INTRODUCTION TO REAL ANALYSIS



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William F. Trench

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Preface

This is a text for a two-term course in introductory real analysis for junior or senior mathematics majors and science students with a serious interest in mathematics. Prospective educators or mathematically gifted high school students can also benefit from the mathematical maturity that can be gained from an introductory real analysis course.

The book is designed to fill the gaps left in the development of calculus as it is usually presented in an elementary course, and to provide the background required for insight into more advanced courses in pure and applied mathematics. The standard elementary calculus sequence is the only specific prerequisite for Chapters 1–5, which deal with real-valued functions. (However, other analysis oriented courses, such as elementary differential equation, also provide useful preparatory experience.) Chapters~6 and 7 require a working knowledge of determinants, matrices and linear transformations, typically available from a first course in linear algebra. Chapter~8 is accessible after completion of Chapters~1–5.

Without taking a position for or against the current reforms in mathematics teaching, I think it is fair to say that the transition from elementary courses such as calculus, linear algebra, and differential equations to a rigorous real analysis course is a bigger step today than it was just a few years ago. To make this step today's students need more help than their predecessors did, and must be coached and encouraged more. Therefore, while striving throughout to maintain a high level of rigor, I have tried to write as clearly and informally as possible. In this connection I find it useful to address the student in the second person. I have included 295 completely worked out examples to illustrate and clarify all major theorems and definitions.

I have emphasized careful statements of definitions and theorems and have tried to be complete and detailed in proofs, except for omissions left to exercises. I give a thorough treatment of real-valued functions before considering vector-valued functions. In making the transition from one to several variables and from real-valued to vector-valued functions, I have left to the student some proofs that are essentially repetitions of earlier theorems. I believe that working through the details of straightforward generalizations of more elementary results is good practice for the student.

- Chapter 1 is concerned with the real number system. Section~1.1 begins with a brief discussion of the axioms for a complete ordered field, but no attempt is made to develop the reals from them; rather, it is assumed that the student is familiar with the consequences of these axioms, except for one: completeness. Since the difference between a rigorous and nonrigorous treatment of calculus can be described largely in terms of the attitude taken toward completeness, I have devoted considerable effort to developing its consequences. Section~1.2 is about induction. Although this may seem out of place in a real analysis course, I have found that the typical beginning real analysis student simply cannot do an induction proof without reviewing the method. Section~1.3 is devoted to elementary set theory and the topology of the real line, ending with the Heine-Borel and Bolzano-Weierstrass theorems.
- Chapter~2 covers the differential calculus of functions of one variable: limits, continuity, differentiablility, L'Hospital's rule, and Taylor's theorem. The emphasis is on rigorous presentation of principles; no attempt is made to develop the properties of specific elementary functions. Even though this may not be done rigorously in most contemporary calculus courses, I believe that the student's time is better spent on principles rather than on reestablishing familiar formulas and relationships.
- Chapter~3 is to devoted to the Riemann integral of functions of one variable. In Section~3.1 the integral is defined in the standard way in terms of Riemann sums. Upper and lower integrals are also defined there and used in Section~3.2 to study the existence of the integral. Section~3.3 is devoted to properties of the integral. Improper integrals are studied in Section~3.4. I believe that my treatment of improper integrals is more detailed than in most comparable textbooks. A more advanced look at the existence of the proper Riemann integral is given in Section~3.5, which concludes with Lebesgue's existence criterion. This section can be omitted without compromising the student's preparedness for subsequent sections.
- Chapter~4 treats sequences and series. Sequences of constant are discussed in Section~4.1. I have chosen to make the concepts of limit inferior and limit superior parts of this development, mainly because this permits greater flexibility and generality, with little extra effort, in the study of infinite series. Section~4.2 provides a brief introduction to the way in which continuity and differentiability can be studied by means of sequences. Sections~4.3–4.5 treat infinite series of constant, sequences and infinite series of functions, and power series, again in greater detail than in most comparable textbooks. The instructor who chooses not to cover these sections completely can omit the less standard topics without loss in subsequent sections.
- Chapter~5 is devoted to real-valued functions of several variables. It begins with a discussion of the toplogy of $\mathbb{R}^n$ in Section~5.1. Continuity and differentiability are discussed in Sections~5.2 and 5.3. The chain rule and Taylor's theorem are

discussed in Section~5.4.

- Chapter~6 covers the differential calculus of vector-valued functions of several variables. Section~6.1 reviews matrices, determinants, and linear transformations, which are integral parts of the differential calculus as presented here. In Section~6.2 the differential of a vector-valued function is defined as a linear transformation, and the chain rule is discussed in terms of composition of such functions. The inverse function theorem is the subject of Section~6.3, where the notion of branches of an inverse is introduced. In Section~6.4. the implicit function theorem is motivated by first considering linear transformations and then stated and proved in general.
- Chapter~7 covers the integral calculus of real-valued functions of several variables. Multiple integrals are defined in Section~7.1, first over rectangular parallelepipeds and then over more general sets. The discussion deals with the multiple integral of a function whose discontinuities form a set of Jordan content zero. Section~7.2 deals with the evaluation by iterated integrals. Section~7.3 begins with the definition of Jordan measurability, followed by a derivation of the rule for change of content under a linear transformation, an intuitive formulation of the rule for change of variables in multiple integrals, and finally a careful statement and proof of the rule. The proof is complicated, but this is unavoidable.
- Chapter~8 deals with metric spaces. The concept and properties of a metric space are introduced in Section~8.1. Section~8.2 discusses compactness in a metric space, and Section~8.3 discusses continuous functions on metric spaces.

Corrections–mathematical and typographical–are welcome and will be incorporated when received.

CHAPTER OVERVIEW

1: The Real Numbers

IN THIS CHAPTER we begin the study of the real number system. The concepts discussed here will be used throughout the book.

- SECTION 1.1 deals with the axioms that define the real numbers, definitions based on them, and some basic properties that follow from them.
- SECTION 1.2 emphasizes the principle of mathematical induction.
- SECTION 1.3 introduces basic ideas of set theory in the context of sets of real numbers. In this section we prove two fundamental theorems: the Heine–Borel and Bolzano–Weierstrass theorems.

1.1: The Real Number System
1.2: Mathematical Induction
1.3: The Real Line

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1.1: The Real Number System

Having taken calculus, you know a lot about the real number system; however, you probably do not know that all its properties follow from a few basic ones. Although we will not carry out the development of the real number system from these basic properties, it is useful to state them as a starting point for the study of real analysis and also to focus on one property, completeness, that is probably new to you.

Field Properties

The real number system (which we will often call simply the *reals*) is first of all a set $\{a, b, c, \cdots\}$ on which the operations of addition and multiplication are defined so that every pair of real numbers has a unique sum and product, both real numbers, with the following properties.

a. (A) a C b D b C a and ab D ba (commutative laws).
b. (B) .a C b/ C c D a C .b C c/ and .ab/c D a.bc/ (associative laws).
c. (C) a.b C c/ D ab C ac (distributive law).
d. (D) There are distinct real numbers 0 and 1 such that a C 0 D a and a1 D a for all a.
e. (E) For each a there is a real number a such that a C .a/ D 0, and if a ¤ 0, there is a real number 1=a such that a.1=a/ D

The manipulative properties of the real numbers, such as the relations

$$\begin{aligned}(a+b)^2 \ar &= a^2+2ab+b^2,\\ (3a+2b)(4c+2d) \ar &= 12ac+6ad+8bc+4bd,\\ (-a) \ar &= (-1)a, \quad a(-b)=(-a)b=-ab,\\ &\arraytext \quad \textit{and}\\ \frac{a}{b}+\frac{c}{d} \ar &= \frac{ad+bc}{bd} \quad (b,d\neq 0),\end{aligned}$$

all follow from –

. We assume that you are familiar with these properties.

A set on which two operations are defined so as to have properties –

is called a {}. The real number system is by no means the only field. The {} (which are the real numbers that can be written as $r = p/q$, where p and q are integers and $q \neq 0$) also form a field under addition and multiplication. The simplest possible field consists of two elements, which we denote by 0 and 1, with addition defined by

$$0+0=1+1=0, \quad 1+0=0+1=1, \tag{1.1.1}$$

and multiplication defined by

$$0\cdot 0=0\cdot 1=1\cdot 0=0, \quad 1\cdot 1=1 \tag{1.1.2}$$

(Exercise~).

The Order Relation

The real number system is ordered by the relation $<$, which has the following properties.

A field with an order relation satisfying –

is an {} field. Thus, the real numbers form an ordered field. The rational numbers also form an ordered field, but it is impossible to define an order on the field with two elements defined by and so as to make it into an ordered field (Exercise~).

We assume that you are familiar with other standard notation connected with the order relation: thus, $a > b$ means that $b < a$; $a \geq b$ means that either $a = b$ or $a > b$; $a \leq b$ means that either $a = b$ or $a < b$; the {} a, denoted by $|a|$, equals a if $a \geq 0$ or $-a$ if $a \leq 0$. (Sometimes we call $|a|$ the {} of a.)

You probably know the following theorem from calculus, but we include the proof for your convenience.

There are four possibilities:

Eq ∘ holds in cases {} and {}, since

Eq.~ holds in cases {} and {}, since

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The triangle inequality appears in various forms in many contexts. It is the most important inequality in mathematics. We will use it often.

Replacing a by $a-b$ in yields

$$|a| \le |a-b| + |b|,$$

so

$$|a-b| \ge |a| - |b|. \tag{1.1.3}$$

Interchanging a and b here yields

$$|b-a| \ge |b| - |a|,$$

which is equivalent to

$$|a-b| \ge |b| - |a|, \tag{1.1.4}$$

since $|b-a| = |a-b|$. Since

Bracket argument to \\ must be a dimension

and imply . Replacing b by $-b$ in yields , since $|-b| = |b|$.

With the real numbers associated in the usual way with the points on a line, these definitions can be interpreted geometrically as follows: b is an upper bound of S if no point of S is to the right of b; $\beta = \sup S$ if no point of S is to the right of β, but there is at least one point of S to the right of any number less than β (Figure~).

This example shows that a supremum of a set may or may not be in the set, since S_1 contains its supremum, but S does not.

A {} set is a set that has at least one member. The {}, denoted by $\emptyset$, is the set that has no members. Although it may seem foolish to speak of such a set, we will see that it is a useful idea.

It is one thing to define an object and another to show that there really is an object that satisfies the definition. (For example, does it make sense to define the smallest positive real number?) This observation is particularly appropriate in connection with the definition of the supremum of a set. For example, the empty set is bounded above by every real number, so it has no supremum. (Think about this.) More importantly, we will see in Example~ that properties –

do not guarantee that every nonempty set that is bounded above has a supremum. Since this property is indispensable to the rigorous development of calculus, we take it as an axiom for the real numbers.

We first show that $\beta = \sup S$ has properties and . Since β is an upper bound of S, it must satisfy . Since any real number a less than β can be written as $\beta - \epsilon$ with $\epsilon = \beta - a > 0$, is just another way of saying that no number less than β is an upper bound of S. Hence, $\beta = \sup S$ satisfies and

.

Now we show that there cannot be more than one real number with properties and . Suppose that $\beta_1 < \beta_2$ and β_2 has property ; thus, if $\epsilon > 0$, there is an x_0 in S such that $x_0 > \beta_2 - \epsilon$. Then, by taking $\epsilon = \beta_2 - \beta_1$, we see that there is an x_0 in S such that

$$x_0 > \beta_2 - (\beta_2 - \beta_1) = \beta_1,$$

so β_1 cannot have property . Therefore, there cannot be more than one real number that satisfies both and

.

We will often define a set S by writing $S = \{x\}\cdots$, which means that S consists of all x that satisfy the conditions to the right of the vertical bar; thus, in Example~,

$$S = \{x\} x < 0 \tag{1.1.5}$$

and

$$S_1 = \{x \mid x \text{ is a negative integer.}$$

We will sometimes abbreviate "x is a member of S" by $x \in S$, and "x is not a member of S" by $x \notin S$. For example, if S is defined by , then

$$-1 \in S \quad \text{but} \quad 0 \notin S.$$

Since $n+1$ is an integer whenever n is, implies that

$$(n+1)\epsilon \le \beta$$

and therefore

$$n\epsilon \le \beta - \epsilon$$

for all integers n. Hence, $\beta - \epsilon$ is an upper bound of S. Since $\beta - \epsilon < \beta$, this contradicts the definition of~β.

From Theorem~ with $\rho = 1$ and $\epsilon = b - a$, there is a positive integer q such that $q(b-a) > 1$. There is also an integer j such that $j > qa$. This is obvious if $a \le 0$, and it follows from Theorem~ with $\epsilon = 1$ and $\rho = qa$ if $a > 0$. Let p be the smallest integer such that $p > qa$. Then $p - 1 \le qa$, so

$$qa < p \le qa + 1.$$

Since $1 < q(b-a)$, this implies that

$$qa < p < qa + q(b-a) = qb,$$

so $qa < p < qb$. Therefore, $a < p/q < b$.

From Theorem~, there are rational numbers r_1 and r_2 such that

$$a < r_1 < r_2 < b. \tag{1.1.6}$$

Let

$$t = r_1 + \frac{1}{\sqrt{2}}(r_2 - r_1).$$

Then t is irrational (why?) and $r_1 < t < r_2$, so $a < t < b$, from .

A set S of real numbers is {} if there is a real number a such that $x \ge a$ whenever $x \in S$. In this case, a is a {} of S. If a is a lower bound of S, so is any smaller number, because of property

. If α is a lower bound of S, but no number greater than α is, then α is an {} of S, and we write

$$\alpha = \inf S.$$

Geometrically, this means that there are no points of S to the left of α, but there is at least one point of S to the left of any number greater than α.

(Exercise~)

A set S is {} if there are numbers a and b such that $a \le x \le b$ for all x in S. A bounded nonempty set has a unique supremum and a unique infimum, and

$$\inf S \le \sup S \tag{1.1.7}$$

(Exercise~).

A nonempty set S of real numbers is {} if it has no upper bound, or {} if it has no lower bound. It is convenient to adjoin to the real number system two fictitious points, $+\infty$ (which we usually write more simply as ∞) and $-\infty$, and to define the order relationships between them and any real number x by

$$-\infty < x < \infty. \tag{1.1.8}$$

We call ∞ and $-\infty$ {}. If S is a nonempty set of reals, we write

$$\sup S = \infty \tag{1.1.9}$$

to indicate that S is unbounded above, and

$$\inf S = -\infty \tag{1.1.10}$$

to indicate that S is unbounded below.

The real number system with ∞ and $-\infty$ adjoined is called the {}, or simply the {}. A member of the extended reals differing from $-\infty$ and ∞ is {}; that is, an ordinary real number is finite. However, the word `finite'' in` finite real number'' is redundant and used only for emphasis, since we would never refer to ∞ or $-\infty$ as real numbers.

The arithmetic relationships among ∞, $-\infty$, and the real numbers are defined as follows.

We also define

$$\infty + \infty = \infty\infty = (-\infty)(-\infty) = \infty$$

and

$$-\infty - \infty = \infty(-\infty) = (-\infty)\infty = -\infty.$$

Finally, we define

$$|\infty| = |-\infty| = \infty.$$

The introduction of ∞ and $-\infty$, along with the arithmetic and order relationships defined above, leads to simplifications in the statements of theorems. For example, the inequality , first stated only for bounded sets, holds for any nonempty set S if it is interpreted properly in accordance with and the definitions of and . Exercises~ and

illustrate the convenience afforded by some of the arithmetic relationships with extended reals, and other examples will illustrate this further in subsequent sections.

It is not useful to define $\infty - \infty$, $0 \cdot \infty$, ∞/∞, and $0/0$. They are called {}, and left undefined. You probably studied indeterminate forms in calculus; we will look at them more carefully in Section~2.4.

These axioms are known as {}. The real numbers can be constructed from the natural numbers by definitions and arguments based on them. This is a formidable task that we will not undertake. We mention it to show how little you need to start with to construct the reals and, more important, to draw attention to postulate

, which is the basis for the principle of mathematical induction.

It can be shown that the positive integers form a subset of the reals that satisfies Peano's postulates (with $\overline{n} = 1$ and $n' = n + 1$), and it is customary to regard the positive integers and the natural numbers as identical. From this point of view, the principle of mathematical induction is basically a restatement of postulate

.

Let

$$\mathbb{M} = \{n \} n \in \mathbb{N} \text{ and } P_n \text{ is true.}$$

From , $1 \in \mathbb{M}$, and from , $n + 1 \in \mathbb{M}$ whenever $n \in \mathbb{M}$. Therefore, $\mathbb{M} = \mathbb{N}$, by postulate

.

A proof based on Theorem~ is an {}, or {}. The assumption that P_n is true is the {}. (Theorem~ permits a kind of induction proof in which the induction assumption takes a different form.)

Induction, by definition, can be used only to verify results conjectured by other means. Thus, in Example~ we did not use induction to {} the sum

$$s_n = 1 + 2 + \cdots + n; \tag{1.1.11}$$

rather, we {} that

$$s_n = \frac{n(n+1)}{2}. \tag{1.1.12}$$

How you guess what to prove by induction depends on the problem and your approach to it. For example, might be conjectured after observing that

$$s_1 = 1 = \frac{1 \cdot 2}{2}, \quad s_2 = 3 = \frac{2 \cdot 3}{2}, \quad s_3 = 6 = \frac{4 \cdot 3}{2}.$$

However, this requires sufficient insight to recognize that these results are of the form for $n = 1$, 2, and 3. Although it is easy to prove by induction once it has been conjectured, induction is not the most efficient way to find s_n, which can be obtained quickly by rewriting as

$$s_n = n + (n-1) + \cdots + 1$$

and adding this to to obtain

$$2s_n = [n+1] + [(n-1)+2] + \cdots + [1+n].$$

There are n bracketed expressions on the right, and the terms in each add up to $n+1$; hence,

$$2s_n = n(n+1),$$

which yields .

The next two examples deal with problems for which induction is a natural and efficient method of solution.

The major effort in an induction proof (after P_1, P_2, , P_n,
have been formulated) is usually directed toward showing that P_n implies P_{n+1}. However, it is important to verify P_1, since P_n may imply P_{n+1} even if some or all of the propositions P_1, P_2, , P_n, are false.

The following formulation of the principle of mathematical induction permits us to start induction proofs with an arbitrary integer, rather than 1, as required in Theorem~.

For $m \geq 1$, let Q_m be the proposition defined by $Q_m = P_{m+n_0-1}$. Then $Q_1 = P_{n_0}$ is true by . If $m \geq 1$ and $Q_m = P_{m+n_0-1}$ is true, then $Q_{m+1} = P_{m+n_0}$ is true by

with n replaced by $m + n_0 - 1$. Therefore, Q_m is true for all $m \geq 1$ by Theorem~ with P replaced by Q and n replaced by m. This is equivalent to the statement that P_n is true for all $n \geq n_0$.

For $n \geq n_0$, let Q_n be the proposition that P_{n_0}, P_{n_0+1}, , P_n are all true. Then Q_{n_0} is true by . Since Q_n implies P_{n+1} by

, and Q_{n+1} is true if Q_n and P_{n+1} are both true, Theorem~ implies that Q_n is true for all $n \geq n_0$. Therefore, P_n is true for all $n \geq n_0$.

One of our objectives is to develop rigorously the concepts of limit, continuity, differentiability, and integrability, which you have seen in calculus. To do this requires a better understanding of the real numbers than is provided in calculus. The purpose of this section is to develop this understanding. Since the utility of the concepts introduced here will not become apparent until we are well into the study of limits and continuity, you should reserve judgment on their value until they are applied. As this occurs, you should reread the applicable parts of this section. This applies especially to the concept of an open covering and to the Heine–Borel and Bolzano–Weierstrass theorems, which will seem mysterious at first.

We assume that you are familiar with the geometric interpretation of the real numbers as points on a line. We will not prove that this interpretation is legitimate, for two reasons: (1) the proof requires an excursion into the foundations of Euclidean geometry, which is not the purpose of this book; (2) although we will use geometric terminology and intuition in discussing the reals, we will base all proofs on properties –

(Section~1.1) and their consequences, not on geometric arguments.

Henceforth, we will use the terms {} and {} synonymously and denote both by the symbol \R; also, we will often refer to a real number as a {} (on the real line).

In this section we are interested in sets of points on the real line; however, we will consider other kinds of sets in subsequent sections. The following definition applies to arbitrary sets, with the understanding that the members of all sets under consideration

in any given context come from a specific collection of elements, called the {}. In this section the universal set is the real numbers.

Every set S contains the empty set $\emptyset$, for to say that $\emptyset$ is not contained in S is to say that some member of $\emptyset$ is not in S, which is absurd since $\emptyset$ has no members. If S is any set, then

$$(S^c)^c = S \quad \text{and} \quad S \cap S^c = \emptyset.$$

If S is a set of real numbers, then $S \cup S^c = \backslash\mathbf{R}$.

The definitions of union and intersection have generalizations: If $\mathcal{F}$ is an arbitrary collection of sets, then $\cup\{S\}S \in \mathcal{F}$ is the set of all elements that are members of at least one of the sets in $\mathcal{F}$, and $\cap\{S\}S \in \mathcal{F}$ is the set of all elements that are members of every set in $\mathcal{F}$. The union and intersection of finitely many sets S_1, , S_n are also written as $\bigcup_{k=1}^{n} S_k$ and $\bigcap_{k=1}^{n} S_k$. The union and intersection of an infinite sequence $\{S_k\}_{k=1}^{\infty}$ of sets are written as $\bigcup_{k=1}^{\infty} S_k$ and $\bigcap_{k=1}^{\infty} S_k$.

1pc 1pc

1pc If a and b are in the extended reals and $a < b$, then the {} (a, b) is defined by

$$(a, b) = \{x\} a < x < b.$$

The open intervals (a, ∞) and $(-\infty, b)$ are {} if a and b are finite, and $(-\infty, \infty)$ is the entire real line.

The idea of neighborhood is fundamental and occurs in many other contexts, some of which we will see later in this book. Whatever the context, the idea is the same: some definition of closeness'' is given (for example, two real numbers are close''if their difference is ``small''), and a neighborhood of a point x_0 is a set that contains all points sufficiently close to x_0.

A {} of a point x_0 is a set that contains every point of some neighborhood of x_0 except for x_0 itself. For example,

$$S = \{x\} 0 < |x - x_0| < \epsilon$$

is a deleted neighborhood of x_0. We also say that it is a {} of x_0.

Let $\mathcal{G}$ be a collection of open sets and

$$S = \cup\{G\} G \in \mathcal{G}.$$

If $x_0 \in S$, then $x_0 \in G_0$ for some G_0 in $\mathcal{G}$, and since G_0 is open, it contains some ϵ-neighborhood of x_0. Since $G_0 \subset S$, this ϵ-neighborhood is in S, which is consequently a neighborhood of x_0. Thus, S is a neighborhood of each of its points, and therefore open, by definition.

Let $\mathcal{F}$ be a collection of closed sets and $T = \cap\{F\} F \in \mathcal{F}$. Then $T^c = \cup\{F^c\} F \in \mathcal{F}$ (Exercise~) and, since each F^c is open, T^c is open, from

. Therefore, T is closed, by definition.

Example~ shows that a set may be both open and closed, and Example~ shows that a set may be neither. Thus, open and closed are not opposites in this context, as they are in everyday speech.

It can be shown that the intersection of finitely many open sets is open, and that the union of finitely many closed sets is closed. However, the intersection of infinitely many open sets need not be open, and the union of infinitely many closed sets need not be closed (Exercises~ and).

The next theorem says that S is closed if and only if $S = \overline{S}$ (Exercise~).

Suppose that S is closed and $x_0 \in S^c$. Since S^c is open, there is a neighborhood of x_0 that is contained in S^c and therefore contains no points of S. Hence, x_0 cannot be a limit point of S. For the converse, if no point of S^c is a limit point of S then every point in S^c must have a neighborhood contained in S^c. Therefore, S^c is open and S is closed.

Theorem~ is usually stated as follows.

Theorem~ and Corollary~ are equivalent. However, we stated the theorem as we did because students sometimes incorrectly conclude from the corollary that a closed set must have limit points. The corollary does not say this. If S has no limit points, then the set of limit points is empty and therefore contained in S. Hence, a set with no limit points is closed according to the corollary, in

agreement with Theorem~. For example, any finite set is closed. More generally, S is closed if there is a $\delta > 0$ such $|x - y| \geq \delta$ for every pair $\{x, y\}$ of distinct points in S.

A collection $\mathcal{H}$ of open sets is an {} of a set S if every point in S is contained in a set H belonging to $\mathcal{H}$; that is, if $S \subset \cup\{H\} H \in \mathcal{H}$.

Since S is bounded, it has an infimum α and a supremum β, and, since S is closed, α and β belong to S (Exercise~). Define

$$S_t = S \cap [\alpha, t] \quad \text{for } t \geq \alpha,$$

and let

$$F = \{t\} \alpha \leq t \leq \beta and finitely many sets from \mathcal{H} cover S_t.$$

Since $S_\beta = S$, the theorem will be proved if we can show that $\beta \in F$. To do this, we use the completeness of the reals.

Since $\alpha \in S$, S_α is the singleton set $\{\alpha\}$, which is contained in some open set H_α from $\mathcal{H}$ because $\mathcal{H}$ covers S; therefore, $\alpha \in F$. Since F is nonempty and bounded above by β, it has a supremum γ. First, we wish to show that $\gamma = \beta$. Since $\gamma \leq \beta$ by definition of F, it suffices to rule out the possibility that $\gamma < \beta$. We consider two cases.

{Case 1}. Suppose that $\gamma < \beta$ and $\gamma \notin S$. Then, since S is closed, γ is not a limit point of S (Theorem~). Consequently, there is an $\epsilon > 0$ such that

$$[\gamma - \epsilon, \gamma + \epsilon] \cap S = \emptyset,$$

so $S_{\gamma - \epsilon} = S_{\gamma + \epsilon}$. However, the definition of γ implies that $S_{\gamma - \epsilon}$ has a finite subcovering from $\mathcal{H}$, while $S_{\gamma + \epsilon}$ does not. This is a contradiction.

{Case 2}. Suppose that $\gamma < \beta$ and $\gamma \in S$. Then there is an open set H_γ in $\mathcal{H}$ that contains γ and, along with γ, an interval $[\gamma - \epsilon, \gamma + \epsilon]$ for some positive ϵ. Since $S_{\gamma - \epsilon}$ has a finite covering $\{H_1, \ldots, H_n\}$ of sets from $\mathcal{H}$, it follows that $S_{\gamma + \epsilon}$ has the finite covering $\{H_1, \ldots, H_n, H_\gamma\}$. This contradicts the definition of γ.

Now we know that $\gamma = \beta$, which is in S. Therefore, there is an open set H_β in $\mathcal{H}$ that contains β and along with β, an interval of the form $[\beta - \epsilon, \beta + \epsilon]$, for some positive ϵ. Since $S_{\beta - \epsilon}$ is covered by a finite collection of sets $\{H_1, \ldots, H_k\}$, S_β is covered by the finite collection $\{H_1, \ldots, H_k, H_\beta\}$. Since $S_\beta = S$, we are finished.

Henceforth, we will say that a closed and bounded set is {}. The Heine–Borel theorem says that any open covering of a compact set S contains a finite collection that also covers S. This theorem and its converse (Exercise~) show that we could just as well define a set S of reals to be compact if it has the Heine–Borel property; that is, if every open covering of S contains a finite subcovering. The same is true of $\R^n$, which we study in Section~5.1. This definition generalizes to more abstract spaces (called {}) for which the concept of boundedness need not be defined.

As an application of the Heine–Borel theorem, we prove the following theorem of Bolzano and Weierstrass.

We will show that a bounded nonempty set without a limit point can contain only a finite number of points. If S has no limit points, then S is closed (Theorem~) and every point x of S has an open neighborhood N_x that contains no point of S other than x. The collection

$$\mathcal{H} = \{N_x\} x \in S$$

is an open covering for S. Since S is also bounded, Theorem~ implies that S can be covered by a finite collection of sets from $\mathcal{H}$, say $N_{x_1}, , N_{x_n}$. Since these sets contain only $x_1, , x_n$ from S, it follows that $S = \{x_1, \ldots, x_n\}$.

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CHAPTER OVERVIEW

2: Differential Calculus of Functions of One Variable

IN THIS CHAPTER we study the differential calculus of functions of one variable.

