PRE-ALGEBRA II (ILLUSTRATIVE MATHEMATICS - GRADE 8)



Illustrative Mathematics

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CHAPTER OVERVIEW

1: Rigid Transformations and Congruence

In this unit, students learn to understand and use the terms “reflection,” “rotation,” “translation,” recognizing what determines each type of transformation, e.g., two points determine a translation. They learn to understand and use the terms “transformation” and “rigid transformation.” They identify and describe translations, rotations, and reflections, and sequences of these, using the terms “corresponding sides” and “corresponding angles,” and recognizing that lengths and angle measures are preserved. They draw images of figures under rigid transformations on and off square grids and the coordinate plane. They use rigid transformations to generate shapes and to reason about measurements of figures. They learn to understand congruence of plane figures in terms of rigid transformations. They recognize when one plane figure is congruent or not congruent to another. Students use the definition of “congruent” and properties of congruent figures to justify claims of congruence or non-congruence.

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1.1: Rigid Transformations

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1.1.1: Moving in the Plane

Lesson

Let's describe ways figures can move in the plane.

Exercise 1.1.1.1: Which One Doesn't Belong: Diagrams

Which one doesn't belong?

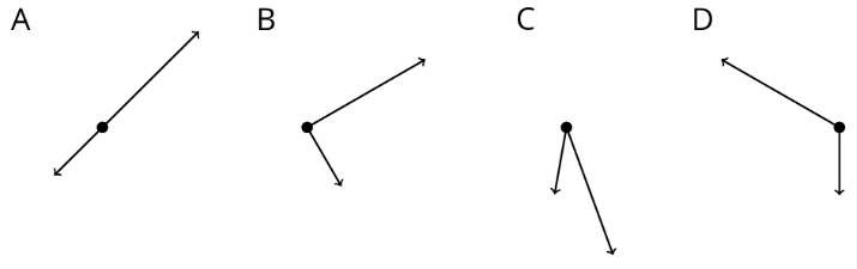


Figure 1.1.1.1: Four figures labeled A, B, C, and D. each figure is composed of two rays and a point that is located at the endpoints of the two rays. In figure A the two rays point in opposite directions. In figure B, one ray extends downward and to the right and the other ray extends upward and to the right. the two rays appear to form a right angle. In figure C, one ray extends downward and slightly to the left and the other ray extends downward and slightly to the right. The two rays appear to form an acute angle. In figure D, one ray extends upward and to the left and the other ray extends directly downward. The two rays appear to form an obtuse angle.

Exercise 1.1.1.2: Triangle Square Dance

Your teacher will give you three pictures. Each shows a different set of dance moves.

1. Arrange the three pictures so you and your partner can both see them right way up. Choose who will start the game.
    1. The starting player mentally chooses A, B, or C and describes the dance to the other player.
    2. The other player identifies which dance is being talked about: A, B, or C.
2. After one round, trade roles. When you have described all three dances, come to an agreement on the words you use to describe the moves in each dance.
3. With your partner, write a description of the moves in each dance.

Dance A:

Dance B:

Dance C:

Are you ready for more?

We could think of each dance as a new dance by running it in reverse, starting in the 6th frame and working backwards to the first.

1. Pick a dance and describe in words one of these reversed dances.
2. How do the directions for running your dance in the forward direction and the reverse direction compare?

Summary

Here are two ways for changing the position of a figure in a plane without changing its shape or size:

- Sliding or shifting the figure without turning it. Shifting Figure A to the right and up puts it in the position of Figure B.

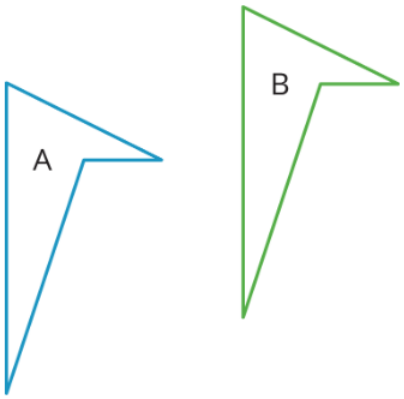


Figure 1.1.1.2

- Turning or rotating the figure around a point. Figure A is rotated around the bottom vertex to create Figure C.

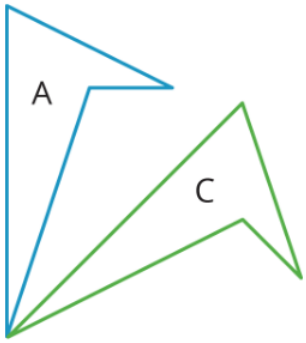


Figure 1.1.1.3

Practice

Exercise 1.1.1.3

The six frames show a shape's different positions.

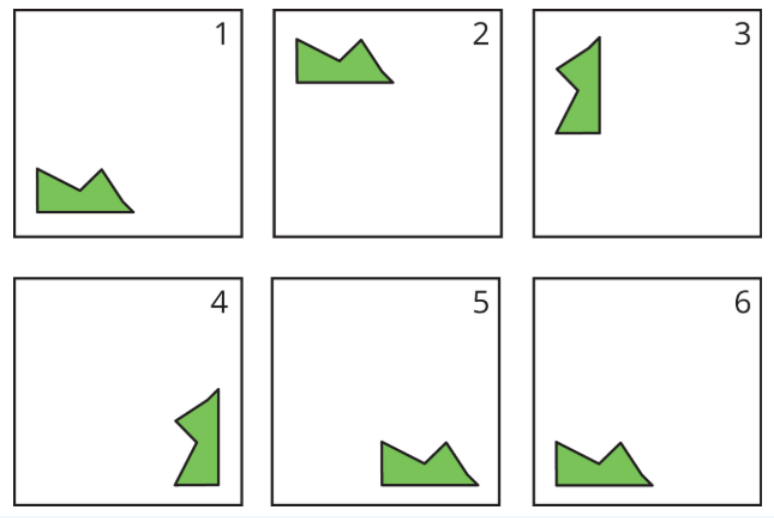


Figure 1.1.1.4

Describe how the shape moves to get from its position in each frame to the next.

Exercise 1.1.1.4

These five frames show a shape's different positions.

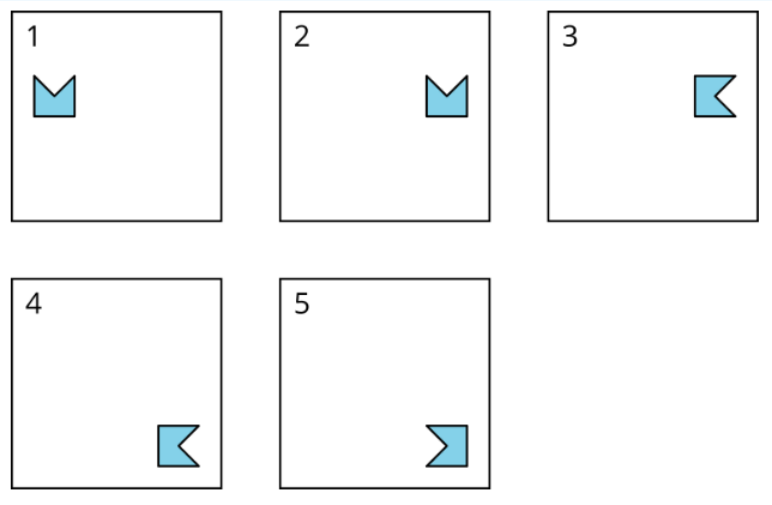


Figure 5 shows figure 4 facing the opposite way and is still on the bottom right side.

Describe how the shape moves to get from its position in each frame to the next.

Exercise 1.1.1.5

Diego started with this shape.

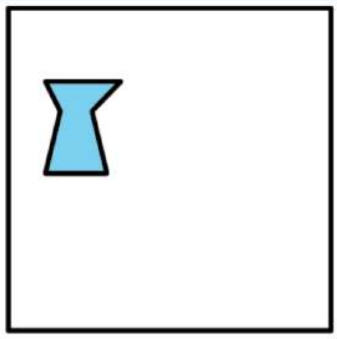


Figure 1.1.1.6

Diego moves the shape down, turns it 90 degrees clockwise, then moves the shape to the right. Draw the location of the shape after each move.

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1.1.2: Naming the Moves

Lesson

Let's be more precise about describing moves of figures in the plane.

Exercise 1.1.2.1: A Pair of Quadrilaterals

Quadrilateral A can be rotated into the position of Quadrilateral B.

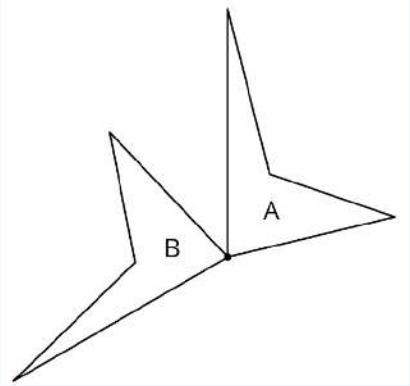


Figure 1.1.2.1

Estimate the angle of rotation.

Exercise 1.1.2.2: How Did You Make That Move?

Here is another set of dance moves.

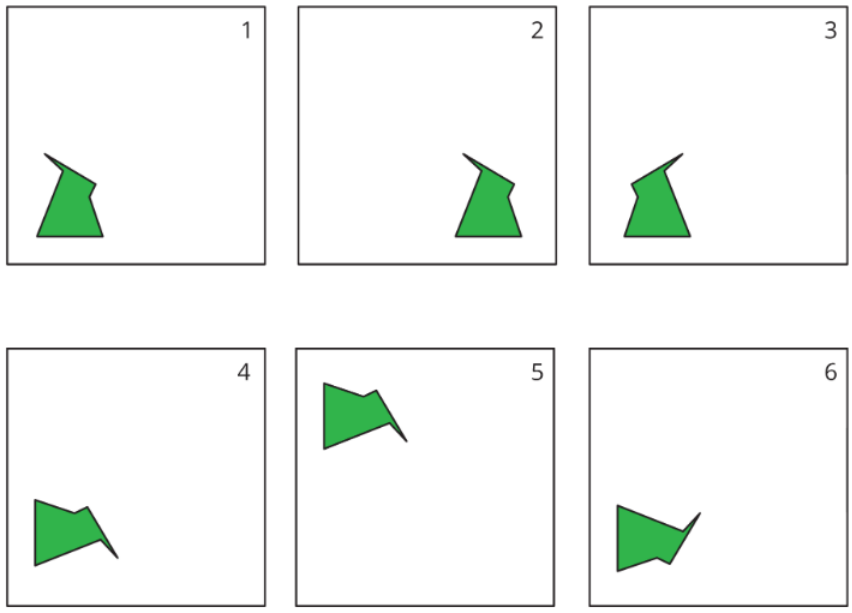


Figure 1.1.2.2

1. Describe each move or say if it is a new move.
   a. Frame 1 to Frame 2
   b. Frame 2 to Frame 3
   c. Frame 3 to Frame 4
   d. Frame 4 to Frame 5
   e. Frame 5 to Frame 6

2. How would you describe the new move?

Exercise 1.1.2.3: Card Sort: Move

Your teacher will give you a set of cards. Sort the cards into categories according to the type of move they show. Be prepared to describe each category and why it is different from the others. You can explore the applets below to see the ways the images move.

Drag the red point. Explore how the image changes.

Click on the box to show the transformed image. Move the yellow points and the red segment to see how the image changes.

Summary

Here are the moves we have learned about so far:

- A **translation** slides a figure without turning it. Every point in the figure goes the same distance in the same direction. For example, Figure A was translated down and to the left, as shown by the arrows. Figure B is a translation of Figure A.

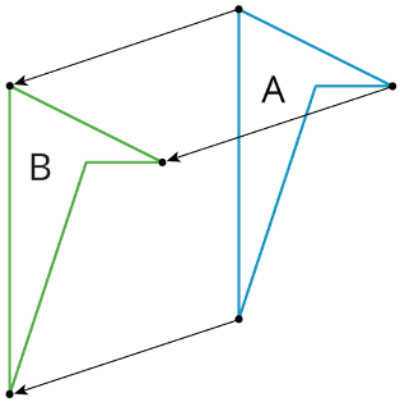


Figure 1.1.2.3

- A **rotation** turns a figure about a point, called the center of the rotation. Every point on the figure goes in a circle around the center and makes the same angle. The rotation can be **clockwise**, going in the same direction as the hands of a clock, or **counterclockwise**, going in the other direction. For example, Figure A was rotated 45° clockwise around its bottom vertex. Figure C is a rotation of Figure A.

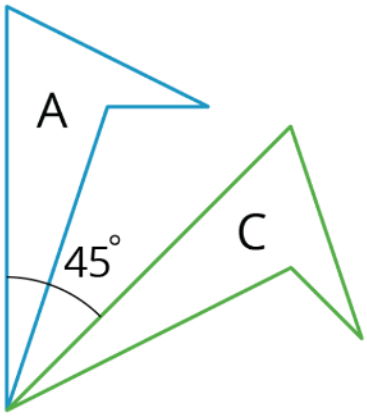


Figure 1.1.2.4

- A **reflection** places points on the opposite side of a reflection line. The mirror image is a backwards copy of the original figure. The reflection line shows where the mirror should stand. For example, Figure A was reflected across the dotted line. Figure D is a reflection of Figure A.

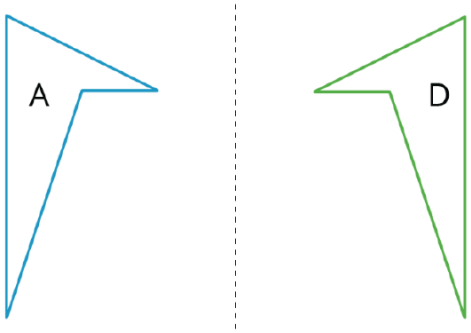


Figure 1.1.2.5

We use the word **image** to describe the new figure created by moving the original figure. If one point on the original figure moves to another point on the new figure, we call them **corresponding points.**

Glossary Entries

Definition: Clockwise

An object is rotating clockwise if it is turning in the same way that the hour or minute hand goes around a clock.

The tilted square is rotated 15° clockwise from the square sitting horizontally on its base.

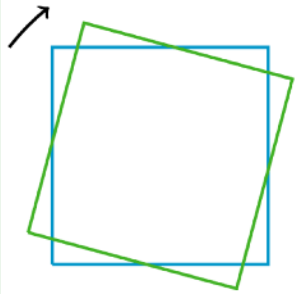


Figure 1.1.2.6

Definition: Corresponding

If a part of the original figure matches up with a part of the copy, we call them corresponding parts. The part could be an angle, point, or side, and you can have corresponding angles, corresponding points, or corresponding sides.

If you have a distance between two points in the original figure, then the distance between the corresponding points in the copy is called the corresponding distance.

Definition: Counterclockwise

An object is rotating counterclockwise if it is turning in the opposite way to the way that the hour or minute hand goes around a clock.

The tilted square is rotated 15° counterclockwise from the square with a horizontal base.

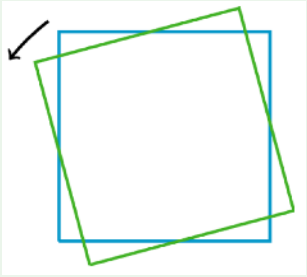


Figure 1.1.2.7

Definition: Image

Translations, rotations, and reflections move objects in the plane. Points, segments, and other parts of the original all have corresponding parts on the "moved object." The moved object is called the **image**.

For example, here is triangle ABC and a translation to the right and up which is labeled DEF.

Point F in the image corresponds to point C, segment EF in the image corresponds to segment BD, and angle DEF corresponds to angle ABC.

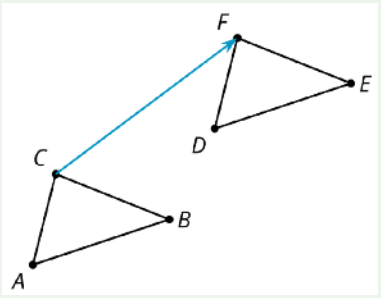


Figure 1.1.2.8

Definition: Reflection

The reflection of a figure across a line takes every point of the figure to a point directly opposite to it on the other side of the line and the same distance from the line. In the figure, the triangle B is the reflection of the triangle A across the line l.

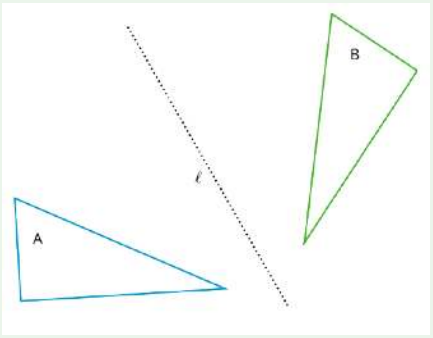


Figure 1.1.2.9

Definition: Rotation

A rotation has a center, an angle, and a direction. It moves every point of a figure in a circle around the center, in the direction specified (clockwise or counterclockwise), and for a distance specified by the angle. For example, in the figure, triangle A is rotated 55° clockwise about center O to get triangle B.

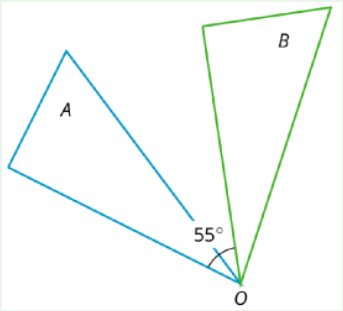


Figure 1.1.2.10

Definition: Translation

A translation has a distance and a direction. It moves every point in a figure the given distance in the given direction.

The figure on the left is translated to the figure on the right in the direction from A to B, using the distance from A to B.

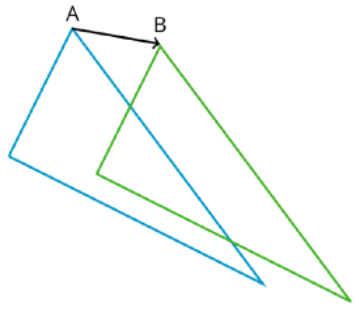


Figure 1.1.2.11

Practice

Exercise 1.1.2.4

Each of the six cards shows a shape.

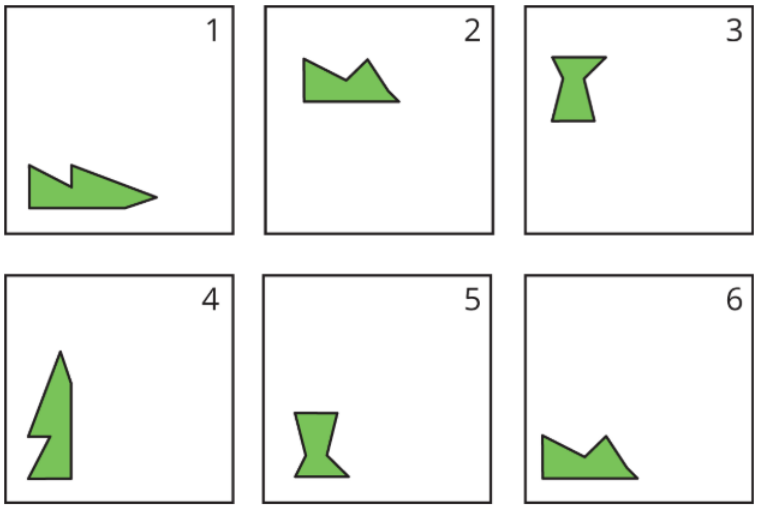


Figure 1.1.2.12

1. Which pair of cards shows a shape and its image after a rotation?
2. Which pair of cards shows a shape and its image after a reflection?

Exercise 1.1.2.5

The five frames show a shape's different positions.

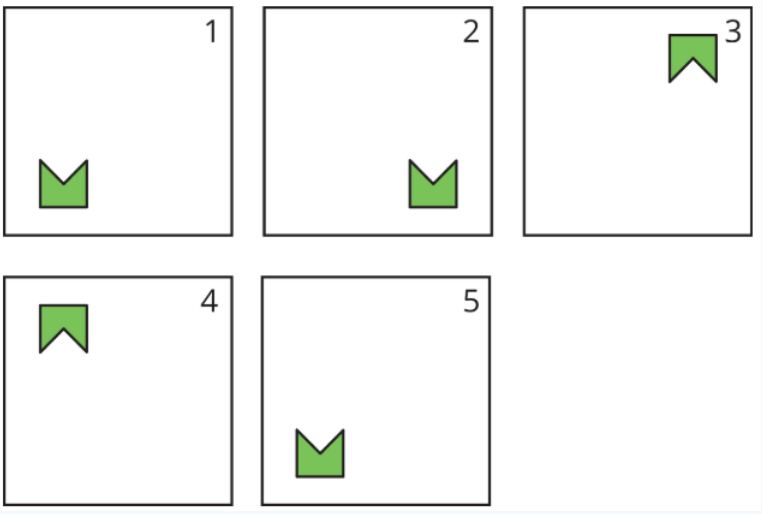


Figure 1.1.2.13

Describe how the shape moves to get from its position in each frame to the next.

Exercise 1.1.2.6

The rectangle seen in Frame 1 is rotated to a new position, seen in Frame 2.

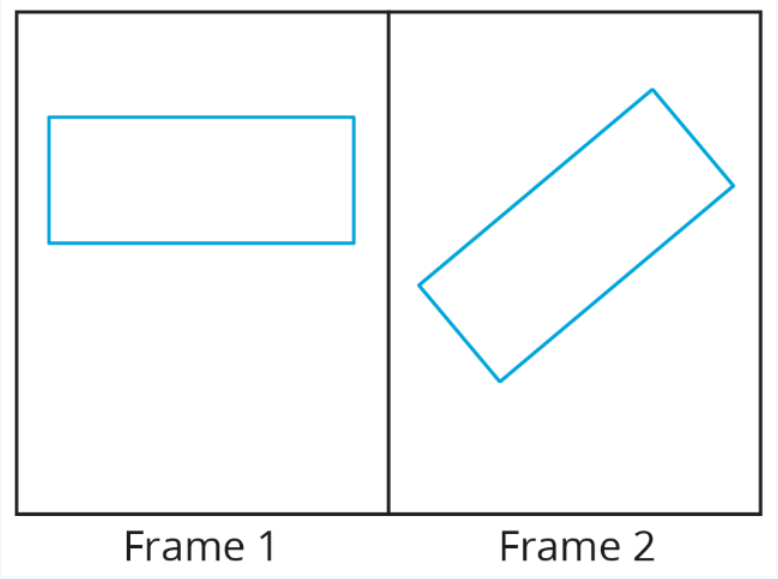


Figure 1.1.2.14: A figure of rectangular Frame 1 and Frame 2, each with a rectangle. Frame 1 rectangle has sides parallel to the frame. Frame 2 rectangle has the left side down, the right side raised and no sides parallel to the frame.

Select **all** the ways the rectangle could have been rotated to get from Frame 1 to Frame 2.

A: 40 degrees clockwise

B: 40 degrees counterclockwise

C: 90 degrees clockwise

D: 90 degrees counterclockwise

E: 140 degrees clockwise

F: 140 degrees counterclockwise

(From Unit 1.1.1)

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1.1.3: Grid Moves

Lesson

Let's transform some figures on grids.

Exercise 1.1.3.1: Notice and Wonder: The Isometric Grid

What do you notice? What do you wonder?

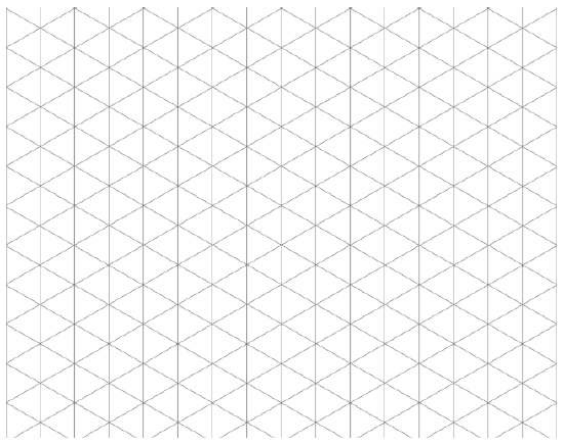


Figure 1.1.3.1

Exercise 1.1.3.2: Transformation Information

Follow the directions below each statement to tell GeoGebra how you want the figure to move. It is important to notice that GeoGebra uses vectors to show translations. A *vector* is a quantity that has magnitude (size) and direction. It is usually represented by an arrow.

These applets are sensitive to clicks. Be sure to make one quick click, otherwise it may count a double-click.

After each example, click the reset button, and then move the slider over for the next question.

1. Translate triangle ABC so that A goes to A'.
   a. Select the Vector tool.

   b. Click on the original point A and then the new point A'. You should see a vector.
   c. Select the Translate by Vector tool.

   d. Click on the figure to translate, and then click on the vector.
2. Translate triangle ABC so that C goes to C'.
3. Rotate triangle ABC 90° counterclockwise using center O.
   a. Select the Rotate around Point tool.

   b. Click on the figure to rotate, and then click on the center point.
   c. A dialog box will open; type the angle by which to rotate and select the direction of rotation.
   d. Click on ok.
4. Reflect triangle ABC using line l.

a. Select the Reflect about Line tool.



b. Click on the figure to reflect, and then click on the line of reflection.

5. Rotate quadrilateral $ABCD$ 60° counterclockwise using center B.
6. Rotate quadrilateral $ABCD$ 60° clockwise using center C.
7. Reflect quadrilateral $ABCD$ using line l.
8. Translate quadrilateral $ABCD$ so that A goes to C.

Try your own translations, reflections, and rotations.

1. Make your own polygon to transform, and choose a transformation.
2. Predict what will happen when you transform the image. Try it - were you right?
3. Challenge your partner! Right click on any vectors or lines and uncheck Show Object. Can they guess what transformation you used?

Visit ggbm.at/eFeE2Veu for the applet.

Summary

When a figure is on a grid, we can use the grid to describe a transformation. For example, here is a figure and an image of the figure after a move.

Quadrilateral $ABCD$ is translated 4 units to the right and 3 units down to the position of quadrilateral $A'B'C'D'$.

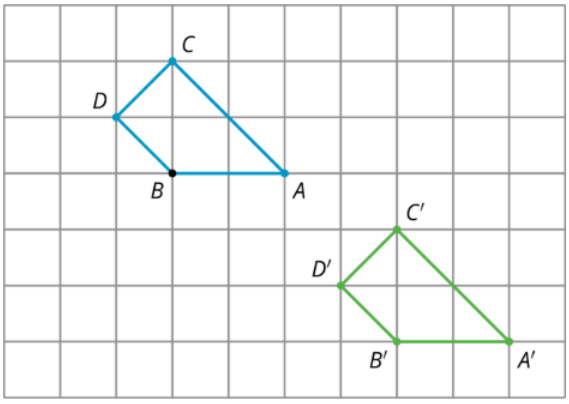


Figure 1.1.3.2: Two identical quadrilaterals on a grid labeled $B\ D\ C\ A$ and B prime D prime C prime A prime. In quadrilateral $B\ D\ C\ A$, point B is 3 units right and 3 units down from the edge of the grid. Point D is 1 unit left and 1 unit up from point B. Point C is 2 units up from point B. Point A is 2 units right from point B. In quadrilateral B prime D prime C prime A prime, point B prime is 3 units down and 4 units right from point B. Point D prime is 3 units down and 4 units right from point D. Point C prime is 3 units down and 4 units right from point C. Point A prime is 3 units down and 4 units right from point A.

A second type of grid is called an *isometric grid*. The isometric grid is made up of equilateral triangles. The angles in the triangles all measure 60 degrees, making the isometric grid convenient for showing rotations of 60 degrees.

Here is quadrilateral $KLMN$ and its image $K'L'M'N'$ after a 60-degree counterclockwise rotation around a point P.

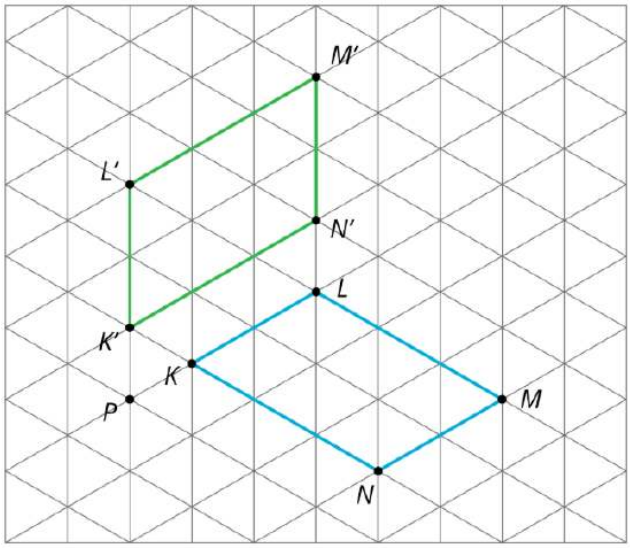


Figure 1.1.3.3

Glossary Entries

Definition: Clockwise

An object is rotating clockwise if it is turning in the same way that the hour or minute hand goes around a clock.

The tilted square is rotated 15° clockwise from the square sitting horizontally on its base.

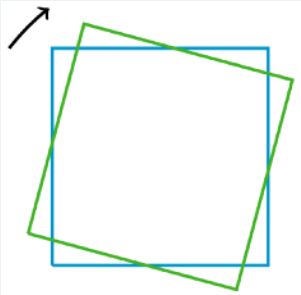


Figure 1.1.3.4

Definition: Corresponding

If a part of the original figure matches up with a part of the copy, we call them corresponding parts. The part could be an angle, point, or side, and you can have corresponding angles, corresponding points, or corresponding sides.

If you have a distance between two points in the original figure, then the distance between the corresponding points in the copy is called the corresponding distance.

Definition: Counterclockwise

An object is rotating counterclockwise if it is turning in the opposite way to the way that the hour or minute hand goes around a clock.

The tilted square is rotated 15° counterclockwise from the square with a horizontal base.

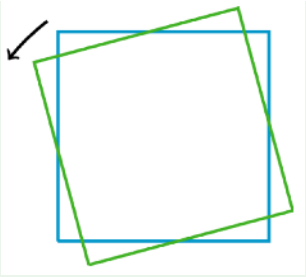


Figure 1.1.3.5

Definition: Image

Translations, rotations, and reflections move objects in the plane. Points, segments, and other parts of the original all have corresponding parts on the "moved object." The moved object is called the **image**.

For example, here is triangle ABC and a translation to the right and up which is labeled DEF.

Point F in the image corresponds to point C, segment EF in the image corresponds to segment BD, and angle DEF corresponds to angle ABC.

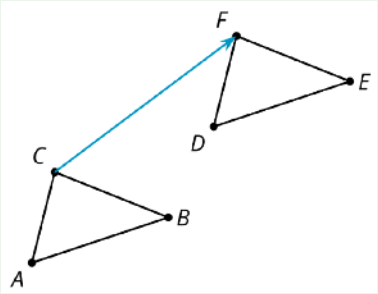


Figure 1.1.3.6

Definition: Reflection

The reflection of a figure across a line takes every point of the figure to a point directly opposite to it on the other side of the line and the same distance from the line. In the figure, the triangle B is the reflection of the triangle A across the line l.

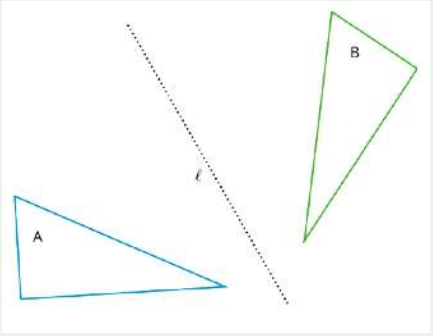


Figure 1.1.3.7

Definition: Rotation

A rotation has a center, an angle, and a direction. It moves every point of a figure in a circle around the center, in the direction specified (clockwise or counterclockwise), and for a distance specified by the angle. For example, in the figure, triangle A is rotated 55° clockwise about center O to get triangle B.

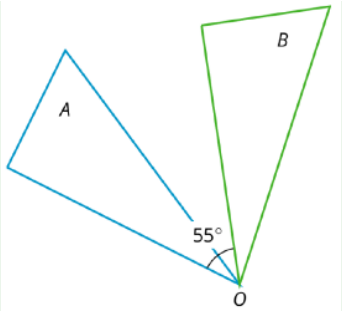


Figure 1.1.3.8

Definition: Translation

A translation has a distance and a direction. It moves every point in a figure the given distance in the given direction.

The figure on the left is translated to the figure on the right in the direction from A to B, using the distance from A to B.

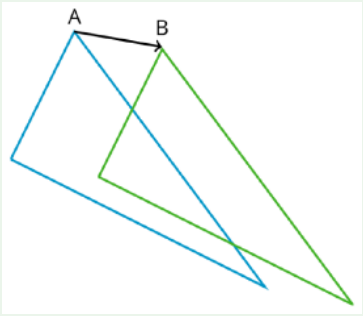


Figure 1.1.3.9

Practice

Exercise 1.1.3.3

Apply each transformation described to Figure A. If you get stuck, try using tracing paper.

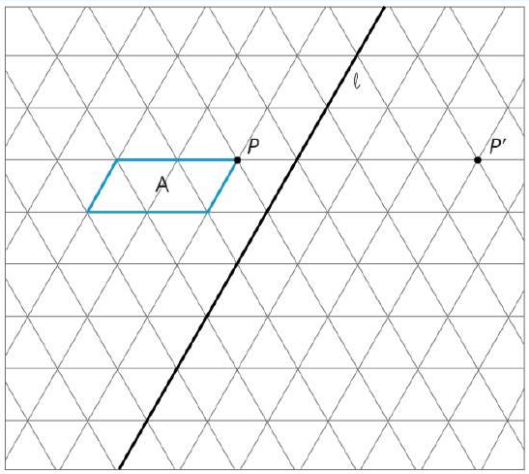


Figure A is a parallelogram and point P is on the upper right corner. Line l is one unit away from and parallel to the right side of figure A. Point P prime is 4 horizontal units from point P.

1. A translation which takes P to P'
2. A counterclockwise rotation of A, using center P, of 60 degrees
3. A reflection of A across line l

Exercise 1.1.3.4

Here is triangle ABC drawn on a grid.

On the grid, draw a rotation of triangle ABC, a translation of triangle ABC, and a reflection of triangle ABC. Describe clearly how each was done.

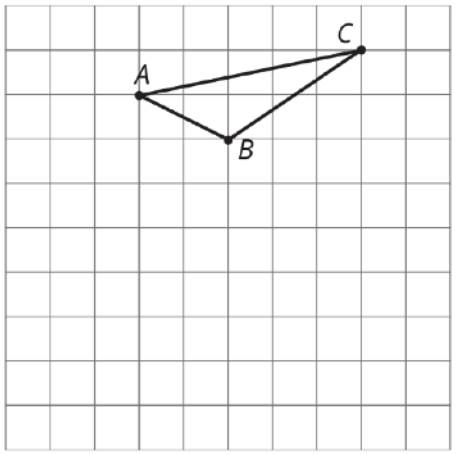


Figure 1.1.3.11

Exercise 1.1.3.5

1. Draw the translated image of $ABCDE$ so that vertex C moves to C'. Tracing paper may be useful.

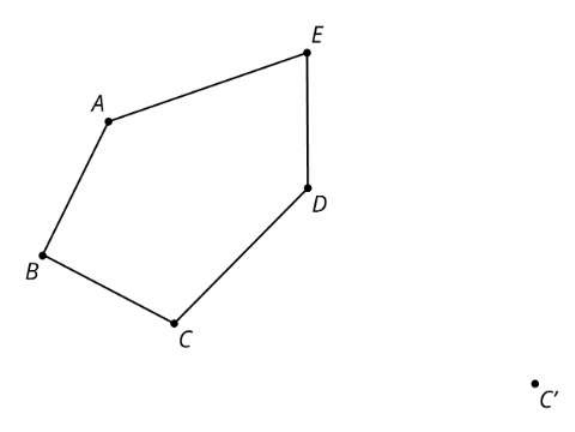


Figure 1.1.3.12

2. Draw the reflected image of Pentagon $ABCDE$ with line of reflection l. Tracing paper may be useful.

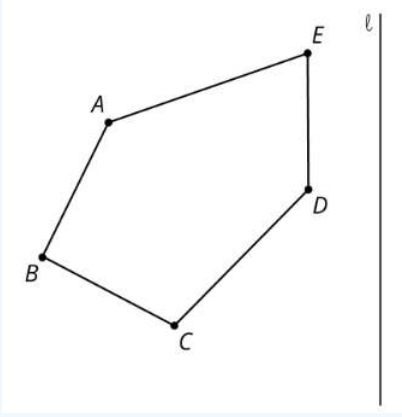


Figure 1.1.3.13

3. Draw the rotation of Pentagon $ABCDE$ around C clockwise by an angle of 150 degrees. Tracing paper and a protractor may be useful.

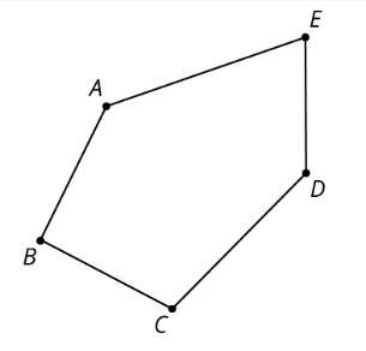


Figure 1.1.3.14

(From Unit 1.1.2)

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1.1.4: Making the Moves

Lesson

Let's draw and describe translations, rotations, and reflections.

Exercise 1.1.4.1: Reflection Quick Image

Here is an incomplete image. Your teacher will display the completed image twice, for a few seconds each time. Your job is to complete the image on your copy.

Exercise 1.1.4.2: Make That Move

Your partner will describe the image of this triangle after a certain **transformation**. Sketch it here.

Exercise 1.1.4.3: A to B to C

Here are some figures on an isometric grid. Explore the transformation tools in the tool bar. (Directions are below the applet if you need them.)

1. Name a transformation that takes Figure A to Figure B. Name a transformation that takes Figure B to Figure C.
2. What is one **sequence of transformations** that takes Figure A to Figure C? Explain how you know.

Translate

1. Select the Vector tool.



2. Click on the original point and then the new point. You should see a vector.
3. Select the Translate by Vector tool.



4. Click on the figure to translate, and then click on the vector.

Rotate

1. Select the Rotate around Point tool.



2. Click on the figure to rotate, and then click on the center point.
3. A dialog box will open. Type the angle by which to rotate and select the direction of rotation.

Reflect

1. Select the Reflect about Line tool.



2. Click on the figure to reflect, and then click on the line of reflection.

Experiment with some other ways to take Figure A to Figure C. For example, can you do it with.

- No rotations?
- No reflections?
- No translations?

Summary

A move, or combination of moves, is called a **transformation**. When we do one or more moves in a row, we often call that a **sequence of transformations**. To distinguish the original figure from its image, points in the image are sometimes labeled with the same letters as the original figure, but with the symbol ' attached, as in A' (pronounced “A prime”).

- A translation can be described by two points. If a translation moves point A to point A', it moves the entire figure the same distance and direction as the distance and direction from A to A'. The distance and direction of a translation can be shown by an arrow.

  For example, here is a translation of quadrilateral $ABCD$ that moves A to A'.

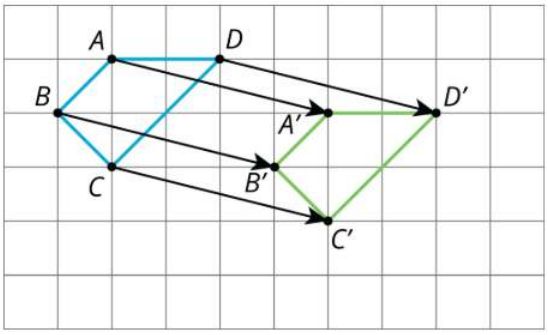


Figure 1.1.4.1

- A rotation can be described by an angle and a center. The direction of the angle can be clockwise or counterclockwise.

  For example, hexagon $ABCDEF$ is rotated 90° counterclockwise using center P.

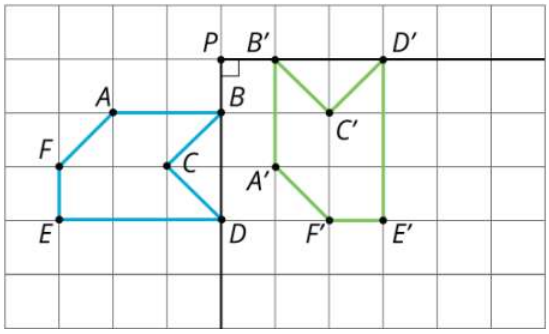


Figure 1.1.4.2

- A reflection can be described by a line of reflection (the “mirror”). Each point is reflected directly across the line so that it is just as far from the mirror line, but is on the opposite side.

  For example, pentagon $ABCDE$ is reflected across line m.

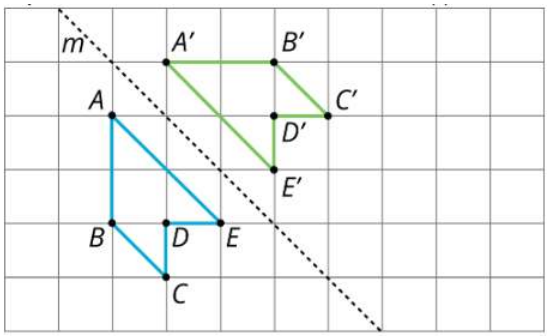


Figure 1.1.4.3

Glossary Entries

Definition: Clockwise

An object is rotating clockwise if it is turning in the same way that the hour or minute hand goes around a clock.

The tilted square is rotated 15° clockwise from the square sitting horizontally on its base.

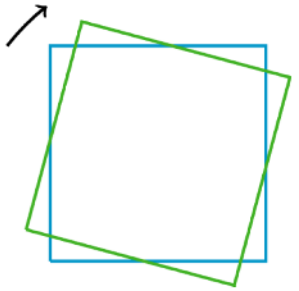


Figure 1.1.4.4

Definition: Corresponding

If a part of the original figure matches up with a part of the copy, we call them corresponding parts. The part could be an angle, point, or side, and you can have corresponding angles, corresponding points, or corresponding sides.

If you have a distance between two points in the original figure, then the distance between the corresponding points in the copy is called the corresponding distance.

Definition: Counterclockwise

An object is rotating counterclockwise if it is turning in the opposite way to the way that the hour or minute hand goes around a clock.

The tilted square is rotated 15° counterclockwise from the square with a horizontal base.

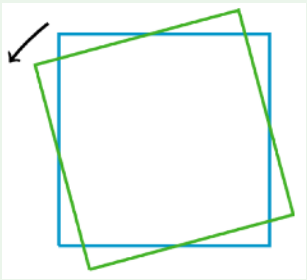


Figure 1.1.4.5

Definition: Image

Translations, rotations, and reflections move objects in the plane. Points, segments, and other parts of the original all have corresponding parts on the "moved object." The moved object is called the **image**.

For example, here is triangle ABC and a translation to the right and up which is labeled DEF.

Point F in the image corresponds to point C, segment EF in the image corresponds to segment BD, and angle DEF corresponds to angle ABC.

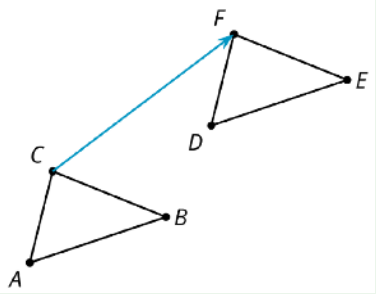


Figure 1.1.4.6

Definition: Reflection

The reflection of a figure across a line takes every point of the figure to a point directly opposite to it on the other side of the line and the same distance from the line. In the figure, the triangle B is the reflection of the triangle A across the line l.

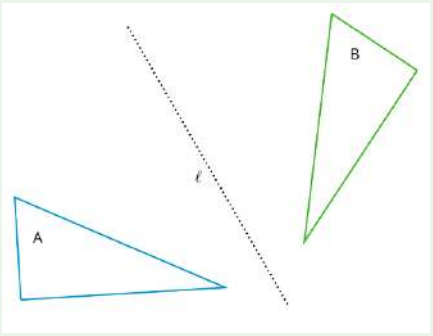


Figure 1.1.4.7

Definition: Rotation

A rotation has a center, an angle, and a direction. It moves every point of a figure in a circle around the center, in the direction specified (clockwise or counterclockwise), and for a distance specified by the angle. For example, in the figure, triangle A is rotated 55° clockwise about center O to get triangle B.

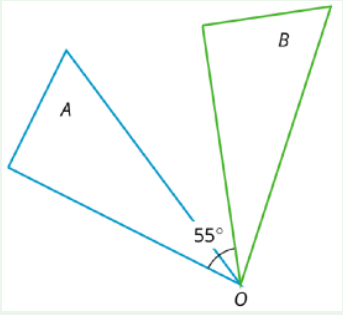


Figure 1.1.4.8

Definition: Sequence of Transformations

A sequence of transformations is a set of translations, rotations, reflections, and dilations performed in a particular order on a geometric figure, resulting in a final figure.

The diagram shows a sequence of transformations consisting of a translation (from A to B) followed by a rotation (from B to C) followed by a reflection (from C to D). The last triangle is the final figure resulting from the sequence.

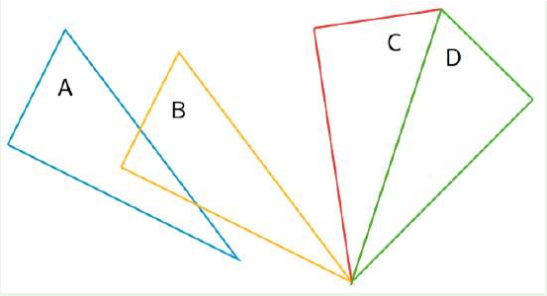


Figure 1.1.4.9

Definition: Transformation

A transformation is a translation, rotation, reflection, or dilation, or combination of these. There is also a more general concept of a transformation of the plane that is not discussed in grade 8.

Definition: Translation

A translation has a distance and a direction. It moves every point in a figure the given distance in the given direction.

The figure on the left is translated to the figure on the right in the direction from A to B, using the distance from A to B.

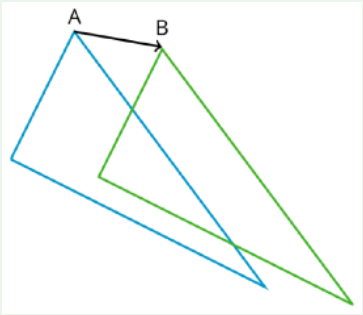


Figure 1.1.4.10

Practice

Exercise 1.1.4.4

For each pair of polygons, describe a sequence of translations, rotations, and reflections that takes Polygon P to Polygon Q.

1.

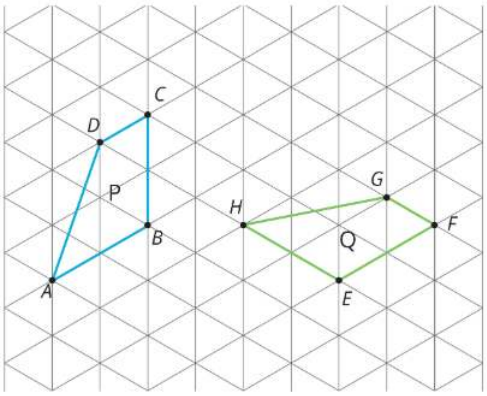


Figure 1.1.4.11

2.

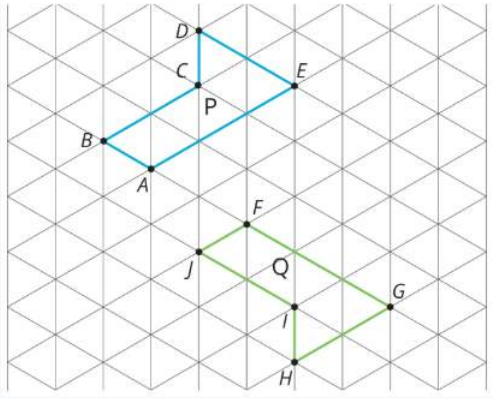


Figure 1.1.4.12

3.

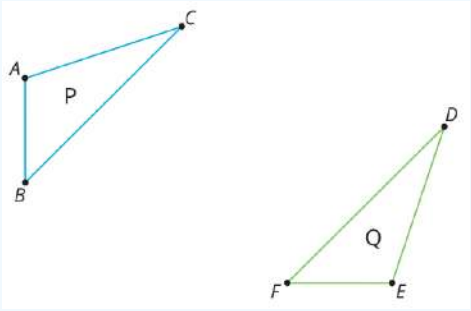


Figure 1.1.4.13

Exercise 1.1.4.5

Here is a quadrilateral $ABCD$ and line l.

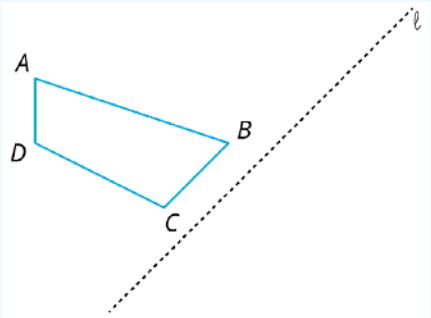


Figure 1.1.4.14

Draw the image of the quadrilateral $ABCD$ after reflecting it across line l.

(From Unit 1.1.2)

Exercise 1.1.4.6

Here is a quadrilateral $ABCD$.

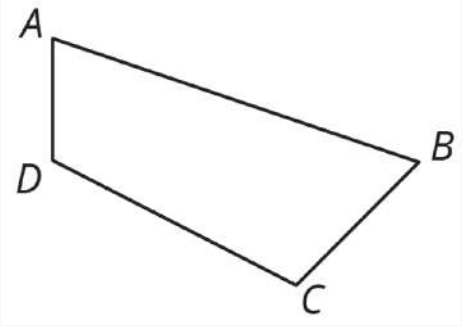


Figure 1.1.4.15

Draw the image of quadrilateral $ABCD$ after each rotation using B as center.

1. 90 degrees clockwise
2. 120 degrees clockwise
3. 30 degrees counterclockwise

(From Unit 1.1.2)

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1.1.5: Coordinate Moves

Lesson

Let's transform some figures and see what happens to the coordinates of points.

Exercise 1.1.5.1: Translating Coordinates

Select all of the translations that take Triangle T to Triangle U. There may be more than one correct answer.

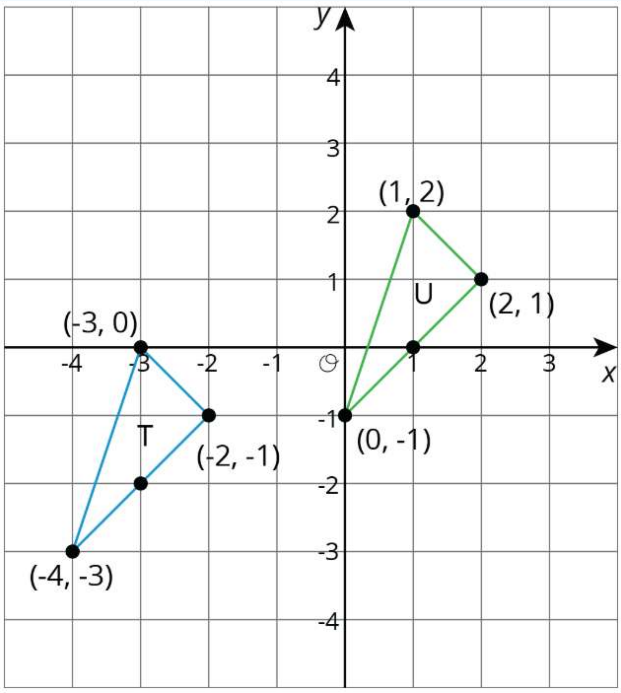


Figure 1.1.5.1

1. Translate $(-3, 0)$ to $(1, 2)$.
2. Translate $(2, 1)$ to $(-2, -1)$.
3. Translate $(-4, -3)$ to $(0, -1)$.
4. Translate $(1, 2)$ to $(2, 1)$.

Exercise 1.1.5.2: Reflecting Points on the Coordinate Plane

1. Five points are plotted on the coordinate plane.
   a. Using the Pen tool or the Text tool, label each with its coordinates.
   b. Using the x-axis as the line of reflection, plot the image of each point.
   c. Label the image of each point with its coordinates.
   d. Include a label using a letter. For example, the image of point A should be labeled A'.
2. If the point $(13, 10)$ were reflected using the x-axis as the line of reflection, what would be coordinates of the image? What about $(13, -20)$? $(13, 570)$? Explain how you know.
3. The point R has coordinates $(3, 2)$.
   a. Without graphing, predict the coordinates of the image of point R if point R were reflected using the y-axis as the line of reflection.

b. Check your answer by finding the image of R on the graph.

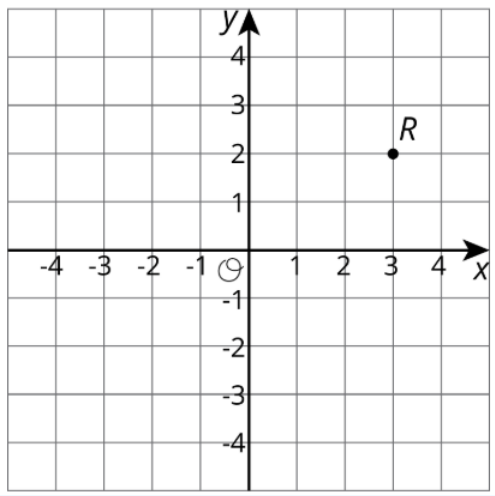


Figure 1.1.5.2

c. Label the image of point R as R'.

d. What are the coordinates of R'?

4. Suppose you reflect a point using the y-axis as line of reflection. How would you describe its image?

Exercise 1.1.5.3: Transformations of a Segment

The applet has instructions for the first 3 questions built into it. Move the slider marked "question" when you are ready to answer the next one. Pause before using the applet to show the transformation described in each question to predict where the new coordinates will be.

Apply each of the following transformations to segment AB. Use the Pen tool to record the coordinates.

1. Rotate segment AB 90 degrees counterclockwise around center B by moving the slider marked 0 degrees. The image of A is named C. What are the coordinates of C?
2. Rotate segment AB 90 degrees counterclockwise around center A by moving the slider marked 0 degrees. The image of B is named D. What are the coordinates of D?
3. Rotate segment AB 90 degrees clockwise around $(0, 0)$ by moving the slider marked 0 degrees. The image of A is named E and the image of B is named F. What are the coordinates of E and F?
4. Compare the two 90-degree counterclockwise rotations of segment AB. What is the same about the images of these rotations? What is different?

Are you ready for more?

Suppose EF and GH are line segments of the same length. Describe a sequence of transformations that moves EF to GH.

Summary

We can use coordinates to describe points and find patterns in the coordinates of transformed points.

We can describe a translation by expressing it as a sequence of horizontal and vertical translations. For example, segment AB is translated right 3 and down 2.

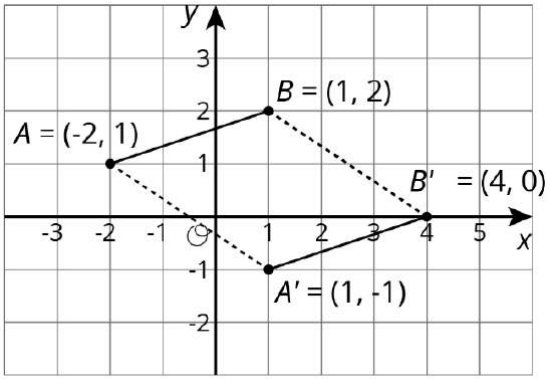


Figure 1.1.5.3: Quadrilateral on a coordinate plane. Horizontal axis scale negative 3 to 5 by 1's. Vertical axis scale negative 2 to 3 by 1's. Quadrilateral A B B prime A prime has coordinates A(negative 2 comma 1), B(1 comma 2), B prime(4 comma 0) and A prime (1 comma negative 1).

Reflecting a point across an axis changes the sign of one coordinate. For example, reflecting the point A whose coordinates are $(2, -1)$ across the x-axis changes the sign of the y-coordinate, making its image the point A' whose coordinates are $(2, 1)$. Reflecting the point A across the y-axis changes the sign of the x-coordinate, making the image the point A" whose coordinates are $(-2, -1)$.

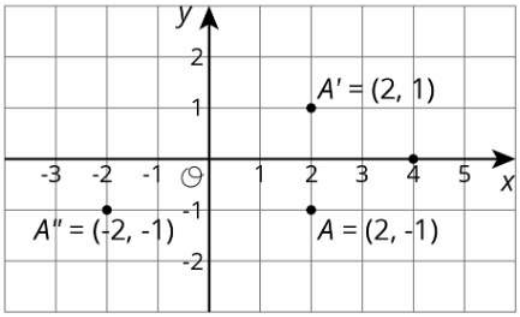


Figure 1.1.5.4: 3 points on a coordinate plane. Horizontal axis scale negative 3 to 5 by 1's. Vertical axis scale negative 2 to 2 by 1's. The points have these coordinates: A(2 comma negative 1), A prime(2 comma 1) and A double prime (negative 2 comma negative 1).

Reflections across other lines are more complex to describe.

We don't have the tools yet to describe rotations in terms of coordinates in general. Here is an example of a $90°$ rotation with center $(0, 0)$ in a counterclockwise direction.

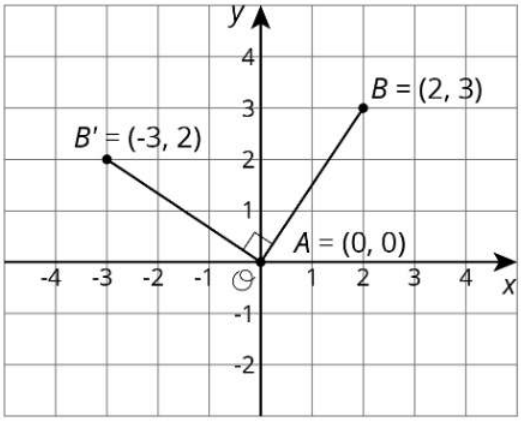


Figure 1.1.5.5: Segment A B rotated on a coordinate plane, origin O. Horizontal axis scale negative 4 to 4 by 1's. Vertical axis scale negative 2 to 4 by 1's. The segments have these coordinates: A(0 comma 0), B(2 comma 3) and B prime (negative 3 comma 2). Angle B A B prime is a right angle.

Point A has coordinates $(0, 0)$. Segment AB was rotated $90°$ counterclockwise around A. Point B with coordinates $(2, 3)$ rotates to point B' whose coordinates are $(-3, 2)$.

Definition: Clockwise

Clockwise means to turn in the same direction as the hands of a clock. The top turns to the right. This diagram shows Figure A turned clockwise to make Figure B.

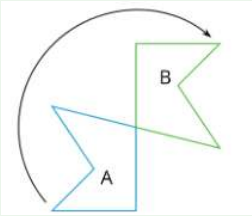


Figure 1.1.5.6

Definition: Coordinate Plane

The coordinate plane is a system for telling where points are. For example. point R is located at $(3, 2)$ on the coordinate plane, because it is three units to the right and two units up.

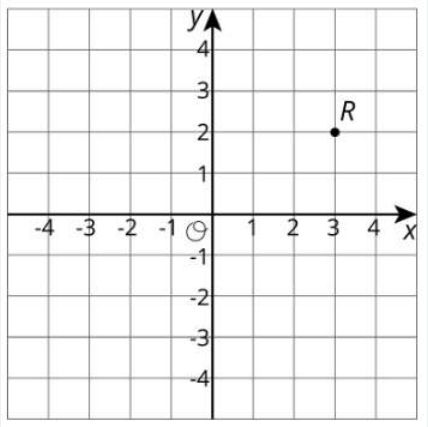


Figure 1.1.5.7

Definition: Counterclockwise

Counterclockwise means to turn opposite of the way the hands of a clock turn. The top turns to the left.

This diagram shows Figure A turned counterclockwise to make Figure B.

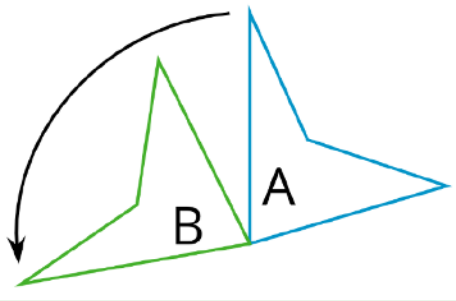


Figure 1.1.5.8

Definition: Image

An image is the result of translations, rotations, and reflections on an object. Every part of the original object moves in the same way to match up with a part of the image.

In this diagram, triangle ABC has been translated up and to the right to make triangle DEF. Triangle DEF is the image of the original triangle ABC.

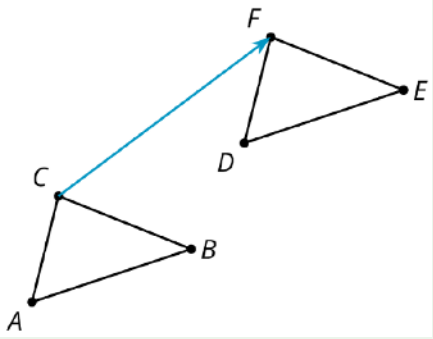


Figure 1.1.5.9

Definition: Reflection

A reflection across a line moves every point on a figure to a point directly on the opposite side of the line. The new point is the same distance from the line as it was in the original figure.

This diagram shows a reflection of A over line l that makes the mirror image B.

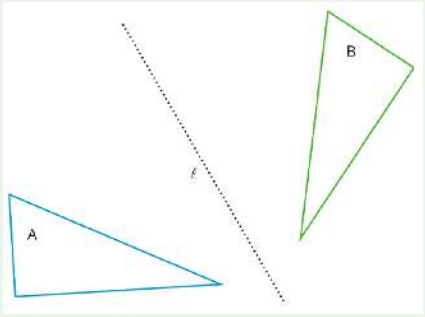


Figure 1.1.5.10

Definition: Rotation

A rotation moves every point on a figure around a center by a given angle in a specific direction.

This diagram shows Triangle A rotated around center O by 55 degrees clockwise to get Triangle B.

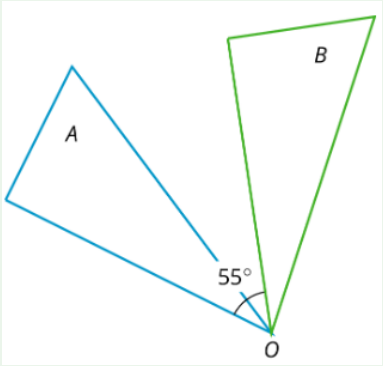


Figure 1.1.5.11

Definition: Sequence of Transformations

A sequence of transformations is a set of translations, rotations, reflections, and dilations on a figure. The transformations are performed in a given order.

This diagram shows a sequence of transformations to move Figure A to Figure C.

First, A is translated to the right to make B. Next, B is reflected across line l to make C.

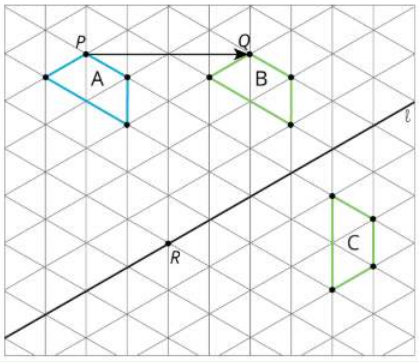


Figure 1.1.5.12

Definition: Transformation

A transformation is a translation, rotation, reflection, or dilation, or a combination of these.

Definition: Translation

A translation moves every point in a figure a given distance in a given direction.

This diagram shows a translation of Figure A to Figure B using the direction and distance given by the arrow.

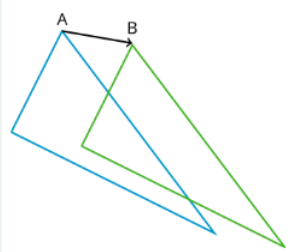


Figure 1.1.5.13

Definition: Vertex

A vertex is a point where two or more edges meet. When we have more than one vertex, we call them vertices.

The vertices in this polygon are labeled A, B, C, D, and E.

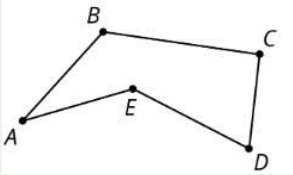


Figure 1.1.5.14

Practice

Exercise 1.1.5.4

1. Here are some points.

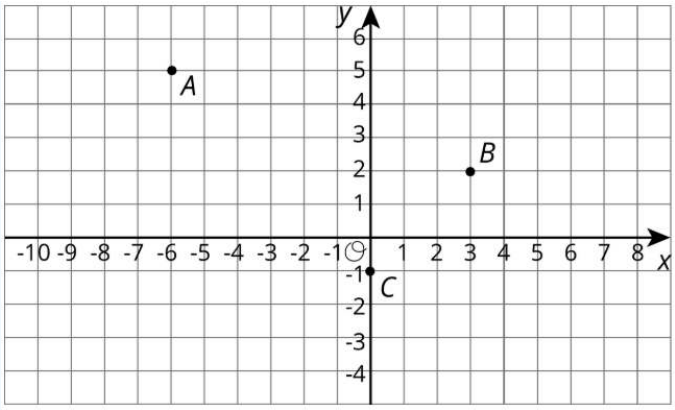


Figure 1.1.5.15: Points, A, B and C on a coordinate plane, origin O. Horizontal axis scale negative 10 to 8 by 1's. Vertical axis scale negative 4 to 6 by 1's. The coordinates of the points are A (negative 6 comma 5), B (3 comma 2) and C (negative 1 comma 0).

What are the coordinates of A, B, and C after a translation to the right by 4 units and up 1 unit? Plot these points on the grid, and label them A', B' and C'.

2. Here are some points.

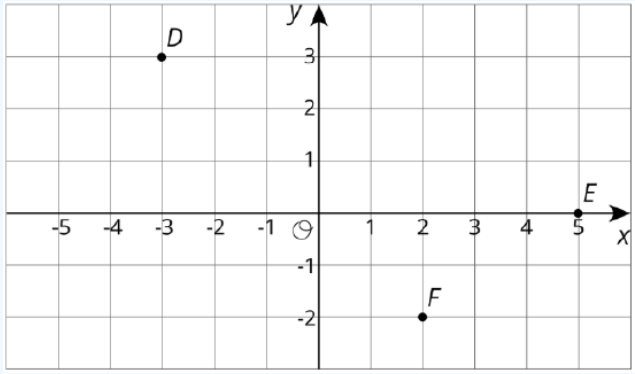


Figure1.1.5.16: Points D, E and F on a coordinate plane, origin O. Horizontal axis scale negative 5 to 5 by 1's. Vertical axis scale negative 2 to 3 by 1's. The coordinates of the points are D (negative 3 comma 3), E(5 comma 0) and F (2 comma negative 2).

What are the coordinates of D, E, and F after a reflection over the axis? Plot these points on the grid, and label them D', E' and F'.

3. Here are some points.

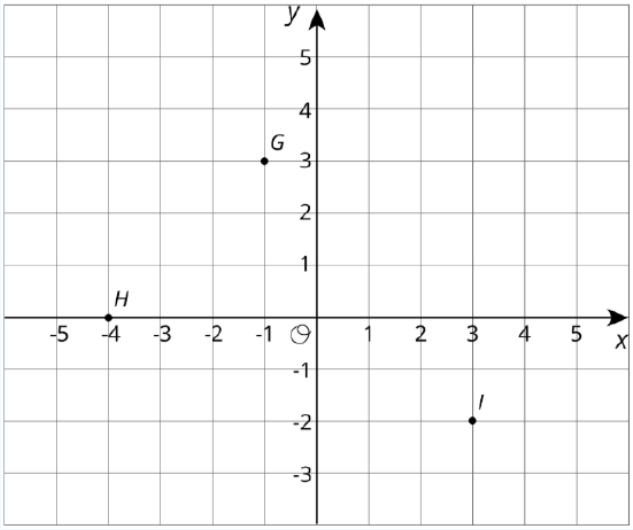


Figure1.1.5.17: Points G, H and I on a coordinate plane, origin O. Horizontal axis scale negative 5 to 5 by 1's. Vertical axis scale negative 3 to 5 by 1's. The coordinates of the points are G(negative 1 comma 3), H(negative 4 comma 0) and I(3 comma negative 2).

What are the coordinates of G, H, and I after a rotation about $(0, 0)$ by 90 degrees clockwise? Plot these points on the grid, and label them G', H' and I'.

Exercise 1.1.5.5

Describe a sequence of transformations that takes trapezoid A to trapezoid B.

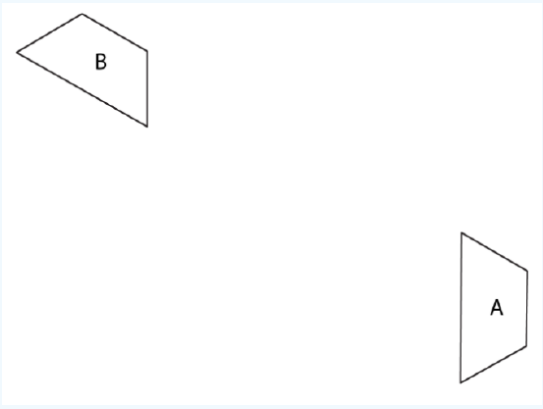


Figure 1.1.5.18: Quadrilateral A and its image quadrilateral B are trapezoids. Quadrilateral A is on the right lower corner, bases are vertical and the smaller base is on the right. Quadrilateral B is in the left upper corner, bases are sloping down to the opposite corner and the smaller base is on top.

(From Unit 1.1.4)

Exercise 1.1.5.6

Reflect polygon P using line l.

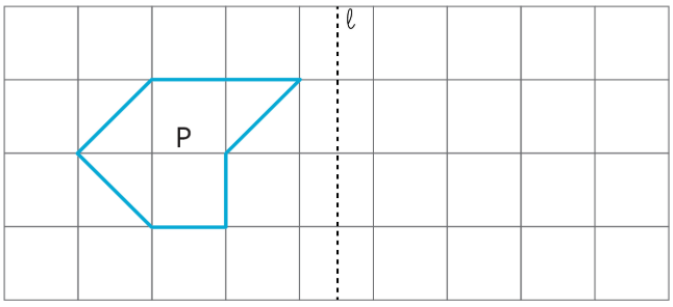


Figure 1.1.5.19: Polygon P and dashed line l on a grid. Let (0 comma 0) be the bottom left corner. Then the coordinates of polygon P are (1 comma 2), (2 comma 3), (4 comma 3), (3 comma 2), (3 comma 1) and (2 comma 1). Line l is the vertical line x = 4 point 5.

(From Unit 1.1.3)

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1.1.6: Describing Transformations

Lesson

Let's transform some polygons in the coordinate plane.

Exercise 1.1.6.1: Finding a Center of Rotation

Andre performs a 90-degree counterclockwise rotation of Polygon P and gets Polygon P', but he does not say what the center of the rotation is. Can you find the center?

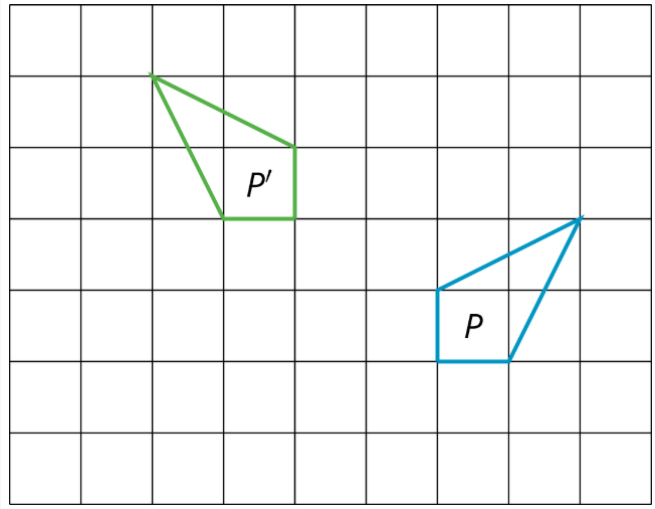


Figure 1.1.6.1

Exercise 1.1.6.2: Info Gap: Transformation Information

Your teacher will give you either a *problem card* or a *data card*. Do not show or read your card to your partner.

If your teacher gives you the *problem card*:

1. Silently read your card and think about what information you need to be able to answer the question.
2. Ask your partner for the specific information that you need.
3. Explain how you are using the information to solve the problem.

   Continue to ask questions until you have enough information to solve the problem.
4. Share the *problem card* and solve the problem independently.
5. Read the *data card* and discuss your reasoning.

If your teacher gives you the *data card*:

1. Silently read your card.
2. Ask your partner "*What specific information do you need?*" and wait for them to *ask* for information.

   If your partner asks for information that is not on the card, do not do the calculations for them. Tell them you don't have that information.
3. Before sharing the information, ask "*Why do you need that information?*" Listen to your partner's reasoning and ask clarifying questions.
4. Read the *problem card* and solve the problem independently.
5. Share the *data card* and discuss your reasoning.

Pause here so your teacher can review your work. Ask your teacher for a new set of cards and repeat the activity, trading roles with your partner.

Are you ready for more?

Sometimes two transformations, one performed after the other, have a nice description as a single transformation. For example, instead of translating 2 units up followed by translating 3 units up, we could simply translate 5 units up. Instead of rotating 20 degrees counterclockwise around the origin followed by rotating 80 degrees clockwise around the origin, we could simply rotate 60 degrees clockwise around the origin.

Can you find a simple description of reflecting across the x-axis followed by reflecting across the y-axis?

Summary

The center of a rotation for a figure doesn't have to be one of the points on the figure. To find a center of rotation, look for a point that is the same distance from two corresponding points. You will probably have to do this for a couple of different pairs of corresponding points to nail it down.

When we perform a sequence of transformations, the order of the transformations can be important. Here is triangle ABC translated up two units and then reflected over the x-axis.

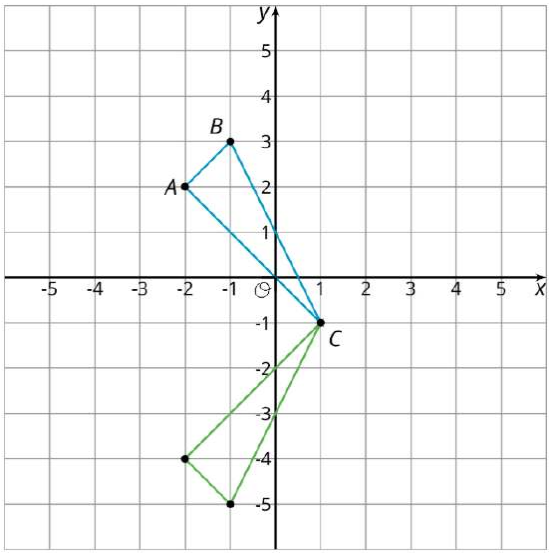


Figure 1.1.6.2: Triangle A B C reflected on a coordinate plane, origin O. Horizontal and vertical axis scale negative 5 to 5 by 1's. Triangle A B C is blue and has coordinates: A(negative 2 comma 2), B(negative 1 comma 3) and C(1 comma negative 1). The green triangle has coordinates: (negative 2 comma negative 4), (negative 1 comma negative 5) and C(1 comma negative 1).

Here is triangle ABC reflected over the x-axis and then translated up two units.

Triangle ABC ends up in two different places when the transformations are applied in the opposite order!

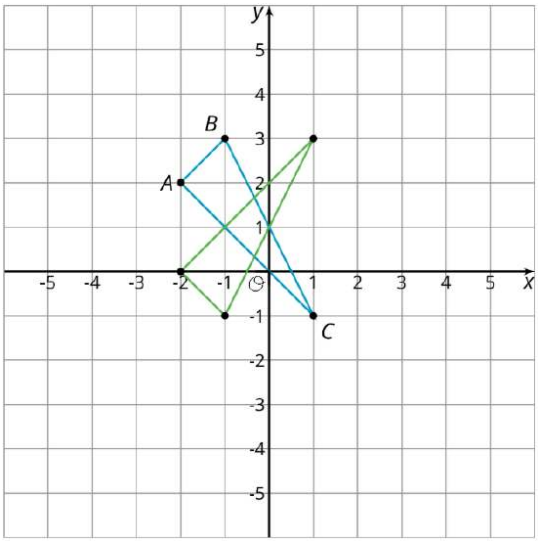


Figure 1.1.6.3: Triangle A B C reflected on a coordinate plane, origin O. Horizontal axis scale negative 5 to 5 by 1's. Vertical axis scale negative 5 to 5 by 1's. Triangle A B C is blue and has coordinates: A(negative 2 comma 2), B(negative 1 comma 3) and C(1 comma negative 1). The green triangle has coordinates: (negative 2 comma 0), (negative 1 comma negative 1) and (1 comma 3).

Glossary Entries

Definition: Clockwise

Clockwise means to turn in the same direction as the hands of a clock. The top turns to the right. This diagram shows Figure A turned clockwise to make Figure B.

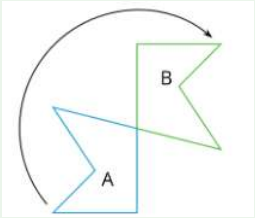


Figure 1.1.6.4

Definition: Coordinate Plane

The coordinate plane is a system for telling where points are. For example. point R is located at $(3, 2)$ on the coordinate plane, because it is three units to the right and two units up.

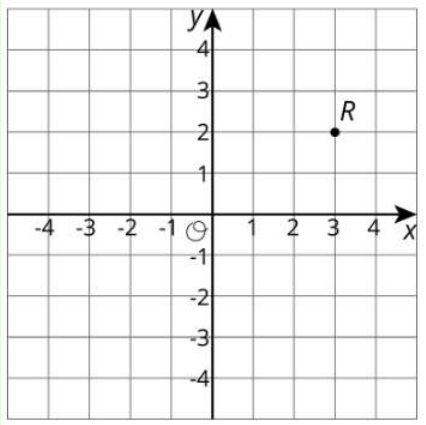


Figure 1.1.6.5

Definition: Counterclockwise

Counterclockwise means to turn opposite of the way the hands of a clock turn. The top turns to the left.

This diagram shows Figure A turned counterclockwise to make Figure B.

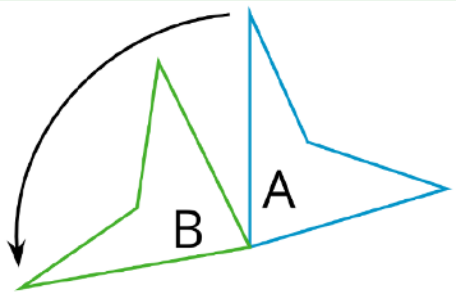


Figure 1.1.6.6

Definition: Image

An image is the result of translations, rotations, and reflections on an object. Every part of the original object moves in the same way to match up with a part of the image.

In this diagram, triangle ABC has been translated up and to the right to make triangle DEF. Triangle DEF is the image of the original triangle ABC.

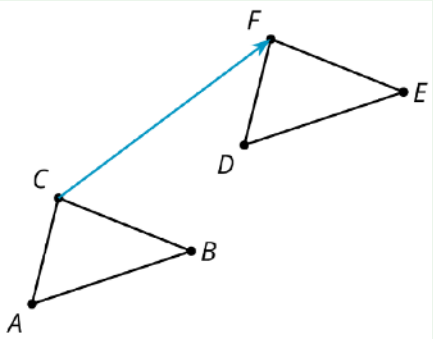


Figure 1.1.6.7

Definition: Reflection

A reflection across a line moves every point on a figure to a point directly on the opposite side of the line. The new point is the same distance from the line as it was in the original figure.

This diagram shows a reflection of A over line l that makes the mirror image B.

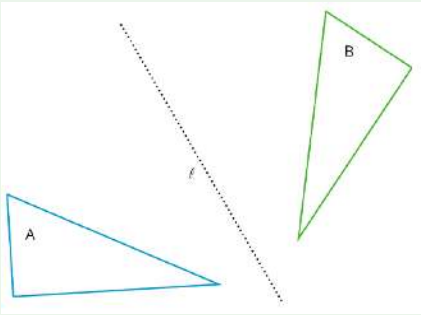


Figure 1.1.6.8

Definition: Rotation

A rotation moves every point on a figure around a center by a given angle in a specific direction.

This diagram shows Triangle A rotated around center O by 55 degrees clockwise to get Triangle B.

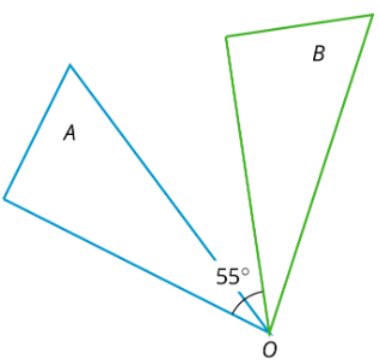


Figure 1.1.6.9

Definition: Sequence of Transformations

A sequence of transformations is a set of translations, rotations, reflections, and dilations on a figure. The transformations are performed in a given order.

This diagram shows a sequence of transformations to move Figure A to Figure C.

First, A is translated to the right to make B. Next, B is reflected across line l to make C.

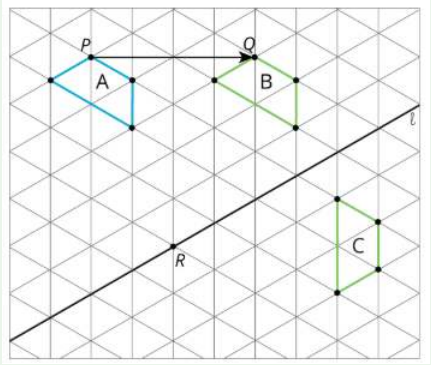


Figure 1.1.6.10

Definition: transformation

A transformation is a translation, rotation, reflection, or dilation, or a combination of these.

Definition: Translation

A translation moves every point in a figure a given distance in a given direction.

This diagram shows a translation of Figure A to Figure B using the direction and distance given by the arrow.

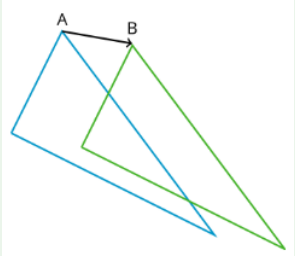


Figure 1.1.6.11

Definition: Vertex

A vertex is a point where two or more edges meet. When we have more than one vertex, we call them vertices.

The vertices in this polygon are labeled A, B, C, D, and E.

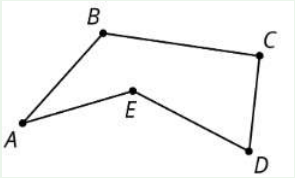


Figure 1.1.6.12

Practice

Exercise 1.1.6.3

Here is Trapezoid A in the coordinate plane:

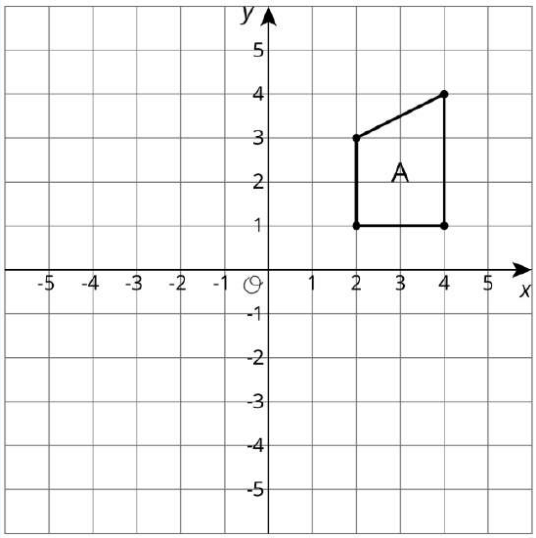


Figure 1.1.6.13

1. Draw Polygon B, the image of A, using the y-axis as the line of reflection.
2. Draw Polygon C, the image of B, using the x-axis as the line of reflection.
3. Draw Polygon D, the image of C, using the x-axis as the line of reflection.

Exercise 1.1.6.4

The point $(-4, 1)$ is rotated 180 degrees counterclockwise using center $(-3, 0)$. What are the coordinates of the image?

A: $(-5, -2)$

B: $(-4, -1)$

C: $(-2, -1)$

D: $(4, -1)$

Exercise 1.1.6.5

Describe a sequence of transformations for which Triangle B is the image of Triangle A.

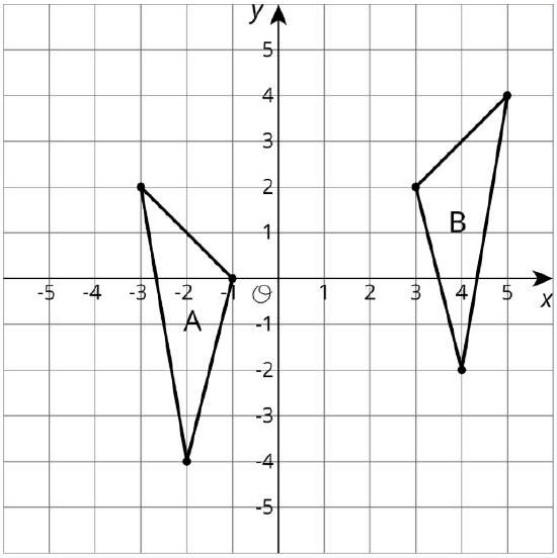


Figure 1.1.6.14: Triangle A and its image triangle B on a coordinate plane, origin O. Horizontal and vertical axis scale negative 5 to 5 by 1's. Triangle A has coordinates (negative 2 comma negative 4), (negative 1 comma 0) and (negative 3 comma 2). Triangle B has coordinates (4 comma negative 2), (3 comma 2) and (5 comma 4).

Exercise 1.1.6.6

Here is a quadrilateral $ABCD$.

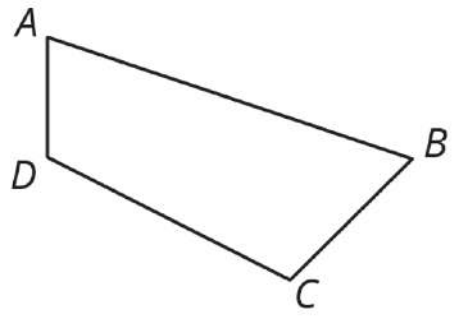


Figure 1.1.6.15

Draw the image of quadrilateral $ABCD$ after each transformation.

1. The translation that takes B to D.
2. The reflection over segment BC.
3. The rotation about point A by angle DAB, counterclockwise.

(From Unit 1.1.2)

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1.2: Properties of Rigid Transformations

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1.2.1: No Bending or Stretching

Lesson

Let's compare measurements before and after translations, rotations, and reflections.

Exercise 1.2.1.1: Measuring Segments

For each question, the unit is represented by the large tick marks with whole numbers.

1. Find the length of this segment to the nearest $\frac{1}{8}$ of a unit.

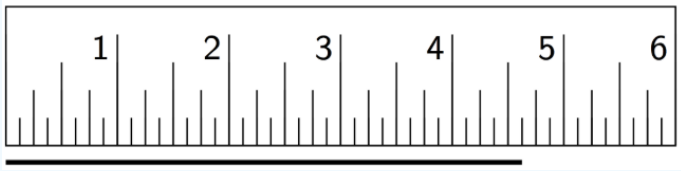


Figure 1.2.1.1: A line segment measured by a ruler above it. The ruler has units represented by large tick marks with whole numbers from 1 to 6. Each unit is evenly divided into 8 parts. The segment measures 4 and the fraction 5 over 8 units.

2. Find the length of this segment to the nearest 0.1 of a unit.

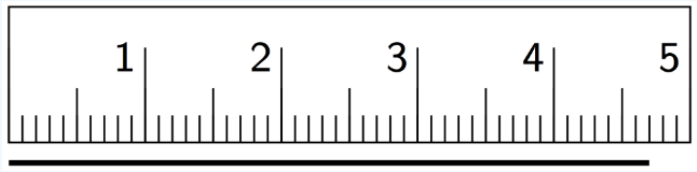


Figure 1.2.1.2: A line segment measured by a ruler above it. The ruler has units represented by large tick marks with whole numbers from 1 to 5. Each unit is evenly divided into 10 parts. The segment measures 4 and the fraction 7 over 10 units.

3. Estimate the length of this segment to the nearest $\frac{1}{8}$ of a unit.

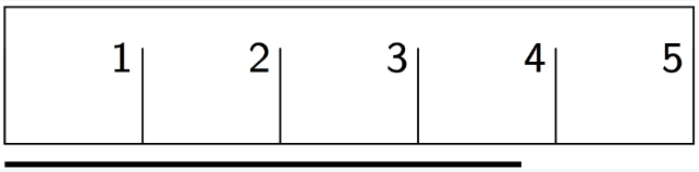


Figure 1.2.1.3: A line segment measured by a ruler above it. The ruler has units represented by large tick marks with whole numbers from 1 to 5. The segment measures near 3 and the fraction 3 over 4 units.

4. Estimate the length of the segment in the prior question to the nearest 0.1 of a unit.

Exercise 1.2.1.2: Sides and Angles

1. Translate Polygon A so point P goes to point P'. In the image, write the length of each side, in grid units, next to the side using the draw tool.
2. Rotate Triangle B 90 degrees clockwise using R as the center of rotation. In the image, write the measure of each angle in its interior using the draw tool.
3. Reflect Pentagon C across line l.
   a. In the image, write the length of each side, in grid units, next to the side.
   b. In the image, write the measure of each angle in the interior.

Exercise 1.2.1.3: Which One?

Here is a grid showing triangle ABC and two other triangles.

You can use a **rigid transformation** to take triangle ABC to *one* of the other triangles.

1. Which one? Explain how you know.

2. Describe a rigid transformation that takes ABC to the triangle you selected.

Are you ready for more?

A square is made up of an L-shaped region and three transformations of the region. If the perimeter of the square is 40 units, what is the perimeter of each L-shaped region?

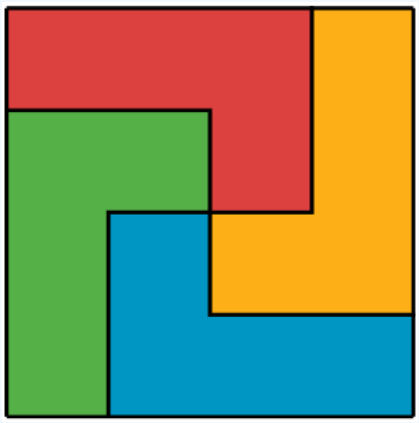


Figure 1.2.1.4

Summary

The transformations we've learned about so far, translations, rotations, reflections, and sequences of these motions, are all examples of **rigid transformations**. A rigid transformation is a move that doesn't change measurements on any figure.

Earlier, we learned that a figure and its image have corresponding points. With a rigid transformation, figures like polygons also have **corresponding** sides and corresponding angles. These corresponding parts have the same measurements.

For example, triangle EFD was made by reflecting triangle ABC across a horizontal line, then translating. Corresponding sides have the same lengths, and corresponding angles have the same measures.

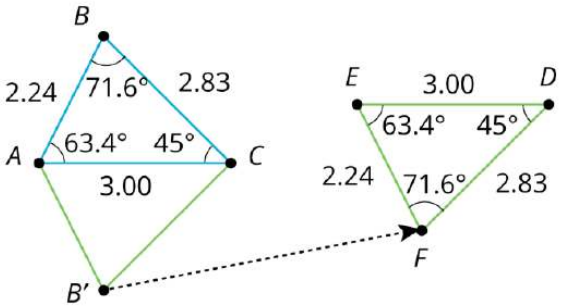


Figure 1.2.1.5

Table 1.2.1.1

measurements in triangle ABC	corresponding measurements in image EFD
$AB = 2.24$	$EF = 2.24$
$BC = 2.83$	$FD = 2.83$
$CA = 3.00$	$DE = 3.00$
$m\angle ABC = 71.6^\circ$	$m\angle EFD = 71.6^\circ$
$m\angle BCA = 45.0^\circ$	$m\angle FDE = 45.0^\circ$
$m\angle CAB = 63.4^\circ$	$m\angle DEF = 63.4^\circ$

Glossary Entries

Definition: Corresponding

When part of an original figure matches up with part of a copy, we call them corresponding parts. These could be points, segments, angles, or distances.

For example, point B in the first triangle corresponds to point E in the second triangle. Segment AC corresponds to segment DF.

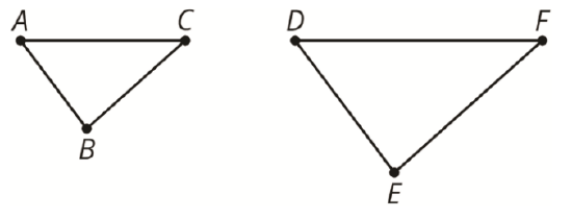


Figure 1.2.1.6

Definition: Rigid Transformation

A rigid transformation is a move that does not change any measurements of a figure. Translations, rotations, and reflections are rigid transformations, as is any sequence of these.

Practice

Exercise 1.2.1.4

Is there a rigid transformation taking Rhombus P to Rhombus Q? Explain how you know.

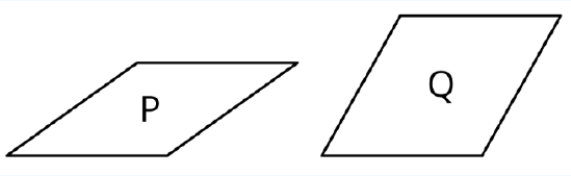


Figure 1.2.1.7

Exercise 1.2.1.5

Describe a rigid transformation that takes Triangle A to Triangle B.

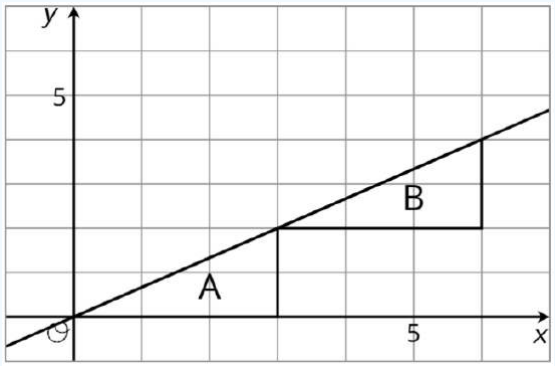


Figure 1.2.1.8: Triangle A and its image triangle B on a coordinate plane, origin O. Horizontal and vertical axis scale negative 5 to 5 by 1's. Triangle A has coordinates (0 comma 0), (3 comma 2) and (3 comma 0). Triangle B has coordinates (3 comma 2), (6 comma 4) and (6 comma 2).

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Exercise 1.2.1.6

Is there a rigid transformation taking Rectangle A to Rectangle B? Explain how you know.

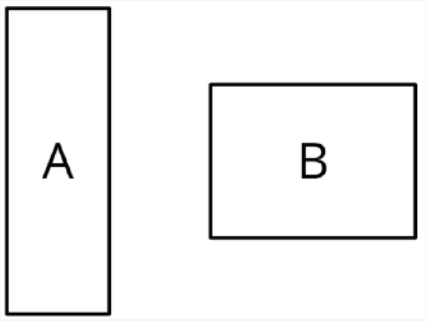


Figure 1.2.1.9:

Exercise 1.2.1.7

For each shape, draw its image after performing the transformation. If you get stuck, consider using tracing paper.

1. Translate the shape so that A goes to A'.

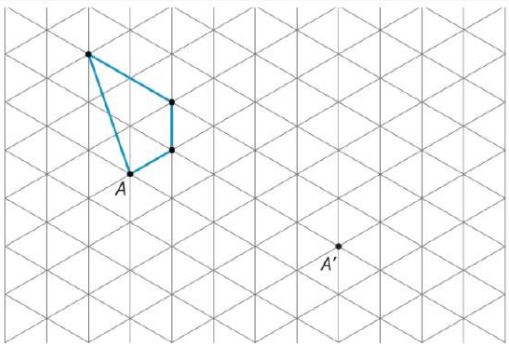


Figure 1.2.1.10

2. Rotate the shape 180 degrees counterclockwise around B.

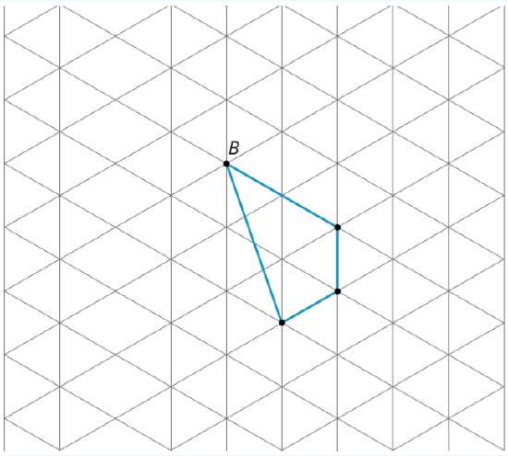


Figure 1.2.1.11

3. Reflect the shape over the line shown.

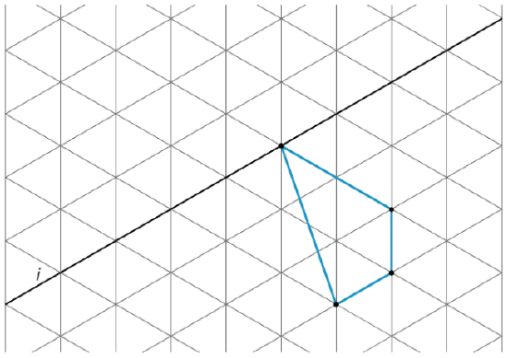


Figure 1.2.1.12

(From Unit 1.1.4)

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1.2.2: Rotation Patterns

Lesson

Let's rotate figures in a plane.

Exercise 1.2.2.1: Building a Quadrilateral

Here is a right isosceles triangle.

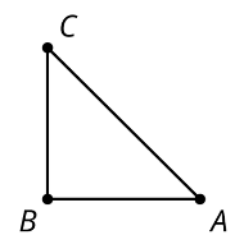


Figure 1.2.2.1

1. Rotate triangle ABC 90 degrees clockwise around B.
2. Rotate triangle ABC 180 degrees clockwise around B.
3. Rotate triangle ABC 270 degrees clockwise around B.
4. What would it look like when you rotate the four triangles 90 degrees clockwise around B? 180 degrees? 270 degrees clockwise?

Exercise 1.2.2.2: Rotating a Segment

Rotating a Segment

Create a segment AB and a point C that is not on segment AB.

1. Rotate segment $AB 180^\circ$ around point B.
2. Rotate segment $AB 180^\circ$ around point C.



Construct the midpoint of segment AB with the Midpoint tool.

1. Rotate segment $AB 180^\circ$ around its midpoint. What is the image of A?
2. What happens when you rotate a segment 180°

Are you ready for more?

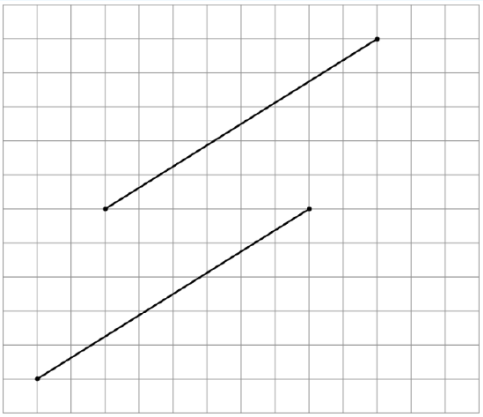


Figure 1.2.2.2

Here are two line segments. Is it possible to rotate one line segment to the other? If so, find the center of such a rotation. If not, explain why not.

Exercise 1.2.2.3: A Pattern of Four Triangles

Here is a diagram built with three different rigid transformations of triangle ABC.

Use the applet to answer the questions. It may be helpful to reset the image after each question.

1. Describe a rigid transformation that takes triangle ABC to triangle CDE.
2. Describe a rigid transformation that takes triangle ABC to triangle EFG.
3. Describe a rigid transformation that takes triangle ABC to triangle GHA.
4. Do segments AC, CE, EG, and GA all have the same length? Explain your reasoning.

Summary

When we apply a 180-degree rotation to a line segment, there are several possible outcomes:

- The segment maps to itself (if the center of rotation is the midpoint of the segment).
- The image of the segment overlaps with the segment and lies on the same line (if the center of rotation is a point on the segment).
- The image of the segment does not overlap with the segment (if the center of rotation is *not* on the segment).

We can also build patterns by rotating a shape. For example, triangle ABC shown here has $m(\angle A) = 60$. If we rotate triangle ABC 60 degrees, 120 degrees, 180 degrees, 240 degrees, and 300 degrees clockwise, we can build a hexagon.

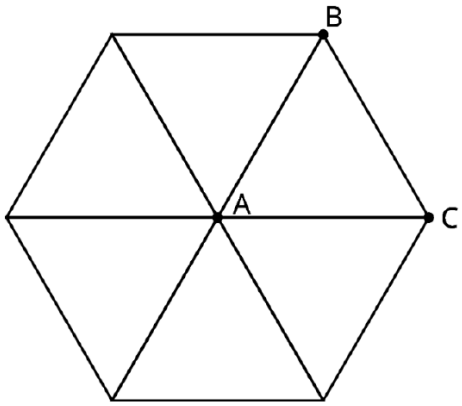


Figure 1.2.2.3: Six identical equilateral triangles are drawn such that each triangle is aligned to another triangle created a hexagon. One of the triangle is labeled A B C and all 6 triangles meet at the common point of A.

Glossary Entries

Definition: Corresponding

When part of an original figure matches up with part of a copy, we call them corresponding parts. These could be points, segments, angles, or distances.

For example, point B in the first triangle corresponds to point E in the second triangle. Segment AC corresponds to segment DF.

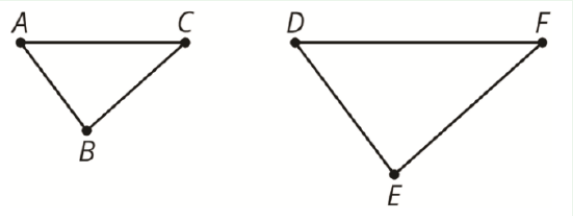


Figure 1.2.2.4

Definition: Rigid Transformation

A rigid transformation is a move that does not change any measurements of a figure. Translations, rotations, and reflections are rigid transformations, as is any sequence of these.

Practice

Exercise 1.2.2.4

For the figure shown here,

1. Rotate segment CD 180° around point D.
2. Rotate segment CD 180° around point E.
3. Rotate segment CD 180° around point M.

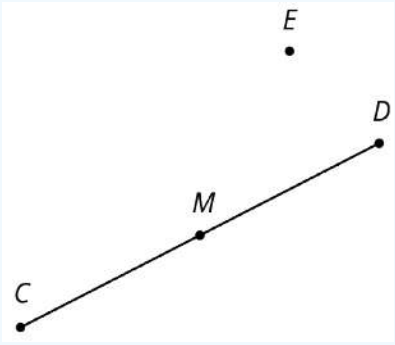


Figure 1.2.2.5

Exercise 1.2.2.5

Here is an isosceles right triangle:

Draw these three rotations of triangle ABC together.

1. Rotate triangle ABC 90 degrees clockwise around A.
2. Rotate triangle ABC 180 degrees around A.
3. Rotate triangle ABC 270 degrees clockwise around A.

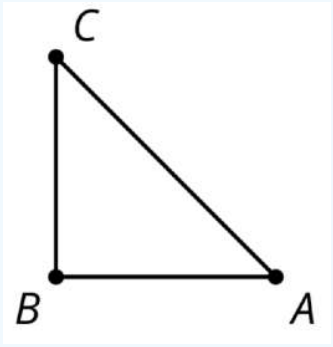


Figure 1.2.2.6

Exercise 1.2.2.6

Each graph shows two polygons $ABCD$ and $A'B'C'D'$. In each case, describe a sequence of transformations that takes $ABCD$ to $A'B'C'D'$.

1.

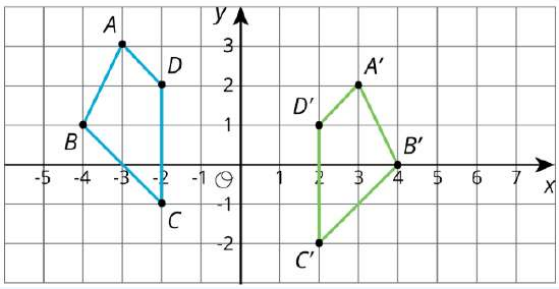


Figure 1.2.2.7: Quadrilateral $ABCD$ and its image quadrilateral $A'B'C'D'$ on a coordinate plane, origin O. Horizontal axis scale negative 5 to 5 by 1's. Vertical axis scale negative 2 to 3 by 1's. The coordinates of $ABCD$ are A(negative 3 comma 3), B(negative 4 comma 1), C(negative 2 comma negative 1)and D(negative 2 comma 2). The coordinates of $A'B'C'D'$ are A'(3 comma 2), B'(4 comma 0), C'(2 comma negative 2)and D'(2 comma 1).

2.

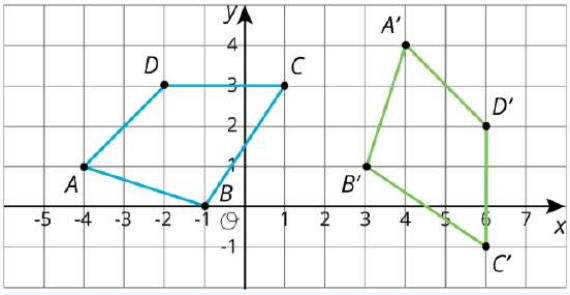


Figure 1.2.2.8: Quadrilateral $ABCD$ and its image quadrilateral $A'B'C'$ and D' on a coordinate plane, origin O. Horizontal axis scale negative 5 to 7 by 1's. Vertical axis scale negative 1 to 4 by 1's. The coordinates of $ABCD$ are A(negative 4 comma 1), B(negative 1 comma 0), C(1 comma 3)and D(negative 2 comma 3). The coordinates of $A'B'C'D'$ are A'(4 comma 4), B'(3 comma 1), C'(6 comma negative 1)and D'(6 comma 2).

(From Unit 1.1.5)

Exercise 1.2.2.7

Lin says that she can map Polygon A to Polygon B using *only* reflections. Do you agree with Lin? Explain your reasoning.

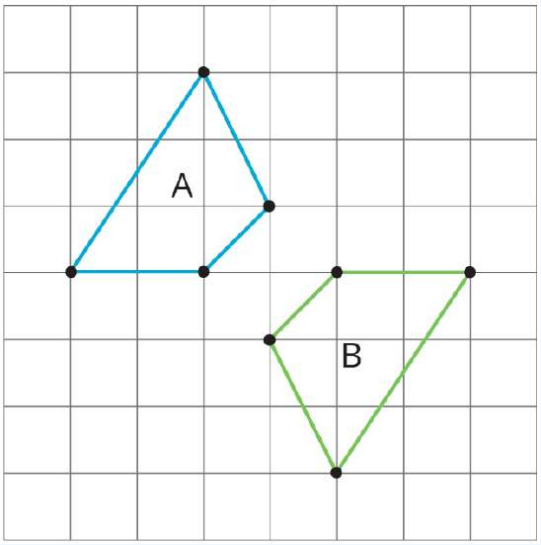


Figure 1.2.2.9: Two quadrilaterals polygon A and B on a grid. Let (0 comma 0) be the bottom left corner. Then the coordinates of polygon A are: (1 comma 4), (3 comma 7), (4 comma 5) and (3 comma 4). The coordinates of polygon B are: (5 comma 1), (7 comma 4), (5 comma 4) and (4 comma 3).

(From Unit 1.1.4)

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1.2.3: Moves in Parallel

Lesson

Let's transform some lines.

Exercise 1.2.3.1: Line Moves

For each diagram, describe a translation, rotation, or reflection that takes line l to line l'. Then plot and label A' and B', the images of A and B.

1.

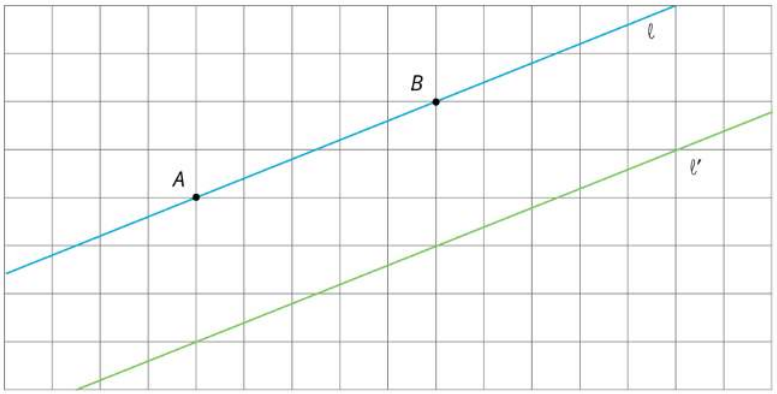


Figure 1.2.3.1

2.

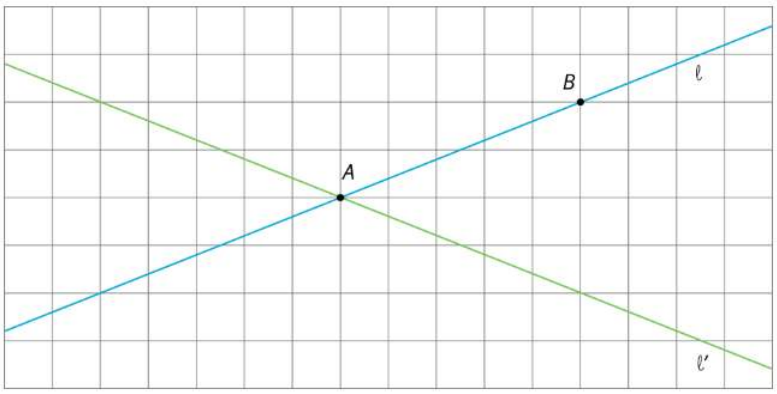


Figure 1.2.3.2

Exercise 1.2.3.2: Parallel Lines

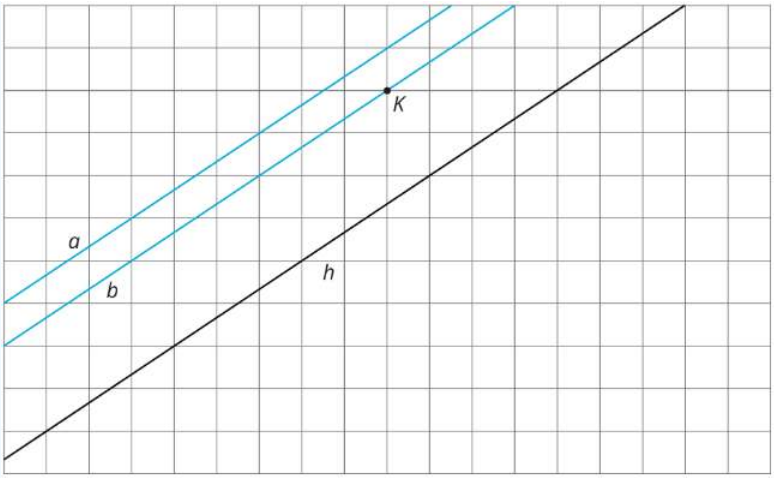


Figure 1.2.3.3

Use a piece of tracing paper to trace lines a and b and point K. Then use that tracing paper to draw the images of the lines under the three different transformations listed.

As you perform each transformation, think about the question:

What is the image of two parallel lines under a rigid transformation?

1. Translate lines a and b 3 units up and 2 units to the right.
    1. What do you notice about the changes that occur to lines a and b after the translation?
    2. What is the same in the original and the image?
2. Rotate lines a and b counterclockwise 180 degrees using K as the center of rotation.
    1. What do you notice about the changes that occur to lines a and b after the rotation?
    2. What is the same in the original and the image?
3. Reflect lines a and b across line h.
    1. What do you notice about the changes that occur to lines a and b after the reflection?
    2. What is the same in the original and the image?

Are you ready for more?

When you rotate two parallel lines, sometimes the two original lines intersect their images and form a quadrilateral. What is the most specific thing you can say about this quadrilateral? Can it be a square? A rhombus? A rectangle that isn't a square? Explain your reasoning.

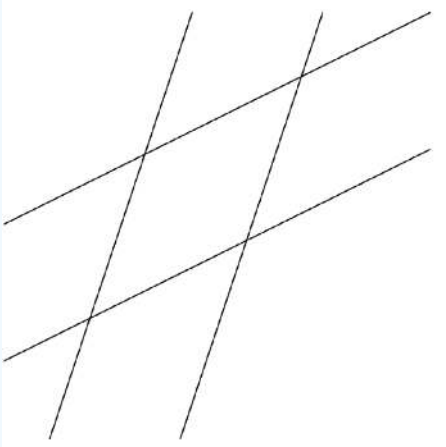


Figure 1.2.3.4

Exercise 1.2.3.3: Let's Do Some 180's

1. The diagram shows a line with points labeled A, C, D, and B.
    a. On the diagram, draw the image of the line and points A, C, and B after the line has been rotated 180 degrees around point D.
    b. Label the images of the points A', B', and C'.
    c. What is the order of all seven points? Explain or show your reasoning.

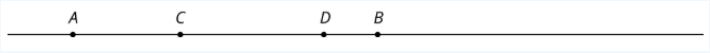


Figure 1.2.3.5

2. The diagram shows a line with points A and C on the line and a segment AD where D is not on the line.
    a. Rotate the figure 180 degrees about point C. Label the image of A as A' and the image of D as D'.
    b. What do you know about the relationship between angle CAD and angle $C'A'D'$? Explain or show your reasoning.

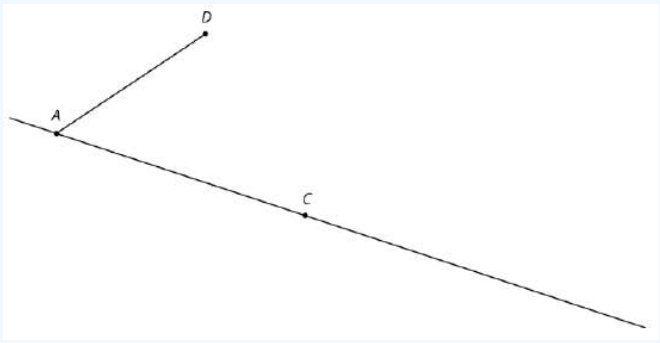


Figure 1.2.3.6

3. The diagram shows two lines l and m that intersect at a point O with point A on l and point D on m.
    a. Rotate the figure 180 degrees around O. Label the image of A as A' and the image of D as D'.
    b. What do you know about the relationship between the angles in the figure? Explain or show your reasoning.

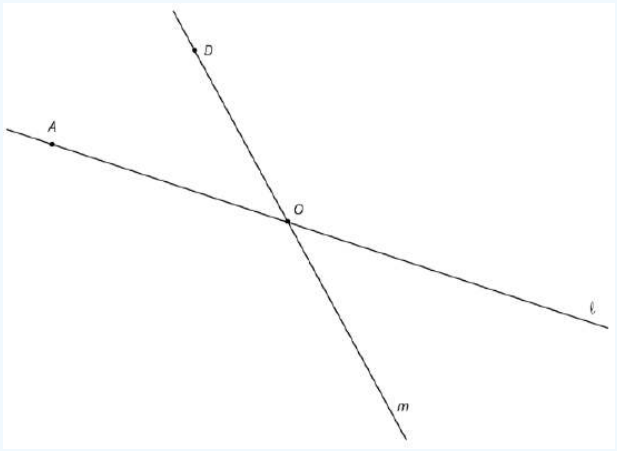


Figure 1.2.3.7

Summary

Rigid transformations have the following properties:

- A rigid transformation of a line is a line.
- A rigid transformation of two parallel lines results in two parallel lines that are the same distance apart as the original two lines.
- Sometimes, a rigid transformation takes a line to itself. For example:

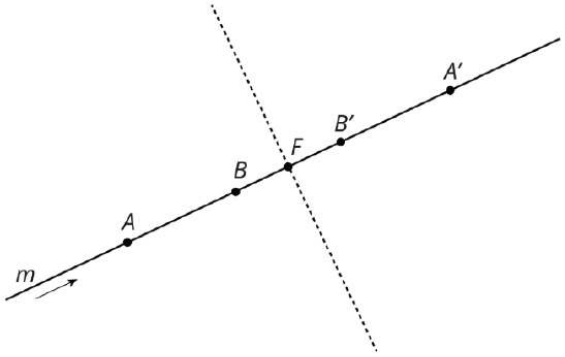


Figure 1.2.3.8

- A translation parallel to the line. The arrow shows a translation of line m that will take m to itself.
- A rotation by 180° around any point on the line. A 180° rotation of line m around point F will take m to itself.
- A reflection across any line perpendicular to the line. A reflection of line m across the dashed line will take m to itself.

These facts let us make an important conclusion. If two lines intersect at a point, which we'll call O, then a 180° rotation of the lines with center O shows that **vertical angles** are congruent. Here is an example:

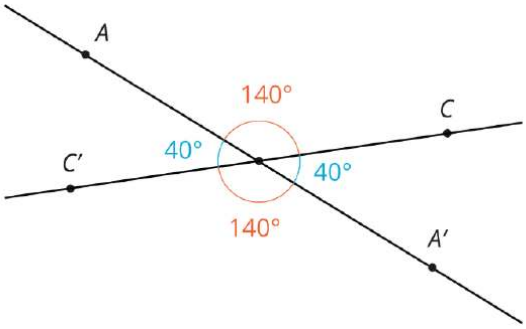


Figure 1.2.3.9

Rotating both lines by 180° around O sends angle AOC to angle $A'OC'$, providing that they have the same measure. The rotation also sends angle AOC' to angle $A'OC$.

Glossary Entries

Definition: Corresponding

When part of an original figure matches up with part of a copy, we call them corresponding parts. These could be points, segments, angles, or distances.

For example, point B in the first triangle corresponds to point E in the second triangle. Segment AC corresponds to segment DF.

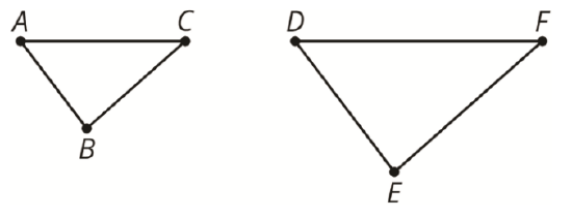


Figure 1.2.3.10

Definition: Rigid Transformation

A rigid transformation is a move that does not change any measurements of a figure. Translations, rotations, and reflections are rigid transformations, as is any sequence of these.

Definition: Vertical Angles

Vertical angles are opposite angles that share the same vertex. They are formed by a pair of intersecting lines. Their angle measures are equal.

For example, angles AEC and DEB are vertical angles. If angle AEC measures $120°$, then angle DEB must also measure $120°$.

Angles AED and BEC are another pair of vertical angles.

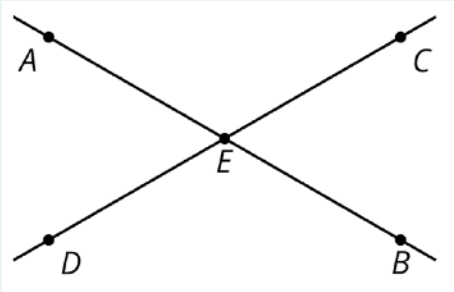


Figure 1.2.3.11

Practice

Exercise 1.2.3.4

1. Draw parallel lines AB and CD.
2. Pick any point E. Rotate AB 90 degrees clockwise around E.
3. Rotate line CD 90 degrees clockwise around E.
4. What do you notice?

Exercise 1.2.3.5

Use the diagram to find the measures of each angle. Explain your reasoning.

1. $m\angle ABC$
2. $m\angle EBD$
3. $m\angle ABE$

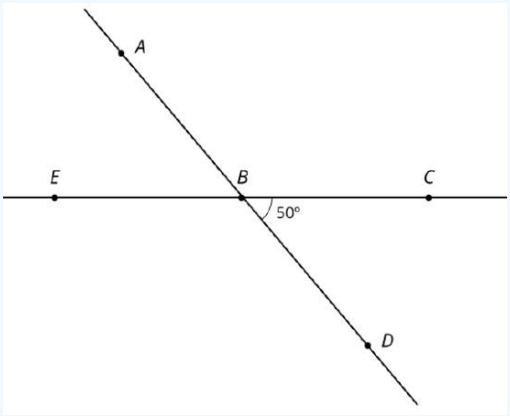


Figure 1.2.3.12

Exercise 1.2.3.6

Points P and Q are plotted on a line.

1. Find a point R so that a 180-degree rotation with center R sends P to Q and Q to P.
2. Is there more than one point R that works for part a?

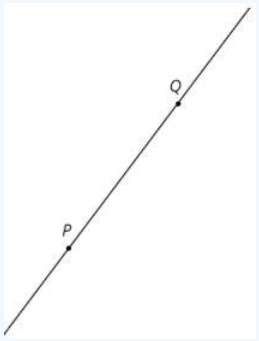


Figure 1.2.3.13

Exercise 1.2.3.7

In the picture triangle $A'B'C'$ is an image of triangle ABC after a rotation. The center of rotation is D.

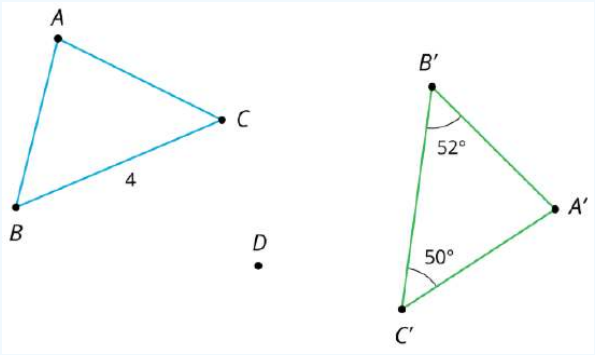


Figure 1.2.3.14

1. What is the length of side $B'C'$? Explain how you know.
2. What is the measure of angle B? Explain how you know.
3. What is the measure of angle C? Explain how you know.

(From Unit 1.2.1)

Exercise 1.2.3.8

The point $(-4, 1)$ is rotated 180 degrees counterclockwise using center $(0, 0)$. What are the coordinates of the image?

A: $(-1, -4)$

B: $(-1, 4)$

C: $(4, 1)$

D: $(4, -1)$

(From Unit 1.1.6)

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1.2.4: Composing Figures

Lesson

Let's use reasoning about rigid transformations to find measurements without measuring.

Exercise 1.2.4.1: Angles of an Isosceles Triangle

Here is a triangle.

1. Reflect triangle ABC over line AB. Label the image of C as C'.
2. Rotate triangle ABC' around A so that C' matches up with B.
3. What can you say about the measures of angles B and C?

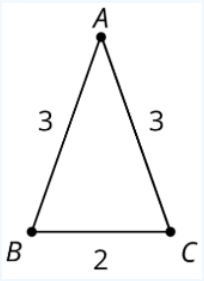


Figure 1.2.4.1

Exercise 1.2.4.2: Triangle Plus One

Here is triangle ABC.

1. Draw midpoint M of side AC.
2. Rotate triangle ABC 180 degrees using center M to form triangle CDA. Draw and label this triangle.
3. What kind of quadrilateral is $ABCD$? Explain how you know.

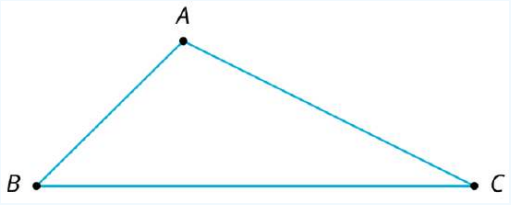


Figure 1.2.4.2

Are you ready for more?

In the activity, we made a parallelogram by taking a triangle and its image under a 180-degree rotation around the midpoint of a side. This picture helps you justify a well-known formula for the area of a triangle. What is the formula and how does the figure help justify it?

Exercise 1.2.4.3: Triangle Plus Two

The picture shows 3 triangles. Triangle 2 and Triangle 3 are images of Triangle 1 under rigid transformations.

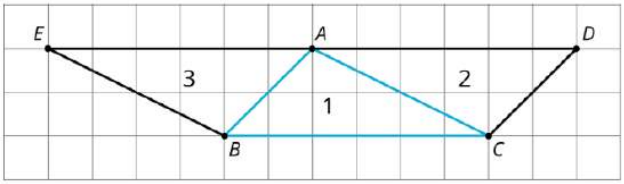


Figure 1.2.4.3

1. Describe a rigid transformation that takes Triangle 1 to Triangle 2. What points in Triangle 2 correspond to points A, B, and C in the original triangle?
2. Describe a rigid transformation that takes Triangle 1 to Triangle 3. What points in Triangle 3 correspond to points A, B, and C in the original triangle?
3. Find two pairs of line segments in the diagram that are the same length, and explain how you know they are the same length.
4. Find two pairs of angles in the diagram that have the same measure, and explain how you know they have the same measure.

Exercise 1.2.4.4: Triangle ONE Plus

Here is isosceles triangle ONE. Its sides ON and OE have equal lengths. Angle O is 30 degrees. The length of ON is 5 units.

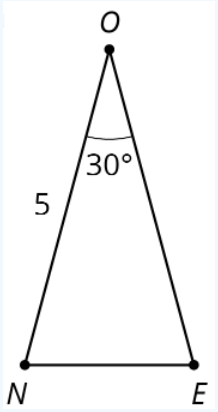


Figure 1.2.4.4

1. Reflect triangle ONE across segment ON. Label the new vertex M.
2. What is the measure of angle MON?
3. What is the measure of angle MOE?
4. Reflect triangle MON across segment OM. Label the point that corresponds to N as T.
5. How long is $\overline{OT}$? How do you know?
6. What is the measure of angle TOE?
7. If you continue to reflect each new triangle this way to make a pattern, what will the pattern look like?

Summary

Earlier, we learned that if we apply a sequence of rigid transformations to a figure, then corresponding sides have equal length and corresponding angles have equal measure. These facts let us figure out things without having to measure them!

For example, here is triangle ABC.

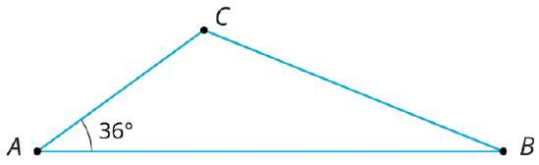


Figure 1.2.4.5

We can reflect triangle ABC across side AC to form a new triangle:

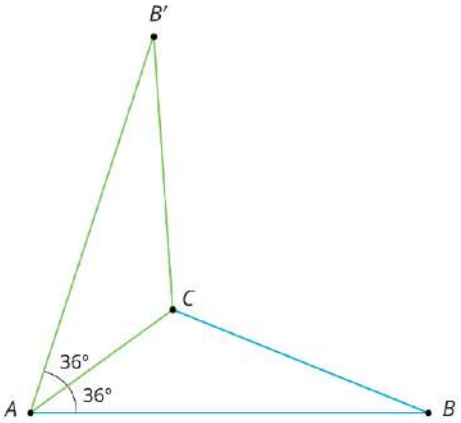


Figure 1.2.4.6

Because points A and C are on the line of reflection, they do not move. So the image of triangle ABC is $AB'C$. We also know that:

- Angle $B'AC$ measures 36° because it is the image of angle BAC.
- Segment AB' has the same length as segment AB.

When we construct figures using copies of a figure made with rigid transformations, we know that the measures of the images of segments and angles will be equal to the measures of the original segments and angles.

Glossary Entries

Definition: Corresponding

When part of an original figure matches up with part of a copy, we call them corresponding parts. These could be points, segments, angles, or distances.

For example, point B in the first triangle corresponds to point E in the second triangle. Segment AC corresponds to segment DF.

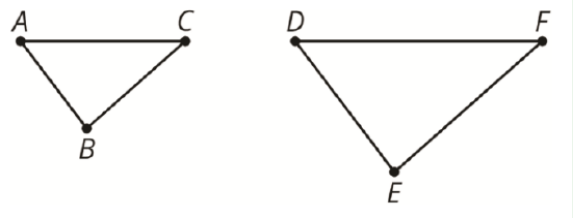


Figure 1.2.4.7

Definition: Rigid Transformation

A rigid transformation is a move that does not change any measurements of a figure. Translations, rotations, and reflections are rigid transformations, as is any sequence of these.

Definition: Vertical Angles

Vertical angles are opposite angles that share the same vertex. They are formed by a pair of intersecting lines. Their angle measures are equal.

For example, angles AEC and DEB are vertical angles. If angle AEC measures 120°, then angle DEB must also measure 120°.

Angles AED and BEC are another pair of vertical angles.

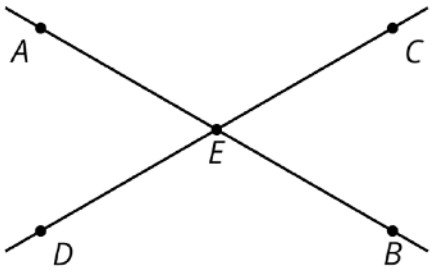


Figure 1.2.4.8

Practice

Exercise 1.2.4.5

Here is the design for the flag of Trinidad and Tobago.



Figure 1.2.4.9: "Flag of Trinidad and Tobago", by Denniss. Public Domain. Wikimedia Commons. Source.

Describe a sequence of translations, rotations, and reflections that take the lower left triangle to the upper right triangle.

Exercise 1.2.4.6

Here is a picture of an older version of the flag of Great Britain. There is a rigid transformation that takes Triangle 1 to Triangle 2, another that takes Triangle 1 to Triangle 3, and another that takes Triangle 1 to Triangle 4.



Figure 1.2.4.10: An image of an older version of the flag of Great Britain. The flag is a rectangle with a vertical length about twice the width. Red stripes divide the flag in half vertically and horizontally. White stripes connect the vertices along diagonals, crossing behind the red stripes. The remaining area is composed of 8 blue right triangles. At the top of the flag, 2 large right triangles line up on either side of the vertical red stripe by their shorter square sides, so that they are mirror images of each other. At the bottom of the flag, 2 large right triangles line up on either side of the vertical red stripe by their shorter square sides, so that they are mirror images of each other. At the left side, 2 small right triangles line up on either side of the horizontal red stripe by their longer square sides so that they are mirror images of each other. The triangle above the red stripe is labeled 1; the triangle below the red strip is labeled 3. At the right side, 2 small right triangles line up on either side of the horizontal red stripe by their longer square sides so that they are mirror images of each other. The triangle above the red stripe is labeled 2; the triangle below the red strip is labeled 4. “Union flag 1606 (Kings Colors)”, by Hoshie. Public Domain. Wikimedia Commons. Source.

1. Measure the lengths of the sides in Triangles 1 and 2. What do you notice?
2. What are the side lengths of Triangle 3? Explain how you know.

3. Do all eight triangles in the flag have the same area? Explain how you know.

Exercise 1.2.4.7

1. Which of the lines in the picture is parallel to line l? Explain how you know.

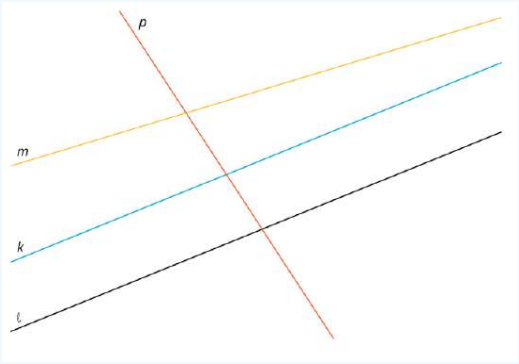


Figure 1.2.4.11: Three lines, m, k and l, cut by a transversal, p. Lines k and l will not intersect no matter how far they extend. Line m appears to be angled towards lines k and l on the right so that it would intersect them at a point not shown.

2. Explain how to translate, rotate or reflect line l to obtain line k.

3. Explain how to translate, rotate or reflect line l to obtain line p.

(From Unit 1.2.3)

Exercise 1.2.4.8

Point A has coordinates $(3, 4)$. After a translation 4 units left, a reflection across the x-axis, and a translation 2 units down, what are the coordinates of the image?

(From Unit 1.1.6)

Exercise 1.2.4.9

Here is a triangle XYZ:

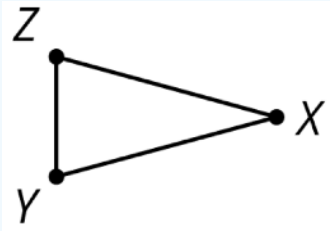


Figure 1.2.4.12

Draw these three rotations of triangle XYZ together.

1. Rotate triangle XYZ 90 degrees clockwise around Z.
2. Rotate triangle XYZ 180 degrees around Z.
3. Rotate triangle XYZ 270 degrees clockwise around Z.

(From Unit 1.2.2)

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1.3: Congruence

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1.3.1: What Is the Same?

Lesson

Let's decide whether shapes are the same.

Exercise 1.3.1.1: Find the Right Hands

A person’s hands are mirror images of each other. In the diagram, a left hand is labeled. Shade all of the right hands.

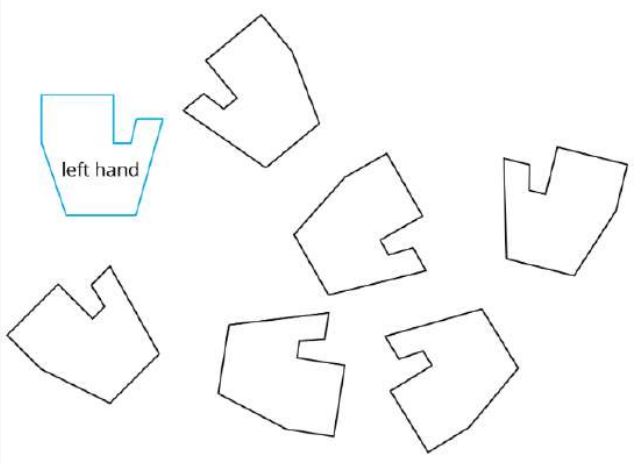


Figure 1.3.1.1

Exercise 1.3.1.2: Are They the Same?

For each pair of shapes, decide whether or not they are the same.

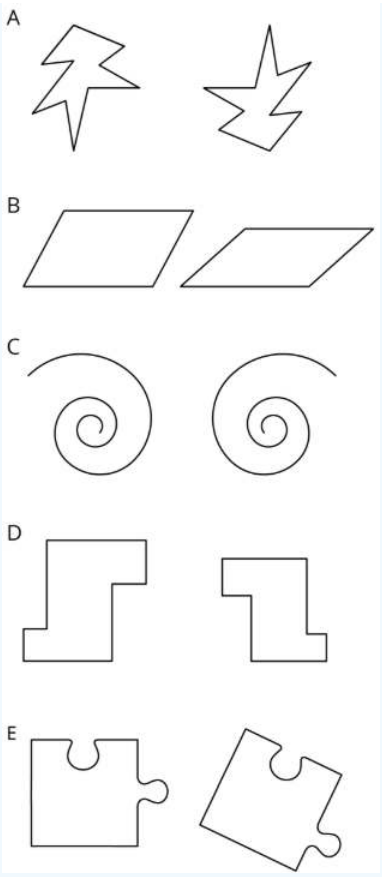


Figure E shapes do not match, the puzzle shape on the right has a lower right side part tab than the same tab of the left puzzle shape.

Exercise 1.3.1.3: Area, Perimeter, and Congruence

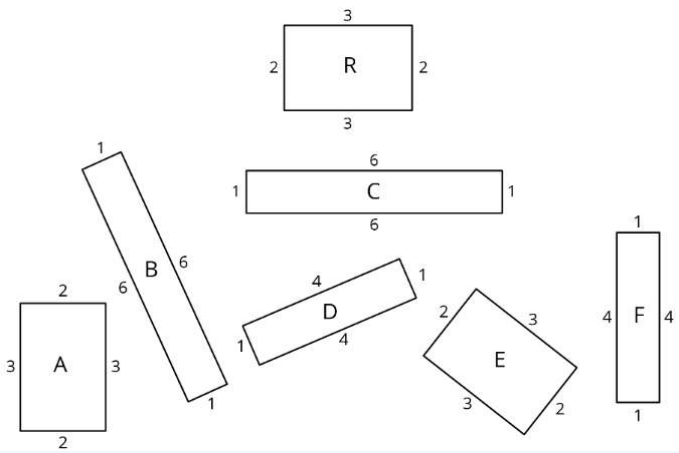


Figure 1.3.1.3

1. Which of these rectangles have the same area as Rectangle R but different perimeter?
2. Which rectangles have the same perimeter but different area?
3. Which have the same area *and* the same perimeter?
4. Use materials from the geometry tool kit to decide which rectangles are **congruent**. Shade congruent rectangles with the same color.

Are you ready for more?

In square $ABCD$, points E, F, G, and H are midpoints of their respective sides. What fraction of square $ABCD$ is shaded? Explain your reasoning.

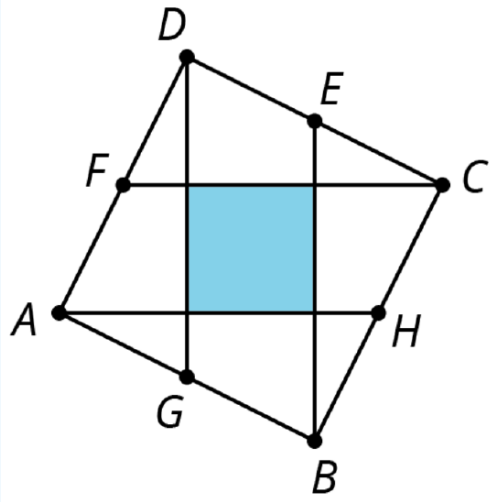


Figure 1.3.1.4

Summary

Congruent is a new term for an idea we have already been using. We say that two figures are congruent if one can be lined up exactly with the other by a sequence of rigid transformations. For example, triangle EFD is congruent to triangle ABC because they can be matched up by reflecting triangle ABC across followed by the translation shown by the arrow. Notice that all corresponding angles and side lengths are equal.

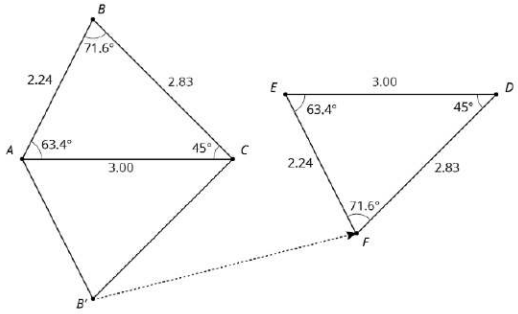


Figure 1.3.1.5: Triangle A B C, triangle A B prime C, dashed arrow from B prime to point F and triangle E F D. Triangle A B prime C is the image of triangle A B C after reflection over A C. Angle A is 63 point 4 degrees, angle B is 71 point 6 degrees and angle C is 45 degrees. Side A B is 2 point 2 4, side B C is 2 point 8 3 and side A C is 3 point 0 0. Angle E is 63 point 4 degrees, angle F is 71 point 6 degrees and angle D is 45 degrees. Side E F is 2 point 2 4, side F D is 2 point 8 3 and side D E is 3 point 0 0.

Here are some other facts about congruent figures:

- We don't need to check all the measurements to prove two figures are congruent; we just have to find a sequence of rigid transformations that match up the figures.
- A figure that looks like a mirror image of another figure can be congruent to it. This means there must be a reflection in the sequence of transformations that matches up the figures.
- Since two congruent polygons have the same area and the same perimeter, one way to show that two polygons are *not* congruent is to show that they have a different perimeter or area.

Glossary Entries

Definition: Congruent

One figure is congruent to another if it can be moved with translations, rotations, and reflections to fit exactly over the other.

In the figure, Triangle A is congruent to Triangles B, C, and D. A translation takes Triangle A to Triangle B, a rotation takes Triangle B to Triangle C, and a reflection takes Triangle C to Triangle D.

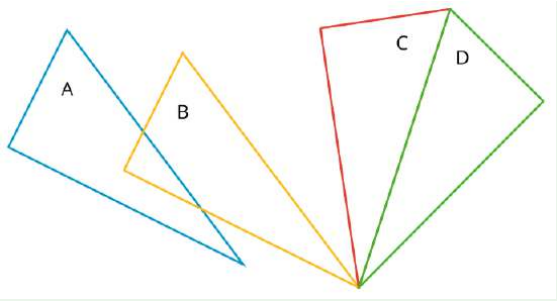


Figure 1.3.1.6

Practice

Exercise 1.3.1.4

If two rectangles have the same perimeter, do they have to be congruent? Explain how you know.

Exercise 1.3.1.5

Draw two rectangles that have the same area, but are *not* congruent.

Exercise 1.3.1.6

For each pair of shapes, decide whether or not the two shapes are congruent. Explain your reasoning.

1.

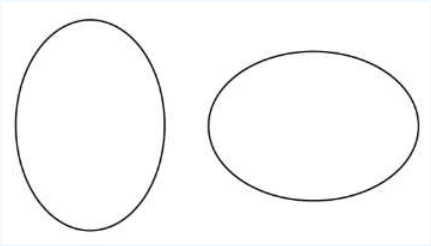


Figure 1.3.1.7

2.

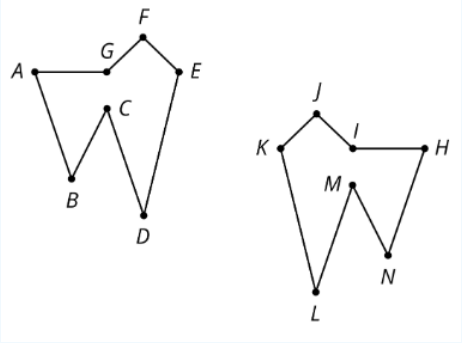


Figure 1.3.1.8

Exercise 1.3.1.7

1. Reflect Quadrilateral A over the x-axis. Label the image quadrilateral B. Reflect Quadrilateral B over the y-axis. Label the image C.

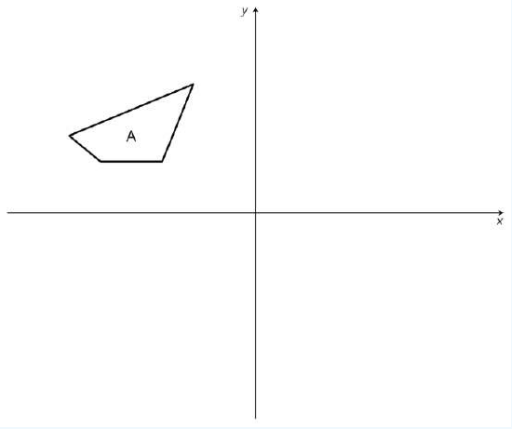


Figure 1.3.1.9

2. Are Quadrilaterals A and C congruent? Explain how you know.

Exercise 1.3.1.8

The point $(-2, -3)$ is rotated 90 degrees counterclockwise using center $(0, 0)$. What are the coordinates of the image?
A: $(-3, -2)$

B: $(-3, 2)$

C: $(3, -2)$

D: $(3, 2)$

(From Unit 1.1.6)

Exercise 1.3.1.9

Describe a rigid transformation that takes Polygon A to Polygon B.

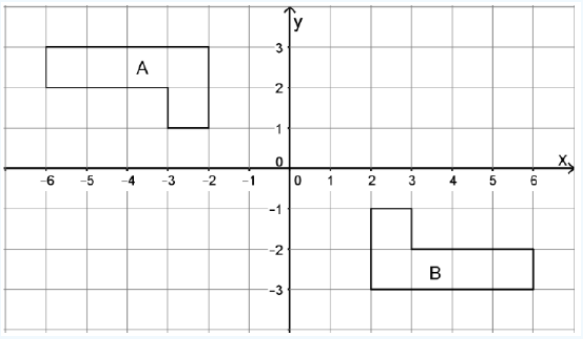


Figure 1.3.1.10: Polygon A and its image polygon B on a coordinate plane, origin O. Horizontal axis scale negative 6 to 6 by 1's. Vertical axis scale negative 3 to 3 by 1's. Polygon A has coordinates (negative 6 comma 2), (negative 6 comma 3), (negative 2 comma 3), (negative 2 comma 1), (negative 3 comma 1)and (negative 3 comma 2). Triangle B has coordinates (6 comma negative 2), (6 comma negative 3), (2 comma negative 3), (2 comma negative 1), (3 comma negative 1) and (3 comma negative 2).

(From Unit 1.2.1)

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1.3.2: Congruent Polygons

Lesson

Let's decide if two figures are congruent.

Exercise 1.3.2.1: Translated Images

All of these triangles are congruent. Sometimes we can take one figure to another with a translation. Shade the triangles that are images of triangle ABC under a translation.

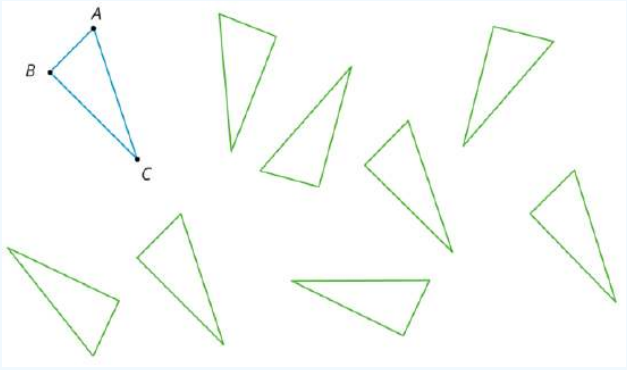


Figure 1.3.2.1

Exercise 1.3.2.2: Congruent Pairs (Part 1)

For each of the following pairs of shapes, decide whether or not they are congruent. Explain your reasoning.

1.

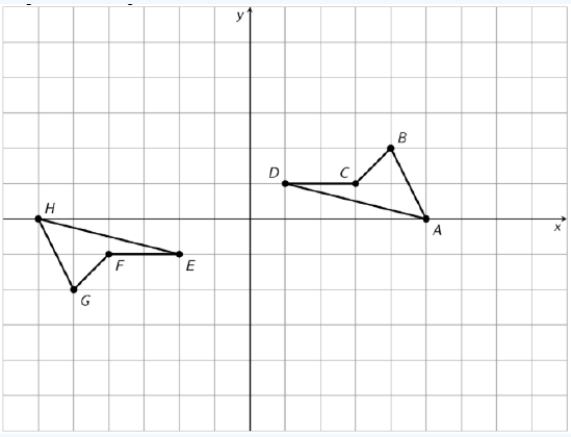


Figure H G F E is the image of figure A B C D after rotation of 180 degrees, followed by a translation left 3 and down 2 units. A B C D has the coordinates A(5 comma 0), B(4 comma 2), C(3 comma 1) and D(1 comma 1). H G F E has the coordinates H(negative 6 comma 0), G(negative 5 comma negative 2), F(negative 4 comma negative 1 and E(negative 2 comma negative 1).

2.

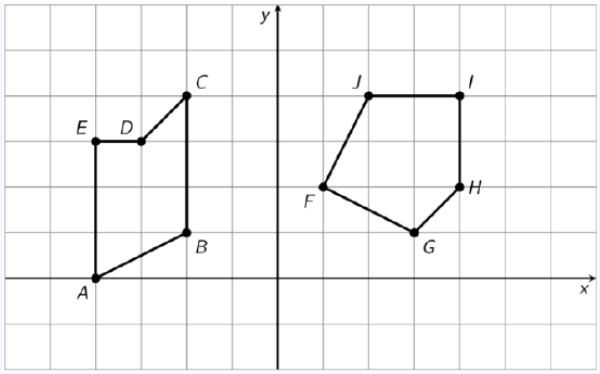


Figure 1.3.2.3: Two pentagons on a coordinate plane. Pentagon ABCDE is non-convex and has coordinates A(negative 4 comma 0), B(negative 2 comma 1), C(negative two comma 4), D(negative 3 comma 3) and E(negative 4 comma 3). Pentagon FGHIJ is convex and has coordinates F(1 comma 2), G(3 comma 1), H(4 comma 2), I(4 comma 4) and J(2 comma 4).

3.

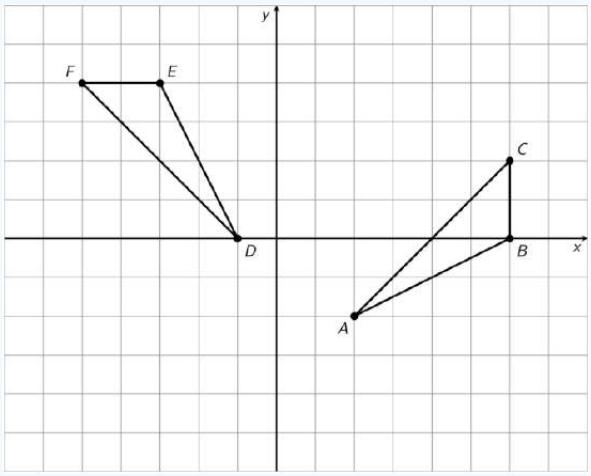


Figure 1.3.2.4: Two triangles D E F and A B C on a coordinate plane. Triangle D E F is the image of triangles A B C after rotation of 90 degrees, followed by a translation left 3 and down 2 units. Triangle A B C has the coordinates A(2 comma negative 2), B(6 comma 0) and C(6 comma 2). Triangle D E F has the coordinates D(negative 1 comma 0), E(negative 3 comma 4) and F(negative 5 comma 4).

4.

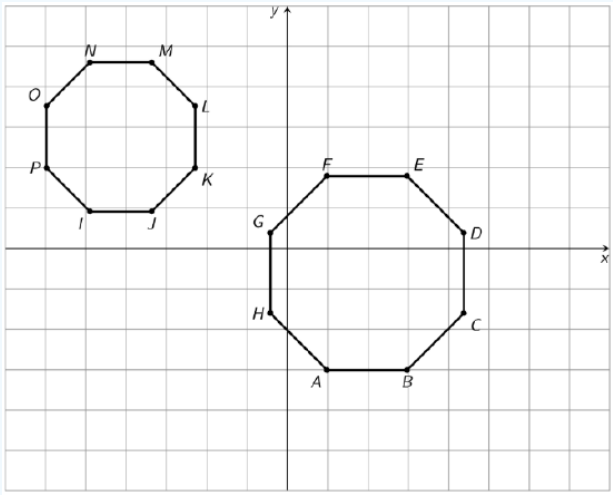


Figure 1.3.2.5

Exercise 1.3.2.3: Congruent Pairs (Part 2)

For each pair of shapes, decide whether or not Shape A is congruent to Shape B. Explain how you know.

1.

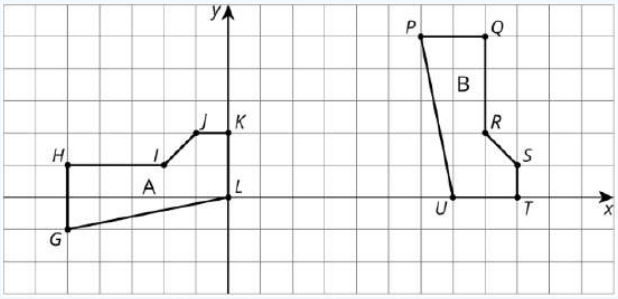


Figure B has points P(6 comma 5), Q(8 comma 5), R(8 comma 2), S(9 comma 1), T(9 comma 0) and U(7 comma 0).

2.

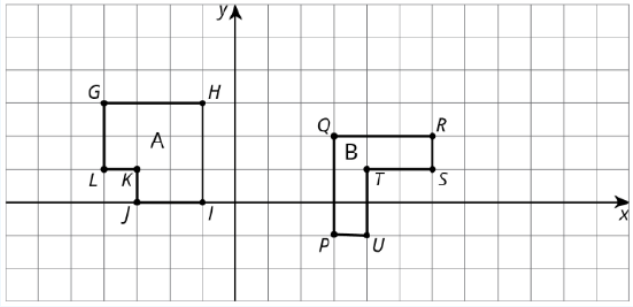


Figure B has points P(3 comma negative 1), Q(3 comma 2), R(6 comma 2), S(6 comma 1), T(4 comma 1) and U(4 comma negative 1).

3.

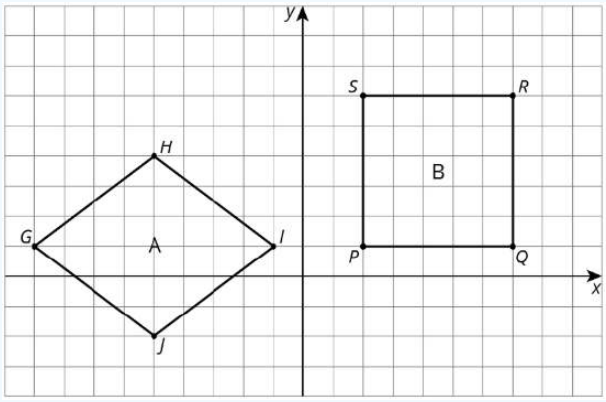


Figure B is a square with side lengths 5 units and has points P(2 comma 1), Q(7 comma 1), R(7 comma 6) and S(2 comma 6).

4.

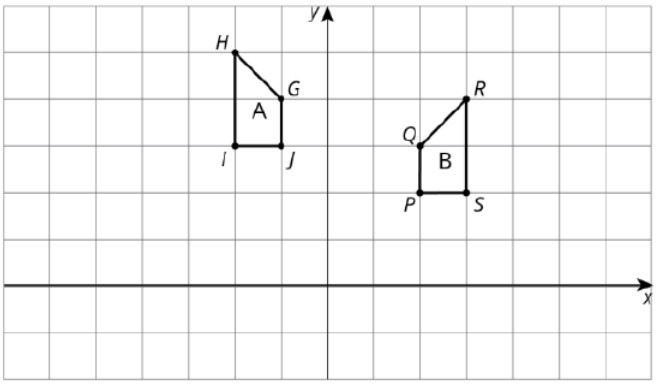


Figure B has points P(3 comma negative 1), Q(3 comma 2), R(6 comma 2) and S(6 comma 1).

5.

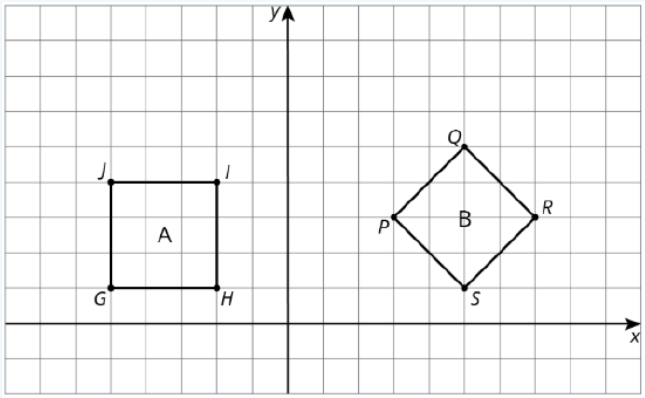


Figure B is a rhombus with side lengths near 2 point 8 units and has points P(3 comma 3), Q(5 comma 5), R(7 comma 3) and S(5 comma 1).

