PINEAPPLE.” Which is more likely, selecting “E” or selecting “P?” Explain your reasoning.

(From Unit 8.1.5)

Exercise 8.2.1.8

On a graph of side length of a square vs. its perimeter, a few points are plotted.

1. Add at least two more ordered pairs to the graph.

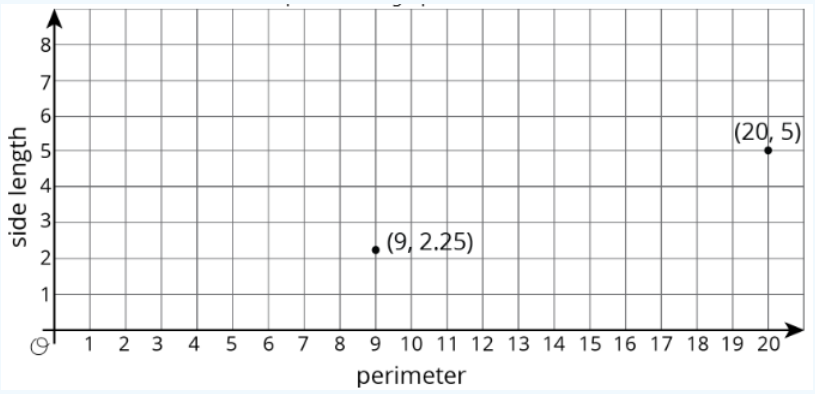


Figure 8.2.1.4: Two points are plotted in the coordinate plane with the origin labeled O. The horizontal axis is labeled perimeter and the numbers 0 through 20 are indicated. The vertical axis is labeled side length and the numbers 0 through 8 are indicated. The two points plotted are 9 comma 2 point 2 5 and 20 comma 5.

2. Is there a proportional relationship between the perimeter and side length? Explain how you know.

(From Unit 2.4.2)

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8.2.2: Keeping Track of All Possible Outcomes

Lesson

Lets explore sample spaces for experiments with multiple parts.

Exercise 8.2.2.1: How Many Different Meals?

How many different meals are possible if each meal includes one main course, one side dish, and one drink?

Table 8.2.2.1

main courses	side dishes	drinks
grilled chicken	salad	milk
turkey sandwich	applesauce	juice
pasta salad	__	water

Exercise 8.2.2.2: Lists, Tables, and Trees

Consider the experiment: Flip a coin, and then roll a number cube.

Elena, Kiran, and Priya each use a different method for finding the sample space of this experiment.

- Elena carefully writes a list of all the options: Heads 1, Heads 2, Heads 3, Heads 4, Heads 5, Heads 6, Tails 1, Tails 2, Tails 3, Tails 4, Tails 5, Tails 6.
- Kiran makes a table:

Table 8.2.2.2

	1	2	3	4	5	6
H	H1	H2	H3	H4	H5	H6
T	T1	T2	T3	T4	T5	T6

- Priya draws a tree with branches in which each pathway represents a different outcome:

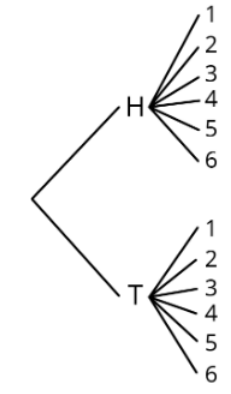


Figure 8.2.2.1

1. Compare the three methods. What is the same about each method? What is different? Be prepared to explain why each method produces all the different outcomes without repeating any.
2. Which method do you prefer for this situation?
   Pause here so your teacher can review your work.
3. Find the sample space for each of these experiments using any method. Make sure you list every possible outcome without repeating any.

a. Flip a dime, then flip a nickel, and then flip a penny. Record whether each lands heads or tails up.
b. Han's closet has: a blue shirt, a gray shirt, a white shirt, blue pants, khaki pants, and black pants. He must select one shirt and one pair of pants to wear for the day.
c. Spin a color, and then spin a number.

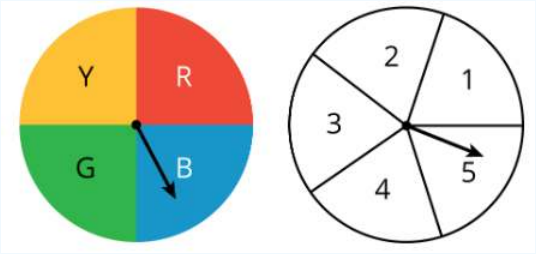


Figure 8.2.2.2

d. Spin the hour hand on an analog clock, and then choose a.m. or p.m.

Exercise 8.2.2.3: How Many Sandwiches?

1. A submarine sandwich shop makes sandwiches with one kind of bread, one protein, one choice of cheese, and *two* vegetables. How many different sandwiches are possible? Explain your reasoning. You do not need to write out the sample space.

- Breads: Italian, white, wheat
- Proteins: Tuna, ham, turkey, beans
- Cheese: Provolone, Swiss, American, none
- Vegetables: Lettuce, tomatoes, peppers, onions, pickles



Figure 8.2.2.3: Submarine sandwich and chips, by jeffreyw. CC BY 2.0. Wikimedia Commons. Source.

2. Andre knows he wants a sandwich that has ham, lettuce, and tomatoes on it. He doesn't care about the type of bread or cheese. How many of the different sandwiches would make Andre happy?

3. If a sandwich is made by randomly choosing each of the options, what is the probability it will be a sandwich that Andre would be happy with?

Are you ready for more?

Describe a situation that involves three parts and has a total of 24 outcomes in the sample space.

Summary

Sometimes we need a systematic way to count the number of outcomes that are possible in a given situation. For example, suppose there are 3 people (A, B, and C) who want to run for the president of a club and 4 different people (1, 2, 3, and 4) who want to run for vice president of the club. We can use a *tree*, a *table*, or an *ordered list* to count how many different combinations are possible for a president to be paired with a vice president.

With a tree, we can start with a branch for each of the people who want to be president. Then for each possible president, we add a branch for each possible vice president, for a total of $3 \cdot 4 = 12$ possible pairs. We can also start by counting vice presidents first and then adding a branch for each possible president, for a total of $3 \cdot 4 = 12$ possible pairs.

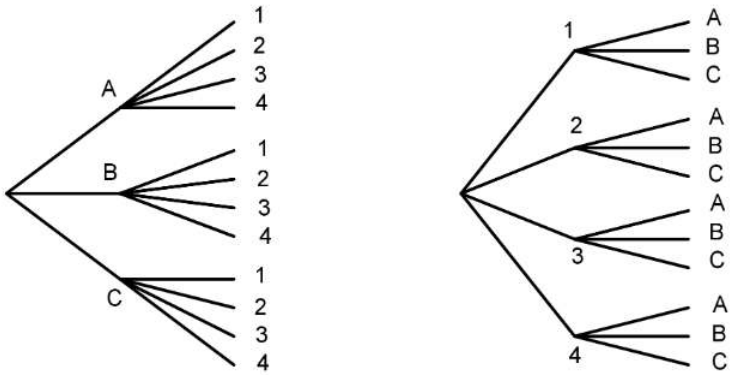


Figure 8.2.2.4: Two tree diagrams. The leftmost tree diagram has three branches for the first choice, labeled A, B, and C. Choices A, B, and C each have four branches labeled with a different number from 1 through 4. The rightmost tree diagram has four branches for the first choice, labeled 1, 2, 3, and 4. Choices 1, 2, 3, and 4 each have three branches, labeled with a different letter A, B, or C.

A table can show the same result:

Table 8.2.2.3

	1	**2**	**3**	**4**
A	$A, 1$	$A, 2$	$A, 3$	$A, 4$
B	$B, 1$	$B, 2$	$B, 3$	$B, 4$
C	$C, 1$	$C, 2$	$C, 3$	$C, 4$

So does this ordered list:

A1, A2, A3, A4, B1, B2, B3, B4, C1, C2, C3, C4

Practice

Exercise 8.2.2.4

Noah is planning his birthday party. Here is a tree showing all of the possible themes, locations, and days of the week that Noah is considering.

1. How many themes is Noah considering?
2. How many locations is Noah considering?
3. How many days of the week is Noah considering?
4. One possibility that Noah is considering is a party with a space theme at the skating rink on Sunday. Write two other possible parties Noah is considering.
5. How many different possible outcomes are in the sample space?

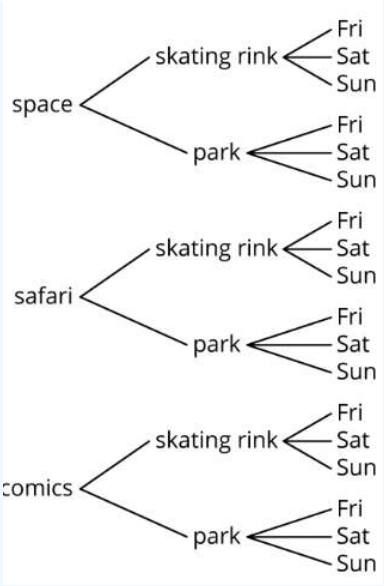


Figure 8.2.2.5: A tree diagram. On the first level of the tree, the branches are for the themes space, safari and comics. On the second level of the tree, there are two branches from each theme for the locations skating rink or park for a total of 6 branches. On the third level of the tree, there are three branches from each location for the days Friday, Saturday, and Sunday, for a total of 18 branches.

Exercise 8.2.2.5

For each event, write the sample space and tell how many outcomes there are.

1. Lin selects one type of lettuce and one dressing to make a salad.
   Lettuce types: iceberg, romaine
   Dressings: ranch, Italian, French
2. Diego chooses rock, paper, or scissors, and Jada chooses rock, paper, or scissors.
3. Spin these 3 spinners.



Figure 8.2.2.6: Three different circular spinners. The leftmost spinner is divided into two equal parts: one labeled B and one labeled M. The pointer is in the part labeled M. The spinner in the center is divided into two equal parts: one labeled A and one labeled E. The pointer is in the part labeled E. The rightmost spinner is divided into three equal parts: one labeled D, one labeled N, and one labeled T. The pointer is in the part labeled T.

Exercise 8.2.2.6

A simulation is done to represent kicking 5 field goals in a single game with a 72% probability of making each one. A 1 represents making the kick and a 0 represents missing the kick.

Table 8.2.2.4

trial	result
1	10101
2	11010
3	00011
4	11111
5	10011

(From Unit 8.2.1)

Exercise 8.2.2.7

There is a bag of 50 marbles.

- Andre takes out a marble, records its color, and puts it back in. In 4 trials, he gets a green marble 1 time.
- Jada takes out a marble, records its color, and puts it back in. In 12 trials, she gets a green marble 5 times.
- Noah takes out a marble, records its color, and puts it back in. In 9 trials, he gets a green marble 3 times.

Estimate the probability of getting a green marble from this bag. Explain your reasoning.

(From Unit 8.1.4)

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8.2.3: Multi-step Experiments

Lesson

Let's look at probabilities of experiments that have multiple steps.

Exercise 8.2.3.1: True or False?

Is each equation true or false? Explain your reasoning.

$8 = (8+8+8+8) \div 3$

$(10+10+10+10+10) \div 5 = 10$

$6+4+6+4+6+4) \div 6 = 5$

Exercise 8.2.3.2: Spinning a Color and Number

The other day, you wrote the sample space for spinning each of these spinners once.

What is the probability of getting:

1. Green and 3?
2. Blue and any odd number?
3. Any color other than red and any number other than 2?

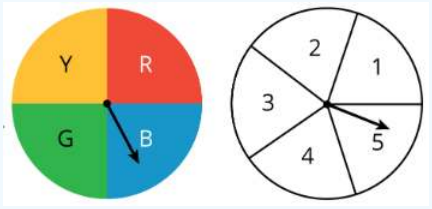


Figure 8.2.3.1: Two different circular spinners. The spinner on the left is divided into four equal parts. The first part is red and labeled R, the second part is blue and labeled B, the third part is green and labeled G, and the fourth part is yellow and labeled Y. The pointer is in the part labeled B. The spinner on the right is divided into five equal parts. Starting from the top right, and moving clockwise, the first part is labeled 1, the second, 2, the third, 3, the fourth, 4, and the fifth, 5. The pointer is in the part labeled 5.

Exercise 8.2.3.3: Cubes and Coins

The other day you looked at a list, a table, and a tree that showed the sample space for rolling a number cube and flipping a coin.

1. Your teacher will assign you one of these three structures to use to answer these questions. Be prepared to explain your reasoning.
   a. What is the probability of getting tails and a 6?
   b. What is the probability of getting heads and an odd number?
   Pause here so your teacher can review your work.
2. Suppose you roll two number cubes. What is the probability of getting:
   a. Both cubes showing the same number?
   b. *Exactly* one cube showing an even number?
   c. *At least* one cube showing an even number?
   d. Two values that have a sum of 8?
   e. Two values that have a sum of 13?
3. Jada flips three quarters. What is the probability that all three will land showing the same side?

Exercise 8.2.3.4: Pick a Card

Imagine there are 5 cards. They are colored red, yellow, green, white, and black. You mix up the cards and select one of them without looking. Then, without putting that card back, you mix up the remaining cards and select another one.

1. Write the sample space and tell how many possible outcomes there are.
2. What structure did you use to write all of the outcomes (list, table, tree, something else)? Explain why you chose that structure.
3. What is the probability that:
   a. You get a white card and a red card (in either order)?
   b. You get a black card (either time)?
   c. You do not get a black card (either time)?
   d. You get a blue card?
   e. You get 2 cards of the same color?
   f. You get 2 cards of different colors?

Are you ready for more?

In a game using five cards numbered 1, 2, 3, 4, and 5, you take two cards and add the values together. If the sum is 8, you win. Would you rather pick a card and put it back before picking the second card, or keep the card in your hand while you pick the second card? Explain your reasoning.

Summary

Suppose we have two bags. One contains 1 star block and 4 moon blocks. The other contains 3 star blocks and 1 moon block.



Figure 8.2.3.2

If we select one block at random from each, what is the probability that we will get two star blocks or two moon blocks?

To answer this question, we can draw a tree diagram to see all of the possible outcomes.

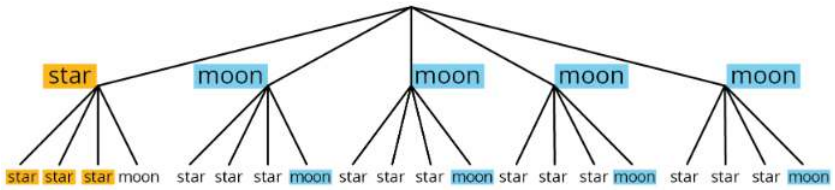


Figure 8.2.3.3: A tree diagram. The first choice has 5 branches, representing the 5 blocks in the bag: one branch is labeled star, the other 4 are labeled moon. Each of these branches has 4 branches, representing the 4 blocks in the second bag. 3 branches are labeled star and one is labeled moon. The word star in the first choice, and the 3 star choices branching from it are highlighted gold. From the first choice, the word moon is highlighted blue on each of the four remaining branches. From each of those branches, the one choice of moon for each is also highlighted blue.

There are $5 \cdot 4 = 20$ possible outcomes. Of these, 3 of them are both stars, and 4 are both moons. So the probability of getting 2 star blocks or 2 moon blocks is $\frac{7}{20}$.

In general, if all outcomes in an experiment are equally likely, then the probability of an event is the fraction of outcomes in the sample space for which the event occurs.

Practice

Exercise 8.2.3.5

A vending machine has 5 colors (white, red, green, blue, and yellow) of gumballs and an equal chance of dispensing each. A second machine has 4 different animal-shaped rubber bands (lion, elephant, horse, and alligator) and an equal chance of dispensing each. If you buy one item from each machine, what is the probability of getting a yellow gumball and a lion band?

Exercise 8.2.3.6

The numbers 1 through 10 are put in one bag. The numbers 5 through 14 are put in another bag. When you pick one number from each bag, what is the probability you get the same number?

Exercise 8.2.3.7

When rolling 3 standard number cubes, the probability of getting all three numbers to match is $\frac{6}{216}$. What is the probability that the three numbers *do not* all match? Explain your reasoning.

Exercise 8.2.3.8

For each event, write the sample space and tell how many outcomes there are.

1. Roll a standard number cube. Then flip a quarter.
2. Select a month. Then select 2020 or 2025.

(From Unit 8.2.2)

Exercise 8.2.3.9

On a graph of the area of a square vs. its perimeter, a few points are plotted.

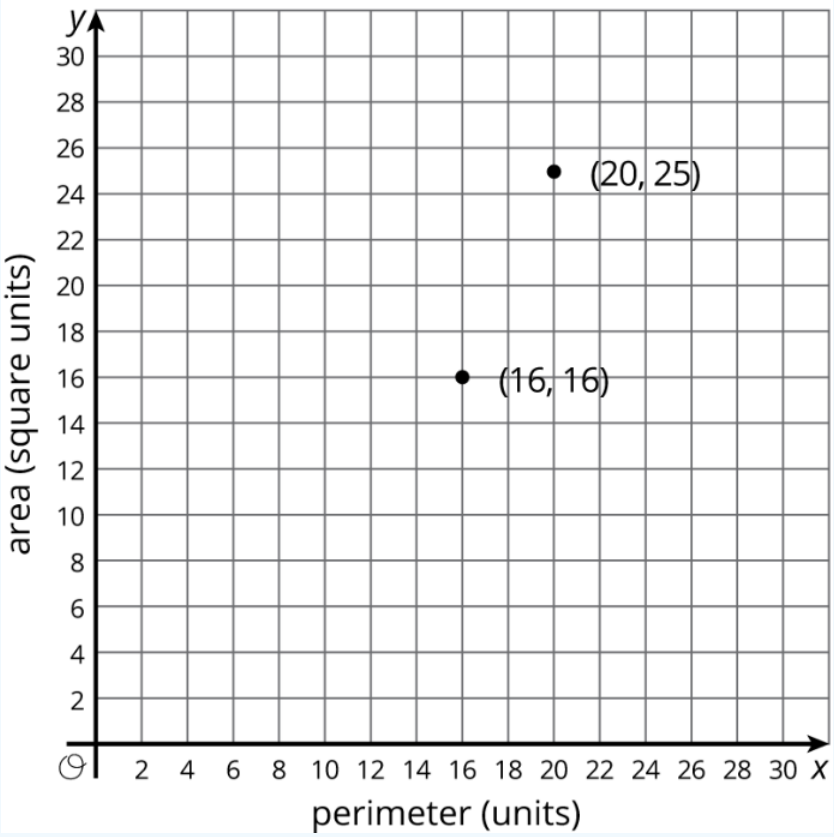


Figure 8.2.3.4: A coordinate grid with the horizontal axis labeled perimeter in units and vertical axis labeled area in square units. On both axes the numbers 0 through 30, in increments of 2, are indicated. Two points with the coordinates (16, 16) and (20, 25) are indicated on the grid.

1. Add some more ordered pairs to the graph.
2. Is there a proportional relationship between the area and perimeter of a square? Explain how you know.

(From Unit 2.4.2)

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8.2.4: Designing Simulations

Lesson

Let's simulate some real-life scenarios.

Exercise 8.2.4.1: Number Talk: Division

Find the value of each expression mentally.

$(4.2+3)\div 2$

$(4.2+2.6+4)\div 3$

$(4.2+2.6+4+3.6)\div 4$

$(4.2+2.6+4+3.6+3.6)\div 5$

Exercise 8.2.4.2: Breeding Mice

A scientist is studying the genes that determine the color of a mouse's fur. When two mice with brown fur breed, there is a 25% chance that each baby will have white fur. For the experiment to continue, the scientist needs at least 2 out of 5 baby mice to have white fur.

To simulate this situation, you can flip two coins at the same time for each baby mouse. If you don't have coins, you can use this applet.

- If both coins land heads up, it represents a mouse with white fur.
- Any other result represents a mouse with brown fur.

1. Have each person in the group simulate a litter of 5 offspring and record their results. Next, determine whether at least 2 of the offspring have white fur.

Table 8.2.4.1

	mouse 1	mouse 2	mouse 3	mouse 4	mouse 5	Do at least 2 have white fur?
simulation 1						
simulation 2						
simulation 3						

2. Based on the results from everyone in your group, estimate the probability that the scientist's experiment will be able to continue.
3. How could you improve your estimate?

Are you ready for more?

For a certain pair of mice, the genetics show that each offspring has a probability of $\frac{1}{16}$ that they will be albino. Describe a simulation you could use that would estimate the probability that at least 2 of the 5 offspring are albino.

Exercise 8.2.4.3: Designing Simulations

Your teacher will give your group a paper describing a situation.

1. Design a simulation that you could use to estimate a probability. Show your thinking. Organize it so it can be followed by others.
2. Explain how you used the simulation to answer the questions posed in the situation.

Summary

Many real-world situations are difficult to repeat enough times to get an estimate for a probability. If we can find probabilities for parts of the situation, we may be able to simulate the situation using a process that is easier to repeat.

For example, if we know that each egg of a fish in a science experiment has a 13% chance of having a mutation, how many eggs do we need to collect to make sure we have 10 mutated eggs? If getting these eggs is difficult or expensive, it might be helpful to have an idea about how many eggs we need before trying to collect them.



Figure 8.2.4.1

We could simulate this situation by having a computer select random numbers between 1 and 100. If the number is between 1 and 13, it counts as a mutated egg. Any other number would represent a normal egg. This matches the 13% chance of each fish egg having a mutation.

We could continue asking the computer for random numbers until we get 10 numbers that are between 1 and 13. How many times we asked the computer for a random number would give us an estimate of the number of fish eggs we would need to collect.

To improve the estimate, this entire process should be repeated many times. Because computers can perform simulations quickly, we could simulate the situation 1,000 times or more.

Practice

Exercise 8.2.4.4

A rare and delicate plant will only produce flowers from 10% of the seeds planted. To see if it is worth planting 5 seeds to see any flowers, the situation is going to be simulated. Which of these options is the best simulation? For the others, explain why it is not a good simulation.