- SECTION 2.1 introduces the concept of function and discusses arithmetic operations on functions, limits, one-sided limits, limits at $\pm\infty$, and monotonic functions.
- SECTION 2.2 defines continuity and discusses removable discontinuities, composite functions, bounded functions, the intermediate value theorem, uniform continuity, and additional properties of monotonic functions.
- SECTION 2.3 introduces the derivative and its geometric interpretation. Topics covered include the interchange of differentiation and arithmetic operations, the chain rule, one-sided derivatives, extreme values of a differentiable function, Rolle's theorem, the intermediate value theorem for derivatives, and the mean value theorem and its consequences.
- SECTION 2.4 presents a comprehensive discussion of L'Hospital's rule.
- SECTION 2.5 discusses the approximation of a function f by the Taylor polynomials of f and applies this result to locating local extrema of f. The section concludes with the extended mean value theorem, which implies Taylor's theorem.

2.1: Functions and Limits
2.2: Continuity
2.3: Differentiable Functions of One Variable
2.4: L'Hospital's Rule
2.5: Taylor's Theorem

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2.1: Functions and Limits

In this section we study limits of real-valued functions of a real variable. You studied limits in calculus. However, we will look more carefully at the definition of limit and prove theorems usually not proved in calculus.

A rule f that assigns to each member of a nonempty set D a unique member of a set Y is a {}. We write the relationship between a member x of D and the member y of Y that f assigns to x as

$$y = f(x).$$

The set D is the {} of f, denoted by D_f. The members of Y are the possible {} of f. If $y_0 \in Y$ and there is an x_0 in D such that $f(x_0) = y_0$ then we say that f {} or {} the value y_0. The set of values attained by f is the {} of f. A {} is a function whose domain and range are both subsets of the reals. Although we are concerned only with real-valued functions of a real variable in this section, our definitions are not restricted to this situation. In later sections we will consider situations where the range or domain, or both, are subsets of vector spaces.

Example 2.1.1

Example 2.1.1 The functions f , g, and h defined on .1; 1/ by f .x/ D x 2 ; g.x/ D sin x; and h.x/ D e x have ranges Œ0; 1/, Œ1; 1 , and .0; 1/, respectively. **Solution**

Add text here.

Example 2.1.2

Example 2.1.2 The equation Œf .x/ 2 D x (2.1.1) does not define a function except on the singleton set f0g. If x < 0, no real number satisfies (2.1.1), while if x > 0, two real numbers satisfy (2.1.1). However, the conditions Œf .x/ 2 D x and f .x/ 0 define a function f on Df D Œ0; 1/ with values f .x/ D p x. Similarly, the conditions Œg.x/ 2 D x and g.x/ 0 define a function g on Dg D Œ0; 1/ with values g.x/ D p x. The ranges of f and g are Œ0; 1/ and .1; 0 , respectively. **Solution**

Add text here.

It is important to understand that the definition of a function includes the specification of its domain and that there is a difference between f , the name of the function, and f .x/, the value of f at x. However, strict observance of these points leads to annoying verbosity, such as "the function f with domain .1; 1/ and values f .x/ D x." We will avoid this in two ways: (1) by agreeing that if a function f is introduced without explicitly defining Df , then Df will be understood to consist of all points x for which the rule defining f .x/ makes sense, and (2) by bearing in mind the distinction between f and f .x/, but not emphasizing it when it would be a nuisance to do so. For example, we will write "consider the function f .x/ D p 1 x 2," rather than "consider the function f defined on Œ1; 1 by f .x/ D p 1 x 2," or "consider the function g.x/ D 1= sin x," rather than "consider the function g defined for x ¤ k (k D integer) by g.x/ D 1= sin x." We will also write f D c (constant) to denote the function f defined by f .x/ D c for all x. Our definition of function is somewhat intuitive, but adequate for our purposes. Moreover, it is the working form of the definition, even if the idea is introduced more rigorously to begin with. For a more precise definition, we first define the Cartesian product X Y of two nonempty sets X and Y to be the set of all ordered pairs .x; y/ such that x 2 X and y 2 Y ; thus, X Y D ˚ .x; y/ ˇ ˇ x 2 X; y

A nonempty subset f of $X \times Y$ is a **function** if no x in X occurs more than once as a first member among the elements of f. Put another way, if (x, y) and (x, y_1) are in f, then $y = y_1$. The set of x's that occur as first members of f is the of f. If x is in the domain of f, then the unique y in Y such that $(x, y) \in f$ is the {}, and we write $y = f(x)$. The set of all such values, a subset of Y, is the **range of** f.

Arithmetic Operations on Functions

The essence of the concept of limit for real-valued functions of a real variable is this: If L is a real number, then $\lim_{x \to x_0} f(x) = L$ means that the value $f(x)$ can be made as close to L as we wish by taking x sufficiently close to x_0. This is made precise in the following definition.

We emphasize that Definition~ does not involve $f(x_0)$, or even require that it be defined, since excludes the case where $x = x_0$.

The next theorem says that a function cannot have more than one limit at a point.

Suppose that holds and let $\epsilon > 0$. From Definition~, there are positive numbers δ_1 and δ_2 such that

$$|f(x) - L_i| < \epsilon \quad \text{if} \quad 0 < |x - x_0| < \delta_i, \quad i = 1, 2.$$

If $\delta = \min(\delta_1, \delta_2)$, then

$$\begin{aligned} |L_1 - L_2| \ar &= |L_1 - f(x) + f(x) - L_2| \\ \ar &\le |L_1 - f(x)| + |f(x) - L_2| < 2\epsilon \quad \text{if} \quad 0 < |x - x_0| < \delta. \end{aligned}$$

We have now established an inequality that does not depend on x; that is,

$$|L_1 - L_2| < 2\epsilon.$$

Since this holds for any positive ϵ, $L_1 = L_2$.

Definition~ is not changed by replacing with

$$|f(x) - L| < K\epsilon, \tag{2.1.1}$$

where K is a positive constant, because if either of or can be made to hold for any $\epsilon > 0$ by making $|x - x_0|$ sufficiently small and positive, then so can the other (Exercise~). This may seem to be a minor point, but it is often convenient to work with rather than , as we will see in the proof of the following theorem.

From and Definition~, if $\epsilon > 0$, there is a $\delta_1 > 0$ such that

$$|f(x) - L_1| < \epsilon \tag{2.1.2}$$

if $0 < |x - x_0| < \delta_1$, and a $\delta_2 > 0$ such that

$$|g(x) - L_2| < \epsilon \tag{2.1.3}$$

if $0 < |x - x_0| < \delta_2$. Suppose that

$$0 < |x - x_0| < \delta = \min(\delta_1, \delta_2), \tag{2.1.4}$$

so that and both hold. Then

$$\begin{aligned} |(f \pm g)(x) - (L_1 \pm L_2)| \ar &= |(f(x) - L_1) \pm (g(x) - L_2)| \\ \ar &\le |f(x) - L_1| + |g(x) - L_2| < 2\epsilon, \end{aligned}$$

which proves and .

To prove , we assume and write

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if $\epsilon < 1$ and x satisfies . This proves .

To prove , we first observe that if $L_2 \neq 0$, there is a $\delta_3 > 0$ such that

$$|g(x) - L_2| < \frac{|L_2|}{2},$$

so

$$|g(x)| > \frac{|L_2|}{2} \tag{2.1.5}$$

if

$$0 < |x - x_0| < \delta_3.$$

To see this, let $L = L_2$ and $\epsilon = |L_2|/2$ in . Now suppose that

$$0 < |x - x_0| < \min(\delta_1, \delta_2, \delta_3),$$

so that , , and all hold. Then

This proves .

Successive applications of the various parts of Theorem~ permit us to find limits without the ϵ–δ arguments required by Definition~.

-.2em The function

$$f(x) = 2x \sin\sqrt{x}$$

satisfies the inequality

$$|f(x)| < \epsilon$$

if $0 < x < \delta = \epsilon/2$. However, this does not mean that $\lim_{x\to 0} f(x) = 0$, since f is not defined for negative x, as it must be to satisfy the conditions of Definition~ with $x_0 = 0$ and $L = 0$. The function

$$g(x) = x + \frac{|x|}{x}, \quad x \neq 0,$$

can be rewritten as

$$g(x) = \left\{\casespace \begin{array}{ll} x+1, & x > 0, \\ x-1, & x < 0; \end{array}\right.$$

hence, every open interval containing $x_0 = 0$ also contains points x_1 and x_2 such that $|g(x_1) - g(x_2)|$ is as close to 2 as we please. Therefore, $\lim_{x\to x_0} g(x)$ does not exist (Exercise~).

Although $f(x)$ and $g(x)$ do not approach limits as x approaches zero, they each exhibit a definite sort of limiting behavior for small positive values of x, as does $g(x)$ for small negative values of x. The kind of behavior we have in mind is defined precisely as follows.

6pt

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Figure~ shows the graph of a function that has distinct left- and right-hand limits at a point x_0.

Left- and right-hand limits are also called {}. We will often simplify the notation by writing

$$\lim_{x\to x_0-} f(x) = f(x_0-) \quad \text{and} \quad \lim_{x\to x_0+} f(x) = f(x_0+).$$

The following theorem states the connection between limits and one-sided limits. We leave the proof to you (Exercise~).

With only minor modifications of their proofs (replacing the inequality $0 < |x - x_0| < \delta$ by $x_0 - \delta < x < x_0$ or $x_0 < x < x_0 + \delta$), it can be shown that the assertions of Theorems~ and remain valid if $\lim_{x\to x_0}$'' is replaced by $\lim_{x\to x_0-}$''or ``$\lim_{x\to x_0+}$'' throughout (Exercise~).

Limits and one-sided limits have to do with the behavior of a function f near a limit point of D_f. It is equally reasonable to study f for large positive values of x if D_f is unbounded above or for large negative values of x if D_f is unbounded below.

Figure~ provides an illustration of the situation described in Definition~.

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We leave it to you to define the statement ``$\lim_{x\to-\infty} f(x) = L$'' (Exercise~) and to show that Theorems~ and remain valid if x_0 is replaced throughout by ∞ or $-\infty$ (Exercise~).

We will sometimes denote $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ by $f(\infty)$ and $f(-\infty)$, respectively.

-.3em The functions

$$f(x) = \frac{1}{x}, \quad g(x) = \frac{1}{x^2}, \quad p(x) = \sin\frac{1}{x},$$

and

$$q(x) = \frac{1}{x^2}\sin\frac{1}{x}$$

do not have limits, or even one-sided limits, at $x_0 = 0$. They fail to have limits in different ways:

The kind of behavior exhibited by f and g near $x_0 = 0$ is sufficiently common and simple to lead us to define {}.

Throughout this book, "$\lim_{x\to x_0} f(x)$ exists" will mean that

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To leave open the possibility that $L = \pm\infty$, we will say that

\/ is only supported in math mode

This convention also applies to one-sided limits and limits as x approaches $\pm\infty$.

We mentioned earlier that Theorems~ and remain valid if `$\lim_{x\to x_0}$'' is replaced by` $\lim_{x\to x_0-}$''or `` $\lim_{x\to x_0+}$.'' They are also valid with x_0 replaced by $\pm\infty$. Moreover, the counterparts of , , and in all these versions of Theorem~ remain valid if either or both of L_1 and L_2 are infinite, provided that their right sides are not indeterminate (Exercises~ and). Equation and its counterparts remain valid if L_1/L_2 is not indeterminate and $L_2 \neq 0$ (Exercise~).

-.3em A function f is {} on an interval I if

$$f(x_1) \le f(x_2) \quad \text{whenever } x_1 \text{ and } x_2 \text{ are in } I \text{ and } x_1 < x_2, \tag{2.1.6}$$

or {} on I if

$$f(x_1) \ge f(x_2) \quad \text{whenever } x_1 \text{ and } x_2 \text{ are in } I \text{ and } x_1 < x_2. \tag{2.1.7}$$

In either case, f is on I. If $\le$ can be replaced by $<$ in , f is {} on I. If $\ge$ can be replaced by $>$ in , f is {} on I. In either of these two cases, f is {} on I.

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In the proof of the following theorem, we assume that you have formulated the definitions called for in Exercise~.

We first show that $f(a+) = \alpha$. If

$M > \alpha$, there is an x_0 in (a, b) such that $f(x_0) < M$. Since f is nondecreasing, $f(x) < M$ if $a < x < x_0$. Therefore, if $\alpha = -\infty$, then $f(a+) = -\infty$. If $\alpha > -\infty$, let $M = \alpha + \epsilon$, where $\epsilon > 0$. Then $\alpha \le f(x) < \alpha + \epsilon$, so

$$|f(x) - \alpha| < \epsilon \quad \text{if} \quad a < x < x_0. \tag{2.1.8}$$

If $a = -\infty$, this implies that $f(-\infty) = \alpha$. If $a > -\infty$, let $\delta = x_0 - a$. Then is equivalent to

$$|f(x) - \alpha| < \epsilon \quad \text{if} \quad a < x < a + \delta,$$

which implies that $f(a+) = \alpha$.

We now show that $f(b-) = \beta$. If $M < \beta$, there is an x_0 in (a, b) such that $f(x_0) > M$. Since f is nondecreasing, $f(x) > M$ if $x_0 < x < b$. Therefore, if $\beta = \infty$, then $f(b-) = \infty$. If $\beta < \infty$, let $M = \beta - \epsilon$, where $\epsilon > 0$. Then $\beta - \epsilon < f(x) \le \beta$, so

$$|f(x) - \beta| < \epsilon \quad \text{if} \quad x_0 < x < b. \tag{2.1.9}$$

If $b = \infty$, this implies that $f(\infty) = \beta$. If $b < \infty$, let $\delta = b - x_0$. Then is equivalent to

$$|f(x) - \beta| < \epsilon \quad \text{if} \quad b - \delta < x < b,$$

which implies that $f(b-) = \beta$.

The proof is similar to the proof of

(Exercise~).

Suppose that f is nondecreasing. Applying

to f on (a, x_0) and (x_0, b) separately shows that

$$f(x_0-) = \sup_{a<x<x_0} f(x) \quad \text{and} \quad f(x_0+) = \inf_{x_0<x<b} f(x).$$

However, if $x_1 < x_0 < x_2$, then

$$f(x_1) \le f(x_0) \le f(x_2);$$

hence,

$$f(x_0-) \le f(x_0) \le f(x_0+).$$

We leave the case where f is nonincreasing to you (Exercise~).

We now introduce some concepts related to limits. We leave the study of these concepts mainly to the exercises.

We say that f is {} on a set S if there is a constant $M < \infty$ such that $|f(x)| \le M$ for all x in S.

Since $S_f(x_1; x_0)$ is the supremum of $\{f(t)\} x_1 < t < x_0$, there is an $\overline{x}$ in $[x_1, x_0)$ such that

$$f(\overline{x}) > S_f(x_1; x_0) - \epsilon/2.$$

This and imply that $f(\overline{x}) > \beta - \epsilon$. Since $\overline{x}$ is in $[a_1, x_0)$, this proves

.

Now we show that there cannot be more than one real number with properties and . Suppose that $\beta_1 < \beta_2$ and β_2 has property ; thus, if $\epsilon > 0$ and a_1 is in $[a, x_0)$, there is an $\overline{x}$ in $[a_1, x_0)$ such that $f(\overline{x}) > \beta_2 - \epsilon$. Letting $\epsilon = \beta_2 - \beta_1$, we see that there is an $\overline{x}$ in $[a_1, b)$ such that

$$f(\overline{x}) > \beta_2 - (\beta_2 - \beta_1) = \beta_1,$$

so β_1 cannot have property . Therefore, there cannot be more than one real number that satisfies both and

.

The proof of the following theorem is similar to this (Exercise~).

In this section we study continuous functions of a real variable. We will prove some important theorems about continuous functions that, although intuitively plausible, are beyond the scope of the elementary calculus course. They are accessible now because of our better understanding of the real number system, especially of those properties that stem from the completeness axiom.

The definitions of

$$f(x_0-) = \lim_{x\to x_0-} f(x), \quad f(x_0+) = \lim_{x\to x_0+} f(x), \quad \text{and} \quad \lim_{x\to x_0} f(x)$$

do not involve $f(x_0)$ or even require that it be defined. However, the case where $f(x_0)$ is defined and equal to one or more of these quantities is important.

The following theorem provides a method for determining whether these definitions are satisfied. The proof, which we leave to you (Exercise~), rests on Definitions~, , and .

From Definition~ and Theorem~, f is

continuous at x_0 if and only if

$$f(x_0-) = f(x_0+) = f(x_0)$$

or, equivalently, if and only if it is continuous from the right and left at x_0 (Exercise~).

6pt

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The function f defined in Example~ (see also Figure~) is continuous on $[0,1)$ and $[1,2]$, but not on any open interval containing 1. The discontinuity of f there is of the simplest kind, described in the following definition.

6pt

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The reason for the adjective ``jump'' can be seen in Figures~ and , where the graphs exhibit a definite jump at each point of discontinuity. The next example shows that not all discontinuities are of this kind.

Theorems~ and imply the next theorem (Exercise~).

Let f be defined on a deleted neighborhood of x_0 and discontinuous (perhaps even undefined) at x_0. We say that f has a at x_0 if $\lim_{x\to x_0} f(x)$ exists. In this case, the function

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is continuous at x_0.

We have seen that the investigation of limits and continuity can be simplified by regarding a given function as the result of addition, subtraction, multiplication, and division of simpler functions. Another operation useful in this connection is {} of functions; that is, substitution of one function into another.

The next theorem says that the composition of continuous functions is continuous.

Suppose that $\epsilon > 0$. Since $g(x_0)$ is an interior point of D_f and f is continuous at $g(x_0)$, there is a $\delta_1 > 0$ such that $f(t)$ is defined and

$$|f(t) - f(g(x_0))| < \epsilon \quad \text{if} \quad |t - g(x_0)| < \delta_1. \tag{2.1.10}$$

Since g is continuous at x_0, there is a $\delta > 0$ such that $g(x)$ is defined and

$$|g(x) - g(x_0)| < \delta_1 \quad \text{if} \quad |x - x_0| < \delta. \tag{2.1.11}$$

Now and imply that

$$|f(g(x)) - f(g(x_0))| < \epsilon \quad \text{if} \quad |x - x_0| < \delta.$$

Therefore, $f \circ g$ is continuous at x_0.

See Exercise~ for a related result concerning limits.

A function f is {} on a set S if there is a real number m such that

$$f(x) \geq m \text{ for all } x \in S.$$

In this case, the set

$$V = \{f(x)\} x \in S$$

has an infimum α, and we write

$$\alpha = \inf_{x\in S} f(x).$$

If there is a point x_1 in S such that $f(x_1) = \alpha$, we say that α is the {}, and write

$$\alpha = \min_{x\in S} f(x).$$

Similarly, f is {} if there is a real number M such that $f(x) \leq M$ for all x in S. In this case, V has a supremum β, and we write

$$\beta = \sup_{x\in S} f(x).$$

If there is a point x_2 in S such that $f(x_2) = \beta$, we say that β is the {}, and write

$$\beta = \max_{x\in S} f(x).$$

If f is bounded above and below on a set S, we say that f is {} on S.

Figure~ illustrates the geometric meaning of these definitions for a function f bounded on an interval $S=[a,b]$. The graph of f lies in the strip bounded by the lines $y=M$ and $y=m$, where M is any upper bound and m is any lower bound for f on $[a,b]$. The narrowest strip containing the graph is the one bounded above by $y=\beta=\sup_{a\le x\le b} f(x)$ and below by $y=\alpha=\inf_{a\le x\le b} f(x)$.

6pt

12pt

6pt

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Suppose that $t\in[a,b]$. Since f is continuous at t, there is an open interval I_t containing t such that

$$|f(x)-f(t)|<1 \quad \text{if} \quad x\in I_t\cap[a,b]. \tag{2.1.12}$$

(To see this, set $\epsilon=1$ in , Theorem~.) The collection $\mathcal{H}=\{I_t\}a\le t\le b$ is an open covering of $[a,b]$. Since $[a,b]$ is compact, the Heine–Borel theorem implies that there are finitely many points $t_1, t_2, , t_n$ such that the intervals $I_{t_1}, I_{t_2}, , I_{t_n}$ cover $[a,b]$. According to with $t=t_i$,

$$|f(x)-f(t_i)|<1 \quad \text{if} \quad x\in I_{t_i}\cap[a,b].$$

Therefore,

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Let

$$M=1+\max_{1\le i\le n}|f(t_i)|.$$

Since $[a,b]\subset\bigcup_{i=1}^n (I_{t_i}\cap[a,b])$, implies that $|f(x)|\le M$ if $x\in[a,b]$.

This proof illustrates the utility of the Heine–Borel theorem, which allows us to choose M as the largest of a {} set of numbers.

Theorem~ and the completeness of the reals imply that

if f is continuous on a finite closed interval $[a,b]$, then f has an infimum and a supremum on $[a,b]$. The next theorem shows that f actually assumes these values at some points in $[a,b]$.

We show that x_1 exists and leave it to you to show that x_2 exists (Exercise~).

Suppose that there is no x_1 in $[a,b]$ such that $f(x_1)=\alpha$. Then $f(x)>\alpha$ for all $x\in[a,b]$. We will show that this leads to a contradiction.

Suppose that $t\in[a,b]$. Then $f(t)>\alpha$, so

$$f(t)>\frac{f(t)+\alpha}{2}>\alpha.$$

Since f is continuous at t, there is an open interval I_t about t such that

$$f(x)>\frac{f(t)+\alpha}{2} \quad \text{if} \quad x\in I_t\cap[a,b] \tag{2.1.13}$$

(Exercise~). The collection $\mathcal{H}=\{I_t\}a\le t\le b$ is an open covering of $[a,b]$. Since $[a,b]$ is compact, the Heine–Borel theorem implies that there are finitely many points $t_1, t_2, , t_n$ such that the intervals $I_{t_1}, I_{t_2}, , I_{t_n}$ cover $[a,b]$. Define

$$\alpha_1=\min_{1\le i\le n}\frac{f(t_i)+\alpha}{2}.$$

Then, since $[a,b]\subset\bigcup_{i=1}^n(I_{t_i}\cap[a,b])$, implies that

$$f(t)>\alpha_1, \quad a\le t\le b.$$

But $\alpha_1>\alpha$, so this contradicts the definition of α. Therefore, $f(x_1)=\alpha$ for some x_1 in $[a,b]$.

The next theorem shows that if f is continuous on a finite closed interval $[a,b]$, then f assumes every value between $f(a)$ and $f(b)$ as x varies from a to b (Figure, page).

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Suppose that $f(a) < \mu < f(b)$. The set

$$S = \{x\} a \le x \le b \quad \text{and} \quad f(x) \le \mu$$

is bounded and nonempty. Let $c = \sup S$. We will show that $f(c) = \mu$. If $f(c) > \mu$, then $c > a$ and, since f is continuous at c, there is an $\epsilon > 0$ such that $f(x) > \mu$ if $c - \epsilon < x \le c$ (Exercise~). Therefore, $c - \epsilon$ is an upper bound for S, which contradicts the definition of c as the supremum of S. If $f(c) < \mu$, then $c < b$ and there is an $\epsilon > 0$ such that $f(x) < \mu$ for $c \le x < c + \epsilon$, so c is not an upper bound for S. This is also a contradiction. Therefore, $f(c) = \mu$.

The proof for the case where $f(b) < \mu < f(a)$ can be obtained by applying this result to $-f$.

Theorem~ and Definition~ imply that a

function f is continuous on a subset S of its domain if for each $\epsilon > 0$ and each x_0 in S, there is a $\delta > 0$, {}, such that

$$|f(x) - f(x_0)| < \epsilon \quad \text{if} \quad |x - x_0| < \delta \quad \text{and} \quad x \in D_f.$$

The next definition introduces another kind of continuity on a set S.

We emphasize that in this definition δ depends only on ϵ and S and not on the particular choice of x and x', provided that they are both in S.

Often a concept is clarified by considering its negation: a function f is {} uniformly continuous on S if there is an $\epsilon_0 > 0$ such that if δ is any positive number, there are points x and x' in S such that

$$|x - x'| < \delta \quad \text{but} \quad |f(x) - f(x')| \ge \epsilon_0.$$

Examples~ and show that a function may be continuous but not uniformly continuous on an interval. The next theorem shows that this cannot happen if the interval is closed and bounded, and therefore compact.

Suppose that $\epsilon > 0$. Since f is continuous on $[a,b]$, for each t in $[a,b]$ there is a positive number δ_t such that

$$|f(x) - f(t)| < \frac{\epsilon}{2} \quad \text{if} \quad |x - t| < 2\delta_t \quad \text{and} \quad x \in [a,b]. \tag{2.1.14}$$

If $I_t = (t - \delta_t, t + \delta_t)$, the collection

$$\mathcal{H} = \{I_t\} t \in [a,b]$$

is an open covering of $[a,b]$. Since $[a,b]$ is compact, the Heine–Borel theorem implies that there are finitely many points $t_1, t_2, , t_n$ in $[a,b]$ such that $I_{t_1}, I_{t_2}, , I_{t_n}$ cover $[a,b]$. Now define

$$\delta = \min\{\delta_{t_1}, \delta_{t_2}, \dots, \delta_{t_n}\}. \tag{2.1.15}$$

We will show that if

$$|x - x'| < \delta \quad \text{and} \quad x, x' \in [a,b], \tag{2.1.16}$$

then $|f(x) - f(x')| < \epsilon$.

From the triangle inequality,

$$\begin{aligned} |f(x) - f(x')| \ar = |\,(f(x) - f(t_r)) + (f(t_r) - f(x'))\,| \\ \ar \le |f(x) - f(t_r)| + |f(t_r) - f(x')|. \end{aligned} \tag{2.1.17}$$

Since $I_{t_1}, I_{t_2}, , I_{t_n}$ cover $[a,b]$, x must be in one of these intervals. Suppose that $x \in I_{t_r}$; that is,

$$|x - t_r| < \delta_{t_r}. \tag{2.1.18}$$

From with $t = t_r$,

$$|f(x)-f(t_r)|<\frac{\epsilon}{2}. \tag{2.1.19}$$

From , , and the triangle inquality,

$$|x'-t_r|=|(x'-x)+(x-t_r)|\le|x'-x|+|x-t_r|<\delta+\delta_{t_r}\le 2\delta_{t_r}.$$

Therefore, with $t=t_r$ and x replaced by x' implies that

$$|f(x')-f(t_r)|<\frac{\epsilon}{2}.$$

This, , and imply that $|f(x)-f(x')|<\epsilon$.

This proof again shows the utility of the Heine–Borel theorem, which allowed us to define δ in as the smallest of a {} set of positive numbers, so that δ is sure to be positive. (An infinite set of positive numbers may fail to have a smallest positive member; for example, consider the open interval $(0,1)$.)

Applied to Example~, Corollary~ implies that the function $g(x)=\cos 1/x$ is uniformly continuous on $[\rho,1]$ if $0<\rho<1$.

Theorem~ implies that if f is monotonic on an interval I, then f is either continuous or has a jump discontinuity at each x_0 in I. This and Theorem~ provide the key to the proof of the following theorem.

We assume that f is nondecreasing, and leave the case where f is nonincreasing to you (Exercise). Theorem

implies that the set $\widetilde{R}_f=\{f(x)\}x\in(a,b)$ is a subset of the open interval $(f(a+),f(b-))$. Therefore,

$$R_f=\{f(a)\}\cup\widetilde{R}_f\cup\{f(b)\}\subset\{f(a)\}\cup(f(a+),f(b-))\cup\{f(b)\}. \tag{2.1.20}$$

Now suppose that f is continuous on $[a,b]$. Then $f(a)=f(a+)$, $f(b-)=f(b)$, so implies that $R_f\subset[f(a),f(b)]$. If $f(a)<\mu<f(b)$, then Theorem~ implies that $\mu=f(x)$ for some x in (a,b). Hence, $R_f=[f(a),f(b)]$.

For the converse, suppose that $R_f=[f(a),f(b)]$. Since $f(a)\le f(a+)$ and $f(b-)\le f(b)$, implies that $f(a)=f(a+)$ and $f(b-)=f(b)$. We know from Theorem~

that if f is nondecreasing and $a<x_0<b$, then

$$f(x_0-)\le f(x_0)\le f(x_0+).$$

If either of these inequalities is strict, R_f cannot be an interval. Since this contradicts our assumption, $f(x_0-)=f(x_0)=f(x_0+)$. Therefore, f is continuous at x_0 (Exercise~). We can now conclude that f is continuous on $[a,b]$.

Theorem~ implies the following theorem.