Are you ready for more?

A polygon has 8 sides: five of length 1, two of length 2, and one of length 3. All sides lie on grid lines. (It may be helpful to use graph paper when working on this problem.)

1. Find a polygon with these properties.
2. Is there a second polygon, not congruent to your first, with these properties?

Exercise 1.3.2.4: Building Quadrilaterals

Your teacher will give you a set of four objects.

1. Make a quadrilateral with your four objects and record what you have made.
2. Compare your quadrilateral with your partner's. Are they congruent? Explain how you know.
3. Repeat steps 1 and 2, forming different quadrilaterals. If your first quadrilaterals were not congruent, can you build a pair that is? If your first quadrilaterals were congruent, can you build a pair that is not? Explain.

Summary

How do we know if two figures are congruent?

- If we copy one figure on tracing paper and move the paper so the copy covers the other figure exactly, then that suggests they are congruent.
- We can prove that two figures are congruent by describing a sequence of translations, rotations, and reflections that move one figure onto the other so they match up exactly.

How do we know that two figures are *not* congruent?

- If there is no correspondence between the figures where the parts have equal measure, that proves that the two figures are *not* congruent. In particular,
  - If two polygons have different sets of side lengths, they can't be congruent. For example, the figure on the left has side lengths 3, 2, 1, 1, 2, 1. The figure on the right has side lengths 3, 3, 1, 2, 2, 1. There is no way to make a correspondence between them where all corresponding sides have the same length.

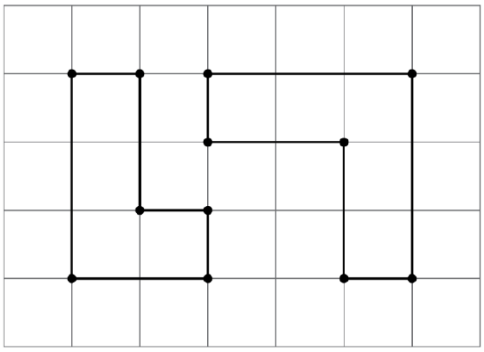


Figure 1.3.2.11

  - If two polygons have the same side lengths, but their orders can't be matched as you go around each polygon, the polygons can't be congruent. For example, rectangle $ABCD$ can't be congruent to quadrilateral $EFGH$. Even though they both have two sides of length 3 and two sides of length 5, they don't correspond in the same order. In $ABCD$, the order is 3, 5, 3, 5 or 5, 3, 5, 3; in $EFGH$, the order is 3, 3, 5, 5 or 3, 5, 5, 3 or 5, 5, 3, 3.

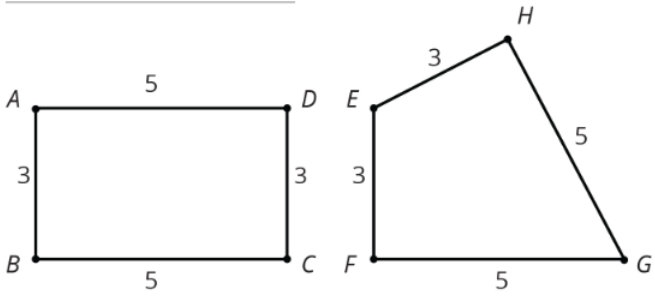


Figure 1.3.2.12: Two figures, A B C D and E F G H. Figure A B C D is a rectangle with side length 3 and base and top length 5. Figure E F G H has base length 5, top length is 3, left side length is 3 and right side length is 5.

  - If two polygons have the same side lengths, in the same order, but different corresponding angles, the polygons can't be congruent. For example, parallelogram $JKLM$ can't be congruent to rectangle $ABCD$. Even though they have the same side lengths in the same order, the angles are different. All angles in $ABCD$ are **right angles**. In $JKLM$, angles J and L are less than 90 degrees and angles K and M are more than 90 degrees.

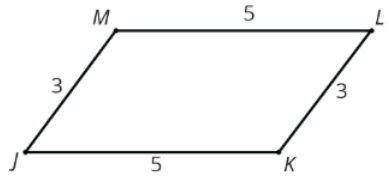


Figure 1.3.2.13

Glossary Entries

Definition: Congruent

One figure is congruent to another if it can be moved with translations, rotations, and reflections to fit exactly over the other.

In the figure, Triangle A is congruent to Triangles B, C, and D. A translation takes Triangle A to Triangle B, a rotation takes Triangle B to Triangle C, and a reflection takes Triangle C to Triangle D.

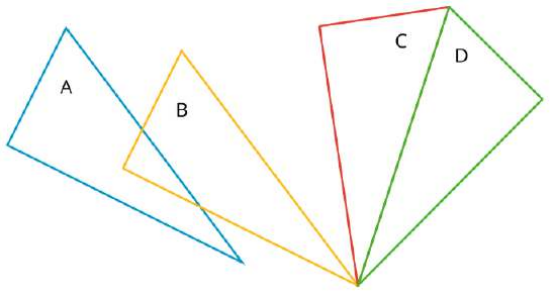


Figure 1.3.2.14

Definition: Right Angle

A right angle is half of a straight angle. It measures 90 degrees.

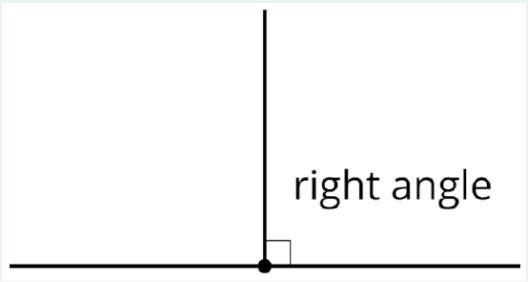


Figure 1.3.2.15

Practice

Exercise 1.3.2.5

1. Show that the two pentagons are congruent.
2. Find the side lengths of $ABCDE$ and the angle measures of $FGHIJ$.

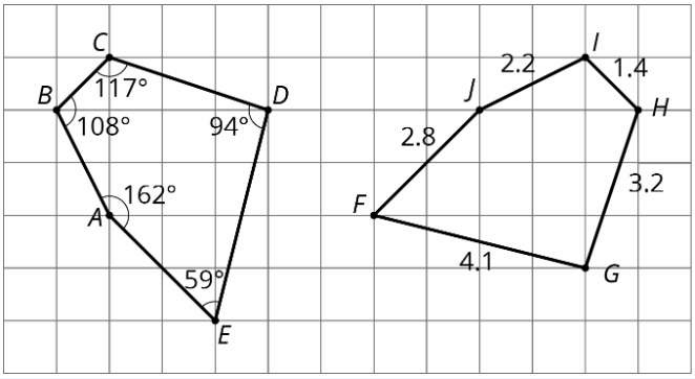


Figure J I H G F appears to be the image of figure A B C D E after a rotation and a translation.

Exercise 1.3.2.6

For each pair of shapes, decide whether or not the two shapes are congruent. Explain your reasoning.

1.

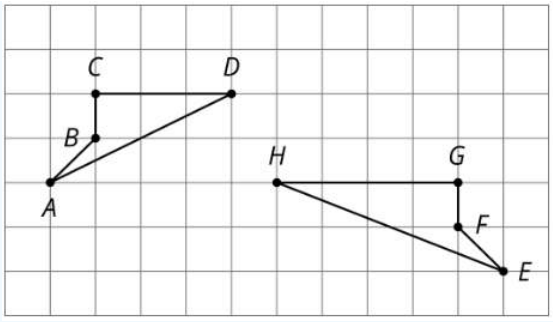


Figure 1.3.2.17: Two quadrilaterals A B C D and E F G H on a square grid. Let the lower left corner be (0 comma 0). Then the coordinates of A B C D are A(1 comma 3), B(2 comma 4), C(2 comma 5)and D(5 comma 5). The coordinates of E F G H are E(11 comma 1), F(10 comma 2), G(10 comma 3)and H(6 comma 3).

2.

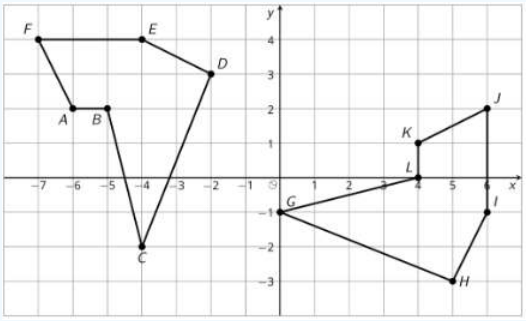


Figure 1.3.2.18: Two figures A B C D E F and G H I J K L on a coordinate plane, origin O. Horizontal axis scale negative 7 to 6 by 1's. Vertical axis scale negative 3 to 4 by 1's. The coordinates of A B C D E F are A(negative 6 comma 2), B(negative 4 comma 2), C(negative 4 comma negative 2), D(negative 2 comma 3), E(negative 4 comma 4) and F(negative 7 comma 4). The coordinates of G H I J K L are G(0 comma negative 1), H(5 comma negative 3), I(6 comma negative1), J(6 comma 2), K(4 comma 1) and L(4 comma 0).

3.

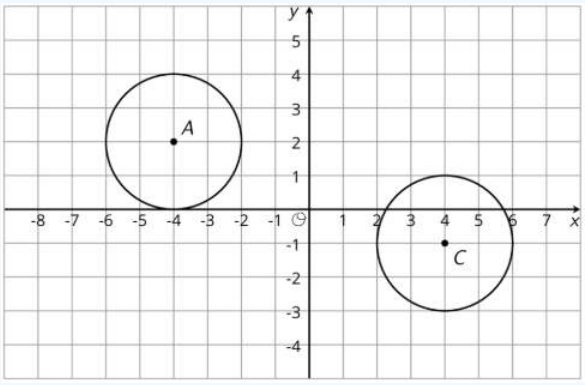


Figure 1.3.2.19: Two circles A and B on a coordinate plane, origin O. Horizontal axis scale negative 8 to 7 by 1's. Vertical axis scale negative 4 to 5 by 1's. The center of circle A is A(negative 4 comma 2). The center of circle C is C(4 comma negative 1). Both circles have a diameter of 2.

Exercise 1.3.2.7

1. Draw segment PQ.
2. When PQ is rotated 180° around point R, the resulting segment is the same as PQ. Where could point R be located?

(From Unit 1.2.2)

Exercise 1.3.2.8

Here is a trapezoid $ABCD$.

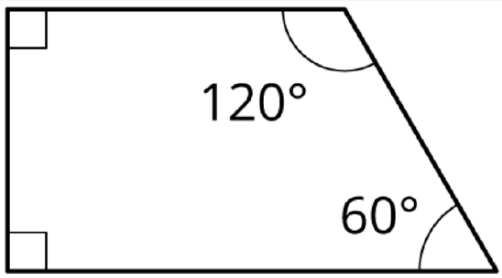


Figure 1.3.2.20

Using rigid transformations on the trapezoid, build a pattern. Describe some of the rigid transformations you used.

(From Unit 1.2.4)

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1.3.3: Congruence

Lesson

Let's find ways to test congruence of interesting figures.

Exercise 1.3.3.1: Not Just the Vertices

Trapezoids $ABCD$ and $A'B'C'D'$ are congruent.

- Draw and label the points on $A'B'C'D'$ that correspond to E and F.
- Draw and label the points on $ABCD$ that correspond to G' and H'.
- Draw and label at least three more pairs of corresponding points.

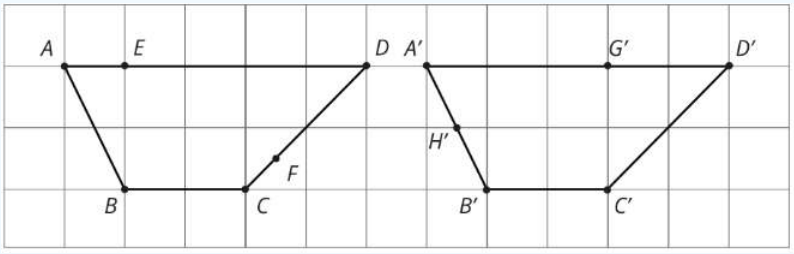


Figure 1.3.3.1: Two trapezoids A B C D and its image A prime B prime C prime D prime on a square grid. Trapezoid A B C D has point E on A D and point F on C D. Trapezoid A prime B prime C prime D prime has point H prime on A prime B prime and point G prime on A prime D prime. Let the lower left corner be (0 comma 0). Then the coordinates of A B C D are A(1 comma 3), B(2 comma 1), C(4 comma 1), F(4 point 5 comma 1 point 5), D(6 comma 3), E(2 comma 3). The coordinates of A prime B prime C prime D prime are A prime(7 comma 3), H prime(7 point 5 comma 2), B prime(8 comma 1), C prime(10 comma 1), D prime(12 comma 3), G prime(10 comma 3).

Exercise 1.3.3.2: Congruent Ovals

Are any of the ovals congruent to one another? Explain how you know.

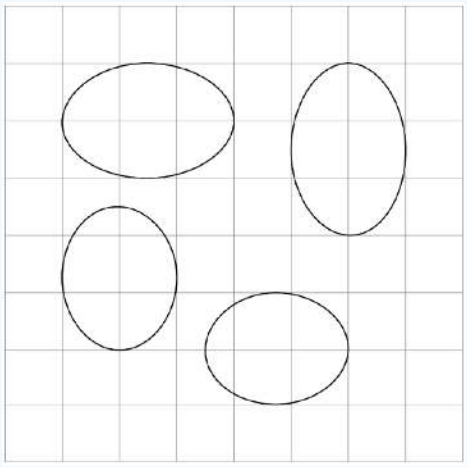


Figure 1.3.3.2

Are you ready for more?

You can use 12 toothpicks to create a polygon with an area of five square toothpicks, like this:

Can you use exactly 12 toothpicks to create a polygon with an area of four square toothpicks?

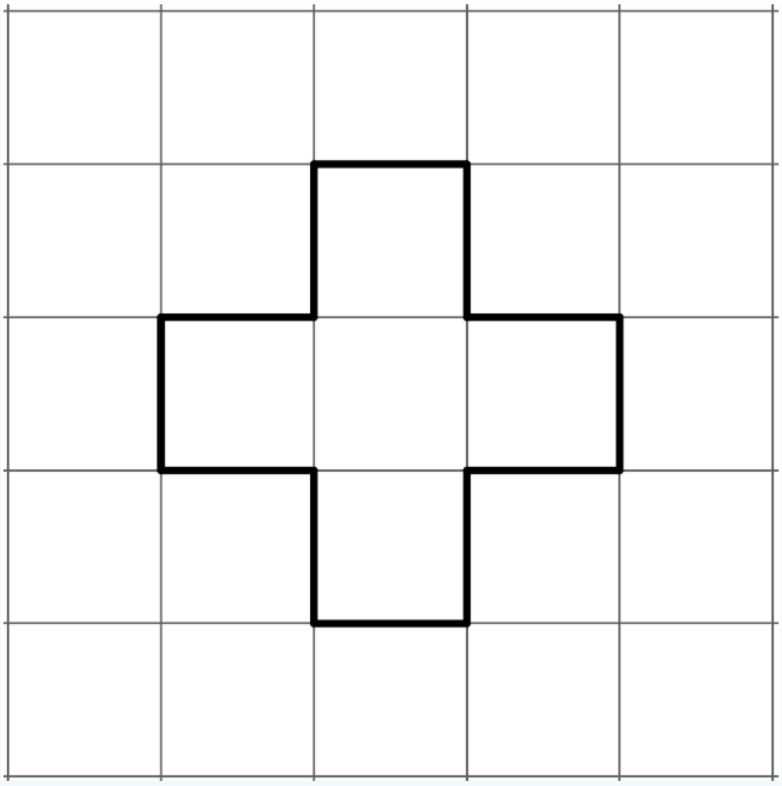


Figure has a vertex at 1 unit right, 2 units up. Another vertex at 1 unit right, 3 units up. Another vertex at 2 units right, 3 units up. Another vertex at 2 units right, 4 units up. Another vertex at 3 units right, 4 units up. Another vertex at 3 units right, 3 units up. Another vertex at 4 units right and 3 units up. Another vertex at 4 units right and 2 units up. Another vertex at 3 units right, 2 units up. Another vertex at 3 units right, 1 unit up. Another vertex at 2 units right, 1 unit up. Another vertex at 2 units right, 2 units up.

Exercise 1.3.3.3: Corresponding Points in Congruent Figures

Here are two congruent shapes with some corresponding points labeled.

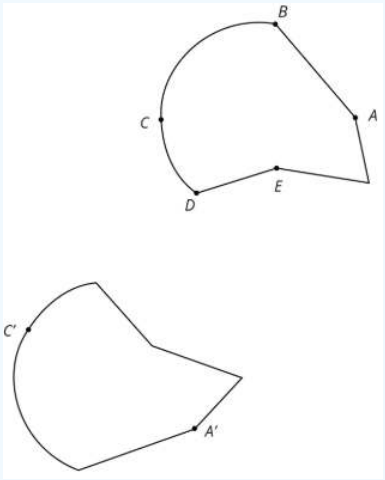


Figure 1.3.3.4

1. Draw the points corresponding to B, D, and E, and label them B', D', and E'.
2. Draw line segments AD and $A'D'$ and measure them. Do the same for segments BC and $B'C'$ and for segments AE and $A'E'$. What do you notice?
3. Do you think there could be a pair of corresponding segments with different lengths? Explain.

Exercise 1.3.3.4: Astonished Faces

Are these faces congruent? Explain your reasoning.

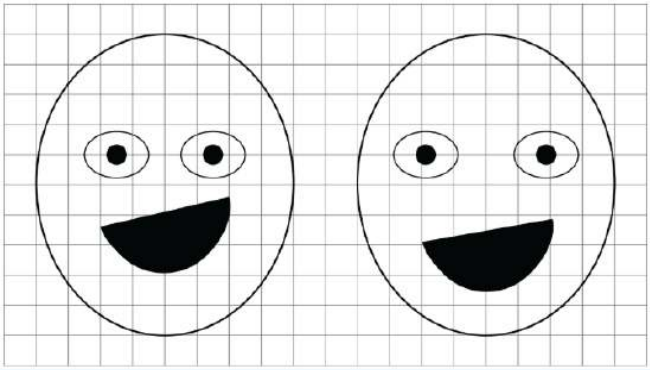


Figure 1.3.3.5

Summary

To show two figures are congruent, you align one with the other by a sequence of rigid transformations. This is true even for figures with curved sides. Distances between corresponding points on congruent figures are always equal, even for curved shapes. For example, corresponding segments AB and $A'B$ on these congruent ovals have the same length:

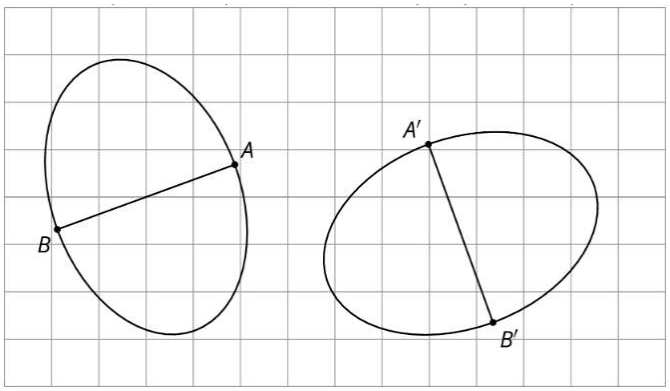


Figure 1.3.3.6: Two congruent ovals on a square grid. In the first oval, two points on opposite longer sides of the oval are labeled A and B and are connected by a line segment. In the second oval, two points on opposite longer sides of the oval are labeled A prime and B prime and are connected by a line segment. Let the lower left corner be (0 comma 0). Then point A is near (5 comma 4 point 75) and point B is near (1 comma 3 point 7 5). Point A prime is near (9 comma 5) and point B prime is near (10 point 5 comma 1 point 3).

To show two figures are not congruent, you can find parts of the figures that should correspond but that have different measurements.

For example, these two ovals don't look congruent.

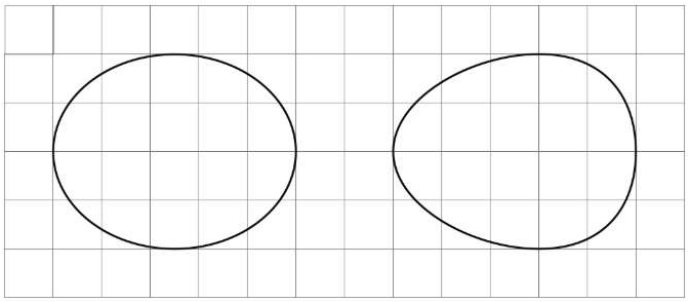


Figure 1.3.3.7

On both, the longest distance is 5 units across, and the longest distance from top to bottom is 4 units. The line segment from the highest to lowest point is in the middle of the left oval, but in the right oval, it's 2 units from the right end and 3 units from the left

end. This proves they are not congruent.

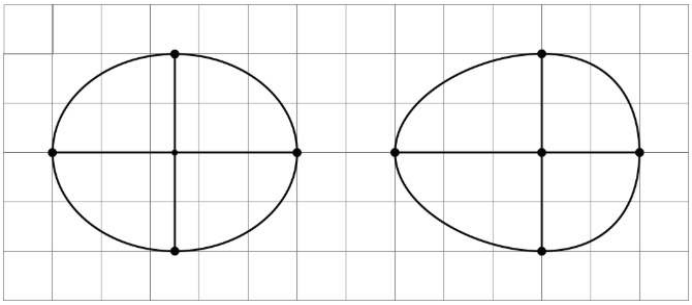


Figure 1.3.3.8

Glossary Entries

Definition: Congruent

One figure is congruent to another if it can be moved with translations, rotations, and reflections to fit exactly over the other.

In the figure, Triangle A is congruent to Triangles B, C, and D. A translation takes Triangle A to Triangle B, a rotation takes Triangle B to Triangle C, and a reflection takes Triangle C to Triangle D.

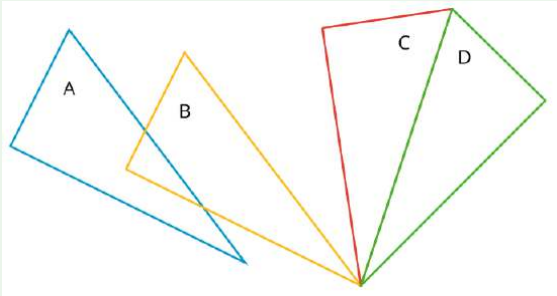


Figure 1.3.3.9

Definition: Right Angle

A right angle is half of a straight angle. It measures 90 degrees.

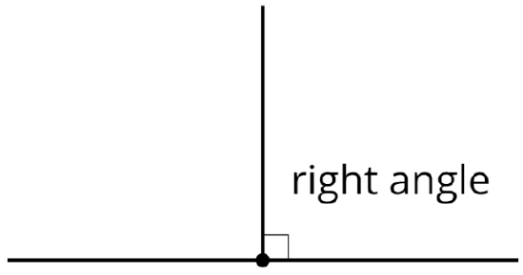


Figure 1.3.3.10

Practice

Exercise 1.3.3.5

Which of these four figures are congruent to the top figure?

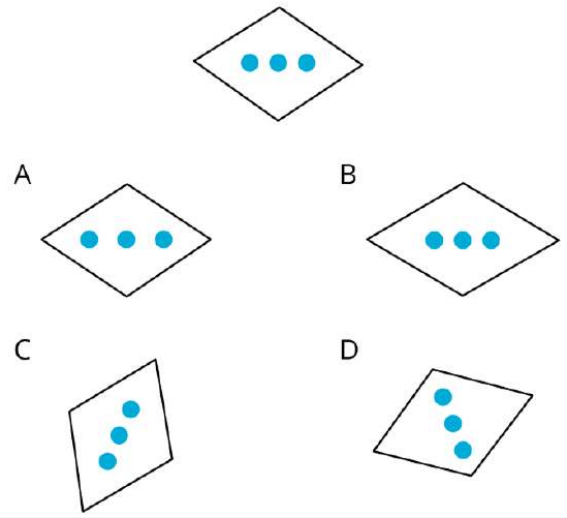


Figure A has dots more widely spread out, figure B has a slightly larger polygon, figure C is rotated the opposite direction and figure D has dots going the opposite direction than the first figure.

A: A

B: B

C: C

D: D

Exercise 1.3.3.6

These two figures are congruent, with corresponding points marked.

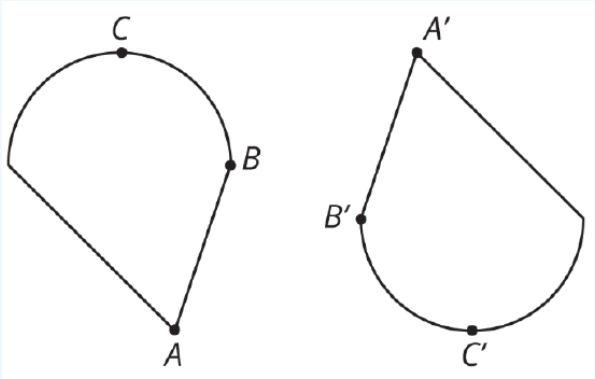


Figure 1.3.3.12

1. Are angles ABC and $A'B'C'$ congruent? Explain your reasoning.
2. Measure angles ABC and $A'B'C'$ to check your answer.

Exercise 1.3.3.7

Here are two figures.

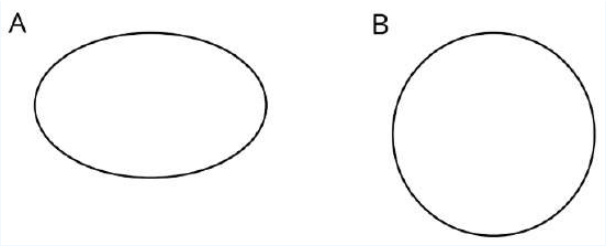


Figure 1.3.3.13

Show, using measurement, that these two figures are *not* congruent.

Exercise 1.3.3.8

Each picture shows two polygons, one labeled Polygon A and one labeled Polygon B. Describe how to move Polygon A into the position of Polygon B using a transformation.

1.

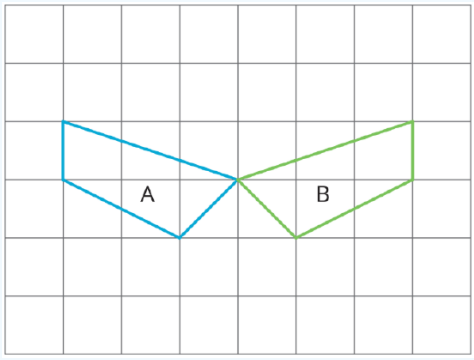


Figure 1.3.3.14

2.

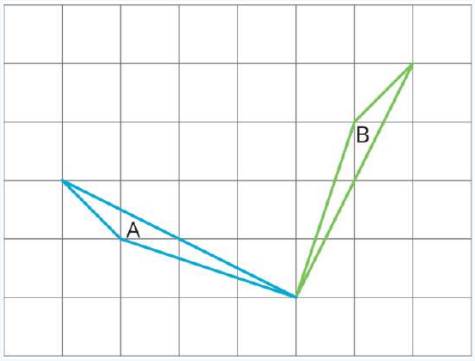


Figure 1.3.3.15

3.

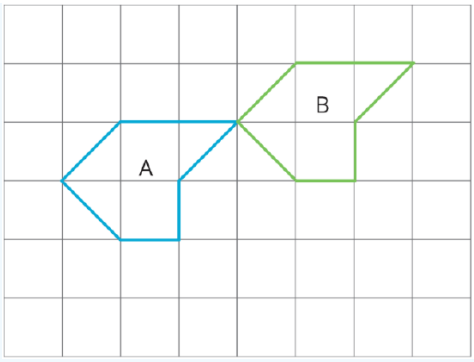


Figure 1.3.3.16

(From Unit 1.1.3)

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1.4: Angles in a Triangle

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1.4.1: Alternate Interior Angles

Lesson

Let's explore why some angles are always equal.

Exercise 1.4.1.1: Angle Pairs

1. Find the measure of angle JGH. Explain or show your reasoning.

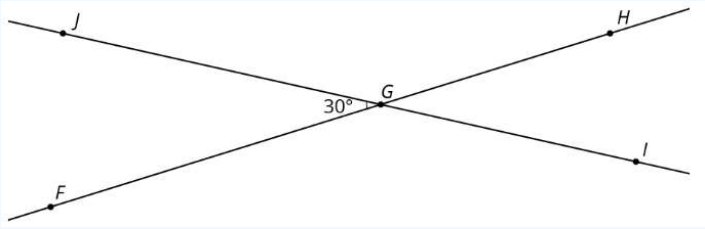


Figure 1.4.1.1

2. Find and label a second 30° degree angle in the diagram. Find and label an angle congruent to angle JGH.

Exercise 1.4.1.2: Cutting Parallel Lines with a Transversal

Lines AC and DF are parallel. They are cut by **transversal** HJ.

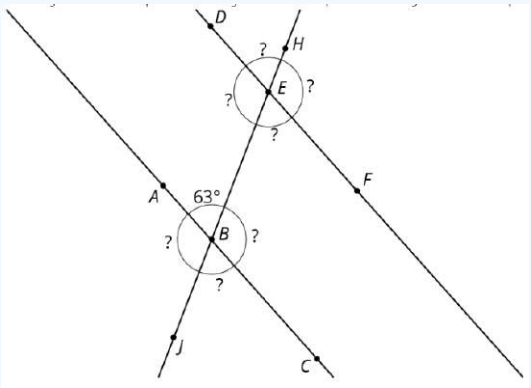


Figure 1.4.1.2: Line A C contains point B. Line D F contains point E. Line J H contains points B and E. Angle A B J is labeled question mark. Angle A B E is labeled 63 degrees. Angle E B C is labeled question mark. Angle C B J is labeled question mark. Angle D E H is labeled question mark. Angle H E F is labeled question mark. Angle F E B is labeled question mark. Angle B E D is labeled question mark.

1. With your partner, find the seven unknown angle measures in the diagram. Explain your reasoning.
2. What do you notice about the angles with vertex B and the angles with vertex E?
3. Using what you noticed, find the measures of the four angles at point B in the second diagram. Lines AC and DF are parallel.

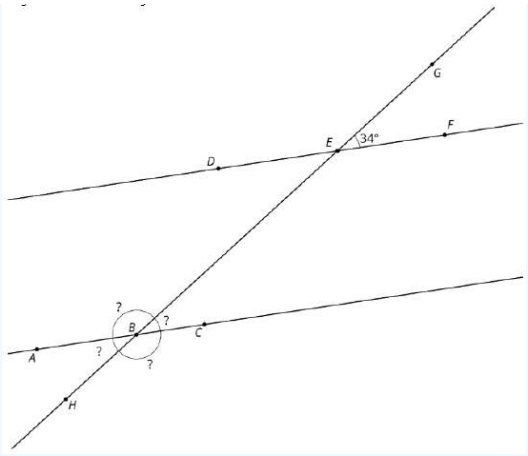


Figure 1.4.1.3: Line A C contains point B. Line D F contains point E. Line H G contains points B and E. Angle A B H is labeled with a question mark. Angle A B E is labeled with a question mark. Angle E B C is labeled with a question mark. Angle C B H is labeled with a question mark. Angle G E F is labeled 34 degrees.

4. The next diagram resembles the first one, but the lines form slightly different angles. Work with your partner to find the six unknown angles with vertices at points B and E.

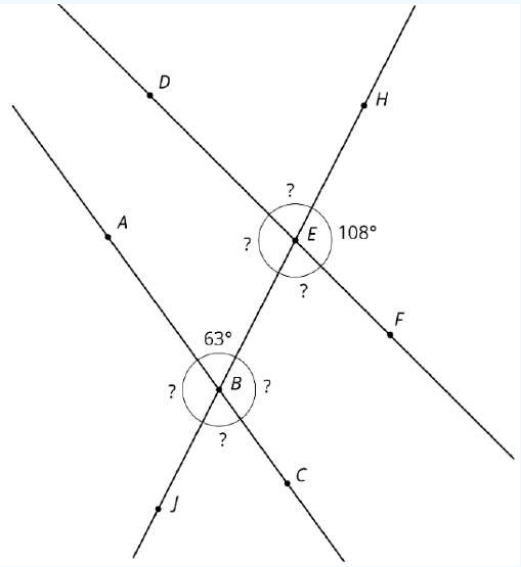


Figure 1.4.1.4: Line A C contains point B. Line D F contains point E. Line J H contains points B and E. Angle A B J is labeled question mark. Angle A B E is labeled 63 degrees. Angle E B C is labeled question mark. Angle C B J is labeled question mark. Angle D E H is labeled question mark. Angle H E F is labeled 108 degrees. Angle F E B is labeled question mark. Angle B E D is labeled question mark.

5. What do you notice about the angles in this diagram as compared to the earlier diagram? How are the two diagrams different? How are they the same?

Are you ready for more?

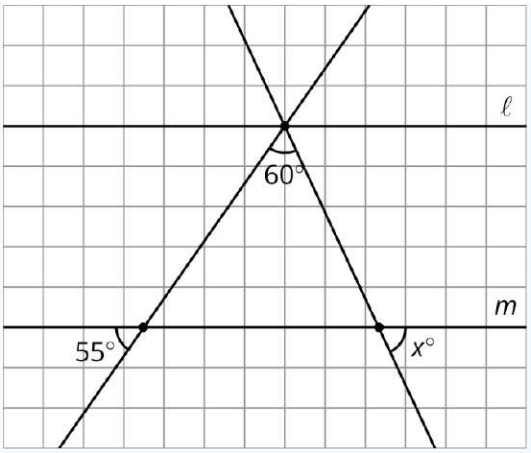


Figure 1.4.1.5: Four lines on a grid. Line l is a horizontal line 8 units up on the grid. Line m is a horizontal line 3 units up on the grid. A line trends up and to the right and crossed line m at a point between 3 and 4 units to the right and 3 units up. The line crosses line l at a point 7 units to the right and 8 units up. Another line crosses line m at a point between 9 and 10 units to the right and 3 units up. It crosses the line l at a point 7 units to the right and 8 units up. The interior angle at the top of the triangle formed is labeled 60 degrees. The bottom left exterior angle of the bottom left vertex is labeled 55 degrees. The bottom right exterior angle of the bottom right vertex is labeled x degrees.

Parallel lines l and m are cut by two transversals which intersect l in the same point. Two angles are marked in the figure. Find the measure x of the third angle.

Exercise 1.4.1.3: Alternate Interior Angles are Congruent

1. Lines l and k are parallel and t is a transversal. Point M is the midpoint of segment PQ.

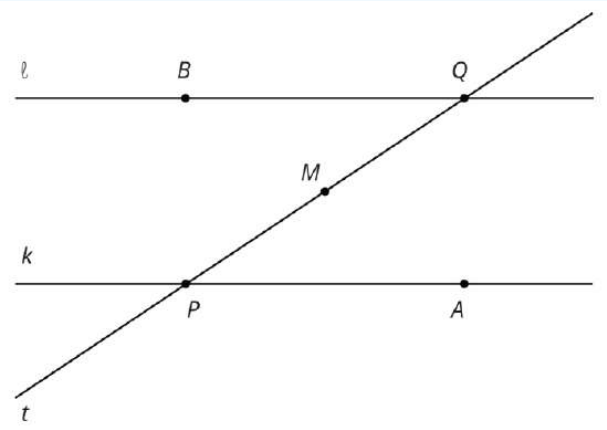


Figure 1.4.1.6

Find a rigid transformation showing that angles MPA and MQB are congruent.

2. In this picture, lines l and k are no longer parallel. M is still the midpoint of segment PQ.

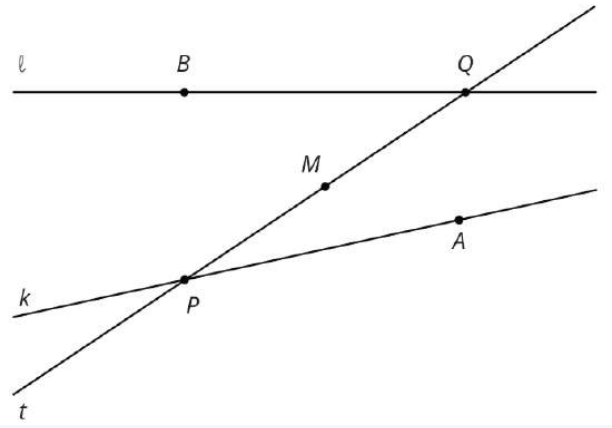


Figure 1.4.1.7

Does your argument in the earlier problem apply in this situation? Explain.

Summary

When two lines intersect, vertical angles are equal and adjacent angles are supplementary, that is, their measures sum to 180°. For example, in this figure angles 1 and 3 are equal, angles 2 and 4 are equal, angles 1 and 4 are supplementary, and angles 2 and 3 are supplementary.

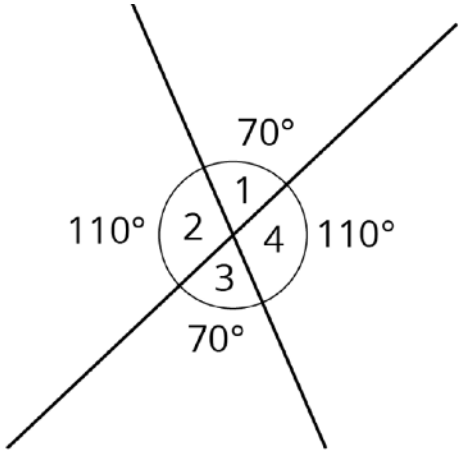


Figure 1.4.1.8: Two intersecting lines. Angle 1 is labeled 70 degrees. Angle 2 is labeled 110 degrees. Angle 3 is labeled 70 degrees. Angle 4 is labeled 110 degrees.

When two parallel lines are cut by another line, called a **transversal**, two pairs of **alternate interior angles** are created. ("Interior" means on the inside, or between, the two parallel lines.) For example, in this figure angles 3 and 5 are alternate interior angles and angles 4 and 6 are also alternate interior angles.

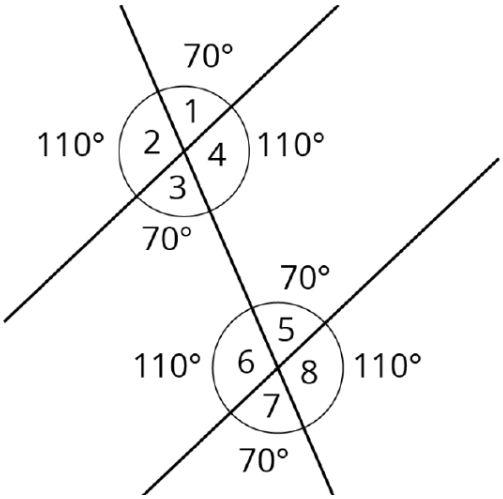


Figure 1.4.1.9: Two lines that do not intersect. A third line intersects with both lines. At the first intersection, angle 1 is labeled 70 degrees. Angle 2 is labeled 110 degrees. Angle 3 is labeled 70 degrees. Angle 4 is labeled 110 degrees. At the second intersection, angle 5 is marked 70 degrees. Angle 6 is marked 110 degrees. Angle 7 is marked 70 degrees. Angle 8 is marked 110 degrees.

Alternate interior angles are equal because a 180° rotation around the midpoint of the segment that joins their vertices takes each angle to the other. Imagine a point M halfway between the two intersections—can you see how rotating 180° about M takes angle 3 to angle 5?

Using what we know about vertical angles, adjacent angles, and alternate interior angles, we can find the measures of any of the eight angles created by a transversal if we know just one of them. For example, starting with the fact that angle 1 is 70° we use vertical angles to see that angle 3 is 70°, then we use alternate interior angles to see that angle 5 is 70°, then we use the fact that angle 5 is supplementary to angle 8 to see that angle 8 is 110° since $180 - 70 = 110$. It turns out that there are only two different measures. In this example, angles 1, 3, 5, and 7 measure 70°, and angles 2, 4, 6, and 8 measure 110°.

Glossary Entries

Definition: Alternate Interior Angles

Alternate interior angles are created when two parallel lines are crossed by another line called a transversal. Alternate interior angles are inside the parallel lines and on opposite sides of the transversal.

This diagram shows two pairs of alternate interior angles. Angles a and d are one pair and angles b and c are another pair.

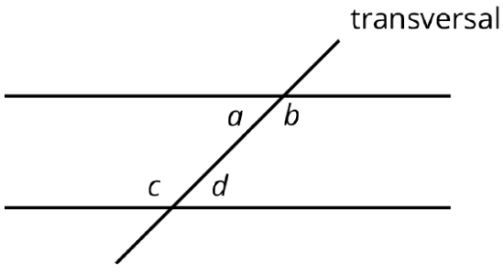


Figure 1.4.1.10: There are two horizontal parallel lines, and a third diagonal line drawn from the bottom left to the upper right, intersecting both horizontal lines. The diagonal line is labeled transversal. There are four angles created by the diagonal line inside the parallel lines. The upper left angle is labeled a, upper right is b, lower left is c, and lower right is d.

Definition: Transversal

A transversal is a line that crosses parallel lines.

This diagram shows a transversal line k intersecting parallel lines m and l.

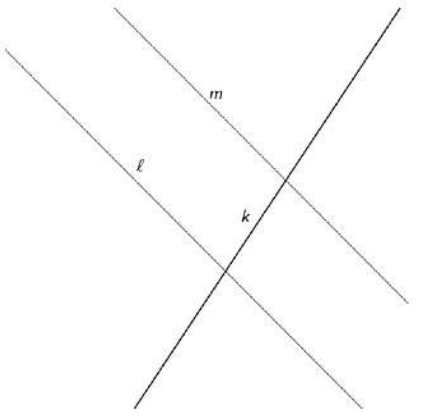


Figure 1.4.1.11

Practice

Exercise 1.4.1.4

Use the daigram to find the measure of each angle.

1. $m\angle ABC$
2. $m\angle EBD$
3. $m\angle ABE$

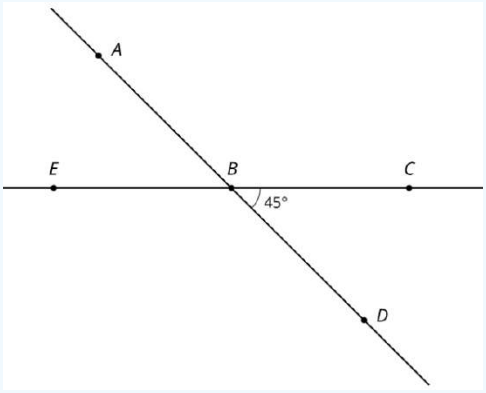


Figure 1.4.1.12: A horizontal line and a line that slopes downward from left to right. The lines intersect at a point labeled B. On the horizontal line, point E is to the left of B and C is to the right of B. On the sloping line, point A is above B and point D is below B. Angle C B D is labeled 45 degrees.

(From Unit 1.2.3)

Exercise 1.4.1.5

Lines k and l are parallel, and the measure of angle ABC is 19 degrees.

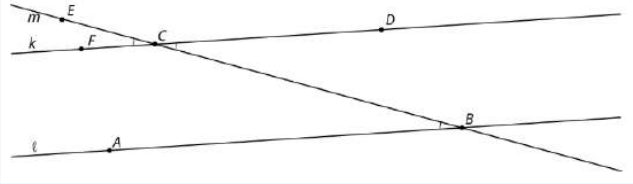


Figure 1.4.1.13: Two parallel lines, k and l, cut by transversal line m. Points F, C and D are on line k. Points A and B are on line l. Points E, C and B are on transversal m. Angle E C F is marked congruent to angle D C B and to angle A B C.

1. Explain why the measure of angle ECF is 19 degrees. If you get stuck, consider translating line by moving B to C.
2. What is the measure of angle BCD? Explain.

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Exercise 1.4.1.6

The diagram shows three lines with some marked angle measures.

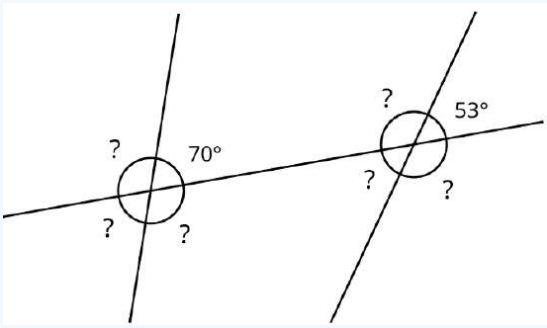


Figure 1.4.1.14: Two lines that do not intersect. A third line intersects with both lines. At the first intersection, angles are marked starting at the top right and going in the clockwise direction, 70 degrees, question mark, question mark, question mark. At the second intersection, angles are marked starting at the top right and going in the clockwise direction, 53 degrees, question mark, question mark, question mark.

Find the missing angle measures marked with question marks.

Exercise 1.4.1.7

Lines s and t are parallel. Find the value of x.

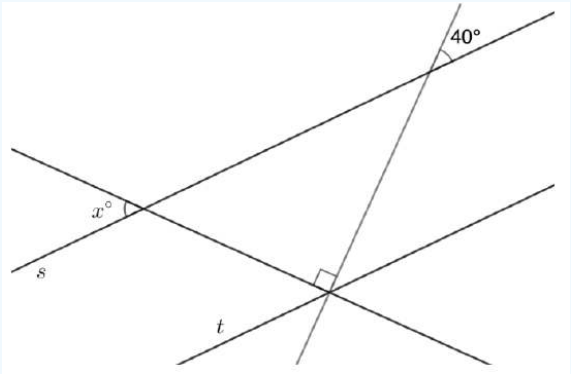


Figure 1.4.1.15

Exercise 1.4.1.8

The two figures are scaled copies of each other.

1. What is the scale factor that takes Figure 1 to Figure 2?
2. What is the scale factor that takes Figure 2 to Figure 1?

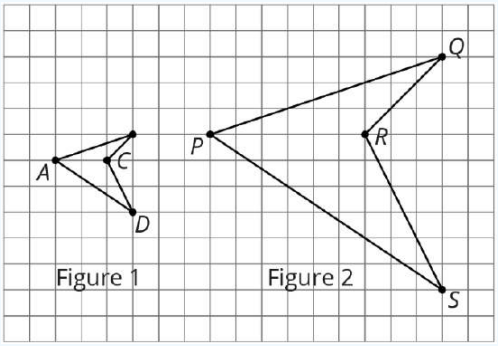


Figure 2.

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1.4.2: Adding the Angles in a Triangle

Lesson

Let's explore angles in triangles.

Exercise 1.4.2.1: Can You Draw It?

1. Complete the table by drawing a triangle in each cell that has the properties listed for its column and row. If you think you cannot draw a triangle with those properties, write “impossible” in the cell.
2. Share your drawings with a partner. Discuss your thinking. If you disagree, work to reach an agreement.

Table 1.4.2.1

	acute (all angles acute)	**right (has a right angle)**	**obtuse (has an obtuse angle)**
scalene (side lengths all different)			
isosceles (at least two side lengths are equal)			
equilateral (three side lengths equal)			

Exercise 1.4.2.2: Find All Three

Your teacher will give you a card with a picture of a triangle.

1. The measurement of one of the angles is labeled. Mentally estimate the measures of the other two angles.
2. Find two other students with triangles congruent to yours but with a different angle labeled. Confirm that the triangles are congruent, that each card has a different angle labeled, and that the angle measures make sense.
3. Enter the three angle measures for your triangle on the table your teacher has posted.

Exercise 1.4.2.3: Tear It Up

Your teacher will give you a page with three sets of angles and a blank space. Cut out each set of three angles. Can you make a triangle from each set that has these same three angles?

Are you ready for more?

1. Draw a quadrilateral. Cut it out, tear off its angles, and line them up. What do you notice?
2. Repeat this for several more quadrilaterals. Do you have a conjecture about the angles?

Summary

A 180° angle is called a **straight angle** because when it is made with two rays, they point in opposite directions and form a straight line.

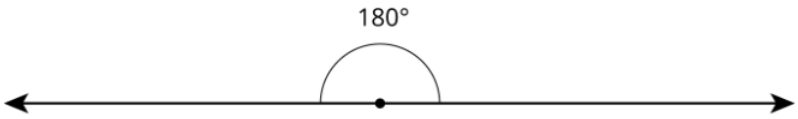


Figure 1.4.2.1

If we experiment with angles in a triangle, we find that the sum of the measures of the three angles in each triangle is 180°—the same as a straight angle!

Through experimentation we find:

- If we add the three angles of a triangle physically by cutting them off and lining up the vertices and sides, then the three angles form a straight angle.
- If we have a line and two rays that form three angles added to make a straight angle, then there is a triangle with these three angles.

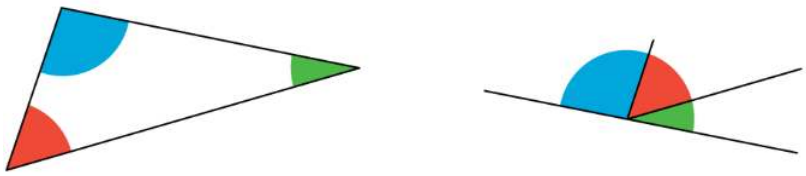


Figure 1.4.2.2

Glossary Entries

Definition: Alternate Interior Angles

Alternate interior angles are created when two parallel lines are crossed by another line called a transversal. Alternate interior angles are inside the parallel lines and on opposite sides of the transversal.

This diagram shows two pairs of alternate interior angles. Angles a and d are one pair and angles b and c are another pair.

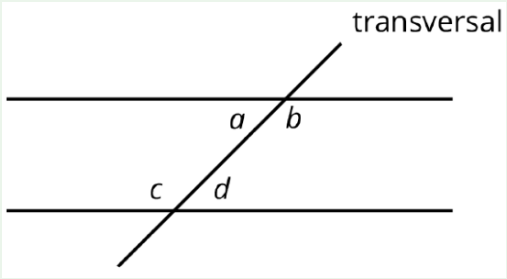


Figure 1.4.2.3: There are two horizontal parallel lines, and a third diagonal line drawn from the bottom left to the upper right, intersecting both horizontal lines. The diagonal line is labeled transversal. There are four angles created by the diagonal line inside the parallel lines. The upper left angle is labeled a, upper right is b, lower left is c, and lower right is d.

Definition: Straight Angle

A straight angle is an angle that forms a straight line. It measures 180 degrees.

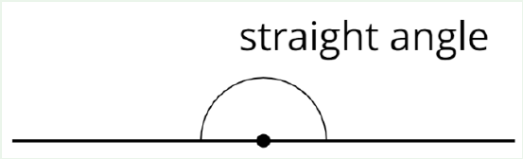


Figure 1.4.2.4

Definition: Transversal

A transversal is a line that crosses parallel lines.

This diagram shows a transversal line k intersecting parallel lines m and l.

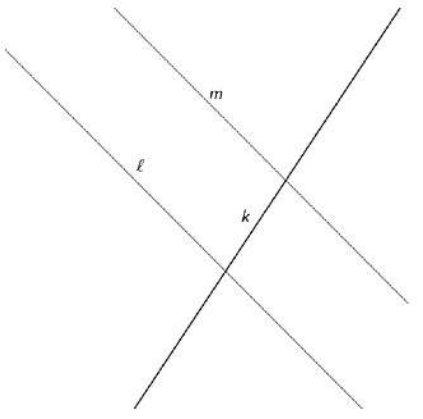


Figure 1.4.2.5

Practice

Exercise 1.4.2.4

In triangle ABC, the measure of angle A is 40°.

1. Give possible measures for angles B and C if triangle ABC is isosceles.
2. Give possible measures for angles B and C if triangle ABC is right.

Exercise 1.4.2.5

For each set of angles, decide if there is a triangle whose angles have these measures in degrees:

1. 60, 60, 60
2. 90, 90, 45
3. 30, 40, 50
4. 90, 45, 45
5. 120, 30, 30

If you get stuck, consider making a line segment. Then use a protractor to measure angles with the first two angle measures.

Exercise 1.4.2.6

Angle A in triangle ABC is obtuse. Can angle B or angle C be obtuse? Explain your reasoning.

Exercise 1.4.2.7

For each pair of polygons, describe the transformation that could be applied to Polygon A to get Polygon B.

1.

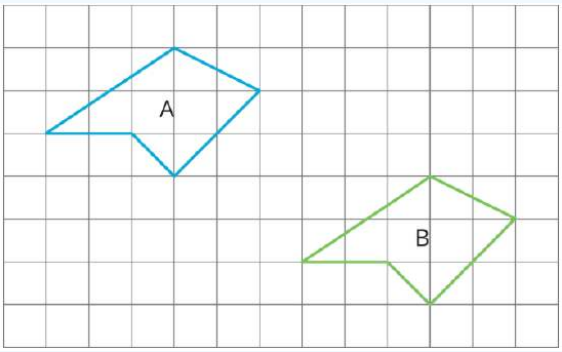


Figure 1.4.2.6

2.

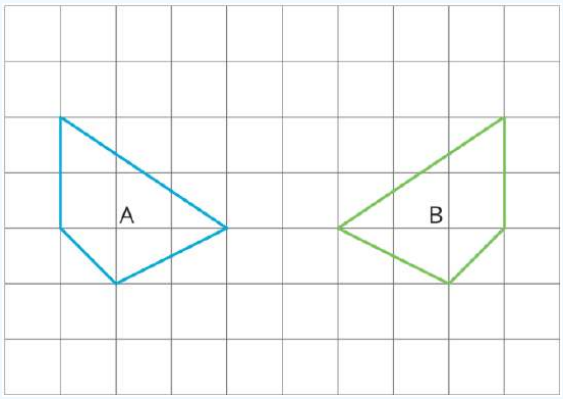


Figure 1.4.2.7

3.

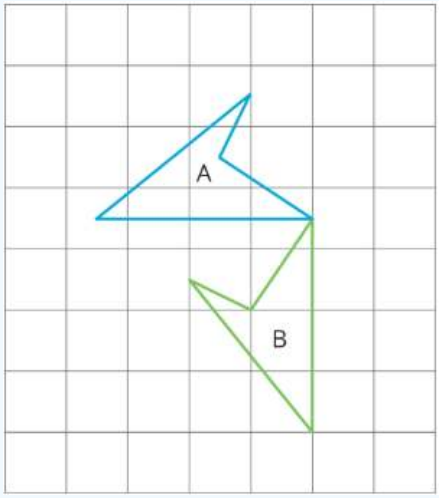


Figure 1.4.2.8

(From Unit 1.1.3)

Exercise 1.4.2.8

On the grid, draw a scaled copy of quadrilateral $ABCD$ using a scale factor of $\frac{1}{2}$.

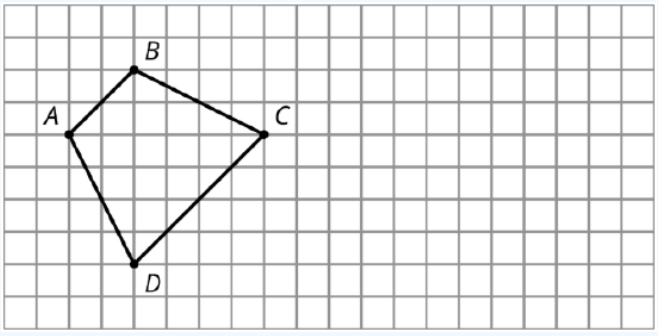


Figure 1.4.2.9: A quadrilateral on a grid. Point A is 2 units right and 6 units up. Point B is 4 units right and 8 units up. Point C is 8 units right and 6 units up. Point D is 4 units right and 2 units up.

(From Unit 1.4.1)

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1.4.3: Parallel Lines and the Angles in a Triangle

Lesson

Let's see why the angles in a triangle add to 180 degrees.

Exercise 1.4.3.1: True or False: Computational Relationships

Is each equation true or false?

$62 - 28 = 60 - 30$

$3 \cdot -8 = (2 \cdot -8) - 8$

$\frac{16}{-2} + \frac{24}{-2} = \frac{40}{-2}$

Exercise 1.4.3.2: Angle Plus Two

Consider triangle ABC. Select the Midpoint tool and click on two points or a segment to find the midpoint.



1. Rotate triangle ABC 180° around the midpoint of side AC. Right click on the point and select Rename to label the new vertex D.
2. Rotate triangle ABC 180° around the midpoint of side AB. Right click on the point and select Rename to label the new vertex E.
3. Look at angles EAB, BAC, and CAD. Without measuring, write what you think is the sum of the measures of these angles. Explain or show your reasoning.
4. Is the measure of angle EAB equal to the measure of any angle in triangle ABC? If so, which one? If not, how do you know?
5. Is the measure of angle CAD equal to the measure of any angle in triangle ABC? If so, which one? If not, how do you know?
6. What is the sum of the measures of angles ABC, BAC, and ACB?

Exercise 1.4.3.3: Every Triangle in the World

Here is ΔABC. Line DE is parallel to line AC.

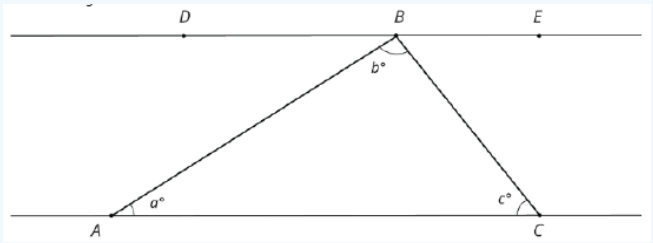


Figure 1.4.3.1: Point B lies on line D E. Triangle B A C is drawn. Angle A B C is labeled b degrees. Angle B A C is labeled a degrees. Angle A C B is labeled c degrees.

1. What is $m\angle DBA + b + m\angle CBE$? Explain how you know.
2. Use your answer to explain why $a + b + c = 180$.
3. Explain why your argument will work for *any* triangle: that is, explain why the sum of the angle measures in *any* triangle is 180°.

Are you ready for more?

1. Using a ruler, create a few quadrilaterals. Use a protractor to measure the four angles inside the quadrilateral. What is the sum of these four angle measures?
2. Come up with an explanation for why anything you notice must be true (hint: draw one diagonal in each quadrilateral).

Exercise 1.4.3.4: Four Triangles Revisited

This diagram shows a square $BDFH$ that has been made by images of triangle ABC under rigid transformations.

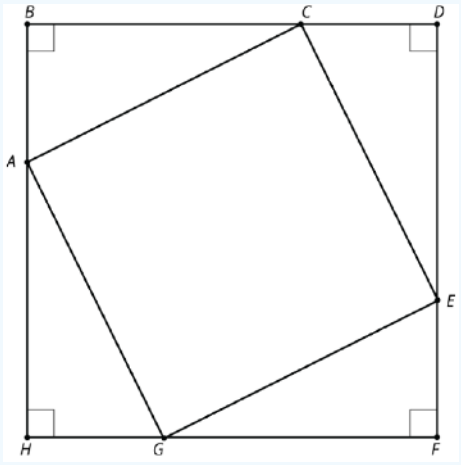


Figure 1.4.3.2: Quadrilateral B D F H. Angles B, D, F, and H are marked as right angles. Point C lies on B D. Point E lies on D F. Point G lies on F H. Point A lies on B H. Segments A C, C E, E G, and G A are drawn.

Given that angle BAC measures 53 degrees, find as many other angle measures as you can.

Summary

Using parallel lines and rotations, we can understand why the angles in a triangle always add to 180°. Here is triangle ABC. Line DE is parallel to AC and contains B.

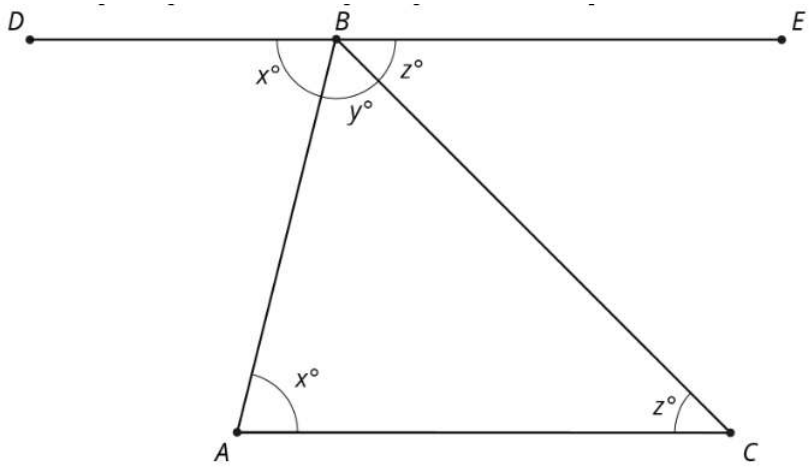


Figure 1.4.3.3: Point B lies on segment D E. Triangle B A C is drawn. Angle D B A is labeled x degrees. Angle A B C is labeled y degrees. Angle E B C is labeled z degrees. Angle B A C is labeled x degrees. Angle B C A is labeled z degrees.

A 180 degree rotation of triangle ABC around the midpoint of AB interchanges angles A and DBA so they have the same measure: in the picture these angles are marked as x°. A 180 degree rotation of triangle ABC around the midpoint of BC interchanges angles C and CBE so they have the same measure: in the picture, these angles are marked as z°. Also, DBE is a straight line because 180 degree rotations take lines to parallel lines. So the three angles with vertex B make a line and they add up to 180° ($x+y+z=180$). But x, y, z are the measures of the three angles in ΔABC so the sum of the angles in a triangle is always 180°!

Glossary Entries

Definition: Alternate Interior Angles

Alternate interior angles are created when two parallel lines are crossed by another line called a transversal. Alternate interior angles are inside the parallel lines and on opposite sides of the transversal.

This diagram shows two pairs of alternate interior angles. Angles a and d are one pair and angles b and c are another pair.

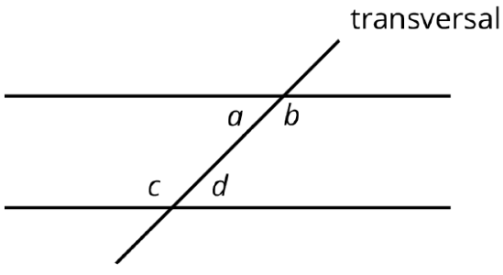


Figure 1.4.3.4: There are two horizontal parallel lines, and a third diagonal line drawn from the bottom left to the upper right, intersecting both horizontal lines. The diagonal line is labeled transversal. There are four angles created by the diagonal line inside the parallel lines. The upper left angle is labeled a, upper right is b, lower left is c, and lower right is d.

Definition: Straight Angle

A straight angle is an angle that forms a straight line. It measures 180 degrees.

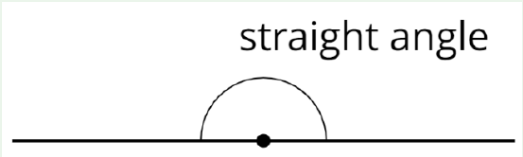


Figure 1.4.3.5

Definition: Transversal

A transversal is a line that crosses parallel lines.

This diagram shows a transversal line k intersecting parallel lines m and l.

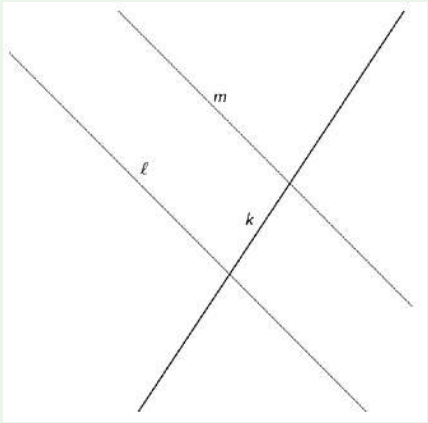


Figure 1.4.3.6

Practice

Exercise 1.4.3.5

For each triangle, find the measure of the missing angle.

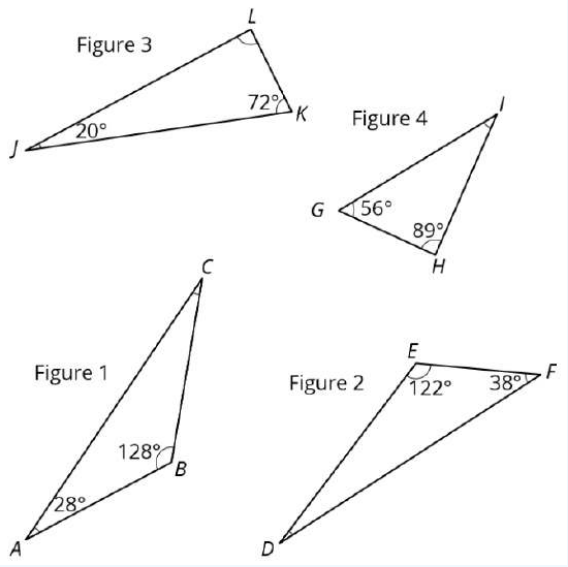


Figure 2 is triangle D E F. Angle E is labeled 122 degrees. Angle F is labeled 38 degrees.

Exercise 1.4.3.6

Is there a triangle with *two* right angles? Explain your reasoning.

Exercise 1.4.3.7

In this diagram, lines AB and CD are parallel.

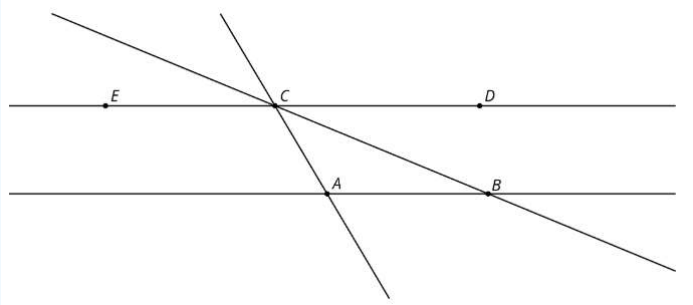


Figure 1.4.3.8

Angle ABC measures 35° and angle BAC measures 115°.

1. What is $m\angle ACE$?
2. What is $m\angle DCB$?
3. What is $m\angle ACB$?

Exercise 1.4.3.8

Here is a diagram of triangle DEF.

1. Find the measures of angles q, r, and s.
2. Find the sum of the measures of angles q, r, and s.
3. What do you notice about these three angles?

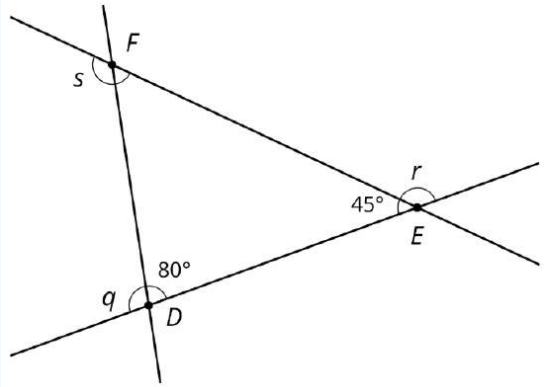


Figure 1.4.3.9: Three lines intersect to form Triangle D E F. The interior angle at D is labeled 80 degrees. The exterior angle at D is labeled q. The interior angle at E is labeled 45 degrees. The exterior angle at E is labeled r. The exterior angle at F is labeled s.

Exercise 1.4.3.9

The two figures are congruent.

1. Label the points A', B', and C' that correspond to A, B, and C in the figure on the right.

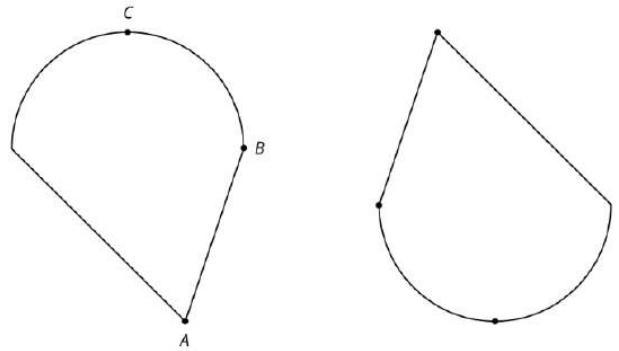


Figure 1.4.3.10: Two congruent figures are semicircles with a connected opposite angle point. The left figure has point A on the bottom angle point, point B on the right side where the semicircle and segment AB meet and point C on the midpoint of the top semicircle. The right figure has the semicircle on the bottom with a point on the midpoint, a point on the left side where the semicircle and segment meet and a point at the top angle point.

2. If segment AB measures 2 cm, how long is segment $A'B'$? Explain.

3. The point D is shown in addition to A and C. How can you find the point D' that corresponds to D? Explain your reasoning.

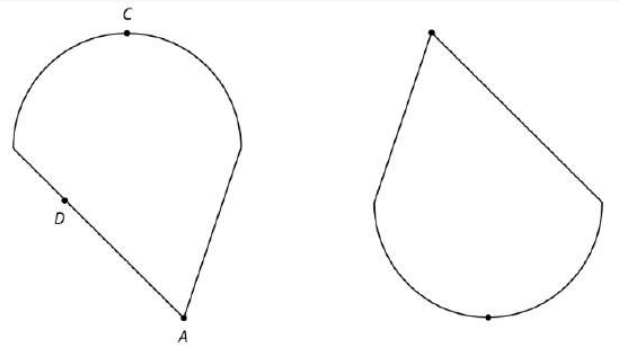


Figure 1.4.3.11: Two congruent figures are semicircles with a connected opposite angle point. The left figure has point A on the bottom angle point, point D on the left side below where the semicircle and segment AB meet and point C on the midpoint of the top semicircle. The right figure has the semicircle on the bottom with a point on the midpoint, and a point at the top angle point.

(From Unit 1.3.3)

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1.5: Let's Put It to Work

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1.5.1: Rotate and Tessellate

Lesson

Let's make complex patterns using transformations.

Exercise 1.5.1.1: Deducing Angle Measures

Your teacher will give you some shapes.

1. How many copies of the equilateral triangle can you fit together around a single vertex, so that the triangles' edges have no gaps or overlaps? What is the measure of each angle in these triangles?
2. What are the measures of the angles in the
    1. square?
    2. hexagon?
    3. parallelogram?
    4. right triangle?
    5. octagon?
    6. pentagon?

Exercise 1.5.1.2: Tessellate This

1. Design your own **tessellation**. You will need to decide which shapes you want to use and make copies. Remember that a tessellation is a repeating pattern that goes on forever to fill up the entire plane.
2. Find a partner and trade pictures. Describe a transformation of your partner's picture that takes the pattern to itself. How many different transformations can you find that take the pattern to itself? Consider translations, reflections, and rotations.
3. If there's time, color and decorate your tessellation.

Exercise 1.5.1.3: Rotate That

Make a design with rotational symmetry.

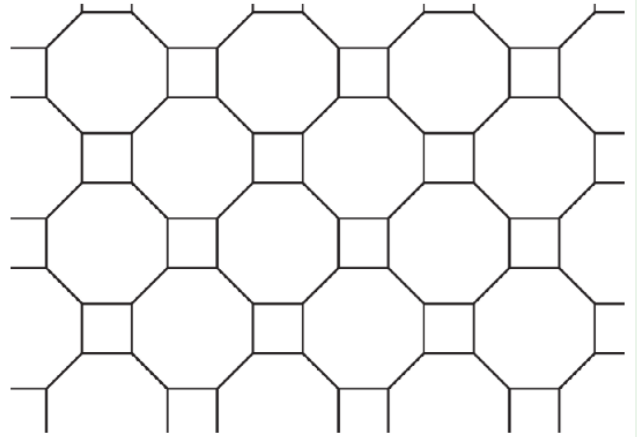
1. Find a partner who has also made a design. Exchange designs and find a transformation of your partner's design that takes it to itself. Consider rotations, reflections, and translations.
2. If there's time, color and decorate your design.

Glossary Entries

Definition: Tessellation

A tessellation is a repeating pattern of one or more shapes. The sides of the shapes fit together perfectly and do not overlap. The pattern goes on forever in all directions.

This diagram shows part of a tessellation.



1.5.1.1

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CHAPTER OVERVIEW

2: Dilations, Similarity, and Introducing Slope

In this unit, students learn to understand and use the term “dilation,” and to recognize that a dilation is determined by a point called the “center” and a number called the “scale factor.” They learn that under a dilation, the image of a circle is a circle and the image of a line is a line parallel to the original. They draw images of figures under dilations on and off the coordinate plane. They use the terms “corresponding sides” and “corresponding angles” to describe correspondences between a figure and its dilated image, and recognizing that angle measures are preserved, but lengths are multiplied by the scale factor. They learn to understand similarity of plane figures in terms of rigid transformations and dilations. They learn to recognize when one plane figure is similar or not similar to another. They use the definition of “similar” and properties of similar figures to justify claims of similarity or non-similarity. Students learn the terms “slope” and “slope triangle,” and use the similarity of slope triangles on the same line to understand that any two distinct points on a line determine the same slope.

2.1: Dilations

2.1.1: Projecting and Scaling

2.1.2: Circular Grid

2.1.3: Dilations with no Grid

2.1.4: Dilations on a Square Grid

2.1.5: More Dilations

2.2: Similarity

2.2.1: Similarity

2.2.2: Similar Polygons

2.2.3: Similar Triangles

2.2.4: Side Length Quotients in Similar Triangles

2.3: Slope

2.3.1: Meet Slope

2.3.2: Writing Equations for Lines

2.3.3: Using Equations for Lines

2.4: Let's Put It to Work

2.4.1: The Shadow Knows

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2.1: Dilations

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2.1.1: Projecting and Scaling

Lesson

Let's explore scaling.

Exercise 2.1.1.1: Number Talk: Remembering Fraction Division

Find each quotient. Write your answer as a fraction or a mixed number.

$6\frac{1}{4} \div 2$

$10\frac{1}{7} \div 5$

$8\frac{1}{2} \div 11$

Exercise 2.1.1.2: Sorting Rectangles

Rectangles were made by cutting an $8\frac{1}{2}$-inch by 11-inch piece of paper in half, in half again, and so on, as illustrated in the diagram. Find the lengths of each rectangle and enter them in the appropriate table.

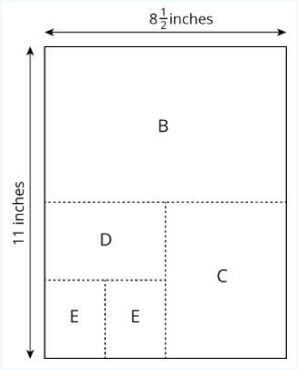


Figure 2.1.1.1: An image of an 8 point 5 by 11 inch rectangle. A dashed line divides the entire rectangle in half and one rectangle is labeled B. A dashed line divides the other half of B and one rectangle is labeled C. A dashed line divides the other half of C and one rectangle is labeled D. A dashed line divides the other half of D and both rectangles are labeled E.

1. Some of the rectangles are scaled copies of the full sheet of paper (Rectangle A). Record the measurements of those rectangles in this table.

Table 2.1.1.1

rectangle	length of short side (inches)	length of long side (inches)
A	$8\frac{1}{2}$	11

2. Some of the rectangles are *not* scaled copies of the full sheet of paper. Record the measurements of those rectangles in this table.

Table 2.1.1.2

rectangle	length of short side (inches)	length of long side (inches)

3. Look at the measurements for the rectangles that are scaled copies of the full sheet of paper. What do you notice about the measurements of these rectangles? Look at the measurements for the rectangles that are *not* scaled copies of the full sheet. What do you notice about these measurements?

4. Stack the rectangles that are scaled copies of the full sheet so that they all line up at a corner, as shown in the diagram. Do the same with the other set of rectangles. On each stack, draw a line from the bottom left corner to the top right corner of the biggest rectangle. What do you notice?

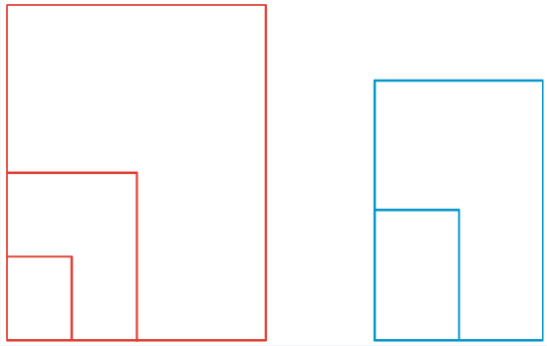


Figure 2.1.1.2

5. Stack *all* of the rectangles from largest to smallest so that they all line up at a corner. Compare the lines that you drew. Can you tell, from the drawn lines, which set each rectangle came from?

Are you ready for more?

In many countries, the standard paper size is not 8.5 inches by 11 inches (called “letter” size), but instead 210 millimeters by 297 millimeters (called “A4” size). Are these two rectangle sizes scaled copies of one another?

Exercise 2.1.1.3: Scaled Rectangles

Here is a picture of Rectangle R, which has been evenly divided into smaller rectangles. Two of the smaller rectangles are labeled B and C.

1. Is B a scaled copy of R? If so, what is the **scale factor**?
2. Is C a scaled copy of B? If so, what is the scale factor?
3. Is C a scaled copy of R? If so, what is the scale factor?

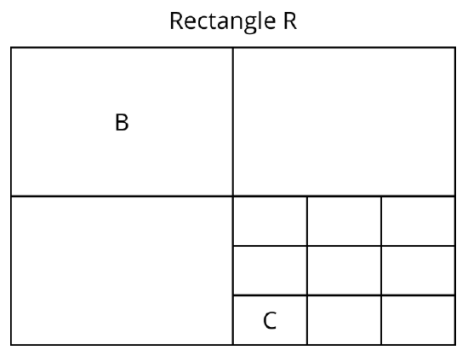


Figure 2.1.1.3

Summary

Scaled copies of rectangles have an interesting property. Can you see what it is?

Here, the larger rectangle is a scaled copy of the smaller one (with a scale factor of $\frac{3}{2}$). Notice how the diagonal of the large rectangle contains the diagonal of the smaller rectangle. This is the case for any two scaled copies of a rectangle if we line them up as shown. If two rectangles are *not* scaled copies of one another, then the diagonals do not match up. In this unit, we will investigate how to make scaled copies of a figure.

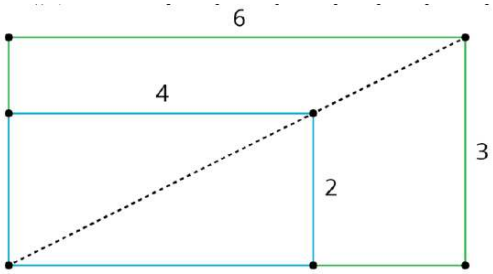


Figure 2.1.1.4: A small rectangle inside a larger rectangle with same lower left point and a dashed diagonal line from that lower left point through both upper points. The small rectangle has length 4 and height 2. The large rectangle has length 6 and height 3.

Glossary Entries

Definition: Scale Factor

To create a scaled copy, we multiply all the lengths in the original figure by the same number. This number is called the scale factor.

In this example, the scale factor is 1.5, because $4 \cdot (1.5) = 6$, $5 \cdot (1.5) = 7.5$, and $6 \cdot (1.5) = 9$.

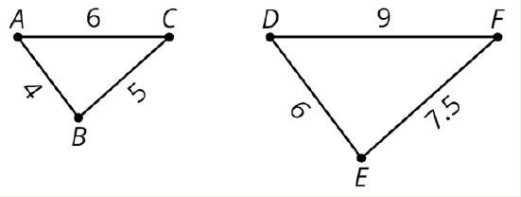


Figure 2.1.1.5

Practice

Exercise 2.1.1.4

Rectangle A measures 12 cm by 3 cm. Rectangle B is a scaled copy of Rectangle A. Select **all** of the measurement pairs that could be the dimensions of Rectangle B.

A. 6 cm by 1.5 cm
B. 10 cm by 2 cm
C. 13 cm by 4 cm
D. 18 cm by 4.5 cm
E. 80 cm by 20 cm

Exercise 2.1.1.5

Rectangle A has length 12 and width 8. Rectangle B has length 15 and width 10. Rectangle C has length 30 and width 15.

1. Is Rectangle A a scaled copy of Rectangle B? If so, what is the scale factor?
2. Is Rectangle B a scaled copy of Rectangle A? If so, what is the scale factor?
3. Explain how you know that Rectangle C is *not* a scaled copy of Rectangle B.
4. Is Rectangle A a scaled copy of Rectangle C? If so, what is the scale factor?

Exercise 2.1.1.6

Here are three polygons.

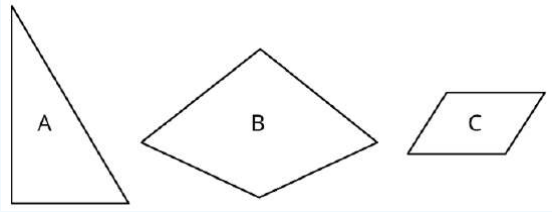


Figure 2.1.1.6

1. Draw a scaled copy of Polygon A with scale factor $\frac{1}{2}$.
2. Draw a scaled copy of Polygon B with scale factor 2.
3. Draw a scaled copy of Polygon C with scale factor $\frac{1}{4}$.

Exercise 2.1.1.7

Which of these sets of angle measures could be the three angles in a triangle?

A. $40^\circ, 50^\circ, 60^\circ$
B. $50^\circ, 60^\circ, 70^\circ$
C. $60^\circ, 70^\circ, 80^\circ$
D. $70^\circ, 80^\circ, 90^\circ$

From (Unit 1.4.2)

Exercise 2.1.1.8

In the picture lines AB and CD are parallel. Find the measures of the following angles. Explain your reasoning.

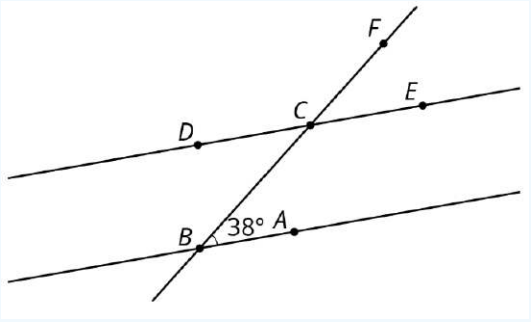


Figure 2.1.1.7: Three lines in a plane. Line A B. Line B F. Line D E. Lines B F and A B intersect at the point B. Lines B F and D E intersect at point C. Line D E is above line A B. Angle A B F is labeled 38 degrees.

1. $\angle BCD$
2. $\angle ECF$
3. $\angle DCF$

(From Unit 1.4.1)

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2.1.2: Circular Grid

Lesson

Let's dilate figures on circular grids.

Exercise 2.1.2.1: Notice and Wonder: Concentric Circles

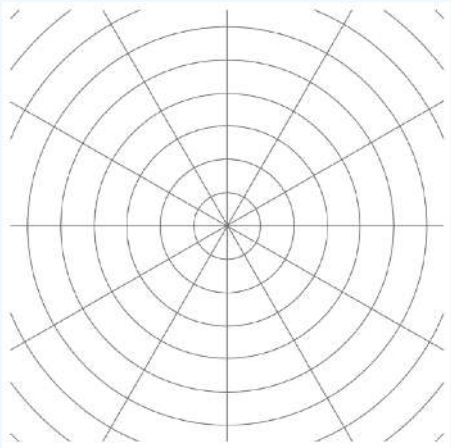


Figure 2.1.2.1

What do you notice? What do you wonder?

Exercise 2.1.2.2: A Droplet on the Surface

The larger Circle d is a **dilation** of the smaller Circle c. P is the **center of dilation**.

1. Draw four points *on* the smaller circle using the Point on Object tool.



2. Draw the rays from P through each of those four points. Select the Ray tool, then point P, and then the second point.



3. Mark the intersection points of the rays and Circle d by selecting the Intersect tool and clicking on the point of intersection.



4. Complete the table. In the row labeled S, write the distance between P and the point on the smaller circle in grid units. In the row labeled L, write the distance between P and the corresponding point on the larger circle in grid units. Measure the distances between pairs of points by selecting the Distance tool, and then clicking on the two points.



Table 2.1.2.1

	A	***B***	***C***	***D***
S				
L				

5. The center of dilation is point P. What is the *scale factor* that takes the smaller circle to the larger circle? Explain your reasoning.

Exercise 2.1.2.3: Quadrilateral on a Circular Grid

Here is a polygon $ABCD$

1. Dilate each vertec of polygon $ABCD$ using P as the center of dilation and a scale factor of 2.
2. Draw segments between the dilated points to create a new polygon.
3. What are some things you notice about the new polygon?
4. Choose a few more points on the sides of the original polygon and transform them using the same dilation. What do you notice?
5. Dilate each vertex of polygon $ABCD$ using P as the center of dilation and a scale factor of $\frac{1}{2}$.
6. What do you notice about this new polygon?

Are you ready for more?

Suppose P is a point not on line segment $\overline{WX}$. Let $\overline{YZ}$ be the dilation of line segment $\overline{WX}$ using P as the center with scale factor 2. Experiment using a circular grid to make predictions about whether each of the following statements must be true, might be true, or must be false.

1. $\overline{YZ}$ is twice as long $\overline{WX}$
2. $\overline{YZ}$ is five units longer than $\overline{WX}$
3. The point P is on $\overline{YZ}$
4. $\overline{YZ}$ and $\overline{WX}$ intersect.

Exercise 2.1.2.4: A Quadrilateral and Concentric Circles

1. Dilate polygon $EFGH$ using Q as the center of dilation and a scale factor of $\frac{1}{3}$. The image of F is already shown on the diagram. (You may need to draw more rays from Q in order to find the images of other points.)

Summary

A circular grid like this one can be helpful for performing **dilations**.

The radius of the smallest circle is one unit, and the radius of each successive circle is one unit more than the previous one.

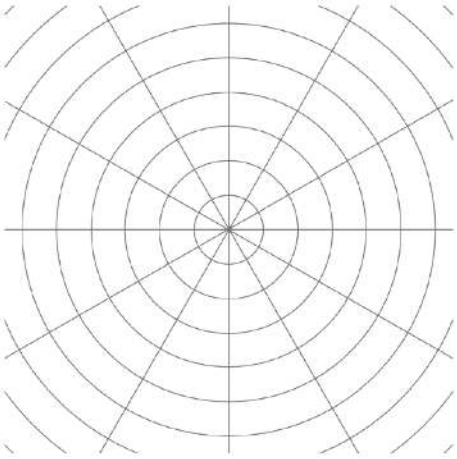


Figure 2.1.2.2

To perform a dilation, we need a **center of dilation**, a scale factor, and a point to dilate. In the picture, P is the center of dilation. With a scale factor of 2, each point stays on the same ray from P, but its distance from P doubles:

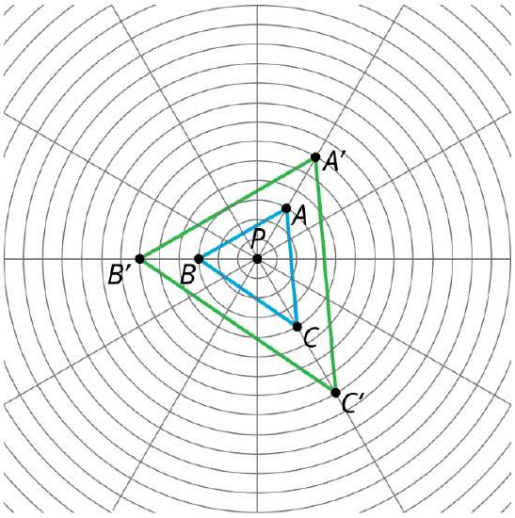


Figure 2.1.2.3

Since the circles on the grid are the same distance apart, segment PA' has twice the length of segment PA, and the same holds for the other points.

Glossary Entries

Definition: Center of a Dilation

The center of a dilation is a fixed point on a plane. It is the starting point from which we measure distances in a dilation.

In this diagram, point P is the center of the dilation.

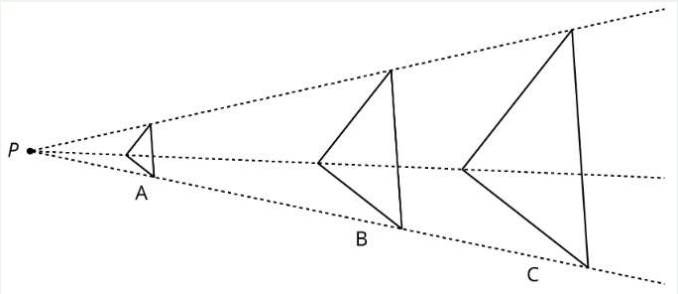


Figure 2.1.2.4

Definition: Dilation

A dilation is a transformation in which each point on a figure moves along a line and changes its distance from a fixed point. The fixed point is the center of the dilation. All of the original distances are multiplied by the same scale factor.

For example, triangle DEF is a dilation of triangle ABC. The center of dilation is O and the scale factor is 3.

This means that every point of triangle DEF is 3 times as far from O as every corresponding point of triangle ABC.

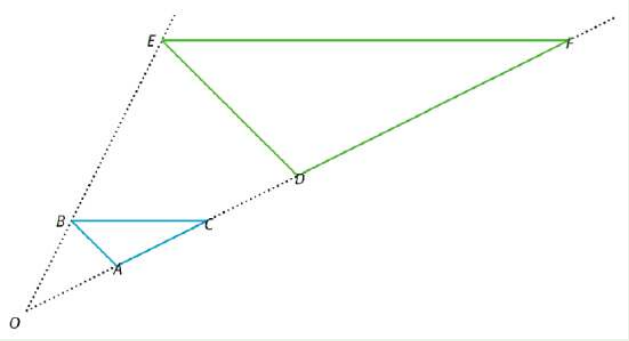


Figure 2.1.2.5

Definition: Scale Factor

To create a scaled copy, we multiply all the lengths in the original figure by the same number. This number is called the scale factor.

In this example, the scale factor is 1.5, because $4 \cdot (1.5) = 6$, $5 \cdot (1.5) = 7.5$, and $6 \cdot (1.5) = 9$.

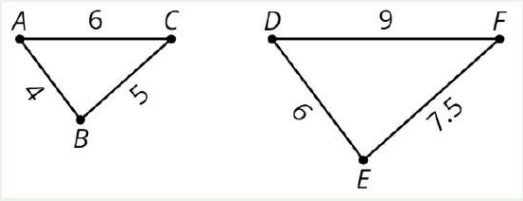


Figure 2.1.2.6

Practice

Exercise 2.1.2.5

Here are Circles c and d. Point O is the center of dilation, and the dilation takes Circle c to Circle d.

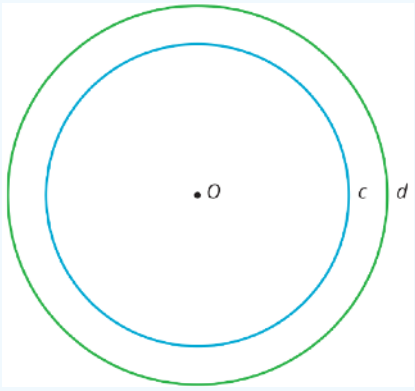


Figure 2.1.2.7

1. Plot a point on Circle c. Label the point P. Plot where P goes when the dilation is applied.
2. Plot a point on Circle d. Label the point Q. Plot a point that the dilation takes to Q.

Exercise 2.1.2.6

Here is a triangle ABC.

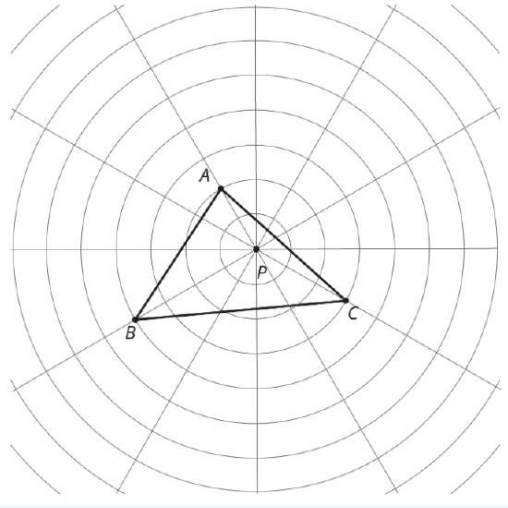


Figure 2.1.2.8

1. Dilate each vertex of triangle ABC using P as the center of dilation and a scale factor of 2. Draw the triangle connecting the three new points.
2. Dilate each vertex of triangle ABC using P as the center of dilation and a scale factor of $\frac{1}{2}$. Draw the triangle connecting the three new points.
3. Measure the longest side of each of the three triangles. What do you notice?
4. Measure the angles of each triangle. What do you notice?

Exercise 2.1.2.7

Describe a rigid transformation that you could use to show the polygons are congruent.

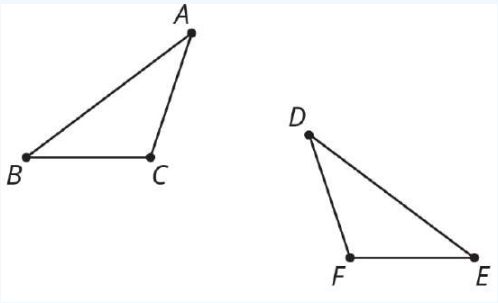


Figure 2.1.2.9

(From Unit 1.3.2)

Exercise 2.1.2.8

The line has been partitioned into three angles.

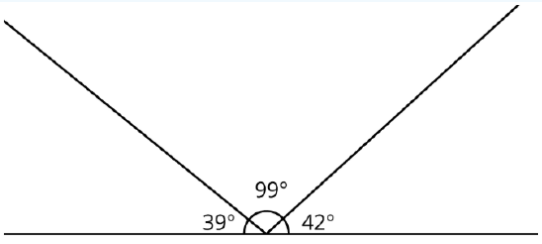


Figure 2.1.2.10: A straight line with two rays coming out of a single point. One slanting up and to the left. One slanting up and to the right. Three angles are formed. 39 degrees. 99 degrees. 42 degrees.

Is there a triangle with these three angle measures? Explain.

(From Unit 1.4.2)

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2.1.3: Dilations with no Grid

Lesson

Let's dilate figures not on grids.

Exercise 2.1.3.1: Points on a Ray

1. Find and label a point C on the ray whose distance from A is twice the distance from B to A.
2. Find and label a point D on the ray whose distance from A is half the distance from B to A.

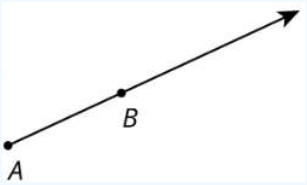


Figure 2.1.3.1

Exercise 2.1.3.2: Dilation Obstacle Course

1. Dilate B using a scale factor of 5 and A as the center of dilation. Which point is its image?
2. Using H as the center of dilation, dilate G so that its image is E. What scale factor did you use?
3. Using H as the center of dilation, dilate E so that its image is G. What scale factor did you use?
4. To dilate F so that its image is B, what point on the diagram can you use as a center?
5. Dilate H using A as the center and a scale factor of \frac{1}{3}\). Which point is its image?
6. Describe a dilation that uses a labeled point as its center and that would take F to H.
7. Using B as the center of dilation, dilate H so that its image is itself. What scale factor did you use?

Exercise 2.1.3.3: Getting Perspective

1. Dilate P using C as the center and a scale factor of 4. Follow the directions to perform the dilations in the applet.
   a. Select the Dilate From Point tool.

   b. Click on the object to dilate, and then click on the center of dilation.
   c. When the dialog box opens, enter the scale factor. Fractions can be written with plain text, ex. 1/2.
   d. Click

   e. Use the Ray tool and the Distance tool to verify.
2. Dilate Q using C as the center and a scale factor of $\frac{1}{2}$.
3. Draw a simple polygon.
   a. Choose a point outside the polygon to use as the center of dilation. Label it C.
   b. Using your center C and the scale factor you were given, draw the image under the dilation of each vertex of the polygon, one at a time. Connect the dilated vertices to create the dilated polygon.
   c. Draw a segment that connects each of the original vertices with its image. This will make your diagram look like a cool three-dimensional drawing of a box! If there's time, you can shade the sides of the box to make it look more realistic.
   d. Compare your drawing to other people's drawings. What is the same and what is different? How do the choices you made affect the final drawing? Was your dilated polygon closer to C than to the original polygon, or farther away? How is that decided?

Are you ready for more?

Here is line segment DE and its image $D'E'$ under a dilation.

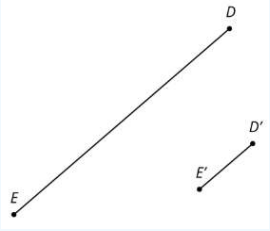


Figure 2.1.3.2

1. Use a ruler to find and draw the center of dilation. Label it F.
2. What is the scale factor of the dilation?

Summary

If A is the center of dilation, how can we find which point is the dilation of B with scale factor 2?



Figure 2.1.3.3

Since the scale factor is larger than 1, the point must be farther away from A than B is, which makes C the point we are looking for. If we measure the distance between A and C, we would find that it is exactly twice the distance between A and B.

A dilation with scale factor less than 1 brings points closer. The point D is the dilation of B with center A and scale factor $\frac{1}{3}$.

Glossary Entries

Definition: Center of a Dilation

The center of a dilation is a fixed point on a plane. It is the starting point from which we measure distances in a dilation.

In this diagram, point P is the center of the dilation.

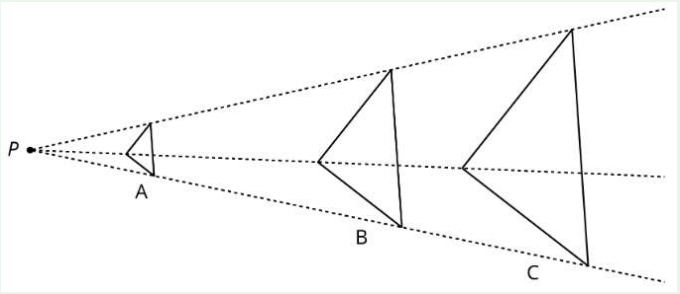


Figure 2.1.3.4

Definition: Dilation

A dilation is a transformation in which each point on a figure moves along a line and changes its distance from a fixed point. The fixed point is the center of the dilation. All of the original distances are multiplied by the same scale factor.

For example, triangle DEF is a dilation of triangle ABC. The center of dilation is O and the scale factor is 3.

This means that every point of triangle DEF is 3 times as far from O as every corresponding point of triangle ABC.

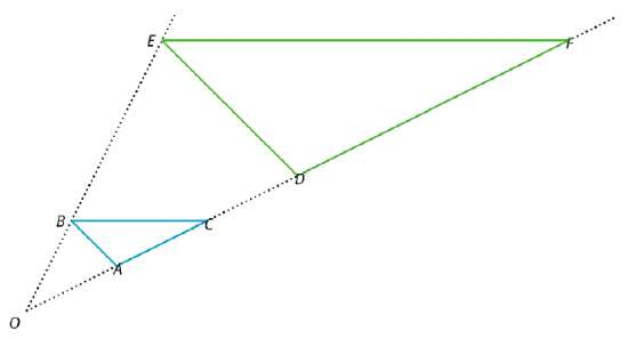


Figure 2.1.3.5

Definition: Scale Factor

To create a scaled copy, we multiply all the lengths in the original figure by the same number. This number is called the scale factor.

In this example, the scale factor is 1.5, because $4 \cdot (1.5) = 6$, $5 \cdot (1.5) = 7.5$, and $6 \cdot (1.5) = 9$.

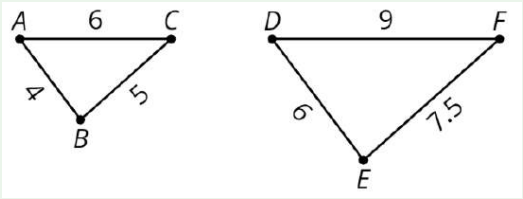


Figure 2.1.3.6

Practice

Exercise 2.1.3.4

Segment AB measures 3 cm. Point O is the center of dilation. How long is the image of AB after a dilation with ...

1. Scale factor 5?
2. Scale factor 3.7?
3. Scale factor $\frac{1}{5}$?
4. Scale factor s?

Exercise 2.1.3.5

Here are points A and B. Plot the points for each dilation described.

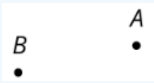


Figure 2.1.3.7

1. C is the image of B using A as the center of dilation and a scale factor of 2.
2. D is the image of A using B as the center of dilation and a scale factor of 2.
3. E is the image of B using A as the center of dilation and a scale factor of $\frac{1}{2}$.
4. F is the image of A using B as the center of dilation and a scale factor of $\frac{1}{2}$.

Exercise 2.1.3.6

Make a perspective drawing. Include in your work the center of dilation, the shape you dilate, and the scale factor you use.

Exercise 2.1.3.7

Triangle ABC is a scaled copy of triangle DEF. Side AB measures 12 cm and is the longest side of ABC. Side DE measures 8 cm and is the longest side of DEF.

1. Triangle ABC is a scaled copy of triangle DEF with what scale factor?
2. Triangle DEF is a scaled copy of triangle ABC with what scale factor?

(From Unit 2.1.1)

Exercise 2.1.3.8

The diagram shows two intersecting lines.

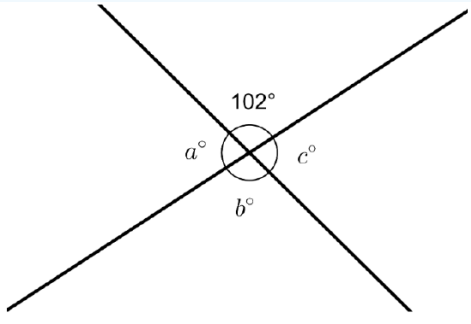


Figure 2.1.3.8: Two intersecting lines, forming a X. The top angle is labeled 102 degrees. The left angle is labeled a degrees. The bottom angle is labeled b degrees. The right angle is labeled c degrees.

Find the missing angle measures.

(From Unit 1.4.1)

Exercise 2.1.3.9

1. Show that the two triangles are congruent.
2. Find the side lengths of DEF and the angle measures of ABC.

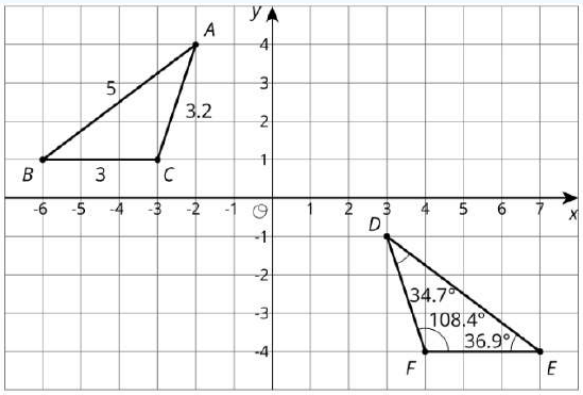


Figure 2.1.3.9: Two triangles in an x y plane. Origin O. Horizontal axis from negative 6 to 6, by 1s. Vertical axis from negative 4 to 4, by 1s. Triangle A B C. Point A at negative 2 comma 4. Point B at negative 6 comma 1. Point C at negative 3 comma 1. Side A C is labeled 3 point 2. Side C B is labeled 3. Side A B is labeled 5. Triangle D E F. Point D at 3 comma negative 1. Point F at 4 comma negative 4. Point E at 7 comma negative 4. Angle F E D is labeled 36 point 9 degrees. Angle E D F is labeled 34 point 7 degrees. Angle D F E is labeled 108 point 4 degrees.

(From Unit 1.2.1)

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2.1.4: Dilations on a Square Grid

Lesson

Let's dilate figures on a square grid.

Exercise 2.1.4.1: Estimating a Scale Factor

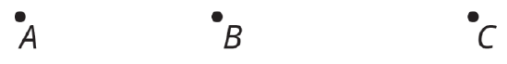


Figure 2.1.4.1

Point C is the dilation of point B with center of dilation and scale factor . Estimate . Be prepared to explain your reasoning.

Exercise 2.1.4.2: Dilations on a Grid

1. Find the dilation of quadrilateral $ABCD$ with center P and scale factor 2.

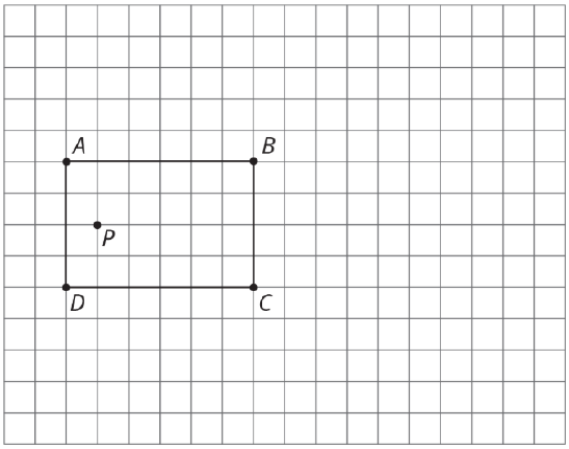


Figure 2.1.4.2

2. Find the dilation of triangle QRS with center T and scale factor 2.

3. Find the dilation of triangle QRS with center T and scale factor $\frac{1}{2}$.

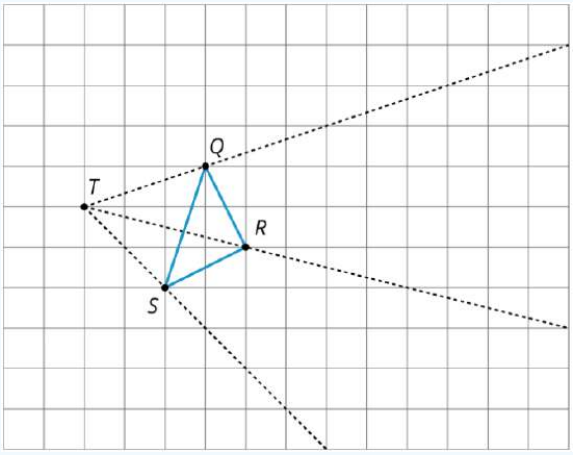


Figure 2.1.4.3: Point T, triangle Q R S and three projection rays on a square grid. Let the lower left corner be (0 comma 0). Then triangle Q R S is Q(5 comma 7), R(6 comma 5), S(4 comma 4) and point T is T(2 comma 6). The dashed projection rays are T Q, T R and T S.

Exercise 2.1.4.3: Card Sort: Matching Dilations on a Coordinate Grid

Your teacher will give you some cards. Each of Cards 1 through 6 shows a figure in the coordinate plane and describes a dilation.

Each of Cards A through E describes the image of the dilation for one of the numbered cards.

Match number cards with letter cards. One of the number cards will not have a match. For this card, you'll need to draw an image.

Are you ready for more?

The image of a circle under dilation is a circle when the center of the dilation is the center of the circle. What happens if the center of dilation is a point on the circle? Using center of dilation $(0, 0)$ and scale factor 1.5, dilate the circle shown on the diagram. This diagram shows some points to try dilating.

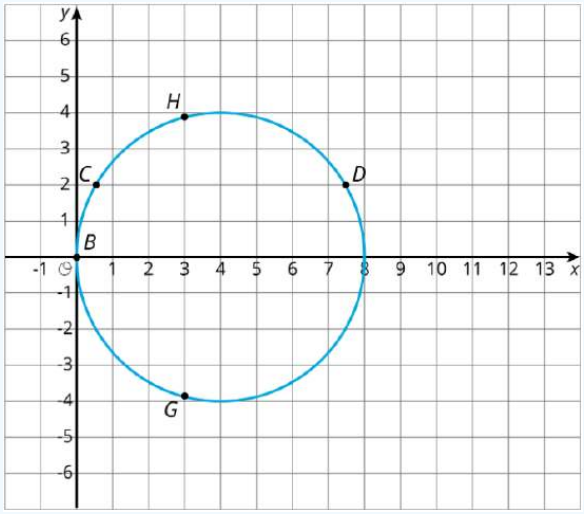


Figure 2.1.4.4: Five points on a circle on a coordinate plane, origin O. Horizontal axis scale negative 1 to 13 by 1's. Vertical axis scale negative 6 to 6 by 1's. Circle has radius 4 and is centered at (4 comma 0). The five points on the circle are B(0 comma 0), C(point 5 comma 2), H(3 comma 4), D(7 point 5 comma 2) and G(3 comma negative 4).

Summary

Square grids can be useful for showing dilations. The grid is helpful especially when the center of dilation and the point(s) being dilated lie at grid points. Rather than using a ruler to measure the distance between the points, we can count grid units.

For example, suppose we want to dilate point Q with center of dilation P and scale factor $\frac{3}{2}$. Since Q is 4 grid squares to the left and 2 grid squares down from P, the dilation will be 6 grid squares to the left and 3 grid squares down from P' (can you see why?). The dilated image is marked as Q' in the picture.

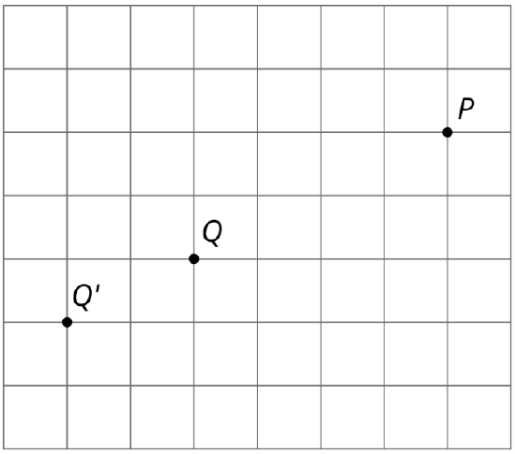


Figure 2.1.4.5

Sometimes the square grid comes with coordinates. The coordinate grid gives us a convenient way to *name* points, and sometimes the coordinates of the image can be found with just arithmetic.

For example, to make a dilation with center $(0, 0)$ and scale factor 2 of the triangle with coordinates $(-1, -2)$, $(3, 1)$, and $(2, -1)$, we can just double the coordinates to get $(-2, -4)$, $(6, 2)$, and $(4, -2)$.

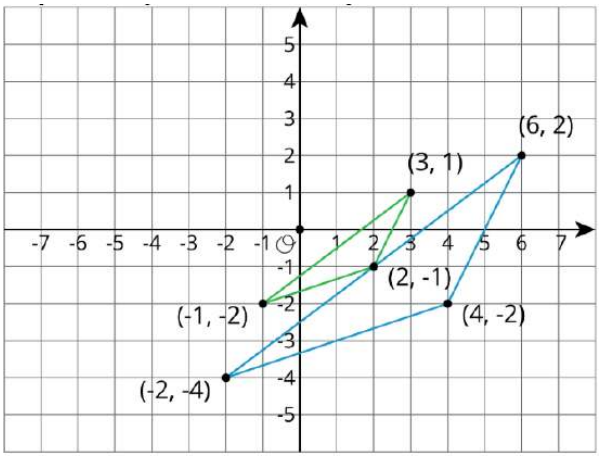


Figure 2.1.4.6: Two triangles on a coordinate plane, origin O. Horizontal axis scale negative 7 to 7 by 1's. Vertical axis scale negative 5 to 5 by 1's. The coordinates of the triangle are (negative 1 comma negative 2), (3 comma 1), (2 comma negative 1). The coordinates of the image are(negative 2 comma negative 4), (6 comma 2), (4 comma negative 2).

Glossary

Definition: Center of a Dilation

The center of a dilation is a fixed point on a plane. It is the starting point from which we measure distances in a dilation.

In this diagram, point P is the center of the dilation.

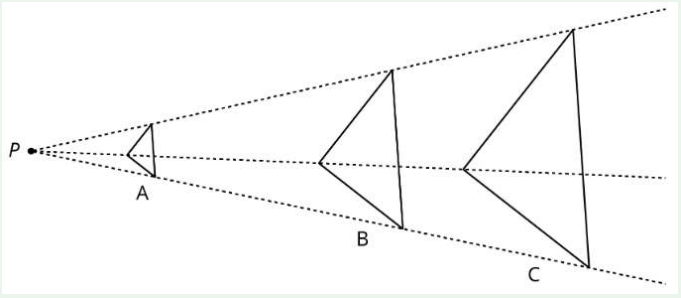


Figure 2.1.4.4

Definition: Dilation

A dilation is a transformation in which each point on a figure moves along a line and changes its distance from a fixed point. The fixed point is the center of the dilation. All of the original distances are multiplied by the same scale factor.

For example, triangle DEF is a dilation of triangle ABC. The center of dilation is O and the scale factor is 3.

This means that every point of triangle DEF is 3 times as far from O as every corresponding point of triangle ABC.

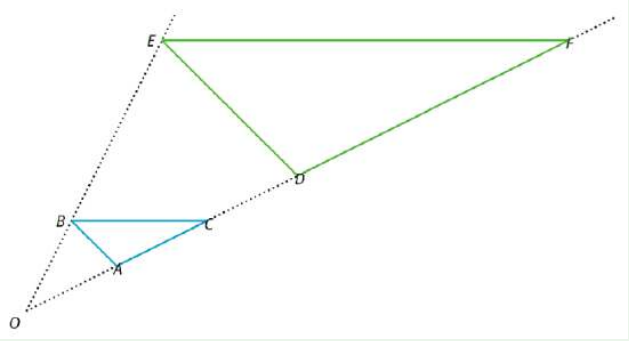


Figure 2.1.4.5

Definition: Scale Factor

To create a scaled copy, we multiply all the lengths in the original figure by the same number. This number is called the scale factor.

In this example, the scale factor is 1.5, because $4 \cdot (1.5) = 6$, $5 \cdot (1.5) = 7.5$, and $6 \cdot (1.5) = 9$.

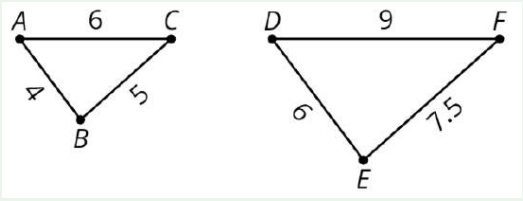


Figure 2.1.4.6

Practice

Exercise 2.1.4.4

Triangle ABC is dilated using D as the center of dilation with scale factor 2.

The image is triangle $A'B'C'$. Clare says the two triangles are congruent, because their angle measures are the same. Do you agree? Explain how you know.

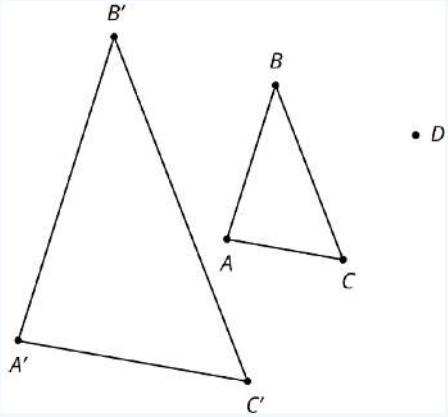


Figure 2.1.4.7: Triangle A B C, it's image after dilation triangle A prime B prime C prime and point D. Point D is on the right, smaller triangle A B C is in the middle and larger image triangle A prime, B prime and C prime is on the right.

Exercise 2.1.4.5

On graph paper, sketch the image of quadrilateral PQRS under the following dilations:

1. The dilation centered at R with scale factor 2.
2. The dilation centered at O with scale factor $\frac{1}{2}$.

3. The dilation centered at S with scale factor $\frac{1}{2}$.

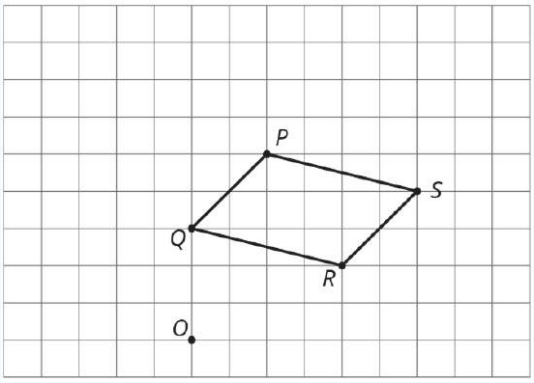


Figure 2.1.4.8: Quadrilateral P Q R S and point O on a square grid. Let the lower left corner be (0 comma 0). Then the coordinates of P Q R S are P(7 comma 6), Q(5 comma 4), R(9 comma 3), S(11 comma 5). The coordinates of point O are O(5 comma 1).

Exercise 2.1.4.6

The diagram shows three lines with some marked angle measures.

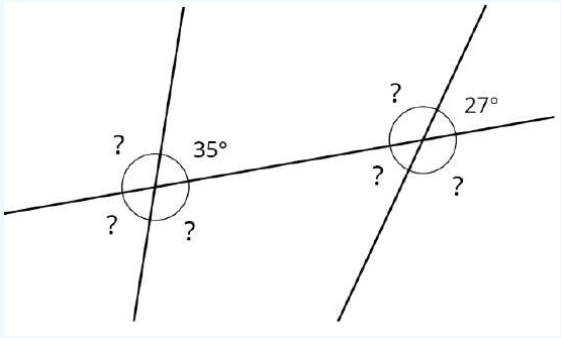


Figure 2.1.4.9:

A line slanting up and to the right. Two lines intersect this line, both slanting up and to the right, but do not intersect. At the first intersection, the top right angle is labeled 35 degrees. The rest are labeled ?. At the second intersection, the top right angle is labeled 27 degrees. The rest are labeled ?.

Find the missing angle measures marked with question marks.

(From Unit 1.4.4)

Exercise 2.1.4.7

Describe a sequence of translations, rotations, and reflections that takes Polygon P to Polygon Q.

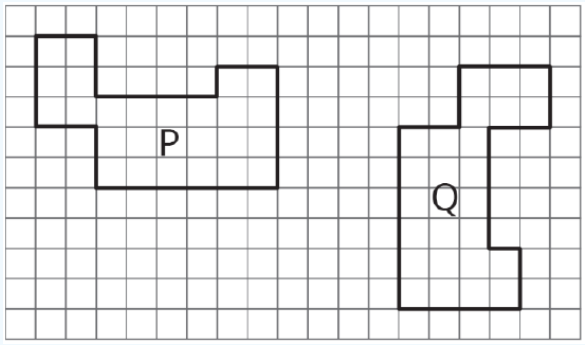


Figure 2.1.4.10

(From Unit 1.1.4)

Exercise 2.1.4.8

Point B has coordinates $(-2, -5)$. After a translation 4 units down, a reflection across the -axis, and a translation 6 units up, what are the coordinates of the image?

(From Unit 1.1.6)

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2.1.5: More Dilations

Lesson

Let's look at dilations in the coordinate plane.

Exercise 2.1.5.1: Many Dilations of a Triangle

Explore the applet and observe the dilation of triangle ABC. The dilation always uses center P, but you can change the scale factor. What connections can you make between the scale factor and the dilated triangle?

Exercise 2.1.5.2: Info Gap: Dilations

Your teacher will give you either a *problem card* or a *data card*. Do not show or read your card to your partner.

If your teacher gives you the *problem card*:

1. Silently read your card and think about what information you need to be able to answer the question.
2. Ask your partner for the specific information that you need.
3. Explain how you are using the information to solve the problem.
4. Continue to ask questions until you have enough information to solve the problem.
5. Share the *problem card* and solve the problem independently.
6. Read the *data card* and discuss your reasoning.

If your teacher gives you the *data card*:

1. Silently read your card.
2. Ask your partner "*What specific information do you need?*" and wait for them to *ask* for information.

   If your partner asks for information that is not on the card, do not do the calculations for them. Tell them you don't have that information.

3. Before sharing the information, ask "*Why do you need that information?*" Listen to your partner's reasoning and ask clarifying questions.
4. Read the *problem card* and solve the problem independently.
5. Share the *data card* and discuss your reasoning.

Pause here so your teacher can review your work. Ask your teacher for a new set of cards and repeat the activity, trading roles with your partner.

Are you ready for more?

Triangle EFG was created by dilating triangle ABC using a scale factor of 2 and center D. Triangle HIJ was created by dilating triangle ABC using a scale factor of $\frac{1}{2}$ and center D.

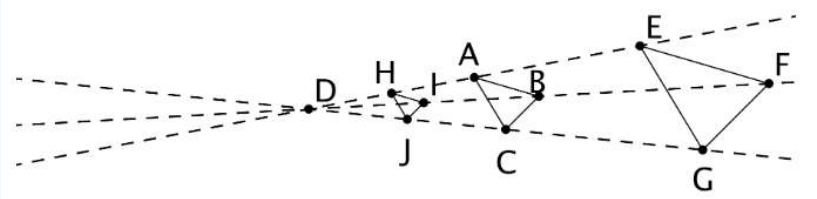


Figure 2.1.5.1: A triangle A B C, two images after dilation, point D and three dashed projection lines. The dashed lines meet near the middle at point D, then to right are the increasing in size triangles, H I J, A B C and E F G. The upper line goes through points H, A and E. The nearly horizontal middle line goes through points I, B and F. The lower line goes through points J, C and G.

1. What would the image of triangle ABC look like under a dilation with scale factor 0?
2. What would the image of the triangle look like under dilation with a scale factor of -1? If possible, draw it and label the vertices A', B', and C'. If it's not possible, explain why not.
3. If possible, describe what happens to a shape if it is dilated with a negative scale factor. If dilating with a negative scale factor is not possible, explain why not.

Summary

One important use of coordinates is to communicate geometric information precisely. Let's consider a quadrilateral $ABCD$ in the coordinate plane. Performing a dilation of $ABCD$ requires three vital pieces of information:

1. The coordinates of A, B, C, and D
2. The coordinates of the center of dilation, P
3. The scale factor of the dilation

With this information, we can dilate the vertices A, B, C, and D and then draw the corresponding segments to find the dilation of $ABCD$. Without coordinates, describing the location of the new points would likely require sharing a picture of the polygon and the center of dilation.

Glossary Entries

Definition: Center of a Dilation

The center of a dilation is a fixed point on a plane. It is the starting point from which we measure distances in a dilation.

In this diagram, point P is the center of the dilation.

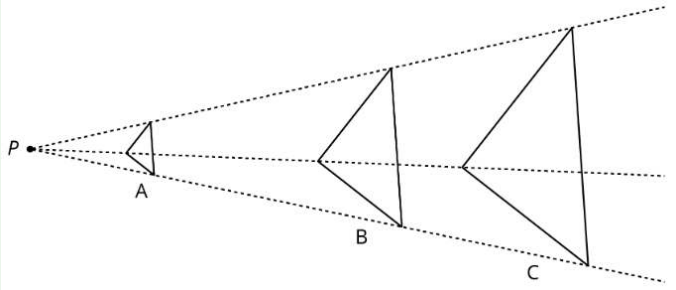


Figure 2.1.5.2

Definition: Dilation

A dilation is a transformation in which each point on a figure moves along a line and changes its distance from a fixed point. The fixed point is the center of the dilation. All of the original distances are multiplied by the same scale factor.

For example, triangle DEF is a dilation of triangle ABC. The center of dilation is O and the scale factor is 3.

This means that every point of triangle DEF is 3 times as far from O as every corresponding point of triangle ABC.

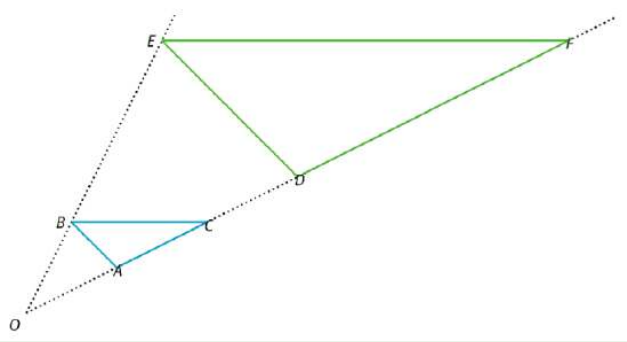


Figure 2.1.5.3

Definition: Scale Factor

To create a scaled copy, we multiply all the lengths in the original figure by the same number. This number is called the scale factor.

In this example, the scale factor is 1.5, because $4 \cdot (1.5) = 6$, $5 \cdot (1.5) = 7.5$, and $6 \cdot (1.5) = 9$.

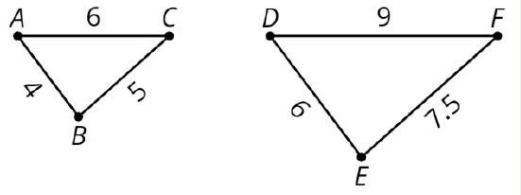


Figure 2.1.5.4

Practice

Exercise 2.1.5.3

Quadrilateral $ABCD$ is dilated with center $(0, 0)$ taking B to B'. Draw $A'B'C'D'$.

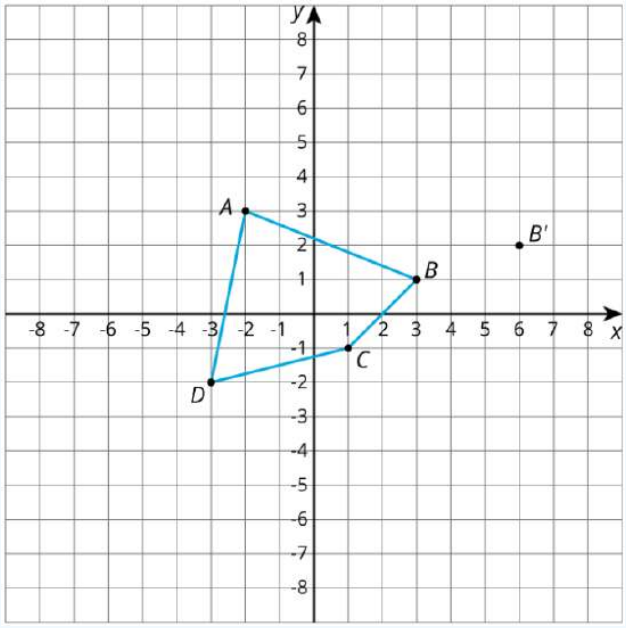


Figure 2.1.5.5

Exercise 2.1.5.4

Triangles B and C have been built by dilating Triangle A.

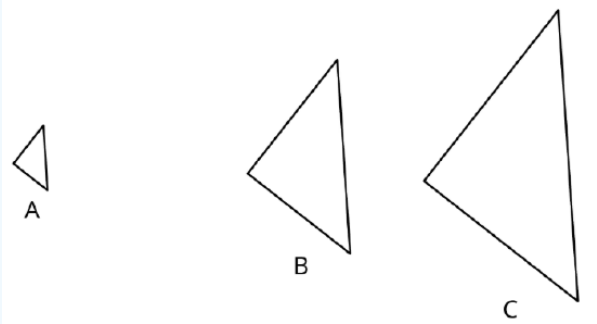


Figure 2.1.5.6

1. Find the center of dilation.
2. Triangle B is a dilation of A with approximately what scale factor?
3. Triangle A is a dilation of B with approximately what scale factor?
4. Triangle B is a dilation of C with approximately what scale factor?

Exercise 2.1.5.5

Here is a triangle.

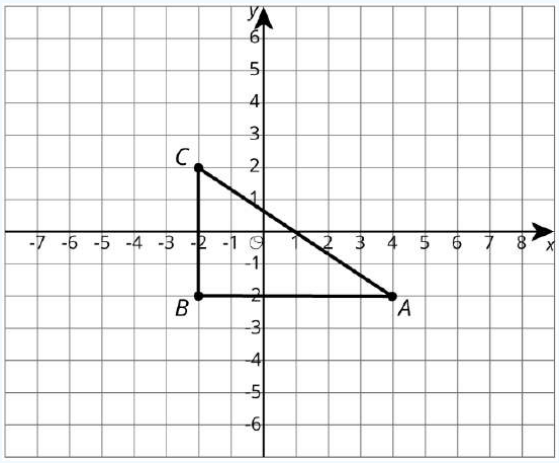


Figure 2.1.5.7

1. Draw the dilation of triangle ABC, with center $(0, 0)$, and scale factor 2. Label this triangle $A'B'C'$.
2. Draw the dilation of triangle ABC, with center $(0, 0)$, and scale factor $\frac{1}{2}$. Label this triangle $A''B''C''$.
3. Is $A''B''C''$ a dilation of triangle $A'B'C'$? If yes, what are the center of dilation and the scale factor?

Exercise 2.1.5.1

Triangle DEF is a right triangle, and the measure of angle D is 28°. What are the measures of the other two angles?

(From Unit 1.4.2)

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2.2: Similarity

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2.2.1: Similarity

Lesson

Let's explore similar figures.

Exercise 2.2.1.1: Equivalent Expressions

Use what you know about operations and their properties to write three expressions equivalent to the expression shown.

$10(2+3)-8\cdot 3$

Exercise 2.2.1.2: Similarity Transformations (Part 1)

1. Triangle EGH and triangle LME are **similar**. Find a sequence of tanslations, rotations, reflections, and dilations that shows this.

2. Hexagon $ABCDEF$ and hexagon $HGLKJI$ are similar. Find a sequence of translations, rotations, reflections, and dilations that shows this.

Are you ready for more?

The same sequence of transformations takes Triangle A to Triangle B, takes Triangle B to Triangle C, and so on. Describe a sequence of transformations with this property.

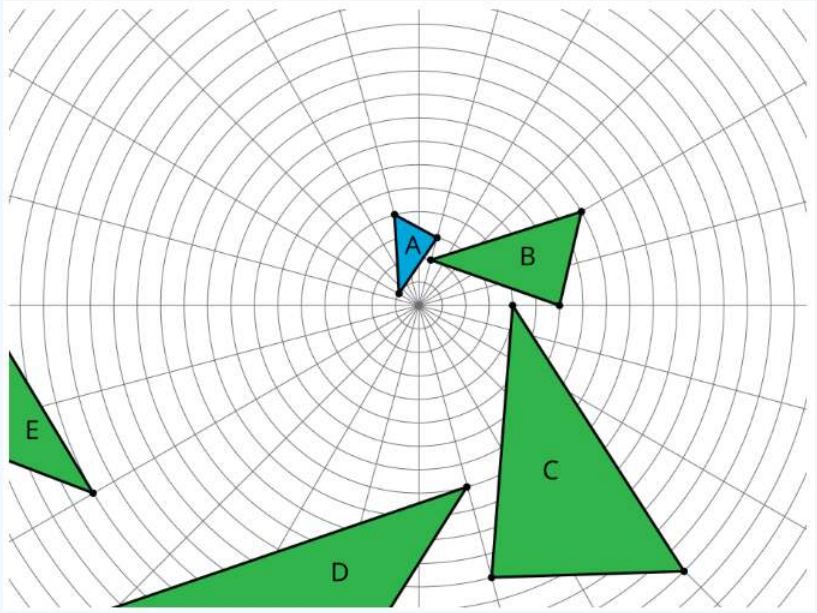


Figure 2.2.1.1

Exercise 2.2.1.3: Similarity Transformations (Part 2)

Sketch figures similar to Figure A that use only the transformations listed to show similarity.

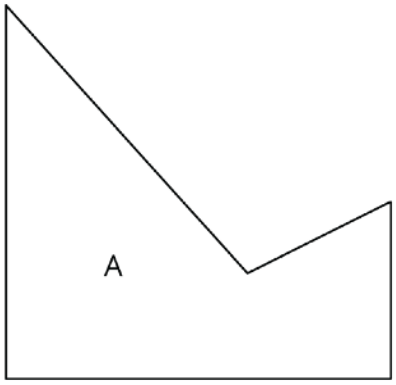


Figure 2.2.1.2

1. A translation and a reflection. Label your sketch Figure B.
   Pause here so your teacher can review your work.
2. A reflection and a dilation with scale factor greater than 1. Label your sketch Figure C.
3. A rotation and a reflection. Label your sketch Figure D.
4. A dilation with scale factor less than 1 and a translation. Label your sketch Figure E.

Exercise 2.2.1.4: Methods for Translations and Dilations

Your teacher will give you a set of five cards and your partner a different set of five cards. Using only the cards you were given, find at least one way to show that triangle ABC and triangle DEF are similar. Compare your method with your partner's method. What is the same about your methods? What is different?

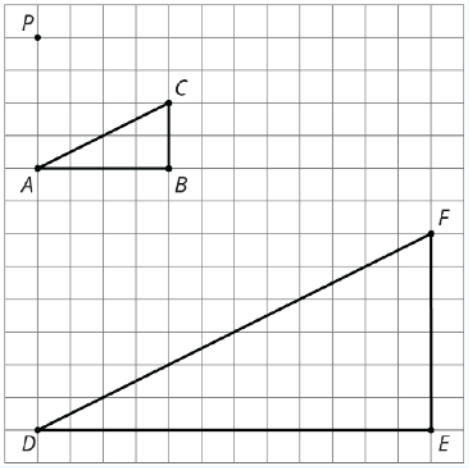


Figure 2.2.1.3

Summary

Let's show that triangle ABC is similar to triangle DEF:

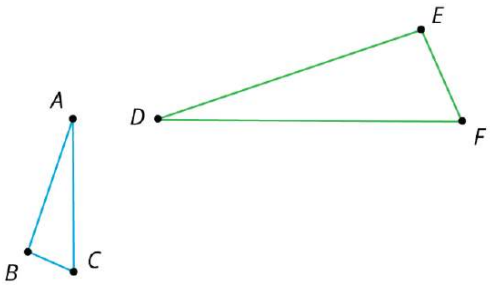


Figure 2.2.1.4

Two figures are **similar** if one figure can be transformed into the other by a sequence of translations, rotations, reflections, and dilations. There are many correct sequences of transformations, but we only need to describe one to show that two figures are similar.

One way to get from ABC to DEF follows these steps:

- step 1: reflect across line f
- step 2: rotate 90° counterclockwise around D
- step 3: dilate with center D and scale factor 2

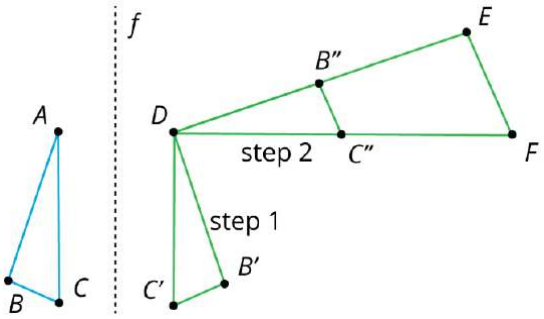


Figure 2.2.1.5

Another way would be to dilate triangle ABC by a scale factor of 2 with center of dilation A, then translate A to D, then reflect over a vertical line through D, and finally rotate it so it matches up with triangle DEF. What steps would you choose to show the two triangles are similar?

Glossary Entries

Definition: Similar

Two figures are similar if one can fit exactly over the other after rigid transformations and dilations.

In this figure, triangle ABC is similar to triangle DEF.

If ABC is rotated around point B and then dilated with center point O, then it will fit exactly over DEF. This means that they are similar.

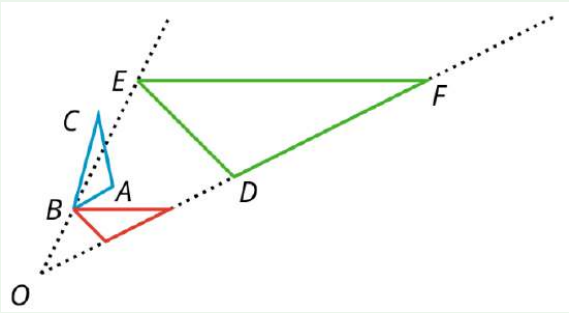


Figure 2.2.1.6

Practice

Exercise 2.2.1.5

Each diagram has a pair of figures, one larger than the other. For each pair, show that the two figures are similar by identifying a sequence of translations, rotations, reflections, and dilations that takes the smaller figure to the larger one.

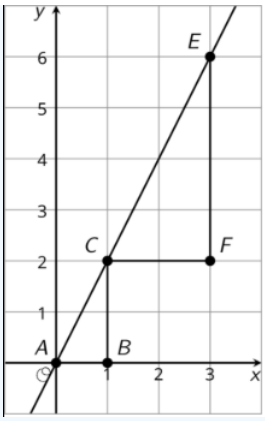


Figure 2.2.1.7

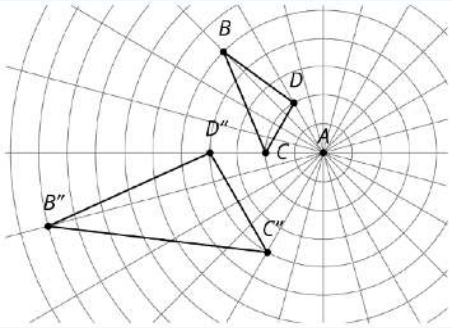


Figure 2.2.1.8

Exercise 2.2.1.6

Here are two similar polygons.

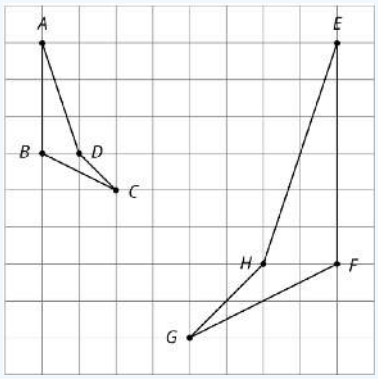


Figure 2.2.1.9: figure ABCD and figure EFGH on a 10 by 10 grid. letting bottom left corner = 0 comma 0, A = 1 comma 9, B = 1 comma 6, C = 3 comma 5, D = 2 comma 6, E = 9 comma 9, F = 9 comma 3, G = 5 comma 1, H = 7 comma 3.

Measure the side lengths and angles of each polygon. What do you notice?

Exercise 2.2.1.7

Each figure shows a pair of similar triangles, one contained in the other. For each pair, describe a point and a scale factor to use for a dilation moving the larger triangle to the smaller one. Use a measurement tool to find the scale factor.

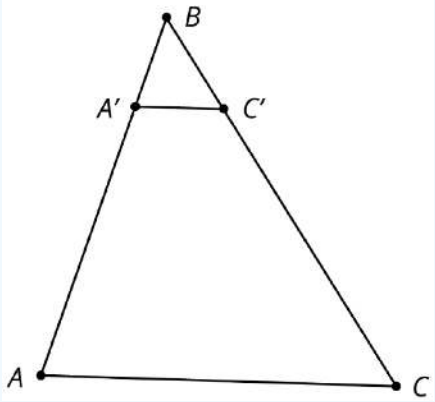


Figure 2.2.1.10

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2.2.2: Similar Polygons

Lesson

Let's look at sides and angles of similar polygons

Exercise 2.2.2.1: All, Some, None: Congruence and Similarity

Choose whether each of the statements is true in *all* cases, in *some* cases, or in *no* cases.

1. If two figures are congruent, then they are similar.
2. If two figures are similar, then they are congruent.
3. If an angle is dilated with the center of dilation at its vertex, the angle measure may change.

Exercise 2.2.2.2: Are They Similar?

1.

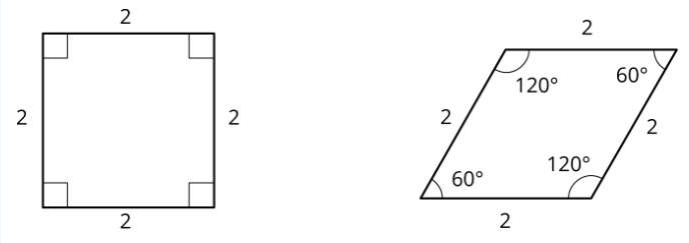


Figure 2.2.2.1

Let's look at a square and a rhombus.

Priya says, "These polygons are similar because their side lengths are all the same." Clare says, "These polygons are not similar because the angles are different." Do you agree with either Priya or Clare? Explain your reasoning.

2. Now let's look at rectangles $ABCD$ and $EFGH$.

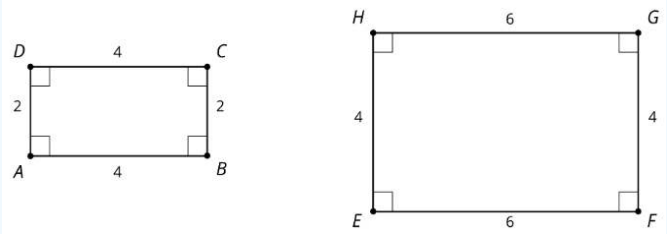


Figure 2.2.2.2

Jada says, "These rectangles are similar because all of the side lengths differ by 2." Lin says, "These rectangles are similar. I can dilate AD and BC using a scale factor of 2 and AB and CD using a scale factor of 1.5 to make the rectangles congruent. Then I can use a translation to line up the rectangles." Do you agree with either Jada or Lin? Explain your reasoning.

Are you ready for more?

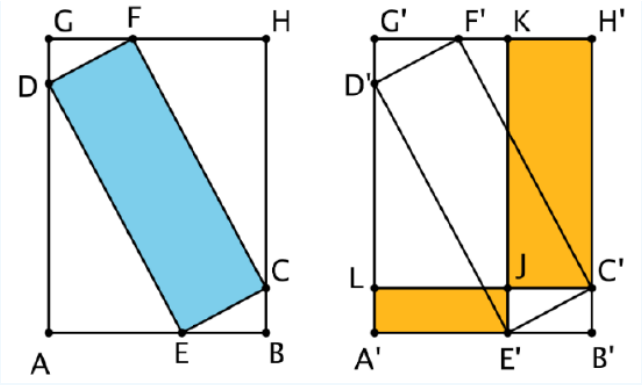


Figure 2.2.2.3: A rectangle is formed by sides G F H and H C B and B E A and A D G. Quadrilateral F C E D is also a rectangle. The shape is translated right to points A, prime through H prime. K lies between F prime and H prime. J is positioned to form rectangle K H prime C prime J and rectangle J E prime A, prime L where L lies between D prime and A prime.

Points A through H are translated to the right to create points A' through H'. All of the following are rectangles: $GHBA$, $FCED$, $KH'C'J$, and $LJE'A'$. Which is greater, the area of blue rectangle $DFCE$ or the total area of yellow rectangles $KH'C'J$ and $LJE'A'$?

Exercise 2.2.2.3: Find Someone Similar

Your teacher will give you a card. Find someone else in the room who has a card with a polygon that is similar but not congruent to yours. When you have found your partner, work with them to explain how you know that the two polygons are similar.

Are you ready for more?

On the left is an equilateral triangle where dashed lines have been added, showing how you can partition an equilateral triangle into smaller similar triangles.

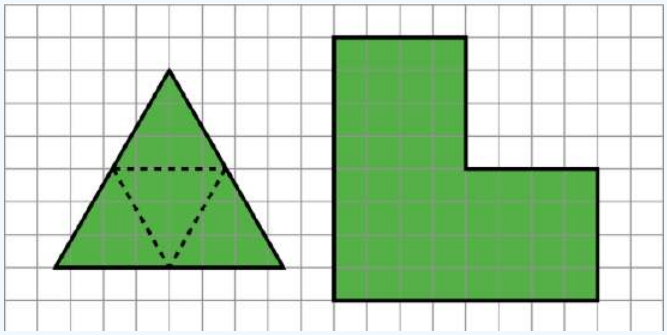


Figure 2.2.2.4: Two figures on a grid. First, an equilateral triangle a dotted line half way between the base and the top vertex parallel to the base. Dotted lines connect each end of the first dotted line to the center of the base. Second, an L shape formed by taking a square with 8 units on each side and removing a 4 by 4 square from the top right corner.

Find a way to do this for the figure on the right, partitioning it into smaller figures which are each similar to that original shape. What's the fewest number of pieces you can use? The most?

Summary

When two polygons are similar:

- Every angle and side in one polygon has a corresponding part in the other polygon.
- All pairs of corresponding angles have the same measure.
- Corresponding sides are related by a single scale factor. Each side length in one figure is multiplied by the scale factor to get the corresponding side length in the other figure.

Consider the two rectangles shown here. Are they similar?

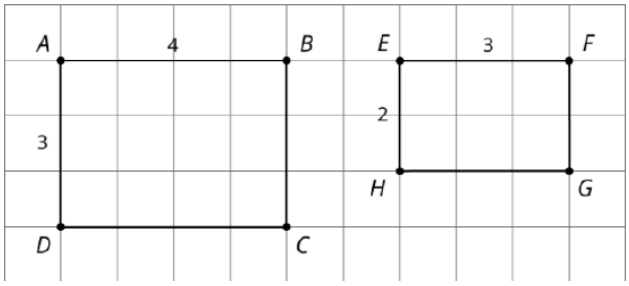


Figure 2.2.2.5

It looks like rectangles $ABCD$ and $EFGH$ could be similar, if you match the long edges and match the short edges. All the corresponding angles are congruent because they are all right angles. Calculating the scale factor between the sides is where we see that "looks like" isn't enough to make them similar. To scale the long side AB to the long side EF, the scale factor must be $\frac{3}{4}$, because $4 \cdot \frac{3}{4} = 3$. But the scale factor to match AD to EH has to be $\frac{2}{3}$, because $3 \cdot \frac{2}{3} = 2$. So, the rectangles are not similar because the scale factors for all the parts must be the same.

Here is an example that shows how sides can correspond (with a scale factor of 1), but the quadrilaterals are not similar because the angles don't have the same measure:

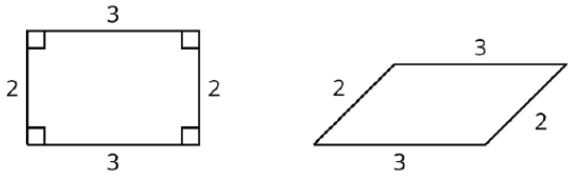


Figure 2.2.2.6

Glossary Entries

Definition: Similar

Two figures are similar if one can fit exactly over the other after rigid transformations and dilations.

In this figure, triangle ABC is similar to triangle DEF.

If ABC is rotated around point B and then dilated with center point O, then it will fit exactly over DEF. This means that they are similar.

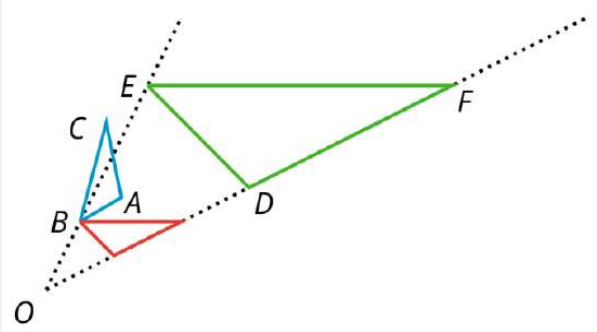


Figure 2.2.2.7

Practice

Exercise 2.2.2.4

Triangle DEF is a dilation of triangle ABC with scale factor 2. In triangle ABC, the largest angle measures $82°$. What is the largest angle measure in triangle DEF?

A. $41°$
B. $82°$

C. 123°
D. 164°

Exercise 2.2.2.5

Draw two polygons that are similar but could be mistaken for not being similar. Explain why they are similar.

Exercise 2.2.2.6

Draw two polygons that are *not* similar but could be mistaken for being similar. Explain why they are not similar.

Exercise 2.2.2.7

These two triangles are similar. Find side lengths a and b. Note: the two figures are not drawn to scale.

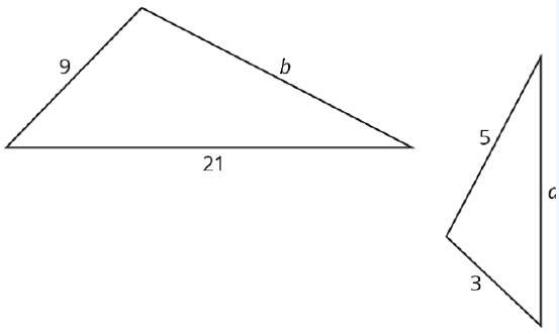


Figure 2.2.2.8

Exercise 2.2.2.8

Jada claims that $B'C'D'$ is a dilation of BCD using A as the center of dilation.

What are some ways you can convince Jada that her claim is not true?

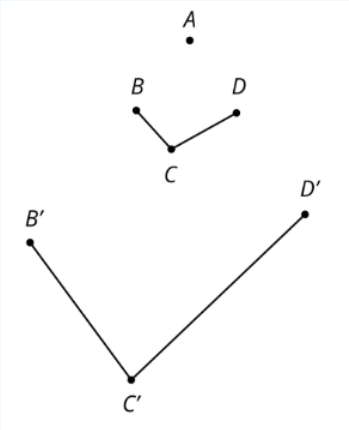


Figure 2.2.2.9: Point A, angle B C D and angle image B prime, C prime, D prime. Point A is above, then B C D and under all is B prime C prime D prime. B C D is smaller and appears to be obtuse. B prime C prime D prime is three times larger and appears to be right.

(From Unit 2.1.3)

Exercise 2.2.2.9

1. Draw a horizontal line segment AB.
2. Rotate segment $AB 90^\circ$ counterclockwise around point A. Label any new points.
3. Rotate segment $AB 90^\circ$ clockwise around point B. Label any new points.

https://math.libretexts.org/@go/page/35699

4. Describe a transformation on segment AB you could use to finish building a square.

(From Unit 1.2.2)

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2.2.3: Similar Triangles

Lesson

Let's look at similar triangles.

Exercise 2.2.3.1: Equivalent Expressions

Create three different expressions that are each equal to 20. Each expression should include only these three numbers: 4, -2, and 10.

Exercise 2.2.3.2: Making Pasta Angles and Triangles

Your teacher will give you some dried pasta and a set of angles.

1. Create a triangle using three pieces of pasta and angle A. Your triangle *must* include the angle you were given, but you are otherwise free to make any triangle you like. Tape your pasta triangle to a sheet of paper so it won't move.
   a. After you have created your triangle, measure each side length with a ruler and record the length on the paper next to the side. Then measure the angles to the nearest 5 degrees using a protractor and record these measurements on your paper.
   b. Find two others in the room who have the same angle A and compare your triangles. What is the same? What is different? Are the triangles congruent? Similar?
   c. How did you decide if they were or were not congruent or similar?
2. Now use more pasta and angles A, B, and C to create another triangle. Tape this pasta triangle on a separate sheet of paper.
   a. After you have created your triangle, measure each side length with a ruler and record the length on the paper next to the side. Then measure the angles to the nearest 5 degrees using a protractor and record these measurements on your paper.
   b. Find two others in the room who used your same angles and compare your triangles. What is the same? What is different? Are the triangles congruent? Similar?
   c. How did you decide if they were or were not congruent or similar?
3. Here is triangle PQR. Break a new piece of pasta, different in length than segment PQ.

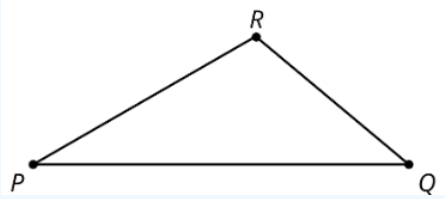


Figure 2.2.3.1

- Tape the piece of pasta so that it lays on top of line PQ with one end of the pasta at P (if it does not fit on the page, break it further). Label the other end of the piece of pasta S.
- Tape a full piece of pasta, with one end at S, making an angle congruent to $\angle PQR$.
- Tape a full piece of pasta on top of line PR with one end of the pasta at P. Call the point where the two full pieces of pasta meet T.
  a. Is your new pasta triangle PST similar to ΔPQR? Explain your reasoning.
  b. If your broken piece of pasta were a different length, would the pasta triangle still be similar to ΔPQR? Explain your reasoning.

Are you ready for more?

Quadrilaterals $ABCD$ and $EFGH$ have four angles measuring 240°, 40°, 40°, and 40°. Do and have to be similar?

Exercise 2.2.3.3: Similar Figures in a Regular Pentagon

1. This diagram has several triangles that are similar to triangle DJI.

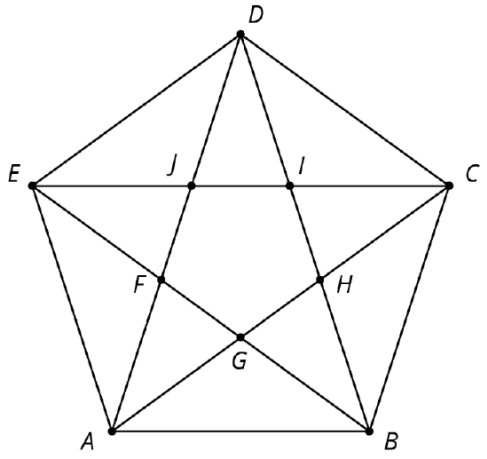


Figure 2.2.3.2

   a. Three different scale factors were used to make triangles similar to DJI. In the diagram, find at least one triangle of each size that is similar to DJI.
   b. Explain how you know each of these three triangles is similar to DJI.

2. Find a triangle in the diagram that is not similar to DJI.

Are you ready for more?

Figure out how to draw some more lines in the pentagon diagram to make more triangles similar to DJI.

Summary

We learned earlier that two polygons are similar when there is a sequence of translations, rotations, reflections, and dilations taking one polygon to the other. When the polygons are triangles, we only need to check that that both triangles have two corresponding angles to show they are similar—can you tell why?

Here is an example. Triangle ABC and triangle DEF each have a 30 degree angle and a 45 degree angle.

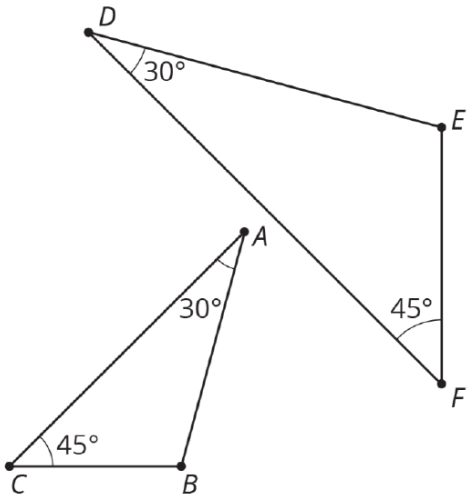


Figure 2.2.3.3

We can translate A to D and then rotate so that the two 30 degree angles are aligned, giving this picture:

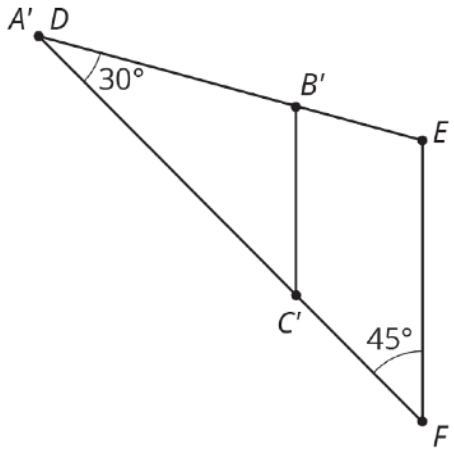


Figure 2.2.3.4

Now a dilation with center D and appropriate scale factor will move C' to F. This dilation also moves B' to E, showing that triangles ABC and DEF are similar.