1. Another plant can be genetically modified to produce flowers 10% of the time. Plant 30 groups of 5 seeds each and wait 6 months for the plants to grow and count the fraction of groups that produce flowers.
2. Roll a standard number cube 5 times. Each time a 6 appears, it represents a plant producing flowers. Repeat this process 30 times and count the fraction of times at least one number 6 appears.
3. Have a computer produce 5 random digits (0 through 9). If a 9 appears in the list of digits, it represents a plant producing flowers. Repeat this process 300 times and count the fraction of times at least one number 9 appears.
4. Create a spinner with 10 equal sections and mark one of them “flowers.” Spin the spinner 5 times to represent the 5 seeds. Repeat this process 30 times and count the fraction of times that at least 1 “flower” was spun.

Exercise 8.2.4.5

Jada and Elena learned that 8% of students have asthma. They want to know the probability that in a team of 4 students, at least one of them has asthma. To simulate this, they put 25 slips of paper in a bag. Two of the slips say “asthma.” Next, they take four papers out of the bag and record whether at least one of them says “asthma.” They repeat this process 15 times.

- Jada says they could improve the accuracy of their simulation by using 100 slips of paper and marking 8 of them.
- Elena says they could improve the accuracy of their simulation by conducting 30 trials instead of 15.

1. Do you agree with either of them? Explain your reasoning.
2. Describe another method of simulation the same scenario.

Exercise 8.2.4.6

The figure on the left is a trapezoidal prism. The figure on the right represents its base. Find the volume of this prism.

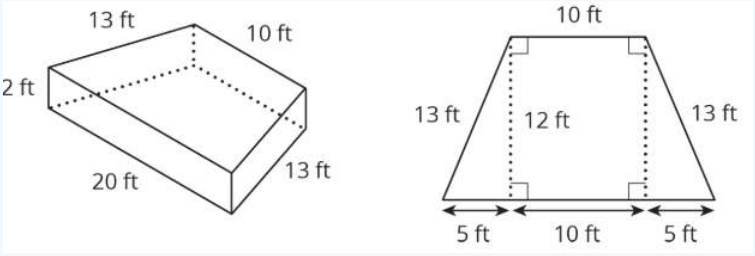


Figure 8.2.4.2

(From Unit 7.3.3)

Exercise 8.2.4.7

Match each expression in the first list with an equivalent expression from the second list.

A. $(8x+6y)+(2x+4y)$
B. $(8x+6y)-(2x+4y)$
C. $(8x+6y)-(2x-4y)$
D. $8x-6y-2x+4y$
E. $8x-6y+2x-4y$
F. $8x-(-6y-2x+4y)$

1. $10(x+y)$
2. $10(x-y)$
3. $6(x-\frac{1}{3}y)$
4. $8x+6y+2x-4y$
5. $8x+6y-2x+4y$
6. $8x-2x+6y-4y$

(From Unit 6.4.5)

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8.3: Sampling

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8.3.1: Comparing Groups

Lesson

Let's compare two groups.

Exercise 8.3.1.1: Notice and Wonder: Comparing Heights

What do you notice? What do you wonder?

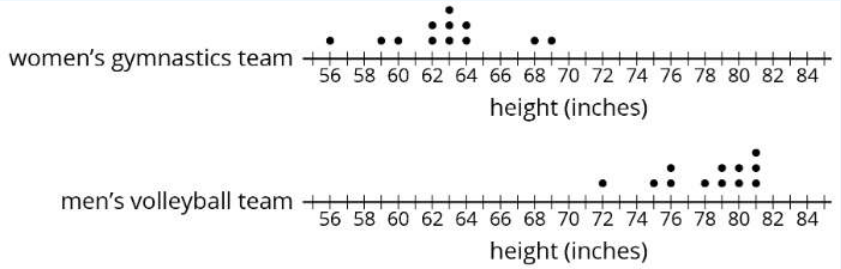


Figure 8.3.1.1

Exercise 8.3.1.2: More Team Heights

1. How much taller is the volleyball team than the gymnastics team?
   - Gymnastics team's heights (in inches) : 56, 59, 60, 62, 62, 63, 63, 63, 64, 64, 68, 69
   - Volleyball team's heights (in inches): 72, 75, 76, 76, 78, 79, 79, 80, 80, 81, 81, 81
2. Make dot plots to compare the heights of the tennis and badminton teams.
   - Tennis team's heights (in inches): 66, 67, 69, 70, 71, 73, 73, 74, 75, 75, 76
   - Badminton team's heights (in inches): 62, 62, 65, 66, 68, 71, 73

   What do you notice about your dot plots?
3. Elena says the members of the tennis team were taller than the badminton team. Lin disagrees. Do you agree with either of them? Explain or show your reasoning.

Exercise 8.3.1.3: Family Heights

Compare the heights of these two families. Explain or show your reasoning.

- The heights (in inches) of Noah's family members: 28, 39, 41, 52, 63, 66, 71
- The heights (in inches) of Jada's family members: 49, 60, 68, 70, 71, 73, 77

Are you ready for more?

If Jada's family adopts newborn twins who are each 18 inches tall, does this change your thinking? Explain your reasoning.

Exercise 8.3.1.4: Track Length

Here are three dot plots that represent the lengths, in minutes, of songs on different albums.

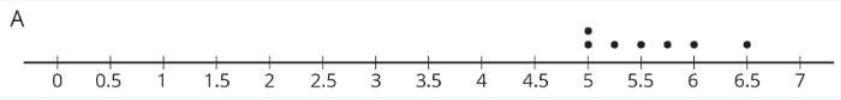


Figure 8.3.1.2: A dot plot labeled A. The numbers 0 through 7, in increments of 0 point 5, are indicated. The data are as follows: 5, 2 dots. 5 point 25, 1 dot. 5 point 5, 1 dot. 5 point 75, 1 dot. 6, 1 dot. 6 point 5, 1 dot.

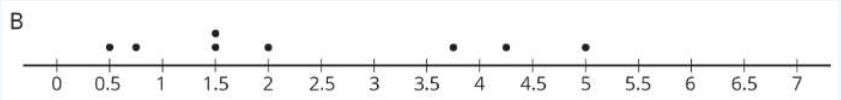


Figure 8.3.1.3: A dot plot labeled B. The numbers 0 through 7, in increments of 0 point 5, are indicated. The data are as follows: 0 point 5, 1 dot. 0 point 7 5, 1 dot. 1 point 5, 2 dots. 2, 1 dot. 3 point 7 5, 1 dot. 4 point 2 5, 1 dot. 5, 1 dot.

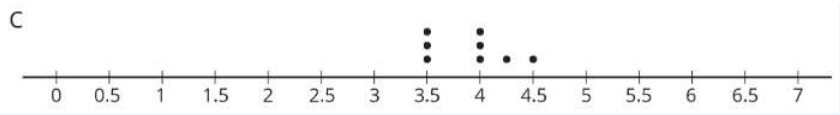


Figure 8.3.1.4

1. One of these data sets has a mean of 5.57 minutes and another has a mean of 3.91 minutes.
   a. Which dot plot shows each of these data sets?
   b. Calculate the mean for the data set on the other dot plot.
2. One of these data sets has a mean absolute deviation of 0.30 and another has a MAD of 0.44.
   a. Which dot plot shows each of these data sets?
   b. Calculate the MAD for the other data set.
3. Do you think the three groups are very different or not? Be prepared to explain your reasoning.
4. A fourth album has a mean length of 8 minutes with a mean absolute deviation of 1.2. Is this data set very different from each of the others?

Summary

Comparing two individuals is fairly straightforward. The question "Which dog is taller?" can be answered by measuring the heights of two dogs and comparing them directly. Comparing two groups can be more challenging. What does it mean for the basketball team to generally be taller than the soccer team?

To compare two groups, we use the distribution of values for the two groups. Most importantly, a measure of center (usually **mean** or **median**) and its associated measure of variability (usually **mean absolute deviation** or *interquartile range)* can help determine the differences between groups.

For example, if the average height of pugs in a dog show is 11 inches, and the average height of the beagles in the dog show is 15 inches, it seems that the beagles are generally taller. On the other hand, if the MAD is 3 inches, it would not be unreasonable to find a beagle that is 11 inches tall or a pug that is 14 inches tall. Therefore the heights of the two dog breeds may not be very different from one another.

Glossary Entries

Definition: Mean

The mean is one way to measure the center of a data set. We can think of it as a balance point. For example, for the data set 7, 9, 12, 13, 14, the mean is 11.

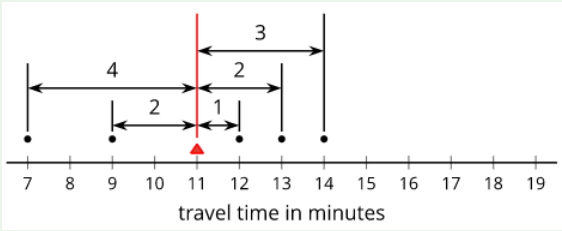


Figure 8.3.1.5

To find the mean, add up all the numbers in the data set. Then, divide by how many numbers there are. $7+9+12+13+14=55$ and $55\div 5=11$.

Definition: Mean Absolute Deviation (MAD)

The mean absolute deviation is one way to measure how spread out a data set is. Sometimes we call this the MAD. For example, for the data set 7, 9, 12, 13, 14, the MAD is 2.4. This tells us that these travel times are typically 2.4 minutes away from the mean, which is 11.

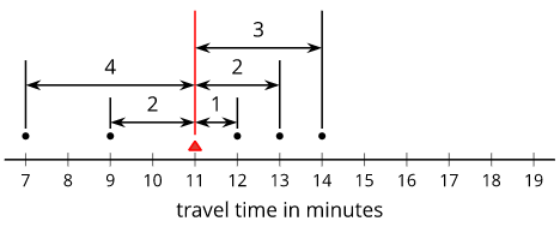


Figure 8.3.1.6

To find the MAD, add up the distance between each data point and the mean. Then, divide by how many numbers there are.

$4+2+1+2+3=12$ and $12 \div 5=2.4$

Definition: Median

The median is one way to measure the center of a data set. It is the middle number when the data set is listed in order.

For the data set 7, 9, 12, 13, 14, the median is 12.

For the data set 3, 5, 6, 8, 11, 12, there are two numbers in the middle. The median is the average of these two numbers. $6+8=14$ and $14 \div 2=7$.

Practice

Exercise 8.3.1.5

Compare the weights of the backpacks for the students in these three classes.

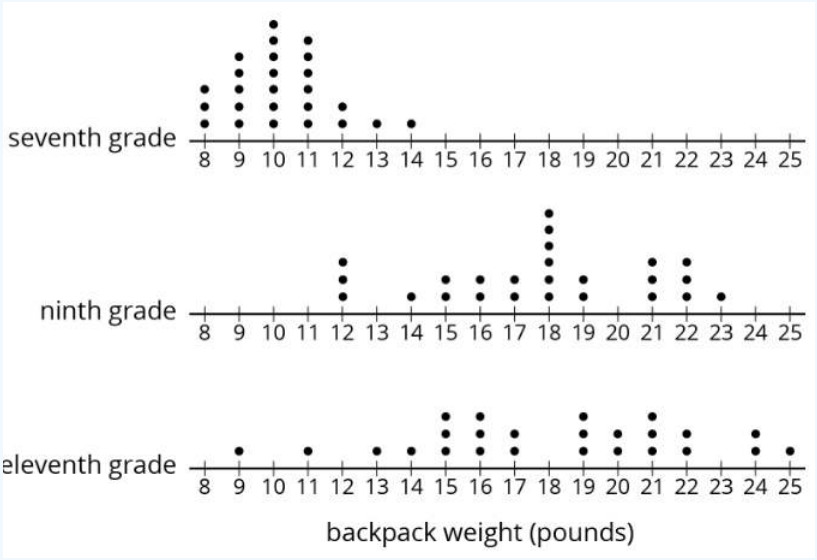


Figure 8.3.1.7

Exercise 8.3.1.6

A bookstore has marked down the price for all the books in a certain series by 15%.

1. How much is the discount on a book that normally costs $18.00?
2. After the discount, how much would the book cost?

(From Unit 4.3.2)

Exercise 8.3.1.7

Match each expression in the first list with an equivalent expression from the second list.

A. $6(x+2y)-2(y-2x)$
B. $2.5(2x+4y)-5(4y-x)$
C. $4(5x-3y)-10x+6y$
D. $5.5(x+y)-2(x+y)+6.5(x+y)$
E. $7.9(5x+3y)-4.2(5x+3y)-1.7(5x+3y)$

1. $10(x-y)$
2. $10(x+y)$
3. $10x+6y$
4. $10x-6y$

(From Unit 6.4.5)

Exercise 8.3.1.8

Angles C and D are complementary. The ratio of the measure of Angle C to the measure of Angle D is $2:3$. Find the measure of each angle. Explain or show your reasoning.

(From Unit 7.1.2)

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8.3.2: Larger Populations

Lesson

Let's compare larger groups.

Exercise 8.3.2.1: First Name versus Last Name

Consider the question: In general, do the students at this school have more letters in their first name or last name? How many more letters?

1. What are some ways you might get some data to answer the question?
2. The other day, we compared the heights of people on different teams and the lengths of songs on different albums. What makes this question about first and last names harder to answer than those questions?

Exercise 8.3.2.2: John Jacobjingleheimerschmidt

Continue to consider the question from the warm-up: In general, do the students at this school have more letters in their first name or last name? How many more letters?

1. How many letters are in your first name? In your last name?
2. Do the number of letters in your own first and last names give you enough information to make conclusions about students' names in your entire school? Explain your reasoning.
3. Your teacher will provide you with data from the class. Record the mean number of letters as well as the mean absolute deviation for each data set.
   a. The first names of the students in your class.
   b. The last names of the students in your class.
4. Which mean is larger? By how much? What does this difference tell you about the situation?
5. Do the mean numbers of letters in the first and last names for everyone in your class give you enough information to make conclusions about students' names in your entire school? Explain your reasoning.

Exercise 8.3.2.3: Siblings and Pets

Consider the question: Do people who are the only child *have more pets*?

1. Earlier, we used information about the people in your class to answer a question about the entire school. Would surveying only the people in your class give you enough information to answer this new question? Explain your reasoning.
2. If you had to have an answer to this question by the end of class today, how would you gather data to answer the question?
3. If you could come back tomorrow with your answer to this question, how would you gather data to answer the question?
4. If someone else in the class came back tomorrow with an answer that was different than yours, what would that mean? How would you determine which answer was better?

Exercise 8.3.2.4: Sampling the Population

For each question, identify the **population** and a possible **sample**.

1. What is the mean number of pages for novels that were on the best seller list in the 1990s?
2. What fraction of new cars sold between August 2010 and October 2016 were built in the United States?
3. What is the median income for teachers in North America?
4. What is the average lifespan of Tasmanian devils?

Are you ready for more?

Political parties often use samples to poll people about important issues. One common method is to call people and ask their opinions. In most places, though, they are not allowed to call cell phones. Explain how this restriction might lead to inaccurate samples of the population.

Summary

A **population** is a set of people or things that we want to study. Here are some examples of populations:

- All people in the world
- All seventh graders at a school
- All apples grown in the U.S.

A **sample** is a subset of a population. Here are some examples of samples from the listed populations:

- The leaders of each country
- The seventh graders who are in band
- The apples in the school cafeteria

When we want to know more about a population but it is not feasible to collect data from everyone in the population, we often collect data from a sample. In the lessons that follow, we will learn more about how to pick a sample that can help answer questions about the entire population.

Glossary Entries

Definition: Mean

The mean is one way to measure the center of a data set. We can think of it as a balance point. For example, for the data set 7, 9, 12, 13, 14, the mean is 11.

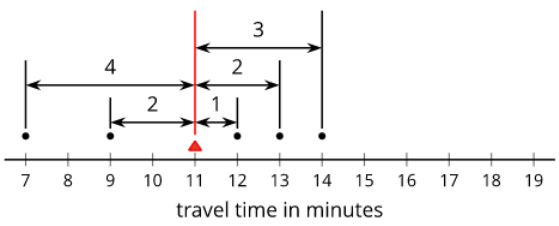


Figure 8.3.2.1

To find the mean, add up all the numbers in the data set. Then, divide by how many numbers there are. $7+9+12+13+14=55$ and $55\div 5=11$.

Definition: Mean Absolute Deviation (MAD)

The mean absolute deviation is one way to measure how spread out a data set is. Sometimes we call this the MAD. For example, for the data set 7, 9, 12, 13, 14, the MAD is 2.4. This tells us that these travel times are typically 2.4 minutes away from the mean, which is 11.

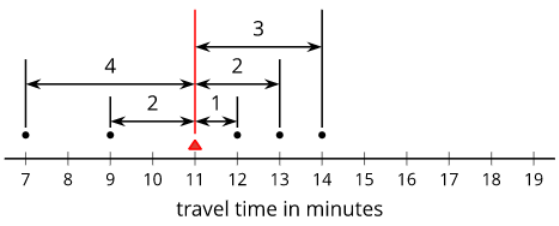


Figure 8.3.2.2

To find the MAD, add up the distance between each data point and the mean. Then, divide by how many numbers there are.

$4+2+1+2+3=12$ and $12\div 5=2.4$

Definition: Median

The median is one way to measure the center of a data set. It is the middle number when the data set is listed in order.

For the data set 7, 9, 12, 13, 14, the median is 12.

For the data set 3, 5, 6, 8, 11, 12, there are two numbers in the middle. The median is the average of these two numbers. $6+8=14$ and $14 \div 2=7$.

Definition: Population

A population is a set of people or things that we want to study.

For example, if we want to study the heights of people on different sports teams, the population would be all the people on the teams.

Definition: Sample

A sample is part of a population. For example, a population could be all the seventh grade students at one school. One sample of that population is all the seventh grade students who are in band.

Practice

Exercise 8.3.2.5

Suppose you are interested in learning about how much time seventh grade students at your school spend outdoors on a typical school day.

Select **all** the samples that are a part of the population you are interested in.

A. The 20 students in a seventh grade math class.
B. The first 20 students to arrive at school on a particular day.
C. The seventh grade students participating in a science fair put on by the four middle schools in a school district.
D. The 10 seventh graders on the school soccer team.
E. The students on the school debate team.

Exercise 8.3.2.6

For each sample given, list two possible populations they could belong to.

1. Sample: The prices for apples at two stores near your house.
2. Sample: The days of the week the students in your math class ordered food during the past week.
3. Sample: The daily high temperatures for the capital cities of all 50 U.S. states over the past year.

Exercise 8.3.2.7

If 6 coins are flipped, find the probability that there is at least 1 heads.

(From Unit 8.2.3)

Exercise 8.3.2.8

A school's art club holds a bake sale on Fridays to raise money for art supplies. Here are the number of cookies they sold each week in the fall and in the spring:

Table 8.3.2.1

fall	20	26	25	24	29	20	19	19	24	24
spring	19	27	29	21	25	22	26	21	25	25

1. Find the mean number of cookies sold in the fall and in the spring.
2. The MAD for the fall data is 2.8 cookies. The MAD for the spring data is 2.6 cookies. Express the difference in means as a multiple of the larger MAD.
3. Based on this data, do you think that sales were generally higher in the spring than in the fall?

(From Unit 8.3.1)

Exercise 8.3.2.9

A school is selling candles for a fundraiser. They keep 40% of the total sales as their commission, and they pay the rest to the candle company.

Table 8.3.2.2

price of candle	number of candles sold
small candle: $11	68
medium candle: $18	45
large candle: $25	21

How much money must the school pay to the candle company?

(From Unit 4.3.2)

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8.3.3: What Makes a Good Sample?

Lesson

Let's see what makes a good sample.

Exercise 8.3.3.1: Number Talk: Division by Powers of 10

Find the value of each quotient mentally.

$34,000 \div 10$

$340 \div 100$

$34 \div 10$

$3.4 \div 100$

Exercise 8.3.3.2: Selling Paintings

Your teacher will assign you to work with either means or medians.

1. A young artist has sold 10 paintings. Calculate the measure of center you were assigned for each of these samples:
   a. The first two paintings she sold were for $50 and $350.
   b. At a gallery show, she sold three paintings for $250, $400, and $1,200.
   c. Her oil paintings have sold for $410, $400, and $375.
2. Here are the selling prices for all 10 of her paintings:
   $\$50 \quad \$200 \quad \$250 \quad \$275 \quad \$280 \quad \$350 \quad \$375 \quad \$400 \quad \$410 \quad \$1,200$
   Calculate the measure of center you were assigned for all of the selling prices.
3. Compare your answers with your partner. Were the measures of center for any of the samples close to the same measure of center for the population?

Exercise 8.3.3.3: Sampling the Fish Market

The price per pound of catfish at a fish market was recorded for 100 weeks.

1. Here are dot plots showing the population and three different samples from that population. What do you notice? What do you wonder?
2. If the goal is to have the sample represent the population, which of the samples would work best? Which wouldn't work so well? Explain your reasoning.

To use this applet, drag the gray bar at the bottom up to see the sample dot plots.

Are you ready for more?

When doing a statistical study, it is important to keep the goal of the study in mind. Representative samples give us the best information about the distribution of the population as a whole, but sometimes a representative sample won't work for the goal of a study!

For example, suppose you want to study how discrimination affects people in your town. Surveying a representative sample of people in your town would give information about how the population generally feels, but might miss some smaller groups. Describe a way you might choose a sample of people to address this question.

Exercise 8.3.3.4: Auditing Sales

An online shopping company tracks how many items they sell in different categories during each month for a year. Three different auditors each take samples from that data. Use the samples to draw dot plots of what the population data might look like for the furniture and electronics categories.