We first show that there is a function g satisfying and . Since f is continuous, Theorem~ implies that for each y_0 in $[c,d]$ there is an x_0 in $[a,b]$ such that

$$f(x_0)=y_0, \tag{2.1.21}$$

and, since f is increasing, there is only one such x_0. Define

$$g(y_0)=x_0. \tag{2.1.22}$$

The definition of x_0 is illustrated in Figure~: with $[c,d]$ drawn on the y-axis, find the intersection of the line $y=y_0$ with the curve $y=f(x)$ and drop a vertical from the intersection to the x-axis to find x_0.

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Substituting into yields

$$f(g(y_0))=y_0,$$

and substituting into yields

$$g(f(x_0))=x_0.$$

Dropping the subscripts in these two equations yields and .

The uniqueness of g follows from our assumption that f is increasing, and therefore only one value of x_0 can satisfy for each y_0.

To see that g is increasing, suppose that $y_1 < y_2$ and let x_1 and x_2 be the points in $[a,b]$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since f is increasing, $x_1 < x_2$. Therefore,

$$g(y_1) = x_1 < x_2 = g(y_2),$$

so g is increasing. Since $R_g = \{g(y) \} y \in [c,d]$ is the interval $[g(c), g(d)] = [a,b]$, Theorem~ with f and $[a,b]$ replaced by g and $[c,d]$ implies that g is continuous on $[c,d]$.

The function g of Theorem~ is the {} of f, denoted by f^{-1}. Since and are symmetric in f and g, we can also regard f as the inverse of g, and denote it by g^{-1}.

In calculus you studied differentiation, emphasizing rules for calculating derivatives. Here we consider the theoretical properties of differentiable functions. In doing this, we assume that you know how to differentiate elementary functions such as x^n, e^x, and $\sin x$, and we will use such functions in examples.

If f is defined on an open set S, we say that f is {} S if f is differentiable at every point of S. If f is differentiable on S, then f' is a function on S. We say that f is {} on S if f' is continuous on S. If f is differentiable on a neighborhood of x_0, it is reasonable to ask if f' is differentiable at x_0. If so, we denote the derivative of f' at x_0 by $f''(x_0)$. This is the {}, and it is also denoted by $f^{(2)}(x_0)$. Continuing inductively, if $f^{(n-1)}$ is defined on a neighborhood of x_0, then the nth {}, denoted by $f^{(n)}(x_0)$, is the derivative of $f^{(n-1)}$ at x_0. For convenience we define the {} of f to be f itself; thus

$$f^{(0)} = f.$$

We assume that you are familiar with the other standard notations for derivatives; for example,

$$f^{(2)} = f'', \quad f^{(3)} = f''',$$

and so on, and

$$\frac{d^n f}{dx^n} = f^{(n)}.$$

To derive differentiation formulas for elementary functions such as $\sin x$, $\cos x$, and e^x directly from Definition~ requires estimates based on the properties of these functions. Since this is done in calculus, we will not repeat it here.

If $f(x)$ is the position of a particle at time $x \neq x_0$, the difference quotient

$$\frac{f(x) - f(x_0)}{x - x_0}$$

is the average velocity of the particle between times x_0 and x. As x approaches x_0, the average applies to shorter and shorter intervals. Therefore, it makes sense to regard the limit , if it exists, as the particle's {}. This interpretation may be useful even if x is not time, so we often regard $f'(x_0)$ as the {}, regardless of the specific nature of the variable x. The derivative also has a geometric interpretation. The equation of the line through two points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ on the curve $y = f(x)$ (Figure~) is

$$y = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0).$$

Varying x_1 generates lines through $(x_0, f(x_0))$ that rotate into the line

$$y = f(x_0) + f'(x_0)(x - x_0) \tag{2.1.23}$$

as x_1 approaches x_0. This is the {} to the curve $y = f(x)$ at the point $(x_0, f(x_0))$. Figure~ depicts the situation for various values of x_1.

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Here is a less intuitive definition of the tangent line: If the function

$$T(x) = f(x_0) + m(x - x_0)$$

approximates f so well near x_0 that

$$\lim_{x \to x_0} \frac{f(x) - T(x)}{x - x_0} = 0,$$

we say that the line $y = T(x)$ is {}.

This tangent line exists if and only if $f'(x_0)$ exists, in which case m is uniquely determined by $m = f'(x_0)$ (Exercise~). Thus, is the equation of the tangent line.

We will use the following lemma to study differentiable functions.

Define

Bracket argument to \\ must be a dimension

Solving for $f(x)$ yields if $x \neq x_0$, and is obvious if $x = x_0$. Definition~ implies that $\lim_{x \to x_0} E(x) = 0$. We defined $E(x_0) = 0$ to make E continuous at x_0.

Since the right side of is continuous at x_0, so is the left. This yields the following theorem.

The converse of this theorem is false, since a function may be continuous at a point without being differentiable at the point.

The following theorem should be familiar from calculus.

The proof is accomplished by forming the appropriate difference quotients and applying Definition~ and Theorem~. We will prove and leave the rest to you (Exercises~, , and).

The trick is to add and subtract the right quantity in the numerator of the difference quotient for $(fg)'(x_0)$; thus,

$$\begin{aligned}\frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} \ar &= \frac{f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)}{x - x_0} \\ \ar &= \frac{f(x) - f(x_0)}{x - x_0} g(x) + f(x_0) \frac{g(x) - g(x_0)}{x - x_0}.\end{aligned}$$

The difference quotients on the right approach $f'(x_0)$ and $g'(x_0)$ as x approaches x_0, and $\lim_{x \to x_0} g(x) = g(x_0)$ (Theorem~). This proves

.

Here is the rule for differentiating a composite function.

Since f is differentiable at $g(x_0)$, Lemma~ implies that

$$f(t) - f(g(x_0)) = [f'(g(x_0)) + E(t)][t - g(x_0)],$$

where

$$\lim_{t \to g(x_0)} E(t) = E(g(x_0)) = 0. \tag{2.1.24}$$

Letting $t = g(x)$ yields

$$f(g(x)) - f(g(x_0)) = [f'(g(x_0)) + E(g(x))][g(x) - g(x_0)].$$

Since $h(x) = f(g(x))$, this implies that

$$\frac{h(x) - h(x_0)}{x - x_0} = [f'(g(x_0)) + E(g(x))] \frac{g(x) - g(x_0)}{x - x_0}. \tag{2.1.25}$$

Since g is continuous at x_0 (Theorem~), and Theorem~ imply that

$$\lim_{x \to x_0} E(g(x)) = E(g(x_0)) = 0.$$

Therefore, implies that

$$h'(x_0) = \lim_{x\to x_0} \frac{h(x) - h(x_0)}{x - x_0} = f'(g(x_0))g'(x_0),$$

as stated.

It may seem reasonable to justify the chain rule by writing

$$\begin{aligned}\frac{h(x) - h(x_0)}{x - x_0} &= \frac{f(g(x)) - f(g(x_0))}{x - x_0}\\ &= \frac{f(g(x)) - f(g(x_0))}{g(x) - g(x_0)}\ \frac{g(x) - g(x_0)}{x - x_0}\end{aligned}$$

and arguing that

$$\lim_{x\to x_0} \frac{f(g(x)) - f(g(x_0))}{g(x) - g(x_0)} = f'(g(x_0))$$

(because $\lim_{x\to x_0} g(x) = g(x_0)$) and

$$\lim_{x\to x_0} \frac{g(x) - g(x_0)}{x - x_0} = g'(x_0).$$

However, this is not a valid proof (Exercise~).

One-sided limits of difference quotients such as and in Example~ are called {} or {}. That is, if f is defined on $[x_0, b)$, the {} is defined to be

$$f'_+(x_0) = \lim_{x\to x_0+} \frac{f(x) - f(x_0)}{x - x_0}$$

if the limit exists, while if f is defined on $(a, x_0]$, the {} is defined to be

$$f'_-(x_0) = \lim_{x\to x_0-} \frac{f(x) - f(x_0)}{x - x_0}$$

if the limit exists. Theorem~ implies that f is differentiable at x_0 if and only if $f'_+(x_0)$ and $f'_-(x_0)$ exist and are equal, in which case

$$f'(x_0) = f'_+(x_0) = f'_-(x_0).$$

In Example~, $f'_+(0) = 1$ and $f'_-(0) = -1$.

This example shows that there is a difference between a one-sided derivative and a one-sided limit of a derivative, since $f'_+(0) = 0$, but, from , $f'(0+) = \lim_{x\to 0+} f'(x)$ does not exist. It also shows that a derivative may exist in a neighborhood of a point x_0 ($= 0$ in this case), but be discontinuous at x_0.

Exercise~ justifies the method used in Example~ to compute $f'(x)$ for $x \neq 0$.

We say that $f(x_0)$ is a {} of f if there is a $\delta > 0$ such that $f(x) - f(x_0)$ does not change sign on

$$(x_0 - \delta, x_0 + \delta) \cap D_f. \tag{2.1.26}$$

More specifically, $f(x_0)$ is a {} of f if

$$f(x) \le f(x_0) \tag{2.1.27}$$

or a {} of f if

$$f(x) \ge f(x_0) \tag{2.1.28}$$

for all x in the set . The point x_0 is called a {} of f, or, more specifically, a {} or {} of f.

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It is geometrically plausible that if the curve $y = f(x)$ has a tangent at a local extreme point of f, then the tangent must be horizontal; that is, have zero slope. (For example, in Figure~, see $x = 1$, $x = 3$, and every x in $(-1, -1/2)$.) The following theorem shows that this must be so. 4pt

4pt

We will show that x_0 is not a local extreme point of f if $f'(x_0) \neq 0$. From Lemma~,

$$\frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) + E(x), \tag{2.1.29}$$

where $\lim_{x \to x_0} E(x) = 0$. Therefore, if $f'(x_0) \neq 0$, there is a $\delta > 0$ such that

$$|E(x)| < |f'(x_0)| \quad \text{if} \quad |x - x_0| < \delta,$$

and the right side of must have the same sign as $f'(x_0)$ for $|x - x_0| < \delta$. Since the same is true of the left side, $f(x) - f(x_0)$ must change sign in every neighborhood of x_0 (since $x - x_0$ does). Therefore, neither nor can hold for all x in any interval about x_0. 4pt

If $f'(x_0) = 0$, we say that x_0 is a {} of f. Theorem~ says that every local extreme point of f at which f is differentiable is a critical point of f. The converse is false. For example, 0 is a critical point of $f(x) = x^3$, but not a local extreme point. 4pt

The use of Theorem~ for finding local extreme points is covered in calculus, so we will not pursue it here. However, we will use Theorem~ to prove the following fundamental theorem, which says that if a curve $y = f(x)$ intersects a horizontal line at $x = a$ and $x = b$ and has a tangent at $(x, f(x))$ for every x in (a, b), then there is a point c in (a, b) such that the tangent to the curve at $(c, f(c))$ is horizontal (Figure~).

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Since f is continuous on $[a, b]$, f attains a maximum and a minimum value on $[a, b]$ (Theorem~). If these two extreme values are the same, then f is constant on (a, b), so $f'(x) = 0$ for all x in (a, b). If the extreme values differ, then at least one must be attained at some point c in the open interval (a, b), and $f'(c) = 0$, by Theorem~.

A derivative may exist on an interval $[a, b]$ without being continuous on $[a, b]$. Nevertheless, an intermediate value theorem similar to Theorem~ applies to derivatives.

Suppose first that

$$f'(a) < \mu < f'(b) \tag{2.1.30}$$

and define

$$g(x) = f(x) - \mu x.$$

Then

$$g'(x) = f'(x) - \mu, \quad a \leq x \leq b, \tag{2.1.31}$$

and implies that

$$g'(a) < 0 \quad \text{and} \quad g'(b) > 0. \tag{2.1.32}$$

Since g is continuous on $[a, b]$, g attains a minimum at some point c in $[a, b]$. Lemma~ and imply that there is a $\delta > 0$ such that

$$g(x) < g(a), \quad a < x < a + \delta, \quad \text{and} \quad g(x) < g(b), \quad b - \delta < x < b$$

(Exercise~), and therefore $c \neq a$ and $c \neq b$. Hence, $a < c < b$, and therefore $g'(c) = 0$, by Theorem~. From , $f'(c) = \mu$.

The proof for the case where $f'(b) < \mu < f'(a)$ can be obtained by applying this result to $-f$.

The function

$$h(x) = [g(b) - g(a)]f(x) - [f(b) - f(a)]g(x)$$

is continuous on $[a, b]$ and differentiable on (a, b), and

$$h(a) = h(b) = g(b)f(a) - f(b)g(a).$$

Therefore, Rolle's theorem implies that $h'(c) = 0$ for some c in (a, b). Since

$$h'(c) = [g(b) - g(a)]f'(c) - [f(b) - f(a)]g'(c),$$

this implies .

The following special case of Theorem~ is important enough to be stated separately.

Apply Theorem~ with $g(x) = x$.

Theorem~ implies that the tangent to the curve $y = f(x)$ at $(c, f(c))$ is parallel to the line connecting the points $(a, f(a))$ and $(b, f(b))$ on the curve (Figure~, page).

If f is differentiable on (a, b) and x_1, $x_2 \in (a, b)$ then f is continuous on the closed interval with endpoints x_1 and x_2 and differentiable on its interior. Hence, the mean value theorem implies that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

for some c between x_1 and x_2. (This is true whether $x_1<x_2 $ or $x_2 < x_1$.) The next three theorems follow from this.

A function that satisfies an inequality like for all x and x' in an interval is said to satisfy a {} on the interval.

6pt

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The method of Theorem~ for finding limits of the sum, difference, product, and quotient of functions breaks down in connection with indeterminate forms. The generalized mean value theorem (Theorem~) leads to a method for evaluating limits of indeterminate forms.

We prove the theorem for finite L and leave the case where $L = \pm\infty$ to you (Exercise~).

Suppose that $\epsilon > 0$. From , there is an x_0 in (a, b) such that

$$\left|\frac{f'(c)}{g'(c)} - L\right| < \epsilon \quad \text{if} \quad x_0 < c < b. \tag{2.1.33}$$

Theorem~ implies that if x and t are in $[x_0, b)$, then there is a c between them, and therefore in (x_0, b), such that

$$[g(x) - g(t)]f'(c) = [f(x) - f(t)]g'(c). \tag{2.1.34}$$

Since g' has no zeros in (a, b), Theorem~ implies that

$$g(x) - g(t) \neq 0 \quad \text{if} \quad x, t \in (a, b).$$

This means that g cannot have more than one zero in (a, b). Therefore, we can choose x_0 so that, in addition to , g has no zeros in $[x_0, b)$. Then can be rewritten as

$$\frac{f(x) - f(t)}{g(x) - g(t)} = \frac{f'(c)}{g'(c)},$$

so implies that

$$\left|\frac{f(x) - f(t)}{g(x) - g(t)} - L\right| < \epsilon \quad \text{if} \quad x, t \in [x_0, b). \tag{2.1.35}$$

If holds, let x be fixed in $[x_0, b)$, and consider the function

$$G(t) = \frac{f(x) - f(t)}{g(x) - g(t)} - L.$$

From ,

$$\lim_{t \to b-} f(t) = \lim_{t \to b-} g(t) = 0,$$

so

$$\lim_{t\to b-} G(t) = \frac{f(x)}{g(x)} - L. \tag{2.1.36}$$

Since

$$|G(t)| < \epsilon \quad \text{if} \quad x_0 < t < b,$$

because of , implies that

$$\left|\frac{f(x)}{g(x)} - L\right| \le \epsilon.$$

This holds for all x in (x_0, b), which implies .

The proof under assumption is more complicated. Again choose x_0 so that holds and g has no zeros in $[x_0, b)$. Letting $t = x_0$ in , we see that

$$\left|\frac{f(x) - f(x_0)}{g(x) - g(x_0)} - L\right| < \epsilon \quad \text{if} \quad x_0 \le x < b. \tag{2.1.37}$$

Since $\lim_{x\to b-} f(x) = \pm\infty$, we can choose $x_1 > x_0$ so that $f(x) \ne 0$ and $f(x) \ne f(x_0)$ if $x_1 < x < b$. Then the function

$$u(x) = \frac{1 - g(x_0)/g(x)}{1 - f(x_0)/f(x)}$$

is defined and nonzero if $x_1 < x < b$, and

$$\lim_{x\to b-} u(x) = 1, \tag{2.1.38}$$

because of .

Since

$$\frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f(x)}{g(x)}\,\frac{1 - f(x_0)/f(x)}{1 - g(x_0)/g(x)} = \frac{f(x)}{g(x)u(x)},$$

implies that

$$\left|\frac{f(x)}{g(x)u(x)} - L\right| < \epsilon \quad \text{if} \quad x_1 < x < b,$$

which can be rewritten as

$$\left|\frac{f(x)}{g(x)} - Lu(x)\right| < \epsilon|u(x)| \quad \text{if} \quad x_1 < x < b. \tag{2.1.39}$$

From this and the triangle inequality,

$$\left|\frac{f(x)}{g(x)} - L\right| \le \left|\frac{f(x)}{g(x)} - Lu(x)\right| + |Lu(x) - L| \le \epsilon|u(x)| + |L|\,|u(x) - 1|. \tag{2.1.40}$$

Because of , there is a point x_2 in (x_1, b) such that

$$|u(x) - 1| < \epsilon \quad \text{and therefore} \quad |u(x)| < 1 + \epsilon \quad \text{if} \quad x_2 < x < b.$$

This, , and imply that

$$\left|\frac{f(x)}{g(x)} - L\right| < \epsilon(1 + \epsilon) + |L|\epsilon \quad \text{if} \quad x_2 < x < b,$$

which proves under assumption .

Theorem~ and the proof given here remain valid if $b = \infty$ and `$x\to b-$'' is replaced by` $x \to \infty$'' throughout. Only minor changes in the proof are required to show that similar theorems are valid for limits from the right, limits at $-\infty$, and ordinary

(two-sided) limits. We will take these as given.

-.4em We say that f/g {} $x \to b-$ if

$$\lim_{x\to b-} f(x) = \lim_{x\to b-} g(x) = 0,$$

or {} if

$$\lim_{x\to b-} f(x) = \pm\infty$$

and

$$\lim_{x\to b-} g(x) = \pm\infty.$$

The corresponding definitions for $x \to b+$ and $x \to \pm\infty$ are similar. If f/g is of one of these forms as $x \to b-$ and as $x \to b+$, then we say that it is of that form as $x \to b$.

In using L'Hospital's rule we usually write, for example,

$$\lim_{x\to b} \frac{f(x)}{g(x)} = \lim_{x\to b} \frac{f'(x)}{g'(x)} \tag{2.1.41}$$

and then try to find the limit on the right. This is convenient, but technically incorrect, since is true only if the limit on the right exists in the extended reals. It may happen that the limit on the left exists but the one on the right does not. In this case, is incorrect.

We say that a product fg {} if one of the factors approaches 0 and the other approaches $\pm\infty$ as $x \to b-$. In this case, it may be useful to apply L'Hospital's rule after writing

$$f(x)g(x) = \frac{f(x)}{1/g(x)} \quad \text{or} \quad f(x)g(x) = \frac{g(x)}{1/f(x)},$$

since one of these ratios is of the form $0/0$ and the other is of the form ∞/∞ as $x \to b-$.

Similar statements apply to limits as $x \to b+$, $x \to b$, and $x \to \pm\infty$.

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A difference $f - g$ {} if

$$\lim_{x\to b-} f(x) = \lim_{x\to b-} g(x) = \pm\infty.$$

In this case, it may be possible to manipulate $f - g$ into an expression that is no longer indeterminate, or is of the form $0/0$ or ∞/∞ as $x \to b-$. Similar remarks apply to limits as $x \to b+$, $x \to b$, or $x \to \pm\infty$.

The function f^g is defined by

$$f(x)^{g(x)} = e^{g(x)\log f(x)} = \exp(g(x)\log f(x))$$

for all x such that $f(x) > 0$. Therefore, if f and g are defined and $f(x) > 0$ on an interval (a, b), Exercise~ implies that

$$\lim_{x\to b-} [f(x)]^{g(x)} = \exp\left(\lim_{x\to b-} g(x)\log f(x)\right) \tag{2.1.42}$$

if $\lim_{x\to b-} g(x)\log f(x)$ exists in the extended reals. (If this limit is $\pm\infty$ then is valid if we define $e^{-\infty} = 0$ and $e^{\infty} = \infty$.) The product $g\log f$ can be of the form $0\cdot\infty$ in three ways as $x \to b-$: 10pt

10pt In these three cases, we say that f^g {}. Similar definitions apply to limits as $x \to b+$, $x \to b$, and $x \to \pm\infty$.

10pt

10pt

\begin{exerciselist}

Prove Theorem~ for the case where $\lim_{x\to b-} f'(x)/g'(x) = \pm\infty$.

$\dst \lim_{x\to 0} \frac{\tan^{-1} x}{\sin^{-1} x}$ & $\dst \lim_{x\to 0} \frac{1-\cos x}{\log(1+x^2)}$ &

$\dst \lim_{x\to 0+} \frac{1+\cos x}{e^x-1}$\ \end{tabular}

$\dst \lim_{x\to \pi} \frac{\sin nx}{\sin x}$ & $\dst \lim_{x\to 0} \frac{\log(1+x)}{x}$ &

$\dst \lim_{x\to \infty} e^x \sin e^{-x^2}$\ \end{tabular}

$\dst \lim_{x\to \infty} x \sin(1/x)$& $\dst \lim_{x\to \infty} \sqrt{x}(e^{-1/x}-1)$&

$\dst \lim_{x\to 0+} \tan x \log x$ \end{tabular}

$\dst \lim_{x\to \pi} \sin x \log(|\tan x|)$&

$\dst \lim_{x\to 0+} \left[\frac{1}{x} + \log(\tan x)\right]$\ \end{tabular}

$\dst \lim_{x\to \infty} (\sqrt{x+1} - \sqrt{x})$&

$\dst \lim_{x\to 0} \left(\frac{1}{e^x-1} - \frac{1}{x}\right)$\ \end{tabular}

$\dst \lim_{x\to 0} (\cot x - \csc x)$&

$\dst \lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$\ \end{tabular}

$\dst \lim_{x\to \pi} |\sin x|^{\tan x}$&

$\dst \lim_{x\to \pi/2} |\tan x|^{\cos x}$\ \end{tabular}

$\dst \lim_{x\to 0} |\sin x|^{x}$&

$\dst \lim_{x\to 0} (1+x)^{1/x}$\ \end{tabular}

$\dst \lim_{x\to \infty} x^{\sin(1/x)}$&

$\dst \lim_{x\to 0} \left(\frac{x}{1-\cos x} - \frac{2}{x}\right)$\ \end{tabular}

$\dst \lim_{x\to 0+} x^{\alpha} \log x$&

$\dst \lim_{x\to e} \frac{\log(\log x)}{\sin(x-e)}$\ \end{tabular}

$\dst \lim_{x\to \infty} \left(\frac{x+1}{x-1}\right)^{\sqrt{x^2-1}}$&

$\dst \lim_{x\to 1+} \left(\frac{x+1}{x-1}\right)^{\sqrt{x^2-1}}$\ \end{tabular}

$\dst \lim_{x\to \infty} \frac{(\log x)^{\beta}}{x}$ &

$\dst \lim_{x\to \infty} (\cosh x - \sinh x)$\ \end{tabular}

$\dst \lim_{x\to \infty} (x^{\alpha} - \log x)$&

$\dst \lim_{x\to -\infty} e^{x^2} \sin(e^x)$\ \end{tabular}

$\dst \lim_{x\to \infty} x(x+1)\left[\log(1+1/x)\right]^2$&

$\dst \lim_{x\to 0} \frac{\sin x - x + x^3/6}{x^5}$\ \end{tabular}

$\dst \lim_{x\to \infty} \frac{e^x}{x^{\alpha}}$ &

$\dst \lim_{x\to 3\pi/2-} e^{\tan x} \cos x$\ \end{tabular}

$\dst \lim_{x\to 1+}$ $\dst (\log x)^{\alpha} \log(\log x)$&

$\dst \lim_{x\to \infty} \frac{x^x}{x \log x}$ \end{tabular}

$\dst \lim_{x\to \pi/2} (\sin x)^{\tan x}$

$\dst \lim_{x\to 0} \frac{e^x - \dst\sum_{r=0}^{n} x^r/r!}{x^n}$ $(n = \text{ integer} \geq 1)$

$\dst \lim_{x\to 0} \frac{\sin x - \dst\sum_{r=0}^{n} (-1)^r \frac{x^{2r+1}}{(2r+1)!}}{x^{2n+1}}$ $(n = \text{ integer} \geq 0)$

\dst$\lim_{x\to 0}\frac{e^{-1/x^2}}{x^n}=0 (n=$ integer)

The {} are defined by $L_0(x)=x$ and

$$L_n(x)=\log(L_{n-1}(x)),\quad x>a_n,\quad n\ge 1,$$

where $a_1=0$ and $a_n=e^{a_{n-1}}, n\ge 1$. Show that

Let f be positive and differentiable on $(0,\infty)$, and suppose that

$$\lim_{x\to\infty}\frac{f'(x)}{f(x)}=L,\quad \text{where}\quad 0<L\le\infty.$$

Define $f_0(x)=x$ and

$$f_n(x)=f\left(f_{n-1}(x)\right),\quad n\ge 1.$$

Use L'Hospital's rule to show that

$$\lim_{x\to\infty}\frac{(f_n(x))^\alpha}{f_{n-1}(x)}=\infty\quad \text{if}\quad \alpha>0\quad \text{and}\quad n\ge 1.$$

Let f be differentiable on some deleted neighborhood N of x_0, and suppose that f and f' have no zeros in N. Find

Suppose that f and g are differentiable and g' has no zeros on (a,b). Suppose also that $\lim_{x\to b-}f'(x)/g'(x)=L$ and either

$$\lim_{x\to b-}f(x)=\lim_{x\to b-}g(x)=0$$

or

$$\lim_{x\to b-}f(x)=\infty\quad \text{and}\quad \lim_{x\to b-}g(x)=\pm\infty.$$

Find $\lim_{x\to b-}(1+f(x))^{1/g(x)}$.

We distinguish between $\infty\cdot\infty\ (=\infty)$ and $(-\infty)\infty\ (=-\infty)$ and between $\infty+\infty\ (=\infty)$ and $-\infty-\infty\ (=-\infty)$. Why don't we distinguish between $0\cdot\infty$ and $0\cdot(-\infty)$, $\infty-\infty$ and $-\infty+\infty$, ∞/∞ and $-\infty/\infty$, and 1^∞ and $1^{-\infty}$?

\end{exerciselist}

A {} is a function of the form

$$p(x)=a_0+a_1(x-x_0)+\cdots+a_n(x-x_0)^n, \tag{2.1.43}$$

where a_0, , a_n and x_0 are constants. Since it is easy to calculate the values of a polynomial, considerable effort has been devoted to using them to approximate more complicated functions. Taylor's theorem is one of the oldest and most important results on this question.

The polynomial is said to be written {} $x-x_0$, and is {} n if $a_n\ne 0$. If we wish to leave open the possibility that $a_n=0$, we say that p is of degree $\le n$. In particular, a constant polynomial $p(x)=a_0$ is of degree zero if $a_0\ne 0$. If $a_0=0$, so that p vanishes identically, then p has no degree according to our definition, which requires at least one coefficient to be nonzero. For convenience we say that the identically zero polynomial p has degree $-\infty$. (Any negative number would do as well as $-\infty$. The point is that with this convention, the statement that p is a polynomial of degree $\le n$ includes the possibility that p is identically zero.)

We saw in Lemma~ that if f is differentiable at x_0, then

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+E(x)(x-x_0),$$

where

$$\lim_{x\to x_0}E(x)=0.$$

To generalize this result, we first restate it: the polynomial

$$T_1(x)=f(x_0)+f'(x_0)(x-x_0),$$

which is of degree ≤ 1 and satisfies

$$T_1(x_0) = f(x_0), \quad T_1'(x_0) = f'(x_0),$$

approximates f so well near x_0 that

$$\lim_{x \to x_0} \frac{f(x) - T_1(x)}{x - x_0} = 0. \tag{2.1.44}$$

Now suppose that f has n derivatives at x_0 and T_n is the polynomial of degree$\sim \leq n$ such that

$$T_n^{(r)}(x_0) = f^{(r)}(x_0), \quad 0 \leq r \leq n. \tag{2.1.45}$$

How well does T_n approximate f near x_0?