Glossary Entries

Definition: Similar

Two figures are similar if one can fit exactly over the other after rigid transformations and dilations.

In this figure, triangle ABC is similar to triangle DEF.

If ABC is rotated around point B and then dilated with center point O, then it will fit exactly over DEF. This means that they are similar.

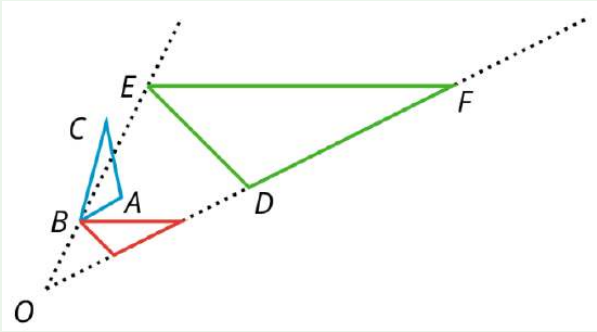


Figure 2.2.3.5

Practice

Exercise 2.2.3.4

In each pair, some of the angles of two triangles in degrees are given. Use the information to decide if the triangles are similar or not. Explain how you know.

- Triangle A: 53, 71, ___; Triangle B: 53, 71, ___
- Triangle C: 90, 37, ___; Triangle D: 90, 53, ___
- Triangle E: 63, 45, ___; Triangle F: 14, 71, ___
- Triangle G: 121, ___, ___; Triangle H: 70, ___, ___

Exercise 2.2.3.5

1. Draw two equilateral triangles that are not congruent.
2. Measure the side lengths and angles of your triangles. Are the two triangles similar?
3. Do you think two equilateral triangles will be similar *always*, *sometimes*, or *never*? Explain your reasoning.

Exercise 2.2.3.6

In the figure, line BC is parallel to line DE.

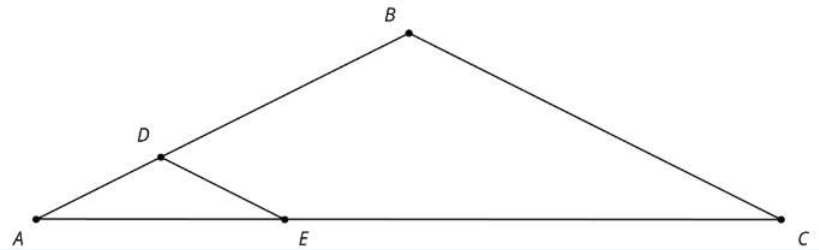


Figure 2.2.3.6

Explain why ΔABC is similar to ΔADE.

Exercise 2.2.3.7

The quadrilateral $PQRS$ in the diagram is a parallelogram. Let $P'Q'R'S'$ be the image of $PQRS$ after applying a dilation centered at a point O (not shown) with scale factor 3.

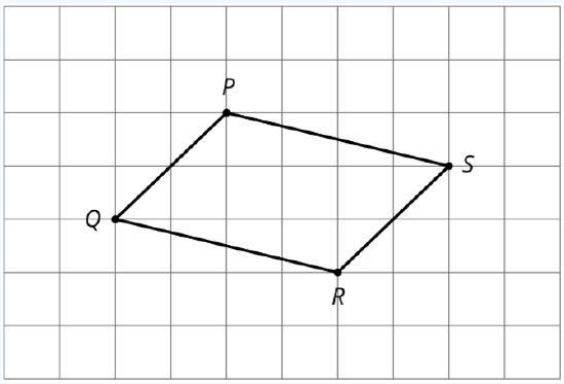


Figure 2.2.3.7

Which of the following is true?

A. $P'Q' = PQ$
B. $P'Q' = 3PQ$
C. $PQ = 3P'Q'$
D. Cannot be determined from the information given

(From Unit 2.1.4)

Exercise 2.2.3.8

Describe a sequence of transformations for which Quadrilateral P is the image of Quadrilateral Q.

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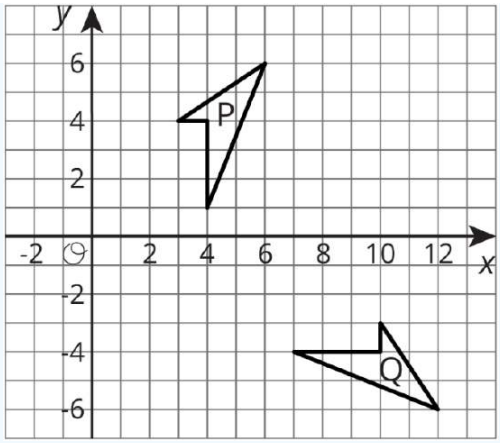


Figure 2.2.3.8: Two quadrilaterals in an x y plane. Origin 0. Vertical axis from negative 5 to 6 by 1s. Horizontal axis from negative 2 to 12 by 1s. Quadrilateral, labeled P, has points at 3 comma 3, 6 comma 6, 4 comma 1, and 4 comma 4. Quadrilateral, labeled Q, has points at 7 comma negative 4, 12 comma negative 6, 10 comma negative 3, 10 comma negative 4.

(From Unit 1.1.6)

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2.2.4: Side Length Quotients in Similar Triangles

Lesson

Let's find missing side lengths in triangles.

Exercise 2.2.4.1: Two-three-four and Four-five-six

Triangle A has side lengths 2, 3, and 4. Triangle B has side lengths 4, 5, and 6. Is Triangle A similar to Triangle B?

Exercise 2.2.4.2: Quotients of Sides Within Similar Triangles

Triangle ABC is similar to triangles DEF, GHI, and JKL. The scale factors for the dilations that show triangle ABC is similar to each triangle are in the table.

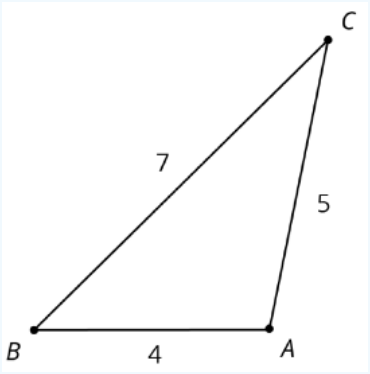


Figure 2.2.4.1

1. Find the side lengths of triangles DEF, GHI, and JKL. Record them in the table.

Table 2.2.4.1

triangle	scale factor	length of short side	length of medium side	length of long side
ABC	1	4	5	7
DEF	2			
GHI	3			
JKL	$\frac{1}{2}$			

2. Your teacher will assign you one of the three columns. For all four triangles, find the quotient of the triangle side lengths assigned to you and record it in the table. What do you notice about the quotients?

Table 2.2.4.2

triangle	(long side) ÷ (short side)	(long side) ÷ (medium side)	(medium side) ÷ (short side)
ABC	$\frac{7}{4}$ or 1.75		
DEF			
GHI			
JKL			

3. Compare your results with your partner's and complete your table.

Are you ready for more?

Triangle ABC and DEF are similar. Explain why $\frac{AB}{BC} = \frac{DE}{EF}$.

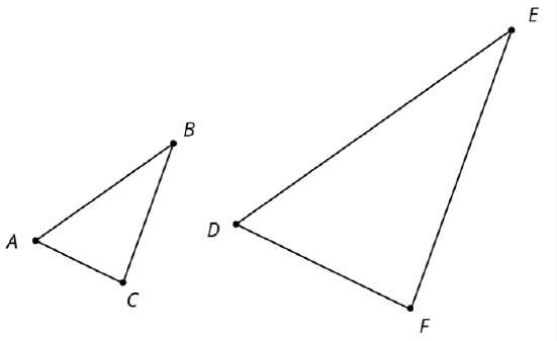


Figure 2.2.4.2

Exercise 2.2.4.3: Using Side Quotients to Find Side Lengths of Similar Triangles

Triangles ABC, EFD, and GHI are all similar. The side lengths of the triangles all have the same units. Find the unknown side lengths.

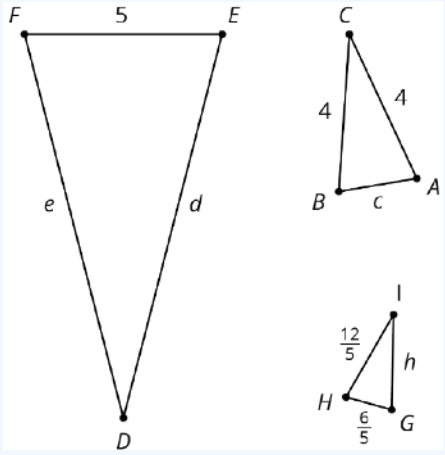


Figure 2.2.4.3: Three triangles. Vertices capital A, B C. Sides B C and A, C length 4. Side A, B length lower-case c. Vertices capital I H G. Side I G length lower-case h. Side I H length fraction 12 over 5. Side H G length fraction 6 over 5. Vertices capital D E F. Side E F 5, Side F D length lower-case e, side E D length lowercase d.

Summary

If two polygons are similar, then the side lengths in one polygon are multiplied by the same scale factor to give the corresponding side lengths in the other polygon.

For these triangles the scale factor is 2:

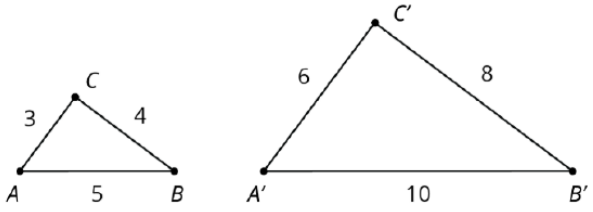


Figure 2.2.4.4

Here is a table that shows relationships between the short and medium length sides of the small and large triangle.

Table 2.2.4.3

	small triangle	large triangle
medium side	4	8
short side	3	6

	small triangle	large triangle
(medium side) ÷ (short side)	$\frac{4}{3}$	$\frac{8}{6} = \frac{4}{3}$

The lengths of the medium side and the short side are in a ratio of $4 : 3$. This means that the medium side in each triangle is $\frac{4}{3}$ as long as the short side. This is true for all similar polygons; the ratio between two sides in one polygon is the same as the ratio of the corresponding sides in a similar polygon.

We can use these facts to calculate missing lengths in similar polygons. For example, triangles $A'B'C'$ and ABC shown here are similar. Let's find the length of segment $B'C'$.

In triangle ABC, side BC is twice as long as side AB, so this must be true for any triangle that is similar to triangle ABC. Since $A'B'$ is 1.2 units long and $2 \cdot 1.2 = 2.4$, the length of side $B'C'$ is 2.4 units.

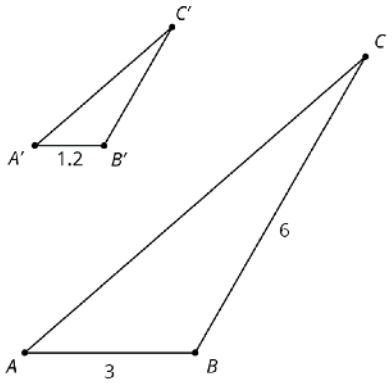


Figure 2.2.4.5

Glossary Entries

Definition: Similar

Two figures are similar if one can fit exactly over the other after rigid transformations and dilations.

In this figure, triangle ABC is similar to triangle DEF.

If ABC is rotated around point B and then dilated with center point O, then it will fit exactly over DEF. This means that they are similar.

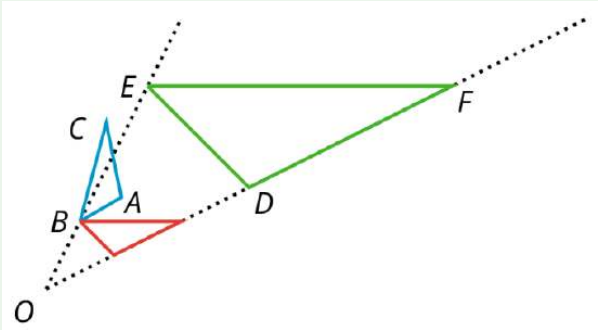


Figure 2.2.4.6

Practice

Exercise 2.2.4.4

These two triangles are similar. What are a and b? Note: the two figures are not drawn to scale.

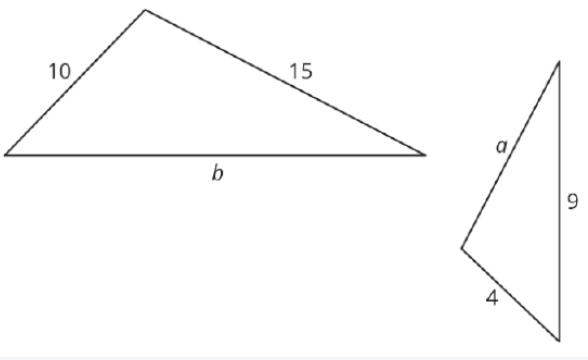


Figure 2.2.4.7

Exercise 2.2.4.5

Here is triangle ABC. Triangle XYZ is similar to ABC with scale factor $\frac{1}{4}$.

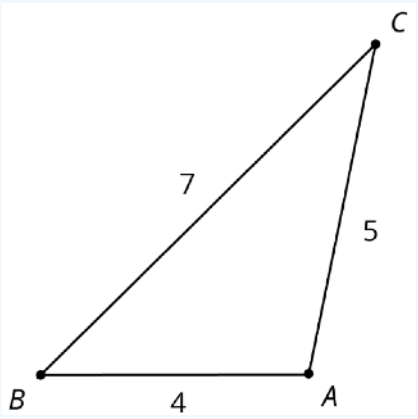


Figure 2.2.4.8

1. Draw what triangle XYZ might look like.
2. How do the angle measures of triangle XYZ compare to triangle ABC? Explain how you know.
3. What are the side lengths of triangle XYZ?
4. For triangle XYZ, calculate (long side) $\div$ (medium side), and compare to triangle ABC.

Exercise 2.2.4.6

The two triangles shown are similar. Find the value of $\frac{d}{c}$.

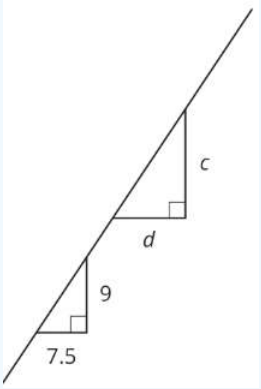


Figure 2.2.4.9

Exercise 2.2.4.7

The diagram shows two nested triangles that share a vertex. Find a center and a scale factor for a dilation that would move the larger triangle to the smaller triangle.

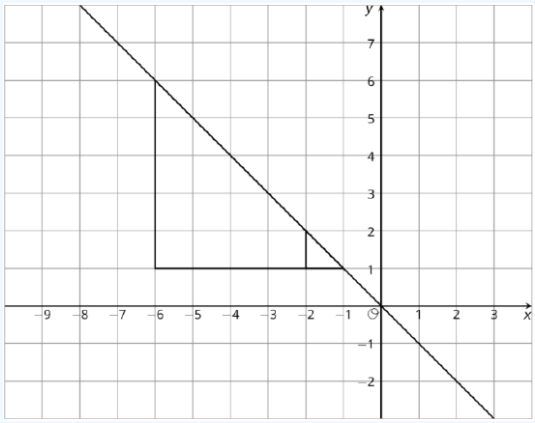


Figure 2.2.4.10: Coordinate plane, x, negative 9 to 3, y, negative 2 to 7. A line through negative 6 comma 6 and the origin. A triangle connecting negative 6 comma 6, to negative 6 comma 1, to negative 1 comma 1. Another triangle connecting negative 2 comma 2, to negative 2 comma 1, to negative 1 comma 1.

(From Unit 2.1.5)

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2.3: Slope

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2.3.1: Meet Slope

Lesson

Let's learn about the slope of a line.

Exercise 2.3.1.1: Equal Quotients

Write some numbers that are equal to $15 \div 12$.

Exercise 2.3.1.2: Similar Triangles on the Same Line

1. The figure shows three right triangles, each with its longest side on the same line. Your teacher will assign you two triangles. Explain why the two triangles are similar.

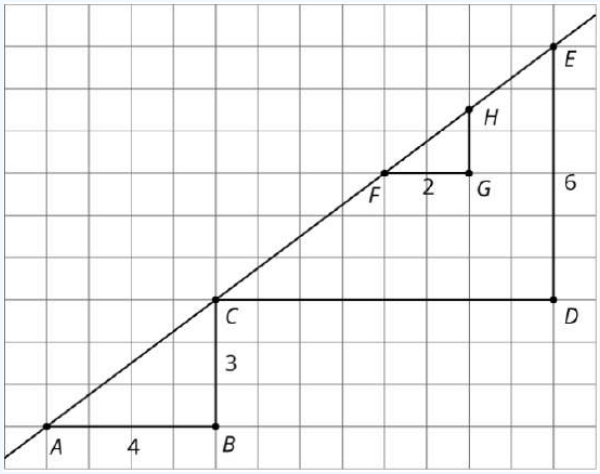


Figure 2.3.1.1

2. Complete the table.

Table 2.3.1.1

triangle	length of vertical side	length of horizontal side	(vertical side) ÷ (horizontal side)
ABC			
CDE			
FGH			

Exercise 2.3.1.3: Multiple Lines with the Same Slope

1. Draw two lines with slope 3. What do you notice about the two lines?
2. Draw two lines with slope $\frac{1}{2}$. What do you notice about the two lines?

Are you ready for more?

As we learn more about lines, we will occasionally have to consider perfectly vertical lines as a special case and treat them differently. Think about applying what you have learned in the last couple of activities to the case of vertical lines. What is the same? What is different?

Exercise 2.3.1.4: Different Slopes of Different Lines

Here are several lines.

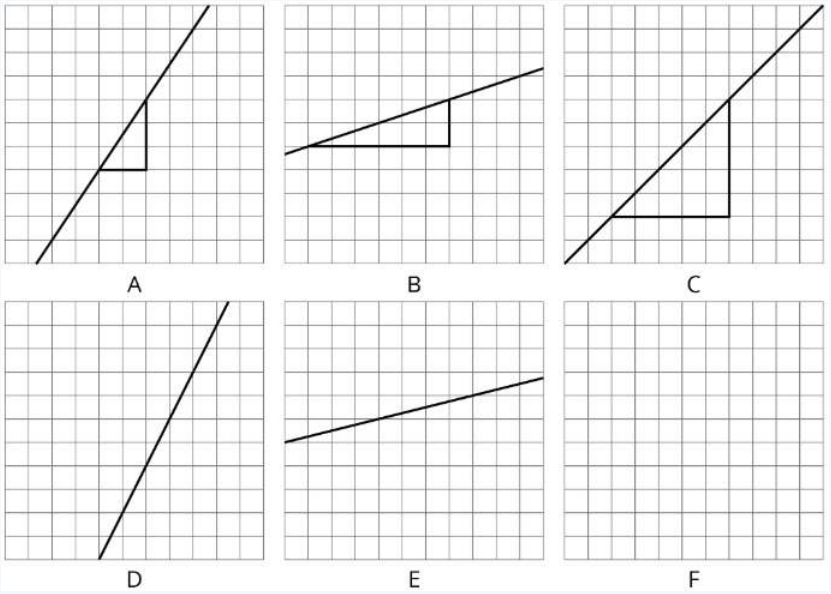


Figure 2.3.1.2

1. Match each line shown with a slope from this list: $\frac{1}{3}, 2, 1, 0.25, \frac{3}{2}, \frac{1}{2}$
2. One of the given slopes does not have a line to match. Draw a line with this slope on the empty grid (F).

Summary

Here is a line drawn on a grid. There are also four right triangles drawn. Do you notice anything the triangles have in common?

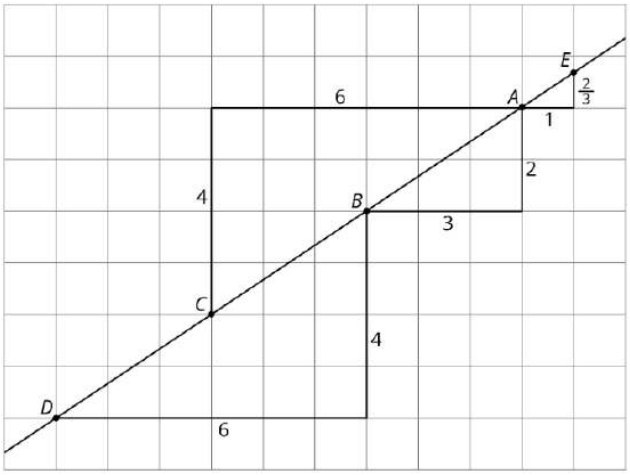


Figure 2.3.1.3: Four right triangles each with hypotenuse on the same line. First horizontal side 6, vertical side 4. Second horizontal side 3, vertical side 2. Third horizontal side 1, vertical side fraction 2 over 3. Fourth horizontal side 6, vertical side 4.

These four triangles are all examples of *slope triangles*. One side of a slope triangle is on the line, one side is vertical, and another side is horizontal. The **slope** of the line is the quotient of the length of the vertical side and the length of the horizontal side of the slope triangle. This number is the same for *all* slope triangles for the same line because all slope triangles for the same line are similar.

In this example, the slope of the line is $\frac{2}{3}$, which is what all four triangles have in common. Here is how the slope is calculated using the slope triangles:

- Points A and B give $2 \div 3 = \frac{2}{3}$
- Points D and B give $4 \div 6 = \frac{2}{3}$

- Points A and C give $4 \div 6 = \frac{2}{3}$
- Points A and E give $\frac{2}{3} \div 1 = \frac{2}{3}$

Glossary Entries

Definition: Similar

Two figures are similar if one can fit exactly over the other after rigid transformations and dilations.

In this figure, triangle ABC is similar to triangle DEF.

If ABC is rotated around point B and then dilated with center point O, then it will fit exactly over DEF. This means that they are similar.

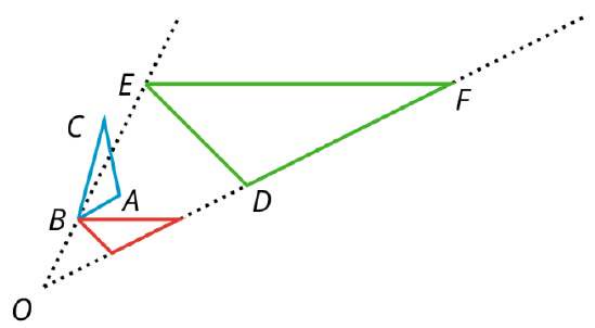


Figure 2.3.1.4

Definition: Slope

The slope of a line is a number we can calculate using any two points on the line. To find the slope, divide the vertical distance between the points by the horizontal distance.

The slope of this line is 2 divided by 3 or $\frac{2}{3}$.

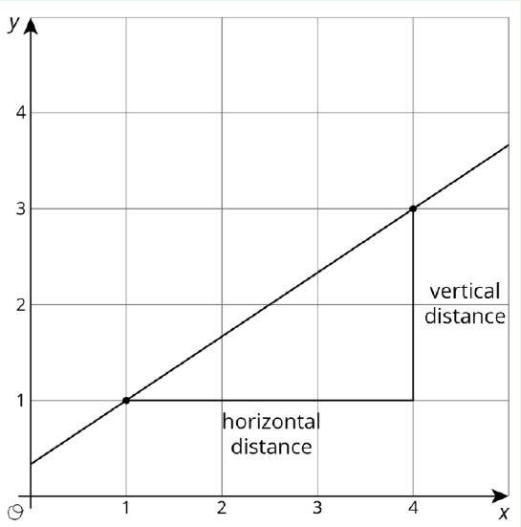


Figure 2.3.1.5

Practice

Exercise 2.3.1.5

Of the three lines in the graph, one has slope 1, one has slope 2, and one has slope Label each line with its slope.

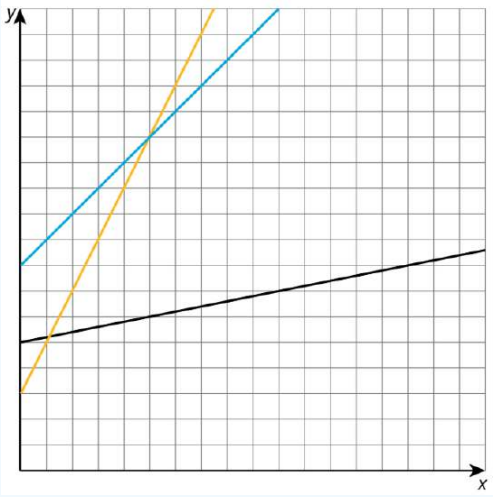


Figure 2.3.1.6

Exercise 2.3.1.6

Draw three lines with slope 2, and three lines with slope $\frac{1}{3}$. What do you notice?

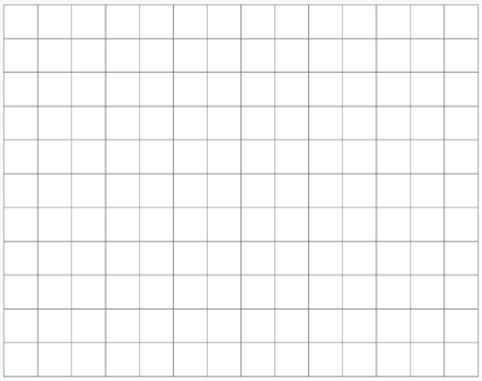


Figure 2.3.1.7

Exercise 2.3.1.7

The figure shows two right triangles, each with its longest side on the same line.

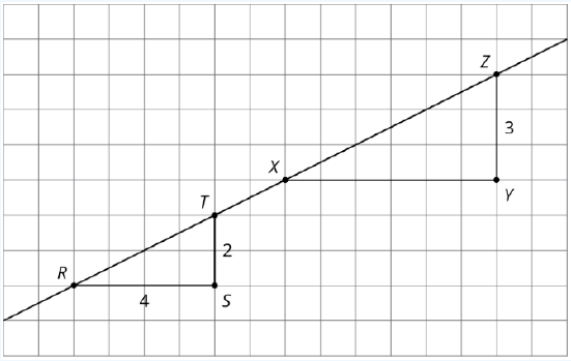


Figure 2.3.1.8

1. Explain how you know the two triangles are similar.
2. How long is XY?
3. For each triangle, calculate (vertical side) ÷ (horizontal side).
4. What is the slope of the line? Explain how you know.

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Exercise 2.3.1.8

Triangle A has side lengths 3, 4, and 5. Triangle B has side lengths 6, 7, and 8.

1. Explain how you know that Triangle B is *not* similar to Triangle A.
2. Give possible side lengths for Triangle B so that it is similar to Triangle A.

(From Unit 2.2.4)

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2.3.2: Writing Equations for Lines

Lesson

Let's explore the relationship between points on a line and the slope of the line.

Exercise 2.3.2.1: Coordinates and Lengths in the Coordinate Plane

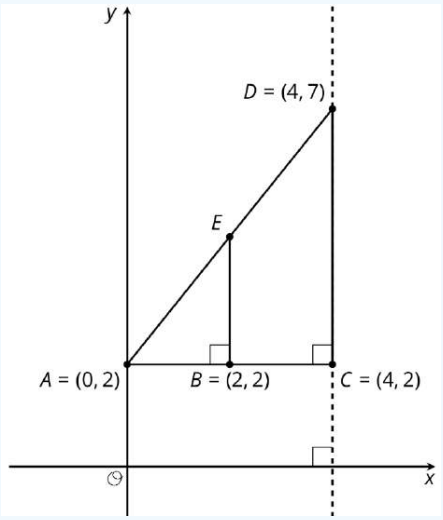


Figure 2.3.2.1

Find each of the following and explain your reasoning:

1. The length of segment BE.
2. The coordinates of E.

Exercise 2.3.2.2: What We Mean by an Equation of a Line

Line j is shown in the coordinate plane.

1. What are the coordinates of B and D?
2. Is point $(20, 15)$ on line j? Explain how you know.
3. Is point $(100, 75)$ on line j? Explain how you know.
4. Is point $(90, 68)$ on line j? Explain how you know.
5. Suppose you know the x- and y-coordinates of a point. Write a rule that would allow you to test whether the point is on line j.

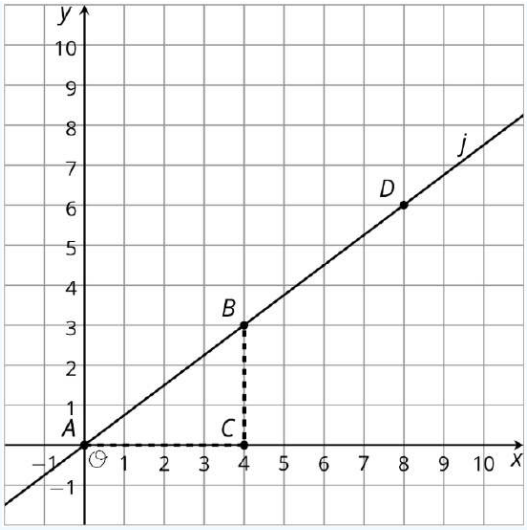


Figure 2.3.2.2

Exercise 2.3.2.3: Writing Relationships from Slope Triangles

Here are two diagrams:

1. Complete each diagram so that all vertical and horizontal segments have expressions for their lengths.
2. Use what you know about similar triangles to find an equation for the quotient of the vertical and horizontal side lengths of ΔDFE in each diagram.

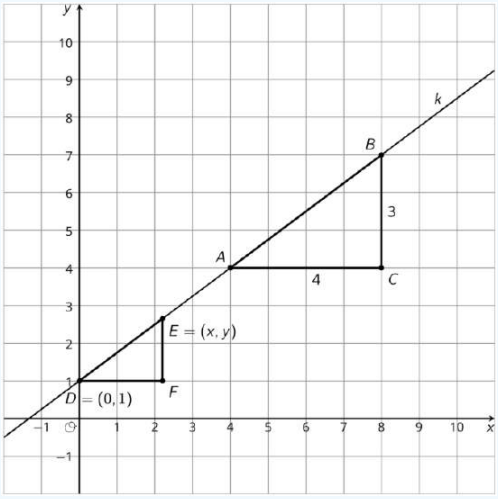


Figure 2.3.2.3

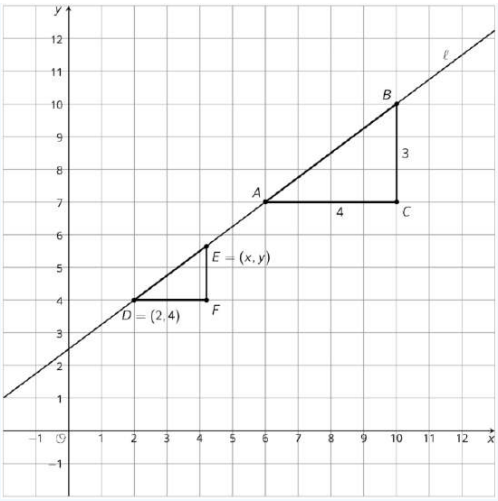


Figure 2.3.2.4

Are you ready for more?

1. Find the area of the shaded region by summing the areas of the shaded triangles.
2. Find the area of the shaded region by subtracting the area of the unshaded region from the large triangle.
3. What is going on here?

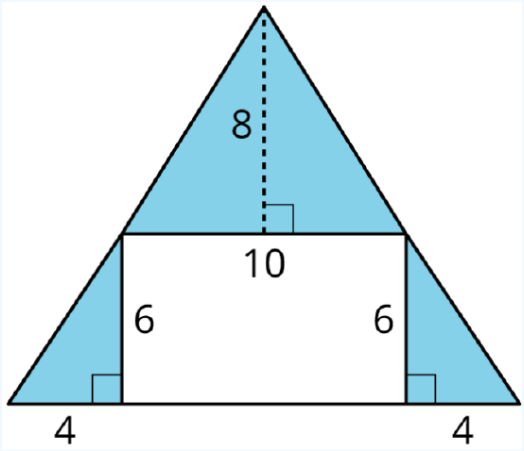


Figure 2.3.2.5

Summary

Here are the points A, C, and E on the same line. Triangles ABC and ADE are slope triangles for the line so we know they are similar triangles. Let's use their similarity to better understand the relationship between x and y, which make up the coordinates of point E.

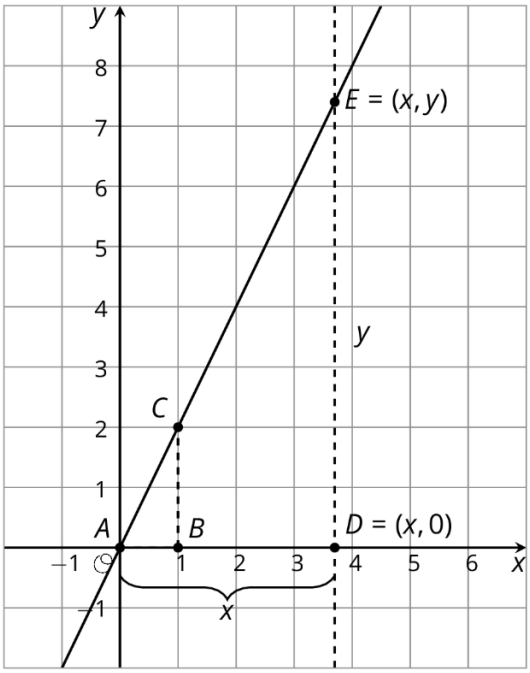


Figure 2.3.2.6: Coordinate plane, x, negative 1 to 6, y negative 1 to 8. A line through 0 comma 0, point C at 1 comma 2, and point E at x comma y. A dotted line connects C to B at 1 comma 0. Another connects E to D at x comma 0. The length of E D is y. The distance from the the origin to D is x.

The slope for triangle ABC is $\frac{2}{1}$ since the vertical side has length 2 and the horizontal side has length 1. The slope we find for triangle ADE is $\frac{y}{x}$ because the vertical side has length y and the horizontal side has length x. These two slopes must be equal since they are from slope triangles for the same line, and so: $\frac{2}{1} = \frac{y}{x}$.

Since $\frac{2}{1} = 2$ this means that the value of y is twice the value of x, or that $y = 2x$. This equation is true for any point (x, y) on the line!

Here are two different slope triangles. We can use the same reasoning to describe the relationship between x and y for this point E.

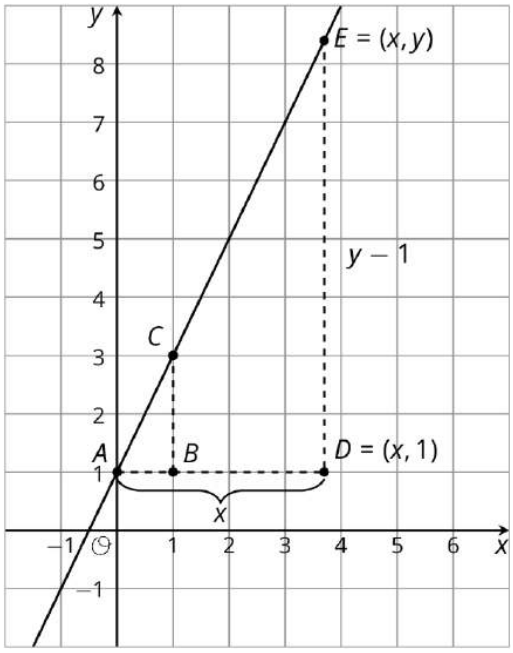


Figure 2.3.2.7: Coordinate plane, x, negative 1 to 6, y negative 1 to 8. A line through point A, at 0 comma 1, point C at 1 comma 3, and point E at x comma y. A dotted line connects C to B at 1 comma 1. Another connects E to D at x comma 1. The length of E D is y minus 1. The length of A, D is x.

The slope for triangle ABC is $\frac{2}{1}$ since the vertical side has length 2 and the horizontal side has length 1. For triangle ADE, the horizontal side has length x. The vertical side has length $y-1$ because the distance from (x, y) to the x-axis is y but the vertical side of the triangle stops 1 unit short of the x-axis. So the slope we find for triangle ADE is $\frac{y-1}{x}$. The slopes for the two slope triangles are equal, meaning: $\frac{2}{1}=\frac{y-1}{x}$

Since $y-1$ is twice x, another way to write this equation is $y-1=2x$. This equation is true for any point (x, y) on the line!

Glossary Entries

Definition: Similar

Two figures are similar if one can fit exactly over the other after rigid transformations and dilations.

In this figure, triangle ABC is similar to triangle DEF.

If ABC is rotated around point B and then dilated with center point O, then it will fit exactly over DEF. This means that they are similar.

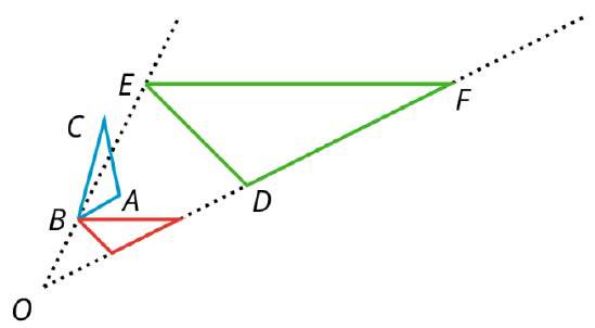


Figure 2.3.2.8

Definition: Slope

The slope of a line is a number we can calculate using any two points on the line. To find the slope, divide the vertical distance between the points by the horizontal distance.

The slope of this line is 2 divided by 3 or $\frac{2}{3}$.

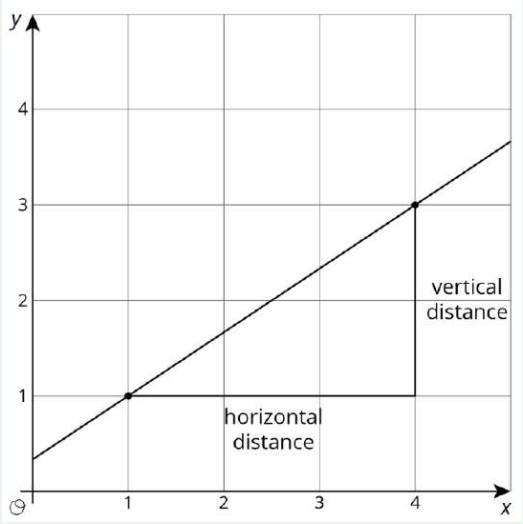


Figure 2.3.2.9

Practice

Exercise 2.3.2.4

For each pair of points, find the slope of the line that passes through both points. If you get stuck, try plotting the points on graph paper and drawing the line through them with a ruler.

1. $(1, 1)$ and $(7, 5)$
2. $(1, 1)$ and $(5, 7)$
3. $(2, 5)$ and $(-1, 2)$
4. $(2, 5)$ and $(-7, -4)$

Exercise 2.3.2.5

Line l is shown in the coordinate plane.

1. What are the coordinates of points B and D?
2. Is the point $(16, 20)$ on line l? Explain how you know.
3. Is the point $(20, 24)$ on line l? Explain how you know.
4. Is the point $(80, 100)$ on line l? Explain how you know.
5. Write a rule that would allow you to test whether (x, y) is on line l.

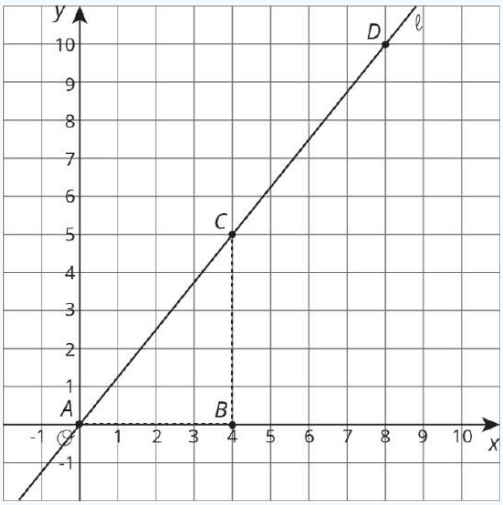


Figure 2.3.2.10

Exercise 2.3.2.6

Consider the graphed line.

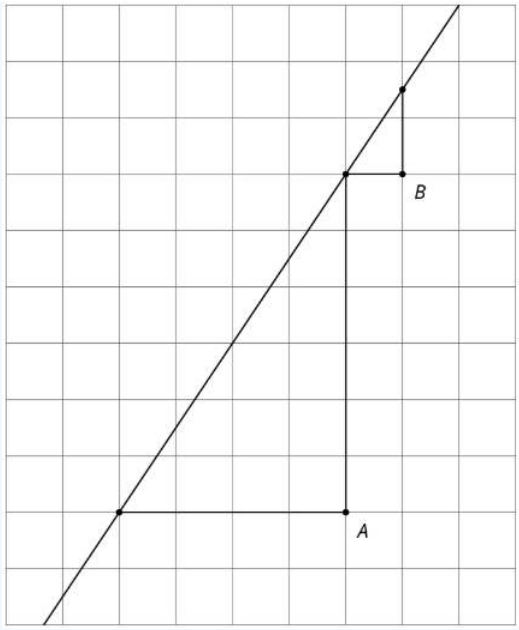


Figure 2.3.2.11

Mai uses Triangle A and says the slope of this line is $\frac{6}{4}$. Elena uses Triangle B and says no, the slope of this line is 1.5. Do you agree with either of them? Explain.

Exercise 2.3.2.7

A rectangle has length 6 and height 4.

Which of these would tell you that quadrilateral $ABCD$ is definitely *not* similar to this rectangle? Select **all** that apply.

A. $AB = BC$
B. $m\angle ABC = 105^\circ$
C. $AB = 8$
D. $BC = 8$
E. $BC = 2 \cdot AB$
F. $2 \cdot AB = 3 \cdot BC$

(From Unit 2.2.1)

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2.3.3: Using Equations for Lines

Lesson

Let's write equations for lines.

Exercise 2.3.3.1: Missing center

A dilation with scale factor 2 sends A to B. Where is the center of the dilation?

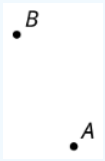


Figure 2.3.3.1

Exercise 2.3.3.2: Writing Relationships from Two Points

Here is a line.

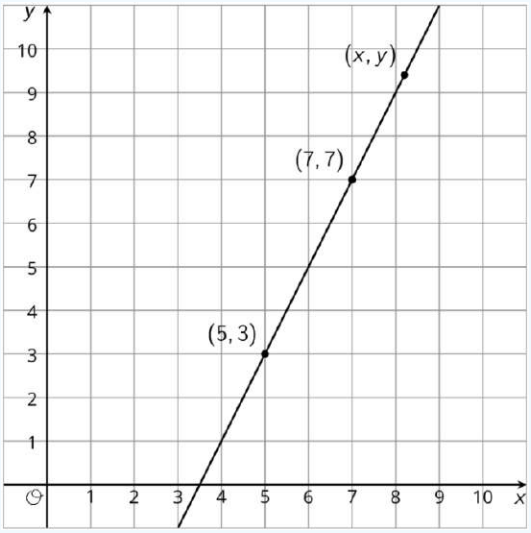


Figure 2.3.3.2

1. Using what you know about similar triangles, find an equation for the line in the diagram.
2. What is the slope of this line? Does it appear in your equation?
3. Is $(9, 11)$ also on the line? How do you know?
4. Is $(100, 93)$ also on the line?

Are you ready for more?

There are many different ways to write down an equation for a line like the one in the problem. Does $\frac{y-3}{x-6} = 2$ represent the line? What about $\frac{y-6}{x-4} = 5$? What about $\frac{y+5}{x-1} = 2$? Explain your reasoning.

Exercise 2.3.3.3: Dilations and Slope Triangles

Here is triangle ABC.

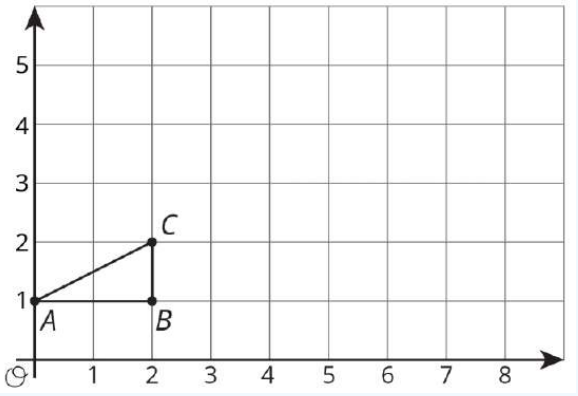


Figure 2.3.3.3

1. Draw the dilation of triangle ABC with center $(0, 1)$ and scale factor 2.
2. Draw the dilation of triangle ABC with center $(0, 1)$ and scale factor 2.5.
3. Where is C mapped by the dilation with center $(0, 1)$ and scale factor ?
4. For which scale factor does the dilation with center $(0, 1)$ send C to $(9, 5.5)$? Explain how you know.

Summary

We can use what we know about slope to decide if a point lies on a line. Here is a line with a few points labeled.

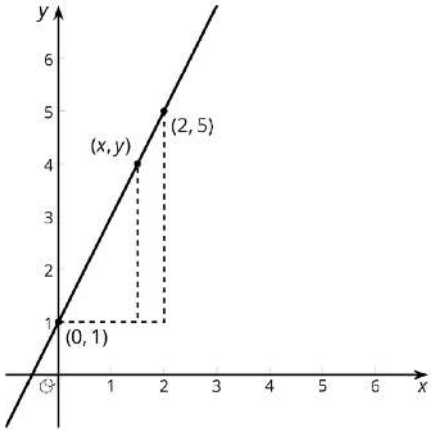


Figure 2.3.3.4

The slope triangle with vertices $(0, 1)$ and $(2, 5)$ gives a slope of $\frac{5-1}{2-0} = 2$. The slope triangle with vertices $(0, 1)$ and (x, y) gives a slope of $\frac{y-1}{x}$. Since these slopes are the same, $\frac{y-1}{x} = 2$ is an equation for the line. So, if we want to check whether or not the point $(11, 23)$ lies on this line, we can check that $\frac{23-1}{11} = 2$. Since $(11, 23)$ is a solution to the equation, it is on the line!

Glossary Entries

Definition: Similar

Two figures are similar if one can fit exactly over the other after rigid transformations and dilations.

In this figure, triangle ABC is similar to triangle DEF.

If ABC is rotated around point B and then dilated with center point O, then it will fit exactly over DEF. This means that they are similar.

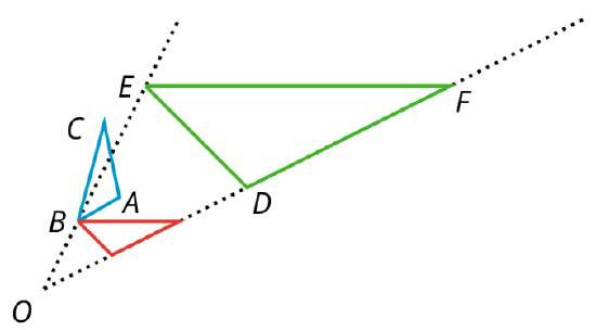


Figure 2.3.3.5

Definition: Slope

The slope of a line is a number we can calculate using any two points on the line. To find the slope, divide the vertical distance between the points by the horizontal distance.

The slope of this line is 2 divided by 3 or $\frac{2}{3}$.

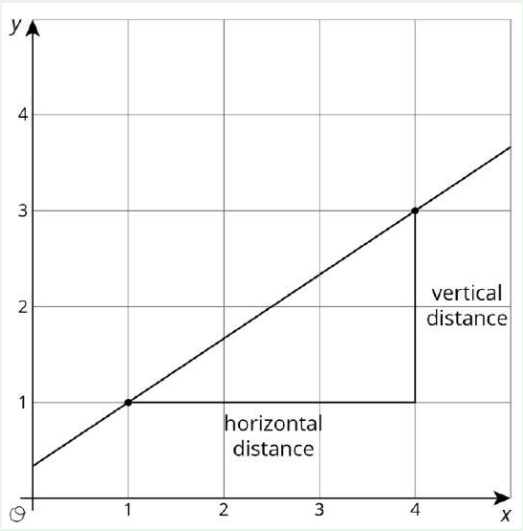


Figure 2.3.3.6

Practice

Exercise 2.3.3.4

Select **all** the points that are on the line through $(0, 5)$ and $(2, 8)$.

A. $(4, 11)$
B. $(5, 10)$
C. $(6, 14)$
D. $(30, 50)$
E. $(40, 60)$

Exercise 2.3.3.5

All three points displayed are on the line. Find an equation relating x and y.

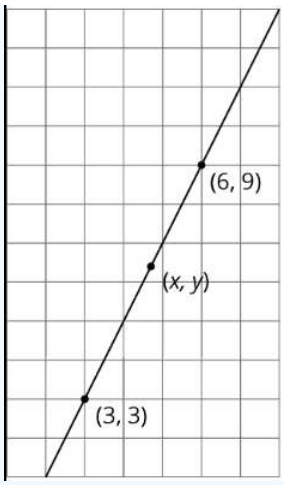


Figure 2.3.3.7

Exercise 2.3.3.6

Here is triangle ABC.

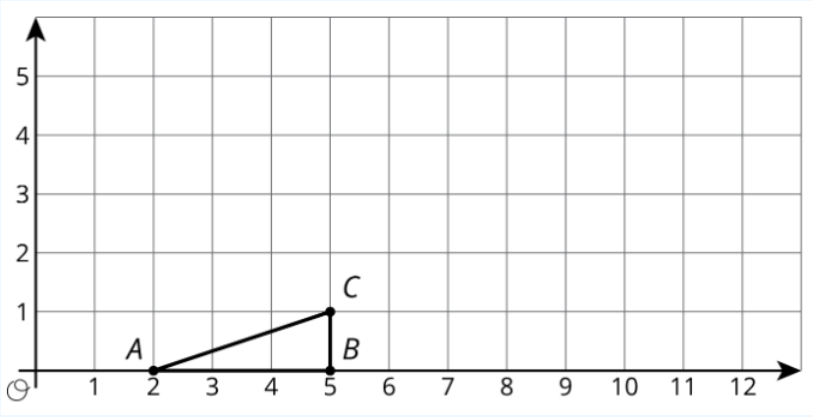


Figure 2.3.3.8

1. Draw the dilation of triangle ABC with center $(2, 0)$ and scale factor 2.
2. Draw the dilation of triangle ABC with center $(2, 0)$ and scale factor 3.
3. Draw the dilation of triangle ABC with center $(2, 0)$ and scale factor $\frac{1}{2}$.
4. What are the coordinates of the image of point C when triangle ABC is dilated with center $(2, 0)$ and scale factor s?
5. Write an equation for the line containing all possible images of point C.

Exercise 2.3.3.7

Here are some line segments.

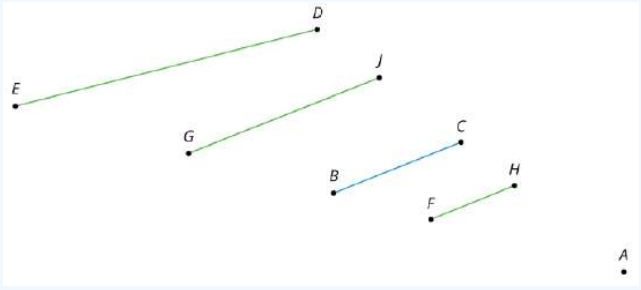


Figure 2.3.3.9: Point A, segment B C and 3 other segments, E D, G J and F H. Segment E D is the largest segment and is not parallel to B C. Segment G J is parallel to and larger than B C. Segment F H is parallel to and smaller than B C.

1. Which segment is a dilation of $\overline{BC}$ using A as the center of dilation and a scale factor of $\frac{2}{3}$?
2. Which segment is a dilation of $\overline{BC}$ using A as the center of dilation and a scale factor of $\frac{3}{2}$?
3. Which segment is not a dilation of $\overline{BC}$, and how do you know?

(From Unit 2.1.4)

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2.4: Let's Put It to Work

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2.4.1: The Shadow Knows

Lesson

Let's use shadows to find the heights of an object.

Exercise 2.4.1.1: Notice and Wonder: Long Shadows and Short Shadows

What do you notice? What do you wonder?



Figure 2.4.1.1



Figure 2.4.1.2

Exercise 2.4.1.2: Objects and Shadows



Figure 2.4.1.3

Here are some measurements that were taken when the photo was taken. It was impossible to directly measure the height of the lamppost, so that cell is blank.

Table 2.4.1.1

	height (inches)	shadow length (inches)
younger boy	43	29
man	72	48
older boy	51	34
lamppost		114

1. What relationships do you notice between an object's height and the length of its shadow?
2. Make a conjecture about the height of the lamppost and explain your thinking.

Exercise 2.4.1.3: Justifying the Relationship

Explain *why* the relationship between the height of these objects and the length of their shadows is approximately proportional.

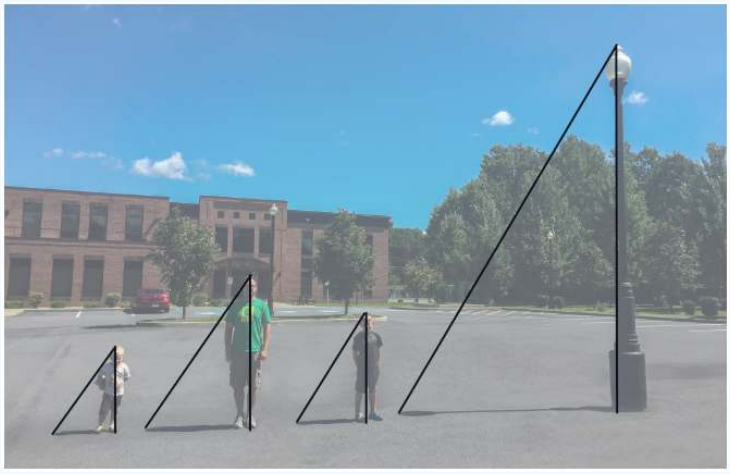


Figure 2.4.1.4

Exercise 2.4.1.4: The Height of a Tall Object

1. Head outside. Make sure that it is a sunny day and you take a measuring device (like a tape measure or meter stick) as well as a pencil and some paper.
2. Choose an object whose height is too large to measure directly. Your teacher may assign you an object.
3. Use what you have learned to figure out the height of the object! Explain or show your reasoning.

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CHAPTER OVERVIEW

3: Linear Relationships

In this unit, students learn to understand and use the terms “rate of change,” “linear relationship,” and “vertical intercept.” They deepen their understanding of slope, and they learn to recognize connections among rate of change, slope, and constant of proportionality, and between linear and proportional relationships. They learn to understand that lines with the same slope are translations of each other. They represent linear relationships with tables, equations, and graphs that include lines with negative slopes or vertical intercepts, and horizontal and vertical lines. They learn to use the term “solution of an equation” when working with one or two linear equations in two variables, and learn to understand the graph of a linear equation as the set of its solutions. Students use these terms and representations in reasoning about situations involving one or two constant rates.

- 3.1: Proportional Relationships
  - 3.1.1: Understanding Proportional Relationships
  - 3.1.2: Graphs of Proportional Relationships
  - 3.1.3: Representing Proportional Relationships
  - 3.1.4: Comparing Proportional Relationships
- 3.2: Representing Linear Relationships
  - 3.2.1: Introduction to Linear Relationships
  - 3.2.2: More Linear Relationships
  - 3.2.3: Representations of Linear Relationships
  - 3.2.4: Translating y=mx+b
- 3.3: Finding Slopes
  - 3.3.1: Slopes Don't Have to be Positive
  - 3.3.2: Calculating Slope
  - 3.3.3: Equations of All Kinds of Lines
- 3.4: Linear Equations
  - 3.4.1: Solutions to Linear Equations
  - 3.4.2: More Solutions to Linear Equations
- 3.5: Let's Put it to Work
  - 3.5.1: Using Linear Relationships to Solve Problems

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3.1: Proportional Relationships

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3.1.1: Understanding Proportional Relationships

Lesson

Let's study some graphs.

Exercise 3.1.1.1: Notice and Wonder: Two Graphs

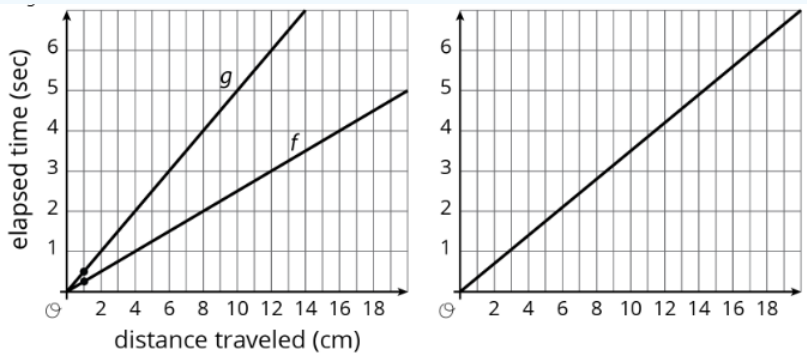


Figure 3.1.1.1: Two graphs. both graphs, horizontal axis, scale 0 to 18, by 2's. vertical axis, scale 0 to 6, by 1's. graph on left horizontal axis labeled distance traveled in centimeters, vertical axis labeled elapsed time in seconds. 2 lines on graph, labeled g and f. line g passes through origin and 6 comma 3. line f passes through origin and 8 comma 2. graph on the right, axes and lines not labeled. line passes through origin and 3 comma 1.

What do you notice? What do you wonder?

Exercise 3.1.1.2: Moving Through Representations

A ladybug and ant move at constant speeds. The diagrams with tick marks show their positions at different times. Each tick mark represents 1 centimeter.

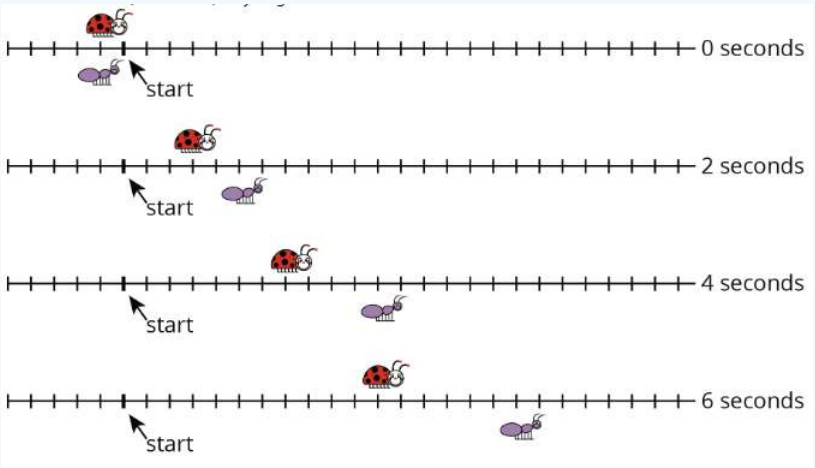


Figure 3.1.1.2: 4 number lines, 30 evenly spaced tick marks. first line, 0 seconds, lady bug and ant both at 5th tick mark, labeled start. second line, 2 seconds, lady bug at 9th tick mark and ant at 11th tick mark. thrid line, 4 seconds, lady bug at 13th tick mark and ant at 17th tick mark. fourth line, 6 seconds, lady bug at 17th tick mark and ant at 23rd tick mark.

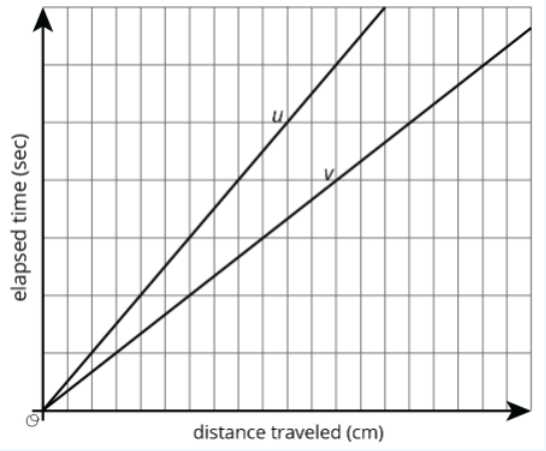


Figure 3.1.1.3: Graph. horizontal axis, distance traveled in centimeters, 20 units. vertical axis, elapsed time in seconds, 6 units. 2 lines graphed, labeled u and v. line u passes through origin and 2nd line to the right and 1st line up. line v passes through origin and 3rd line to the right and 1st line up.

1. Lines u and v also show the positions of the two bugs. Which line shows the ladybug's movement? Which line shows the ant's movement? Explain your reasoning.
2. How long does it take the ladybug to travel 12 cm? The ant?
3. Scale the vertical and horizontal axes by labeling each grid line with a number. You will need to use the time and distance information shown in the tick-mark diagrams.
4. Mark and label the point on line u and the point on line v that represent the time and position of each bug after traveling 1 cm.

Are you ready for more?

1. How fast is each bug traveling?
2. Will there ever be a time when the ant is twice as far away from the start as the ladybug? Explain or show your reasoning.

Exercise 3.1.1.3: Moving Twice as Fast

Refer to the tick-mark diagrams and graph in the earlier activity when needed.

1. Imagine a bug that is moving twice as fast as the ladybug. On each tick-mark diagram, mark the position of this bug.
2. Plot this bug's positions on the coordinate axes with lines u and v, and connect them with a line.
3. Write an equation for each of the three lines.

Summary

Graphing is a way to help us make sense of relationships. But the graph of a line on a coordinate axes without scale or labels isn't very helpful. For example, let's say we know that on longer bike rides Kiran can ride 4 miles every 16 minutes and Mai can ride 4 miles every 12 minutes. Here are the graphs of these relationships:

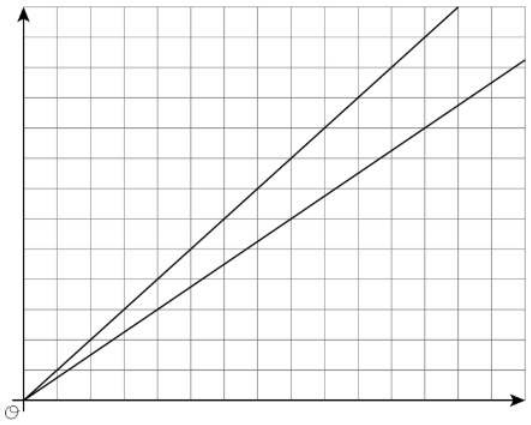


Figure 3.1.1.4: Graph. horizontal axis, 15 units. vertical axis, 12 units. 2 lines graphed. first line passes through origin and 5th unit over and 4th unit up. second line passes through origin and 4th unit over 3rd unit up.

Without labels we can't even tell which line is Kiran and which is Mai! Without labels and a scale on the axes, we can't use these graphs to answer questions like:

1. Which graph goes with which rider?
2. Who rides faster?
3. If Kiran and Mai start a bike trip at the same time, how far are they after 24 minutes?
4. How long will it take each of them to reach the end of the 12 mile bike path?

Here are the same graphs, but now with labels and scale:

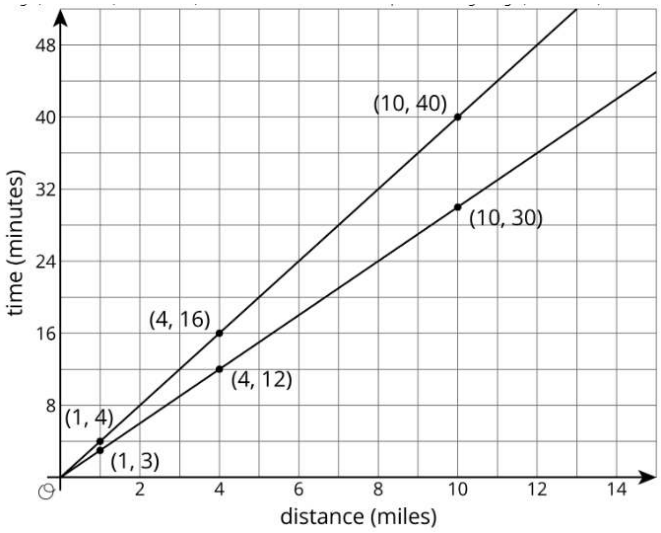


Figure 3.1.1.5: Graph. horizontal axis, distance in miles, scale 0 to 14, by 2's. vertical axis, time in minutes, scale 0 to 48, by 8's. first line passes through origin, 1 comma 4, 4 comma 16, and 10 comma 40. second line passes through origin, 1 comma 3, 4 comma 12, and 10 comma 30.

Revisiting the questions from earlier:

1. Which graph goes with each rider? If Kiran rides 4 miles in 16 minutes, then the point is on his graph. If he rides for 1 mile, it will take 4 minutes. 10 miles will take 40 minutes. So the upper graph represents Kiran's ride. Mai's points for the same distances are , , and , so hers is the lower graph. (A letter next to each line would help us remember which is which!)
2. Who rides faster? Mai rides faster because she can ride the same distance as Kiran in a shorter time.
3. If Kiran and Mai start a bike trip at the same time, how far are they after 20 minutes? The points on the graphs at height 20 are 5 miles for Kiran and a little less than 7 miles for Mai.
4. How long will it take each of them to reach the end of the 12 mile bike path? The points on the graphs at a horizontal distance of 12 are 36 minutes for Mai and 48 minutes for Kiran. (Kiran's time after 12 miles is almost off the grid!)

Glossary Entries

Definition: Constant of Proportionality

In a proportional relationship, the values for one quantity are each multiplied by the same number to get the values for the other quantity. This number is called the constant of proportionality.

In this example, the constant of proportionality is 3, because $2 \cdot 3 = 6$, $3 \cdot 3 = 9$, and $5 \cdot 3 = 15$. This means that there are 3 apples for every 1 orange in the fruit salad.

Table 3.1.1.1

number of oranges	number of apples
2	6
3	9
5	15

Practice

Exercise 3.1.1.4

Priya jogs at a constant speed. The relationship between her distance and time is shown on the graph. Diego bikes at a constant speed twice as fast as Priya. Sketch a graph showing the relationship between Diego's distance and time.

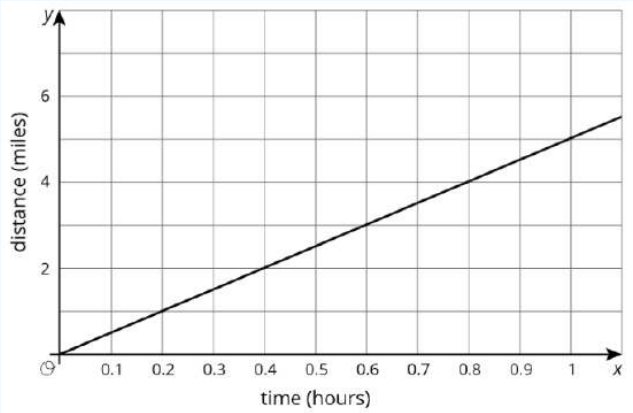


Figure 3.1.1.6

Exercise 3.1.1.5

A you-pick blueberry farm offers 6 lbs of blueberries for $16.50.

Sketch a graph of the relationship between cost and pounds of blueberries.

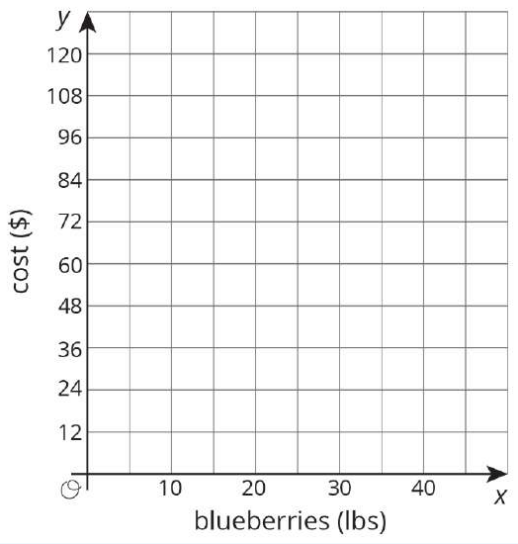


Figure 3.1.1.7

Exercise 3.1.1.6

A line contains the points $(-4, 1)$ and $(4, 6)$. Decide whether or not each of these points is also on the line:

1. $(0, 3.5)$
2. $(12, 11)$
3. $(80, 50)$
4. $(-1, 2.875)$

(From Unit 2.3.3)

Exercise 3.1.1.7

The points $(2, -4)$, (x, y), A, and B all lie on the line. Find an equation relating x and y.

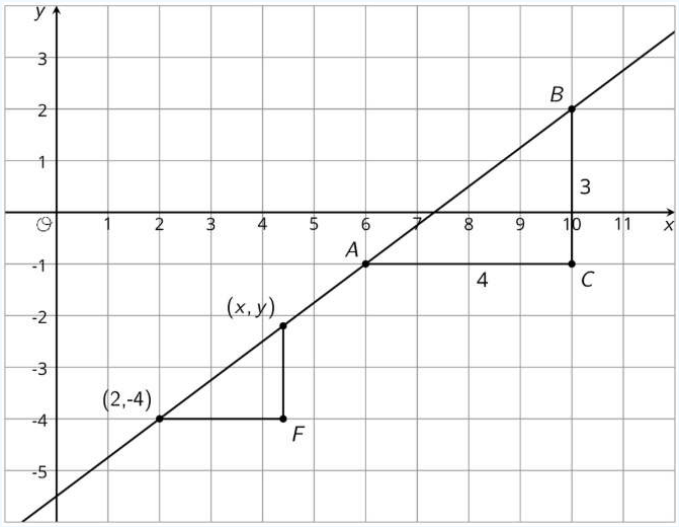


Figure 3.1.1.8: Graph. horizontal axis, scale 0 to 11, by 1's. vertical axis, scale -5 to 3, by 1's. line passing through 2 comma -4, x comma y, 6 comma -1, labeled A, and 10 comma 2, labeled B. right triangles drawn between 1st and 2nd points and points A and B. vertex where 90 degree angle is, labeled F in first triangle and C in second triangle. C is at 10 comma -1.

(From Unit 2.3.2)

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3.1.2: Graphs of Proportional Relationships

Lesson

Let's think about scale.

Exercise 3.1.2.1: An Unknown Situation

Here is a graph that could represent a variety of different situations.

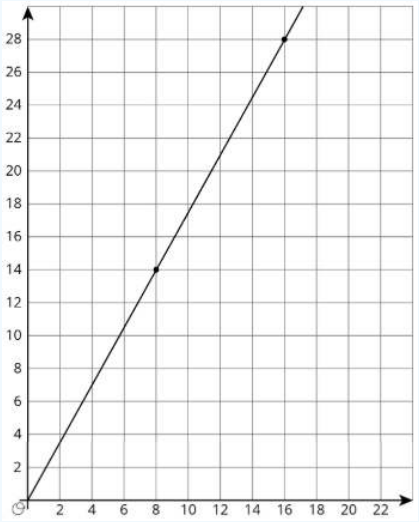


Figure 3.1.2.1

1. Write an equation for the graph.
2. Sketch a new graph of this relationship.

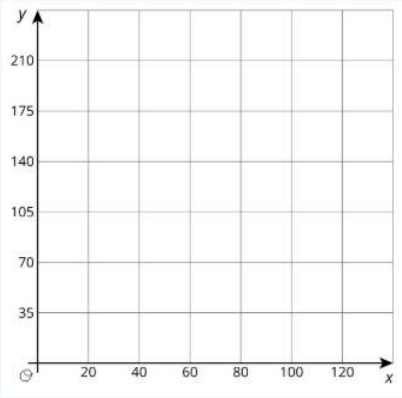


Figure 3.1.2.2

Exercise 3.1.2.2: Card Sort: Proportional Relationships

Your teacher will give you 12 graphs of proportional relationships.

1. Sort the graphs into groups based on what proportional relationship they represent.
2. Write an equation for each *different* proportional relationship you find.

Exercise 3.1.2.3: Different Scales

Two large water tanks are filling with water. Tank A is not filled at a constant rate, and the relationship between its volume of water and time is graphed on each set of axes. Tank B is filled at a constant rate of $\frac{1}{2}$ liters per minute. The relationship

between its volume of water and time can be described by the equation $v = \frac{1}{2}t$, where t is the time in minutes and v is the total volume in liters of water in the tank.

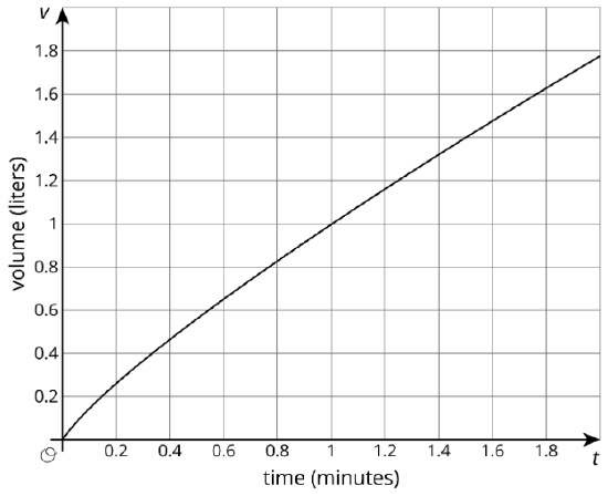


Figure 3.1.2.3: Graph, horizontal axis, time in minutes, scale 0 to 1 and 8 tenths, by 2 tenth's. vertical axis, volume in liters, 0 to 1 and 8 tenths, by 2 tenth's. curved line passing through origin, 1 comma 1 and 1 and 4 tenths comma 1 and 3 tenths.

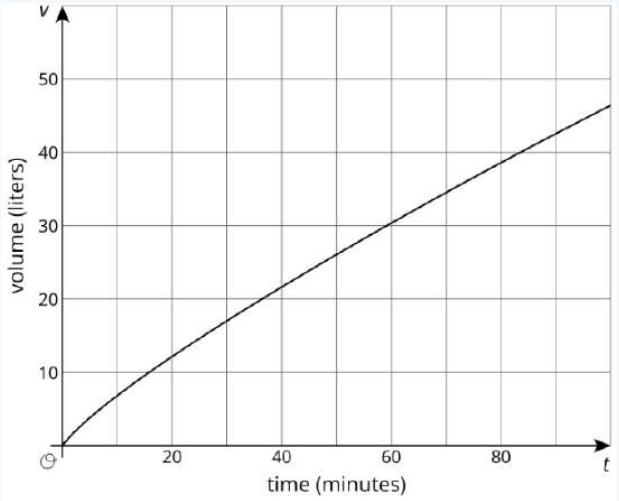


Figure 3.1.2.4: Graph, horizontal axis, time in minutes, scale 0 to 80, by 20's. vertical axis, volume in liters, 0 to 50, by 10's. curved line passing through origin, 20 comma 12 and 60 comma 30.

1. Sketch and label a graph of the relationship between the volume of water v and time t for Tank B on each of the axes.
2. Answer the following questions and say which graph you used to find your answer.
   A. After 30 seconds, which tank has the most water?
   B. At approximately what times do both tanks have the same amount of water?
   C. At approximately what times do both tanks contain 1 liter of water? 20 liters?

Are you ready for more?

A giant tortoise travels at 0.17 miles per hour and an arctic hare travels at 37 miles per hour.

1. Draw separate graphs that show the relationship between time elapsed, in hours, and distance traveled, in miles, for both the tortoise and the hare.
2. Would it be helpful to try to put both graphs on the same pair of axes? Why or why not?
3. The tortoise and the hare start out together and after half an hour the hare stops to take a rest. How long does it take the tortoise to catch up?

Summary

The scales we choose when graphing a relationship often depend on what information we want to know. For example, say two water tanks are filled at different constant rates. The relationship between time in minutes t and volume in liters v of tank A is given by $v = 2.2t$.

For tank B the relationship is $v = 2.75t$.

These equations tell us that tank A is being filled at a constant rate of 2.2 liters per minute and tank B is being filled at a constant rate of 2.75 liters per minute.

If we want to use graphs to see at what times the two tanks will have 110 liters of water, then using an axis scale from 0 to 10, as shown here, isn't very helpful.

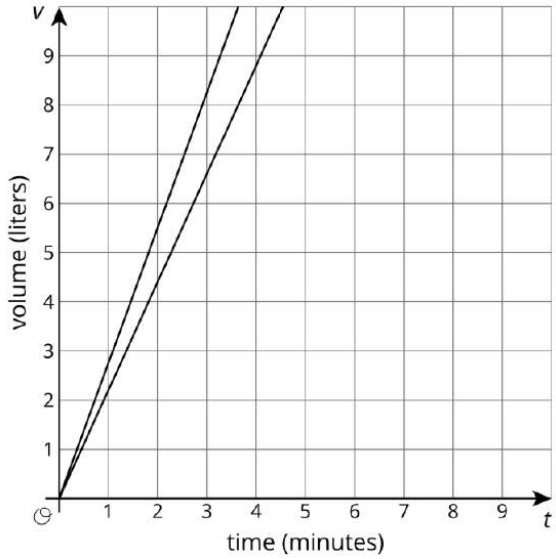


Figure 3.1.2.5: Graph, horizontal axis, time in minutes, scale 0 to 9, by 1's. vertical axis, volume in liters, 0 to 9, by 1's. lines, first line passing through origin and 2 comma 5 and 5 tenths. second line passing through origin and 2 comma 4 and 5 tenths.

If we use a vertical scale that goes to 150 liters, a bit beyond the 110 we are looking for, and a horizontal scale that goes to 100 minutes, we get a much more useful set of axes for answering our question.

Now we can see that the two tanks will reach 110 liters 10 minutes apart—tank B after 40 minutes of filling and tank A after 50 minutes of filling.

It is important to note that both of these graphs are correct, but one uses a range of values that helps answer the question. In order to always pick a helpful scale, we should consider the situation and the questions asked about it.

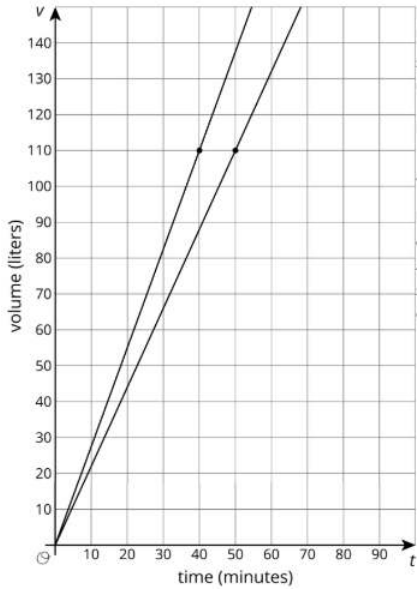


Figure 3.1.2.6: Graph, horizontal axis, time in minutes, scale 0 to 90, by 10's. vertical axis, volume in liters, 0 to 140, by 10's. 2 lines, first line passing through origin and 40 comma 110. second line passing through origin and 50 comma 110.

Glossary Entries

Definition: Constant of Proportionality

In a proportional relationship, the values for one quantity are each multiplied by the same number to get the values for the other quantity. This number is called the constant of proportionality.

In this example, the constant of proportionality is 3, because $2 \cdot 3 = 6$, $3 \cdot 3 = 9$, and $5 \cdot 3 = 15$. This means that there are 3 apples for every 1 orange in the fruit salad.

Table 3.1.2.1

number of oranges	number of apples
2	6
3	9
5	15

Practice

Exercise 3.1.2.4

The tortoise and the hare are having a race. After the hare runs 16 miles the tortoise has only run 4 miles.

The relationship between the distance x the tortoise "runs" in miles for every y miles the hare runs is $y = 4x$. Graph this relationship.

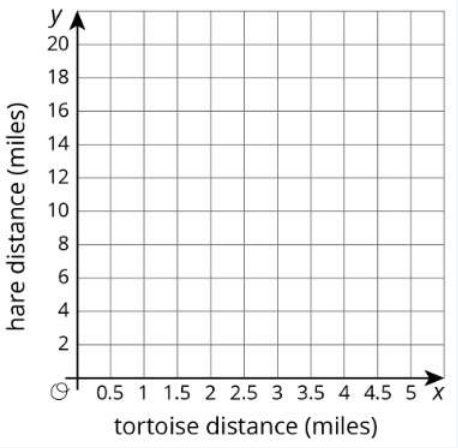


Figure 3.1.2.7

Exercise 3.1.2.5

The table shows a proportional relationship between the weight on a spring scale and the distance the spring has stretched.

1. Complete the table.
2. Describe the scales you could use on the x and y axes of a coordinate grid that would show all the distances and weights in the table.

Table 3.1.2.2

distance (cm)	weight (newtons)
20	28
55	
	140
1	

Exercise 3.1.2.6

Find a sequence of rotations, reflections, translations, and dilations showing that one figure is similar to the other. Be specific: give the amount and direction of a translation, a line of reflection, the center and angle of a rotation, and the center and scale factor of a dilation.

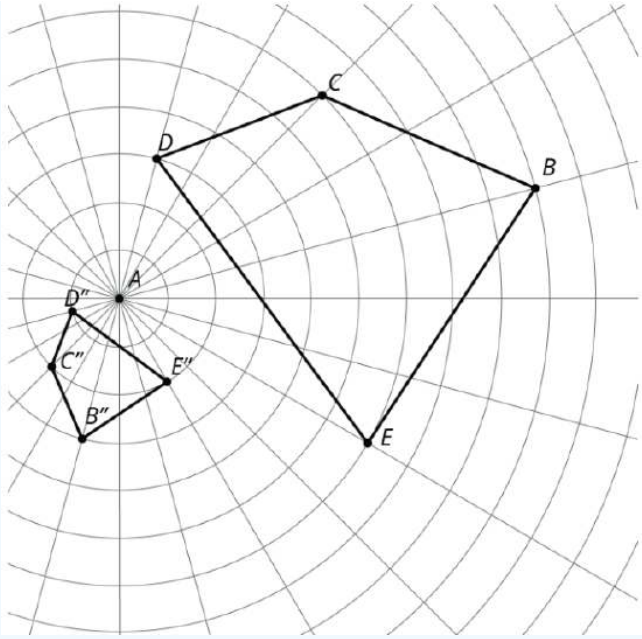


Figure 3.1.2.8: polar coordinate plane with center at A. quadrilateral BCDE and quadrilateral B prime prime, C prime prime, D prime prime, E prime prime graphed. B at 9 comma pi over 12. C at 6 comma pi over 4. D at 3 comma 5 pi over 12, E at 6 comma 11 pi over 6. B prime prime at 3 comma 17 pi over 12. C prime prime at 2 comma 5 pi over 4. D prime prime at 1 comma 13 pi over 12. E prime prime at 2 comma 5 pi over 3.

(From Unit 2.2.1)

Exercise 3.1.2.7

Andre said, “I found two figures that are congruent, so they can’t be similar.”

Diego said, “No, they are similar! The scale factor is 1.”

Do you agree with either of them? Use the definition of similarity to explain your answer.

(From Unit 2.2.1)

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3.1.3: Representing Proportional Relationships

Lesson

Let's graph proportional relationships.

Exercise 3.1.3.1: Number Talk: Multiplication

Find the value of each product mentally.

$15 \cdot 2$

$15 \cdot 0.5$

$15 \cdot 0.25$

$15 \cdot (2.25)$

Exercise 3.1.3.2: Representations of Proportional Relationships

1. Here are two ways to represent a situation.

Description:

Jada and Noah counted the number of steps they took to walk a set distance. To walk the same distance, Jada took 8 steps while Noah took 10 steps. Then they found that when Noah took 15 steps, Jada took 12 steps.

Equation:

Let x represent the number of steps Jada takes and let y represent the number of steps Noah takes $y = \frac{5}{4}x$.

a. Create a table that represents this situation with at least 3 pairs of values.
b. Graph this relationship and label the axes.

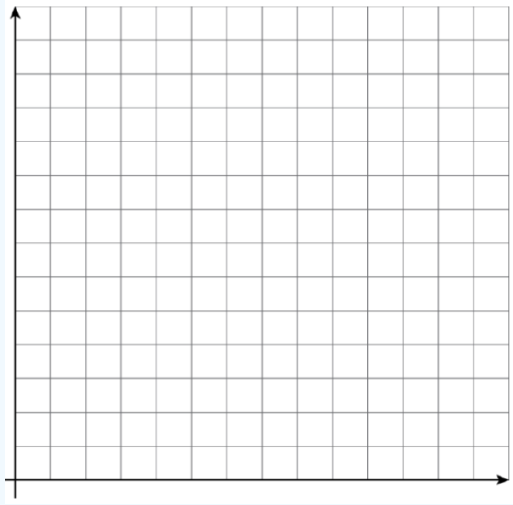


Figure 3.1.3.1

c. How can you see or calculate the constant of proportionality in each representation? What does it mean?

d. Explain how you can tell that the equation, description, graph, and table all represent the same situation.

2. Here are two ways to represent a situation.

Description:

The Origami Club is doing a car wash fundraiser to raise money for a trip. They charge the same price for every car. After 11 cars, they raised a total of $93.50. After 23 cars, they raised a total of $195.50.

Table:

Table 3.1.3.1

number of cars	amount raised in dollars
11	93.50
23	195.50

a. Write an equation that represents this situation. (Use c to represent number of cars and use m to represent amount raised in dollars.)

b. Create a graph that represents this situation.

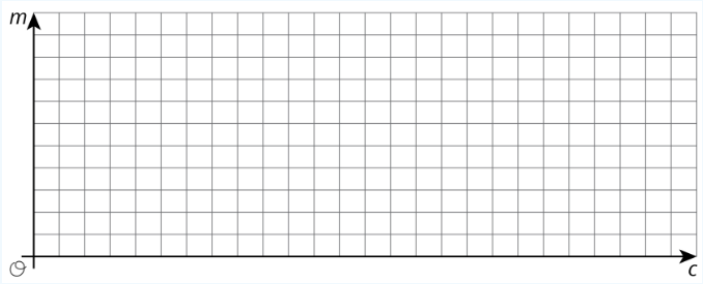


Figure 3.1.3.2

c. How can you see or calculate the constant of proportionality in each representation? What does it mean?

d. Explain how you can tell that the equation, description, graph, and table all represent the same situation.

Exercise 3.1.3.3: Info Gap: Proportional Relationships

Your teacher will give you either a *problem card* or a *data card*. Do not show or read your card to your partner.

If your teacher gives you the *problem card*:

1. Silently read your card and think about what information you need to be able to answer the question.
2. Ask your partner for the specific information that you need.
3. Explain how you are using the information to solve the problem.Continue to ask questions until you have enough information to solve the problem.
4. Share the *problem card* and solve the problem independently.
5. Read the *data card* and discuss your reasoning.

If your teacher gives you the *data card*:

1. Silently read your card.
2. Ask your partner "*What specific information do you need?*" and wait for them to *ask* for information.

   If your partner asks for information that is not on the card, do not do the calculations for them. Tell them you don't have that information.

3. Before sharing the information, ask "*Why do you need that information?*" Listen to your partner's reasoning and ask clarifying questions.
4. Read the *problem card* and solve the problem independently.
5. Share the *data card* and discuss your reasoning.

Pause here so your teacher can review your work. Ask your teacher for a new set of cards and repeat the activity, trading roles with your partner.

Are you ready for more?

Ten people can dig five holes in three hours. If n people digging at the same rate dig m holes in d hours:

1. Is n proportional to m when $d = 3$?
2. Is n proportional to d when $m = 5$?
3. Is m proportional to d when $n = 10$?

Summary

Proportional relationships can be represented in multiple ways. Which representation we choose depends on the purpose. And when we create representations we can choose helpful values by paying attention to the context. For example, a stew recipe calls for 3 carrots for every 2 potatoes. One way to represent this is using an equation. If there are p potatoes and c carrots, then $c = \frac{3}{2}p$.

Suppose we want to make a large batch of this recipe for a family gathering, using 150 potatoes. To find the number of carrots we could just use the equation: $\frac{3}{2} \cdot 150 = 225$ carrots.

Now suppose the recipe is used in a restaurant that makes the stew in large batches of different sizes depending on how busy a day it is, using up to 300 potatoes at at time.

Then we might make a graph to show how many carrots are needed for different amounts of potatoes. We set up a pair of coordinate axes with a scale from 0 to 300 along the horizontal axis and 0 to 450 on the vertical axis, because $450 = \frac{3}{2} \cdot 300$. Then we can read how many carrots are needed for any number of potatoes up to 300.

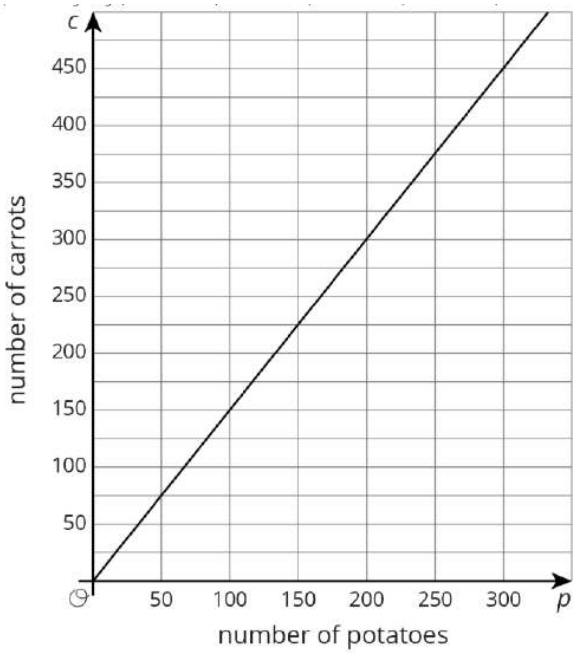


Figure 3.1.3.3: Graph, horizontal axis, number of potatoes, scale 0 to 300, by 50's. vertical axis, number of carrots, scale 0 to 450, by 50's. line passes through origin, 100 comma 150, 150 comma 225, 200 comma 300, 250 comma 375, and 300 comma 450.

Or if the recipe is used in a food factory that produces very large quantities and the potatoes come in bags of 150, we might just make a table of values showing the number of carrots needed for different multiplies of 150.

Table 3.1.3.2

number of potatoes	number of carrots
150	225
300	450
450	675
600	900

No matter the representation or the scale used, the constant of proportionality, $\frac{3}{2}$, is evident in each. In the equation it is the number we multiply p by; in the graph, it is the slope; and in the table, it is the number we multiply values in the left column to get numbers in the right column. We can think of the constant of proportionality as a **rate of change** of c with respect to p. In this case the rate of change is $\frac{3}{2}$ carrots per potato.

Glossary Entries

Definition: Constant of Proportionality

In a proportional relationship, the values for one quantity are each multiplied by the same number to get the values for the other quantity. This number is called the constant of proportionality.

In this example, the constant of proportionality is 3, because $2 \cdot 3 = 6$, $3 \cdot 3 = 9$, and $5 \cdot 3 = 15$. This means that there are 3 apples for every 1 orange in the fruit salad.

Table 3.1.3.3

number of oranges	number of apples
2	6
3	9
5	15

Definition: Rate of Change

The rate of change in a linear relationship is the amount y changes when x increases by 1. The rate of change in a linear relationship is also the slope of its graph.

In this graph, y increases by 15 dollars when x increases by 1 hour. The rate of change is 15 dollars per hour.

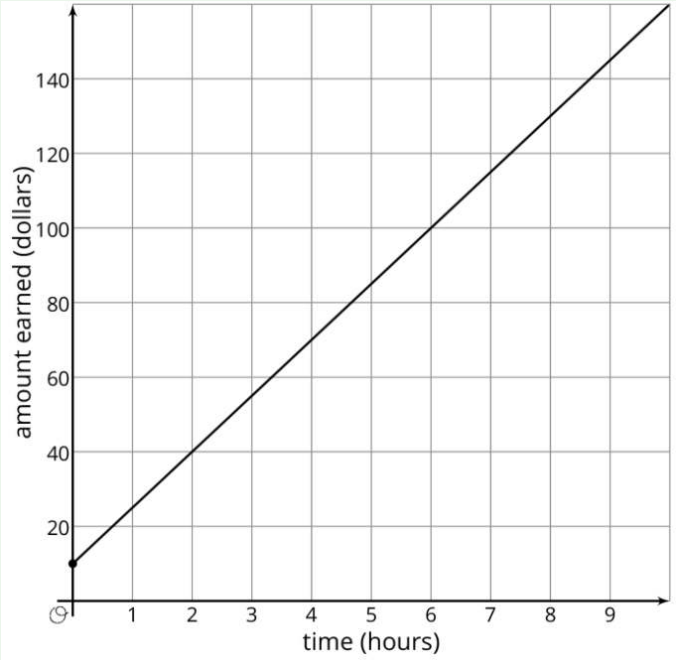


Figure 3.1.3.4: Graph, horizontal axis, time in hours, scale 0 to 9, by 1's. vertical axis, amount earned in dollars, scale 0 to 140, by 20's. line starting at 0 comma 10, passing through 2 comma 40 and 60 comma 100.

Practice

Exercise 3.1.3.4

Here is a graph of the proportional relationship between calories and grams of fish:

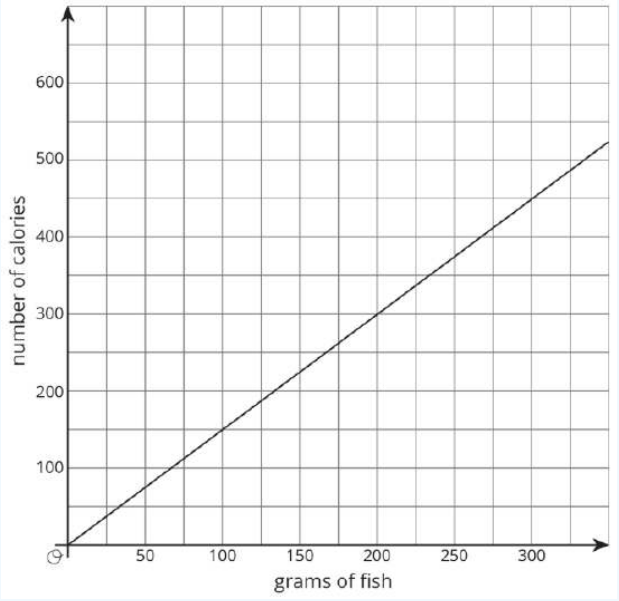


Figure 3.1.3.5: Graph, horizontal axis, grams of fish, scale 0 to 300, by 50's. vertical axis, number of calories, scale 0 to 600, by 100's. line passes through origin, 100 comma 150, 200 comma 300, 300 comma 450.

1. Write an equation that reflects this relationship using x to represent the amount of fish in grams and y to represent the number of calories.
2. Use your equation to complete the table:

Table 3.1.3.4

grams of fish	number of calories
1000	
	2001
1	

Exercise 3.1.3.5

Students are selling raffle tickets for a school fundraiser. They collect $24 for every 10 raffle tickets they sell.

1. Suppose M is the amount of money the students collect for selling R raffle tickets. Write an equation that reflects the relationship between M and R.
2. Label and scale the axes and graph this situation with M on the vertical axis and R on the horizontal axis. Make sure the scale is large enough to see how much they would raise if they sell 1000 tickets.

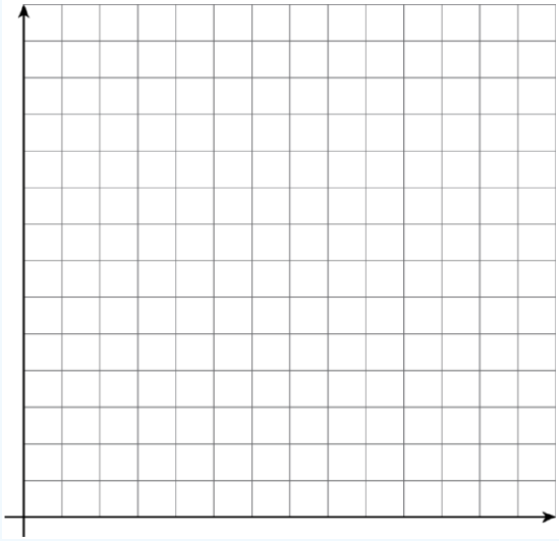


Figure 3.1.3.6

Exercise 3.1.3.6

Describe how you can tell whether a line's slope is greater than 1, equal to 1, or less than 1.

(From Unit 2.3.1)

Exercise 3.1.3.7

A line is represented by the equation $\frac{y}{x-2} = \frac{3}{11}$. What are the coordinates of some points that lie on the line? Graph the line on graph paper.

(From Unit 2.3.3)

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3.1.4: Comparing Proportional Relationships

Lesson

Let's compare proportional relationships.

Exercise 3.1.4.1: What's the Relationship?

The equation $y = 4.2x$ could represent a variety of different situations.

1. Write a description of a situation represented by this equation. Decide what quantities x and y represent in your situation.
2. Make a table and a graph that represent the situation.

Exercise 3.1.4.2: Comparing Two Different Representations

1. Elena babysits her neighbor's children. Her earnings are given by the equation $y = 8.40x$, where x represents the number of hours she worked and y represents the amount of money she earned.
   Jada earns $7 per hour mowing her neighbors' lawns.
   a. Who makes more money after working 12 hours? How much more do they make? Explain your reasoning by creating a graph or a table.
   b. What is the rate of change for each situation and what does it mean?
   c. Using your graph or table, determine how long it would take each person to earn $150.
2. Clare and Han have summer jobs stuffing envelopes for two different companies.
   a. By creating a graph, show how much money each person makes after stuffing 1,500 envelopes.
   b. What is the rate of change for each situation and what does it mean?
   c. Using your graph, determine how much more money one person makes relative to the other after stuffing 1,500 envelopes. Explain or show your reasoning.

   Han earns $15 for every 300 envelopes he finishes.

   Clare's earnings can be seen in the table.

Table 3.1.4.1

3.

number of envelopes	money in dollars
400	40
900	90

4. Tyler plans to start a lemonade stand and is trying to perfect his recipe for lemonade. He wants to make sure the recipe doesn't use too much lemonade mix (lemon juice and sugar) but still tastes good.

   Lemonade Recipe 1 is given by the equation $y = 4x) where \ ($ x represents the amount of lemonade mix in cups and y represents the amount of water in cups.

   Lemonade Recipe 2 is given in the table.

Table \(\PageIndex{2}\

lemonade mix (cups)	water (cups)
10	50
13	65
21	105

a. If Tyler had 16 cups of lemonade mix, how many cups of water would he need for each recipe? Explain your reasoning by creating a graph or a table.
b. What is the rate of change for each situation and what does it mean?
c. Tyler has a 5-gallon jug (which holds 80 cups) to use for his lemonade stand and 16 cups of lemonade mix. Which lemonade recipe should he use? Explain or show your reasoning.

Are you ready for more?

Han and Clare are still stuffing envelopes. Han can stuff 20 envelopes in a minute, and Clare can stuff 10 envelopes in a minute. They start working together on a pile of 1,000 envelopes.

1. How long does it take them to finish the pile?
2. Who earns more money?

Summary

When two proportional relationships are represented in different ways, we compare them by finding a common piece of information.

For example, Clare's earnings are represented by the equation $y = 14.5x$, where y is her earnings in dollars for working x hours.

The table shows some information about Jada's pay.

Table 3.1.4.3

time worked (hours)	earnings (dollars)
7	92.75
4.5	59.63
37	490.25

Who is paid at a higher rate per hour? How much more does that person have after 20 hours?

In Clare's equation we see that the rate of change (how many dollars she earns every hour) is 14.50.

We can calculate Jada's rate of change by dividing a value in the earnings column by the value in the same row in the time worked column. Using the last row, the rate of change for Jada is 13.25, since $490.25 \div 37 = 13.25$. An equation representing Jada's earnings is $y = 13.25x$. This means she earns \$13.25 per hour.

So Clare is paid at a higher rate than Jada. Clare earns \$1.25 more per hour than Jada. After 20 hours of work, she earns \$25 more than Jada because $20 \cdot (1.25) = 25$.

Glossary Entries

Definition: Constant of Proportionality

In a proportional relationship, the values for one quantity are each multiplied by the same number to get the values for the other quantity. This number is called the constant of proportionality.

In this example, the constant of proportionality is 3, because $2 \cdot 3 = 6$, $3 \cdot 3 = 9$, and $5 \cdot 3 = 15$. This means that there are 3 apples for every 1 orange in the fruit salad.

Table 3.1.4.4

number of oranges	number of apples
2	6
3	9
5	15

Definition: Rate of Change

The rate of change in a linear relationship is the amount y changes when x increases by 1. The rate of change in a linear relationship is also the slope of its graph.

In this graph, y increases by 15 dollars when x increases by 1 hour. The rate of change is 15 dollars per hour.

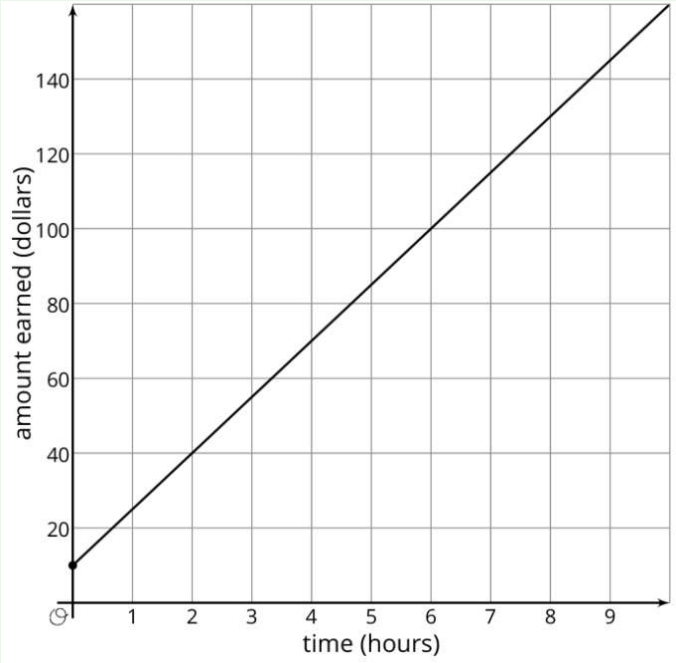


Figure 3.1.4.1: Graph, horizontal axis, time in hours, scale 0 to 9, by 1's. vertical axis, amount earned in dollars, scale 0 to 140, by 20's. line starting at 0 comma 10, passing through 2 comma 40 and 60 comma 100.

Practice

Exercise 3.1.4.3

A contractor must haul a large amount of dirt to a work site. She collected information from two hauling companies.

EZ Excavation gives its prices in a table.

Table 3.1.4.5

dirt (cubic yards)	cost (dollars)
8	196
20	490
26	637

Happy Hauling Service gives its prices in a graph.

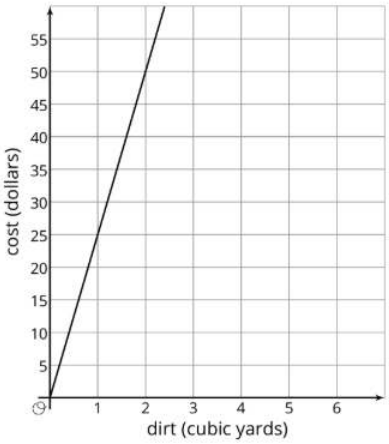


Figure 3.1.4.2

1. How much would each hauling company charge to haul 40 cubic yards of dirt? Explain or show your reasoning.
2. Calculate the rate of change for each relationship. What do they mean for each company?
3. If the contractor has 40 cubic yards of dirt to haul and a budget of $1000, which hauling company should she hire? Explain or show your reasoning.

Exercise 3.1.4.4

Andre and Priya are tracking the number of steps they walk. Andre records that he can walk 6000 steps in 50 minutes. Priya writes the equation $y = 118x$, where y is the number of steps and x is the number of minutes she walks, to describe her step rate. This week, Andre and Priya each walk for a total of 5 hours. Who walks more steps? How many more?

Exercise 3.1.4.5

Find the coordinates of point D in each diagram:

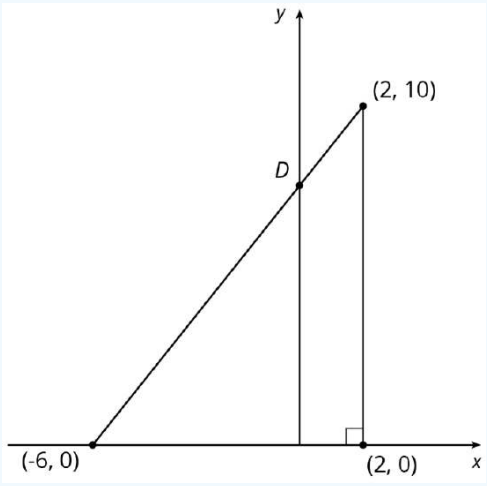


Figure 3.1.4.3

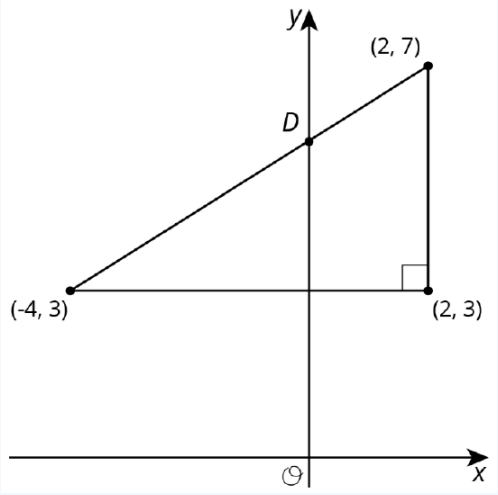


Figure 3.1.4.4

(From Unit 2.3.2)

Exercise 3.1.4.6

Select **all** the pairs of points so that the line between those points has slope $\frac{2}{3}$.

A. $(0, 0)$ and $(2, 3)$
B. $(0, 0)$ and $(3, 2)$
C. $(1, 5)$ and $(4, 7)$
D. $(-2, -2)$ and $(4, 2)$
E. $(20, 30)$ and $(-20, -30)$

(From Unit 2.3.2)

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3.2: Representing Linear Relationships

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3.2.1: Introduction to Linear Relationships

Lesson

Let's explore some relationships between two variables.

Exercise 3.2.1.1: Number Talk: Fraction Division

Find the value of $2\frac{5}{8} \div \frac{1}{2}$.

Exercise 3.2.1.2: Stacking Cups

We have two stacks of Styrofoam cups.

- One stack has 6 cups, and its height is 15 cm.
- The other stack has 12 cups, and its height is 23 cm.

How many cups are needed for a stack with a height of 50 cm?



Figure 3.2.1.1

Exercise 3.2.1.3: Connecting Slope to Rate of Change

1. If you didn't create your own graph of the situation before, do so now.

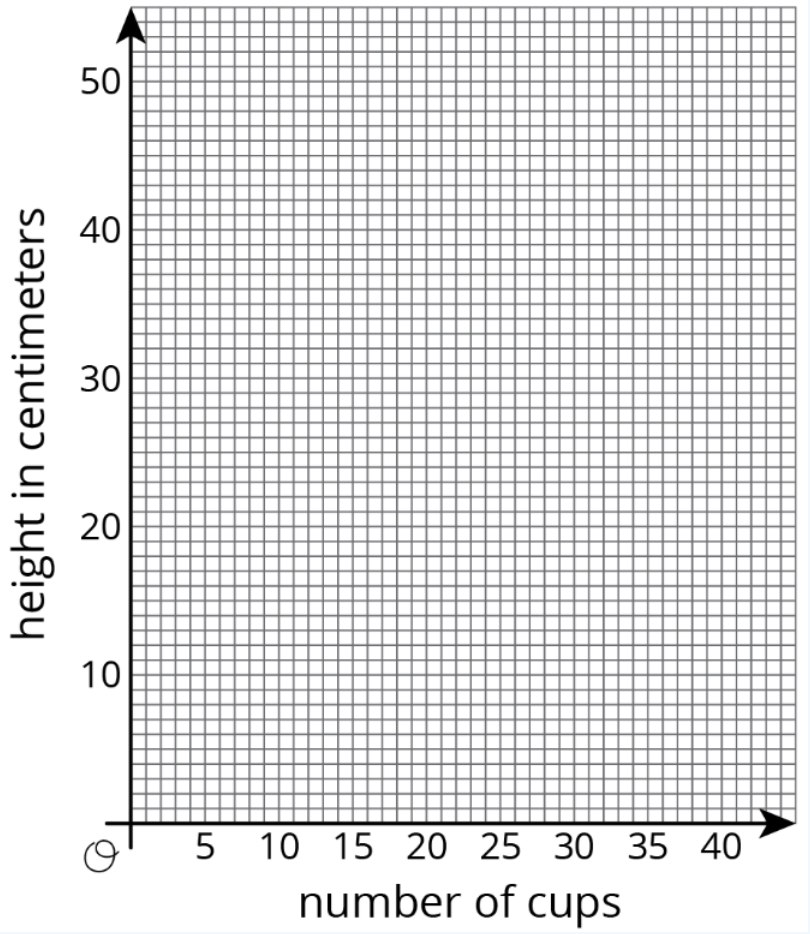


Figure 3.2.1.2

2. What are some ways you can tell that the number of cups is not proportional to the height of the stack?

3. What is the **slope** of the line in your graph? What does the slope mean in this situation?

4. At what point does your line intersect the vertical axis? What do the coordinates of this point tell you about the cups?

5. How much height does each cup after the first add to the stack?

Summary

Andre starts babysitting and charges $10 for traveling to and from the job, and $15 per hour. For every additional hour he works he charges another $15. If we graph Andre's earnings based on how long he works, we have a line that starts at $10 on the vertical axis and then increases by $15 each hour. A **linear relationship** is any relationship between two quantities where one quantity has a constant **rate of change** with respect to the other.

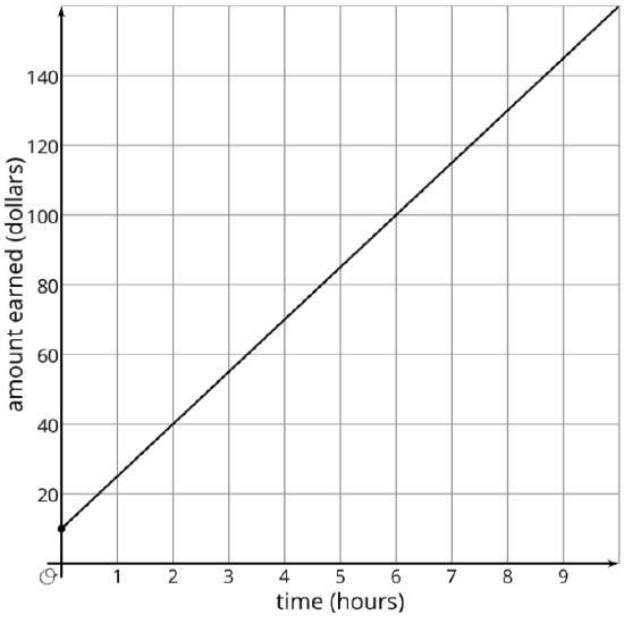


Figure 3.2.1.3: Graph, horizontal axis, time in hours, scale 0 to 9, by 1's. vertical axis, amount earned in dollars, scale 0 to 140, by 20's. line starting at 0 comma 10, passing through 2 comma 40 and 60 comma 100.

We can figure out the rate of change using the graph. Because the rate of change is constant, we can take any two points on the graph and divide the amount of vertical change by the amount of horizontal change. For example, take the points $(2, 40)$ and $(6, 100)$. They mean that Andre earns \$40 for working 2 hours and \$100 for working 6 hours. The rate of change is $\frac{100-40}{6-2} = 15$ dollars per hour. Andre's earnings go up \$15 for each hour of babysitting.

Notice that this is the same way we calculate the **slope** of the line. That's why the graph is a line, and why we call this a linear relationship. The rate of change of a linear relationship is the same as the slope of its graph.

With proportional relationships we are used to graphs that contain the point $(0, 0)$. But proportional relationships are just one type of linear relationship. In the following lessons, we will continue to explore the other type of linear relationship where the quantities are not both 0 at the same time.

Glossary Entries

Definition: Linear Relationship

A linear relationship between two quantities means they are related like this: When one quantity changes by a certain amount, the other quantity always changes by a set amount. In a linear relationship, one quantity has a constant rate of change with respect to the other.

The relationship is called linear because its graph is a line.

The graph shows a relationship between number of days and number of pages read.

When the number of days increases by 2, the number of pages read always increases by 60. The rate of change is constant, 30 pages per day, so the relationship is linear.

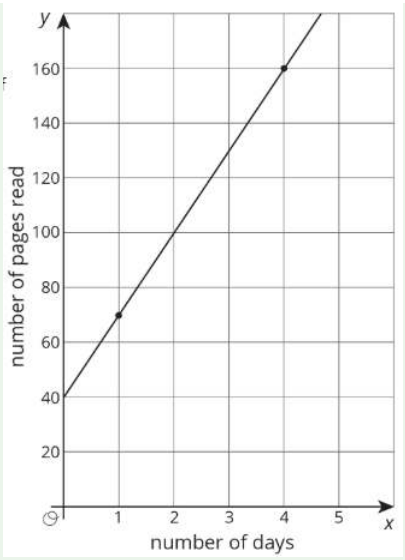


Figure 3.2.1.4

Practice

Exercise 3.2.1.4

A restaurant offers delivery for their pizzas. The total cost is a delivery fee added to the price of the pizzas. One customer pays \$25 to have 2 pizzas delivered. Another customer pays \$58 for 5 pizzas. How many pizzas are delivered to a customer who pays \$80?

Exercise 3.2.1.5

To paint a house, a painting company charges a flat rate of \$500 for supplies, plus \$50 for each hour of labor.

1. How much would the painting company charge to paint a house that needs 20 hours of labor? A house that needs 50 hours?
2. Draw a line representing the relationship between x, the number of hours it takes the painting company to finish the house, and y, the total cost of painting the house. Label the two points from the earlier question on your graph.

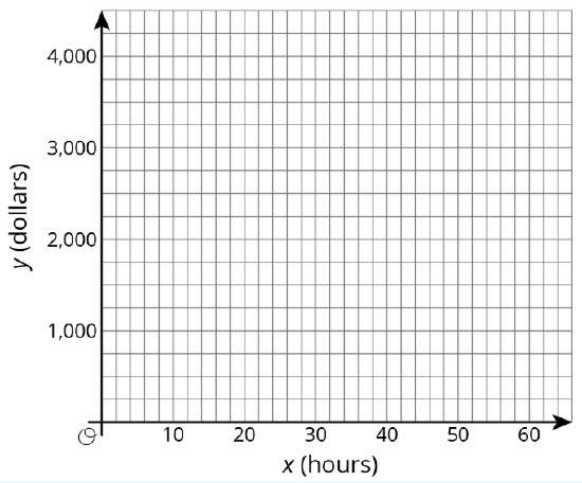


Figure 3.2.1.5

3. Find the slope of the line. What is the meaning of the slope in this context?

Exercise 3.2.1.6

Tyler and Elena are on the cross country team.

Tyler's distances and times for a training run are shown on the graph.

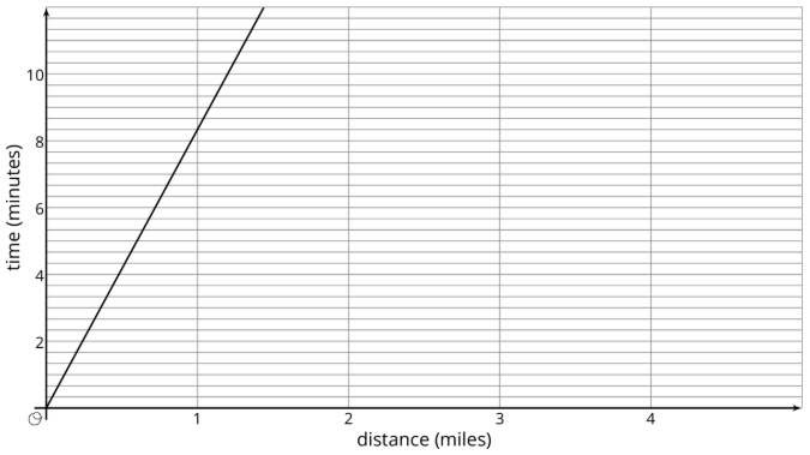


Figure 3.2.1.6: Graph, horizontal axis, distance in miles, scale 0 to 4, by 1's. vertical axis, time in minutes, scale 0 to 190, by 2's. line passes through the origin and 1 comma 8 and the fraction 1 over 3.

Elena's distances and times for a training run are given by the equation $y = 8.5x$, where x represents distance in miles and y represents time in minutes.

1. Who ran farther in 10 minutes? How much farther? Explain how you know.
2. Calculate each runner's pace in minutes per mile.
3. Who ran faster during the training run? Explain or show your reasoning.

(From Unit 3.1.4)

Exercise 3.2.1.7

Write an equation for the line that passes through $(2, 5)$ and $(6, 7)$.

(From Unit 2.3.3)

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3.2.2: More Linear Relationships

Lesson

Let's explore some more relationships between two variables.

Exercise 3.2.2.1: Growing

Look for a growing pattern. Describe the pattern you see.

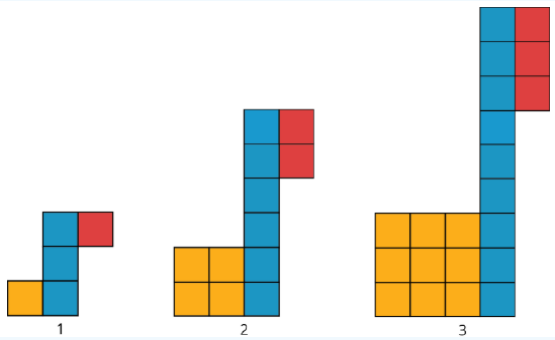


Figure 3.2.2.1

1. If your pattern continues growing in the same way, how many tiles of each color will be in the 4th and 5th diagram? The 10th diagram?
2. How many tiles of each color will be in the nth diagram? Be prepared to explain how your reasoning.

Exercise 3.2.2.2: Slopes, Vertical Intercepts, and Graphs

Your teacher will give you 6 cards describing different situations and 6 cards with graphs.

1. Match each situation to a graph.
2. Pick one proportional relationship and one non-proportional relationship and answer the following questions about them.
   a. How can you find the slope from the graph? Explain or show your reasoning.
   b. Explain what the slope means in the situation.
   c. Find the point where the line crosses the vertical axis. What does that point tell you about the situation?

Exercise 3.2.2.3: Summer Reading

Lin has a summer reading assignment. After reading the first 30 pages of the book, she plans to read 40 pages each day until she finishes. Lin makes the graph shown here to track how many total pages she'll read over the next few days.

After day 1, Lin reaches page 70, which matches the point $(1, 70)$ she made on her graph. After day 4, Lin reaches page 190, which does not match the point $(4, 160)$ she made on her graph. Lin is not sure what went wrong since she knows she followed her reading plan.

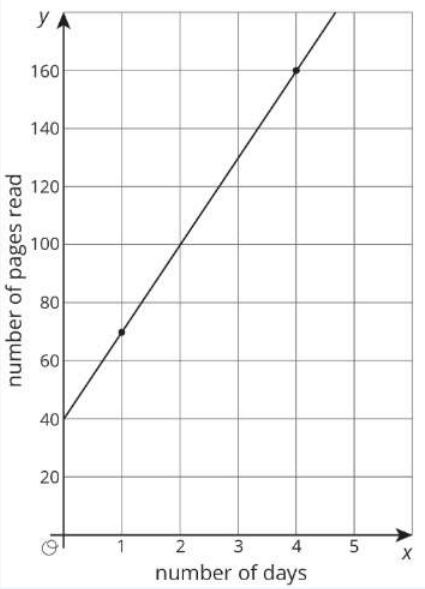


Figure 3.2.2.2

1. Sketch a line showing Lin's original plan on the axes.
2. What does the **vertical intercept** mean in this situation? How do the vertical intercepts of the two lines compare?
3. What does the slope mean in this situation? How do the slopes of the two lines compare?

Are you ready for more?

Jada's grandparents started a savings account for her in 2010. The table shows the amount in the account each year.

If this relationship is graphed with the year on the horizontal axis and the amount in dollars on the vertical axis, what is the vertical intercept? What does it mean in this context?

Table 3.2.2.1

year	amount in dollars
2010	600
2012	750
2014	900
2016	1050

Summary

At the start of summer break, Jada and Lin decide to save some of the money they earn helping out their neighbors to use during the school year. Jada starts by putting $20 into a savings jar in her room and plans to save $10 a week. Lin starts by putting $10 into a savings jar in her room plans to save $20 a week. Here are graphs of how much money they will save after 10 weeks if they each follow their plans:

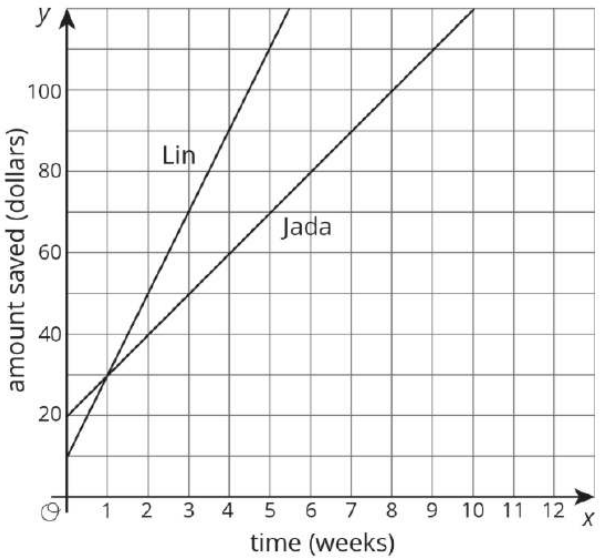


Figure 3.2.2.3: Graph of 2 lines in quadrant 1. Horizontal axis, time in weeks, scale 0 to 12, by 1's. Vertical axis, amount saved in dollars, scale 0 to 100, by 20's. Lin line, y intercept = 10. Slope = 20. Jada line, y intercept = 20, slope= 10.

The value where a line intersects the vertical axis is called the **vertical intercept**. When the vertical axis is labeled with a variable like y, this value is also often called the *y-intercept*. Jada's graph has a vertical intercept of \$20 while Lin's graph has a vertical intercept of \$10. These values reflect the amount of money they each started with. At 1 week they will have saved the same amount, \$30. But after week 1, Lin is saving more money per week (so she has a larger rate of change), so she will end up saving more money over the summer if they each follow their plans.

Glossary Entries

Definition: Linear Relationship

A linear relationship between two quantities means they are related like this: When one quantity changes by a certain amount, the other quantity always changes by a set amount. In a linear relationship, one quantity has a constant rate of change with respect to the other.

The relationship is called linear because its graph is a line.

The graph shows a relationship between number of days and number of pages read.

When the number of days increases by 2, the number of pages read always increases by 60. The rate of change is constant, 30 pages per day, so the relationship is linear.

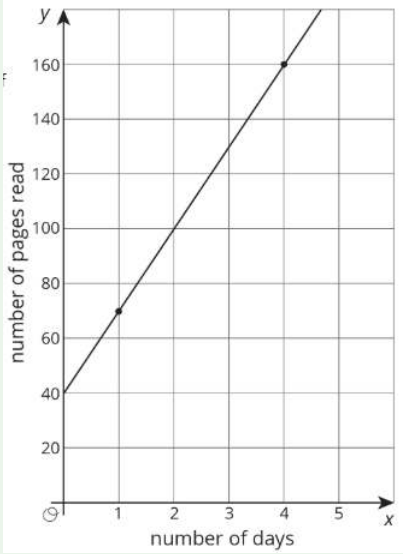


Figure 3.2.2.4

Definition: Vertical Intercept

The vertical intercept is the point where the graph of a line crosses the vertical axis.

The vertical intercept of this line is $(0, -6)$ or just -6.

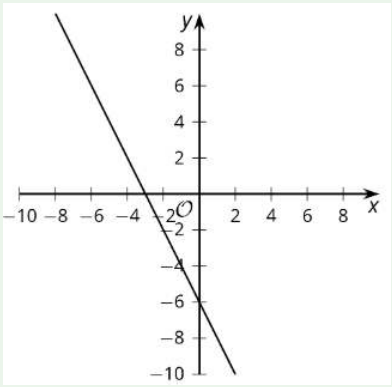


Figure 3.2.2.5

Practice

Exercise 3.2.2.4

Explain what the slope and intercept mean in each situation.

1. A graph represents the perimeter, y, in units, for an equilateral triangle with side length x units. The slope of the line is 3 and the y-intercept is 0.
2. The amount of money, y, in a cash box after x tickets are purchased for carnival games. The slope of the line is $\frac{1}{4}$ and the y-intercept is 8.
3. The number of chapters read, y, after x days. The slope of the line is $\frac{5}{4}$ and the y-intercept is 2.
4. The graph shows the cost in dollars, y, of a muffin delivery and the number of muffins, x, ordered. The slope of the line is 2 and the y-intercept is 3.

Exercise 3.2.2.5

Customers at the gym pay a membership fee to join and then a fee for each class they attend. Here is a graph that represents the situation.

1. What does the slope of the line shown by the points mean in this situation?
2. What does the vertical intercept mean in this situation?

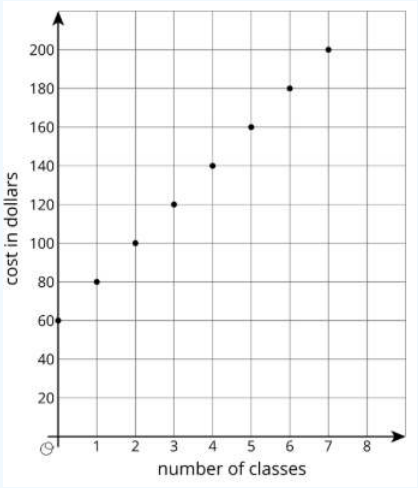


Figure 3.2.2.6

Exercise 3.2.2.6

The graph shows the relationship between the number of cups of flour and the number of cups of sugar in Lin's favorite brownie recipe.

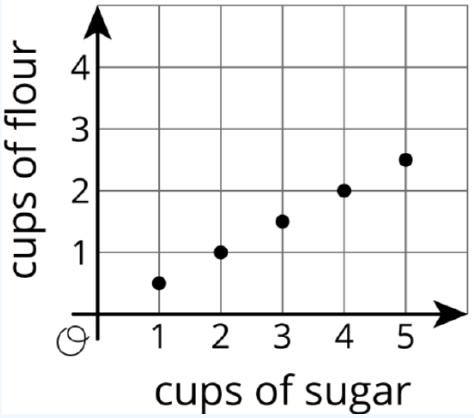


Figure 3.2.2.7: Graph, horizontal axis, cups of sugar, scale 0 to 5, by 1's. vertical axis, cups of flour, scale 0 to 4, by 1's. points plotted at 1 comma 5 tenths, 2 comma 1, 3 comma 1 and 5 tenths, 4 comma 2, 5 comma 2 and 5 tenths.

The table shows the amounts of flour and sugar needed for Noah's favorite brownie recipe.

Table 3.2.2.2

cups of sugar	cups of flour
$\frac{3}{2}$	1
3	2
$4\frac{1}{2}$	3

1. Noah and Lin buy a 12-cup bag of sugar and divide it evenly to make their recipes. If they each use all their sugar, how much flour do they each need?

2. Noah and Lin buy a 10-cup bag of flour and divide it evenly to make their recipes. If they each use all their flour, how much sugar do they each need?

(From Unit 3.1.4)

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3.2.3: Representations of Linear Relationships

Lesson

Let's write equations from real solutions.

Exercise 3.2.3.1: Estimation: Which Holds More?

Which glass will hold the most water? The least?

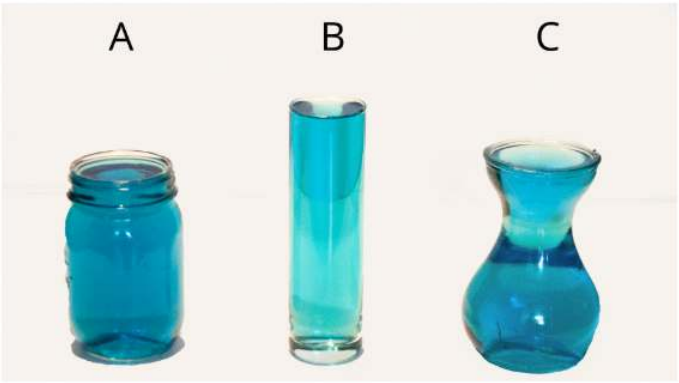


Figure 3.2.3.1

Exercise 3.2.3.2: Rising Water Levels

Move the green circle to set the starting level of water to a value you or your teacher choose.

1. What is the volume, V, in the cylinder after you add:
   a. 3 objects?
   b. 7 objects?
   c. x objects? Explain your reasoning.
2. If you wanted to make the water reach the highest mark on the cylinder, how many objects would you need?
3. Plot and label points that show your measurements from the experiment.
4. Plot and label a point that shows the depth of the water before you added any objects.
5. The points should fall on a line. Use the Line tool to draw this line.
6. Compute the slope of the line using several different triangles. Does it matter which triangle you use to compute the slope? Why or why not?
7. The equation of the line in the experiment has two numbers and two variables. What physical quantities do the two numbers represent? What does V represent and what does x represent?

Are you ready for more?

A situation is represented by the equation $y = 5 + \frac{1}{2}x$.

1. Invent a story for this situation.
2. Graph the equation.
3. What do the $\frac{1}{2}$ and the 5 represent in your situation?
4. Where do you see the $\frac{1}{2}$ and 5 on the graph?

Exercise 3.2.3.3: Calculate the Slope

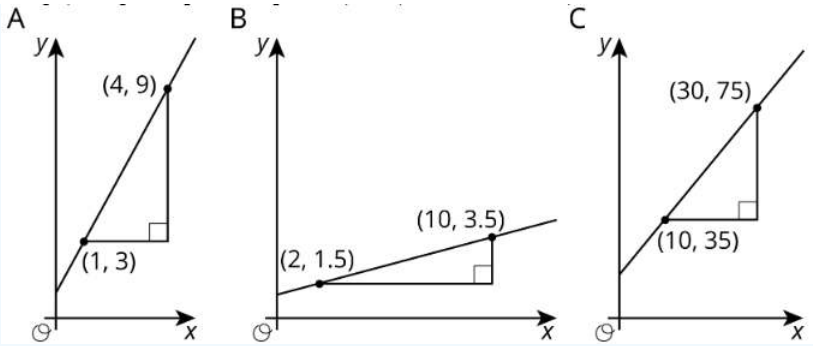


Figure 3.2.3.2: 3 graphs of lines labeled A, B, C. graph A, right triangle drawn right and up from point 1 comma 3 to point 4 comma 9, both on the line. graph B, right triangle drawn right and up from point 2 comma 1 and 5 tenths to point 10 comma 3 and 5 tenths, both on the line. graph C, right triangle drawn right and up from point 10 comma 35 to point 30 comma 75, both on the line.

1. For each graph record:

Table 3.2.3.1

vertical change	horizontal change	slope

2. Describe a procedure for finding the slope between any two points on a line.
3. Write an expression for the slope of the line in the graph using the letters u, v, s, and t.

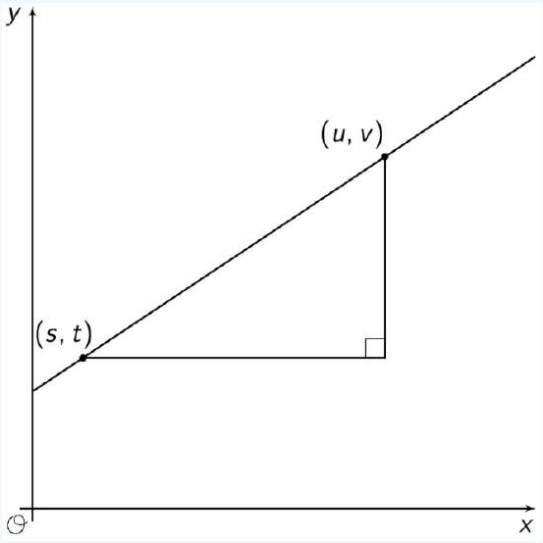


Figure 3.2.3.3

Summary

Let's say we have a glass cylinder filled with 50 ml of water and a bunch of marbles that are 3 ml in volume. If we drop marbles into the cylinder one at a time, we can watch the height of the water increase by the same amount, 3 ml, for each one added. This constant rate of change means there is a linear relationship between the number of marbles and the height of the water. Add one marble, the water height goes up 3 ml. Add 2 marbles, the water height goes up 6 ml. Add x marbles, the water height goes up $3x$ ml.

Reasoning this way, we can calculate that the height, y, of the water for x marbles is $y = 3x + 50$. Any linear relationships can be expressed in the form $y = mx + b$ using just the rate of change, m, and the initial amount, b. The 3 represents the rate of change, or slope of the graph, and the 50 represents the initial amount, or vertical intercept of the graph. We'll learn about some more ways to think about this equation in future lessons.

Now what if we didn't have a description to use to figure out the slope and the vertical intercept? That's okay so long as we can find some points on the line! For the line graphed here, two of the points on the line are $(3, 3)$ and $(9, 5)$ and we can use these points to

draw in a slope triangle as shown:

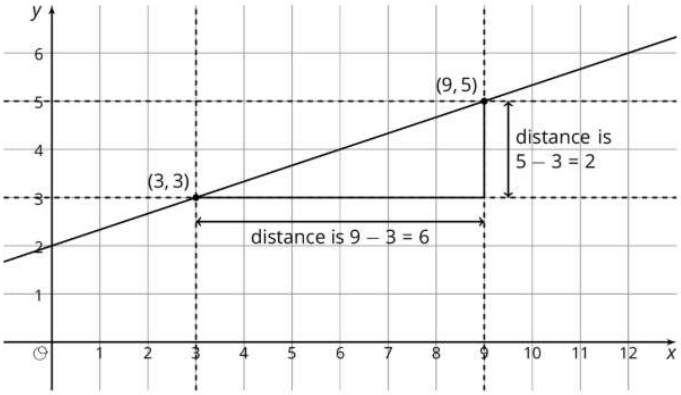


Figure 3.2.3.4

The slope of this line is the quotient of the length of the vertical side of the slope triangle and the length of the horizontal side of the slope triangle. So the slope, m, is $\frac{vertical change}{horizontal change} = \frac{2}{6} = \frac{1}{3}$. We can also see from the graph that the vertical intercept, b, is 2. Putting these together, we can say that the equation for this line is $y = \frac{1}{3}x + 2$.

Glossary Entries

Definition: Linear Relationship

A linear relationship between two quantities means they are related like this: When one quantity changes by a certain amount, the other quantity always changes by a set amount. In a linear relationship, one quantity has a constant rate of change with respect to the other.

The relationship is called linear because its graph is a line.

The graph shows a relationship between number of days and number of pages read.

When the number of days increases by 2, the number of pages read always increases by 60. The rate of change is constant, 30 pages per day, so the relationship is linear.

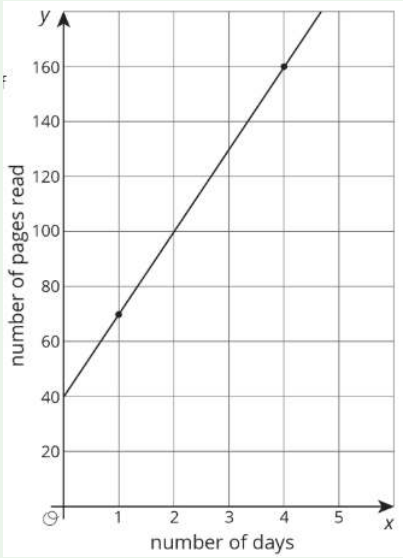


Figure 3.2.3.5

Definition: Vertical Intercept

The vertical intercept is the point where the graph of a line crosses the vertical axis.

The vertical intercept of this line is $(0, -6)$ or just -6.

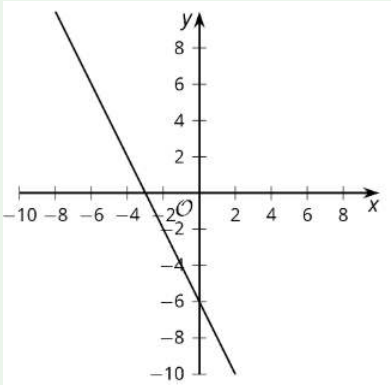


Figure 3.2.3.6

Practice

Exercise 3.2.3.4

Create a graph that shows three linear relationships with different -intercepts using the following slopes, and write an equation for each line.

Slopes:

- $\frac{1}{5}$
- $\frac{3}{5}$
- $\frac{6}{5}$

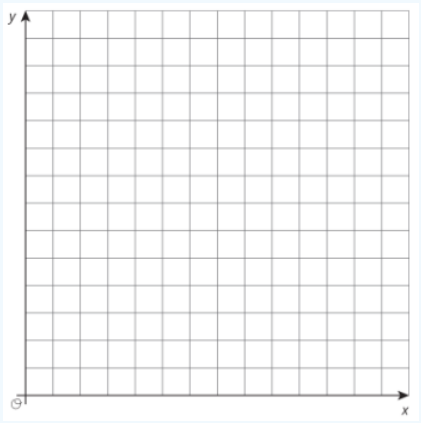


Figure 3.2.3.7

Exercise 3.2.3.5

The graph shows the height in inches, h, of a bamboo plant t months after it has been planted.

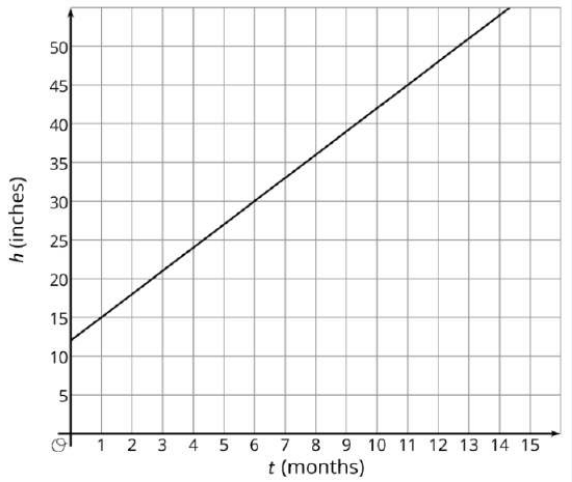


Figure 3.2.3.8

1. Write an equation that describes the relationship between h and t.
2. After how many months will the bamboo plant be 66 inches tall? Explain or show your reasoning.

Exercise 3.2.3.6

Here are recipes for two different banana cakes. Information for the first recipe is shown in the table.

Table 3.2.3.2

sugar (cups)	flour (cups)
$\frac{1}{2}$	$\frac{3}{4}$
$2\frac{1}{2}$	$3\frac{3}{4}$
3	$4\frac{1}{2}$

The relationship between cups of flour y and cups of sugar x in the second recipe is $y = \frac{7}{4}x$.

1. If you used 4 cups of sugar, how much flour does each recipe need?
2. What is the constant of proportionality for each situation and what does it mean?

(From Unit 3.1.4)

Exercise 3.2.3.7

Show that the two figures are similar by identifying a sequence of translations, rotations, reflections, and dilations that takes the larger figure to the smaller one.

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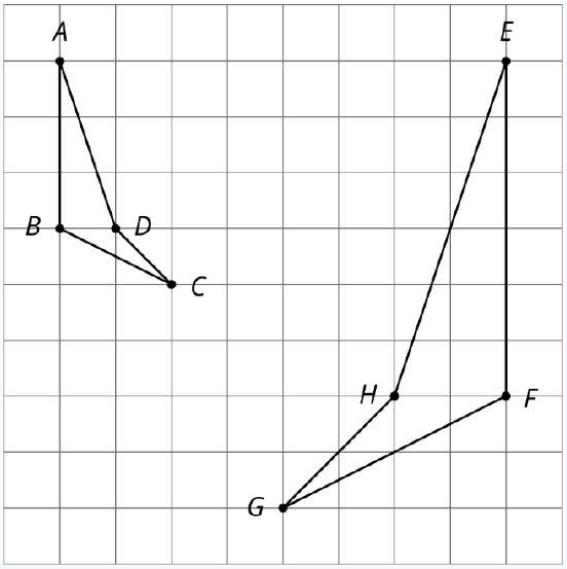


Figure ABCD and figure EFGH on a 10 by 10 grid. letting bottom left corner = 0 comma 0, A = 1 comma 9, B = 1 comma 6, C = 3 comma 5, D = 2 comma 6, E = 9 comma 9, F = 9 comma 3, G = 5 comma 1, H = 7 comma 3.

(From Unit 2.2.1)

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3.2.4: Translating y=mx+b

Lesson

Let's see what happens to the equations of translated lines.

Exercise 3.2.4.1: Lines that Are Translations

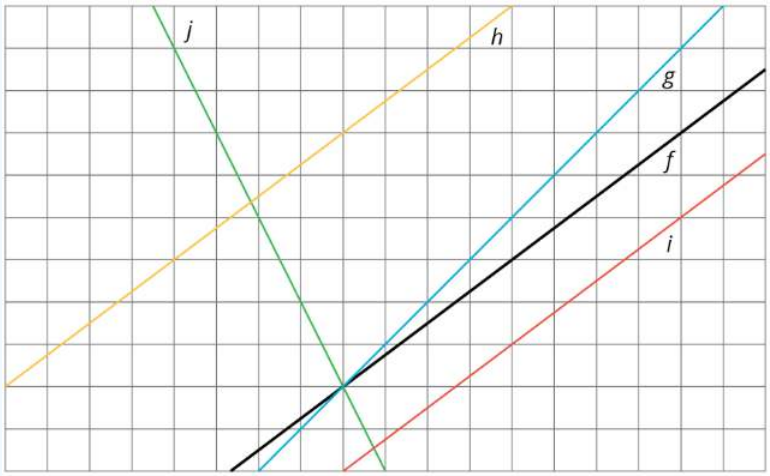


Figure 3.2.4.1: Lines j, h, g, f, i graphed on grid. j is green with a negative slope, h is yellow and parallel to f which is black, h is 6 unit up from f. i is red and parallel to f and h. i is 2 unit below f. g is blue and has a more steep positive slope than h, f, and i.

The diagram shows several lines. You can only see part of the lines, but they actually continue forever in both directions.

1. Which lines are images of line f under a translation?
2. For each line that is a translation of f, draw an arrow on the grid that shows the vertical translation distance.

Exercise 3.2.4.2: Increased Savings

1. Diego earns $10 per hour babysitting. Assume that he has no money saved before he starts babysitting and plans to save all of his earnings. Graph how much money, y, he has after x hours of babysitting.
2. Now imagine that Diego started with $30 saved before he starts babysitting. On the same set of axes, graph how much money, y, he would have after x hours of babysitting.
3. Compare the second line with the first line. How much *more* money does Diego have after 1 hour of babysitting? 2 hours? 5 hours? x hours?
4. Write an equation for each line.

Exercise 3.2.4.3: Translating a Line

1. Experiment with moving point A.
   a. Place point A in three different locations *above* the x-axis. For each location, write the equation of the line and the coordinates of point A.
   b. Place point A in three different locations *below* the x-axis. For each location, write the equation of the line and the coordinates of point A.
   c. In the equations, what changes as you move the line? What stays the same?
   d. If the line passes through the origin, what equation is displayed? Why do you think this is the case?
2. Your teacher will give you 12 cards. There are 4 pairs of lines, A–D, showing the graph, a, of a proportional relationship and the image, h, of a under a translation. Match each line with an equation and either a table or description. For the line with no matching equation, write one on the blank card.

Are you ready for more?

A student says that the graph of the equation $y = 3(x + 8)$ is the same as the graph of $y = 3x$, only translated upwards by 8 units. Do you agree? Why or why not?

Summary

During an early winter storm, the snow fell at a rate of $\frac{1}{2}$ inch per hour. We can see the rate of change, $\frac{1}{2}$, in both the equation that represents this storm, $y = \frac{1}{2}x$, and in the slope of the line representing this storm.

In addition to being a linear relationship between the time since the beginning of the storm and the depth of the snow, we can also call this as a proportional relationship since the depth of snow was 0 at the beginning of the storm.

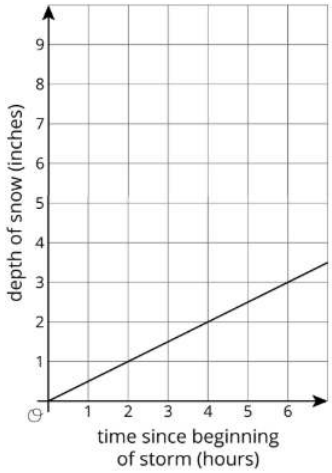


Figure 3.2.4.2: Graph of line. Horizontal axis, time since beginning of storm in hours, scale 0 to 6, by 1's. Vertical axis, depth of snow in inches, scale 0 to 9, by 1's. Points on line include 0 comma 0, 2 comma 1 and 4 comma 2.

During a mid-winter storm, the snow again fell at a rate of inch per hour, but this time there was already 5 inches of snow on the ground.

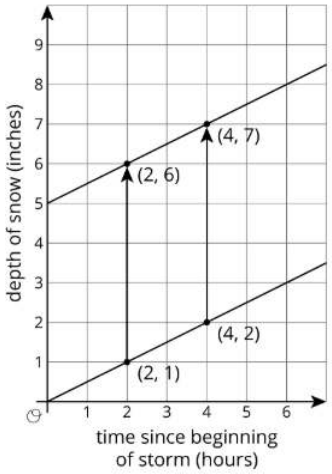


Figure 3.2.4.3

We can graph this storm on the same axes as the first storm by taking all the points on the graph of the first storm and translating them up 5 inches.

Two hours after each storm begins, 1 inch of new snow has fallen. For the first storm, this means there is now 1 inch of snow on the ground. For the second storm, this means there are now 6 inches of snow on the ground.

Unlike the first storm, the second is not a proportional relationship since the line representing the second storm has a vertical intercept of 5. The equation representing the storm, $y = \frac{1}{2}x + 5$, is of the form $y = mx + b$, where m is the rate of change, also the slope of the graph, and b is the initial amount, also the vertical intercept of the graph.

Glossary Entries

Definition: Linear Relationship

A linear relationship between two quantities means they are related like this: When one quantity changes by a certain amount, the other quantity always changes by a set amount. In a linear relationship, one quantity has a constant rate of change with respect to the other.

The relationship is called linear because its graph is a line.

The graph shows a relationship between number of days and number of pages read.

When the number of days increases by 2, the number of pages read always increases by 60. The rate of change is constant, 30 pages per day, so the relationship is linear.

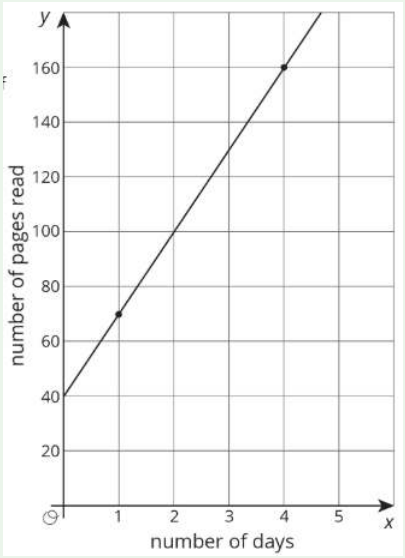


Figure 3.2.4.4

Definition: Vertical Intercept

The vertical intercept is the point where the graph of a line crosses the vertical axis.

The vertical intercept of this line is $(0, -6)$ or just -6.

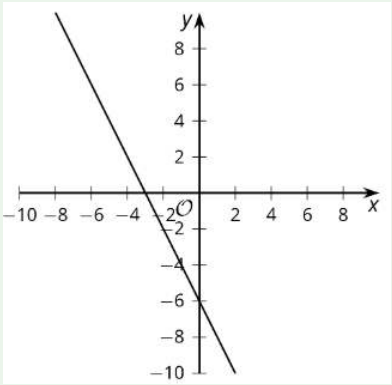


Figure 3.2.4.5

Practice

Exercise 3.2.4.4

Select **all** the equations that have graphs with the same y-intercept.

A. $y = 3x - 8$
B. $y = 3x - 9$
C. $y = 3x + 8$
D. $y = 5x - 8$
E. $y = 2x - 8$
F. $y = \frac{1}{3}x - 8$

Exercise 3.2.4.5

Create a graph showing the equations $y = \frac{1}{4}x$ and $y = \frac{1}{4}x - 5$. Explain how the graphs are the same and how they are different.

Exercise 3.2.4.6

A cable company charges $70 per month for cable service to existing customers.

1. Find a linear equation representing the relationship between x, the number of months of service, and y, the total amount paid in dollars by an existing customer.
2. For new customers, there is an additional one-time $100 service fee. Repeat the previous problem for new customers.
3. When the two equations are graphed in the coordinate plane, how are they related to each other geometrically?

Exercise 3.2.4.7

A mountain road is 5 miles long and gains elevation at a constant rate. After 2 miles, the elevation is 5500 feet above sea level. After 4 miles, the elevation is 6200 feet above sea level.

1. Find the elevation of the road at the point where the road begins.
2. Describe where you would see the point in part (a) on a graph where y represents the elevation in feet and x represents the distance along the road in miles.

(From Unit 3.2.2.)

Exercise 3.2.4.8

Match each graph to a situation.

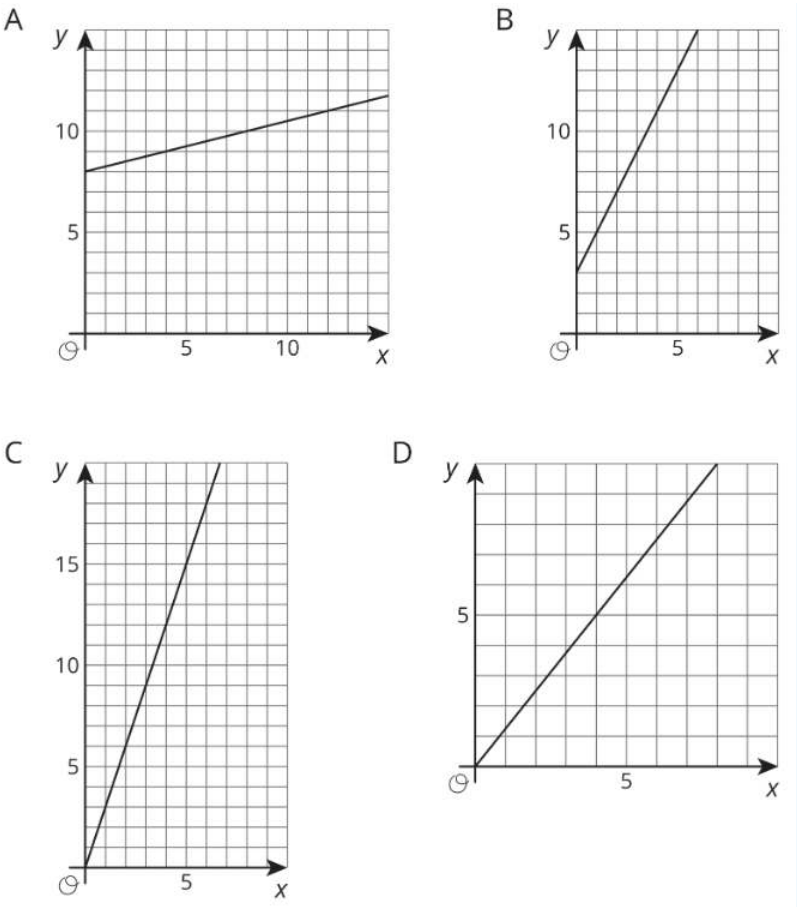


Figure 3.2.4.6: 4 graphs of lines labeled A, B, C. D. A, y intercept = 8, slope = the fraction 1 over 4. B, y intercept = 3, slope = 2. C, y intercept = 0, slope = 3. D, y intercept = 0, slope = the fraction 5 over 4.

A. Graph A
B. Graph B
C. Graph C
D. Graph D

1. The graph represents the perimeter, y, in units, for an equilateral triangle with side length of x units. The slope of the line is 3.
2. The amount of money, y, in a cash box after x tickets are purchased for carnival games. The slope of the line is $\frac{1}{4}$.
3. The number of chapters read, y, after x days. The slope of the line is $\frac{5}{4}$.
4. The graph shows the cost in dollars, y, of a muffin delivery and the number of muffins, x, ordered. The slope of the line is 2.

(From Unit 3.2.2)

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3.3: Finding Slopes

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3.3.1: Slopes Don't Have to be Positive

Lesson

Let's find out what a negative slope means.

Exercise 3.3.1.1: Which One Doesn't Belong: Odd Line Out

Which line doesn't belong?

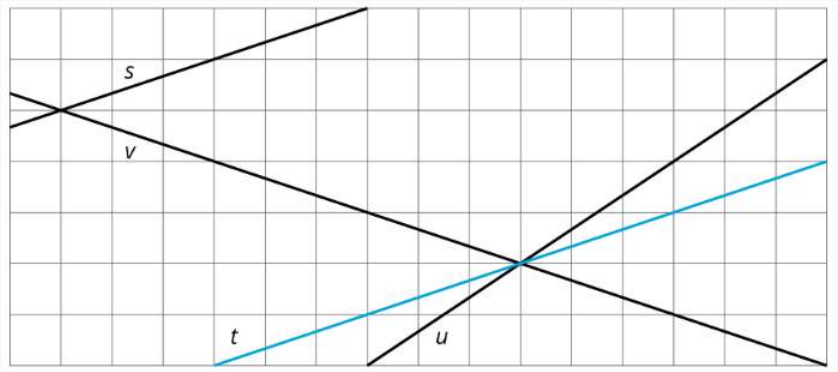


Figure 3.3.1.1: 4 lines plotted on a grid, lines labeled s, t, u, v. line s, posiitve slope, intersects line v. line v, negative slope, intersects lines s, t, u. line t, positive slope, intersects lines v and u. line u, positive slope, intersects lines t and v.

Exercise 3.3.1.2: Stand Clear of the Closing Doors, Please

Noah put $40 on his fare card. Every time he rides public transportation, $2.50 is subtracted from the amount available on his card.

1. How much money, in dollars, is available on his card after he takes
   a. 0 rides?
   b. 1 ride?
   c. 2 rides?
   d. rides?
2. Graph the relationship between amount of money on the card and number of rides.
3. How many rides can Noah take before the card runs out of money? Where do you see this number of rides on your graph?