Auditor 1's sample

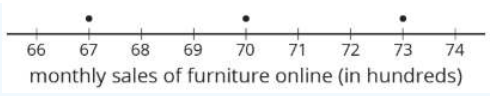


Figure 8.3.3.1: A dot plot for monthly sales of furniture online in hundreds. The numbers 66 through 74 are indicated. The data titled Auditor ones sample are as follows: 67 hundred, 1 dot. 70 hundred, 1 dot. 73 hundred, 1 dot.

Auditor 2's sample

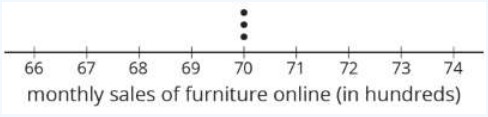


Figure 8.3.3.2

Auditor 3’s sample

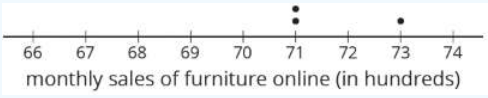


Figure 8.3.3.3: A dot plot for monthly sales of furniture online in hundreds. The numbers 66 through 74 are indicated. The data titled Auditor threes sample are as follows: 71 hundred, 2 dots. 73 hundred, 1 dot.

Population

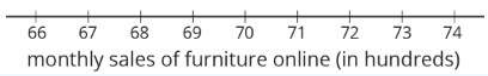


Figure 8.3.3.4

Auditor 1’s sample

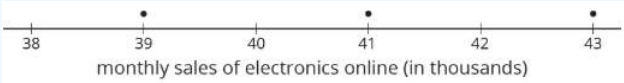


Figure 8.3.3.5: A dot plot for monthly sales of electronics online in thousands. The numbers 38 through 43 are indicated. The data titled Auditor one's sample are as follows: 39 thousand, 1 dot. 41 thousand, 1 dot. 43 thousand, 1 dot.

Auditor 2's sample



Figure 8.3.3.6: A dot plot for monthly sales of electronics online in thousands. The numbers 38 through 43 are indicated. The data titled Auditor two's sample are as follows: 41 thousand, 1 dot. 43 thousand, 2 dots.

Auditor 3's sample



Figure 8.3.3.7: A dot plot for monthly sales of electronics online in thousands. The numbers 38 through 43 are indicated. The data titled Auditor three's sample are as follows: 40 thousand, 1 dot. 41 thousand, 1 dot. 43 thousand, 1 dot.

Population



Figure 8.3.3.8

Summary

A sample that is **representative** of a population has a distribution that closely resembles the distribution of the population in shape, center, and spread.

For example, consider the distribution of plant heights, in cm, for a population of plants shown in this dot plot. The mean for this population is 4.9 cm, and the MAD is 2.6 cm.

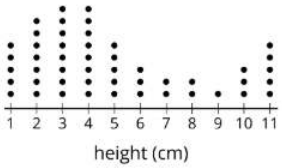


Figure 8.3.3.9: A dot plot for height in centimeters. The numbers 1 through 11 are indicated. The data are as follows: 1 centimeter, 5 dots; 2 centimeters, 7 dots; 3 centimeters, 8 dots; 4 centimeters, 8 dots; 5 centimeters, 5 dots; 6 centimeters, 3 dots; 7 centimeters, 2 dots; 8 centimeters, 2 dots; 9 centimeters, 1 dot; 10 centimeters, 3 dots; 11 centimeters, 5 dots.

A representative sample of this population should have a larger peak on the left and a smaller one on the right, like this one. The mean for this sample is 4.9 cm, and the MAD is 2.3 cm.

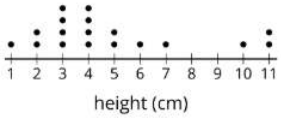


Figure 8.3.3.10: A dot plot for height in centimeters. The numbers 1 through 11 are indicated. The data are as follows: 1 centimeter, 1 dot; 2 centimeters, 2 dots; 3 centimeters, 4 dots; 4 centimeters, 4 dots; 5 centimeters, 2 dots; 6 centimeters, 1 dot; 7 centimeters, 1 dot; 10 centimeters, 1 dot; 11 centimeters, 2 dots.

Here is the distribution for another sample from the same population. This sample has a mean of 5.7 cm and a MAD of 1.5 cm. These are both very different from the population, and the distribution has a very different shape, so it is not a representative sample.

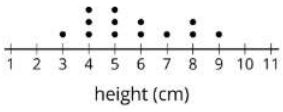


Figure 8.3.3.11: A dot plot for height in centimeters. The numbers 1 through 11 are indicated. The data are as follows: 3 centimeters, 1 dot; 4 centimeters, 3 dots; 5 centimeters, 3 dots; 6 centimeters, 2 dots; 7 centimeters, 1 dot; 8 centimeters, 2 dots; 9 centimeters, 1 dot.

Glossary Entries

Definition: Mean

The mean is one way to measure the center of a data set. We can think of it as a balance point. For example, for the data set 7, 9, 12, 13, 14, the mean is 11.

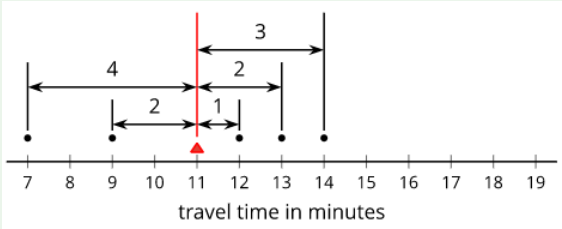


Figure 8.3.3.12

To find the mean, add up all the numbers in the data set. Then, divide by how many numbers there are. $7+9+12+13+14=55$ and $55\div 5=11$.

Definition: Mean Absolute Deviation (MAD)

The mean absolute deviation is one way to measure how spread out a data set is. Sometimes we call this the MAD. For example, for the data set 7, 9, 12, 13, 14, the MAD is 2.4. This tells us that these travel times are typically 2.4 minutes away from the mean, which is 11.

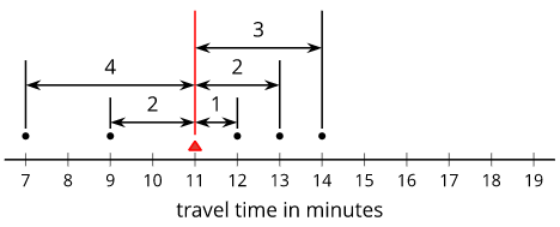


Figure 8.3.3.13

To find the MAD, add up the distance between each data point and the mean. Then, divide by how many numbers there are.

$4+2+1+2+3=12$ and $12 \div 5=2.4$

Definition: Median

The median is one way to measure the center of a data set. It is the middle number when the data set is listed in order.

For the data set 7, 9, 12, 13, 14, the median is 12.

For the data set 3, 5, 6, 8, 11, 12, there are two numbers in the middle. The median is the average of these two numbers. $6+8=14$ and $14 \div 2=7$.

Definition: Population

A population is a set of people or things that we want to study.

For example, if we want to study the heights of people on different sports teams, the population would be all the people on the teams.

Definition: Representative

A sample is representative of a population if its distribution resembles the population's distribution in center, shape, and spread.

For example, this dot plot represents a population.

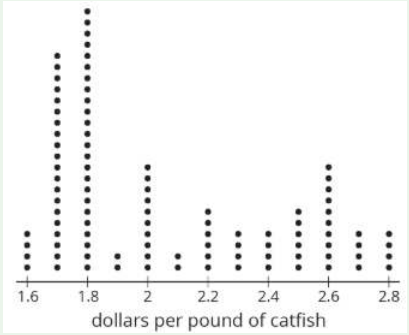


Figure 8.3.3.14

This dot plot shows a sample that is representative of the population.



Figure 8.3.3.15

Definition: Sample

A sample is part of a population. For example, a population could be all the seventh grade students at one school. One sample of that population is all the seventh grade students who are in band.

Practice

Exercise 8.3.3.5

Suppose 45% of all the students at Andre's school brought in a can of food to contribute to a canned food drive. Andre picks a representative sample of 25 students from the school and determines the sample's percentage.

He expects the percentage for this sample will be 45%. Do you agree? Explain your reasoning.

Exercise 8.3.3.6

This is a dot plot of the scores on a video game for a population of 50 teenagers.

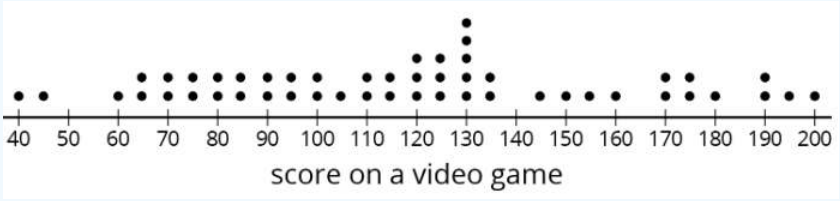


Figure 8.3.3.16: A dot plot for score on a video game. The numbers 40 through 200, in increments of 10, are indicated. The data are as follows: Score of 40, 1 dot. Score of 45, 1 dot. Score of 60, 1 dot. Score of 65, 2 dots. Score of 70, 2 dots. Score of 75, 2 dots. Score of 80, 2 dots. Score of 85, 2 dots. Score of 90, 2 dots. Score of 95, 2 dots. Score of 100, 2 dots. Score of 105, 1 dot. Score of 110, 2 dots. Score of 115, 2 dots. Score of 120, 3 dots. Score of 125, 3 dots. Score of 130, 5 dots. Score of 135, 2 dots. Score of 145, 1 dot. Score of 150, 1 dot. Score of 155, 1 dot. Score of 160, 1 dot. Score of 170, 2 dots. Score of 175, 2 dots. Score of 180, 1 dot. Score of 190, 2 dots. Score of 195, 1 dot. Score of 200, 1 dot.

The three dot plots together are the scores of teenagers in three samples from this population. Which of the three samples is most representative of the population? Explain how you know.

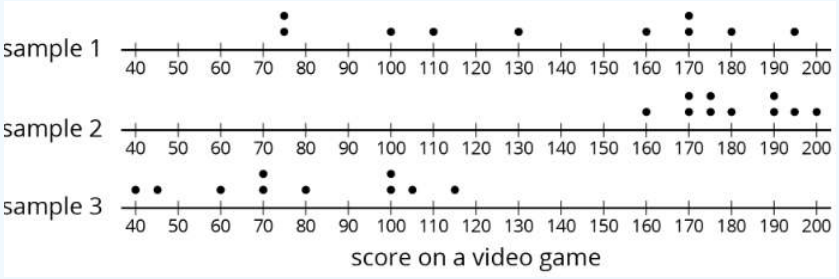


Figure 8.3.3.17: Three dot plots for score on a video game are labeled sample 1, sample 2, and sample 3. The numbers 40 through 200, in increments of 10, are indicated. The data are as follows: Sample 1: Score of 75, 2 dots. Score of 100, 1 dot. Score of 110, 1 dot. Score of 130, 1 dot. Score of 160, 1 dot. Score of 170, 2 dots. Score of 180, 1 dot. Score of 195, 1 dot. Sample 2: Score of 160, 1 dot. Score of 170, 2 dots. Score of 175, 2 dots. Score of 180, 1 dot. Score of 190, 2 dots. Score of 195, 1 dot. Score of 200, 1 dot. Sample 3: Score of 40, 1 dot. Score of 45, 1 dot. Score of 60, 1 dot. Score of 70, 2 dots. Score of 80, 1 dot. Score of 100, 2 dots. Score of 105, 1 dot. Score of 115, 1 dot.

Exercise 8.3.3.7

This is a dot plot of the number of text messages sent one day for a sample of the students at a local high school. The sample consisted of 30 students and was selected to be representative of the population.

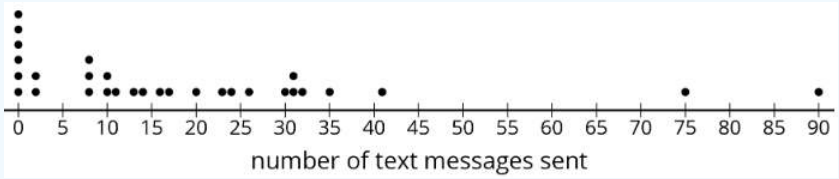


Figure 8.3.3.18: A dot plot for number of text messages sent. The numbers 0 through 90, in increments of 5, are indicated. The data are as follows: 0 text messages, 6 dots. 2 text messages, 2 dots. 8 text messages, 3 dots. 10 text messages, 2 dots. 11 text messages, 1 dot. 13 text messages, 1 dot. 14 text messages, 1 dot. 16 text messages, 1 dot. 17 text messages, 1 dot. 20 text messages, 1 dot. 23 text messages, 1 dot. 24 text messages, 1 dot. 26 text messages, 1 dot. 30 text messages, 1 dot. 31 text messages, 2 dots. 32 text messages, 1 dot. 35 text messages, 1 dot. 41 text messages, 1 dot. 75 text messages, 1 dot. 90 text messages, 1 dot.

1. What do the six values of 0 in the dot plot represent?
2. Since this sample is representative of the population, describe what you think a dot plot for the entire population might look like.

Exercise 8.3.3.8

A doctor suspects you might have a certain strain of flu and wants to test your blood for the presence of markers for this strain of virus. Why would it be good for the doctor to take a sample of your blood rather than use the population?

(From Unit 8.3.2)

Exercise 8.3.3.9

How many different outcomes are in each sample space? Explain your reasoning. (You do not need to write out the actual options, just provide the number and your reasoning.)

1. A letter of the English alphabet is followed by a digit from 0 to 9.
2. A baseball team's cap is selected from 3 different colors, 2 different clasps, and 4 different locations for the team logo. A decision is made to include or not to include reflective piping.
3. A locker combination like 7-23-11 uses three numbers, each from 1 to 40. Numbers can be used more than once, like 7-23-7.

(From Unit 8.2.2)

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8.3.4: Sampling in a Fair Way

Lesson

Let's explore ways to get representative samples.

Exercise 8.3.4.1: Ages of Moviegoers

A survey was taken at a movie theater to estimate the average age of moviegoers.

Here is a dot plot showing the ages of the first 20 people surveyed.

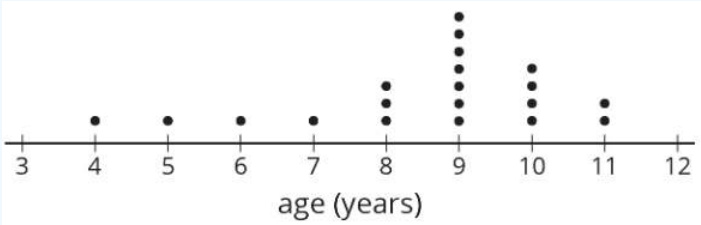


Figure 8.3.4.1: A dot plot for age in years. The numbers 3 through 12 are indicated. The data are as follows: 3 years old, 0 dots. 4 years old, 1 dot. 5 years old, 1 dot. 6 years old, 1 dot. 7 years old, 1 dot. 8 years old, 3 dots. 9 years old, 7 dots. 10 years old, 4 dots. 11 years old, 2 dots. 12 years old, 0 dots.

1. What questions do you have about the data from survey?
2. What assumptions would you make based on these results?

Exercise 8.3.4.2: Comparing Methods for Selecting Samples

Take turns with your partner reading each option aloud. For each situation, discuss:

- Would the different methods for selecting a sample lead to different conclusions about the population?
- What are the benefits of each method?
- What might each method overlook?
- Which of the methods listed would be the most likely to produce samples that are representative of the population being studied?
- Can you think of a better way to select a sample for this situation?

1. Lin is running in an election to be president of the seventh grade. She wants to predict her chances of winning. She has the following ideas for surveying a sample of the students who will be voting:
   a. Ask everyone on her basketball team who they are voting for.
   b. Ask every third girl waiting in the lunch line who they are voting for.
   c. Ask the first 15 students to arrive at school one morning who they are voting for.
2. A nutritionist wants to collect data on how much caffeine the average American drinks per day. She has the following ideas for how she could obtain a sample:
   a. Ask the first 20 adults who arrive at a grocery store after 10:00 a.m. about the average amount of caffeine they consume each day.
   b. Every 30 minutes, ask the first adult who comes into a coffee shop about the average amount of caffeine they consume each day.

Exercise 8.3.4.3: That's the First Straw

Your teacher will have some students draw straws from a bag.

1. As each straw is taken out and measured, record its length (in inches) in the table.

Table 8.3.4.1

	straw 1	straw 2	straw 3	straw 4	straw5

	straw 1	straw 2	straw 3	straw 4	straw5
sample 1					
sample 2					

2. Estimate the mean length of all the straws in the bag based on:
   a. the mean of the first sample.
   b. the mean of the second sample.
3. Were your two estimates the same? Did the mean length of all the straws in the bag change in between selecting the two samples? Explain your reasoning.
4. The actual mean length of all of the straws in the bag is about 2.37 inches. How do your estimates compare to this mean length?
5. If you repeated the same process again but you selected a larger sample (such as 10 or 20 straws, instead of just 5), would your estimate be more accurate? Explain your reasoning.

Exercise 8.3.4.4: That's the Last Straw

There were a total of 35 straws in the bag. Suppose we put the straws in order from shortest to longest and then assigned each straw a number from 1 to 35. For each of these methods, decide whether it would be fair way to select a sample of 5 straws. Explain your reasoning.

1. Select the straws numbered 1 through 5.
2. Write the numbers 1 through 35 on pieces of paper that are all the same size. Put the papers into a bag. Without looking, select five papers from the bag. Use the straws with those numbers for your sample.
3. Using the same bag as the previous question, select one paper from the bag. Use the number on that paper to select the first straw for your sample. Then use the next 4 numbers in order to complete your sample. (For example, if you select number 17, then you also use straws 18, 19, 20, and 21 for your sample.)
4. Create a spinner with 35 sections that are all the same size, and number them 1 through 35. Spin the spinner 5 times and use the straws with those numbers for your sample.

Are you ready for more?

Computers accept inputs, follow instructions, and produce outputs, so they cannot produce truly random numbers. If you knew the input, you could predict the output by following the same instructions the computer is following. When truly random numbers are needed, scientists measure natural phenomena such as radioactive decay or temperature variations. Before such measurements were possible, statisticians used random number tables, like this:

85 67 95 02 42 61 21 35 15 34 41
85 94 61 72 53 24 15 67 85 94 12
67 88 15 32 42 65 75 98 46 25 13
07 53 60 75 82 34 67 44 20 42 33
99 37 40 33 40 88 90 50 75 22 90
00 03 84 57 91 15 70 08 90 03 02
78 07 16 51 13 89 67 64 54 05 26
62 06 61 43 02 60 73 58 38 53 88
02 50 88 44 37 05 13 54 78 97 30

Figure 8.3.4.2: 11 by 9 table of random numbers.

Use this table to select a sample of 5 straws. Pick a starting point at random in the table. If the number is between 01 and 35, include that number straw in your sample. If the number has already been selected, or is not between 01 and 35, ignore it, and move on to the next number.

Summary

A sample is *selected at random* from a population if it has an equal chance of being selected as every other sample of the same size. For example, if there are 25 students in a class, then we can write each of the students' names on a slip of paper and select 5 papers from a bag to get a sample of 5 students selected at random from the class.

Other methods of selecting a sample from a population are likely to be *biased*. This means that it is less likely that the sample will be representative of the population as a whole. For example, if we select the first 5 students who walk in the door, that will not give us a random sample because students who typically come late are not likely to be selected. A sample that is selected at random may not always be a representative sample, but it is more likely to be representative than using other methods.

It is not always possible to select a sample at random. For example, if we want to know the average length of wild salmon, it is not possible to identify each one individually, select a few at random from the list, and then capture and measure those exact fish. When a sample cannot be selected at random, it is important to try to reduce bias as much as possible when selecting the sample.

Glossary Entries

Definition: Mean

The mean is one way to measure the center of a data set. We can think of it as a balance point. For example, for the data set 7, 9, 12, 13, 14, the mean is 11.

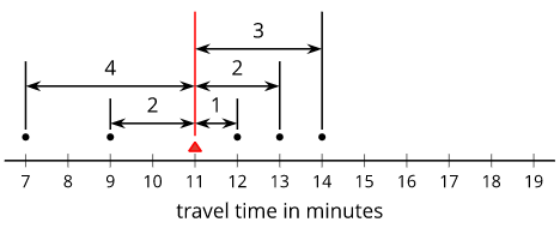


Figure 8.3.4.3

To find the mean, add up all the numbers in the data set. Then, divide by how many numbers there are. $7+9+12+13+14=55$ and $55\div 5=11$.

Definition: Mean Absolute Deviation (MAD)

The mean absolute deviation is one way to measure how spread out a data set is. Sometimes we call this the MAD. For example, for the data set 7, 9, 12, 13, 14, the MAD is 2.4. This tells us that these travel times are typically 2.4 minutes away from the mean, which is 11.

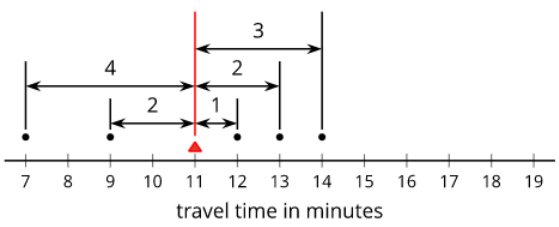


Figure 8.3.4.4

To find the MAD, add up the distance between each data point and the mean. Then, divide by how many numbers there are.

$4+2+1+2+3=12$ and $12\div 5=2.4$

Definition: Median

The median is one way to measure the center of a data set. It is the middle number when the data set is listed in order.

For the data set 7, 9, 12, 13, 14, the median is 12.

For the data set 3, 5, 6, 8, 11, 12, there are two numbers in the middle. The median is the average of these two numbers. $6+8=14$ and $14\div 2=7$.

Definition: Population

A population is a set of people or things that we want to study.

For example, if we want to study the heights of people on different sports teams, the population would be all the people on the teams.

Definition: Representative

A sample is representative of a population if its distribution resembles the population's distribution in center, shape, and spread.