To answer this question, we must first find T_n. Since T_n is a polynomial of degree$\sim \leq n$, it can be written as

$$T_n(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n, \tag{2.1.46}$$

where $a_0, , a_n$ are constants. Differentiating yields

$$T_n^{(r)}(x_0) = r!a_r, \quad 0 \leq r \leq n,$$

so determines a_r uniquely as

$$a_r = \frac{f^{(r)}(x_0)}{r!}, \quad 0 \leq r \leq n.$$

Therefore,

$$\begin{aligned} T_n(x) \ar &= f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \\ \ar &= \sum_{r=0}^{n} \frac{f^{(r)}(x_0)}{r!}(x - x_0)^r. \end{aligned}$$

We call T_n {} {}.

The following theorem describes how T_n approximates f near x_0.

The proof is by induction. Let P_n be the assertion of the theorem. From we know that is true if $n = 1$; that is, P_1 is true. Now suppose that P_n is true for some integer $n \geq 1$, and $f^{(n+1)}$ exists. Since the ratio

$$\frac{f(x) - T_{n+1}(x)}{(x - x_0)^{n+1}}$$

is indeterminate of the form $0/0$ as $x \to x_0$, L'Hospital's rule implies that

$$\lim_{x \to x_0} \frac{f(x) - T_{n+1}(x)}{(x - x_0)^{n+1}} = \frac{1}{n+1} \lim_{x \to x_0} \frac{f'(x) - T_{n+1}'(x)}{(x - x_0)^n} \tag{2.1.47}$$

if the limit on the right exists. But f' has an nth derivative at x_0, and

$$T_{n+1}'(x) = \sum_{r=0}^{n} \frac{f^{(r+1)}(x_0)}{r!}(x - x_0)^r$$

is the nth Taylor polynomial of f' about x_0. Therefore, the induction assumption, applied to f', implies that

$$\lim_{x \to x_0} \frac{f'(x) - T_{n+1}'(x)}{(x - x_0)^n} = 0.$$

This and imply that

$$\lim_{x \to x_0} \frac{f(x) - T_{n+1}(x)}{(x - x_0)^{n+1}} = 0,$$

which completes the induction.

It can be shown (Exercise~) that if

$$p_n = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n$$

is a polynomial of degree $\le n$ such that

$$\lim_{x \to x_0} \frac{f(x) - p_n(x)}{(x - x_0)^n} = 0,$$

then

$$a_r = \frac{f^{(r)}(x_0)}{r!};$$

that is, $p_n = T_n$. Thus, T_n is the only polynomial of degree $\le n$ that approximates f near x_0 in the manner indicated in .

Theorem~ can be restated as a generalization of Lemma~.

Define

$$E_n(x) = \begin{cases} \dst \frac{f(x) - T_n(x)}{(x - x_0)^n}, & x \in D_f - \{x_0\}, \\ 0, & x = x_0. \end{cases}$$

Then implies that $\lim_{x \to x_0} E_n(x) = E_n(x_0) = 0$, and it is straightforward to verify .

If q is a nonnegative integer, then $\dst \binom{q}{n}$ is the binomial coefficient defined in Exercise~. In this case, we see from that

$$T_n(x) = (1 + x)^q = f(x), \quad n \ge q.$$

Lemma~ yields the following theorem.

Since $f^{(r)}(x_0) = 0$ for $1 \le r \le n - 1$, implies that

$$f(x) - f(x_0) = \left[\frac{f^{(n)}(x_0)}{n!} + E_n(x) \right] (x - x_0)^n \tag{2.1.48}$$

in some interval containing x_0. Since $\lim_{x \to x_0} E_n(x) = 0$ and $f^{(n)}(x_0) \ne 0$, there is a $\delta > 0$ such that

$$|E_n(x)| < \left| \frac{f^{(n)}(x_0)}{n!} \right| \quad \text{if} \quad |x - x_0| < \delta.$$

This and imply that

$$\frac{f(x) - f(x_0)}{(x - x_0)^n} \tag{2.1.49}$$

has the same sign as $f^{(n)}(x_0)$ if $0 < |x - x_0| < \delta$. If n is odd the denominator of changes sign in every neighborhood of x_0, and therefore so must the numerator (since the ratio has constant sign for $0 < |x - x_0| < \delta$). Consequently, $f(x_0)$ cannot be a local extreme value of f. This proves . If n is even, the denominator of is positive for $x \ne x_0$, so $f(x) - f(x_0)$ must have the same sign as $f^{(n)}(x_0)$ for $0 < |x - x_0| < \delta$. This proves

.

For $n = 2$,

is called the {} for local extreme points.

Theorem~ implies that the error in approximating $f(x)$ by $T_n(x)$ approaches zero faster than $(x - x_0)^n$ as x approaches x_0; however, it gives no estimate of the error in approximating $f(x)$ by $T_n(x)$ for a {} x. For instance, it provides no estimate of the error in the approximation

$$e^{0.1} \approx T_2(0.1) = 1 + \frac{0.1}{1!} + \frac{(0.1)^2}{2!} = 1.105 \tag{2.1.50}$$

obtained by setting $n=2$ and $x=0.1$ in . The following theorem provides a way of estimating errors of this kind under the additional assumption that $f^{(n+1)}$ exists in a neighborhood of x_0.

This theorem follows from an extension of the mean value theorem that we will prove below. For now, let us assume that Theorem~ is correct, and apply it.

We now consider the extended mean value theorem, which implies Theorem~ (Exercise~). In the following theorem, a and b are the endpoints of an interval, but we do not assume that $a<b$.

The proof is by induction. The mean value theorem (Theorem~) implies the conclusion for $n=0$. Now suppose that $n\ge 1$, and assume that the assertion of the theorem is true with n replaced by $n-1$. The left side of can be written as

$$f(b)-\sum_{r=0}^{n}\frac{f^{(r)}(a)}{r!}(b-a)^r=K\frac{(b-a)^{n+1}}{(n+1)!} \tag{2.1.51}$$

for some number K. We must prove that $K=f^{(n+1)}(c)$ for some c in I^0. To this end, consider the auxiliary function

$$h(x)=f(x)-\sum_{r=0}^{n}\frac{f^{(r)}(a)}{r!}(x-a)^r-K\frac{(x-a)^{n+1}}{(n+1)!},$$

which satisfies

$$h(a)=0,\quad h(b)=0,$$

(the latter because of) and is continuous on the closed interval I and differentiable on I^0, with

$$h'(x)=f'(x)-\sum_{r=0}^{n-1}\frac{f^{(r+1)}(a)}{r!}(x-a)^r-K\frac{(x-a)^n}{n!}. \tag{2.1.52}$$

Therefore, Rolle's theorem (Theorem~) implies that $h'(b_1)=0$ for some b_1 in I^0; thus, from ,

$$f'(b_1)-\sum_{r=0}^{n-1}\frac{f^{(r+1)}(a)}{r!}(b_1-a)^r-K\frac{(b_1-a)^n}{n!}=0.$$

If we temporarily write $f'=g$, this becomes

$$g(b_1)-\sum_{r=0}^{n-1}\frac{g^{(r)}(a)}{r}(b_1-a)^r-K\frac{(b_1-a)^n}{n!}=0. \tag{2.1.53}$$

Since $b_1\in I^0$, the hypotheses on f imply that g is continuous on the closed interval J with endpoints a and b_1, $g^{(n)}$ exists on J^0, and, if $n\ge 1$, g', , $g^{(n-1)}$ exist and are continuous at a (also at b_1, but this is not important). The induction hypothesis, applied to g on the interval J, implies that

$$g(b_1)-\sum_{r=0}^{n-1}\frac{g^{(r)}(a)}{r!}(b_1-a)^r=\frac{g^{(n)}(c)}{n!}(b_1-a)^n$$

for some c in J^0. Comparing this with and recalling that $g=f'$ yields

$$K=g^{(n)}(c)=f^{(n+1)}(c).$$

Since c is in I^0, this completes the induction.

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CHAPTER OVERVIEW

3: Integral Calculus of Functions of One Variable

IN THIS CHAPTER we discuss the Riemann on a finite interval $[a, b]$, and improper integrals in which either the function or the interval of integration is unbounded.

- SECTION 3.1 begins with the definition of the Riemann integral and presents the geometrical interpretation of the Riemann integral as the area under a curve. We show that an unbounded function cannot be Riemann integrable. Then we define upper and lower sums and upper and lower integrals of a bounded function. The section concludes with the definition of the Riemann–Stieltjes integral.
- SECTION 3.2 presents necessary and sufficient conditions for the existence of the Riemann integral in terms of upper and lower sums and upper and lower integrals. We show that continuous functions and bounded monotonic functions are Riemann integrable.
- SECTION 3.3 begins with proofs that the sum and product of Riemann integrable functions are integrable, and that $|f|$ is Riemann integrable if f is Riemann integrable. Other topics covered include the first mean value theorem for integrals, antiderivatives, the fundamental theorem of calculus, change of variables, integration by parts, and the second mean value theorem for integrals.
- SECTION 3.4 presents a comprehensive discussion of improper integrals. Concepts defined and considered include absolute and conditional convergence of an improper integral, Dirichlet's test, and change of variable in an improper integral.
- SECTION 3.5 defines the notion of a set with Lebesgue measure zero, and presents a necessary and sufficient condition for a bounded function f to be Riemann integrable on an interval $[a, b]$; namely, that the discontinuities of f form a set with Lebesgue masure zero.

3.1: Definition of the Integral
3.2: Existence of the Integral
3.3: Properties of the Integral
3.4: Improper Integrals
3.5: A More Advanced Look at the Existence of the Proper Riemann Integral

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3.1: Definition of the Integral

The integral that you studied in calculus is the {}, named after the German mathematician , who provided a rigorous formulation of the integral to

replace the intuitive notion due to and . Since Riemann's time, other kinds of integrals have been defined and studied; however, they are all generalizations of the Riemann integral, and it is hardly possible to understand them or appreciate the reasons for developing them without a thorough understanding of the Riemann integral. In this section we deal with functions defined on a finite interval $[a,b]$. A {} is a set of subintervals

$$[x_0,x_1],\ [x_1,x_2],\dots,[x_{n-1},x_n], \tag{3.1.1}$$

where

$$a=x_0<x_1\cdots<x_n=b. \tag{3.1.2}$$

Thus, any set of $n+1$ points satisfying defines a partition P of $[a,b]$, which we denote by

$$P=\{x_0,x_1,\dots,x_n\}.$$

The points x_0, x_1, , x_n are the {} of P. The largest of the lengths of the subintervals is the {} of P, written as $\|P\|$; thus,

$$\|P\|=\max_{1\le i\le n}(x_i-x_{i-1}).$$

If P and P' are partitions of $[a,b]$, then P' is a {} if every partition point of P is also a partition point of P'; that is, if P' is obtained by inserting additional points between those of P. If f is defined on $[a,b]$, then a sum

$$\sigma=\sum_{j=1}^{n}f(c_j)(x_j-x_{j-1}),$$

where

$$x_{j-1}\le c_j\le x_j,\quad 1\le j\le n,$$

is a . (Occasionally we will say more simply that σ is a Riemann sum of f over $[a,b]$.) Since c_j can be chosen arbitrarily in $[x_j,x_{j-1}]$, there are infinitely many Riemann sums for a given function f over a given partition P.

We leave it to you (Exercise~) to show that $\int_a^b f(x)\,dx$ is unique, if it exists; that is, there cannot be more than one number L that satisfies Definition~.

For brevity we will say integrable'' and integral''when we mean Riemann integrable'' and Riemann integral.'' Saying that $\int_a^b f(x)\,dx$ exists is equivalent to saying that f is integrable on $[a,b]$.

-.3em An important application of the integral, indeed, the one invariably used to motivate its definition, is the computation of the area bounded by a curve $y=f(x)$, the x-axis, and the lines $x=a$ and $x=b$ (``the area under the curve''), as in Figure~.

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For simplicity, suppose that $f(x)>0$. Then $f(c_j)(x_j-x_{j-1})$ is the area of a rectangle with base x_j-x_{j-1} and height $f(c_j)$, so the Riemann sum

$$\sum_{j=1}^{n}f(c_j)(x_j-x_{j-1})$$

can be interpreted as the sum of the areas of rectangles related to the curve $y=f(x)$, as shown in Figure~.

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An apparently plausible argument, that the Riemann sums approximate the area under the curve more and more closely as the number of rectangles increases and the largest of their widths is made smaller, seems to support the assertion that $\int_a^b f(x)\,dx$ equals

the area under the curve. This argument is useful as a motivation for Definition~, which without it would seem mysterious. Nevertheless, the logic is incorrect, since it is based on the assumption that the area under the curve has been previously defined in some other way. Although this is true for certain curves such as, for example, those consisting of line segments or circular arcs, it is not true in general. In fact, the area under a more complicated curve is {} to be equal to the integral, if the integral exists. That this new definition is consistent with the old one, where the latter applies, is evidence that the integral provides a useful generalization of the definition of area.

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We will show that if f is unbounded on $[a,b]$, P is any partition of $[a,b]$, and $M>0$, then there are Riemann sums σ and σ' of f over P such that

$$|\sigma-\sigma'|\ge M. \tag{3.1.3}$$

We leave it to you (Exercise~) to complete the proof by showing from this that f cannot satisfy Definition~.

Let

$$\sigma=\sum_{j=1}^{n}f(c_j)(x_j-x_{j-1})$$

be a Riemann sum of f over a partition P of $[a,b]$. There must be an integer i in $\{1,2,\dots,n\}$ such that

$$|f(c)-f(c_i)|\ge\frac{M}{x_i-x_{i-1}} \tag{3.1.4}$$

for some c in $[x_{i-1}x_i]$, because if there were not so, we would have

$$|f(x)-f(c_j)|<\frac{M}{x_j-x_{j-1}},\quad x_{j-1}\le x\le x_j,\quad 1\le j\le n.$$

Then

$$\begin{aligned}|f(x)|\ar&=|f(c_j)+f(x)-f(c_j)|\le|f(c_j)|+|f(x)-f(c_j)|\\ \ar&\le|f(c_j)|+\frac{M}{x_j-x_{j-1}},\quad x_{j-1}\le x\le x_j,\quad 1\le j\le n.\end{aligned}$$

which implies that

$$|f(x)|\le\max_{1\le j\le n}|f(c_j)|+\frac{M}{x_j-x_{j-1}},\quad a\le x\le b,$$

contradicting the assumption that f is unbounded on $[a,b]$.

Now suppose that c satisfies , and consider the Riemann sum

$$\sigma'=\sum_{j=1}^{n}f(c_j')(x_j-x_{j-1})$$

over the same partition P, where

$$c_j'=\begin{cases}\casespace c_j, & j\ne i,\\ c, & j=i.\end{cases}$$

Since

$$|\sigma-\sigma'|=|f(c)-f(c_i)|(x_i-x_{i-1}),$$

implies .

Because of Theorem~, we consider only bounded functions throughout the rest of this section.

To prove directly from Definition~ that $\int_a^b f(x)\,dx$ exists, it is necessary to discover its value L in one way or another and to show that L has the properties required by the definition. For a specific function it may happen that this can be done by straightforward calculation, as in Examples~ and . However, this is not so if the objective is to find general conditions which imply that $\int_a^b f(x)\,dx$ exists. The following approach avoids the difficulty of having to discover L in advance, without knowing whether it exists in the first place, and requires only that we compare two numbers that must exist if f is bounded on $[a,b]$. We will see that $\int_a^b f(x)\,dx$ exists if and only if these two numbers are equal.

If $m \le f(x) \le M$ for all x in $[a,b]$, then

$$m(b-a) \le s(P) \le S(P) \le M(b-a)$$

for every partition P; thus, the set of upper sums of f over all partitions P of $[a,b]$ is bounded, as is the set of lower sums. Therefore, Theorems~ and imply that $\overline{\int_a^b} f(x)\,dx$ and $\underline{\int_a^b} f(x)\,dx$ exist, are unique, and satisfy the inequalities

$$m(b-a) \le \overline{\int_a^b} f(x)\,dx \le M(b-a)$$

and

$$m(b-a) \le \underline{\int_a^b} f(x)\,dx \le M(b-a).$$

If $P = \{x_0, x_1, \dots, x_n\}$, then

$$S(P) = \sum_{j=1}^{n} M_j(x_j - x_{j-1}),$$

where

$$M_j = \sup_{x_{j-1} \le x \le x_j} f(x).$$

An arbitrary Riemann sum of f over P is of the form

$$\sigma = \sum_{j=1}^{n} f(c_j)(x_j - x_{j-1}),$$

where $x_{j-1} \le c_j \le x_j$. Since $f(c_j) \le M_j$, it follows that $\sigma \le S(P)$.

Now let $\epsilon > 0$ and choose $\bar{c}_j$ in $[x_{j-1}, x_j]$ so that

$$f(\bar{c}_j) > M_j - \frac{\epsilon}{n(x_j - x_{j-1})}, \quad 1 \le j \le n.$$

The Riemann sum produced in this way is

$$\overline{\sigma} = \sum_{j=1}^{n} f(\bar{c}_j)(x_j - x_{j-1}) > \sum_{j=1}^{n} \left[M_j - \frac{\epsilon}{n(x_j - x_{j-1})}) \right] (x_j - x_{j-1}) = S(P) - \epsilon.$$

Now Theorem~ implies that $S(P)$ is the supremum of the set of Riemann sums of f over P.

Exercise~.

Unknown environment 'example'

The motivation for Definition~ can be seen by again considering the idea of area under a curve. Figure~ shows the graph of a positive function $y = f(x)$, $a \le x \le b$, with $[a,b]$ partitioned into four subintervals.

6pt

12pt

The upper and lower sums of f over this partition can be interpreted as the sums of the areas of the rectangles surmounted by the solid and dashed lines, respectively. This indicates that a sensible definition of area A under the curve must admit the inequalities

$$s(P) \le A \le S(P)$$

for every partition P of $[a,b]$. Thus, A must be an upper bound for all lower sums and a lower bound for all upper sums of f over partitions of $[a,b]$. If

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx, \tag{3.1.5}$$

there is only one number, the common value of the upper and lower integrals, with this property, and we define A to be that number; if does not hold, then A is not defined. We will see below that this definition of area is consistent with the definition stated earlier in terms of Riemann sums.

The {} is an important generalization of the Riemann integral. We define it here, but confine our study of it to the exercises in this and other sections of this chapter.

\begin{exerciselist}

Show that there cannot be more than one number L that satisfies Definition~.

2pt

Suppose that $\int_a^b f(x)\,dx$ exists and there is a number A such that, for every $\epsilon > 0$ and $\delta > 0$, there is a partition P of $[a,b]$ with $\|P\| < \delta$ and a Riemann sum σ of f over P that satisfies the inequality $|\sigma - A| < \epsilon$. Show that $\int_a^b f(x)\,dx = A$.

2pt

Prove directly from Definition~ that

$$\int_a^b x^2\,dx = \frac{b^3 - a^3}{3}.$$

Do not assume in advance that the integral exists. The proof of this is part of the problem.

2pt

Generalize the proof of Exercise~ to show directly from Definition~ that

$$\int_a^b x^m\,dx = \frac{b^{m+1} - a^{m+1}}{m+1}$$

if m is an integer ≥ 0.

2pt

Prove directly from Definition~ that $f(x)$ is integrable on $[a,b]$ if and only if $f(-x)$ is integrable on $[-b,-a]$, and, in this case,

$$\int_a^b f(x)\,dx = \int_{-b}^{-a} f(-x)\,dx.$$

2pt

Let f be bounded on $[a,b]$ and let P be a partition of $[a,b]$. Prove: The lower sum $s(P)$ of f over P is the infimum of the set of all Riemann sums of f over P.

2pt Let f be defined on $[a,b]$ and let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a,b]$.

Find $\underline{\int_0^1} f(x)\,dx$ and $\overline{\int_0^1} f(x)\,dx$ if

-.5em

Given that $\int_a^b e^x\,dx$ exists, evaluate it by using the formula

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r} \quad (r \ne 1)$$

to calculate certain Riemann sums.

Given that $\int_0^b \sin x\,dx$ exists, evaluate it by using the identity

$$\cos(j-1)\theta - \cos(j+1)\theta = 2\sin\theta\sin j\theta$$

to calculate certain Riemann sums.

Given that $\int_0^b \cos x\,dx$ exists, evaluate it by using the identity

$$\sin(j+1)\theta - \sin(j-1)\theta = 2\sin\theta\cos j\theta$$

to calculate certain Riemann sums.

Show that if $g(x) = x + c$ (c=constant), then $\int_a^b f(x)\,dg(x)$ exists if and only if $\int_a^b f(x)\,dx$ exists, in which case

$$\int_a^b f(x)\,dg(x) = \int_a^b f(x)\,dx.$$

Suppose that $-\infty < a < d < c < \infty$ and

$$g(x) = \begin{cases} g_1, & a < x < d, \\ g_2, & d < x < b, \end{cases} \quad (g_1, g_2 = \text{constants}),$$

and let $g(a)$, $g(b)$, and $g(d)$ be arbitrary. Suppose that f is defined on $[a,b]$, continuous from the right at a and from the left at b, and continuous at d. Show that $\int_a^b f(x)\,dg(x)$ exists, and find its value.

Suppose that $-\infty < a = a_0 < a_1 < \cdots < a_p = b < \infty$, let $g(x) = g_m$ (constant) on (a_{m-1}, a_m), $1 \le m \le p$, and let $g(a_0)$, $g(a_1)$, , $g(a_p)$ be arbitrary. Suppose that f is defined on $[a,b]$, continuous from the right at a and from the left at b, and continuous at a_1, a_2, , a_{p-1}. Evaluate $\int_a^b f(x)\,dg(x)$.

For the case where g is nondecreasing and f is bounded on $[a,b]$, define upper and lower Riemann–Stieltjes integrals in a way analogous to Definition~.

\end{exerciselist}

The following lemma is the starting point for our study of the integrability of a bounded function f on a closed interval $[a,b]$.

We will prove and leave the proof of to you (Exercise~). First suppose that $r = 1$, so P' is obtained by adding one point c to the partition $P = \{x_0, x_1, \dots, x_n\}$; then $x_{i-1} < c < x_i$ for some i in $\{1, 2, \dots, n\}$. If $j \ne i$, the product $M_j(x_j - x_{j-1})$ appears in both $S(P)$ and $S(P')$ and cancels out of the difference $S(P) - S(P')$. Therefore, if

$$M_{i1} = \sup_{x_{i-1} \le x \le c} f(x) \quad \text{and} \quad M_{i2} = \sup_{c \le x \le x_i} f(x),$$

then

Bracket argument to \\ must be a dimension

Since implies that

$$0 \le M_i - M_{ir} \le 2M, \quad r = 1, 2,$$

implies that

$$0 \le S(P) - S(P') \le 2M(x_i - x_{i-1}) \le 2M\|P\|.$$

This proves for $r = 1$.

Now suppose that $r > 1$ and P' is obtained by adding points c_1, c_2, , c_r to P. Let $P^{(0)} = P$ and, for $j \ge 1$, let $P^{(j)}$ be the partition of $[a,b]$ obtained by adding c_j to $P^{(j-1)}$. Then the result just proved implies that

$$0 \le S(P^{(j-1)}) - S(P^{(j)}) \le 2M\|P^{(j-1)}\|, \quad 1 \le j \le r.$$

Adding these inequalities and taking account of cancellations yields

$$0 \le S(P^{(0)}) - S(P^{(r)}) \le 2M(\|P^{(0)}\| + \|P^{(1)}\| + \cdots + \|P^{(r-1)}\|). \tag{3.1.6}$$

Since $P^{(0)} = P$, $P^{(r)} = P'$, and $\|P^{(k)}\| \le \|P^{(k-1)}\|$ for $1 \le k \le r-1$, implies that

$$0 \le S(P) - S(P') \le 2Mr\|P\|,$$

which is equivalent to .

Suppose that P_1 and P_2 are partitions of $[a,b]$ and P' is a refinement of both. Letting $P = P_1$ in and $P = P_2$ in shows that

$$s(P_1) \le s(P') \quad \text{and} \quad S(P') \le S(P_2).$$

Since $s(P') \le S(P')$, this implies that $s(P_1) \le S(P_2)$. Thus, every lower sum is a lower bound for the set of all upper sums. Since $\overline{\int_a^b} f(x)\,dx$ is the infimum of this set, it follows that

$$s(P_1) \le \overline{\int_a^b} f(x)\,dx$$

for every partition P_1 of $[a,b]$. This means that $\overline{\int_a^b} f(x)\,dx$ is an upper bound for the set of all lower sums. Since $\underline{\int_a^b} f(x)\,dx$ is the supremum of this set, this implies .

We prove that $\overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx$ and leave it to you to show that $\underline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx$ (Exercise~).

Suppose that P is a partition of $[a,b]$ and σ is a Riemann sum of f over P. Since

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the triangle inequality implies that

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Now suppose that $\epsilon > 0$. From Definition~, there is a partition P_0 of $[a,b]$ such that

$$\overline{\int_a^b} f(x)\,dx \le S(P_0) < \overline{\int_a^b} f(x)\,dx + \frac{\epsilon}{3}. \tag{3.1.7}$$

From Definition~, there is a $\delta > 0$ such that

$$\left|\sigma - \int_a^b f(x)\,dx\right| < \frac{\epsilon}{3} \tag{3.1.8}$$

if $\|P\| < \delta$. Now suppose that $\|P\| < \delta$ and P is a refinement of P_0. Since $S(P) \le S(P_0)$ by Lemma~, implies that

$$\overline{\int_a^b} f(x)\,dx \le S(P) < \overline{\int_a^b} f(x)\,dx + \frac{\epsilon}{3},$$

so

$$\left|S(P) - \overline{\int_a^b} f(x)\,dx\right| < \frac{\epsilon}{3} \tag{3.1.9}$$

in addition to . Now , , and imply that

$$\left|\overline{\int_a^b} f(x)\,dx - \int_a^b f(x)\,dx\right| < \frac{2\epsilon}{3} + |S(P) - \sigma| \tag{3.1.10}$$

for every Riemann sum σ of f over P. Since $S(P)$ is the supremum of these Riemann sums (Theorem~), we may choose σ so that

$$|S(P) - \sigma| < \frac{\epsilon}{3}.$$

Now implies that

$$\left|\overline{\int_a^b} f(x)\,dx - \int_a^b f(x)\,dx\right| < \epsilon.$$

Since ϵ is an arbitrary positive number, it follows that

$$\overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx.$$

-6.5ex6.5ex

We show that holds if $\|P\|$ is sufficiently small, and leave the rest of the proof to you (Exercise~).

The first inequality in follows immediately from Definition~. To establish the second inequality, suppose that $|f(x)| \le K$ if $a \le x \le b$. From Definition~, there is a partition $P_0 = \{x_0, x_1, \dots, x_{r+1}\}$ of $[a,b]$ such that

$$S(P_0) < \overline{\int_a^b} f(x)\,dx + \frac{\epsilon}{2}. \tag{3.1.11}$$

If P is any partition of $[a,b]$, let P' be constructed from the partition points of P_0 and P. Then

$$S(P') \le S(P_0), \tag{3.1.12}$$

by Lemma~. Since P' is obtained by adding at most r points to P, Lemma~ implies that

$$S(P') \ge S(P) - 2Kr\|P\|. \tag{3.1.13}$$

Now , , and imply that

$$\begin{aligned} S(P)\ar &\le S(P') + 2Kr\|P\| \\ \ar &\le S(P_0) + 2Kr\|P\| \\ &< \overline{\int_a^b} f(x)\,dx + \frac{\epsilon}{2} + 2Kr\|P\|. \end{aligned}$$

Therefore, holds if

$$\|P\| < \delta = \frac{\epsilon}{4Kr}.$$

-4.5ex4.5ex

If $\epsilon > 0$, there is a $\delta > 0$ such that

$$\underline{\int_a^b} f(x)\,dx - \epsilon < s(P) \le S(P) < \overline{\int_a^b} f(x)\,dx + \epsilon \tag{3.1.14}$$

if $\|P\| < \delta$ (Lemma~). If σ is a Riemann sum of f over P, then

$$s(P) \le \sigma \le S(P),$$

so and imply that

$$L - \epsilon < \sigma < L + \epsilon$$

if $\|P\| < \delta$. Now Definition~ implies .

Theorems~ and imply the following theorem.

The next theorem translates this into a test that can be conveniently applied.