Exercise 3.3.1.3: Travel Habits in July

Here is a graph that shows the amount on Han's dare card for every day of last july.

1. Describe what happened with the amount on Han’s fare card in July.
2. Plot and label 3 different points on the line.
3. Write an equation that represents the amount on the card in July, y, after x days.
4. What value makes sense for the slope of the line that represents the amounts on Han’s fare card in July?

Are you ready for more?

Let’s say you have taken out a loan and are paying it back. Which of the following graphs have positive slope and which have negative slope?

1. Amount paid on the vertical axis and time since payments started on the horizontal axis.
   Amount owed on the vertical axis and time remaining until the loan is paid off on the horizontal axis.
2. Amount paid on the vertical axis and time remaining until the loan is paid off on the horizontal axis.

Exercise 3.3.1.4: Payback Plan

Elena borrowed some money from her brother. She pays him back by giving him the same amount every week. The graph shows how much she owes after each week.

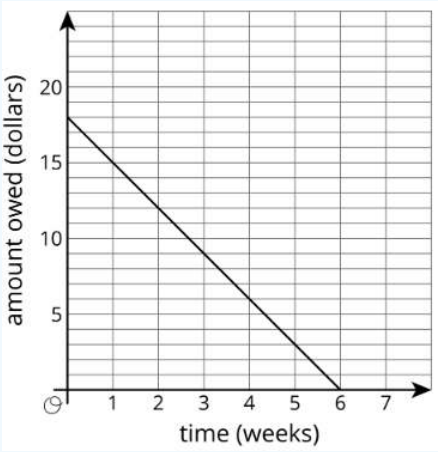


Figure 3.3.1.2: Graph of a line on grid, origin O. horizontal axis, time in weeks, scale 0 to 7, by 1's. vertical axis, scale 0 to 20, by 5's. line crosses y axis at 0 comma 18 and crosses the x axis at 6 comma 0.

Answer and explain your reasoning for each question.

1. What is the slope of the line?
2. Explain how you know whether the slope is positive or negative.
3. What does the slope represent in this situation?
4. How much did Elena borrow?
5. How much time will it take for Elena to pay back all the money she borrowed?

Summary

At the end of winter in Maine, the snow on the ground was 30 inches deep. Then there was a particularly warm day and the snow melted at the rate of 1 inch per hour. The graph shows the relationship between the time since the snow started to melt and the depth of the snow.

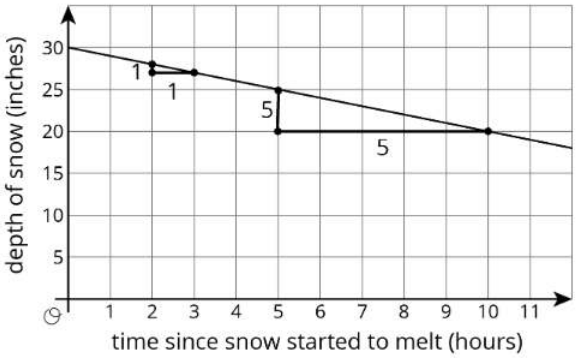


Figure 3.3.1.3: Graph on grid, origin O. horizontal axis, time since snow started to melt in hours, scale 0 to 11, by 1's. vertical axis, depth of snow in inches, scale 0 to 30, by 5's. line with negative slope drawn that goes through points 2 comma 28, 3 comma 27. triange with down 1 right 1 drawn below these 2 points. line also goes through 5 comma 25 and 10 comma 20. triangle with down 5 right 5 drawn below these 2 points.

The slope of the graph is -1 since the rate of change is -1 inch per hour. That is, the depth goes *down* 1 inch per hour. The vertical intercept is 30 since the snow was 30 inches deep when the warmth started to melt the snow. The two slope triangles show how the rate of change is constant. It just also happens to be negative in this case since after each hour that passes, there is 1 inch *less* snow.

Graphs with negative slope often describe situations where some quantity is decreasing over time, like the depth of snow on warm days or the amount of money on a fare card being used to take rides on buses.

Slopes can be positive, negative, or even zero! A slope of 0 means there is no change in the y-value even though the x-value may be changing. For example, Elena won a contest where the prize was a special pass that gives her free bus rides for a year. Her fare card had $5 on it when she won the prize. Here is a graph of the amount of money on her fare card after winning the prize:

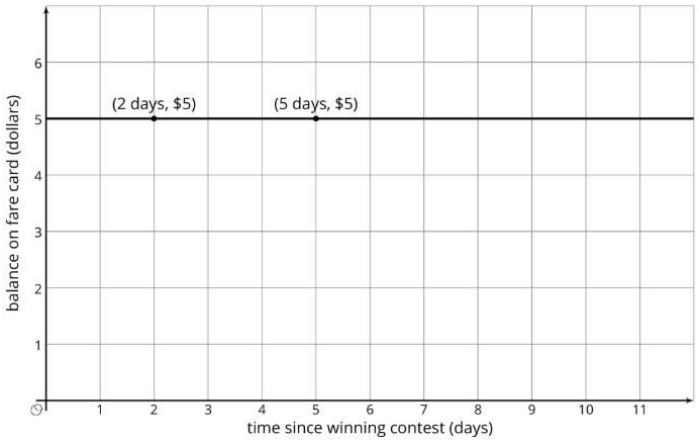


Figure 3.3.1.4: Graph on a grid, origin O. horizontal axis, time since winning contest in days, scale 0 to 11, by 1's. vertical axis, balance on fare card in dollars, 0 to 6, by 1's. horizontal line through 2 days comma 5 dollars and 5 days comma 5 dollars.

The vertical intercept is 5, since the graph starts when she has $5 on her fare card. The slope of the graph is 0 since she doesn't use her fare card for the next year, meaning the amount on her fare card doesn't change for a year. In fact, all graphs of linear relationships with slopes equal to 0 are horizontal—a rate of change of 0 means that, from one point to the next, the y-values remain the same.

Practice

Exercise 3.3.1.5

Suppose that during its flight, the elevation e (in feet) of a certain airplane and its time t, in minutes since takeoff, are related by a linear equation. Consider the graph of this equation, with time represented on the horizontal axis and elevation on the vertical axis. For each situation, decide if the slope is positive, zero, or negative.

1. The plane is cruising at an altitude of 37,000 feet above sea level.
2. The plane is descending at rate of 1000 feet per minute.
3. The plane is ascending at a rate of 2000 feet per minute.

Exercise 3.3.1.6

A group of hikers park their car at a trail head and walk into the forest to a campsite. The next morning, they head out on a hike from their campsite walking at a steady rate. The graph shows their distance in miles, d, from the car after h hours of hiking.

1. How far is the campsite from their car? Explain how you know.
2. Write an equation that describes the relationship between d and h.
3. After how many hours of hiking will they be 16 miles from their car? Explain or show your reasoning.

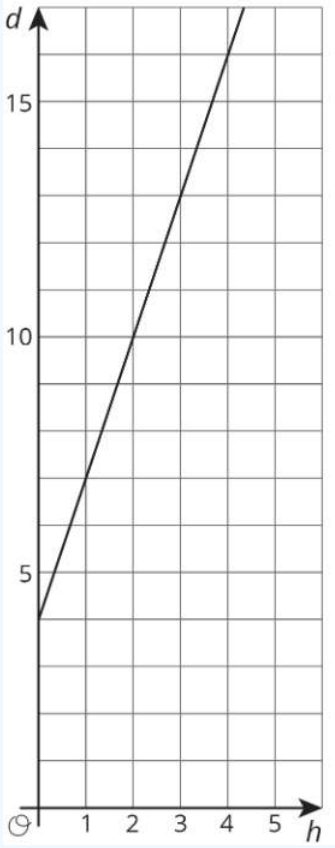


Figure 3.3.1.5

(From Unit 3.2.3)

Exercise 3.3.1.7

Elena's aunt pays her $1 for each call she makes to let people know about her aunt's new business.

The table shows how much money Diego receives for washing windows for his neighbors.

Table 3.3.1.1

number of windows	number of dollars
27	30
45	50
81	90

Select **all** the statements about the situation that are true.

A. Elena makes more money for making 10 calls than Diego makes for washing 10 windows.
B. Diego makes more money for washing each window than Elena makes for making each call.
C. Elena makes the same amount of money for 20 calls as Diego makes for 18 windows.
D. Diego needs to wash 35 windows to make as much money as Elena makes for 40 calls.
E. The equation $y = \frac{9}{10}x$, where y is number of dollars and x is number of windows, represents Diego's situation.
F. The equation $y = x$, where y is the number of dollars and x is the number of calls, represents Elena's situation.

(From Unit 3.1.4)

Exercise 3.3.1.8

Each square on a grid represents 1 unit on each side. Match the graphs with the slopes of the lines.

- $-\frac{1}{4}$
- $\frac{1}{4}$
- -4

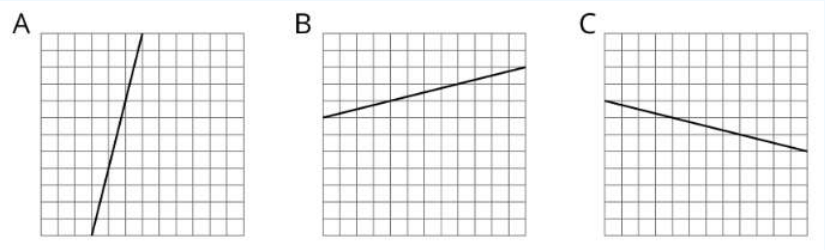


Figure 3.3.1.6

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3.3.2: Calculating Slope

Lesson

Let's calculate slope from two points.

Exercise 3.3.2.1: Number Talk: Integer Operations

Find values for a and b that make each side have the same value.

$\frac{a}{b} = -2$

$\frac{a}{b} = 2$

$a - b = -2$

Exercise 3.3.2.2: Toward a More General Slope Formula

1. Plot the points $(1, 11)$ and $(8, 2)$, and use a ruler to draw the line that passes through them.
2. Without calculating, do you expect the slope of the line through $(1, 11)$ and $(8, 2)$ to be positive or negative? How can you tell?
3. Calculate the slope of this line.

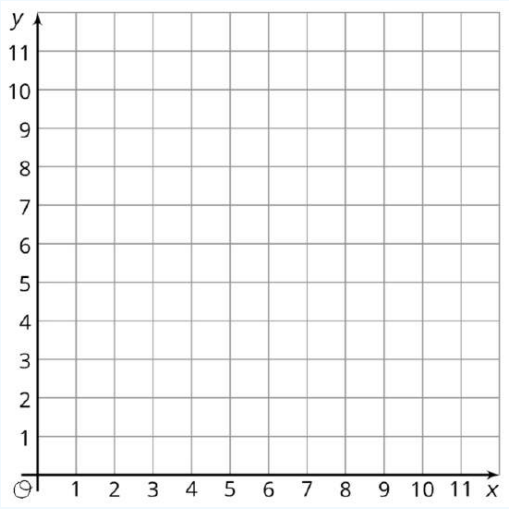


Figure 3.3.2.1

Are you ready for more?

Find the value of k so that the line passing through each prair of points has the given slope.

1. $(k, 2)$ and $(11, 14)$, slope =2
2. $(1, k)$ and $(4, 1)$, slope =-2
3. $(3, 5)$ and $(k, 9)$, slope =$\frac{1}{2}$
4. $(-1, 4)$ and $(-3, k)$, slope =-$\frac{1}{2}$
5. $(-\frac{15}{2}, \frac{3}{16})$ and $(-\frac{13}{22}, k)$, slope =0

Exercise 3.3.2.3: Making Designs

Your teacher will give you either a design or a blank graph. Do not show your card to your partner.

If your teacher gives you the design:

1. Look at the design silently and think about how you could communicate what your partner should draw. Think about ways that you can describe what a line looks like, such as its slope or points that it goes through.
2. Describe each line, one at a time, and give your partner time to draw them.

3. Once your partner thinks they have drawn all the lines you described, only then should you show them the design.

If your teacher gives you the blank graph:

1. Listen carefully as your partner describes each line, and draw each line based on their description.
2. You are not allowed to ask for more information about a line than what your partner tells you.
3. Do not show your drawing to your partner until you have finished drawing all the lines they describe.

When finished, place the drawing next to the card with the design so that you and your partner can both see them. How is the drawing the same as the design? How is it different? Discuss any miscommunication that might have caused the drawing to look different from the design.

Pause here so your teacher can review your work. When your teacher gives you a new set of cards, switch roles for the second problem.

Summary

We learned earlier that one way to find the slope of a line is by drawing a slope triangle. For example, using the slope triangle shown here, the slope of the line is $-\frac{2}{4}$, or $-\frac{1}{2}$ (we know the slope is negative because the line is decreasing from left to right).

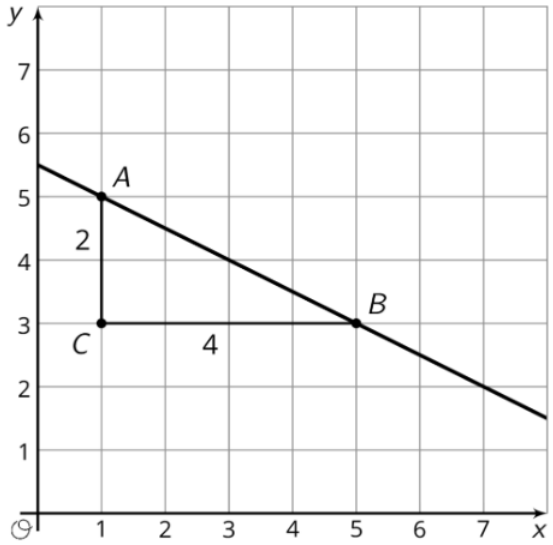


Figure 3.3.2.2

But slope triangles are only one way to calculate the slope of a line. Let's compute the slope of this line a different way using just the points $A = (1, 5)$ and $B = (5, 3)$. Since we know the slope is the vertical change divided by the horizontal change, we can calculate the change in the y-values and then the change in the x-values. Between points A and B, the y-value change is $3 - 5 = -2$ and the x-value change is $5 - 1 = 4$. This means the slope is $-\frac{2}{4}$, or $-\frac{1}{2}$, which is the same as what we found using the slope triangle.

Notice that in each of the calculations, We subtracted the value from point A from the value from point B. If we had done it the other way around, then the y-value change would have been $5 - 3 = 2$ and the x-value change would have been $1 - 5 = -4$, which still gives us a slope of $-\frac{1}{2}$. But what if we were to mix up the orders? If that had happened, we would think the slope of the line is *positive* $\frac{1}{2}$ since we would either have calculated $\frac{-2}{-4}$ or $\frac{2}{4}$. Since we already have a graph of the line and can see it has a negative slope, this is clearly incorrect. It we don't have a graph to check our calculation, we could think about how the point on the left, $(1, 5)$, is higher than the point on the right, $(5, 3)$, meaning the slope of the line must be negative.

Practice

Exercise 3.3.2.4

For each graph, calculate the slope of the line.

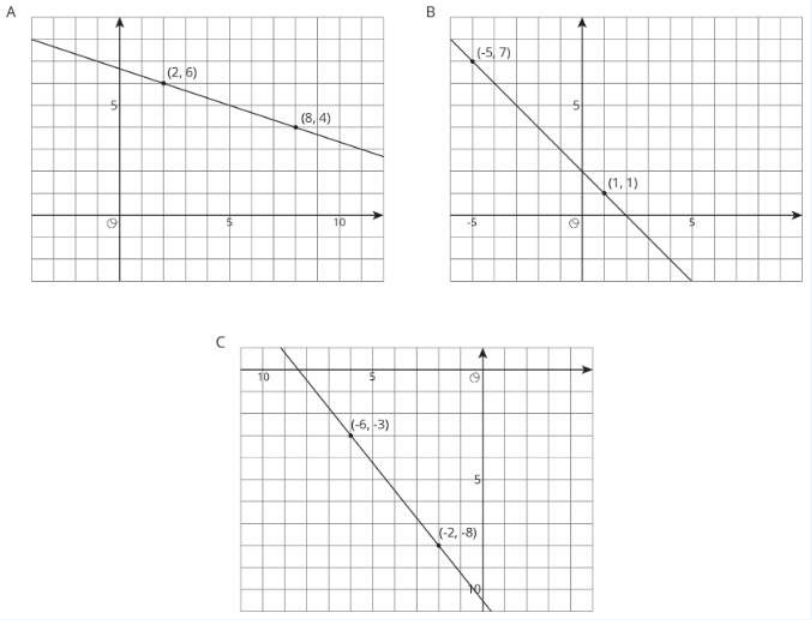


Figure 3.3.2.3: 3 graphs of lines labeled A, B, C. Graph A, line through points 2 comma 6 and 8 comma 4. Graph B, line through points -5 comma 7 and 1 comma 1. Graph C, line through points -6 comma -3 and -2 comma -8.

Exercise 3.3.2.5

Match each pair of points to the slope of the line that joins them.

A. $(9, 10)$ and $(7, 2)$
B. $(-8, -11)$ and $(-1, -5)$
C. $(5, -6)$ and $(2, 3)$
D. $(6, 3)$ and $(5, -1)$
E. $(4, 7)$ and $(6, 2)$

1. 4
2. -3
3. $-\frac{5}{2}$
4. $\frac{6}{7}$

Exercise 3.3.2.6

Draw a line with the given slope through the given point. What other point lies on that line?

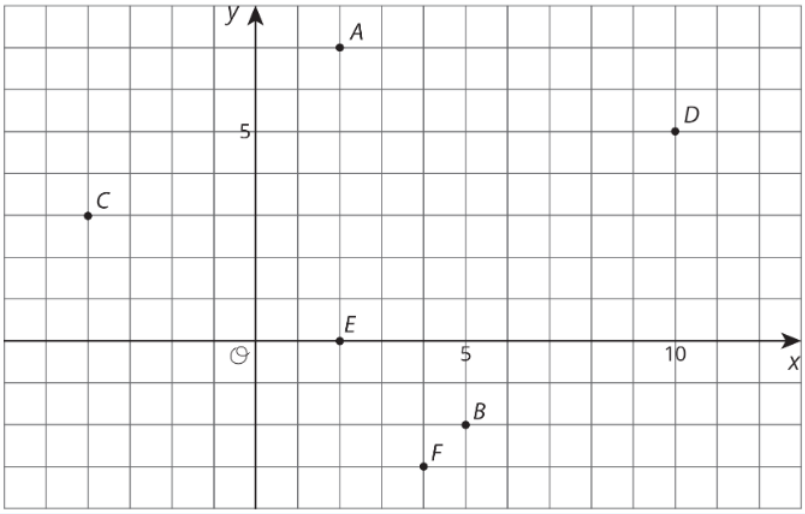


Figure 3.3.2.4: Coordinate grid with points A, B, C, D, E, F plotted. A is at 2 comma 7. B is at 5 comma -2. C is at -4 comma 3. D is at 10 comma 5. E is at 2 comma 0. F is at 4 comma -3.

1. Point A, slope $=-3$
2. Point A, slope $=-\frac{1}{4}$
3. Point C, slope $=-\frac{1}{2}$
4. Point E, slope $=-\frac{2}{3}$

Exercise 3.3.2.7

Make a sketch of a linear relationship with a slope of 4 and a negative y-intercept. Show how you know the slope is 4 and write an equation for the line.

(From Unit 3.2.4)

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3.3.3: Equations of All Kinds of Lines

Lesson

Let's write equations for vertical and horizontal lines.

Exercise 3.3.3.1: Which One Doesn't Belong: Pairs of Lines

Which one doesn't belong?

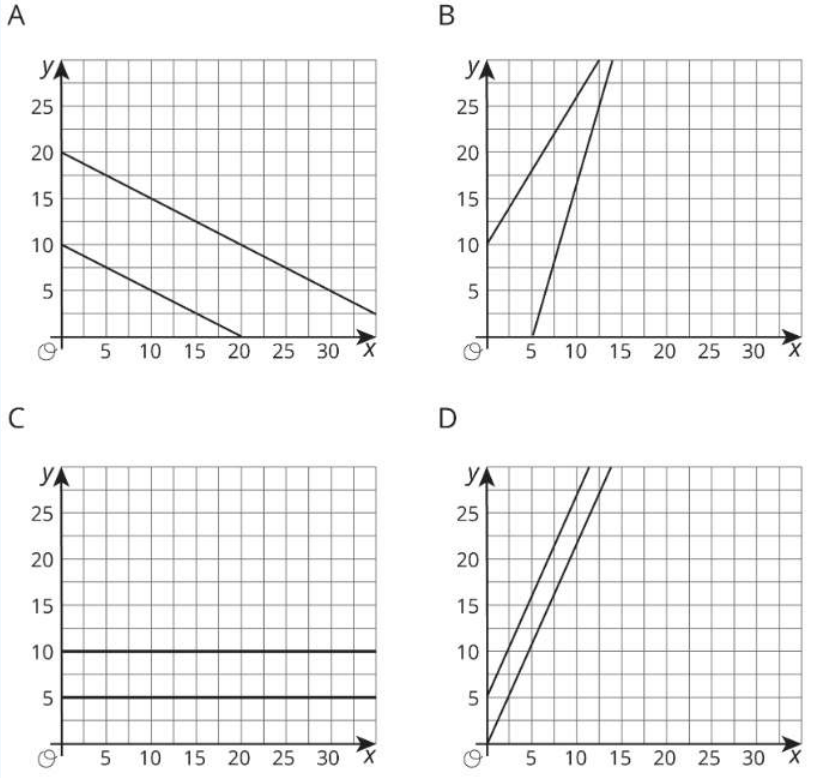


Figure 3.3.3.1: Four graphs labeled A, B, C, D. graph A, two parallel lines with negative slope. Graph b, two non parallel lines with positive slope. graph c, two parallel lines with slope of 0. graph d, two parallel lines with positive slope.

Exercise 3.3.3.2: All the Same

All the Same

1. Plot at least 10 points whose y-coordinate is -4. What do you notice about them?
2. Which equation makes the most ense to represent all of the points with y-coordinate -4? Explain how you know.
   $x=-4$ $x=-4x$ $y=-4$ $x+y=-4$
3. Plot at least 10 points whose x-coordinate is 3. What do you notice about them?
4. Which equation makes the most sense to represent all of the points with -coordinate 3? Explain how you know.
   $x=3$ $y=3x$ $y=3$ $x+y=3$
5. Graph the equation $x=-2$.
6. Graph the equation $y=5$.

Are you ready for more?

1. Draw the rectangle with vertices $(2,1),(5,1),(5,3)(2,3)$
2. For each of the four sides of the rectangle, write an equation for a line containing the side.
3. A rectangle has sides on the graphs of $x=-1, x=3, y=-1y=1$. Find the coordinates of each vertex.

Exercise 3.3.3.3: Same Perimeter

1. There are many possible rectangles whose perimeter is 50 units. Complete the table with lengths, l, and widths, w, of at least 10 such rectangles.

Table 3.3.3.1

l										
w										

2. The graph shows one rectangle whose perimeter is 50 units, and has its lower left vertex at the origin and two sides on the axes. On the same graph, draw more rectangles with perimeter 50 units using the values from your table. Make sure that each rectangle has a lower left vertex at the origin and two sides on the axes
3. Each rectangle has a vertex that lies in the first quadrant. These vertices lie on a line. Draw in this line, and write an equation for it.
4. What is the the slope of this line? How does the slope describe how the width changes as the length changes (or vice versa)?

Summary

Horizontal lines in the coordinate plane represent situations where the y value doesn't change at all while the x value changes. For example, the horizontal line that goes through the point $(0, 13)$ can be described in words as "for all points on the line, the y value is always 13." An equation that says the same thing is $y = 13$.

Vertical lines represent situations where the x value doesn't change at all while the y value changes. The equation $x = -4$ describes a vertical line through the point $(-4, 0)$.

Practice

Exercise 3.3.3.4

Suppose you wanted to graph the equation $y = -4x - 1$.

1. Describe the steps you would take to draw the graph.
2. How would you check that the graph you drew is correct?

Exercise 3.3.3.5

Draw the following lines and then write an equation for each.

1. Slope is 0, y-intercept is 5
2. Slope is 2, y-intercept is -1
3. Slope is -2, y-intercept is 1
4. Slope is $-\frac{1}{2}$, y-intercept is -1

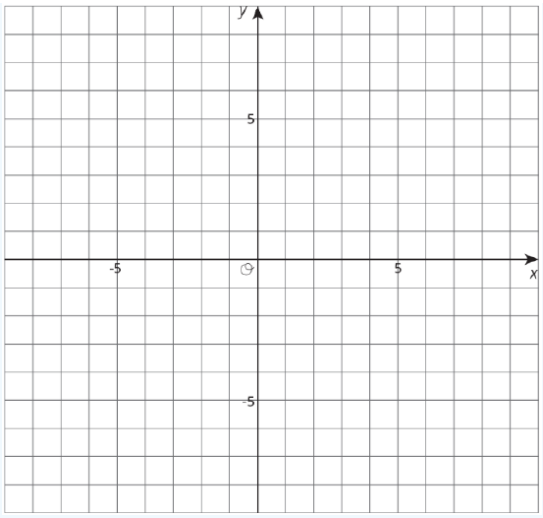


Figure 3.3.3.2

Exercise 3.3.3.6

Write an equation for each line.

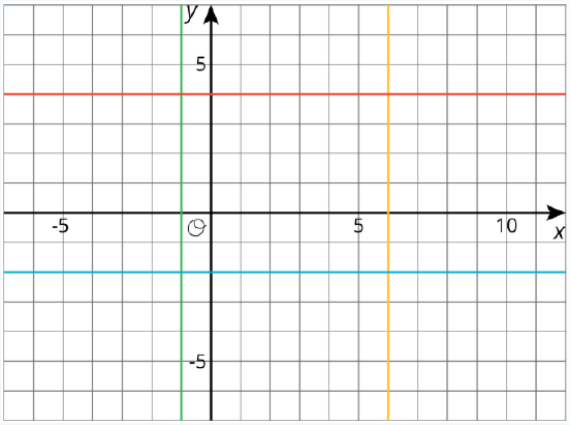


Figure 3.3.3.3: 4 lines on coordinate grid colored red, blue, green, yellow. red line, y intercept = 4, slope =0. green line, x intercept = -1, no slope. blue line, y intercept = -2, slope = 0. yellow line, x intercept = 6, no slope

Exercise 3.3.3.7

A publisher wants to figure out how thick their new book will be. The book has a front cover and a back cover, each of which have a thickness of $\frac{1}{4}$ of an inch. They have a choice of which type of paper to print the book on.

1. Bond paper has a thickness of $\frac{1}{4}$ inch per one hundred pages. Write an equation for the width of the book, y, if it has x hundred pages, printed on bond paper.
2. Ledger paper has a thickness of $\frac{2}{5}$ inch per one hundred pages. Write an equation for the width of the book, y, if it has x hundred pages, printed on ledger paper.
3. If they instead chose front and back covers of thickness $\frac{1}{3}$ of an inch, how would this change the equations in the previous two parts?

(From Unit 3.2.3)

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3.4: Linear Equations

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3.4.1: Solutions to Linear Equations

Lesson

Let's think about what it means to be a solution to a linear equation with two variables in it.

Exercise 3.4.1.1: Estimate Area

Which figure has the largest shaded region?

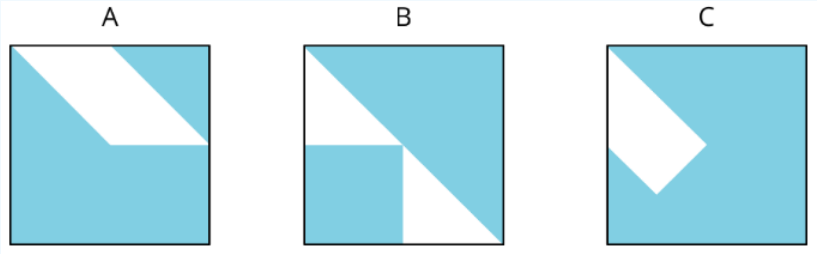


Figure 3.4.1.1: 3 squares labeled A, B, C. square A has one half plus one quarter shaded. square b has one half plus one quarter shaded. square c has the fraction 3 over 4 plus the fraction 1 over 16 shaded

Exercise 3.4.1.2: Apples and Oranges

At the corner produce market, apples cost \$1 each and oranges cost \$2 each.

1. Find the cost of:
   a. 6 apples and 3 oranges
   b. 4 apples and 4 oranges
   c. 5 apples and 4 oranges
   d. 8 apples and 2 oranges
2. Noah has \$10 to spend at the produce market. Can he buy 7 apples and 2 oranges? Explain or show your reasoning.
3. What combinations of apples and oranges can Noah buy if he spends all of his \$10?
4. Use two variables to write an equation that represents \$10-combinations of apples and oranges. Be sure to say what each variable means.
5. What are 3 combinations of apples and oranges that make your equation true? What are three combinations of apples and oranges that make it false?

Are you ready for more?

1. Graph the equation you wrote relating the number of apples and the number of oranges.
2. What is the slope of the graph? What is the meaning of the slope in terms of the context?
3. Suppose Noah has \$20 to spend. Graph the equation describing this situation. What do you notice about the relationship between this graph and the earlier one?

Exercise 3.4.1.3: Solutions and Everything Else

You have two numbers. If you double the first number and add it to the second number, the sum is 10.

1. Let x represent the first number and let y represent the second number. Write an equation showing the relationship between x, y, and 10.
2. Draw and label a set of x- and y-axes. Plot at least five points on this coordinate plane that make the statement and your equation true. What do you notice about the points you have plotted?
3. List ten points that do *not* make the statement true. Using a different color, plot each point in the same coordinate plane. What do you notice about these points compared to your first set of points?

Summary

Think of all the rectangles whose perimeters are 8 units. If x represents the width and y represents the length, then $2x+2y=8$ expresses the relationship between the width and length for all such rectangles.

For example, the width and length could be 1 and 3, since $2\cdot 1+2\cdot 3=8$ or the width and length could be 2.75 and 1.25, since $2\cdot(2.75)+2\cdot(1.25)=8$.

We could find many other possible pairs of width and length, (x,y), that make the equation true—that is, pairs (x,y) that when substituted into the equation make the left side and the right side equal.

A **solution to an equation with two variables** is any pair of values (x,y) that make the equation true.

We can think of the pairs of numbers that are solutions of an equation as points on the coordinate plane. Here is a line created by all the points (x,y) that are solutions to $2x+2y=8$. Every point on the line represents a rectangle whose perimeter is 8 units. All points not on the line are not solutions to $2x+2y=8$.

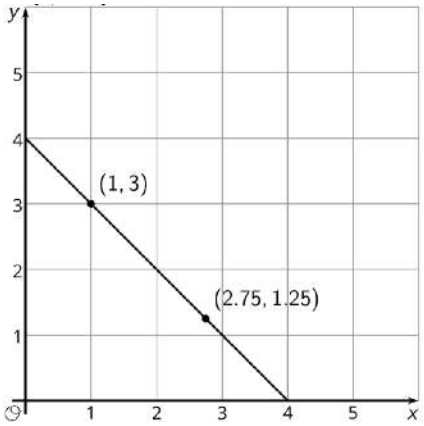


Figure 3.4.1.2: Graph of a line, origin O, with grid. Horizontal axis, scale 0 to 5, by 1's. Vertical axis, scale 0 to 5, by 1's. Line begins on vertical axis above origin, passes through 1 comma 3 and 2 and 75 hundredths comma 1 and 25 hundredths.

Glossary Entries

Definition: Solution to an Equation with Two Variables

A solution to an equation with two variables is a pair of values of the variables that make the equation true.

For example, one possible solution to the equation $4x+3y=24$ is $(6,0)$. Substituting 6 for x and 0 for y makes this equation true because $4(6)+3(0)=24$.

Practice

Exercise 3.4.1.4

Select **all** of the ordered pairs (x,y) that are solutions to the linear equation $2x+3y=6$.

A. $(0,2)$
B. $(0,6)$
C. $(2,3)$
D. $(3,-2)$
E. $(3,0)$
F. $(6,-2)$

Exercise 3.4.1.5

The graph shows a linear relationship between x and y.

x represents the number of comic books Priya buys at the store, all at the same price, and y represents the amount of money (in dollars) Priya has after buying the comic books.

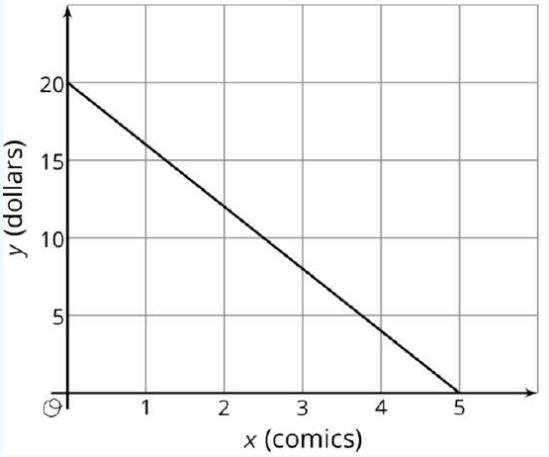


Figure 3.4.1.3: Graph of a line, origin O, with grid. Horizontal axis, comics, scale 0 to 5, by 1's. Vertical axis, dollars, scale 0 to 20, by 5's. Line begins at 0 comma 20 and ends at 5 comma 0.

1. Find and interpret the x- and y-intercepts of this line.
2. Find and interpret the slope of this line.
3. Find an equation for this line.
4. If Priya buys 3 comics, how much money will she have remaining?

Exercise 3.4.1.6

Match each equation with its three solutions.

A. $y = 1.5x$
B. $2x + 3y = 7$
C. $x - y = 4$
D. $3x = \frac{y}{2}$
E. $y = -x + 1$

1. $(14, 21), (2, 3), (8, 12)$
2. $(-3, -7), (0, -4), (-1, -5)$
3. $(\frac{1}{8}, \frac{7}{8}), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{4}, \frac{3}{4})$
4. $(1, 1\frac{2}{3}), (-1, 3), (0, 2\frac{1}{3})$
5. $(0.5, 3), (1, 6), (1.2, 7.2)$

Exercise 3.4.1.7

A container of fuel dispenses fuel at the rate of 5 gallons per second. If y represents the amount of fuel remaining in the container, and x represents the number of seconds that have passed since the fuel started dispensing, then x and y satisfy a linear relationship.

In the coordinate plane, will the slope of the line representing that relationship have a positive, negative, or zero slope? Explain how you know.

(From Unit 3.3.2)

Exercise 3.4.1.8

A sandwich store charges a delivery fee to bring lunch to an office building. One office pays $33 for 4 turkey sandwiches. Another office pays $61 for 8 turkey sandwiches. How much does each turkey sandwich add to the cost of the delivery? Explain how you know.

(From Unit 3.2.1)

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3.4.2: More Solutions to Linear Equations

Lesson

Let's find solutions to more linear equations.

Exercise 3.4.2.1: Coordinate Pairs

For each equation choose a value for x and then solve to find the corresponding y value that makes that equation true.

1. $6x = 7y$
2. $5x + 3y = 9$
3. $y + 5 - \frac{1}{3}x = 7$

Exercise 3.4.2.2: True or False: Solutions in the Coordinate Plane

Here are graphs representing three linear relationships. These relationships could also be represented with equations.

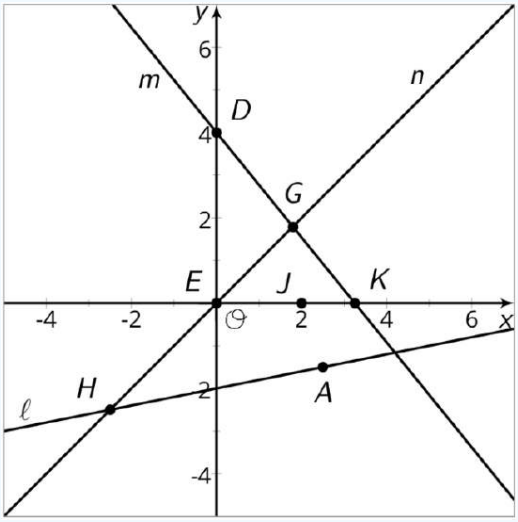


Figure 3.4.2.1: Graph, origin O, no grid. Lines l, m, n. Line l, y intercept -2. Line m, y intercept 4, labeled D. Line n, y intercept 0, labeled E. Point A on line l. Point H on lines l and n. Point E on line n. Point G on lines n and m. Point K on line m. Point J not on any line plotted at 2 comma 0.

For each statement below, decide if it is true or false. Explain your reasoning.

1. $(4, 0)$ is a solution of the equation for line m.
2. The coordinates of the point G make both the equations for line m and the equation for line n is true.
3. $x = 0$ is a solution of the equation for line n.
4. $(2, 0)$ makes both the equation for line m and the equation for line n true.
5. There is no solution for the equation for line l that has $y = 0$.
6. The coordinates of point H are solutions to the equations for line l.
7. There are exactly two solutions of the equation for line l.
8. There is a point whose coordinates make the equations of all three lines true.

After you finish discussing the eight statements, find another group and check your answers against theirs. Discuss any disagreements.

Exercise 3.4.2.3: I'll Take an X, Please

One partner has 6 cards labeled A through F and one partner has 6 cards labeled a through f. In each pair of cards (for example, Cards A and a), there is an equation on one card and a coordinate pair, (x, y), that makes the equation true on the other card.

1. The partner with the equation asks the partner with a solution for either the x-value or the y-value and explains why they chose the one they did.
2. The partner with the equation uses this value to find the other value, explaining each step as they go.
3. The partner with the coordinate pair then tells the partner with the equation if they are right or wrong. If they are wrong, both partners should look through the steps to find and correct any errors. If they are right, both partners move onto the next set of cards.
4. Keep playing until you have finished Cards A through F.

Are you ready for more?

Consider the equation $ax + by = c$, where a, b, and c are positive numbers.

1. Find the coordinates of the x- and y-intercepts of the graph of the equation.
2. Find the slope of the graph.

Summary

Let's think about the linear equation $2x - 4y = 12$. If we know $(0, -3)$ is a solution to the equation, then we also know $(0, -3)$ is a point on the graph of the equation. Since this point is on the y-axis, we also know that it is the vertical intercept of the graph. But what about the coordinate of the horizontal intercept, when $y = 0$? Well, we can use the equation to figure it out.

$$\begin{aligned} 2x - 4y &= 12 \\ 2x - 4(0) &= 12 \\ 2x &= 12 \\ x &= 6 \end{aligned}$$

Since $x = 6$ when $y = 0$, we know the point $(6, 0)$ is on the graph of the line. No matter the form a linear equation comes in, we can always find solutions to the equation by starting with one value and then solving for the other value.

Glossary Entries

Definition: Solution to an Equation with Two Variables

A solution to an equation with two variables is a pair of values of the variables that make the equation true.

For example, one possible solution to the equation $4x + 3y = 24$ is $(6, 0)$. Substituting 6 for x and 0 for y makes this equation true because $4(6) + 3(0) = 24$.

Practice

Exercise 3.4.2.4

For each equation, find y when $x = -3$. Then find x when $y = 2$

1. $y = 6x + 8$
2. $y = \frac{2}{3}x$
3. $y = -x + 5$
4. $y = \frac{3}{4}x - 2\frac{1}{2}$
5. $y = 1.5x + 11$

Exercise 3.4.2.5

True or false: The points $(6, 13)$, $(21, 33)$, and $(99, 137)$ all lie on the same line. The equation of the line is $y = \frac{4}{3}x + 5$. Explain or show your reasoning.

Exercise 3.4.2.6

Here is a linear equation: $y = \frac{1}{4}x + \frac{5}{4}$

1. Are ((1,1.5)\) and $(12, 4)$ solutions to the equation? Explain or show your reasoning.
2. Find the x-intercept of the graph of the equation. Explain or show your reasoning.

Exercise 3.4.2.7

Find the coordinates of B, C, and D given that $AB = 5$ and $BC = 10$.

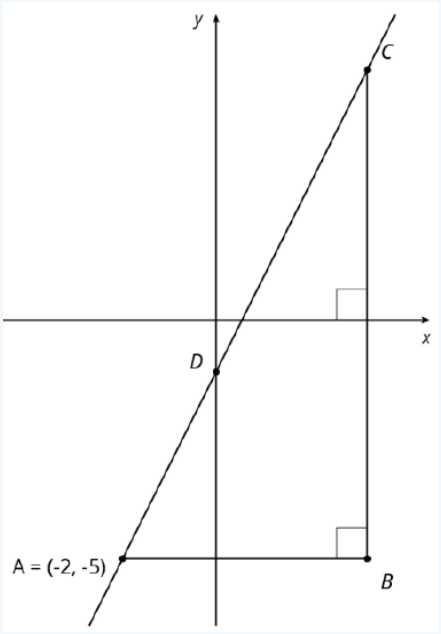


Figure 3.4.2.2: Line graphed on xy plane with no grid. line passes through point A at -2 comma -5, point D, unlabeled on y axis below origin, and C, unlabeled in quadrant 1. right triangle drawn from point c, down to point B, where right angle is located, left to point A.

(From Unit 2.3.2)

Exercise 3.4.2.8

Match each graph of a linear relationship to a situation that most reasonably reflects its context.

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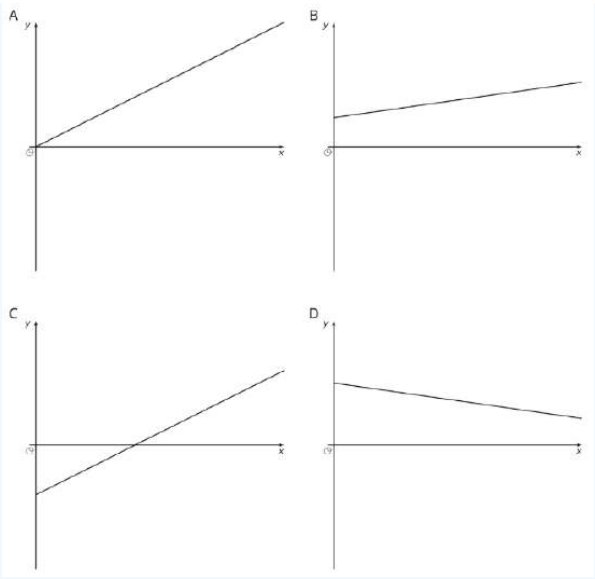


Figure 3.4.2.3: 4 graphs of lines labeled A, B, C. D. A, y intercept = 0, positive slope. B, y intercept = positive, positive slope, less steep than A. C, y intercept = negative, positive slope. D, y intercept = positive, slope = negative.

A. Graph A
B. Graph B
C. Graph C
D. Graph D

1. y is the weight of a kitten x days after birth.
2. y is the distance left to go in a car ride after x hours of driving at a constant rate towards its destination.
3. y is the temperature, in degrees C, of a gas being warmed in a laboratory experiment.
4. y is the amount of calories consumed eating x crackers.

(From Unit 3.3.1)

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3.5: Let's Put it to Work

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3.5.1: Using Linear Relationships to Solve Problems

Lesson

Let's write equations for real-world situations and think about their solutions.

Exercise 3.5.1.1: Buying Fruit

For each relationship described, write an equation to represent the relationship.

1. Grapes cost \$2.39 per pound. Bananas cost \$0.59 per pound. You have \$15 to spend on g pounds of grapes and b pounds of bananas.
2. A savings account has \$50 in it at the start of the year and \$20 is deposited each week. After x weeks, there are y dollars in the account.

Exercise 3.5.1.2: Five Savings Accounts

Each line represents one person's weekly savings account balance from the start of the year.

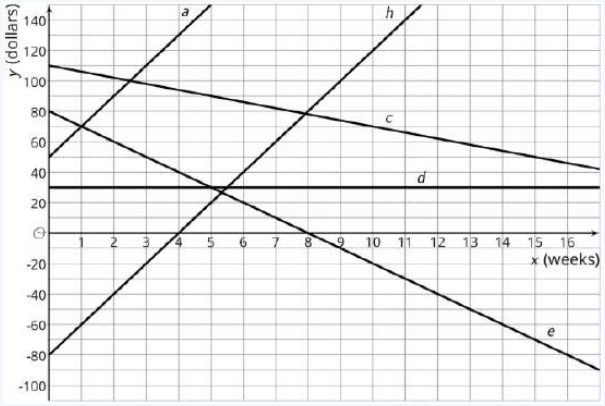


Figure 3.5.1.1: Coordinate grid. Lines a, c, d, e, h. Line a, slope = 20, y intercept = 50. Line c, slope = -5, y intercept = 110. Line d, slope= 0, y intercept = 30. Line e, slope = -10, y intercept = 80. Line h, slope = 20, y intercept -80.

1. Choose one line and write a description of what happens to that person's account over the first 17 weeks of the year. Do not tell your group which line you chose.
2. Share your story with your group and see if anyone can guess your line.
3. Write an equation for each line on the graph. What do the slope, m, and vertical intercept, b, in each equation mean in the situation?
4. For which equation is $(1, 70)$ a solution? Interpret this solution in terms of your story.
5. Predict the balance in each account after 20 weeks.

Exercise 3.5.1.3: Fabulous Fish

The Fabulous Fish Market orders tilapia, which costs \$3 per pound, and salmon, which costs \$5 per pound. The market budgets \$210 to spend on this order each day.

1.What are five different combinations of salmon and tilapia that the market can order?

2. Define variables and write an equation representing the relationship between the amount of each fish bought and how much the market spends.

3. Sketch a graph of the relationship. Label your axes.

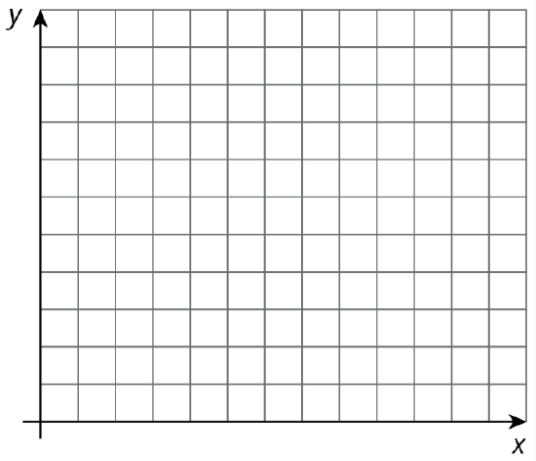


Figure 3.5.1.2

4. On your graph, plot and label the combinations A-F.

Table 3.5.1.1

	A	B	C	D	E	F
pounds of tilapia	5	19	27	25	65	55
pounds of salmon	36	30.6	25	27	6	4

5. Which of these combinations can the market order? Explain or show your reasoning.

6. List two ways you can tell if a pair of numbers is a solution to an equation.

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CHAPTER OVERVIEW

4: Linear Equations and Linear Systems

In this unit, students write and solve linear equations in one variable. These include equations in which the variable occurs on both sides of the equal sign, and equations with no solutions, exactly one solution, and infinitely many solutions. They learn that any one such equation is false, true for one value of the variable, or true for all values of the variable. They interpret solutions in the contexts from which the equations arose. Students write and solve systems of linear equations in two variables and interpret the solutions in the contexts from which the equations arose. They learn what is meant by a solution for a system of equations, namely that a solution of the system is a solution for each equation in the system. Students use the understanding that each pair of values that make an equation true are coordinates of a point on the graph of the equation and conversely that the coordinates of each point on the graph of an equation make the equation true. Thus, a pair of values that satisfies a system of equations are coordinates of a point that lies on the graphs of all the equations in the system, and, conversely, a point that lies on the graphs of all the equations in the system has coordinates that satisfy all the equations in the system. Students learn to understand and use the terms “system of equations,” “solution for the system of equations,” “zero solutions,” “no solution,” “one solution,” and “infinitely many solutions.”

4.1: Puzzle Problems

4.1.1: Number Puzzles

4.2: Linear Equations in One Variable

4.2.1: Keeping the Equation Balanced

4.2.2: Balanced Moves

4.2.3: More Balanced Moves

4.2.4: Solving Any Linear Equation

4.2.5: Strategic Solving

4.2.6: All, Some, or No Solutions

4.2.7: How Many Solutions?

4.2.8: When Are They the Same?

4.3: Systems of Linear Equations

4.3.1: On or Off the Line?

4.3.2: On Both of the Lines

4.3.3: Systems of Equations

4.3.4: Solving Systems of Equations

4.3.5: Solving More Systems

4.3.6: Writing Systems of Equations

4.4: Let's Put it to Work

4.4.1: Solving Problems with Systems of Equations

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4.1: Puzzle Problems

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4.1.1: Number Puzzles

Lesson

Let's solve some puzzles!

Exercise 4.1.1.1: Notice and Wonder: A Number Line

What do you notice? What do you wonder?

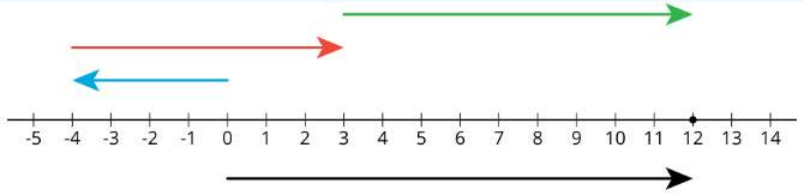


Figure 4.1.1.1: A number line from negative 5 to 14, by 1s. A blue arrow, pointing to the left, goes from 0 to negative 4. A red arrow, pointing to the right, goes from negative 4 to 3. A black arrow, pointing to the right, goes from 0 to 12. A green arrow, pointing to the right, goes from 3 to 12. A point is plotted on the number line at 12.

Exercise 4.1.1.2: Telling Temperatures

Solve each puzzle. Show your thinking. Organize it so it can be followed by others.

1. The temperature was very cold. Then the temperature doubled.
   Then the temperature dropped by 10 degrees. Then the temperature increased by 40 degrees. The temperature is now 16 degrees. What was the starting temperature?
2. Lin ran twice as far as Diego. Diego ran 300 m farther than Jada. Jada ran $\frac{1}{3}$ the distance that Noah ran. Noah ran 1200 m. How far did Lin run?

Exercise 4.1.1.3: Making a Puzzle

Write another number puzzle with at least three steps. On a different piece of paper, write a solution to your puzzle.

Trade puzzles with your partner and solve theirs. Make sure to show your thinking.

With your partner, compare your solutions to each puzzle. Did they solve them the same way you did? Be prepared to share with the class which solution strategy you like best.

Are you ready for more?

Here is a number puzzle that uses math. Some might call it a magic trick!

1. Think of a number.
2. Double the number.
3. Add 9.
4. Subtract 3.
5. Divide by 2.
6. Subtract the number you started with.
7. The answer should be 3.

Why does this always work? Can you think of a different number puzzle that uses math (like this one) that will always result in 5?

Summary

Here is an example of a puzzle problem: Twice a number plus 4 is 18. What is the number?

There are many different ways to represent and solve puzzle problems.

- We can reason through it.

  Twice a number plus 4 is 18.
  Then twice the number is $18 - 4 = 14$.
  That means the number is 7.
- We can draw a diagram.

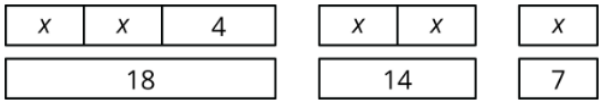


Figure 4.1.1.2: Three tape diagrams. First diagram is 3 parts labeled x, x, 4. Below it, a tape diagram of equal length with 1 part labeled 18. Second diagram is 2 parts labeled x, x. Below it, a tape diagram of the same width with 1 part labeled 14. Third diagram is 1 part labeled x. Below it, a tape diagram of equal length, 1 part, labeled 7.

- We can write and solve an equation, $2x + 4 = 18$ $2x = 14x = 7$

Reasoning and diagrams help us see what is going on and why the answer is what it is. But as number puzzles and story problems get more complex, those methods get harder, and equations get more and more helpful. We will use different kinds of diagrams to help us understand problems and strategies in future lessons, but we will also see the power of writing and solving equations to answer increasingly more complex mathematical problems.

Practice

Exercise 4.1.1.4

Tyler reads $\frac{2}{15}$ of a book on Monday, $\frac{1}{3}$ of it on Tuesday, $\frac{2}{9}$ of it on Wednesday, and $\frac{3}{4}$ of the remainder on Thursday. If he still has 14 pages left to read on Friday, how many pages are there in the book?

Exercise 4.1.1.5

Clare asks Andre to play the following number puzzle:

- Pick a number
- Add 2
- Multiply by 3
- Subtract 7
- Add your original number

Andre's final result is 27.
Which number did he start with?

Exercise 4.1.1.6

In a basketball game, Elena scores twice as many points as Tyler. Tyler scores four points fewer than Noah, and Noah scores three times as many points as Mai. If Mai scores 5 points, how many points did Elena score? Explain your reasoning.

Exercise 4.1.1.7

Select **all** of the given points in the coordinate plane that lie on the graph of the linear equation $4x - y = 3$.

A. $(-1, -7)$
B. $(0, 3)$
C. $(\frac{3}{4}, 0)$
D. $(1, 1)$
E. $(2, 5)$
F. $(4, -1)$

(From Unit 3.4.1)

Exercise 4.1.1.8

A store is designing the space for rows of nested shopping carts. Each row has a starting cart that is 4 feet long, followed by the nested carts (so 0 nested carts means there's just the starting cart). The store measured a row of 13 nested carts to be 23.5 feet long, and a row of 18 nested carts to be 31 feet long.

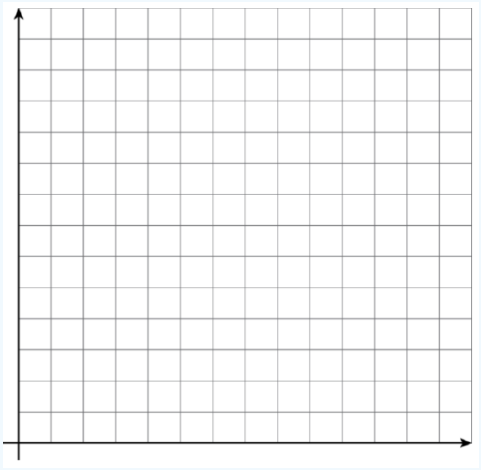


Figure 4.1.1.3

1. Create a graph of the situation.
2. How much does each nested cart add to the length of the row? Explain your reasoning.
3. If the store design allows for 43 feet for each row, how many total carts fit in a row?

(From Unit 3.2.1)

Exercise 4.1.1.9

Triangle A is an isosceles triangle with two angles of measure x degrees and one angle of measure y degrees.

1. Find three combinations of x and y that make this sentence true.
2. Write an equation relating x and y.
3. If you were to sketch the graph of this linear equation, what would its slope be? How can you interpret the slope in the context of the triangle?

(From Unit 3.4.2)

Exercise 4.1.1.10

Consider the following graphs of linear equations. Decide which line has a positive slope, and which has a negative slope. Then calculate each line's exact slope.

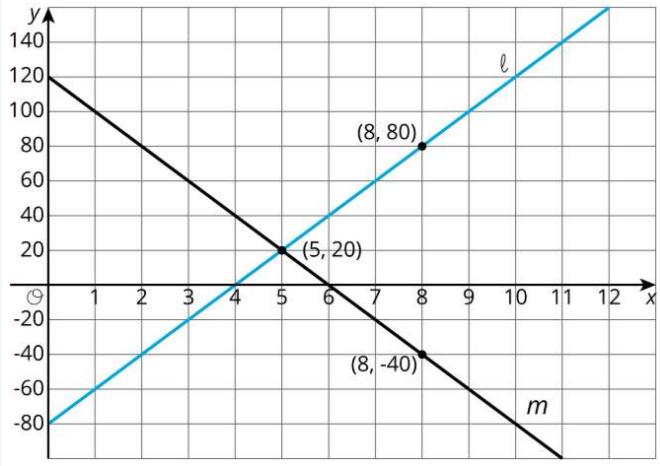


Figure 4.1.1.4: Graph of two lines, l and m, origin O, with grid. Horizontal axis, scale 0 to 12, by 1's. Vertical axis, scale -80 to 140, by 20's. Line l begins at 0 comma -80, passes through 8 comma 80. Line m begins at 0 comma 120, passes through 8 comma -40.

(From Unit 3.3.2)

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4.2: Linear Equations in One Variable

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4.2.1: Keeping the Equation Balanced

Lesson

Let's figure out unknown weights on balanced hangers.

Exercise 4.2.1.1: Notice and Wonder: Hanging Socks

What do you notice? What do you wonder?



Figure 4.2.1.1: Two clothing hangers, each with two socks. The first hanger is level and has two pink socks, one on each end. The second hanger is lower on the left than the right and has two blue socks, one on each end.

Exercise 4.2.1.2: Hanging Blocks

This picture represents a hanger that is balanced because the weight on both sides is the same.

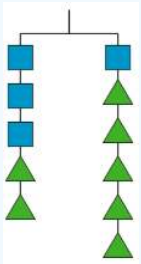


Figure 4.2.1.2

1. Elena takes two triangles off of the left side and three triangles off of the right side. Will the hanger still be in balance, or will it tip to one side? Which side? Explain how you know.
2. Use the applet to see if your answer to question [1] was correct. Can you find another way to make the hanger balance?
3. If a triangle weighs 1 gram, how much does a square weigh? After you make a prediction, use the applet to see if you were right. Can you find another pair of values that makes the hanger balance?

Are you ready for more?

Try your own Hanger Balances!

GeoGebra Applet RQ58fzCE

Exercise 4.2.1.3: More Hanging Blocks

A triangle weighs 3 grams, and a circle weighs 6 grams.

1. Find the weight of a square.
2. Find the weight of a pentagon.
3. Write an equation to represent each hanger.

Are you ready for more?

Try your own!

GeoGebra Applet hDfcuwNb

Summary

If we have equal weights on the ends of a hanger, then the hanger will be in balance. If there is more weight on one side than the other, the hanger will tilt to the heavier side.

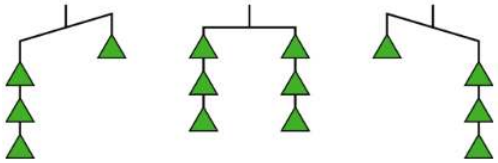


Figure 4.2.1.3: Three hangers. First hanger is unbalanced. Left side, 3 triangles. Right side, 1 triangle. Left lower than right. Second hanger is balanced. Left side, 3 triangles. Right side, 3 triangles. Third hanger is unbalanced. Left side, 1 triangle. Right side, 3 triangles.

We can think of a balanced hanger as a metaphor for an equation. An equation says that the expressions on each side have equal value, just like a balanced hanger has equal weights on each side.

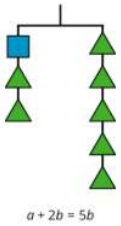


Figure 4.2.1.4

If we have a balanced hanger and add or remove the same amount of weight from each side, the result will still be in balance.

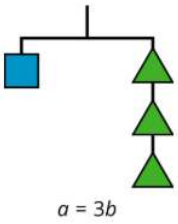


Figure 4.2.1.5

We can do these moves with equations as well: adding or subtracting the same amount from each side of an equation maintains the equality.

Practice

Exercise 4.2.1.4

Which of the changes would keep the hanger in balance?

Select all that apply.

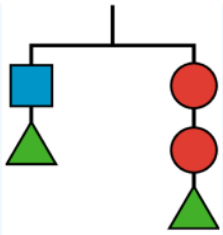


Figure 4.2.1.6

A. Adding two circles on the left and a square on the right
B. Adding 2 triangles to each side
C. Adding two circles on the right and a square on the left
D. Adding a circle on the left and a square on the right
E. Adding a triangle on the left and a square on the right

Exercise 4.2.1.5

Here is a balanced hanger diagram.

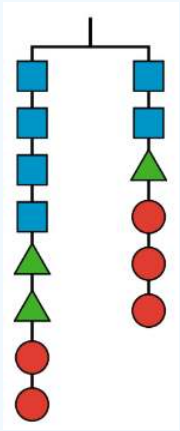


Figure 4.2.1.7

Each triangle weighs 2.5 pounds, each circle weighs 3 pounds, and x represents the weight of each square. Select *all* equations that represent the hanger.

A. $x+x+x+x+11=x+11.5$
B. $2x=0.5$
C. $4x+5+6=2x+2.5+6$
D. $2x+2.5=3$
E. $4x+2.5+2.5+3+3=2x+2.5+3+3+3$

Exercise 4.2.1.6

What is the weight of a square if a triangle weighs 4 grams?

Explain your reasoning.

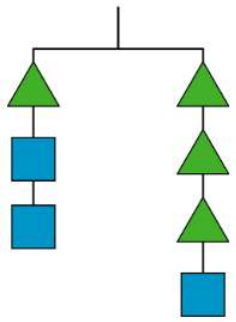


Figure 4.2.1.8

Exercise 4.2.1.7

Andre came up with the following puzzle. “I am three years younger than my brother, and I am 2 years older than my sister. My mom's age is one less than three times my brother's age. When you add all our ages, you get 87. What are our ages?”

1. Try to solve the puzzle.
2. Jada writes this equation for the sum of the ages:
   $(x)+(x+3)+(x-2)+3(x+3)-1=87$
   Explain the meaning of the variable and each term of the equation.
3. Write the equation with fewer terms.
4. Sovle the puzzle if you haven't already.

(From Unit 4.1.1)

Exercise 4.2.1.8

These two lines are parallel. Write an equation for each.

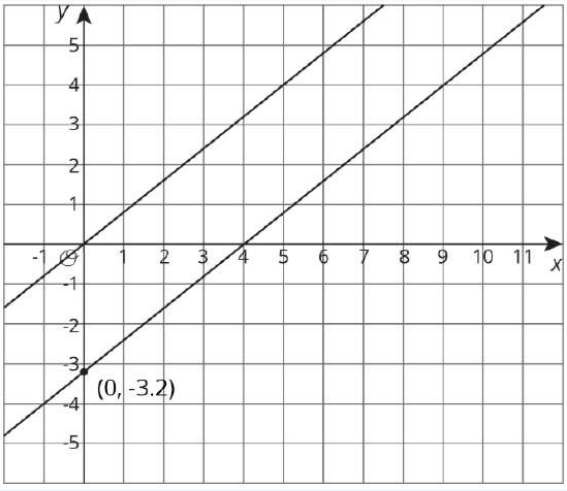


Figure 4.2.1.9: Two lines in an x y plane. A line passes through the origin and 5 comma 4. Another line passes through 0 comma negative 3 point 2 and crosses the x axis at 4.

(From Unit 3.2.4)

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4.2.2: Balanced Moves

Lesson

Let's rewrite equations while keeping the same solutions.

Exercise 4.2.2.1: Matching Hangers

Figures A, B, C, and D show the result of simplifying the hanger in Figure A by removing equal weights from each side.

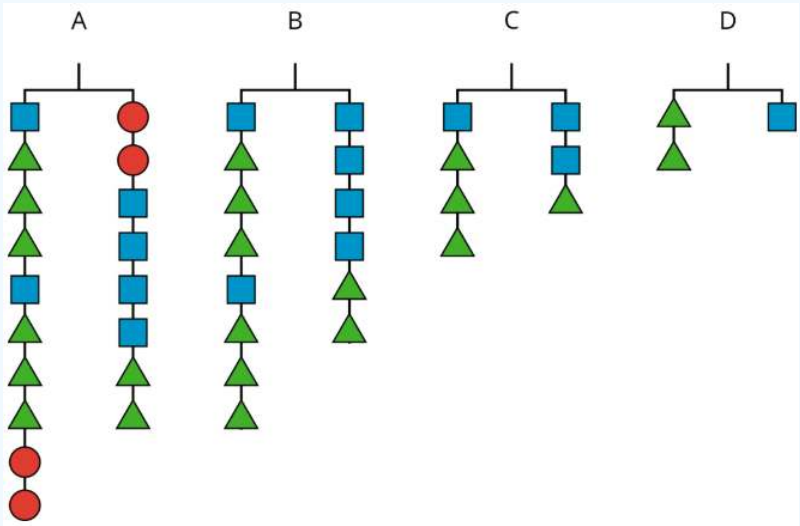


Figure 4.2.2.1: Four balanced hangers. A. Left side, 1 square, 3 triangles, 1 square, 3 triangles, 2 circles. Right side, 2 circles, 4 squares, 2 triangles. B. Left side, 1 square, 3 triangles, 1 square, 3 triangles. Right side, 4 squares, 2 triangles. C. Left side, 1 square, 3 triangles, Right side, 2 squares, 1 triangle. D. Left side, 2 triangles. Right side, 1 square.

Here are some equations. Each equation represents one of the hanger diagrams.

$(2(x+3y)=4x+2y \qquad 2y=x \qquad 2(x+3y)+2z=2z+4x+2y \qquad x+3y=2x+y$

1. Write the equation that goes with each figure:
   A.
   B.
   C.
   D.
2. Each variable (x, y, and z) represents the weight of one shape. Which goes with which?
3. Explain what was done to each equation to create the next equation. If you get stuck, think about how the hangers changed.

Exercise 4.2.2.2: Matching Equation Moves

Your teacher will give you some cards. Each of the cards 1 through 6 show two equations. Each of the cards A through E describe a move that turns one equation into another.

1. Match each number card with a letter card.
2. One of the letter cards will not have a match. For this card, write two equations showing the described move.

Exercise 4.2.2.3: Keeping Equality

1. Noah and Lin both solved the equation $14a=2(a-3)$.

Do you agree with either of them? Why?

Noah's solution:

$$\begin{aligned} 14a &= 2(a-3) \\ 14a &= 2a-6 \\ 12a &= -6 \\ a &= -\frac{1}{2} \end{aligned}$$

Lin's solution:

$$\begin{aligned} 14a &= 2(a-3) \\ 7a &= a-3 \\ 6a &= -3 \\ a &= -\frac{1}{2} \end{aligned}$$

2. Elena is asked to solve $15-10x=5(x+9)$. What do you recommend she does to each side first?

3. Diego is asked to sole $3x-8=4(x+5)$. What do you recommend he does to each side first?

Are you ready for more?

In a cryptarithmetic puzzle, the digits 0–9 are represented with letters of the alphabet. Use your understanding of addition to find which digits go with the letters A, B, E, G, H, L, N, and R.

HANGER + HANGER + HANGER = ALGEBRA

Summary

An equation tells us that two expressions have equal value. For example, if $4x+9$ and $-2x-3$ have equal value, we can write the equation

$$4x+9=-2x-3$$

Earlier, we used hangers to understand that if we add the same positive number to each side of the equation, the sides will still have equal value. It also works if we add *negative numbers*! For example, we can add -9 to each side of the equation.

$$\begin{aligned} 4x+9+-9 &= -2x-3+-9 && \text{add -9 to each side} \\ 4x &= -2x-12 && \text{combine like terms} \end{aligned}$$

Since expressions represent numbers, we can also add *expressions* to each side of an equation. For example, we can add $2x$ to each side and still maintain equality.

$$\begin{aligned} 4x+2x &= -2x-12+2x && \text{add 2x to each side} \\ 6x &= -12 && \text{combine like terms} \end{aligned}$$

If we multiply or divide the expressions on each side of an equation by the same number, we will also maintain the equality (so long as we do not divide by zero).

$$6x \cdot \frac{1}{6} = -12 \cdot \frac{1}{6} \qquad \text{multiply each side by } \frac{1}{6}$$

or

$$6x \div 6 = -12 \div 6 \qquad \text{divide each side by } 6$$

Now we can see that $x=-2$ is the solution to our equation.

We will use these moves in systematic ways to solve equations in future lessons.

Practice

Exercise 4.2.2.4

In this hanger, the weight of the triangle is x and the weight of the square is y.

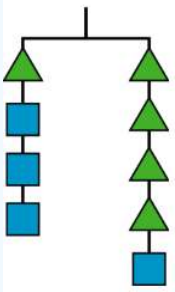


Figure 4.2.2.2

1. Write an equation using x and y to represent the hanger.
2. If x is 6, what is y?

Exercise 4.2.2.5

Andre and Diego were each trying to solve $2x + 6 = 3x - 8$. Describe the first step they each make to the equation.

1. The result of Andre's first step was $-x + 6 = -8$.
2. The result of Diego's first step was $6 = x - 8$.

Exercise 4.2.2.6

1. Complete the table with values for x or y that make this equation true: $3x + y = 15$.

Table 4.2.2.1

x	2		6	0	3		
y		3				0	8

2. Create a graph, plot these points, and find the slope of the line that goes through them.

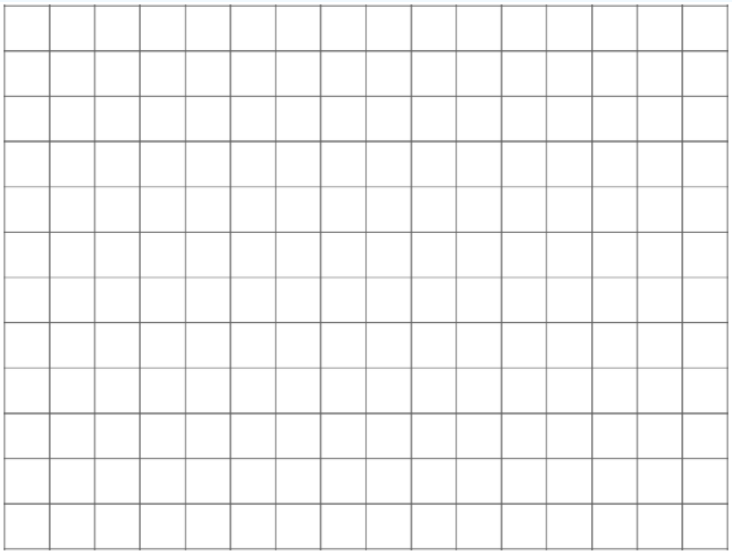


Figure 4.2.2.3

(From Unit 3.3.3)

Exercise 4.2.2.7

Match each set of equations with the move that turned the first equation into the second.

A.

B.
$$\begin{aligned} -4(5x-7) &= -18 \\ 5x-7 &= 4.5 \end{aligned}$$

C.
$$\begin{aligned} 8-10x &= 7+5x \\ 4-10x &= 3+5x \end{aligned}$$

D.
$$\begin{aligned} \frac{-5x}{4} &= 4 \\ 5x &= -16 \end{aligned}$$

E.
$$\begin{aligned} 12x+4 &= 20x+24 \\ 3x+1 &= 6x+6 \end{aligned}$$

1. Multiply both sides by $\frac{-1}{4}$
2. Multiply both sides by -4
3. Multiply both sides by $\frac{1}{4}$
4. Add $-4x$ to both sides
5. Add -4 to both sides

Exercise 4.2.2.8

Select **all** the situations for which only zero or positive solutions make sense.

A. Measuring temperature in degrees Celsius at an Arctic outpost each day in January.
B. The height of a candle as it burns over an hour.
C. The elevation above sea level of a hiker descending into a canyon.
D. The number of students remaining in school after 6:00 p.m.
E. A bank account balance over a year.
F. The temperature in degrees Fahrenheit of an oven used on a hot summer day.

(From Unit 3.5.1)

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4.2.3: More Balanced Moves

Lesson

Let's rewrite some more equations while keeping the same solutions.

Exercise 4.2.3.1: Different Equations?

Equation 1

$$x - 3 = 2 - 4x$$

Which of these have the same solution as Equation 1? Be prepared to explain your reasoning.

Equation A	Equation B	Equation C	Equation D
$2x - 6 = 4 - 8x$	$x - 5 = -4x$	$2(1 - 2x) = x - 3$	$-3 = 2 - 5x$

Exercise 4.2.3.2: Step by Step by Step by Step

Here is an equation, and then all the steps Clare wrote to solve it:

$$\begin{aligned} 14x - 2x + 3 &= 3(5x + 9) \\ 12x + 3 &= 3(5x + 9) \\ 3(4x + 1) &= 3(5x + 9) \\ 4x + 1 &= 5x + 9 \\ 1 &= x + 9 \\ -8 &= x \end{aligned}$$

Here is the same equation, and the steps Lin wrote to solve it:

$$\begin{aligned} 14x - 2x + 3 &= 3(5x + 9) \\ 12x + 3 &= 3(5x + 9) \\ 12x + 3 &= 15x + 27 \\ 12x &= 15x + 24 \\ -3x &= 24 \\ x &= -8 \end{aligned}$$

1. Are both of their *solutions* correct? Explain your reasoning.
2. Describe some ways the steps they took are alike and different.
3. Mai and Noah also solved the equation, but some of their steps have errors. Find the incorrect step in each solution and explain why it is incorrect.

Mai:

$$\begin{aligned} 14x - 2x + 3 &= 3(5x + 9) \\ 12x + 3 &- 3(5x + 9) \\ 7x + 3 &= 3(9) \\ 7x + 3 &= 27 \\ 7x &= 24 \\ x &= \frac{24}{7} \end{aligned}$$

Noah:

$$\begin{aligned} 14x-2x+3 &= 3(5x+9) \\ 12x+3 &= 15x+27 \\ 27x+3 &= 27 \\ 27x &= 24 \\ x &= \frac{24}{27} \end{aligned}$$

Exercise 4.2.3.3: Make Your Own Steps

Solve these equations for x.

1. $\frac{12+6x}{3} = \frac{5-9}{2}$
2. $x-4 = \frac{1}{3}(6x-54)$
3. $-(3x-12) = 9x-4$

Are you ready for more?

I have 24 pencils and 3 cups. The second cup holds one more pencil than the first. The third holds one more than the second. How many pencils does each cup contain?

Summary

How do we make sure the solution we find for an equation is correct? Accidentally adding when we meant to subtract, missing a negative when we distribute, forgetting to write an x from one line to the next–there are many possible mistakes to watch out for!

Fortunately, each step we take solving an equation results in a new equation with the same solution as the original. This means we can check our work by substituting the value of the solution into the original equation. For example, say we solve the following equation:

$$\begin{aligned} 2x &= -3(x+5) \\ 2x &= -3x+15 \\ 5x &= 15 \\ x &= 3 \end{aligned}$$

Substituting 3 in place of x into the original equation,

$$\begin{aligned} 2(3) &= -3(3+5) \\ 6 &= -3(8) \\ 6 &= -24 \end{aligned}$$

we get a statement that isn't true! This tells us we must have made a mistake somewhere. Checking our original steps carefully, we made a mistake when distributing -3. Fixing it, we now have

$$\begin{aligned} 2x &= -3(x+5) \\ 2x &= -3x-15 \\ 5x &= -15 \\ x &= -3 \end{aligned}$$

Substituting -3 in place of x into the original equation to make sure we didn't make another mistake:

$$\begin{aligned} 2(-3) &= -3(-3+5) \\ -6 &= -3(2) \\ -6 &= -6 \end{aligned}$$

This equation is true, so $x = -3$ is the solution.

Practice

Exercise 4.2.3.4

Mai and Tyler work on the equation $\frac{2}{5}b+1=-11$ together. Mai's solution is $b=-25$ and Tyler's is $b=-28$. Here is their work. Do you agree with their solutions? Explain or show your reasoning.

Mai:

$$\begin{aligned}\frac{2}{5}b+1 &= -11\\ \frac{2}{5}b &= -10\\ b &= -10\cdot\frac{5}{2}\\ b &= -25\end{aligned}$$

Tyler:

$$\begin{aligned}\frac{2}{5}b+1 &= -11\\ 2b+1 &= -55\\ 2b &= -56\\ b &= -28\end{aligned}$$

Exercise 4.2.3.5

Solve $3(x-4)=12x$

Exercise 4.2.3.6

Describe what is being done in each step while solving the equation.

1. $2(-3x+4)=5x+2$
2. $-6x+8=5x+2$
3. $8=11x+2$
4. $6=11x$
5. $x=\frac{6}{11}$

Exercise 4.2.3.7

Andre solved an equation, but when he checked his answer he saw his solution was incorrect. He knows he made a mistake, but he can't find it. Where is Andre's mistake and what is the solution to the equation?

$$\begin{aligned}-2(3x-5) &= 4(x+3)+8\\ -6x+10 &= 4x+12+8\\ -6x+10 &= 4x+20\\ 10 &= -2x+20\\ -10 &= -2x\\ 5 &= x\end{aligned}$$

Exercise 4.2.3.8

Choose the equation that has solutions $(5,7)$ and $(8,13)$.

A. $3x-y=8$
B. $y=x+2$
C. $y-x=5$
D. $y=2x-3$

(From Unit 3.4.1)

Exercise 4.2.3.9

A length of ribbon is cut into two pieces to use in a craft project. The graph shows the length of the second piece, x, for each length of the first piece, y.

1. How long is the ribbon? Explain how you know.
2. What is the slope of the line?
3. Explain what the slope of the line represents and why it fits the story.

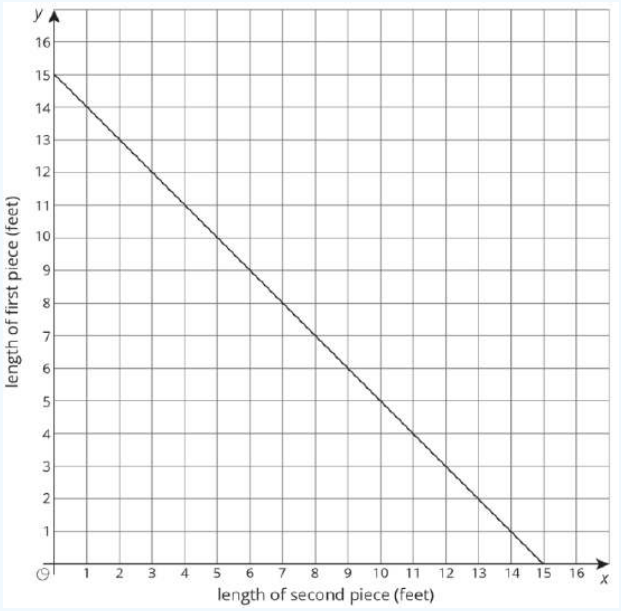


Figure 4.2.3.1: The graph of a line in the x y plane, origin O, with grid. Horizontal axis, length of second piece in feet, scale 0 to 16, by 1's, Vertical axis, length of first piece in feet, scale 0 to 16, by 1s. The line crosses the y axis at 15 and the x axis at 15.

(From Unit 3.3.1)

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4.2.4: Solving Any Linear Equation

Lesson

Let's solve linear equations.

Exercise 4.2.4.1: Equation Talk

Solve each equation mentally.

$5 - x = 8$

$-1 = x - 2$

$-3x = 9$

$-10 = -5x$

Exercise 4.2.4.2: Trading Moves

Your teacher will give you 4 cards, each with an equation.

1. With your partner, select a card and choose who will take the first turn.
2. During your turn, decide what the next move to solve the equation should be, explain your choice to your partner, and then write it down once you both agree. Switch roles for the next move. This continues until the equation is solved.
3. Choose a second equation to solve in the same way, trading the card back and forth after each move.
4. For the last two equations, choose one each to solve and then trade with your partner when you finish to check one another's work.

Exercise 4.2.4.3: A Puzzling Puzzle

Tyler says he invented a number puzzle. He asks Clare to pick a number, and then asks her to do the following:

- Triple the number
- Subtract 7
- Double the result
- Subtract 22
- Divide by 6

Clare says she now has a -3. Tyler says her original number must have been a 3. How did Tyler know that? Explain or show your reasoning. Be prepared to share your reasoning with the class.

Summary

When we have an equation in one variable, there are many different ways to solve it. We generally want to make moves that get us closer to an equation like

variable = some number.

For example, $x = 5$ or $t = \frac{7}{3}$. Since there are many ways to do this, it helps to choose moves that leave fewer **terms** or factors. If we have an equation like $3t + 5 = 7$,

adding -5 to each side will leave us with fewer terms. The equation then becomes $3t = 2$.

Dividing each side of this equation by 3 will leave us with t by itself on the left and that $t = \frac{2}{3}$.

Or, if we have an equation like $4(5 - a) = 12$,

dividing each side by 4 will leave us with fewer factors on the left, $5 - a = 3$.

Some people use the following steps to solve a linear equation in one variable:

1. Use the distributive property so that all the expressions no longer have parentheses.

2. Collect like terms on each side of the equation.
3. Add or subtract an expression so that there is a variable on just one side.
4. Add or subtract an expression so that there is just a number on the other side.
5. Multiply or divide by a number so that you have an equation that looks like *variable some number*.

For example, suppose we want to solve $9-2b+6=-3(b+5)+4b$.

$$\begin{aligned} 9-2b+6 &= -3b-15+4b && \text{Use the distributive property} \\ 15-2b &= b-15 && \text{Gather like terms} \\ 15 &= 3b-15 && \text{Add 2b to each side} \\ 30 &= 3b && \text{Add 15 to each side} \\ 10 &= b && \text{Divide each side by 3} \end{aligned}$$

Following these steps will always work, although it may not be the most efficient method. From lots of experience, we learn when to use different approaches.

Definition: Term

A term is a part of an expression. It can be a single number, a variable, or a number and a variable that are multiplied together. For example, the expression $5x+18$ has two terms. The first term is $5x$ and the second term is 18.

Practice

Exercise 4.2.4.4

Solve each of these equations. Explain or show your reasoning.

$$2(x+5)=3x+1 \qquad 3y-4=6-2y \qquad 3(n+2)=9(6-n)$$

Exercise 4.2.4.5

Clare was solving an equation, but when she checked her answer she saw her solution was incorrect. She knows she made a mistake, but she can't find it. Where is Clare's mistake and what is the solution to the equation?

$$\begin{aligned} 12(5+2y) &= 4y-(5-9y) \\ 72+24y &= 4y-5-9y \\ 72+24y &= -5y-5 \\ 24y &= -5y-77 \\ 29y &= -77 \\ y &= -\frac{77}{29} \end{aligned}$$

Exercise 4.2.4.6

Solve each equation, and check your solution.

$$\frac{1}{9}(2m-16)=\frac{1}{3}(2m+4) \qquad -4(r+2)=4(2-2r) \qquad 12(5+2y)=4y-(6-9y)$$

Exercise 4.2.4.7

Here is the graph of a linear equation.

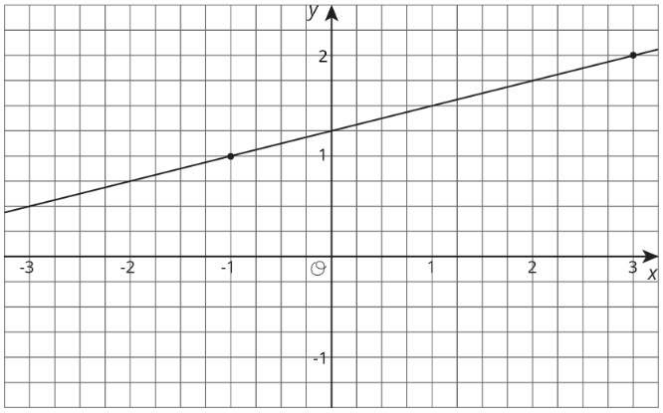


Figure 4.2.4.1

Select **all** true statements about the line and its equation.

A. One solution of the equation is $(3, 2)$.
B. One solution of the equation is $(-1, 1)$.
C. One solution of the equation is $(1, \frac{3}{2})$.
D. There are 2 solutions.
E. There are infinitely many solutions.
F. The equation of the line is $y = \frac{1}{4}x + \frac{5}{4}$.
G. The equation of the line is $y = \frac{5}{4}x + \frac{1}{4}$.

(From Unit 3.4.2)

Exercise 4.2.4.8

A participant in a 21-mile walkathon walks at a steady rate of 3 miles per hour. He thinks, "The relationship between the number of miles left to walk and the number of hours I already walked can be represented by a line with slope -3." Do you agree with his claim? Explain your reasoning.

(From Unit 3.3.1)

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4.2.5: Strategic Solving

Lesson

Let's solve linear equations like a boss.

Exercise 4.2.5.1: Equal Perimeters

The triangle and the square have equal perimeters.

1. Find the value of x.
2. What is the perimeter of each of the figures?

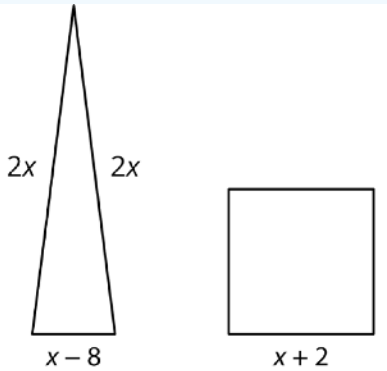


Figure 4.2.5.1

Exercise 4.2.5.2: Predicting Solutions

Without solving, identify whether these equations have a solution that is positive, negative, or zero.

1. $\frac{x}{6} = \frac{3x}{4}$
2. $7x = 3.25$
3. $7x = 32.5$
4. $3x + 11 = 11$
5. $9 - 4x = 4$
6. $-8 + 5x = -20$
7. $-\frac{1}{2}(-8 + 5x) = -20$

Exercise 4.2.5.3: Which Would You Rather Solve?

Here are a lot of equations:

A. $-\frac{5}{6}(8 + 5b) = 75 + \frac{5}{3}b$
B. $-\frac{1}{2}(t + 3) - 10 = -6.5$
C. $\frac{10 - v}{4} = 2(v + 17)$
D. $2(4k + 3) - 13 = 2(18 - k) - 13$
E. $\frac{n}{7} - 12 = 5n + 5$
F. $3(c - 1) + 2(3c - 1) = -(3c + 1)$
G. $\frac{4m - 3}{4} = -\frac{9 + 4m}{8}$
H. $p - 5(p + 4) = p - (8 - p)$
I. $2(2q + 1.5) = 18 - q$
J. $2r + 49 = -8(-r - 5)$

1. Without solving, identify 3 equations that you think would be least difficult to solve and 3 equations you think would be most difficult to solve. Be prepared to explain your reasoning.

2. Choose 3 equations to solve. At least one should be from your "least difficult" list and one should be from your "most difficult" list.

Are you ready for more?

Mai gave half of her brownies, and then half a brownie more, to Kiran. Then she gave half of what was left, and half a brownie more, to Tyler. That left her with one remaining brownie. How many brownies did she have to start with?

Summary

Sometimes we are asked to solve equations with a lot of things going on on each side. For example,

$$x-2(x+5)=\frac{3(2x-20)}{6}$$

This equation has variables on each side, parentheses, and even a fraction to think about. Before we start distributing, let's take a closer look at the fraction on the right side. The expression $2x-20$ is being multiplied by 3 and divided by 6, which is the same as just dividing by 2, so we can re-write the equation as

$$x-2(x+5)=\frac{2x-20}{2}$$

But now it's easier to see that all the terms on the numerator of right side are divisible by 2, which means we can re-write the right side again as

$$x-2(x+5)=x-10$$

At this point, we could do some distribution and then collect like terms on each side of the equation. Another choice would be to use the structure of the equation. Both the left and the right side have something being subtracted from x. But, if the two sides are equal, that means the "something" being subtracted on each side must also be equal. Thinking this way, the equation can now be re-written with less terms as

$$2(x+5)=10$$

Only a few steps left! But what can we tell about the solution to this problem right now? Is it positive? Negative? Zero? Well, the 2 and the 5 multiplied together are 10, so that means the 2 and the x multiplied together cannot have a positive or a negative value. Finishing the steps we have:

$$\begin{aligned} 2(x+5) &= 10 & \\ x+5 &= 5 & \text{Divide each side by 2.} \\ x &= 0 & \text{Subtract 5 from each side} \end{aligned}$$

Neither positive nor negative. Just as predicted.

Glossary Entries

Definition: Term

A term is a part of an expression. It can be a single number, a variable, or a number and a variable that are multiplied together. For example, the expression $5x+18$ has two terms. The first term is $5x$ and the second term is 18.

Practice

Exercise 4.2.5.4

Solve each of these equations. Explain or show your reasoning.

$2b+8-5b+3=-13+8b-5$

$2x+7-5x+8=3(5+6x)-12x$

$2c-3=2(6-c)+7c$

Exercise 4.2.5.5

Solve each equation and check your solution.

$-3w-4=w+3$

$3(3-3x)=2(x+3)-30$

$\frac{1}{3}(z+4)-6=\frac{2}{3}(5-z)$

Exercise 4.2.5.6

Elena said the equation $9x+15=3x+15$ has no solutions because $9x$ is greater than $3x$. Do you agree with Elena? Explain your reasoning.

Exercise 4.2.5.7

The table gives some sample data for two quantities, x and y, that are in a proportional relationship.

x	y
14	21
64	
	39
1	

Table4.2.5.1

1. Complete the table.
2. Write an equation that represents the relationship between x and y shown in the table.
3. Graph the relationship. Use a scale for the axes that shows all the points in the table.

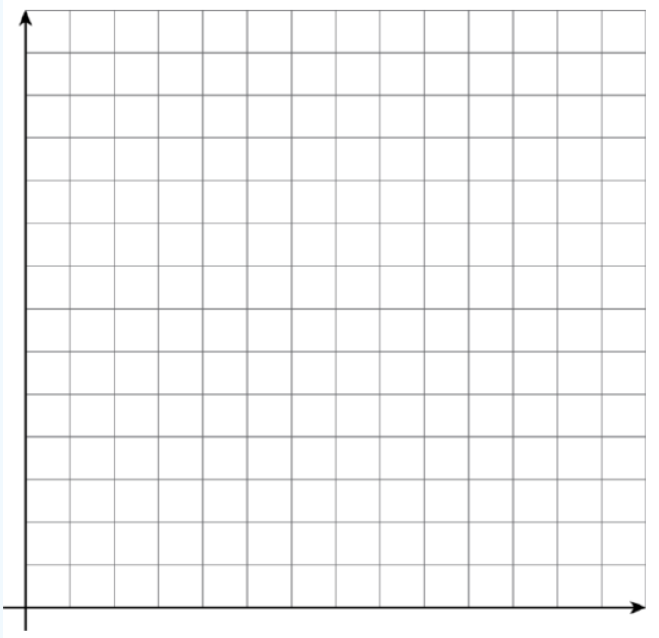


Figure 4.2.5.2

(From Unit 3.1.3)

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4.2.6: All, Some, or No Solutions

Lesson

Let's think about how many solutions an equation can have.

Exercise 4.2.6.1: Which one doesn't belong?

1. $5+7=7+5$
2. $5 \cdot 7=7 \cdot 5$
3. $2=7-5$
4. $5-7=7-5$

Exercise 4.2.6.2: Thinking About Solutions

$n=n$

$2t+6=2(t+3)$

$3(n+1)=3n+1$

$\frac{1}{4}(20d+4)=5d$

$5-9+3x=-10+6+3x$

$\frac{1}{2}+x=\frac{1}{3}+x$

$y \cdot -6 \cdot -3=2 \cdot y \cdot 9$

$v+2=v-2$

1. Sort these equations into the two types: true for all values and true for no values.
2. Write the other side of this equation so that this equation is true for all values of u. $6(u-2)+2=$
3. Write the other side of this equation so that this equation is true for no values of u. $6(u-2)+2=$

Are you ready for more?

Consecutive numbers follow one right after the other. An example of three consecutive numbers is 17, 18, and 19. Another example is -100, -99, -98.

How many sets of two or more consecutive positive integers can be added to obtain a sum of 100?

Exercise 4.2.6.3: What's the Equation?

1. Complete each equation so that it is true for all values of x.
   a. $3x+6=3(x+\_)$
   b. $x-2=-(\_-x)$
   c. $\frac{15x-10}{5}=\_-2$
2. Complete each equation so that it is true for no values of x.
   a. $3x+6=3(x+\_)$
   b. $x-2=-(\_-x)$
   c. $\frac{15x-10}{5}=\_-2$
3. Describe how you know whether an equation will be true for all values of x or true for no values of x.

Summary

An equation is a statement that two expressions have an equal value. The equation

$2x=6$

is a true statement if x is 3:

$2 \cdot 3 = 6$

It is a false statement if x is 4:

$2 \cdot 4 = 6$

The equation $2x = 6$ has *one and only one solution*, because there is only one number (3) that you can double to get 6.

Some equations are true no matter what the value of the variable is. For example:

$2x = x + x$

is always true, because if you double a number, that will always be the same as adding the number to itself. Equations like $2x = x + x$ have an *infinite number of solutions*. We say it is true for all values of x.

Some equations have *no solutions*. For example:

$x = x + 1$

has no solutions, because no matter what the value of x is, it can't equal one more than itself.

When we solve an equation, we are looking for the values of the variable that make the equation true. When we try to solve the equation, we make allowable moves assuming it *has* a solution. Sometimes we make allowable moves and get an equation like this:

$8 = 7$

This statement is false, so it must be that the original equation had no solution at all.

Glossary Entries

Definition: Term

A term is a part of an expression. It can be a single number, a variable, or a number and a variable that are multiplied together. For example, the expression $5x + 18$ has two terms. The first term is $5x$ and the second term is 18.

Practice

Exercise 4.2.6.4

For each equation, decide if it is alwaus true or never true.

1. $x - 13 = x + 1$
2. $x + \frac{1}{2} = x - \frac{1}{2}$
3. $2(x + 3) = 5x + 6 - 3x$
4. $x - 3 = 2x - 3 - x$
5. $3(x - 5) = 2(x - 5) + x$

Exercise 4.2.6.5

Mai says that the equation $2x + 2 = x + 1$ has no solution because the left hand side is double the right hand side. Do you agree with Mai? Explain your reasoning.

Exercise 4.2.6.6

1. Write the other side of this equation so it's true for all values of x:
   $\frac{1}{2}(6x - 10) - x =$
2. Write the other side of this equation so it's true for no values of x:
   $\frac{1}{2}(6x - 10) - x =$

Exercise 4.2.6.7

Here is an equation that is true for all values of x: $5(x+2) = 5x+10$. Elena saw this equation and says she can tell $20(x+2)+31 = 4(5x+10)+31$ is also true for any value of x. How can she tell? Explain your reasoning.

Exercise 4.2.6.8

Elena and Lin are trying to solve $\frac{1}{2}x+3 = \frac{7}{2}x+5$. Describe the change they each make to each side of the equation.

1. Elena's first step is to write $3 = \frac{7}{2}x - \frac{1}{2}x + 5$.
2. Lin's first step is to write $x+6 = 7x+10$.

(From Unit 4.2.3)

Exercise 4.2.6.9

Solve each equation and check your solution.

$$3x-6 = 4(2-3x)-8x \qquad \frac{1}{2}z+6 = \frac{3}{2}(z+6) \qquad 9-7w = 8w+8$$

(From Unit 4.2.5)

Exercise 4.2.6.10

The point $(-3, 6)$ is on a line with a slope of 4.

1. Find two more points on the line.
2. Write an equation for the line.

(From Unit 3.4.1)

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4.2.7: How Many Solutions?

Lesson

Let's solve equations with different numbers of solutions.

Exercise 4.2.7.1: Matching Solutions

Consider the unfinished equation $12(x-3)+18=\_$. Match the following expressions with the number of solutions the equation would have with that expression on the right hand side.

1. $6(2x-3)$
2. $4(3x-3)$
3. $4(2x-3)$

- one solution
- no solutions
- all solutions

Exercise 4.2.7.2: Thinking About Solutions Some More

Your teacher will give you some cards.

1. With your partner, solve each equation.
2. Then, sort them into categories.
3. Describe the defining characteristics of those categories and be prepared to share your reasoning with the class.

Exercise 4.2.7.3: Make Use of Structure

For each equation, determine whether it has no solutions, exactly one solution, or is true for all values of x (and has infinitely many solutions). If an equation has one solution, solve to find the value of x that makes the statement true.

1. a. $6x+8=7x+13$
   b. $6x+8=2(3x+4)$
   c. $6x+8=6x+13$
2. a. $\frac{1}{4}(12-4x)=3-x$
   b. $x-3=3-x$
   c. $x-3=3+x$
3. a. $-5x-3x+2=-8x+2$
   b. $-5x-3x-4=-8x+2$
   c. $-5x-4x-2=-8x+2$
4. a. $4(2x-2)+2=4(x-2)$
   b. $4x+2(2x-3)=8(x-1)$
   c. $4x+2(2x-3)=4(2x-2)+2$
5. a. $x-3(2-3x)=2(5x+3)$
   b. $x-3(2+3x)=2(5x-3)$
   c. $x-3(2-3x)=2(5x-3)$
6. What do you notice about equations with one solution? How is this different from equations with no solutions and equations that are true for every x?

Are you ready for more?

Consecutive numbers follow one right after the other. An example of three consecutive numbers is 17, 18, and 19. Another example is -100, -99, -98.

1. Choose any set of three consecutive numbers. Find their average. What do you notice?
2. Find the average of another set of three consecutive numbers. What do you notice?

3. Explain why the thing you noticed must always work, or find a counterexample.

Summary

Sometimes it's possible to look at the structure of an equation and tell if it has infinitely many solutions or no solutions. For example, look at

$$2(12x+18)+6=18x+6(x+7)$$

.

Using the distributive property on the left and right sides, we get

$$24x+36+6=18x+6x+42$$

.

From here, collecting like terms gives us

$$24x+42=24x+42$$

.

Since the left and right sides of the equation are the same, we know that this equation is true for any value of x without doing any more moves!

Similarly, we can sometimes use structure to tell if an equation has no solutions. For example, look at

$$6(6x+5)=12(3x+2)+12$$

.

If we think about each move as we go, we can stop when we realize there is no solution:

$$\begin{aligned} \frac{1}{6}\cdot 6(6x+5) &= \frac{1}{6}\cdot(12(3x+2)+12) && \text{Multiply each side by } \frac{1}{6}. \\ 6x+5 &= 2(3x+2)+2 && \text{Distribute } \frac{1}{6} \text{ on the right side.} \\ 6x+5 &= 6x+4+2 && \text{Distribute 2 on the right side.} \end{aligned}$$

The last move makes it clear that the **constant terms** on each side, 5 and $4+2$, are not the same. Since adding 5 to an amount is always less than adding $4+2$ to that same amount, we know there are no solutions.

Doing moves to keep an equation balanced is a powerful part of solving equations, but thinking about what the structure of an equation tells us about the solutions is just as important.

Glossary Entries

Definition: Coefficient

A coefficient is a number that is multiplied by a variable.

For example, in the expression $3x+5$, the coefficient of x is 3. In the expression $y+5$, the coefficient of y is 1, because $y=1\cdot y$.

Definition: Constant Term

In an expression like $5x+2$, the number 2 is called the constant term because it doesn't change when x changes.

In the expression $7x+9$, 9 is the constant term.
In the expression $5x+(-8)$, -8 is the constant term.
In the expression $12-4x$, 12 is the constant term.

Definition: Term

A term is a part of an expression. It can be a single number, a variable, or a number and a variable that are multiplied together. For example, the expression $5x+18$ has two terms. The first term is $5x$ and the second term is 18.

Practice

Exercise 4.2.7.4

Lin was looking at the equation $2x-32+4(3x-2462)=14x$. She said, "I can tell right away there are no solutions, because on the left side, you will have $2x+12x$ and a bunch of constants, but you have just $14x$ on the right side." Do you agree with Lin? Explain your reasoning.

Exercise 4.2.7.5

Han was looking at the equation $6x-4+2(5x+2)=16x$. He said, "I can tell right away there are no solutions, because on the left side, you will have $6x+10x$ and a bunch of constants, but you have just $16x$ on the right side." Do you agree with Han? Explain your reasoning.

Exercise 4.2.7.6

Decide whether each equation is true for all, one, or no values of x.

1. $6x=-4=-4+6x$
2. $4x-6=4x+3$
3. $-2x+4=-3x+4$

Exercise 4.2.7.7

Solve each of these equations. Explain or show your reasoning.

1. $3(x-5)=6$
2. $2(x-\frac{2}{3})=0$
3. $4x-5=2-x$

(From Unit 4.2.3)

Exercise 4.2.7.8

The points $(-2,0)$ and $(0,-6)$ are each on the graph of a linear equation. Is $(2,6)$ also on the graph of this linear equation? Explain your reasoning.

(From Unit 3.4.2)

Exercise 4.2.7.9

In the picture triangle $A'B'C'$ is an image of triangle ABC after a rotation. The center of rotation is E.

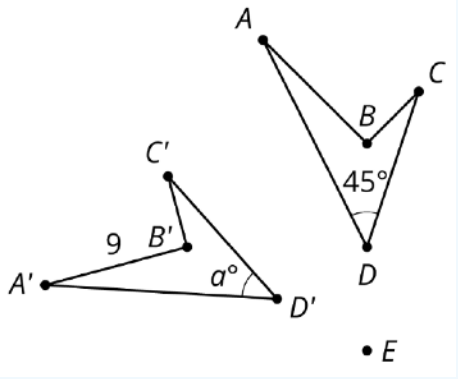


Figure 4.2.7.1

1. What is the length of side AB? Explain how you know.
2. What is the measure of angle D'? Explain how you know.

(From Unit 1.2.1)

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4.2.8: When Are They the Same?

Lesson

Let's use equations to think about situations.

Exercise 4.2.8.1: Which Would You Choose?

If you were babysitting, would you rather

- Charge \$5 for the first hour and \$8 for each additional hour?

Or

- Charge \$15 for the first hour and \$6 for each additional hour?

Explain your reasoning.

Exercise 4.2.8.2: Water Tanks

The amount of water in two tanks every 5 minutes is shown in the table.

Table 4.2.8.1

time (minutes)	tank 1 (liters)	tank 2 (liters)
0	25	1000
5	175	900
10	325	800
15	475	700
20	625	600
25	775	500
30	925	400
35	1075	300
40	1225	200
45	1375	100
50	1525	0

1. Describe what is happening in each tank. Either draw a picture, say it verbally, or write a few sentences.
2. Use the table to estimate when the tanks will have the same amount of water.
3. The amount of water (in liters) in tank 1 after t minutes is $30t + 25$. The amount of water (in liters) in tank 2 after t minutes is $-20t + 1000$. Find the time when the amount of water will be equal.

Exercise 4.2.8.3: Elevators

A building has two elevators that both go above and below ground.

At a certain time of the day, the travel time it takes elevator A to reach height h in meters is $0.8h + 16$ seconds.

The travel time it take elevator B to reach height h in meters is $-0.8h + 12$ seconds.

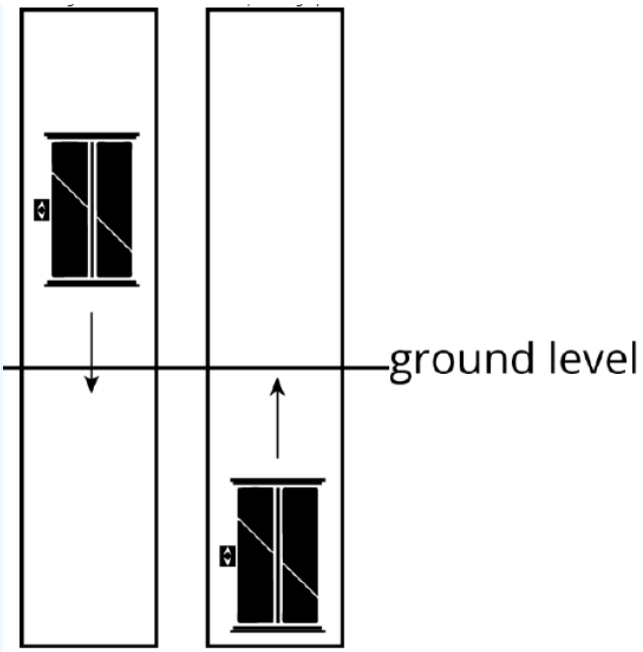


Figure 4.2.8.1: Two elevators. One elevator is above a line labeled ground level with an arrow pointing down. Another elevator is below a line labeled ground level with an arrow pointing up.

1. What is the height of each elevator at this time?
2. How long would it take each elevator to reach ground level at this time?
3. If the two elevators travel toward one another, at what height do they pass each other? How long would it take?
4. If you are on an underground parking level 14 meters below ground, which elevator would reach you first?

Are you ready for more?

1. In a two-digit number, the ones digit is twice the tens digit. If the digits are reversed, the new number is 36 more than the original number. Find the number.
2. The sum of the digits of a two-digit number is 11. If the digits are reversed, the new number is 45 less than the original number. Find the number.
3. The sum of the digits in a two-digit number is 8. The value of the number is 4 less than 5 times the ones digit. Find the number.

Summary

Imagine a full 1,500 liter water tank that springs a leak, losing 2 liters per minute. We could represent the number of liters left in the tank with the expression $-2x+1,500$, where x represents the number of minutes the tank has been leaking.

Now imagine at the same time, a second tank has 300 liters and is being filled at a rate of 6 liters per minute. We could represent the amount of water in liters in this second tank with the expression $6x+300$, where x represents the number of minutes that have passed.

Since one tank is losing water and the other is gaining water, at some point they will have the same amount of water—but when? Asking when the two tanks have the same number of liters is the same as asking when $-2x+1,500$ (the number of liters in the first tank after x minutes) is equal to $6x+300$ (the number of liters in the second tank after x minutes),

$$-2x+1,500=6x+300.$$

Solving for x gives us $x=150$ minutes. So after 150 minutes, the number of liters of the first tank is equal to the number of liters of the second tank. But how much water is actually in each tank at that time? Since both tanks have the same number of liters after 150 minutes, we could substitute $x=150$ minutes into either expression.

Using the expression for the first tank, we get $-2(150)+1,500$ which is equal to $-300+1,500$, or 1,200 liters.

If we use the expression for the second tank, we get $6(150)+300$, or just $900+300$, which is also 1,200 liters. That means that after 150 minutes, each tank has 1,200 liters.

Glossary Entries

Definition: Coefficient

A coefficient is a number that is multiplied by a variable.

For example, in the expression $3x+5$, the coefficient of x is 3. In the expression $y+5$, the coefficient of y is 1, because $y=1\cdot y$.

Definition: Constant Term

In an expression like $5x+2$, the number 2 is called the constant term because it doesn't change when x changes.

In the expression $7x+9$, 9 is the constant term.
In the expression $5x+(-8)$, -8 is the constant term.
In the expression $12-4x$, 12 is the constant term.

Definition: Term

A term is a part of an expression. It can be a single number, a variable, or a number and a variable that are multiplied together. For example, the expression $5x+18$ has two terms. The first term is $5x$ and the second term is 18.

Practice

Exercise 4.2.8.4

Cell phone Plan A costs $70 per month and comes with a free $500 phone. Cell phone Plan B costs $50 per month but does not come with a phone. If you buy the $500 phone and choose Plan B, how many months is it until your cost is the same as Plan A's?

Exercise 4.2.8.5

Priya and Han are biking in the same direction on the same path.

1. Han is riding at a constant speed of 16 miles per hour. Write an expression that shows how many miles Han has gone after t hours.
2. Priya started riding a half hour before Han. If Han has been riding for t hours, how long has Priya been riding?
3. Priya is riding at a constant speed of 12 miles per hour. Write an expression that shows how many miles Priya has gone after Han has been riding for t hours.
4. Use your expressions to find when Han and Priya meet.

Exercise 4.2.8.6

Which story matches the equation $-6+3x=2+4x$?

A:

At 5 p.m., the temperatures recorded at two weather stations in Antarctica are -6 degrees and 2 degrees. The temperature changes at the same constant rate, x degrees per hour, throughout the night at both locations. The temperature at the first station 3 hours after this recording is the same as the temperature at the second station 4 hours after this recording.

B:

Elena and Kiran play a card game. Every time they collect a pair of matching cards, they earn x points. At one point in the game, Kiran has -6 points and Elena has 2 points. After Elena collects 3 pairs and Kiran collects 4 pairs, they have the same number of points.

Exercise 4.2.8.7

For what value of x do the expressions $\frac{2}{3}x+2$ and $\frac{4}{3}x-6$ have the same value?

Exercise 4.2.8.8

Decide whether each equation is true for all, one, or no values of x.

1. $2x+9=-3.5x+19$
2. $9(x-2)=7x+5$
3. $3(3x+2)-2x=7x+6$

(From Unit 4.2.7)

Exercise 4.2.8.9

Solve each equation. Explain your reasoning.

$3d+16=-2(5-3d$

$2k-3(4-k)=3k+4$

$\frac{3y-6}{9}=\frac{4-2y}{-3}$

(From Unit 4.2.5)

Exercise 4.2.8.10

Describe a rigid transformation that takes Polygon A to Polygon B.

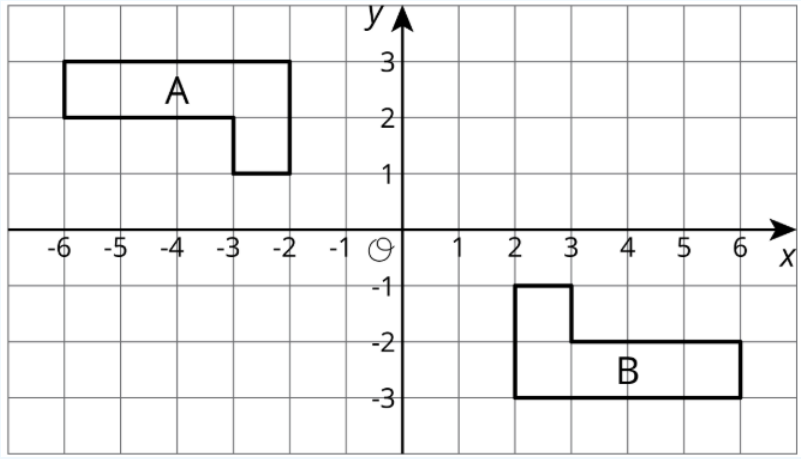


Figure B is L shaped and has vertices at 2 comma negative 1, 2 comma negative 3, 6 comma negative 3, 6 comma negative 2, 3 comma negative 2, 3 comma negative 1.

(From Unit 1.2.1)

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4.3: Systems of Linear Equations

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4.3.1: On or Off the Line?

Lesson

Let's interpret the meaning of points in a coordinate plane.

Exercise 4.3.1.1: Which One Doesn't Belong: Lines in the Plane

Which one doesn't belong? Explain your reasoning.

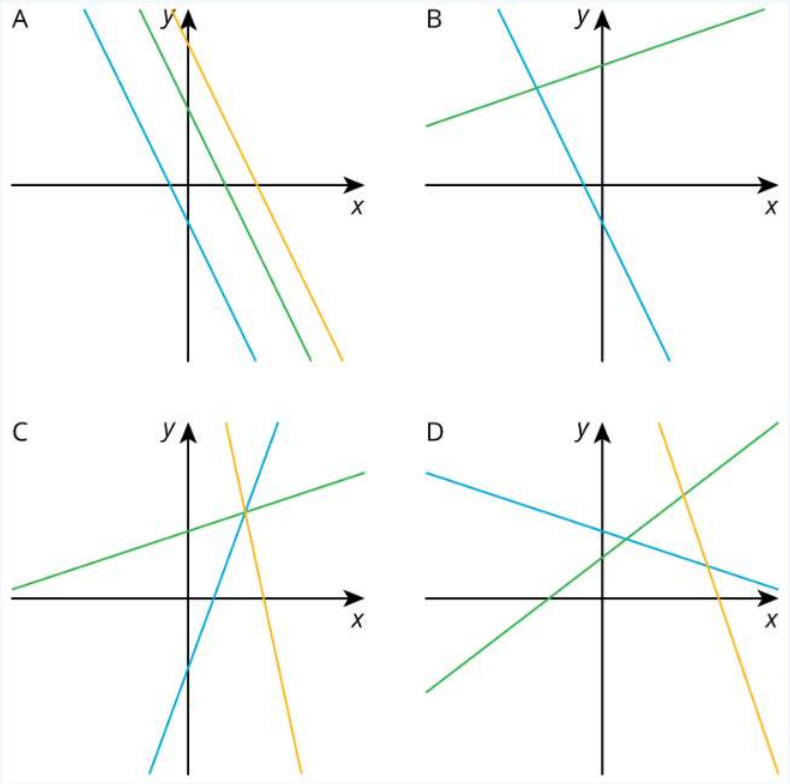


Figure 4.3.1.1: Four graphs, each in the x y plane. Graph A. 3 lines that do not intersect. One line crosses the x axis to the left of the origin and the y axis below the origin. Another line crosses the y axis above the origin and the x axis to the right of the origin. The third line crosses the y axis above the origin and the x axis to the right of the origin. Graph B. 2 intersecting lines. One line crosses the x axis to the left of the origin and the y axis below the origin. Another line crosses the y axis above the origin. Graph C. Three lines that intersect at a single point. One line crosses the y axis above the origin. Another line crosses the x axis to the right of the origin and the y axis below the origin. The third line crosses the x axis to the right of the origin. Graph D. Three lines. There are 3 points of intersection between two lines each. One line crosses the y axis above the origin. Another line crosses the x axis to the left of the origin and the y axis above the origin. The third line crosses the x axis to the right of the origin.

Exercise 4.3.1.2: Pocket Full of Change

Jada told Noah that she has $2 worth of quarters and dimes in her pocket and 17 coins all together. She asked him to guess how many of each type of coin she has.

1. Here is a table that shows some combinations of quarters and dimes that are worth $2. Complete the table.

Table 4.3.1.1

number of quarters	number of dimes
0	20
4	
	0

number of quarters	number of dimes
	5

2. Here is a graph of the relationship between the number of quarters and the number of dimes when there are a total of 17 coins.
   a. What does Point A represent?
   b. How much money, in dollars, is the combination represented by Point A worth?

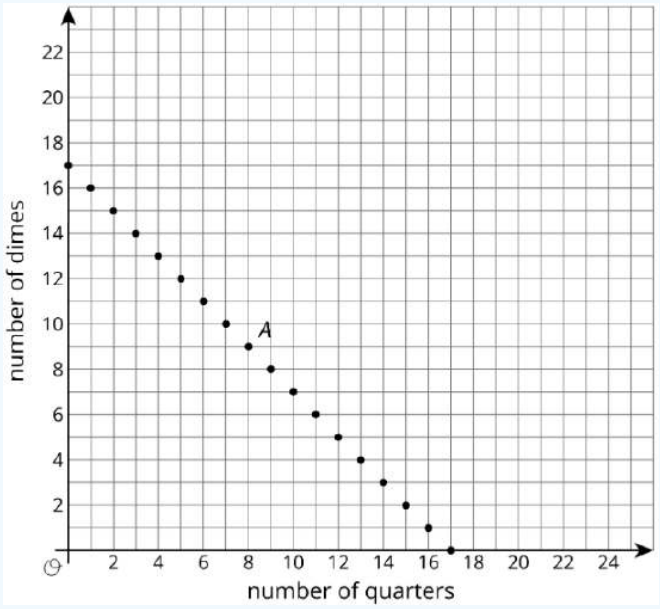


Figure 4.3.1.2: Graph of points in the x y plane, origin O, with grid. Horizontal axis, number of quarters, scale 0 to 24 by 1s. Vertical axis, number of dimes, scale 0 to 24 by 1s. Points plotted are 0 comma 17, 1 comma 16, 2 comma 15, 3 comma 14, 4 comma 13, 5 comma 12, 6 comma 11, 7 comma 10, 8 comma 9 labeled A, 9 comma 8, 10 comma 7, 11 comma 6, 12 comma 5, 13 comma 4, 14 comma 3, 15 comma 2, 16 comma 1, and 17 comma 0.

3. Is it possible for Jada to have 4 quarters and 13 dimes in her pocket? Explain how you know.

4. How many quarters and dimes must Jada have? Explain your reasoning.

Exercise 4.3.1.3: Making Signs

Clare and Andre are making signs for all the lockers as part of the decorations for the upcoming spirit week. Yesterday, Andre made 15 signs and Clare made 5 signs. Today, they need to make more signs. Each person's progress today is shown in the coordinate plane.

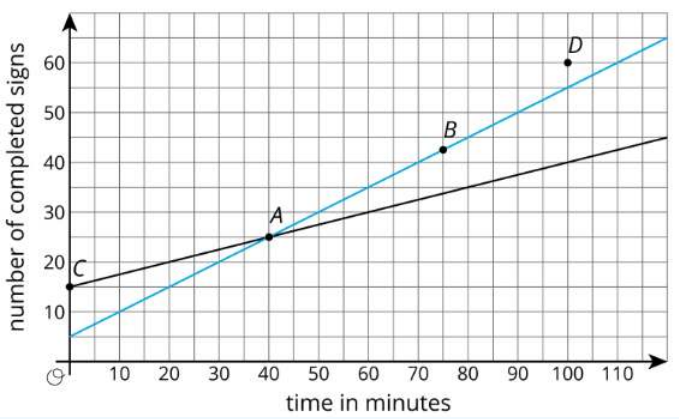


Figure 4.3.1.3: Graph of two lines in the x y plane, origin 0, with grid. Horizontal axis, time in minutes, scale 0 to 110, by 5s. Vertical axis, number of completed signs, scale 0 to 70 by 5s. A line crosses the y axis at 5 and passes through the points A and B. Another line crosses the y axis at 15 and passes through the points C and A. Starting at the origin, point A is 8 unit to the right and 5 units up. Starting at the origin, point B is 15 units to the right and 8 point 5 units up. Starting at the origin, point C is 0 units to the right, and 3 units up. Starting at the origin, point D is 20 units to the right and 12 units up.

Based on the lines, mark the statements as true or false for each person.

Table 4.3.1.2

point	what it says	Clare	Andre
A	At 40 minutes, I have 25 signs completed.		
B	At 75 minutes, I ahve 42 and a half signs completed.		
C	At 0 minutes, I have 15 signs completed.		
D	At 100 minutes, I have 60 signs completed.		

Are you ready for more?

- 4 toothpicks make 1 square
- 7 toothpicks make 2 squares
- 10 toothpicks make 3 squares



Figure 4.3.1.4

Do you see a pattern? If so, how many toothpicks would you need to make 10 squares according to your pattern? Can you represent your pattern with an expression?

Summary

We studied linear relationships in an earlier unit. We learned that values of x and y that make an equation true correspond to points (x, y) on the graph. For example, if we have x pounds of flour that costs \$0.80 per pound and y pounds of sugar that costs \$0.50 per pound, and the total cost is \$9.00, then we can write an equation like this to represent the relationship between x and y :

$$0.8x + 0.5y = 9$$

Since 5 pounds of flour costs \$4.00 and 10 pounds of sugar costs \$5.00, we know that $x = 5$, $y = 10$ is a solution to the equation, and the point $(5, 10)$ is a point on the graph. The line shown is the graph of the equation:

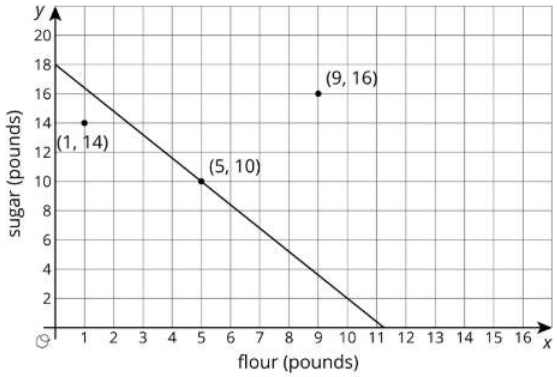


Figure 4.3.1.5: The graph of a line in the x y plane. The line slants downward and right, crosses the y axis at 18, and passes through the point 5 comma 10. Two additional points, 9 comma 16 and 1 comma 14, are labeled on the graph.

Notice that there are two points shown that are not on the line. What do they mean in the context? The point $(1, 14)$ means that there is 1 pound of flour and 14 pounds of sugar. The total cost for this is $(0.8 \cdot 1 + 0.5 \cdot 14$ or \$7.80. Since the cost is not \$9.00, this point is not on the graph. Likewise, 9 pounds of flour and 16 pounds of sugar costs $0.8 \cdot 9 + 0.5 \cdot 16$ or \$15.20, so the other point is not on the graph either.

Suppose we also know that the flour and sugar together weigh 15 pounds. This means that

$$x + y = 15$$

If we draw the graph of this equation on the same coordinate plane, we see it passes through two of the three labeled points:

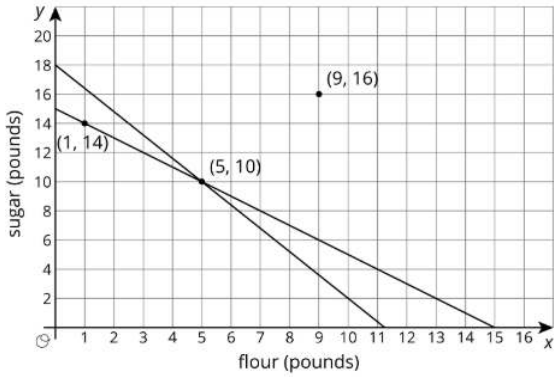


Figure 4.3.1.6: The graph of two intersecting lines in the x y plane. The first line slants downward and right, crosses the y axis at 18, and passes through the point 5 comma 10. The second line slants downward and to the right and passes through the points 1 comma 14 and 5 comma 10. An additional point, 9 comma 16, is labeled on the graph.

The point $(1, 14)$ is on the graph of $x + y = 15$ because $1 + 14 = 15$. Similarly, $5 + 10 = 15$. But $9 + 16 \neq 15$, so $(9, 16)$ is *not* on the graph of $x + y = 15$. In general, if we have two lines in the coordinate plane,

- The coordinates of a point that is on both lines makes both equations true.
- The coordinates of a point on only one line makes only one equation true.
- The coordinates of a point on neither line make both equations false.

Practice

Exercise 4.3.1.4

1. Match the lines m and n to the statements they represent:

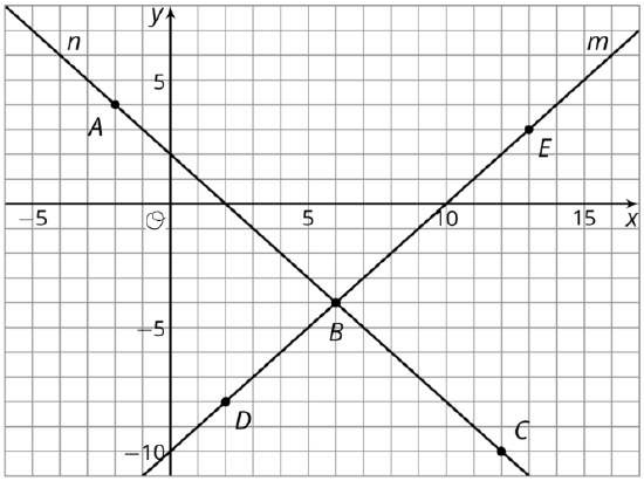


Figure 4.3.1.7: Two lines in an x y plane. Line m passes through points E, B, and D. Point E is at 13 comma 3. Point B is at 6 comma negative 4. Point D is at 2 comma negative 8. Line n passes through points A, B, and C. Point A is at negative 2 comma 4. Point B is at 6 comma negative 4. Point C is at 12 comma negative 10.

a. A set of points where the coordinates of each point have a sum of 2
b. A set of points where the y-coordinate of each point is 10 less than its x-coordinate

2. Match the labeled points on the graph to statements about their coordinates:

a. Two numbers with a sum of 2
b. Two numbers where the y-coordinate is 10 less than the x-coordinate
c. Two numbers with a sum of 2 and where the y-coordinate is 10 less than the x-coordinate

Exercise 4.3.1.5

Here is an equation: $4x - 4 = 4x + \_$. What could you write in the blank so the equation would be true for:

1. No values of x
2. All values of x
3. One value of x

(From Unit 4.2.6)

Exercise 4.3.1.6

Mai earns $7 per hour mowing her neighbors' lawns. She also earned $14 for hauling away bags of recyclables for some neighbors.

Priya babysits her neighbor's children. The table shows the amount of money m she earns in h hours. Priya and Mai have agreed to go to the movies the weekend after they have earned the *same* amount of money for the *same* number of work hours.

Table 4.3.1.3

h	m
1	$8.40
2	$16.80
4	$33.60

1. How many hours do they each have to work before they go to the movies?
2. How much will each of them have earned?
3. Explain where the solution can be seen in tables of values, graphs, and equations that represent Priya's and Mai's hourly earnings.

Exercise 4.3.1.7

For each equation, explain what you could do first to each side of the equation so that there would be no fractions. You do not have to solve the equations (unless you want more practice).

1. $\frac{3x-4}{8} = \frac{x+2}{3}$
2. $\frac{3(2-4)}{4} = \frac{3+r}{6}$
3. $\frac{4p+3}{8} = \frac{p+2}{4}$
4. $\frac{2(a-7)}{15} = \frac{a+4}{6}$

(From Unit 4.2.5)

Exercise 4.3.1.8

The owner of a new restaurant is ordering tables and chairs. He wants to have only tables for 2 and tables for 4. The total number of people that can be seated in the restaurant is 120.

1. Describe some possible combinations of 2-seat tables and 4-seat tables that will seat 120 customers. Explain how you found them.
2. Write an equation to represent the situation. What do the variables represent?
3. Create a graph to represent the situation.

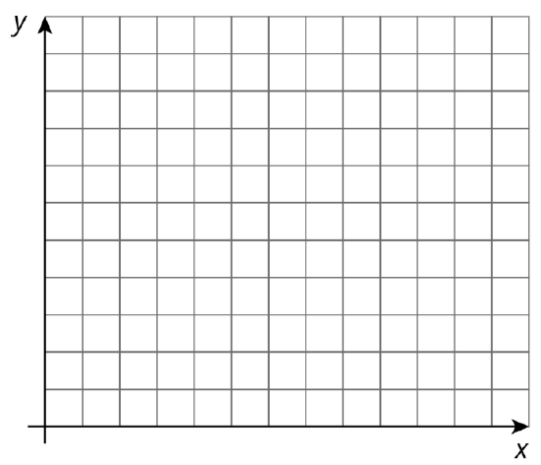


Figure 4.3.1.8

4. What does the slope tell us about the situation?
5. Interpret the x and y intercepts in the situation.

(From Unit 3.5.1)

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4.3.2: On Both of the Lines

Lesson

Let's use lines to think about situations.

Exercise 4.3.2.1: Notice and Wonder: Bugs Passing in the Night

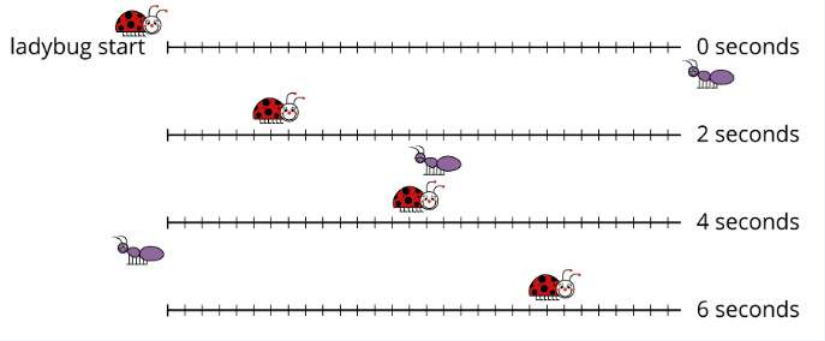


Figure 4.3.2.1: Four blank number lines. Each number line has a ladybug facing right and an ant facing left. Number line 1 is labeled ladybug start on the left end and 0 seconds. The ladybug is to the left of the number line and the ant is to the right of the number line. Number line 2 is labeled 2 seconds. The ladybug is at the 8th tick mark from the right. The ant is at the 16th tick mark from the left. Number line 3 is labeled 4 seconds. The ladybug is at the 16th tick mark from the right. The ant is at the 32nd tick mark from the left. Number line 4 is labeled 6 seconds. The ladybug is at the 24th tick mark. The ant is not shown in the number line.

Exercise 4.3.2.2: Bugs Passing in the Night, Continued

A different ant and ladybug are a certain distance apart, and they start walking toward each other. The graph shows the ladybug's distance from its starting point over time and the labeled point $(2.5, 10)$ indicates when the ant and the ladybug pass each other.

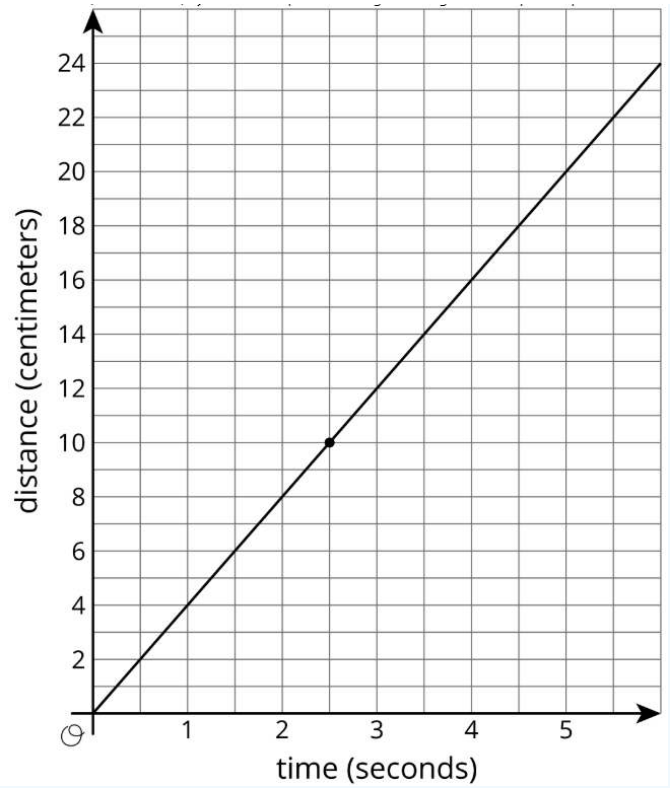


Figure 4.3.2.2: Graph of a line, origin O, with grid. Horizontal axis, time in seconds, scale 0 to 5, by 1's. Vertical axis, distance in centimeters, scale 0 to 24, by 2's. The line passes through the origin and the point 2 point 5 comma 10.

The ant is walking 2 centimeters per second.

1. Write an equation representing the relationship between the ant's distance from the ladybug;s starting point and the amount of time that has passed.
2. If you haven't already, draw the graph of your equation on the same coordinate plane.

Exercise 4.3.2.3: A Close Race

Elena and Jada were racing 100 meters on their bikes. Both racers started at the same time and rode at constant speed. Here is a table that gives information about Jada's bike race:

Table 4.3.2.1

time from start (seconds)	distance from start (meters)
6	36
9	54

1. Graph the relationship between distance and time for Jada's bike race. Make sure to label and scale the axes appropriately.

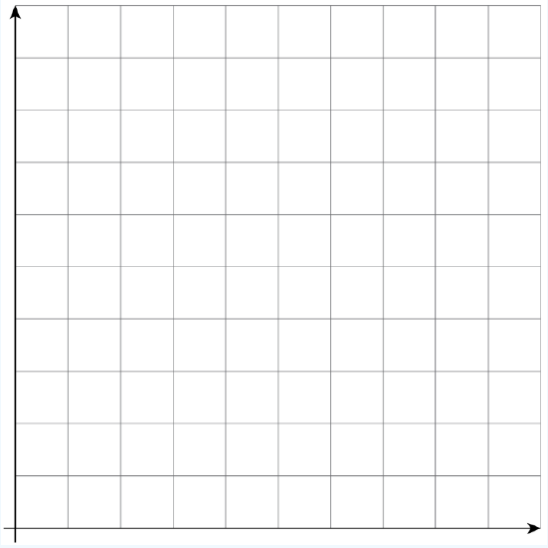


Figure 4.3.2.3

2. Elena traveled the entire race at a steady 6 meters per second. On the same set of axes, graph the relationship between distance and time for Elena's bike race.

3. Who won the race?

Summary

The solutions to an equation correspond to points on its graph. For example, if Car A is traveling 75 miles per hour and passes a rest area when $t = 0$, then the distance in miles it has traveled from the rest area after t hours is

$$d = 75t$$

The point $(2, 150)$ is on the graph of this equation because $150 = 75 \cdot 2$: two hours after passing the rest area, the car has traveled 150 miles.

If you have *two* equations, you can ask whether there is an ordered pair that is a solution to *both* equations simultaneously. For example, if Car B is traveling towards the rest area and its distance from the rest area is

$$d = 14 - 65t$$

We can ask if there is ever a time when the distance of Car A from the rest area is the same as the distance of Car B from the rest area. If the answer is "yes", then the solution will correspond to a point that is on both lines.

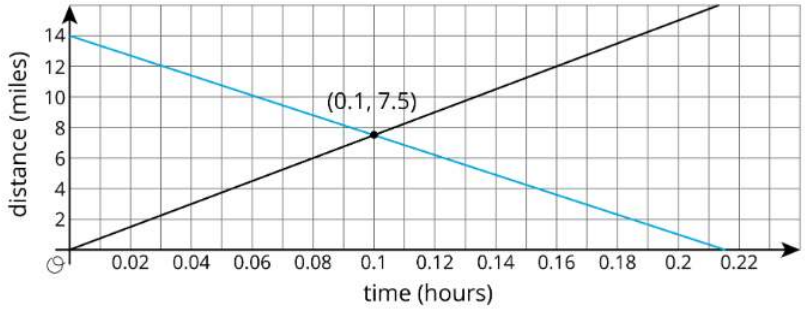


Figure 4.3.2.4: Graph of 2 lines, origin O, with grid. Horizontal axis, time in hours, scale 0 to point 22, by point 0 2's. Vertical axis, distance in miles, scale 0 to 14, by 2's. One line passes through the origin and the point 0 point 1 comma 7 point 5. Another line crosses the y axis at 14 and passes through the point 0 point 1 comma 7 point 5.

Looking at the coordinates of the intersection point, we see that Car A and Car B will both be 7.5 miles from the rest area after 0.1 hours (which is 6 minutes).

Now suppose another car, Car C, had also passed the rest stop at time $t = 0$ and traveled in the same direction as Car A, also going 75 miles per hour. It's equation would also be $d = 75t$. Any solution to the equation for Car A would also be a solution for Car C,

and any solution to the equation for Car C would also be a solution for Car A. The line for Car C would land right on top of the line for Car A. In this case, every point on the graphed line is a solution to both equations, so that there are infinitely many solutions to the question "when are Car A and Car C the same distance from the rest stop?" This would mean that Car A and Car C were side by side for their whole journey.

When we have two linear equations that are equivalent to each other, like $y = 3x + 2$ and $2y = 6x + 4$, we will get two lines that are "right on top" of each other. Any solution to one equation is also solution to the other, so these two lines intersect at infinitely many points.

Practice

Exercise 4.3.2.4

Diego has \$11 and begins saving \$5 each week toward buying a new phone. At the same time that Diego begins saving, Lin has \$60 and begins spending \$2 per week on supplies for her art class. Is there a week when they have the same amount of money? How much do they have at that time?

Exercise 4.3.2.5

Use a graph to find x and y values that make both $y = \frac{-2}{3}x + 3$ and $y = 2x - 5$ true.

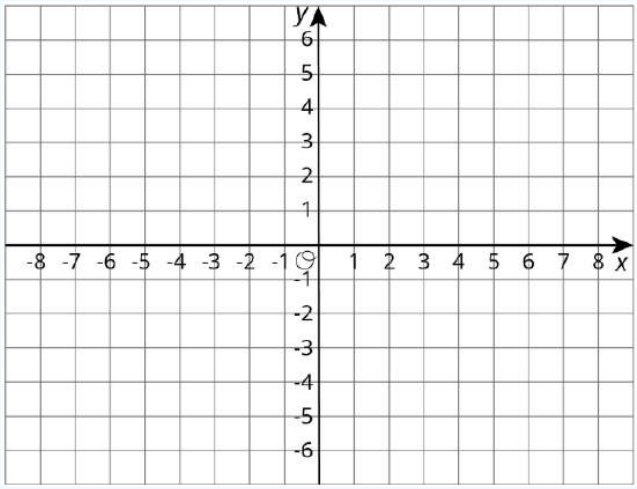


Figure 4.3.2.5

Exercise 4.3.2.6

The point where the graphs of two equations intersect has y-coordinate 2. One equation is $y = -3x + 5$. Find the other equation if its graph has a slope of 1.

Exercise 4.3.2.7

A farm has chickens and cows. All the cows have 4 legs and all the chickens have 2 legs. All together, there are 82 cow and chicken legs on the farm. Complete the table to show some possible combinations of chickens and cows to get 82 total legs.

Table 4.3.2.2

number of chickens (x)	number of cows (y)
35	
7	
	10
19	

number of chickens (x)	number of cows (y)
	5

Here is a graph that shows possible combinations of chickens and cows that add up to 30 animals:

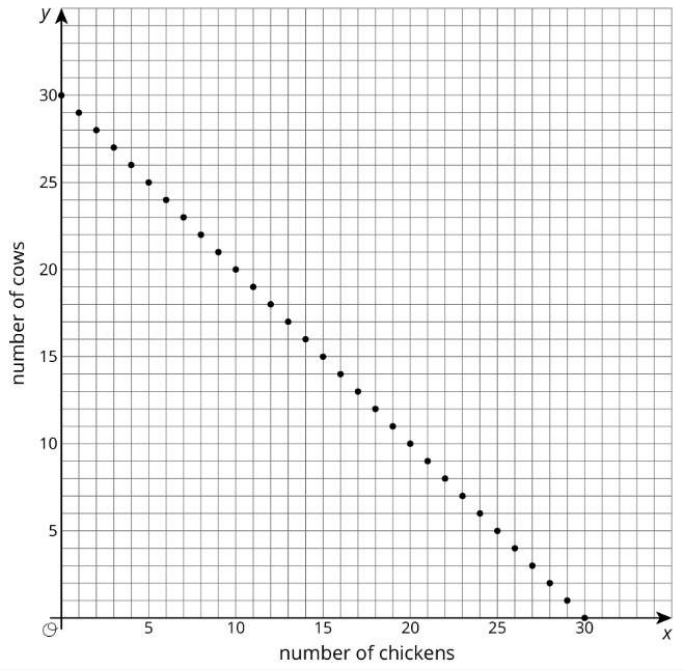


Figure 4.3.2.6: Graph of points in the x y plane, origin O, with grid. Horizontal axis, number of chickens, scale 0 to 30 by 1s. Vertical axis, number of cows, scale 0 to 30 by 1s. Points plotted are 0 comma 30, 1 comma 29, 2 comma 28, 3 comma 27, 4 comma 26, 5 comma 25, 6 comma 24, 7 comma 23, 8 comma 22, 9 comma 21, 10 comma 20, 11 comma 19, 12 comma 18, 13 comma 17, 14 comma 16, 15 comma 15, 16 comma 14, 17 comma 13, 18 comma 12, 19 comma 11, 20 comma 10, 21 comma 9, 22 comma 8, 23 comma 7, 24 comma 6, 25 comma 5, 26 comma 4, 27 comma 3, 28 comma 2, 29 comma 1, 30 comma 0.

If the farm has 30 chickens and cows, and there are 82 chicken and cow legs all together, then how many chickens and how many cows could the farm have?

(From Unit 4.3.2)

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4.3.3: Systems of Equations

Lesson

Let's learn what a system of equation is.

Exercise 4.3.3.1: Milkshakes

Diego and Lin are drinking milkshakes. Lin starts with 12 ounces and drinks $\frac{1}{4}$ an ounce per second. Diego starts with 20 ounces and drinks $\frac{2}{3}$ an ounce per second.

1. How long will it take Lin and Diego to finish their milkshakes?
2. Without graphing, explain what the graphs in this situation would look like. Think about slope, intercepts, axis labels, units, and intersection points to guide your thinking.
3. Discuss your description with your partner. If you disagree, work to reach an agreement.

Exercise 4.3.3.2: Passing on the Trail

There is a hiking trail near the town where Han and Jada live that starts at a parking lot and ends at a lake. Han and Jada both decide to hike from the parking lot to the lake and back, but they start their hikes at different times.

At the time that Han reaches the lake and starts to turn back, Jada is 0.6 miles away from the parking lot and hiking at a constant speed of 3.2 miles per hour toward the lake. Han's distance, d, from the parking lot can be expressed as $d = -2.4t + 4.8$, where t represents the time in hours since he left the lake.

1. What is an equation for Jada's distance from the parking lot as she heads toward the lake?
2. Draw both graphs: one representing Han's equation and one representing Jada's equation. It is important to be very precise.
3. Fnd the point where the two graphs intersect each other. What are the coordinates of this point?
4. What do the coordinates mean in this situation?
5. What has to be true about the relationship between these coordinates and Jada's equation?
6. What has to be true about the relationship between these coordinates and Han's equation?

Exercise 4.3.3.3: Stacks of Cups

A stack of n small cups has a height, h, in centimeters of $h = 1.5n + 6$. A stack of n large cups has a height, h, in centimeters of $h = 1.5n + 9$.

1. Graph the equations for each cup on the same set of axis. Make sure to label the axes and decide on an appropriate scale.
2. For what number of cops will the two stacks have the same height?

Summary

A **system of equations** is a set of 2 (or more) equations where the variables represent the same unknown values. For example, suppose that two different kinds of bamboo are planted at the same time. Plant A starts at 6 ft tall and grows at a constant rate of $\frac{1}{4}$ foot each day. Plant B starts at 3 ft tall and grows at a constant rate of $\frac{1}{2}$ foot each day. We can write equations $y = \frac{1}{4}x + 6$ for Plant A and $y = \frac{1}{2}x + 3$ for Plant B, where x represents the number of days after being planted, and represents height. We can write this system of equations.

$$\begin{cases} y = \frac{1}{4}x + 6 \\ y = \frac{1}{2}x + 3 \end{cases}$$

Solving a system of equations means to find the values of x and y that make both equations true at the same time. One way we have seen to find the solution to a system of equations is to graph both lines and find the intersection point. The intersection point represents the pair of x and y values that make both equations true. Here is a graph for the bamboo example:

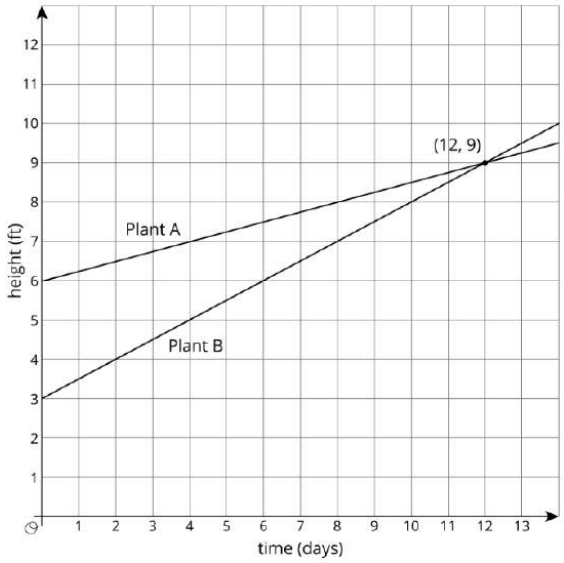


Figure 4.3.3.1: Graph of two lines, origin O, with grid. Horizontal axis, time in days, scale 0 to 13, by 1's. Vertical axis, height in feet, scale 0 to 12, by 1's. A line, labeled Plant A, crosses the y axis at 6. A line, labeled Plant B, crosses the y axis at 3. The lines intersect at the point 12 comma 9.

The solution to this system of equations is $(12, 9)$, which means that both bamboo plants will be 9 feet tall after 12 days.

We have seen systems of equations that have no solutions, one solution, and infinitely many solutions.

- When the lines do not intersect, there is no solution. (Lines that do not intersect are *parallel*.)
- When the lines intersect once, there is one solution.
- When the lines are right on top of each other, there are infinitely many solutions.

In future lessons, we will see that some systems cannot be easily solved by graphing, but can be easily solved using algebra.

Glossary Entries

Definition: System of Equations

A system of equations is a set of two or more equations. Each equation contains two or more variables. We want to find values for the variables that make all the equations true.

These equations make up a system of equations:

$$\begin{cases} x+y=-2 \\ x-y=12 \end{cases}$$

The solution to this system is $x=5$ and $y=-7$ because when these values are substituted for x and y, each equation is true: $5+(-7)=-2$ and $5-(-7)=12$.

Practice

Exercise 4.3.3.4

Here is the graph for one equation in a system of equations:

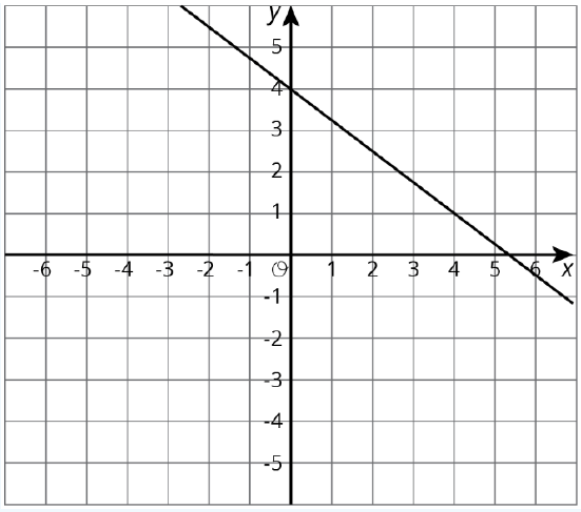


Figure 4.3.3.2: Graph of a line in the x y plane, origin O, with grid. Horizontal axis, scale negative 6 to 6, by 1's. Vertical axis, scale negative 6 to 6, by 1's. The line crosses the y axis at 4. It passes through the point 4 comma 1.

1. Write a second equation for the system so it has infinitely many solutions.
2. Write a second equation whose graph goes through $(0, 1)$ so the system has no solutions.
3. Write a second equation whose graph goes through $(0, 2)$ so the system has one solution at $(4, 1)$.

Exercise 4.3.3.5

Create a second equation so the system has no solutions.

$$\begin{cases} y = \frac{3}{4}x - 4 \\ \end{cases}$$

Exercise 4.3.3.6

Andre is in charge of cooking broccoli and zucchini for a large group. He has to spend all \$17 he has and can carry 10 pounds of veggies. Zucchini costs \$1.50 per pound and broccoli costs \$2 per pound. One graph shows combinations of zucchini and broccoli that weigh 10 pounds and the other shows combinations of zucchini and broccoli that cost \$17.

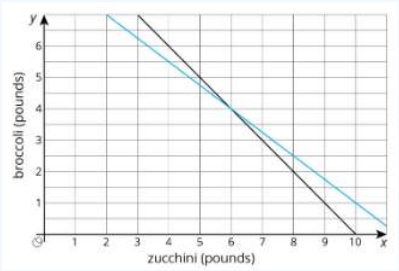


Figure 4.3.3.3: The graph of two intersecting lines in the x y plane. The first line slants downward and right and passes through the point 4 comma 5 point 5. The second line slants downward and to the right and passes through the point 5 comma 5. The lines intersect at the point 6 comma 4.

1. Name one combination of veggies that weighs 10 pounds but does not cost \$17.
2. Name one combination of veggies that costs \$17 but does not weigh 10 pounds.
3. How many pounds each of zucchini and broccoli can Andre get so that he spends all \$17 and gets 10 pounds of veggies?

(From Unit 4.3.2)

Exercise 4.3.3.7

The temperature in degrees Fahrenheit, F, is related to the temperature in degrees Celsius, C, by the equation $F = \frac{9}{5}C + 32$

1. In the Sahara desert, temperatures often reach 50 degrees Celsius. How many degrees Fahrenheit is this?
2. In parts of Alaska, the temperatures can reach -60 degrees Fahrenheit. How many degrees Celsius is this?
3. There is one temperature where the degrees Fahrenheit and degrees Celsius are the same, so that $C = F$. Use the expression from the equation, where F is expressed in terms of C, to solve for this temperature.

(From Unit 4.3.1)

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4.3.4: Solving Systems of Equations

Lesson

Let's solve systems of equations.

Exercise 4.3.4.1: True or False: Two Lines

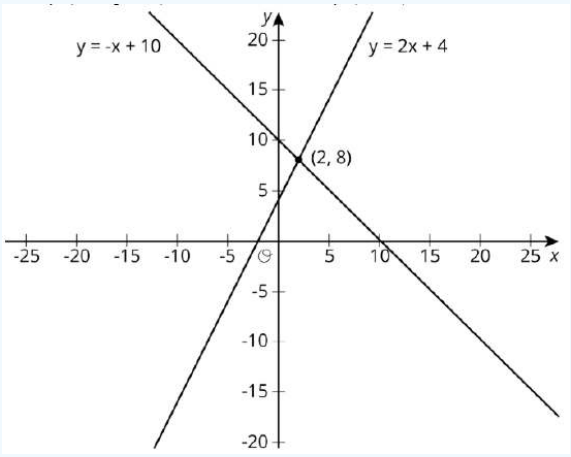


Figure 4.3.4.1: Graph of 2 lines, origin O. Horizontal axis, scale negative 25 to 25, by 5's. Vertical axis, scale negative 20 to 20, by 5's. A line is labeled y equals negative x plus 10. Another line is labeled y equals 2 x plus 4. The lines intersect at the point 2 comma 8.

Use the lines to decide whether each statement is true or false. Be prepared to explain your reasoning using the lines.

1. A solution to $8 = -x + 10$ is 2.
2. A solution to $2 = 2x + 4$ is 8.
3. A solution to $-x + 10 = 2x + 4$ is 8.
4. A solution to $-x + 10 = 2x + 4$ is 2.
5. There are no values of x and y that make $y = -x + 10$ and $y = 2x + 4$ true at the same time.

Exercise 4.3.4.2: Matching Graphs to Systems

Here are three systems of equations graphed on a coordinate plane:

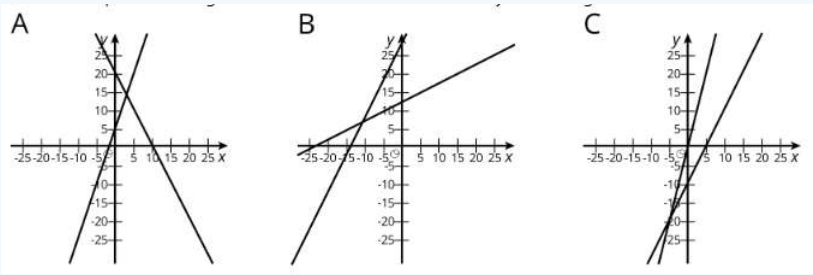


Figure 4.3.4.2: Three graphs, with two lines each, in the x y plane, origin O. All graphs have a scale of negative 25 to 25 on both the x and y axis. Graph A. One line slants downward and right. It crosses the x axis at 10 and the y axis at 20. Another line slants upward and right. It crosses the y axis at 5. It crosses the x axis to the left of the origin. Graph B. One line slants upward and right. It crosses the y axis between 25 and 30. It crosses the x axis between negative 10 and negative 15. Another line slants upward and right. It crosses the y axis between 10 and 15. It crosses the x axis between negative 20 and negative 25. Graph C. One line slants upward and right. It crosses the x axis between 0 and 5. It crosses the y axis between 0 and negative 5. Another line slants upward and right. It crosses the x axis at 5 and the y axis at negative 10.

1. Match each figure to one of the systems of equations shown here.
   a. $\begin{cases} y = 3x + 5 \\ y = -2x + 20 \end{cases}$
   b. $\begin{cases} y = 2x - 10 \\ y = 4x - 1 \end{cases}$

c. $\begin{cases} y = 0.5x + 12 \\ y = 2x + 27 \end{cases}$

2. Find the solution to each system and then check that your solution is reasonable on the graph.

- Notice that the sliders set the values of the coefficient and the constant term in each equation.
- Change the sliders to the values of the coefficient and the constant term in the next pair of equations.
- Click on the spot where the lines intersect and a labeled point should appear.

Exercise 4.3.4.3: Different Types of Systems

Your teacher will give you a page with 6 systems of equations.

1. Graph each system of equations by typing each pair of the equations in the applet, one at a time.
2. Describe what the graph of a system of equations looks like when it has . . .
   a. 1 solution
   b. 0 solutions
   c. infinitely many solutions

Use the applet to confirm your answer to question 2.

Are you ready for more?

The graphs of the equations $Ax + By = 15$ and $Ax - By = 9$ intersect at $(2, 1)$. Find A and B. Show or explain your reasoning.

Summary

Sometimes it is easier to solve a system of equations without having to graph the equations and look for an intersection point. In general, whenever we are solving a system of equations written as

$$\begin{cases} y = [\text{some stuff}] \\ y = [\text{some other stuff}] \end{cases}$$

we know that we are looking for a pair of values (x, y) that makes both equations true. In particular, we know that the value for y will be the same in both equations. That means that

$$[\text{some stuff}] = [\text{some other stuff}]$$

For example, look at this system of equations:

$$\begin{cases} y = 2x + 6 \\ y = -3x - 4 \end{cases}$$

Since the y value of the solution is the same in both equations, then we know $2x + 6 = -3x - 4$

We can solve this equation for x:

$$\begin{aligned} 2x + 6 &= -3x - 4 & \\ 5x + 6 &= -4 & \text{add 3x to each side} \\ 5x &= -10 & \text{subtract 6 from each side} \\ x &= -2 & \text{divide each side by 5} \end{aligned}$$

But this is only half of what we are looking for: we know the value for x, but we need the corresponding value for y. Since both equations have the same y value, we can use either equation to find the y-value:

$$y = 2(-2) + 6$$

Or

$$y = -3(-2) - 4$$

In both cases, we find that $y = 2$. So the solution to the system is $(-2, 2)$. We can verify this by graphing both equations in the coordinate plane.

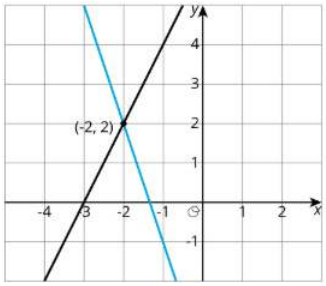


Figure 4.3.4.3: Graph of two lines line, origin O, with grid. Horizontal axis, x, scale negative 4 to 1, by 1s. Vertical axis, y, scale negative 1 to 4, by 1's. The lines intersect at the point negative 2 comma 2.

In general, a system of linear equations can have:

- No solutions. In this case, the lines that correspond to each equation never intersect.
- Exactly one solution. The lines that correspond to each equation intersect in exactly one point.
- An infinite number of solutions. The graphs of the two equations are the same line!

Glossary Entries

Definition: System of Equations

A system of equations is a set of two or more equations. Each equation contains two or more variables. We want to find values for the variables that make all the equations true.

These equations make up a system of equations:

$$\begin{cases} x+y=-2 \\ x-y=12 \end{cases}$$

The solution to this system is $x=5$ and $y=-7$ because when these values are substituted for x and y, each equation is true: $5+(-7)=-2$ and $5-(-7)=12$.