For example, this dot plot represents a population.

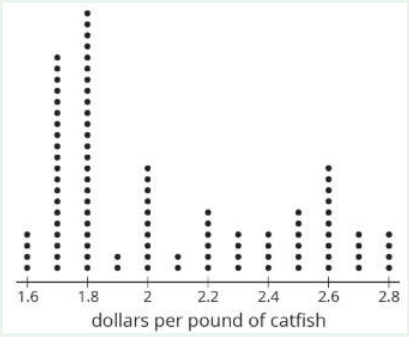


Figure 8.3.4.5

This dot plot shows a sample that is representative of the population.

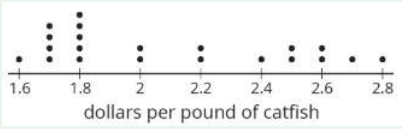


Figure 8.3.4.6

Definition: Sample

A sample is part of a population. For example, a population could be all the seventh grade students at one school. One sample of that population is all the seventh grade students who are in band.

Practice

Exercise 8.3.4.5

The meat department manager at a grocery store is worried some of the packages of ground beef labeled as having one pound of meat may be under-filled. He decides to take a sample of 5 packages from a shipment containing 100 packages of ground beef. The packages were numbered as they were put in the box, so each one has a different number between 1 and 100.

Describe how the manager can select a fair sample of 5 packages.

Exercise 8.3.4.6

Select **all** the reasons why random samples are preferred over other methods of getting a sample.

A. If you select a random sample, you can determine how many people you want in the sample.
B. A random sample is always the easiest way to select a sample from a population.
C. A random sample is likely to give you a sample that is representative of the population.
D. A random sample is a fair way to select a sample, because each person in the population has an equal chance of being selected.
E. If you use a random sample, the sample mean will always be the same as the population mean.

Exercise 8.3.4.7

Jada is using a computer's random number generator to produce 6 random whole numbers between 1 and 100 so she can use a random sample. The computer produces the numbers: 1, 2, 3, 4, 5, and 6. Should she use these numbers or have the computer generate a new set of random numbers? Explain your reasoning.

Exercise 8.3.4.8

A group of 100 people is divided into 5 groups with 20 people in each. One person's name is chosen, and everyone in their group wins a prize. Noah simulates this situation by writing 100 different names on papers and putting them in a bag, then drawing one out. Kiran suggests there is a way to do it with fewer paper slips. Explain a method that would simulate this situation with fewer than 100 slips of paper.

(From Unit 8.1.6)

Exercise 8.3.4.9

Data collected from a survey of American teenagers aged 13 to 17 was used to estimate that 29% of teens believe in ghosts. This estimate was based on data from 510 American teenagers. What is the population that people carrying out the survey were interested in?

A. All people in the United States.
B. The 510 teens that were surveyed.
C. All American teens who are between the ages of 13 and 17.
D. The 29% of the teens surveyed who said they believe in ghosts.

(From Unit 8.3.2)

Exercise 8.3.4.10

A computer simulates flipping a coin 100 times, then counts the longest string of heads in a row.

Based on these results, estimate the probability that there will be at least 15 heads in a row.

Table 8.3.4.2

trial	most heads in a row
1	8
2	6
3	5
4	11
5	13

(From Unit 8.2.1)

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8.4: Using Samples

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8.4.1: Estimating Population Measures of Center

Lesson

Let's use samples to estimate measures of center for the population.

Exercise 8.4.1.1: Describing the Center

Would you use the median or mean to describe the center of each data set? Explain your reasoning.

Heights of 50 basketball players

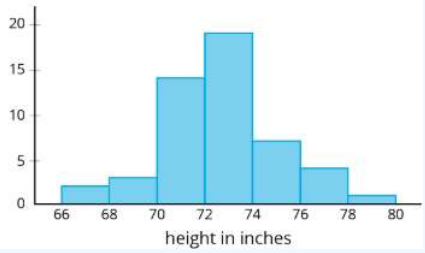


Figure 8.4.1.1

Ages of 30 people at a family dinner party

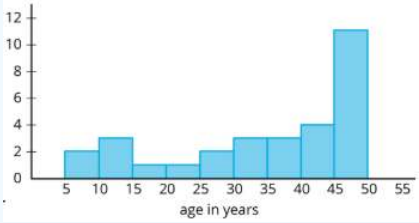


Figure 8.4.1.2

Backpack weights of sixth-grade students

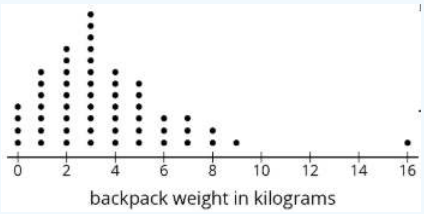


Figure 8.4.1.3

How many books students read over summer break

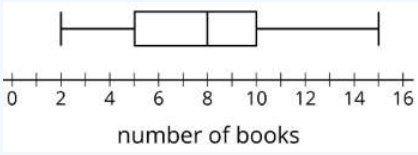


Figure 8.4.1.4

Exercise 8.4.1.2: Three Different TV Shows

Here are the ages (in years) of a random sample of 10 viewers for 3 different television shows. The shows are titled, “Science Experiments YOU Can Do,” “Learning to Read,” and “Trivia the Game Show.”

Table 8.4.1.1

sample 1	6	6	5	4	8	5	7	8	6	6
sample 2	15	14	12	13	12	10	12	11	10	8

sample 3	43	60	50	36	58	50	73	59	69	51

1. Calculate the mean for *one* of the samples. Make sure each person in your group works with a different sample. Record the answers for all three samples.
2. Which show do you think each sample represents? Explain your reasoning

Exercise 8.4.1.3: Who's Watching What?

Here are three more samples of viewer ages collected for these same 3 television shows.

Table 8.4.1.2

sample 4	57	71	5	54	52	13	59	65	10	71
sample 5	15	5	4	5	4	3	25	2	8	3
sample 6	6	11	9	56	1	3	11	10	11	2

1. Calculate the mean for *one* of these samples. Record all three answers.
2. Which show do you think each of these samples represents? Explain your reasoning.
3. For each show, estimate the mean age for all the show's viewers.
4. Calculate the mean absolute deviation for *one* of the shows' samples. Make sure each person in your group works with a different sample. Record all three answers.

Table 8.4.1.3

	Learning to Read	**Science Experiments YOU Can Do**	**Trivia the Game Show**
Which sample?			
MAD			

5. What do the different values for the MAD tell you about each group?
6. An advertiser has a commercial that appeals to 15- to 16-year-olds. Based on these samples, are any of these shows a good fit for this commercial? Explain or show your reasoning.

Exercise 8.4.1.4: Movie Reviews

A movie rating website has many people rate a new movie on a scale of 0 to 100. Here is a dot plot showing a random sample of 20 of these reviews.

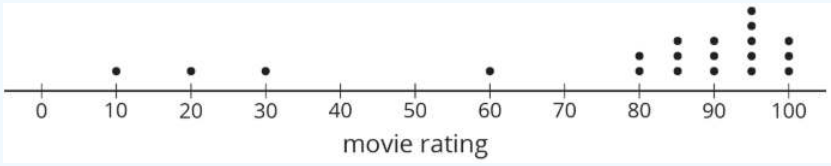


Figure 8.4.1.5

1. Would the mean or median be a better measure for the center of this data? Explain your reasoning.
2. Use the sample to estimate the measure of center that you chose for *all* the reviews.
3. For this sample, the mean absolute deviation is 19.6, and the interquartile range is 15. Which of these values is associated with the measure of center that you chose?
4. Movies must have an average rating of 75 or more from all the reviews on the website to be considered for an award. Do you think this movie will be considered for the award? Use the measure of center and measure of variability that you chose to justify your answer.

Are you ready for more?

Estimate typical temperatures in the United States today by looking up current temperatures in several places across the country. Use the data you collect to decide on the appropriate measure of center for the country, and calculate the related measure of variation for your sample.

Summary

Some populations have greater variability than others. For example, we would expect greater variability in the weights of dogs at a dog park than at a beagle meet-up.

Dog park:



Figure 8.4.1.6

Mean Weight: 12.8 kg MAD: 2.3 kg

Beagle meet-up:



Mean Weight: 10.1 kg MAD: 0.8 kg

The lower MAD indicates there is less variability in the weights of the beagles. We would expect that the mean weight from a sample that is randomly selected from a group of beagles will provide a more accurate estimate of the mean weight of all the beagles than a sample of the same size from the dogs at the dog park.

In general, a sample of a similar size from a population with *less* variability is *more likely* to have a mean that is close to the population mean.

Glossary Entries

Definition: Interquartile range (IQR)

The interquartile range is one way to measure how spread out a data set is. We sometimes call this the IQR. To find the interquartile range we subtract the first quartile from the third quartile.

For example, the IQR of this data set is 20 because $50 - 30 = 20$.

Table 8.4.1.4

22	29	30	31	32	43	44	45	50	50	59
		Q1			Q2			Q3		

Practice

Exercise 8.4.1.5

A random sample of 15 items were selected.

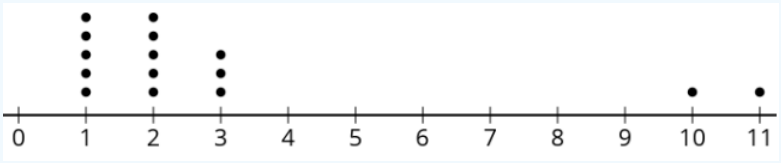


Figure 8.4.1.6

For this data set, is the mean or median a better measure of center? Explain your reasoning.

Exercise 8.4.1.6

A video game developer wants to know how long it takes people to finish playing their new game. They surveyed a random sample of 13 players and asked how long it took them (in minutes).

$$1,235 \quad 952 \quad 457 \quad 1,486 \quad 1,759 \quad 1,148 \quad 548 \quad 1,037 \quad 1,864 \quad 1,245 \quad 976 \quad 866 \quad 1,431$$

1. Estimate the median time it will take *all* players to finish this game.
2. Find the interquartile range for this sample.

Exercise 8.4.1.7

Han and Priya want to know the mean height of the 30 students in their dance class. They each select a random sample of 5 students.

- The mean height for Han's sample is 59 inches.
- The mean height for Priya's sample is 61 inches.

Does it surprise you that the two sample means are different? Are the population means different? Explain your reasoning.

Exercise 8.4.1.8

Clare and Priya each took a random sample of 25 students at their school.

- Clare asked each student in her sample how much time they spend doing homework each night. The sample mean was 1.2 hours and the MAD was 0.6 hours.
- Priya asked each student in her sample how much time they spend watching TV each night. The sample mean was 2 hours and the MAD was 1.3 hours.

1. At their school, do you think there is more variability in how much time students spend doing homework or watching TV? Explain your reasoning.
2. Clare estimates the students at her school spend an average of 1.2 hours each night doing homework. Priya estimates the students at her school spend an average of 2 hours each night watching TV. Which of these two estimates is likely to be closer to the actual mean value for all the students at their school? Explain your reasoning.

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8.4.2: Estimating Population Proportions

Lesson

Let's estimate population proportions using samples.

Exercise 8.4.2.1: Getting to School

A teacher asked all the students in one class how many minutes it takes them to get to school. Here is a list of their responses:

20 10 15 8 5 15 10 5 20 5 15 10 3 10 18 5 25 5 5 12 10 30 5 10

1. What fraction of the students in this class say:
   a. it takes them 5 minutes to get to school?
   b. it takes them more than 10 minutes to get to school?
2. If the whole school has 720 students, can you use this data to estimate how many of them would say that it takes them more than 10 minutes to get to school?
   Be prepared to explain your reasoning.

Exercise 8.4.2.2: Reacting Times

The track coach at a high school needs a student whose reaction time is less than 0.4 seconds to help out at track meets. All the twelfth graders in the school measured their reaction times. Your teacher will give you a bag of papers that list their results.

1. Work with your partner to select a random sample of 20 reaction times, and record them in the table.

Table 8.4.2.1

2. What **proportion** of your sample is less than 0.4 seconds?
3. Estimate the proportion of all twelfth graders at this school who have a reaction time of less than 0.4 seconds. Explain your reasoning.
4. There are 120 twelfth graders at this school. Estimate how many of them have a reaction time of less than 0.4 seconds.
5. Suppose another group in your class comes up with a different estimate than yours for the previous question.
   a. What is another estimate that would be *reasonable*?
   b. What is an estimate you would consider *unreasonable*?

Exercise 8.4.2.3: A New Comic Book Hero

Here are the results of a survey of 20 people who read *The Adventures of Super Sam* regarding what special ability they think the new hero should have.



Figure 8.4.2.1: Flame 002, by Will Eisner (pencils) and Lou Fine (inks) uploaded by Roygbiv666 (Public Domain Super Heroes). Public Domain. Wikimedia Commons. Source.

Table 8.4.2.2

response	what new ability?
1	fly

response	what new ability?
2	freeze
3	freeze
4	fly
5	fly
6	freeze
7	fly
8	super strength
9	freeze
10	fly
11	freeze
12	freeze
13	fly
14	invisibility
15	freeze
16	fly
17	freeze
18	fly
19	super strength
20	freeze

1. What proportion of this sample want the new hero to have the ability to fly?
2. If there are 2,024 dedicated readers of *The Adventures of Super Sam*, estimate the number of readers who want the new hero to fly.
   Two other comic books did a similar survey of their readers.
   - In a survey of people who read *Beyond Human*, 42 out of 60 people want a new hero to be able to fly.
   - In a survey of people who read *Mysterious Planets*, 14 out of 40 people want a new hero to be able to fly.
3. Do you think the proportion of all readers who want a new hero that can fly are nearly the same for the three different comic books? Explain your reasoning.
4. If you were in charge of these three comics, would you give the ability to fly to any of the new heroes? Explain your reasoning using the proportions you calculated.

Exercise 8.4.2.4: Flying to the Shelves

The authors of *The Adventures of Super Sam* chose 50 different random samples of readers. Each sample was of size 20. They looked at the sample proportions who prefer the new hero to fly.

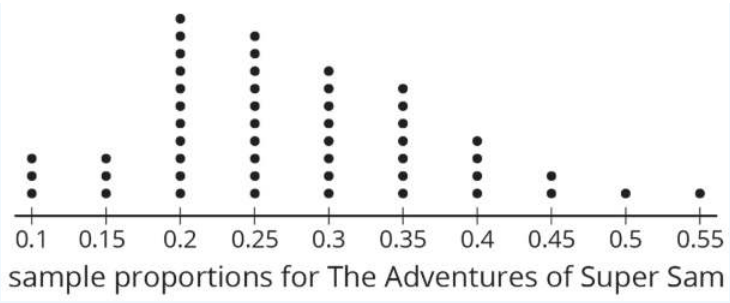


Figure 8.4.2.2: A dot plot for sample proportions for The Adventures of Super Sam with the numbers 0 point 1 through 0 point 55, in increments of zero point zero 5, indicated. The data are as follows: 0 point 1, 3 dots. 0 point 1 5, 3 dots. 0 point 2, 11 dots. 0 point 2 5, 10 dots. 0 point 3, 8 dots. 0 point 3 5, 7 dots. 0 point 4, 4 dots. 0 point 4 5, 2 dots. 0 point 5, 1 dot. 0 point 5 5, 1 dot.

1. What is a good estimate of the proportion of *all* readers who want the new hero to be able to fly?
2. Are most of the sample proportions within 0.1 of your estimate for the population proportion?
3. If the authors of *The Adventures of Super Sam* give the new hero the ability to fly, will that please most of the readers? Explain your reasoning.

The authors of the other comic book series created similar dot plots.

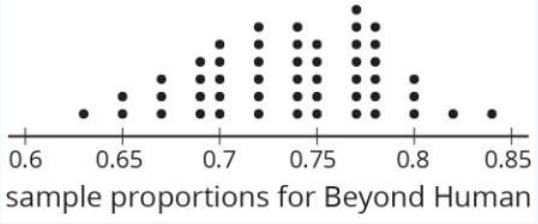


Figure 8.4.2.3: A dot plot for sample proportions for Beyond Human with the numbers 0 point 6 through 0 point 8 5, in increments of zero point zero 5, indicated. The data are as follows: 0 point 6 3, 1 dot. 0 point 6 5, 2 dots. 0 point 6 7, 3 dots. 0 point 6 9, 4 dots. 0 point 7, 5 dots. 0 point 7 2, 6 dots. 0 point 7 4, 6 dots. 0 point 7 5, 5 dots. 0 point 7 7, 7 dots. 0 point 7 8, 6 dots. 0 point 8, 3 dots 0 point 8 2, 1 dot. 0 point 8 4, 1 dot.

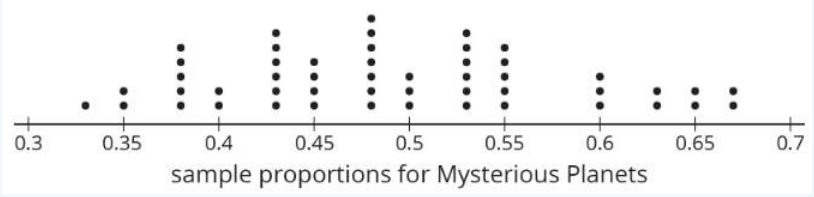


Figure 8.4.2.4: A dot plot for sample proportions for Mysterious Planets with the numbers 0 point 3 through 0 point 7, in increments of 0 point 5, indicated. The data are as follows: 0 point 3 3, 1 dot. 0 point 3 5, 2 dots. 0 point 3 8, 5 dots. 0 point 4, 2 dots. 0 point 4 3, 6 dots. 0 point 4 5, 4 dots. 0 point 4 8, 7 dots. 0 point 5, 3 dots. 0 point 5 3, 6 dots. 0 point 5 5, 5 dots. 0 point 6, 3 dots. 0 point 6 3, 2 dots. 0 point 6 5, 2 dots. 0 point 6 7, 2 dots.

4. For each of these series, estimate the proportion of all readers who want the new hero to fly.

- *Beyond Human*:
- *Mysterious Planets*:

5. Should the authors of either of these series give their new hero the ability to fly?

6. Why might it be more difficult for the authors of *Mysterious Planets* to make the decision than the authors of the other series?

Are you ready for more?

Draw an example of a dot plot with at least 20 dots that represent the sample proportions for different random samples that would indicate that the population proportion is above 0.6, but there is a lot of uncertainty about that estimate.

Summary

Sometimes a data set consists of information that fits into specific categories. For example, we could survey students about whether they have a pet cat or dog. The categories for these data would be {neither, dog only, cat only, both}. Suppose we surveyed 10

students. Here is a table showing possible results:

Table 8.4.2.3

option	number of responses
neither dog nor cat	2
dog only	4
cat only	1
both dog and cat	3

In this sample, 3 of the students said they have both a dog and a cat. We can say that the **proportion** of these students who have a both a dog and a cat is $\frac{3}{10}$ or 0.3. If this sample is representative of all 720 students at the school, we can predict that about $\frac{3}{10}$ of 720, or about 216 students at the school have both a dog and a cat.

In general, a proportion is a number from 0 to 1 that represents the fraction of the data that belongs to a given category.

Glossary Entries

Definition: Interquartile range (IQR)

The interquartile range is one way to measure how spread out a data set is. We sometimes call this the IQR. To find the interquartile range we subtract the first quartile from the third quartile.

For example, the IQR of this data set is 20 because $50 - 30 = 20$.

Table 8.4.2.4

22	29	30	31	32	43	44	45	50	50	59
		Q1			Q2			Q3		

Definition: Proportion

A proportion of a data set is the fraction of the data in a given category.

For example, a class has 18 students. There are 2 left-handed students and 16 right-handed students in the class. The proportion of students who are left-handed is $\frac{2}{20}$, or 0.1.

Practice

Exercise 8.4.2.5

Tyler wonders what proportion of students at his school would dye their hair blue, if they were allowed to. He surveyed a random sample of 10 students at his school, and 2 of them said they would. Kiran didn't think Tyler's estimate was very accurate, so he surveyed a random sample of 100 students, and 17 of them said they would.

1. Based on Tyler's sample, estimate what proportion of the students would dye their hair blue.
2. Based on Kiran's sample, estimate what proportion of the students would dye their hair blue.
3. Whose estimate is more accurate? Explain how you know.

Exercise 8.4.2.6

Han surveys a random sample of students about their favorite pasta dish served by the cafeteria and makes a bar graph of the results.

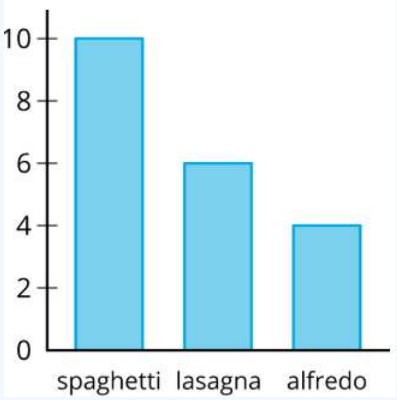


Figure 8.4.2.5

Estimate the proportion of the students who like lasagna as their favorite pasta dish.

Exercise 8.4.2.7

Elena wants to know what proportion of people have cats as pets. Describe a process she could use to estimate an answer to her question.

Exercise 8.4.2.8

The science teacher gives daily homework. For a random sample of days throughout the year, the median number of problems is 5 and the IQR is 2. The Spanish teacher also gives daily homework. For a random sample of days throughout the year, the median number of problems is 10 and the IQR is 1. If you estimate the median number of science homework problems to be 5 and the median number of Spanish problems to be 10, which is more likely to be accurate? Explain your reasoning.

(From Unit 8.4.1)

Exercise 8.4.2.9

Diego wants to survey a sample of students at his school to learn about the percentage of students who are satisfied with the food in the cafeteria. He decides to go to the cafeteria on a Monday and ask the first 25 students who purchase a lunch at the cafeteria if they are satisfied with the food.