We leave it to you (Exercise~) to show that if $\int_a^b f(x)\,dx$ exists, then holds for $\|P\|$ sufficiently small. This implies that the stated condition is necessary for integrability. To show that it is sufficient, we observe that since

$$s(P) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le S(P)$$

for all P, implies that

$$0 \le \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx < \epsilon.$$

Since ϵ can be any positive number, this implies that

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx.$$

Therefore, $\int_a^b f(x)\,dx$ exists, by Theorem~.

The next two theorems are important applications of Theorem~.

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. Since f is continuous on $[a,b]$, there are points c_j and c_j' in $[x_{j-1}, x_j]$ such that

$$f(c_j) = M_j = \sup_{x_{j-1} \le x \le x_j} f(x)$$

and

$$f(c_j') = m_j = \inf_{x_{j-1} \le x \le x_j} f(x)$$

(Theorem~). Therefore,

$$S(P) - s(P) = \sum_{j=1}^{n} \left[f(c_j) - f(c_j')\right](x_j - x_{j-1}). \tag{3.1.15}$$

Since f is uniformly continuous on $[a,b]$ (Theorem~), there is for each $\epsilon > 0$ a $\delta > 0$ such that

$$|f(x') - f(x)| < \frac{\epsilon}{b-a}$$

if x and x' are in $[a,b]$ and $|x - x'| < \delta$. If $\|P\| < \delta$, then $|c_j - c_j'| < \delta$ and, from ,

$$S(P) - s(P) < \frac{\epsilon}{b-a} \sum_{j=1}^{n} (x_j - x_{j-1}) = \epsilon.$$

Hence, f is integrable on $[a,b]$, by Theorem~.

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. Since f is nondecreasing,

$$f(x_j)\ar = M_j = \sup_{x_{j-1} \le x \le x_j} f(x)$$
$$\arraytextand$$
$$f(x_{j-1})\ar = m_j = \inf_{x_{j-1} \le x \le x_j} f(x).$$

Hence,

$$S(P) - s(P) = \sum_{j=1}^{n} (f(x_j) - f(x_{j-1}))(x_j - x_{j-1}).$$

Since $0 < x_j - x_{j-1} \le \|P\|$ and $f(x_j) - f(x_{j-1}) \ge 0$,

$$S(P) - s(P)\ar \le \|P\| \sum_{j=1}^{n} (f(x_j) - f(x_{j-1}))$$
$$\ar = \|P\|(f(b) - f(a)).$$

Therefore,

$$S(P) - s(P) < \epsilon \quad \text{if} \quad \|P\|(f(b) - f(a)) < \epsilon,$$

so f is integrable on $[a,b]$, by Theorem~.

The proof for nonincreasing f is similar.

We will also use Theorem~ in the next section to establish properties of the integral. In Section~3.5 we will study more general conditions for integrability.

We now use the results of Sections~3.1 and 3.2 to establish the properties of the integral. You are probably familiar with most of these properties, but not with their proofs.

4pt

Any Riemann sum of $f+g$ over a partition $P=\{x_0,x_1,\dots,x_n\}$ of $[a,b]$ can be written as

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where σ_f and σ_g are Riemann sums for f and g. Definition~ implies that if $\epsilon>0$ there are positive numbers δ_1 and δ_2 such that

$$\left|\sigma_f-\int_a^b f(x)\,dx\right| \ar < \frac{\epsilon}{2} \quad \text{if} \quad \|P\|<\delta_1$$
$$\arraytextand$$
$$\left|\sigma_g-\int_a^b g(x)\,dx\right| \ar < \frac{\epsilon}{2} \quad \text{if} \quad \|P\|<\delta_2.$$

If $\|P\|<\delta=\min(\delta_1,\delta_2)$, then

$$\begin{aligned}\left|\sigma_{f+g}-\int_a^b f(x)\,dx-\int_a^b g(x)\,dx\right| &\ar\\ =\left|\left(\sigma_f-\int_a^b f(x)\,dx\right)+\left(\sigma_g-\int_a^b g(x)\,dx\right)\right| &\\ \ar\le\left|\sigma_f-\int_a^b f(x)\,dx\right|+\left|\sigma_g-\int_a^b g(x)\,dx\right| &\\ <\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon,&\end{aligned}$$

so the conclusion follows from Definition~.

The next theorem also follows from Definition~ (Exercise~).

Theorems~ and and induction yield the following result (Exercise~).

Since $g(x)-f(x)\ge 0$, every lower sum of $g-f$ over any partition of $[a,b]$ is nonnegative. Therefore,

$$\underline{\int_a^b}(g(x)-f(x))\,dx\ge 0.$$

Hence,

Bracket argument to \\ must be a dimension

which yields . (The first equality in follows from Theorems~ and ; the second, from Theorem~.)

Let P be a partition of $[a,b]$ and define

$$\begin{aligned}M_j\ar&=\sup\{f(x)\}x_{j-1}\le x\le x_j,\\ m_j\ar&=\inf\{f(x)\}x_{j-1}\le x\le x_j,\\ \overline{M}_j\ar&=\sup\{|f(x)|\}x_{j-1}\le x\le x_j,\\ \overline{m}_j\ar&=\inf\{|f(x)|\}x_{j-1}\le x\le x_j.\end{aligned}$$

Then

$$\begin{aligned}\overline{M}_j-\overline{m}_j\ar&=\dst\sup\{|f(x)|-|f(x')|\}x_{j-1}\le x,x'\le x_j\\ \ar&\le\dst\sup\{|f(x)-f(x')|\}x_{j-1}\le x,x'\le x_j\\ \ar&=M_j-m_j.\end{aligned}\tag{3.1.16}$$

Therefore,

$$\overline{S}(P)-\overline{s}(P)\le S(P)-s(P),$$

where the upper and lower sums on the left are associated with $|f|$ and those on the right are associated with f. Now suppose that $\epsilon>0$. Since f is integrable on $[a,b]$, Theorem~ implies that there is a partition P of $[a,b]$ such that $S(P)-s(P)<\epsilon$. This inequality and imply that $\overline{S}(P)-\overline{s}(P)<\epsilon$. Therefore, $|f|$ is integrable on $[a,b]$, again by Theorem~.

Since

$$f(x)\le|f(x)|\quad\text{and}\quad -f(x)\le|f(x)|,\quad a\le x\le b,$$

Theorems~ and imply that

$$\int_a^b f(x)\,dx\le\int_a^b|f(x)|\,dx\quad\text{and}\quad -\int_a^b f(x)\,dx\le\int_a^b|f(x)|\,dx,$$

which implies .

We consider the case where f and g are nonnegative, and leave the rest of the proof to you (Exercise~). The subscripts f, g, and fg in the following argument identify the functions with which the various quantities are associated. We assume that neither f nor g is identically zero on $[a,b]$, since the conclusion is obvious if one of them is.

If $P=\{x_0,x_1,\dots,x_n\}$ is a partition of $[a,b]$, then

$$S_{fg}(P)-s_{fg}(p)=\sum_{j=1}^{n}(M_{fg,j}-m_{fg,j})(x_j-x_{j-1}).\tag{3.1.17}$$

Since f and g are nonnegative, $M_{fg,j}\le M_{f,j}M_{g,j}$ and $m_{fg,j}\ge m_{f,j}m_{g,j}$. Hence,

Bracket argument to \\ must be a dimension

where M_f and M_g are upper bounds for f and g on $[a,b]$. From and the last inequality,

$$S_{fg}(P)-s_{fg}(P)\le M_g[S_f(P)-s_f(P)]+M_f[S_g(P)-s_g(P)].\tag{3.1.18}$$

Now suppose that $\epsilon>0$. Theorem~ implies that there are partitions P_1 and P_2 of $[a,b]$ such that

$$S_f(P_1)-s_f(P_1)<\frac{\epsilon}{2M_g}\quad\text{and}\quad S_g(P_2)-s_g(P_2)<\frac{\epsilon}{2M_f}.\tag{3.1.19}$$

If P is a refinement of both P_1 and P_2, then and Lemma~ imply that

$$S_f(P)-s_f(P)<\frac{\epsilon}{2M_g}\quad\text{and}\quad S_g(P)-s_g(P)<\frac{\epsilon}{2M_f}.$$

This and yield

$$S_{fg}(P)-s_{fg}(P)<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.$$

Therefore, fg is integrable on $[a,b]$, by Theorem~.

3pt From Theorem~, u is integrable on $[a,b]$. Therefore, Theorem~ implies that the integral on the left exists. If $m=\min\{u(x)\}a\le x\le b$ and $M=\max\{u(x)\}a\le x\le b$ (recall Theorem~), then

$$m\le u(x)\le M$$

and, since $v(x)\ge0$,

$$mv(x)\le u(x)v(x)\le Mv(x).$$

Therefore, Theorems~ and imply that 2pt

$$m\int_a^b v(x)\,dx\le\int_a^b u(x)v(x)\,dx\le M\int_a^b v(x)\,dx.\tag{3.1.20}$$

2pt This implies that holds for any c in $[a,b]$ if $\int_a^b v(x)\,dx=0$. If $\int_a^b v(x)\,dx\ne0$, let 1pt

$$\overline{u} = \frac{\dst \int_a^b u(x)v(x)\,dx}{\dst \int_a^b v(x)\,dx} \tag{3.1.21}$$

1pt Since $\int_a^b v(x)\,dx > 0$ in this case (why?), implies that $m \le \overline{u} \le M$, and the intermediate value theorem (Theorem~) implies that $\overline{u} = u(c)$ for some c in $[a,b]$. This implies .

1pt If $v(x) \equiv 1$, then reduces to

$$\overline{u} = \frac{1}{b-a}\int_a^b u(x)\,dx,$$

so $\overline{u}$ is the average of $u(x)$ over $[a,b]$. More generally, if v is any nonnegative integrable function such that $\int_a^b v(x)\,dx \neq 0$, then $\overline{u}$ in is the {}. Theorem~ says that a continuous function assumes any such weighted average at some point in $[a,b]$.

3pt

Suppose that $\epsilon > 0$. From Theorem~, there is a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ such that

$$S(P) - s(P) = \sum_{j=1}^{n} (M_j - m_j)(x_j - x_{j-1}) < \epsilon. \tag{3.1.22}$$

We may assume that a_1 and b_1 are partition points of P, because if not they can be inserted to obtain a refinement P' such that $S(P') - s(P') \le S(P) - s(P)$ (Lemma~). Let $a_1 = x_r$ and $b_1 = x_s$. Since every term in is nonnegative,

$$\sum_{j=r+1}^{s} (M_j - m_j)(x_j - x_{j-1}) < \epsilon.$$

Thus, $\overline{P} = \{x_r, x_{r+1}, \ldots, x_s\}$ is a partition of $[a_1, b_1]$ over which the upper and lower sums of f satisfy

$$S(\overline{P}) - s(\overline{P}) < \epsilon.$$

Therefore, f is integrable on $[a_1, b_1]$, by Theorem~.

We leave the proof of the next theorem to you (Exercise~).

So far we have defined $\int_\alpha^\beta f(x)\,dx$ only for the case where $\alpha < \beta$. Now we define

$$\int_\beta^\alpha f(x)\,dx = -\int_\alpha^\beta f(x)\,dx$$

if $\alpha < \beta$, and

$$\int_\alpha^\alpha f(x)\,dx = 0.$$

With these conventions, holds no matter what the relative order of a, b, and c, provided that f is integrable on some closed interval containing them (Exercise~).

Theorem~ and these definitions enable us to define a function $ F(x)=_c^x f(t),dt$, where c is an arbitrary, but fixed, point in $[a,b]$.

If x and x' are in $[a,b]$, then

$$F(x) - F(x') = \int_c^x f(t)\,dt - \int_c^{x'} f(t)\,dt = \int_{x'}^x f(t)\,dt,$$

by Theorem~ and the conventions just adopted. Since $|f(t)| \le K$ $(a \le t \le b)$ for some constant K,

$$\left|\int_{x'}^x f(t)\,dt\right| \le K|x - x'|, \quad a \le x,\, x' \le b$$

(Theorem~), so

$$|F(x) - F(x')| \le K|x - x'|, \quad a \le x,\, x' \le b.$$

-2em2em

We consider the case where $a < x_0 < b$ and leave the rest to you (Exercise~). Since

$$\frac{1}{x-x_0}\int_{x_0}^{x} f(x_0)\,dt = f(x_0),$$

we can write

$$\frac{F(x)-F(x_0)}{x-x_0} - f(x_0) = \frac{1}{x-x_0}\int_{x_0}^{x}[f(t)-f(x_0)]\,dt.$$

From this and Theorem~,

$$\left|\frac{F(x)-F(x_0)}{x-x_0} - f(x_0)\right| \le \frac{1}{|x-x_0|}\left|\int_{x_0}^{x}|f(t)-f(x_0)|\,dt\right|. \tag{3.1.23}$$

(Why do we need the absolute value bars outside the integral?) Since f is continuous at x_0, there is for each $\epsilon > 0$ a $\delta > 0$ such that

$$|f(t)-f(x_0)| < \epsilon \quad \text{if} \quad |x-x_0| < \delta$$

and t is between x and x_0. Therefore, from ,

$$\left|\frac{F(x)-F(x_0)}{x-x_0} - f(x_0)\right| < \epsilon\frac{|x-x_0|}{|x-x_0|} = \epsilon \quad \text{if} \quad 0 < |x-x_0| < \delta.$$

Hence, $F'(x_0) = f(x_0)$.

The next theorem relates integration and differentiation in another way.

If $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a,b]$, then

$$F(b)-F(a) = \sum_{j=1}^{n}(F(x_j)-F(x_{j-1})). \tag{3.1.24}$$

From Theorem~, there is in each open interval (x_{j-1}, x_j) a point c_j such that

$$F(x_j)-F(x_{j-1}) = f(c_j)(x_j-x_{j-1}).$$

Hence, can be written as

$$F(b)-F(a) = \sum_{j=1}^{n} f(c_j)(x_j-x_{j-1}) = \sigma,$$

where σ is a Riemann sum for f over P. Since f is integrable on $[a,b]$, there is for each $\epsilon > 0$ a $\delta > 0$ such that

$$\left|\sigma - \int_a^b f(x)\,dx\right| < \epsilon \quad \text{if} \quad \|P\| < \delta.$$

Therefore,

$$\left|F(b)-F(a)-\int_a^b f(x)\,dx\right| < \epsilon$$

for every $\epsilon > 0$, which implies .

Apply Theorem~ with F and f replaced by f and f', respectively.

A function F is an {} of f on $[a,b]$ if F is continuous on $[a,b]$ and differentiable on (a,b), with

$$F'(x) = f(x), \quad a < x < b.$$

If F is an antiderivative of f on $[a,b]$, then so is $F+c$ for any constant c. Conversely, if F_1 and F_2 are antiderivatives of f on $[a,b]$, then $F_1 - F_2$ is constant on $[a,b]$ (Theorem$_{\text{). Theorem}}$ shows that antiderivatives can be used to evaluate integrals.

The function $F_0(x) = \int_a^x f(t)\,dt$ is continuous on $[a,b]$ by Theorem~, and $F_0'(x) = f(x)$ on (a,b) by Theorem~. Therefore, F_0 is an antiderivative of f on $[a,b]$. Now let $F = F_0 + c$ ($c =$ constant) be an arbitrary antiderivative of f on $[a,b]$. Then -2pt

$$F(b) - F(a) = \int_a^b f(x)\,dx + c - \int_a^a f(x)\,dx - c = \int_a^b f(x)\,dx.$$

-2.5em2.5em

When applying this theorem, we will use the familiar notation

$$F(b) - F(a) = F(x)\Big|_a^b.$$

Since u and v are continuous on $[a,b]$ (Theorem~), they are integrable on $[a,b]$. Therefore, Theorems~ and imply that the function

$$(uv)' = u'v + uv'$$

is integrable on $[a,b]$, and Theorem~ implies that

$$\int_a^b [u(x)v'(x) + u'(x)v(x)]\,dx = u(x)v(x)\Big|_a^b,$$

which implies .

We will use Theorem~ here and in the next section to obtain other results.

Since f is differentiable on $[a,b]$, it is continuous on $[a,b]$ (Theorem~). Since g is continuous on $[a,b]$, so is fg (Theorem~). Therefore, Theorem~ implies that the integrals in exist. If

$$G(x) = \int_a^x g(t)\,dt, \tag{3.1.25}$$

then $G'(x) = g(x),\ a < x < b$ (Theorem~). Therefore, Theorem~ with $u = f$ and $v = G$ yields

$$\int_a^b f(x)g(x)\,dx = f(x)G(x)\Big|_a^b - \int_a^b f'(x)G(x)\,dx. \tag{3.1.26}$$

Since f' is nonnegative and G is continuous, Theorem~ implies that

$$\int_a^b f'(x)G(x)\,dx = G(c)\int_a^b f'(x)\,dx \tag{3.1.27}$$

for some c in $[a,b]$. From Corollary~,

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

From this and , can be rewritten as

$$\int_a^b f'(x)G(x)\,dx = (f(b) - f(a))\int_a^c g(x)\,dx.$$

Substituting this into and noting that $G(a) = 0$ yields

$$\begin{aligned}\int_a^b f(x)g(x)\,dx \ar &= f(b)\int_a^b g(x)\,dx - (f(b) - f(a))\int_a^c g(x)\,dx,\\ \ar &= f(a)\int_a^c g(x)\,dx + f(b)\left(\int_a^b g(x)\,dx - \int_c^a g(x)\,dx\right)\\ \ar &= f(a)\int_a^c g(x)\,dx + f(b)\int_c^b g(x)\,dx.\end{aligned}$$

-2.5em2.5em

-.4em The following theorem on change of variable is useful for evaluating integrals.

Both integrals in exist: the one on the left by Theorem~, the one on the right by Theorems~ and and the continuity of $f(\phi(t))$. By Theorem~, the function

$$F(x) = \int_a^x f(y)\,dy$$

is an antiderivative of f on $[a,b]$ and, therefore, also on the closed interval with endpoints α and β. Hence, by Theorem~,

$$\int_\alpha^\beta f(x)\,dx = F(\beta) - F(\alpha). \tag{3.1.28}$$

By the chain rule, the function

$$G(t) = F(\phi(t))$$

is an antiderivative of $f(\phi(t))\phi'(t)$ on $[c,d]$, and Theorem~ implies that

$$\begin{aligned}\int_c^d f(\phi(t))\phi'(t)\,dt \ar &= G(d) - G(c) = F(\phi(d)) - F(\phi(c)) \\ \ar &= F(\beta) - F(\alpha).\end{aligned}$$

Comparing this with yields .

These examples illustrate two ways to use Theorem~. In Example~ we evaluated the left side of by transforming it to the right side with a suitable substitution $x = \phi(t)$, while in Example~ we evaluated the right side of by recognizing that it could be obtained from the left side by a suitable substitution.

The following theorem shows that the rule for change of variable remains valid under weaker assumptions on f if ϕ is monotonic.

We consider the case where f is nonnegative and ϕ is nondecreasing, and leave the the rest of the proof to you (Exercises~ and).

First assume that ϕ is increasing. We show first that

$$\overline{\int_a^b} f(x)\,dx = \overline{\int_c^d} f(\phi(t))\phi'(t)\,dt. \tag{3.1.29}$$

Let $\overline{P} = \{t_0, t_1, \dots, t_n\}$ be a partition of $[c,d]$ and $P = \{x_0, x_1, \dots, x_n\}$ with $x_j = \phi(t_j)$ be the corresponding partition of $[a,b]$. Define

$$\begin{aligned}U_j \ar = \sup\{\phi'(t)\}t_{j-1} \le t \le t_j, \\ u_j \ar = \inf\{\phi'(t)\}t_{j-1} \le t \le t_j, \\ M_j \ar = \sup\{f(x)\}x_{j-1} \le x \le x_j, \\ &\arraytext{and} \\ \overline{M}_j \ar = \sup\{f(\phi(t))\phi'(t)\}t_{j-1} \le t \le t_j.\end{aligned}$$

Since ϕ is increasing, $u_j \ge 0$. Therefore,

$$0 \le u_j \le \phi'(t) \le U_j, \quad t_{j-1} \le t \le t_j.$$

Since f is nonnegative, this implies that

$$0 \le f(\phi(t))u_j \le f(\phi(t))\phi'(t) \le f(\phi(t))U_j, \quad t_{j-1} \le t \le t_j.$$

Therefore,

$$M_j u_j \le \overline{M}_j \le M_j U_j,$$

which implies that

$$\overline{M}_j = M_j \rho_j, \tag{3.1.30}$$

where

$$u_j \le \rho_j \le U_j. \tag{3.1.31}$$

Now consider the upper sums

$$\overline{S}(\overline{P}) = \sum_{j=1}^n \overline{M}_j(t_j - t_{j-1}) \quad \text{and} \quad S(P) = \sum_{j=1}^n M_j(x_j - x_{j-1}). \tag{3.1.32}$$

From the mean value theorem,

$$x_j - x_{j-1} = \phi(t_j) - \phi(t_{j-1}) = \phi'(\tau_j)(t_j - t_{j-1}), \tag{3.1.33}$$

where $t_{j-1} < \tau_j < t_j$, so

$$u_j \le \phi'(\tau_j) \le U_j. \tag{3.1.34}$$

From , , and ,

$$\overline{S}(\overline{P}) - S(P) = \sum_{j=1}^{n} M_j(\rho_j - \phi'(\tau_j))(t_j - t_{j-1}). \tag{3.1.35}$$

Now suppose that $|f(x)| \le M$, $a \le x \le b$. Then , , and imply that

$$\left|\overline{S}(\overline{P}) - S(P)\right| \le M \sum_{j=1}^{n} (U_j - u_j)(t_j - t_{j-1}).$$

The sum on the right is the difference between the upper and lower sums of ϕ' over $\overline{P}$. Since ϕ' is integrable on $[c, d]$, this can be made as small as we please by choosing $\|\overline{P}\|$ sufficiently small (Exercise~).

From , $\|P\| \le K\|\overline{P}\|$ if $|\phi'(t)| \le K$, $c \le t \le d$. Hence, Lemma~ implies that

$$\left|S(P) - \overline{\int_a^b} f(x)\,dx\right| < \frac{\epsilon}{3} \quad \text{and} \quad \left|\overline{S}(\overline{P}) - \overline{\int_c^d} f(\phi(t))\phi'(t)\,dt\right| < \frac{\epsilon}{3} \tag{3.1.36}$$

if $\|\overline{P}\|$ is sufficiently small. Now

$$\begin{aligned}\left|\overline{\int_a^b} f(x)\,dx - \overline{\int_c^d} f(\phi(t))\,\phi'(t)\,dt\right| &\ar \\ \le \left|\overline{\int_a^b} f(x)\,dx - S(P)\right| + |S(P) - \overline{S}(\overline{P})| & \\ + \left|\overline{S}(\overline{P}) - \overline{\int_c^d} f(\phi(t))\phi'(t)\,dt\right|.&\end{aligned}$$

Choosing $\overline{P}$ so that $|S(P) - \overline{S}(\overline{P}| < \epsilon/3$ in addition to yields

$$\left|\int_a^b f(x)\,dx - \int_c^d f(\phi(t))\phi'(t)\,dt\right| < \epsilon.$$

Since ϵ is an arbitrary positive number, this implies .

If ϕ is nondecreasing (rather than increasing), it may happen that $x_{j-1} = x_j$ for some values of j; however, this is no real complication, since it simply means that some terms in $S(P)$ vanish.

By applying to $-f$, we infer that

$$\begin{aligned}\underline{\int_a^b} f(x)\,dx\ar &= \underline{\int_c^d} f(\phi(t))\phi'(t)\,dt, \\ [2\jot]\arraytext{since} & \\ \overline{\int_a^b} (-f)(x)\,dx\ar &= -\underline{\int_a^b} f(x)\,dx \\ \arraytext{and} & \\ \overline{\int_c^d} (-f(\phi(t)\phi'(t))\,dt\ar &= -\underline{\int_c^d} f(\phi(t))\phi'(t)\,dt.\end{aligned} \tag{3.1.37}$$

Now suppose that f is integrable on $[a, b]$. Then

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx,$$

by Theorem~. From this, , and ,

$$\underline{\int_c^d} f(\phi(t))\phi'(t)\,dt = \overline{\int_c^d} f(\phi(t))\phi'(t)\,dt = \int_a^b f(x)\,dx.$$

This and Theorem~ (applied to $f(\phi(t))\phi'(t)$) imply that $f(\phi(t))\phi'(t)$ is integrable on $[c,d]$ and

$$\int_a^b f(x)\,dx = \int_c^d f(\phi(t))\phi'(t)\,dt. \tag{3.1.38}$$

A similar argument shows that if $f(\phi(t))\phi'(t)$ is integrable on $[c,d]$, then f is integrable on $[a,b]$, and holds.

\begin{exerciselist}

Prove Theorem~. 4pt

Prove Theorem~.

4pt

Can $|f|$ be integrable on $[a,b]$ if f is not? 4pt

Complete the proof of Theorem~.

4pt

Prove: If f is integrable on $[a,b]$ and $|f(x)| \ge \rho > 0$ for $a \le x \le b$, then $1/f$ is integrable on $[a,b]$

Suppose that f is integrable on $[a,b]$ and define

Bracket argument to \\ must be a dimension

Show that f^+ and f^- are integrable on $[a,b]$, and

$$\int_a^b f(x)\,dx = \int_a^b f^+(x)\,dx + \int_a^b f^-(x)\,dx.$$

Find the weighted average $\overline{u}$ of $u(x)$ over $[a,b]$ with respect to v, and find a point c in $[a,b]$ such that $u(c) = \overline{u}$.

Prove Theorem~.

Show that

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

for all possible relative orderings of a, b, and c, provided that f is integrable on a closed interval containing them.

Prove: If f is integrable on $[a,b]$ and $a = a_0 < a_1 < \cdots < a_n = b$, then

$$\int_a^b f(x)\,dx = \int_{a_0}^{a_1} f(x)\,dx + \int_{a_1}^{a_2} f(x)\,dx + \cdots + \int_{a_{n-1}}^{a_n} f(x)\,dx.$$

Suppose that f is continuous on $[a,b]$ and $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a,b]$. Show that there is a Riemann sum of f over P that {} $\int_a^b f(x)\,dx$.

Suppose that f' exists and $|f'(x)| \le M$ on $[a,b]$. Show that any Riemann sum σ of f over any partition P of $[a,b]$ satisfies

$$\left|\sigma - \int_a^b f(x)\,dx\right| \le M(b-a)\|P\|.$$

Prove: If f is integrable and $f(x) \ge 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \ge 0$, with strict inequality if f is continuous and positive at some point in $[a,b]$.

Complete the proof of Theorem~.

State theorems analogous to Theorems~ and for the function

$$G(x) = \int_x^c f(t)\,dt,$$

and show how your theorems can be obtained from them.

The symbol $\int f(x)\,dx$ denotes an antiderivative of f. A plausible analog of Theorem~ would state that if f and g have antiderivatives on $[a,b]$, then so does $f+g$, which is true, and

$$\int (f+g)(x)\,dx = \int f(x)\,dx + \int g(x)\,dx.\eqno(A)$$

However, this is not true in the usual sense.

(See Exercise~.) Formulate a valid interpretation of the relation

$$\int (cf)(x)\,dx = c\int f(x)\,dx \quad (c \neq 0).$$

Is your interpretation valid if $c = 0$?

In addition to the assumptions of Theorem~, suppose that $f(a) = 0$, $f \not\equiv 0$, and $g(x) > 0$ $(a < x < b)$. Show that there is only one point c in $[a,b]$ with the property stated in Theorem~.

Assuming that Theorem~ is true under the additional assumption that f is nonnegative on $[a,b]$, show that it is true without this assumption.

Assuming that the conclusion of Theorem~ is true if ϕ is nondecreasing, show that it is true if ϕ is nonincreasing.

Suppose g' is integrable and f is continuous on $[a,b]$. Show that $\int_a^b f(x)\,dg(x)$ exists and equals $\int_a^b f(x)g'(x)\,dx$.

Suppose f and g'' are bounded and fg' is integrable on $[a,b]$. Show that $\int_a^b f(x)\,dg(x)$ exists and equals $\int_a^b f(x)g'(x)\,dx$.

\end{exerciselist}

So far we have confined our study of the integral to bounded functions on finite closed intervals. This was for good reasons:

In this section we extend the definition of integral to include cases where f is unbounded or the interval is unbounded, or both.

We say f is {} on an interval I if f is integrable on every finite closed subinterval of I. For example,

$$f(x) = \sin x$$

is locally integrable on $(-\infty,\infty)$;

$$g(x) = \frac{1}{x(x-1)}$$

is locally integrable on $(-\infty,0)$, $(0,1)$, and $(1,\infty)$; and

$$h(x) = \sqrt{x}$$

is locally integrable on $[0,\infty)$.