Practice

Exercise 4.3.4.4

1. Write equations for the lines shown.

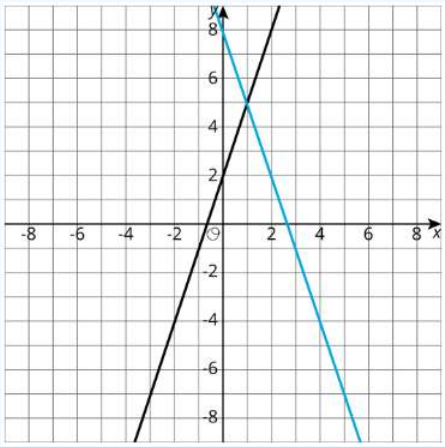


Figure 4.3.4.4: Graph of two intersecting lines in the xy-plane. Scale negative 8 through 8 on both axes. The first line slants upward and right, crosses the y axis at 2, and passes through the point 1 comma 5. The second line slants downward and to the right, crosses the y axis at 8, and passes through the point 1 comma 5.

2. Describe how to find the solution to the corresponding system by looking at the graph.

3. Describe how to find the solution to the corresponding system by using the equations.

Exercise 4.3.4.5

The solution to a system of equations is $(5, -19)$. Choose two equations that might make up the system.

A. $y = -3x - 6$
B. $y = 2x - 23$
C. $y = -7x + 16$
D. $y = x - 17$
E. $y = -2x - 9$

Exercise 4.3.4.6

Solve the system of equations:

$$\begin{cases} y = 4x - 3 \\ y = -2x + 9 \end{cases}$$

Exercise 4.3.4.7

Solve the system of equations:

$$\begin{cases} y = \frac{5}{4}x - 2 \\ y = -\frac{1}{4}x + 19 \end{cases}$$

Exercise 4.3.4.8

Here is an equation: $\frac{15(x-3)}{5} = 3(2x - 3)$

1. Solve the equation by using the distributive property first.
2. Solve the equation without using the distributive property.
3. Check your solution.

(From Unit 4.2.5)

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4.3.5: Solving More Systems

Lesson

Let's solve systems of equations.

Exercise 4.3.5.1: Algebra Talk: Solving Systems Mentally

Solve these without writing anything down:

$$\begin{cases} x = 5 \\ y = x - 7 \end{cases}$$

$$\begin{cases} y = 4 \\ y = x + 3 \end{cases}$$

$$\begin{cases} x = 8 \\ y = -11 \end{cases}$$

Exercise 4.3.5.2: Challenge Yourself

Here are a lot of systems of equations:

A. $\begin{cases} y = 4 \\ x = -5y + 6 \end{cases}$

B. $\begin{cases} y = 7 \\ x = 3y - 4 \end{cases}$

C. $\begin{cases} y = \frac{3}{2}x + 7 \\ x = -4 \end{cases}$

D. $\begin{cases} y = -3x + 10 \\ y = -2x + 6 \end{cases}$

E. $\begin{cases} y = -3x - 5 \\ y = 4x + 30 \end{cases}$

F. $\begin{cases} y = 3x - 2 \\ y = -2x + 8 \end{cases}$

G. $\begin{cases} y = 3x \\ x = -2y + 56 \end{cases}$

H. $\begin{cases} x = 2y - 15 \\ y = -2x \end{cases}$

I. $\begin{cases} 3x + 4y = 10 \\ x = 2y \end{cases}$

J. $\begin{cases} y = 3x + 2 \\ 2x + y = 47 \end{cases}$

K. $\begin{cases} y = -2x + 5 \\ 2x + 3y = 31 \end{cases}$

L. $\begin{cases} x + y = 10 \\ x = 2y + 1 \end{cases}$

1. Without solving, identify 3 systems that you think would be the least difficult to solve and 3 systems that you think would be the most difficult to solve. Be prepared to explain your reasoning.
2. Choose 4 systems to solve. At least one should be from your "least difficult" list and one should be from your "most difficult" list.

Exercise 4.3.5.3: Five Does Not Equal Seven

Tyler was looking at this system of equations:

$$\begin{cases} x+y=5 \\ x+y=7 \end{cases}$$

He said, "Just looking at the system, I can see it has no solution. If you add two numbers, that sum can't be equal to two different numbers."

Do you agree with Tyler?

Are you ready for more?

In rectangle $ABCD$, side AB is 8 centimeters and side BC is 6 centimeters. F is a point on BC and E is a point on AB. The area of triangle DFC is 20 square centimeters, and the area of triangle DEF is 16 square centimeters. What is the area of triangle AED?

Summary

When we have a system of linear equations where one of the equations is of the form $y =$ [stuff] or $x =$ [stuff], we can solve it algebraically by using a technique called *substitution*. The basic idea is to replace a variable with an expression it is equal to (so the expression is like a substitute for the variable). For example, let's start with the system:

$$\begin{cases} y=5x \\ 2x-y=9 \end{cases}$$

Since we know that $y=5x$, we can substitute $5x$ for y in the equation $2x-y=9$,

$$2x-(5x)=9,$$

and then solve the equation for x,

$$x=-3.$$

We can find y using either equation. Using the first one: $y=5\cdot -3$. So

$(-3,15)$

is the solution to this system. We can verify this by looking at the graphs of the equations in the system:

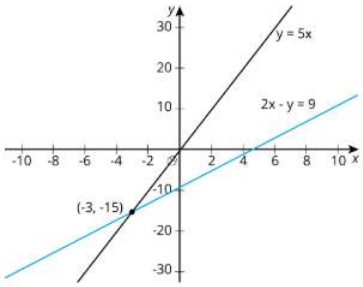


Figure 4.3.5.1: Graph of two lines, origin O, with grid. Horizontal axis, x, scale negative 10 to 10, by 2's. Vertical axis, y, scale negative 30 to 30, by 10's. One line is labeled as y equals 5 x. Another line is labeled as 2 x minus y equals 9. The lines intersect at negative 3 comma negative 15.

Sure enough! They intersect at $(-3,-15)$.

We didn't know it at the time, but we were actually using substitution in the last lesson as well. In that lesson, we looked at the system

$$\begin{cases} y=2x+6 \\ y=-3x-4 \end{cases}$$

and we substituted $2x+6$ for y into the second equation to get $2x+6=-3x-4$. Go back and check for yourself!

Glossary Entries

Definition: System of Equations

A system of equations is a set of two or more equations. Each equation contains two or more variables. We want to find values for the variables that make all the equations true.

These equations make up a system of equations:

$$\begin{cases} x+y=-2 \\ x-y=12 \end{cases}$$

The solution to this system is $x=5$ and $y=-7$ because when these values are substituted for x and y, each equation is true: $5+(-7)=-2$ and $5-(-7)=12$.

Practice

Exercise 4.3.5.4

Solve:

$$\begin{cases} y=6x \\ 4x+y=7 \end{cases}$$

Exercise 4.3.5.5

Solve:

$$\begin{cases} y=3x \\ x=-2y+70 \end{cases}$$

Exercise 4.3.5.6

A. $y=-1.5x+6$
B. $y=-1.5x$
C. $2y=-3x+6$
D. $2y+3x=6$
E. $y=-2x+3$

Exercise 4.3.5.7

The system $x=6y=4$, $3x-18y=4$ has no solution.

1. Change one constant or coefficient to make a new system with one solution.
2. Change one constant or coefficient to make a new system with an infinite number of solutions.

Exercise 4.3.5.8

Match each graph to its equation.

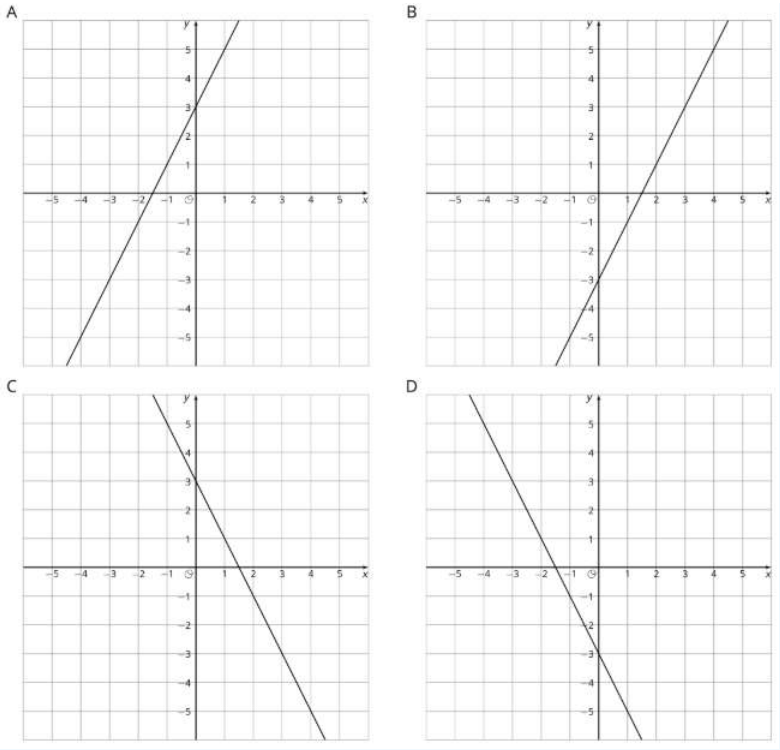


Figure 4.3.5.2: Four graphs, each with a line in the x y plane. Graph A. The line crosses the y axis at 3 and passes through the point negative 1 comma 1. Graph B. The line crosses the y axis at negative 3 and passes through the point 1 comma negative 1. Graph C. The line crosses the y axis at 3 and passes through the point 1 comma 1. Graph D. The line crosses the y axis at negative 3 and passes through the point negative 1 comma negative 1.

1. $y = 2x + 3$
2. $y = -2x + 3$
3. $y = 2x - 3$
4. $y = -2x - 3$

(From Unit 3.3.3)

Exercise 4.3.5.9

Here are two points: $(-3, 4)$, $(1, 7)$. What is the slope of the line between them?

A. $\frac{4}{3}$
B. $\frac{3}{4}$
C. $\frac{1}{6}$
D. $\frac{2}{3}$

(From Unit 3.3.2)

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4.3.6: Writing Systems of Equations

Lesson

Let's write systems of equations from real-world situations.

Exercise 4.3.6.1: How Many Solutions? Matching

Match each system of equations with the number of solutions the system has.

1. $\begin{cases} y = -\frac{4}{3}x + 4 \\ y = -\frac{4}{3}x - 1 \end{cases}$
2. $\begin{cases} y = 4x - 5 \\ y = -2x + 7 \end{cases}$
3. $\begin{cases} 2x + 3y = 8 \\ 4x + 6y = 17 \end{cases}$
4. $\begin{cases} y = 5x - 15 \\ y = 5(x - 3) \end{cases}$

1. No solutions
2. One solution
3. Infinitely many solutions

Exercise 4.3.6.2: Situations and Systems

For each situation:

- Create a system of equations.
- Then, without solving, interpret what the solution to the system would tell you about the situation.

1. Lin's family is out for a bike ride when her dad stops to take a picture of the scenery. He tells the rest of the family to keep going and that he'll catch up. Lin's dad spends 5 minutes taking the photo and then rides at 0.24 miles per minute until he meets up with the rest of the family further along the bike path. Lin and the rest were riding at 0.18 miles per minute.
2. Noah is planning a kayaking trip. Kayak Rental A charges a base fee of $15 plus $4.50 per hour. Kayak Rental B charges a base fee of $12.50 plus $5 per hour.
3. Diego is making a large batch of pastries. The recipe calls for 3 strawberries for every apple. Diego used 52 fruits all together.
4. Flour costs $0.80 per pound and sugar costs $0.50 per pound. An order of flour and sugar weighs 15 pounds and costs $9.00.

Exercise 4.3.6.3: Info Gap: Racing and Play Tickets

Your teacher will give you either a *problem card* or a *data card*. Do not show or read your card to your partner.

If your teacher gives you the *problem card*:

1. Silently read your card and think about what information you need to be able to answer the question.
2. Ask your partner for the specific information that you need.
3. Explain how you are using the information to solve the problem.
   Continue to ask questions until you have enough information to solve the problem.
4. Share the *problem card* and solve the problem independently.
5. Read the *data card* and discuss your reasoning.

If your teacher gives you the *data card*:

1. Silently read your card.

2. Ask your partner *"What specific information do you need?"* and wait for them to *ask* for information.
   If your partner asks for information that is not on the card, do not do the calculations for them. Tell them you don't have that information.
3. Before sharing the information, ask "*Why do you need that information?*" Listen to your partner's reasoning and ask clarifying questions.
4. Read the *problem card* and solve the problem independently.
5. Share the *data card* and discuss your reasoning.

Exercise 4.3.6.4: Solving Systems Practice

Here are a lot of systems of equations:

- $\begin{cases} y = -2x + 6 \\ y = x - 3 \end{cases}$
- $\begin{cases} y = 5x - 4 \\ y = 4x + 12 \end{cases}$
- $\begin{cases} y = \frac{2}{3}x - 4 \\ y = -\frac{4}{3}x + 9 \end{cases}$
- $\begin{cases} 4y + 7x = 6 \\ 4y + 7x = -5 \end{cases}$
- $\begin{cases} y = x - 6 \\ x = 6 + y \end{cases}$
- $\begin{cases} y = 0.24x \\ y = 0.18x + 0.9 \end{cases}$
- $\begin{cases} y = 4.5x + 15 \\ y = 5x + 12.5 \end{cases}$
- $\begin{cases} y = 3x \\ +y = 52 \end{cases}$

1. Without solving, identify 3 systems that you think would be the least difficult for you to solve and 3 systems you think would be the most difficult. Be prepared to explain your reasoning.
2. Choose 4 systems to solve. At least one should be from your "least difficult" list and one should be from your "most difficult" list.

Summary

We have learned how to solve many kinds of systems of equations using algebra that would be difficult to solve by graphing. For example, look at

$$\begin{cases} y = 2x - 3 \\ x + 2y = 7 \end{cases}$$

The first equation says that $y = 2x - 3$, so wherever we see y, we can substitute the expression $2x - 3$ instead. So the second equation becomes $x + 2(2x - 3) = 7$.

We can solve for x:

$$\begin{aligned} x + 4x - 6 &= 7 && \text{distributive property} \\ 5x - 6 &= 7 && \text{combine like terms} \\ 5x &= 13 && \text{add 6 to each side} \\ x &= \frac{13}{5} && \text{multiply each side by } \frac{1}{5} \end{aligned}$$

We know that the y value for the solution is the same for either equation, so we can use either equation to solve for it. Using the first equation, we get:

$$
\begin{aligned}
y &= 2\left(\frac{13}{5}\right) - 3 && \text{substitute x=}\frac{13}{5}\text{into the equation} \\
y &= \frac{26}{5} - 3 && \text{multiply } 2\left(\frac{13}{5}\right) \text{ to make } \frac{26}{5} \\
y &= \frac{26}{5} - \frac{15}{5} && \text{rewrite 3 as } \frac{15}{5} \\
y &= \frac{11}{5}
\end{aligned}
$$

If we substitute $x = \frac{13}{5}$ into the other equation, $x + 2y = 7$, we get the same y value. So the solution to the system is $\left(\frac{13}{5}, \frac{11}{5}\right)$.

There are many kinds of systems of equations that we will learn how to solve in future grades, like

$$
\begin{cases} 2x + 3y = 6 \\ -x + 2y = 3 \end{cases}
$$

Or even

$$
\begin{cases} y = x^2 + 1 \\ y = 2x + 3 \end{cases}
$$

Glossary Entries

Definition: System of Equations

A system of equations is a set of two or more equations. Each equation contains two or more variables. We want to find values for the variables that make all the equations true.

These equations make up a system of equations:

$$
\begin{cases} x + y = -2 \\ x - y = 12 \end{cases}
$$

The solution to this system is $x = 5$ and $y = -7$ because when these values are substituted for x and y, each equation is true: $5 + (-7) = -2$ and $5 - (-7) = 12$.

Practice

Exercise 4.3.6.5

Kiran and his cousin work during the summer for a landscaping company. Kiran's cousin has been working for the company longer, so his pay is 30% more than Kiran's. Last week his cousin worked 27 hours, and Kiran worked 23 hours. Together, they earned $493.85. What is Kiran's hourly pay? Explain or show your reasoning.

Exercise 4.3.6.6

Decide which story can be represented by the system of equations $y = x + 6$ and $x + y = 100$. Explain your reasoning.

1. Diego's teacher writes a test worth 100 points. There are 6 more multiple choice questions than short answer questions.
2. Lin and her younger cousin measure their heights. They notice that Lin is 6 inches taller, and their heights add up to exactly 100 inches.

Exercise 4.3.6.7

Clare and Noah play a game in which they earn the same number of points for each goal and lose the same number of points for each penalty. Clare makes 6 goals and 3 penalties, ending the game with 6 points. Noah earns 8 goals and 9 penalties and ends the game with -22 points.

1. Write a system of equations that describes Clare and Noah's outcomes. Use x to represent the number of points for a goal and y to represent the number of points for a penalty.

2. Solve the system. What does your solution mean?

Exercise 4.3.6.8

Solve:

$$\begin{cases} y = 6x - 8 \\ y = -3x + 10 \end{cases}$$

(From Unit 4.3.5)

Exercise 4.3.6.9

1. Estimate the coordinates of the point where the two lines meet.

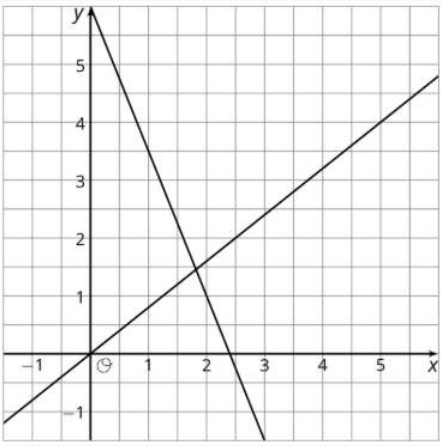


Figure 4.3.6.1: The graph of two intersecting lines in the xy-plane. Scale negative 1 through 5 on both axes. The first line slants upward and right, crosses y axis at 0, and passes through the point 3 comma 2. The second line slants downward and to the right, crosses the y axis at 6, and passes through the point 2 comma 1.

2. Choose two equations that make up the system represented by the graph.

a. $y = \frac{5}{4}x$
b. $y = 6 - 2.5x$
c. $y = 2.5x + 6$
d. $y = 6 - 3x$
e. $y = 0.8x$

Exercise 4.3.6.10

Solve the system of equations and confirm the accuracy of your estimate.

(From Unit 4.3.4)

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4.4: Let's Put it to Work

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4.4.1: Solving Problems with Systems of Equations

Lesson

Let's solve some gnarly problems.

Exercise 4.4.1.1: Are We There Yet?

A car is driving towards home at 0.5 miles per minute. If the car is 4 miles from home at $t = 0$, which of the following can represent the distance that the car has left to drive?

- $0.5t$
- $4 + 0.5t$
- $4 - 0.5t$
- $4 \cdot (0.5t)$

___?

Solve each problem. Explain or show your reasoning.

1. Two friends live 7 miles apart. One Saturday, the two friends set out on their bikes at 8 am and started riding towards each other. One rides at 0.2 miles per minute, and the other rides at 0.15 miles per minute. At what time will the two friends meet?
2. Students are selling grapefruits and nuts for a fundraiser. The grapefruits cost $1 each and a bag of nuts cost $10 each. They sold 100 items and made $307. How many grapefruits did they sell?
3. Jada earns $7 per hour mowing her neighbors' lawns. Andre gets paid $5 per hour for the first hour of babysitting and $8 per hour for any additional hours he babysits. What is the number of hours they both can work so that they get paid the same amount?
4. Pause here so your teacher can review your work. Then, invent another problem that is like one of these, but with different numbers. Solve your problem.
5. Create a visual display that includes:
   a. Enough work space for someone to show a solution.
   b. The new problem you wrote, without the solution.
6. Trade your display with another group, and solve each other's new problem. Make sure that you explain your solution carefully. Be prepared to share this solution with the class.
7. When the group that got the problem you invented shares their solution, check that their answer is correct.

Are you ready for more?

On a different Saturday, two friends set out on bikes at 8:00 am and met up at 8:30 am. (The same two friends who live 7 miles apart.) If one was riding at 10 miles per hour, how fast was the other riding?

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CHAPTER OVERVIEW

5: Functions and Volume

In this unit, students are introduced to the concept of a function. They learn to understand and use the terms “input,” “output,” and “function,” e.g., “temperature is a function of time.” They describe functions as increasing or decreasing between specific numerical inputs, and they consider the inputs of a function to be values of its independent variable and its outputs to be values of its dependent variable. (The terms “Independent variable” and “dependent variable” were introduced in grade 6.) They use tables, equations, and graphs to represent functions, and describe information presented in tables, equations, or graphs in terms of functions. In working with linear functions, students coordinate and synthesize their understanding of “constant of proportionality” (which was introduced in grade 7), “rate of change” and “slope” (which were introduced earlier in grade 8), and increasing and decreasing. Students perceive similarities in structure between pairs of known and new volume formulas: for a rectangular prism and a cylinder; and for a cylinder and a cone. Students rearrange these formulas to show functional relationships and use them to reason about how the volume of a figure changes as another measurement changes, e.g., the height of a cylinder is proportional to its volume; if the radius of a cylinder triples, its volume becomes nine times larger.

- 5.1: Inputs and Outputs
  - 5.1.1: Inputs and Outputs
  - 5.1.2: Introduction to Functions
- 5.2: Representing and Interpreting Functions
  - 5.2.1: Equations for Functions
  - 5.2.2: Tables, Equations, and Graphs of Functions
  - 5.2.3: More Graphs of Functions
  - 5.2.4: Even More Graphs of Functions
  - 5.2.5: Connecting Representations of Functions
- 5.3: Linear Functions and Rates of Change
  - 5.3.1: Linear Functions
  - 5.3.2: Linear Models
  - 5.3.3: Piecewise Linear Functions
- 5.4: Cylinders and Cones
  - 5.4.1: Filling Containers
  - 5.4.2: How Much Will Fit?
  - 5.4.3: The Volume of a Cylinder
  - 5.4.4: Finding Cylinder Dimensions
  - 5.4.5: The Volume of a Cone
  - 5.4.6: Finding Cone Dimensions
- 5.5: Dimensions and Spheres
  - 5.5.1: Scaling One Dimension
  - 5.5.2: Scaling Two Dimensions
  - 5.5.3: Estimating a Hemisphere
  - 5.5.4: The Volume of a Sphere
  - 5.5.5: Cylinders, Cones, and Spheres
- 5.6: Let's Put It to Work
  - 5.6.1: Volume as a Function of ...

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5.1: Inputs and Outputs

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5.1.1: Inputs and Outputs

Lesson

Let's make some rules.

Exercise 5.1.1.1: Dividing by 0

Study the statements carefully.

- $12 \div 3 = 4$ because $12 = 4 \cdot 3$
- $6 \div 0 = x$ because $6 = x \cdot 0$

What value can be used in place of x to create true statements? Explain your reasoning.

Exercise 5.1.1.2: Guess My Rule

Try to figure out what's happening in the “black box.”

Note: You must hit enter or return before you click GO.

Are you ready for more?

If you have a rule, you can apply it several times in a row and look for patterns. For example, if your rule was "add 1" and you started with the number 5, then by applying that rule over and over again you would get 6, then 7, then 8, etc., forming an obvious pattern.

Try this for the rules in this activity. That is, start with the number 5 and apply each of the rules a few times. Do you notice any patterns? What if you start with a different starting number?

Exercise 5.1.1.3: Making Tables

For each input-output rule, fill in the table with the outputs that go with a given input. Add two more input-output pairs to the table.

1.

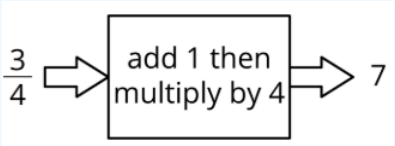


Figure 5.1.1.1

Table 5.1.1.1

input	output
$\frac{3}{4}$	7
2.35	
42	

2.

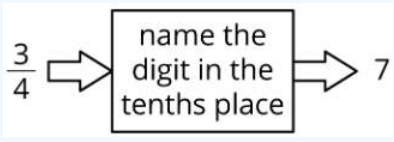


Figure 5.1.1.2

Table 5.1.1.2

input	output
$\frac{3}{4}$	7
2.35	
42	

3.

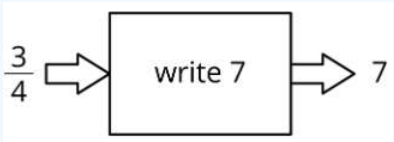


Figure 5.1.1.3

Table 5.1.1.3

input	output
$\frac{3}{4}$	7
2.35	
42	

Pause here until your teacher directs you to the last rule.

4.

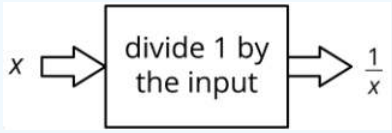


Figure 5.1.1.4

Table 5.1.1.4

input	output
$\frac{3}{7}$	$\frac{7}{3}$
1	
0	

Summary

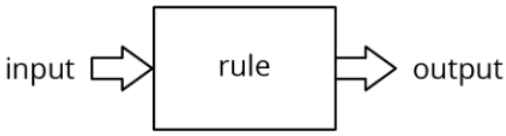


Figure 5.1.1.5

An *input-output rule* is a rule that takes an allowable input and uses it to determine an output. For example, the following diagram represents the rule that takes any number as an input, then adds 1, multiplies by 4, and gives the resulting number as an output.

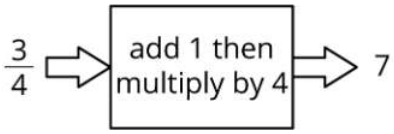


Figure 5.1.1.6

In some cases, not all inputs are allowable, and the rule must specify which inputs will work. For example, this rule is fine when the input is 2:

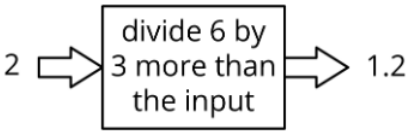


Figure 5.1.1.7

But if the input is -3, we would need to evaluate $6 \div 0$ to get the output.

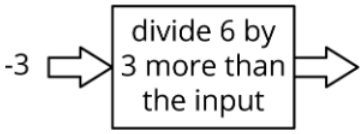


Figure 5.1.1.8

So, when we say that the rule is “divide 6 by 3 more than the input,” we also have to say that -3 is not allowed as an input.

Practice

Exercise 5.1.1.4

Given the rule:

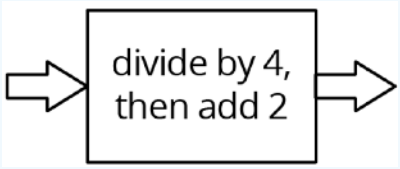


Figure 5.1.1.9

Complete the table for the function rule for the following input values:

Table 5.1.1.1

input	0	2	4	6	8	10
output						

Exercise 5.1.1.5

Here is an input-output rule:

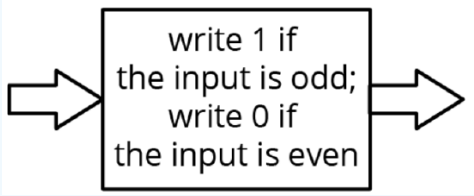


Figure 5.1.1.10

Complete the table for the input-output rule:

Table 5.1.1.2

input	-3	-2	-1	0	1	2	3
output							

Exercise 5.1.1.6

Andre's school orders some new supplies for the chemistry lab. The online store shows a pack of 10 test tubes costs $4 less than a set of nested beakers. In order to fully equip the lab, the school orders 12 sets of beakers and 8 packs of test tubes.

1. Write an equation that shows the cost of a pack of test tubes, t, in terms of the cost of a set of beakers, b.
2. The school office receives a bill for the supplies in the amount of $348. Write an equation with t and b that describes this situation.
3. Since t is in terms of b from the first equation, this expression can be substituted into the second equation where t appears. Write an equation that shows this substitution.
4. Solve the equation for b.
5. How much did the school pay for a set of beakers? For a pack of test tubes?

(From Unit 4.3.6)

Exercise 5.1.1.7

Solve:

$$\begin{cases} y = x - 4 \\ y = 6x - 10 \end{cases}$$

Exercise 5.1.1.8

For what value of x do the expressions $2x + 3$ and $3x - 6$ have the same value?

(From Unit 4.2.8)

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5.1.2: Introduction to Functions

Lesson

Let's learn what a function is.

Exercise 5.1.2.1: Square Me

Here are some numbers in a list:

$$1, -3, -\frac{1}{2}, 3, 2, \frac{1}{4}, 0.5$$

1. How many different numbers are in the list?
2. Make a new list containing the squares of all these numbers.
3. How many different numbers are in the new list?
4. Explain why the two lists do not have the same number of different numbers.

Exercise 5.1.2.2: You Know This, Do You Know That?

Say yes or no for each question. If yes, draw an input-output diagram. If no, give examples of two different outputs that are possible for the same input.

1. A person is 5.5 feet tall. Do you know their height in inches?
2. A number is 5. Do you know its square?
3. The square of a number is 16. Do you know the number?
4. A square has a perimeter of 12 cm. Do you know its area?
5. A rectangle has an area of 16 cm^2. Do you know its length?
6. You are given a number. Do you know the number that is $\frac{1}{5}$ as big?
7. You are given a number. Do you know its reciprocal?

Exercise 5.1.2.3: Using Function Languages

Here are the questions from the previous activity. For the ones you said yes to, write a statement like, “The height a rubber ball bounces to depends on the height it was dropped from” or “Bounce height is a **function** of drop height.” For all of the ones you said no to, write a statement like, “The day of the week does not determine the temperature that day” or “The temperature that day is not a function of the day of the week.”

1. A person is 5.5 feet tall. Do you know their height in inches?
2. A number is 5. Do you know its square?
3. The square of a number is 16. Do you know the number?
4. A square has a perimeter of 12 cm. Do you know its area?
5. A rectangle has an area of 16 cm^2. Do you know its length?
6. You are given a number. Do you know the number that is $\frac{1}{5}$ as big?
7. You are given a number. Do you know its reciprocal?

Exercise 5.1.2.4: Same Function, Different Rule?

Which input-output rules could describe the same function (if any)? Be prepared to explain your reasoning.

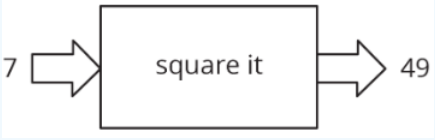


Figure 5.1.2.1

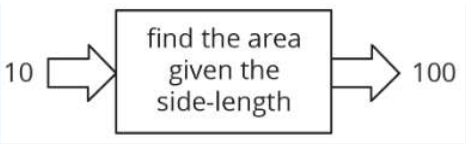


Figure 5.1.2.2

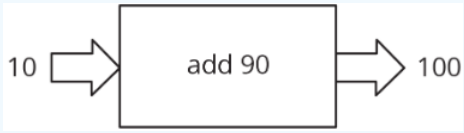


Figure 5.1.2.3

Are you ready for more?

The phrase “is a function of” gets used in non-mathematical speech as well as mathematical speech in sentences like, “The range of foods you like is a function of your upbringing.” What is that sentence trying to convey? Is it the same use of the word “function” as the mathematical one?

Summary

Let's say we have an input-output rule that for each allowable input gives exactly one output. Then we say the output *depends* on the input, or the output is a **function** of the input.

For example, the area of a square is a function of the side length, because you can find the area from the side length by squaring it. So when the input is 10 cm, the output is 100 cm^2.

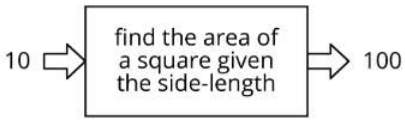


Figure 5.1.2.4

Sometimes we might have two different rules that describe the same function. As long as we always get the same, single output from the same input, the rules describe the same function.

Glossary Entries

Definition: Function

A function is a rule that assigns exactly one output to each possible input.

The function $y = 6x + 4$ assigns one value of the output, y, to each value of the input, x. For example, when x is 5, then $y = 6(5) + 4$ or 34.

Practice

Exercise 5.1.2.5

Here are several function rules. Calculate the output for each rule when you use -6 as the input.

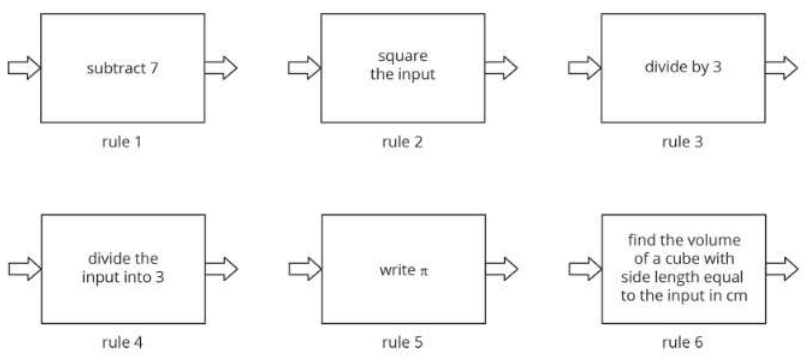


Figure 5.1.2.5: Six function rule diagrams, no input or output given for any. Rule 1, subtract 7. Rule 2, square the input. Rule 3, divide by 3. Rule 4, divide the input into 3. Rule 5, write pi. Rule 6, find the volume of a cube with side length equal to the input in centimeters.

Exercise 5.1.2.6

A group of students is timed while sprinting 100 meters. Each student's speed can be found by dividing 100 m by their time. Is each statement true or false? Explain your reasoning.

1. Speed is a function of time.
2. Time is a function of distance.
3. Speed is a function of number of students racing.
4. Time is a function of speed.

Exercise 5.1.2.7

Diego's history teacher writes a test for the class with 26 questions. The test is worth 123 points and has two types of questions: multiple choice worth 3 points each, and essays worth 8 points each. How many essay questions are on the test? Explain or show your reasoning.

(From Unit 4.3.6)

Exercise 5.1.2.8

These tables correspond to inputs and outputs. Which of these input and output tables could represent a function rule, and which ones could not? Explain or show your reasoning.

Table A:

Table 5.1.2.1

input	output
-2	4
-1	1
0	0
1	1
2	4

Table B:

Table 5.1.2.2

input	output
4	-2

input	output
1	-1
0	0
1	1
4	2

Table C:

Table 5.1.2.3

input	output
1	0
2	0
3	0

Table D:

Table 5.1.2.4

input	output
0	1
0	2
0	3

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5.2: Representing and Interpreting Functions

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5.2.1: Equations for Functions

Lesson

Let's find outputs from equations.

Exercise 5.2.1.1: A Square's Area

Fill in the table of input-output pairs for the given rule. Write an algebraic expression for the rule in the box in the diagram.



Figure 5.2.1.1

Table 5.2.1.1

input	output
8	
2.2	
$12\frac{1}{4}$	
s	

Exercise 5.2.1.2: Diagrams, Equations, and Descriptions

Record your answers to these questions in the table provided.

1. Match each of these descriptions with a diagram:
   a. the circumference, C, of a circle with **radius**, r
   b. the distance in miles, d, that you would travel in t hours if you drive at 60 miles per hour
   c. the output when you triple the input and subtract 4
   d. the volume of a cube, v given its edge length, s
2. Write an equation for each description that expresses the output as a function of the input.
3. Find the output when the input is 5 for each equation.
4. Name the **independent** and **dependent variables** of each equation.

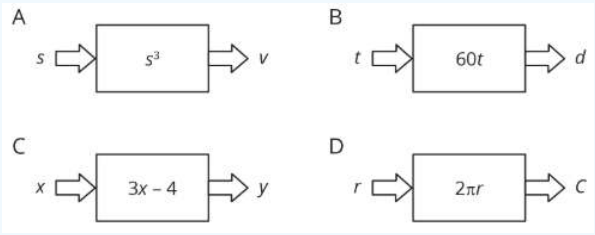


Figure 5.2.1.2: Four input-output rule diagrams. Diagram A, s, right arrow, rule is, s cubed, right arrow, v. Diagram B, t, rule is, 60 t, right arrow, d. Diagram C, x, right arrow, rule is, 3 x minus 4, right arrow, y. Diagram D, r, right arrow, 2 pi r, right arrow, C.

Table 5.2.1.2

description	a	b	c	d
diagram				
equation				

description	a	b	c	d
input = 5 **output = ?**				
independent variable				
dependent variable				

Are you ready for more?

Choose a 3-digit number as an input.

Apply the following rule to it, one step at a time:

- Multiply your number by 7.
- Add one to the result.
- Multiply the result by 11.
- Subtract 5 from the result.
- Multiply the result by 13
- Subtract 78 from the result to get the output.

Can you describe a simpler way to describe this rule? Why does this work?

Exercise 5.2.1.3: Dimes and Quarters

Jada had some dimes and quarters that had a total value of \$12.50. The relationship between the number of dimes, d, and the number of quarters, q, can be expressed by the equation $0.1d + 0.25q = 12.5$.

1. If Jada has 4 quarters, how many dimes does she have?
2. If Jada has 10 quarters, how many dimes does she have?
3. Is the number of dimes a function of the number of quarters? If yes, write a rule (that starts with $d =$...) that you can use to determine the output, d, from a given input, q. If no, explain why not.
4. If Jada has 25 dimes, how many quarters does she have?
5. If Jada has 30 dimes, how many quarters does she have?
6. Is the number of quarters a function of the number of dimes? If yes, write a rule (that starts with $q =$...) that you can use to determine the output, q, from a given input, d. If no, explain why not.

Summary

We can sometimes represent functions with equations. For example, the area, A, of a circle is a function of the radius, r, and we can express this with an equation: $A = \pi r^2$

We can also draw a diagram to represent this function:

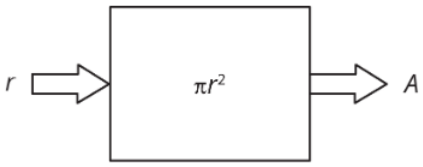


Figure 5.2.1.3

In this case, we think of the radius, r, as the input, and the area of the circle, A, as the output. For example, if the input is a radius of 10 cm, then the output is an area of 100π cm^2, or about 314 square cm. Because this is a function, we can find the area, A, for any given radius, r.

Since it is the input, we say that r is the **independent variable** and, as the output, A is the **dependent variable**.

Sometimes when we have an equation we get to choose which variable is the independent variable. For example, if we know that

$$10A - 4B = 120$$

then we can think of A as a function of B and write

$$A = 0.4B + 12$$

or we can think of B as a function of A and write

$$B = 2.5A - 30$$

Glossary Entries

Definition: Dependent Variable

A dependent variable represents the output of a function.

For example, suppose we need to buy 20 pieces of fruit and decide to buy apples and bananas. If we select the number of apples first, the equation $b = 20 - a$ shows the number of bananas we can buy. The number of bananas is the dependent variable because it depends on the number of apples.

Definition: Independent Variable

An independent variable represents the input of a function.

For example, suppose we need to buy 20 pieces of fruit and decide to buy some apples and bananas. If we select the number of apples first, the equation $b = 20 - a$ shows the number of bananas we can buy. The number of apples is the independent variable because we can choose any number for it.

Definition: Radius

A radius is a line segment that goes from the center to the edge of a circle. A radius can go in any direction. Every radius of the circle is the same length. We also use the word *radius* to mean the length of this segment.

For example, r is the radius of this circle with center O.

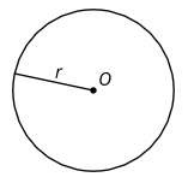


Figure 5.2.1.4

Practice

Exercise 5.2.1.4

Here is an equation that represents a function: $72x + 12y = 60$.

Select **all** the different equations that describe the same function:

A. $120y + 720x = 600$
B. $y = 5 - 6x$
C. $2y + 12x = 10$
D. $y = 5 + 6x$
E. $x = \frac{5}{6} - \frac{y}{6}$
F. $7x + 2y = 6$
G. $x = \frac{5}{6} + \frac{y}{6}$

Exercise 5.2.1.5

1. Graph a system of linear equations with no solutions.
2. Write an equation for each line you graph.

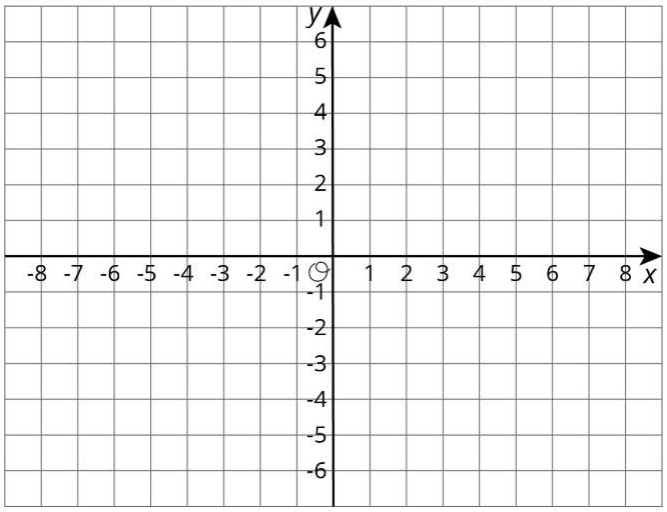


Figure 5.2.1.5

(From Unit 4.3.4)

Exercise 5.2.1.6

Brown rice costs \$2 per pound, and beans cost \$1.60 per pound. Lin has \$10 to spend on these items to make a large meal of beans and rice for a potluck dinner. Let b be the number of pounds of beans Lin buys and r be the number of pounds of rice she buys when she spends all her money on this meal.

1. Write an equation relating the two variables.
2. Rearrange the equation so b is the independent variable.
3. Rearrange the equation so r is the independent variable.

Exercise 5.2.1.7

Solve each equation and check your answer.

$2x+4(3-2x)=\frac{3(2x+2)}{6}+4$ $\quad$ $4x+5=-3x-8$ $\quad$ $\frac{1}{2}-\frac{1}{8}q=\frac{q-1}{4}$

(From Unit 4.2.5)

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5.2.2: Tables, Equations, and Graphs of Functions

Lesson

Let's connect equations and graphs of functions.

Exercise 5.2.2.1: Notice and Wonder: Doubling Back

What do you notice? What do you wonder?

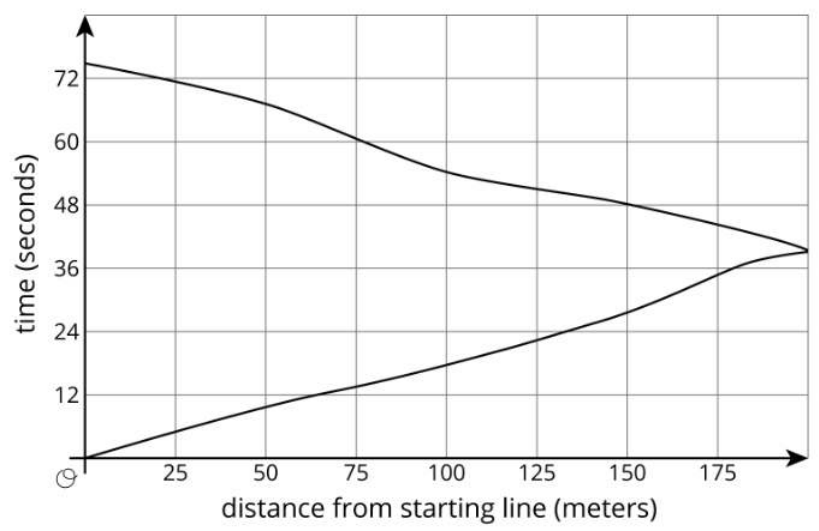


Figure 5.2.2.1: A graph in a coordinate plane. Horizontal axis, distance from starting line in meters, vertical axis, time in seconds. The graph begins at the origin and increases steadily as it moves right until it reaches 200 comma 40. The graph then turns and increases steadily as it moves left until it reaches the point 0 comma 74.

Exercise 5.2.2.2: Equations and Graphs of Functions

The graphs of three functions are shown.

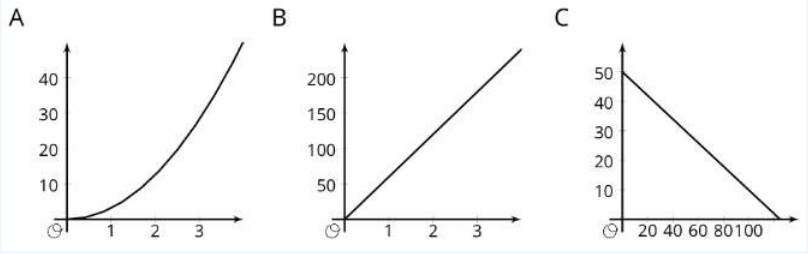


Figure 5.2.2.2: Graphs of 3 functions. Graph A, horizontal 0 to 3 by ones, vertical 0 to 40 by tens, beginning at the origin and increases, slope also increasing. Graph B, horizontal 0 to 3 by ones, vertical o to 200 by fifties, beginning at the origin, graph increases linearly. Graph C, horizontal, 0 to 100 by twenties, vertical 0 to 50 by tens, beginning at 0 comma 50, decreases linearly to 120 comma 0.

1. Match one of these equations to each of the graphs.
   a. $d = 60t$, where d is the distance in miles that you would travel in t hours if you drove at 60 miles per hour.
   b. $q = 50 - 0.4d$, where q is the number of quarters, and d is the number of dimes, in a pile of coins worth $12.50.
   c. $A = \pi r^2$, where A is the area in square centimeters of a circle with radius r centimeters.
2. Label each of the axes with the independent and dependent variables and the quantities they represent.
3. For each function: What is the output when the input is 1? What does this tell you about the situation? Label the corresponding point on the graph.
4. Find two more input-output pairs. What do they tell you about the situation? Label the corresponding points on the graph.

Are you ready for more?

A function inputs fractions $\frac{a}{b}$ between 0 and 1 where a and b have no common factors, and outputs the fraction $\frac{1}{b}$. For

example, given the input $\frac{3}{4}$ the function outputs $\frac{1}{4}$, and to the input $\frac{1}{2}$ the function outputs $\frac{1}{2}$. These two input-output pairs are shown on the graph.

Plot at least 10 more points on the graph of this function. Are most points on the graph above or below a height of 0.3? Of height 0.01?

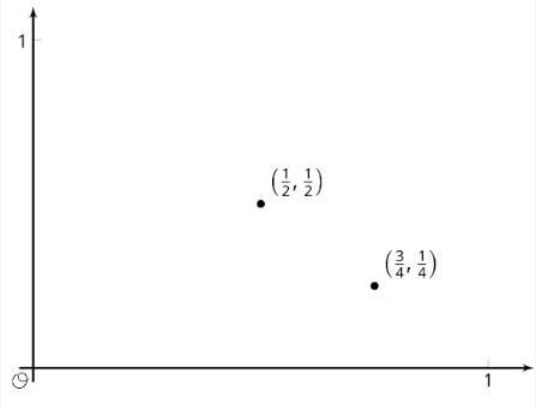


Figure 5.2.2.3

Exercise 5.2.2.3: Running around a Track

1. Kiran was running around the track. The graph shows the time, t, he took to run various distances, d. The table shows his time in seconds after every three meters.

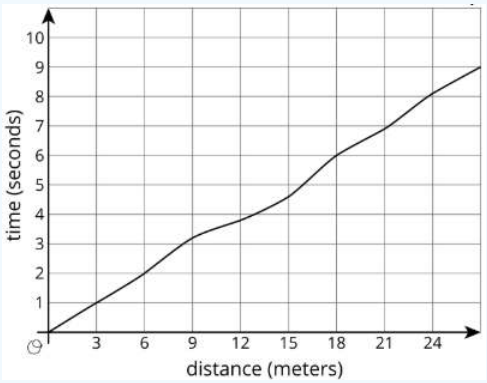


Figure 5.2.2.4: A graph in a coordinate plane, horizontal axis, distance in meters, 0 to 24 by threes, vertical axis, time in seconds, 0 to 10 by ones. Graph begins at the origin and moves steadily upward and to the right, passes through (3 comma 1) and (18 comma 6).

Table 5.2.2.1

d	0	3	6	9	12	15	18	21	24	27
t	0	1.0	2.0	3.2	3.8	4.6	6.0	6.9	8.09	9.0

   a. How long did it take Kiran to run 6 meters?
   b. How far had he gone after 6 seconds?
   c. Estimate when he had run 19.5 meters.
   d. Estimate how far he ran in 4 seconds.
   e. Is Kiran's time a function of the distance he has run? Explain how you know.

2. Priya is running once around the track. The graph shows her time given how far she is from her starting point.

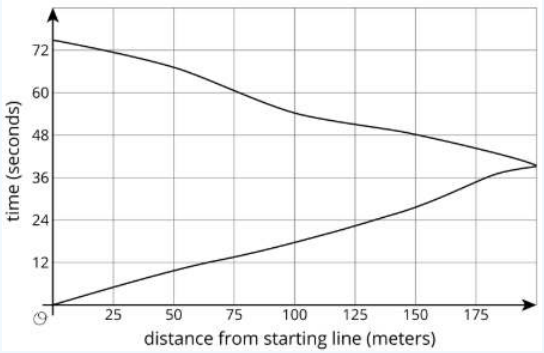


Figure 5.2.2.5: A graph in a coordinate plane. Horizontal axis, distance from starting line in meters, vertical axis, time in seconds. The graph begins at the origin and increases steadly as it moves right until it reaches 200 comma 40. The graph then turns and increases steadily as it moves left until it reaches the point 0 comma 74.

a. What was her farthest distance from her starting point?
b. Estimate how long it took her to run around the track.
c. Estimate when she was 100 meters from her starting point.
d. Estimate how far she was from the starting line after 60 seconds.
e. Is Priya's time a function of her distance from her starting point? Explain how you know.

Summary

Here is the graph showing Noah's run.

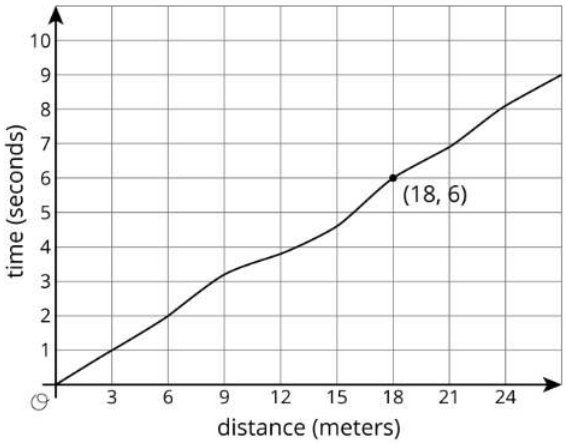


Figure 5.2.2.6: A graph in the coordinate plane, horizontal, distance in meters, 0 to 24 by threes, vertical, time in seconds, 0 to 10 by ones. The graph begins at the origin and steadily increases as it moves right, passing through the labeled point at (18 comma 6).

The time in seconds since he started running is a function of the distance he has run. The point (18,6) on the graph tells you that the time it takes him to run 18 meters is 6 seconds. The input is 18 and the output is 6.

The graph of a function is all the coordinate pairs, (input, output), plotted in the coordinate plane. By convention, we always put the input first, which means that the inputs are represented on the horizontal axis and the outputs, on the vertical axis.

Glossary Entries

Definition: Dependent Variable

A dependent variable represents the output of a function.

For example, suppose we need to buy 20 pieces of fruit and decide to buy apples and bananas. If we select the number of apples first, the equation $b = 20 - a$ shows the number of bananas we can buy. The number of bananas is the dependent variable because it depends on the number of apples.

Definition: Independent Variable

An independent variable represents the input of a function.

For example, suppose we need to buy 20 pieces of fruit and decide to buy some apples and bananas. If we select the number of apples first, the equation $b = 20 - a$ shows the number of bananas we can buy. The number of apples is the independent variable because we can choose any number for it.

Definition: Radius

A radius is a line segment that goes from the center to the edge of a circle. A radius can go in any direction. Every radius of the circle is the same length. We also use the word *radius* to mean the length of this segment.

For example, r is the radius of this circle with center O.

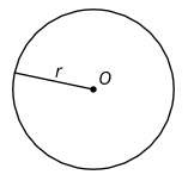


Figure 5.2.2.7

Practice

Exercise 5.2.2.4

The graph and the table show the high temperatures in a city over a 10-day period.

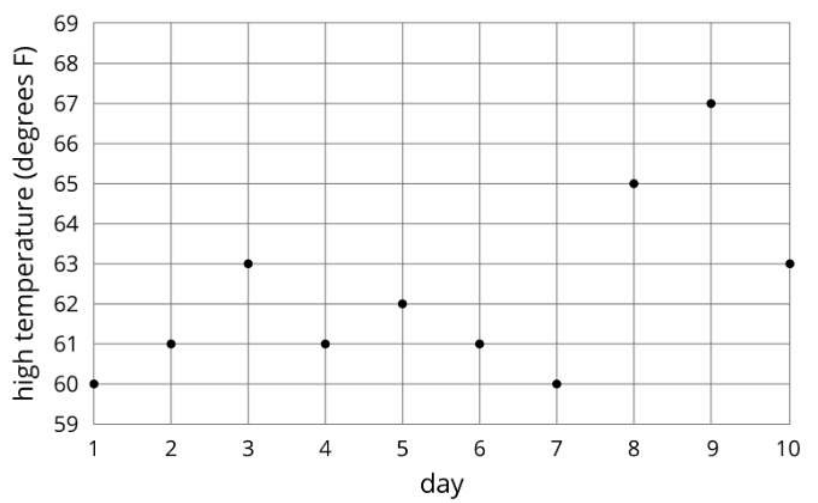


Figure 5.2.2.8

Table 5.2.2.2

day	1	2	3	4	5	6	7	8	9	10
temperature (degrees F)	60	61	63	61	62	61	60	65	67	63

1. What was the high temperature on Day 7?
2. On which days was the high temperature 61 degrees?
3. Is the high temperature a function of the day? Explain how you know.
4. Is the day a function of the high temperature? Explain how you know.

Exercise 5.2.2.5

The amount Lin's sister earns at her part-time job is proportional to the number of hours she works. She earns $9.60 per hour.

1. Write an equation in the form $y = kx$ to describe this situation, where x represents the hours she works and y represents the dollars she earns.
2. Is y a function of x? Explain how you know.
3. Write an equation describing x as a function of y.

Exercise 5.2.2.6

Use the equation $2m + 4s = 16$ to complete the table, then graph the line using s as the dependent variable.

Table 5.2.2.3

m	0		-2	
s		3		0

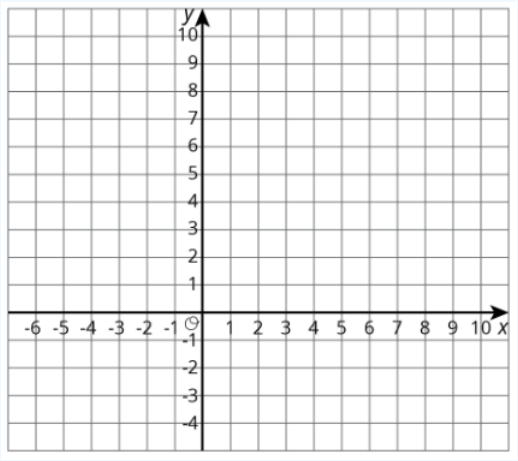


Figure 5.2.2.9

Exercise 5.2.2.7

Solve the system of equations:

$$\begin{cases} y = 7x + 10 \\ y = -4x - 23 \end{cases}$$

(From Unit 4.3.4)

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5.2.3: More Graphs of Functions

Lesson

Let's interpret graphs of functions.

Exercise 5.2.3.1: Which One Doesn't Belong: Graphs

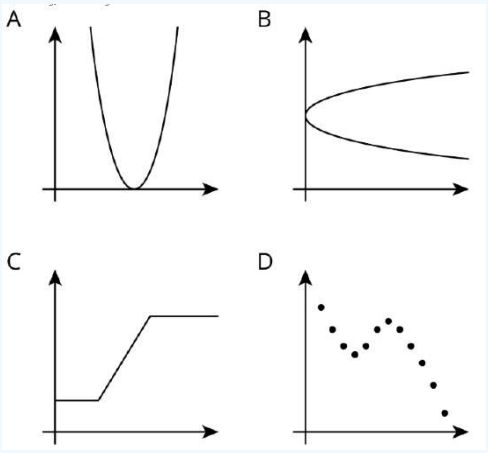


Figure 5.2.3.1: Four graphs, all quadrant 1. Graph A, a parabola, open up, vertex on the horizontal axis. Graph B, parabola, open right, vertex on the vertical axis. Graph C, piecewise graph, constant, positive slope, constant. Graph D, scatterplot, decreasing, increasing, decreasing.

Exercise 5.2.3.2: Time and Temperature

The graph shows the temperature between noon and midnight in one day in a certain city.

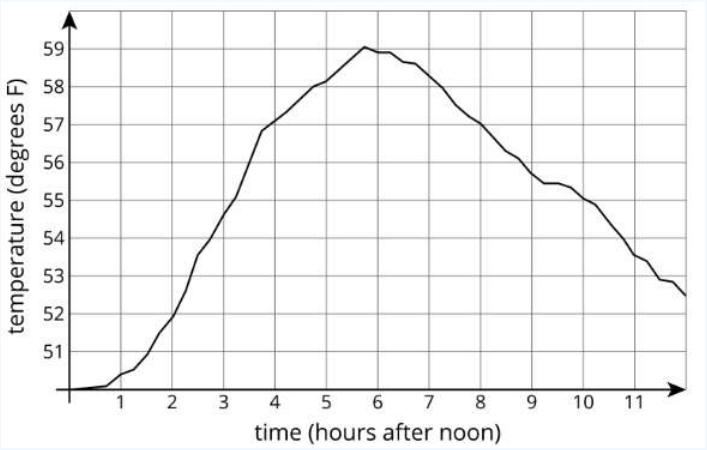


Figure 5.2.3.2: Graph in a coordinate plane, horizontal, time in hours after noon, 0 to 12 by ones, vertical, temperature in degrees Farenheit, 50 to 60 by ones. The graph starts at 0 comma 50 and climbs slowly, the more quickly as it moves right, going through 2 comma 52 and 4 comma 57 before peaking at 5 point 8 comma 59. The graph then steadily declines as it moves right until it reaches the point 12 comma 52 point 5.

1. Was it warmer at 3:00 p.m. or 9:00 p.m.?
2. Approximately when was the temperature highest?
3. Find another time that the temperature was the same as it was at 4:00 p.m.
4. Did the temperature change more between 1:00 p.m. and 3:00 p.m. or between 3:00 p.m. and 5:00 p.m.?
5. Does this graph show that temperature is a function of time, or time is a function of temperature?
6. When the input for the function is 8, what is the output? What does that tell you about the time and temperature?

Exercise 5.2.3.3: Garbage

1. The graph shows the amount of garbage produced in the US each year between 1991 and 2013.

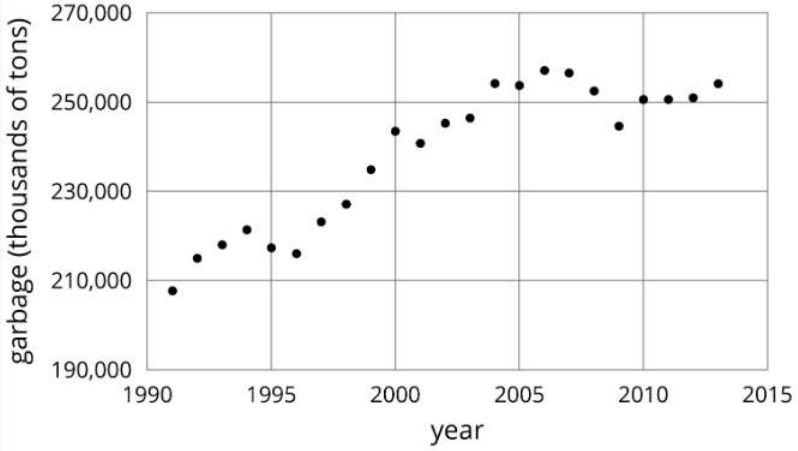


Figure 5.2.3.3: Scatterplot, horizontal, year, 1990 to 2015 by fives, vertical, garbage in thousands of tons, 190,000 to 270,000 by 20,000. Points start at 1991 comma 208,000, and increase steadily to 2007 comma 255,000 except for slight decreases at 1995 and 2001. Between 2006 and 2009, the points decrease, but increase again from 2009 to 2014.

a. Did the amount of garbage increase or decrease between 1999 and 2000?
b. Did the amount of garbage increase or decrease between 2005 and 2009?
c. Between 1991 and 1995, the garbage increased for three years, and then it decreased in the fourth year. Describe how the amount of garbage changed in the years between 1995 and 2000.



Figure 5.2.3.4: **Attribution:** By PublicDomainPictures. Public Domain. Pixabay. Source.

2. The graph shows the percentage of garbage that was recycled between 1991 and 2013.

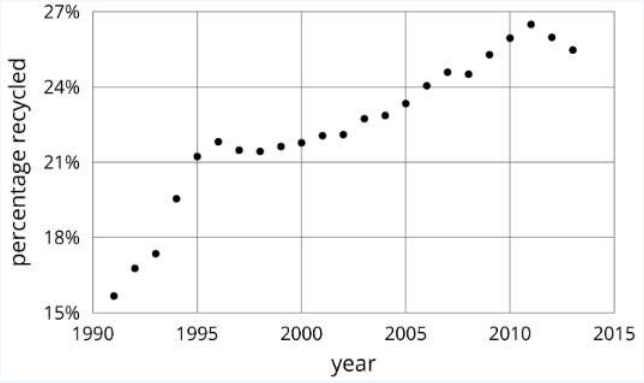


Figure 5.2.3.5: A scatterplot, horizontal, year, 1900 to 2015 by fives, vertical, percentage recycled, 15 % to 27 % by threes. Beginning with 1991 comma 16%, points trend linearly upward to 1996 comma 22%, then linearly upward at a less steep rate to 2011 comma 26 %, then two points trend down.

a. When was it increasing?
b. When was it decreasing?
c. Tell the story of the change in the percentage of garbage recycled in the US over this time period.

Are you ready for more?

Refer to the graph in the first part of the activity.

1. Find a year where the amount of garbage produced increased from the previous year, but not by as much it increased the following year.
2. Find a year where the amount of garbage produced increased from the previous year, and then increased by a smaller amount the following year.
3. Find a year where the amount of garbage produced decreased from the previous year, but not by as much it decreased the following year.
4. Find a year where the amount of garbage produced decreased from the previous year, and then decreased by a smaller amount the following year.

Summary

Here is a graph showing the temperature in a town as a function of time after 8:00 p.m.

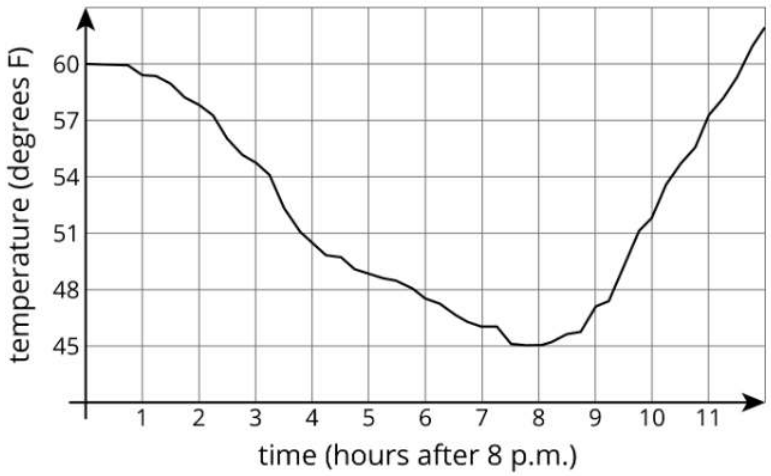


Figure 5.2.3.6: The graph of a curve on the coordinate plane. The horizontal axis is labeled “time in hours after 8 pm” and the numbers 1 through 11 are indicated. The vertical axis is labeled “temperature in degrees Fahrenheit” and the numbers 45 through 60, in increments of 3 are indicated. The curve starts on the vertical axis at the point 0 comma 60, and moves downwards and to the right. It continues downward until reaching a minimum point of 8 comma 45, turns, and then moves upward and to the right, passing through the point 11 comma 57.

The graph of a function tells us what is happening in the context the function represents. In this example, the temperature starts out at 60° F at 8:00 p.m. It decreases during the night, reaching its lowest point at 8 hours after 8:00 p.m., or 4:00 a.m. Then it starts to increase again.

Glossary Entries

Definition: Dependent Variable

A dependent variable represents the output of a function.

For example, suppose we need to buy 20 pieces of fruit and decide to buy apples and bananas. If we select the number of apples first, the equation $b = 20 - a$ shows the number of bananas we can buy. The number of bananas is the dependent variable because it depends on the number of apples.

Definition: Independent Variable

An independent variable represents the input of a function.

For example, suppose we need to buy 20 pieces of fruit and decide to buy some apples and bananas. If we select the number of apples first, the equation $b = 20 - a$ shows the number of bananas we can buy. The number of apples is the independent variable because we can choose any number for it.

Definition: Radius

A radius is a line segment that goes from the center to the edge of a circle. A radius can go in any direction. Every radius of the circle is the same length. We also use the word *radius* to mean the length of this segment.

For example, r is the radius of this circle with center O.

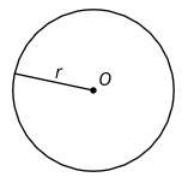


Figure 5.2.3.7

Practice

Exercise 5.2.3.4

The solution to a system of equations is $(6, -3)$. Choose two equations that might make up the system.

A. $y = -3x + 6$
B. $y = 2x - 9$
C. $y = -5x + 27$
D. $y = 2x - 15$
E. $y = -4x + 27$

(From Unit 4.3.4)

Exercise 5.2.3.5

A car is traveling on a small highway and is either going 55 miles per hour or 35 miles per hour, depending on the speed limits, until it reaches its destination 200 miles away. Letting x represent the amount of time in hours that the car is going 55 miles per hour, and y being the time in hours that the car is going 35 miles per hour, an equation describing the relationship is: $55x + 35y = 200$

1. If the car spends 2.5 hours going 35 miles per hour on the trip, how long does it spend going 55 miles per hour?
2. If the car spends 3 hours going 55 miles per hour on the trip, how long does it spend going 35 miles per hour?
3. If the car spends no time going 35 miles per hour, how long would the trip take? Explain your reasoning.

(From Unit 5.2.1)

Exercise 5.2.3.6

The graph represents an object that is shot upwards from a tower and then falls to the ground. The independent variable is time in seconds and the dependent variable is the object's height above the ground in meters.

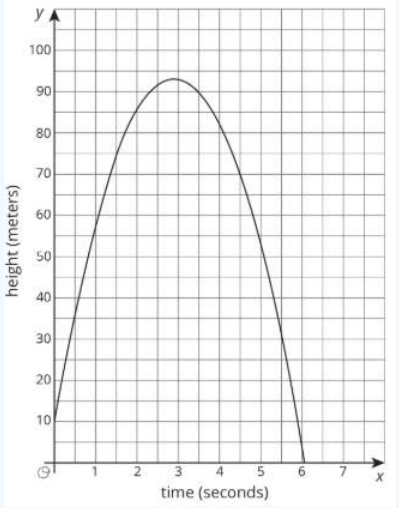


Figure 5.2.3.8: Coordinate plane, x, time in seconds, 0 to 7 by ones, y, height in meters, 0 to 100 by tens. Graph of a curve begins at (0 comma 10), climbs steeply as it moves right through (1 point 5 comma 75), to (2 point 9 comma 93), then back down through (4 point 5 comma 70), to (6 point 1 comma 0).

1. How tall is the tower from which the object was shot?
2. When did the object hit the ground?
3. Estimate the greatest height the object reached and the time it took to reach that height. Indicate this situation on the graph.

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5.2.4: Even More Graphs of Functions

Lesson

Let's draw a graph from a story.

Exercise 5.2.4.1: Dog Run

Here are five pictures of a dog taken at equal intervals of time.



Figure 5.2.4.1

Diego and Lin drew different graphs to represent this situation:

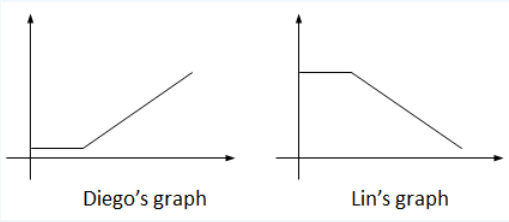


Figure 5.2.4.2

They both used time as the independent variable. What do you think each one used for the dependent variable? Explain your reasoning.

Exercise 5.2.4.2: Which Graph is It?

For each situation,

- name the independent and dependent variables
- pick the graph that best fits the situation, or sketch the graph if one isn't provided
- label the axes
- answer the question: which quantity is a function of which? Be prepared to explain your reasoning.

1. Jada is training for a swimming race. The more she practices, the less time it takes for her to swim one lap.

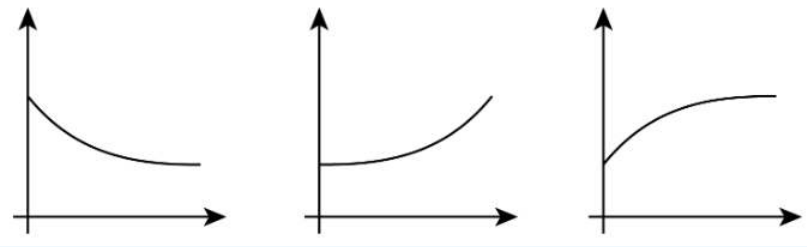


Figure 5.2.4.3

2. Andre adds some money to a jar in his room each week for 3 weeks and then takes some out in week 4.

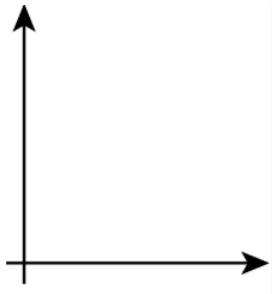


Figure 5.2.4.4

Exercise 5.2.4.3: Sketching a Story about a Boy and a Bike

You will use the tools in the applet for creating a visual display that shows your response to each question.

Here is a story: “Noah was at home. He got on his bike and rode to his friend’s house and stayed there for awhile. Then he rode home again. Then he rode to the park. Then he rode home again.”

1. Sketch a graph of this story.
2. What are the two quantities? Label the axes with their names and units of measure. (For example, if this were a story about pouring water into a pitcher, one of your labels might say “volume (liters).”)
3. Which quantity is a function of which? Explain your reasoning.
4. Based on your graph, is his friend’s house or the park closer to Noah's home? Explain how you know.
5. Read the story and all your responses again. Does everything make sense? If not, make changes to your work.

Note: To change the values on the axes, select the Move Graphics tool, click near the end of the axis you want to change, and drag it to grow or shrink it.



Are you ready for more?

It is the year 3000. Noah’s descendants are still racing around the park, but thanks to incredible technological advances, now with much more powerful gadgets at their disposal. How might their newfound access to teleportation and time-travel devices alter the graph of stories of their daily adventures? Could they affect whether or not the distance from home is a function of the time elapsed?

Summary

Here is a graph showing Andre's distance as a function of time.

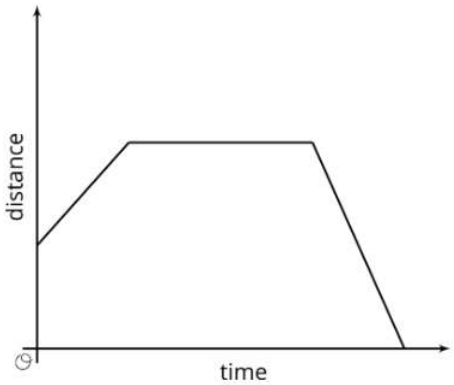


Figure 5.2.4.5

For a graph representing a context, it is important to specify the quantities represented on each axis. For example, if this is showing distance from home, then Andre starts at some distance from home (maybe at his friend’s house), moves further away (maybe to a

park), then returns home. If instead the graph is showing distance from school, the story may be Andre starts out at home, moves further away (maybe to a friend's house), then goes to school. What could the story be if the graph is showing distance from a park?

Glossary Entries

Definition: Dependent Variable

A dependent variable represents the output of a function.

For example, suppose we need to buy 20 pieces of fruit and decide to buy apples and bananas. If we select the number of apples first, the equation $b = 20 - a$ shows the number of bananas we can buy. The number of bananas is the dependent variable because it depends on the number of apples.

Definition: Independent Variable

An independent variable represents the input of a function.

For example, suppose we need to buy 20 pieces of fruit and decide to buy some apples and bananas. If we select the number of apples first, the equation $b = 20 - a$ shows the number of bananas we can buy. The number of apples is the independent variable because we can choose any number for it.

Definition: Radius

A radius is a line segment that goes from the center to the edge of a circle. A radius can go in any direction. Every radius of the circle is the same length. We also use the word *radius* to mean the length of this segment.

For example, r is the radius of this circle with center O.

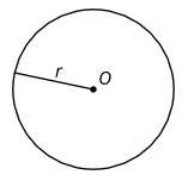


Figure 5.2.4.6

Practice

Exercise 5.2.4.4

Match the graph to the following situations (you can use a graph multiple times). For each match, name possible independent and dependent variables and how you would label the axes.



Figure 5.2.4.7

1. Tyler pours the same amount of milk from a bottle every morning.
2. A plant grows the same amount every week.
3. The day started very warm but then it got colder.
4. A carnival has an entry fee of $5 and tickets for rides cost $1 each.

Exercise 5.2.4.5

Jada fills her aquarium with water.

The graph shows the height of the water, in cm, in the aquarium as a function of time in minutes. Invent a story of how Jada fills the aquarium that fits the graph.

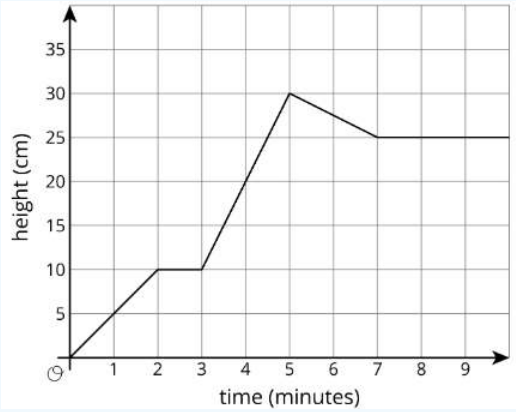


Figure 5.2.4.8

Exercise 5.2.4.6

Recall the formula for area of a circle.

1. Write an equation relating a circle's radius, r, and area, A.
2. Is area a function of the radius? Is radius a function of the area?
3. Fill in the missing parts of the table.

Table 5.2.4.1

r	3		$\frac{1}{2}$	
A		16π		100π

(From Unit 5.2.2)

Exercise 5.2.4.7

The points with coordinates $4, 8)$, $(2, 10)$, and $(5, 7)$ all lie on the line $2x + 2y = 24$.

1. Create a graph, plot the points, and sketch the line.
2. What is the slope of the line you graphed?
3. What does this slope tell you about the relationship between lengths and widths of rectangles with perimeter 24?

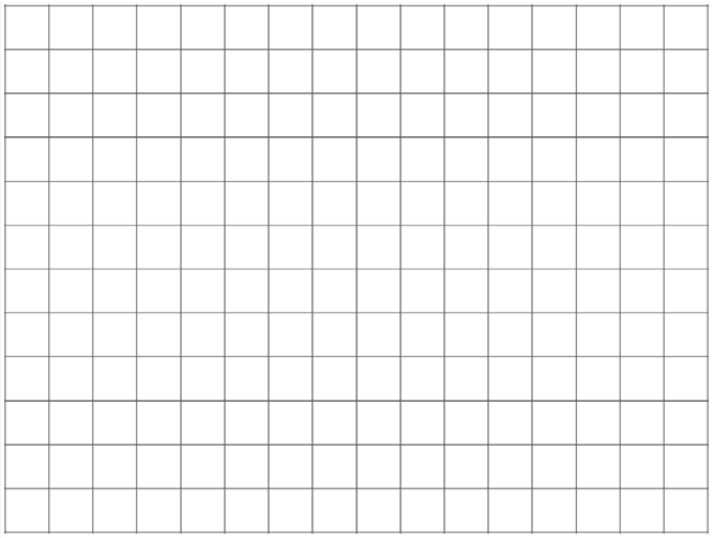


Figure 5.2.4.9

(From Unit 3.3.3)

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5.2.5: Connecting Representations of Functions

Lesson

Let's connect tables, equations, graphs, and stories of functions.

Exercise 5.2.5.1: Which are the Same? Which are Different?

Here are three different ways of representing functions. How are they alike? How are they different? $y = 2x$

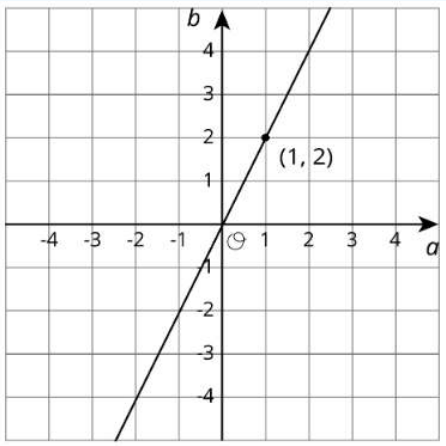


Figure 5.2.5.1

Table 5.2.5.1

p	-2	-1	0	1	2	3
q	4	2	0	-2	-4	-6

Exercise 5.2.5.2: Comparing Temperatures

The graph shows the temperature between noon and midnight in City A on a certain day.

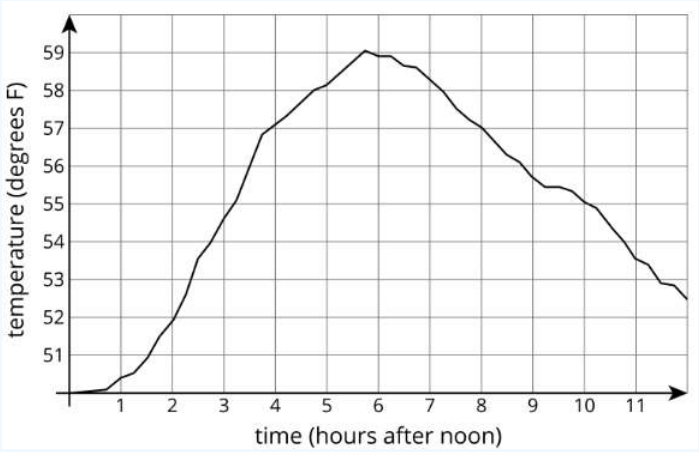


Figure 5.2.5.2: Graph in a coordinate plane, horizontal, time in hours after noon, 0 to 12 by ones, vertical, temperature in degrees Farenheit, 50 to 60 by ones. The graph starts at 0 comma 50 and climbs slowly, the more quickly as it moves right, going through 2 comma 52 and 4 comma 57 before peaking at 5 point 8 comma 59. The graph then steadily declines as it moves right until it reaches the point 12 comma 52 point 5.

The table shows the temperature, T, in degrees Fahrenheit, for h hours after noon, in City B.

Table 5.2.5.2

h	1	2	3	4	5	6

T	82	78	75	62	58	59

1. Which city was warmer at 4:00 p.m.?
2. Which city had a bigger change in temperature between 1:00 p.m. and 5:00 p.m.?
3. How much greater was the highest recorded temperature in City B than the highest recorded temperature in City A during this time?
4. Compare the outputs of the functions when the input is 3.

Exercise 5.2.5.3: Comparing Volumes

The volume, V, of a cube with edge length s cm is given by the equation $V = s^3$. The volume of a sphere is a function of its radius (in centimeters), and the graph of this relationship is shown here.

1. Is the volume of a cube with edge length $s = 3$ greater or less than the volume of a sphere with radius 3?
2. If a sphere has the same volume as a cube with edge length 5, estimate the radius of the sphere.
3. Compare the outputs of the two volume functions when the inputs are 2.

Here is an applet to use if you choose.

Are you ready for more?

Estimate the edge length of a cube that has the same volume as a sphere with radius 2.5.

Exercise 5.2.5.4: It's Not a Race

Elena's family is driving on the freeway at 55 miles per hour.

Andre's family is driving on the same freeway, but not at a constant speed. The table shows how far Andre's family has traveled, d, in miles, every minute for 10 minutes.

Table 5.2.5.3

t	1	2	3	4	5	6	7	8	9	10
d	0.9	1.9	3.0	4.1	5.1	6.2	6.8	7.4	8	9.1

1. How many miles per minute is 55 miles per hour?
2. Who had traveled farther after 5 minutes? After 10 minutes?
3. How long did it take Elena's family to travel as far as Andre's family had traveled after 8 minutes?
4. For both families, the distance in miles is a function of time in minutes. Compare the outputs of these functions when the input is 3.

Summary

Functions are all about getting outputs from inputs. For each way of representing a function—equation, graph, table, or verbal description—we can determine the output for a given input.

Let's say we have a function represented by the equation $y = 3x + 2$ where y is the dependent variable and x is the independent variable. If we wanted to find the output that goes with 2, we can input 2 into the equation for x and finding the corresponding value of y. In this case, when x is 2, y is 8 since $3 \cdot 2 + 2 = 8$.

If we had a graph of this function instead, then the coordinates of points on the graph are the input-output pairs. So we would read the y-coordinate of the point on the graph that corresponds to a value of 2 for x. Looking at the graph of this function here, we can see the point $(2, 8)$ on it, so the output is 8 when the input is 2.

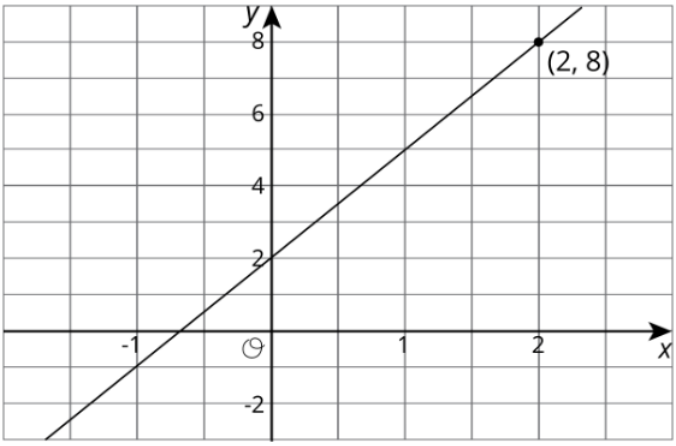


Figure 5.2.5.3: The graph of a line in the coordinate plane with the origin labeled “O”. The horizontal axis has the numbers negative 1 through 2 indicated and there are vertical gridlines between each integer. The vertical axis has the numbers negative 2 through 8, in increments of 2, indicated, and there are horizontal grid lines in between each integer. The line begins to the right of the y axis and below the x axis. It slants upward and to the right passing through the point with coordinates negative 1 comma negative 1, crosses the y axis at 2, and passes through the indicated point labeled 2 comma 8.

A table representing this function shows the input-output pairs directly (although only for select inputs).

Table 5.2.5.4

x	-1	0	1	2	3
y	-1	2	5	8	11

Again, the table shows that if the input is 2, the output is 8.

Glossary Entries

Definition: Dependent Variable

A dependent variable represents the output of a function.

For example, suppose we need to buy 20 pieces of fruit and decide to buy apples and bananas. If we select the number of apples first, the equation $b = 20 - a$ shows the number of bananas we can buy. The number of bananas is the dependent variable because it depends on the number of apples.

Definition: Independent Variable

An independent variable represents the input of a function.

For example, suppose we need to buy 20 pieces of fruit and decide to buy some apples and bananas. If we select the number of apples first, the equation $b = 20 - a$ shows the number of bananas we can buy. The number of apples is the independent variable because we can choose any number for it.

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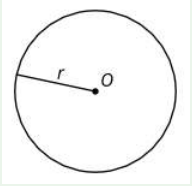


Figure 5.2.5.4

Definition: Volume

Volume is the number of cubic units that fill a three-dimensional region, without any gaps or overlaps.

For example, the volume of this rectangular prism is 60 units^3, because it is composed of 3 layers that are each 20 units^3.

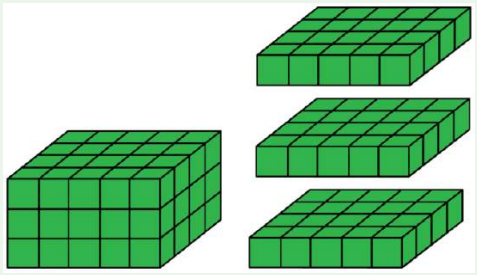


Figure 5.2.5.5

Practice

Exercise 5.2.5.5

The equation and the tables represent two different functions. Use the equation $b = 4a - 5$ and the table to answer the questions. This table represents c as a function of a.

Table 5.2.5.5

a	-3	0	2	5	10	12
c	-20	7	3	21	19	45

1. When a is -3, is b or c greater?
2. When c is 21, what is the value of a? What is the value of b that goes with this value of a?
3. When a is 6, is b or c greater?
4. For what values of a do we know that c is greater than b?

Exercise 5.2.5.6

Elena and Lin are training for a race. Elena runs her mile at a constant speed of 7.5 miles per hour.

Lin's total distances are recorded every minute:

Table 5.2.5.6

time (minutes)	1	2	3	4	5	6	7	8	9
distance (miles)	0.11	0.21	0.32	0.41	0.53	0.62	0.73	0.85	1

1. Who finished their mile first?
2. This is a graph of Lin's progress. Draw a graph to represent Elena's mile on the same axes.

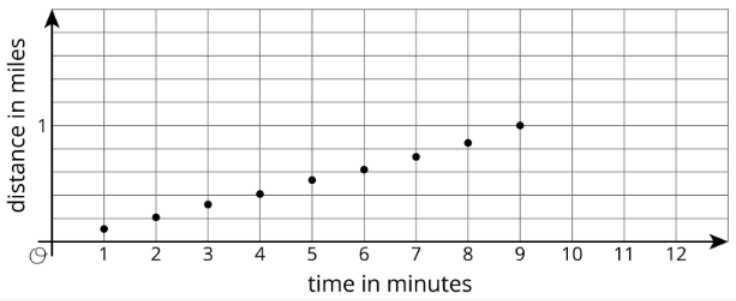


Figure 5.2.5.6

3. For these models, is distance a function of time? Is time a function of distance? Explain how you know.

Exercise 5.2.5.7

Match each function rule with the value that could not be a possible input for that function.

A. 3 divided by the input
B. Add 4 to the input, then divide this value into 3
C. Subtract 3 from the input, then divide this value into 1

1. 3
2. 4
3. -4
4. 0
5. 1

(From Unit 5.1.2)

Exercise 5.2.5.8

Find a value of x that makes the equation true. Explain your reasoning, and check that your answer is correct.

$$-(-2x+1) = 9 - 14x$$

(From Unit 4.2.3)

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5.3: Linear Functions and Rates of Change

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5.3.1: Linear Functions

Lesson

Let's investigate linear functions.

Exercise 5.3.1.1: Bigger and Smaller

Diego said that these graphs are ordered from smallest to largest. Mai said they are ordered from largest to smallest. But these are graphs, not numbers! What do you think Diego and Mai are thinking?

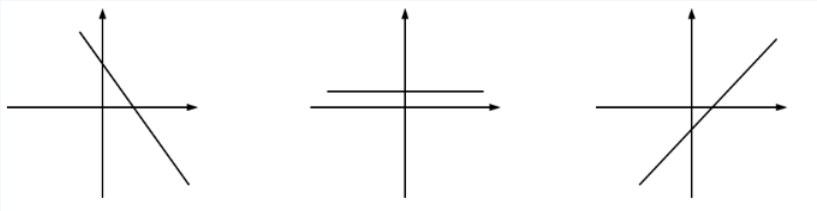


Figure 5.3.1.1

Exercise 5.3.1.2: Proportional Relationships Define Linear Functions

1. Jada earns $7 per hour mowing her neighbor's lawns.

    a. Name the two quantities in this situation that are in a functional relationship. Which did you choose to be the independent variable? What is the variable that depends on it?
    b. Here is a graph of the function. Label the axes. Label at least two points with input-output pairs.
    c. Write an equation that represents the function.

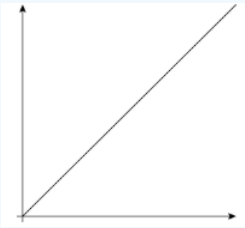


Figure 5.3.1.2

2. To convert feet to yards, you multiply the number of feet by $\frac{1}{3}$.

    a. Name the two quantities in this situation that are in a functional relationship. Which did you choose to be the independent variable? What is the variable that depends on it?
    b. Write an equation that represents the function.
    c. Draw the graph of the function. Label at least two points with input-output pairs.

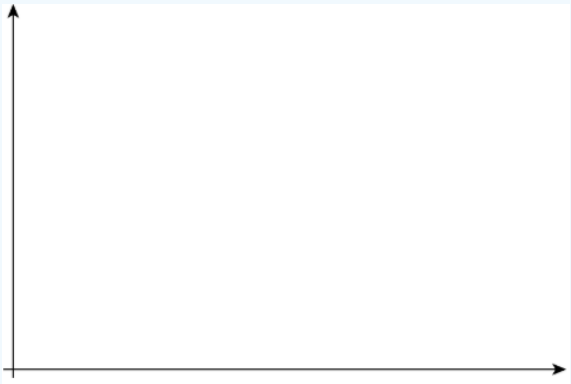


Figure 5.3.1.3

Exercise 5.3.1.3: Is it Filling Up or Draining Out?

There are four tanks of water.

- The amount of water in gallons, A, in Tank A is given by the function $A = 200 + 8t$, where t is in minutes.
- The amount of water in gallons B, in Tank B starts at 400 gallons and is decreasing at 5 gallons per minute. These functions work when $t \geq 0$ and $t \leq 80$.

1. Which tank started out with more water?
2. Write an equation representing the relationship between B and t.
3. One tank is filling up. The other is draining out. Which is which? How can you tell?
4. The amount of water in gallons, C, in Tank C is given by the function $C = 800 - 7t$. Is it filling up or draining out? Can you tell just by looking at the equation?
5. The graph of the function for the amount of water in gallons, D, in Tank D at time t is shown. Is it filling up or draining out? How do you know?

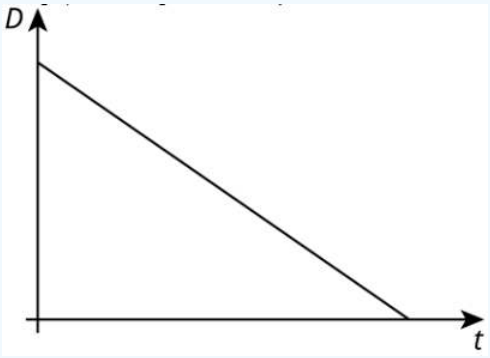


Figure 5.3.1.4

Are you ready for more?

- Pick a tank that was draining out. How long did it take for that tank to drain? What percent full was the tank when 30% of that time had elapsed? When 70% of the time had elapsed?
- What point in the plane is 30% of the way from $(0, 15)$ to $(5, 0)$? 70% of the way?
- What point in the plane is 30% of the way from $(3, 5)$ to $(8, 6)$? 70% of the way?

Exercise 5.3.1.4: Which is Growing Faster?

Noah is depositing money in his account every week to save money. The graph shows the amount he has saved as a function of time since he opened his account.

Elena opened an account the same day as Noah. The amount of money E in her account is given by the function $E = 8w + 60$, where w is the number of weeks since the account was opened.

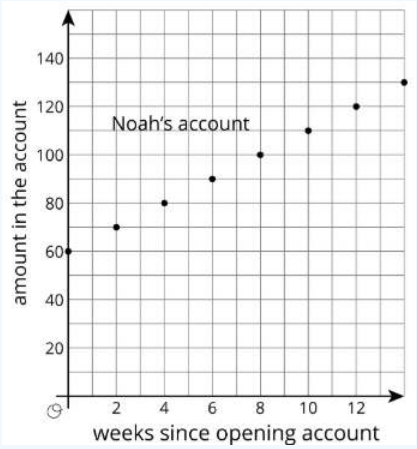


Figure 5.3.1.5

1. Who started out with more money in their account? Explain how you know.
2. Who is saving money at a faster rate? Explain how you know.
3. How much will Noah save over the course of a year if he does not make any withdrawals? How long will it take Elena to save that much?

Summary

Suppose a car is traveling at 30 miles per hour. The relationship between the time in hours and the distance in miles is a proportional relationship. We can represent this relationship with an equation of the form $d = 30t$, where distance is a function of time (since each input of time has exactly one output of distance). Or we could write the equation $t = \frac{1}{30}d$ instead, where time is a function of distance (since each input of distance has exactly one output of time).

More generally, if we represent a linear function with an equation like $y = mx + b$, then b is the initial value (which is 0 for proportional relationships), and m is the rate of change of the function. If m is positive, the function is increasing. If m is negative, the function is decreasing. If we represent a linear function in a different way, say with a graph, we can use what we know about graphs of lines to find the m and b values and, if needed, write an equation.

Practice

Exercise 5.3.1.5

Two cars drive on the same highway in the same direction. The graphs show the distance, d, of each one as a function of time, t. Which car drives faster? Explain how you know.

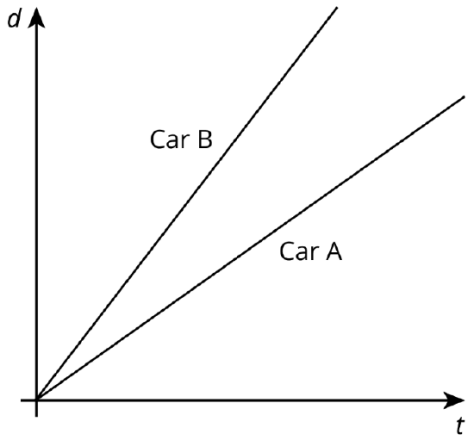


Figure 5.3.1.6

Exercise 5.3.1.6

Two car services offer to pick you up and take you to your destination. Service A charges 40 cents to pick you up and 30 cents for each mile of your trip. Service B charges $1.10 to pick you up and charges c cents for each mile of your trip.

1. Match the services to the Lines l and m.
2. For Service B, is the additional charge per mile greater or less than 30 cents per mile greater or less than 30 cents per mile of the trip? Explain your reasoning.

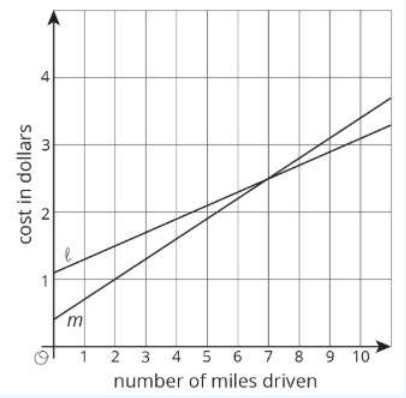


Figure 5.3.1.7

Exercise 5.3.1.7

Kiran and Clare like to race each other home from school. They run at the same speed, but Kiran's house is slightly closer to school than Clare's house. On a graph, their distance from their homes in meters is a function of the time from when they begin the race in seconds.

1. As you read the graphs left to right, would the lines go up or down?
2. What is different about the lines representing Kiran's run and Clare's run?
3. What is the same about the liens representing Kiran's run and Clare's run?

Exercise 5.3.1.8

Write an equation for each line.

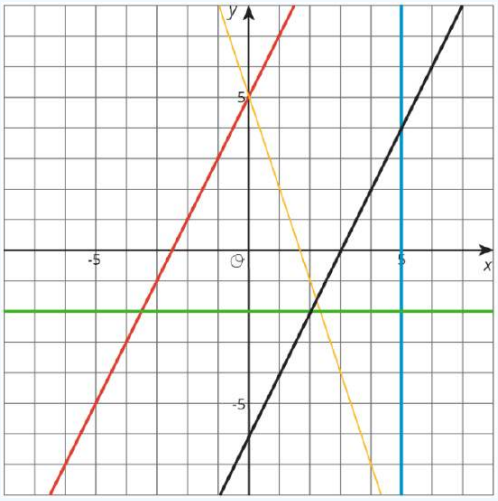


Figure 5.3.1.8: Five lines. Green line, horizontal through (0 comma negative 2). Red line through (0 comma 5) and (1 comma 7). Yellow line through (0 comma 5) and (1 comma 2). Black line through (0 comma negative 5) and (5 comma 4). Blue line, vertical, through (5 comma 0).

(From Unit 3.3.3)

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5.3.2: Linear Models

Lesson

Let's model situations with linear functions.

Exercise 5.3.2.1: Candlelight

A candle is burning. It starts out 12 inches long. After 1 hour, it is 10 inches long. After 3 hours, it is 5.5 inches long.

1. When do you think the candle will burn out completely?
2. Is the height of the candle a function of time? If yes, is it a linear function? Explain your thinking.

This tool is here for you to use if you choose. To plot a point, type its coordinates. For example, try typing $(1, 2)$. To graph a line, type its equation. Try typing $y = 2x - 3$. You can delete anything by clicking on the X next to it.

Exercise 5.3.2.2: Shadows

When the Sun was directly overhead, the stick had no shadow. After 20 minutes, the shadow was 10.5 cm long. After 60 minutes, it was 26 cm long.



Figure 5.3.2.1

1. Based on this information, estimate how long it will be after 95 minutes.
2. After 95 minutes, the shadow measured 38.5 cm. How does this compare to your estimate?
3. Is the length of the shadow a function of time? If so, is it linear? Explain your reasoning.

This tool is here for you to use if you choose. To plot a point, type its coordinates. For example, try typing $(3, 5)$. To graph a line, type its equation. Try typing $y = 2x + 7$. You can delete anything by clicking on the X next to it.

Exercise 5.3.2.3: Recycling

In an earlier lesson, we saw this graph that shows the percentage of all garbage in the U.S. that was recycled between 1991 and 2013.



Figure 5.3.2.2: **Attribution:** By Shirley810. Public Domain. Pixabay. Source.

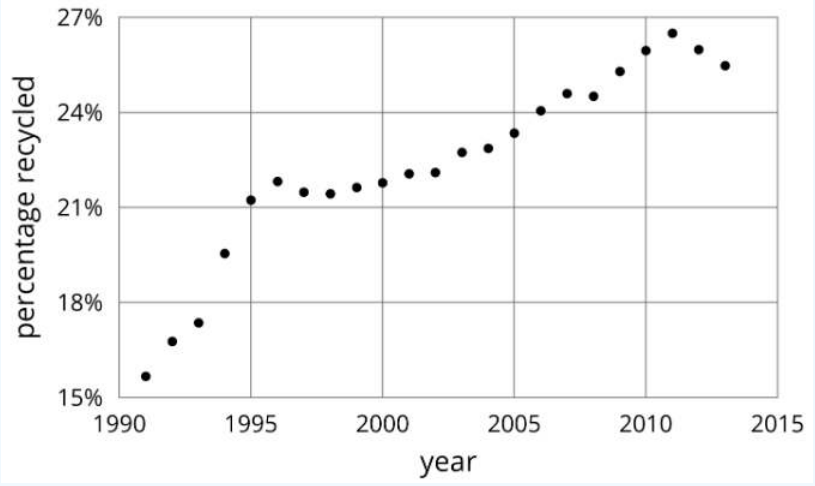


Figure 5.3.2.3: A scatterplot, horizontal, year, 1900 to 2015 by fives, vertical, percentage recycled, 15 % to 27 % by threes. Beginning with 1991 comma 16%, points trend linearly upward to 1996 comma 22%, then linearly upward at a less steep rate to 2011 comma 26 %, then two points trend down.

1. Sketch a linear function that models the change in the percentage of garbage that was recycled between 1991 and 1995. For which years is the model good at predicting the percentage of garbage that is produced? For which years is it not as good?
2. Pick another time period to model with a sketch of a linear function. For which years is the model good at making predictions? For which years is it not very good?

Summary

Water has different boiling points at different elevations. At 0 m above sea level, the boiling point is 100° C. At 2,500 m above sea level, the boiling point is 91.3° C. If we assume the boiling point of water is a linear function of elevation, we can use these two data points to calculate the slope of the line: $m = \frac{91.3-100}{2,500-0} = \frac{-8.7}{2,500}$

This slope means that for each increase of 2,500 m, the boiling point of water decreases by 8.7° C. Next, we already know the y-intercept is 100° C from the first point, so a linear equation representing the data is $y = \frac{-8.7}{2,500}x + 100$

This equation is an example of a mathematical *model*. A mathematical model is a mathematical object like an equation, a function, or a geometric figure that we use to represent a real-life situation. Sometimes a situation can be modeled by a linear function. We have to use judgment about whether this is a reasonable thing to do based on the information we are given. We must also be aware that the model may make imprecise predictions, or may only be appropriate for certain ranges of values.

Testing our model for the boiling point of water, it accurately predicts that at an elevation of 1,000 m above sea level (when $x = 1,000$), water will boil at 96.6° C since $y = \frac{-8.7}{2,500} \cdot 1000 + 100 = 96.5$. For higher elevations, the model is not as accurate, but it is still close. At 5,000 m above sea level, it predicts 82.6° C, which is 0.6° C off the actual value of 83.2° C. At 9,000 m above sea level, it predicts 69.7° C, which is about 3° C less than the actual value of 71.5° C. The model continues to be less accurate at even higher elevations since the relationship between the boiling point of water and elevation isn't linear, but for the elevations in which most people live, it's pretty good.

Practice

Exercise 5.3.2.4

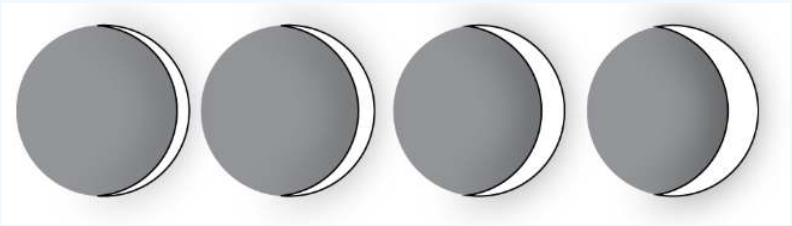


Figure 5.3.2.4

On the first day after the new moon, 2% of the Moon's surface is illuminated. On the second day, 6% is illuminated.

1. Based on this information, predict the day on which the Moon's surface is 50% illuminated and 100% illuminated.
2. The Moon's surface is 100% illuminated on day 14. Does this agree with the prediction you made?
3. Is the percentage illumination of the Moon's surface a linear function of the day?

Exercise 5.3.2.5

In science class, Jada uses a graduated cylinder with water in it to measure the volume of some marbles. After dropping in 4 marbles so they are all under water, the water in the cylinder is at a height of 10 milliliters. After dropping in 6 marbles so they are all under water, the water in the cylinder is at a height of 11 milliliters.

1. What is the volume of 1 marble?
2. How much water was in the cylinder before any marbles were dropped in?
3. What should be the height of the water after 13 marbles are dropped in?
4. Is the relationship between volume of water and number of marbles a linear relationship? If so, what does the slope of a line representing this relationship mean? If not, explain your reasoning.

Exercise 5.3.2.6

Solve each of these equations. Explain or show your reasoning.

$$2(3x+2)=2x+28 \qquad 5y+13=-43-3y \qquad 4(2a+2)=8(2-3a)$$

(From Unit 4.2.4)

Exercise 5.3.2.7

For a certain city, the high temperatures (in degrees Celsius) are plotted against the number of days after the new year.

Based on this information, is the high temperature in this city a linear function of the number of days after the new year?

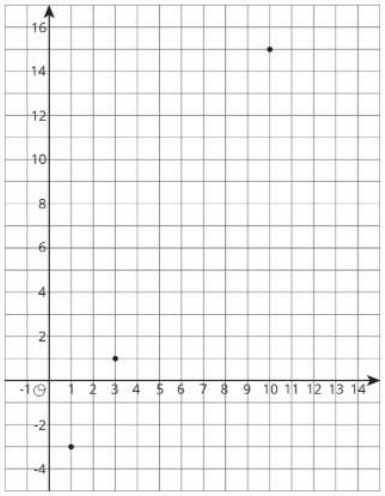


Figure 5.3.2.5

Exercise 5.3.2.8

The school designed their vegetable garden to have a perimeter of 32 feet with the length measuring two feet more than twice the width.

1. Using l to represent the length of the garden and w to represent its width, write and solve a system of equations that describes this situation.
2. What are the dimensions of the garden?

(From Unit 4.3.6)

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5.3.3: Piecewise Linear Functions

Lesson

Let's explore functions built out of linear pieces.

Exercise 5.3.3.1: Notice and Wonder: Lines on Dots

What do you notice? What do you wonder?

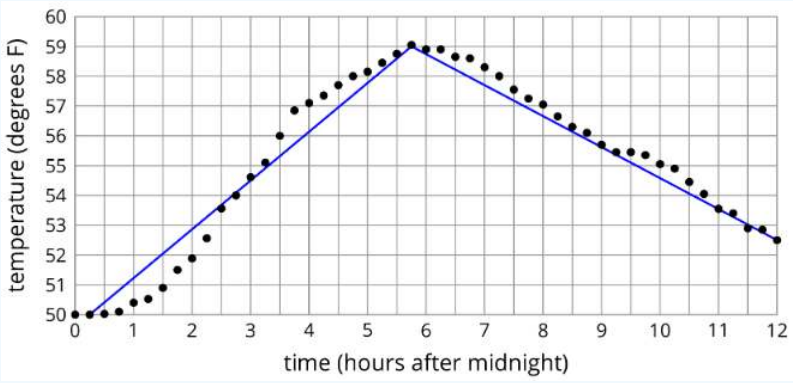


Figure 5.3.3.1: Scatterplot, horizontal, time in hours after midnight, 0 to 12 by ones, vertical, temperature in degrees Fahrenheit. Fifty points approximate a straight line from point 2 5 comma 50 increasing to 5 point 75 comma 59 and then decreasing from there to 12 comma 52 point 5.

Exercise 5.3.3.2: Modeling Recycling

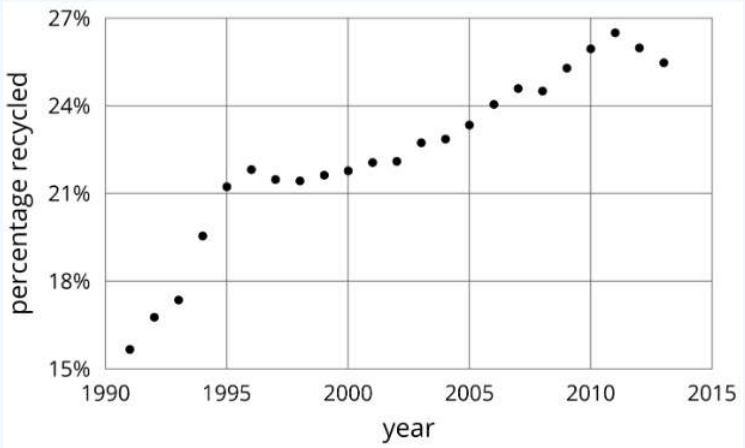


Figure 5.3.3.2: A scatterplot, horizontal, year, 1900 to 2015 by fives, vertical, percentage recycled, 15 % to 27 % by threes. Beginning with 1991 comma 16%, points trend linearly upward to 1996 comma 22%, then linearly upward at a less steep rate to 2011 comma 26 %, then two points trend down.

1. Approximate the percentage recycled each year with a piecewise linear function by drawing between three and five line segments to approximate the graph.
2. Find the slope for each piece. What do these slopes tell you?

Exercise 5.3.3.3: Dog Bath

Elena filled up the tub and gave her dog a bath. Then she let the water out of the tub.

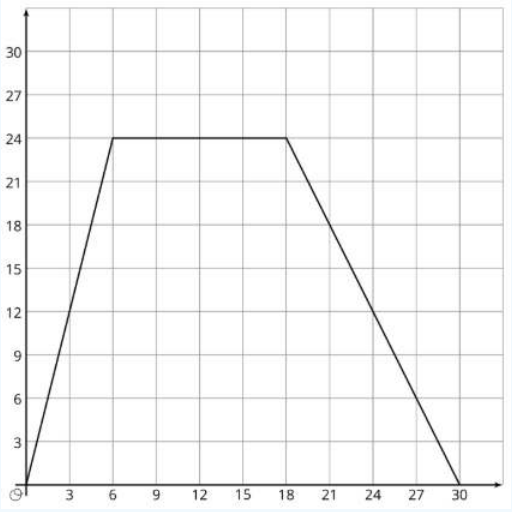


Figure 5.3.3.3

1. The graph shows the amount of water in the tub, in gallons, as a function of time, in minutes. Add labels to the graph to show this.
2. When did she turn off the water faucet?
3. How much water was in the tub when she bathed her dog?
4. How long did it take for the tub to drain completely?
5. At what rate did the faucet fill the tub?
6. At what rate did the water drain from the tub?

Exercise 5.3.3.4: Distance and Speed

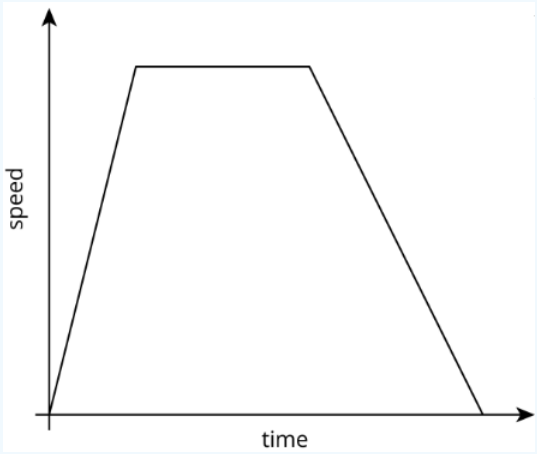


Figure 5.3.3.4

The graph shows the speed of a car as a function of time. Describe what a person watching the car would see.

Are you ready for more?

The graph models the speed of a car over a function of time during a 3-hour trip. How far did the car go over the course of the trip?

There is a nice way to visualize this quantity in terms of the graph. Can you find it?

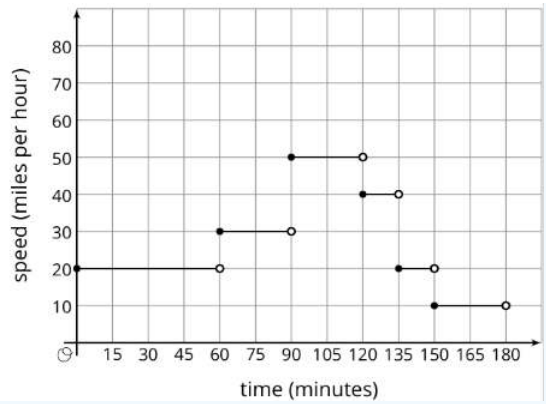


Figure 5.3.3.5: Coordinate plane, horizontal axis, time in minutes, 0 to 180 by fifteens. Vertival axis, speed in miles per hour, 0 to 80 by tens. A segment from closed circle at (0 comma 20) to open circle at (0 comma 60). segment from closed circle at (60 comma 30) to open circle at (90 comma 30). Segment from closed circle at (90 comma 50) to open circle at (120 comma 50). Segment from closed circle at (120 comma 40) to open circle at (135 comma 40). Segment from closed circle at (135 comma 20) to open circle (150 comma 20). Segment from closed circle (150 comma 10) to open circle at (180 comma 10).

Summary

This graph shows Andre biking to his friend's house where he hangs out for a while. Then they bike together to the store to buy some groceries before racing back to Andre's house for a movie night. Each line segment in the graph represents a different part of Andre's travels.

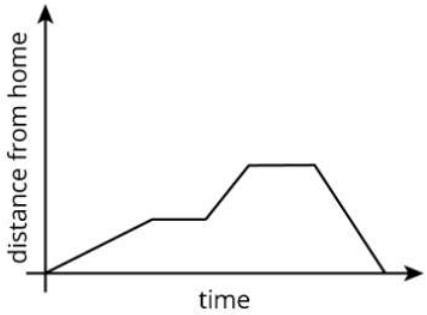


Figure 5.3.3.6: Graph composed of 5 linear sections. Horizontal axis, time, vertical axis, distance from home. Beginning at the origin, first segment slopes up as it moves right. Second segment horizontal, third segment slopes up and right, more steeply than the first segment. Fourth segment horizontal, fifth segment slopes down and right back to the horizontal axis.

This is an example of a piecewise linear function, which is a function whose graph is pieced together out of line segments. It can be used to model situations in which a quantity changes at a constant rate for a while, then switches to a different constant rate.

We can use piecewise functions to represent stories, or we can use them to model actual data. In the second example, temperature recordings at several times throughout a day are modeled with a piecewise function made up of two line segments. Which line segment do you think does the best job of modeling the data?

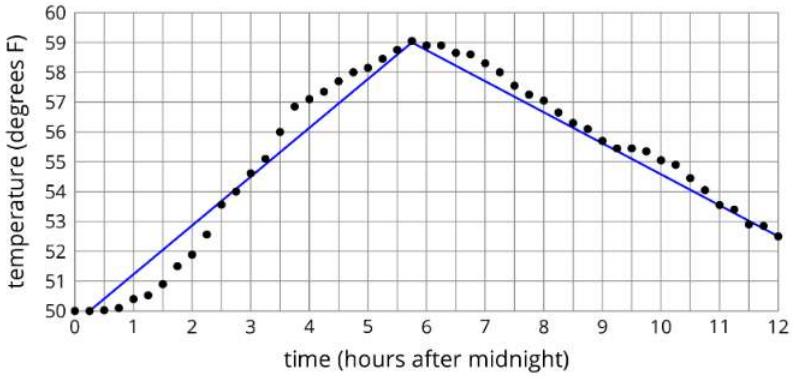


Figure 5.3.3.7: Scatterplot, horizontal, time in hours after midnight, 0 to 12 by ones, vertical, temperature in degrees Fahrenheit. Fifty points approximate a straight line from point 2 5 comma 50 increasing to 5 point 75 comma 59 and then decreasing from there to 12 comma 52 point 5.

Practice

Exercise 5.3.3.5

The graph shows the distance of a car from home as a function of time.

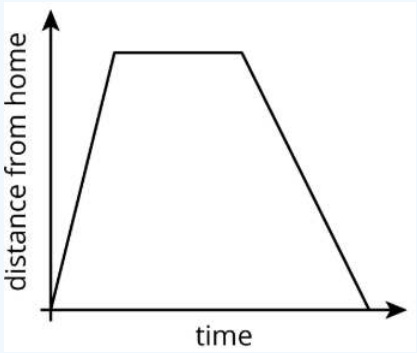


Figure 5.3.3.8

Describe what a person watching the car may be seeing.

Exercise 5.3.3.6

The equation and the graph represent two functions. Use the equation $y = 4$ and the graph to answer the questions.

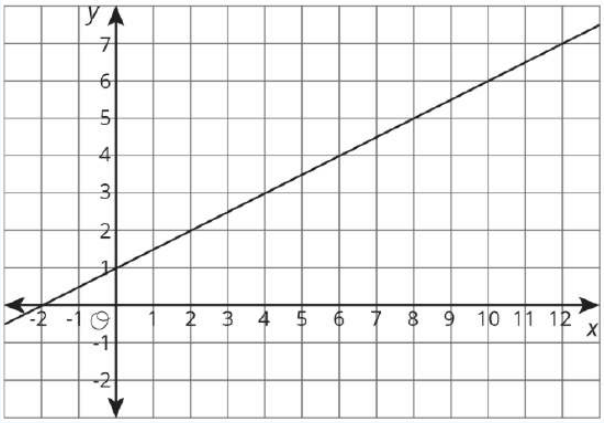


Figure 5.3.3.9

1. When x is 4, is the output of the equation or the graph greater?
2. What value for x produces the same output in both the graph and the equation?

https://math.libretexts.org/@go/page/36033

(From Unit 5.2.5)

Exercise 5.3.3.7

This graph shows a trip on a bike trail. The trail has markers every 0.5 km showing the distance from the beginning of the trail.

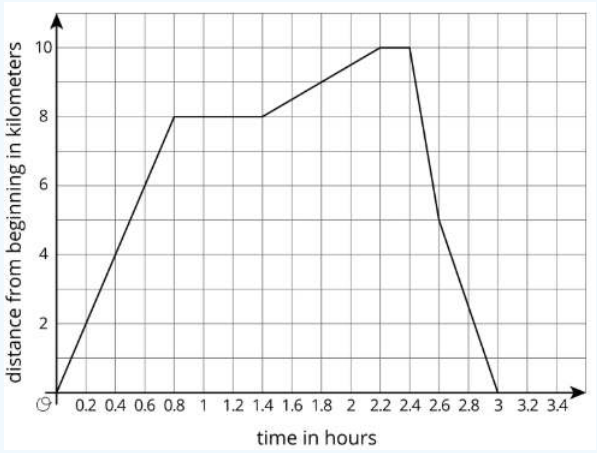


Figure 5.3.3.10: Coordinate plane, x, time in hours, 0 to 3 point 4 by point 2, y, distance from beginning in kilometers, 0 to 10 by 2. Line segments connecting the origin and (point 8 comma 8), (1 point 4 comma 8), (2 point 2 comma 10), (2 point 4 comma 10), (2 point 6 comma 5), (3 comma 0).

1. When was the bike rider going the fastest?
2. When was the bike rider going the slowest?
3. During what times was the rider going away from the beginning of the trail?
4. During what times was the rider going back towards the beginning of the trail?
5. During what times did the rider stop?

Exercise 5.3.3.8

The expression $-25t+1250$ represents the volume of liquid of a container after t seconds. The expression $50t+250$ represents the volume of liquid of another container after t seconds. What does the equation $-25t+1250=50t+250$ mean in this situation?

(From Unit 4.2.8)

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5.4: Cylinders and Cones

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5.4.1: Filling Containers

Lesson

Let's fill containers with water.

Exercise 5.4.1.1: Which One Doesn't Belong: Solids

These are drawings of three-dimensional objects. Which one doesn't belong? Explain your reasoning.

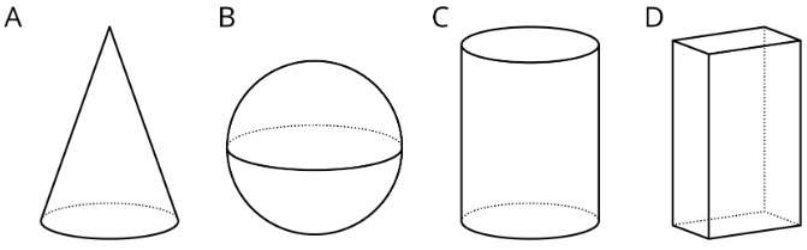


Figure 5.4.1.1

Exercise 5.4.1.2: Height and Volume

Use the applet to investigate the height of water in the cylinder as a function of the water volume.

1. Before you get started, make a prediction about the shape of the graph.
2. Check Reset and set the radius and height of the graduated cylinder to values you choose.
3. Let the cylinder fill with different amounts of water and record the data in the table.
4. Create a graph that shows the height of the water in the cylinder as a function of the water volume.
5. Choose a point on the graph and explain its meaning in the context of the situation.

Exercise 5.4.1.3: What is the Shape?

1. The graph shows the height vs. volume function of an unknown container. What shape could this container have? Explain how you know and draw a possible container.

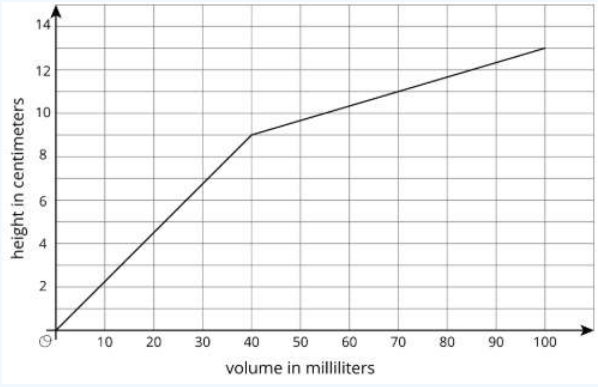


Figure 5.4.1.2

2. The graph shows the height vs. volume function of a different unknown container. What shape could this container have? Explain how you know and draw a possible container.

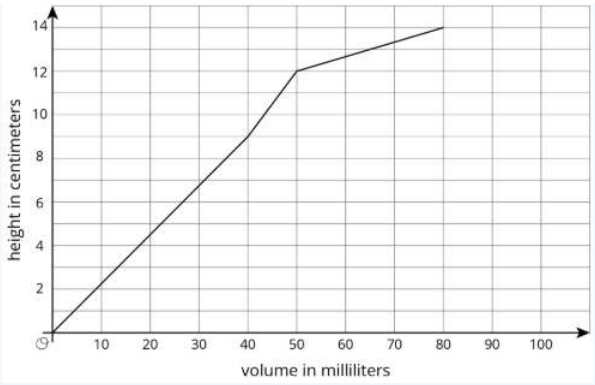


Figure 5.4.1.3

3. How are the two containers similar? How are they different?

Are you ready for more?

The graph shows the height vs. volume function of an unknown container. What shape could this container have? Explain how you know and draw a possible container.

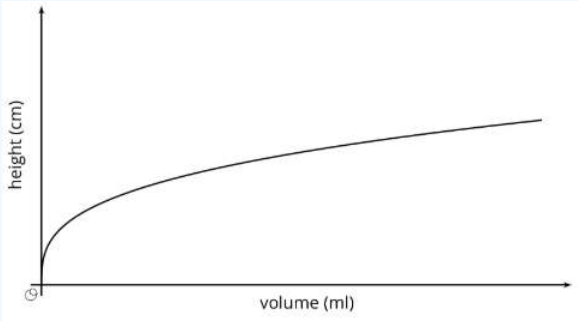


Figure 5.4.1.4

Summary

When filling a shape like a cylinder with water, we can see how the dimensions of the cylinder affect things like the changing height of the water. For example, let's say we have two cylinders, D and E, with the same height, but D has a radius of 3 cm and E has a radius of 6 cm.

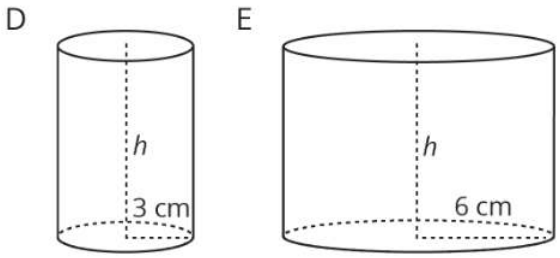


Figure 5.4.1.5

If we pour water into both cylinders at the same rate, the height of water in D will increase faster than the height of water in E due to its smaller radius. This means that if we made graphs of the height of water as a function of the volume of water for each cylinder, we would have two lines and the slope of the line for cylinder D would be greater than the slope of the line for cylinder E.

Glossary Entries

Definition: Cylinder

A cylinder is a three-dimensional figure like a prism, but with bases that are circles.

Practice

Exercise 5.4.1.4

Cylinder A, B, and C have the same radius but different heights. Put the cylinders in order of their volume from least to greatest.

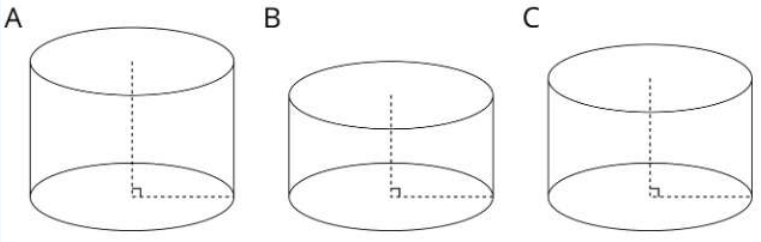


Figure 5.4.1.1

Exercise 5.4.1.5

Two cylinders, a and b, each started with different amounts of water. The graph shows how the height of the water changed as the volume of water increased in each cylinder. Match the graphs of a and b to Cylinders P and Q. Explain your reasoning.

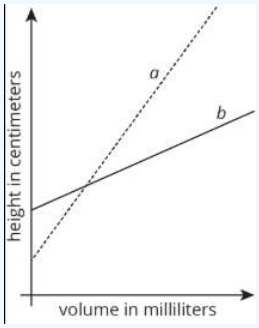


Figure 5.4.1.2

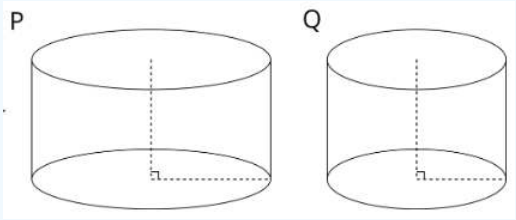


Figure 5.4.1.3

Exercise 5.4.1.6

Which of the following graphs could represent the volume of water in a cylinder as a function of its height? Explain your reasoning.

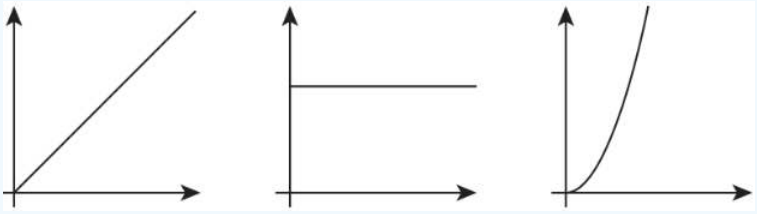


Figure 5.4.1.4

Exercise 5.4.1.7

Together, the areas of the rectangles sum to 30 square centimeters.

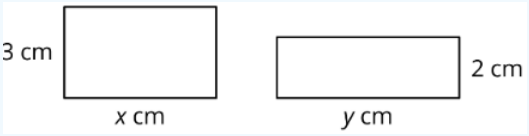


Figure 5.4.1.5

1. Write an equation showing the relationship between x and y.
2. Fill in the table with the missing values.

Table 5.4.1.1

x	3		8		12
y		5		10	

(From Unit 5.2.1)

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5.4.2: How Much Will Fit?

Lesson

Let's reason about the volume of different shapes.

Exercise 5.4.2.1: Two Containers

Your teacher will show you some containers. The small container holds 200 beans. Estimate how many beans the large jar holds.

Exercise 5.4.2.2: What's Your Estimate?

Your teacher will show you some containers.

1. If the pasta box holds 8 cups of rice, how much rice would you need for the other rectangular prisms?
2. If the pumpkin can holds 15 fluid ounces of rice, how much do the other cylinders hold?
3. If the small **cone** holds 2 fluid ounces of rice, how much does the large cone hold?
4. If the golf ball were hollow, it would hold about 0.2 cups of water. If the baseball were hollow, how much would the **sphere** hold?

Exercise 5.4.2.3: Do You Know These Figures?

- What shapes are the faces of each type of object shown here? For example, all six faces of a cube are squares

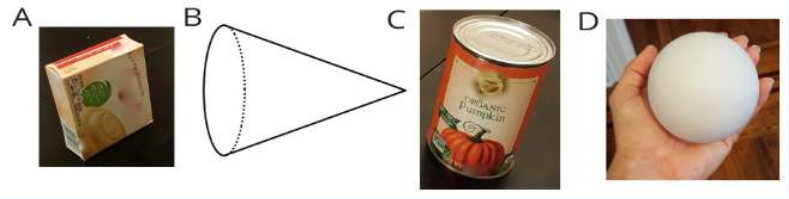


Figure 5.4.2.1

1. Which faces could be referred to as a "base" of the object?
2. Here is a method for quickly sketching a cylinder:

- Draw two ovals.
- Connect the edges.
- Which parts of your drawing would be hidden behind the cylinder? Make these parts dashed lines.

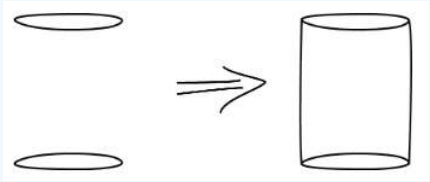


Figure 5.4.2.2

- Practice sketching some cylinders. Sketch a few different sizes, including short, tall, narrow, wide, and sideways. Label the radius r and height h on each cylinder.

Are you ready for more?



Figure 5.4.2.3: **Attribution:** "football / soccer ball", by Tobbi. Public Domain. OpenClipArt. Source.

A soccer ball is a polyhedron with 12 black pentagonal faces and 20 white hexagonal faces. How many edges in total are on this polyhedron?

Summary

The volume of a three-dimensional figure, like a jar or a room, is the amount of space the shape encloses. We can measure volume by finding the number of equal-sized volume units that fill the figure without gaps or overlaps. For example, we might say that a room has a volume of 1,000 cubic feet, or that a pitcher can carry 5 gallons of water. We could even measure volume of a jar by the number of beans it could hold, though a bean count is not really a measure of the volume in the same way that a cubic centimeter is because there is space between the beans. (The number of beans that fit in the jar do depend on the volume of the jar, so it is an okay estimate when judging the relative sixes of containers.)

In earlier grades, we studied three-dimensional figures with flat faces that are polygons. We learned how to calculate the volumes of rectangular prisms. Now we will study three-dimensional figures with circular faces and curved surfaces: cones, cylinders, and spheres.

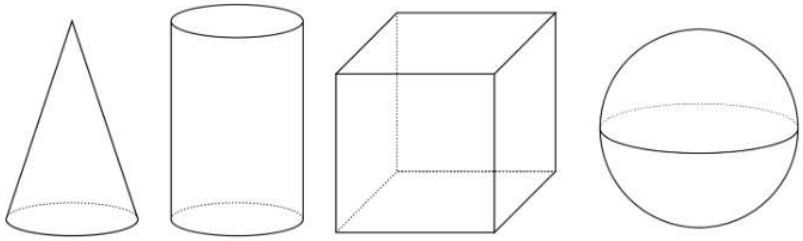


Figure 5.4.2.4

To help us see the shapes better, we can use dotted lines to represent parts that we wouldn't be able to see if a solid physical object were in front of us. For example, if we think of the cylinder in this picture as representing a tin can, the dotted arc in the bottom half of that cylinder represents the back half of the circular base of the can. What objects could the other figures in the picture represent?

Glossary Entries

Definition: Cone

A cone is a three-dimensional figure like a pyramid, but the base is a circle.

Definition: Cylinder

A cylinder is a three-dimensional figure like a prism, but with bases that are circles.

Definition: Sphere

A sphere is a three-dimensional figure in which all cross-sections in every direction are circles.

Practice

Exercise 5.4.2.4

1. Sketch a cube and label its side length as 4 cm (this will be Cube A).
2. Sketch a cube with sides that are twice as long as Cube A and label its side length (this will be Cube B).
3. Find the volumes of Cube A and Cube B.

Exercise 5.4.2.5

Two paper drink cups are shaped like cones. The small cone can hold 6 oz of water. The large cone is $\frac{4}{3}$ the height and $\frac{4}{3}$ the diameter of the small cone. Which of these could be the amount of water the large cone holds?

A. 8 cm

B. 14 oz
C. 4.5 oz
D. 14 cm

Exercise 5.4.2.6

The graph represents the volume of a cylinder with a height equal to its radius.

1. When the diameter is 2 cm, what is the radius of the cylinder?
2. Express the volume of a cube of side length s as an equation.
3. Make a table for volume of the cube at $s = 0$ cm, $s = 1$ cm, $s = 2$ cm, and $s = 3$ cm.
4. Which volume is greater: the volume of the cube when $s = 3$ cm, or the volume of the cylinder when its diameter is 3 cm?

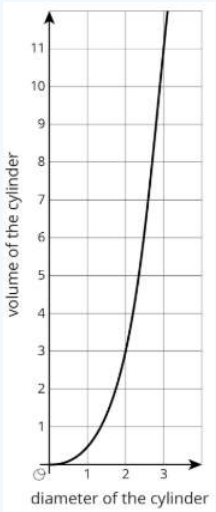


Figure 5.4.2.5

(From Unit 5.2.5)

Exercise 5.4.2.7

Select **all** the points that are on a line with slope 2 that also contains the point $(2, -1)$.

A. $(3, 1)$
B. $(1, 1)$
C. $(1, -3)$
D. $(4, 0)$
E. $(6, 7)$

(From Unit 3.3.2)

Exercise 5.4.2.8

Several glass aquariums of various sizes are for sale at a pet shop. They are all shaped like rectangular prisms. A 15-gallon tank is 24 inches long, 12 inches wide, and 12 inches tall. Match the dimensions of the other tanks with the volume of water they can each hold.

A. Tank 1: 36 inches long, 18 inches wide, and 12 inches tall
B. Tank 2: 16 inches long, 8 inches wide, and 10 inches tall
C. Tank 3: 30 inches long, 12 inches wide, and 12 inches tall
D. Tank 4: 20 inches long, 10 inches wide, and 12 inches tall

1. 5 gallons
2. 10 gallons

3. 20 gallons
4. 30 gallons

Exercise 5.4.2.9

Solve:

$$\begin{cases} y = -2x - 20 \\ y = x + 4 \end{cases}$$

(From Unit 4.3.5)

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5.4.3: The Volume of a Cylinder

Lesson

Let's explore cylinders and their volumes.

Exercise 5.4.3.1: A Circle's Dimensions

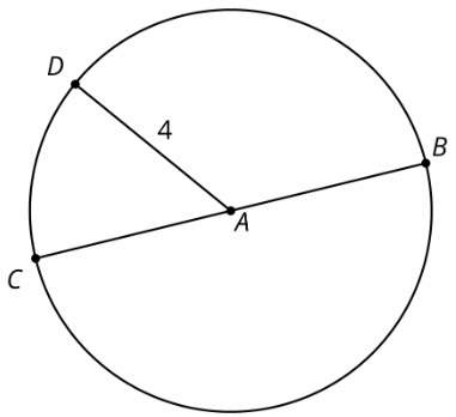


Figure 5.4.3.1

Here is a circle. Points A, B, C, and D are drawn, as well as Segments AD and BC.

1. What is the area of the circle, in square units? Select all that apply.
   a. 4π
   b. $\pi 8$
   c. 16π
   d. $\pi 4^2$
   e. approximately 25
   f. approximately 50
2. If the area of a circle is 49π square units, what is its radius? Explain your reasoning.

Exercise 5.4.3.2: Circular Volumes

What is the volume of each figure, in cubic units? Even if you aren't sure, make a reasonable guess.

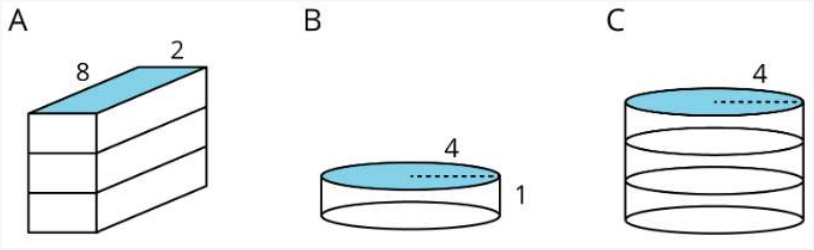


Figure 5.4.3.2

1. Figure A: A rectangular prism whose base has an area of 16 square units and whose height is 3 units.
2. Figure B: A cylinder whose base has an area of 16π square units and whose height is 1 unit.
3. Figure C: A cylinder whose base has an area of 16π square units and whose height is 3 units.

Are you ready for more?

Table 5.4.3.1

prism	prism	prism	cylinder
base: square	base: hexagon	base: octagon	base: circle

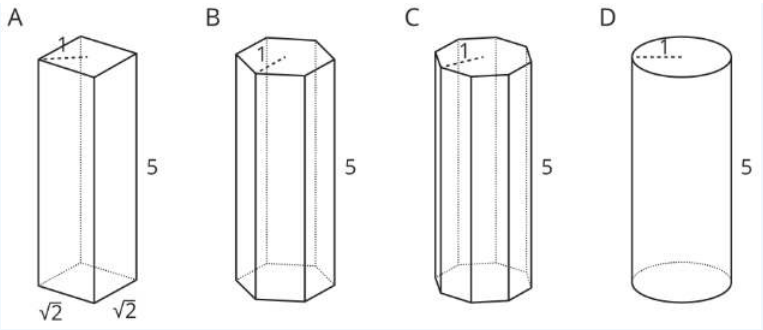


Figure 5.4.3.3

Here are solids that are related by a common measurement. In each of these solids, the distance from the center of the base to the furthest edge of the base is 1 unit, and the height of the solid is 5 units. Use 3.14 as an approximation for π to solve these problems.

1. Find the area of the square base and the circular base.
2. Use these areas to compute the volumes of the rectangular prism and the cylinder. How do they compare?
3. Without doing any calculations, list the figures from smallest to largest by volume. Use the images and your knowledge of polygons to explain your reasoning.
4. The area of the hexagon is approximately 2.6 square units, and the area of the octagon is approximately 2.83 square units. Use these areas to compute the volumes of the prisms with the hexagon and octagon bases. How does this match your explanation to the previous question?

Exercise 5.4.3.3: A Cylinder's Dimensions

1. For cylinders A–D, sketch a radius and the height. Label the radius with an r and the height with an h.

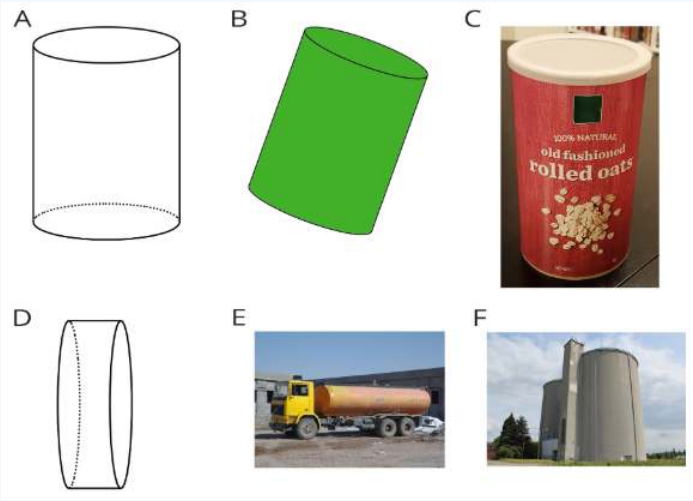


Figure 5.4.3.4: **Attribution:** "Volvo water tank truck in Iraq" / silo photo, by Jum Gordon / N3dling. Public Domain. Wikimedia Commons/ Pixabay. Source.

2. Earlier you learned how to sketch a cylinder. Sketch cylinders for E and F and label each one's radius and height.

Exercise 5.4.3.4: A Cylinder's Volume

1. Here is a cylinder with height 4 units and diameter 10 units.

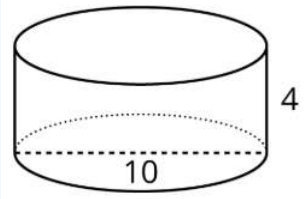


Figure 5.4.3.5

a. Shade the cylinder's base.
b. What is the area of the cylinder's base? Express your answer in terms of π.
c. What is the volume of this cylinder? Express your answer in terms of π.

2. A silo is a cylindrical container that is used on farms to hold large amounts of goods, such as grain. On a particular farm, a silo has a height of 18 feet and diameter of 6 feet. Make a sketch of this silo and label its height and radius. How many cubic feet of grain can this silo hold? Use 3.14 as an approximation for π.

Are you ready for more?

One way to construct a cylinder is to take a rectangle (for example, a piece of paper), curl two opposite edges together, and glue them in place.

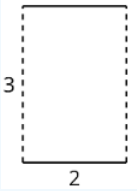


Figure 5.4.3.6

Which would give the cylinder with the greater volume: Gluing the two dashed edges together, or gluing the two solid edges together?

Summary

We can find the volume of a cylinder with radius r and height h using two ideas we've seen before:

- The volume of a rectangular prism is a result of multiplying the area of its base by its height.
- The base of the cylinder is a circle with radius r, so the base area is πr^2.

Remember that π is the number we get when we divide the circumference of any circle by its diameter. The value of π is approximately 3.14.

Just like a rectangular prism, the volume of a cylinder is the area of the base times the height. For example, take a cylinder whose radius is 2 cm and whose height is 5 cm.

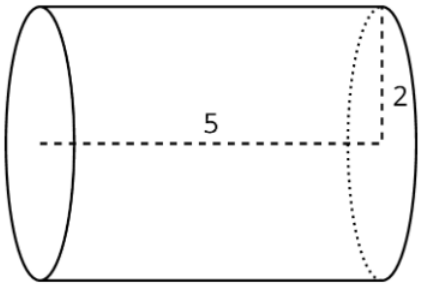


Figure 5.4.3.7

The base has an area of 4π cm^2 (since $\pi \cdot 2^2 = 4\pi$), so the volume is 20π cm^3 (since $4\pi \cdot 5 = 20\pi$). Using 3.14 as an approximation for , we can say that the volume of the cylinder is approximately 62.8 cm^3.

In general, the base of a cylinder with radius r units has area πr^2 square units. If the height is h units, then the volume V in cubic units is $V = \pi r^2 h$

Glossary Entries

Definition: Cone

A cone is a three-dimensional figure like a pyramid, but the base is a circle.

Definition: Cylinder

A cylinder is a three-dimensional figure like a prism, but with bases that are circles.

Definition: Sphere

A sphere is a three-dimensional figure in which all cross-sections in every direction are circles.

Practice

Exercise 5.4.3.5

- Sketch a cylinder.
- Label its radius 3 and its height 10.
- Shade in one of its bases.

Exercise 5.4.3.6

At a farm, animals are fed bales of hay and buckets of grain. Each bale of hay is in the shape a rectangular prism. The base has side lengths 2 feet and 3 feet, and the height is 5 feet. Each bucket of grain is a cylinder with a diameter of 3 feet. The height of the bucket is 5 feet, the same as the height of the bale.

1. Which is larger in area, the rectangular base of the bale or the circular base of the bucket? Explain how you know.
2. Which is larger in volume, the bale or the bucket? Explain how you know.

Exercise 5.4.3.7

Three cylinders have a height of 8 cm. Cylinder 1 has a radius of 1 cm. Cylinder 2 has a radius of 2 cm. Cylinder 3 has a radius of 3 cm. Find the volume of each cylinder.

Exercise 5.4.3.8

A one-quart container of tomato soup is shaped like a rectangular prism. A soup bowl shaped like a hemisphere can hold 8 oz of liquid. How many bowls will the soup container fill? Recall that 1 quart is equivalent to 32 fluid ounces (oz).

(From Unit 5.4.2)

Exercise 5.4.3.9

Match each set of information about a circle with the area of that circle.

A. Circle A has a radius of 4 units.
B. Circle B has a radius of 10 units.
C. Circle C has a radius of 16 units.
D. Circle D has a radius of 4π units.

1. 4π square units
2. approximately 314 square units
3. 64π square units
4. 16π square units

Exercise 5.4.3.10

Two students join a puzzle solving club and get faster at finishing the puzzles as they get more practice. Student A improves their times faster than Student B.

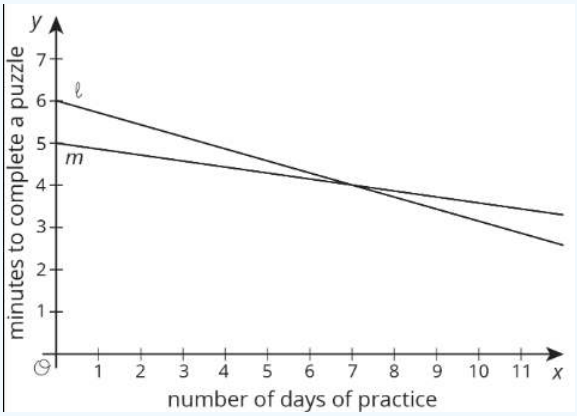


Figure 5.4.3.8

1. Match the students to the Lines l and m.
2. Which student was faster at puzzle solving before practice?

(From Unit 5.3.1)

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5.4.4: Finding Cylinder Dimensions

Lesson

Let's figure out the dimensions of cylinders.

Exercise 5.4.4.1: A Cylinder of Unknown Height

What is a possible volume for this cylinder if the diameter is 8 cm? Explain your reasoning.

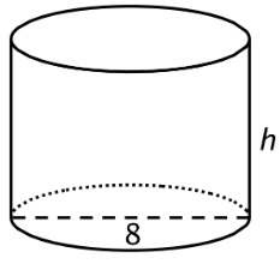


Figure 5.4.4.1

Exercise 5.4.4.2: What's the Dimension?

The volume V of a cylinder with radius r is given by the formula $V = \pi r^2 h$.

1. The volume of this cylinder with radius 5 units is 50π cubic units. This statement is true: $50\pi = 5^2\pi h$

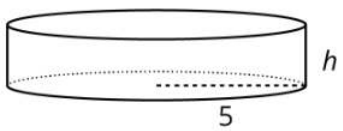


Figure 5.4.4.2

What does the height of this cylinder have to be? Explain how you know.

2. The volume of this cylinder with height 4 units is 36π cubic units. This statement is true: $36\pi = 4^2\pi 4$

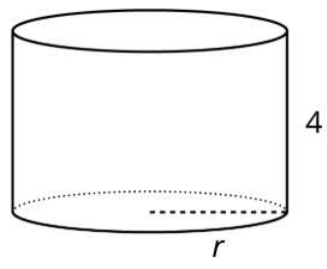


Figure 5.4.4.3

What does the radius of this cylinder have to be? Explain how you know.

Are you ready for more?

Suppose a cylinder has a volume of 36π cubic inches, but it is not the same cylinder as the one you found earlier in this activity.

1. What are some possibilities for the dimensions of the cylinder?
2. How many different cylinders can you find that have a volume of 36π cubic inches?

Exercise 5.4.4.3: Cylinders with Unknown Dimensions

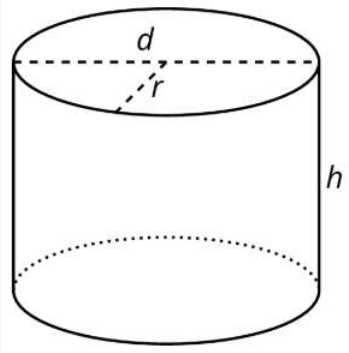


Figure 5.4.4.4

Each row of the table has information about a particular cylinder. Complete the table with the missing dimensions.

Table 5.4.4.1

diameter (units)	radius (units)	area of the base (square units)	height (units)	volume (cubic units)
	3		5	
12				108π
			11	99π
8				16π
			100	16π
	10			20π
20				314
			b	$\pi \cdot b \cdot a^2$

Summary

In an earlier lesson we learned that the volume, V, of a cylinder with radius r and height h is

$$V = \pi r^2 h$$

We say that the volume depends on the radius and height, and if we know the radius and height, we can find the volume. It is also true that if we know the volume and one dimension (either radius or height), we can find the other dimension.

For example, imagine a cylinder that has a volume of $500\pi\,\text{cm}^3$ and a radius of 5 cm, but the height is unknown. From the volume formula we know that

$$500\pi = \pi \cdot 25 \cdot h$$

must be true. Looking at the structure of the equation, we can see that $500 = 25h$. That means that the height has to be 20 cm, since $500 \div 25 = 20$.

Now imagine another cylinder that also has a volume of $500\pi\,\text{cm}^3$ with an unknown radius and a height of 5 cm. Then we know that

$$500\pi = \pi \cdot r^2 \cdot 5$$

must be true. Looking at the structure of this equation, we can see that . So the radius must be 10 cm.

Glossary Entries

Definition: Cone

A cone is a three-dimensional figure like a pyramid, but the base is a circle.

Definition: Cylinder

A cylinder is a three-dimensional figure like a prism, but with bases that are circles.

Definition: Sphere

A sphere is a three-dimensional figure in which all cross-sections in every direction are circles.

Practice

Exercise 5.4.4.4

Complete the table with all of the missing information about three different cylinders.

Table 5.4.4.2

diameter of base (units)	area of base (square units)	height (units)	volume (cubic units)
4		10	
6			63π
	25π	6	

Exercise 5.4.4.5

A cylinder has volume 45π and radius 3. What is its height?

Exercise 5.4.4.6

Three cylinders have a volume of 2826 cm^3. Cylinder A has a height of 900 cm. Cylinder B has a height of 225 cm. Cylinder C has a height of 100 cm. Find the radius of each cylinder. Use 3.14 as an approximation for π.

Exercise 5.4.4.7

A gas company's delivery truck has a cylindrical tank that is 14 feet in diameter and 40 feet long.

1. Sketch the tank, and mark the radius and the height.
2. How much gas can fit in the tank?

(From Unit 5.4.3)

Exercise 5.4.4.8

Here is a graph that shows the water height of the ocean between September 22 and September 24, 2016 in Bodega Bay, CA.

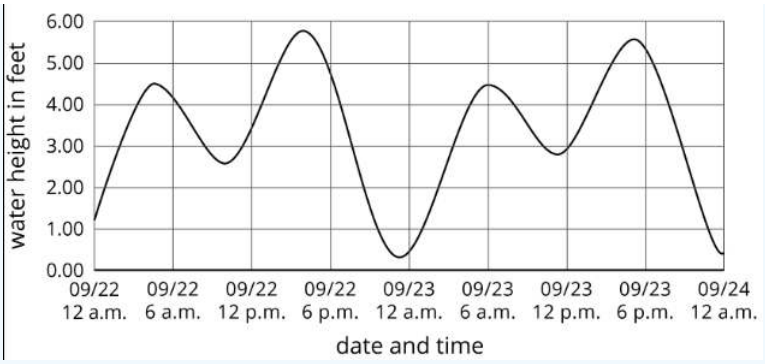


Figure 5.4.4.5

1. Estimate the water height at 12 p.m. on September 22.
2. How many times was the water height 5 feet? Find two times when this happens.
3. What was the lowest the water got during this time period? When does this occur?
4. Does the water ever reach a height of 6 feet?

(From Unit 5.2.3)

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5.4.5: The Volume of a Cone

Lesson

Let's explore cones and their volumes.

Exercise 5.4.5.1: Which Has a Larger Volume

The cone and cylinder have the same height, and the radii of their bases are equal.

1. Which figure has a larger volume?
2. Do you think the volume of the smaller one is more or less than $\frac{1}{2}$ the volume of the larger one? Explain your reasoning.
3. Sketch two different sized cones. The oval doesn't have to be on the bottom! For each drawing, label the cone's radius with r and height with h.

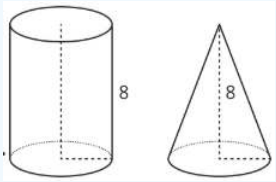


Figure 5.4.5.1

Here is a method for quickly sketching a cone:



Figure 5.4.5.2

- Draw an oval.
- Draw a point centered above the oval.
- Connect the edges of the oval to the point.
- Which parts of your drawing would be hidden behind the object? Make these parts dashed lines.

Exercise 5.4.5.2: From Cylinders to Cones

A cone and cylinder have the same height and their bases are congruent circles.

1. If the volume of the cylinder is 90 cm^3, what is the volume of the cone?
2. If the volume of the cone is 120 cm^3, what is the volume of the cylinder?
3. If the volume of the cylinder is $v = \pi r^2 h$, what is the volume of the cone? Either write an expression for the cone or explain the relationship in words.

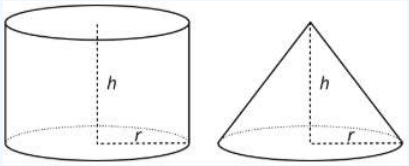


Figure 5.4.5.3

Exercise 5.4.5.3: Calculate That Cone

1. Here is a cylinder and cone that have the same height and the same base area. What is the volume of each figure? Express your answers in terms of π.

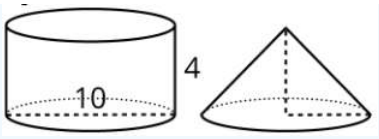


Figure 5.4.5.4

2. Here is a cone.

a. What is the area of the base? Express your answer in terms of π.
b. What is the volume of the cone? Express your answer in terms of π.

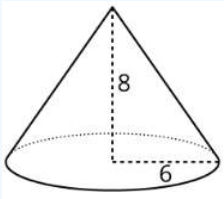


Figure 5.4.5.5

3. A cone-shaped popcorn cup has a radius of 5 centimeters and a height of 9 centimeters. How many cubic centimeters of popcorn can the cup hold? Use 3.14 as an approximation for π, and give a numerical answer.

Are you ready for more?

A grain silo has a cone shaped spout on the bottom in order to regulate the flow of grain out of the silo. The diameter of the silo is 8 feet. The height of the cylindrical part of the silo above the cone spout is 12 feet while the height of the entire silo is 16 feet.

How many cubic feet of grain are held in the cone spout of the silo? How many cubic feet of grain can the entire silo hold?

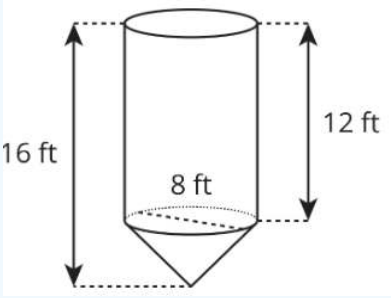


Figure 5.4.5.6

Summary

If a cone and a cylinder have the same base and the same height, then the volume of the cone is $\frac{1}{3}$ of the volume of the cylinder. For example, the cylinder and cone shown here both have a base with radius 3 feet and a height of 7 feet.

The cylinder has a volume of 63π cubic feet since $\pi \cdot 3^2 \cdot 7 = 63\pi$. The cone has a volume that is $\frac{1}{3}$ of that, or 21π cubic feet.

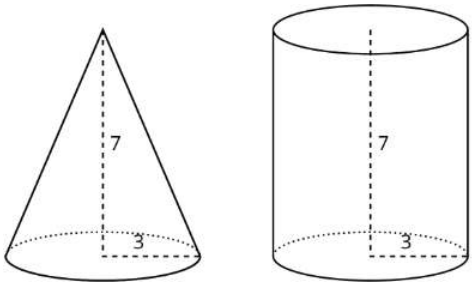


Figure 5.4.5.7

If the radius for both is r and the height for both is h, then the volume of the cylinder is pir^2h. That means that the volume, V, of the cone is $V = \frac{1}{3}\pi r^2 h$

Glossary Entries

Definition: Cone

A cone is a three-dimensional figure like a pyramid, but the base is a circle.