Do you think this is a good way for Diego to select his sample? Explain your reasoning.

(From Unit 8.3.4)

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8.4.3: More about Sampling Variability

Lesson

Let's compare samples from the same population.

Exercise 8.4.3.1: Average Reactions

The other day, you worked with the reaction times of twelfth graders to see if they were fast enough to help out at the track meet. Look back at the sample you collected.

1. Calculate the mean reaction time for your sample.
2. Did you and your partner get the same sample mean? Explain why or why not.

Exercise 8.4.3.2: Reaction Population

Your teacher will display a blank dot plot.

1. Plot your sample mean from the previous activity on your teacher's dot plot.
2. What do you notice about the distribution of the sample means from the class?
   a. Where is the center?
   b. Is there a lot of variability?
   c. Is it approximately symmetric?
3. The population mean is 0.442 seconds. How does this value compare to the sample means from the class?
   Pause here so your teacher can display a dot plot of the population of reaction times.
4. What do you notice about the distribution of the population?
   a. Where is the center?
   b. Is there a lot of variability?
   c. Is it approximately symmetric?
5. Compare the two displayed dot plots.
6. Based on the distribution of sample means from the class, do you think the mean of a random sample of 20 items is likely to be:
   a. within 0.01 seconds of the actual population mean?
   b. within 0.1 seconds of the actual population mean?

Explain or show your reasoning.

Exercise 8.4.3.3: How Much Do You Trust the Answer?

The other day you worked with 2 different samples of viewers from each of 3 different television shows. Each sample included 10 viewers. Here are the mean ages for 100 different samples of viewers from each show.

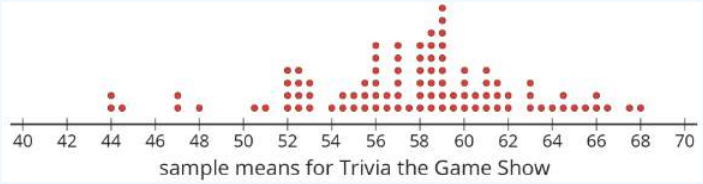


Figure 8.4.3.1: A dot plot for sample means for Trivia the Game Show with the numbers 40 through 70, in increments of 2, is indicated. The data are as follows: 44 years old, 2 dots. 44 point 5 years old, 1 dot. 47 years old, 2 dots. 48 years old, 1 dot. 50 point 5 years old, 1 dot. 51 years old, 1 dot. 52 years old, 4 dots. 52 point 5 years old, 4 dots. 53 years old, 3 dots. 54 years old, 1 dot. 54 point 5 years old, 2 dots. 55 years old, 2 dots. 55 point 5 years old, 3 dots. 56 years old, 6 dots. 56 point 5 years old, 2 dots. 57 years old, 6 dots. 57 point 5 years old, 1 dot. 58 years old, 6 dots. 58 point 5 years old, 7 dots. 59 years old, 9 dots. 59 point 5 years old, 2 dots. 60 years old, 4 dots. 60 point 5 years old, 2 dots. 61 years old, 4 dots. 61 point 5 years old, 3 dots. 62 years old, 2 dots. 63 years old, 3 dots. 63 point 5 years old, 1 dot. 64 years old, 1 dot. 64 point 5 years old, 2 dots. 65 years old, 1 dot. 65 point 5 years old, 1 dot. 66 years old, 2 dots. 66 point 5 years old, 1 dot. 67 point 5 years old, 1 dot. 68 years old, 1 dot.

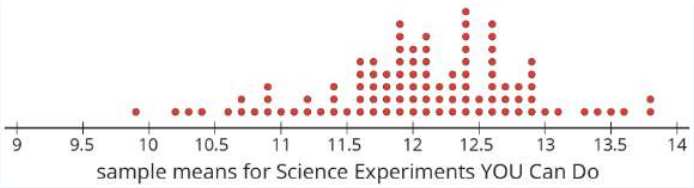


Figure 8.4.3.2: A dot plot for sample means for Science Experiments YOU Can Do with the numbers 9 through 14, in increments of zero point 5, indicated. The data are as follows: 9 point 8 years old, 1 dot. 10 point 2 years old, 1 dot. 10 point 3 years old, 1 dot. 10 point 4 years old, 1 dot. 10 point 6 years old, 1 dot. 10 point 7 years old, 2 dots. 10 point 8 years old, 1 dot. 10 point 9 years old, 3 dots. 11 years old, 1 dot. 11 point 1 years old, 1 dot. 11 point 2 years old, 2 dots. 11 point 3 years old, 1 dot. 11 point 4 years old, 3 dots. 11 point 5 years old, 1 dot. 11 point 6 years old, 5 dots. 11 point 7 years old, 5 dots. 11 point 8 years old, 4 dots. 11 point 9 years old, 8 dots. 12 years old, 6 dots. 12 point 1 years old, 7 dots. 12 point 2 years old, 3 dots. 12 point 3 years old, 4 dots. 12 point 4 years old, 9 dots. 12 point 5 years old, 2 dots. 12 point 6 years old, 8 dots. 12 point 7 years old, 3 dots. 12 point 8 years old, 3 dots. 12 point 9 years old, 5 dots. 13 years old, 1 dot. 13 point 1 years old, 1 dot. 13 point 3 years old, 1 dot. 13 point 4 years old, 1 dot. 13 point 5 years old, 1 dot. 13 point 6 years old, 1 dot. 13 point 8 years old, 2 dots.

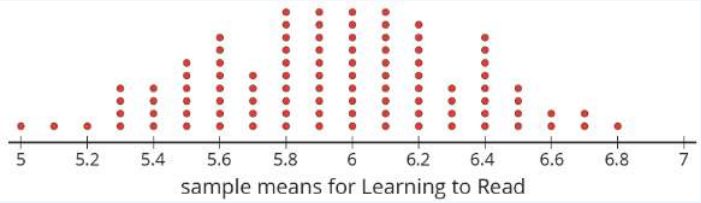


Figure 8.4.3.3: A dot plot for sample means for Learning to Read with the numbers 5 through 7, in increments of zero point 2, indicated. The data are as follows: 5 years old, 1 dot. 5 point 1 years old, 1 dot. 5 point 2 years old, 1 dot. 5 point 3 years old, 4 dots. 5 point 4 years old, 4 dots. 5 point 5 years old, 6 dots. 5 point 6 years old, 8 dots. 5 point 7 years old, 5 dots. 5 point 8 years old, 10 dots. 5 point 9 years old, 10 dots. 6 years old, 10 dots. 6 point 1 years old, 10 dots. 6 point 2 years old, 9 dots. 6 point 3 years old, 4 dots. 6 point 4 years old, 8 dots. 6 point 5 years old, 4 dots. 6 point 6 years old, 2 dots. 6 point 7 years old, 2 dots. 6 point 8 years old, 1 dot.

1. For each show, use the dot plot to estimate the *population* mean.
   a. Trivia the Game Show
   b. Science Experiments YOU Can Do
   c. Learning to Read
2. For each show, are most of the sample means within 1 year of your estimated population mean?
3. Suppose you take a new random sample of 10 viewers for each of the 3 shows. Which show do you expect to have the new sample mean closest to the population mean? Explain or show your reasoning.

Are you ready for more?

Market research shows that advertisements for retirement plans appeal to people between the ages of 40 and 55. Younger people are usually not interested and older people often already have a plan. Is it a good idea to advertise retirement plans during any of these three shows? Explain your reasoning.

Summary

This dot plot shows the weights, in grams, of 18 cookies. The triangle indicates the mean weight, which is 11.6 grams.

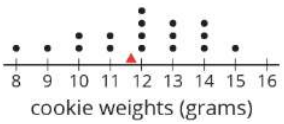


Figure 8.4.3.4: A dot plot labeled cookie weights in grams. The numbers 8 through 16 are indicated. A triangle at approximately 11.6 is indicated. The data are as follows: 8 grams, 1 dot; 9 grams, 1 dot; 10 grams, 2 dots; 11 grams, 2 dots; 12 grams, 4 dots; 13 grams, 3 dots; 14 grams, 3 dots; 15 grams, 1 dot.

This dot plot shows the *means* of 20 samples of 5 cookies, selected at random. Again, the triangle shows the mean for the *population* of cookies. Notice that most of the sample means are fairly close to the mean of the entire population.

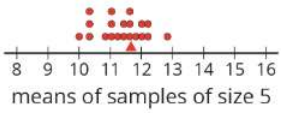


Figure 8.4.3.5: A dot plot for means of samples of size 5 with the numbers 8 through 16 indicated. The data are as follows: 10, 1 dot. 10 point 3, 3 dots. 10 point 9, 1 dot. 11, 3 dots. 11 point 2, 1 dot. 11 point 4, 2 dots. 11 point 6, 3 dots. 11 point 8, 1 dot. 12, 2 dots. 12 point 2, 2 dots. 12 point 8, 1 dot. There is a triangle located at 11.6.

This dot plot shows the means of 20 samples of 10 cookies, selected at random. Notice that the means for these samples are even closer to the mean for the entire population.

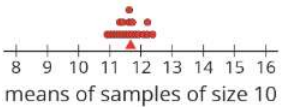


Figure 8.4.3.6: A dot plot labeled means of samples of size 10. The numbers 8 through 16 are indicated. The data are as follows: 10 point 9, 1 dot. 11, 1 dot. 11 point 1, 1 dot. 11 point 2, 1 dot. 11 point 3, 2 dots. 11 point 4, 2 dots. 11 point 5, 1 dot. 11 point 6, 3 dots. 11 point 7, 2 dots. 11 point 8, 1 dot. 11 point 9, 1 dot. 12 point 1, 1 dot. 12 point 3, 2 dots. 12 point 5, 1 dot. A triangle is located at 11 point 6.

In general, as the sample size gets bigger, the mean of a sample is more likely to be closer to the mean of the population.

Glossary Entries

Definition: Interquartile range (IQR)

The interquartile range is one way to measure how spread out a data set is. We sometimes call this the IQR. To find the interquartile range we subtract the first quartile from the third quartile.

For example, the IQR of this data set is 20 because $50 - 30 = 20$.

Table 8.4.3.1

22	29	30	31	32	43	44	45	50	50	59
		Q1			Q2			Q3		

Definition: Proportion

A proportion of a data set is the fraction of the data in a given category.

For example, a class has 18 students. There are 2 left-handed students and 16 right-handed students in the class. The proportion of students who are left-handed is $\frac{2}{20}$, or 0.1.

Practice

Exercise 8.4.3.4

One thousand baseball fans were asked how far they would be willing to travel to watch a professional baseball game. From this population, 100 different samples of size 40 were selected. Here is a dot plot showing the mean of each sample.

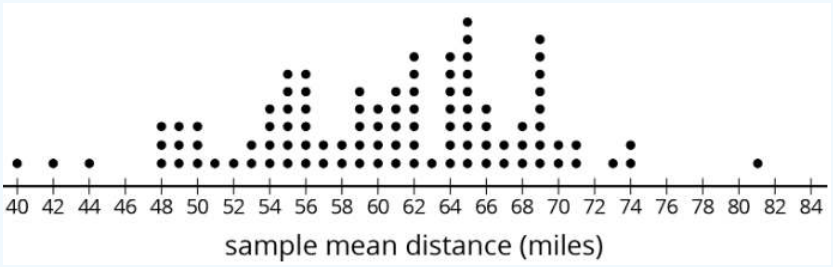


Figure 8.4.3.7

Based on the distribution of sample means, what do you think is a reasonable estimate for the mean of the population?

Exercise 8.4.3.5

Last night, everyone at the school music concert wrote their age on a slip of paper and placed it in a box. Today, each of the students in a math class selected a random sample of size 10 from the box of papers. Here is a dot plot showing their sample means, rounded to the nearest year.

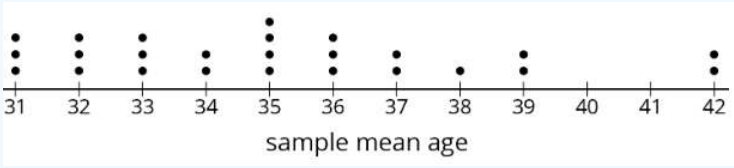


Figure 8.4.3.8: A dot plot for sample mean age. The numbers 31 through 42 are indicated. The data are as follows: 31 years old, 3 dots. 32 years old, 3 dots. 33 years old, 3 dots. 34 years old, 2 dots. 35 years old, 4 dots. 36 years old, 3 dots. 37 years old, 2 dots. 38 years old, 1 dot. 39 years old, 2 dots. 42 years old, 2 dots.

1. Does the number of dots on the dot plot tell you how many people were at the concert or how many students are in the math class?
2. The mean age for the population was 35 years. If Elena picks a new sample of size 10 from this population, should she expect her sample mean to be within 1 year of the population mean? Explain your reasoning.
3. What could Elena do to select a random sample that is more likely to have a sample mean within 1 year of the population mean?

Exercise 8.4.3.6

A random sample of people were asked which hand they prefer to write with. “l” means they prefer to use their left hand, and “r” means they prefer to use their right hand. Here are the results:

$$l \quad r \quad r \quad r \quad r \quad r \quad r \quad r \quad r \quad r \quad l \quad r \quad r \quad r \quad r$$

Based on this sample, estimate the proportion of the population that prefers to write with their left hand.

(From Unit 8.4.2)

Exercise 8.4.3.7

Andre would like to estimate the mean number of books the students at his school read over the summer break. He has a list of the names of all the students at the school, but he doesn’t have time to ask every student how many books they read.

What should Andre do to estimate the mean number of books?

(From Unit 8.4.1)

Exercise 8.4.3.8

A hockey team has a 75% chance of winning against the opposing team in each game of a playoff series. To win the series, the team must be the first to win 4 games.

1. Design a simulation for this event.
2. What counts as a successful outcome in your simulation?
3. Estimate the probability using your simulation.

(From Unit 8.2.4)

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8.4.4: Comparing Populations Using Samples

Lesson

Let's compare different populations using samples.

Exercise 8.4.4.1: Same Mean? Same MAD?

Without calculating, tell whether each pair of data sets have the same mean and whether they have the same mean absolute deviation.

1.

Table 8.4.4.1

set A	1	3	3	5	6	8	10	14
set B	21	23	23	25	26	28	30	34

2.

Table 8.4.4.2

set X	1	2	3	4	5	
set Y	1	2	3	4	5	6

3.

Table 8.4.4.3

set P	47	53	58	62
set Q	37	43	68	72

Exercise 8.4.4.2: With a Heavy Load

Consider the question: Do tenth-grade students' backpacks generally weigh more than seventh-grade students' backpacks?

Here are dot plots showing the weights of backpacks for a random sample of students from these two grades:

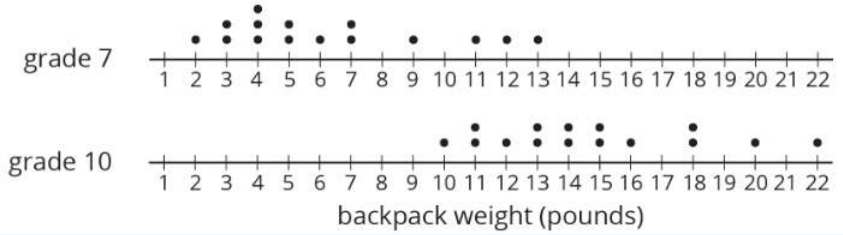


Figure 8.4.4.1: Two dot plots for backpack weight in pounds are labeled grade 7 and grade 10, and the numbers 1 through 22 are indicated. The data are as follows: Grade 7: 2 pounds, 1 dot. 3 pounds, 2 dots. 4 pounds, 3 dots. 5 pounds, 2 dots. 6 pounds, 1 dot. 7 pounds, 2 dots. 9 pounds, 1 dot. 11 pounds, 1 dot. 12 pounds, 1 dot. 13 pounds, 1 dot. Grade 10: 10 pounds, 1 dot. 11 pounds, 2 dots. 12 pounds, 1 dot. 13 pounds, 2 dots. 14 pounds, 2 dots. 15 pounds, 2 dots. 16 pounds, 1 dot. 18 pounds, 2 dots. 20 pounds, 1 dot. 22 pounds, 1 dot.

1. Did any seventh-grade backpacks in this sample weigh more than a tenth-grade backpack?
2. The mean weight of this sample of seventh-grade backpacks is 6.3 pounds. Do you think the mean weight of backpacks for *all* seventh-grade students is exactly 6.3 pounds?
3. The mean weight of this sample of tenth-grade backpacks is 14.8 pounds. Do you think there is a meaningful difference between the weight of all seventh-grade and tenth-grade students' backpacks? Explain or show your reasoning.

Exercise 8.4.4.3: Do They Carry More?

Here are 10 more random samples of seventh-grade students' backpack weights.

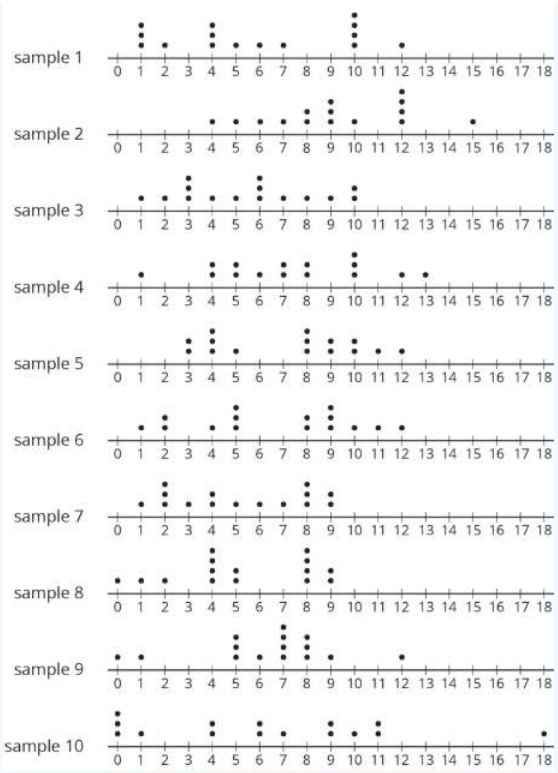


Figure 8.4.4.2: Ten dot plots titled sample one through sample ten, each with the numbers 0 through 18 indicated.

Table 8.4.4.4

sample number	mean weight (pounds)
1	5.8
2	9.2
3	5.5
4	7.3
5	7.2
6	6.6
7	5.2
8	5.3
9	6.3
10	6.4

1. a. Which sample has the highest mean weight?
   b. Which sample has the lowest mean weight?
   c. What is the difference between these two sample means?
2. All of the samples have a mean absolute deviation of about 2.8 pounds. Express the difference between the highest and lowest sample means as a multiple of the MAD.
3. Are these samples very different? Explain or show your reasoning.
4. Remember our sample of tenth-grade students' backpacks had a mean weight of 14.8 pounds. The MAD for this sample is 2.7 pounds. Your teacher will assign you one of the samples of seventh-grade students' backpacks to use.
   a. What is the difference between the sample means for the the tenth-grade students' backpacks and the seventh-grade students' backpacks?

b. Express the difference between these two sample means as a multiple of the larger of the MADs.

5. Do you think there is a meaningful difference between the weights of all seventh-grade and tenth-grade students' backpacks? Explain or show your reasoning.

Exercise 8.4.4.4: Steel From Different Regions

When anthropologists find steel artifacts, they can test the amount of carbon in the steel to learn about the people that made the artifacts. Here are some box plots showing the percentage of carbon in samples of steel that were found in two different regions:

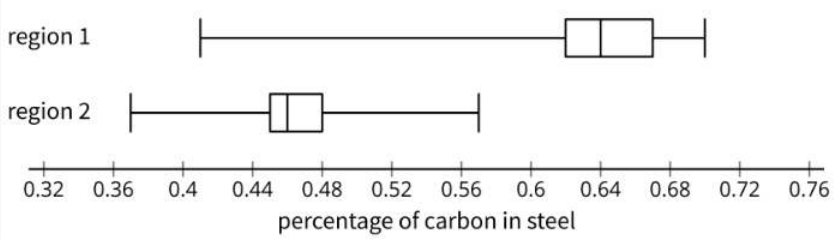


Figure 8.4.4.3: Two box plots from 0 point 32 to 0 point 76 by 0 point 4s. Top box plot labeled region 1. Whisker from about 0 point 41 to 48. Box from about 0 point 62 to about 0 point 67 with vertical line at 0 point 64. Whisker from about 0 point 67 to about 0 point 7. Bottom box plot labeled region 2. Whisker from about 0 point 37 to 45. Box from about 0 point 45 to 0 point 48 with vertical line at about 0 point 46. Whisker from 0 point 48 to about 0 point 57.

1. Was there any steel found in region 1 that had:
   a. *more* carbon than some of the steel found in region 2?
   b. *less* carbon than some of the steel found in region 2?
2. Do you think there is a meaningful difference between all the steel artifacts found in regions 1 and 2?
3. Which sample has a distribution that is *not* approximately symmetric?
4. What is the difference between the sample medians for these two regions?

Table 8.4.4.5

	sample median (%)	IQR (%)
region 1	0.64	0.05
region 2	0.47	0.03

5. Express the difference between these two sample medians as a multiple of the larger interquartile range.
6. The anthropologists who conducted the study concluded that there was a meaningful difference between the steel from these regions. Do you agree? Explain or show your reasoning.