The limit in always exists if $[a,b)$ is finite and f is locally integrable and bounded on $[a,b)$. In this case, Definitions~ and assign the same value to $\int_a^b f(x)\,dx$ no matter how $f(b)$ is defined (Exercise~). However, the limit may also exist in cases where $b = \infty$ or $b < \infty$ and f is unbounded as x approaches b from the left. In these cases, Definition~ assigns a value to an integral that does not exist in the sense of Definition~, and $\int_a^b f(x)\,dx$ is said to be an {} that {} to the limit in . We also say in this case that f is {} $[a,b)$ and that $\int_a^b f(x)\,dx$ {}. If the limit in does not exist (finite), we say that the improper integral $\int_a^b f(x)\,dx$ {}, and f is {} $[a,b)$. In particular, if $\lim_{c\to b-}\int_a^c f(x)\,dx = \pm\infty$, we say that $\int_a^b f(x)\,dx$ {} $\pm\infty$, and we write

$$\int_a^b f(x)\,dx = \infty \quad \text{or} \quad \int_a^b f(x)\,dx = -\infty,$$

whichever the case may be.

Similar comments apply to the next two definitions.

The existence and value of $\int_a^b f(x)\,dx$ according to Definition~ do not depend on the particular choice of α in (a,b) (Exercise~).

When we wish to distinguish between improper integrals and integrals in the sense of Definition~, we will call the latter {}.

In stating and proving theorems on improper integrals, we will consider integrals of the kind introduced in Definition~. Similar results apply to the integrals of Definitions~ and . We leave it to you to formulate and use them in the examples and exercises as the need arises.

If $a < c < b$, then

$$\int_a^c (c_1f_1 + c_2f_2 + \cdots + c_nf_n)(x)\,dx \ar = c_1\int_a^c f_1(x)\,dx + c_2\int_a^c f_2(x)\,dx \ar + \cdots + c_n\int_a^c f_n(x)\,dx,$$

by Theorem~. Letting $c \to b-$ yields the stated result.

The theory of improper integrals of nonnegative functions is particularly simple.

2pt Since F is nondecreasing on $[a,b)$, Theorem~

implies the conclusion.

We often write 1pt

$$\int_a^b f(x)\,dx < \infty$$

2pt

to indicate that an improper integral of a nonnegative function converges. Theorem~ justifies this convention, since it asserts that a divergent integral of this kind can only diverge to ∞. Similarly, if f is nonpositive and $\int_a^b f(x)\,dx$ converges, we write 1pt

$$\int_a^b f(x)\,dx > -\infty$$

2pt

because a divergent integral of this kind can only diverge to $-\infty$. (To see this, apply Theorem~ to $-f$.) These conventions do not apply to improper integrals of functions that assume both positive and negative values in (a,b), since they may diverge without diverging to $\pm\infty$.

4pt

Assumption implies that

$$\int_a^x f(t)\,dt \le \int_a^x g(t)\,dt, \quad a \le x < b$$

(Theorem~), so

$$\sup_{a\le x<b} \int_a^x f(t)\,dt \le \sup_{a\le x\le b} \int_a^x g(t)\,dt.$$

If $\int_a^b g(x)\,dx < \infty$, the right side of this inequality is finite by Theorem~, so the left side is also. This implies that $\int_a^b f(x)\,dx < \infty$, again by Theorem~.

The proof is by contradiction. If $\int_a^b g(x)\,dx < \infty$, then

implies that $\int_a^b f(x)\,dx < \infty$, contradicting the assumption that $\int_a^b f(x)\,dx = \infty$.

The comparison test is particularly useful if the integrand of the improper integral is complicated but can be compared with a function that is easy to integrate.

If f is any function (not necessarily nonnegative) locally integrable on $[a,b)$, then

$$\int_a^c f(x)\,dx = \int_a^{a_1} f(x)\,dx + \int_{a_1}^c f(x)\,dx$$

if a_1 and c are in $[a,b)$. Since $\int_a^{a_1} f(x)\,dx$ is a proper integral, on letting $c \to b-$ we conclude that if either of the improper integrals $\int_a^b f(x)\,dx$ and $\int_{a_1}^b f(x)\,dx$ converges then so does the other, and in this case

$$\int_a^b f(x)\,dx = \int_a^{a_1} f(x)\,dx + \int_{a_1}^b f(x)\,dx.$$

This means that any theorem implying convergence or divergence of an improper integral $\int_a^b f(x)\,dx$ in the sense of Definition~ remains valid if its hypotheses are satisfied on a subinterval $[a_1,b)$ of $[a,b)$ rather than on all of $[a,b)$. For example, Theorem~ remains valid if is replaced by

$$0 \le f(x) \le g(x), \quad a_1 \le x < b,$$

where a_1 is any point in $[a,b)$.

From this, you can see that if $f(x) \ge 0$ on some subinterval $[a_1,b)$ of $[a,b)$, but not necessarily for all x in $[a,b)$, we can still use the convention introduced earlier for positive functions; that is, we can write $\int_a^b f(x)\,dx < \infty$ if the improper integral converges or $\int_a^b f(x)\,dx = \infty$ if it diverges.

From , there is a point a_2 in $[a_1,b)$ such that

$$0 < \frac{M}{2} < \frac{f(x)}{g(x)} < \frac{3M}{2}, \quad a_2 \le x < b,$$

and therefore

$$\frac{M}{2}g(x) < f(x) < \frac{3M}{2}g(x), \quad a_2 \le x < b. \tag{3.1.39}$$

Theorem~ and the first inequality in imply that

$$\int_{a_2}^b g(x)\,dx < \infty \quad \text{if} \quad \int_{a_2}^b f(x)\,dx < \infty.$$

Theorem~ and the second inequality in imply that

$$\int_{a_2}^b f(x)\,dx < \infty \quad \text{if} \quad \int_{a_2}^b g(x)\,dx < \infty.$$

Therefore, $\int_{a_2}^b f(x)\,dx$ and $\int_{a_2}^b g(x)\,dx$ converge or diverge together, and in the latter case they must diverge to ∞, since their integrands are nonnegative (Theorem~).

If $M = \infty$, there is a point a_2 in $[a_1,b)$ such that

$$f(x) \ge g(x), \quad a_2 \le x \le b,$$

so Theorem~

implies that $\int_a^b f(x)\,dx = \infty$.

If $M = 0$, there is a point a_2 in $[a_1,b)$ such that

$$f(x) \le g(x), \quad a_2 \le x \le b,$$

so Theorem~

implies that $\int_a^b f(x)\,dx < \infty$.

The hypotheses of Theorem~ and do not imply that $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ necessarily converge or diverge together. For example, if $b = \infty$, then $f(x) = 1/x$ and $g(x) = 1/x^2$ satisfy the hypotheses of Theorem~ , while $f(x) = 1/x^2$ and $g(x) = 1/x$ satisfy the hypotheses of Theorem~

. However, $\int_1^\infty 1/x\,dx = \infty$, while $\int_1^\infty 1/x^2\,dx < \infty$.

1em

{}

1em

1em

If

$$g(x) = |f(x)| - f(x),$$

then

$$0 \le g(x) \le 2|f(x)|$$

and $\int_a^b g(x)\,dx < \infty$, because of Theorem~ and the absolute integrability of f. Since

$$f = |f| - g,$$

Theorem~ implies that $\int_a^b f(x)\,dx$ converges.

-.15em We say that f is {} at $b-$ $(= \infty \quad \text{if} \quad b = \infty)$ if f is defined on $[a,b)$ and does not change sign on some subinterval $[a_1,b)$ of $[a,b)$. If f changes sign on every such subinterval, f is {} at $b-$. For a function that is locally integrable on $[a,b)$ and nonoscillatory at $b-$, convergence and absolute convergence of $\int_a^b f(x)\,dx$ amount to the same thing (Exercise~), so absolute convergence is not an interesting concept in connection with such functions. However, an oscillatory function may be integrable, but not absolutely integrable, on $[a,b)$, as the next example shows. We then say that f is {} integrable on $[a,b)$, and that $\int_a^b f(x)\,dx$ converges {}.

The method used in Example~ is a special case of the following test for convergence of improper integrals.

The continuous function fg is locally integrable on $[a,b)$. Integration by parts yields

$$\int_a^c f(x)g(x)\,dx = F(c)g(c) - \int_a^c F(x)g'(x)\,dx, \quad a \le c < b. \tag{3.1.40}$$

Theorem~ implies that the integral on the right converges absolutely as $c \to b-$, since $\int_a^b |g'(x)|\,dx < \infty$ by assumption, and

$$|F(x)g'(x)| \le M|g'(x)|,$$

where M is an upper bound for $|F|$ on $[a,b)$. Moreover, and the boundedness of F imply that $\lim_{c\to b-} F(c)g(c) = 0$. Letting $c \to b-$ in yields

$$\int_a^b f(x)g(x)\,dx = -\int_a^b F(x)g'(x)\,dx,$$

where the integral on the right converges absolutely.

Dirichlet's test is useful only if f is oscillatory at $b-$, since it can be shown that if f is nonoscillatory at $b-$ and F is bounded on $[a,b)$, then $\int_a^b |f(x)g(x)|\,dx < \infty$ if only g is locally integrable and bounded on $[a,b)$ (Exercise~).

The method used in Example~ is a special case of the following test for divergence of improper integrals.

The proof is by contradiction. Let $f = uv$ and $g = 1/v$, and suppose that $\int_a^b u(x)v(x)\,dx$ converges. Then f has the bounded antiderivative $F(x) = \int_a^x u(t)v(t)\,dt$ on $[a,b)$, $\lim_{x\to\infty} g(x) = 0$ and $g' = -v'/v^2$ is absolutely integrable on $[a,b)$. Therefore, Theorem~ implies that $\int_a^b u(x)\,dx$ converges, a contradiction.

If Dirichlet's test shows that $\int_a^b f(x)g(x)\,dx$ converges, there remains the question of whether it converges absolutely or conditionally. The next theorem sometimes answers this question. Its proof can be modeled after the method of Example~ (Exercise~). The idea of an infinite sequence, which we will discuss in Section~4.1, enters into the statement of this theorem. We assume that you recall the concept sufficiently well from calculus to understand the meaning of the theorem.

-.4em The next theorem enables us to investigate an improper integral by transforming it into another whose convergence or divergence is known. It follows from Theorem~ and Definitions~, , and . We omit the proof.

In Section~3.2 we found necessary and sufficient conditions for existence of the proper Riemann integral, and in Section~3.3 we used them to study the properties of the integral. However, it is awkward to apply these conditions to a specific function and determine whether it is integrable, since they require computations of upper and lower sums and upper and lower integrals, which

may be difficult. The main result of this section is an integrability criterion due to that does not require computation, but has to do with how badly discontinuous a function may be and still be integrable.

We emphasize that we are again considering proper integrals of bounded functions on finite intervals.

For a fixed x in (a,b), $W_f(x-h,x+h)$ is a nonnegative and nondecreasing function of h for $0<h<\min(x-a,b-x)$; therefore, $w_f(x)$ exists and is nonnegative, by Theorem~. Similar arguments apply to $w_f(a)$ and $w_f(b)$.

Suppose that $a<x_0<b$. First, suppose that $w_f(x_0)=0$ and $\epsilon>0$. Then

$$W_f[x_0-h,x_0+h]<\epsilon$$

for some $h>0$, so

$$|f(x)-f(x')|<\epsilon \quad \text{if} \quad x_0-h\le x,x'\le x_0+h.$$

Letting $x'=x_0$, we conclude that

$$|f(x)-f(x_0)|<\epsilon \quad \text{if} \quad |x-x_0|<h.$$

Therefore, f is continuous at x_0.

Conversely, if f is continuous at x_0 and $\epsilon>0$, there is a $\delta>0$ such that

$$|f(x)-f(x_0)|<\frac{\epsilon}{2} \quad \text{and} \quad |f(x')-f(x_0)|<\frac{\epsilon}{2}$$

if $x_0-\delta\le x$, $x'\le x_0+\delta$. From the triangle inequality,

$$|f(x)-f(x')|\le|f(x)-f(x_0)|+|f(x')-f(x_0)|<\epsilon,$$

so

$$W_f[x_0-h,x_0+h]\le\epsilon \quad \text{if} \quad h<\delta;$$

therefore, $w_f(x_0)=0$. Similar arguments apply if $x_0=a$ or $x_0=b$.

We use the Heine–Borel theorem (Theorem~). If $w_f(x)<\epsilon$, there is an $h_x>0$ such that

$$|f(x')-f(x'')|<\epsilon \tag{3.1.41}$$

if

$$x-2h_x<x',x''<x+2h_x \quad \text{and} \quad x',x''\in[a,b]. \tag{3.1.42}$$

If $I_x=(x-h_x,x+h_x)$, then the collection

$$\mathcal{H}=\{I_x\}a\le x\le b$$

is an open covering of $[a,b]$, so the Heine–Borel theorem implies that there are finitely many points x_1, x_2, , x_n in $[a,b]$ such that I_{x_1}, I_{x_2}, , I_{x_n} cover $[a,b]$. Let

$$h=\min_{1\le i\le n}h_{x_i}$$

and suppose that $[a_1,b_1]\subset[a,b]$ and $b_1-a_1<h$. If x' and x'' are in $[a_1,b_1]$, then $x'\in I_{x_r}$ for some r $(1\le r\le n)$, so

$$|x'-x_r|<h_{x_r}.$$

Therefore,

$$\begin{aligned}|x''-x_r|\ar\le|x''-x'|+|x'-x_r|<b_1-a_1+h_{x_r}\\ &<h+h_{x_r}\le 2h_{x_r}.\end{aligned}$$

Thus, any two points x' and x'' in $[a_1,b_1]$ satisfy with $x=x_r$, so they also satisfy . Therefore, ϵ is an upper bound for the set

$$\{|f(x')-f(x'')|\}x',x''\in[a_1,b_1],$$

which has the supremum $W_f[a_1,b_1]$. Hence, $W_f[a_1,b_1]\le\epsilon$.

In the following, $L(I)$ is the length of the interval I.

We first show that E_ρ is closed. Suppose that x_0 is a limit point of E_ρ. If $h > 0$, there is an $\overline{x}$ from E_ρ in $(x_0 - h, x_0 + h)$. Since $[\overline{x} - h_1, \overline{x} + h_1] \subset [x_0 - h, x_0 + h]$ for sufficiently small h_1 and $W_f[\overline{x} - h_1, \overline{x} + h_1] \geq \rho$, it follows that $W_f[x_0 - h, x_0 + h] \geq \rho$ for all $h > 0$. This implies that $x_0 \in E_\rho$, so E_ρ is closed (Corollary~).

Now we will show that the stated condition in necessary for integrability. Suppose that the condition is not satisfied; that is, there is a $\rho > 0$ and a $\delta > 0$ such that

$$\sum_{j=1}^{p} L(I_j) \geq \delta$$

for every finite set $\{I_1, I_2, \dots, I_p\}$ of open intervals covering E_ρ. If $P = \{x_0, x_1, \dots, x_n\}$ is a partition of $[a, b]$, then

$$S(P) - s(P) = \sum_{j \in A} (M_j - m_j)(x_j - x_{j-1}) + \sum_{j \in B} (M_j - m_j)(x_j - x_{j-1}), \tag{3.1.43}$$

where

$$A = \{j\}[x_{j-1}, x_j] \cap E_\rho \neq \emptyset \quad \text{and} \quad B = \{j\}[x_{j-1}, x_j] \cap E_\rho = \emptyset$$

Since $\bigcup_{j \in A} (x_{j-1}, x_j)$ contains all points of E_ρ except any of x_0, x_1, , x_n that may be in E_ρ, and each of these finitely many possible exceptions can be covered by an open interval of length as small as we please, our assumption on E_ρ implies that

$$\sum_{j \in A} (x_j - x_{j-1}) \geq \delta.$$

Moreover, if $j \in A$, then

$$M_j - m_j \geq \rho,$$

so implies that

$$S(P) - s(P) \geq \rho \sum_{j \in A} (x_j - x_{j-1}) \geq \rho\delta.$$

Since this holds for every partition of $[a, b]$, f is not integrable on $[a, b]$, by Theorem~. This proves that the stated condition is necessary for integrability.

For sufficiency, let ρ and δ be positive numbers and let I_1, I_2, , I_p be open intervals that cover E_ρ and satisfy . Let

$$\tilde{I}_j = [a, b] \cap \overline{I}_j.$$

($\overline{I}_j$ = closure of I.) After combining any of $\tilde{I}_1$, $\tilde{I}_2$, , $\tilde{I}_p$ that overlap, we obtain a set of pairwise disjoint closed subintervals

$$C_j = [\alpha_j, \beta_j], \quad 1 \leq j \leq q \ (\leq p),$$

of $[a, b]$ such that

$$a \leq \alpha_1 < \beta_1 < \alpha_2 < \beta_2 \cdots < \alpha_{q-1} < \beta_{q-1} < \alpha_q < \beta_q \leq b, \tag{3.1.44}$$

$$\sum_{i=1}^{q} (\beta_i - \alpha_i) < \delta \tag{3.1.45}$$

and

$$w_f(x) < \rho, \quad \beta_j \leq x \leq \alpha_{j+1}, \quad 1 \leq j \leq q - 1.$$

Also, $w_f(x) < \rho$ for $a \leq x \leq \alpha_1$ if $a < \alpha_1$ and for $\beta_q \leq x \leq b$ if $\beta_q < b$.

Let P_0 be the partition of $[a, b]$ with the partition points indicated in , and refine P_0 by partitioning each subinterval $[\beta_j, \alpha_{j+1}]$ (as well as $[a, \alpha_1]$ if $a < \alpha_1$ and $[\beta_q, b]$ if $\beta_q < b$) into subintervals on which the oscillation of f is not greater than ρ. This is possible by Lemma~. In this way, after renaming the entire collection of partition points, we obtain a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ for which $S(P) - s(P)$ can be written as in , with

$$\sum_{j\in A}(x_j-x_{j-1})=\sum_{i=1}^{q}(\beta_i-\alpha_i)<\delta$$

(see) and

$$M_j-m_j\le\rho,\quad j\in B.$$

For this partition,

$$\sum_{j\in A}(M_j-m_j)(x_j-x_{j-1})\le 2K\sum_{j\in A}(x_j-x_{j-1})<2K\delta,$$

where K is an upper bound for $|f|$ on $[a,b]$ and

$$\sum_{j\in B}(M_j-m_j)(x_j-x_{j-1})\le\rho(b-a).$$

We have now shown that if ρ and δ are arbitrary positive numbers, there is a partition P of $[a,b]$ such that

$$S(P)-s(P)<2K\delta+\rho(b-a). \tag{3.1.46}$$

If $\epsilon>0$, let

$$\delta=\frac{\epsilon}{4K}\quad\text{and}\quad\rho=\frac{\epsilon}{2(b-a)}.$$

Then yields

$$S(P)-s(P)<\epsilon,$$

and Theorem~ implies that f is integrable on $[a,b]$.

We need the next definition to state Lebesgue's integrability condition.

-3em

Note that any subset of a set of Lebesgue measure zero is also of Lebesgue measure zero. (Why?)

-2em

There are also nondenumerable sets of Lebesgue measure zero, but it is beyond the scope of this book to discuss examples.

The next theorem is the main result of this section.

From Theorem~,

$$S=\{x\in[a,b]\}w_f(x)>0$$

Since $w_f(x)>0$ if and only if $w_f(x)\ge 1/i$ for some positive integer i, we can write

$$S=\bigcup_{i=1}^{\infty}S_i, \tag{3.1.47}$$

where

$$S_i=\{x\in[a,b]\}w_f(x)\ge 1/i.$$

Now suppose that f is integrable on $[a,b]$ and $\epsilon>0$. From Lemma~, each S_i can be covered by a finite number of open intervals I_{i1}, I_{i2}, , I_{in} of total length less than $\epsilon/2^i$. We simply renumber these intervals consecutively; thus,

$$I_1,I_2,\cdots=I_{11},\dots,I_{1n_1},I_{21},\dots,I_{2n_2},\dots,I_{i1},\dots,I_{in_i},\dots.$$

Now and hold because of and , and we have shown that the stated condition is necessary for integrability.

For sufficiency, suppose that the stated condition holds and $\epsilon>0$. Then S can be covered by open intervals $I_1,I_2,\dots$ that satisfy . If $\rho>0$, then the set

$$E_\rho=\{x\in[a,b]\}w_f(x)\ge\rho$$

of Lemma~ is contained in S (Theorem~), and therefore E_ρ is covered by $I_1, I_2, \ldots$. Since E_ρ is closed (Lemma~) and bounded, the Heine–Borel theorem implies that E_ρ is covered by a finite number of intervals from $I_1, I_2, \ldots$. The sum of the lengths of the latter is less than ϵ, so Lemma~ implies that f is integrable on $[a, b]$.

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3.5: A More Advanced Look at the Existence of the Proper Riemann Integral

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CHAPTER OVERVIEW

4: Infinite Sequences and Series

IN THIS CHAPTER we consider infinite sequences and series of constants and functions of a real variable.

- SECTION 4.1 introduces infinite sequences of real numbers. The concept of a limit of a sequence is defined, as is the concept of divergence of a sequence to $\pm\infty$. We discuss bounded sequences and monotonic sequences. The limit inferior and limit superior of a sequence are defined. We prove the Cauchy convergence criterion for sequences of real numbers.
- SECTION 4.2 defines a subsequence of an infinite sequence. We show that if a sequence converges to a limit or diverges to $\pm\infty$, then so do all subsequences of the sequence. Limit points and boundedness of a set of real numbers are discussed in terms of sequences of members of the set. Continuity and boundedness of a function are discussed in terms of the values of the function at sequences of points in its domain.
- SECTION 4.3 introduces concepts of convergence and divergence to $\pm\infty$ for infinite series of constants. We prove Cauchy's convergence criterion for a series of constants. In connection with series of positive terms, we consider the comparison test, the integral test, the ratio test, and Raabe's test. For general series, we consider absolute and conditional convergence, Dirichlet's test, rearrangement of terms, and multiplication of one infinite series by another.
- SECTION 4.4 deals with pointwise and uniform convergence of sequences and series of functions. Cauchy's uniform convergence criteria for sequences and series are proved, as is Dirichlet's test for uniform convergence of a series. We give sufficient conditions for the limit of a sequence of functions or the sum of an infinite series of functions to be continuous, integrable, or differentiable.
- SECTION 4.5 considers power series. It is shown that a power series that converges on an open interval defines an infinitely differentiable function on that interval. We define the Taylor series of an infinitely differentiable function, and give sufficient conditions for the Taylor series to converge to the function on some interval. Arithmetic operations with power series are discussed.

4.1: Sequences of Real Numbers

4.2: Earlier Topics Revisited With Sequences

4.3: Infinite Series of Constants

4.4: Sequences and Series of Functions

4.5: Power Series

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4.1: Sequences of Real Numbers

An {} (more briefly, a {}) of real numbers is a real-valued function defined on a set of integers $\{n\}n \geq k$. We call the values of the function the {} of the sequence. We denote a sequence by listing its terms in order; thus,

$$\{s_n\}_k^\infty = \{s_k, s_{k+1}, \ldots\}. \tag{4.1.1}$$

For example,

$$\left\{\frac{1}{n^2+1}\right\}_0^\infty \ar = \left\{1, \frac{1}{2}, \frac{1}{5}, \ldots, \frac{1}{n^2+1}, \ldots\right\},$$
$$\{(-1)^n\}_0^\infty \ar = \{1, -1, 1, \ldots, (-1)^n, \ldots\},$$
$$\arraytext{and}$$
$$\left\{\frac{1}{n-2}\right\}_3^\infty \ar = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n-2}, \ldots\right\}.$$

The real number s_n is the nth {} of the sequence. Usually we are interested only in the terms of a sequence and the order in which they appear, but not in the particular value of k in . Therefore, we regard the sequences

$$\left\{\frac{1}{n-2}\right\}_3^\infty \quad \text{and} \quad \left\{\frac{1}{n}\right\}_1^\infty$$

as identical.

We will usually write $\{s_n\}$ rather than $\{s_n\}_k^\infty$. In the absence of any indication to the contrary, we take $k = 0$ unless s_n is given by a rule that is invalid for some nonnegative integer, in which case k is understood to be the smallest positive integer such that s_n is defined for all $n \geq k$. For example, if

$$s_n = \frac{1}{(n-1)(n-5)},$$

then $k = 6$.

The interesting questions about a sequence $\{s_n\}$ concern the behavior of s_n for large n.

As we saw in Section~2.1 when discussing limits of functions, Definition~ is not changed by replacing with

$$|s_n - s| < K\epsilon \quad \text{if} \quad n \geq N,$$

where K is a positive constant.

Definition~ does not require that there be an integer N such that holds for all ϵ; rather, it requires that for each positive ϵ there be an integer N that satisfies for that particular ϵ. Usually, N depends on ϵ and must be increased if ϵ is decreased. The constant sequences (Example~) are essentially the only ones for which N does not depend on ϵ (Exercise~).

We say that the terms of a sequence $\{s_n\}_k^\infty$ satisfy a given condition {} n if s_n satisfies the condition for all $n \geq k$, or {} n if there is an integer $N > k$ such that s_n satisfies the condition whenever $n \geq N$. For example, the terms of $\{1/n\}_1^\infty$ are positive for all n, while those of $\{1 - 7/n\}_1^\infty$ are positive for large n (take $N = 8$).

Suppose that

$$\lim_{n\to\infty} s_n = s \quad \text{and} \quad \lim_{n\to\infty} s_n = s'.$$

5pt We must show that $s = s'$. Let $\epsilon > 0$. From Definition~, there are integers N_1 and N_2 such that

$$|s_n - s| < \epsilon \quad \text{if} \quad n \geq N_1$$

5pt (because $\lim_{n\to\infty} s_n = s$), and

$$|s_n - s'| < \epsilon \quad \text{if} \quad n \geq N_2$$

(because $\lim_{n\to\infty} s_n = s'$). These inequalities both hold if $n \geq N = \max(N_1, N_2)$, which implies that

$$|s-s'|\ar=|(s-s_N)+(s_N-s')|$$
$$\ar\le |s-s_N|+|s_N-s'|<\epsilon+\epsilon=2\epsilon.$$

Since this inequality holds for every $\epsilon>0$ and $|s-s'|$ is independent of ϵ, we conclude that $|s-s'|=0$; that is, $s=s'$.

We say that

$$\lim_{n\to\infty} s_n=\infty$$

if for any real number a, $s_n>a$ for large n. Similarly,

$$\lim_{n\to\infty} s_n=-\infty$$

if for any real number a, $s_n<a$ for large n. However, we do not regard $\{s_n\}$ as convergent unless $\lim_{n\to\infty} s_n$ is finite, as required by Definition~. To emphasize this distinction, we say that $\{s_n\}$ {} ∞ $(-\infty)$ if $\lim_{n\to\infty} s_n=\infty$ $(-\infty)$.

Unknown environment 'definition'

By taking $\epsilon=1$ in , we see that if $\lim_{n\to\infty} s_n=s$, then there is an integer N such that

$$|s_n-s|<1 \quad \text{if} \quad n\ge N.$$

Therefore,

$$|s_n|=|(s_n-s)+s|\le |s_n-s|+|s|<1+|s| \quad \text{if} \quad n\ge N,$$

and

$$|s_n|\le \max\{|s_0|,|s_1|,\dots,|s_{N-1}|,1+|s|\}$$

for all n, so $\{s_n\}$ is bounded.

. Let $\beta=\sup\{s_n\}$. If $\beta<\infty$, Theorem~ implies that if $\epsilon>0$ then

$$\beta-\epsilon<s_N\le\beta$$

for some integer N. Since $s_N\le s_n\le\beta$ if $n\ge N$, it follows that

$$\beta-\epsilon<s_n\le\beta \quad \text{if} \quad n\ge N.$$

This implies that $|s_n-\beta|<\epsilon$ if $n\ge N$, so $\lim_{n\to\infty} s_n=\beta$, by Definition~. If $\beta=\infty$ and b is any real number, then $s_N>b$ for some integer N. Then $s_n>b$ for $n\ge N$, so $\lim_{n\to\infty} s_n=\infty$.

We leave the proof of

to you (Exercise~)

The next theorem enables us to apply the theory of limits developed in Section~2.1 to some sequences. We leave the proof to you (Exercise~).