Definition: Cylinder

A cylinder is a three-dimensional figure like a prism, but with bases that are circles.

Definition: Sphere

A sphere is a three-dimensional figure in which all cross-sections in every direction are circles.

Practice

Exercise 5.4.5.4

A cylinder and cone have the same height and radius. The height of each is 5 cm, and the radius is 2 cm. Calculate the volume of the cylinder and the cone.

Exercise 5.4.5.5

The volume of this cone is 36π cubic units.

What is the volume of a cylinder that has the same base area and the same height?

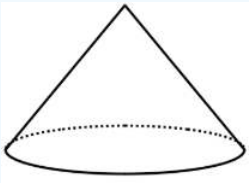


Figure 5.4.5.8

Exercise 5.4.5.6

A cylinder has a diameter of 6 cm and a volume of 36π cm^3.

1. Sketch the cylinder.
2. Find its height and radius in centimeters.
3. Label your sketch with the cylinder's height and radius.

(From Unit 5.4.4)

Exercise 5.4.5.7

Lin wants to get some custom T-shirts printed for her basketball team. Shirts cost $10 each if you order 10 or fewer shirts and $9 each if you order 11 or more shirts.

1. Make a graph that shows the total cost of buying shirts, for 0 through 15 shirts.
2. There are 10 people on the team. Do they save money if they buy an extra shirt? Explain your reasoning.
3. What is the slope of the graph between 0 and 10? What does it mean in the story?
4. What is the slope of the graph between 11 and 15? What does it mean in the story?

(From Unit 5.3.3)

Exercise 5.4.5.8

In the following graphs, the horizontal axis represents time and the vertical axis represents distance from school. Write a possible story for each graph.

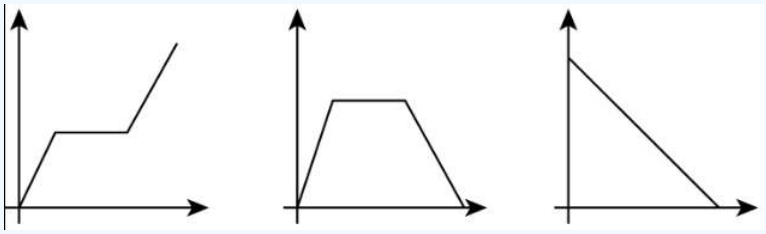


Figure 5.4.5.9

(From Unit 5.2.4)

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5.4.6: Finding Cone Dimensions

Lesson

Let's figure out the dimensions of cones.

Exercise 5.4.6.1: Number Talk: Thirds

For each equation, decide what value, if any, would make it true.

$27 = \frac{1}{3}h$

$27 = \frac{1}{3}r^2$

$12\pi = \frac{1}{3}\pi a$

$12\pi = \frac{1}{3}\pi b^2$

Exercise 5.4.6.2: An Unknown Radius

The volume V of a cone with radius r is given by the formula $V = \frac{1}{3}\pi r^2 h$.

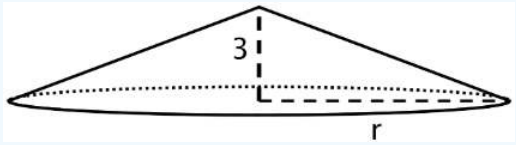


Figure 5.4.6.1

The volume of this cone with height 3 units and radius r is $V = 64\pi$ cubic units. This statement is true:

$$64\pi = \frac{1}{3}\pi r^2 \cdot 3$$

What does the radius of this cone have to be? Explain how you know.

Exercise 5.4.6.3: Cones with Unknown Dimensions

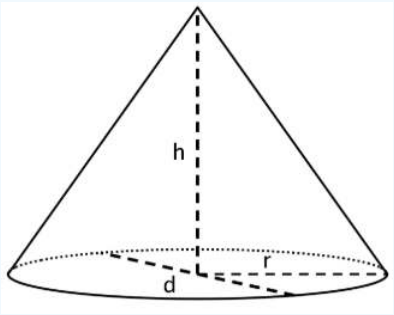


Figure 5.4.6.2

Each row of the table has some information about a particular cone. Complete the table with the missing dimensions.

Table 5.4.6.1

diameter (units)	radius (units)	area of the base (square units)	height (units)	volume of cone (cubic units)
	4		3	
	$\frac{1}{3}$		6	
		36π	$\frac{1}{4}$	
20				200π

diameter (units)	radius (units)	area of the base (square units)	height (units)	volume of cone (cubic units)
			12	64π
			3	3.14

Are you ready for more?

A *frustum* is the result of taking a cone and slicing off a smaller cone using a cut parallel to the base.



Figure 5.4.6.3

Find a formula for the volume of a frustum, including deciding which quantities you are going to include in your formula.

Exercise 5.4.6.4: Popcorn Deals

A movie theater offers two containers:

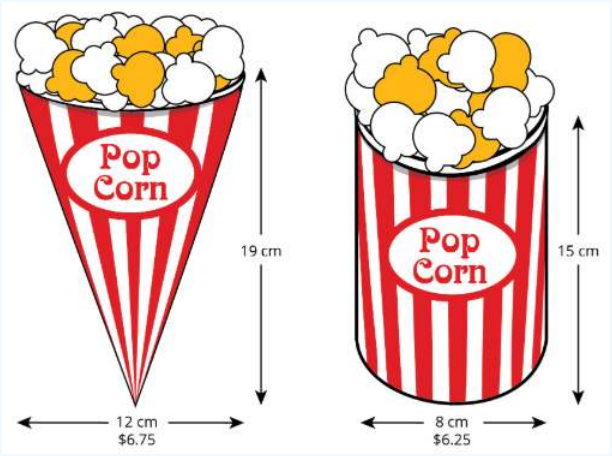


Figure 5.4.6.4

Which container is the better value? Use 3.14 as an approximation for π.

Summary

As we saw with cylinders, the volume V of a cone depends on the radius r of the base and the height h:

$$V = \frac{1}{3}\pi r^2 h$$

If we know the radius and height, we can find the volume. If we know the volume and one of the dimensions (either radius or height), we can find the other dimension.

For example, imagine a cone with a volume of $64\pi\,\text{cm}^3$, a height of 3 cm, and an unknown radius r. From the volume formula, we know that

$$64\pi = \frac{1}{3}\pi r^2 \cdot 3$$

Looking at the structure of the equation, we can see that $r^2 = 64$, so the radius must be 8 cm.

Now imagine a different cone with a volume of $18\pi\,\text{cm}^3$, a radius of 3 cm, and an unknown height h. Using the formula for the volume of the cone, we know that

$$18\pi = \tfrac{1}{3}\pi 3^2 h$$

so the height must be 6 cm. Can you see why?

Glossary Entries

Definition: Cone

A cone is a three-dimensional figure like a pyramid, but the base is a circle.

Definition: Cylinder

A cylinder is a three-dimensional figure like a prism, but with bases that are circles.

Definition: Sphere

A sphere is a three-dimensional figure in which all cross-sections in every direction are circles.

Practice

Exercise 5.4.6.5

The volume of this cylinder is 175π cubic units.

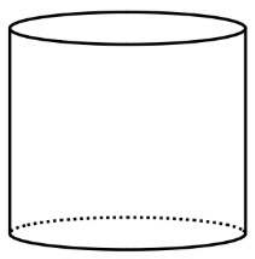


Figure 5.4.6.5

What is the volume of a cone that has the same base area and the same height?

(From Unit 5.4.5)

Exercise 5.4.6.6

A cone has volume 12π cubic inches. Its height is 4 inches. What is its radius?

Exercise 5.4.6.7

A cone has volume 3π.

1. If the cone's radius is 1, what is its height?
2. If the cone's radius is 2, what is its height?
3. If the cone's radius is 5, what is its height?
4. If the cone's radius is $\frac{1}{2}$, what is its height?
5. If the cone's radius in r, then what is the height?

Exercise 5.4.6.8

Three people are playing near the water. Person A stands on the dock. Person B starts at the top of a pole and ziplines into the water, then climbs out of the water. Person C climbs out of the water and up the zipline pole. Match the people to the graphs where the horizontal axis represents time in seconds and the vertical axis represents height above the water level in feet.

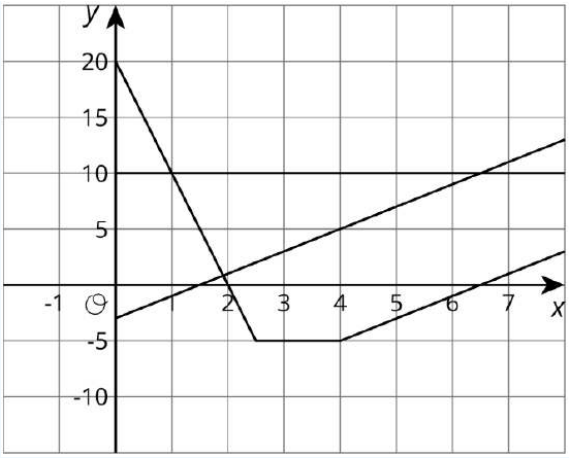


Figure 5.4.6.6: Coordinate plane, x, negative 1 to 8 by 1, y, negative 10 to 20 by 5. Three lines. First line starts at 0 comma 20, declines to 2 point 5 comma negative 5, horizontal to 4 comma negative 5, then increases to 3 comma 5. Second line begins at 0 comma 10 and remains horizontal to 8 comma 10. Third line begins at 0 comma negative 3 and increases to 8 comma 13.

(From Unit 5.2.4)

Exercise 5.4.6.9

A room is 15 feet tall. An architect wants to include a window that is 6 feet tall. The distance between the floor and the bottom of the window is b feet. The distance between the ceiling and the top of the window is a feet. This relationship can be described by the equation $a = 15 - (b + 6)$

1. Which variable is independent based on the equation given?
2. If the architect wants b to be 3, what does this mean? What value of a would work with the given value for b?
3. The customer wants the window to have 5 feet of space above it. Is the customer describing a or b? What is the value of the other variable?

(From Unit 5.2.1)

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5.5: Dimensions and Spheres

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5.5.1: Scaling One Dimension

Lesson

Let's see how changing one dimension changes the volume of a shape.

Exercise 5.5.1.1: Driving the Distance

Here is a graph of the amount of gas burned during a trip by a tractor-trailer truck as it drives at a constant speed down a highway:

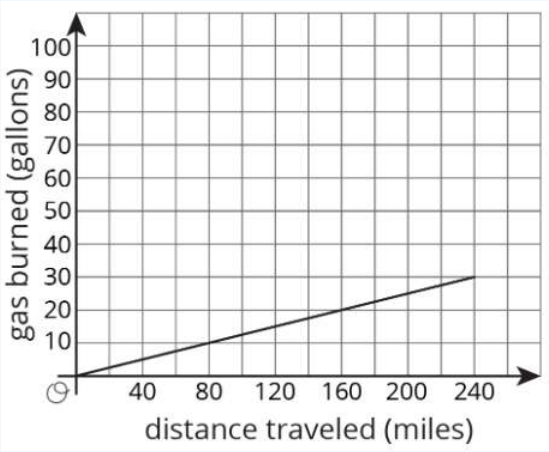


Figure 5.5.1.1

1. At the end of the trip, how far did the truck drive, and how much gas did it use?
2. If a truck traveled half this distance at the same rate, how much gas would it use?
3. If a truck traveled double this distance at the same rate, how much gas would it use?
4. Complete the sentence: _____ is a function of _____.

Exercise 5.5.1.2: Double the Edge

There are many right rectangular prisms with one edge of length 5 units and another edge of length 3 units. Let s represent the length of the third edge and V represent the volume of these prisms.

1. Write an equation that represents the relationship between V and s.
2. Graph this equation.
3. What happens to the volume if you double the edge length s? Where do you see this in the graph? Where do you see it algebraically?

Exercise 5.5.1.3: Halve the Height

There are many cylinders with radius 5 units. Let h represent the height and V represent the volume of these cylinders.

1. Write an equation that represents the relationship between V and h. Use 3.14 as an approximation of π.
2. Graph this equation.
3. What happens to the volume if you halve the height, h? Where can you see this in the graph? How can you see it algebraically?

Are you ready for more?

Suppose we have a rectangular prism with dimensions 2 units by 3 units by 6 units, and we would like to make a rectangular prism of volume 216 cubic units by stretching *one* of the three dimensions.

- What are the three ways of doing this? Of these, which gives the prism with the smallest surface area?
- Repeat this process for a starting rectangular prism with dimensions 2 units by 6 units by 6 units.

- Can you give some general tips to someone who wants to make a box with a certain volume, but wants to save cost on material by having as small a surface area as possible?

Exercise 5.5.1.4: Figuring Out Cone Dimensions

Here is a graph of the relationship between the height and the volume of some cones that all have the same radius:

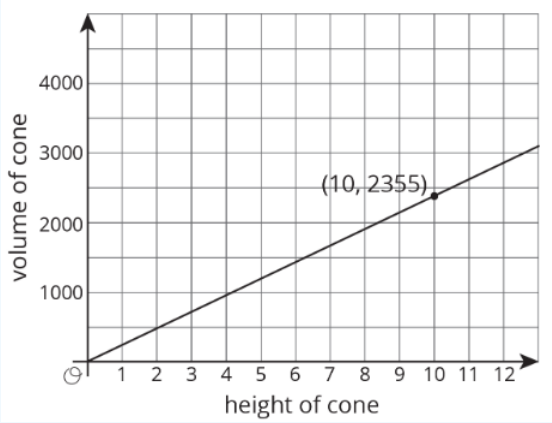


Figure 5.5.1.2

1. What do the coordinates of the labeled point represent?
2. What is the volume of the cone with height 5? With height 30?
3. Use the labeled point to find the radius of these cones. Use 3.14 as an approximation for π.
4. Write an equation that relates the volume V and height h.

Summary

Imagine a cylinder with a radius of 5 cm that is being filled with water. As the height of the water increases, the volume of water increases.

We say that the volume of the water in the cylinder, V, depends on the height of the water h. We can represent this relationship with an equation: $V = \pi \cdot 5^2 h$ or just

$$V = 25\pi h$$

This equation represents a *proportional relationship* between the height and the volume. We can use this equation to understand how the volume changes when the height is tripled.

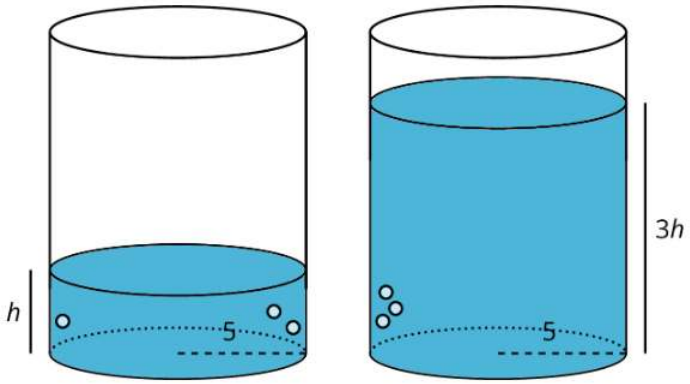


Figure 5.5.1.3

The new volume would be $V = 25\pi(3h) = 75\pi h$, which is precisely 3 times as much as the old volume of $25\pi h$. In general, when one quantity in a proportional relationship changes by a given factor, the other quantity changes by the same factor.

Remember that proportional relationships are examples of linear relationships, which can also be thought of as functions. So in this example V, the volume of water in the cylinder, is a function of the height h of the water.

Practice

Exercise 5.5.1.5

A cylinder has a volume of 48π cm^3 and height h. Complete this table for volume of cylinders with the same radius but different heights.

Table 5.5.1.1

height (cm)	volume (cm^3)
h	48π
$2h$	
$5h$	
$\frac{h}{2}$	
$\frac{h}{5}$	

Exercise 5.5.1.6

A cylinder has a radius of 3 cm and a height of 5 cm.

1. What is the volume of the cylinder?
2. What is the volume of the cylinder when its height is tripled?
3. What is the volume of the cylinder when its height is halved?

Exercise 5.5.1.7

A graduated cylinder that is 24 cm tall can hold 1 L of water. What is the radius of the cylinder? What is the height of the 500 ml mark? The 250 ml mark? Recall that 1 liter (L) is equal to 1000 milliliters (ml), and that 1 liter (L) is equal to 1,000 cm^3.

Exercise 5.5.1.8

An ice cream shop offers two ice cream cones. The waffle cone holds 12 ounces and is 5 inches tall. The sugar cone also holds 12 ounces and is 8 inches tall. Which cone has a larger radius?

(From Unit 5.4.6)

Exercise 5.5.1.9

A 6 oz paper cup is shaped like a cone with a diameter of 4 inches. How many ounces of water will a plastic cylindrical cup with a diameter of 4 inches hold if it is the same height as the paper cup?

(From Unit 5.4.5)

Exercise 5.5.1.10

Lin's smart phone was fully charged when she started school at 8:00 a.m. At 9:20 a.m., it was 90% charged, and at noon, it was 72% charged.

1. When do you think her battery will die?
2. Is battery life a function of time? If yes, is it a linear function? Explain your reasoning.

(From Unit 5.3.2)

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5.5.2: Scaling Two Dimensions

Lesson

Let's change more dimensions of shapes.

Exercise 5.5.2.1: Tripling Statements

m, n, a, b, and c all represent positive integers. Consider these two equations: $m = a + b + c \quad n = abc$

1. Which of these statements are true? Select **all** that apply.
   a. If a is tripled, m is tripled.
   b. If a, b, and c are all tripled, then m is tripled.
   c. If a is tripled, n is tripled.
   d. If a, b, and c are all tripled, then n is tripled.
2. Create a true statement of your own about one of the equations.

Exercise 5.5.2.2: A Square Base

Clare sketches a rectangular prism with a height of 11 and a square base and labels the edges of the base s. She asks Han what he thinks will happen to the volume of the rectangular prism if she triples s.

Han says the volume will be 9 times bigger. Is he right? Explain or show your reasoning.

Are you ready for more?

A cylinder can be constructed from a piece of paper by curling it so that you can glue together two opposite edges (the dashed edges in the figure).

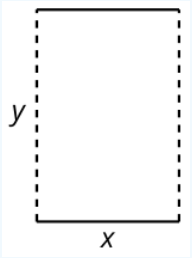


Figure 5.5.2.1

1. If you wanted to increase the volume inside the resulting cylinder, would it make more sense to double x, y, or does it not matter?
2. If you wanted to increase the surface area of the resulting cylinder, would it make more sense to double x, y, or does it not matter?
3. How would your answers to these questions change if we made a cylinder by gluing together the solid lines instead of the dashed lines?

Exercise 5.5.2.3: Playing with Cones

There are many cones with a height of 7 units. Let r represent the radius and V represent the volume of these cones.

1. Write an equation that expresses the relationship between V and r. Use 3.14 as an approximation for π.
2. Predict what happens to the volume if you triple the value of r.
3. Graph this equation.
4. What happens to the volume if you triple r? Where do you see this in the graph? How can you see it algebraically?

Summary

There are many rectangular prisms that have a length of 4 units and width of 5 units but differing heights. If h represents the height, then the volume V of such a prism is

$$V = 20h$$

The equation shows us that the volume of a prism with a base area of 20 square units is a linear function of the height. Because this is a proportional relationship, if the height gets multiplied by a factor of a, then the volume is also multiplied by a factor of a:

$$V = 20(ah)$$

What happens if we scale *two* dimensions of a prism by a factor of a? In this case, the volume gets multiplied by a factor of a twice, or a^2.

For example, think about a prism with a length of 4 units, width of 5 units, and height of 6 units. Its volume is 120 cubic units since $4 \cdot 5 \cdot 6 = 120$. Now imagine the length and width each get scaled by a factor of a, meaning the new prism has a length of $4a$, width of $5a$, and a height of 6. The new volume is $120a^2$ cubic units since $4a \cdot 5a \cdot 6 = 120a^2$.

A similar relationship holds for cylinders. Think of a cylinder with a height of 6 and a radius of 5. The volume would be 150π cubic units since $\pi \cdot 5^2 \cdot 6 = 150\pi$. Now, imagine the radius is scaled by a factor of a. Then the new volume is $\pi \cdot (5a)^2 \cdot 6 = \pi \cdot 25a^2 \cdot 6$ or $150a^2\pi$ cubic units. So scaling the radius by a factor of a has the effect of multiplying the volume by a^2!

Why does the volume multiply by a^2 when only the radius changes? This makes sense if we imagine how scaling the radius changes the base area of the cylinder. As the radius increases, the base area gets larger in two dimensions (the circle gets wider and also taller), while the third dimension of the cylinder, height, stays the same.

Practice

Exercise 5.5.2.4

There are many cylinders with a height of 18 meters. Let r represent the radius in meters and V represent the volume in cubic meters.

1. Write an equation that represents the volume V as a function of the radius r.
2. Complete this table, giving three possible examples.

Table 5.5.2.1

r	V
1	

3. If the radius of a cylinder is doubled, does the volume double? Explain how you know.
4. Is the graph of this function a line? Explain how you know.

Exercise 5.5.2.5

As part of a competition, Diego must spin around in a circle 6 times and then run to a tree. The time he spends on each spin is represented by s and the time he spends running is r. He gets to the tree 21 seconds after he starts spinning.

1. Write an equation showing the relationship between s and r.
2. Rearrange the equation so that it shows r as a function of s.
3. If it takes Diego 1.2 seconds to spin around each time, how many seconds did he spend running?

(From Unit 5.2.1)

Exercise 5.5.2.6

The table and graph represent two functions. Use the table and graph to answer the questions.

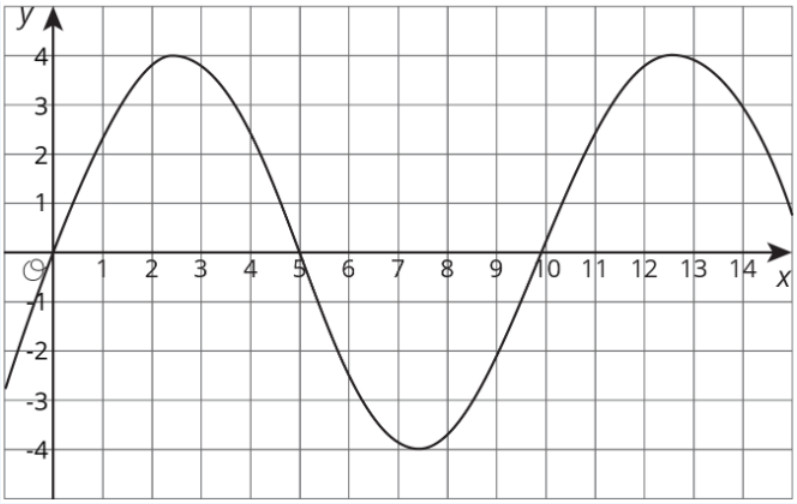


Figure 5.5.2.2: A coordinate plane, x, 0 to 14 by ones, y, negative 4 to 4. The graph begins at the origin, increases through approximately (2 point 5 comma 4), then decreases through (5 comma 0) to (7 point 5 comma negative 4), then increases through (10 comma 0) to (12 point 5 comma 4) then begins to decrease again.

Table 5.5.2.2

x	1	2	3	4	5	6
y	3	-1	0	4	5	-1

1. For which values of x is the output from the table less than the output from the graph?
2. In the graphed function, which values of x give an output of 0?

(From Unit 5.2.5)

Exercise 5.5.2.7

A cone has a radius of 3 units and a height of 4 units.

1. What is this volume of this cone?
2. Another cone has quadruple the radius, and the same height. How many times larger is the new cone's volume?

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5.5.3: Estimating a Hemisphere

Lesson

Let's estimate volume of hemispheres with figures we know.

Exercise 5.5.3.1: Notice and Wonder: Two Shapes

Here are two shapes.

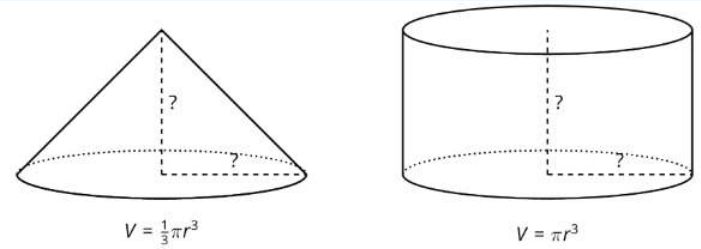


Figure 5.5.3.1

What do you notice? What do you wonder?

Exercise 5.5.3.2: Hemispheres in Boxes

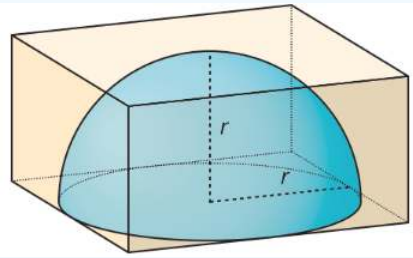


Figure 5.5.3.2

1. Mai has a dome paperweight that she can use as a magnifier. The paperweight is shaped like a hemisphere made of solid glass, so she wants to design a box to keep it in so it won't get broken. Her paperweight has a radius of 3 cm.
   a. What should the dimensions of the inside of box be so the box is as small as possible?
   b. What is the volume of the box?
   c. What is a reasonable estimate for the volume of the paperweight?
2. Tyler has a different box with side lengths that are twice as long as the sides of Mai's box. Tyler's box is just large enough to hold a different glass paperweight.
   a. What is the volume of the new box?
   b. What is a reasonable estimate for the volume of this glass paperweight?
   c. How many times bigger do you think the volume of the paperweight in this box is than the volume of Mai's paperweight? Explain your thinking.

Exercise 5.5.3.3: Estimating Hemispheres

1. A hemisphere with radius 5 units fits snugly into a cylinder of the same radius and height.

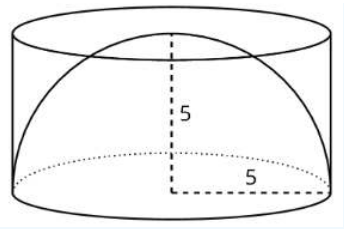


Figure 5.5.3.3

a. Calculate the volume of the cylinder.
b. Estimate the volume of the hemisphere. Explain your reasoning.

2. A cone fits snugly inside a hemisphere, and they share a radius of 5.

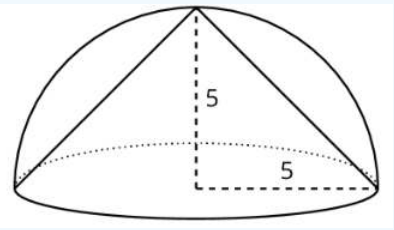


Figure 5.5.3.4

a. What is the volume of the cone?
b. Estimate the volume of the hemisphere. Explain your reasoning.

3. Compare your estimate for the hemisphere with the cone inside to your estimate of the hemisphere inside the cylinder. How do they compare to the volumes of the cylinder and the cone?

Are you ready for more?

Estimate what fraction of the volume of the cube is occupied by the pyramid that shares the base and a top vertex with the cube, as in the figure.

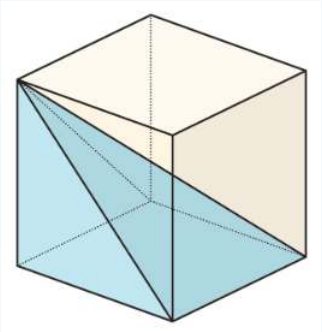


Figure 5.5.3.5

Summary

We can estimate the volume of a hemisphere by comparing it to other shapes for which we know the volume. For example, a hemisphere of radius 1 unit fits inside a cylinder with a radius of 1 unit and height of 1 unit.

Since the hemisphere is *inside* the cylinder, it must have a smaller volume than the cylinder making the cylinder's volume a reasonable over-estimate for the volume of the hemisphere.

The volume of this particular cylinder is about 3.14 units^3 since $\pi(1)^2(1) = \pi$, so we know the volume of the hemisphere is less than 3.14 cubic units.

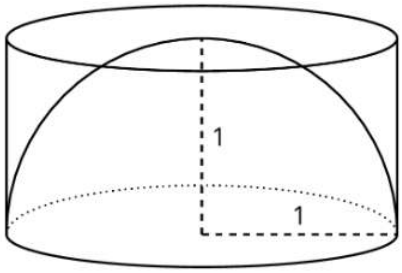


Figure 5.5.3.6

Using similar logic, a cone of radius 1 unit and height 1 unit fits inside of the hemisphere of radius 1 unit.

Since the cone is *inside* the hemisphere, the cone must have a smaller volume than the hemisphere making the cone's volume a reasonable under-estimate for the volume of the hemisphere.

The volume of this particular cone is about 1.05 units^3 since $\frac{1}{3}\pi(1)^2(1) = \frac{1}{3}\pi \approx 1.05$, so we know the volume of the hemisphere is more than 1.05 cubic units.

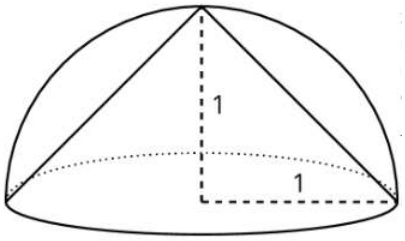


Figure 5.5.3.7

Averaging the volumes of the cylinder and the cone, we can estimate the volume of the hemisphere to be about 2.10 units^3 since $\frac{3.14+1.05}{2} \approx 2.10$. And, since a hemisphere is half of a sphere, we can also estimate that a sphere with radius of 1 would be double this volume, or about 4.20 units^3.

Practice

Exercise 5.5.3.4

A baseball fits snugly inside a transparent display cube. The length of an edge of the cube is 2.9 inches.

Is the baseball's volume greater than, less than, or equal to 2.9^3 cubic inches? Explain how you know.

Exercise 5.5.3.5

There are many possible cones with a height of 18 meters. Let r represent the radius in meters and V represent the volume in cubic meters.

1. Write an equation that represents the volume V as a function of the radius r.
2. Complete this table for the function, giving three possible examples.

Table 5.5.3.1

r	V
2	

3. If you double the radius of a cone, does the volume double? Explain how you know.
4. Is the graph of this function a line? Explain how you know.

(From Unit 5.5.2)

Exercise 5.5.3.6

A hemisphere fits snugly inside a cylinder with a radius of 6 cm. A cone fits snugly inside the same hemisphere.

1. What is the volume of the cylinder?
2. What is the volume of the cone?
3. Estimate the volume of the hemisphere by calculating the average of the volumes of the cylinder and cone.

Exercise 5.5.3.7

1. Find the hemisphere's diameter if its radius is 6 cm.
2. Find the hemisphere's diameter if its radius is $\frac{1000}{3}$ m.
3. Find the hemisphere's diameter if its radius is 9.008 ft.
4. Find the hemisphere's radius if its diameter is 6 cm.
5. Find the hemisphere's radius if its diameter is $\frac{1000}{3}$ m.
6. Find the hemisphere's radius if its diameter is 9.008 ft.

Exercise 5.5.3.8

After almost running out of space on her phone, Elena checks with a couple of friends who have the same phone to see how many pictures they have on their phones and how much memory they take up. The results are shown in the table.

Table 5.5.3.2

number of photos			
memory used in MB			

1. Could this information be reasonably modeled with a linear function? Explain your reasoning.
2. Elena needs to delete photos to create 1,200 MB of space. Estimate the number of photos she should delete.

(From Unit 5.3.2)

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5.5.4: The Volume of a Sphere

Lesson

Let's explore spheres and their volumes.

Exercise 5.5.4.1: Sketch a Sphere

Here is a method for quickly sketching a sphere:

- Draw a circle.
- Draw an oval in the middle whose edges touch the sphere.

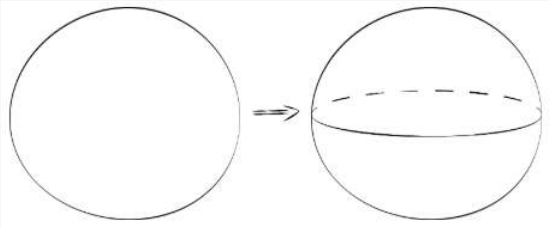


Figure 5.5.4.1

1. Practice sketching some spheres. Sketch a few different sizes.
2. For each sketch, draw a radius and label it r.

Exercise 5.5.4.2: A Sphere in a Cylinder

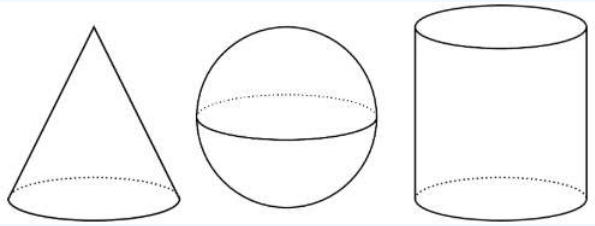


Figure 5.5.4.2

Here are a cone, a sphere, and a cylinder that all have the same radii and heights. The radius of the cylinder is 5 units. When necessary, express all answers in terms of π.

1. What is the height of the cylinder?
2. What is the volume of the cylinder?
3. What is the volume of the cone?
4. What is the volume of the sphere? Explain your reasoning.

Exercise 5.5.4.3: Spheres in Cylinders

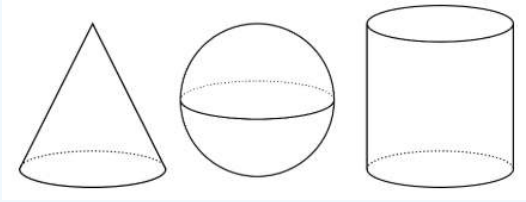


Figure 5.5.4.3

Here are a cone, a sphere, and a cylinder that all have the same radii and heights. Let the radius of the cylinder be r units. When necessary, express answers in terms of π.

1. What is the height of the cylinder in terms of r?
2. What is the volume of the cylinder in terms of r?

3. What is the volume of the cone in terms of r?
4. What is the volume of the sphere in terms of r?
5. A volume of the cone is $\frac{1}{3}$ the volume of a cylinder. The volume of the sphere is what fraction of the volume of the cylinder?

Summary

Think about a sphere with radius r units that fits snugly inside a cylinder. The cylinder must then also have a radius of r units and a height of $2r$ units. Using what we have learned about volume, the cylinder has a volume of $\pi r^2 h = \pi r^2 \cdot (2r)$, which is equal to $2\pi r^3$ cubic units.

We know from an earlier lesson that the volume of a cone with the same base and height as a cylinder has $\frac{1}{3}$ of the volume. In this example, such a cone has a volume of $\frac{1}{3} \cdot \pi r^2 \cdot 2r$ or just $\frac{2}{3}\pi r^3$ cubic units.

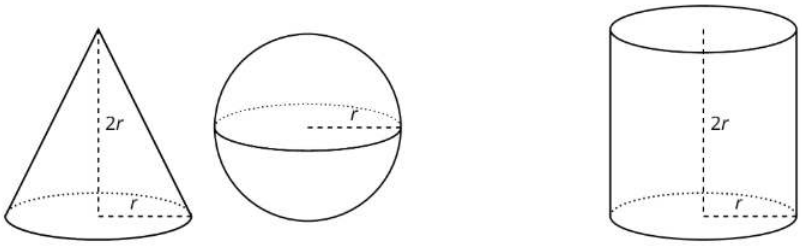


Figure 5.5.4.4

If we filled the cone and sphere with water, and then poured that water into the cylinder, the cylinder would be completely filled. That means the volume of the sphere and the volume of the cone add up to the volume of the cylinder. In other words, if V is the volume of the sphere, then

$$V + \tfrac{2}{3}\pi r^3 = 2\pi r^3$$

This leads to the formula for the volume of the sphere,

$$V = \tfrac{4}{3}\pi r^3$$

Practice

Exercise 5.5.4.4

1. A cube's volume is 512 cubic units. What is the length of its edge?
2. If a sphere fits snugly inside this cube, what is its volume?
3. What fration of the cube is taken up by the sphere? What percentage is this? Explain or show your reasoning.

Exercise 5.5.4.5

Sphere A has radius 2 cm. Sphere B has radius 4 cm.

1. Calculate the volume of each sphere.
2. The radius of Sphere B is double that of Sphere A. How many times greater is the volume of B?

Exercise 5.5.4.6

Three cones have a volume of $192\pi\,\text{cm}^3$. Cone A has a radius of 2 cm. Cone B has a radius of 3 cm. Cone C has a radius of 4 cm. Find the height of each cone.

(From Unit 5.4.6)

Exercise 5.5.4.7

The graph represents the average price of regular gasoline in the United States in dollars as a function of the number of months after January 2014.

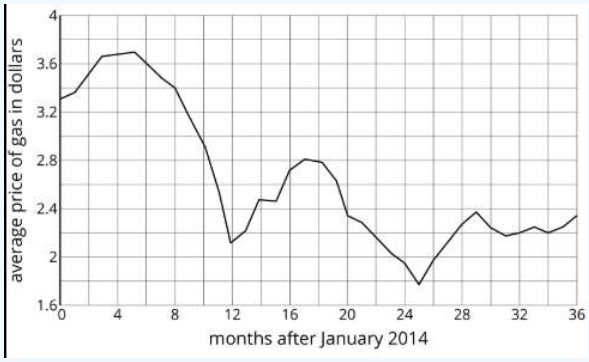


Figure 5.5.4.5

1. How many months after January 2014 was the price of gas the greatest?
2. Did the average price of gas ever get below $2?
3. Describe what happened to the average price of gas in 2014.

(From Unit 5.2.3)

Exercise 5.5.4.8

Match the description of each sphere to its correct volume.

A. Sphere A: radius of 4 cm
B. Sphere B: diameter of 6 cm
C. Sphere C: radius of 8 cm
D. Sphere D: radius of 6 cm

1. 288π cm^3
2. $\frac{256}{3}\pi$ cm^3
3. 36π cm^3
4. $\frac{2048}{3}\pi$ cm^3

Exercise 5.5.4.9

While conducting an inventory in their bicycle shop, the owner noticed the number of bicycles is 2 fewer than 10 times the number of tricycles. They also know there are 410 wheels on all the bicycles and tricycles in the store. Write and solve a system of equations to find the number of bicycles in the store.

(From Unit 4.3.6)

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5.5.5: Cylinders, Cones, and Spheres

Lesson

Let's find the volume of shapes.

Exercise 5.5.5.1: Sphere Adjustments

Four students each calculated the volume of a sphere with a radius of 9 centimeters and they got four different answers.

- Han thinks it is 108 cubic centimeters.
- Jada got 108π cubic centimeters.
- Tyler calculated 972 cubic centimeters.
- Mai says it is 972π cubic centimeters.

Do you agree with any of them? Explain your reasoning.

Exercise 5.5.5.2: Sphere's Radius

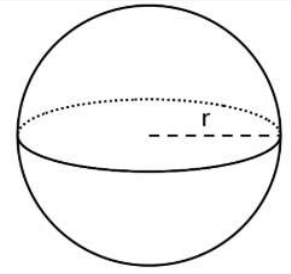


Figure 5.5.5.1

The volume of this sphere with radius r is $V = 288\pi$. This statement is true:

$288\pi = \frac{4}{3}r^3\pi$. What is the value of r for this sphere? Explain how you know.

Exercise 5.5.5.3: Info Gap: Unknown Dimensions

Your teacher will give you either a *problem card* or a *data card*. Do not show or read your card to your partner.

If your teacher gives you the *problem card*:

1. Silently read your card and think about what information you need to be able to answer the question.
2. Ask your partner for the specific information that you need.
3. Explain how you are using the information to solve the problem.
   Continue to ask questions until you have enough information to solve the problem.
4. Share the *problem card* and solve the problem independently.
5. Read the *data card* and discuss your reasoning.

If your teacher gives you the *data card*:

1. Silently read your card.
2. Ask your partner "*What specific information do you need?*" and wait for them to *ask* for information.
   If your partner asks for information that is not on the card, do not do the calculations for them. Tell them you don't have that information.
3. Before sharing the information, ask "*Why do you need that information?*" Listen to your partner's reasoning and ask clarifying questions.
4. Read the *problem card* and solve the problem independently.
5. Share the *data card* and discuss your reasoning.

Pause here so your teacher can review your work. Ask your teacher for a new set of cards and repeat the activity, trading roles with your partner.

Exercise 5.5.5.4: The Right Fit

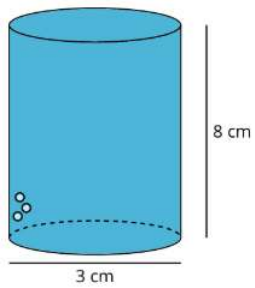


Figure 5.5.5.2

A cylinder with diameter 3 centimeters and height 8 centimeters is filled with water. Decide which figures described here, if any, could hold all of the water from the cylinder. Explain your reasoning.

1. Cone with a height of 8 centimeters and a radius of 3 centimeters.
2. Cylinder with a diameter of 6 centimeters and height of 2 centimeters.
3. Rectangular prism with a length of 3 centimeters, width of 4 centimeters, and height of 8 centimeters.
4. Sphere with a radius of 2 centimeters.

Are you ready for more?

A thirsty crow wants to raise the level of water in a cylindrical container so that it can reach the water with its beak.

- The container has diameter of 2 inches and a height of 9 inches.
- The water level is currently at 6 inches.
- The crow can reach the water if it is 1 inch from the top of the container.

In order to raise the water level, the crow puts spherical pebbles in the container. If the pebbles are approximately $\frac{1}{2}$ inch in diameter, what is the fewest number of pebbles the crow needs to drop into the container in order to reach the water?

Summary

The formula

$$V = \frac{4}{3}\pi r^3$$

gives the volume of a sphere with radius r. We can use the formula to find the volume of a sphere with a known radius. For example, if the radius of a sphere is 6 units, then the volume would be

$$\frac{4}{3}\pi(6)^3 = 288\pi$$

or approximately 904 cubic units. We can also use the formula to find the radius of a sphere if we only know its volume. For example, if we know the volume of a sphere is 36π cubic units but we don't know the radius, then this equation is true:

$$36\pi = \frac{4}{3}\pi r^3$$

That means that $r^3 = 27$, so the radius r has to be 3 units in order for both sides of the equation to have the same value.

Many common objects, from water bottles to buildings to balloons, are similar in shape to rectangular prisms, cylinders, cones, and spheres—or even combinations of these shapes! Using the volume formulas for these shapes allows us to compare the volume of different types of objects, sometimes with surprising results.

For example, a cube-shaped box with side length 3 centimeters holds less than a sphere with radius 2 centimeters because the volume of the cube is 27 cubic centimeters ($3^3 = 27$), and the volume of the sphere is around 33.51 cubic centimeters ($\frac{4}{3}\pi \cdot 2^3 \approx 33.51$).

Practice

Exercise 5.5.5.5

A scoop of ice cream has a 3-inch diameter. How tall should the ice cream cone of the same diameter be in order to contain all of the ice cream inside the cone?

Exercise 5.5.5.6

Calculate the volume of the following shapes with the given information. For the first three questions, give each answer both in terms of π and by using 3.14 to approximate π. Make sure to include units.

1. Sphere with a diameter of 6 inches
2. Cylinder with a height of 6 inches and a diameter of 6 inches
3. Cone with a height of 6 inches and a radius of 3 inches
4. How are these three volumes related?

Exercise 5.5.5.7

A coin-operated bouncy ball dispenser has a large glass sphere that holds many spherical balls. The large glass sphere has a radius of 9 inches. Each bouncy ball has radius of 1 inch and sits inside the dispenser.

If there are 243 bouncy balls in the large glass sphere, what proportion of the large glass sphere's volume is taken up by bouncy balls? Explain how you know.

Exercise 5.5.5.8

A farmer has a water tank for cows in the shape of a cylinder with radius of 7 ft and a height of 3 ft. The tank comes equipped with a sensor to alert the farmer to fill it up when the water falls to 20% capacity. What is the volume of the tank be when the sensor turns on?

(From Unit 5.4.3)

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5.6: Let's Put It to Work

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5.6.1: Volume as a Function of ...

Lesson

Let's compare water heights in different containers.

Exercise 5.6.1.1: Missing Information

A cylinder and sphere have the same height.

1. If the sphere has a volume of 36π cubic units, what is the height of the cylinder?
2. What is a possible volume for the cylinder? Be prepared to explain your reasoning.

Exercise 5.6.1.2: Scaling Volume of a Sphere

1. Fill in the missing volumes in terms of π. Add two more radius and volume pairs of your choosing.

Table 5.6.1.1

radius	1	2	3	$\frac{1}{2}$	$\frac{1}{3}$	100			r
volume	$\frac{4}{3}\pi$								

   a. How does the volume of a sphere with radius 2 cm compare to the volume of a sphere with radius 1 cm
   b. How does the volume of a sphere with radius $\frac{1}{2}$ cm compare to the volume of a sphere with radius 1 cm?

2. A sphere has a radius of length r.
   a. What happens to the volume of this sphere if its radius is doubled?
   b. What happens to the volume of this sphere if its radius is halved?
3. Sphere Q has a volume of 500 cm^3. Sphere S has a radius $\frac{1}{5}$ as large as Sphere Q. What is the volume of Sphere S?

Exercise 5.6.1.3: A Cylinder, a Cone, and a Sphere

Three containers of the same height were filled with water at the same rate. One container is a cylinder, one is a cone, and one is a sphere. As they were filled, the relationship between the volume of water and the height of the water was recorded in different ways, shown here:

- Cylinder: $h = \frac{V}{4\pi}$
- Cone:

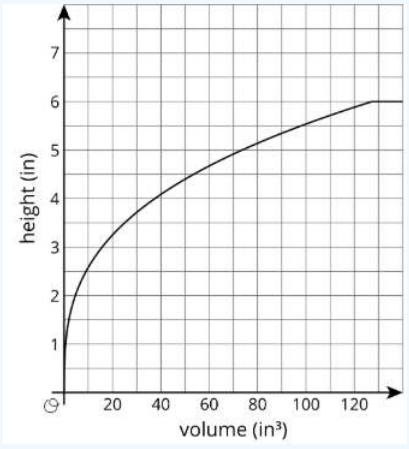


Figure 5.6.1.1: Coordinate plane, horizontal, volume, inches cubed, 0 to 120 by twenties, vertical, height, inches, 0 to 7 by ones. Curve starts at origin, increases steeply through volume = 10, decreases less steeply to volume = 125, then remains constant at height = 6 to the edge of the graph.

- Sphere:

Table 5.6.1.2

volume (in^3)	height (in)
0	0
8.38	1
29.32	2
56.55	3
83.76	4
104.72	5
113.04	6
120	6
200	6

1. The maximum volume of water the cylinder can hold is 24π. What is the radius of the cylinder?
2. Graph the relationship between the volume of water poured into the cylinder and the height of water in the cylinder on the same axes as the cone. What does the slope of this line represent?
3. Which container can fit the largest volume of water? The smallest?
4. About how much water does it take for the cylinder and the sphere to have the same height? The cylinder and the cone? Explain how you know.
5. For what approximate range of volumes is the height of the water in the cylinder greater than the height of the water in the cone? Explain how you know.
6. For what approximate range of volumes is the height of the water in the sphere less than the height of the water in the cylinder? Explain how you know.

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CHAPTER OVERVIEW

6: Associations in Data

In this unit, students generate and work with bivariate data sets that has more variability than in previous units. They learn to understand and use the terms “scatter plot” and “association,” and describe associations as “positive” or “negative” and “linear” or “non-linear.” Students describe scatter plots, using a term previously used to describe univariate data “cluster,” and the new term “outlier.” They fit lines to scatter plots and informally assess their goodness of fit by judging the closeness of the data points to the lines, and compare predicted and actual values. Students learn to understand and use the terms “two-way table,” “bar graph,” and “segmented bar graph,” using two-way tables to investigate categorical data.

6.1: Does This Predict That?

6.1.1: Organizing Data

6.1.2: Plotting Data

6.2: Associations in Numerical Data

6.2.1: What a Point in a Scatter Plot Means

6.2.2: Fitting a Line to Data

6.2.3: Describing Trends in Scatter Plots

6.2.4: The Slope of a Fitted Line

6.2.5: Observing More Patterns in Scatter Plots

6.2.6: Analyzing Bivariate Data

6.3: Associations in Categorical Data

6.3.1: Looking for Associations

6.3.2: Using Data Displays to Find Associations

6.4: Let's Put it to Work

6.4.1: Gone in 30 Seconds

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6.1: Does This Predict That?

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6.1.1: Organizing Data

Lesson

Let's find ways to show patterns in data

Exercise 6.1.1.1: Notice and Wonder: Messy Data

Here is a table of data. Each row shows two measurements of a triangle.

Table 6.1.1.1

length of short side (cm)	length of perimeter (cm)
0.25	1
2	7.5
6.5	22
3	9.5
0.5	2
1.25	3.5
3.5	12.5
1.5	5
4	14
1	2.5

What do you notice? What do you wonder?

Exercise 6.1.1.2: Seeing the Data

Here is the table of isosceles right triangle measurements from the warm-up and an empty table.

Table 6.1.1.2

length of short sides (cm)	length of perimeter (cm)	length of short sides (cm)	length of perimeter (cm)
0.25	1		
2	7.5		
6.5	22		
3	9.5		
0.5	2		
1.25	3.5		
3.5	12.5		
1.5	5		
4	14		
1	2.5		

1. How can you organize the measurements from the first table so that any patterns are easier to see? Write the organized measurements in the empty table.

2. For each of the following lengths, estimate the perimeter of an isosceles right triangle whose short sides have that length. Explain your reasoning for each triangle.
   a. length of short sides is 0.75 cm
   b. length of short sides is 5 cm
   c. length of short sides is 10 cm

Are you ready for more?

In addition to the graphic representations of data you have learned, there are others that make sense in other situations. Examine the maps showing the results of the elections for United States president for 2012 and 2016. In red are the states where a majority of electorate votes were cast for the Republican nominee. In blue are the states where a majority of the electorate votes were cast for the Democrat nominee.

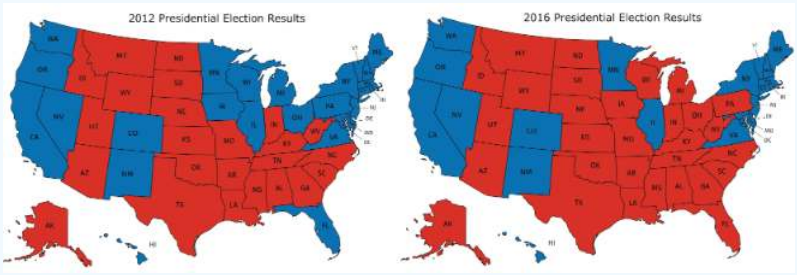


Figure 6.1.1.1

1. What information can you see in these maps that would be more difficult to see in a bar graph showing the number of electorate votes for the 2 main candidates?
2. Why are these representations appropriate for the data that is shown?

Exercise 6.1.1.3: Tables and Their Scatter Plots

Here are four **scatter plots**. Your teacher will give you four tables of data.

- Match each table with one of the scatter plots.
- Use information from the tables to label the axes for each scatter plot.

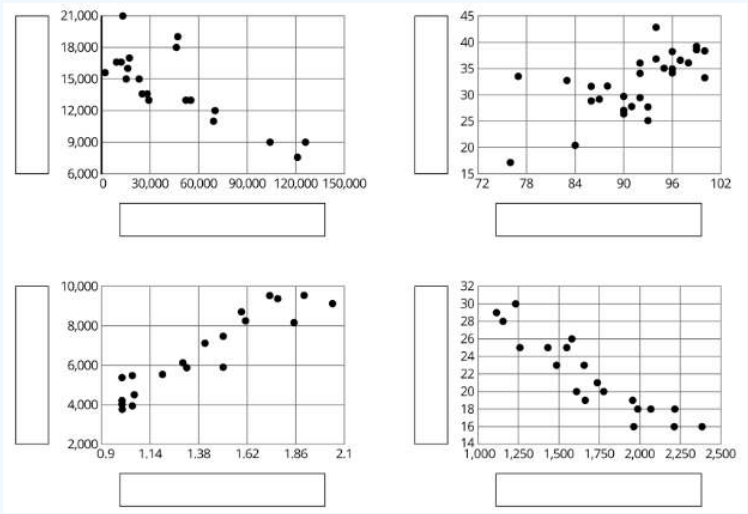


Figure 6.1.1.2: Four scatterplots. First, x, 0 to 150,000 by 30,000, y, 6,000 to 21,000 by 3,000. Points begin around 2,000 comma 16,000 and trend down and right toward 120,000 comma 7,000. Second, x, 72 to 102 by 6, y, 15 to 45 by 5. Points begin around 76 comma 18 and trend up and right toward 100 comma 40. Third, x, point 9 to 2 point 1 by point 2 4, y 2,000 to 10,000 by 2,000. Points begin around point 1 and trend up and right toward 1 point 8 6 comma 8,000. Fourth, x, 1,000 to 2,500 by 250, y, 14 to 32 by 2. Points begin around 1,100 comma 29 and trend down and right toward 2,300 comma 16.

Summary

Consider the data collected from pulling back a toy car and then letting it go forward. In the first table, the data may not seem to have an obvious pattern. The second table has the same data and shows that both values are increasing together.

Table 6.1.1.3: Unorganized Table

distance pulled back (in)	distance traveled (in)
8	23.57
4	18.48
10	38.66
8	31.12
2	13.86
1	8.95

Table 6.1.1.4: Organized Table

distance pulled back (in)	distance traveled (in)
1	8.95
2	13.86
4	18.48
6	23.57
8	31.12
10	38.66

A scatter plot of the data makes the pattern clear enough that we can estimate how far the car will travel when it is pulled back 5 inches.

Patterns in data can sometimes become more obvious when reorganized in a table or when represented in **scatter plots** or other diagrams. If a pattern is observed, it can sometimes be used to make predictions.

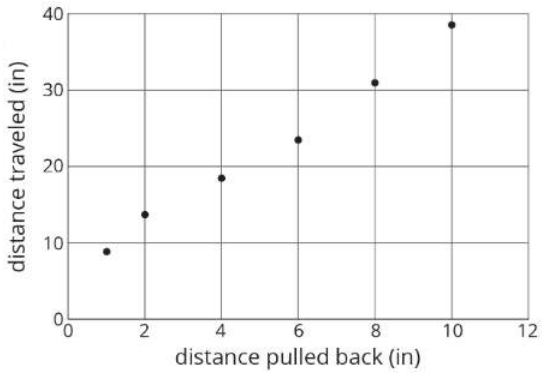


Figure 6.1.1.3

Glossary Entries

Definition: Scatter Plot

A scatter plot is a graph that shows the values of two variables on a coordinate plane. It allows us to investigate connections between the two variables.

Each plotted point corresponds to one dog. The coordinates of each point tell us the height and weight of that dog.

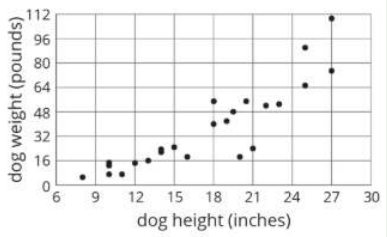


Figure 6.1.1.4

Practice

Exercise 6.1.1.4

Here is data on the number of cases of whooping cough from 1939 to 1955.

Table 6.1.1.5

year	number of cases
1941	222,202
1950	120,718
1945	133,792
1942	191,383
1953	37,129
1939	103,188
1951	68,687
1948	74,715
1955	62,786
1952	45,030
1940	183,866
1954	60,866
1944	109,873
1946	109,860
1943	191,890
1949	69,479
1947	156,517

1. Make a new table that orders the data by year.
2. Circle the years in your table that had fewer than 100,000 cases of whooping cough.
3. Based on this data, would you expect 1956 to have closer to 50,000 cases or closer to 100,000 cases?

Exercise 6.1.1.5

In volleyball statistics, a block is recorded when a player deflects the ball hit from the opposing team. Additionally, scorekeepers often keep track of the average number of blocks a player records in a game. Here is part of a table that records the number of blocks and blocks per game for each player in a women's volleyball tournament. A scatter plot that goes with the table follows.

Table 6.1.1.6

blocks	blocks per game
13	1.18
1	0.17
5	0.42
0	0
0	0
7	0.64

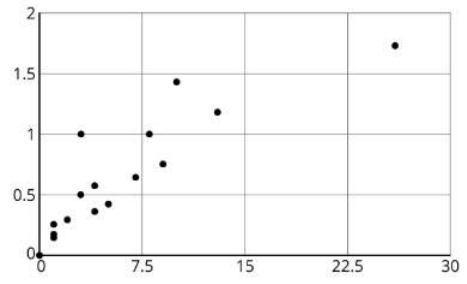


Figure 6.1.1.5

Label the axes of the scatter plot with the necessary information.

Exercise 6.1.1.6

A cylinder has a radius of 4 cm and a height of 5 cm.

1. What is the volume of the cylinder?
2. What is the volume of the cylinder when its radius is tripled?
3. What is the volume of the cylinder when its radius is halved?

(From Unit 5.5.2)

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6.1.2: Plotting Data

Lesson

Let's collect and display some data about the class.

Exercise 6.1.2.1: Representing Data

Lin surveyed 30 students about the longest time they had ever run. Andre asked them about their favorite color. How could Lin and Andre represent their data sets? Would they represent them in the same way? Why or why not?

Exercise 6.1.2.2: Gathering Data

Are older students always taller? Do taller students tend to have bigger hands? To investigate these questions, the class will gather data.

- A person's *arm span* is the distance between the tips of their index fingers, when their arms are fully spread out.
- A person's *hand span* is the distance from the tip of their thumb to the tip of their little finger, when their fingers are fully spread out.

1. Each partner should:
   - Measure the other partner's height, arm span, and hand span for their right hand to the nearest centimeter.
   - Record the other partner's measurements and age (in months) in the table.

Table 6.1.2.1

	height (cm)	arm span (cm)	hand span (cm)	age (months)
partner A				
partner B				

2. One partner records the data from your table in a table of data for the entire class.

Exercise 6.1.2.3: Scatter Plots

1. What types of graphical representations could be used to show the class's height measurements? Make a graphical representation of the class's height measurements using these directions for the applet:
   - Enter the class height data in column A. Note: enter only one value in each cell, just the height of each student.
   - Click on the column header to highlight it.
   - Select the One-Variable Analysis tool (the one that looks like a histogram), and a new frame will appear.
   - Drag the window open and you will see a histogram of the data.
   - Change the type of graph by choosing from the drop-down menu.
2. Make a scatter plot of the heights and hand spans of each student in your class. Enter the class height data into one column and the corresponding hand span data into the other column. The points will appear on the graph as you type them in. To see more of the graph after you have entered in the data, click the double left arrows to minimize the data view.
3. Based on your scatter plot, answer these questions:
   a. Do taller students in your class tend to have bigger hands? Explain how you know.
   b. Is hand span a linear function of height? Explain how you know.

Are you ready for more?

Although the data may be accurate, displaying the data incorrectly can tell the wrong story. What is wrong with each of these graphic representations of the data?

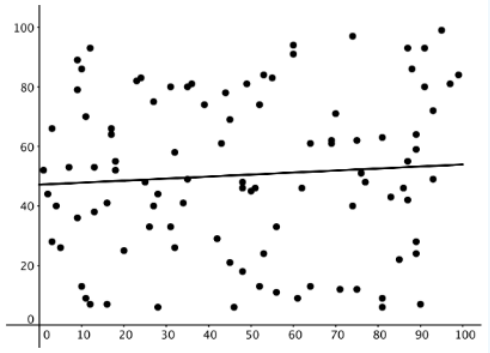


Figure 6.1.2.1

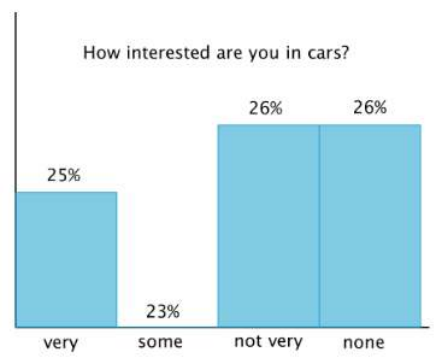


Figure 6.1.2.2

Summary

Histograms show us how measurements of a single attribute are distributed. For example, a veterinarian saw 25 dogs in her clinic one week. She measured the height and weight of each dog.

This histogram shows how the weights of the dogs are distributed.

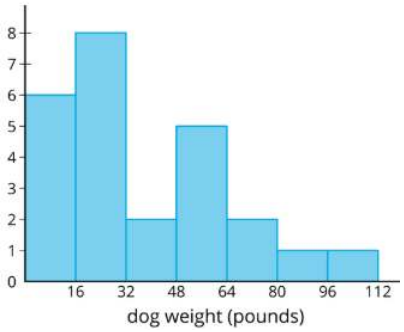


Figure 6.1.2.3

This histogram shows how the heights of the dogs are distributed.

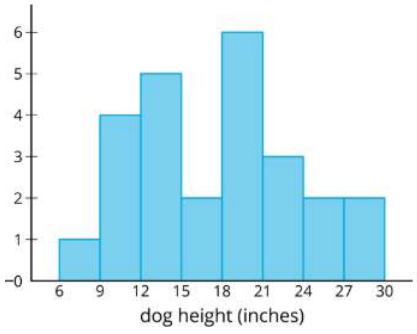


Figure 6.1.2.4

These histograms tell us how the weights of the dogs and how the heights of dogs were distributed. But, they do not give any evidence of a connection between a dog's height and its weight.

Scatter plots allow us to investigate possible connections between two attributes. In this example, each plotted point corresponds to one of the 25 dogs, and its coordinates tell us the height and weight of that dog. Examination of the scatter plot allows us to see a connection between height and weight for the dogs.

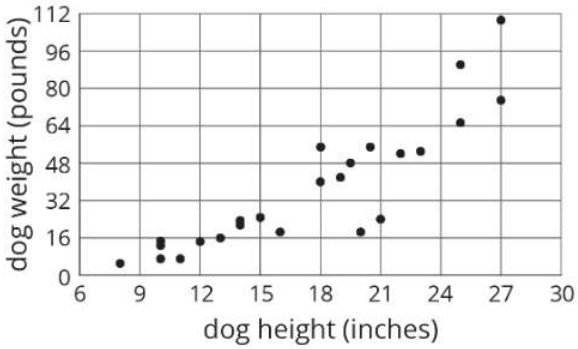


Figure 6.1.2.5

Glossary Entries

Definition: Scatter Plot

A scatter plot is a graph that shows the values of two variables on a coordinate plane. It allows us to investigate connections between the two variables.

Each plotted point corresponds to one dog. The coordinates of each point tell us the height and weight of that dog.

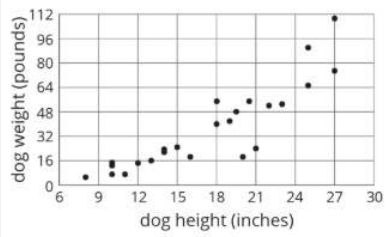


Figure 6.1.2.4

Practice

Exercise 6.1.2.4

In hockey, a player gets credited with a "point" in their statistics when they get an assist or goal. The table shows the number of assists and number of points for 15 hockey players after a season.

Table 6.1.2.2

assists	points
22	28
16	18
46	72
19	29
13	26
9	13
16	22

assists	points
8	18
12	13
12	17
37	50
7	12
17	34
27	58
18	34

Make a scatter plot of this data. Make sure to scale and label the axes.

Exercise 6.1.2.5

Select **all** the representations that are appropriate for comparing bite strength to weight for different carnivores.

A. Histogram
B. Scatter plot
C. Dot plot
D. Table
E. Box plot

Exercise 6.1.2.6

When is it better to use a table? When is it better to use a scatter plot?

Exercise 6.1.2.7

There are many cylinders with radius 6 meters. Let h represent the height in meters and V represent the volume in cubic meters.

1. Write an equation that represents the volume V as a function of the height h.
2. Sketch the graph of the function, using 3.14 as an approximation for π.
3. If you double the height of a cylinder, what happens to the volume? Explain this using the equation.
4. If you multiply the height of a cylinder by $\frac{1}{3}$, what happens to the volume? Explain this using the graph.

(From Unit 5.5.1)

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6.2: Associations in Numerical Data

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6.2.1: What a Point in a Scatter Plot Means

Lesson

Let's investigate points in scatter plots.

Exercise 6.2.1.1: The Giant Panda

A giant panda lives in a zoo. What does the point on the graph tell you about the panda?



Figure 6.2.1.1: "Giant Panda 2", by Sheila Lau (Own work) . Public Domain. Wikimedia Commons. Source.

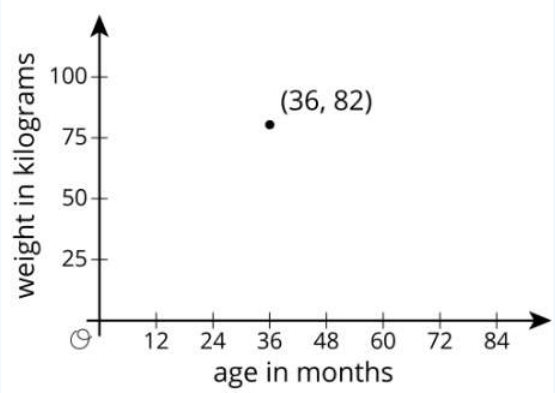


Figure 6.2.1.2

Exercise 6.2.1.2: Weight and Fuel Efficiency

The table shows the weight and fuel efficiency of 18 different cars.

Table 6.2.1.1

car	weight (kg)	fuel efficiency (mpg)
A		
B		
C		
D		
E		
F		
G		
H		
I		
J		
K		
L		
M		

car	weight (kg)	fuel efficiency (mpg)
N		
O		
P		
Q		
R		

The data points in the table are shown in this scatter plot.

You can hide the expressions list using the double arrow. You can click on a point to see its coordinates.

1. Which point in the scatter plot represents Car L's measurements? Drag the circle around the correct point.
2. What is the fuel efficiency of the car with the greatest weight?
3. What is the weight of the car with the greatest fuel efficiency?
4. Car S weighs 1,912 kilograms and gets 16 miles per gallon. On the scatter plot, plot a point that represents Car S's measurements.
5. Cars N and O, shown in the scatter plot, are made by the same company. Compare their weights and fuel efficiencies. Does anything surprise you about these cars?
6. A different company makes Cars F and G. Compare their weights and fuel efficiencies. Does anything surprise you about these cars?

Are you ready for more?

After a board game competition, the tournament director collects 50 dice from the games played and rolls each one until he gets bored and tries a different one. The scatter plot shows the number of times he rolled each die and the number of 6s that resulted during those rolls.

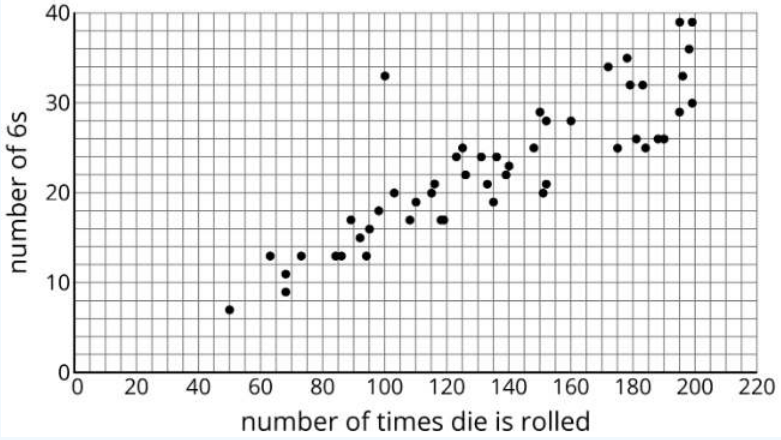


Figure 6.2.1.3

Select a point in the scatter plot and give its approximate coordinates, then tell the story of that point in the context of the problem.

Exercise 6.2.1.3: Coat Sales

A clothing store keeps track of the average monthly temperature in degrees Celsius and coat sales in dollars.

Table 6.2.1.2

temperature (degrees Celsius)	coat sales (dollars)
-5	1,550
-3	1,340

temperature (degrees Celsius)	coat sales (dollars)
3	1,060
8	1,070
15	680
21	490
23	410
21	510
17	600
11	740
6	940
-2	1,390

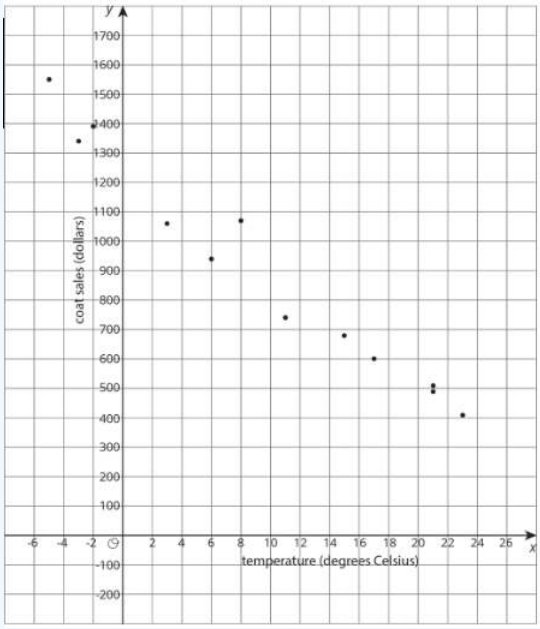


Figure 6.2.1.4

1. What does the point $(15, 680)$ represent?
2. For the month with the lowest average temperature, estimate the total amount made from coat sales. Explain how you used the table to find this information.
3. For the month with the smallest coat sales, estimate the average monthly temperature. Explain how you used the scatter plot to find this information.
4. If there were a point at $(0, A)$ what would it represent? Use the scatter plot to estimate a value for A.
5. What would a point at $((B, 0)$ represent? Use the scatter plot to estimate a value for B.
6. Would it make sense to use this trend to estimate the value of sales when the average monthly temperature is 60 degrees Celsius? Explain your reasoning.

Summary

Scatter plots show two measurements for each individual from a group. For example, this scatter plot shows the weight and height for each dog from a group of 25 dogs.

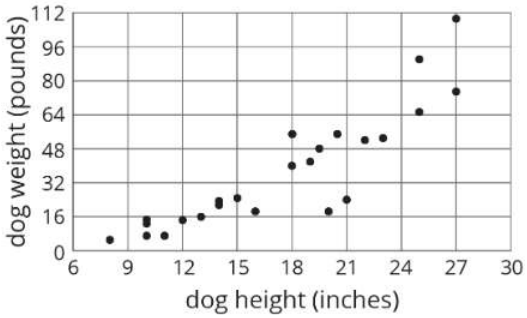


Figure 6.2.1.5

We can see that the tallest dogs are 27 inches, and that one of those tallest dogs weighs about 75 pounds while the other weighs about 110 pounds. This shows us that dog weight is not a function of dog height because there would be two different outputs for the same input. But we can see a general trend: Taller dogs tend to weigh more than shorter dogs. There are exceptions. For example, there is a dog that is 18 inches tall and weighs over 50 pounds, and there is another dog that is 21 inches tall but weighs less than 30 pounds.

When we collect data by measuring attributes like height, weight, area, or volume, we call the data *numerical data* (or measurement data), and we say that height, weight, area, or volume is a *numerical variable*. Upcoming lessons will discuss how to identify and describe trends in data that has been collected.

Practice

Exercise 6.2.1.4

Here is a table and a scatter plot that compares points per game to free throw attempts for a basketball team during a tournament.

Table 6.2.1.3

player	free throw attempts	points
player A	5.5	28.3
player B	2.1	18.6
player C	4.1	13.7
player D	1.6	10.6
player E	3.1	10.4
player F	1	5
player G	1.2	5
player H	0.7	4.7
player I	1.5	3.7
player J	1.5	3.5
player K	1.2	3.1
player L	0	1
player M	0	0.8
player N	0	0.6

1. Circle the point that represents the data for player E.
2. What does the point $(2.1, 18.6)$represent?

3. In that same tournament, Player O on another team scored 14.3 points per game with 4.8 free throw attempts per game. Plot a point on the graph that shows this information.

Exercise 6.2.1.5

Select **all** the representations that are appropriate for comparing exam score to number of hours of sleep the night before the exam.

A. Histogram
B. Scatter plot
C. Dot plot
D. Table
E. Box plot

(From Unit 6.1.2)

Exercise 6.2.1.6

A cylinder has a volume of 36π cm^3 and height h. Complete this table for the volume of other cylinders with the same radius but different heights.

Table 6.2.1.4

height (cm)	volume (cm^3)
h	36π
$2h$	
$5h$	
$\frac{h}{2}$	
$\frac{h}{5}$	

(From Unit 5.5.1)

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6.2.2: Fitting a Line to Data

Lesson

Let's look at the scatter plots as a whole.

Exercise 6.2.2.1: Predict This

Here is a scatter plot that shows weights and fuel efficiencies of 20 different types of cars.

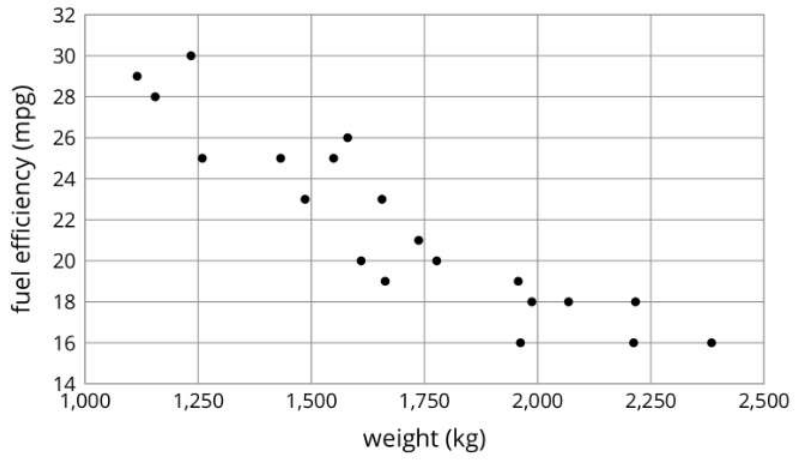


Figure 6.2.2.1

If a car weighs 1,750 kg, would you expect its fuel efficiency to be closer to 22 mpg or to 28 mpg? Explain your reasoning.

Exercise 6.2.2.2: Shine Bright

Here is a table that shows weights and prices of 20 different diamonds.

Table 6.2.2.1

weight (carats)	actual price (dollars)	predicted price (dollars)
1	3,772	4,429
1	4,221	4,429
1	4,032	4,429
1	5,385	4,429
1.05	3,942	4,705
1.05	4,480	4,705
1.06	4,511	4,760
1.2	5,544	5533
1.3	6,131	6,085
1.32	5,872	6,195
1.41	7,122	6,692
1.5	7,474	7,189
1.5	5,904	7,189
1.59	8,706	7,686

weight (carats)	actual price (dollars)	predicted price (dollars)
1.61	8,252	7,796
1.73	9,530	8,459
1.77	9,374	8,679
1.85	8,169	9,121
1.9	9,541	9,397
2.04	9,125	10,170

The scatter plot shows the prices and weights of the 20 diamonds together with the graph of $y = 5,520x - 1,091$.

The function described by the equation $y = 5,520x - 1,091$ is a *model* of the relationship between a diamond's weight and its price.

This model *predicts* the price of a diamond from its weight. These predicted prices are shown in the third column of the table.

1. Two diamonds that both weigh 1.5 carats have different prices. What are their prices? How can you see this in the table? How can you see this in the graph?
2. The *model* predicts that when the weight is 1.5 carats, the price will be $7,189. How can you see this in the graph? How can you see this using the equation?
3. One of the diamonds weighs 1.9 carats. What does the model predict for its price? How does that compare to the actual price?
4. Find a diamond for which the model makes a very good prediction of the actual price. How can you see this in the table? In the graph?
5. Find a diamond for which the model's prediction is not very close to the actual price. How can you see this in the table? In the graph?

Exercise 6.2.2.3: The Agony of the Feet

Here is a scatter plot that shows lengths and widths of 20 different left feet. Use the double arrows to show or hide the expressions list.

1. Estimate the widths of the longest foot and the shortest foot.
2. Estimate the lengths of the widest foot and the narrowest foot.
3. Click on the gray circle next to the words "The Line" in the expressions list. The graph of a linear model should appear. Find the data point that seems weird when compared to the model. What length and width does that point represent?

Summary

Sometimes, we can use a linear function as a model of the relationship between two variables. For example, here is a scatter plot that shows heights and weights of 25 dogs together with the graph of a linear function which is a model for the relationship between a dog's height and its weight.

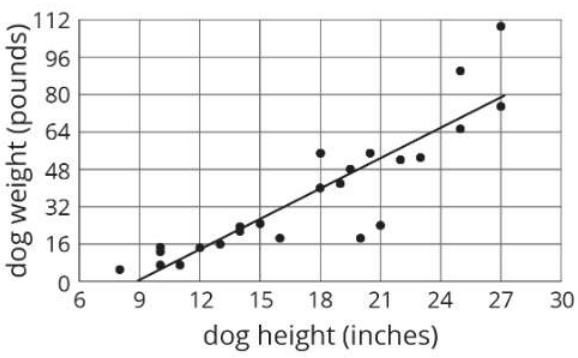


Figure 6.2.2.2

We can see that the model does a good job of predicting the weight given the height for some dogs. These correspond to points on or near the line. The model doesn't do a very good job of predicting the weight given the height for the dogs whose points are far from the line.

For example, there is a dog that is about 20 inches tall and weighs a little more than 16 pounds. The model predicts that the weight would be about 48 pounds. We say that the model *overpredicts* the weight of this dog. There is also a dog that is 27 inches tall and weighs about 110 pounds. The model predicts that its weight will be a little less than 80 pounds. We say the model *underpredicts* the weight of this dog.

Sometimes a data point is far away from the other points or doesn't fit a trend that all the other points fit. We call these **outliers**.

Glossary Entries

Definition: Outlier

An outlier is a data value that is far from the other values in the data set.

Here is a scatter plot that shows lengths and widths of 20 different left feet. The foot whose length is 24.5 cm and width is 7.8 cm is an outlier.

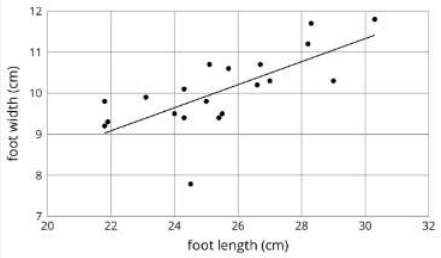


Figure 6.2.2.3

Practice

Exercise 6.2.2.4

The scatter plot shows the number of hits and home runs for 20 baseball players who had at least 10 hits last season. The table shows the values for 15 of those players.

The model, represented by $y = 0.15x - 1.5$, is graphed with a scatter plot.

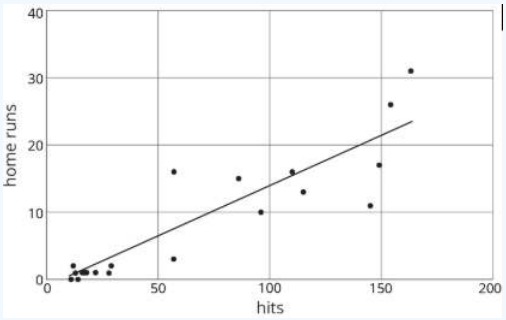


Figure 6.2.2.4

hits	home runs	predicted home runs
12	2	0.3
22	1	1.8
154	26	21.6
145	11	20.3
110	16	15

hits	home runs	predicted home runs
57	3	7.1
149	17	20.9
29	2	2.9
13	1	0.5
18	1	1.2
86	15	11.4
163	31	23
115	13	15.8
57	16	7.1
96	10	12.9

Table 6.2.2.2

Use the graph and the table to answer the questions.

1. Player A had 154 hits in 2015. How many home runs did he have? How many was he predicted to have?
2. Player B was the player who most outperformed the prediction. How many hits did Player B have last season?
3. What would you expect to see in the graph for a player who hit many fewer home runs than the model predicted?

Exercise 6.2.2.5

Here is a scatter plot that compares points per game to free throw attempts per game for basketball players in a tournament. The model, represented by $y = 4.413x + 0.377$, is graphed with the scatter plot. Here, x represents free throw attempts per game, and y represents points per game.

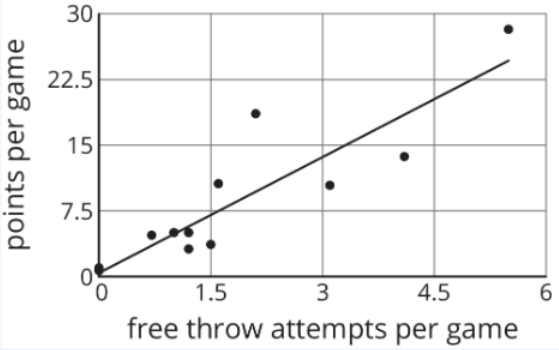


Figure 6.2.2.5

1. Circle any data points that appear to be outliers.
2. What does it mean for a point to be far above the line in this situation?
3. Based on the model, how many points per game would you expect a player who attempts 4.5 free throws per game to have? Round your answer to the nearest tenth of a point per game.
4. One of the players scored 13.7 points per game with 4.1 free throw attempts per game. How does this compare to what the model predicts for this player?

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6.2.3: Describing Trends in Scatter Plots

Lesson

Let's look for associations between variables.

Exercise 6.2.3.1: Which One Doesn't Belong - Scatter Plots

Which one doesn't belong?

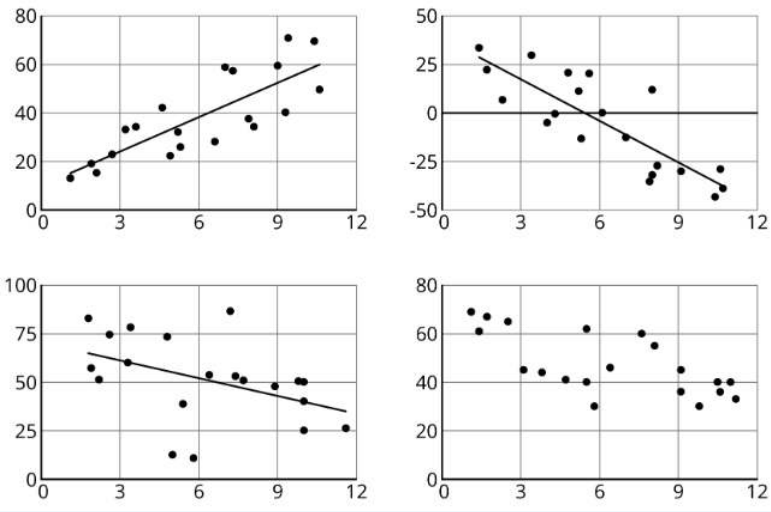


Figure 6.2.3.1

Exercise 6.2.3.2: Fitting Lines

Experiment with finding lines to fit the data. Drag the points to move the line. You can close the expressions list by clicking on the double arrow.

1. Here is a scatter plot. Experiment with different lines to fit the data. Pick the line that you think best fits the data. Compare it with a partner's.
2. Here is a different scatter plot. Experiment with drawing lines to fit the data. Pick the line that you think best fits the data. Compare it with a partner's.
3. In your own words, describe what makes a line fit a data set well.

Exercise 6.2.3.3: Good Fit Bad Fit

The scatter plots both show the year and price for the same 17 used cars. However, each scatter plot shows a different model for the relationship between year and price.

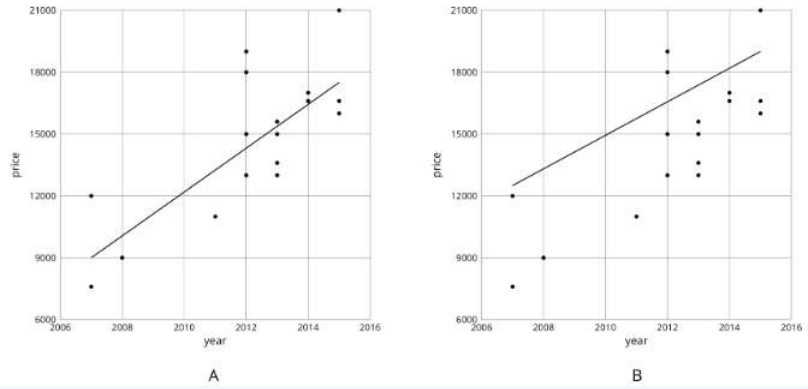


Figure 6.2.3.2

1. Look at Diagram A.
    a. For how many cars does the model in Diagram A make a good prediction of its price?
    b. For how many cars does the model underestimate the price?
    c. For how many cars does it overestimate the price?
2. Look at Diagram B.
    a. For how many cars does the model in Diagram B make a good prediction of its price?
    b. For how many cars does the model underestimate the price?
    c. For how many cars does it overestimate the price?
3. For how many cars does the prediction made by the model in Diagram A differ by more than $3,000? What about the model in Diagram B?
4. Which model does a better job of predicting the price of a used car from its year?

Exercise 6.2.3.4: Practice Fitting Lines

1. Is this line a good fit for the data? Explain your reasoning.

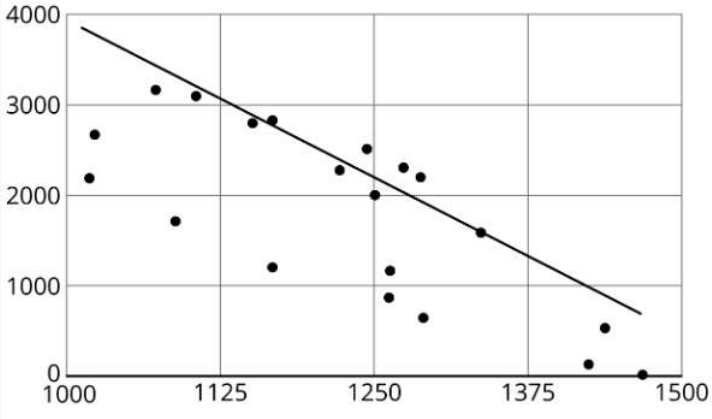


Figure 6.2.3.3

2. Draw a line that fits the data better.

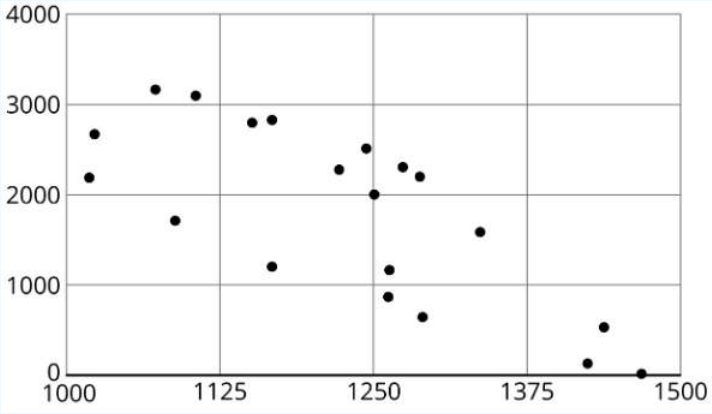


Figure 6.2.3.4

3. Is this line a good fit for the data? Explain your reasoning.

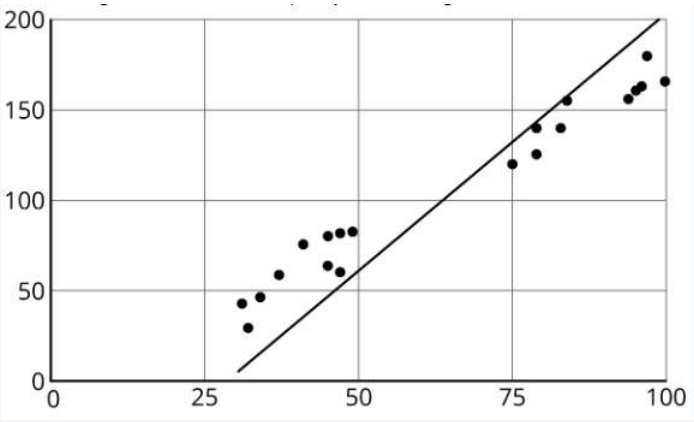


Figure 6.2.3.5

4. Draw a line that fits the data better.

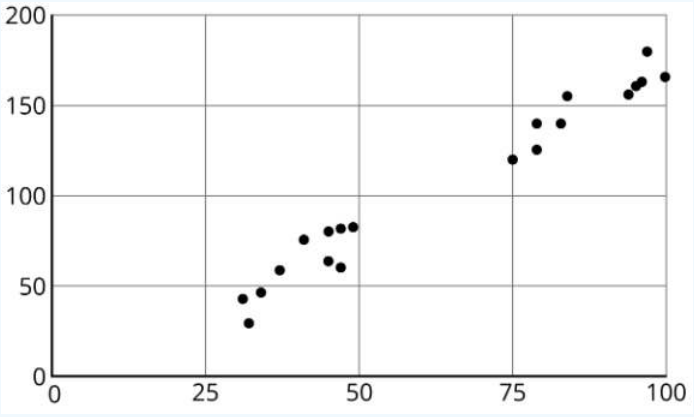


Figure 6.2.3.6

Are you ready for more?

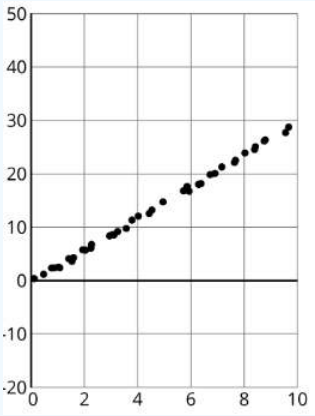


Figure 6.2.3.7

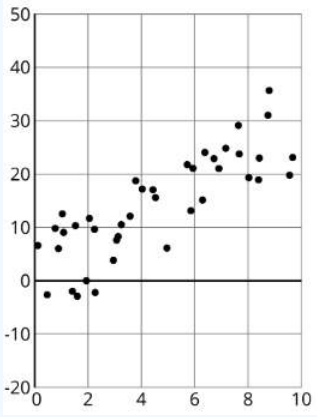


Figure 6.2.3.8

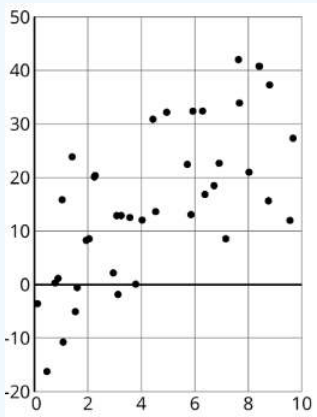


Figure 6.2.3.9

These scatter plots were created by multiplying the x-coordinate by 3 then adding a random number between two values to get the y-coordinate. The first scatter plot added a random number between -0.5 and 0.5 to the y-coordinate. The second scatter plot added a random number between -2 and 2 to the -coordinate. The third scatter plot added a random number between -10 and 10 to the y-coordinate.

1. For each scatter plot, draw a line that fits the data.
2. Explain why some were easier to do than others.

Summary

When a linear function fits data well, we say there is a *linear association* between the variables. For example, the relationship between height and weight for 25 dogs with the linear function whose graph is shown in the scatter plot.

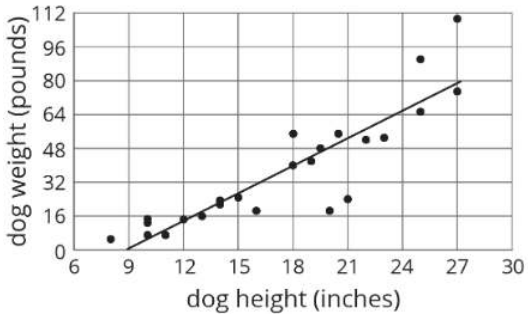


Figure 6.2.3.10

Because the model fits the data well and because the slope of the line is positive, we say that there is a **positive association** between dog height and dog weight.

What do you think the association between the weight of a car and its fuel efficiency is?

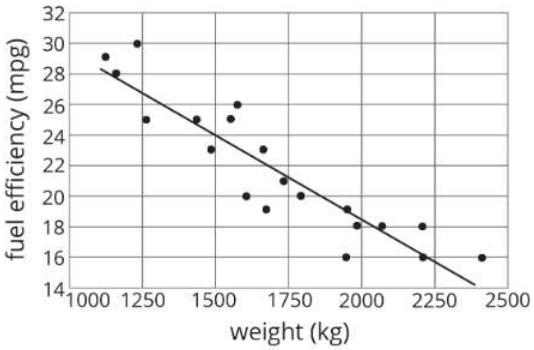


Figure 6.2.3.11

Because the slope of a line that fits the data well is negative, we say that there is a **negative association** between the fuel efficiency and weight of a car.

Glossary Entries

Definition: Negative Association

A negative association is a relationship between two quantities where one tends to decrease as the other increases. In a scatter plot, the data points tend to cluster around a line with negative slope.

Different stores across the country sell a book for different prices.

The scatter plot shows that there is a negative association between the the price of the book in dollars and the number of books sold at that price.



Figure 6.2.3.12

Definition: Outlier

An outlier is a data value that is far from the other values in the data set.

Here is a scatter plot that shows lengths and widths of 20 different left feet. The foot whose length is 24.5 cm and width is 7.8 cm is an outlier.

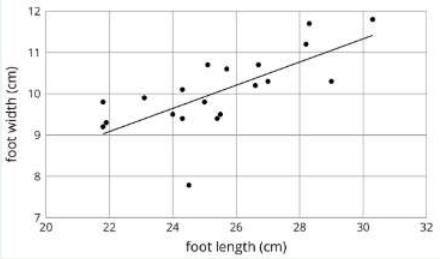


Figure 6.2.3.13

Definition: Positive Association

A positive association is a relationship between two quantities where one tends to increase as the other increases. In a scatter plot, the data points tend to cluster around a line with positive slope.

The relationship between height and weight for 25 dogs is shown in the scatter plot. There is a positive association between dog height and dog weight.

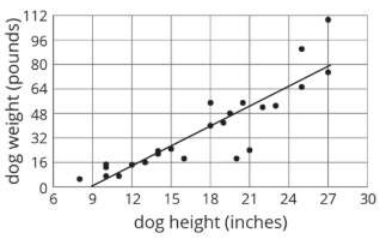


Figure 6.2.3.14

Practice

Exercise 6.2.3.5

1. Draw a line that you think is a good fit for this data. For this data, the inputs are the horizontal values, and the outputs are the vertical values.

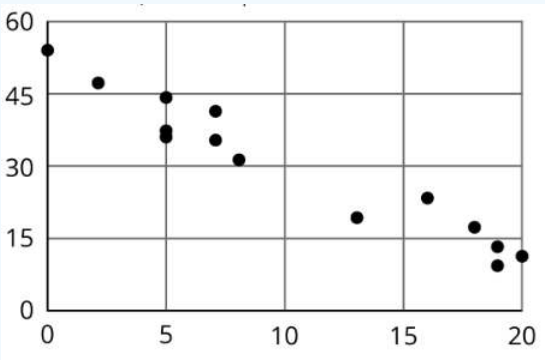


Figure 6.2.3.15

2. Use your line of fit to estimate what you would expect the output value to be when the input is 10.

Exercise 6.2.3.6

Here is a scatter plot that shows the most popular videos in a 10-year span.

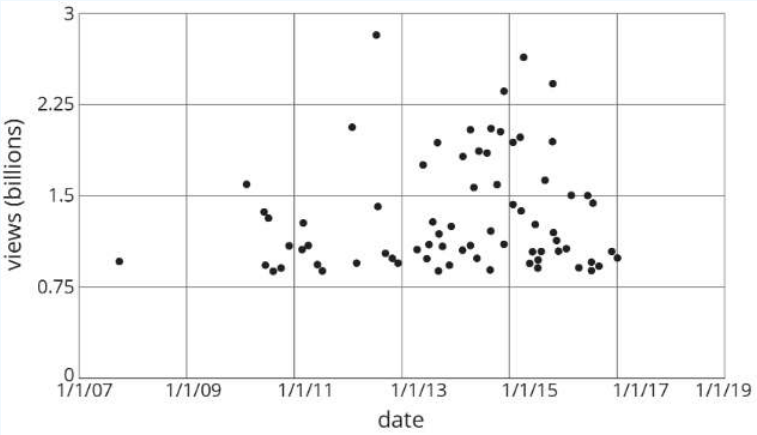


Figure 6.2.3.16

1. Use the scatter plot to estimate the number of views for the most popular video in this 10-year span.
2. Estimate when the 4th most popular video was released.

(From Unit 6.2.1)

Exercise 6.2.3.7

A recipe for bread calls for 1 teaspoon of yeast for every 2 cups of flour.

1. Name two quantities in this situation that are in a functional relationship.
2. Write an equation that represents the function.
3. Draw the graph of the function. Label at least two points with input-output pairs.

(From Unit 5.3.1)

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6.2.4: The Slope of a Fitted Line

Lesson

Let's look at how changing one variable changes another.

Exercise 6.2.4.1: Estimating Slope

Estimate the slope of the line.

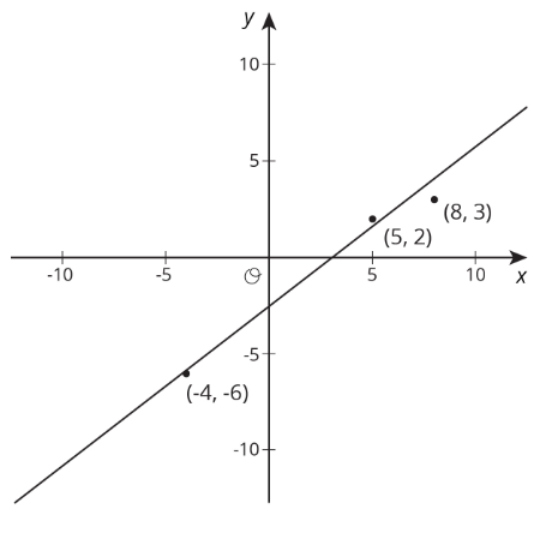


Figure 6.2.4.1

Exercise 6.2.4.2: Describing Linear Associations

For each scatter plot, decide if there is an association between the two variables, and describe the situation using one of these sentences:

- For these data, as _____ increases, _____ tends to increase.
- For these data, as _____ increases, _____ tends to decrease.
- For these data, _____ and _____ do not appear to be related.

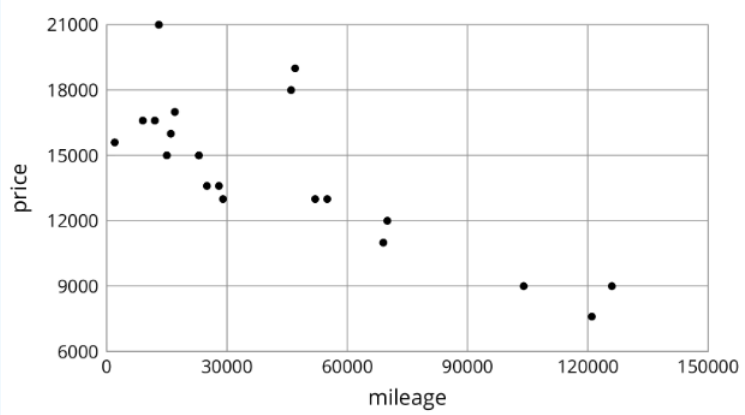


Figure 6.2.4.2

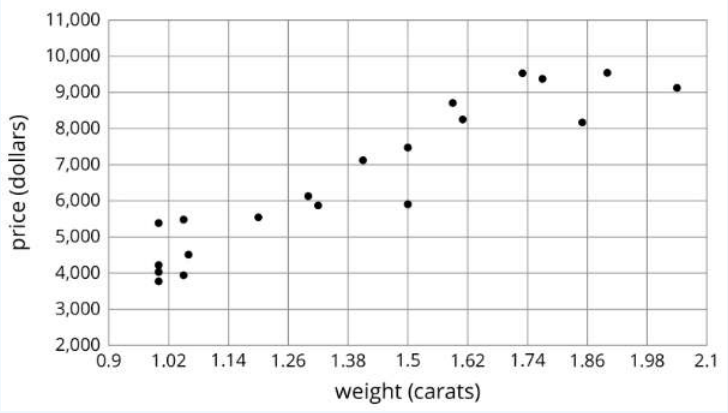


Figure 6.2.4.3

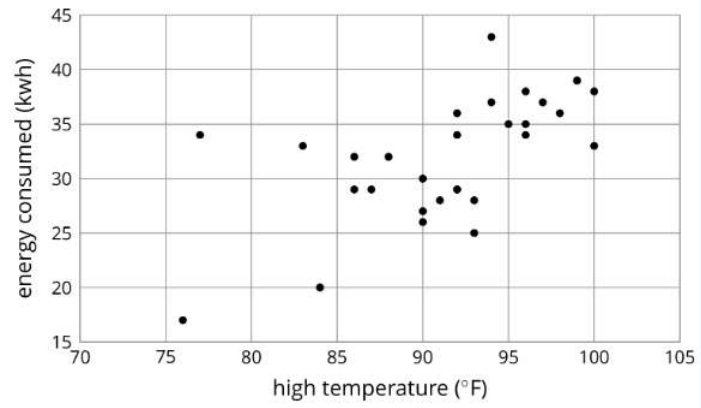


Figure 6.2.4.4

Exercise 6.2.4.3: Interpreting Slopes

For each of the situations, a linear model for some data is shown.

1. What is the slope of the line in the scatter plot for each situation?
2. What is the meaning of the slope in that situation?

$$y = 5,520.619x - 1,091.393$$

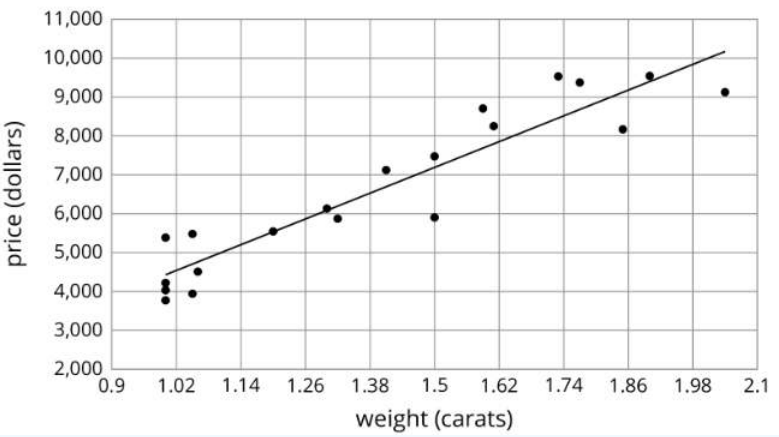


Figure 6.2.4.5

$$y = -.011x + 40.604$$

https://math.libretexts.org/@go/page/36716

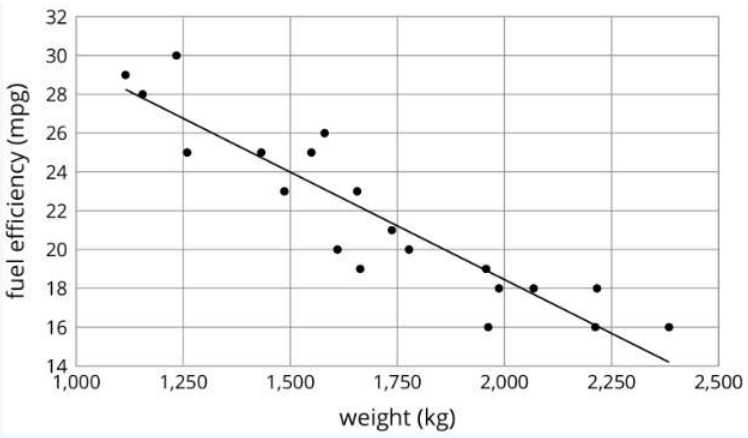


Figure 6.2.4.6

$$y = 0.59x - 21.912$$

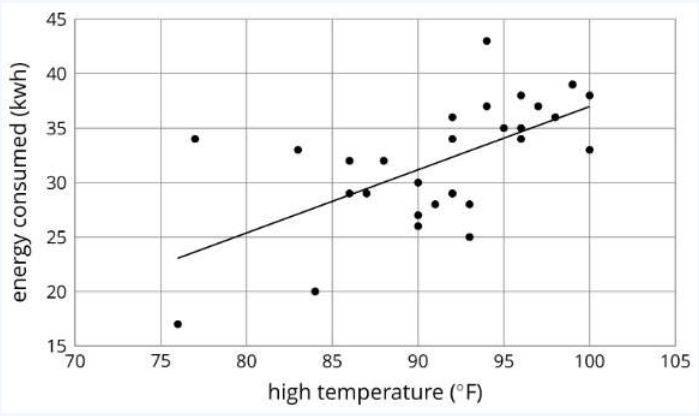


Figure 6.2.4.7

Are you ready for more?

The scatter plot shows the weight and fuel efficiency data used in an earlier lesson along with a linear model represented by the equation $y = -0.0114x + 41.3021$

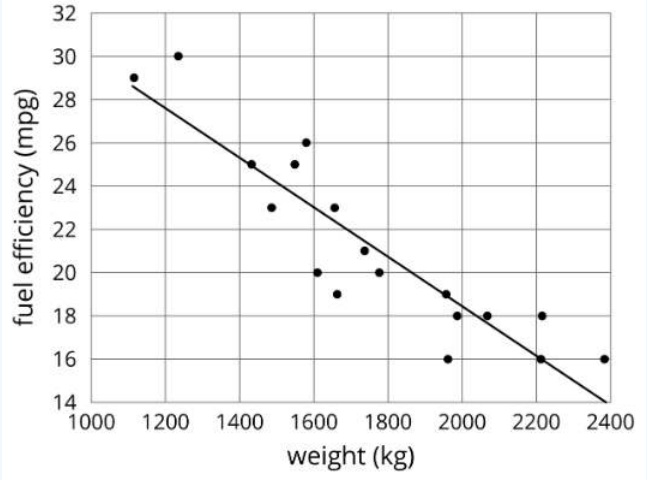


Figure 6.2.4.8

1. What is the value of the slope and what does it mean in this context?
2. What does the other number in the equation represent on the graph? What does it mean in context?
3. Use the equation to predict the fuel efficiency of a car that weighs 100 kilograms.
4. Use the equation to predict the weight of a car that has a fuel efficiency of 22 mpg.

5. Which of these two predictions probably fits reality better? Explain.

Exercise 6.2.4.4: Positive or Negative?

1. For each of the scatter plots, decide whether it makes sense to fit a linear model to the data. If it does, would the graph of the model have a positive slope, a negative slope, or a slope of zero?

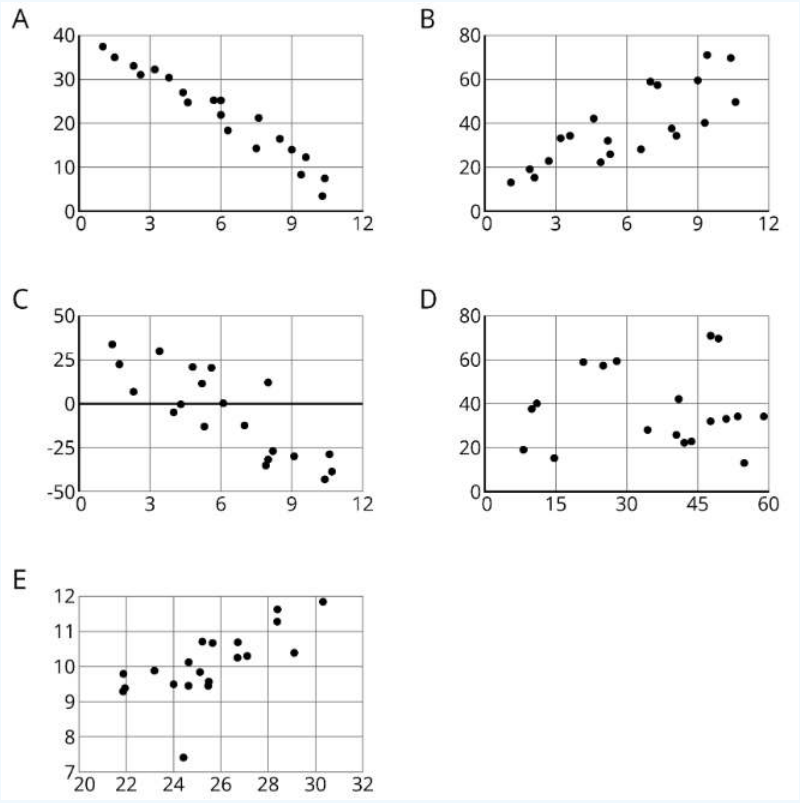


Figure 6.2.4.9

2. Which of these scatter plots show evidence of a positive association between the variables? Of a negative association? Which do not appear to show an association?

Summary

Here is a scatter plot that we have seen before. As noted earlier, we can see from the scatter plot that taller dogs tend to weigh more than shorter dogs. Another way to say it is that weight tends to increase as height increases. When we have a positive association between two variables, an increase in one means there tends to be an increase in the other.

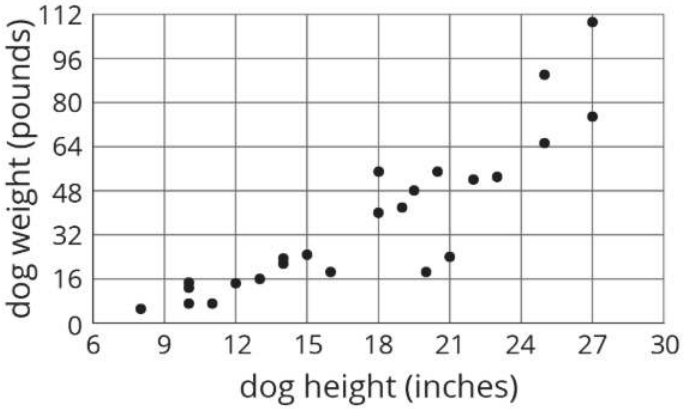


Figure 6.2.4.10

We can quantify this tendency by fitting a line to the data and finding its slope. For example, the equation of the fitted line is $w = 4.27h - 37$ where h is the height of the dog and w is the predicted weight of the dog.

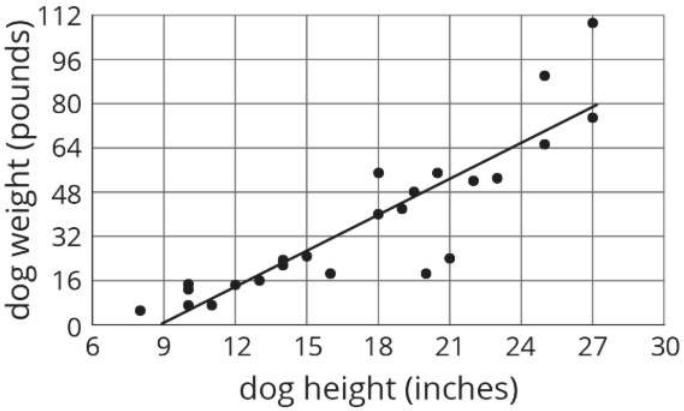


Figure 6.2.4.11

The slope is 4.27, which tells us that for every 1-inch increase in dog height, the weight is predicted to increase by 4.27 pounds.

In our example of the fuel efficiency and weight of a car, the slope of the fitted line shown is -0.01.

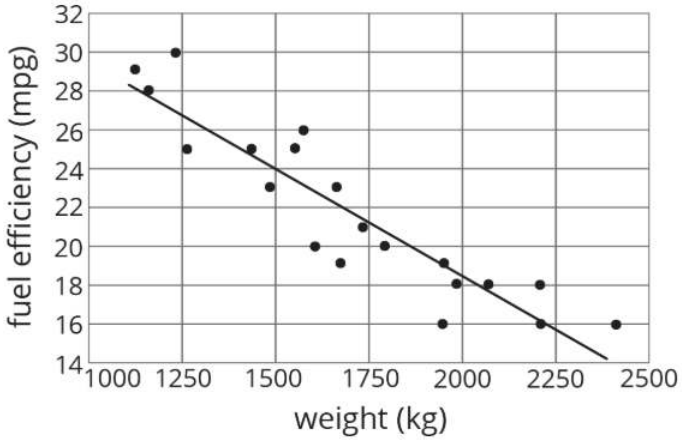


Figure 6.2.4.12

This tells us that for every 1-kilogram increase in the weight of the car, the fuel efficiency is predicted to decrease by 0.01 miles per gallon. When we have a negative association between two variables, an increase in one means there tends to be a decrease in the other.

Glossary Entries

Definition: Negative Association

A negative association is a relationship between two quantities where one tends to decrease as the other increases. In a scatter plot, the data points tend to cluster around a line with negative slope.

Different stores across the country sell a book for different prices.

The scatter plot shows that there is a negative association between the the price of the book in dollars and the number of books sold at that price.



Figure 6.2.4.13

Definition: Outlier

An outlier is a data value that is far from the other values in the data set.

Here is a scatter plot that shows lengths and widths of 20 different left feet. The foot whose length is 24.5 cm and width is 7.8 cm is an outlier.

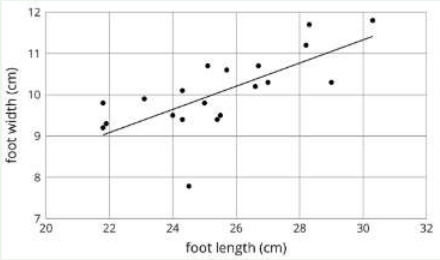


Figure 6.2.4.14

Definition: Positive Association

A positive association is a relationship between two quantities where one tends to increase as the other increases. In a scatter plot, the data points tend to cluster around a line with positive slope.

The relationship between height and weight for 25 dogs is shown in the scatter plot. There is a positive association between dog height and dog weight.

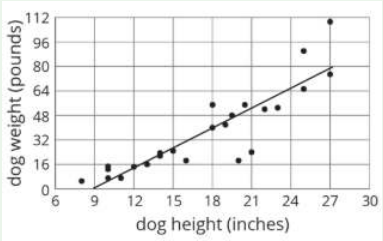


Figure 6.2.4.15

Practice

Exercise 6.2.4.5

Which of these statements is true about the data in the scatter plot?

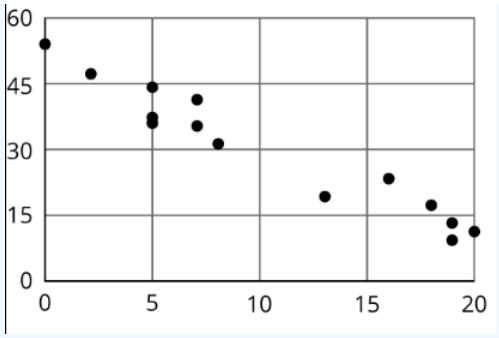


Figure 6.2.4.16

A. As x increases, y tends to increase.
B. As x increases, y tends to decrease.
C. Asx increases, y tends to stay unchanged.
D. x and y are unrelated.

Exercise 6.2.4.6

Here is a scatter plot that compares hits to at bats for players on a baseball team.

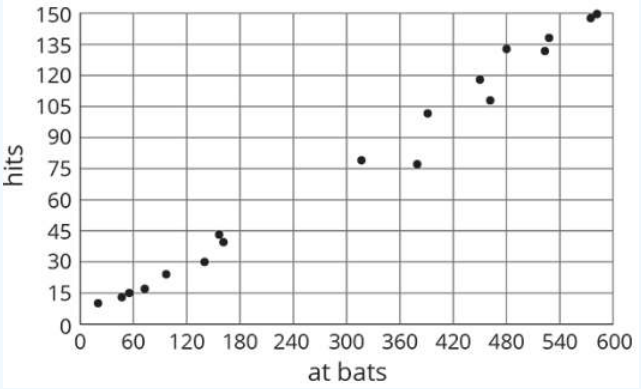


Figure 6.2.4.17

Describe the relationship between the number of at bats and the number of hits using the data in the scatter plot.

Exercise 6.2.4.7

The linear model for some butterfly data is given by the equation $y = 0.238x + 4.642$. Which of the following best describes the slope of the model?



Figure 6.2.4.18: **Attribution:** By Couleur. Public Domain. Pixabay. Source.

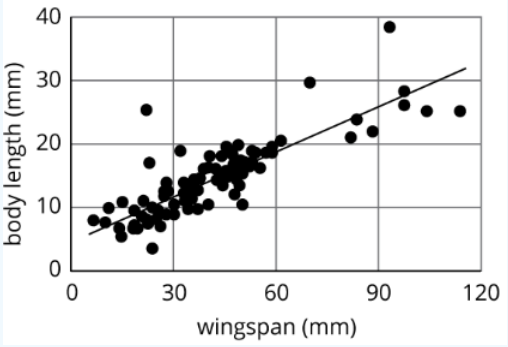


Figure 6.2.4.19

A. For every 1 mm the wingspan increases, the length of the butterfly increases 0.238 mm.
B. For every 1 mm the wingspan increases, the length of the butterfly increases 4.642 mm.
C. For every 1 mm the length of the butterfly increases, the wingspan increases 0.238 mm.
D. For every 1 mm the length of the butterfly increases, the wingspan increases 4.642 mm.

Exercise 6.2.4.8

Nonstop, one-way flight times from O'Hare Airport in Chicago and prices of a one-way ticket are shown in the scatter plot.

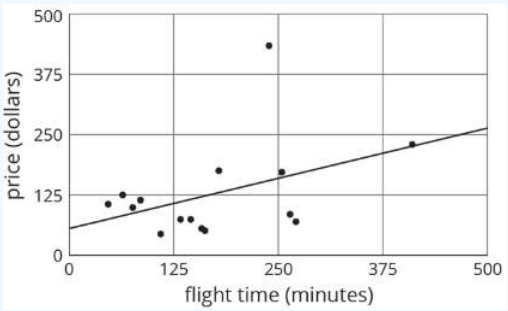


Figure 6.2.4.20

1. Circle any data that appear to be outliers.
2. Use the graph to estimate the difference between any outliers and their predicted values.

(From Unit 6.2.2)

Exercise 6.2.4.9

Solve:

$$\begin{cases} y = -3x + 13 \\ y = -2x + 1 \end{cases}$$

(From Unit 4.3.5)

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6.2.5: Observing More Patterns in Scatter Plots

Lesson

Let's look for other patterns in data.

Exercise 6.2.5.1: Notice and Wonder: Nonlinear Scatter Plot

What do you notice? What do you wonder?

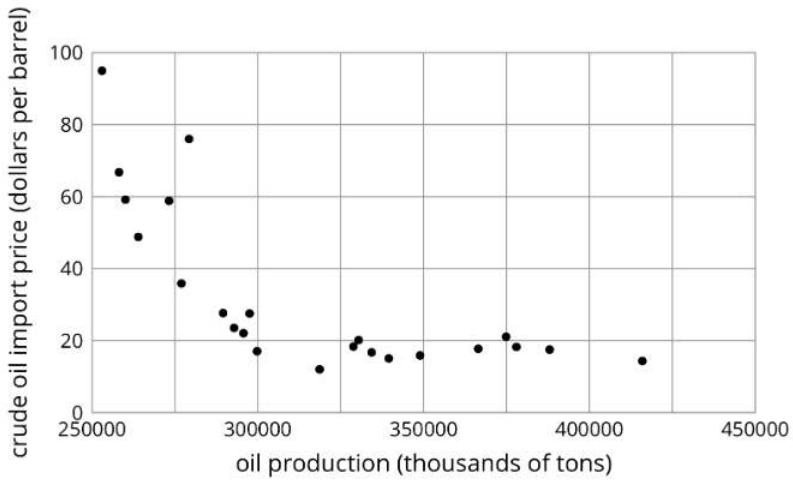


Figure 6.2.5.1

Exercise 6.2.5.2: Scatter Plot City

Your teacher will give you a set of cards. Each card shows a scatter plot.

1. Sort the cards into categories and describe each category.
2. Explain the reasoning behind your categories to your partner. Listen to your partner's reasoning for their categories.
3. Sort the cards into two categories: positive associations and negative associations. Compare your sorting with your partner's and discuss any disagreements.
4. Sort the cards into two categories: linear associations and non-linear associations. Compare your sorting with your partner's and discuss any disagreements.

Exercise 6.2.5.3: Clustering

How are these scatter plots alike? How are they different?

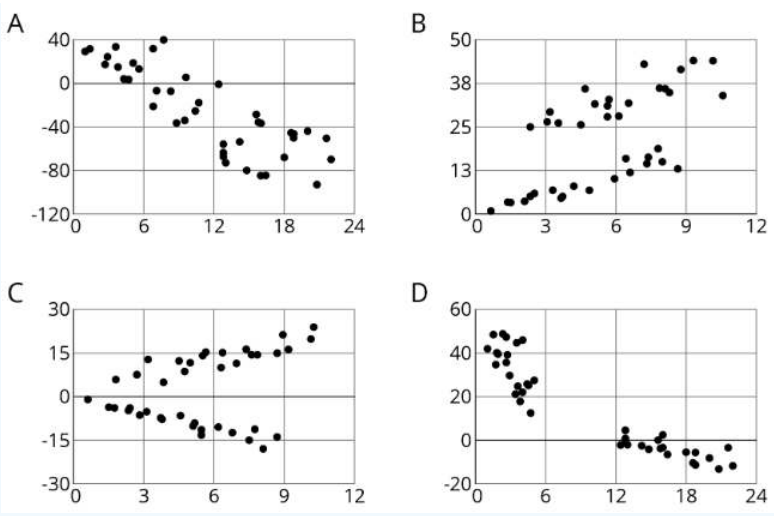


Figure 6.2.5.2: Four scatterplots. Plot A, points start at 0 comma 40 and trend down and to the right. Plot B has 2 sets of points, 1 starts near the origin and trends up and to the right. The second starts near 3 comma 25 and trends up and to the right. Plot C has 2 sets of points, 1 starts near the origin and trends up and to the right. The second starts near the origin and trends down and to the right. Plot D points starts near 0 comma 40 and trends down and to the right toward 24 negative 20, but there are no points between x = 6 and x = 12.

Summary

Sometimes a scatter plot shows an association that is *not* linear:

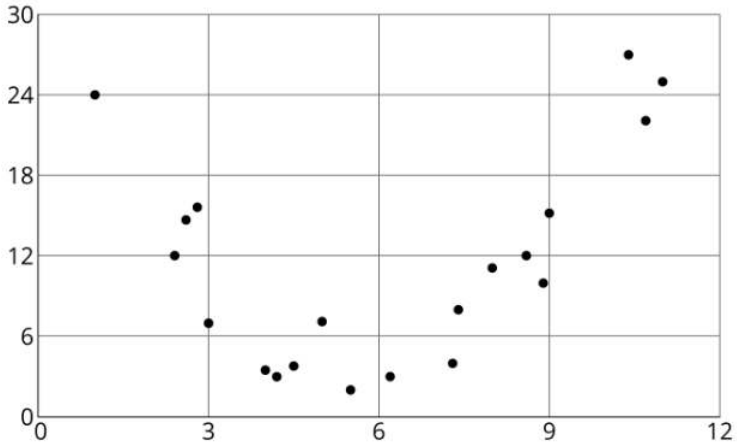


Figure 6.2.5.3

We call such an association a *non-linear association*. In later grades, you will study functions that can be models for non-linear associations.

Sometimes in a scatter plot we can see separate groups of points.

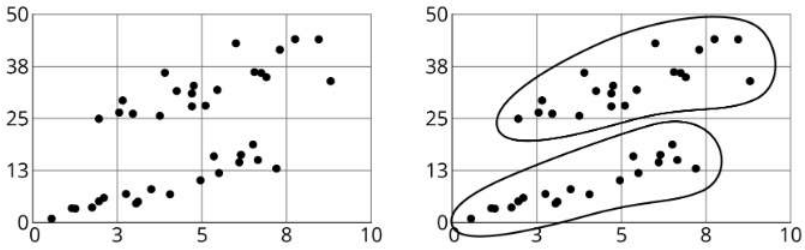


Figure 6.2.5.4: A scatterplot with two groups of points. The first begins near the origin and trends up and to the right toward 8 comma 13. Second group begins near 3 comma 25 and trends up and right toward 9 comma 45. The second image is the same scatterplot, with each group circled.

We call these groups *clusters*.

Glossary Entries

Definition: Negative Association

A negative association is a relationship between two quantities where one tends to decrease as the other increases. In a scatter plot, the data points tend to cluster around a line with negative slope.

Different stores across the country sell a book for different prices.

The scatter plot shows that there is a negative association between the the price of the book in dollars and the number of books sold at that price.



Figure 6.2.5.5

Definition: Outlier

An outlier is a data value that is far from the other values in the data set.

Here is a scatter plot that shows lengths and widths of 20 different left feet. The foot whose length is 24.5 cm and width is 7.8 cm is an outlier.

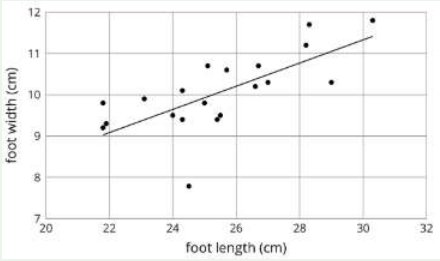


Figure 6.2.5.6

Definition: Positive Association

A positive association is a relationship between two quantities where one tends to increase as the other increases. In a scatter plot, the data points tend to cluster around a line with positive slope.

The relationship between height and weight for 25 dogs is shown in the scatter plot. There is a positive association between dog height and dog weight.

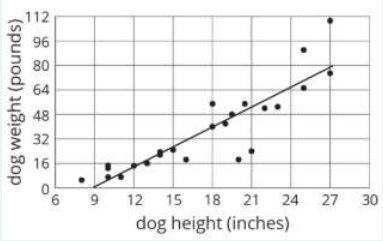


Figure 6.2.5.7

Practice

Exercise 6.2.5.4

Literacy rate and population for the 12 countries with more than 100 million people are shown in the scatter plot. Circle any clusters in the data.

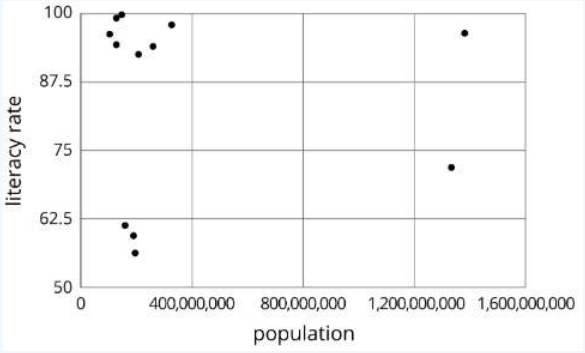


Figure 6.2.5.8

Exercise 6.2.5.5

Here is a scatter plot:

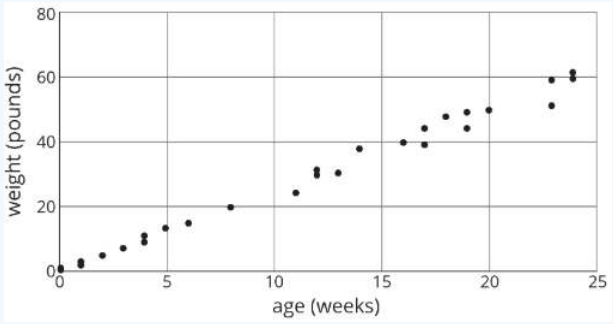


Figure 6.2.5.9

Select **all** the following that describe the association in the scatter plot:

A. Linear association
B. Non-linear association
C. Positive association
D. Negative association
E. No association

Exercise 6.2.5.6

For the same data, two different models are graphed. Which model more closely matches the data? Explain your reasoning.

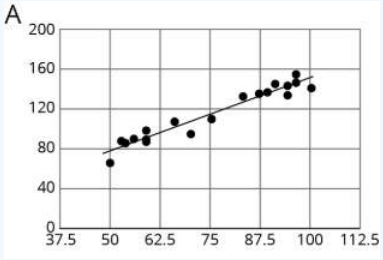


Figure 6.2.5.10

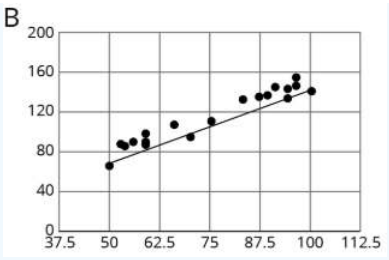


Figure 6.2.5.11

(From Unit 6.2.3)

Exercise 6.2.5.7

Here is a scatter plot of data for some of the tallest mountains on Earth.

The heights in meters and year of first recorded ascent is shown. Mount Everest is the tallest mountain in this set of data.

1. Estimate the height of Mount Everest.
2. Estimate the year of the first recorded ascent of Mount Everest.

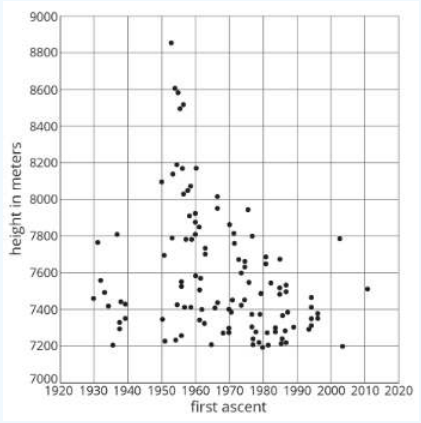


Figure 6.2.5.12

(From Unit 6.2.1)

Exercise 6.2.5.8

A cone has a volume V, radius r, and a height of 12 cm.

1. A cone has the same height and $\frac{1}{3}$ of the radius of the original cone. Write an expression for its volume.
2. A cone has the same height and 3 times the radius of the original cone. Write an expression for its volume.

(From Unit 5.5.2)

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6.2.6: Analyzing Bivariate Data

Lesson

Let's analyze data like a pro.

Exercise 6.2.6.1: Speed vs Step Length

A researcher found an association between a dog's stride length and its speed: the longer a dog's steps, the faster it goes. The predicted speed in meters per second, s, as a function of step length in meters, l, is

$s = 4l - 1.6$

What does the rate of change of the function tell you about the association between stride length and speed?

Exercise 6.2.6.2: Animal Brains

Is there an association between the weight of an animal's body and the weight of the animal's brain?

Table 6.2.6.1

animal	body weight (kg)	brain weight (kg)
cow	465	423
grey wolf	36	120
goat	28	115
donkey	187	419
horse	521	655
potar monkey	10	115
cat	3	26
giraffe	529	680
gorilla	207	406
human	62	1320
rhesus monkey	7	179
kangaroo	35	56
sheep	56	175
jaguar	100	157
chimpanzee	52	440
pig	192	180

Use the data to make a scatter plot. Are there any outliers?

Experiment with the line to fit the data. Drag the points to move the line. You can close the expressions list by clicking on the double arrow.

1. Without including any outliers, does there appear to be an association between body weight and brain weight? Describe the association in a sentence.
2. Adjust the line by moving the green points, fitting the line to your scatter plot, and estimate its slope. What does this slope mean in the context of brain and body weight?
3. Does the fitted line help you identify any other outliers?

Are you ready for more?

Use one of the suggestions or find another set of data that interested you to look for associations between the variables.

- Number of wins vs number of points per game for your favorite sports team in different seasons
- Amount of money grossed vs critic rating for your favorite movies
- Price of a ticket vs stadium capacity for popular bands on tour

After you have collected the data,

1. Create a scatter plot for the data.
2. Are any of the points very far away from the rest of the data?
3. Would a linear model fit the data in your scatter plot? If so, draw it. If not, explain why a line would be a bad fit.
4. Is there an association between the two variables? Explain your reasoning.

Exercise 6.2.6.3: Equal Body Dimensions

1. Sometimes a person's arm span is the same as their height. Is this true for anyone in the class?
2. Make a scatter plot for the arm span and height data, and describe any association. Click on the plus sign to get a menu and add a table, if you choose.
3. Is the line $y = x$ a good fit for the data? If so, explain why. If not, find the equation of a line that fits the data better.
4. Examine the scatter plot. Which person in your class has the *largest* ratio between their arm span and their height? Explain or show your reasoning.

Summary

People often collect data in two variables to investigate possible associations between two numerical variables and use the connections that they find to predict more values of the variables. Data analysis usually follows these steps:

1. Collect data.
2. Organize and represent the data, and look for an association.
3. Identify any outliers and try to explain why these data points are exceptions to the trend that describes the association.
4. Find a function that fits the data well.

Although computational systems can help with data analysis by graphing the data, finding a function that might fit the data, and using that function to make predictions, it is important to understand the process and think about what is happening. A computational system may find a function that does not make sense or use a line when the situation suggests that a different model would be more appropriate.

Glossary Entries

Definition: Negative Association

A negative association is a relationship between two quantities where one tends to decrease as the other increases. In a scatter plot, the data points tend to cluster around a line with negative slope.

Different stores across the country sell a book for different prices.

The scatter plot shows that there is a negative association between the the price of the book in dollars and the number of books sold at that price.



Figure 6.2.6.1

Definition: Outlier

An outlier is a data value that is far from the other values in the data set.

Here is a scatter plot that shows lengths and widths of 20 different left feet. The foot whose length is 24.5 cm and width is 7.8 cm is an outlier.

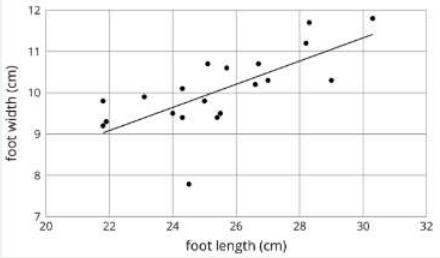


Figure 6.2.6.2

Definition: Positive Association

A positive association is a relationship between two quantities where one tends to increase as the other increases. In a scatter plot, the data points tend to cluster around a line with positive slope.

The relationship between height and weight for 25 dogs is shown in the scatter plot. There is a positive association between dog height and dog weight.

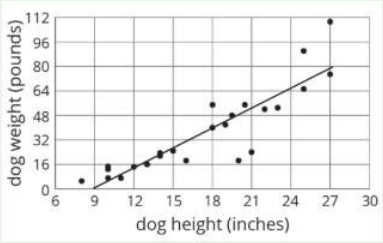


Figure 6.2.6.3

Practice

Exercise 6.2.6.4

Different stores across the country sell a book for different prices. The table shows the price of the book in dollars and the number of books sold at that price.

Table 6.2.6.2

price in dollars	number sold
11.25	53
10.50	60
12.10	30
8.45	81
9.25	70
9.75	80
7.25	120
12	37
9.99	130

price in dollars	number sold
7.99	100
8.75	90

1. Draw a scatter plot of this data. Label the axes.
2. Are there any outliers? Explain your reasoning.
3. If there is a relationship between the variables, explain what it is.
4. Remove any outliers, and draw a line that you think is a good fit for the data.

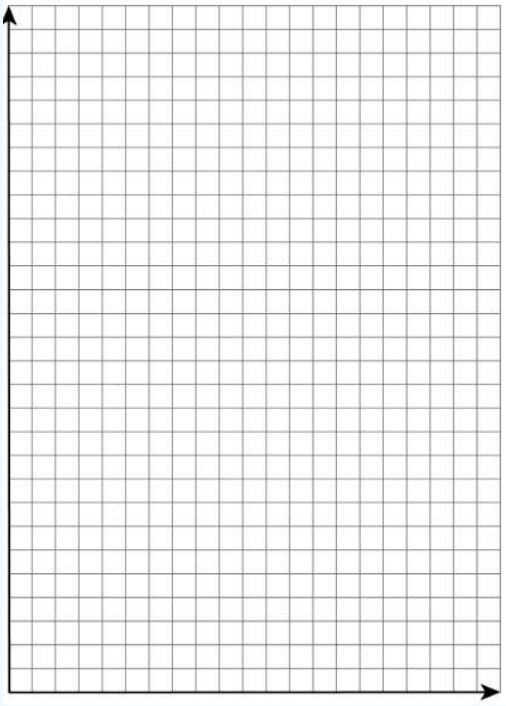


Figure 6.2.6.4

Exercise 6.2.6.5

Here is a scatter plot:

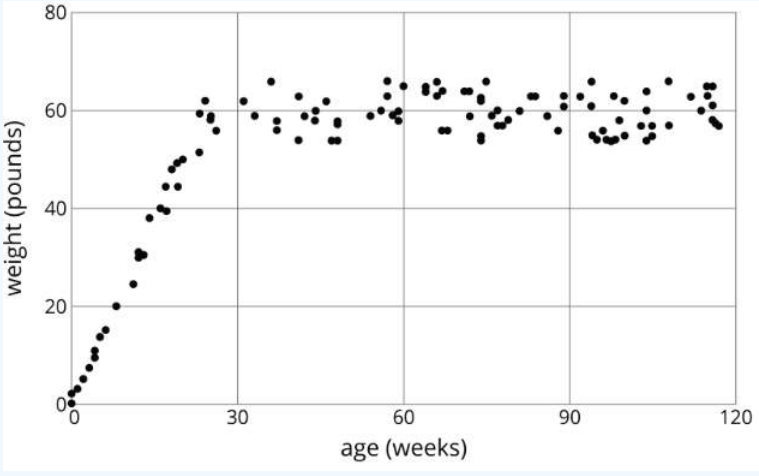


Figure 6.2.6.5

Select **all** the following that describe the association in the scatter plot:

A. Linear association
B. Non-linear association
C. Positive association
D. Negative association
E. No association

(From Unit 6.2.5)

Exercise 6.2.6.6

Using the data in the scatter plot, what can you tell about the slope of a good model?

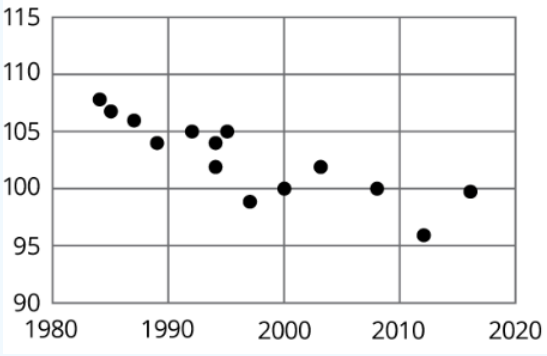


Figure 6.2.6.6

A. The slope is positive.
B. The slope is zero.
C. The slope is negative.
D. There is no association.

(From Unit 6.2.4)

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6.3: Associations in Categorical Data

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6.3.1: Looking for Associations

Lesson

Let's look for associations in data.

Exercise 6.3.1.1: Notice and Wonder: Bar Association

What do you notice? What do you wonder?

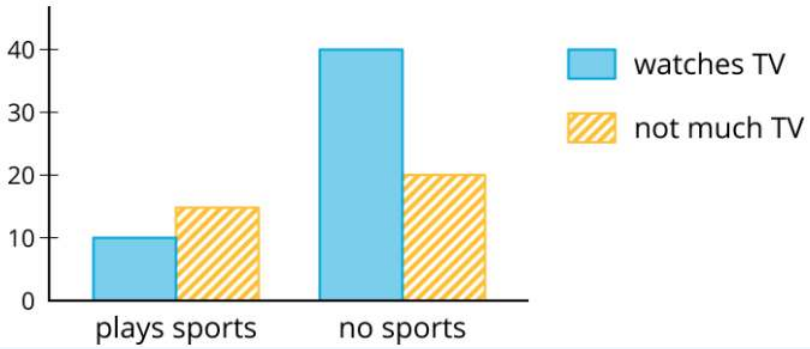


Figure 6.3.1.1: Double bar graphs in blue and yellow stripes. Horizontal labeled plays sports and no sports. Vertical labeled 0 to 40. Blue represents watches TV. Yellow stripes represents not much TV.

Exercise 6.3.1.2: Card Sort: Matching Representations

Your teacher will hand out some cards.

Some cards show **two-way tables** like this:

Table 6.3.1.1

	has cell phone	does not have cell phone	total
10 to 12 years old	25	35	60
13 to 15 years old	40	10	50
16 to 18 years old	50	10	60
total	115	55	170

Some cards show bar graphs like this:

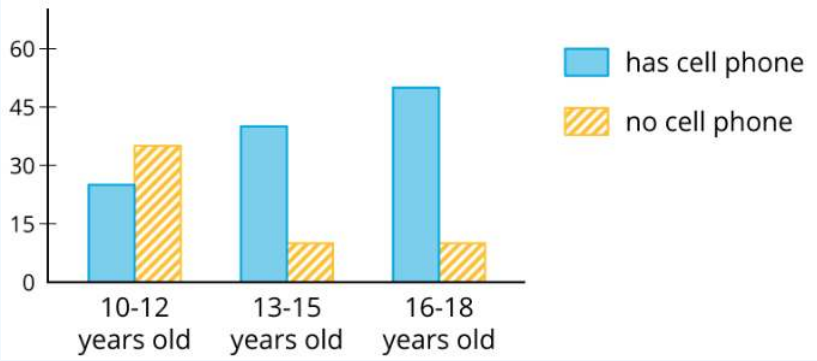


Figure 6.3.1.2: Double bar graph in blue and yellow stripes. Horizontal labeled 10 to 12 years old, 13 to 15 years old and 16 to 18 years old. Vertical labeled 0 to 60. Blue section means has cell phone. Yellow stripes means has no cell phone.

Some cards show **segmented bar graphs** like this:

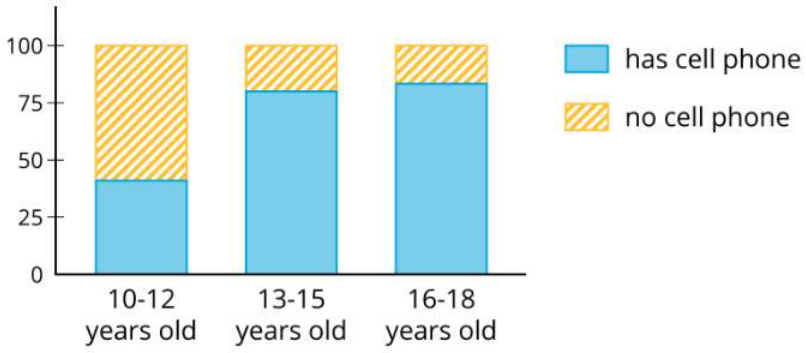


Figure 6.3.1.3: Stacked bar graph in blue and yellow stripes. Horizontal labeled 10 to 12 years old, 13 to 15 years old and 16 to 18 years old. Vertical labeled 0 to 100. Blue section means has cell phone. Yellow stripes means has no cell phone.

The bar graphs and segmented bar graphs have their labels removed.

1. Put all the cards that describe the same situation in the same group.
2. One of the groups does not have a two-way table. Make a two-way table for the situation described by the graphs in the group.
3. Label the bar graphs and segmented bar graphs so that the categories represented by each bar are indicated.
4. Describe in your own words the kind of information shown by a segmented bar graph.

Are you ready for more?

One of the segmented bar graphs is missing. Construct a segmented bar graph that matches the other representations.

Exercise 6.3.1.3: Building Another Type of Two-Way Table

Here is a two-way table that shows data about cell phone usage among children aged 10 to 18.

Table 6.3.1.2

	has cell phone	does not have cell phone	total
10 to 12 years old	25	35	60
13 to 15 years old	40	10	50
16 to 18 years old	50	10	60
total	115	55	170

1. Complete the table. In each row, the entries for “has cell phone” and “does not have cell phone” should have the total 100%. Round entries to the nearest percentage point.

Table 6.3.1.3

	has cell phone	does not have cell phone	total
10 to 12 years old	42%		
13 to 15 years old			100%
16 to 18 years old		17%	

This is still a two-way table. Instead of showing *frequency*, this table shows **relative frequency**.

2. Two-way tables that show relative frequencies often don’t include a “total” row at the bottom. Why?

3. Is there an association between age and cell phone use? How does the two-way table of relative frequencies help to illustrate this?

Are you ready for more?

A pollster attends a rally and surveys many of the participants about whether they associate with political Party A or political Party B and whether they are for or against Proposition 3.14 going up for vote soon. The results are sorted into the table shown.

Table 6.3.1.4

	for	against
party A	832	165
party B	80	160

- A news station reports these results by saying, “A poll shows that about the same number of people from both parties are voting against Proposition 3.14.”
- A second news station shows this graphic.

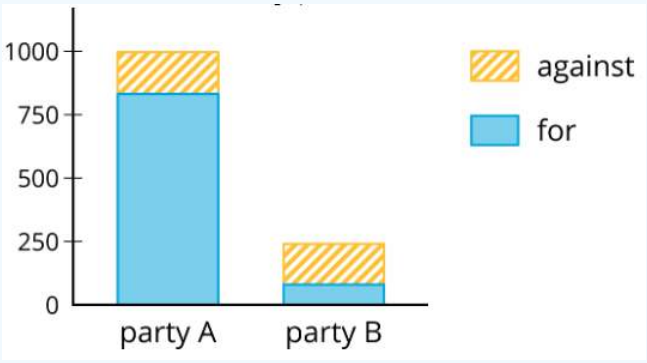


Figure 6.3.1.4

1. Are any of the news reports misleading? Explain your reasoning.
2. Create a headline, graphic, and short description that more accurately represents the data in the table.

Summary

When we collect data by counting things in various categories, like red, blue, or yellow, we call the data *categorical data*, and we say that color is a *categorical variable*.

We can use **two-way tables** to investigate possible connections between two categorical variables. For example, this two-way table of frequencies shows the results of a study of meditation and state of mind of athletes before a track meet.

Table 6.3.1.5

	meditated	did not meditate	total
calm	45	8	53
agitated	23	21	44
total	68	29	97

If we are interested in the question of whether there is an association between meditating and being calm, we might present the frequencies in a bar graph, grouping data about meditators and grouping data about non-meditators, so we can compare the numbers of calm and agitated athletes in each group.

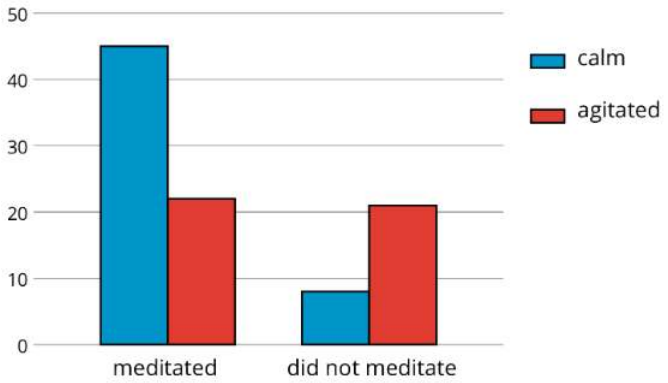


Figure 6.3.1.5: Double bar graph in blue and red. Horizontal labeled meditated and did not meditate. Vertical labeled 0 to 50. Blue represents calm. Red represents agitated.

Notice that the number of athletes who did not meditate is small compared to the number who meditated (29 as compared to 68, as shown in the table).

If we want to know the proportions of calm meditators and calm non-meditators, we can make a two-way table of **relative frequencies** and present the relative frequencies in a **segmented bar graph**.

Table 6.3.1.6

	meditated	did not meditate
calm	66%	28%
agitated	34%	72%
total	100%	100%

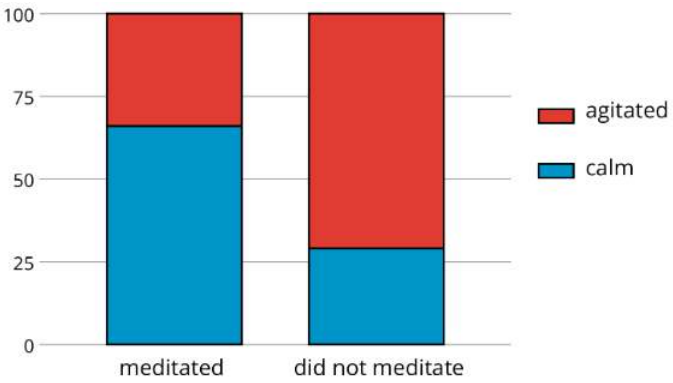


Figure 6.3.1.6: Stacked bar graph in blue and red. Horizontal labeled meditated and did not meditate. Vertical labeled 0 to 100. Blue represents calm. Red represents agitated.

Glossary Entries

Definition: Relative Frequency

The relative frequency of a category tells us the proportion at which the category occurs in the data set. It is expressed as a fraction, a decimal, or a percentage of the total number.

For example, suppose there were 21 dogs in the park, some white, some brown, some black, and some multi-color. The table shows the frequency and the relative frequency of each color.

Table 6.3.1.7

color	frequency	relative frequency
white	5	$\frac{5}{21}$

color	frequency	relative frequency
brown	7	$\frac{7}{21}$
black	3	$\frac{3}{21}$
multi-color	6	$\frac{6}{21}$

Definition: Segmented Bar Graph

A segmented bar graph compares two categories within a data set. The whole bar represents all the data within one category. Then, each bar is separated into parts (segments) that show the percentage of each part in the second category.

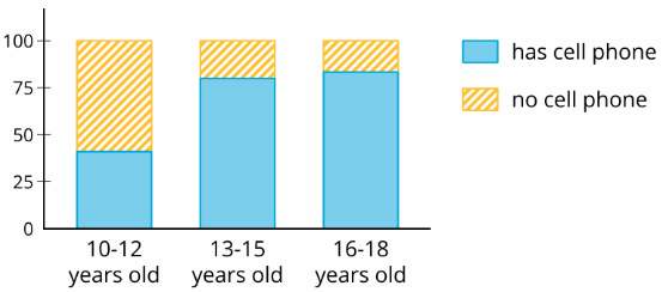


Figure 6.3.1.7

This segmented bar graph shows the percentage of people in different age groups that do and do not have a cell phone. For example, among people ages 10 to 12, about 40% have a cell phone and 60% do not have a cell phone.

Definition: Two-Way Table

A two-way table provides a way to compare two categorical variables.

It shows one of the variables across the top and the other down one side. Each entry in the table is the frequency or relative frequency of the category shown by the column and row headings.

A study investigates the connection between meditation and the state of mind of athletes before a track meet. This two-way table shows the results of the study.

Table 6.3.1.8

	meditated	did not meditate	total
calm	45	8	53
agitated	23	21	44
total	68	29	97

Practice

Exercise 6.3.1.4

A scientist wants to know if the color of the water affects how much animals drink. The average amount of water each animal drinks was recorded in milliliters for a week and then graphed. Is there evidence to suggest an association between water color and animal?

Table 6.3.1.9

	cat intake (mL)	dog intake (mL)	total (mL)
blue water	210	1200	1410

	cat intake (mL)	dog intake (mL)	total (mL)
green water	200	1100	1300
total	410	2300	2710

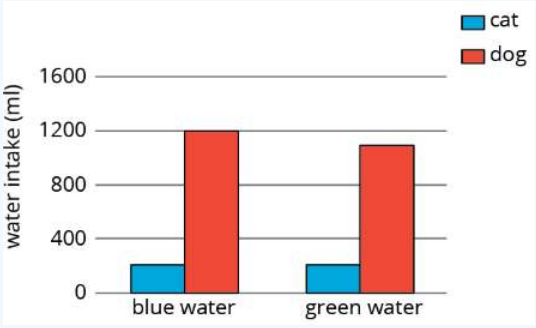


Figure 6.3.1.8

Exercise 6.3.1.5

A farmer brings his produce to the farmer's market and records whether people buy lettuce, apples, both, or something else.

Table 6.3.1.10

	bought apples	did not buy apples
bought lettuce	14	58
did not buy lettuce	8	29

Make a table that shows the relative frequencies for each row. Use this table to decide if there is an association between buying lettuce and buying apples.

Exercise 6.3.1.6

Researchers at a media company want to study news-reading habits among different age groups. They tracked print and online subscription data and made a 2-way table.

Table 6.3.1.11

	internet media	print media
18-25 year olds	151	28
26-45 year olds	132	72
46-65 year olds	48	165

1. Create a segmented bar graph using one bar for each row of the table.
2. Is there an association between age groups and the method they use to read articles? Explain your reasoning.

Exercise 6.3.1.7

Using the data in the scatter plot, what is a reasonable slope of a model that fits this data?

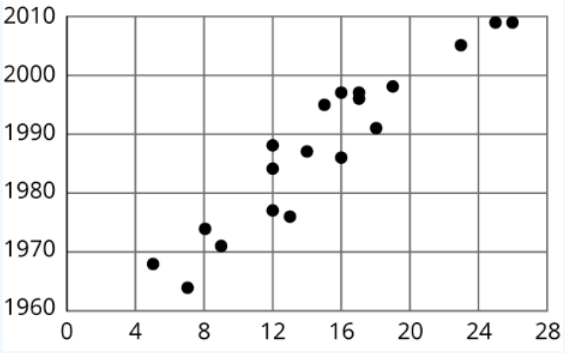


Figure 6.3.1.9

A. -2.5
B. -1
C. 1
D. 2.5

(From Unit 6.2.4)

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6.3.2: Using Data Displays to Find Associations

Lesson

Let's use data displays to find associations.

Exercise 6.3.2.1: Sports and Musical Instruments

For a survey, students in a class answered these questions:

- Do you play a sport?
- Do you play a musical instrument?

1. Here is a two-way table that gives some results from the survey. Complete the table, assuming that all students answered both questions.

Table 6.3.2.1

	plays instrument	does not play instrument	total
plays sport	5		16
does not play sport			
total		15	25

2. To the nearest percentage point, what percentage of students who play a sport *don't* play a musical instrument?

3. To the nearest percentage point, what percentage of students who *don't* play a sport also *don't* play a musical instrument?

Exercise 6.3.2.2: Sports and Music Association

Your teacher will give you a two-way table with information about the number of people in your class who play sports or musical instruments.

1. Complete this table to make a two-way table for the data from earlier. The table will show relative frequencies *by row*.

Table 6.3.2.2

	plays instrument	does not play instrument	row total
plays sport			100%
does not play sport			100%

2. Make a segmented bar graph for the table. Use one bar of the graph for each row of the table.



Figure 6.3.2.1

3. Complete the table to make a two-way table for the data from earlier. The table will show relative frequencies *by column*.

Table 6.3.2.3

	plays instrument	**does not play instrument**
plays sport		
does not play sport		
column total	100%	100%

4. Using the values in the table, make a segmented bar graph. Use one bar of the graph for each column of the table.



Figure 6.3.2.2

5. Based on the two-way tables and segmented bar graphs, do you think there is an association between playing a sport and playing a musical instrument? Explain how you know.

Exercise 6.3.2.3: Colored Erasers

An eraser factory has five machines. One machine makes the eraser shapes. Then each shape goes through the red machine, blue machine, yellow machine, or green machine to have a side colored.