Summary

Sometimes we want to compare two different populations. For example, is there a meaningful difference between the weights of pugs and beagles? Here are histograms showing the weights for a sample of dogs from each of these breeds:

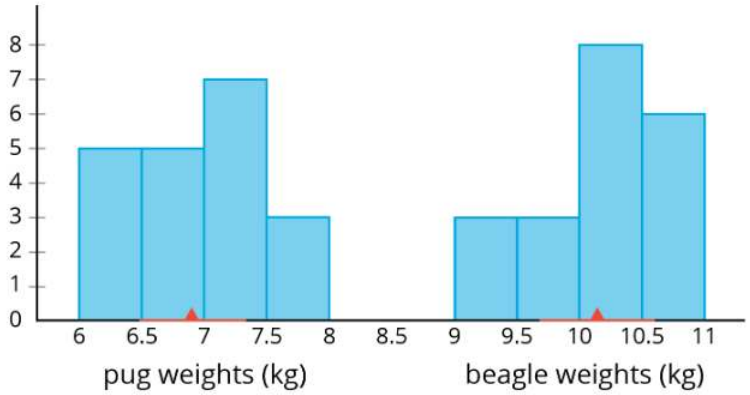


Figure 8.4.4.4: A histogram for two different populations: On the horizontal axis, the numbers 6 through 11, in increments of zero point 5, are indicated. The label pug weights in kilograms is indicated for the numbers 6 through 8 and beagle weights in kilograms is indicated for the numbers 9 through 11. On the vertical axis, the numbers 0 through 8 are indicated. The data represented by the bars are as follows: Pug weights in kilograms: Weight from 6 up to 6 point 5, 5. Weight from 6 point 5 up to 7, 5. Weight from 7 up to 7 point 5, 7. Weight from 7 point 5 up to 8, 3. A triangle is located at 6 point 9 kilograms. Beagle weights in kilograms: Weight from 9 up to 9 point 5, 3. Weight from 9 point 5 up to 10, 3. Weight from 10 up to 10 point 5, 8. Weight from 10 point 5 up to 11, 6. A triangle is located at 10 point 1.

The red triangles show the mean weight of each sample, 6.9 kg for the pugs and 10.1 kg for the beagles. The red lines show the weights that are within 1 MAD of the mean. We can think of these as "typical" weights for the breed. These typical weights do not overlap. In fact, the distance between the means is $10.1 - 6.9$ or 3.2 kg, over 6 times the larger MAD! So we can say there *is* a meaningful difference between the weights of pugs and beagles.

Is there a meaningful difference between the weights of male pugs and female pugs? Here are box plots showing the weights for a sample of male and female pugs:

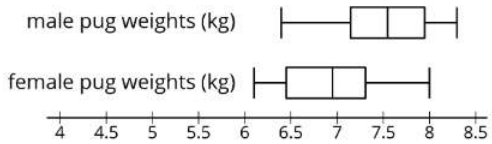


Figure 8.4.4.5: Two box plots labeled male pug weights in kilograms and female pug weights in kilograms are indicated. The numbers 4 through 8 point 5, in increments of zero point 5, are indicated. The five-number summary for the box plots are as follows: Male pug weights in kilograms: Minimum value, 6 point 4. Maximum value, 8 point 3. Q1, 7 point 2. Q2, 7 point 6. Q3, 7 point 9. Female pug weights in kilograms: Minimum value, 6 point 2. Maximum value, 8. Q1, 6 point 4. Q2, 6 point 9. Q3, 7 point 3.

We can see that the medians are different, but the weights between the first and third quartiles overlap. Based on these samples, we would say there is *not* a meaningful difference between the weights of male pugs and female pugs.

In general, if the measures of center for two samples are at least two measures of variability apart, we say the difference in the measures of center is meaningful. Visually, this means the range of typical values does not overlap. If they are closer, then we don't consider the difference to be meaningful.

Glossary Entries

Definition: Interquartile range (IQR)

The interquartile range is one way to measure how spread out a data set is. We sometimes call this the IQR. To find the interquartile range we subtract the first quartile from the third quartile.

For example, the IQR of this data set is 20 because $50 - 30 = 20$.

Table 8.4.4.6

22	29	30	31	32	43	44	45	50	50	59
		Q1			Q2			Q3		

Definition: Proportion

A proportion of a data set is the fraction of the data in a given category.

For example, a class has 18 students. There are 2 left-handed students and 16 right-handed students in the class. The proportion of students who are left-handed is $\frac{2}{20}$, or 0.1.

Practice

Exercise 8.4.4.5

Lin wants to know if students in elementary school generally spend more time playing outdoors than students in middle school. She selects a random sample of size 20 from each population of students and asks them how many hours they played outdoors last week. Suppose that the MAD for each of her samples is about 3 hours.

Select **all** pairs of sample means for which Lin could conclude there is a meaningful difference between the two populations.

A. elementary school: 12 hours, middle school: 10 hours
B. elementary school: 14 hours, middle school: 9 hours
C. elementary school: 13 hours, middle school: 6 hours
D. elementary school: 13 hours, middle school: 10 hours
E. elementary school: 7 hours, middle school: 15 hours

Exercise 8.4.4.6

These two box plots show the distances of a standing jump, in inches, for a random sample of 10-year-olds and a random sample of 15-year-olds.

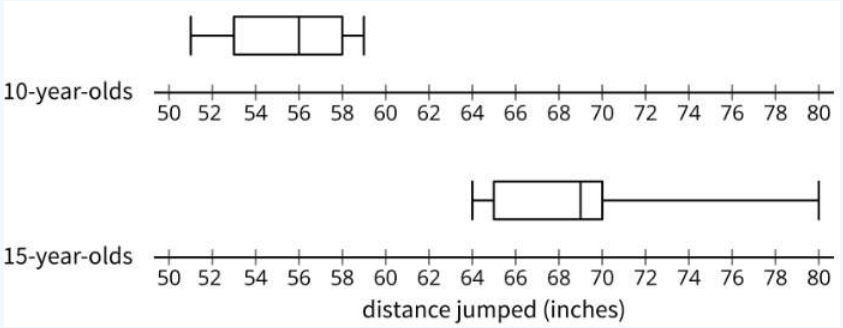


Figure 8.4.4.6: Two box plots from 50 to 80 by 2s. Top box plot labeled 10 year olds. Whisker from 51 to 53. Box from 53 to 58 with vertical line at 56. Whisker from 58 to 59. Bottom box plot labeled 15 year olds. Whisker from 64 to 65. Box from 65 to 70 with vertical line at 69. Whisker from 70 to 80.

Is there a meaningful difference in median distance for the two populations? Explain how you know.

Exercise 8.4.4.7

The median income for a sample of people from Chicago is about $60,000 and the median income for a sample of people from Kansas City is about $46,000, but researchers have determined there is not a meaningful difference in the medians. Explain why the researchers might be correct.

Exercise 8.4.4.8

A farmer grows 5,000 pumpkins each year. The pumpkins are priced according to their weight, so the farmer would like to estimate the mean weight of the pumpkins he grew this year. He randomly selects 8 pumpkins and weighs them. Here are the weights (in pounds) of these pumpkins:

$$2.9 \quad 6.8 \quad 7.3 \quad 7.7 \quad 8.9 \quad 10.6 \quad 12.3 \quad 15.3$$

1. Estimate the mean weight of the pumpkins the farmer grew.

This dot plot shows the mean weight of 100 samples of eight pumpkins, similar to the one above.

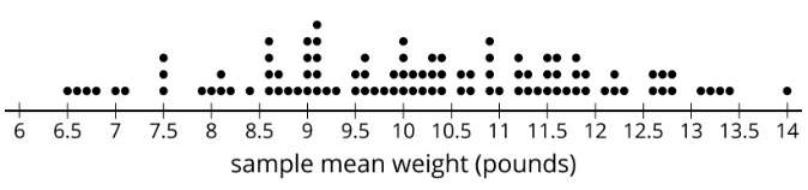


Figure 8.4.4.7

2. What appears to be the mean weight of the 5,000 pumpkins?

3. What does the dot plot of the sample means suggest about how accurate an estimate based on a single sample of 8 pumpkins might be?

4. What do you think the farmer might do to get a more accurate estimate of the population mean?

(From Unit 8.4.3)

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8.4.5: Comparing Populations With Friends

Lesson

Let's ask important questions to compare groups.

Exercise 8.4.5.1: Features of Graphic Representations

Dot plots, histograms, and box plots are different ways to represent a data set graphically.

Which of those displays would be the easiest to use to find each feature of the data?

1. the mean
2. the median
3. the mean absolute deviation
4. the interquartile range
5. the symmetry

Exercise 8.4.5.2: Info Gap: Comparing Populations

Your teacher will give you either a *problem card* or a *data card*. Do not show or read your card to your partner.

If your teacher gives you the *problem card*:

1. Silently read your card and think about what information you need to be able to answer the question.
2. Ask your partner for the specific information that you need.
3. Explain how you are using the information to solve the problem.
   Continue to ask questions until you have enough information to solve the problem.
4. Share the *problem card* and solve the problem independently.
5. Read the *data card* and discuss your reasoning.

If your teacher gives you the *data card*:

1. Silently read your card.
2. Ask your partner "*What specific information do you need?*" and wait for them to *ask* for information.
   If your partner asks for information that is not on the card, do not do the calculations for them. Tell them you don't have that information.
3. Before sharing the information, ask "*Why do you need that information?*" Listen to your partner's reasoning and ask clarifying questions.
4. Read the *problem card* and solve the problem independently.
5. Share the *data card* and discuss your reasoning.

Pause here so your teacher can review your work. Ask your teacher for a new set of cards and repeat the activity, trading roles with your partner.

Are you ready for more?

Is there a meaningful difference between top sports performance in two different decades? Choose a variable from your favorite sport (for example, home runs in baseball, kills in volleyball, aces in tennis, saves in soccer, etc.) and compare the leaders for each year of two different decades. Is the performance in one decade meaningfully different from the other?

Exercise 8.4.5.3: Comparing to Known Characteristics

1. A college graduate is considering two different companies to apply to for a job. Acme Corp lists this sample of salaries on their website:
   $\$45,000 \quad \$55,000 \quad \$140,000 \quad \$70,000 \quad \$60,000 \quad \$50,000$
   What typical salary would Summit Systems need to have to be meaningfully different from Acme Corp? Explain your reasoning.

2. A factory manager is wondering whether they should upgrade their equipment. The manager keeps track of how many faulty products are created each day for a week.
   $6 \quad 7 \quad 8 \quad 6 \quad 7 \quad 5 \quad 7$
   The new equipment guarantees an average of 4 or fewer faulty products per day. Is there a meaningful difference between the new and old equipment? Explain your reasoning.

Summary

When using samples to comparing two populations, there are a lot of factors to consider.

- Are the samples representative of their populations? If the sample is biased, then it may not have the same center and variability as the population.
- Which characteristic of the populations makes sense to compare—the mean, the median, or a proportion?
- How variable is the data? If the data is very spread out, it can be more difficult to make conclusions with certainty.

Knowing the correct questions to ask when trying to compare groups is important to correctly interpret the results.

Glossary Entries

Definition: Interquartile Range (IQR)

The interquartile range is one way to measure how spread out a data set is. We sometimes call this the IQR. To find the interquartile range we subtract the first quartile from the third quartile.

For example, the IQR of this data set is 20 because $50 - 30 = 20$.

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		Q1			Q2			Q3		

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A proportion of a data set is the fraction of the data in a given category.

For example, a class has 18 students. There are 2 left-handed students and 16 right-handed students in the class. The proportion of students who are left-handed is $\frac{2}{20}$, or 0.1.

Practice

Exercise 8.4.5.4

An agent at an advertising agency asks a random sample of people how many episodes of a TV show they watch each day. The results are shown in the dot plot.

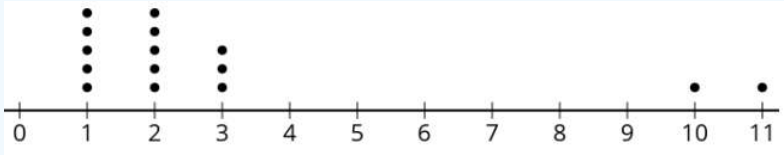


Figure 8.4.5.1

The agency currently advertises on a different show, but wants to change to this one as long as the typical number of episodes is not meaningfully less.

1. What measure of center and measure of variation would the agent need to find for their current show to determine if there is a meaningful difference? Explain your reasoning.
2. What are the values for these same characteristics for the data in the dot plot?
3. What numbers for these characteristics would be meaningfully different if the measure of variability for the current show is similar? Explain your reasoning.

Exercise 8.4.5.5

Jada wants to know if there is a meaningful difference in the mean number of friends on social media for teens and adults. She looks at the friend count for the 10 most popular of her friends and the friend count for 10 of her parents' friends. She then computes the mean and MAD of each sample and determines there is a meaningful difference.

Jada's dad later tells her he thinks she has not come to the right conclusion. Jada checks her calculations and everything is right. Do you agree with her dad? Explain your reasoning.

Exercise 8.4.5.6

The mean weight for a sample of a certain kind of ring made from platinum is 8.21 grams. The mean weight for a sample of a certain kind of ring made from gold is 8.61 grams. Is there a meaningful difference in the weights of the two types of rings? Explain your reasoning.

Exercise 8.4.5.7

The lengths in feet of a random sample of 20 male and 20 female humpback whales were measured and used to create the box plot.

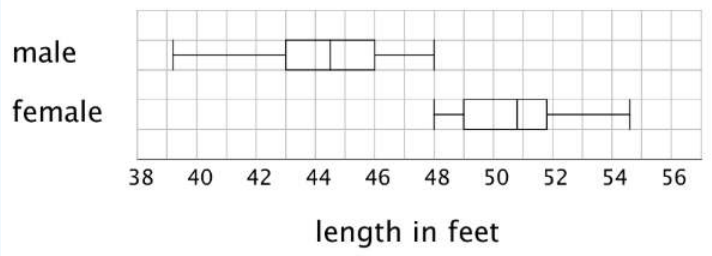


Figure 8.4.5.2: Two box plots, male and female. From 38 to 56 by 2s. Male box plot is: whisker from just above 39 to 43. Box from 43 to 46 with vertical line at 44 point 5. Whisker from 46 to 48. Female box plot is: whisker from just above 48 to 49. Box from 49 to just under 52 with vertical line at about 50 point 8. Whisker from 50 point 8 to 54 point 5.

Estimate the median lengths of male and female humpback whales based on these samples.

(From Unit 8.4.1)

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8.5: Let's Put it to Work

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8.5.1: Memory Test

Lesson

Let's put it all together.

Exercise 8.5.1.1: Collecting a Sample

You teacher will give you a paper that lists a data set with 100 numbers in it. Explain whether each method of obtaining a sample of size 20 would produce a random sample.

Option 1: A spinner has 10 equal sections on it. Spin once to get the row number and again to get the column number for each member of your sample. Repeat this 20 times.

Option 2: Since the data looks random already, use the first two rows.

Option 3: Cut up the data and put them into a bag. Shake the bag to mix up the papers, and take out 20 values.

Option 4: Close your eyes and point to one of the numbers to use as your first value in your sample. Then, keep moving one square from where your finger is to get a path of 20 values for your sample.

Exercise 8.5.1.2: Sample Probabilities

Continue working with the data set your teacher gave you in the previous activity. The data marked with a star all came from students at Springfield Middle School.

1. When you select the first value for your random sample, what is the probability that it will be a value that came from a student at Springfield Middle School?
2. What proportion of your entire sample would you expect to be from Springfield Middle School?
3. If you take a random sample of size 10, how many scores would you expect to be from Springfield Middle School?
4. Select a random sample of size 10.
5. Did your random sample have the expected number of scores from Springfield Middle School?

Exercise 8.5.1.3: Estimating a Measure of Center for the Population

1. Decide which measure of center makes the most sense to use based on the distribution of your sample. Discuss your thinking with your partner. If you disagree, work to reach an agreement.
2. Estimate this measure of center for your population based on your sample.
3. Calculate the measure of variability for your sample that goes with the measure of center that you found.

Exercise 8.5.1.4: Comparing Populations

Using only the values you computed in the previous two activities, compare your sample to your partner's.

Is it reasonable to conclude that the measures of center for each of your populations are meaningfully different? Explain or show your reasoning.

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9.1: Running a Restaurant

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9.1.1: Planning Recipes

Lesson

Let's choose some recipes for a restaurant.

Exercise 9.1.1.1: A Recipe for Your Restaurant

Imagine you could open a restaurant.

1. Select a recipe for a main dish you would like to serve at your restaurant.
2. Record the amount of each ingredient from your recipe in the first two columns of the table. You may not need to use every row.

Table 9.1.1.1

ingredient	amount	amount per serving	calories per serving

3. How many servings does this recipe make? Determine the amount of each ingredient in one serving, and record it in the third column of the table.
4. Restaurants are asked to label how many calories are in each meal on their menu.
   a. Use the information to calculate the amount of calories from each ingredient in your meal, and record it in the last column of the table.
   b. Next, find the total calories in one serving of your meal.
5. If a person wants to eat 2,000 calories per day, what percentage of their daily calorie intake would one serving of your meal be?

Grains

Table 9.1.1.2

	common measure	mass (g)	calories	fat (g)	sodium (mg)
bagels	1 bagel	99	261	1.31	418
biscuit dough, refrigerated	1 biscuit	58	178	6.14	567
biscuits, dry mix	1 cup	120	514	18.48	1531
bread crumbs	1 oz	28.35	112	1.5	208
bread stuffing, dry mix	1 oz	28.35	109	0.96	398
cornbread, dry mix	1 oz	28.35	119	3.46	232
cornmeal	1 cup	157	581	2.75	11
couscous	1 cup	173	650	1.11	17
croissants	1 oz	28.35	115	5.95	132
croutons	$\frac{1}{2}$ oz	14.2	66	2.6	155
egg noodles	1 cup	38	146	1.69	8
English muffins	1 oz	28.35	67	0.51	132

	common measure	mass (g)	calories	fat (g)	sodium (mg)
focaccia bread	1 piece	57	142	4.5	320
French bread	1 oz	28.35	77	0.69	171
frybread	1 piece	152	502	18.57	500
grits	1 tbsp	9.7	36	0.12	0
hamburger or hotdog buns	1 roll	44	123	1.72	217
millet	1 cup	200	756	8.44	10
multigrain bread	1 oz	28.35	75	1.2	108
naan bread	1 piece	90	262	5.09	418
oats	1 cup	81	307	5.28	5
pasta	1 cup	91	338	1.37	5
phyllo dough	1 oz	28.35	85	1.7	137
pie crust, refrigerated	1 crust	229	1019	58.3	937
pita bread	1 oz	28.35	8	0.34	152
potato chips	1 oz	28	149	9.51	148
potato flour	1 cup	160	571	0.54	88
pretzels	1 oz	28.35	109	0.83	352
puff pastry, frozen	1 oz	28.35	156	10.8	71
puffed rice cereal	1 cup	14	56	0.07	0
quinoa	1 cup	170	626	10.32	8
ramen noodles	1 package	81	356	14.25	1503
rice flour	1 cup	158	578	2.24	0
rice noodles	2 oz	57	207	0.32	104
rice, brown	1 cup	185	679	5.92	9
rice, white	1 cup	185	675	1.22	9
rice, wild	1 cup	160	571	1.73	11
rye flour	1 cup	128	416	2.84	3
saltine crackers	5 crackers	14.9	62	1.29	140
taco shells	1 shell	12.9	61	2.81	42
tortilla chips	1 oz	28.35	134	5.86	93
tortillas	1 tortilla	49	146	3.71	364
tostada shells	1 piece	12.3	58	2.88	81
wheat bread	1 slice	29	79	1.31	137
wheat flour	1 cup	125	455	1.22	2

	common measure	mass (g)	calories	fat (g)	sodium (mg)
wheat rolls	1 roll	28	76	1.76	147
white bread	1 slice	29	77	0.97	142
wonton wrappers	1 oz	28.35	82	0.43	162