The next theorem enables us to investigate convergence of sequences by examining simpler sequences. It is analogous to Theorem~.

We prove and and leave the rest to you (Exercises~ and). For , we write

$$s_nt_n-st=s_nt_n-st_n+st_n-st=(s_n-s)t_n+s(t_n-t);$$

hence,

$$|s_nt_n-st|\le |s_n-s|\,|t_n|+|s|\,|t_n-t|. \tag{4.1.2}$$

Since $\{t_n\}$ converges, it is bounded (Theorem~). Therefore, there is a number R such that $|t_n|\le R$ for all n, and implies that

$$|s_nt_n-st|\le R|s_n-s|+|s|\,|t_n-t|. \tag{4.1.3}$$

From , if $\epsilon>0$ there are integers N_1 and N_2 such that

$$|s_n - s|\ar < \epsilon \quad \text{if} \quad n \geq N_1 \tag{4.1.4}$$
$$\arraytext{and}$$
$$|t_n - t|\ar < \epsilon \quad \text{if} \quad n \geq N_2. \tag{4.1.5}$$

If $N = \max(N_1, N_2)$, then and both hold when $n \geq N$, and implies that

$$|s_n t_n - st| \leq (R + |s|)\epsilon \quad \text{if} \quad n \geq N.$$

This proves .

Now consider in the special case where $s_n = 1$ for all n and $t \neq 0$; thus, we want to show that

$$\lim_{n\to\infty} \frac{1}{t_n} = \frac{1}{t}.$$

First, observe that since $\lim_{n\to\infty} t_n = t \neq 0$, there is an integer M such that $|t_n| \geq |t|/2$ if $n \geq M$. To see this, we apply Definition~ with $\epsilon = |t|/2$; thus, there is an integer M such that $|t_n - t| < |t/2|$ if $n \geq M$. Therefore,

$$|t_n| = |t + (t_n - t)| \geq ||t| - |t_n - t|| \geq \frac{|t|}{2} \quad \text{if} \quad n \geq M.$$

If $\epsilon > 0$, choose N_0 so that $|t_n - t| < \epsilon$ if $n \geq N_0$, and let $N = \max(N_0, M)$. Then

$$\left|\frac{1}{t_n} - \frac{1}{t}\right| = \frac{|t - t_n|}{|t_n|\,|t|} \leq \frac{2\epsilon}{|t|^2} \quad \text{if} \quad n \geq N;$$

hence, $\lim_{n\to\infty} 1/t_n = 1/t$. Now we obtain in the general case from with $\{t_n\}$ replaced by $\{1/t_n\}$.

Equations – are valid even if s and t are arbitrary extended reals, provided that their right sides are defined in the extended reals (Exercises~, , and); is valid if s/t is defined in the extended reals and $t \neq 0$ (Exercise~).

Requiring a sequence to converge may be unnecessarily restrictive in some situations. Often, useful results can be obtained from assumptions on the {} and {} of a sequence, which we consider next.

We will prove and leave the proof of

to you (Exercise~). Since $\{s_n\}$ is bounded above, there is a number β such that $s_n < \beta$ for all n. Since $\{s_n\}$ does not diverge to $-\infty$, there is a number α such that $s_n > \alpha$ for infinitely many n. If we define

$$M_k = \sup\{s_k, s_{k+1}, \ldots, s_{k+r}, \ldots\},$$

then $\alpha \leq M_k \leq \beta$, so $\{M_k\}$ is bounded. Since $\{M_k\}$ is nonincreasing (why?), it converges, by Theorem~. Let

$$\overline{s} = \lim_{k\to\infty} M_k. \tag{4.1.6}$$

If $\epsilon > 0$, then $M_k < \overline{s} + \epsilon$ for large k, and since $s_n \leq M_k$ for $n \geq k$, $\overline{s}$ satisfies .

If were false for some positive ϵ, there would be an integer K such that

$$s_n \leq \overline{s} - \epsilon \quad \text{if} \quad n \geq K.$$

However, this implies that

$$M_k \leq \overline{s} - \epsilon \quad \text{if} \quad k \geq K,$$

which contradicts . Therefore, $\overline{s}$ has the stated properties.

Now we must show that $\overline{s}$ is the only real number with the stated properties. If $t < \overline{s}$, the inequality

$$s_n < t + \frac{\overline{s} - t}{2} = \overline{s} - \frac{\overline{s} - t}{2}$$

cannot hold for all large n, because this would contradict with $\epsilon = (\overline{s} - t)/2$. If $\overline{s} < t$, the inequality

$$s_n > t - \frac{t - \overline{s}}{2} = \overline{s} + \frac{t - \overline{s}}{2}$$

cannot hold for infinitely many n, because this would contradict with $\epsilon = (t - \overline{s})/2$. Therefore, $\overline{s}$ is the only real number with the stated properties.

The existence and uniqueness of $\overline{s}$ and $\underline{s}$ follow from Theorem~ and Definition~. If $\overline{s}$ and $\underline{s}$ are both finite, then and imply that

$$\underline{s} - \epsilon < \overline{s} + \epsilon$$

for every $\epsilon > 0$, which implies . If $\underline{s} = -\infty$ or $\overline{s} = \infty$, then is obvious. If $\underline{s} = \infty$ or $\overline{s} = -\infty$, then follows immediately from Definition~.

If $s = \pm\infty$, the equivalence of and follows immediately from their definitions. If $\lim_{n\to\infty} s_n = s$ (finite), then Definition~ implies that – hold with $\overline{s}$ and $\underline{s}$ replaced by s. Hence, follows from the uniqueness of $\overline{s}$ and $\underline{s}$. For the converse, suppose that $\overline{s} = \underline{s}$ and let s denote their common value. Then and imply that

$$s - \epsilon < s_n < s + \epsilon$$

for large n, and follows from Definition~ and the uniqueness of $\lim_{n\to\infty} s_n$ (Theorem~).

To determine from Definition~ whether a sequence has a limit, it is necessary to guess what the limit is. (This is particularly difficult if the sequence diverges!) To use Theorem~ for this purpose requires finding $\overline{s}$ and $\underline{s}$. The following convergence criterion has neither of these defects.

Suppose that $\lim_{n\to\infty} s_n = s$ and $\epsilon > 0$. By Definition~, there is an integer N such that

$$|s_r - s| < \frac{\epsilon}{2} \quad \text{if} \quad r \geq N.$$

Therefore,

$$|s_n - s_m| = |(s_n - s) + (s - s_m)| \leq |s_n - s| + |s - s_m| < \epsilon \quad \text{if} \quad n, m \geq N.$$

Therefore, the stated condition is necessary for convergence of $\{s_n\}$. To see that it is sufficient, we first observe that it implies that $\{s_n\}$ is bounded (Exercise~), so $\overline{s}$ and $\underline{s}$ are finite (Theorem~). Now suppose that $\epsilon > 0$ and N satisfies . From and ,

$$|s_n - \overline{s}| < \epsilon, \tag{4.1.7}$$

for some integer $n > N$ and, from and ,

$$|s_m - \underline{s}| < \epsilon \tag{4.1.8}$$

for some integer $m > N$. Since

$$\begin{aligned} |\overline{s} - \underline{s}| \ar &= |(\overline{s} - s_n) + (s_n - s_m) + (s_m - \underline{s})| \\ \ar &\leq |\overline{s} - s_n| + |s_n - s_m| + |s_m - \underline{s}|, \end{aligned}$$

– imply that

$$|\overline{s} - \underline{s}| < 3\epsilon.$$

Since ϵ is an arbitrary positive number, this implies that $\overline{s} = \underline{s}$, so $\{s_n\}$ converges, by Theorem~.

To see that cannot have more than one solution, suppose that $x = f(x)$ and $x' = f(x')$. From and the mean value theorem (Theorem~),

$$x - x' = f'(c)(x - x')$$

for some c between x and x'. This and imply that

$$|x - x'| \leq r|x - x'|.$$

Since $r < 1$, $x = x'$.

We will now show that has a solution. With x_0 arbitrary, define

$$x_n = f(x_{n-1}), \quad n \geq 1. \tag{4.1.9}$$

We will show that $\{x_n\}$ converges. From and the mean value theorem,

$$x_{n+1}-x_n=f(x_n)-f(x_{n-1})=f'(c_n)(x_n-x_{n-1}),$$

where c_n is between x_{n-1} and x_n. This and imply that

$$|x_{n+1}-x_n|\le r|x_n-x_{n-1}| \quad \text{if} \quad n\ge 1. \tag{4.1.10}$$

The inequality

$$|x_{n+1}-x_n|\le r^n|x_1-x_0| \quad \text{if} \quad n\ge 0, \tag{4.1.11}$$

follows by induction from . Now, if $n>m$,

$$\begin{aligned}|x_n-x_m|\ar &= |(x_n-x_{n-1})+(x_{n-1}-x_{n-2})+\cdots+(x_{m+1}-x_m)|\\ \ar &\le |x_n-x_{n-1}|+|x_{n-1}-x_{n-2}|+\cdots+|x_{m+1}-x_m|,\end{aligned}$$

and yields

$$|x_n-x_m|\le |x_1-x_0|\,r^m(1+r+\cdots+r^{n-m-1}). \tag{4.1.12}$$

In Example~ we saw that the sequence $\{s_k\}$ defined by

$$s_k=1+r+\cdots+r^k$$

converges to $1/(1-r)$ if $|r|<1$; moreover, since we have assumed here that $0<r<1$, $\{s_k\}$ is nondecreasing, and therefore $s_k<1/(1-r)$ for all k. Therefore, yields

$$|x_n-x_m|<\frac{|x_1-x_0|}{1-r}r^m \quad \text{if} \quad n>m.$$

Now it follows that

$$|x_n-x_m|<\frac{|x_1-x_0|}{1-r}r^N \quad \text{if} \quad n,m>N,$$

and, since $\lim_{N\to\infty} r^N=0$, $\{x_n\}$ converges, by Theorem~. If $\hat{x}=\lim_{n\to\infty}x_n$, then and the continuity of f imply that $\hat{x}=f(\hat{x})$.

In Chapter~2.3 we used ϵ–δ definitions and arguments to develop the theory of limits, continuity, and differentiability; for example, f is continuous at x_0 if for each $\epsilon>0$ there is a $\delta>0$ such that $|f(x)-f(x_0)|<\epsilon$ when $|x-x_0|<\delta$. The same theory can be developed by methods based on sequences. Although we will not carry this out in detail, we will develop it enough to give some examples. First, we need another definition about sequences.

Note that $\{s_n\}$ is a subsequence of itself, as can be seen by taking $n_k=k$. All other subsequences of $\{s_n\}$ are obtained by deleting terms from $\{s_n\}$ and leaving those remaining in their original relative order.

Since a subsequence $\{s_{n_k}\}$ is again a sequence (with respect to k), we may ask whether $\{s_{n_k}\}$ converges.

The sequence in this example has subsequences that converge to different limits. The next theorem shows that if a sequence converges to a finite limit or diverges to $\pm\infty$, then all its subsequences do also.

We consider the case where s is finite and leave the rest to you (Exercise~). If holds and $\epsilon>0$, there is an integer N such that

$$|s_n-s|<\epsilon \quad \text{if} \quad n\ge N.$$

Since $\{n_k\}$ is an increasing sequence, there is an integer K such that $n_k\ge N$ if $k\ge K$. Therefore,

$$|s_{n_k}-L|<\epsilon \quad \text{if} \quad k\ge K,$$

which implies .

We consider the case where $\{s_n\}$ is nondecreasing and leave the rest to you (Exercise~). Since $\{s_{n_k}\}$ is also nondecreasing in this case, it suffices to show that

$$\sup\{s_{n_k}\}=\sup\{s_n\} \tag{4.1.13}$$

and then apply Theorem~

. Since the set of terms of $\{s_{n_k}\}$ is contained in the set of terms of $\{s_n\}$,

$$\sup\{s_n\} \geq \sup\{s_{n_k}\}. \tag{4.1.14}$$

Since $\{s_n\}$ is nondecreasing, there is for every n an integer n_k such that $s_n \leq s_{n_k}$. This implies that

$$\sup\{s_n\} \leq \sup\{s_{n_k}\}.$$

This and imply .

In Section~1.3 we defined {} in terms of neighborhoods: $\overline{x}$ is a limit point of a set S if every neighborhood of $\overline{x}$ contains points of S distinct from $\overline{x}$. The next theorem shows that an equivalent definition can be stated in terms of sequences.

For sufficiency, suppose that the stated condition holds. Then, for each $\epsilon > 0$, there is an integer N such that $0 < |x_n - x| < \epsilon$ if $n \geq N$. Therefore, every ϵ-neighborhood of $\overline{x}$ contains infinitely many points of S. This means that $\overline{x}$ is a limit point of S.

For necessity, let $\overline{x}$ be a limit point of S. Then, for every integer $n \geq 1$, the interval $(\overline{x} - 1/n, \overline{x} + 1/n)$ contains a point x_n $(\neq \overline{x})$ in S. Since $|x_m - \overline{x}| \leq 1/n$ if $m \geq n$, $\lim_{n\to\infty} x_n = \overline{x}$.

We will use the next theorem to show that continuity can be defined in terms of sequences.

We prove and leave and

to you (Exercise~). Let S be the set of distinct numbers that occur as terms of $\{x_n\}$. (For example, if $\{x_n\} = \{(-1)^n\}$, $S = \{1, -1\}$; if $\{x_n\} = \{1, \frac{1}{2}, 1, \frac{1}{3}, \dots, 1, 1/n, \dots\}$, $S = \{1, \frac{1}{2}, \dots, 1/n, \dots\}$.) If S contains only finitely many points, then some $\overline{x}$ in S occurs infinitely often in $\{x_n\}$; that is, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that $x_{n_k} = \overline{x}$ for all k. Then $\lim_{k\to\infty} x_{n_k} = \overline{x}$, and we are finished in this case.

If S is infinite, then, since S is bounded (by assumption), the Bolzano–Weierstrass theorem (Theorem~) implies that S has a limit point $\overline{x}$. From Theorem~, there is a sequence of points $\{y_j\}$ in S, distinct from $\overline{x}$, such that

$$\lim_{j\to\infty} y_j = \overline{x}. \tag{4.1.15}$$

Although each y_j occurs as a term of $\{x_n\}$, $\{y_j\}$ is not necessarily a subsequence of $\{x_n\}$, because if we write

$$y_j = x_{n_j},$$

there is no reason to expect that $\{n_j\}$ is an increasing sequence as required in Definition~. However, it is always possible to pick a subsequence $\{n_{j_k}\}$ of $\{n_j\}$ that is increasing, and then the sequence $\{y_{j_k}\} = \{s_{n_{j_k}}\}$ is a subsequence of both $\{y_j\}$ and $\{x_n\}$. Because of and Theorem~ this subsequence converges to~$\overline{x}$.

-.3em We now show that continuity can be defined and studied in terms of sequences.

Assume that $a < \overline{x} < b$; only minor changes in the proof are needed if $\overline{x} = a$ or $\overline{x} = b$. First, suppose that f is continuous at $\overline{x}$ and $\{x_n\}$ is a sequence of points in $[a, b]$ satisfying . If $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(x) - f(\overline{x})| < \epsilon \quad \text{if} \quad |x - \overline{x}| < \delta. \tag{4.1.16}$$

From , there is an integer N such that $|x_n - \overline{x}| < \delta$ if $n \geq N$. This and imply that $|f(x_n) - f(\overline{x})| < \epsilon$ if $n \geq N$. This implies , which shows that the stated condition is necessary.

For sufficiency, suppose that f is discontinuous at $\overline{x}$. Then there is an $\epsilon_0 > 0$ such that, for each positive integer n, there is a point x_n that satisfies the inequality

$$|x_n - \overline{x}| < \frac{1}{n}$$

while

$$|f(x_n) - f(\overline{x})| \geq \epsilon_0.$$

The sequence $\{x_n\}$ therefore satisfies , but not . Hence, the stated condition cannot hold if f is discontinuous at $\overline{x}$. This proves sufficiency.

Armed with the theorems we have proved so far in this section, we could develop the theory of continuous functions by means of definitions and proofs based on sequences and subsequences. We give one example, a new proof of Theorem~, and leave others for exercises.

The proof is by contradiction. If f is not bounded on $[a,b]$, there is for each positive integer n a point x_n in $[a,b]$ such that $|f(x_n)| > n$. This implies that

$$\lim_{n\to\infty} |f(x_n)| = \infty. \tag{4.1.17}$$

Since $\{x_n\}$ is bounded, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ (Theorem~

). If

$$\overline{x} = \lim_{k\to\infty} x_{n_k},$$

then $\overline{x}$ is a limit point of $[a,b]$, so $\overline{x} \in [a,b]$. If f is continuous on $[a,b]$, then

$$\lim_{k\to\infty} f(x_{n_k}) = f(\overline{x})$$

by Theorem~, so

$$\lim_{k\to\infty} |f(x_{n_k})| = |f(\overline{x})|$$

(Exercise~), which contradicts . Therefore, f cannot be both continuous and unbounded on $[a,b]$

The theory of sequences developed in the last two sections can be combined with the familiar notion of a finite sum to produce the theory of infinite series. We begin the study of infinite series in this section.

2pt We will usually refer to infinite series more briefly as {}. 2pt

The series $\sum_{n=0}^{\infty} r^n$ is called the {}. It occurs in many applications.

An infinite series can be viewed as a generalization of a finite sum

$$A = \sum_{n=k}^{N} a_n = a_k + a_{k+1} + \cdots + a_N$$

by thinking of the finite sequence $\{a_k, a_{k+1}, \ldots, a_N\}$ as being extended to an infinite sequence $\{a_n\}_k^{\infty}$ with $a_n = 0$ for $n > N$. Then the partial sums of $\sum_{n=k}^{\infty} a_n$ are

$$A_n = a_k + a_{k+1} + \cdots + a_n, \quad k \le n < N,$$

and

$$A_n = A, \quad n \ge N;$$

that is, the terms of $\{A_n\}_k^{\infty}$ equal the finite sum A for $n \ge k$. Therefore, $\lim_{n\to\infty} A_n$
$= A$.

The next two theorems can be proved by applying Theorems~ and to the partial sums of the series in question (Exercises~ and).

Dropping finitely many terms from a series does not alter convergence or divergence, although it does change the sum of a convergent series if the terms dropped have a nonzero sum. For example, suppose that we drop the first k terms of a series $\sum_{n=0}^{\infty} a_n$, and consider the new series $\sum_{n=k}^{\infty} a_n$. Denote the partial sums of the two series by

$$\begin{aligned} A_n &= a_0 + a_1 + \cdots + a_n, \quad n \ge 0, \\ &\text{and} \\ A'_n &= a_k + a_{k+1} + \cdots + a_n, \quad n \ge k. \end{aligned}$$

Since

$$A_n = (a_0 + a_1 + \cdots + a_{k-1}) + A'_n, \quad n \ge k,$$

it follows that $A = \lim_{n\to\infty} A_n$ exists (in the extended reals) if and only if $A' = \lim_{n\to\infty} A'_n$ does, and in this case

$$A = (a_0 + a_1 + \cdots + a_{k-1}) + A'.$$

An important principle follows from this.

We will soon give several conditions concerning convergence of a series $\sum_{n=k}^{\infty} a_n$ with nonnegative terms. According to Lemma~, these results apply to series that have at most finitely many negative terms, as long as a_n is nonnegative and satisfies the conditions for n sufficiently large.

When we are interested only in whether $\sum_{n=k}^{\infty} a_n$ converges or diverges and not in its sum, we will simply say `$\sum a_n$ converges'' or` $\sum a_n$ diverges.'' Lemma~ justifies this convention, subject to the understanding that $\sum a_n$ stands for $\sum_{n=k}^{\infty} a_n$, where k is an integer such that a_n is defined for $n \geq k$. (For example,

$$\sum \frac{1}{(n-6)^2} \quad \text{stands for} \quad \sum_{n=k}^{\infty} \frac{1}{(n-6)^2},$$

-.3em where $k \geq 7$.) We write $\sum a_n = \infty$ $(-\infty)$ if $\sum a_n$ diverges to ∞ $(-\infty)$. Finally, let us agree that

$$\sum_{n=k}^{\infty} a_n \quad \text{and} \quad \sum_{n=k-j}^{\infty} a_{n+j}$$

-.35em (where we obtain the second expression by shifting the index in the first) both represent the same series.

The Cauchy convergence criterion for sequences (Theorem~) yields a useful criterion for convergence of series.

In terms of the partial sums $\{A_n\}$ of $\sum a_n$,

$$a_n + a_{n+1} + \cdots + a_m = A_m - A_{n-1}.$$

Therefore, can be written as

$$|A_m - A_{n-1}| < \epsilon \quad \text{if} \quad m \geq n \geq N.$$

Since $\sum a_n$ converges if and only if $\{A_n\}$ converges, Theorem~ implies the conclusion.

Intuitively, Theorem~ means that $\sum a_n$ converges if and only if arbitrarily long sums

$$a_n + a_{n+1} + \cdots + a_m, \quad m \geq n,$$

can be made as small as we please by picking n large enough.

Letting $m = n$ in yields the following important corollary of Theorem~.

It must be emphasized that Corollary~ gives a {} condition for convergence; that is, $\sum a_n$ cannot converge unless $\lim_{n\to\infty} a_n = 0$. The condition is {}; $\sum a_n$ may diverge even if $\lim_{n\to\infty} a_n = 0$. We will see examples below.

We leave the proof of the following corollary of Theorem~ to you (Exercise~).

The theory of series $\sum a_n$ with terms that are nonnegative for sufficiently large n is simpler than the general theory, since such a series either converges to a finite limit or diverges to ∞, as the next theorem shows.

Since $A_n = A_{n-1} + a_n$ and $a_n \geq 0$ $(n \geq k)$, the sequence $\{A_n\}$ is nondecreasing, so the conclusion follows from Theorem~

and Definition~.

If $a_n \geq 0$ for sufficiently large n, we will write $\sum a_n < \infty$ if $\sum a_n$ converges. This convention is based on Theorem~, which says that such a series diverges only if $\sum a_n = \infty$. The convention does not apply to series with infinitely many negative terms, because such series may diverge without diverging to ∞; for example, the series $\sum_{n=0}^{\infty} (-1)^n$ oscillates, since its partial sums are alternately 1 and 0.

If

$$A_n = a_k + a_{k+1} + \cdots + a_n \quad \text{and} \quad B_n = b_k + b_{k+1} + \cdots + b_n, \quad n \geq k,$$

then, from ,

$$A_n \leq B_n. \tag{4.1.18}$$

Now we use Theorem~. If $\sum b_n < \infty$, then $\{B_n\}$ is bounded above and implies that $\{A_n\}$ is also; therefore, $\sum a_n < \infty$. On the other hand, if $\sum a_n = \infty$, then $\{A_n\}$ is unbounded above and implies that $\{B_n\}$ is also; therefore, $\sum b_n = \infty$.

We leave it to you to show that implies

.

The comparison test is useful if we have a collection of series with nonnegative terms and known convergence properties. We will now use the comparison test to build such a collection.

We first observe that holds if and only if

$$\sum_{n=k}^{\infty}\int_{n}^{n+1} f(x)\,dx < \infty \tag{4.1.19}$$

(Exercise~), so it is enough to show that holds if and only if does. From and the assumption that f is nonincreasing,

$$c_{n+1} = f(n+1) \le f(x) \le f(n) = c_n, \quad n \le x \le n+1, \quad n \ge k.$$

Therefore,

$$c_{n+1} = \int_{n}^{n+1} c_{n+1}\,dx \le \int_{n}^{n+1} f(x)\,dx \le \int_{n}^{n+1} c_n\,dx = c_n, \quad n \ge k$$

(Theorem~). From the first inequality and Theorem~ with $a_n = c_{n+1}$ and $b_n = \int_n^{n+1} f(x)\,dx$, implies that $\sum c_{n+1} < \infty$, which is equivalent to . From the second inequality and Theorem~

with $a_n = \int_n^{n+1} f(x)\,dx$ and $b_n = c_n$, implies .

This example provides an infinite family of series with known convergence properties that can be used as standards for the comparison test.

Except for the series of Example~, the integral test is of limited practical value, since convergence or divergence of most of the series to which it can be applied can be determined by simpler tests that do not require integration. However, the method used to prove the integral test is often useful for estimating the rate of convergence or divergence of a series. This idea is developed in Exercises~ and .

The next theorem is often applicable where the integral test is not. It does not require the kind of trickery that we used in Example~.

If $\limsup_{n\to\infty} a_n/b_n < \infty$, then $\{a_n/b_n\}$ is bounded, so there is a constant M and an integer k such that

$$a_n \le Mb_n, \quad n \ge k.$$

Since $\sum b_n < \infty$, Theorem~ implies that $\sum(Mb_n) < \infty$. Now $\sum a_n < \infty$, by the comparison test.

If $\liminf_{n\to\infty} a_n/b_n > 0$, there is a constant m and an integer k such that

$$a_n \ge mb_n, \quad n \ge k.$$

Since $\sum b_n = \infty$, Theorem~ implies that $\sum(mb_n) = \infty$. Now $\sum a_n = \infty$, by the comparison test.

The following corollary of Theorem~ is often useful, although it does not apply to the series of Example~.

It is sometimes possible to determine whether a series with positive terms converges by comparing the ratios of successive terms with the corresponding ratios of a series known to converge or diverge.

Rewriting as

$$\frac{a_{n+1}}{b_{n+1}} \le \frac{a_n}{b_n},$$

we see that $\{a_n/b_n\}$ is nonincreasing. Therefore, $\limsup_{n\to\infty} a_n/b_n < \infty$, and Theorem~ implies

.

To prove , suppose that $\sum a_n = \infty$. Since $\{a_n/b_n\}$ is nonincreasing, there is a number ρ such that $b_n \ge \rho a_n$ for large n. Since $\sum(\rho a_n) = \infty$ if $\sum a_n = \infty$, Theorem~

(with a_n replaced by ρa_n) implies that $\sum b_n = \infty$.

We will use this theorem to obtain two other widely applicable tests: the ratio test and Raabe's test.

If

$$\limsup_{n\to\infty}\frac{a_{n+1}}{a_n}<1,$$

there is a number r such that $0<r<1$ and

$$\frac{a_{n+1}}{a_n}<r$$

for n sufficiently large. This can be rewritten as

$$\frac{a_{n+1}}{a_n}<\frac{r^{n+1}}{r^n}.$$

Since $\sum r^n<\infty$, Theorem~

with $b_n=r^n$ implies that $\sum a_n<\infty$.

If

$$\liminf_{n\to\infty}\frac{a_{n+1}}{a_n}>1,$$

there is a number r such that $r>1$ and

$$\frac{a_{n+1}}{a_n}>r$$

for n sufficiently large. This can be rewritten as

$$\frac{a_{n+1}}{a_n}>\frac{r^{n+1}}{r^n}.$$

Since $\sum r^n=\infty$, Theorem~

with $a_n=r^n$ implies that $\sum b_n=\infty$.

To see that no conclusion can be drawn if holds, consider

$$\sum a_n=\sum\frac{1}{n^p}.$$

This series converges if $p>1$ or diverges if $p\le 1$; however,

$$\limsup_{n\to\infty}\frac{a_{n+1}}{a_n}=\liminf_{n\to\infty}\frac{a_{n+1}}{a_n}=1$$

for every p.

The following corollary of the ratio test is the familiar ratio rest from calculus.

The ratio test does not imply that $\sum a_n<\infty$ if merely

$$\frac{a_{n+1}}{a_n}<1 \tag{4.1.20}$$

for large n, since this could occur with $\lim_{n\to\infty}a_{n+1}/a_n=1$, in which case the test is inconclusive. However, the next theorem shows that $\sum a_n<\infty$ if is replaced by the stronger condition that

$$\frac{a_{n+1}}{a_n}\le 1-\frac{p}{n}$$

for some $p>1$ and large n. It also shows that $\sum a_n=\infty$ if

$$\frac{a_{n+1}}{a_n}\ge 1-\frac{q}{n}$$

for some $q<1$ and large n.

We need the inequality

$$\frac{1}{(1+x)^p} > 1-px, \quad x>0,\ p>0. \tag{4.1.21}$$

This follows from Taylor's theorem (Theorem~), which implies that

$$\frac{1}{(1+x)^p} = 1-px+\frac{1}{2}\frac{p(p+1)}{(1+c)^{p+2}}x^2,$$

where $0<c<x$. (Verify.) Since the last term is positive if $p>0$, this implies .