The manager notices that an uncolored side of some erasers is flawed at the end of the process and wants to know which machine needs to be fixed: the shape machine or some of the color machines. The manager collected data on the number of

flawed and unflawed erasers of each color.

Table 6.3.2.4

	unflawed	flawed	total
red	285	15	300
blue	223	17	240
yellow	120	80	200
green	195	65	260
total	823	177	1000

1. Work with a partner. Each of you should make one segmented bar graph for the data in the table. One segmented bar graph should have a bar for each *row* of the table. The other segmented bar graph should have one bar for each *column* of the table.
2. Are the flawed erasers associated with certain colors? If so, which colors? Explain your reasoning.

Are you ready for more?

Based on the federal budgets for 2009, the table shows where some of the federal money was expected to go. The values are in billions of U.S. Dollars.

Table 6.3.2.5

	United States	Japan	United Kingdom
defense	718.4	42.8	49.2
education	44.9	47.5	113.9

1. Why would a segmented bar graph be more useful than the table of data to see any associations between the country and where the money is spent?
2. Create a segmented bar graph that represents the data from the table.
3. Is there an association between the country's budget and their spending in these areas? Explain your reasoning.

Summary

In an earlier lesson, we looked at data on meditation and state of mind in athletes.

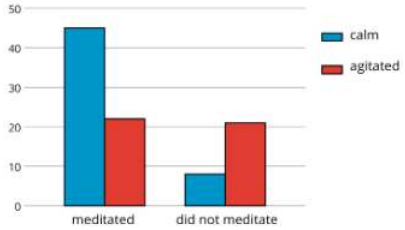


Figure 6.3.2.3: Double bar graph in blue and red. Horizontal labeled meditated and did not meditate. Vertical labeled 0 to 50. Blue represents calm. Red represents agitated.

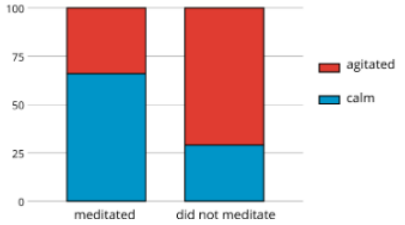


Figure 6.3.2.4: Stacked bar graph in blue and red. Horizontal labeled meditated and did not meditate. Vertical labeled 0 to 100. Blue represents calm. Red represents agitated.

Is there an association between meditation and state of mind?

The bar graph shows that more athletes were calm than agitated among the group that meditated, and more athletes were agitated than calm among the group that did not. We can see the proportions of calm meditators and calm non-meditators from the segmented bar graph, which shows that about 66% of athletes who meditated were calm, whereas only about 27% of those who did not meditate were calm.

This does not necessarily mean that meditation causes calm; it could be the other way around, that calm athletes are more inclined to meditate. But it does suggest that there is an association between meditating and calmness.

Glossary Entries

Definition: Relative Frequency

The relative frequency of a category tells us the proportion at which the category occurs in the data set. It is expressed as a fraction, a decimal, or a percentage of the total number.

For example, suppose there were 21 dogs in the park, some white, some brown, some black, and some multi-color. The table shows the frequency and the relative frequency of each color.

Table 6.3.2.6

color	frequency	relative frequency
white	5	$\frac{5}{21}$
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multi-color	6	$\frac{6}{21}$

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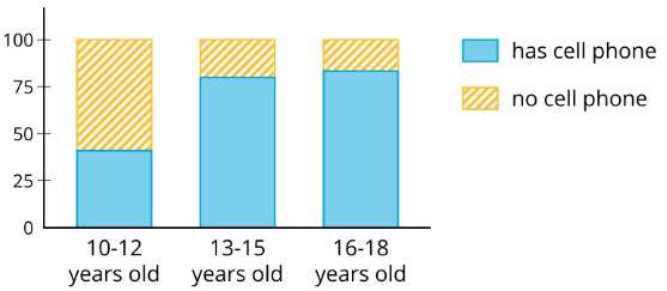


Figure 6.3.2.5

This segmented bar graph shows the percentage of people in different age groups that do and do not have a cell phone. For example, among people ages 10 to 12, about 40% have a cell phone and 60% do not have a cell phone.

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A study investigates the connection between meditation and the state of mind of athletes before a track meet. This two-way table shows the results of the study.

Table 6.3.2.7

	meditated	did not meditate	total
calm	45	8	53
agitated	23	21	44
total	68	29	97

Practice

Exercise 6.3.2.4

An ecologist is studying a forest with a mixture of tree types. Since the average tree height in the area is 40 feet, he measures the height of the tree against that. He also records the type of tree. The results are shown in the table and segmented bar graph. Is there evidence of an association between tree height and tree type? Explain your reasoning.

Table 6.3.2.8

	under 40 ft	40 feet or taller	total
deciduous	45	30	75
evergreen	14	10	24
total	59	40	99

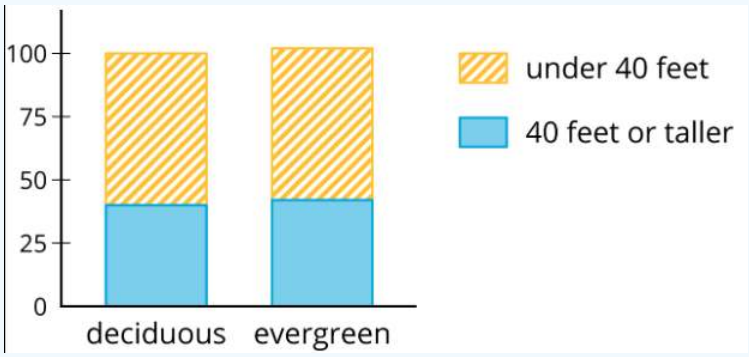


Figure 6.3.2.6

Exercise 6.3.2.5

Workers at an advertising agency are interested in people's TV viewing habits. They take a survey of people in two cities to try to find patterns in the types of shows they watch. The results are recorded in a table and shown in a segmented bar graph. Is there evidence of different viewing habits? If so, explain.

Table 6.3.2.9

	reality	news	comedy	drama
Chicago	50	40	90	20
Topeka	45	70	40	45

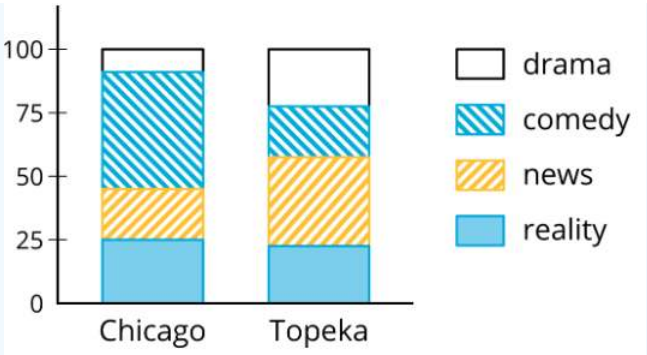


Figure 6.3.2.7

Exercise 6.3.2.6

A scientist is interested in whether certain species of butterflies like certain types of local flowers. The scientist captures butterflies in two zones with different flower types and records the number caught. Do these data show an association between butterfly type and zone? Explain your reasoning.

Table 6.3.2.10

	zone 1	zone 2
eastern tiger swallowtail	16	34
monarch	24	46

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6.4: Let's Put it to Work

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6.4.1: Gone in 30 Seconds

Lesson

Let's gather and analyze some timing data.

Exercise 6.4.1.1: Measuring 30 Seconds

In this activity, you'll get two chances to guess at how long 30 seconds is, then look for an association between the two guesses of all students.

1. Work with a partner. Follow the instructions listed here to gather your data.

- One of you will hold a stopwatch where the other person cannot see it.
- The person holding the stopwatch says "go" and starts the timer.
- The other person says "stop" when they think 30 seconds have passed.
- The person holding the stopwatch will stop the timer, then report and record the time to the nearest second.
- The person holding the stopwatch will give a second chance, repeating the experiment.
- After *both* times are recorded, switch roles.

2. Record the group data in this table. When you finish, a group member should give the data to the teacher.

Table 6.4.1.1

name	time 1	time 2

3. Look at your data. Comparing Time 1 to Time 2, do you think there is a positive association, a negative association, or no association? Discuss your thinking with your group.

4. What are some ways you could organize and represent the entire class's data?

5. Make a scatter plot of the entire class's data and look for patterns. Identify any outliers and the type of any association you observe.

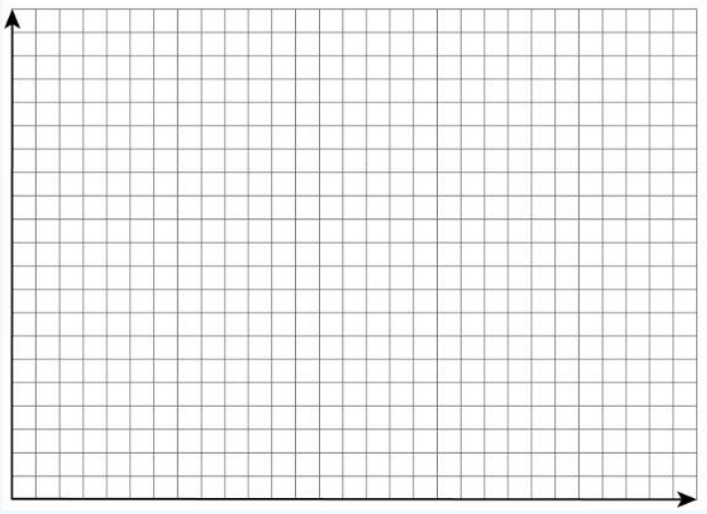


Figure 6.4.1.1

6. Draw two lines on your scatter plot: a vertical line and a horizontal line, each representing 30 seconds for one trial. Use the table for the class's data to complete this two-way table.

Table 6.4.1.2

	time 2 $<$ 30 sec	time 2 $=$ 30 sec	time 2 $>$ 30 sec	total

	time 2 $<$ 30 sec	time 2 $=$ 30 sec	time 2 $>$ 30 sec	total
time 1 $<$ 30 sec				
time 1 $=$ 30 sec				
time 1 $>$ 30 sec				
total				

7. Use the two-way table to decide whether there is an association between Time 1 and Time 2. Explain how you know.

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CHAPTER OVERVIEW

7: Exponents and Scientific Notation

In grade 6, students studied whole-number exponents. In this unit, they extend the definition of exponents to include all integers, and in the process codify the properties of exponents. They apply these concepts to the base-ten system, and learn about orders of magnitude and scientific notation in order to represent and compute with very large and very small quantities.

7.1: Exponent Review

7.1.1: Exponent Review

7.2: Exponent Rules

7.2.1: Multiplying Powers of 10

7.2.2: Powers of Powers of 10

7.2.3: Dividing Powers of 10

7.2.4: Negative Exponents with Powers of 10

7.2.5: What about Other Bases?

7.2.6: Practice with Rational Bases

7.2.7: Combining Bases

7.3: Scientific Notation

7.3.1: Describing Large and Small Numbers Using Powers of 10

7.3.2: Representing Large Numbers on the Number Line

7.3.3: Representing Small Numbers on the Number Line

7.3.4: Applications of Arithmetic with Powers of 10

7.3.5: Definition of Scientific Notation

7.3.6: Multiplying, Dividing, and Estimating with Scientific Notation

7.3.7: Adding and Subtracting with Scientific Notation

7.4: Let's Put It to Work

7.4.1: Is a Smartphone Smart Enough to Go to the Moon?

Thumbnail: Expansion of small numbers expressed in scientific notation (CC BY SA 3.0 Unported; Brian Brondel via Wikipedia)

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7.1: Exponent Review

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7.1.1: Exponent Review

Lesson

Let's review exponents.

Exercise 7.1.1.1: Which One Doesn't Belong: Twos

Which expression does not belong? Be prepared to share your reasoning.

2^3	3^2
8	$2^2 \cdot 2^1$

Exercise 7.1.1.2: Return of the Genie

Mai and Andre found an old, brass bottle that contained a magical genie. They freed the genie, and it offered them each a magical $1 coin as thanks.

- The magic coin turned into 2 coins on the first day.
- The 2 coins turned into 4 coins on the second day.
- The 4 coins turned into 8 coins, on the third day.

This doubling pattern continued for 28 days.

Click on the arrow to see the coins start to magically multiply.

Mai was trying to calculate how many coins she would have and remembered that instead of writing $1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot$ for the number of coins on the 6th day, she could write 2^6.

1. The number of coins Mai had on the 28th day is very, very large. Write an expression to represent this number without computing its value.
2. Andre's coins lost their magic on the 25th day, so Mai has a lot more coins than he does. How many times more coins does Mai have than Andre?

Exercise 7.1.1.3: Broken Coin

After a while, Jada picks up a coin that seems different than the others. She notices that the next day, only half of the coin is left!

- On the second day, only $\frac{1}{4}$ of the coin is left.
- On the third day, $\frac{1}{8}$ of the coin remains.

1. What fraction of the coin remains after 6 days?
2. What fraction of the coin remains after 28 days? Write an expression to describe this without computing its value.
3. Does the coin disappear completely? If so, after how many days?

Watch the magical coin changing for ten days with this applet.

Are you ready for more?

Every animal has two parents. Each of its parents also has two parents.

1. Draw a family tree showing an animal, its parents, its grandparents, and its great-grandparents.
2. We say that the animal's eight great-grandparents are "three generations back" from the animal. At which generation back would an animal have 262,144 ancestors?

Summary

Exponents make it easy to show repeated multiplication. For example, $2^6 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot$ One advantage to writing 2^6 is that we can see right away that this is 2 to the *sixth* power. When this is written out using multiplication, $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot$, we need to count the number of factors. Imagine writing out 2^{100} using multiplication!

Let's say you start out with one grain of rice and that each day the number of grains of rice you have doubles. So on day one, you have 2 grains, on day two, you have 4 grains, and so on. When we write 2^{25}, we can see from the expression that the rice has doubled 25 times. So this notation is not only convenient, but it also helps us see structure: in this case, we can see right away that it is on the 25th day that the number of grains of rice has doubled! That's a lot of rice (more than a cubic meter)!

Glossary Entries

Definition: Exponent

In expressions like 5^3 and 8^2, the 3 and the 2 are called exponents. They tell you how many factors to multiply. For example, $5^3 = 5 \cdot 5 \cdot 5 \cdot$, and $8^2 = 8 \cdot 8$.

Practice

Exercise 7.1.1.4

Write each expression using an exponent:

1. $1 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7$
2. $1 \cdot \left(\frac{4}{5}\right) \cdot \left(\frac{4}{5}\right) \cdot \left(\frac{4}{5}\right) \cdot \left(\frac{4}{5}\right) \cdot \left(\frac{4}{5}\right)$
3. $1 \cdot (9.3) \cdot (9.3) \cdot (9.3) \cdot (9.3) \cdot (9.3) \cdot (9.3) \cdot (9.3) \cdot (9.3)$
4. The number of coins Jada will have on the eighth day, if Jada starts with one coin and the number of coins doubles every day. (She has two coins on the first day of the doubling.)

Exercise 7.1.1.5

Evaluate each expression:

1. 2^5
2. 3^3
3. 4^3

1. 6^2
2. $\left(\frac{1}{2}\right)^4$
3. $\left(\frac{1}{3}\right)^2$

Exercise 7.1.1.6

Clare made \$160 babysitting last summer. She put the money in a savings account that pays 3% interest per year. If Clare doesn't touch the money in her account, she can find the amount she'll have the next year by multiplying her current amount by 1.03.

1. How much money will Clare have in her account after 1 year? After 2 years?
2. How much money will Clare have in her account after 5 years? Explain your reasoning.
3. Write an expression for the amount of money Clare would have after 30 years if she never withdraws money from the account.

Exercise 7.1.1.7

The equation $y = 5,280x$ gives the number of feet, y, in x miles. What does the number 5,280 represent in this relationship?

(From Unit 3.1.1)

Exercise 7.1.1.8

The points $(2, 4)$ and $(6, 7)$ lie on a line. What is the slope of the line?

A. 2
B. 1
C. $\frac{4}{3}$
D. $\frac{3}{4}$

(From Unit 3.2.1)

Exercise 7.1.1.9

The diagram shows a pair of similar figures, one contained in the other. Name a point and a scale factor for a dilation that moves the larger figure to the smaller one.

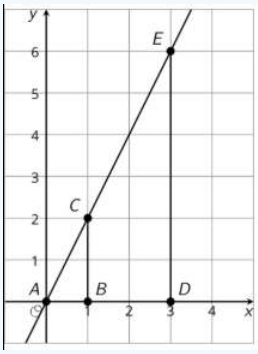


Figure 7.1.1.1

(From Unit 2.2.1)

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7.2: Exponent Rules

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7.2.1: Multiplying Powers of 10

Lesson

Let's explore patterns with exponents when we multiply powers of 10.

Exercise 7.2.1.1: 100, 1, or $\frac{1}{100}$?

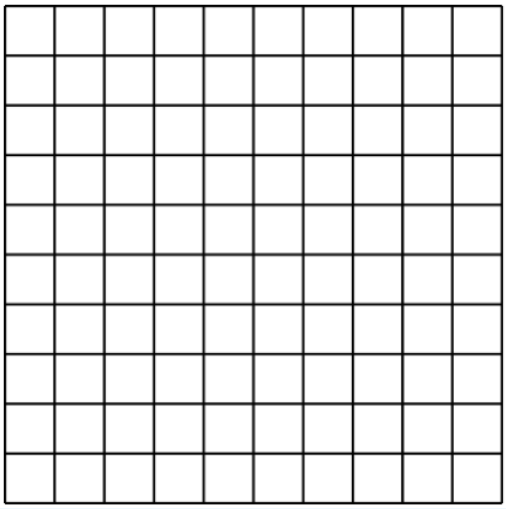


Figure 7.2.1.1

Clare said she sees 100.

Tyler says he sees 1.

Mai says she sees $\frac{1}{100}$.

Who do you agree with?

Exercise 7.2.1.2: Picture a Power of 10

In the diagram, the medium rectangle is made up of 10 small squares. The large square is made up of 10 medium rectangles.

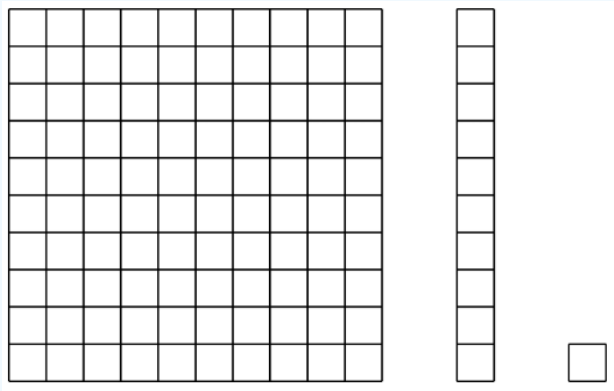


Figure 7.2.1.2

1. How could you represent the large square as a power of 10?
2. If each small square represents 10^2, then what does the medium rectangle represent? The large square?
3. If the medium rectangle represents 10^5, then what does the large square represent? The small square?
4. If the large square represents 10^{100}, then what does the medium rectangle represent? The small square?

Exercise 7.2.1.3: Multiplying Powers of Ten

1. a. Complete the table to explore patterns in the exponents when multiplying powers of 10. You may skip a single box in the table, but if you do, be prepared to explain why you skipped it.

Table 7.2.1.1

expression	expanded	single power of 10
$10^2 \cdot 10^3$	$(10 \cdot 10)(10 \cdot 10 \cdot 10)$	10^5
$10^4 \cdot 10^3$		
$10^4 \cdot 10^4$		
	$10 \cdot 10 \cdot 10)(10 \cdot 10 \cdot 10 \cdot 10 \cdot 10)$	
$10^{18} \cdot 10^{23}$		

   b. If you chose to skip one entry in the table, which entry did you skip? Why?

2. a. Use the patterns you found in the table to rewrite $10^n \cdot 10^m$ as an equivalent expression with a single exponent, like 10^x.
   b. Use your rule to write $10^4 \cdot 10^0$ with a single exponent. What does this tell you about the value of 10^0?
3. The state of Georgia has roughly 10^7 human residents. Each human has roughly 10^{13} bacteria cells in his or her digestive tract. How many bacteria cells are there in the digestive tracts of all the humans in Georgia?

Are you ready for more?

There are four ways to make 10^4 by multiplying powers of 10 with smaller, positive exponents.

$10^1 \cdot 10^1 \cdot 10^1 \cdot 10^1$

$10^1 \cdot 10^1 \cdot 10^2$

$10^1 \cdot 10^3$

$10^2 \cdot 10^2$

(This list is complete if you don't pay attention to the order you write them in. For example, we are only counting $10^1 \cdot 10^3$ and $10^3 \cdot 10^1$ once.)

1. How many ways are there to make 10^6 by multiplying smaller powers of 10 together?
2. How about 10^7? 10^8?

Summary

In this lesson, we developed a rule for multiplying powers of 10: multiplying powers of 10 corresponds to adding the exponents together. To see this, multiply 10^5 and 10^2. We know that 10^5 has five factors that are 10 and 10^2 has two factors that are 10. That means that $10^5 \cdot 10^2$ has 7 factors that are 10. $10^5 \cdot 10^2 = (10 \cdot 10 \cdot 10 \cdot 10 \cdot 10) \cdot (10 \cdot 10) = 10^7$. This will work for other powers of 10 too. So $10^{14} \cdot 10^{47} = 10^{61}$.

This rule makes it easier to understand and work with expressions that have exponents.

Practice

Exercise 7.2.1.4

Write each expression with a single exponent:

1. $10^3 \cdot 10^9$
2. $10 \cdot 10^4$
3. $10^{10} \cdot 10^7$
4. $10^3 \cdot 10^3$
5. $10^5 \cdot 10^{12}$

6. $10^6 \cdot 10^6 \cdot 10^6$

Exercise 7.2.1.5

A large rectangular swimming pool is 1,000 feet long, 100 feet wide, and 10 feet deep. The pool is filled to the top with water.

1. What is the area of the surface of the water in the pool?
2. How much water does the pool hold?
3. Express your answers to the previous two questions as powers of 10.

Exercise 7.2.1.6

Here is triangle ABC. Triangle DEF is similar to triangle ABC, and the length of EF is 5 cm. What are the lengths of sides DE and DF, in centimeters?

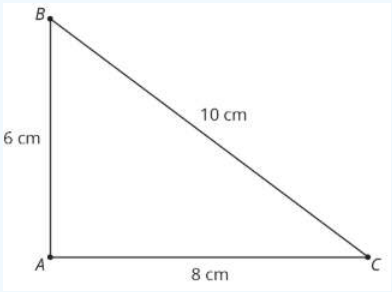


Figure 7.2.1.3

(From Unit 2.2.2)

Exercise 7.2.1.7

Elena and Jada distribute flyers for different advertising companies. Elena gets paid 65 cents for every 10 flyers she distributes, and Jada gets paid 75 cents for every 12 flyers she distributes.

Draw graphs on the coordinate plane representing the total amount each of them earned, y, after distributing x flyers. Use the graph to decide who got paid more after distributing 14 flyers.

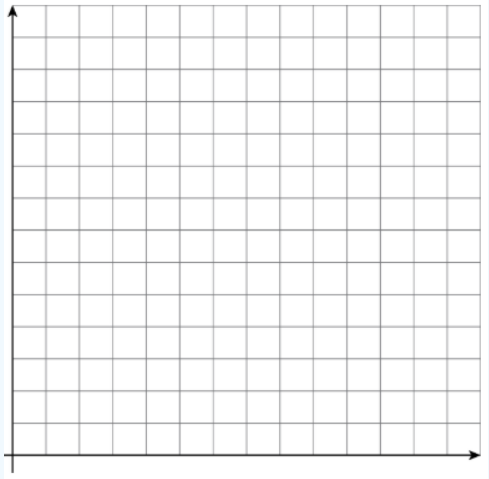


Figure 7.2.1.4

(From Unit 3.1.3)

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7.2.2: Powers of Powers of 10

Lesson

Let's look at powers of powers of 10.

Exercise 7.2.2.1: Big Cube

What is the volume of a giant cube that measures 10,000 km on each side?

Exercise 7.2.2.2: Raising Powers of 10 to Another Power

1. a. Complete the table to explore patterns in the exponents when raising a power of 10 to a power. You may skip a single box in the table, but if you do, be prepared to explain why you skipped it.

Table 7.2.2.1

expression	expanded	single power of 10
$(10^3)^2$	$(10 \cdot 10 \cdot 10)(10 \cdot 10 \cdot 10)$	10^6
$(10^2)^5$	$(10 \cdot 10)(10 \cdot 10)(10 \cdot 10)(10 \cdot 10)(10 \cdot 10)$	
	$(10 \cdot 10 \cdot 10)(10 \cdot 10 \cdot 10)(10 \cdot 10 \cdot 10)(10 \cdot 10 \cdot 10)$	
$(10^4)^2$		
$(10^8)^{11}$		

   b. If you chose to skip one entry in the table, which entry did you skip? Why?

2. Use the patterns you found in the table to rewrite $(10^m)^n$ as an equivalent expression with a single exponent, like 10^x.
3. you took the amount of oil consumed in 2 months in 2013 worldwide, you could make a cube of oil that measures 10^3 meters on each side. How many cubic meters of oil is this? Do you think this would be enough to fill a pond, a lake, or an ocean?

Exercise 7.2.2.3: How Do the Rules Work?

Andre and Elena want to write $10^2 \cdot 10^2 \cdot 10^2$ with a single exponent.

- Andre says, "When you multiply powers with the same **base**, it just means you add the exponents, so $10^2 \cdot 10^2 \cdot 10^2 = 10^{2+2+2} = 10^6$."
- Elena says, "10^2 is multiplied by itself 3 times, so $10^2 \cdot 10^2 \cdot 10^2 = (10^2)^3 = 10^{2+3} = 10^5$."

Do you agree with either of them? Explain your reasoning.

Are you ready for more?

$2^{12} = 4,096$. How many other whole numbers can you raise to a power and get 4,096? Explain or show your reasoning.

Summary

In this lesson, we developed a rule for taking a power of 10 to another power: Taking a power of 10 and raising it to another power is the same as multiplying the exponents. See what happens when raising 10^4 to the power of 3.

$$(10^4)^3 = 10^4 \cdot 10^4 \cdot 10^4 = 10^{12}$$

This works for any power of powers of 10. For example, $(10^6)^{11} = 10^{66}$. This is another rule that will make it easier to work with and make sense of expressions with exponents.

Glossary Entries

Definition: Base (of an exponent)

In expressions like 5^3 and 8^2, the 5 and the 8 are called bases. They tell you what factor to multiply repeatedly. For example, $5^3 = 5 \cdot 5 \cdot 5$, and $8^2 = 8 \cdot 8$.

Practice

Exercise 7.2.2.4

Write each expression with a single exponent:

1. $(10^7)^2$
2. $(10^9)^3$
3. $(10^6)^3$
4. $(10^2)^3$
5. $(10^3)^2$
6. $(10^5)^7$

Exercise 7.2.2.5

You have 1,000,000 number cubes, each measuring one inch on a side.

1. If you stacked the cubes on top of one another to make an enormous tower, how high would they reach? Explain your reasoning.
2. If you arranged the cubes on the floor to make a square, would the square fit in your classroom? What would its dimensions be? Explain your reasoning.
3. If you layered the cubes to make one big cube, what would be the dimensions of the big cube? Explain your reasoning.

Exercise 7.2.2.6

An amoeba divides to form two amoebas after one hour. One hour later, each of the two amoebas divides to form two more. Every hour, each amoeba divides to form two more.

1. How many amoebas are there after 1 hour?
2. How many amoebas are there after 2 hours?
3. Write an expression for the number of amoebas after 6 hours.
4. Write an expression for the number of amoebas after 24 hours.
5. Why might exponential notation be preferable to answer these questions?

(From Unit 7.1.1)

Exercise 7.2.2.7

Elena noticed that, nine years ago, her cousin Katie was twice as old as Elena was then. Then Elena said, “In four years, I’ll be as old as Katie is now!” If Elena is currently e years old and Katie is k years old, which system of equations matches the story?

A. $\begin{cases} k-9=2e \\ e+4=k \end{cases}$

B. $\begin{cases} 2k=e-9 \\ e=k+4 \end{cases}$

C. $\begin{cases} k=2e-9 \\ e+4=k+4 \end{cases}$

D. $\begin{cases} k-9=2(e-9) \\ e+4=k \end{cases}$

(From Unit 4.3.6)

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7.2.3: Dividing Powers of 10

Lesson

Let's explore patterns with exponents when we divide powers of 10.

Exercise 7.2.3.1: A Surprising One

What is the value of the expression?

$$\frac{2^5 \cdot 3^4 \cdot 3^2}{2 \cdot 3^6 \cdot 2^4}$$

Exercise 7.2.3.2: Dividing Powers of Ten

1. a. Complete the table to explore patterns in the exponents when dividing powers of 10. Use the "expanded" column to show why the given expression is equal to the single power of 10. You may skip a single box in the table, but if you do, be prepared to explain why you skipped it.

Table 7.2.3.1

expression	expanded	single power
$10^4 \div 10^2$	$\frac{10 \cdot 10 \cdot 10 \cdot 10}{10 \cdot 10} = \frac{10 \cdot 10}{10 \cdot 10} \cdot 10 \cdot 10 = 1 \cdot 10 \cdot 10$	10^2
	$\frac{10 \cdot 10 \cdot 10 \cdot 10 \cdot 10}{10 \cdot 10} = \frac{10 \cdot 10}{10 \cdot 10} \cdot 10 \cdot 10 \cdot 10 = 1 \cdot 10 \cdot 10 \cdot 10$	
$10^6 \div 10^3$		
$10^{43} \div 10^{17}$		

   b. If you chose to skip one entry in the table, which entry did you skip? Why?

2. se the patterns you found in the table to rewrite $\frac{10^n}{10^m}$ as an equivalent expression of the form 10^x.
3. It is predicted that by 2050, there will be 10^{10} people living on Earth. At that time, it is predicted there will be approximately 10^{12} trees. How many trees will there be for each person?

Are you ready for more?

Table 7.2.3.2

expression	expanded	single power
$10^4 \div 10^6$		

Exercise 7.2.3.3: Zero Exponent

So far we have looked at powers of 10 with exponents greater than 0. What would happen to our patterns if we included 0 as a possible exponent?

1. a. Write $10^{12} \cdot 10^0$ with a power of 10 with a single exponent using the appropriate exponent rule. Explain or show your reasoning.
   b. What number could you multiply 10^{12} by to get this same answer?
2. a. Write $\frac{10^8}{10^0}$ with a single power of 10 using the appropriate exponent rule. Explain or show your reasoning.
   b. What number could you divide 10^8 by to get this same answer?
3. If we want the exponent rules we found to work even when the exponent is 0, then what does the value of 10^0 have to be?
4. Noah says, “If I try to write 10^0 expanded, it should have zero factors that are 10, so it must be equal to 0.” Do you agree? Discuss with your partner.

Exercise 7.2.3.4: Making Millions

Write as many expressions as you can that have the same value as 10^6. Focus on using exponents, multiplication, and division. What patterns do you notice with the exponents?

Summary

In an earlier lesson, we learned that when multiplying powers of 10, the exponents add together. For example, $10^6 \cdot 10^3 = 10^9$ because 6 factors that are 10 multiplied by 3 factors that are 10 makes 9 factors that are 10 all together. We can also think of this multiplication equation as division: $10^6 = \frac{10^9}{10^3}$ So when dividing powers of 10, the exponent in the denominator is subtracted from the exponent in the numerator. This makes sense because $\frac{10^9}{10^3} = \frac{10^3 \cdot 10^6}{10^3} = \frac{10^3}{10^3} \cdot 10^6 = 1 \cdot 10^6 = 10^6$ This rule works for other powers of 10 too. For example, $\frac{10^{56}}{10^{23}} = 10^{33}$ because 23 factors that are 10 in the numerator and in the denominator are used to make 1, leaving 33 factors remaining.

This gives us a new exponent rule: $\frac{10^n}{10^m} = 10^{n-m}$ So far, this only makes sense when n and m are positive exponents and $n > m$, but we can extend this rule to include a new power of 10, 10^0. If we look at $\frac{10^6}{10^0}$, using the exponent rule gives 10^{6-0}, which is equal to 10^6. So dividing 10^6 by 10^0 doesn't change its value. That means that if we want the rule to work when the exponent is 0, then it must be that $10^0 = 1$.

Glossary Entries

Definition: Base (of an exponent)

In expressions like 5^3 and 8^2, the 5 and the 8 are called bases. They tell you what factor to multiply repeatedly. For example, $5^3 = 5 \cdot 5 \cdot 5$, and $8^2 = 8 \cdot 8$.

Practice

Exercise 7.2.3.5

Evaluate:

1. 10^0
2. $\frac{10^3}{10^3}$
3. $10^2 + 10^1 + 10^0$

Exercise 7.2.3.6

Write each expression as a single power of 10.

1. $\frac{10^3 \cdot 10^4}{10^5}$
2. $(10^4) \cdot \frac{10^{12}}{10^7}$
3. $\left(\frac{10^5}{10^3}\right)^4$
4. $\frac{10^4 \cdot 10^5 \cdot 10^6}{10^3 \cdot 10^7}$
5. $\frac{(10^5)^2}{(10^2)^3}$

Exercise 7.2.3.7

The Sun is roughly 10^2 times as wide as Earth. The star KW Sagittarii is roughly 10^5 times as wide as Earth. About how many times as wide as the Sun is KW Sagittarii? Explain how you know.

Exercise 7.2.3.8

Bananas cost $1.50 per pound, and guavas cost $3.00 per pound. Kiran spends $12 on fruit for a breakfast his family is hosting. Let b be the number of pounds of bananas Kiran buys and g be the number of pounds of guavas he buys.

1. Write an equation relating the two variables.
2. Rearrange the equation so b is the independent variable.
3. Rearrange the equation so g is the independent variable.

(From Unit 5.2.1)

Exercise 7.2.3.9

Lin's mom bikes at a constant speed of 12 miles per hour. Lin walks at a constant speed $\frac{1}{3}$ of the speed her mom bikes. Sketch a graph of both of these relationships.

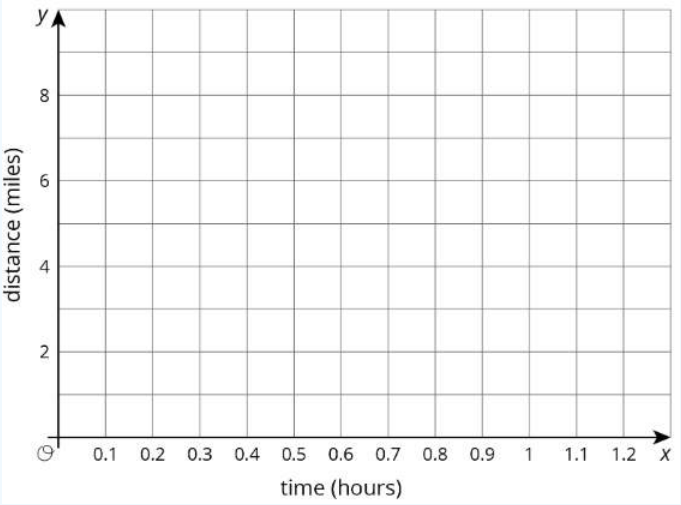


Figure 7.2.3.1

(From Unit 3.1.1)

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7.2.4: Negative Exponents with Powers of 10

Lesson

Let's see what happens when exponents are negative.

Exercise 7.2.4.1: Number Talk: What's That Exponent?

Solve each equation mentally.

$\frac{100}{1} = 10^x$

$\frac{100}{x} = 10^1$

$\frac{x}{100} = 10^0$

$\frac{10}{1000} = 10^x$

Exercise 7.2.4.2: Negative Exponent Table

Complete the table to explore what negative exponents mean.

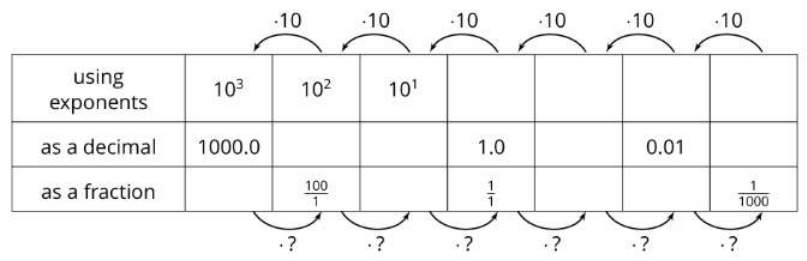


Figure 7.2.4.1: A table with three rows. Top row, using exponents, 10 cubed, 10 squared, 10 to the first power, blank, blank, blank, blank. Second row, as a decimal, 1000 point 0, blank, blank, 1 point 0, blank, 0 point 0 1, blank. Third row, as a fraction, blank, fraction 100 over 1, blank, fraction 1 over 1, blank, blank, fraction 1 over one thousand. Above the row, arrows point from column to column, right to left, each labeled times 10. Below the bottom row, arrows point from column to column, left to right, each labeled time question mark.

1. As you move toward the left, each number is being multiplied by 10. What is the multiplier as you move right?
2. How does a multiplier of 10 affect the placement of the decimal in the product? How does the other multiplier affect the placement of the decimal in the product?
3. Use the patterns you found in the table to write 10^{-7} as a fraction.
4. Use the patterns you found in the table to write 10^{-5} as a decimal.
5. Write $\frac{1}{100{,}000{,}000}$ using a single exponent.
6. Use the patterns in the table to write 10^{-n} as a fraction.

Exercise 7.2.4.3: Follow the Exponential Rules

1. a. Match each exponential expression with an equivalent multiplication expression:

$(10^2)^3$	$\frac{1}{(10 \cdot 10} \cdot \frac{1}{(10 \cdot 10} \cdot \frac{1}{(10 \cdot 10}$
$(10^2)^{-3}$	$\left(\frac{1}{10} \cdot \frac{1}{10}\right)\left(\frac{1}{10} \cdot \frac{1}{10}\right)\left(\frac{1}{10} \cdot \frac{1}{10}\right)$
$(10^{-2})^3$	$\frac{1}{\frac{1}{10} \cdot \frac{1}{10}} \cdot \frac{1}{\frac{1}{10} \cdot \frac{1}{10}} \cdot \frac{1}{\frac{1}{10} \cdot \frac{1}{10}}$
$(10^{-2})^{-3}$	$(10 \cdot 10)(10 \cdot 10)(10 \cdot 10)$

   b. Write $(10^2)^{-3}$ as a power of 10 with a single exponent. Be prepared to explain your reasoning.

2. a. Match each exponential expression with an equivalent multiplication expression:

$\frac{10^2}{10^5}$	$\frac{\frac{1}{10}\cdot\frac{1}{10}}{\frac{1}{10}\cdot\frac{1}{10}\cdot\frac{1}{10}\cdot\frac{1}{10}\cdot\frac{1}{10}}$
$\frac{10^2}{10^{-5}}$	$\frac{10\cdot10}{10\cdot10\cdot10\cdot10\cdot10}$
$\frac{10^{-2}}{10^5}$	$\frac{\frac{1}{10}\cdot\frac{1}{10}}{10\cdot10\cdot10\cdot10\cdot10}$
$\frac{10^{-2}}{10^{-5}}$	$\frac{10\cdot10}{\frac{1}{10}\cdot\frac{1}{10}\cdot\frac{1}{10}\cdot\frac{1}{10}\cdot\frac{1}{10}}$

b. Write $\frac{10^{-2}}{10^{-5}}$ as a power of 10 with a single exponent. Be prepared to explain your reasoning.

3. a. Match each exponential expression with an equivalent multiplication expression:

b.

$10^4 \cdot 10^3$	$(10\cdot10\cdot10\cdot10)\cdot\left(\frac{1}{10}\cdot\frac{1}{10}\cdot\frac{1}{10}\right)$
$10^4 \cdot 10^{-3}$	$\left(\frac{1}{10}\cdot\frac{1}{10}\cdot\frac{1}{10}\cdot\frac{1}{10}\right)\cdot\left(\frac{1}{10}\cdot\frac{1}{10}\cdot\frac{1}{10}\right)$
$10^{-4} \cdot 10^3$	$\left(\frac{1}{10}\cdot\frac{1}{10}\cdot\frac{1}{10}\frac{1}{10}\right)\cdot(10\cdot10\cdot10)$
$10^{-4} \cdot 10^{-3}$	$(10\cdot10\cdot10\cdot10)\cdot(10\cdot10\cdot10)$

c. Write $10^{-4} \cdot 10^3$ as a power of 10 with a single exponent. Be prepared to explain your reasoning.

Are you ready for more?

Priya, Jada, Han, and Diego stand in a circle and take turns playing a game.

Priya says, SAFE. Jada, standing to Priya's left, says, OUT and leaves the circle. Han is next: he says, SAFE. Then Diego says, OUT and leaves the circle. At this point, only Priya and Han are left. They continue to alternate. Priya says, SAFE. Han says, OUT and leaves the circle. Priya is the only person left, so she is the winner.

Priya says, "I knew I'd be the only one left, since I went first."

1. Record this game on paper a few times with different numbers of players. Does the person who starts always win?
2. Try to find as many numbers as you can where the person who starts always wins. What patterns do you notice?

Summary

When we multiply a positive power of 10 by $\frac{1}{10}$, the exponent *decreases* by 1:$10^8 \cdot \frac{1}{10} = 10^7$ This is true for *any* positive power of 10. We can reason in a similar way that multiplying by 2 factors that are $\frac{1}{10}$ *decreases* the exponent by 2: $\left(\frac{1}{10}\right)^2 \cdot 10^8 = 10^6$

That means we can extend the rules to use negative exponents if we make $10^{-2} = \left(\frac{1}{10}\right)^2$. Just as 10^2 is two factors that are 10, we have that 10^{-2} is two factors that are $\frac{1}{10}$. More generally, the exponent rules we have developed are true for *any* integers n and m if we make $10^{-n} = \left(\frac{1}{10}\right)^n = \frac{1}{10^n}$

Here is an example of extending the rule $\frac{10^n}{10^m} = 10^{n-m}$ to use negative exponents:$\frac{10^3}{10^5} = 10^{3-5} = 10^{-2}$ To see why, notice that $\frac{10^3}{10^5} = \frac{10^3}{10^3\cdot10^2} = \frac{10^3}{10^3}\cdot\frac{1}{10^2} = \frac{1}{10^2}$ which is equal to 10^{-2}.

Here is an example of extending the rule $(10^m)^n = 10^{m\cdot n}$ to use negative exponents:$(10^{-2})^3 = 10^{(-2)(3)} = 10^{-6}$ To see why, notice that $10^{-2} = \frac{1}{10}\cdot\frac{1}{10}$. This means that $(10^{-2})^3 = \left(\frac{1}{10}\cdot\frac{1}{10}\right)^3 = \left(\frac{1}{10}\cdot\frac{1}{10}\right)\cdot\left(\frac{1}{10}\cdot\frac{1}{10}\right)\cdot\left(\frac{1}{10}\cdot\frac{1}{10}\right) = \frac{1}{10^6} = 10^{-6}$

Glossary Entries

Definition: Base (of an exponent)

In expressions like 5^3 and 8^2, the 5 and the 8 are called bases. They tell you what factor to multiply repeatedly. For example, $5^3 = 5\cdot5\cdot5$, and $8^2 = 8\cdot8$.

Practice

Exercise 7.2.4.4

Write with a single exponent: (ex: $\frac{1}{10} \cdot \frac{1}{10} = 10^{-2}$)

1. $\frac{1}{10} \cdot \frac{1}{10} \cdot \frac{1}{10}$
2. $\frac{1}{10} \cdot \frac{1}{10} \cdot \frac{1}{10} \cdot \frac{1}{10} \cdot \frac{1}{10} \cdot \frac{1}{10} \cdot \frac{1}{10}$
3. $\left(\frac{1}{10} \cdot \frac{1}{10} \cdot \frac{1}{10} \cdot \frac{1}{10}\right)^{2}$
4. $\left(\frac{1}{10} \cdot \frac{1}{10} \cdot \frac{1}{10}\right)^{3}$
5. $(10 \cdot 10 \cdot 10)^{-2}$

Exercise 7.2.4.5

Write each expression as a single power of 10.

1. $10^{-3} \cdot 10^{-2}$
2. $10^{4} \cdot 10^{-1}$
3. $\frac{10^5}{10^7}$
4. $\left(10^{-4}\right)^{5}$
5. $10^{-3} \cdot 10^{2}$
6. $\frac{10^{-9}}{10^5}$

Exercise 7.2.4.6

Select **all** of the following that are equivalent to $\frac{1}{10,000}$:

A. $(10,000)^{-1}$
B. $(-10,000)$
C. $(100)^{-2}$
D. $(10)^{-4}$
E. $(-10)^{2}$

Exercise 7.2.4.7

Match each equation to the situation it describes. Explain what the constant of proportionality means in each equation.

Equations:

1. $y = 3x$
2. $\frac{1}{2}x = y$
3. $y = 3.5x$
4. $y = \frac{5}{2}x$

Situations:

- A dump truck is hauling loads of dirt to a construction site. After 20 loads, there are 70 square feet of dirt.
- I am making a water and salt mixture that has 2 cups of salt for every 6 cups of water.
- A store has a “4 for $10” sale on hats.
- For every 48 cookies I bake, my students get 24.

(From Unit 3.1.2)

Exercise 7.2.4.8

1. Explain why triangle ABC is similar to EDC.

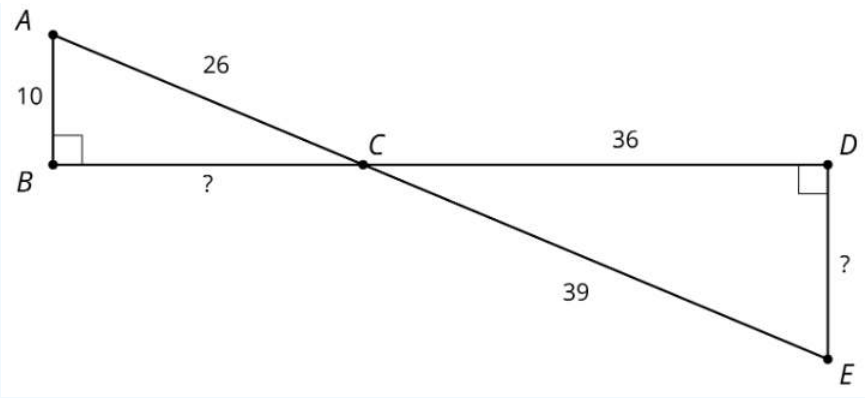


Figure 7.2.4.2

2. Find the missing side lengths.

(From Unit 2.2.3)

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7.2.5: What about Other Bases?

Lesson

Let's explore exponent patterns with bases other than 10.

Exercise 7.2.5.1: True or False: Comparing Expressions with Exponents

Is each statement true or false? Be prepared to explain your reasoning.

1. $3^5 < 4^6$
2. $(-3)^2 < 3^2$
3. $(-3)^3 = 3^3$
4. $(-5)^2 > -5^2$

Exercise 7.2.5.2: What Happens with Zero and Negative Exponents?

Complete the table to show what it means to have an exponent of zero or a negative exponent.

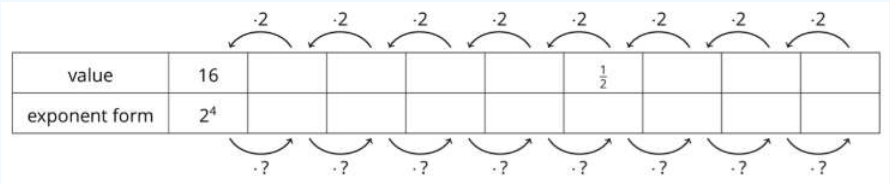


Figure 7.2.5.1: A table with two rows. First row, value, 16, 4 blanks, fraction 1 over 2, three blanks. Second row, exponent form, 2 to the fourth power, 8 blanks. Above the top row, arrows point column to column right to left, labeled time 2. Below the bottom row, arrows point column to column, left to right, labeled times question mark.

1. As you move toward the left, each number is being multiplied by 2. What is the multiplier as you move toward the right?
2. Use the patterns you found in the table to write 2^{-6} as a fraction.
3. Write $\frac{1}{32}$ as a power of 2 with a single exponent.
4. What is the value of 2^0?
5. From the work you have done with negative exponents, how would you write 5^{-3} as a fraction?
6. How would you write 3^{-4} as a fraction?

Are you ready for more?

1. Find an expression equivalent to $\left(\frac{2}{3}\right)^{-3}$ but with positive exponents.
2. Find an expression equivalent to $\left(\frac{4}{5}\right)^{-8}$ but with positive exponents.
3. What patterns do you notice when you start with a fraction raised to a negative exponent and rewrite it using a single positive exponent? Show or explain your reasoning.

Exercise 7.2.5.3: Exponent Rules with Bases Other than 10

Lin, Noah, Diego, and Elena decide to test each other's knowledge of exponents with bases other than 10. They each chose an expression to start with and then came up with a new list of expressions; some of which are equivalent to the original and some of which are not.

Choose 2 of the 4 lists to analyze. For each list of expressions you choose to analyze, decide which expressions are *not* equivalent to the original. Be prepared to explain your reasoning.

1. Lin's original expression is 5^{-9} and her list is:

$$(5^3)^{-3} \qquad -5^9 \qquad \frac{5^{-6}}{5^3} \qquad (5^3)^{-2} \qquad \frac{5^{-4}}{5^{-5}} \qquad 5^{-4} \cdot 5^{-5}$$

2. Noah's original expression is 3^{10} and his list is:

$3^5 \cdot 3^2$ $(3^5)^2$ $(3 \cdot 3)(3 \cdot 3)(3 \cdot 3)(3 \cdot 3)$ $\left(\frac{1}{3}\right)^{-10}$ $3^7 \cdot 3^3$ $\frac{3^{20}}{3^{10}}$ $\frac{3^{20}}{3^2}$

3. Diego's original expression is x^4 and his list is:

$\frac{x^8}{x^4}$ $x \cdot x \cdot x \cdot x$ $\frac{x^{-4}}{x^{-8}}$ $\frac{x^{-4}}{x^8}$ $(x^2)^2$ $4 \cdot x$ $x \cdot x^3$

4. Elena's original expression is 8^0 and her list is:

1 0 $8^3 \cdot 8^{-3}$ $\frac{8^2}{8^2}$ 10^0 11^0

Summary

Earlier we focused on powers of 10 because 10 plays a special role in the decimal number system. But the exponent rules that we developed for 10 also work for other bases. For example, if $2^0 = 1$ and $2^{-n} = \frac{1}{2^n}$, then

$$\begin{aligned} 2^m \cdot 2^n &= 2^{m+n} \\ (2^m)^n &= 2^{m \cdot n} \\ \frac{2^m}{2^n} &= 2^{m-n} \end{aligned}$$

These rules also work for powers of numbers less than 1. For example, $\left(\frac{1}{3}\right)^2 = \frac{1}{3} \cdot \frac{1}{3}$ and $\left(\frac{1}{3}\right)^4 = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}$. We can also check that $\left(\frac{1}{3}\right)^2 \cdot \left(\frac{1}{3}\right)^4 = \left(\frac{1}{3}\right)^{2+4}$.

Using a variable x helps to see this structure. Since $x^2 \cdot x^5 = x^7$ (both sides have 7 factors that are x), if we let $x = 4$, we can see that $4^2 \cdot 4^5 = 4^7$. Similarly, we could let $x = \frac{2}{3}$ or $x = 11$ or any other positive value and show that these relationships still hold.

Glossary Entries

Definition: Base (of an exponent)

In expressions like 5^3 and 8^2, the 5 and the 8 are called bases. They tell you what factor to multiply repeatedly. For example, $5^3 = 5 \cdot 5 \cdot 5$, and $8^2 = 8 \cdot 8$.

Practice

Exercise 7.2.5.4

Priya says "I can figure out 5^0 by looking at other powers of 5. 5^3 is 125, 5^2 is 25, then 5^1 is 5."

1. What pattern do you notice?
2. If this pattern continues, what should be the value of 5^0? Explain how you know.
3. If this pattern continues, what should be the value of 5^{-1}? Explain how you know.

Exercise 7.2.5.5

Select **all** the expressions that are equivalent to 4^{-3}.

A. -12
B. 2^{-6}
C. $\frac{1}{4^3}$
D. $\left(\frac{1}{4}\right) \cdot \left(\frac{1}{4}\right) \cdot \left(\frac{1}{4}\right)$
E. 12
F. $(-4) \cdot (-4) \cdot (-4)$
G. $\frac{8^{-1}}{2^2}$

Exercise 7.2.5.6

Write each expression using a single exponent.

1. $\frac{5^3}{5^6}$
2. $(14^3)^6$
3. $8^3 \cdot 8^6$
4. $\frac{16^6}{16^3}$
5. $(21^3)^{-6}$

Exercise 7.2.5.7

Andre sets up a rain gauge to measure rainfall in his back yard. On Tuesday, it rains off and on all day.

- He starts at 10 a.m. with an empty gauge when it starts to rain.
- Two hours later, he checks, and the gauge has 2 cm of water in it.
- It starts raining even harder, and at 4 p.m., the rain stops, so Andre checks the rain gauge and finds it has 10 cm of water in it.
- While checking it, he accidentally knocks the rain gauge over and spills most of the water, leaving only 3 cm of water in the rain gauge.
- When he checks for the last time at 5 p.m., there is no change.

Graph A

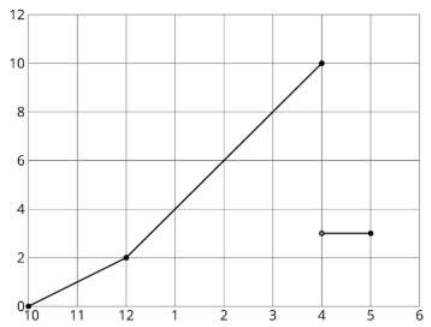


Figure 7.2.5.2: Line graph, horizontal axis labeled 10, 11, 12, 1, 2, 3, 4, 5, 6. Vertical, 0 to 12 by 2. Segments, 10 comma 0 to 12 comma 2, then to 4 comma 10, closed circles. An open circle at 4 comma 3 connects to a closed circle at 5 comma 3.

Graph B

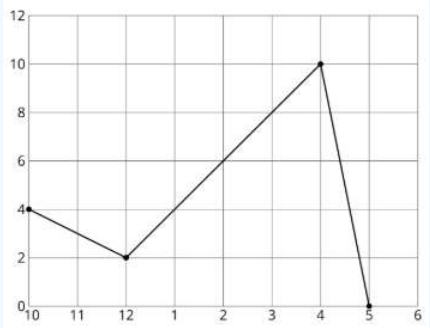


Figure 7.2.5.3

1. Which of the two graphs could represent Andre's story? Explain your reasoning.
2. Label the axes of the correct graph with appropriate units.
3. Use the graph to determine how much total rain fell on Tuesday.

(From Unit 5.2.4)

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7.2.6: Practice with Rational Bases

Lesson

Let's practice with exponents.

Exercise 7.2.6.1: Which One Doesn't Belong: Exponents

Which expression doesnt't belong?

$\frac{2^8}{2^5}$ $\left(\frac{3}{4}\right)^{-5} \cdot \left(\frac{3}{4}\right)^8$

$(4^{-5})^8$ $\frac{10^8}{10^5}$

Exercise 7.2.6.2: Exponent Rule Practie

1. Choose 6 of the equations to write using a single exponent:
   - $7^5 \cdot 7^6$
   - $3^{-3} \cdot 3^8$
   - $2^{-4} \cdot 2^{-3}$
   - $\left(\frac{5}{6}\right)^4 \left(\frac{5}{6}\right)^5$
   - $\frac{3^5}{3^{28}}$
   - $\frac{2^{-5}}{2^4}$
   - $\frac{6^5}{6^{-8}}$
   - $\frac{10^{-12}}{10^{-20}}$
   - $(7^2)^3$
   - $(4^3)^{-3}$
   - $(2^{-8})^{-4}$
   - $(6^{-3})^5$
2. Which problems did you want to skip in the previous question? Explain your thinking.
3. Choose 3 of the following to write using a single, positive exponent:
   - 2^{-7}
   - 3^{-23}
   - $(11^{-8}$
   - 4^{-9}
   - 2^{-32}
   - 8^{-3}
4. Choose 3 of the following to evaluate:
   - $\frac{10^5}{10^5}$
   - $\left(\frac{2}{3}\right)^3$
   - $2^8 \cdot 2^{-8}$
   - $\left(\frac{5}{4}\right)^2$
   - $(3^4)^0$
   - $\left(\frac{7}{2}\right)^2$

Exercise 7.2.6.3: Inconsistent Bases

Mark each equation as true or false. What could you change about the false equations to make them true?

1. $\left(\frac{1}{3}\right)^2 \cdot \left(\frac{1}{3}\right)^4 = \left(\frac{1}{3}\right)^6$

2. $3^2 \cdot 5^3 = 15^5$
3. $5^4 + 5^5 = 5^9$
4. $\left(\frac{1}{2}\right)^4 \cdot 10^3 = 5^7$
5. $3^2 \cdot 5^2 = 15^2$

Are you ready for more?

Solve this equation: $3^{x-5} = 9^{x+4}$. Explain or show your reasoning.

Summary

In the past few lessons, we found rules to more easily keep track of repeated factors when using exponents. We also extended these rules to make sense of negative exponents as repeated factors of the **reciprocal** of the base, as well as defining a number to the power of 0 to have a value of 1. These rules can be written symbolically as:

$$x^n \cdot x^m = x^{n+m}, \qquad (x^n)^m = x^{n \cdot m}, \qquad \frac{x^n}{x^m} = x^{n-m}, \qquad x^{-n} = \frac{1}{x^n}, \quad \text{and} \quad x^0 = 1$$

where the base x can be any positive number. In this lesson, we practiced using these exponent rules for different bases and exponents.

Glossary Entries

Definition: Base (of an exponent)

In expressions like 5^3 and 8^2, the 5 and 8 are called bases. They tell you what factor to multiply repeatedly. For example, $5^3 = 5 \cdot 5 \cdot 5$, and $8^2 = 8 \cdot 8$.

Definition: Reciprocal

Dividing 1 by a number gives the reciprocal of that number. For example, the reciprocal of 12 is $\frac{1}{12}$, and the reciprocal of $\frac{2}{5}$ is $\frac{5}{2}$.

Practice

Exercise 7.2.6.4

Write with a single exponent:

1. $\frac{7^6}{7^2}$
2. $(11^4)^5$
3. $4^2 \cdot 4^6$
4. $6 \cdot 6^8$
5. $(12^2)^7$
6. $\frac{3^{10}}{3}$
7. $(0.173)^9 \cdot (0.173)^2$
8. $\frac{0.87^5}{0.87^3}$
9. $\frac{\left(\frac{5}{2}\right)^8}{\left(\frac{5}{2}\right)^6}$

Exercise 7.2.6.5

Noah says that $2^4 \cdot 3^2 = 6^6$. Tyler says that $2^4 \cdot 4^2 = 16^2$.

1. Do you agree with Noah? Explain or show your reasoning.
2. Do you agree with Tyler? Explain or show your reasoning.

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7.2.7: Combining Bases

Lesson

Let's multiply expressions with different bases.

Exercise 7.2.7.1: Same Exponent, Different Base

1. Evaluate $5^3 \cdot 2^3$
2. Evaluate 10^3

Exercise 7.2.7.2: Power of Products

1. The table contains products of expressions with different bases and the same exponent. Complete the table to see how we can rewrite them. Use the "expanded" column to work out how to combine the factors into a new base.

Table 7.2.7.1

expression	expanded	exponent
$5^3 \cdot 2^3$	$(5 \cdot 5 \cdot 5) \cdot (2 \cdot 2 \cdot 2) = (5 \cdot 2)(5 \cdot 2)(5 \cdot 2) = 10 \cdot 10 \cdot 10$	10^3
$3^2 \cdot 7^2$		21^2
$2^4 \cdot 3^4$		
		15^3
		30^4
$2^4 \cdot x^4$		
$a^n \cdot b^n$		
$7^4 \cdot 2^4 \cdot 5^4$		

2. Can you write $2^3 \cdot 3^4$ with a single exponent? What happens if neither the exponents nor the bases are the same? Explain or show your reasoning.

Exercise 7.2.7.3: How Many Ways Can You Make 3,600?

Your teacher will give your group tools for creating a visual display to play a game. Divide the display into 3 columns, with these headers:

$$a^n \cdot a^m = a^{n+m} \qquad \frac{a^n}{a^m} = a^{n-m} \qquad a^n \cdot b^n = (a \cdot b)^n$$

How to play:

When the time starts, you and your group will write as many expressions as you can that equal a specific number using one of the exponent rules on your board. When the time is up, compare your expressions with another group to see how many points you earn.

- Your group gets 1 point for every *unique* expression you write that is equal to the number and follows the exponent rule you claimed.
- If an expression uses negative exponents, you get 2 points instead of just 1.
- You can challenge the other group's expression if you think it is not equal to the number or if it does not follow one of the three exponent rules.

Are you ready for more?

You have probably noticed that when you square an odd number, you get another odd number, and when you square an even number, you get another even number. Here is a way to expand the concept of odd and even for the number 3. Every integer is either divisible by 3, one *more* than a multiple of 3, or one *less* than a multiple of 3.

1. Examples of numbers that are one more than a multiple of 3 are 4, 7, and 25. Give three more examples.
2. Examples of numbers that are one less than a multiple of 3 are 2, 5, and 32. Give three more examples.
3. Do you think it's true that when you square a number that is a multiple of 3, your answer will still be a multiple of 3? How about for the other two categories? Try squaring some numbers to check your guesses.

Summary

Before this lesson, we made rules for multiplying and dividing expressions with exponents that only work when the expressions have the *same* base. For example, $10^3 \cdot 10^2 = 10^5$ or $2^6 \div 2^2 = 2^4$

In this lesson, we studied how to combine expressions with the same exponent, but *different* bases. For example, we can write $2^3 \cdot 5^3$ as $2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5$. Regrouping this as $(2 \cdot 5) \cdot (2 \cdot 5) \cdot (2 \cdot 5)$ shows that

$$\begin{aligned} 2^3 \cdot 5^3 &= (2 \cdot 5)^3 \\ &= 10^3 \end{aligned}$$

Notice that the 2 and 5 in the previous example could be replaced with different numbers or even variables. For example, if a and b are variables then $a^3 \cdot b^3 = (a \cdot b)^3$. More generally, for a positive number n, $a^n \cdot b^n = (a \cdot b)^n$ because both sides have exactly n factors that are a and n factors that are b.

Glossary Entries

Definition: Base (of an exponent)

In expressions like 5^3 and 8^2, the 5 and 8 are called bases. They tell you what factor to multiply repeatedly. For example, $5^3 = 5 \cdot 5 \cdot 5$, and $8^2 = 8 \cdot 8$.

Definition: Reciprocal

Dividing 1 by a number gives the reciprocal of that number. For example, the reciprocal of 12 is $\frac{1}{12}$, and the reciprocal of $\frac{2}{5}$ is $\frac{5}{2}$.

Practice

Exercise 7.2.7.4

Select **all** the true statements:

A. $2^8 \cdot 2^9 = 2^{17}$
B. $8^2 \cdot 9^2 = 72^2$
C. $8^2 \cdot 9^2 = 72^4$
D. $2^8 \cdot 2^9 = 4^{17}$

Exercise 7.2.7.5

Find x, y, and x if $(3 \cdot 5)^4 \cdot (2 \cdot 3)^5 \cdot (2 \cdot 5)^7 = 2^x \cdot 3^y \cdot 5^z$.

Exercise 7.2.7.6

Han found a way to compute complicated expressions more easily. Since $2 \cdot 5 = 10$, he looks for pairings of 2s and 5s that he knows equal 10. For example, $3 \cdot 2^4 \cdot 5^5 = 3 \cdot 2^4 \cdot 5^4 \cdot 5 = (3 \cdot 5) \cdot (2 \cdot 5)^4 = 15 \cdot 10^4 = 150,000$ Use Han's technique to compute the following:

1. $2^4 \cdot 5 \cdot (3 \cdot 5)^3$

2. $\frac{2^3 \cdot 5^2 \cdot (2 \cdot 3)^2 \cdot (3 \cdot 5)^2}{3^2}$

Exercise 7.2.7.7

The cost of cheese at three stores is a function of the weight of the cheese. The cheese is not prepackaged, so a customer can buy any amount of cheese.

- Store A sells the cheese for a dollars per pound.
- Store B sells the same cheese for b dollars per pound and a customer has a coupon for $5 off the total purchase at that store.
- Store C is an online store, selling the same cheese at c dollar per pound, but with a $10 delivery fee.

This graph shows the price functions for stores A, B, and C.

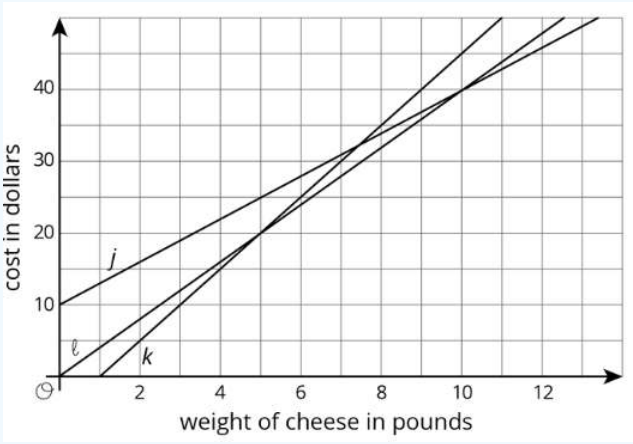


Figure 7.2.7.1: Coordinate plane, horizontal, weight of cheese in pounds, 0 to 12 by 2, vertical, cost in dollars, 0 to 40 by 10. Line j, from 0 comma 10 to 13 point 5 to 50. Line l, origin to 12 point 5. Line k from 1 comma 0 to 11 comma 60. Lines j and l meet at 10 comma 40, lines k and l meet at 5 comma 20, lines j and k meet at 7 comma 35.

1. Match Stores A, B, and C with Graphs j, k, and l.
2. How much does each store charge for the cheese per pound?
3. How many pounds of cheese does the coupon for Store B pay for?
4. Which store has the lowest price for a half a pound of cheese?
5. If a customer wants to buy 5 pounds of cheese for a party, which store has the lowest price?
6. How many pounds would a customer need to order to make Store C a good option?

(From Unit 5.3.1)

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7.3: Scientific Notation

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7.3.1: Describing Large and Small Numbers Using Powers of 10

Lesson

Let's find out how to use powers of 10 to write large or small numbers.

Exercise 7.3.1.1: Thousand Million Billion Trillion

1. Match each expression with its corresponding value and word.

expression	value	word
10^{-3}	1,000,000,000,000	billion
10^{6}	$\frac{1}{100}$	milli-
10^{9}	1,000	million
10^{-2}	1,000,000,000	thousand
10^{12}	1,000,000	centi-
10^{3}	$\frac{1}{1,000}$	trillion

Figure 7.3.1.1

2. For each of the numbers, think of something in the world that is described by that number.

Exercise 7.3.1.2: Base-ten Representations Matching

1. Match each expression to one or more diagrams that could represent it. For each match, explain what the value of a single small square would have to be.

a. $2 \cdot 10^{-1} + 4 \cdot 10^{-2}$
b. $2 \cdot 10^{-1} + 4 \cdot 10^{-3}$
c. $2 \cdot 10^{3} + 4 \cdot 10^{1}$
d. $2 \cdot 10^{3} + 4 \cdot 10^{2}$

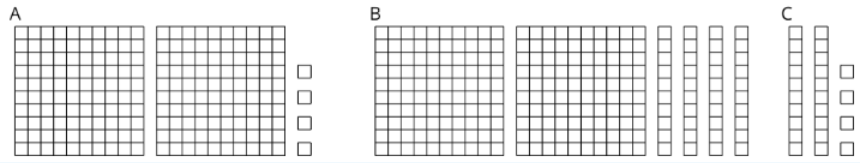


Figure C, 2 rectangles composed of 10 blocks, 10 by 1, 4 small squares, each 1 block.

2.

a. Write an expression to describe the base-ten diagram if each small square represents 10^{-4}. What is the value of this expression?

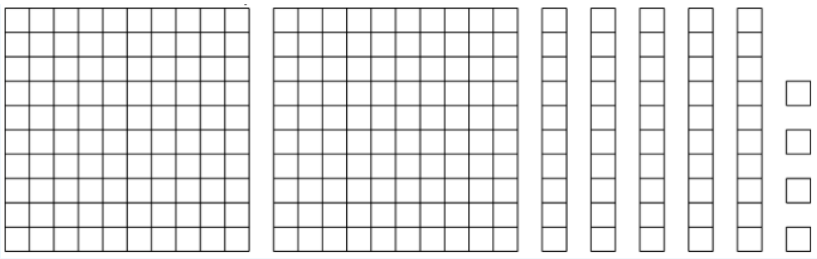


Figure 7.3.1.3

b. How does changing the value of the small square change the value of the expression? Explain or show your thinking.

c. Select at least two different powers of 10 for the small square, and write the corresponding expressions to describe the base-ten diagram. What is the value of each of your expressions?

Exercise 7.3.1.3: Using Powers of 10 to Describe Large and Small Numbers

Your teacher will give you a card that tells you whether you are Partner A or B and gives you the information that is missing from your partner's statements. Do not show your card to your partner.

Read each statement assigned to you, ask your partner for the missing information, and write the number your partner tells you.

Partner A's statements:

1. Around the world, about _____ pencils are made each year.
2. The mass of a proton is _____ kilograms.
3. The population of Russia is about _____ people.
4. The diameter of a bacteria cell is about _____ meter.

Partner B's statements:

1. Light waves travel through space at a speed of _____ meters per second.
2. The population of India is about _____ people.
3. The wavelength of a gamma ray is _____ meters.
4. The tardigrade (water bear) is _____ meters long.

Are you ready for more?

A "googol" is a name for a really big number: a 1 followed by 100 zeros.

1. If you square a googol, how many zeros will the answer have? Show your reasoning.
2. If you raise a googol to the googol power, how many zeros will the answer have? Show your reasoning.

Summary

Sometimes powers of 10 are helpful for expressing quantities, especially very large or very small quantities. For example, the United States Mint has made over

$$500,000,000,000$$

pennies. In order to understand this number, we have to count all the zeros. Since there are 11 of them, this means there are 500 billion pennies. Using powers of 10, we can write this as: $500 \cdot 10^9$ (five hundred times a billion), or even as: $5 \cdot 10^{11}$ The advantage to using powers of 10 to write a large number is that they help us see right away how large the number is by looking at the exponent.

The same is true for small quantities. For example, a single atom of carbon weighs about

$$0.0000000000000000000000199$$

grams. We can write this using powers of 10 as $199 \cdot 10^{-25}$ or, equivalently, $(1.99) \cdot 10^{-23}$ Not only do powers of 10 make it easier to write this number, but they also help avoid errors since it would be very easy to write an extra zero or leave one out when writing out the decimal because there are so many to keep track of!

Practice

Exercise 7.3.1.4

Match each number to its name.

1. $1,000,000$
2. 0.01
3. $1,000,000,000$
4. 0.000001
5. 0.001
6. $10,000$

- One hundredth
- One thousandth
- One millionth
- Ten thousand
- One million
- One billion

Exercise 7.3.1.5

Write each expression as a multiple of a power of 10:

1. $42,300$
2. $2,000$
3. $9,200,000$
4. Four thousand
5. 80 million
6. 32 billion

Exercise 7.3.1.6

Each statement contains a quantity. Rewrite each quantity using a power of 10.

1. There are about 37 trillion cells in an average human body.
2. The Milky Way contains about 300 billion stars.
3. A sharp knife is 23 millionths of a meter thick at its tip.
4. The wall of a certain cell in the human body is 4 nanometers thick. (A nanometer is one billionth of a meter.)

Exercise 7.3.1.7

A fully inflated basketball has a radius of 12 cm. Your basketball is only inflated halfway. How many more cubic centimeters of air does your ball need to fully inflate? Express your answer in terms of π. Then estimate how many cubic centimeters this is by using 3.14 to approximate π.

(From Unit 5.5.4)

Exercise 7.3.1.8

Solve each of these equations. Explain or show your reasoning.

$$2(3-2c)=30 \qquad 3x-2=7-6x \qquad 31=5(b-2)$$

(From Unit 4.2.4)

Exercise 7.3.1.9

Graph the line going through $(-6, 1)$ with a slope of $\frac{-2}{3}$ and write its equation.

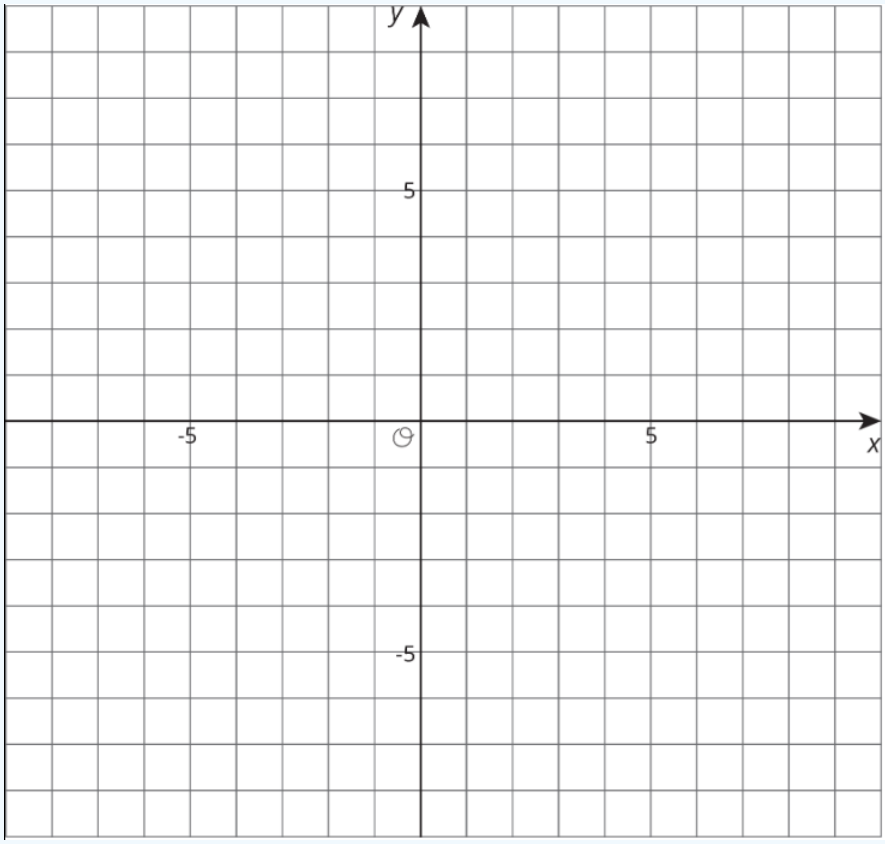


Figure 7.3.1.4

(From Unit 3.3.2)

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7.3.2: Representing Large Numbers on the Number Line

Lesson

Let's visualize large numbers on the number line using powers of 10.

Exercise 7.3.2.1: Labeling Tick Marks on a Number Line

Label the tick marks on the number line. Be prepared to explain your reasoning.

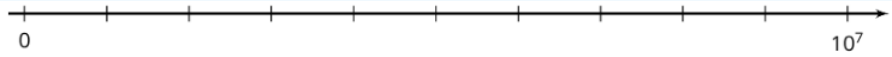


Figure 7.3.2.1

Exercise 7.3.2.2: Comparing Large Numbers with a Number Line

1. Drag the points to their proper places on the number line. Be prepared to explain your reasoning.
2. Discuss with a partner how you decided where each point should go.
3. Which is larger, 4,000,000 or $75 \cdot 10^5$? Estimate how many times larger.

Exercise 7.3.2.3: The Speeds of Light

The table shows how fast light waves or electricity can travel through different materials:

Table 7.3.2.1

material	speed of light (meters per second)
space	$300,000,000$
water	$(2.25) \cdot 10^8$
copper wire (electricity)	$280,000,000$
diamond	$124 \cdot 10^6$
ice	$(2.3) \cdot 10^8$
olive oil	$200,000,000$

1. Which is faster, light through diamond or light through ice? How can you tell from the expressions for speed
   Let's zoom in to highlight the values between $(2.0) \cdot 10^8$ and $(3.0) \cdot 10^8$.
2. Plot a point for each speed on both number lines, and label it with the corresponding material.
3. There is one speed that you cannot plot on the bottom number line. Which is it? Plot it on the top number line instead.
4. Which is faster, light through ice or light through diamond? How can you tell from the number line?

Are you ready for more?

Find a four-digit number using only the digits 0, 1, 2, or 3 where:

- the first digit tells you how many zeros are in the number,
- the second digit tells you how many ones are in the number,
- the third digit tells you how many twos are in the number, and
- the fourth digit tells you how many threes are in the number.

The number 2,100 is close, but doesn't quite work. The first digit is 2, and there are 2 zeros. The second digit is 1, and there is 1 one. The fourth digit is 0, and there are no threes. But the third digit, which is supposed to count the number of 2's, is zero.

1. Can you find more than one number like this?
2. How many solutions are there to this problem? Explain or show your reasoning.

Summary

There are many ways to compare two quantities. Suppose we want to compare the world population, about

7.4 billion

to the number of pennies the U.S. made in 2015, about

8,900,000,000

There are many ways to do this. We could write 7.4 billion as a decimal, 7,400,000,000, and then we can tell that there were more pennies made in 2015 than there are people in the world! Or we could use powers of 10 to write these numbers: $7.4 \cdot 10^9$ for people in the world and $8.9 \cdot 10^9$ for the number of pennies.

For a visual representation, we could plot these two numbers on a number line. We need to carefully choose our end points to make sure that the numbers can both be plotted. Since they both lie between 10^9 and 10^{10}, if we make a number line with tick marks that increase by one billion, or 10^9, we start the number line with 0 and end it with $10 \cdot 10^9$, or 10^{10}. Here is a number line with the number of pennies and world population plotted:

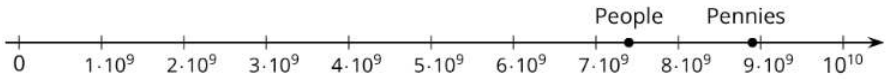


Figure 7.3.2.2

Practice

Exercise 7.3.2.4

Find three different ways to write the number 437,000 using powers of 10.

Exercise 7.3.2.5

For each pair of numbers below, circle the number that is greater. Estimate how many times greater.

1. $17 \cdot 10^8$ or $4 \cdot 10^8$
2. $2 \cdot 10^6$ or $7.839 \cdot 10^6$
3. $42 \cdot 10^7$ or $8.5 \cdot 10^8$

Exercise 7.3.2.6

What number is represented by point A? Explain or show how you know.

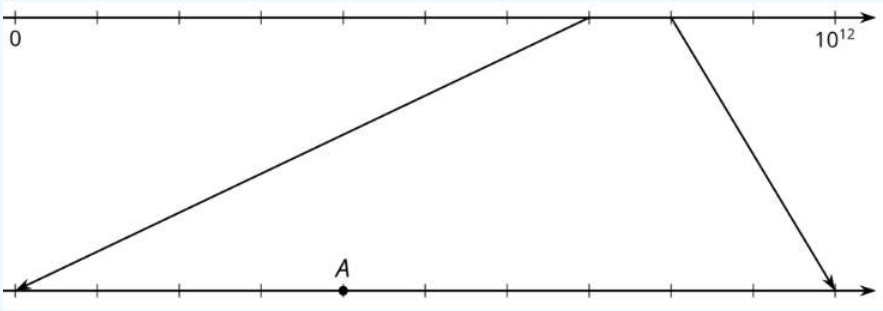


Figure 7.3.2.3

Exercise 7.3.2.7

Here is a scatter plot that shows the number of points and assists by a set of hockey players. Select **all** the following that describe the association in the scatter plot:

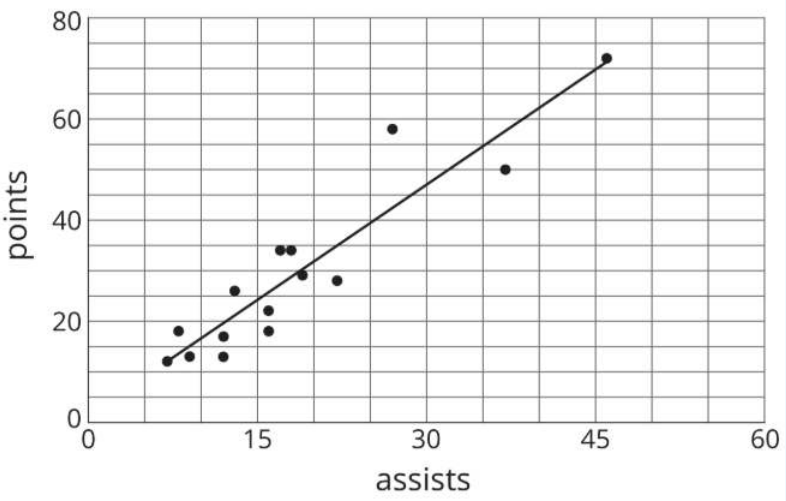


Figure 7.3.2.4

A. Linear association
B. Non-linear association
C. Positive association
D. Negative association
E. No association

(From Unit 6.2.5)

Exercise 7.3.2.8

Here is the graph of days and the predicted number of hours of sunlight, h, on the d-th day of the year.

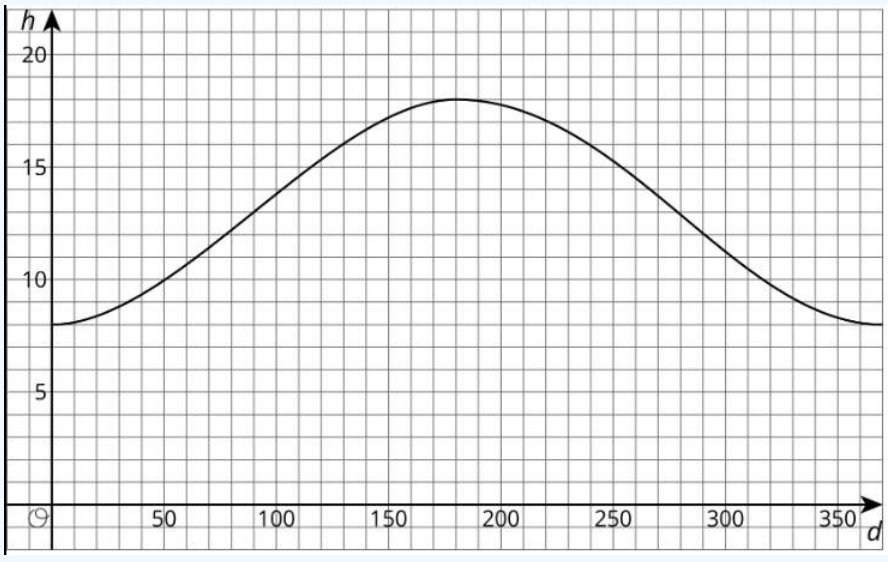


Figure 7.3.2.5

1. Is hours of sunlight a function of days of the year? Explain how you know.
2. For what days of the year is the number of hours of sunlight increasing? For what days of the year is the number of hours of sunlight decreasing?
3. Which day of the year has the greatest number of hours of sunlight?

(From Unit 5.2.3)

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7.3.3: Representing Small Numbers on the Number Line

Lesson

Let's visualize small numbers on the number line using power of 10.

Exercise 7.3.3.1: Small Numbers on a Number Line

Kiran drew this number line.

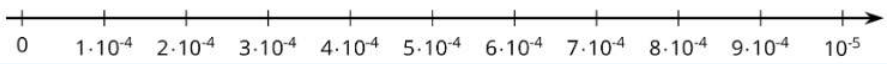


Figure 7.3.3.1: A number line, 11 tick marks, 0, 1 times 10 to the negative 4 power, 2 times 10 to the negative 4 power, 3 times 10 to the negative 4 power, 4 times 10 to the negative 4 power, 5 times 10 to the negative 4 power, 6 times 10 to the negative 4 power, 7 times 10 to the negative 4 power, 8 times 10 to the negative 4 power, 9 times 10 to the negative 4 power, 10 to the negative 5 power.

Andre said, “That doesn’t look right to me.”

Explain why Kiran is correct or explain how he can fix the number line.

Exercise 7.3.3.2: Comparing Small Numbers on a Number Line

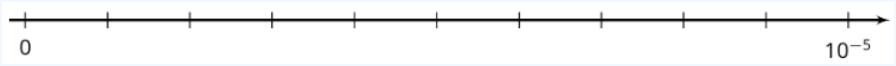


Figure 7.3.3.2

1. Label the tick marks on the number line.
2. Plot the following numbers on the number line:
   A. $6 \cdot 10^{-6}$
   B. $6 \cdot 10^{-7}$
   C. $29 \cdot 10^{-7}$
   D. $(0.7) \cdot 10^{-5}$
3. Which is larger, $29 \cdot 10^{-7}$ or $6 \cdot 10^{-6}$? Estimate how many times larger.
4. Which is larger, $7 \cdot 10^{-8}$ or $3 \cdot 10^{-9}$? Estimate how many times larger.

Exercise 7.3.3.3: Atomic Scale

1. The radius of an electron is about 0.0000000000003 cm.
   a. Write this number as a multiple of power 10.
   b. Decide what power of 10 to put on the right side of this number line and label it.
   c. Label each tick mark as a multiple of a power of 10.

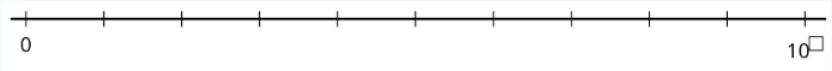


Figure 7.3.3.3

   d. Plot the radius of the electron in centimeters on the number line.
2. The mass of a proton is about 0.000000000000000000000000017 grams.
   a. Write this number as a multiple of a power of 10.
   b. Decide what power of 10 to put on the right side of this number line and label it.
   c. Label each tick mark as a multiple of a power of 10.

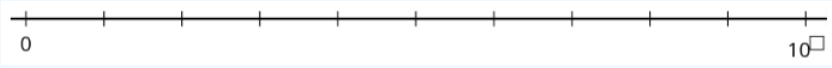


Figure 7.3.3.4

   d. Plot the mass of the proton in grams on the number line.

3. Point A on the zoomed-in-number line describes the wavelength of a certain X-ray in meters.

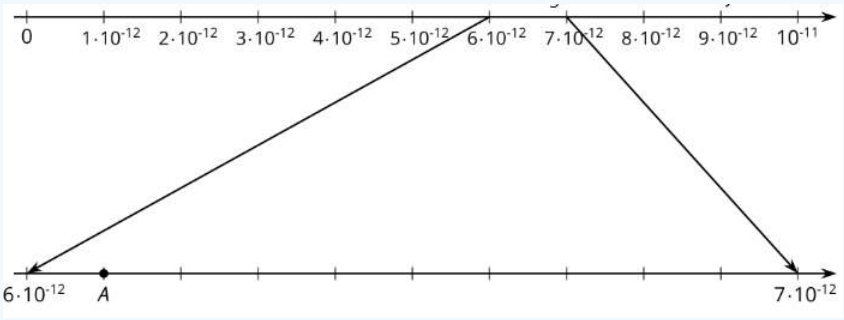


Figure 7.3.3.5

a. Write the wavelength of the X-ray as a multiple power of 10.
b. Write the wavelength of the X-ray as a decimal.

Summary

The width of a bacterium cell is about $2 \cdot 10^{-6}$ meters. If we want to plot this on a number line, we need to find which two powers of 10 it lies between. We can see that $2 \cdot 10^{-6}$ is a multiple of 10^{-6}. So our number line will be labeled with multiples of 10^{-6}

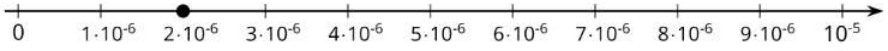


Figure 7.3.3.6

Note that the right side is labeled $10 \cdot 10^{-6} = 10^{-5}$

The power of ten on the right side of the number line is always *greater* than the power on the left. This is true for powers with positive or negative exponents.

Practice

Exercise 7.3.3.4

Select **all** the expressions that are equal to

A. $4 \cdot \left(\frac{1}{10}\right) \cdot \left(\frac{1}{10}\right) \cdot \left(\frac{1}{10}\right)$
B. $4 \cdot (-10) \cdot (-10) \cdot (-10)$
C. $4 \cdot 0.001$
D. $4 \cdot 0.0001$
E. 0.004
F. 0.0004

Exercise 7.3.3.5

Write each expression as a multiple of a power of 10:

1. 0.04
2. 0.072
3. 0.0000325
4. Three thousandths
5. 23 hundredths
6. 729 thousandths
7. 41 millionths

Exercise 7.3.3.6

A family sets out on a road trip to visit their cousins. They travel at a steady rate. The graph shows the distance remaining to their cousins' house for each hour of the trip.

1. How fast are they traveling?
2. Is the slope positive or negative? Explain how you know and why that fits the situation.
3. How far is the trip and how long did it take? Explain how you know.

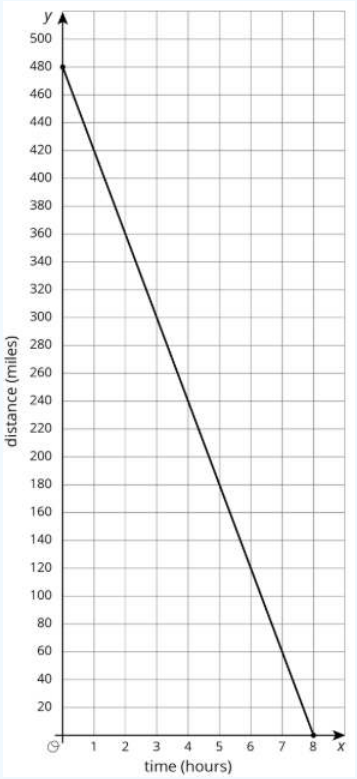


Figure 7.3.3.7

(From Unit 3.3.1)

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7.3.4: Applications of Arithmetic with Powers of 10

Lesson

Let's use powers of 10 to help us make calculations with large and small numbers.

Exercise 7.3.4.1: What Information Do You Need?

What information would you need to answer these questions?

1. How many meter sticks does it take to equal the mass of the Moon?
2. If all of these meter sticks were lined up end to end, would they reach the Moon?

Exercise 7.3.4.2: Meter Sticks to the Moon

1. How many meter sticks does it take to equal the mass of the moon? Explain or show your reasoning.

2. Label the number line and plot your answer for the number of meter sticks.

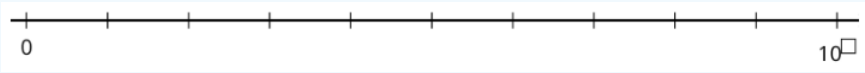


Figure 7.3.4.1

3. If you took all the meter sticks from the last question and lined them up end to end, will they reach the Moon? Will they reach beyond the Moon? If yes, how many times farther will they reach? Explain your reasoning.

4. One light year is approximately 10^{16} meters. How many light years away would the meter sticks reach? Label the number line and plot your answer.

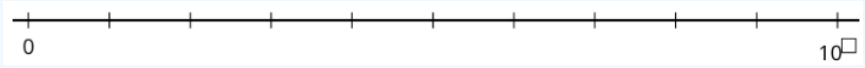


Figure 7.3.4.2

Are you ready for more?

Here is a problem that will take multiple steps to solve. You may not know all the facts you need to solve the problem. That is okay. Take a guess at reasonable answers to anything you don't know. Your final answer will be an estimate.

If everyone alive on Earth right now stood very close together, how much area would they take up?

Exercise 7.3.4.3: That's a Tall Stack of Cash

In 2016, the Burj Khalifa was the tallest building in the world. It was very expensive to build.

Consider the question: Which is taller, the Burj Khalifa or a stack of the money it cost to build the Burj Khalifa?

1.What information would you need to be able to solve the problem?

2. Record the information your teacher shares with the class.

3. Answer the question "Which is taller, the Burj Khalifa or a stack of the money it cost to build the Burj Khalifa?" and explain or show your reasoning.

4. Decide what power of 10 to use to label the rightmost tick mark of the number line, and plot the height of the stack of money and the height of the Burj Khalifa.

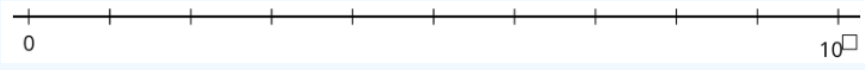


Figure 7.3.4.3

5. Which has more mass, the Burj Khalifa or the mass of the pennies it cost to build the Burj Khalifa? What information do you need to answer this?

6. Decide what power of 10 to use to label the rightmost tick mark of the number line, and plot the mass of the Burj Khalifa and the mass of the pennies it cost to build the Burj Khalifa.

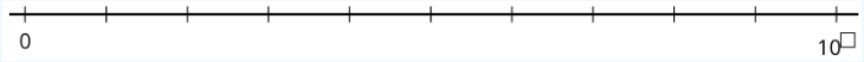


Figure 7.3.4.4

Summary

Powers of 10 can be helpful for making calculations with large or small numbers. For example, in 2014, the United States had

318,586,495

people who used the equivalent of

2,203,799,778,107

kilograms of oil in energy. The amount of energy per person is the total energy divided by the total number of people. We can use powers of 10 to estimate the total energy as $2 \cdot 10^{12}$ and the population as $3 \cdot 10^{8}$ So the amount of energy per person in the U.S. is roughly $(2 \cdot 10^{12}) \div (3 \cdot 10^{8})$ That is the equivalent of $\frac{2}{3} \cdot 10^{4}$ kilograms of oil in energy. That's a lot of energy—the equivalent of almost 7,000 kilograms of oil per person!

In general, when we want to perform arithmetic with very large or small quantities, estimating with powers of 10 and using exponent rules can help simplify the process. If we wanted to find the exact quotient of 2,203,799,778,107 by 318,586,495, then using powers of 10 would not simplify the calculation.

Practice

Exercise 7.3.4.4

Which is larger: the number of meters across the Milky Way, or the number of cells in all humans? Explain or show your reasoning.

Some useful information:

- The Milky Way is about 100,000 light years across.
- There are about 37 trillion cells in a human body.
- One light year is about 10^{16} meters.
- The world population is about 7 billion.

Exercise 7.3.4.5

Ecologists measure the body length and wingspan of 127 butterfly specimens caught in a single field.

1. Draw a line that you think is a good fit for the data.
2. Write an equation for the line.
3. What does the slope of the line tell you about the wingspans and lengths of these butterflies?

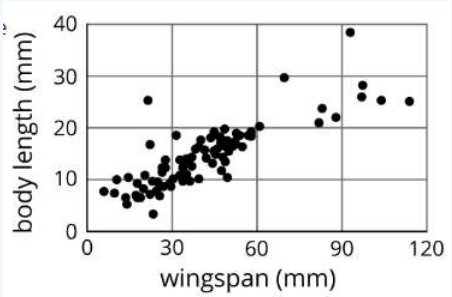


Figure 7.3.4.5

(From Unit 6.2.3)

Exercise 7.3.4.6

Diego was solving an equation, but when he checked his answer, he saw his solution was incorrect. He knows he made a mistake, but he can't find it. Where is Diego's mistake and what is the solution to the equation?

$$\begin{aligned} -4(7-2x) &= 3(x+4) \\ -28-8x &= 3x+12 \\ -28 &= 11x+12 \\ -40 &= 11x \\ -\frac{40}{11} &= x \end{aligned}$$

(From Unit 4.2.4)

Exercise 7.3.4.7

The two triangles are similar. Find x.

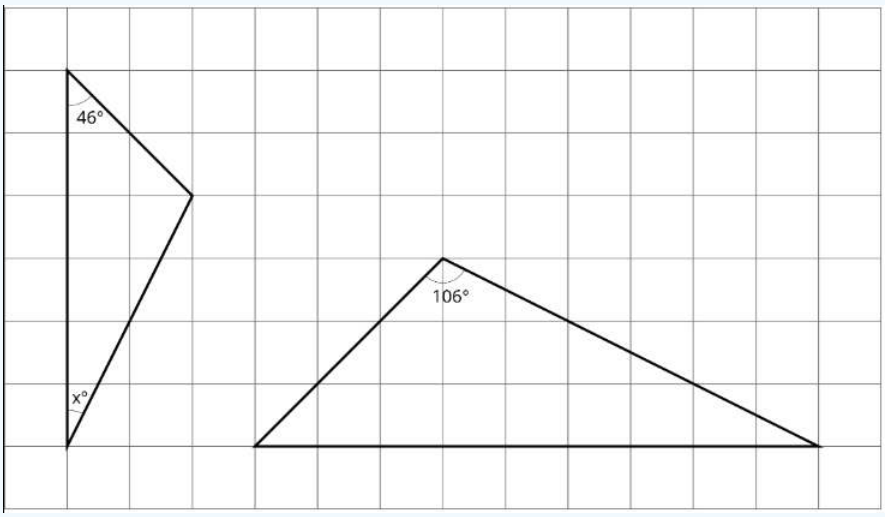


Figure 7.3.4.6

(From Unit 2.2.2)

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7.3.5: Definition of Scientific Notation

Lesson

Let's use scientific notation to describe large and small numbers.

Exercise 7.3.5.1: Number Talk: Multiplying by Powers of 10

Find the value of each expression mentally.

$123 \cdot 10,000$

$(3.4) \cdot 1,000$

$(0.6) \cdot 100$

$(7.3) \cdot (0.01)$

Exercise 7.3.5.2: The "Science" of Scientific Notation

The table shows the speed of light or electricity through different materials.

Table 7.3.5.1

material	speed (meters per second)
space	300, 000, 000
water	2.25×10^8
copper (electricity)	280, 000, 000
diamond	124×10^6
ice	2.3×10^8
olive oil	0.2×10^9

Circle the speeds that are written in scientific notation. Write the others using scientific notation.

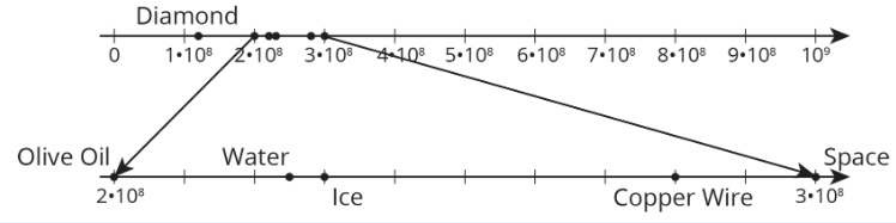


Figure 7.3.5.1: A number line, 11 tick marks, 0, 1 times 10 to the power 8, 2 times 10 to the power 8, 3 times 10 to the power 8, 4 times 10 to the power 8, 5 times 10 to the power 8, 6 times 10 to the power 8, 7 times 10 to the power 8, 8 times 10 to the power 8, 9 times 10 to the power 8, 10 to the power 9. Two times 10 to the power 8 to 3 times 10 to the power 8 is zoomed out to a new number line with 9 tick marks between them. There are 3 points on the new line, 1 between the third and fourth tick mark, 1 at the fourth, and one at the ninth.

Exercise 7.3.5.3: Scientific Notation Matching

Your teacher will give you and your partner a set of cards. Some of the cards show numbers in scientific notation, and other cards show numbers that are not in scientific notation.

1. Shuffle the cards and lay them facedown.
2. Players take turns trying to match cards with the same value.
3. On your turn, choose two cards to turn faceup for everyone to see. Then:

   a. If the two cards have the same value *and* one of them is written in scientific notation, whoever says "Science!" first gets to keep the cards, and it becomes that player's turn. If it's already your turn when you call "Science!", that means you get to go again. If you say "Science!" when the cards do not match or one is not in scientific notation, then your opponent gets a point.
   b. If both partners agree the two cards have the same value, then remove them from the board and keep them. You get a point for each card you keep.
   c. If the two cards do not have the same value, then set them facedown in the same position and end your turn.
4. If it is not your turn:
   a. If the two cards have the same value *and* one of them is written in scientific notation, then whoever says "Science!" first gets to keep the cards, and it becomes that player's turn. If you call "Science!" when the cards do not match or one is not in scientific notation, then your opponent gets a point.
   b. Make sure both of you agree the cards have the same value.
   If you disagree, work to reach an agreement.
5. Whoever has the most points at the end wins.

Are you ready for more?

1. What is $9 \times 10^{-1} + 9 \times 10^{-2}$? Express your answer as:
   a. A decimal
   b. A fraction
2. What is $9 \times 10^{-1} + 9 \times 10^{-2} + 9 \times 10^{-3} + 9 \times 10^{-4}$? Express your answer as:
   a. A decimal
   b. A fraction
3. The answers to the two previous questions should have been close to 1. What power of 10 would you have to go up to if you wanted your answer to be so close to 1 that it was only $\frac{1}{1{,}000{,}000}$ off?
4. What power of 10 would you have to go up to if you wanted your answer to be so close to 1 that it was only $\frac{1}{1{,}000{,}000{,}000}$ off? Can you keep adding numbers in this pattern to get as close to 1 as you want? Explain or show your reasoning.
5. Imagine a number line that goes from your current position (labeled 0) to the door of the room you are in (labeled 1). In order to get to the door, you will have to pass the points 0.9, 0.99, 0.999, etc. The Greek philosopher Zeno argued that you will never be able to go through the door, because you will first have to pass through an infinite number of points. What do you think? How would you reply to Zeno?

Summary

The total value of all the quarters made in 2014 is 400 million dollars. There are many ways to express this using powers of 10. We could write this as $400 \cdot 10^6$ dollars, $40 \cdot 10^7$ dollars, $0.4 \cdot 10^9$ dollars, or many other ways. One special way to write this quantity is called **scientific notation**. In scientific notation,

400 million

dollars would be written as 4×10^8 dollars. For scientific notation, the $\times$ symbol is the standard way to show multiplication instead of the $\cdot$ symbol. Writing the number this way shows exactly where it lies between two consecutive powers of 10. The 10^8 shows us the number is between 10^8 and 10^9. The 4 shows us that the number is 4 tenths of the way to 10^9.

Some other examples of scientific notation are 1.2×10^{-8}, 9.99×10^{16}, and 7×10^{12}. The first factor is a number greater than or equal to 1, but less than 10. The second factor is an integer power of 10.

Thinking back to how we plotted these large (or small) numbers on a number line, scientific notation tells us which powers of 10 to place on the left and right of the number line. For example, if we want to plot 3.4×10^{11} on a number line, we know that the number is larger than 10^{11}, but smaller than 10^{12}. We can find this number by zooming in on the number line:

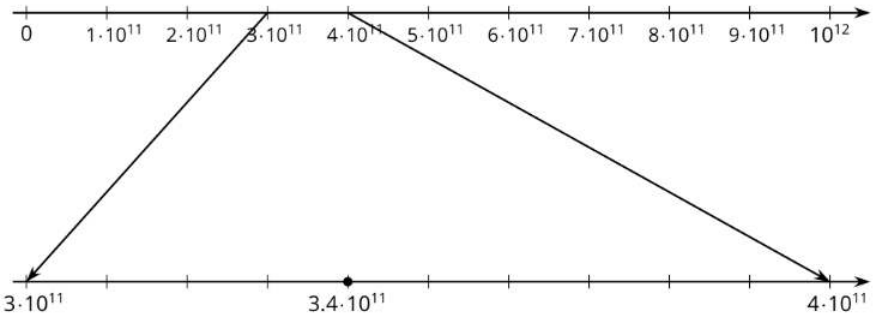


Figure 7.3.5.2: A number line, 11 tick marks, 0, 1 times 10 to the power 11, 2 times 10 to the power 11, 3 times 10 to the power 11, 4 times 10 to the power 11, 5 times 10 to the power 11, 6 times 10 to the power 11, 7 times 10 to the power 11, 8 times 10 to the power 11, 9 times 10 to the power 11, 10 to the power 12. Three times 10 to the power 11 to 4 times 10 to the power 11 is zoomed out, to 11 tick marks labeled 3 times 10 to the power 11, blank, blank, blank, 3 point 4 times 10 to the power 11, blank, blank, blank, blank, blank, 4 times 10 to the power 11.

Glossary Entries

Definition: Scientific Notation

Scientific notation is a way to write very large or very small numbers. We write these numbers by multiplying a number between 1 and 10 by a power of 10.

For example, the number 425,000,000 in scientific notation is 4.25×10^8. The number 0.0000000000783 in scientific notation is 7.83×10^{-11}.

Practice

Exercise 7.3.5.4

Write each number in scientific notation.

1. 14,700
2. 0.00083
3. 760,000,000
4. 0.038
5. 0.38
6. 3.8
7. 3,800,000,000,000
8. 0.0000000009

Exercise 7.3.5.5

Perform the following calculations. Express your answers in scientific notation.

1. $(2 \times 10^5) + (6 \times 10^5)$
2. $(4.1 \times 10^7) \cdot 2$
3. $(1.5 \times 10^{11}) \cdot 3$
4. $(3 \times 10^3)^2$
5. $(9 \times 10^6) \cdot (3 \times 10^6)$

Exercise 7.3.5.6

Jada is making a scale model of the solar system. The distance from Earth to the Moon is about 2.389×10^5 miles. The distance from Earth to the Sun is about 9.296×10^7 miles. She decides to put Earth on one corner of her dresser and the Moon on another corner, about a foot away. Where should she put the sun?

- On a windowsill in the same room?

- In her kitchen, which is down the hallway?
- A city block away?

Explain your reasoning.

Exercise 7.3.5.7

Here is the graph for one equation in a system of equations.

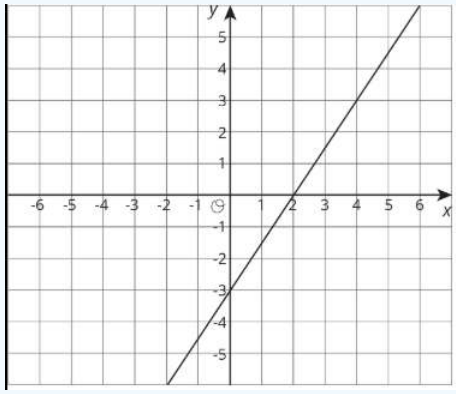


Figure 7.3.5.3

1. Write a second equation for the system so it has infinitely many solutions.
2. Write a second equation whose graph goes through $(0, 2)$ so that the system has no solutions.
3. Write a second equation whose graph goes through $(2, 2)$ so that the system has one solution at $(4, 3)$.

(From Unit 4.3.3)

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7.3.6: Multiplying, Dividing, and Estimating with Scientific Notation

Lesson

Let's multiply and divide with scientific notation to answer questions about animals, careers, and planets.

Exercise 7.3.6.1: True or False: Equations

Is each equation true or false? Explain your reasoning.

1. $4 \times 10^5 \times 4 \times 10^4 = 4 \times 10^{20}$
2. $\frac{7 \times 10^6}{2 \times 10^4} = (7 \div 2) \times 10^{(6-4)}$
3. $8.4 \times 10^3 \times 2 = (8.4 \times 2) \times 10^{(3 \times 2)}$

Exercise 7.3.6.2: Biomass

Use the table to answer questions about different creatures on the planet. Be prepared to explain your reasoning.

Table 7.3.6.1

creature	number	mass of one individual (kg)
humans	7.5×10^9	6.2×10^1
cows	1.3×10^9	4×10^2
sheep	1.75×10^9	6×10^1
chickens	2.4×10^{10}	2×10^0
ants	5×10^{16}	3×10^{-6}
blue whales	4.7×10^3	1.9×10^5
Antarctic krill	7.8×10^{14}	4.86×10^{-4}
zooplankton	1×10^{20}	5×10^{-8}
bacteria	5×10^{30}	1×10^{-12}

1. Which creature is least numerous? Estimate how many times more ants there are.
2. Which creature is the least massive? Estimate how many times more massive a human is.
3. Which is more massive, the total mass of all the humans or the total mass of all the ants? About how many times more massive is it?
4. Which is more massive, the total mass of all the krill or the total mass of all the blue whales? About how many times more massive is it?

Exercise 7.3.6.3: Info Gap: Distances in the Solar System

Your teacher will give you either a problem card or a data card. Do not show or read your card to your partner.

If your teacher gives you the *problem card*:

1. Silently read your card and think about what information you need to answer the question.
2. Ask your partner for the specific information that you need.
3. Explain to your partner how you are using the information to solve the problem.
4. Solve the problem and explain your reasoning to your partner.

If your teacher gives you the *data card*:

1. Silently read the information on your card.

2. Ask your partner “What specific information do you need?” and wait for your partner to *ask* for information. *Only* give information that is on your card. (Do not figure out anything for your partner!)
3. Before telling your partner the information, ask “Why do you need that information?”
4. After your partner solves the problem, ask them to explain their reasoning and listen to their explanation.

Pause here so your teacher can review your work. Ask your teacher for a new set of cards and repeat the activity, trading roles with your partner.

Are you ready for more?

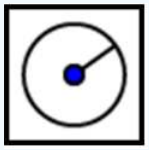
Choose two celestial objects and create a scale drawing of them in the applet below.

Table 7.3.6.4

object	distance to earth (km)	diameter (km)	mass (kg)
Sun	1.496×10^8	1.392×10^6	1.989×10^{30}
Mercury	7.73×10^7	4.878×10^3	3.3×10^{23}
Venus	4×10^7	1.21×10^4	4.87×10^{24}
Earth	N/A	1.28×10^4	5.98×10^{24}
Mars	5.46×10^7	6.785×10^3	6.4×10^{23}
Jupiter	5.88×10^8	1.428×10^5	1.898×10^{27}
Saturn	1.2×10^9	1.199×10^5	5.685×10^{26}
Uranus	2.57×10^9	5.149×10^4	8.68×10^{25}
Neptune	4.3×10^9	4.949×10^4	1.024×10^{26}



Figure 7.3.6.1


Figure 7.3.6.2

Plot a point for the center of each circle. Select the *Circle with Center and Radius* tool and click on a point. When the dialog box opens, enter the radius.

GeoGebra Applet esEdKmyB

Exercise 7.3.6.4: Professions in the United States

Use the table to answer questions about professions in the United States as of 2012.

Table 7.3.6.3

profession	number	typical annual salary (U.S. dollars)
architect	1.074×10^5	7.3×10^4
artist	5.14×10^4	4.4×10^4
programmer	1.36×10^6	8.85×10^4

profession	number	typical annual salary (U.S. dollars)
doctor	6.9×10^5	1.87×10^5
engineer	6.17×10^5	8.6×10^4
firefighter	3.07×10^5	4.5×10^4
military-enlisted	1.16×10^6	4.38×10^4
military-officer	2.5×10^5	1×10^5
nurse	3.45×10^6	6.03×10^4
police officer	7.8×10^5	5.7×10^4
college professor	1.27×10^6	6.9×10^4
retail sales	4.67×10^6	2.14×10^4
truck driver	1.7×10^6	3.82×10^4

Answer the following questions about professions in the United States. Express each answer in scientific notation.

1. Estimate how many times more nurses there are than doctors.
2. Estimate how much money all doctors make put together.
3. Estimate how much money all police officers make put together.
4. Who makes more money, all enlisted military put together or all military officers put together? Estimate how many times more.

Summary

Multiplying numbers in scientific notation extends what we do when we multiply regular decimal numbers. For example, one way to find $(80)(60)$ is to view 80 as 8 tens and to view 60 as 6 tens. The product $(80)(60)$ is 48 hundreds or 4,800. Using scientific notation, we can write this calculation as $(8 \times 10^1)(6 \times 10^1) = 48 \times 10^2$. To express the product in scientific notation, we would rewrite it as 4.8×10^3.

Calculating using scientific notation is especially useful when dealing with very large or very small numbers. For example, there are about 39 million or 3.9×10^7 residents in California. Each Californian uses about 180 gallons of water a day. To find how many gallons of water Californians use in a day, we can find the product $(180)(3.9 \times 10^7) = 702 \times 10^7$, which is equal to 7.02×10^9. That's about 7 billion gallons of water each day!

Comparing very large or very small numbers by estimation also becomes easier with scientific notation. For example, how many ants are there for every human? There are 5×10^{16} ants and 7×10^9 humans. To find the number of ants per human, look at $\frac{5 \times 10^{16}}{7 \times 10^9}$. Rewriting the numerator to have the number 50 instead of 5, we get $\frac{50 \times 10^{15}}{7 \times 10^9}$. This gives us $\frac{50}{7} \times 10^7$. Since $\frac{50}{7}$ is roughly equal to 7, there are about 7×10^6 or 7 million ants per person!

Glossary Entries

Definition: Scientific Notation

Scientific notation is a way to write very large or very small numbers. We write these numbers by multiplying a number between 1 and 10 by a power of 10.

For example, the number 425,000,000 in scientific notation is 4.25×10^8. The number 0.0000000000783 in scientific notation is 7.83×10^{-11}.

Practice

Exercise 7.3.6.5

Evaluate each expression. Use scientific notation to express your answer.

1. $(1.5 \times 10^2)(5 \times 10^{10})$
2. $\frac{4.8 \times 10^{-8}}{3 \times 10^{-3}}$
3. $(5 \times 10^8)(4 \times 10^3)$
4. $(7.2 \times 10^3) \div (1.2 \times 10^5)$

Exercise 7.3.6.6

How many bucketloads would it take to bucket out the world's oceans? Write your answer in scientific notation.

Some useful information:

- The world's oceans hold roughly 1.4×10^9 cubic kilometers of water.
- A typical bucket holds roughly 20,000 cubic centimeters of water.
- There are 10^{15} cubic centimeters in a cubic kilometer.

Exercise 7.3.6.7

The graph represents the closing price per share of stock for a company each day for 28 days.

1. What variable is represented on the horizontal axis?
2. In the first week, was the stock price generally increasing or decreasing?
3. During which period did the closing price of the stock decrease for at least 3 days in a row?

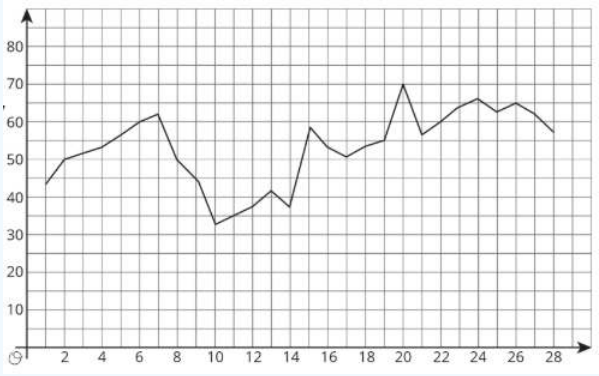


Figure 7.3.6.3

(From Unit 5.2.3)

Exercise 7.3.6.8

Write an equation for the line that passes through $(-8.5, 11)$ and $(5, -2.5)$.

(From Unit 3.3.3)

Exercise 7.3.6.9

Explain why triangle ABC is similar to triangle CFE.

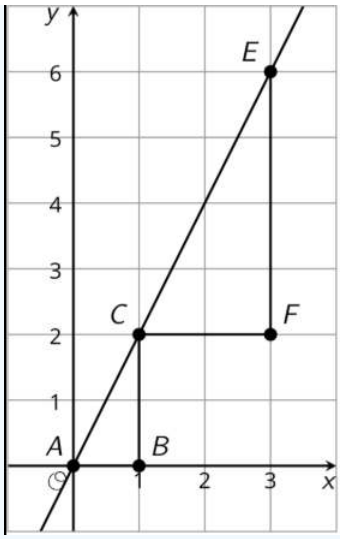


Figure 7.3.6.4

(From Unit 2.2.1)

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7.3.7: Adding and Subtracting with Scientific Notation

Lesson

Let's add and subtract using scientific notation to answer questions about animals and the solar system.

Exercise 7.3.7.1: Number Talk: Non-Zero Digits

Mentally decide how many non-zero digits each number will have.

$(3 \times 10^9)(2 \times 10^7)$

$(3 \times 10^9) \div (2 \times 10^7)$

$3 \times 10^9 + 2 \times 10^7$

$3 \times 10^9 - 2 \times 10^7$

Exercise 7.3.7.2: Measuring the Planets

Diego, Kiran, and Clare were wondering:

"If Neptune and Saturn were side by side, would they be wider than Jupiter?"

1. They try to add the diameters, 4.7×10^4 km and 1.2×10^5 km. Here are the ways they approached the problem. Do you agree with any of them? Explain your reasoning.
   a. Diego says, "When we add the distances, we will get $4.7 + 1.2 = 5.9$. The exponent will be 9. So the two planets are 5.9×10^9 km side by side."
   b. Kiran wrote 4.7×10^4 as $47,000$ and 1.2×10^5 as $120,000$ and added them:

$$\begin{array}{r} 120,000 \\ +\,47,000 \\ \hline 167,000 \end{array}$$

   c. Clare says, "I think you can't add unless they are the same power of 10." She adds 4.7×10^4 km and 12×10^4 to get 16.7×10^4.
2. Jupiter has a diameter of 1.43×10^5. Which is wider, Neptune and Saturn put side by side, or Jupiter?

Exercise 7.3.7.3: A Celestial Dance

Table 7.3.7.1

object	diameter (km)	distance from the Sun (km)
Sun	1.392×10^6	0×10^0
Mercury	4.878×10^3	5.79×10^7
Venus	1.21×10^4	1.08×10^8
Earth	1.28×10^4	1.47×10^8
Mars	6.785×10^3	2.28×10^8
Jupiter	1.428×10^5	7.79×10^8

1. When you add the distances of Mercury, Venus, Earth, and Mars from the Sun, would you reach as far as Jupiter?
2. Add all the diameters of all the planets except the Sun. Which is wider, all of these objects side by side, or the Sun? Draw a picture that is close to scale.

Are you ready for more?

The emcee at a carnival is ready to give away a cash prize! The winning contestant could win anywhere from $1 to $100. The emcee only has 7 envelopes and she wants to make sure she distributes the 100 $1 bills among the 7 envelopes so that no matter what the contestant wins, she can pay the winner with the envelopes without redistributing the bills. For example, it's possible to divide 6 $1 bills among 3 envelopes to get any amount from $1 to $6 by putting $1 in the first envelope, $2 in the second envelope, and $3 in the third envelope (Go ahead and check. Can you make $4? $5? $6?).

How should the emcee divide up the 100 $1 bills among the 7 envelopes so that she can give away any amount of money, from $1 to $100, just by handing out the right envelopes?

Exercise 7.3.7.4: Old McDonald's Massive Farm

Use the table to answer questions about different life forms on the planet.

Table 7.3.7.2

creature	number	mass of one individual (kg)
humans	7.5×10^9	6.2×10^1
cows	1.3×10^9	4×10^2
sheep	1.75×10^9	6×10^1
chickens	2.4×10^{10}	2×10^0
ants	5×10^{16}	3×10^{-6}
blue whales	4.7×10^3	1.9×10^5
antarctic krill	7.8×10^{14}	4.86×10^{-4}
zooplankton	1×10^{20}	5×10^{-8}
bacteria	5×10^{30}	1×10^{-12}

1. On a farm there was a cow. And on the farm there were 2 sheep. There were also 3 chickens. What is the total mass of the 1 cow, the 2 sheep, the 3 chickens, and the 1 farmer on the farm?
2. Make a conjecture about how many ants might be on the farm. If you added all these ants into the previous question, how would that affect your answer for the total mass of all the animals?
3. What is the total mass of a human, a blue whale, and 6 ants all together?
4. Which is greater, the number of bacteria, or the number of all the other animals in the table put together?

Summary

When we add decimal numbers, we need to pay close attention to place value. For example, when we calculate $13.25 + 6.7$, we need to make sure to add hundredths to hundredths (5 and 0), tenths to tenths (2 and 7), ones to ones (3 and 6), and tens to tens (1 and 0). The result is 19.95.

We need to take the same care when we add or subtract numbers in scientific notation. For example, suppose we want to find how much further Earth is from the Sun than Mercury. Earth is about 1.5×10^8 km from the Sun, while Mercury is about 5.8×10^7 km. In order to find $1.5 \times 10^8 - 5.8 \times 10^7$ we can rewrite this as $1.5 \times 10^8 - 0.58 \times 10^8$ Now that both numbers are written in terms of 10^8, we can subtract 0.58 from 1.5 to find 0.92×10^8 Rewriting this in scientific notation, Earth is 9.2×10^7 km further from the Sun than Mercury.

Glossary Entries

Definition: Scientific Notation

Scientific notation is a way to write very large or very small numbers. We write these numbers by multiplying a number between 1 and 10 by a power of 10.

For example, the number 425,000,000 in scientific notation is 4.25×10^8. The number 0.0000000000783 in scientific notation is 7.83×10^{-11}.

Practice

Exercise 7.3.7.5

Evaluate each expression, giving the answer in scientific notation:

1. $5.3 \times 10^4 + 4.7 \times 10^4$
2. $3.7 \times 10^6 - 3.3 \times 10^6$
3. $4.8 \times 10^{-3} + 6.3 \times 10^{-3}$
4. $6.6 \times 10^{-5} - 6.1 \times 10^{-5}$

Exercise 7.3.7.6

1. Write a scenario that describes what is happening in the graph.
2. What is happening at 5 minutes?
3. What does the slope of the line between 6 and 8 minutes mean?

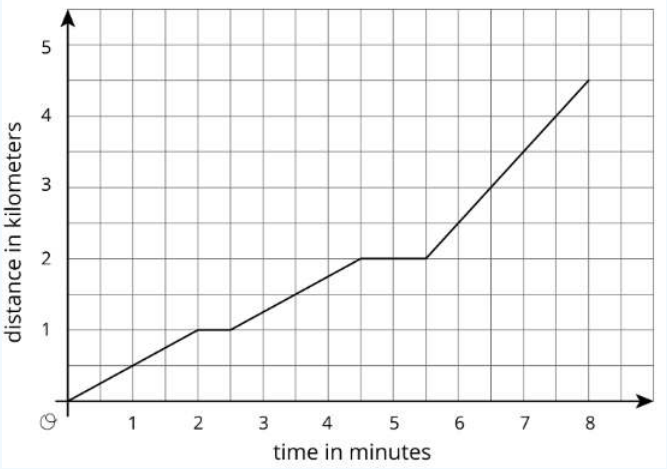


Figure 7.3.7.1

(From Unit 5.3.3)

Exercise 7.3.7.7

Apples cost \$1 each. Oranges cost \$2 each. You have \$10 and want to buy 8 pieces of fruit. One graph shows combinations of apples and oranges that total to \$10. The other graph shows combinations of apples and oranges that total to 8 pieces of fruit.

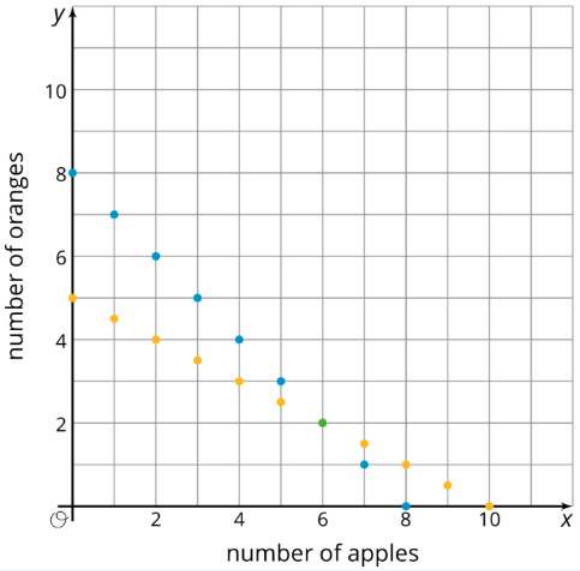


Figure 7.3.7.2: Coordinate plane, x, number of apples, 0 to 10 by 2, y, number of oranges, 0 to 10 by 2. Yellow points at 0 comma 5, 1 comma 4 point 5, 2 comma 4, 3 comma 3 point 5, 4 comma 3, 5 comma 2 point 5, 7 comma 1 point 5, 8 comma 1, 9 comma point 5, 10 comma 0. Blue points at 0 comma 8, 1 comma 7, 2 comma 6, 3 comma 5, 4 comma 4, 5 comma 3, 7 comma 1, 8 comma 0. Green point at 6 comma 2.

1. Name one combination of 8 fruits shown on the graph that whose cost does *not* total to $10.
2. Name one combination of fruits shown on the graph whose cost totals to $10 that are *not* 8 fruits all together.
3. How many apples and oranges would you need to have 8 fruits that cost $10 at the same time?

(From Unit 4.3.1)

Exercise 7.3.7.8

Solve each equation and check your solution.

$$-2(3x-4)=4(x+3)+6 \qquad \tfrac{1}{2}(z+4)-6=-2x+8 \qquad 4w-7=6w+31$$

(From Unit 4.2.4)

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7.4: Let's Put It to Work

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7.4.1: Is a Smartphone Smart Enough to Go to the Moon?

Lesson

Let's compare digital media and computer hardware using scientific notation.

Exercise 7.4.1.1: Old Hardware, New Hardware

In 1966, the Apollo Guidance Computer was developed to make the calculations that would put humans on the Moon.

Your teacher will give you advertisements for different devices from 1966 to 2016. Choose one device and compare that device with the Apollo Guidance Computer. If you get stuck, consider using scientific notation to help you do your calculations.

For reference, storage is measured in bytes, processor speed is measured in hertz, and memory is measured in bytes. Kilo stands for 1,000, mega stands for 1,000,000, giga stands for 1,000,000,000, and tera stands for 1,000,000,000,000.



Figure 7.4.1.1: "Apollo 11 Saturn V lifting off on July 16, 1969", by NASA. Public Domain. Wikimedia Commons. Source.

1. Which one can store more information? How many times more information?
2. Which one has a faster processor? How many times faster?
3. Which one has more memory? How many times more memory?

Exercise 7.4.1.2

For each question, think about what information you would need to figure out an answer. Your teacher may provide some of the information you ask for. Give your answers using scientific notation.

1. Mai found an 80's computer magazine with an advertisement for a machine with hundreds of kilobytes of storage! Mai was curious and asked, "How many kilobytes would my dad's new 2016 computer hold?"
2. The old magazine showed another ad for a 750-kilobyte floppy disk, a device used in the past to store data. How many gigabytes is this?
3. Mai and her friends are actively involved on a social media service that limits each message to 140 characters. She wonders about how the size of a message compares to other media.
   Estimate how many messages it would take for Mai to fill up a floppy disk with her 140-character messages. Explain or show your reasoning.
4. Estimate how many messages it would take for Mai to fill a floppy disk with messages that only use emojis (each message being 140 emojis). Explain or show your reasoning.
5. Mai likes to go to the movies with her friends and knows that a high-definition film takes up a lot of storage space on a computer.
   Estimate how many floppy disks it would take to store a high-definition movie. Explain or show your reasoning.
6. How many seconds of a high-definition movie would one floppy disk be able to hold?
7. If you fall asleep watching a movie streaming service and it streams movies all night while you sleep, how many floppy disks of information would that be?

Are you ready for more?

Humans tend to work with numbers using powers of 10, but computers work with numbers using powers of 2. A "binary kilobyte" is 1,024 bytes instead of 1,000, because $1,024 = 2^{10}$. Similarly, a "binary megabyte" is 1,024 binary kilobytes, and a "binary gigabyte" is 1,024 binary megabytes.

1. Which is bigger, a binary gigabyte or a regular gigabyte? How many more bytes is it?
2. Which is bigger, a binary terabyte or a regular terabyte? How many more bytes is it?

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CHAPTER OVERVIEW

8: Pythagorean Theorem and Irrational Numbers

In this unit, students work with geometric and symbolic representations of square and cube roots. They understand and use notation such as $\sqrt{2}$ and $\sqrt[3]{5}$ for square and cube roots. They understand the terms “rational number” and “irrational number,” using long division to express fractions as decimals. They use their understanding of fractions to plot rational numbers on the number line and their understanding of approximation of irrationals by rationals to approximate the number-line location of a given irrational. Students learn (without proof) that $\sqrt{2}$ is irrational. They understand two proofs of the Pythagorean Theorem—an algebraic proof that involves manipulation of two expressions for the same area and a geometric proof that involves decomposing and rearranging two squares. They use the Pythagorean Theorem in two and three dimensions, e.g., to determine lengths of diagonals of rectangles and right rectangular prisms, and to estimate distances between points in the coordinate plane.

8.1: Side Lengths and Areas of Squares

8.1.1: The Areas of Squares and Their Side Lengths

8.1.2: Side Lengths and Areas

8.1.3: Rational and Irrational Numbers

8.1.4: Square Roots on the Number Line

8.1.5: Reasoning About Square Roots

8.2: The Pythagorean Theorem

8.2.1: Finding Side Lengths of Triangles

8.2.2: A Proof of the Pythagorean Theorem

8.2.3: Finding Unknown Side Lengths

8.2.4: The Converse

8.2.5: Applications of the Pythagorean Theorem

8.2.6: Finding Distances in the Coordinate Plane

8.3: Side Lengths and Volumes of Cubes

8.3.1: Edge Lengths and Volumes

8.3.2: Cube Roots

8.4: Decimal Representation of Rational and Irrational Numbers

8.4.1: Decimal Representation of Rational Numbers

8.4.2: Infinite Decimal Expansions

8.5: Let's Put it to Work

8.5.1: When Is the Same Size Not the Same Size?

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8.1: Side Lengths and Areas of Squares

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8.1.1: The Areas of Squares and Their Side Lengths

Lesson

Let's investigate the squares and their side lengths.

Exercise 8.1.1.1: Two Regions

Which shaded region is larger? Explain your reasoning.

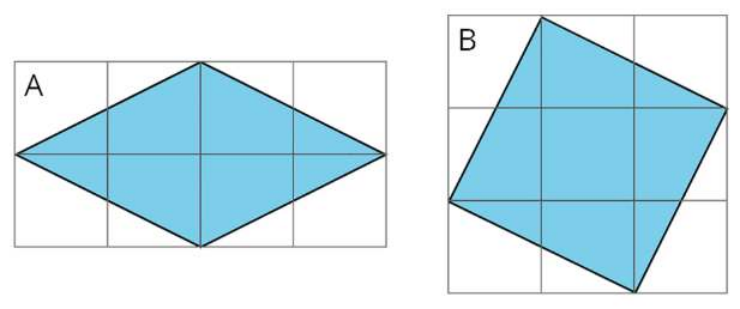


Figure 8.1.1.1

Exercise 8.1.1.2: Decomposing to Find Area

Find the area of each shaded square (in square units).

1.

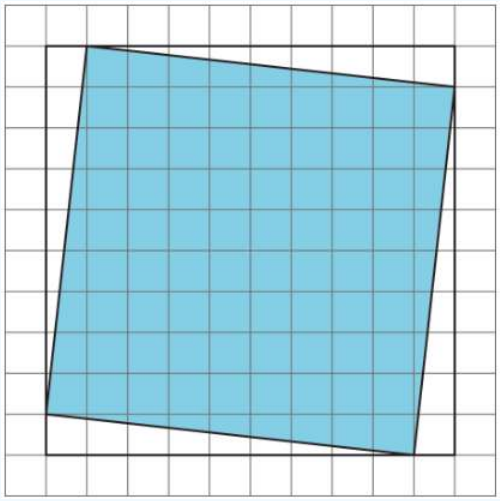


Figure 8.1.1.2

2.

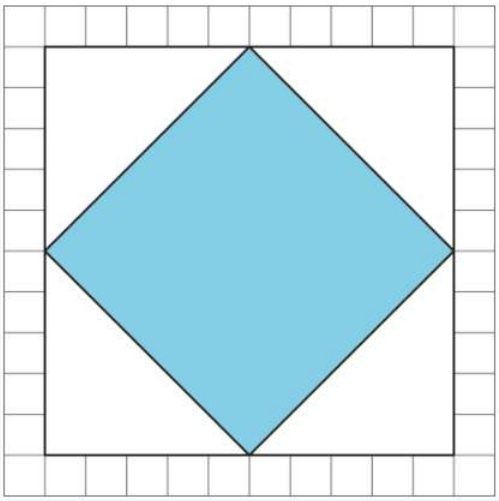


Figure 8.1.1.3

3.

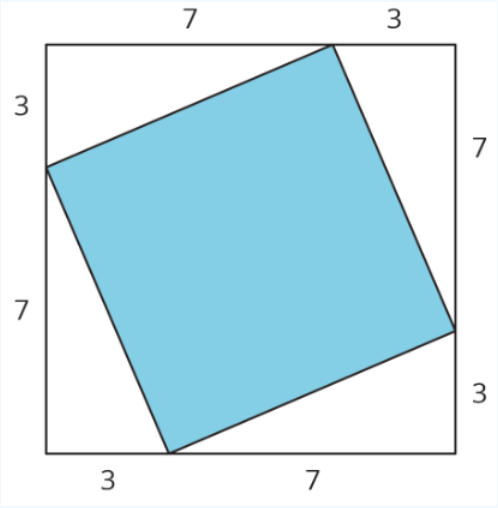


Figure 8.1.1.4

Are you ready for more?

Any triangle with a base of 13 and a height of 5 has an area of $\frac{65}{2}$.

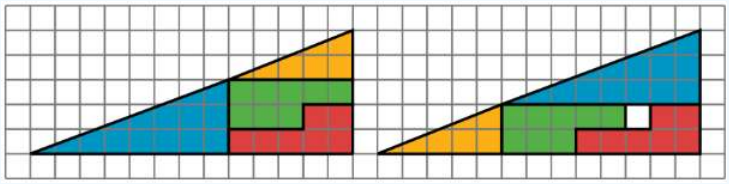


Figure 8.1.1.5

Both shapes in the figure have been partitioned into the same four pieces. Find the area of each of the pieces, and verify the corresponding parts are the same in each picture. There appears to be one extra square unit of area in the right figure. If all of the pieces have the same area, how is this possible?

Exercise 8.1.1.3: Estimating Side Lengths from Areas

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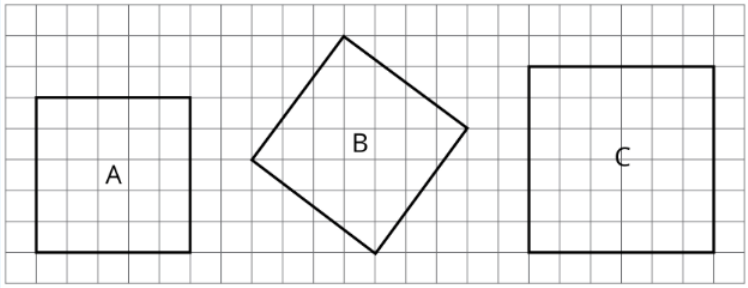


Figure 8.1.1.6

1. What is the side length of square A? What is its area?
2. What is the side length of square C? What is its area?
3. What is the area of square B? What is its side length? (Use tracing paper to check your answer to this.)
4. Find the areas of squares D, E, and F. Which of these squares must have a side length that is greater than 5 but less than 6? Explain how you know.

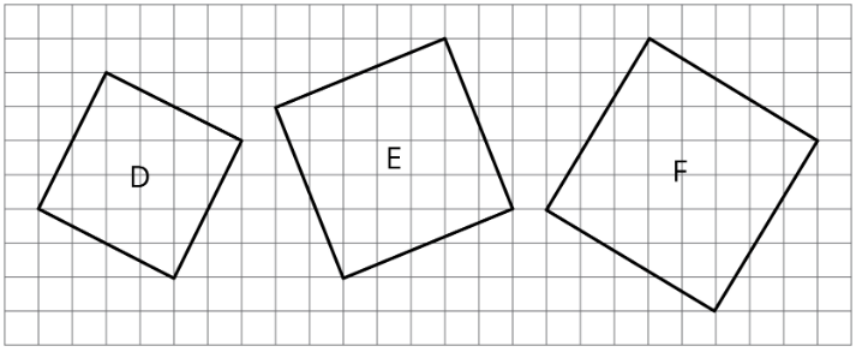


Figure 8.1.1.7

Exercise 8.1.1.4: Making Squares

Use the applet to determine the total area of the five shapes, D, E, F, G, and H. Assume each small square is equal to 1 square unit.

Summary

The area of a square with side length 12 units is 12^2 or 144 units^2.

The side length of a square with area 900 units^2 is 30 units because $30^2 = 900$.

Sometimes we want to find the area of a square but we don't know the side length. For example, how can we find the area of square $ABCD$?

One way is to enclose it in a square whose side lengths we do know.

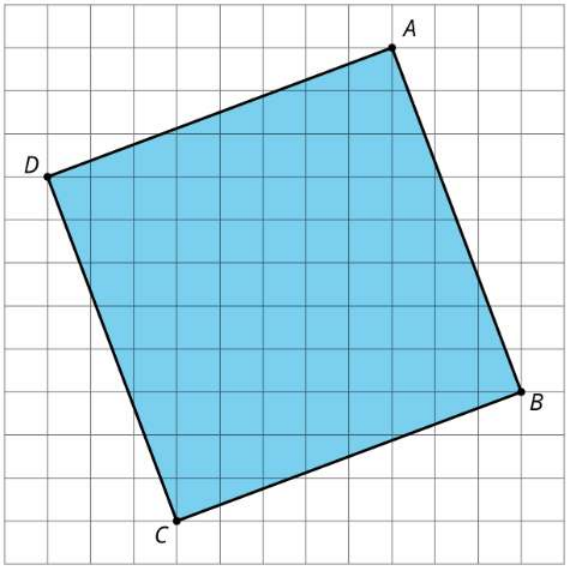


Figure 8.1.1.8

The outside square $EFGH$ has side lengths of 11 units, so its area is 121 units^2. The area of each of the four triangles is $\frac{1}{2} \cdot 8 \cdot 3 = 12$, so the area of all four together is $4 \cdot 12 = 48$ units^2. To get the area of the shaded square, we can take the area of the outside square and subtract the areas of the 4 triangles. So the area of the shaded square $ABCD$ is $121 - 48 = 73$ or 73 units^2.

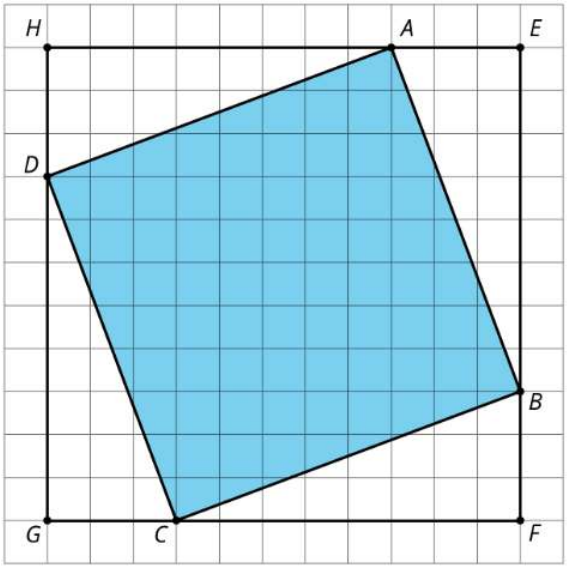


Figure 8.1.1.9

Practice

Exercise 8.1.1.5

Find the area of each square. Each grid square rpresents 1 square unit.

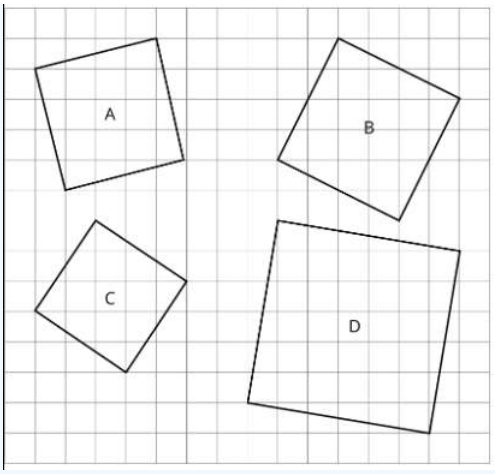


Figure 8.1.1.10

Exercise 8.1.1.6

Find the length of a side of a square if its area is:

1. 81 square inches
2. $\frac{4}{25}$ cm^2
3. 0.49 square units
4. m^2 square units

Exercise 8.1.1.7

Find the area of a square if its side length is:

1. 3 inches
2. 7 units
3. 100 cm
4. 40 inches
5. x units

Exercise 8.1.1.8

Evaluate $(3.1 \times 10^4) \cdot (2 \times 10^6)$. Choose the correct answer.

A. 5.1×10^{10}
B. 5.1×10^{24}
C. 6.2×10^{10}
D. 6.2×10^{24}

(From Unit 7.3.6)

Exercise 8.1.1.9

Noah reads the problem, "Evaluate each expression, giving the answer in scientific notation." The first problem part is: $5.4 \times 10^5 + 2.3 \times 10^4$.

Noah says, "I can rewrite 5.4×10^5 as 54×10^4. Now I can add the numbers: $54 \times 10^4 + 2.3 \times 10^4 = 57.3 \times 10^4$."

Do you agree with Noah's solution to the problem? Explain your reasoning.

(From Unit 7.3.7)

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Exercise 8.1.1.10

Select **all** the expressions that are equivalent to 3^8.

A. $(3^2)^4$
B. 8^3
C. $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$
D. $(3^4)^2$
E. $\frac{3^6}{3^{-2}}$
F. $3^6 \cdot 10^2$

(From Unit 7.2.5)

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8.1.2: Side Lengths and Areas

Lesson

Let's investigate some more squares.

Exercise 8.1.2.1: Notice and Wonder: Intersecting Circles

What do you notice? What do you wonder?

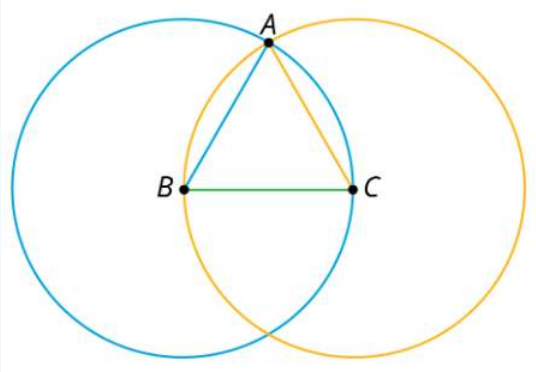


Figure 8.1.2.1

Exercise 8.1.2.2: One Square

1. Use the circle to estimate the area of the square shown here:

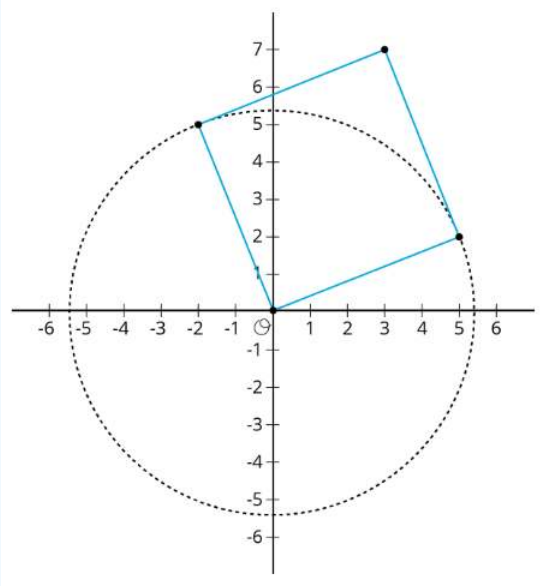


Figure 8.1.2.2

2. Use the grid to check your answer to the first problem.

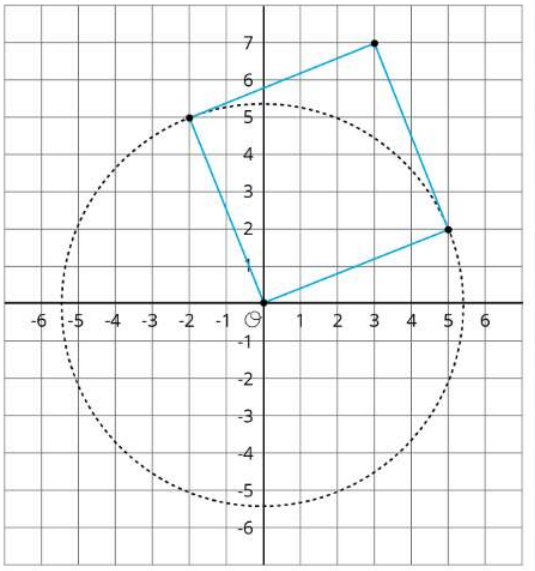


Figure 8.1.2.3

Are you ready for more?

One vertex of the equilateral triangle is in the center of the square, and one vertex of the square is in the center of the equilateral triangle. What is x?

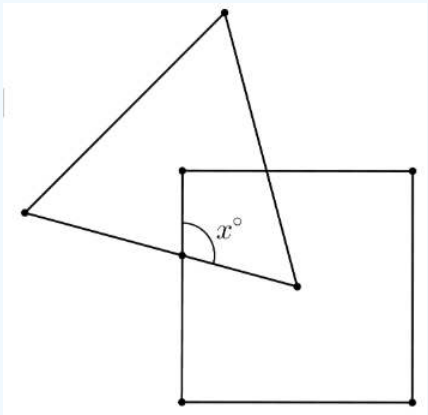


Figure 8.1.2.4

Exercise 8.1.2.3: The Sides and Areas of Tilted Squares

1. Find the area of each square and estimate the side lengths using your geometry toolkit. Then write the exact lengths for the sides of each square.

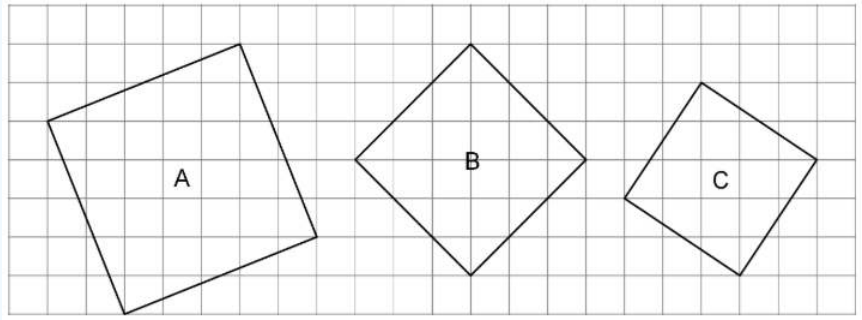


Figure 8.1.2.5

2. Complete the tables with the missing side lengths and areas.

Table 8.1.2.1

	side length, s	0.5		1.5		2.5		3.5	
row 1	**area, a**		1		4		9		16

Table 8.1.2.2

	side length, s	4.5		5.5		6.5		7.5	
row 1	**area, a**		25		36		49		64

3. Plot the points, (s, a), on the coordinate plane shown here.

4. Use this graph to estimate the side lengths of the squares in the first question. How do your estimates from the graph compare to the estimates you made initially using your geometry toolkit?

5. Use the graph to approximate $\sqrt{45}$.

Summary

We saw earlier that the area of square ABCD is 73 units^2.

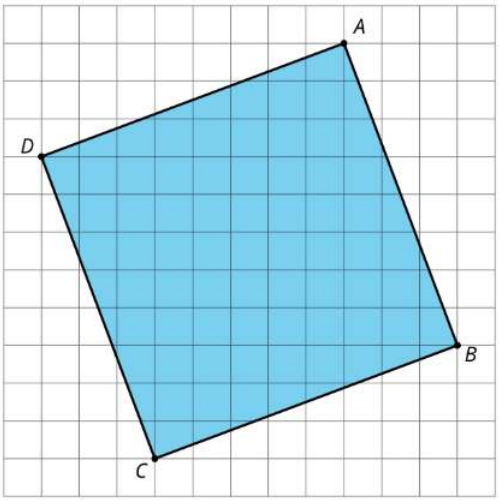


Figure 8.1.2.6

What is the side length? The area is between $8^2 = 64$ and $9^2 = 81$, so the side length must be between 8 units and 9 units. We can also use tracing paper to trace a side length and compare it to the grid, which also shows the side length is between 8 units and 9 units. But we want to be able to talk about its *exact* length. In order to write "the side length of a square whose area is 73 square units," we use the **square root** symbol. "The square root of 73" is written $\sqrt{73}$, and it means "the length of a side of a square whose area is 73 square units."

We say the side length of a square with area 73 units^2 is $\sqrt{73}$ units. This means that $\left(\sqrt{73}\right)^2 = 73$

All of these statements are also true:

$\sqrt{9} = 3$ because $3^2 = 9$

$\sqrt{16} = 4$ because $4^2 = 16$

$\sqrt{10}$ units is the side length of a square whose area is 10 units^2, and $\left(\sqrt{10}\right)^2 = 10$

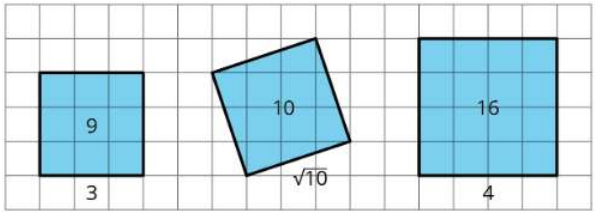


Figure 8.1.2.7: There are 3 squares on a square grid, arranged in order of area, from smallest, on the left, to largest, on the right. The left most square is aligned to the grid and has side lengths of 3 with an area of 9. The middle square is tilted on the grid so that its sides are diagonal to the grid. The square is labeled with a side length of square root of 10 and an area of 10. The right most square is aligned to the grid and has side lengths of 4 with an area of 16.

Glossary Entries

Definition: Square Root

The square root of a positive number n is the positive number whose square is n. It is also the the side length of a square whose area is n. We write the square root of n as $\sqrt{n}$.

For example, the square root of 16, written as $\sqrt{16}$, is 4 because 4^2 is 16.

$\sqrt{16}$ is also the side length of a square that has an area of 16.

Practice

Exercise 8.1.2.4

A square has an area of 81 square feet. Select **all** the expressions that equal the side length of this square, in feet.

A. $\frac{81}{2}$
B. $\sqrt{81}$
C. 9
D. $\sqrt{9}$
E. 3

Exercise 8.1.2.5

Write the exact value of the side length, in units, of a square whose area in square units is:

1. 36
2. 37
3. $\frac{100}{9}$
4. $\frac{2}{5}$
5. 0.0001
6. 0.11

Exercise 8.1.2.6

Square A is smaller than Square B. Square B is smaller than Square C.

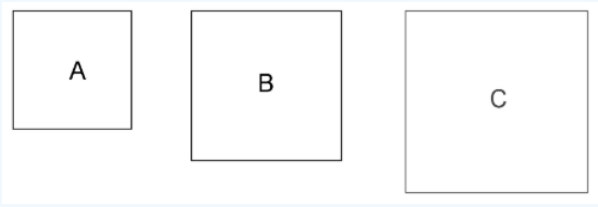


Figure 8.1.2.8

The three squares' side lengths are $\sqrt{26}$, 4.2, and $\sqrt{11}$.

What is the side length of Square A? Square B? Square C? Explain how you know.

Exercise 8.1.2.7

Find the area of a square if its side length is:

1. $\frac{1}{5}$ cm
2. $\frac{3}{7}$ units
3. $\frac{11}{8}$ inches
4. 0.1 meters
5. 3.5 cm

(From Unit 8.1.1)

Exercise 8.1.2.8

Here is a table showing the areas of the seven largest countries.

Table 8.1.2.3

country	area (in km^2)
Russia	1.71×10^7
Canada	9.98×10^6
China	9.60×10^6
United States	9.53×10^6
Brazil	8.52×10^6
Australia	6.79×10^6
India	3.29×10^6

1. How much larger is Russia than Canada?
2. The Asian countries on this list are Russia, China, and India. The American countries are Canada, the United States, and Brazil. Which has the greater total area: the three Asian countries, or the three American countries?

(From Unit 7.3.7)

Exercise 8.1.2.9

Select **all** the expressions that are equivalent to 10^{-6}.

A. $\frac{1}{1000000}$
B. $\frac{-1}{1000000}$
C. $\frac{1}{10^6}$
D. $10^8 \cdot 10^{-2}$
E. $\left(\frac{1}{10}\right)^6$
F. $\frac{1}{10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10}$

(From Unit 7.2.4)

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8.1.3: Rational and Irrational Numbers

Lesson

Let's learn about irrational numbers.

Exercise 8.1.3.1: algebra Talk: Positive Solutions

Find a positive solution to each equation:

$x^2 = 36$

$x^2 = \frac{9}{4}$

$x^2 = \frac{1}{4}$

$x^2 = \frac{49}{25}$

Exercise 8.1.3.2: Three Squares

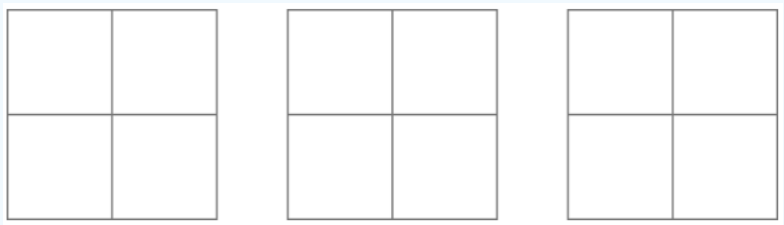


Figure 8.1.3.1

1. Draw 3 squares of different sizes with vertices aligned to the vertices of the grid.
2. For each square:
   a. Label the area.
   b. Label the side length.
   c. Write an equation that shows the relationship between the side length and the area.

Exercise 8.1.3.3: Looking for a Solution

Are any of these numbers a solution to the equation $x^2 = 2$? Explain your reasoning.

- 1
- $\frac{1}{2}$
- $\frac{3}{2}$
- $\frac{7}{5}$

Exercise 8.1.3.4: Looking for $\sqrt{2}$

A **rational number** is a fraction or its opposite (or any number equivalent to a fraction or its opposite).

1. Find some more rational numbers that are close to $\sqrt{2}$.
2. Can you find a rational number that is exactly $\sqrt{2}$?

Are you ready for more?

If you have an older calculator evaluate the expression $\left(\frac{577}{408}\right)^2$ and it will tell you that the answer is 2, which might lead you to think that $\sqrt{2} = \frac{577}{408}$.

1. Explain why you might be suspicious of the calculator's result.
2. Find an explanation for why $408^2 \cdot 2$ could not possibly equal 577^2. How does this show that $\left(\frac{577}{408}\right)^2$ could not equal 2?