Fruits and Vegetables

Table 9.1.1.3

	common measure	mass (g)	calories	fat (g)	sodium (mg)
acorn squash	1 cup	140	56	0.14	4
ancho peppers, dried	1 pepper	17	48	1.39	7
apple juice	1 cup	248	114	0.32	10
apple juice, frozen concentrate	6 fl oz	211	350	0.78	53
apples	1 cup	110	53	0.14	0
applesauce	1 cup	246	167	0.42	5
apricots	1 cup	155	74	0.6	2
apricots, canned	1 cup	219	182	0.24	9
artichokes	1 artichoke	128	60	0.19	120
asparagus	1 cup	134	27	0.16	3
avocados	1 cup	150	240	22	10
bamboo shoots, canned	1 cup	131	25	0.52	9
bananas	1 cup	225	200	0.74	2
beets	1 cup	136	58	0.23	106
beets, canned	1 cup	246	74	0.22	352
bell peppers, green	1 cup	149	30	0.25	4
bell peppers, red	1 cup	149	46	0.45	6
blackberries	1 cup	144	62	0.71	1
blueberries	1 cup	148	84	0.49	1
blueberries, frozen	1 cup	155	79	0.99	2
brocolli	1 cup	91	31	0.34	30
broccoli, frozen	10 oz	95	28	0.32	16
brussels sprouts	1 cup	88	38	0.26	22
brussels sprouts, frozen	$\frac{1}{2}$ cup	95	39	0.39	10
butternut squash	1 cup	140	63	0.14	6
cabbage	$\frac{1}{2}$ cup	35	8	0.06	6

	common measure	mass (g)	calories	fat (g)	sodium (mg)
cantaloupe	1 cup	177	60	0.34	28
carrots	1 cup	128	52	0.31	88
cauliflower	1 cup	107	27	0.3	32
celery	1 cup	101	16	0.17	81
chayote	1 cup	132	25	0.17	3
cherries	1 cup	138	87	0.28	0
chives	1 tbsp	3	1	0.02	0
clementines	1 fruit	74	35	0.11	1
coconut	1 cup	80	283	26.79	16
collards	1 cup	36	12	0.22	6
corn	1 cup	145	125	1.96	22
corn, frozen	1 cup	136	120	1.06	4
cranberries	1 cup	110	51	0.14	2
cranberries, dried	$\frac{1}{4}$ cup	40	123	0.44	2
cranberry sauce, canned	1 cup	277	440	0.42	14
cucumber	1 cup	133	16	0.21	3
currants	1 cup	112	63	0.22	1
dates	1 date	24	66	0.04	0
edamame, frozen	1 cup	118	129	5.58	7
eggplant	1 cup	82	20	0.15	2
feijoa	1 cup	243	148	1.02	7
figs	1 large	64	47	0.19	1
fruit cocktail, canned	1 cup	242	138	0.17	15
grape juice	1 cup	253	152	0.33	13
grapefruit	1 cup	230	74	0.23	0
grapes	1 cup	151	104	0.24	3
green beans	1 cup	100	31	0.22	6
green beans, canned	1 cup	135	28	0.55	362
green beans, frozen	1 cup	121	47	0.25	4
guavas	1 cup	165	112	1.57	3
hearts of palms, canned	1 cup	146	41	0.91	622

	common measure	mass (g)	calories	fat (g)	sodium (mg)
hominy, canned	1 cup	165	119	1.45	569
honeydew	1 cup	170	61	0.24	31
jalapenos	1 cup	90	26	0.33	3
jicama	1 cup	120	46	0.11	5
kale	1 cup	16	8	0.15	6
leeks	1 cup	89	54	0.27	18
lemon juice	1 cup	244	54	0.59	2
lemon peel	1 tbsp	6	3	0.02	0
lemonade, frozen concentrate	1 fl oz	36.5	72	0.26	3
lemons	1 cup	212	61	0.64	4
lettuce, butterhead	1 cup	55	7	0.12	3
lettuce, green leaf	1 cup	36	5	0.05	10
lettuce, iceberg	1 cup	72	10	0.1	7
lettuce, red leaf	1 cup	28	4	0.06	7
lettuce, romaine	1 cup	47	8	0.14	4
limes	1 lime	67	20	0.13	1
Mandarin oranges, canned	1 cup	252	154	0.25	15
mangoes	1 cup	165	99	0.63	2
maraschino cherries	1 cherry	5	8	0.01	0
mashed potatoes, dehydrated flakes	1 cup	60	212	0.25	46
mushrooms	1 cup	70	15	0.24	4
nopales	1 cup	86	14	0.08	18
okra	1 cup	100	33	0.19	7
olives, canned	1 tbsp	8.4	10	0.9	62
onions	1 cup	160	64	0.16	6
onions, frozen, chopped	10 oz	95	28	0.1	11
orange juice	1 cup	249	122	0.3	5
orange juice, frozen concentrate	1 cup	262	388	0.66	18
orange peel	1 tbsp	6	6	0.01	0
oranges	1 cup	180	85	0.22	0
papayas	1 cup	145	62	0.38	12

	common measure	mass (g)	calories	fat (g)	sodium (mg)
parsnips	1 cup	133	100	0.4	13
peaches	1 cup	154	60	0.38	0
peaches, canned	1 cup	251	136	0.08	13
pears	1 cup	140	80	0.2	1
pears, canned	1 cup	251	143	0.08	13
peas	1 cup	145	117	0.58	7
peas, frozen	1 cup	134	103	0.54	145
pickles, dill	1 spear	35	4	0.1	283
pickles, sweet	1 cup	160	146	0.66	731
pineapple	1 cup	165	82	0.2	2
pineapple, canned	1 cup	181	109	0.2	2
plantains	1 cup	148	181	0.55	6
plums	1 cup	165	76	0.46	0
pomegranate juice	1 cup	1249	134	0.72	22
pomegranates	$\frac{1}{2}$ cup	87	72	1.02	3
portabella mushrooms	1 cup	86	19	0.3	8
potatoes, frozen hash browns	$\frac{1}{2}$ cup	105	86	0.65	23
potatoes, red	$\frac{1}{2}$ cup	75	52	0.11	14
potatoes, russet	$\frac{1}{2}$ cup	75	59	0.06	4
prickly pears	1 unit	149	61	0.76	7
pumpkin	1 cup	116	30	0.12	1
pumpkin, canned	1 cup	245	83	0.69	12
radishes	1 cup	116	19	0.12	45
raisins	1 cup	165	493	0.76	18
raspberries	1 cup	123	64	0.8	1
rhubarb	1 cup	122	26	0.24	5
sauerkraut	1 cup	142	27	0.2	939
serrano peppers	1 cup	105	34	0.46	10
shallots	1 tbsp	10	7	0.01	1
shiitake mushrooms	1 piece	19	6	0.09	2
soybeans	1 cup	256	376	17.41	38
spinach	1 cup	30	7	0.12	24

	common measure	mass (g)	calories	fat (g)	sodium (mg)
spinach, frozen	1 cup	156	45	0.89	115
strawberries	1 cup	152	49	0.46	2
strawberries, frozen	1 cup	221	77	0.24	4
summer squash	1 cup	113	18	0.2	2
sweet onions	1 serving	148	47	0.12	12
sweet potato	1 cup	133	114	0.07	73
sweet potato, canned	1 cup	196	212	0.63	76
Swiss chard	1 cup	36	7	0.07	77
tangerines	1 cup	195	103	0.6	4
taro	1 cup	104	116	0.21	11
tomatillos	1 tomatillo	34	11	0.35	0
tomato juice, canned	1 cup	243	41	0.7	615
tomato sauce, canned	1 cup	245	59	0.74	1161
tomato soup, canned, condensed	1 cup	148	98	0.65	558
tomatoes	1 cup	149	27	0.3	7
tomatoes, sun-dried	1 cup	54	139	1.6	58
turnip greens	1 cup	55	18	0.16	22
turnips	1 cup	130	36	0.13	87
water chestnuts, canned	$\frac{1}{2}$ cup	70	35	0.04	6
watermelon	1 cup	154	46	0.23	2
winter squash	1 cup	116	39	0.15	5
yams	1 cup	150	177	0.26	14
zucchini	1 cup	124	21	0.4	10

Meat

Table 9.1.1.4

	common measure	mass (g)	calories	fat (g)	sodium (mg)
bacon	1 slice	26	106	10.21	122
bratwurst	2.33 oz	66	196	17.38	560
chicken leg	3 oz	85	182	13.56	71
chicken thigh	1 piece	193	427	32.06	156

	common measure	mass (g)	calories	fat (g)	sodium (mg)
chicken wing	1 piece	107	204	13.75	90
chicken, light meat	1 lb	88	100	1.45	60
chorizo	1 oz	28.35	129	10.85	350
cod	3 oz	85	61	0.17	93
crab	3 oz	85	73	0.82	251
crayfish	3 oz	85	61	0.82	53
elk	1 oz	28.35	31	0.41	16
ground beef, 70% lean	4 oz	113	375	33.9	75
ground beef, 90% lean	4 oz	113	199	11.3	75
ground pork, 84% lean	4 oz	113	246	18.08	77
ground pork, 96% lean	4 oz	113	137	4.52	76
ground turkey, 85% lean	4 oz	85	153	10.66	46
ground turkey, 93% lean	1 oz	28.35	43	2.36	20
halibut	3 oz	85	77	1.13	58
ham	1 oz	28.35	38	1.53	319
hot dogs	1 piece	51	141	12.33	498
kielbasa	3 oz	85	276	25.19	789
lamb	1 oz	28.35	38	1.5	18
lobster	1 lobster	150	116	1.12	634
mahimahi	3 oz	85	72	0.6	75
pepperoni	3 oz	85	428	39.34	1345
pork ribs	1 rib	128	242	15.13	81
pork sausage	1 piece	25	72	6.2	185
pork tenderloin	3 oz	85	102	3	44
salmon	1 fillet	108	373	12.34	55
shrimp	3 oz	85	72	0.43	101
tofu	$\frac{1}{2}$ cup	126	98	5.25	15
trout	1 fillet	79	111	4.88	40
tuna, canned	1 oz	28.35	24	0.27	70
turkey	1 serving	85	92	2.12	105

Nuts, Beans, and Seeds

Table 9.1.1.5

	common measure	mass (g)	calories	fat (g)	sodium (mg)
almonds	1 cup	143	828	71.4	1
baked beans, canned	1 cup	254	239	0.94	871
black beans, canned	1 cup	240	218	0.7	331
black beans, dry	1 cup	194	662	2.75	10
balck-eyed peas, canned	1 cup	240	185	1.32	703
black-eyed peas, dry	1 cup	167	561	2.1	27
broadbeans (fava beans), canned	1 cup	256	182	0.56	1160
broadbeans (fava beans), dry	1 cup	150	512	2.3	20
cashews	1 oz	28.35	157	12.43	3
chia seeds	1 oz	28.35	138	8.71	5
chickpeas (garbanzo beans), canned	1 cup	240	211	4.68	667
chickpeas (garbanzo beans), dry	1 cup	200	756	12.08	48
flaxseed	1 tbsp	10.3	55	4.34	3
great northern beans, canned	1 cup	262	299	1.02	969
great northern beans, dry	1 cup	183	620	2.09	26
kidney beans, canned	1 cup	256	215	1.54	758
kidney beans, dry	1 cup	184	613	1.53	44
lentils, dry	1 cup	192	676	2.04	12
lima beans, canned	1 cup	241	190	0.41	810
lima beans, frozen	1 cup	164	216	0.72	85
macadamia nuts	1 cup	134	962	101.53	7
navy beans, canned	1 cup	262	296	1.13	880
navy beans, dry	1 cup	208	701	3.12	10

	common measure	mass (g)	calories	fat (g)	sodium (mg)
peanut butter	2 tbsp	32	191	16.22	136
peanuts	1 oz	28.35	166	14.08	116
pecans	1 cup	109	753	78.45	0
pine nuts	1 cup	135	909	92.3	3
pinto beans, canned	1 cup	240	197	1.34	643
pinto beans, dry	1 cup	193	670	2.37	23
pistachios	1 cup	123	689	55.74	1
pumpkin seeds	1 cup	129	721	63.27	9
sesame seeds	1 cup	144	825	71.52	16
sunflower seeds	1 cup	46	269	23.67	4

Dairy

Table 9.1.1.6

	common measure	mass (g)	calories	fat (g)	sodium (mg)
almond milk	1 cup	262	39	2.88	186
blue cheese	1 oz	28.35	100	8.15	325
butter	1 pat	5	36	4.06	1
cheddar cheese	1 cup	132	533	43.97	862
coconut milk	1 cup	226	445	48.21	29
cottage cheese	4 oz	113	111	4.86	411
cream cheese	1 tbsp	14.5	51	4.99	46
egg white	1 large	33	17	0.06	55
egg yolk	1 large	17	55	4.51	8
eggs	1 large	50	72	4.76	71
evaporated milk	1 cup	252	270	5.04	252
feta cheese	1 cup	150	396	31.92	1376
heavy whipped cream	1 cup	120	408	43.3	32
margarine	1 tbsp	14.2	101	11.38	4
milk, 1%	1 cup	244	102	2.37	107
milk, 2%	1 cup	244	122	4.83	115
milk, skim	1 cup	245	83	0.2	103
milk, whole	1 cup	244	149	7.93	105
mozzarella cheese	1 cup	132	389	26.11	879
Parmesan cheese	1 cup	100	420	27.84	1804

	common measure	mass (g)	calories	fat (g)	sodium (mg)
powdered milk	1 cup	68	243	0.49	373
queso fresco cheese	1 cup	122	365	29.06	916
rice milk	1 cup	240	113	2.33	94
sour cream, light	1 tbsp	12	16	1.27	10
soymilk	1 cup	243	80	3.91	90
Swiss cheese	1 cup	132	519	40.91	247
yogurt	6 oz	170	107	2.64	119

Other

Table 9.1.1.7

	common measure	mass (g)	calories	fat (g)	sodium (mg)
agave syrup	1 tsp	6.9	21	0.03	0
allspice	1 tsp	1.9	5	0.17	1
baking powder	1 tsp	4.6	2	0	488
baking soda	1 tsp	4.6	0	0	1259
balsamic vinegar	1 tbsp	16	14	0	4
barbecue sauce	1 tbsp	17	29	0.11	175
basil	1 tsp	0.7	2	0.03	1
bay leaf	1 tsp	0.6	2	0.05	0
black pepper	1 tsp	2.3	6	0.07	0
brown sugar	1 tsp	3	11	0	1
canola oil	1 tbsp	14	124	14	0
chicken bouillon	1 cube	4.8	10	0.23	1152
chicken broth	1 cup	249	15	0.52	924
chili powder	1 tsp	2.7	8	0.39	77
chocolate syrup	1 cup	304	1064	27.06	1052
cilantro	$\frac{1}{4}$ cup	4	1	0.02	2
cinnamon	1 tsp	2.6	6	0.03	0
cloves	1 tsp	2.1	6	0.27	6
cocoa powder	1 cup	86	196	11.78	18
coconut oil	1 tbsp	13.6	121	13.47	0
coffee	1 fl oz	29.6	0	0.01	1
corn syrup	1 cup	341	965	0.68	211
cornstarch	1 cup	138	488	0.06	12

	common measure	mass (g)	calories	fat (g)	sodium (mg)
cream of chicken soup	$\frac{1}{2}$ cup	126	113	7.27	885
cream of mushroom soup	$\frac{1}{2}$ cup	126	100	6.68	871
cream of tartar	1 tsp	3	8	0	2
cumin	1 tsp	2.1	8	0.47	4
curry powder	1 tsp	2	6	0.28	1
dill	1 tsp	2.1	6	0.31	0
dulce de leche	1 tbsp	19	60	1.4	25
enchilada sauce	$\frac{1}{4}$ cup	56	17	0.51	306
garlic	1 clove	3	4	0.01	0.5
garlic powder	1 tsp	3.1	10	0.02	2
ginger	1 tsp	1.8	6	0.08	0
gravy, canned	1 cup	233	1	5.5	1305
honey	1 cup	339	25	0	14
horseradish	1 tsp	5	35	0.03	21
hot sauce	1 tsp	4.7	56	0.02	124
hummus	1 tbsp	15	17	1.44	57
Italian dressing	1 tbsp	14.7	35	3.1	146
jams, jellies, and preserves	1 tbsp	20	56	0.01	6
ketchup	1 tbsp	17	17	0.02	154
lard	1 tbsp	12.8	115	12.8	0
maple syrup	1 tbsp	20	52	0.01	2
marshmallows, mini	1 cup	50	159	0.1	40
mayonnaise	1 tbsp	15	103	11.67	73
molasses	1 cup	337	977	0.34	125
mustard	1 tsp	5	3	0.17	55
mustard powder	1 tsp	2	10	0.72	0
olive oil	1 tbsp	13.5	119	13.5	0
onion powder	1 tsp	2.4	8	0.02	2
onion soup mix	1 tbsp	7.5	22	0.03	602
oregano	1 tsp	1	3	0.04	0
paprika	1 tsp	2.3	6	0.3	2
parsley	1 tsp	0.5	1	0.03	2

	common measure	mass (g)	calories	fat (g)	sodium (mg)
pasta sauce	$\frac{1}{2}$ cup	132	66	2.13	577
pesto	$\frac{1}{4}$ cup	63	263	23.69	380
pizza sauce	$\frac{1}{4}$ cup	63	34	0.72	219
poultry seasoning	1 tsp	1.5	5	0.11	0
powdered sugar	1 cup	120	467	0	2
ranch dressing	1 tbsp	15	64	6.68	135
rosemary	1 tsp	1.2	4	0.18	1
sage	1 tsp	0.7	2	0.09	0
salsa	2 tbsp	36	10	0.06	256
salt	1 tsp	6	0	0	2325
shortening	1 tbsp	12.8	113	12.8	0
soy sauce	1 tbsp	16	8	0.09	879
sugar	1 packet	2.8	11	0	0
sweet and sour sauce	2 tbsp	35	52	0.01	130
taco seasoning	2 tsp	5.7	18	0	411
tartar sauce	2 tbsp	30	63	5.01	200
tea, black	1 fl oz	29.6	0	0	1
tea, green	1 cup	245	2	0	0
teriyaki sauce	1 tbsp	18	16	0	690
thyme	1 tsp	1	3	0.01	1
vanilla extract	1 tsp	4.2	12	0	0
vegetable broth	1 cup	221	11	0.15	654
vinegar	1 tbsp	14.9	3	0	0
wasabi	1 tbsp	20	58	2.18	678
water	1 fl oz	29.6	0	0	1
watercress	1 cup	34	4	0.03	14
Worcestershire sauce	1 tbsp	17	13	0	167
yeast	1 tsp	4	13	0.3	2

The labels on packaged foods tell how much of different nutrients they contain. Here is what some different food labels say about their sodium content.

- cheese crackers, 351 mg, 14% daily value
- apple chips, 15 mg, <1% daily value
- granola bar, 82 mg, 3% daily value

Estimate the maximum recommended amount of sodium intake per day (100% daily value). Explain your reasoning.

Exercise 9.1.1.2: Health Claims

For a meal to be considered:

- “low calorie”—it must have 120 calories or less per 100 grams of food.
- “low fat”—it must have 3 grams of fat or less per 100 grams of food.
- “low sodium”—it must have 140 milligrams of sodium or less per 100 grams of food.

g

1. Does the meal you chose in the previous activity meet the requirements to be considered:
   a. low calorie?
   b. low fat?
   c. low sodium?
   Be prepared to explain your reasoning.
2. Select or invent another recipe you would like to serve at your restaurant that does meet the requirements to be considered either low calorie, low fat, or low sodium. Show that your recipe meets that requirement. Organize your thinking so it can be followed by others.

Table 9.1.1.8

ingredient	amount per serving	calories per serving	fat per serving	sodium per serving

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9.1.2: Costs of Running a Restaurant

Lesson

Let's explore how much the food will cost.

Exercise 9.1.2.1: Introducing Spreadsheets

1. Type each formula into the cells of a spreadsheet program and press enter. Record what the cell displays. Make sure to type each formula exactly as it is written here.

Table 9.1.2.1

	A	B	C	D
1	$=40-32$	$=1.5+3.6$	$=14/7$	$=0.5*6$

2. a. Predict what will happen if you type the formula =A1*C1 into cell C2 of your spreadsheet.
   b. Type in the formula, and press enter to check your prediction.
3. a. Predict what will happen next if you delete the formula in cell A1 and replace it with the number 100.
   b. Replace the formula with the number, and press enter to check your prediction.
4. a. Predict what will happen if you copy cell C2 and paste it into cell D2 of your spreadsheet.
   b. Copy and paste the formula to check your prediction.

Exercise 9.1.2.2: Cost per Serving

1. Set up a spreadsheet with these column labels in the first row.

	A	B	C	D
1	ingredient	unit in recipe	amount in recipe	amount per serving
2				

Table 9.1.2.2

   a. Type the information about the ingredients in your recipe into the first 3 columns of the spreadsheet.
   b. Type a formula into cell D2 to automatically calculate the amount per serving for your first ingredient.
   c. Copy cell D2 and paste it into the cells beneath it to calculate the amount per serving for the rest of your ingredients. Pause here so your teacher can review your work.

2. Add these column labels to your spreadsheet.

E	F	G	H
purchase price	purchase amount	purchase unit	cost per purchase unit

Table 9.1.2.3

   a. Research the cost of each ingredient in your meal, and record the information in columns E, F, and G.
   b. Type a formula into cell H2 to automatically calculate the cost per purchase unit for your first ingredient.
   c. Copy cell H2, and paste it into the cells beneath it to calculate the cost per purchase unit for the rest of your ingredients.

3. Add these column labels to your spreadsheet.

I	J	K
conversion from purchase units to recipe units	cost per recipe unit	cost per serving

Table 9.1.2.4

a. Complete column I with how many of your recipe unit are in 1 of your purchase unit for each ingredient. For example, if your recipe unit was cups and your purchase unit was gallons, then your conversion would be 16 because there are 16 cups in 1 gallon.
b. Type a formula into cell J2 to calculate the cost per recipe unit for your first ingredient.
c. Type a formula into cell K2 to calculate the cost per serving for your first ingredient.
d. Compare formulas with your partner. Discuss your thinking. If you disagree, work to reach an agreement.
e. Copy cells J2 and K2, and paste them into the cells beneath them to calculate the cost per recipe unit and cost per serving for the rest of your ingredients.

4. Type a formula into the first empty cell below your last ingredient in column K to calculate the total cost per serving for all of the ingredients in your recipe. Record the answer here.

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9.1.3: More Costs of Running a Restaurant

Lesson

Let's explore how much it costs to run a restaurant.

Exercise 9.1.3.1: Are We Making Money?

1. Restaurants have many more expenses than just the cost of the food.
   a. Make a list of other items you would have to spend money on if you were running a restaurant.
   b. Identify which expenses on your list depend on the number of meals ordered and which are independent of the number of meals ordered.
   c. Identify which of the expenses that are independent of the number of meals ordered only have to be paid once and which are ongoing.
   d. Estimate the monthly cost for each of the ongoing expenses on your list. Next, calculate the total of these monthly expenses.
2. Tell whether each restaurant is making a profit or losing money if they have to pay the amount you predicted in ongoing expenses per month. Organize your thinking so it can be followed by others.
   a. Restaurant A sells 6,000 meals in one month, at an average price of $17 per meal and an average cost of $4.60 per meal.
   b. Restaurant B sells 8,500 meals in one month, at an average price of $8 per meal and an average cost of $2.20 per meal.
   c. Restaurant C sells 4,800 meals in one month, at an average price of $29 per meal and an average cost of $6.90 per meal.
3. a. Predict how many meals your restaurant would sell in one month.
   b. How much money would you need to charge for each meal to be able to cover all the ongoing costs of running a restaurant?
4. What percentage of the cost of the ingredients is the markup on your meal?

Exercise 9.1.3.2: Disposable or Reusable?

A sample of full service restaurants and a sample of fast food restaurants were surveyed about the average number of customers they serve per day.