3. Repeat these questions for $\left(\frac{14142135623 75}{10000000000000}\right)^2 \neq 2$, an equation that even many modern calculators and computers will get wrong.

Summary

In an earlier activity, we learned that square root notation is used to write the length of a side of a square given its area. For example, a square whose area is 2 square units has a side length of $\sqrt{2}$ units, which means that $\sqrt{2} \cdot \sqrt{2} = 2$.

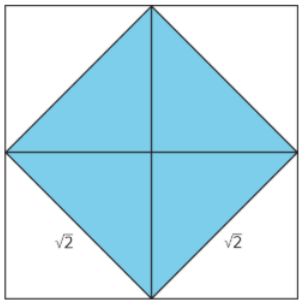


Figure 8.1.3.2

A square whose area is 25 square units has a side length of $\sqrt{25}$ units, which means that $\sqrt{25} \cdot \sqrt{25} = 25$. Since $5 \cdot 5 = 25$, we know that $\sqrt{25} = 5$

$\sqrt{25}$ is an example of a rational number. A **rational number** is a fraction or its opposite. Remember that a fraction $\frac{a}{b}$ is a point on the number line found by dividing the segment from 0 to 1 into b equal intervals and going a of those intervals to the right of 0. We can always write a fraction in the form $\frac{a}{b}$ where a and b are whole numbers (and b is not 0), but there are other ways to write them. For example, we can write $\sqrt{25} = \frac{5}{1}$. You first learned about fractions in earlier grades, and at that time, you probably didn't know about negative numbers. Rational numbers are fractions, but they can be positive or negative. So, -5 is also a rational number. Because fractions and ratios are closely related ideas, fractions and their opposites are called RATIOnal numbers.

Here are some examples of rational numbers: $\frac{7}{4}, 0, \frac{6}{3}, 0.2, -\frac{1}{3}, -5, \sqrt{9}, -\frac{\sqrt{16}}{\sqrt{100}}$ Can you see why they are each examples of "a fraction or its opposite?"

An **irrational number** is a number that is not rational. That is, it is a number that is not a fraction or its opposite. $\sqrt{2}$ is an example of an irrational number. It has a location on the number line, and its location can be approximated by rational numbers (it's a tiny bit to the right of $\frac{7}{5}$), but $\sqrt{2}$ can not be found on a number line by dividing the segment from 0 to 1 into b equal parts and going a of those parts away from 0 (if a and b are whole numbers).

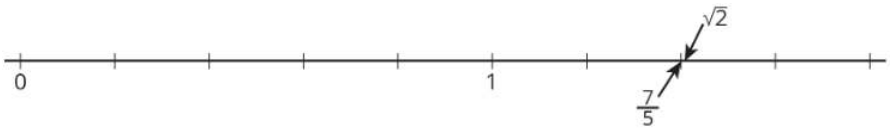


Figure 8.1.3.3

$\frac{17}{12}$ is also close to $\sqrt{2}$, because $\left(\frac{17}{12}\right)^2 = \frac{289}{144}$. $\frac{289}{144}$ is very close to 2, since $\frac{288}{144} = 2$. But we could keep looking forever for solutions to $x^2 = 2$ that are rational numbers, and we would not find any. $\sqrt{2}$ is not a rational number! It is irrational.

In your future studies, you may have opportunities to understand or write a proof that $\sqrt{2}$ is irrational, but for now, we just take it as a fact that $\sqrt{2}$ is irrational. Similarly, the square root of any whole number is either a whole number ($\sqrt{36} = 6$, $\sqrt{64} = 8$, etc.) or irrational ($\sqrt{17}$, $\sqrt{65}$, etc.). Here are some other examples of irrational numbers: $\sqrt{10}, -\sqrt{3}, -\frac{\sqrt{5}}{2}, \pi$

Glossary Entries

Definition: Irrational Number

An irrational number is a number that is not a fraction or the opposite of a fraction.

Pi (π) and $\sqrt{2}$ are examples of irrational numbers.

Definition: Rational Number

A rational number is a fraction or the opposite of a fraction.

Some examples of rational numbers are: $\frac{7}{4}, 0, \frac{6}{3}, 0.2, -\frac{1}{3}, -5, \sqrt{9}$

Definition: Square Root

The square root of a positive number n is the positive number whose square is n. It is also the the side length of a square whose area is n. We write the square root of n as $\sqrt{n}$.

For example, the square root of 16, written as $\sqrt{16}$, is 4 because 4^2 is 16.

$\sqrt{16}$ is also the side length of a square that has an area of 16.

Practice

Exercise 8.1.3.5

Decide whether each number in this list is *rational* or *irrational*.

$\frac{-13}{3}, 0.1234, \sqrt{37}, -77, -\sqrt{100}, -\sqrt{12}$

Exercise 8.1.3.6

Which value is an exact solution of the equation $m^2 = 14$?

A. 7
B. $\sqrt{14}$
C. 3.74
D. $\sqrt{3.74}$

Exercise 8.1.3.7

A square has vertices $(0, 0), (5.2), (3, 7)$ and $(-2, 5)$. Which of these statements is true?

A. The square's side length is 5.
B. The square's side length is between 5 and 6.
C. The square's side length is between 6 and 7.
D. The square's side length is 7.

(From Unit 8.1.2)

Exercise 8.1.3.8

Rewrite each expression in an equivalent form that uses a single exponent.

1. $(10^2)^{-3}$
2. $(3^{-3})^2$
3. $3^{-5} \cdot 4^{-5}$
4. $2^5 \cdot 3^{-5}$

(From Unit 7.2.7)

Exercise 8.1.3.9

The graph represents the area of arctic sea ice in square kilometers as a function of the day of the year in 2016.

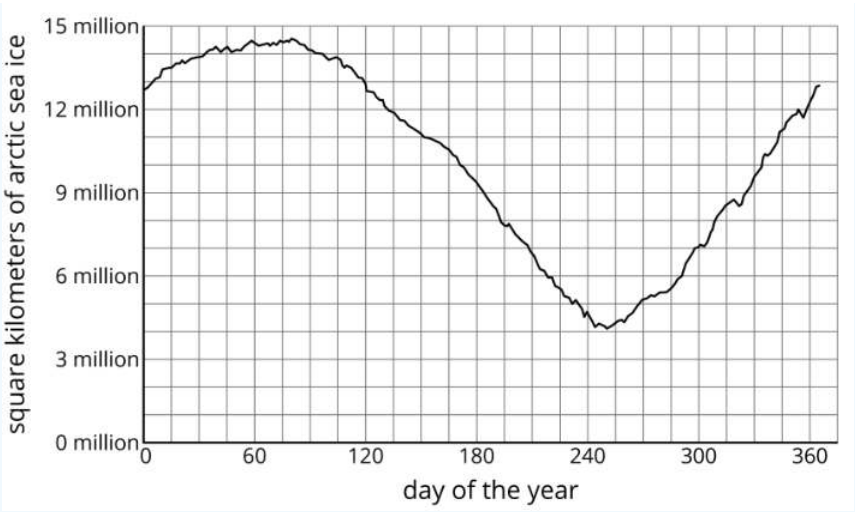


Figure 8.1.3.4

1. Give an approximate interval of days when the area of arctic sea ice was decreasing.
2. On which days was the area of arctic sea ice 12 million square kilometers?

(From Unit 5.2.3)

Exercise 8.1.3.10

The high school is hosting an event for seniors but will also allow some juniors to attend. The principal approved the event for 200 students and decided the number of juniors should be 25% of the number of seniors. How many juniors will be allowed to attend? If you get stuck, try writing two equations that each represent the number of juniors and seniors at the event.

(From Unit 4.3.5)

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8.1.4: Square Roots on the Number Line

Lesson

Let's explore square roots.

Exercise 8.1.4.1: Notice and Wonder: Diagonals

What do you notice? What do you wonder?

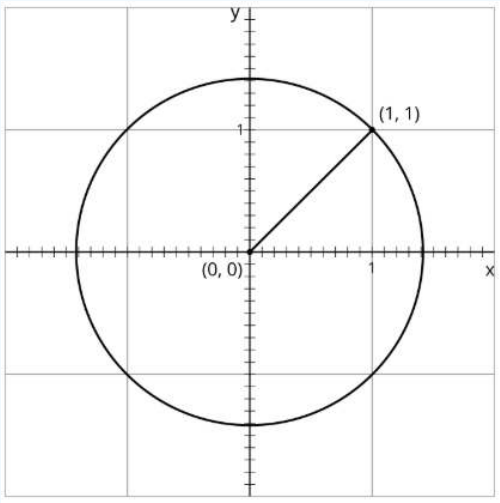


Figure 8.1.4.1

Exercise 8.1.4.2: Squaring Lines

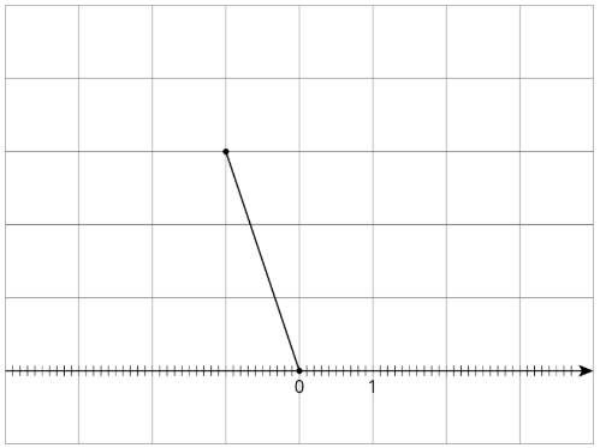


Figure 8.1.4.2

1. Estimate the length of the line segment to the nearest tenth of a unit (each grid square is 1 square unit).
2. Find the exact length of the segment.

Exercise 8.1.4.3: Square Root of 3

Diego said that he thinks that $\sqrt{3} \approx 2.5$.

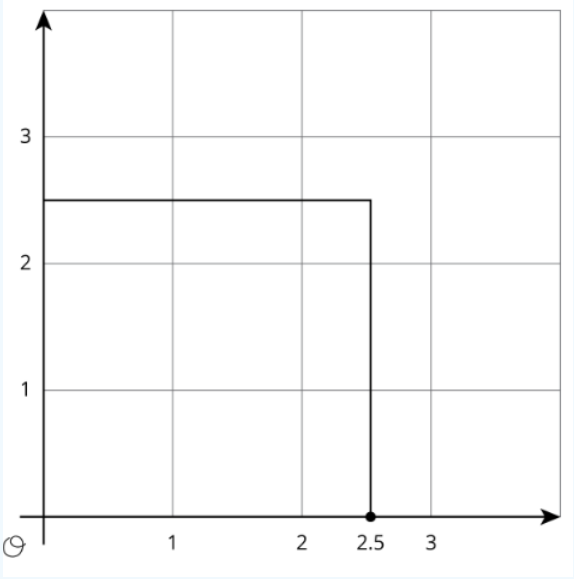


Figure 8.1.4.3

1. Use the square to explain why 2.5 is not a very good approximation for $\sqrt{3}$. Find a point on the number line that is closer to $\sqrt{3}$. Draw a new square on the axes and use it to explain how you know the point you plotted is a good approximation for $\sqrt{3}$.
2. Use the fact that $\sqrt{3}$ is a solution to the equation $x^2 = 3$ to find a decimal approximation of $\sqrt{3}$ whose square is between 2.9 and 3.1.

Are you ready for more?

A farmer has a grassy patch of land enclosed by a fence in the shape of a square with a side length of 4 meters. To make it a suitable home for some animals, the farmer would like to carve out a smaller square to be filled with water, as in the figure.

What should the side length of the smaller square be so that half of the area is grass and half is water?

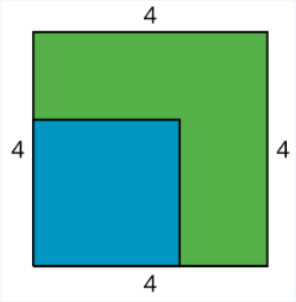


Figure 8.1.4.4

Summary

Here is a line segment on a grid. What is the length of this line segment?

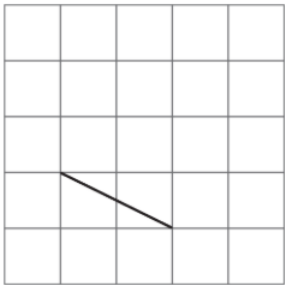


Figure 8.1.4.5

By drawing some circles, we can tell that it's longer than 2 units, but shorter than 3 units.

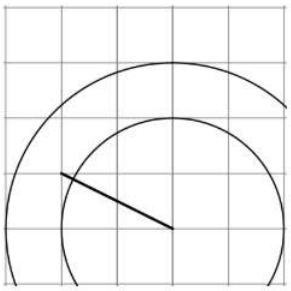


Figure 8.1.4.6

To find an exact value for the length of the segment, we can build a square on it, using the segment as one of the sides of the square.

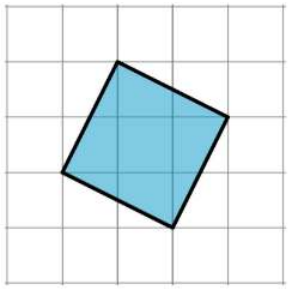


Figure 8.1.4.7

The area of this square is 5 square units. (Can you see why?) That means the exact value of the length of its side is $\sqrt{5}$ units.

Notice that 5 is greater than 4, but less than 9. That means that $\sqrt{5}$ is greater than 2, but less than 3. This makes sense because we already saw that the length of the segment is in between 2 and 3.

With some arithmetic, we can get an even more precise idea of where $\sqrt{5}$ is on the number line. The image with the circles shows that $\sqrt{5}$ is closer to 2 than 3, so let's find the value of 2.1^2 and 2.2^2 and see how close they are to 5. It turns out that $2.1^2 = 4.41$ and $2.2^2 = 4.84$, so we need to try a larger number. If we increase our search by a tenth, we find that $2.3^2 = 5.29$. This means that $\sqrt{5}$ is greater than 2.2, but less than 2.3. If we wanted to keep going, we could try 2.25^2 and eventually narrow the value of $\sqrt{5}$ to the hundredths place. Calculators do this same process to many decimal places, giving an approximation like $\sqrt{5} \approx 2.2360679775$ Even though this is a lot of decimal places, it is still not exact because $\sqrt{5}$ is irrational.

Glossary Entries

Definition: Irrational Number

An irrational number is a number that is not a fraction or the opposite of a fraction.

Pi (π) and $\sqrt{2}$ are examples of irrational numbers.

Definition: Rational Number

A rational number is a fraction or the opposite of a fraction.

Some examples of rational numbers are: $\frac{7}{4}, 0, \frac{6}{3}, 0.2, -\frac{1}{3}, -5, \sqrt{9}$

Definition: Square Root

The square root of a positive number n is the positive number whose square is n. It is also the the side length of a square whose area is n. We write the square root of n as $\sqrt{n}$.

For example, the square root of 16, written as $\sqrt{16}$, is 4 because 4^2 is 16.

$\sqrt{16}$ is also the side length of a square that has an area of 16.

Practice

Exercise 8.1.4.4

1. Find the exact length of each line segment.

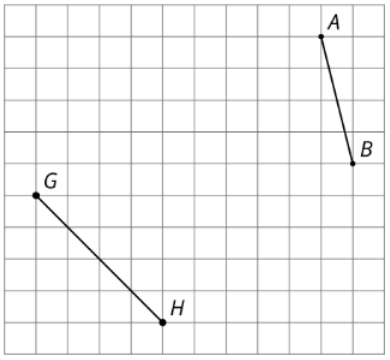


Figure 8.1.4.8

2. Estimate the length of each line segment to the nearest tenth of a unit. Explain your reasoning.

Exercise 8.1.4.5

Plot each number on the x-axis: $\sqrt{16}$, $\sqrt{35}$, $\sqrt{66}$. Consider using the grid to help.

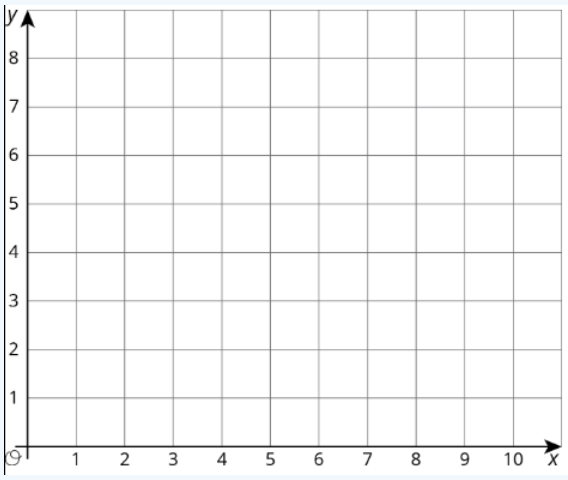


Figure 8.1.4.9

Exercise 8.1.4.6

Use the fact that $\sqrt{7}$ is a solution to the equation $x^2 = 7$ to find a decimal approximation of $\sqrt{7}$ whose square is between 6.9 and 7.1.

Exercise 8.1.4.7

Graphite is made up of layers of graphene. Each layer of graphene is about 200 picometers, or 200×10^{-12} meters, thick. How many layers of graphene are there in a 1.6-mm-thick piece of graphite? Express your answer in scientific notation.

(From Unit 7.3.6)

Exercise 8.1.4.8

Here is a scatter plot that shows the number of assists and points for a group of hockey players. The model, represented by $y = 1.5x + 1.2$, is graphed with the scatter plot.

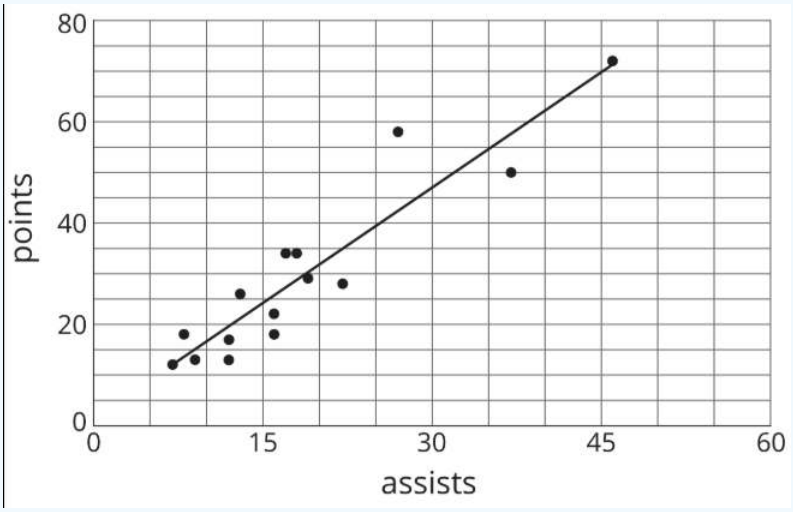


Figure 8.1.4.10

1. What does the slope mean in this situation?
2. Based on the model, how many points will a player have if he has 30 assists?

(From Unit 6.2.4)

Exercise 8.1.4.9

The points $(12, 23)$ and $(14, 45)$ lie on a line. What is the slope of the line?

(From Unit 3.2.1)

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8.1.5: Reasoning About Square Roots

Lesson

Let's approximate square roots.

Exercise 8.1.5.1: True or False: Squared

Decide if each statement is true or false.

$(\sqrt{5})^2 = 5$

$(\sqrt{10})^2 = 100$

$(\sqrt{9})^2 = 3$

$(\sqrt{16}) = 2^2$

$7 = (\sqrt{7})^2$

Exercise 8.1.5.2: Square Root Values

What two whole numbers does each square root lie between? Be prepared to explain your reasoning.

1. $\sqrt{7}$
2. $\sqrt{23}$
3. $\sqrt{50}$
4. $\sqrt{98}$

Are you ready for more?

Can we do any better than "between 3 and 4" for $\sqrt{12}$? Explain a way to figure out if the value is closer to 3.1 or closer to 3.9.

Exercise 8.1.5.3: Solutions on a Number Line

The numbers x, y, and z are positive, and $x^2 = 3$, $y^2 = 16$, and $z^2 = 30$.



Figure 8.1.5.1

1. Plot x, y, and z on the number line. Be prepared to share your reasoning with the class.
2. Plot $-\sqrt{2}$ on the number line.

Summary

In general, we can approximate the values of square roots by observing the whole numbers around it, and remembering the relationship between square roots and squares. Here are some examples:

- $\sqrt{65}$ is a little more than 8, because $\sqrt{65}$ is a little more than $\sqrt{64}$ and $\sqrt{64} = 8$.
- $\sqrt{80}$ is a little more than 9, because $\sqrt{80}$ is a little less than $\sqrt{81}$ and $\sqrt{81} = 9$.
- $\sqrt{75}$ is between 8 and 9 (it's 8 point something), because 75 is between 64 and 81.
- $\sqrt{75}$ is approximately 8.67, because $8.67^2 = 75.1689$.

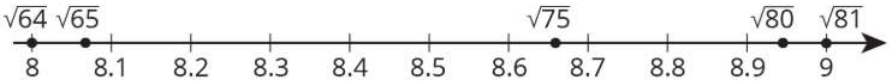


Figure 8.1.5.2: A number line with the numbers 8 through 9, in increments of zero point 1, are indicated. The square root of 64 is indicated at 8. The square root of 65 is indicated between 8 and 8 point 1, where the square root of 65 is closer to 8 point 1. The square root of 75 is indicated between 8 point 6 and 8 point 7, the square root of 75 is closer to 8 point 7. The square root of 80 is indicated between 8 point 9 and 9, where the square root of 80 is closer to 8 point 9. The square root of 81 is indicated at 9.

If we want to find a square root between two whole numbers, we can work in the other direction. For example, since $22^2 = 484$ and $23^2 = 529$, then we know that $\sqrt{500}$ (to pick one possibility) is between 22 and 23.

Many calculators have a square root command, which makes it simple to find an approximate value of a square root.

Glossary Entries

Definition: Irrational Number

An irrational number is a number that is not a fraction or the opposite of a fraction.

Pi (π) and $\sqrt{2}$ are examples of irrational numbers.

Definition: Rational Number

A rational number is a fraction or the opposite of a fraction.

Some examples of rational numbers are: $\frac{7}{4}, 0, \frac{6}{3}, 0.2, -\frac{1}{3}, -5, \sqrt{9}$

Definition: Square Root

The square root of a positive number n is the positive number whose square is n. It is also the the side length of a square whose area is n. We write the square root of n as $\sqrt{n}$.

For example, the square root of 16, written as $\sqrt{16}$, is 4 because 4^2 is 16.

$\sqrt{16}$ is also the side length of a square that has an area of 16.

Practice

Exercise 8.1.5.4

1. Explain how you know that $\sqrt{37}$ is a little more than 6.
2. Explain how you know that $\sqrt{95}$ is a little less than 10.
3. Explain how you know that $\sqrt{30}$ is between 5 and 6.

Exercise 8.1.5.5

Plot each number on the number line: $6, \sqrt{83}, \sqrt{40}, \sqrt{64}, 7.5$

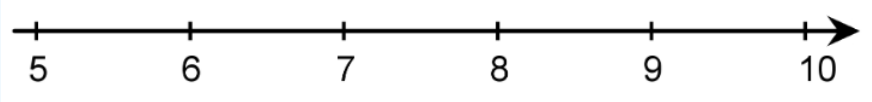


Figure 8.1.5.3

Exercise 8.1.5.6

The equation $x^2 = 25$ has *two* solutions. This is because both $5 \cdot 5 = 25$, and also $-5 \cdot -5 = 25$. So, 5 is a solution, and also -5 is a solution.

Select **all** the equations that have a solution of -4:

A. $10 + x = 6$
B. $10 - x = 6$
C. $-3x = -12$
D. $-3x = 12$
E. $8 = x^2$
F. $x^2 = 16$

Exercise 8.1.5.7

Find all the solutions to each equation.

1. $x^2 = 81$
2. $x^2 = 100$
3. $\sqrt{x} = 12$

Exercise 8.1.5.8

Select all the irrational numbers in the list. $\frac{2}{3}, \frac{-123}{45}, \sqrt{14}, \sqrt{64}, \sqrt{\frac{9}{1}}, -\sqrt{99}, -\sqrt{100}$

(From Unit 8.1.3)

Exercise 8.1.5.9

Each grid square represents 1 square unit. What is the exact side length of the shaded square?

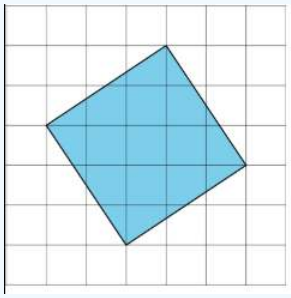


Figure 8.1.5.4

(From Unit 8.1.2)

Exercise 8.1.5.10

For each pair of numbers, which of the two numbers is larger? Estimate how many times larger.

1. $0.37 \cdot 10^6$ and $700 \cdot 10^4$
2. $4.87 \cdot 10^4$ and $15 \cdot 10^5$
3. $500,000$ and $2.3 \cdot 10^8$

(From Unit 7.3.2)

Exercise 8.1.5.11

The scatter plot shows the heights (in inches) and three-point percentages for different basketball players last season.

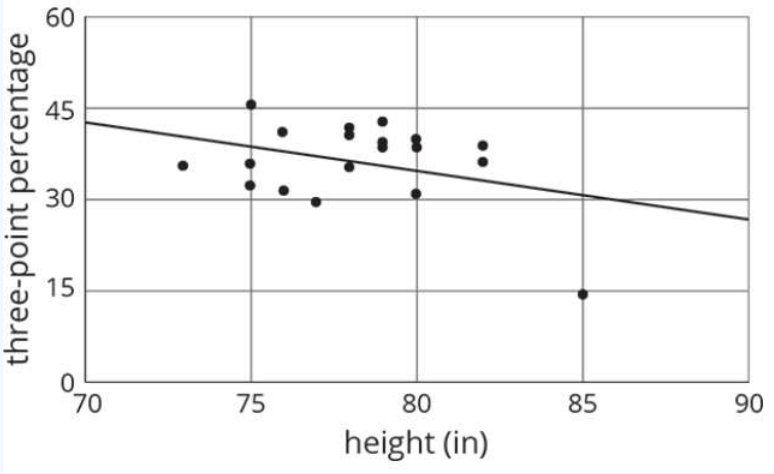


Figure 8.1.5.5

1. Circle any data points that appear to be outliers.
2. Compare any outliers to the values predicted by the model.

(From Unit 6.2.2)

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8.2: The Pythagorean Theorem

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8.2.1: Finding Side Lengths of Triangles

Lesson

Let's find triangle side lengths.

Exercise 8.2.1.1: Which One Doesn't Belong: Triangles

Which triangle doesn't belong?

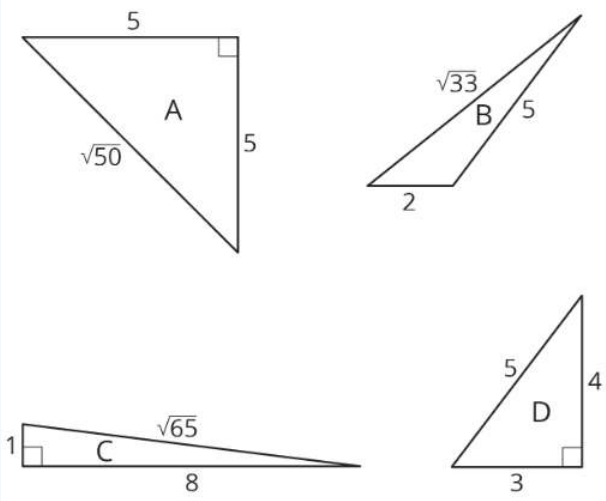


Figure 8.2.1.1

Exercise 8.2.1.2: A Table of Triangles

1. Complete the tables for these three triangles:

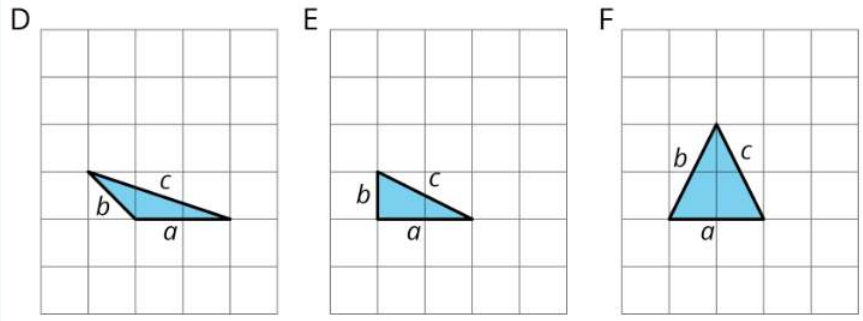


Figure 8.2.1.2

Table 8.2.1.1

triangle	a	b	c
D			
E			
F			

Table 8.2.1.2

triangle	a^2	b^2	c^2
D			
E			

triangle	a^2	b^2	c^2
F			

2. What do you notice about the values in the table for Triangle E but not for Triangles D and F?

3. Complete the tables for these three more triangles:

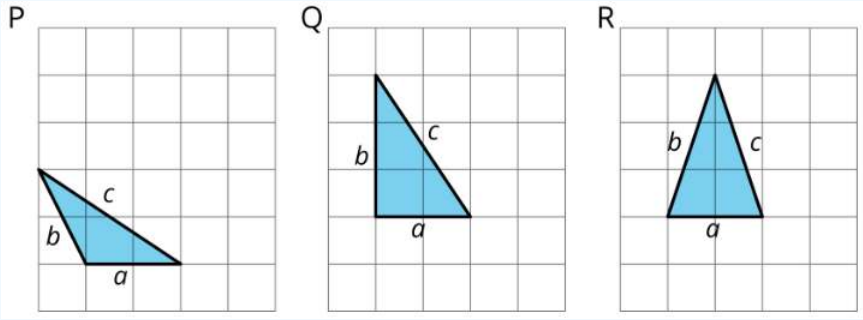


Figure 8.2.1.3

Table 8.2.1.3

triangle	a	b	c
P			
Q			
R			

Table 8.2.1.4

triangle	a^2	b^2	c^2
P			
Q			
R			

4. What do you notice about the values in the table for Triangle Q but not for Triangles P and R?

5. What do Triangle E and Triangle Q have in common?

Exercise 8.2.1.3: Meet the Pythagorean Theorem

1. Find the missing side lengths. Be prepared to explain your reasoning.
2. For which triangles does $a^2 + b^2 = c^2$?

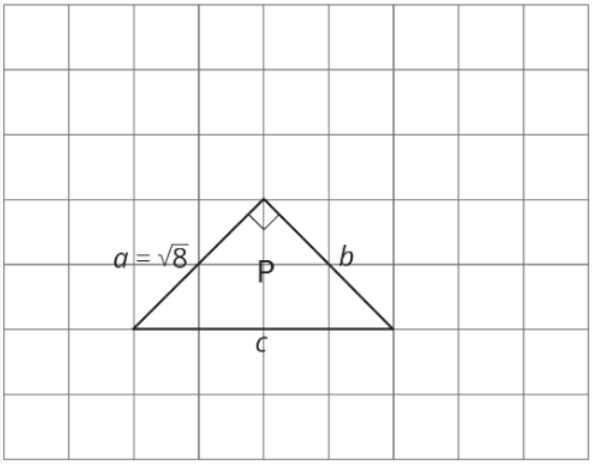


Figure 8.2.1.4

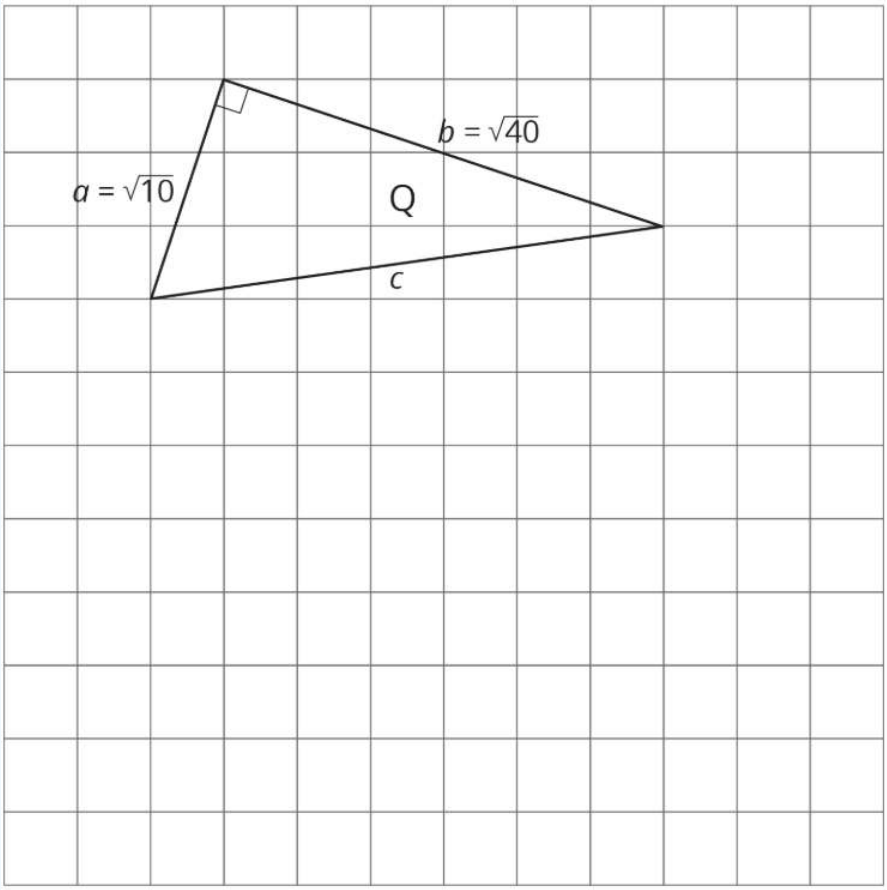


Figure 8.2.1.5

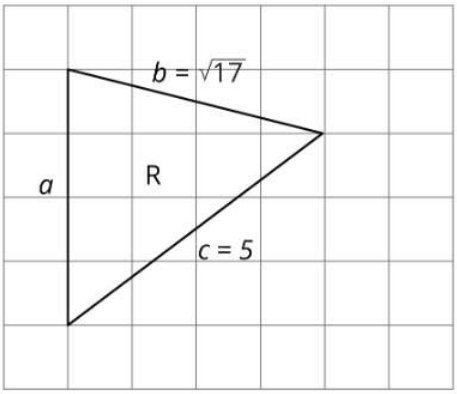


Figure 8.2.1.6

Are you ready for more?

If the four shaded triangles in the figure are congruent right triangles, does the inner quadrilateral have to be a square? Explain how you know.

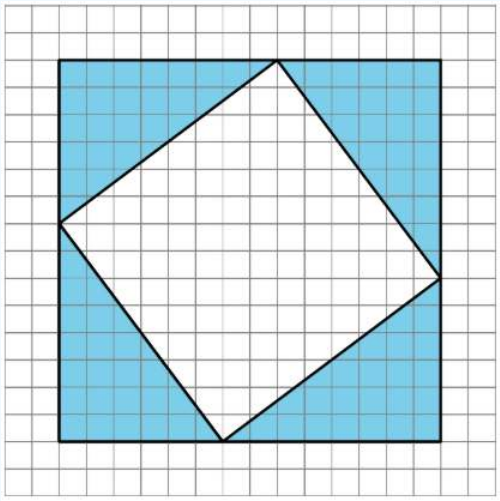


Figure 8.2.1.7: A square with side lengths of 14 units on a square grid. there is a second square inside the square. Each of the vertices of the inside square divides the side lengths of the large square into two lengths: 8 units and 6 units creating 4 right triangles.

Summary

A *right triangle* is a triangle with a right angle. In a right triangle, the side opposite the right angle is called the **hypotenuse**, and the two other sides are called its **legs**. Here are some right triangles with the hypotenuse and legs labeled:

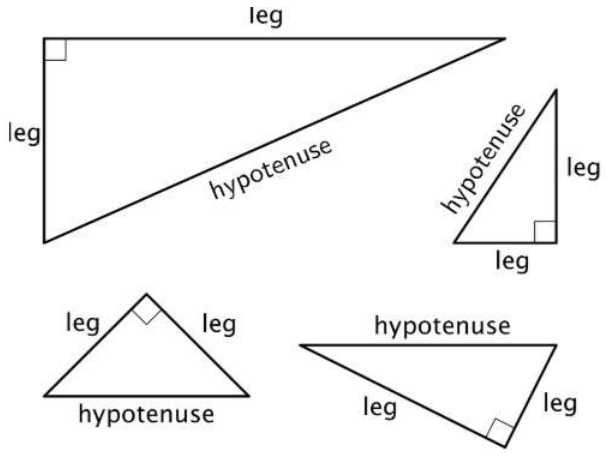


Figure 8.2.1.8

We often use the letters a and b to represent the lengths of the shorter sides of a triangle and c to represent the length of the longest side of a right triangle. If the triangle is a right triangle, then a and b are used to represent the lengths of the legs, and c is used to represent the length of the hypotenuse (since the hypotenuse is always the longest side of a right triangle). For example, in this right triangle, $a = \sqrt{20}$, $b = \sqrt{5}$, and $c = 5$.

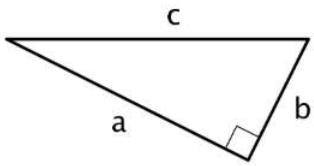


Figure 8.2.1.9

Here are some right triangles:

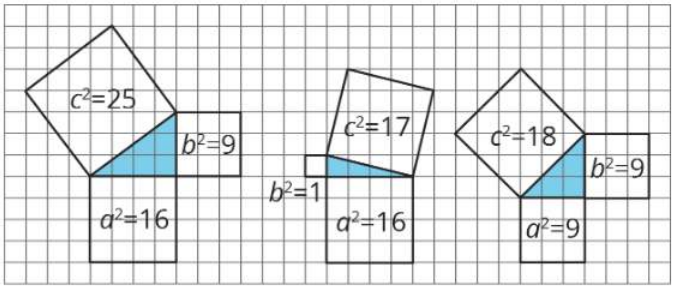


Figure 8.2.1.10

Notice that for these examples of right triangles, the square of the hypotenuse is equal to the sum of the squares of the legs. In the first right triangle in the diagram, $9 + 16 = 25$, in the second, $1 + 16 = 17$, and in the third, $9 + 9 = 18$. Expressed another way, we have $a^2 + b^2 = c^2$. This is a property of all right triangles, not just these examples, and is often known as the **Pythagorean Theorem**. The name comes from a mathematician named Pythagoras who lived in ancient Greece around 2,500 BCE, but this property of right triangles was also discovered independently by mathematicians in other ancient cultures including Babylon, India, and China. In China, a name for the same relationship is the Shang Gao Theorem. In future lessons, you will learn some ways to explain why the Pythagorean Theorem is true for *any* right triangle.

It is important to note that this relationship does not hold for *all* triangles. Here are some triangles that are not right triangles, and notice that the lengths of their sides do not have the special relationship $a^2 + b^2 = c^2$. That is, $16 + 10$ does not equal 18, and $2 + 10$ does not equal 16.

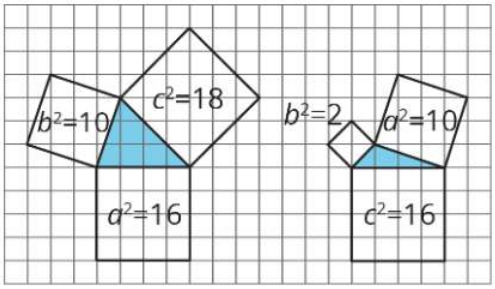


Figure 8.2.1.11

Glossary Entries

Definition: Pythagorean Theorem

The Pythagorean Theorem describes the relationship between the side lengths of right triangles.

The diagram shows a right triangle with squares built on each side. If we add the areas of the two small squares, we get the area of the larger square.

The square of the hypotenuse is equal to the sum of the squares of the legs. This is written as $a^2 + b^2 = c^2$.

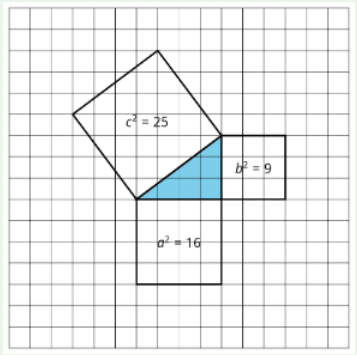


Figure 8.2.1.12

Definition: Hypotenuse

The hypotenuse is the side of a right triangle that is opposite the right angle. It is the longest side of a right triangle.

Here are some right triangles. Each hypotenuse is labeled.

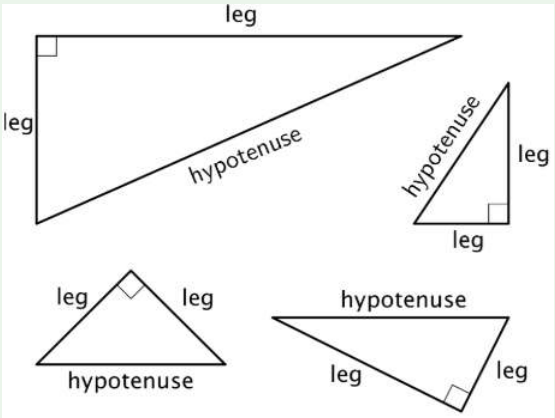


Figure 8.2.1.13

Definition: LEgs

The legs of a right triangle are the sides that make the right angle.

Here are some right triangles. Each leg is labeled.

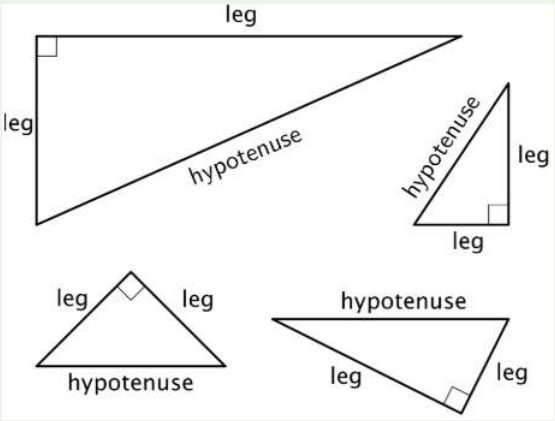


Figure 8.2.1.14

Practice

Exercise 8.2.1.4

Here is a diagram of an acute triangle and three squares.

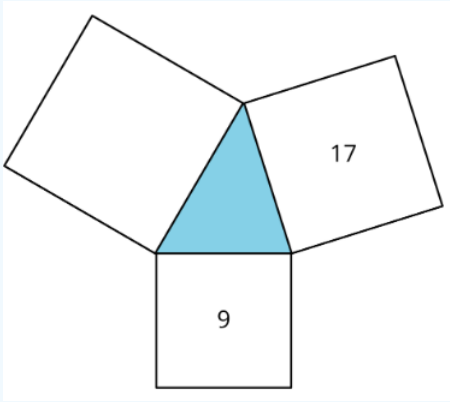


Figure 8.2.1.15: An acute triangle with squares along each side of the triangle. Each square has sides equal to the length of the side of the triangle it touches. The square on the bottom is touching the shortest side and is labeled 9. The square on the top right is touching the next longest side and is labeled 17. The square on the top left is touching the longest side and is unlabeled.

Priya says the area of the large unmarked square is 26 square units because $9+17=26$. Do you agree? Explain your reasoning.

Exercise 8.2.1.5

m, p, and z represent the lengths of the three sides of this right triangle.

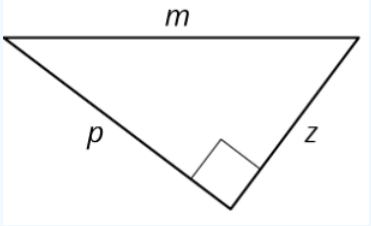


Figure 8.2.1.16

Select **all** the equations that represent the relationship between m, p, and z.

A. $m^2 + p^2 = z^2$
B. $m^2 = p^2 + z^2$
C. $m^2 = z^2 + p^2$
D. $p^2 + m^2 = z^2$
E. $z^2 + p^2 = m^2$
F. $p^2 + z^2 = m^2$

Exercise 8.2.1.6

The lengths of the three sides are given for several right triangles. For each, write an equation that expresses the relationship between the lengths of the three sides.

1. $10, 6, 8$
2. $\sqrt{5}, \sqrt{3}, \sqrt{8}$
3. $5, \sqrt{5}, \sqrt{30}$
4. $1, \sqrt{37}, 6$
5. $3, \sqrt{2}, \sqrt{7}$

Exercise 8.2.1.7

Order the following expressions from least to greatest.

$$25 \div 10 \qquad 250,000 \div 1,000 \qquad 2.5 \div 1,000 \qquad 0.025 \div 1$$

(From Unit 4.1.1)

Exercise 8.2.1.8

What is the best explanation for why $-\sqrt{10}$ is irrational?

A. $-\sqrt{10}$ is irrational because it is not rational.
B. $-\sqrt{10}$ is irrational because it is less than zero.
C. $-\sqrt{10}$ is irrational because it is not a whole number.
D. $-\sqrt{10}$ is irrational because if I put $-\sqrt{10}$ into a calculation, I get -3.16227766 which does not make a repeating pattern.

(From Unit 8.1.3)

Exercise 8.2.1.9

A teacher tells her students she is just over 1 and $\frac{1}{2}$ billion seconds old.

1. Write her age in seconds using scientific notation.
2. What is a more reasonable unit of measurement for this situation?
3. How old is she when you use a more reasonable unit of measurement?

(From Unit 7.3.7)

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8.2.2: A Proof of the Pythagorean Theorem

Lesson

Let's prove the Pythagorean Theorem.

Exercise 8.2.2.1: Notice and Wonder: A Square and Four Triangles

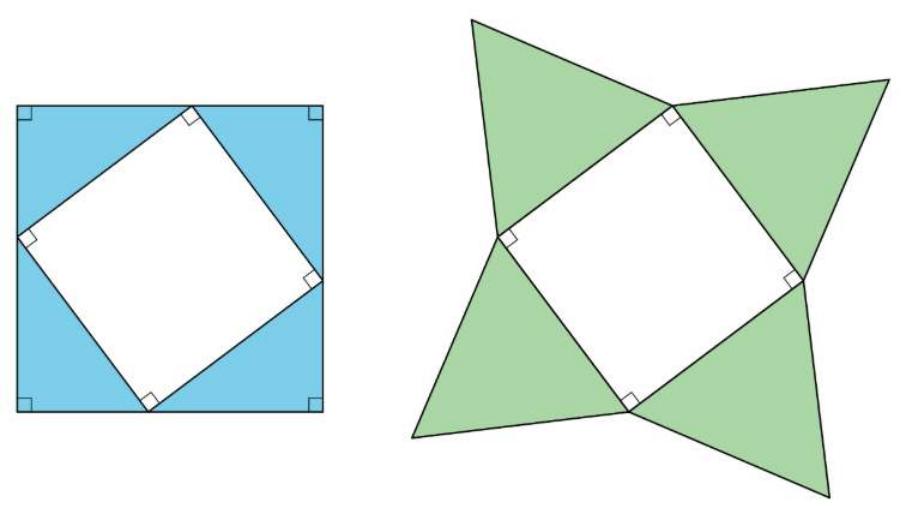


Figure 8.2.2.1

What do you notice? What do you wonder?

Exercise 8.2.2.2: Adding Up Areas

Both figures shown here are squares with a side length of $a+b$. Notice that the first figure is divided into two squares and two rectangles. The second figure is divided into a square and four right triangles with legs of lengths a and b. Let's call the hypotenuse of these triangles c.

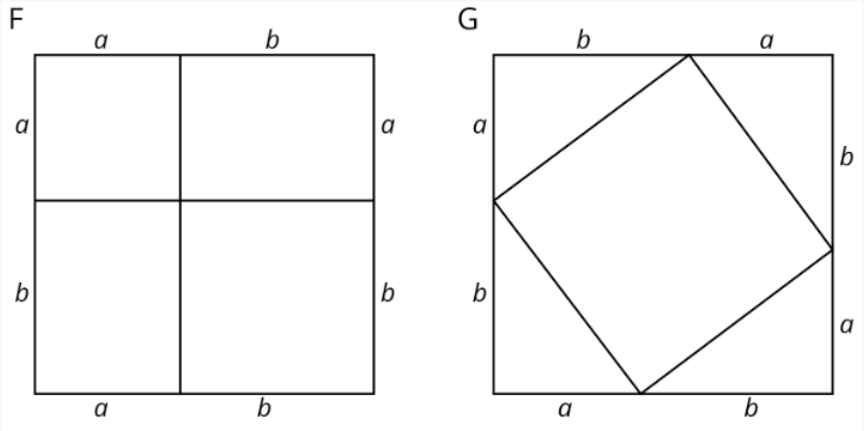


Figure 8.2.2.2

1. What is the total area of each figure?
2. Find the area of each of the 9 smaller regions shown the figures and label them.
3. Add up the area of the four regions in Figure F and set this expression equal to the sum of the areas of the five regions in Figure G. If you rewrite this equation using as few terms as possible, what do you have?

Are you ready for more?

Take a 3-4-5 right triangle, add on the squares of the side lengths, and form a hexagon by connecting vertices of the squares as in the image. What is the area of this hexagon?

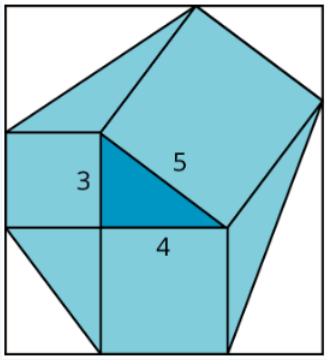


Figure 8.2.2.3

Exercise 8.2.2.3: Let's Take it for a Spin

Find the unknown side lengths in these right triangles.

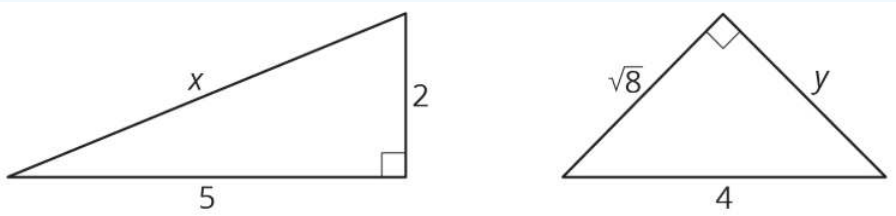


Figure 8.2.2.4

Exercise 8.2.2.4: A Transformational Proof

Use the applets to explore the relationship between areas.

- Consider Squares A and B.
- Check the box to show the pieces.
- Check the box to turn C.
- Arrange the five pieces to fit inside Square C.
- Check the box to see the right triangle.
- Arrange the figures so the squares are adjacent to the sides of the triangle.

1. If the right triangle has legs a and b and hypotenuse c, what have you just demonstrated to be true?
2. Try it again with different squares. Estimage the areas of the new Squares, A, B, and C and explain what you observe.
3. Estiamte the areas of these new Squares, A, B, and C, and then explain what you observe as you complete the activity.
4. What do you think we may be able to conclude?

Summary

The figures shown here can be used to see why the Pythagorean Theorem is true. Both large squares have the same area, but they are broken up in different ways. (Can you see where the triangles in Square G are located in Square F? What does that mean about the smaller squares in F and H?) When the sum of the four areas in Square F are set equal to the sum of the 5 areas in Square G, the result is $a^2 + b^2 = c^2$, where c is the hypotenuse of the triangles in Square G and also the side length of the square in the middle. Give it a try!

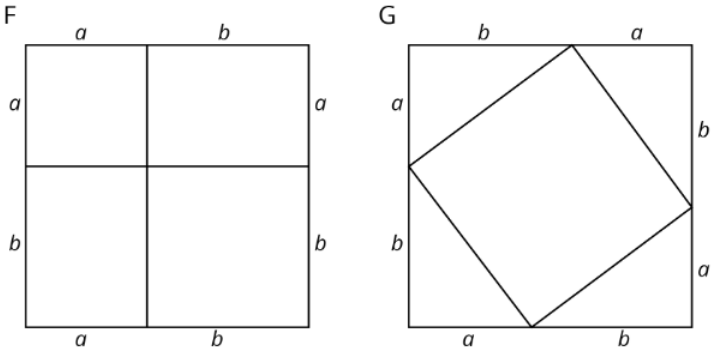


Figure 8.2.2.5

This is true for any right triangle. If the legs are a and b and the hypotenuse is c, then $a^2+b^2=c^2$. This property can be used any time we can make a right triangle. For example, to find the length of this line segment:

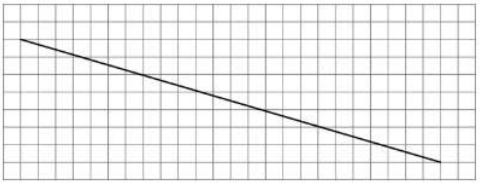


Figure 8.2.2.6

The grid can be used to create a right triangle, where the line segment is the hypotenuse and the legs measure 24 units and 7 units:

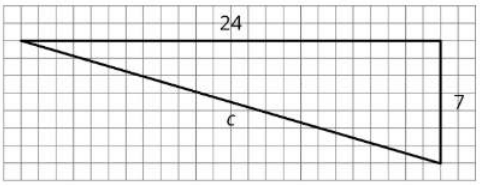


Figure 8.2.2.7

Since this is a right triangle, $24^2+7^2=c^2$. The solution to this equation (and the length of the line segment) is $c=25$.

Glossary Entries

Definition: Pythagorean Theorem

The Pythagorean Theorem describes the relationship between the side lengths of right triangles.

The diagram shows a right triangle with squares built on each side. If we add the areas of the two small squares, we get the area of the larger square.

The square of the hypotenuse is equal to the sum of the squares of the legs. This is written as $a^2+b^2=c^2$.

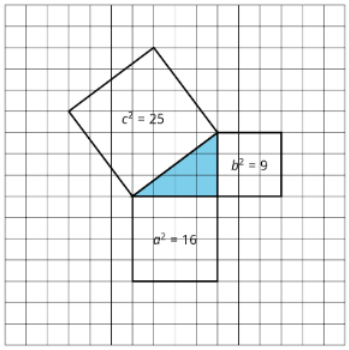


Figure 8.2.2.8

Definition: Hypotenuse

The hypotenuse is the side of a right triangle that is opposite the right angle. It is the longest side of a right triangle.

Here are some right triangles. Each hypotenuse is labeled.

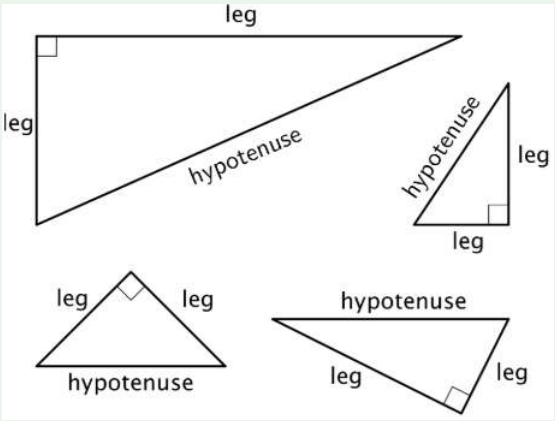


Figure 8.2.2.9

Definition: LEgs

The legs of a right triangle are the sides that make the right angle.

Here are some right triangles. Each leg is labeled.

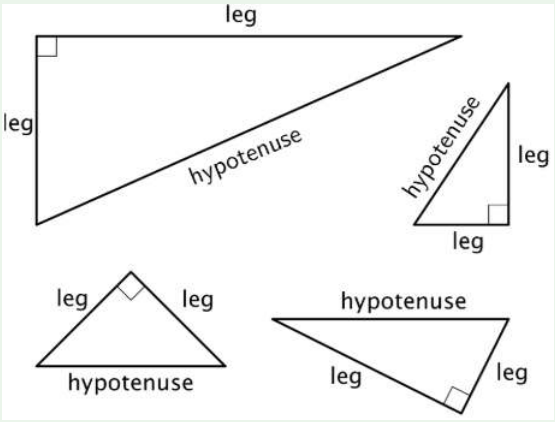


Figure 8.2.2.10

Practice

Exercise 8.2.2.5

1. Find the lengths of the unlabeled sides.

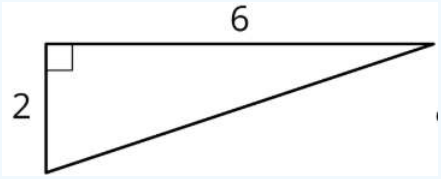


Figure 8.2.2.11

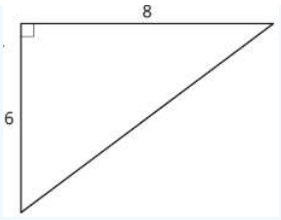


Figure 8.2.2.12

2. One segment is n units long and the other is p units long. Find the value of n and p. (Each small grid square is 1 square unit.)

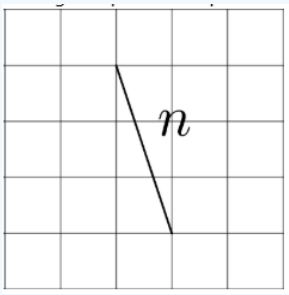


Figure 8.2.2.13

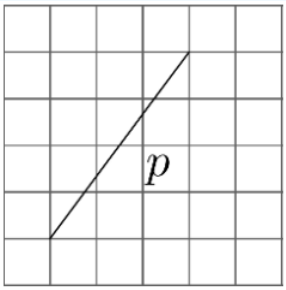


Figure 8.2.2.14

Exercise 8.2.2.6

Use the areas of the two identical squares to explain why $5^2 + 12^2 = 13^2$ without doing any calculations.

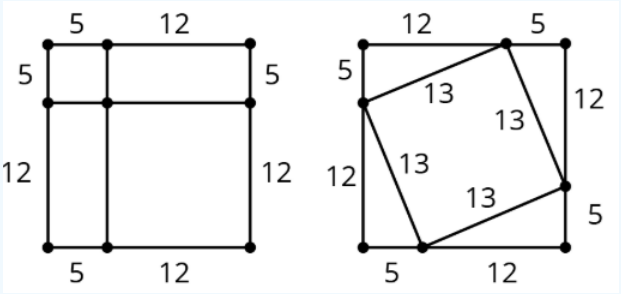


Figure 8.2.2.15

Exercise 8.2.2.7

Each number is between which two consecutive integers?

1. $\sqrt{10}$
2. $\sqrt{54}$
3. $\sqrt{18}$
4. $\sqrt{99}$
5. $\sqrt{41}$

(From Unit 8.1.5)

Exercise 8.2.2.8

1. Give an example of a rational number, and explain how you know it is rational.
2. Give three examples of irrational numbers.

(From Unit 8.1.3)

Exercise 8.2.2.9

Write each expression as a single power of 10.

1. $10^5 \cdot 10^0$
2. $\frac{10^9}{10^0}$

(From Unit 7.2.3)

Exercise 8.2.2.10

Andre is ordering ribbon to make decorations for a school event. He needs a total of exactly 50.25 meters of blue and green ribbon. Andre needs 50% more blue ribbon than green ribbon for the basic design, plus an extra 6.5 meters of blue ribbon for accents. How much of each color of ribbon does Andre need to order?

(From Unit 4.3.6)

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8.2.3: Finding Unknown Side Lengths

Lesson

Let's find missing side lengths of right triangles.

Exercise 8.2.3.1: Which One Doesn't Belong: Equations

Which one doesn't belong?

$3^2 + b^2 = 5^2$

$b^2 = 5^2 - 3^2$

$3^2 + 5^2 = b^2$

$3^2 + 4^2 = 5^2$

Exercise 8.2.3.2: Which One is the Hypotenuse?

Label all the hypotenuses with c.

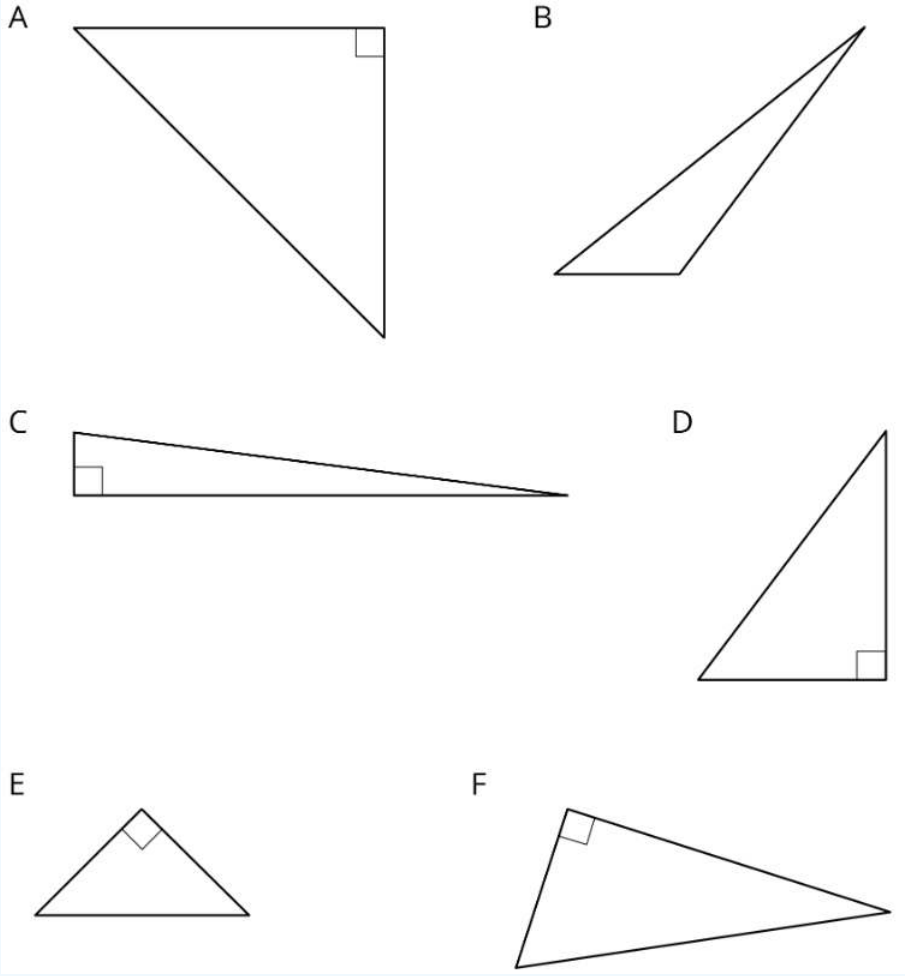


Figure 8.2.3.1

Exercise 8.2.3.3: Find the Missing Side Lengths

1. Find c.

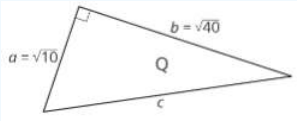


Figure 8.2.3.2

2. Find b.

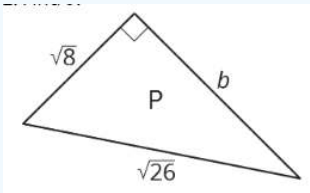


Figure 8.2.3.3

3. A right triangle has sides of length 2.4 cm and 6.5 cm. What is the length of the hypotenuse?

4. A right triangle has a side of length $\frac{1}{4}$ and a hypotenuse of length $\frac{1}{3}$. What is the length of the other side?

5. Find the value of x in the figure.

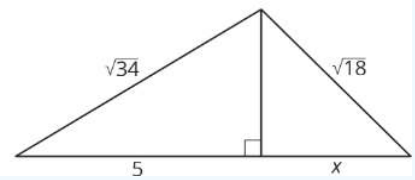


Figure 8.2.3.4

Are you ready for more?

The spiral in the figure is made by starting with a right triangle with both legs measuring one unit each. Then a second right triangle is built with one leg measuring one unit, and the other leg being the hypotenuse of the first triangle. A third right triangle is built on the second triangle's hypotenuse, again with the other leg measuring one unit, and so on.

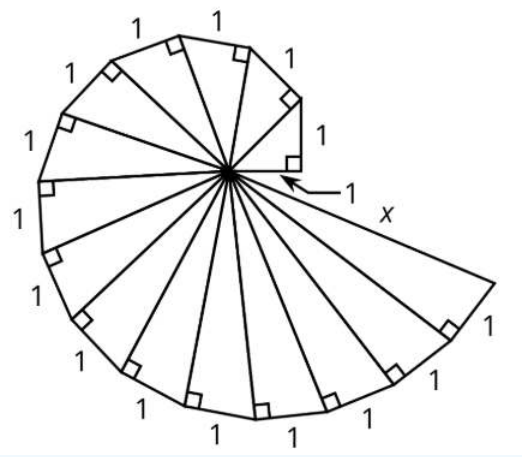


Figure 8.2.3.5

Find the length, x, of the hypotenuse of the last triangle constructed in the figure.

Summary

There are many examples where the lengths of two legs of a right triangle are known and can be used to find the length of the hypotenuse with the Pythagorean Theorem. The Pythagorean Theorem can also be used if the length of the hypotenuse and one leg

is known, and we want to find the length of the other leg. Here is a right triangle, where one leg has a length of 5 units, the hypotenuse has a length of 10 units, and the length of the other leg is represented by g.

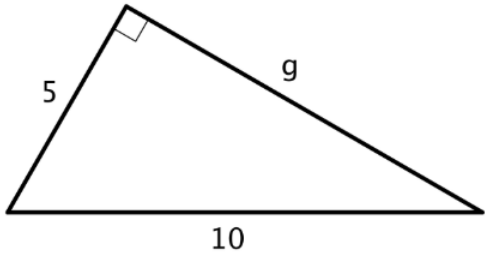


Figure 8.2.3.6

Start with $a^2+b^2=c^2$, make substitutions, and solve for the unknown value. Remember that c represents the hypotenuse: the side opposite the right angle. For this triangle, the hypotenuse is 10.

$$\begin{aligned} a^2+b^2 &= c^2 \\ 5^2+g^2 &= 10^2 \\ g^2 &= 10^2-5^2 \\ g^2 &= 100-25 \\ g^2 &= 75 \\ g &= \sqrt{25} \end{aligned}$$

Use estimation strategies to know that the length of the other leg is between 8 and 9 units, since 75 is between 64 and 81. A calculator with a square root function gives $\sqrt{75} \approx 8.66$.

Glossary Entries

Definition: Pythagorean Theorem

The Pythagorean Theorem describes the relationship between the side lengths of right triangles.

The diagram shows a right triangle with squares built on each side. If we add the areas of the two small squares, we get the area of the larger square.

The square of the hypotenuse is equal to the sum of the squares of the legs. This is written as $a^2+b^2=c^2$.

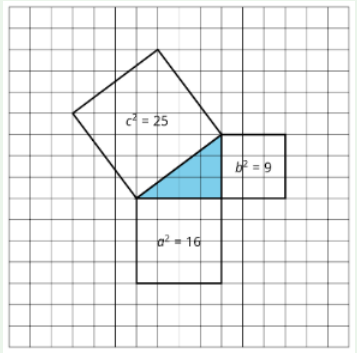


Figure 8.2.3.7

Definition: Hypotenuse

The hypotenuse is the side of a right triangle that is opposite the right angle. It is the longest side of a right triangle.

Here are some right triangles. Each hypotenuse is labeled.

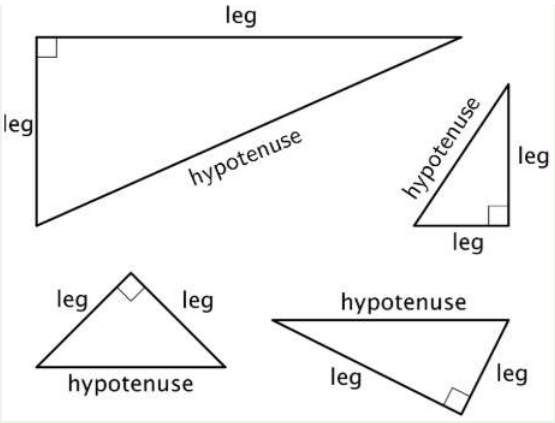


Figure 8.2.3.8

Definition: LEgs

The legs of a right triangle are the sides that make the right angle.

Here are some right triangles. Each leg is labeled.

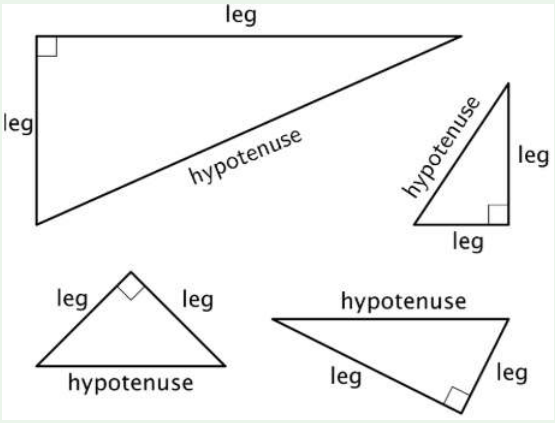


Figure 8.2.3.9

Practice

Exercise 8.2.3.4

Find the exact value of each variable that represents a side length in a right triangle.

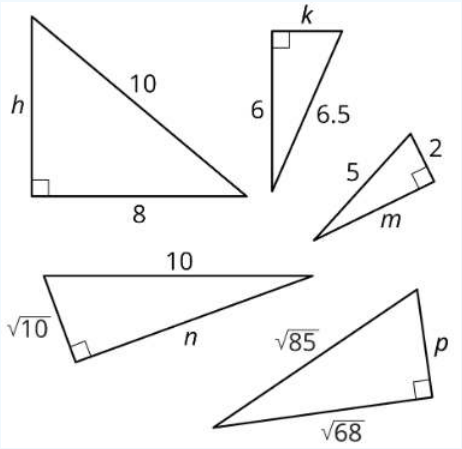


Figure 8.2.3.10

Exercise 8.2.3.5

A right triangle has side lengths of a, b, and c units. The longest side has a length of c units. Complete each equation to show three relations among a, b, and c.

- $c^2 =$
- $a^2 =$
- $b^2 =$

(From Unit 8.2.2)

Exercise 8.2.3.6

What is the exact length of each line segment? Explain or show your reasoning. (Each grid square represents 1 square unit.)

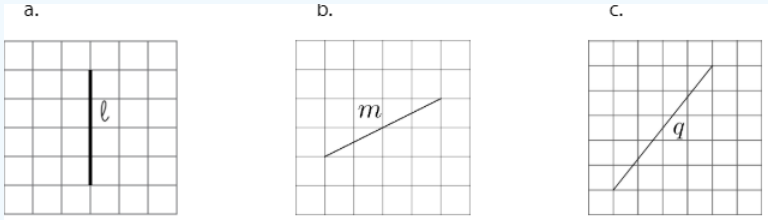


Figure 8.2.3.11

(From Unit 8.2.2)

Exercise 8.2.3.7

In 2015, there were roughly 1×10^6 high school football players and 2×10^3 professional football players in the United States. About how many times more high school football players are there? Explain how you know.

(From Unit 7.3.7)

Exercise 8.2.3.8

Evaluate:

1. $\left(\frac{1}{2}\right)^3$
2. $\left(\frac{1}{2}\right)^{-3}$

(From Unit 7.2.5)

Exercise 8.2.3.9

Here is a scatter plot of weight vs. age for different Dobermans. The model, represented by $y = 2.45x + 1.22$, is graphed with the scatter plot. Here, x represents age in weeks, and represents weight in pounds.

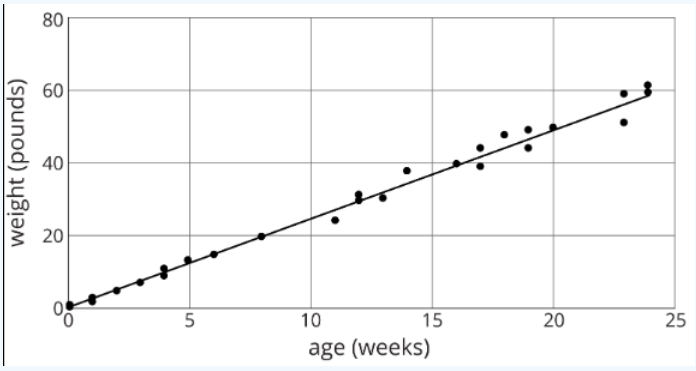


Figure 8.2.3.12

1. What does the slope mean in this situation?
2. Based on this model, how heavy would you expect a newborn Doberman to be?

(From Unit 6.2.4)

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8.2.4: The Converse

Lesson

Let's figure out if a triangle is a right triangle.

Exercise 8.2.4.1: The Hands of a Clock

Consider the tips of the hands of an analog clock that has an hour hand that is 3 centimeters long and a minute hand that is 4 centimeters long.

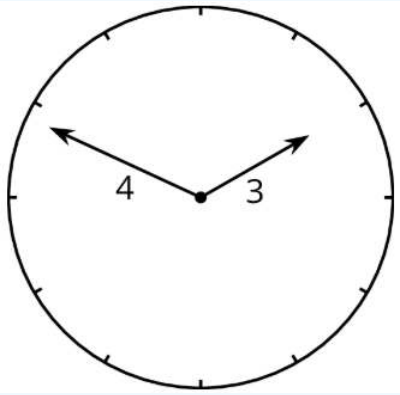


Figure 8.2.4.1: The image of a circle that represent an analog clock. On the circle are 12 evenly spaced tick marks. There are two hands on the clock. One hand is labeled 3, begins in the center of the circle and extends upward and to the right, and points to the third tick mark from the top. The other hand is labeled 4, begins in the center of the circle and extends upward and to the left. It points to the eleventh tick mark from the top.

Over the course of a day:

1. What is the farthest apart the two tips get?
2. What is the closest the two tips get?
3. Are the two tips ever exactly five centimeters apart?

Exercise 8.2.4.2: Proving the Converse

Here are three triangles with two side lengths measuring 3 and 4 units, and the third side of unknown length.

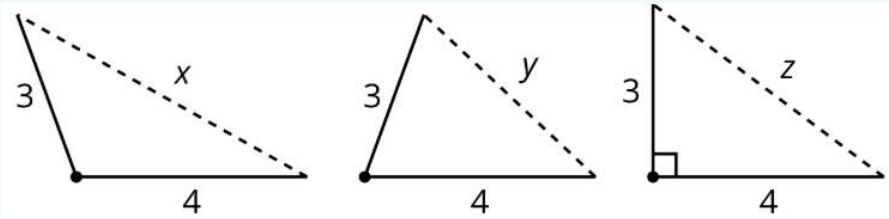


Figure 8.2.4.2: A figure of three triangles each with 2 given side lengths and one unknown side length. The first triangle has a horizontal side of 4, a side length slanted upward and to the left of 3, and the third side length labeled x. The middle triangle has a horizontal side length of 4, a second side length slanted upward and to the right of 3, and the third side length labeled y. The third triangle is a right triangle with a horizontal side length of 4, a vertical side length of 3, and the third side is labeled z.

Sort the following six numbers from smallest to largest. Put an equal sign between any you know to be equal. Be ready to explain your reasoning.

$$1 \quad 5 \quad 7 \quad x \quad y \quad z$$

Are you ready for more?

A related argument also lets us distinguish acute from obtuse triangles using only their side lengths.

Decide if triangles with the following side lengths are acute, right, or obtuse. In right or obtuse triangles, identify which side length is opposite the right or obtuse angle.

- $x = 15, \quad y = 20, \quad z = 8$
- $x = 8, \quad y = 15, \quad z = 13$
- $x = 17, \quad y = 8, \quad z = 15$

Exercise 8.2.4.3: Calculating Legs of Right Triangles

1. Given the information provided for the right triangles shown here, find the unknown leg lengths to the nearest tenth.

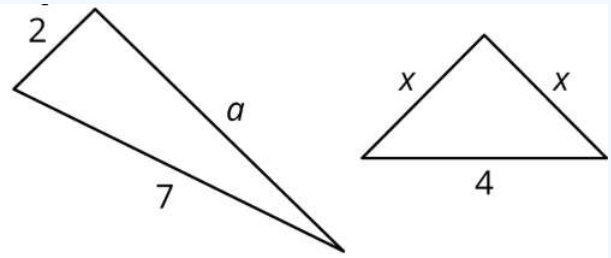


Figure 8.2.4.3: Two right triangles are indicated. The triangle on the left has two leg with lengths of 2 and a. The hypotenuse has a length of 7. The triangle on the right has two legs with length x and a hypotenuse of length 4.

2. The triangle shown here is not a right triangle. What are two different ways you change *one* of the values so it would be a right triangle? Sketch these new right triangles, and clearly label the right angle.

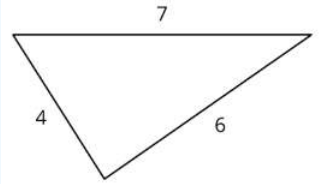


Figure 8.2.4.4

Summary

What if it isn't clear whether a triangle is a right triangle or not? Here is a triangle:

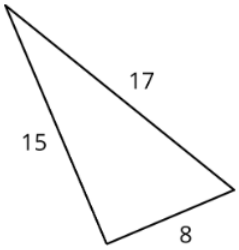


Figure 8.2.4.5

Is it a right triangle? It's hard to tell just by looking, and it may be that the sides aren't drawn to scale.

If we have a triangle with side lengths a, b, and c, with c being the longest of the three, then the *converse* of the Pythagorean Theorem tells us that any time we have $a^2 + b^2 = c^2$, we *must* have a right triangle. Since $8^2 + 15^2 = 64 + 225 = 289 = 17^2$, any triangle with side lengths 8, 15, and 17 *must* be a right triangle.

Together, the Pythagorean Theorem and its converse provide a one-step test for checking to see if a triangle is a right triangle just using its side lengths. If $a^2 + b^2 = c^2$, it is a right triangle. If $a^2 + b^2 \neq c^2$, it is not a right triangle.

Glossary Entries

Definition: Pythagorean Theorem

The Pythagorean Theorem describes the relationship between the side lengths of right triangles.

The diagram shows a right triangle with squares built on each side. If we add the areas of the two small squares, we get the area of the larger square.

The square of the hypotenuse is equal to the sum of the squares of the legs. This is written as $a^2 + b^2 = c^2$.

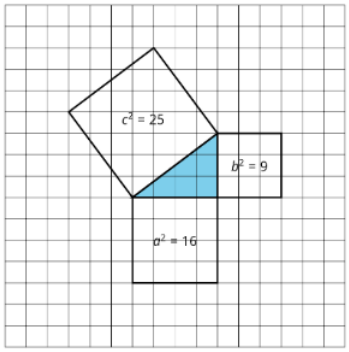


Figure 8.2.4.6

Definition: Hypotenuse

The hypotenuse is the side of a right triangle that is opposite the right angle. It is the longest side of a right triangle.

Here are some right triangles. Each hypotenuse is labeled.

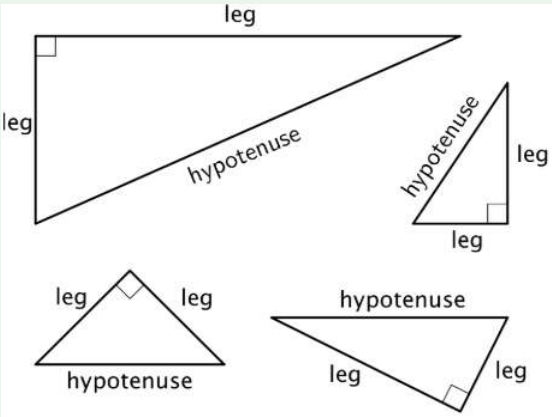


Figure 8.2.4.7

Definition: LEgs

The legs of a right triangle are the sides that make the right angle.

Here are some right triangles. Each leg is labeled.

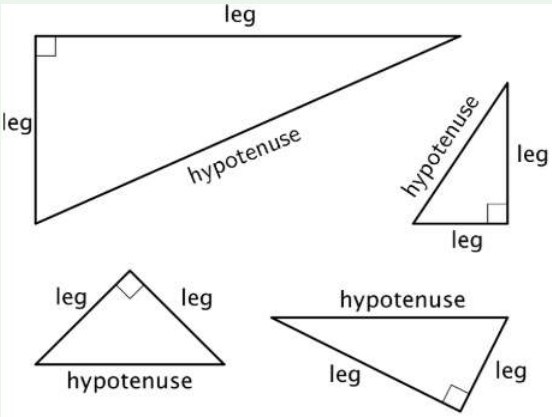


Figure 8.2.4.8

Practice

Exercise 8.2.4.4

Which of these triangles are definitely right triangles? Explain how you know. (Note that not all triangles are drawn to scale.)

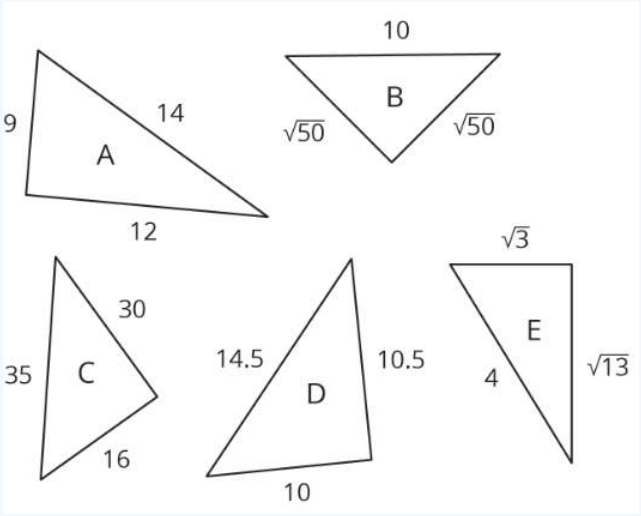


Figure 8.2.4.9

Exercise 8.2.4.5

A right triangle has a hypotenuse of 15 cm. What are possible lengths for the two legs of the triangle? Explain your reasoning.

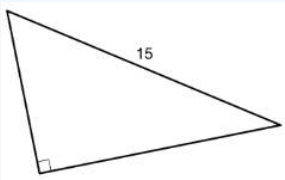


Figure 8.2.4.10

Exercise 8.2.4.6

In each part, a and b represent the length of a leg of a right triangle, and c represents the length of its hypotenuse. Find the missing length, given the other two lengths.

1. $a = 12, \quad b = 5, \quad c = ?$
2. $a = ?, \quad b = 21, \quad c = 29$

(From Unit 8.2.3)

Exercise 8.2.4.7

For which triangle does the Pythagorean Theorem express the relationship between the lengths of its three sides?

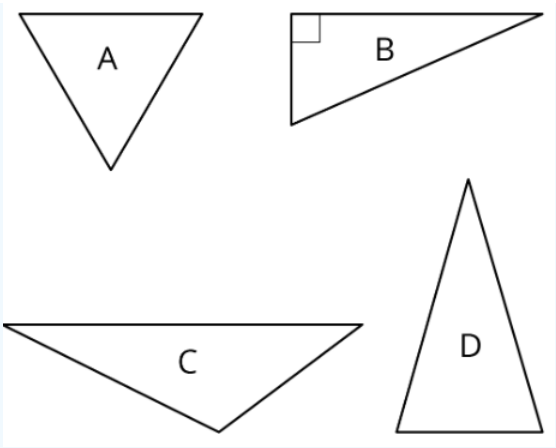


Figure 8.2.4.11

(From Unit 8.2.1)

Exercise 8.2.4.8

Andre makes a trip to Mexico. He exchanges some dollars for pesos at a rate of 20 pesos per dollar. While in Mexico, he spends 9000 pesos. When he returns, he exchanges his pesos for dollars (still at 20 pesos per dollar). He gets back $\frac{1}{10}$ the amount he started with. Find how many dollars Andre exchanged for pesos and explain your reasoning. If you get stuck, try writing an equation representing Andre's trip using a variable for the number of dollars he exchanged.

(From Unit 4.2.4)

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8.2.5: Applications of the Pythagorean Theorem

Lesson

Let's explore some applications of the Pythagorean Theorem.

Exercise 8.2.5.1: Closest Estimate: Square Roots

Which estimate is closest to the actual value of the expression? Explain your reasoning.

1. $\sqrt{24}$
   - 4
   - 4.5
   - 5
2. $\sqrt{7}$
   - 2
   - 2.5
   - 3
3. $\sqrt{42}$
   - 6
   - 6.5
   - 7
4. $\sqrt{10}+\sqrt{97}$
   - 13
   - 13.25
   - 13.5

Exercise 8.2.5.2: Cutting Corners

Mai and Tyler were standing at one corner of a large rectangular field and decided to race to the opposite corner. Since Mai had a bike and Tyler did not, they thought it would be a fairer race if Mai rode along the sidewalk that surrounds the field while Tyler ran the shorter distance directly across the field. The field is 100 meters long and 80 meters wide. Tyler can run at around 5 meters per second, and Mai can ride her bike at around 7.5 meters per second.

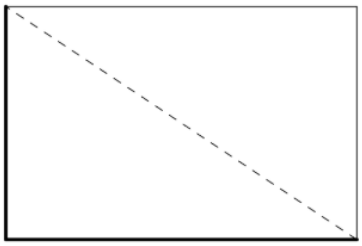


Figure 8.2.5.1

1. Before making any calculations, who do you think will win? By how much? Explain your thinking.
2. Who wins? Show your reasoning.

Are you ready for more?

A calculator may be necessary to answer the following questions. Round answers to the nearest hundredth.

1. If you could give the loser of the race a head start, how much time would they need in order for both people to arrive at the same time?
2. If you could make the winner go slower, how slow would they need to go in order for both people to arrive at the same time?

Exercise 8.2.5.3: Internal Dimensions

Here are two rectangular prisms:

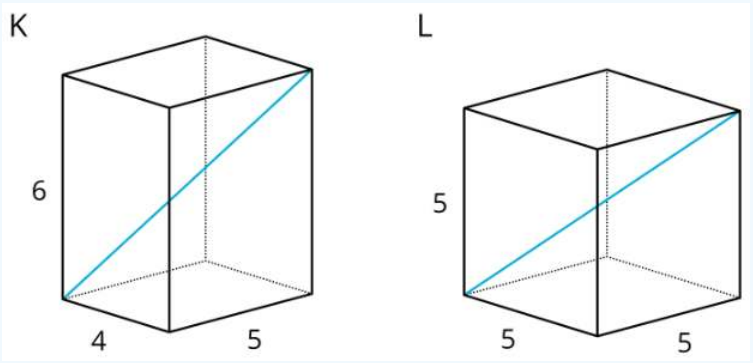


Figure 8.2.5.2

1. Which figure do you think has the longer diagonal? Note that the figures are not drawn to scale.
2. Calculate the lengths of both diagonals. Which one is actually longer?

Summary

The Pythagorean Theorem can be used to solve any problem that can be modeled with a right triangle where the lengths of two sides are known and the length of the other side needs to be found. For example, let's say a cable is being placed on level ground to support a tower. It's a 17-foot cable, and the cable should be connected 15 feet up the tower. How far away from the bottom of the tower should the other end of the cable connect to the ground?

It is often very helpful to draw a diagram of a situation, such as the one shown here:

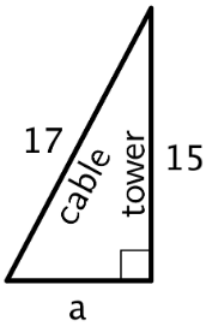


Figure 8.2.5.3

It's assumed that the tower makes a right angle with the ground. Since this is a right triangle, the relationship between its sides is $a^2+b^2=c^2$, where c represents the length of the hypotenuse and a and b represent the lengths of the other two sides. The hypotenuse is the side opposite the right angle. Making substitutions gives $a^2+15^2=17^2$. Solving this for a gives $a=8$. So, the other end of the cable should connect to the ground 8 feet away from the bottom of the tower.

Glossary Entries

Definition: Pythagorean Theorem

The Pythagorean Theorem describes the relationship between the side lengths of right triangles.

The diagram shows a right triangle with squares built on each side. If we add the areas of the two small squares, we get the area of the larger square.

The square of the hypotenuse is equal to the sum of the squares of the legs. This is written as $a^2+b^2=c^2$.

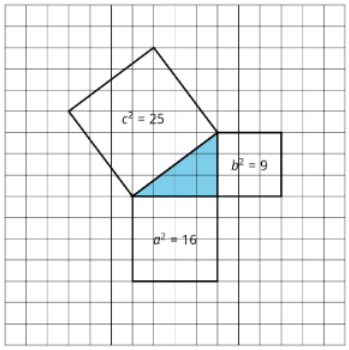


Figure 8.2.5.4

Definition: Hypotenuse

The hypotenuse is the side of a right triangle that is opposite the right angle. It is the longest side of a right triangle.

Here are some right triangles. Each hypotenuse is labeled.

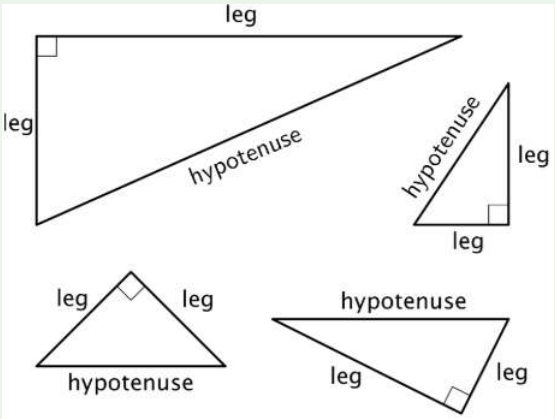


Figure 8.2.5.5

Definition: LEgs

The legs of a right triangle are the sides that make the right angle.

Here are some right triangles. Each leg is labeled.

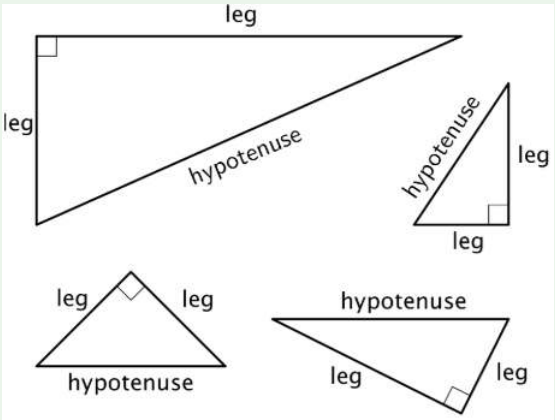


Figure 8.2.5.6

Practice

Exercise 8.2.5.4

A man is trying to zombie-proof his house. He wants to cut a length of wood that will brace a door against a wall. The wall is 4 feet away from the door, and he wants the brace to rest 2 feet up the door. About how long should he cut the brace?

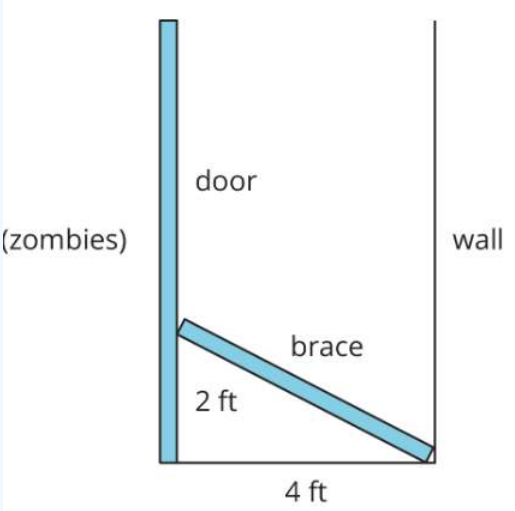


Figure 8.2.5.7

Exercise 8.2.5.5

At a restaurant, a trash can's opening is rectangular and measures 7 inches by 9 inches. The restaurant serves food on trays that measure 12 inches by 16 inches. Jada says it is impossible for the tray to accidentally fall through the trash can opening because the shortest side of the tray is longer than either edge of the opening. Do you agree or disagree with Jada's explanation? Explain your reasoning.

Exercise 8.2.5.6

Select **all** the sets that are the three side lengths of right triangles.

A. $8, 7, 15$
B. $4, 10, \sqrt{84}$
C. $\sqrt{8}, 11, \sqrt{129}$
D. $\sqrt{1}, 2, \sqrt{3}$

(From Unit 8.2.4)

Exercise 8.2.5.7

For each pair of numbers, which of the two numbers is larger? How many times larger?

1. $12 \cdot 10^9$ and $4 \cdot 10^9$
2. $1.5 \cdot 10^{12}$ and $3 \cdot 10^{12}$
3. $20 \cdot 10^4$ and $6 \cdot 10^5$

(From Unit 7.3.2)

Exercise 8.2.5.8

A line contains the point $(3, 5)$. If the line has negative slope, which of these points could also be on the line?

A. $(2, 0)$
B. $(4, 7)$
C. $(5, 4)$

D. $(6, 5)$

(From Unit 3.3.2)

Exercise 8.2.5.9

Noah and Han are preparing for a jump rope contest. Noah can jump 40 times in 0.5 minutes. Han can jump y times in x minutes, where $y = 78x$. If they both jump for 2 minutes, who jumps more times? How many more?

(From Unit 3.1.4)

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8.2.6: Finding Distances in the Coordinate Plane

Lesson

Let's find distances in the coordinate plane.

Exercise 8.2.6.1: Closest Distance

1. Order the following pairs of coordinates from closest to farthest apart. Be prepared to explain your reasoning.
   a. $(2, 4)$ and $(2, 10)$
   b. $(-3, 6)$ and $(5, 6)$
   c. $(-12, -12)$ and $(-12, -1)$
   d. $(7, 0)$ and $(7, -9)$
   e. $(1, -10)$ and $(-4, -10)$
2. Name another pair of coordinates that would be closer together than the first pair on your list.
3. Name another pair of coordinates that would be farther apart than the last pair on your list.

Exercise 8.2.6.2: How Far Apart?

Find the distances between the three points shown.

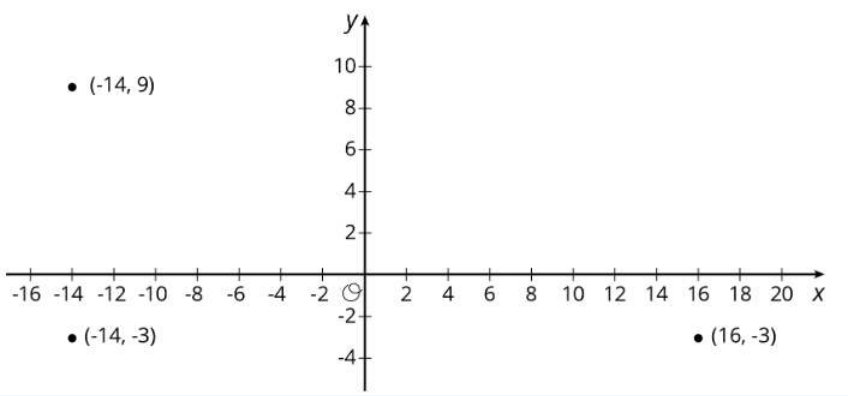


Figure 8.2.6.1

Exercise 8.2.6.3: Perimeters with Pythagoras

1. Which figure do you think has the longer perimeter?
2. Select one figure and calculate its perimeter. Your partner will calculate the perimeter of the other. Were you correct about which figure had the longer perimeter?

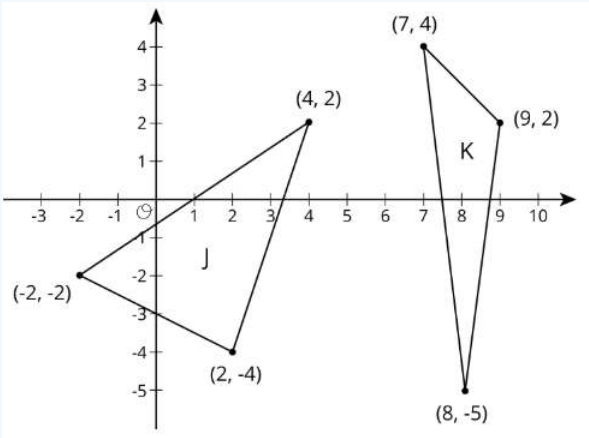


Figure 8.2.6.2

Are you ready for more?

Quadrilateral $ABCD$ has vertices at $A = (-5, 1)$, $B = (-4, 4)$, $C = (2, 2)$, and $D = (1, -1)$.

1. Use the Pythagorean Theorem to find the lengths of sides AB, BC, CD, and AD.
2. Use the Pythagorean Theorem to find the lengths of the two diagonals, AC and BD.
3. Explain why quadrilateral $ABCD$ is a rectangle.

Exercise 8.2.6.4: Finding the Right Distance

Have each person in your group select one of the sets of coordinate pairs shown here. Then calculate the length of the line segment between those two coordinates. Once the values are calculated, have each person in the group briefly share how they did their calculations.

- $(-3, 1)$ and $(5, 7)$
- $(-1, -6)$ and $(5, 2)$
- $(-1, -2)$ and $(5, -6)$
- $(-2, -5)$ and $(6, 1)$

1. How does the value you found compare to the rest of your group?
2. In your own words, write an explanation to another student for how to find the distance between any two coordinate pairs.

Summary

We can use the Pythagorean Theorem to find the distance between any two points on the coordinate plane. For example, if the coordinates of point A are $(-2, -3)$, and the coordinates of point B are $(-8, 4)$, let's find the distance between them. This distance is also the length of line segment AB. It is a good idea to plot the points first.

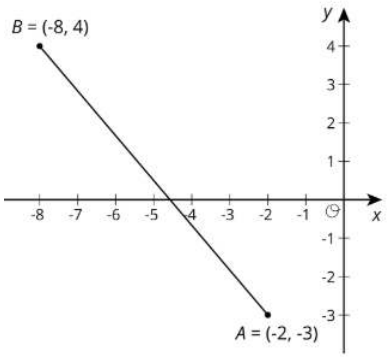


Figure 8.2.6.3

Think of the distance between A and B, or the length of segment AB, as the hypotenuse of a right triangle. The lengths of the legs can be deduced from the coordinates of the points.

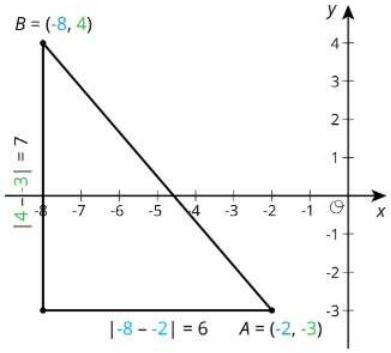


Figure 8.2.6.4

The length of the horizontal leg is 6, which can be seen in the diagram, but it is also the distance between the x-coordinates of A and B since $|-8 - -2| = 6$. The length of the vertical leg is 7, which can be seen in the diagram, but this is also the distance

between the y-coordinates of A and B since $|-4--3|=7$.

Once the lengths of the legs are known, we use the Pythagorean Theorem to find the length of the hypotenuse, AB, which we can represent with c. Since c is a positive number, there is only one value it can take:

$$\begin{aligned} 6^2+7^2 &= c^2 \\ 36+49 &= c^2 \\ 85 &= c^2 \\ \sqrt{85} &= c \end{aligned}$$

This length is a little longer than 9, since 85 is a little longer than 81. Using a calculator gives a more precise answer, $\sqrt{85} \approx 9.22$.

Glossary Entries

Definition: Pythagorean Theorem

The Pythagorean Theorem describes the relationship between the side lengths of right triangles.

The diagram shows a right triangle with squares built on each side. If we add the areas of the two small squares, we get the area of the larger square.

The square of the hypotenuse is equal to the sum of the squares of the legs. This is written as $a^2+b^2=c^2$.

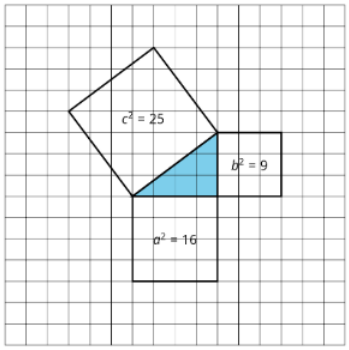


Figure 8.2.6.5

Definition: Hypotenuse

The hypotenuse is the side of a right triangle that is opposite the right angle. It is the longest side of a right triangle.

Here are some right triangles. Each hypotenuse is labeled.

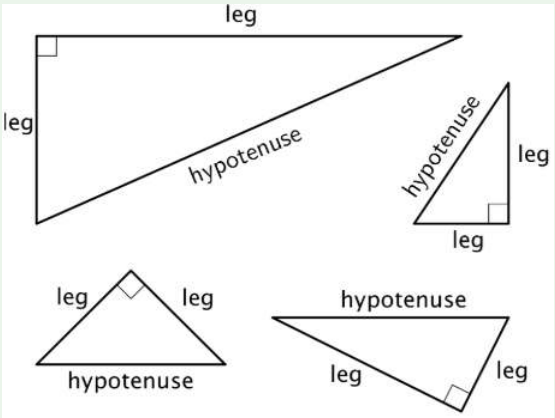


Figure 8.2.6.6

Definition: LEgs

The legs of a right triangle are the sides that make the right angle.

Here are some right triangles. Each leg is labeled.

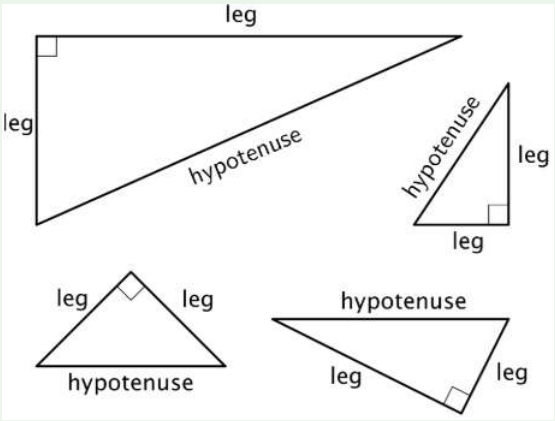


Figure 8.2.6.7

Practice

Exercise 8.2.6.5

The right triangles are drawn in the coordinate plane, and the coordinates of their vertices are labeled. For each right triangle, label each leg with its length.

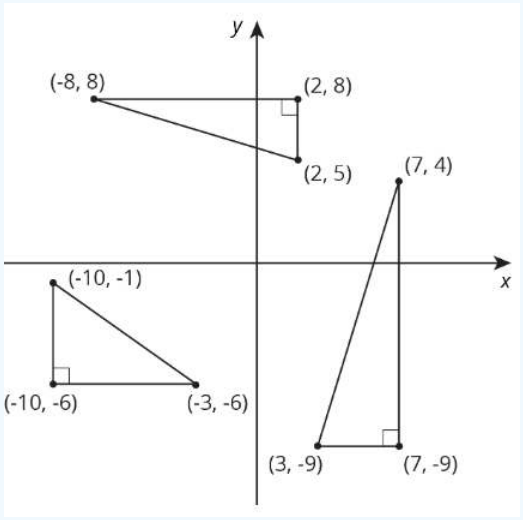


Figure 8.2.6.8

Exercise 8.2.6.6

Find the distance between each pair of points. If you get stuck, try plotting the points on graph paper.

1. $M = (0, -11)$ and $P = (0, 2)$
2. $A = (0, 0)$ and $B = (-3, -4)$
3. $C = (9, 0)$ and $D = (0, -6)$

Exercise 8.2.6.7

1. Find an object that contains a right angle. This can be something in nature or something that was made by humans or machines.
2. Measure the two sides that make the right angle. Then measure the distance from the end of one side to the end of the other.
3. Draw a diagram of the object, including the measurements.
4. Use the Pythagorean Theorem to show that your object really does have a right angle.

(From Unit 8.2.4)

Exercise 8.2.6.8

Which line has a slope of 0.625, and which line has a slope of 1.6? Explain why the slopes of these lines are 0.625 and 1.6.

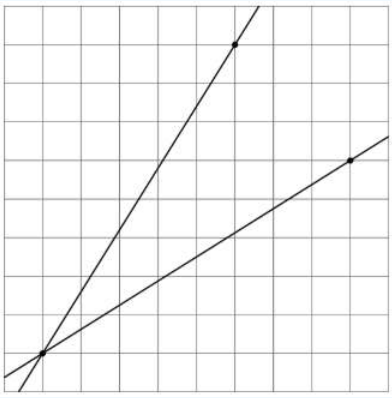


Figure 8.2.6.9

(From Unit 2.3.1)

Exercise 8.2.6.9

Write an equation for the graph.

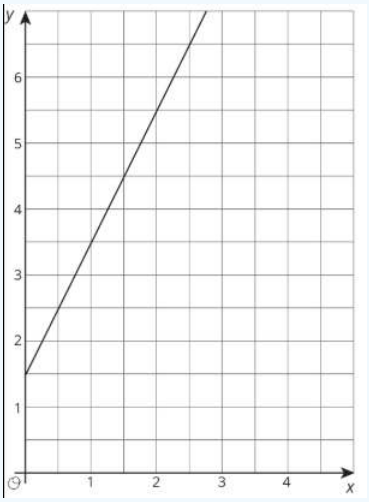


Figure 8.2.6.10

(From Unit 3.2.3)

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8.3: Side Lengths and Volumes of Cubes

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8.3.1: Edge Lengths and Volumes

Lesson

Let's explore the relationship between volume and edge lengths of cubes.

Exercise 8.3.1.1: Ordering Squares and Cubes

Let a, b, c, d, e, and f be positive numbers.

Given these equations, arrange a, b, c, d, e, and f from least to greatest. Explain your reasoning.

- $a^2 = 9$
- $b^3 = 8$
- $c^2 = 10$
- $d^3 = 9$
- $e^2 = 8$
- $f^3 = 7$

Exercise 8.3.1.2: Name That Edge Length!

Fill in the missing values using the information provided:

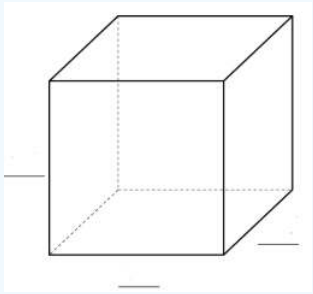


Figure 8.3.1.1

Table 8.3.1.1

sides	volume	volume equation
	$27\ \text{in}^3$	
$\sqrt[3]{5}$		
		$\left(\sqrt[3]{16}\right)^3 = 16$

Are you ready for more?

A cube has a volume of 8 cubic centimeters. A square has the same value for its area as the value for the surface area of the cube. How long is each side of the square?

Exercise 8.3.1.3: Card Sort: Rooted in the Number Line

Your teacher will give your group a set of cards. For each card with a letter and value, find the two other cards that match. One shows the location on a number line where the value exists, and the other shows an equation that the value satisfies. Be prepared to explain your reasoning.

Summary

To review, the side length of the square is the square root of its area. In this diagram, the square has an area of 16 units and a side length of 4 units.

These equations are both true: $4^2 = 16$ $\sqrt{16} = 4$

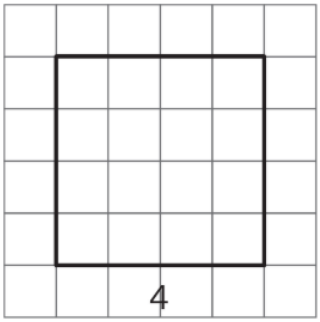


Figure 8.3.1.2

Now think about a solid cube. The cube has a volume, and the edge length of the cube is called the **cube root** of its volume. In this diagram, the cube has a volume of 64 units and an edge length of 4 units:

These equations are both true:

$4^3 = 64$

$\sqrt[3]{64} = 4$

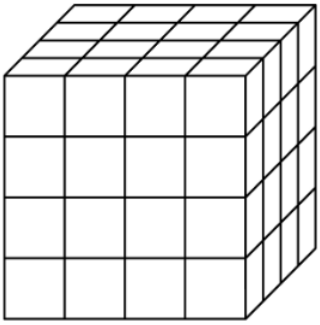


Figure 8.3.1.3

$\sqrt[3]{64}$ is pronounced "The cube root of 64." Here are some other values of cube roots:

$$\sqrt[3]{8} = 2\text{, because } 2^3 = 8$$

$$\sqrt[3]{27} = 3\text{, because } 3^3 = 27$$

$$\sqrt[3]{125} = 5\text{, because } 5^3 = 125$$

Glossary Entries

Definition: Cube Root

The cube root of a number n is the number whose cube is n. It is also the edge length of a cube with a volume of n. We write the cube root of n as $\sqrt[3]{n}$.

For example, the cube root of 64, written as $\sqrt[3]{64}$, is 4 because 4^3 is 64. $\sqrt[3]{64}$ is also the edge length of a cube that has a volume of 64.

Practice

Exercise 8.3.1.4

1. What is the volume of a cube with a side length of
   a. 4 centimeters?
   b. $\sqrt[3]{11}$ feet?
   c. s units?
2. What is the side length of a cube with a volume of
   a. $1,000$ cubic centimeters?

b. 23 cubic inches?
c. v cubic units?

Exercise 8.3.1.5

Write an equivalent expression that doesn't use a cube root symbol.

1. $\sqrt[3]{1}$
2. $\sqrt[3]{216}$
3. $\sqrt[3]{8000}$
4. $\sqrt[3]{\frac{1}{64}}$
5. $\sqrt[3]{\frac{27}{125}}$
6. $\sqrt[3]{0.027}$
7. $\sqrt[3]{0.000125}$

Exercise 8.3.1.6

Find the distance between each pair of points. If you get stuck, try plotting the points on graph paper.

1. $X = (5, 0)$ and $Y = (-4, 0)$
2. $K = (-21, -29)$ and $L = (0, 0)$

(From Unit 8.2.6)

Exercise 8.3.1.7

Here is a 15-by-8 rectangle divided into triangles. Is the shaded triangle a right triangle? Explain or show your reasoning.

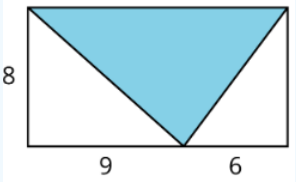


Figure 8.3.1.4

(From Unit 8.2.4)

Exercise 8.3.1.8

Here is an equilateral triangle. The length of each side is 2 units. A height is drawn. In an equilateral triangle, the height divides the opposite side into two pieces of equal length.

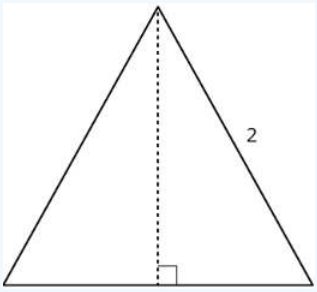


Figure 8.3.1.5

1. Find the exact height.
2. Find the area of the equilateral triangle.
3. (Challenge) Using x for the length of each side in an equilateral triangle, express its area in terms of x.

(From Unit 8.2.5)

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8.3.2: Cube Roots

Lesson

Let's compare cube roots.

Exercise 8.3.2.1: True or False: Cubed

Decide if each statement is true or false.

$\left(\sqrt[3]{5}\right)^3 = 5$

$\left(\sqrt[3]{27}\right)^3 = 3$

$7 = \left(\sqrt[3]{7}\right)^3$

$\left(\sqrt[3]{10}\right)^3 = 1,000$

$\left(\sqrt[3]{64}\right) = 2^3$

Exercise 8.3.2.2: Cube Root Values

What two whole numbers does each cube root lie between? Be prepared to explain your reasoning.

1. $\sqrt[3]{5}$
2. $\sqrt[3]{23}$
3. $\sqrt[3]{81}$
4. $\sqrt[3]{999}$

Exercise 8.3.2.3: Solutions on a Number Line

The numbers x, y, and z are positive, and:

$x^3 = 5 \qquad y^3 = 27 \qquad z^3 = 700$

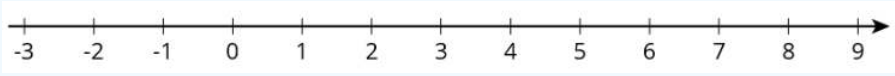


Figure 8.3.2.1

1. Plot x, y, and z on the number line. Be prepared to share your reasoning with the class.
2. Plot $-\sqrt[3]{2}$ on the number line.

Are you ready for more?

Diego knows that $8^2 = 64$ and that $4^3 = 64$. He says that this means the following are all true:

- $\sqrt{64} = 8$
- $\sqrt[3]{64} = 4$
- $\sqrt{-64}$
- $\sqrt[3]{-64}$

Is he correct? Explain how you know.

Summary

Remember that square roots of whole numbers are defined as side lengths of squares. For example, $\sqrt{17}$ is the side length of a square whose area is 17. We define cube roots similarly, but using cubes instead of squares. The number $\sqrt[3]{17}$, pronounced "the cube root of 17," is the edge length of a cube which has a volume of 17.

We can approximate the values of cube roots by observing the whole numbers around it and remembering the relationship between cube roots and cubes. For example, $\sqrt[3]{20}$ is between 2 and 3 since $2^3 = 8$ and $3^3 = 27$, and 20 is between 8 and 27. Similarly, since 100 is between 4^3 and 5^3, we know $\sqrt[3]{100}$ is between 4 and 5. Many calculators have a cube root function which can be used

to approximate the value of a cube root more precisely. Using our numbers from before, a calculator will show that $\sqrt[3]{20} \approx 2.7144$ and that $\sqrt[3]{100} \approx 4.6416$.

Also like square roots, most cube roots of whole numbers are irrational. The only time the cube root of a number is a whole number is when the original number is a perfect cube.

Glossary Entries

Definition: Cube Root

The cube root of a number n is the number whose cube is n. It is also the edge length of a cube with a volume of n. We write the cube root of n as $\sqrt[3]{n}$.

For example, the cube root of 64, written as $\sqrt[3]{64}$, is 4 because 4^3 is 64. $\sqrt[3]{64}$ is also the edge length of a cube that has a volume of 64.

Practice

Exercise 8.3.2.4

Find the positive solution to each equation. If the solution is irrational, write the solution using square root or cube root notation.

1. $t^3 = 216$
2. $a^2 = 15$
3. $m^3 = 8$
4. $c^3 = 343$
5. $f^3 = 181$

Exercise 8.3.2.5

For each cube root, find the two whole numbers that it lies between.

1. $\sqrt[3]{11}$
2. $\sqrt[3]{80}$
3. $\sqrt[3]{120}$
4. $\sqrt[3]{250}$

Exercise 8.3.2.6

Order the following values from least to greatest:

$\sqrt[3]{530}, \sqrt{48}, \pi, \sqrt{121}, \sqrt[3]{27}, \frac{19}{2}$

Exercise 8.3.2.7

Select **all** the equations that have a solution of $\frac{2}{7}$

A. $x^2 = \frac{2}{7}$
B. $x^2 = \frac{4}{14}$
C. $x^2 = \frac{4}{49}$
D. $x^3 = \frac{6}{21}$
E. $x^3 = \frac{8}{343}$
F. $x^3 = \frac{6}{7}$

Exercise 8.3.2.8

The equation $x^2 = 25$ has *two* solutions. This is because both $5 \cdot 5 = 25$, and also $-5 \cdot -5 = 25$. So, 5 is a solution, and also -5 is a solution. But! The equation $x^3 = 125$ only has one solution, which is 5. This is because $5 \cdot 5 \cdot 5 = 125$, and there are no other numbers you can cube to make 125. (Think about why -5 is not a solution!)

Find all the solutions to each equation.

1. $x^3 = 8$
2. $\sqrt[3]{x} = 3$
3. $x^2 = 49$
4. $x^3 = \frac{64}{125}$

Exercise 8.3.2.9

Find the value of each variable, to the nearest tenth.

1.

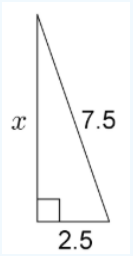


Figure 8.3.2.2

2.

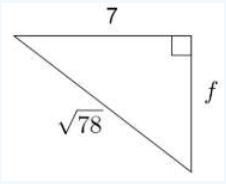


Figure 8.3.2.3

3.

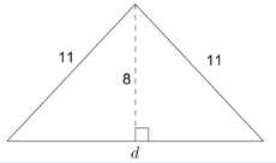


Figure 8.3.2.4

(From Unit 8.2.3)

Exercise 8.3.2.10

A standard city block in Manhattan is a rectangle measuring 80 m by 270 m. A resident wants to get from one corner of a block to the opposite corner of a block that contains a park. She wonders about the difference between cutting across the diagonal through the park compared to going around the park, along the streets. How much shorter would her walk be going through the park? Round your answer to the nearest meter.

(From Unit 8.2.5)

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8.4: Decimal Representation of Rational and Irrational Numbers

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8.4.1: Decimal Representation of Rational Numbers

Lesson

Let's learn more about how rational numbers can be represented.

Exercise 8.4.1.1: Notice and Wonder: Shaded Bars

What do you notice? What do you wonder?

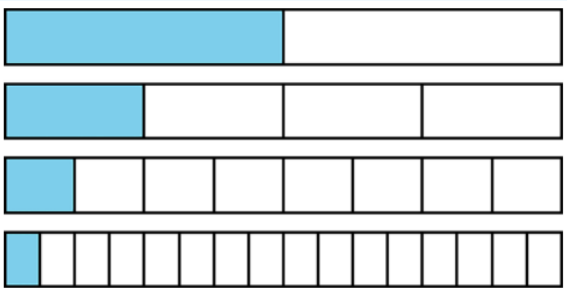


Figure 8.4.1.1: Four rectangular bars of equal length, aligned vertically. The first rectangular bar is divided into two equal sized parts and the first part is shaded. The second rectangular bar is divided into 4 equal sized parts and the first part is shaded. The third rectangular bar is divided into 8 equal sized parts and the first part is shaded. The fourth rectangular bar is divided into 16 equal sized parts and the first bar is shaded.

Exercise 8.4.1.2: Halving The Length

Here is a number line from 0 to 1.

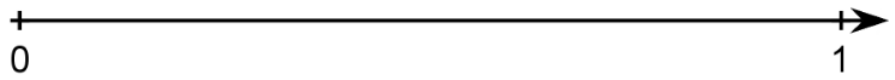


Figure 8.4.1.2

1. Mark the midpoint between 0 and 1. What is the decimal representation of that number?
2. Mark the midpoint between 0 and the newest point. What is the decimal representation of that number?
3. Repeat step two. How did you find the value of this number?
4. Describe how the value of the midpoints you have added to the number line keep changing as you find more. How do the decimal representations change?

Exercise 8.4.1.3: Recalculating Rational Numbers

1. Rational numbers are fractions and their opposites. All of these numbers are rational numbers. Show that they are rational by writing them in the form $\frac{a}{b}$ or $-\frac{a}{b}$.
   a. 0.2
   b. $-\sqrt{4}$
   c. 0.333
   d. $\sqrt[3]{1000}$
   e. -1.000001
   f. $\sqrt{\frac{1}{9}}$
2. All rational numbers have decimal representations, too. Find the decimal representation of each of these rational numbers.
   a. $\frac{3}{8}$
   b. $\frac{7}{5}$
   c. $\frac{999}{1000}$
   d. $\frac{111}{2}$
   e. $\sqrt[3]{\frac{1}{8}}$

Exercise 8.4.1.4: Zooming In On $\frac{2}{11}$

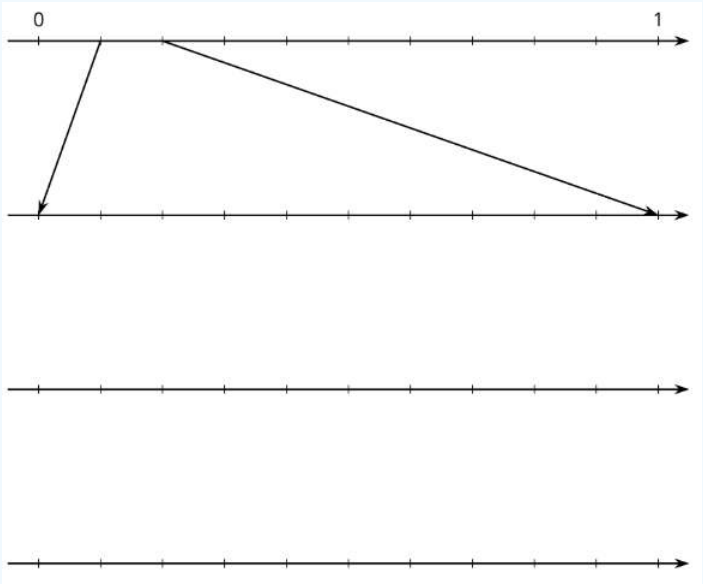


Figure 8.4.1.3

1. On the topmost number line, label the tick marks. Next, find the first decimal place of $\frac{2}{11}$ using long division and estimate where $\frac{2}{11}$ should be placed on the top number line.
2. Label the tick marks of the second number line. Find the next decimal place of $\frac{2}{11}$ by continuing the long division and estimate where $\frac{2}{11}$ should be placed on the second number line. Add arrows from the second to the third number line to zoom in on the location of $\frac{2}{11}$.
3. Repeat the earlier step for the remaining number lines.
4. What do you think the decimal expansion of $\frac{2}{11}$ is?

Are you ready for more?

Let $x = \frac{25}{11} = 2.272727\ldots$ and $y = \frac{58}{33} = 1.75757575\ldots$

For each of the following questions, first decide whether the fraction or decimal representations of the numbers are more helpful to answer the question, and then find the answer.

- Which of x or y is closer to 2?
- Find x^2.

Summary

We learned earlier that rational numbers are a fraction or the opposite of a fraction. For example, $\frac{3}{4}$ and $-\frac{5}{2}$ are both rational numbers. A complicated-looking numerical expression can also be a rational number as long as the value of the expression is a positive or negative fraction. For example, $\sqrt{64}$ and $-\sqrt[3]{\frac{1}{8}}$ are rational numbers because $\sqrt{64} = 8$ and $-\sqrt[3]{\frac{1}{8}} = -\frac{1}{2}$.

Rational numbers can also be written using decimal notation. Some have finite decimal expansions, like 0.75, -2.5, or -0.5. Other rational numbers have infinite decimal expansions, like 0.7434343 . . . where the 43s repeat forever. To avoid writing the **repeating** part over and over, we use the notation $0.7\overline{43}$ for this number. The bar over part of the expansion tells us the part which is to repeat forever.

A decimal expansion of a number helps us plot it accurately on a number line divided into tenths. For example, $0.7\overline{43}$ should be between 0.7 and 0.8. Each further decimal digit increases the accuracy of our plotting. For example, the number $0.7\overline{43}$ is between 0.743 and 0.744.

Glossary Entries

Definition: Repeating Decimal

A repeating decimal has digits that keep going in the same pattern over and over. The repeating digits are marked with a line above them.

For example, the decimal representation for $\frac{1}{3}$ is $0.\overline{3}$, which means 0.3333333 . . . The decimal representation for $\frac{25}{22}$ is $1.1\overline{36}$ which means 1.136363636 . . .

Practice

Exercise 8.4.1.5

Andre and Jada are discussing how to write $\frac{17}{20}$ as a decimal.

Andre says he can use long division to divide 17 by 20 to get the decimal.

Jada says she can write an equivalent fraction with a denominator of 100 by multiplying by $\frac{5}{5}$, then writing the number of hundredths as a decimal.

1. Do both of these strategies work?
2. Which strategy do you prefer? Explain your reasoning.
3. Write $\frac{17}{20}$ as a decimal. Explain or show your reasoning.

Exercise 8.4.1.6

Write each fraction as a decimal.

1. $\sqrt{\frac{9}{100}}$
2. $\frac{99}{100}$
3. $\sqrt{\frac{9}{16}}$
4. $\frac{23}{10}$

Exercise 8.4.1.7

Write each decimal as a fraction.

1. $\sqrt{0.81}$
2. 0.0276
3. $\sqrt{0.04}$
4. 10.01

Exercise 8.4.1.8

Find the positive solution to each equation. If the solution is irrational, write the solution using square root or cube root notation.

1. $x^2 = 90$
2. $p^3 = 90$
3. $z^2 = 1$
4. $y^3 = 1$
5. $w^2 = 36$
6. $h^3 = 64$

(From Unit 8.3.2)

Exercise 8.4.1.9

Here is a right square pyramid.

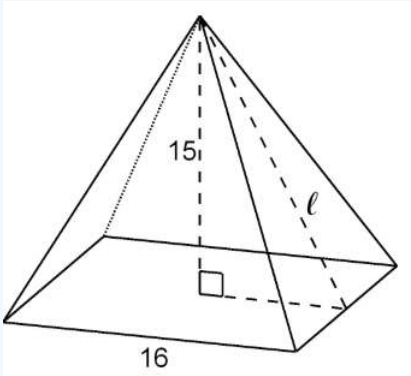


Figure 8.4.1.4

1. What is the measurement of the slant height l of the triangular face of the pyramid? If you get stuck, use a cross section of the pyramid.
2. What is the surface area of the pyramid?

(From Unit 8.2.5)

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8.4.2: Infinite Decimal Expansions

Lesson

Let's think about infinite decimals.

Exercise 8.4.2.1: Searching for Digits

The first 3 digits after the decimal for the decimal expansion of $\frac{3}{7}$ have been calculated. Find the next 4 digits.

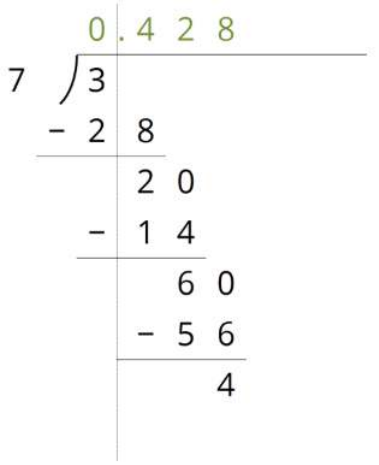


Figure 8.4.2.1

Exercise 8.4.2.2: Some Numbers are Rational

Your teacher will give your group a set of cards. Each card will have a calculations side and an explanation side.

1. The cards show Noah's work calculating the fraction representation of $0.4\overline{85}$. Arrange these in order to see how he figured out that $0.4\overline{85} = \frac{481}{990}$ without needing a calculator.
2. Use Noah's method to calculate the fraction representation of:
   a. $0.1\overline{86}$
   b. $0.7\overline{88}$

Are you ready for more?

Use this technique to find fractional representations for $0.\overline{3}$ and $0.\overline{9}$.

Exercise 8.4.2.3: Some Numbers are Not Rational

1.
   a. Why is $\sqrt{2}$ between 1 and 2 on the number line?
   b. Why is $\sqrt{2}$ between 1.4 and 1.5 on the number line?
   c. How can you figure out an approximation for $\sqrt{2}$ accurate to 3 decimal places?
   d. Label all of the tick marks. Plot $\sqrt{2}$ on all three number lines. Make sure to add arrows from the second to the third number lines.

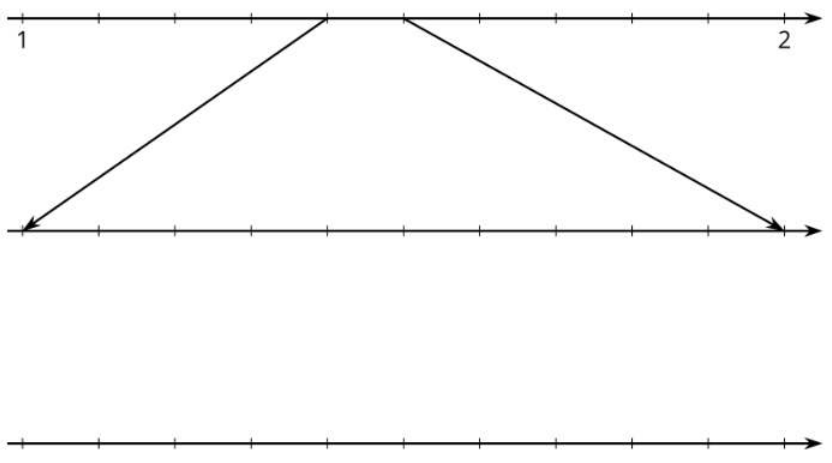


Figure 8.4.2.2

2.

a. Elena notices a beaker in science class says it has a diameter of 9 cm and measures its circumference to be 28.3 cm. What value do you get for π using these values and the equation for circumference, $C = 2\pi r$?
b. Diego learned that one of the space shuttle fuel tanks had a diameter of 840 cm and a circumference of 2,639 cm. What value do you get for π using these values and the equation for circumference, $C = 2\pi r$?
c. Label all of the tick marks on the number lines. Use a calculator to get a very accurate approximation of π and plot that number on all three number lines.

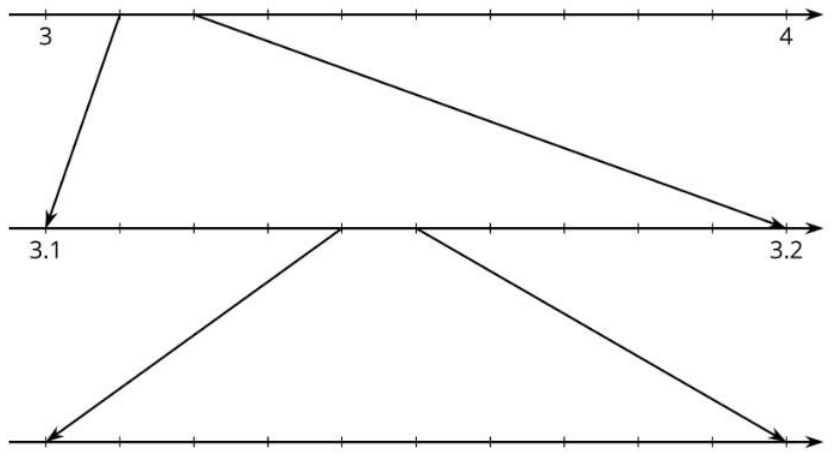


Figure 8.4.2.3

d. How can you explain the differences between these calculations of π?

Summary

Not every number is rational. Earlier we tried to find a fraction whose square is equal to 2. That turns out to be impossible, although we can get pretty close (try squaring $\frac{7}{5}$). Since there is no fraction equal to $\sqrt{2}$ it is not a rational number, which is why we call it an irrational number. Another well-known irrational number is π.

Any number, rational or irrational, has a decimal expansion. Sometimes it goes on forever. For example, the rational number $\frac{2}{11}$ has the decimal expansion $0.181818\ldots$with the 18s repeating forever. Every rational number has a decimal expansion that either stops at some point or ends up in a repeating pattern like $\frac{2}{11}$. Irrational numbers also have infinite decimal expansions, but they don't end up in a repeating pattern. From the decimal point of view we can see that rational numbers are pretty special. Most numbers are irrational, even though the numbers we use on a daily basis are more frequently rational.

Glossary Entries

Definition: Repeating Decimal

A repeating decimal has digits that keep going in the same pattern over and over. The repeating digits are marked with a line above them.

For example, the decimal representation for $\frac{1}{3}$ is $0.\overline{3}$, which means 0.3333333 . . . The decimal representation for $\frac{25}{22}$ is $1.1\overline{36}$ which means 1.136363636 . . .

Practice

Exercise 8.4.2.4

Elena and Han are discussing how to write the repeating decimal $x = 0.1\overline{37}$ as a fraction. Han says that $0.1\overline{37}$ equals $\frac{13764}{99900}$. "I calculated $1000x = 137.7\overline{77}$ because the decimal begins repeating after 3 digits. Then I subtracted to get $999x = 137.64$ Then I multiplied by 100 to get rid of the decimal: $99900x = 13764$ And finally I divided to get $x = \frac{13764}{99900}$." Elena says that $0.1\overline{37}$ equals $\frac{124}{900}$. "I calculated $10x = 1.37\overline{7}$ because one digit repeats. Then I subtracted to get $9x = 1.24$. Then I did what Han did to get $900x = 124$ and $x = \frac{124}{900}$."

Do you agree with either of them? Explain your reasoning.

Exercise 8.4.2.5

How are the numbers 0.444 and $0.\overline{4}$ the same? How are they different?

Exercise 8.4.2.6

1. Write each fraction as a decimal.
   a. $\frac{2}{3}$
   b. $\frac{126}{37}$
2. Write each decimal as a fraction.
   a. $0.\overline{75}$
   b. $0.\overline{3}$

Exercise 8.4.2.7

Write each fraction as a decimal.

1. $\frac{5}{9}$
2. $\frac{5}{4}$
3. $\frac{48}{99}$
4. $\frac{5}{99}$
5. $\frac{7}{100}$
6. $\frac{53}{90}$

Exercise 8.4.2.8

Write each decimal as a fraction.

1. $0.\overline{7}$
2. $0.\overline{2}$
3. $0.1\overline{3}$
4. $0.\overline{14}$

5. $0.\overline{03}$
6. $0.6\overline{38}$
7. $0.52\overline{4}$
8. $0.1\overline{5}$

Exercise 8.4.2.9

$2.2^2 = 4.84$ and $2.3^2 = 5.29$. This gives some information about $\sqrt{5}$.

Without directly calculating the square root, plot $\sqrt{5}$ on all three number lines using successive approximation.

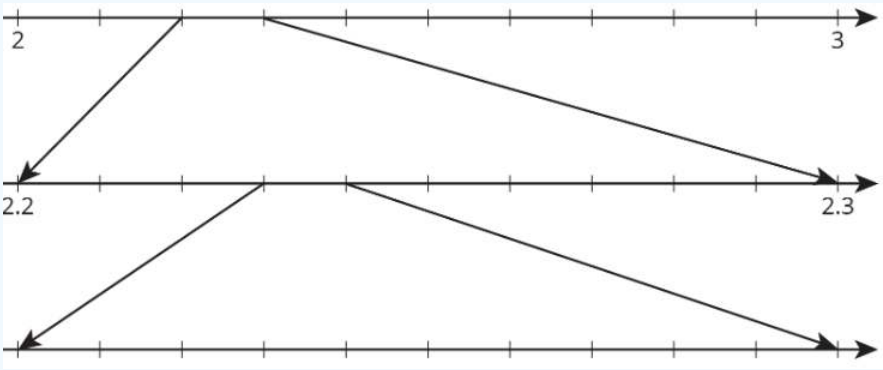


Figure 8.4.2.4

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8.5: Let's Put it to Work

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8.5.1: When Is the Same Size Not the Same Size?

Lesson

- Let's figure out how aspect ratio affects screen area.

Exercise 8.5.1.1: Three Figures

How are these shapes the same? How are they different?

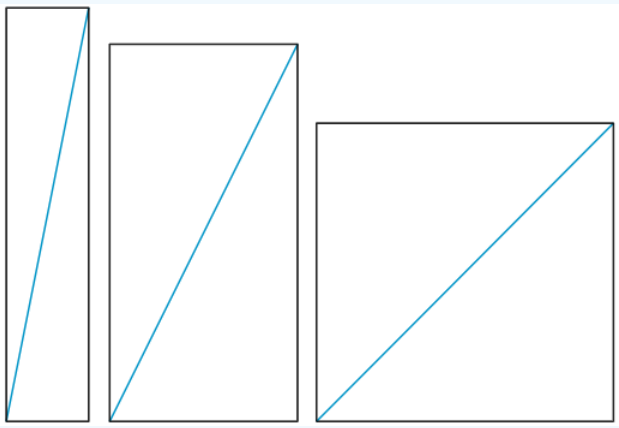


Figure 8.5.1.1: Three quadrilaterals with the diagonal drawn from the bottom left corner to the top right corner. The first figure is a rectangle and is tall and narrow. The second figure is a rectangle and is shorter and wider. The third figure looks like a square.

Exercise 8.5.1.2: A $4:3$ Rectangle

A typical aspect ratio for photos is $4:3$. Here's a rectangle with a $4:3$ aspect ratio.

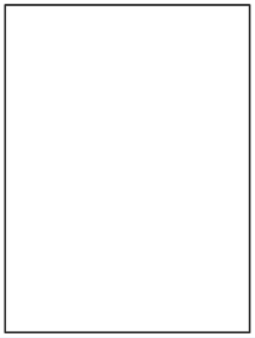


Figure 8.5.1.2

1. What does it mean that the aspect ratio is $4:3$? Mark up the diagram to show what that means.
2. If the shorter side of the rectangle measures 15 inches:
   a. What is the length of the longer side?
   b. What is the length of the rectangle's diagonal?
3. If the diagonal of the $4:3$ rectangle measures 10 inches, how long are its sides?
4. If the diagonal of the $4:3$ rectangle measures 6 inches, how long are its sides?

Exercise 8.5.1.3: The Screen Is that Same Size . . . or is it?

Before 2017, a smart phone manufacturer's phones had a diagonal length of 5.8 inches and an aspect ratio of $16:9$. In 2017, they released a new phone that also had a 5.8-inch diagonal length, but an aspect ratio of $18.5:9$. Some customers complained that the new phones had a smaller screen. Were they correct? If so, how much smaller was the new screen compared to the old screen?

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CHAPTER OVERVIEW

9: Putting It All Together

In these optional lessons, students solve complex problems. In the first several lessons, they consider tessellations of the plane, understanding and using the terms “tessellation” and “regular tessellation” in their work, and using properties of shapes (for example, the sum of the interior angles of a quadrilateral is 360 degrees) to make inferences about regular tessellations. These lessons need to come after unit 8.1 has been done. In the later lessons, they investigate relationships of temperature and latitude, climate, season, cloud cover, or time of day. In particular, they use scatter plots and lines of best fit to investigate the question of modeling temperature as a function of latitude. These lessons need to come after units 8.5 and 8.6 have been done.

9.1: Tessellations

9.1.1: Tessellations of the Plane

9.1.2: Regular Tessellations

9.1.3: Tessellating Polygons

9.2: The Weather

9.2.1: What Influences Temperature?

9.2.2: Plotting the Weather

9.2.3: Using and Interpreting a Mathematical Model

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9.1: Tessellations

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9.1.1: Tessellations of the Plane

Lesson

Let's explore geometric patterns!

Exercise 9.1.1.1: Notice and Wonder: Polygon Patterns

What do you notice? What do you wonder?

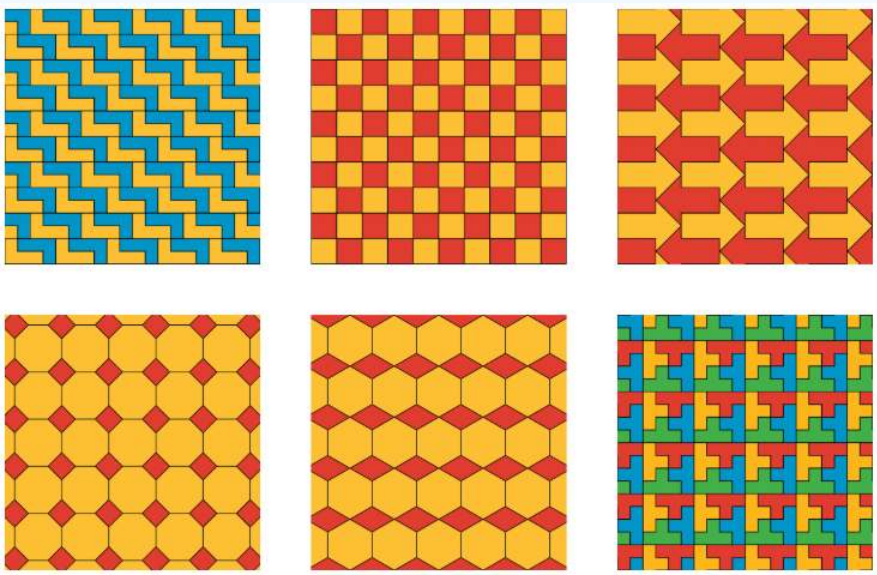


Figure 9.1.1.1

Exercise 9.1.1.2: Tessellations

1. With your partner, choose one of the six shapes in the toolbar that you will both use.
   - Select the shape tool by clicking on it. Create copies of your shape by clicking in the work space.
   - When you have enough to work with, click on the Move tool (the arrow) to drag or turn them.
   - If you have trouble aligning the shapes, right click to turn on the grid.
   - When you are finished making your tessellation, you can hide the points.
2. Compare your tessellation to your partner's. How are they similar? How are they different?
3. If possible, make a third tessellation of the plane with your shape (different from the ones you and your partner already created). If not possible, explain why it is not possible.

Exercise 9.1.1.3: Describing a Tessellation

1. Pick one of the figures and describe the tessellation. Your partner will identify which tessellation you are describing. Then trade roles so your partner describes the tessellation and you identify the figure.

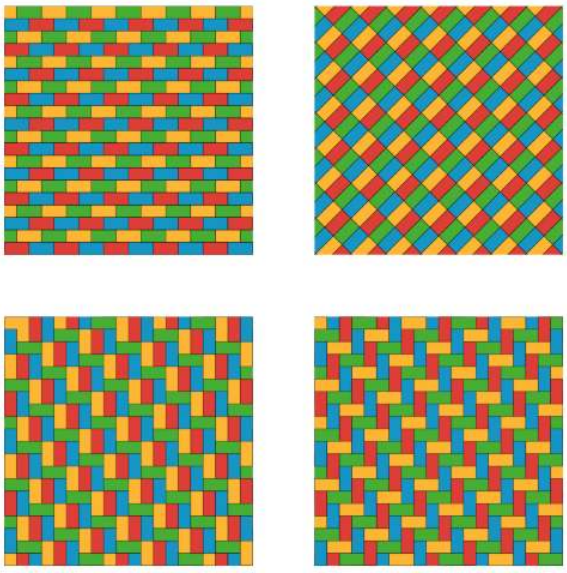


Figure 9.1.1.2

2. You and your partner each have a card with a tessellation. Describe what is on your card so that your partner can produce the tessellation (this should be done so that you cannot see your partner's work until it is complete).

3. Check together to see if your partner's tessellation agrees with your card and discuss any differences.

4. Change roles so your partner describes a tessellation, which you attempt to produce.

5. Check the accuracy of your construction and discuss any discrepancies.

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9.1.2: Regular Tessellations

Lesson

Let's make some regular tessellations.

Exercise 9.1.2.1: Regular Tessellations

1. For each shape (triangle, square, pentagon, hexagon, and octagon), decide if you can use that shape to make a regular tessellation of the plane. Explain your reasoning.
2. For the polygons that do not work what goes wrong? Explain your reasoning.

Exercise 9.1.2.2: Equilateral Triangle Tessellation

1. What is the measure of each angle in an equilateral triangle? How do you know?
2. How many triangles can you fit together at one vertex? Explain why there is no space between the triangles.
3. Explain why you can continue the pattern of triangles to tessellate the plane.
4. How can you use your triangular tessellation of the plane to show that regular hexagons can be used to give a regular tessellation of the plane?

Exercise 9.1.2.3: Regular Tessellation for Other Polygons

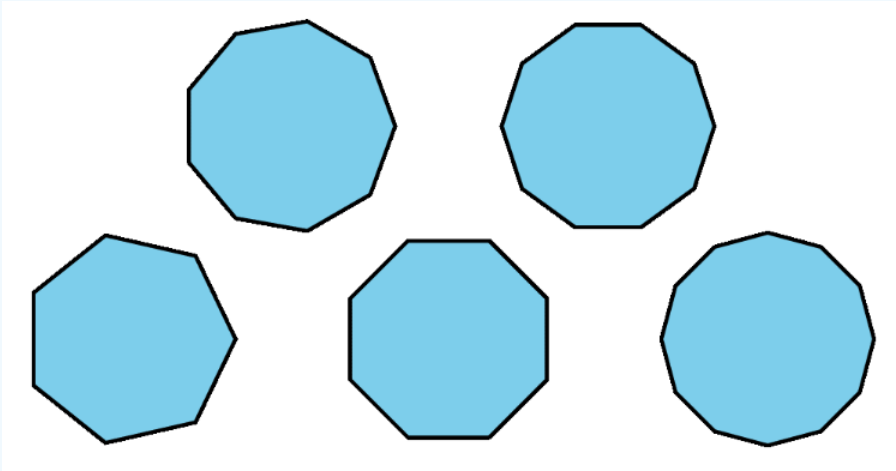


Figure 9.1.2.1

1. Can you make a regular tessellation of the plane using regular polygons with 7 sides? What about 9 sides? 10 sides? 11 sides? 12 sides? Explain.
2. How does the measure of each angle in a square compare to the measure of each angle in an equilateral triangle? How does the measure of each angle in a regular 8-sided polygon compare to the measure of each angle in a regular 7-sided polygon?
3. What happens to the angles in a regular polygon as you add more sides?
4. Which polygons can be used to make regular tessellations of the plane?

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9.1.3: Tessellating Polygons

Lesson

Let's make tessellations with different polygons.

Exercise 9.1.3.1: Triangle Tessellations

Your teacher will assign you one of the three triangles. Your goal is to find a tessellation of the plane with copies of the triangle.

Exercise 9.1.3.2: Quadrilateral Tessellations

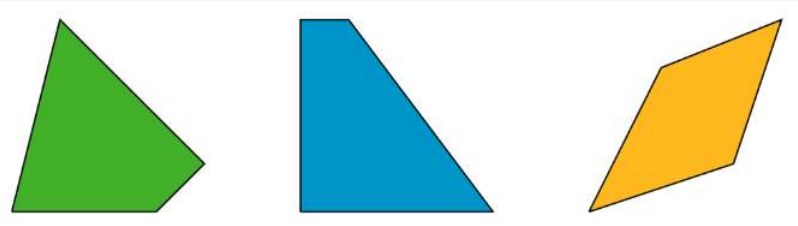


Figure 9.1.3.1

1. Can you make a tessellation of the plane with copies of the trapezoid? Explain.
2. Choose one of the other two quadrilaterals. Next, rotate the quadrilateral 180 degrees around the midpoint of each side. What do you notice?
3. Can you make a tessellation of the plane with copies of the quadrilateral from the previous problem? Explain your reasoning.

Exercise 9.1.3.3: Pentagonal Tessellations

1. Can you tessellate the plane with copies of the pentagon? Explain. Note that the two sides making angle A are congruent.

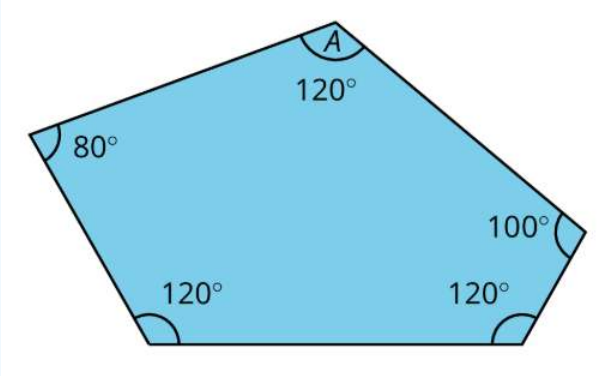


Figure 9.1.3.2

Pause your work here.

2. Take one pentagon and rotate it 120 degrees clockwise about the vertex at angle A, and trace the new pentagon. Next, rotate the pentagon 240 degrees clockwise about the vertex at angle A, and trace the new pentagon.

3. Explain why the three pentagons make a full circle at the central vertex.

4. Explain why the shape that the three pentagons make is a hexagon (that is, the sides that look like they are straight really are straight).

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9.2: The Weather

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9.2.1: What Influences Temperature?

Lesson

Let's see if we can predict the weather.

Exercise 9.2.1.1: Temperature Changes

What factors or variables can influence the outside temperature?

- Make a list of different factors.
- Write a sentence for each factor describing how changing it could change the temperature.

Example: One factor is time of day. Often, after sunrise, the temperature increases, reaches a peak in the early afternoon, and then decreases.

Exercise 9.2.1.2: Is Temperature a Function of Latitude?

1. Andre and Lin are wondering if temperature is a function of latitude.

Andre says, “I think it is, as long as we fix the time when we are measuring the temperature.”

Lin says, “But what if you have two places with the same latitude? Look at this weather map for Washington State. Seattle and Spokane have the same latitude but different temperatures right now.”

What do Andre and Lin mean?



Figure 9.2.1.1

2. Andre and Lin are discussing whether it is possible to define latitude and temperature in a way that makes sense to talk about temperature as a function of latitude. They are considering different options. What are some advantages and disadvantages of each option?

Here are the options:

a. Finding the temperature right now in cities with different latitudes.
b. Finding the daily high temperature in cities that have different latitudes.
c. Finding the average high temperature in a specific month, for example, September in cities that have different latitudes.
d. Finding the average yearly temperature in cities that have have different latitudes.

Exercise 9.2.1.3: Is There an Association Between Latitude and Temperature?

Lin and Andre decided that modeling temperature as a function of latitude doesn’t really make sense. They realized that they can ask whether there is an *association* between latitude and temperature.

1. What information could they gather to determine whether temperature is related to latitude?

2. What should they do with that information to answer the question?

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9.2.2: Plotting the Weather

Lesson

Let's construct a model.

Exercise 9.2.2.1: California Rain

What do you notice? What do you wonder?

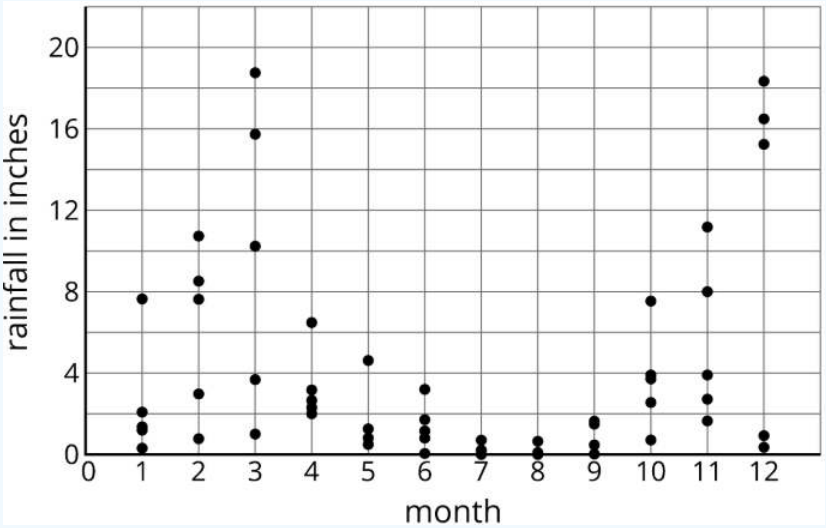


Figure 9.2.2.1

Exercise 9.2.2.2: Data Snooping

The table shows the average high temperature in September for cities with different latitudes. Examine the data in the table.

Table 9.2.2.1

city	latitude (degrees North)	temperature (degrees Fahrenheit)
Atlanta, GA	33.38	82
Portland, ME	43.38	69
Boston, MA	42.22	73
Dallas, TX	32.51	88
Denver, CO	39.46	77
Edmonton, AB	53.34	62
Fairbanks, AK	64.48	55
Juneau, AK	58.22	56
Kansas City, MO	39.16	78
Lincoln, NE	40.51	77
Miami, FL	25.45	88
Minneapolis, MN	44.53	71
New York City, NY	40.38	75
Orlando, FL	28.26	90

city	latitude (degrees North)	temperature (degrees Fahrenheit)
Philadelphia, PA	39.53	78
San Antonio, TX	29.32	89
San Francisco, CA	37.37	74
Seattle, WA	47.36	69
Tampa, FL	27.57	89
Tucson, AZ	32.13	93
Yellowknife, NT	62.27	50

1. What information does each row contain?
2. What is the range for each variable?
3. Do you see an association between the two variables? If so, describe the association.

Exercise 9.2.2.3: Temperature vs. Latitude

1. Make a scatter plot of the data.

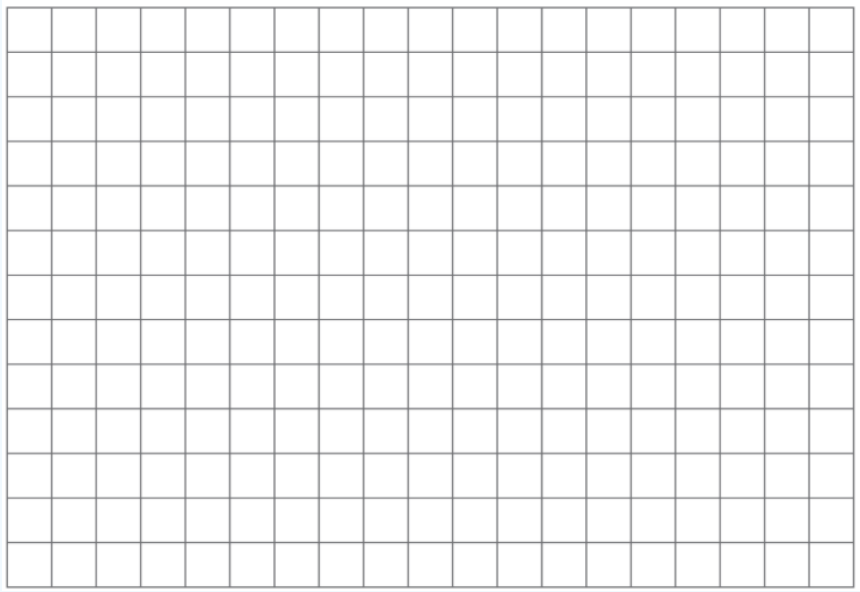


Figure 9.2.2.2

2. Describe any patterns of association that you notice.
3. Draw a line that fits the data. Write an equation for this line.

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9.2.3: Using and Interpreting a Mathematical Model

Lesson

Let's use a model to make some predictions.

Exercise 9.2.3.1: Using a Mathematical Model

In the previous activity, you found the equation of a line to represent the association between latitude and temperature. This is a *mathematical model*.

1. Use your model to predict the average high temperature in September at the following cities that were not included in the original data set:
   a. Detroit (Lat: 42.14)
   b. Albuquerque (Lat: 35.2)
   c. Nome (Lat: 64.5)
   d. Your own city (if available)
2. Draw points that represent the predicted temperatures for each city on the scatter plot.
3. The actual average high temperature in September in these cities were:
   - Detroit: $74^\circ F$
   - Albuquerque: $82^\circ F$
   - Nome: $49^\circ F$
   - Your own city (if available):
     How well does you model predict the temperature? Compare the predicted and actual temperatures.
4. If you added the actual temperatures for these four cities to the scatter plot, would you move your line?
5. Are there any outliers in the data? What might be the explanation?

Exercise 9.2.3.2: Interpreting a Mathematical Model

Refer to your equation for the line that models the association between latitude and temperature of the cities.

1. What does the slope mean in the context of this situation?
2. Find the vertical and horizontal intercepts and interpret them in the context of the situation.
3. Can you think of a city or a location that could not be represented using this same model? Explain your thinking.

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