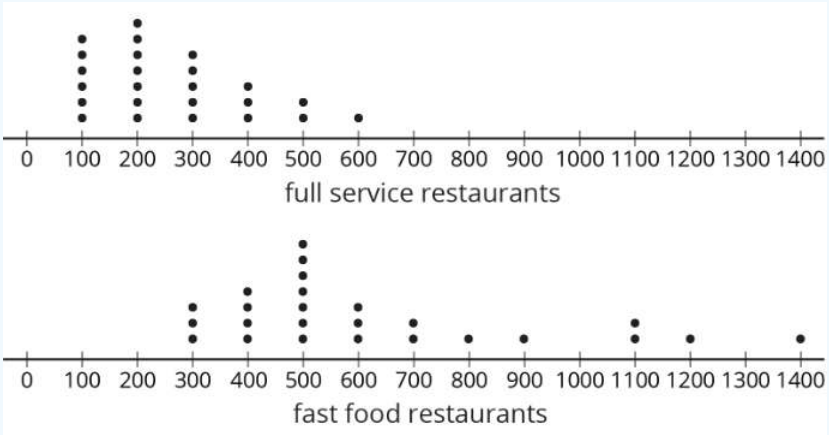


Figure 9.1.3.1: Two dot plots labeled full service restaurants and fast food restaurants each with the numbers 0 through 1400, in increments of 100, indicated. The data are as follows: Full service restaurants: 100 restaurants, 6 dots. 200 restaurants, 7 dots. 300 restaurants, 5 dots. 400 restaurants, 3 dots. 500 restaurants, 2 dots. 600 restaurants, 1 dot. Fast food restaurants: 300 restaurants, 3 dots. 400 restaurants, 4 dots. 500 restaurants, 7 dots. 600 restaurants, 3 dots. 700 restaurants, 2 dots. 800 restaurants, 1 dot. 900 restaurants, 1 dot. 1100 restaurants, 2 dots. 1200 restaurants, 1 dot. 1400 restaurants, 1 dot.

1. How does the average number of customers served per day at a full service restaurant generally compare to the number served at a fast food restaurant? Explain your reasoning.
2. About how many customers do you think your restaurant will serve per day? Explain your reasoning.
3. Here are prices for plates and forks:

	plates	forks
disposable	165 paper plates for $12.50	600 plastic forks for $10
reusable	12 ceramic plates for $28.80	24 metal forks for $30

Table 9.1.3.1

a. Using your predicted number of customers per day from the previous question, write an equation for the total cost, d, of using disposable plates and forks for every customer for n days.
b. Is d proportional to n? Explain your reasoning.
c. Use your equation to predict the cost of using disposable plates and forks for 1 year. Explain any assumptions you make with this calculation.

4. a. How much would it cost to buy enough reusable plates and forks for your predicted number of customers per day?
   b. If it costs $10.75 a day to wash the reusable plates and forks, write an expression that represents the total cost, r, of buying and washing reusable plates and forks after n days.
   c. Is r proportional to n? Explain your reasoning.
   d. How many days can you use the reusable plates and forks for the same cost that you calculated for using disposable plates and forks for 1 year?

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9.1.4: Restaurant Floor Plan

Lesson

Let's design the floor plan for a restaurant.

Exercise 9.1.4.1: Dining Area

1. Restaurant owners say it is good for each customer to have about 300 in^2 of space at their table. How many customers would you seat at each table?

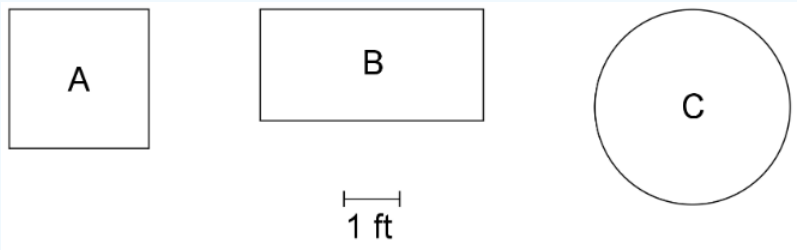


Figure 9.1.4.1: A square labeled A, a rectangle labeled B, and a circle labeled C are indicated. A scale of one defined unit equals 1 foot is also indicated. The square has side lengths of 2 point 5 units. The rectangle has side lengths of 2 units and 4 units. The circle has a diameter of 3 point 5 units.

2. It is good to have about 15 ft^2 of floor space per customer in the dining area.
   a. How many customers would you like to be able to seat at one time?
   b. What size and shape dining area would be large enough to fit that many customers?
   c. Select an appropriate scale, and create a scale drawing of the outline of your dining area.
3. Using the same scale, what size would each of the tables from the first question appear on your scale drawing?
4. To ensure fast service, it is good for all of the tables to be within 60 ft of the place where the servers bring the food out of the kitchen. Decide where the food pickup area will be, and draw it on your scale drawing. Next, show the limit of how far away tables can be positioned from this place.
5. It is good to have at least $1\frac{1}{2}$ ft between each table and at least $1\frac{1}{2}$ ft between the sides of tables where the customers will be sitting. On your scale drawing, show one way you could arrange tables in your dining area.

Are you ready for more?

The dining area usually takes up about 60% of the overall space of a restaurant because there also needs to be room for the kitchen, storage areas, office, and bathrooms. Given the size of your dining area, how much more space would you need for these other areas?

Exercise 9.1.4.2: Cold Storage

Some restaurants have very large refrigerators or freezers that are like small rooms. The energy to keep these rooms cold can be expensive.

- A standard walk-in refrigerator (rectangular, 10 feet wide, 10 feet long, and 7 feet tall) will cost about $150 per month to keep cold.
- A standard walk-in freezer (rectangular, 8 feet wide, 10 feet long, and 7 feet tall) will cost about $372 per month to keep cold.

Here is a scale drawing of a walk-in refrigerator and freezer. About how much would it cost to keep them both cold? Show your reasoning.

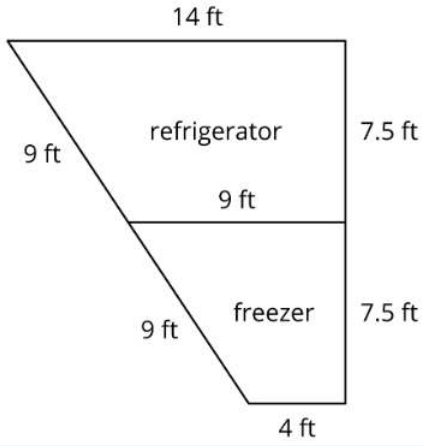


Figure 9.1.4.2: A large right trapezoid, that represents the shape of a refrigerator and freezer, with a longer top horizontal base. A horizontal line is drawn such that it divides the large trapezoid into two smaller trapezoids labeled refrigerator and freezer. The top trapezoid labeled refrigerator has a top horizontal base of 14 feet, a bottom horizontal base of 9 feet, a vertical leg of 7 point 5 feet, and a leg of 9 feet. The bottom trapezoid labeled freezer has a top horizontal base of 9 feet, a bottom horizontal base of 4 feet, a vertical leg of 7 point 5 feet, and a leg of 9 feet.

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9.2: Making Connections

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9.2.1: How Crowded Is this Neighborhood?

Lesson

Let's see how proportional relationships apply to where people live.

Exercise 9.2.1.1: Dot Density

The figure shows four squares. Each square encloses an array of dots. Squares A and B have side length 2 inches. Squares C and D have side length 1 inch.

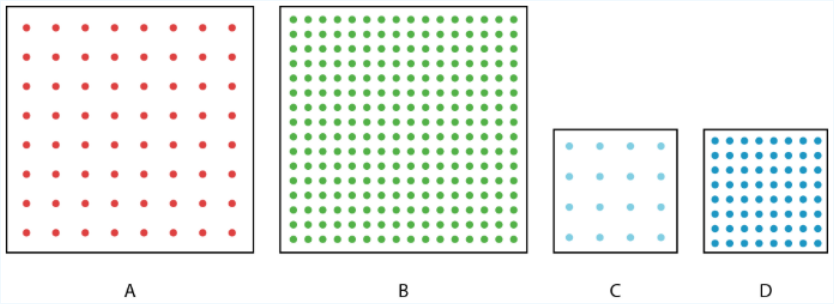


Figure 9.2.1.1

1. Complete the table with information about each square.

Table 9.2.1.1

square	area of the square in square inches	number of dots	number of dots per square inch
A			
B			
C			
D			

2. Compare each square to the others. What is the same and what is different?

Exercise 9.2.1.2: Dot Density with a Twist

The figure shows two arrays, each enclosed by a square that is 2 inches wide.

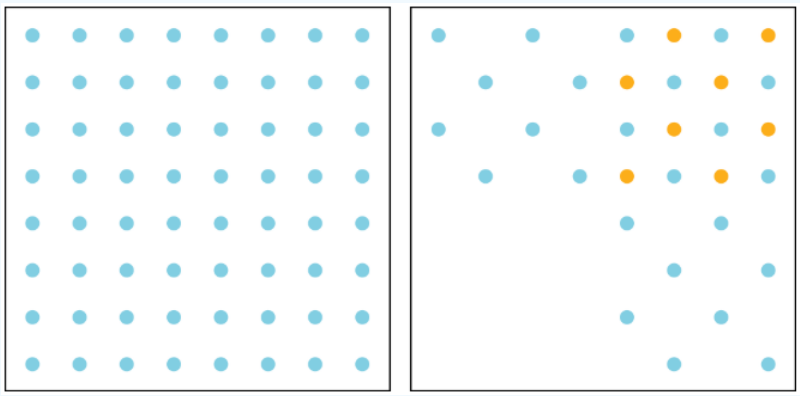


Figure 9.2.1.2: Two equal sized squares with an array of dots inside each. The left square contains an 8 by 8 array of dots. In the right square, the array of dots are as follows: Row 1: 4 blue dots, 2 yellow dots. Row 2: 4 blue dots, 2 yellow dots. Row 3: 4 blue dots, 2 yellow dots. Row 4: 4 blue dots, 2 yellow dots. Row 5: 2 blue dots. Row 6: 2 blue dots. Row 7: 2 blue dots. Row 8: 2 blue dots.

1. Let a be the area of the square and d be the number of dots enclosed by the square. For each square, plot a point that represents its values of a and d. Start by choosing an appropriate scale for each axis, using the sliders to adjust the intervals.
2. Draw lines from $(0, 0)$ to each point. For each line, write an equation that represents the proportional relationship.
3. What is the constant of proportionality for each relationship? What do the constants of proportionality tell us about the dots and squares?

Exercise 9.2.1.3: Housing Density

Here are pictures of two different neighborhoods.

This image depicts an area that is 0.3 kilometers long and 0.2 kilometers wide.



Figure 9.2.1.3

This image depicts an area that is 0.4 kilometers long and 0.2 kilometers wide.



Figure 9.2.1.4: An aerial image of a neighborhood displaying the roofs of houses. A length of 0 point 1 kilometer indicating one fourth of the length of the neighborhood is indicated. There are a total of 9 or 10 houses displayed.

For each neighborhood, find the number of houses per square kilometer.

Exercise 9.2.1.4: Population Density

- New York City has a population of 8,406 thousand people and covers an area of 1,214 square kilometers.
- Los Angeles has a population of 3,884 thousand people and covers an area of 1,302 square kilometers.

1. The points labeled A and B each correspond to one of the two cities. Which is which? Label them on the graph.

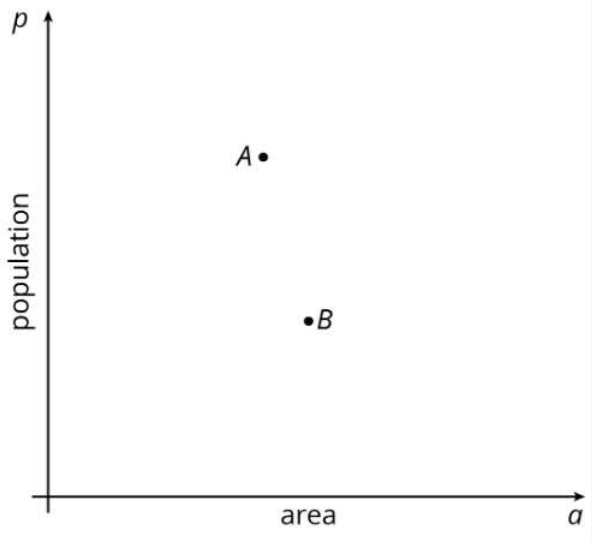


Figure 9.2.1.5: Two points labeled A and B plotted on a coordinate plane. The a-axis is labeled area. The p-axis is labeled population. Point A is to the right of the p-axis and high above the a-axis. Point B is to the right of point A and above the a-axis.

2. Write an equation for the line that passes through $(0, 0)$ and A. What is the constant of proportionality?
3. Write an equation for the line that passes through $(0, 0)$ and B. What is the constant of proportionality?
4. What do the constants of proportionality tell you about the crowdedness of these two cities?

Are you ready for more?

1. Predict where these types of regions would be shown on the graph:
   a. a suburban region where houses are far apart, with big yards
   b. a neighborhood in an urban area with many high-rise apartment buildings
   c. a rural state with lots of open land and not many people
2. Next, use this data to check your predictions:

Table 9.2.1.2

place	description	population	area (km^2)
Chalco	a suburb of Omaha, Nebraska	10, 994	7.5
Anoka County	a county in Minnesota, near Minneapolis/ St. Paul	339, 534	1, 155
Guttenberg	a city in New Jersey	11, 176	0.49
New York	a state	19, 746, 227	141, 300
Rhode Island	a state	1, 055, 173	3, 140
Alaska	a state	736, 732	1, 717, 856
Tok	a community in Alaska	1, 258	342.7

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9.2.2: Fermi Problems

Lesson

Let's estimate some quantities.

Exercise 9.2.2.1: How Old Are You?

What is your *exact* age at this moment?

Exercise 9.2.2.2: A Heart Stoppingly Large Number

How many times has your heart beat in your lifetime?

Exercise 9.2.2.3: All the Hairs on Your Head

How many strands of hair do you have on your head?

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9.2.3: More Expressions and Equations

Lesson

Let's solve harder problems by writing equivalent expressions.

Exercise 9.2.3.1: Tickets for the School Play

Student tickets for the school play cost $2 less than adult tickets.

1. If a represents the price of one adult ticket, write an expression for the price of a student ticket.
2. Write an expression that represents the amount of money they collected each night:
   a. The first night, the school sold 60 adult tickets and 94 student tickets.
   b. The second night, the school sold 83 adult tickets and 127 student tickets.
3. Write an expression that represents the total amount of money collected from ticket sales on both nights.
4. Over these two nights, they collected a total of $1,651 in ticket sales.
   a. Write an equation that represents this situation.
   b. What was the cost of each type of ticket?
5. Is your solution reasonable? Explain how you know.

Exercise 9.2.3.2: A Souvenir Strand

The souvenir stand sells hats, postcards, and magnets. They have twice as many postcards as hats, and 100 more magnets than post cards.

1. Let h represent the total number of hats. Write an expression in terms of h for the total number of items they have to sell.
2. The owner of the stand pays $8 for each hat, $0.10 for each post card, and $0.50 for each magnet. Write an expression for the total cost of the items.
3. The souvenir stand sells the hats for $11.75 each, the postcards for $0.25 each, and the magnets for $3.50 each. Write an expression for the total amount of money they would take in if they sold all the items.
4. Profits are calculated by subtracting costs from income. Write an expression for the profits of the souvenir stand if they sell all the items they have. Use properties to write an equivalent expression with fewer terms.
5. The souvenir stand sells all these items and makes a total profit of $953.25.
   a. Write an equation that represents this situation.
   b. How many of each item does the souvenir stand sell? Explain or show your reasoning.

Exercise 9.2.3.3: Jada Crochets a Scarf

Basic crochet stitches are called single, double, and triple. Jada measures her average stitch size and sees that a "double crochet" stitch is not really twice as long; it uses $\frac{1}{2}$ inch less than twice as much yarn as a single crochet stitch. Jada's "triple crochet" stitch uses 1 inch less than three times as much yarn as a single crochet stitch.

1. Write an expression that represents the amount of yarn Jada needs to crochet a scarf that includes 800 single crochet stitches, 400 double crochet stitches, and 200 triple crochet stitches.
2. Write an equivalent expression with as few terms as possible.
3. If Jada uses 5540 inches of yarn for the entire scarf, what length of yarn does she use for a single crochet stitch?

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9.2.4: Measurement Error (Part 1)

Lesson

Let's check how accurate our measurements are.

Exercise 9.2.4.1: How Long Are These Pencils?

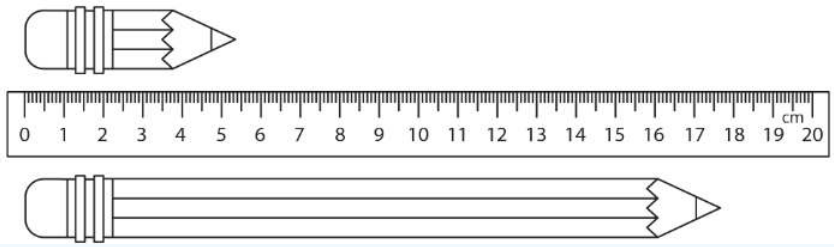


Figure 9.2.4.1: Two pencils, one significantly shorter than the other, are indicated by a ruler with millimeter and centimeter markings. The top pencils eraser lines up with 0 and the tip ends between 5 point 3 and 5 point 5 centimeters. The bottom pencils eraser lines up with the 0 and the tip ends between 17 point 6 and 17 point 8 centimeters.

1. Estimate the length of each pencil.
2. How accurate are your estimates?
3. For each estimate, what is the largest possible percent error?

Exercise 9.2.4.2: How Long Are These Floor Boards?

A wood floor is made by laying multiple boards end to end. Each board is measured with a maximum percent error of 5%. What is the maximum percent error for the total length of the floor?

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9.2.5: Measurement Error (Part 2)

Lesson

Let's check how accurate our calculations are.

Exercise 9.2.5.1: Measurement Error for Area

Imagine that you measure the length and width of a rectangle and you know the measurements are accurate within 5% of the actual measurements. If you use your measurements to find the area, what is the maximum percent error for the area of the rectangle?

Exercise 9.2.5.2: Measurement Error for Volume

1. The length, width, and height of a rectangular prism were measured to be 10 cm, 12 cm, and 25 cm. Assuming that these measurements are accurate to the nearest cm, what is the largest percent error possible for:
   a. each of the dimensions?
   b. the volume of the prism?
2. If the length, width, and height of a right rectangular prism have a maximum percent error of 1%, what is the largest percent error possible for the volume of the prism?

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9.3: Designing a Course

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9.3.1: Measuring Long Distances Over Uneven Terrain

Lesson

Let's measure long distances over uneven terrain.

Exercise 9.3.1.1: How Far Is It?

How do people measure distances in different situations? What tools do they use? Come up with at least three different methods and situations where those methods are used.

Exercise 9.3.1.2: Planning a 5K Course

The school is considering holding a 5K fundraising walk on the school grounds. Your class is supposed to design the course for the walk.

1. What will you need to do to design the course for the walk?
2. Come up with a method to measure the course. Pause here so your teacher can review your plan.

Exercise 9.3.1.3: Comparing Methods

Let’s see how close different measuring methods are to each other. Your teacher will show you a path to measure.

1. Use your method to measure the length of the path at least two times.
2. Decide what distance you will report to the class.
3. Compare your results with those of two other groups. Express the differences between the measurements in terms of percentages.
4. Discuss the advantages and disadvantages of each group’s method.

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9.3.2: Building a Trundle Wheel

Lesson

Let's build a trundle wheel.

Exercise 9.3.2.1: What Is a Trundle Wheel?

A tool that surveyors use to measure distances is called a trundle wheel.

1. How does a trundle wheel measure distance?
2. Why is this method of measuring distances better than the methods we used in the previous lesson?
3. How could we construct a simple trundle wheel? What materials would we need?



Figure 9.3.2.1

Exercise 9.3.2.2: Building a Trundle Wheel

Your teacher will give you some supplies. Construct a trundle wheel and use it to measure the length of the classroom. Record:

1. the diameter of your trundle wheel
2. the number of clicks across classroom
3. the length of the classroom (Be prepared to explain your reasoning.)

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9.3.3: Using a Trundle Wheel to Measure Distances

Lesson

Let's use our trundle wheels.

Exercise 9.3.3.1: Measuring Distances with the Trundle Wheel

Earlier you made trundle wheels so that you can measure long distances.

If you do not have a trundle wheel, you can use an applet to simulate measuring. Your teacher will tell you a path whose distance you will measure.

1. Measure the path with your trundle wheel three times and calculate the distance. Record your results in the table.

Table 9.3.3.1

	diameter	number of rotations	computation	distance
trial 1				
trial 2				
trial 3				

2. Decide what distance you will report to the class. Be prepared to explain your reasoning.
3. Compare this distance with the distance you measured the other day for this same path.
4. Compare your results with the results of two other groups. Express the differences between the measurements in terms of percentages.

To use this applet:

- Choose a diameter for the wheel, and enter a number 1 -- 5 for the path you will measure.
- Watch carefully to keep track of the number of rotations that the wheel makes before it stops.
- Try three different diameters for the same path, and compare your results.
- Answer the questions.

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9.3.4: Designing a 5K Course

Lesson

Let's map out the 5K course.

Exercise 9.3.4.1: Make a Proposal

Your teacher will give you a map of the school grounds.

1. On the map, draw in the path you measured earlier with your trundle wheel and label its length.
2. Invent another route for a walking course and draw it on your map. Estimate the length of the course you drew.
3. How many laps around your course must someone complete to walk 5 km?

Exercise 9.3.4.2: Measuring and Finalizing the Course

1. Measure your proposed race course with your trundle wheel at least two times. Decide what distance you will report to the class.
2. Revise your course, if needed.
3. Create a visual display that includes:
   - A map of your final course
   - The starting and ending locations
   - The number of laps needed to walk 5 km
   - Any other information you think would be helpful to the race organizers

Are you ready for more?

The map your teacher gave you didn't include a scale. Create one.

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