Now suppose that $M<-p<-1$. Then there is an integer k such that

$$n\left(\frac{a_{n+1}}{a_n}-1\right) < -p, \quad n\ge k,$$

so

$$\frac{a_{n+1}}{a_n} < 1-\frac{p}{n}, \quad n\ge k.$$

Hence,

$$\frac{a_{n+1}}{a_n} < \frac{1}{(1+1/n)^p}, \quad n\ge k,$$

as can be seen by letting $x=1/n$ in . From this,

$$\frac{a_{n+1}}{a_n} < \frac{1}{(n+1)^p}\Big/\frac{1}{n^p}, \quad n\ge k.$$

Since $\sum 1/n^p<\infty$ if $p>1$, Theorem~

implies that $\sum a_n<\infty$.

Here we need the inequality

$$(1-x)^q < 1-qx, \quad 0<x<1, \quad 0<q<1. \tag{4.1.22}$$

This also follows from Taylor's theorem, which implies that

$$(1-x)^q = 1-qx+q(q-1)(1-c)^{q-2}\frac{x^2}{2},$$

where $0<c<x$.

Now suppose that $-1<-q<m$. Then there is an integer k such that

$$n\left(\frac{a_{n+1}}{a_n}-1\right) > -q, \quad n\ge k,$$

so

$$\frac{a_{n+1}}{a_n} \ge 1-\frac{q}{n}, \quad n\ge k.$$

If $q\le 0$, then $\sum a_n=\infty$, by Corollary~. Hence, we may assume that $0<q<1$, so the last inequality implies that

$$\frac{a_{n+1}}{a_n} > \left(1-\frac{1}{n}\right)^q, \quad n\ge k,$$

as can be seen by setting $x=1/n$ in . Hence,

$$\frac{a_{n+1}}{a_n} > \frac{1}{n^q}\Big/\frac{1}{(n-1)^q}, \quad n\ge k.$$

Since $\sum 1/n^q=\infty$ if $q<1$, Theorem~

implies that $\sum a_n=\infty$.

The next theorem, which will be useful when we study power series (Section~4.5), concludes our discussion of series with nonnegative terms.

If $\limsup_{n\to\infty} a_n^{1/n} < 1$, there is an r such that $0 < r < 1$ and $a_n^{1/n} < r$ for large n. Therefore, $a_n < r^n$ for large n. Since $\sum r^n < \infty$, the comparison test implies that $\sum a_n < \infty$.

If $\limsup_{n\to\infty} a_n^{1/n} > 1$, then $a_n^{1/n} > 1$ for infinitely many values of n, so $\sum a_n = \infty$, by Corollary~.

We now drop the assumption that the terms of $\sum a_n$ are nonnegative for large n. In this case, $\sum a_n$ may converge in two quite different ways. The first is defined as follows.

Any test for convergence of a series with nonnegative terms can be used to test an arbitrary series $\sum a_n$ for absolute convergence by applying it to $\sum |a_n|$. We used the comparison test this way in Examples~ and .

The proof of the next theorem is analogous to the proof of Theorem~. We leave it to you (Exercise~).

For example, Theorem~ implies that

$$\sum \frac{\sin n\theta}{n^p}$$

converges if $p > 1$, since it then converges absolutely (Example~).

The converse of Theorem~ is false; a series may converge without converging absolutely. We say then that the series converges {}, or is {}; thus, $\sum (-1)^n/n^p$ converges conditionally if $0 < p \le 1$.

Except for Theorem~ and Corollary~, the convergence tests we have studied so far apply only to series whose terms have the same sign for large n. The following theorem does not require this. It is analogous to Dirichlet's test for improper integrals (Theorem~).

The proof is similar to the proof of Dirichlet's test for integrals. Define

$$B_n = b_k + b_{k+1} + \cdots + b_n, \quad n \ge k$$

and consider the partial sums of $\sum_{n=k}^{\infty} a_n b_n$:

$$S_n = a_k b_k + a_{k+1} b_{k+1} + \cdots + a_n b_n, \quad n \ge k. \tag{4.1.23}$$

By substituting

$$b_k = B_k \quad \text{and} \quad b_n = B_n - B_{n-1}, \quad n \ge k+1,$$

into , we obtain

$$S_n = a_k B_k + a_{k+1}(B_{k+1} - B_k) + \cdots + a_n(B_n - B_{n-1}),$$

which we rewrite as

$$\begin{aligned} S_n \ar &= (a_k - a_{k+1})B_k + (a_{k+1} - a_{k+2})B_{k+1} + \cdots \\ \ar &+ (a_{n-1} - a_n)B_{n-1} + a_n B_n. \end{aligned} \tag{4.1.24}$$

(The procedure that led from to is called {}. It is analogous to integration by parts.) Now can be viewed as

$$S_n = T_{n-1} + a_n B_n, \tag{4.1.25}$$

where

$$T_{n-1} = (a_k - a_{k+1})B_k + (a_{k+1} - a_{k+2})B_{k+1} + \cdots + (a_{n-1} - a_n)B_{n-1};$$

that is, $\{T_n\}$ is the sequence of partial sums of the series

$$\sum_{j=k}^{\infty} (a_j - a_{j+1})B_j. \tag{4.1.26}$$

Since

$$|(a_j - a_{j+1})B_j| \le M|a_j - a_{j+1}|$$

from , the comparison test and imply that the series converges absolutely. Theorem~ now implies that $\{T_n\}$ converges. Let $T = \lim_{n\to\infty} T_n$. Since $\{B_n\}$ is bounded and $\lim_{n\to\infty} a_n = 0$, we infer from that

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} T_{n-1} + \lim_{n\to\infty} a_n B_n = T + 0 = T.$$

Therefore, $\sum a_n b_n$ converges.

\begin{example}\rm To apply Dirichlet's test to

$$\sum_{n=2}^{\infty} \frac{\sin n\theta}{n + (-1)^n}, \quad \theta \neq k\pi \quad (k = \text{integer}),$$

we take

$$a_n = \frac{1}{n + (-1)^n} \quad \text{and} \quad b_n = \sin n\theta.$$

Then $\lim_{n\to\infty} a_n = 0$, and

$$|a_{n+1} - a_n| < \frac{3}{n(n-1)}$$

(verify), so

$$\sum |a_{n+1} - a_n| < \infty.$$

Now

$$B_n = \sin 2\theta + \sin 3\theta + \cdots + \sin n\theta.$$

To show that $\{B_n\}$ is bounded, we use the trigonometric identity

$$\sin r\theta = \frac{\cos\left(r - \frac{1}{2}\right)\theta - \cos\left(r + \frac{1}{2}\right)\theta}{2\sin(\theta/2)}, \quad \theta \neq 2k\pi,$$

to write

Bracket argument to \\ must be a dimension

which implies that

$$|B_n| \leq \left|\frac{1}{\sin(\theta/2)}\right|, \quad n \geq 2.$$

Since $\{a_n\}$ and $\{b_n\}$ satisfy the hypotheses of Dirichlet's theorem, $\sum a_n b_n$ converges. \end{example}

Dirichlet's test takes a simpler form if $\{a_n\}$ is nonincreasing, as follows.

If $a_{n+1} \leq a_n$, then

$$\sum_{n=k}^{m} |a_{n+1} - a_n| = \sum_{n=k}^{m} (a_n - a_{n+1}) = a_k - a_{m+1}.$$

Since $\lim_{m\to\infty} a_{m+1} = 0$, it follows that

$$\sum_{n=k}^{\infty} |a_{n+1} - a_n| = a_k < \infty.$$

Therefore, the hypotheses of Dirichlet's test are satisfied, so $\sum a_n b_n$ converges.

The alternating series test from calculus follows easily from Abel's test.

Let $b_n = (-1)^n$; then $\{|B_n|\}$ is a sequence of zeros and ones and therefore bounded. The conclusion now follows from Abel's test.

The terms of a finite sum can be grouped by inserting parentheses arbitrarily. For example,

$$(1+7)+(6+5)+4=(1+7+6)+(5+4)=(1+7)+(6+5+4).$$

According to the next theorem, the same is true of an infinite series that converges or diverges to $\pm\infty$.

If T_r is the rth partial sum of $\sum_{j=1}^{\infty} b_{n_j}$ and $\{A_n\}$ is the nth partial sum of $\sum_{s=k}^{\infty} a_s$, then

$$\begin{aligned} T_r &= b_1+b_2+\cdots+b_r \\ &= (a_1+\cdots+a_{n_1})+(a_{n_1+1}+\cdots+a_{n_2})+\cdots+(a_{n_{r-1}+1}+\cdots+a_{n_r}) \\ &= A_{n_r}. \end{aligned}$$

Thus, $\{T_r\}$ is a subsequence of $\{A_n\}$, so $\lim_{r\to\infty} T_r=\lim_{n\to\infty} A_n=A$ by Theorem~.

-.4em A finite sum is not changed by rearranging its terms; thus,

$$1+3+7=1+7+3=3+1+7=3+7+1=7+1+3=7+3+1.$$

This is not true of all infinite series. Let us say that $\sum b_n$ is a {} of $\sum a_n$ if the two series have the same terms, written in possibly different orders. Since the partial sums of the two series may form entirely different sequences, there is no apparent reason to expect them to exhibit the same convergence properties, and in general they do not.

We are interested in what happens if we rearrange the terms of a convergent series. We will see that every rearrangement of an absolutely convergent series has the same sum, but that conditionally convergent series fail, spectacularly, to have this property.

Let

$$\overline{A}_n=|a_1|+|a_2|+\cdots+|a_n| \quad \text{and} \quad \overline{B}_n=|b_1|+|b_2|+\cdots+|b_n|.$$

For each $n\ge 1$, there is an integer k_n such that $b_1, b_2, , b_n$ are included among $a_1, a_2, , a_{k_n}$, so $\overline{B}_n\le\overline{A}_{k_n}$. Since $\{\overline{A}_n\}$ is bounded, so is $\{\overline{B}_n\}$, and therefore $\sum|b_n|<\infty$ (Theorem~).

Now let

$$\begin{aligned} A_n &= a_1+a_2+\cdots+a_n, \quad B_n=b_1+b_2+\cdots+b_n, \\ A &= \sum_{n=1}^{\infty} a_n, \quad \text{and} \quad B=\sum_{n=1}^{\infty} b_n. \end{aligned}$$

We must show that $A=B$. Suppose that $\epsilon>0$. From Cauchy's convergence criterion for series and the absolute convergence of $\sum a_n$, there is an integer N such that

$$|a_{N+1}|+|a_{N+2}|+\cdots+|a_{N+k}|<\epsilon, \quad k\ge 1.$$

-.3em Choose N_1 so that $a_1, a_2, , a_N$ are included among $b_1, b_2, , b_{N_1}$. If $n\ge N_1$, then A_n and B_n both include the terms $a_1, a_2, , a_N$, which cancel on subtraction; thus, $|A_n-B_n|$ is dominated by the sum of the absolute values of finitely many terms from $\sum a_n$ with subscripts greater than N. Since every such sum is less than~ϵ,

$$|A_n-B_n|<\epsilon \quad \text{if} \quad n\ge N_1.$$

Therefore, $\lim_{n\to\infty}(A_n-B_n)=0$ and $A=B$.

To investigate the consequences of rearranging a conditionally convergent series, we need the next theorem, which is itself important.

If both series in converge, then $\sum a_n$ converges absolutely, while if one converges and the other diverges, then $\sum a_n$ diverges to ∞ or $-\infty$. Hence, both must diverge.

The next theorem implies that a conditionally convergent series can be rearranged to produce a series that converges to any given number, diverges to $\pm\infty$, or oscillates.

We consider the case where μ and ν are finite and leave the other cases to you (Exercise~). We may ignore any zero terms that occur in $\sum_{n=1}^{\infty} a_n$. For convenience, we denote the positive terms by $P=\{\alpha_i\}_1^{\infty}$ and and the negative terms by $Q=\{-\beta_j\}_1^{\infty}$. We construct the sequence

$$\{b_n\}_1^{\infty}=\{\alpha_1,\dots,\alpha_{m_1},-\beta_1,\dots,-\beta_{n_1},\alpha_{m_1+1},\dots,\alpha_{m_2},-\beta_{n_1+1},\dots,-\beta_{n_2},\dots\}, \tag{4.1.27}$$

with segments chosen alternately from P and Q. Let $m_0 = n_0 = 0$. If $k \ge 1$, let m_k and n_k be the smallest integers such that $m_k > m_{k-1}$, $n_k > n_{k-1}$,

$$\sum_{i=1}^{m_k} \alpha_i - \sum_{j=1}^{n_{k-1}} \beta_j \ge \nu, \quad \text{and} \quad \sum_{i=1}^{m_k} \alpha_i - \sum_{j=1}^{n_k} \beta_j \le \mu.$$

Theorem~ implies that this construction is possible: since $\sum \alpha_i = \sum \beta_j = \infty$, we can choose m_k and n_k so that

$$\sum_{i=m_{k-1}}^{m_k} \alpha_i \quad \text{and} \quad \sum_{j=n_{k-1}}^{n_k} \beta_j$$

are as large as we please, no matter how large m_{k-1} and n_{k-1} are (Exercise~). Since m_k and n_k are the smallest integers with the specified properties,

$$\nu \le B_{m_k+n_{k-1}} \ar < \nu + \alpha_{m_k}, \quad k \ge 2, \tag{4.1.28}$$

\arraytext{and}

$$\mu - \beta_{n_k} \ar < B_{m_k+n_k} \le \mu, \quad k \ge 2. \tag{4.1.29}$$

From , $b_n < 0$ if $m_k + n_{k-1} < n \le m_k + n_k$, so

$$B_{m_k+n_k} \le B_n \le B_{m_k+n_{k-1}}, \quad m_k + n_{k-1} \le n \le m_k + n_k, \tag{4.1.30}$$

while $b_n > 0$ if $m_k + n_k < n \le m_{k+1} + n_k$, so

$$B_{m_k+n_k} \le B_n \le B_{m_{k+1}+n_k}, \quad m_k + n_k \le n \le m_{k+1} + n_k. \tag{4.1.31}$$

Because of and , and imply that

$$\mu - \beta_{n_k} \ar < B_n < \nu + \alpha_{m_k}, \quad m_k + n_{k-1} \le n \le m_k + n_k, \tag{4.1.32}$$

\arraytext{and}

$$\mu - \beta_{n_k} \ar < B_n < \nu + \alpha_{m_{k+1}}, \quad m_k + n_k \le n \le m_{k+1} + n_k. \tag{4.1.33}$$

From the first inequality of , $B_n \ge \nu$ for infinitely many values of n. However, since $\lim_{i\to\infty} \alpha_i = 0$, the second inequalities in and imply that if $\epsilon > 0$ then $B_n > \nu + \epsilon$ for only finitely many values of n. Therefore, $\limsup_{n\to\infty} B_n = \nu$. From the second inequality in , $B_n \le \mu$ for infinitely many values of n. However, since $\lim_{j\to\infty} \beta_j = 0$, the first inequalities in and imply that if $\epsilon > 0$ then $B_n < \mu - \epsilon$ for only finitely many values of n. Therefore, $\liminf_{n\to\infty} B_n = \mu$.

-.5em The product of two finite sums can be written as another finite sum: for example,

$$\begin{aligned}(a_0 + a_1 + a_2)(b_0 + b_1 + b_2) \ar = a_0b_0 + a_0b_1 + a_0b_2 \\ \ar + a_1b_0 + a_1b_1 + a_1b_2 \\ \ar + a_2b_0 + a_2b_1 + a_2b_2,\end{aligned}$$

where the sum on the right contains each product a_ib_j $(i, j = 0, 1, 2)$ exactly once. These products can be rearranged arbitrarily without changing their sum. The corresponding situation for series is more complicated.

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Given two series 1pc

$$\sum_{n=0}^{\infty} a_n \quad \text{and} \quad \sum_{n=0}^{\infty} b_n$$

1pc (because of applications in Section~4.5, it is convenient here to start the summation index at zero), we can arrange all possible products a_ib_j $(i, j \ge 0)$ in a two-dimensional array: 1pc

$$\begin{array}{cccc} a_0b_0 & a_0b_1 & a_0b_2 & a_0b_3 & \cdots \\ a_1b_0 & a_1b_1 & a_1b_2 & a_1b_3 & \cdots \\ a_2b_0 & a_2b_1 & a_2b_2 & a_2b_3 & \cdots \\ a_3b_0 & a_3b_1 & a_3b_2 & a_3b_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{array} \tag{4.1.34}$$

1pc

where the subscript on a is constant in each row and the subscript on b is constant in each column. Any sensible definition of the product 1pc

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right)$$

1pc clearly must involve every product in this array exactly once; thus, we might define the product of the two series to be the series $\sum_{n=0}^{\infty} p_n$, where $\{p_n\}$ is a sequence obtained by ordering the products in according to some method that chooses every product exactly once. One way to do this is indicated by 2pc

$$\begin{array}{ccccccccc}
a_0b_0 & \to & a_0b_1 & & a_0b_2 & \to & a_0b_3 & \cdots \\
 & & \downarrow & & \uparrow & & \downarrow & \\
a_1b_0 & \leftarrow & a_1b_1 & & a_1b_2 & & a_1b_3 & \cdots \\
\downarrow & & & & \uparrow & & \downarrow & \\
a_2b_0 & \to & a_2b_1 & \to & a_2b_2 & & a_2b_3 & \cdots \\
 & & & & & & \downarrow & \\
a_3b_0 & \leftarrow & a_3b_1 & \leftarrow & a_3b_2 & \leftarrow & a_3b_3 & \cdots \\
\downarrow & & & & & & & \\
\vdots & & \vdots & & \vdots & & \vdots &
\end{array} \tag{4.1.35}$$

and another by

$$\begin{array}{ccccccccc}
a_0b_0 & \to & a_0b_1 & & a_0b_2 & \to & a_0b_3 & & a_0b_4 & \cdots \\
 & \swarrow & & \nearrow & & \swarrow & & \nearrow & & \\
a_1b_0 & & a_1b_1 & & a_1b_2 & & a_1b_3 & & \cdots & \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & & & \\
a_2b_0 & & a_2b_1 & & a_2b_2 & & a_2b_3 & & \cdots & \\
 & \swarrow & & \nearrow & & & & & & \\
a_3b_0 & & a_3b_1 & & a_3b_2 & & a_3b_3 & & \cdots & \\
\downarrow & \nearrow & & & & & & & & \\
a_4b_0 & & \vdots & & \vdots & & \vdots & & &
\end{array} \tag{4.1.36}$$

There are infinitely many others, and to each corresponds a series that we might consider to be the product of the given series. This raises a question: If

$$\sum_{n=0}^{\infty} a_n = A \quad \text{and} \quad \sum_{n=0}^{\infty} b_n = B$$

where A and B are finite, does every product series $\sum_{n=0}^{\infty} p_n$ constructed by ordering the products in converge to AB?

The next theorem tells us when the answer is yes.

First, let $\{p_n\}$ be the sequence obtained by arranging the products $\{a_ib_j\}$ according to the scheme indicated in , and define

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From , we see that

$$P_0 = A_0B_0, \quad P_3 = A_1B_1, \quad P_8 = A_2B_2,$$

and, in general,

$$P_{(m+1)^2-1} = A_mB_m. \tag{4.1.37}$$

Similarly,

$$\overline{P}_{(m+1)^2-1} = \overline{A}_m\overline{B}_m. \tag{4.1.38}$$

If $\sum |a_n| < \infty$ and $\sum |b_n| < \infty$, then $\{\overline{A}_m \overline{B}_m\}$ is bounded and, since $\overline{P}_m \le \overline{P}_{(m+1)^2-1}$, implies that $\{\overline{P}_m\}$ is bounded. Therefore, $\sum |p_n| < \infty$, so $\sum p_n$ converges. Now

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Since any other ordering of the products in produces a a rearrangement of the absolutely convergent series $\sum_{n=0}^{\infty} p_n$, Theorem~ implies that $\sum |q_n| < \infty$ for every such ordering and that $\sum_{n=0}^{\infty} q_n = AB$. This shows that the stated condition is sufficient.

For necessity, again let $\sum_{n=0}^{\infty} p_n$ be obtained from the ordering indicated in , and suppose that $\sum_{n=0}^{\infty} p_n$ and all its rearrangements converge to AB. Then $\sum p_n$ must converge absolutely, by Theorem~. Therefore, $\{\overline{P}_{m^2-1}\}$ is bounded, and implies that $\{\overline{A}_m\}$ and $\{\overline{B}_m\}$ are bounded. (Here we need the assumption that neither $\sum a_n$ nor $\sum b_n$ consists entirely of zeros. Why?) Therefore, $\sum |a_n| < \infty$ and $\sum |b_n| < \infty$.

The following definition of the product of two series is due to Cauchy. We will see the importance of this definition in Section~4.5.

-3em3em

Henceforth, $\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right)$ should be interpreted as the Cauchy product. Notice that

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \left(\sum_{n=0}^{\infty} b_n\right)\left(\sum_{n=0}^{\infty} a_n\right),$$

and that the Cauchy product of two series is defined even if one or both diverge. In the case where both converge, it is natural to inquire about the relationship between the product of their sums and the sum of the Cauchy product. Theorem~ yields a partial answer to this question, as follows.

Let C_n be the nth partial sum of the Cauchy product; that is,

$$C_n = c_0 + c_1 + \cdots + c_n$$

(see). Let $\sum_{n=0}^{\infty} p_n$ be the series obtained by ordering the products $\{a_i, b_j\}$ according to the scheme indicated in , and define P_n to be its nth partial sum; thus,

$$P_n = p_0 + p_1 + \cdots + p_n.$$

Inspection of shows that c_n is the sum of the $n+1$ terms connected by the diagonal arrows. Therefore, $C_n = P_{m_n}$, where

$$m_n = 1 + 2 + \cdots + (n+1) - 1 = \frac{n(n+3)}{2}.$$

From Theorem~, $\lim_{n\to\infty} P_{m_n} = AB$, so $\lim_{n\to\infty} C_n = AB$. To see that $\sum |c_n| < \infty$, we observe that

$$\sum_{r=0}^{n} |c_r| \le \sum_{s=0}^{m_n} |p_s|$$

and recall that $\sum |p_s| < \infty$, from Theorem~.

The Cauchy product of two series may converge under conditions weaker than those of Theorem~. If one series converges absolutely and the other converges conditionally, the Cauchy product of the two series converges to the product of the two sums (Exercise~). If two series and their Cauchy product all converge, then the sum of the Cauchy product equals the product of the sums of the two series (Exercise~). However, the next example shows that the Cauchy product of two conditionally convergent series may diverge.

Until now we have considered sequences and series of constants. Now we turn our attention to sequences and series of real-valued functions defined on subsets of the reals. Throughout this section, `subset'' means` nonempty subset.''

If F_k, F_{k+1}, , F_n, ... are real-valued functions defined on a subset D of the reals, we say that $\{F_n\}$ is an {} or (simply a {}) {} D. If the sequence of values $\{F_n(x)\}$ converges for each x in some subset S of D, then $\{F_n\}$ defines a limit function on S. The formal definition is as follows.

\begin{example}

If x is irrational, then $x \notin S_n$ for any n, so $F_n(x) = 0$, $n \geq 1$. If x is rational, then $x \in S_n$ and $F_n(x) = 1$ for all sufficiently large n. Therefore,

$$\lim_{n\to\infty} F_n(x) = F(x) = \left\{\casespace \begin{array}{ll} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{array}\right.$$

\end{example}

10pt

18pt

-.4em The pointwise limit of a sequence of functions may differ radically from the functions in the sequence. In Example~, each F_n is continuous on $(-\infty, 1]$, but F is not. In Example~, the graph of each F_n has two triangular spikes with heights that tend to ∞ as $n \to \infty$, while the graph of F (the x-axis) has none. In Example~, each F_n is integrable, while F is nonintegrable on every finite interval. (Exercise~). There is nothing in Definition~ to preclude these apparent anomalies; although the definition implies that for each x_0 in S, $F_n(x_0)$ approximates $F(x_0)$ if n is sufficiently large, it does not imply that any particular F_n approximates F well over $\{\}$ of S. To formulate a definition that does, it is convenient to introduce the notation

$$\|g\|_S = \sup_{x\in S} |g(x)|$$

and to state the following lemma. We leave the proof to you (Exercise~).

-1em1em

If $S = [a, b]$ and F is the function with graph shown in Figure~, then implies that the graph of

$$y = F_n(x), \quad a \leq x \leq b,$$

lies in the shaded band

$$F(x) - \epsilon < y < F(x) + \epsilon, \quad a \leq x \leq b,$$

if $n \geq N$.

From Definition~, if $\{F_n\}$ converges uniformly on S, then $\{F_n\}$ converges uniformly on any subset of S (Exercise~).

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12pt

The next theorem provides alternative definitions of pointwise and uniform convergence. It follows immediately from Definitions~ and .

\begin{theorem} Let $\{F_n\}$ be defined on S. Then \begin{alist} % (a) $\{F_n\}$ converges pointwise to F on S if and only if there is, for each $\epsilon > 0$ and $x \in S$, an integer N (which may depend on x as well as ϵ) such that

$$|F_n(x) - F(x)| < \epsilon \quad \text{if} \quad n \geq N.$$

% (b) $\{F_n\}$ converges uniformly to F on S if and only if there is for each $\epsilon > 0$ an integer N (which depends only on ϵ and not on any particular x in S) such that

$$|F_n(x) - F(x)| < \epsilon \quad \text{for all } x \text{ in } S \text{ if } n \geq N.$$

\end{alist} \end{theorem}

6pt

12pt

The next theorem follows immediately from Theorem~ and Example~.

The next theorem enables us to test a sequence for uniform convergence without guessing what the limit function might be. It is analogous to Cauchy's convergence criterion for sequences of constants (Theorem~).

For necessity, suppose that $\{F_n\}$ converges uniformly to F on S. Then, if $\epsilon > 0$, there is an integer N such that

$$\|F_k - F\|_S < \frac{\epsilon}{2} \quad \text{if} \quad k \geq N.$$

Therefore,

$$\begin{aligned} \|F_n - F_m\|_S \ar &= \|(F_n - F) + (F - F_m)\|_S \\ \ar &\leq \|F_n - F\|_S + \|F - F_m\|_S \quad (\text{Lemma ???}) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{if} \quad m, n \geq N. \end{aligned}$$

For sufficiency, we first observe that implies that

$$|F_n(x) - F_m(x)| < \epsilon \quad \text{if} \quad n, m \geq N,$$

for any fixed x in S. Therefore, Cauchy's convergence criterion for sequences of constants (Theorem~) implies that $\{F_n(x)\}$ converges for each x in S; that is, $\{F_n\}$ converges pointwise to a limit function F on S. To see that the convergence is uniform, we write

$$\begin{aligned} |F_m(x) - F(x)| \ar = |[F_m(x) - F_n(x)] + [F_n(x) - F(x)]| \\ \ar \leq |F_m(x) - F_n(x)| + |F_n(x) - F(x)| \\ \ar \leq \|F_m - F_n\|_S + |F_n(x) - F(x)|. \end{aligned}$$

This and imply that

$$|F_m(x) - F(x)| < \epsilon + |F_n(x) - F(x)| \quad \text{if} \quad n, m \geq N. \tag{4.1.39}$$

Since $\lim_{n\to\infty} F_n(x) = F(x)$,

$$|F_n(x) - F(x)| < \epsilon$$

for some $n \geq N$, so implies that

$$|F_m(x) - F(x)| < 2\epsilon \quad \text{if} \quad m \geq N.$$

But this inequality holds for all x in S, so

$$\|F_m - F\|_S \leq 2\epsilon \quad \text{if} \quad m \geq N.$$

Since ϵ is an arbitrary positive number, this implies that $\{F_n\}$ converges uniformly to F on~S.

The next example is similar to Example~.

We now study properties of the functions of a uniformly convergent sequence that are inherited by the limit function. We first consider continuity.

Suppose that each F_n is continuous at x_0. If $x \in S$ and $n \geq 1$, then

$$\begin{aligned} |F(x) - F(x_0)| \ar \leq |F(x) - F_n(x)| + |F_n(x) - F_n(x_0)| + |F_n(x_0) - F(x_0)| \\ \ar \leq |F_n(x) - F_n(x_0)| + 2\|F_n - F\|_S. \end{aligned} \tag{4.1.40}$$

Suppose that $\epsilon > 0$. Since $\{F_n\}$ converges uniformly to F on S, we can choose n so that $\|F_n - F\|_S < \epsilon$. For this fixed n, implies that

$$|F(x) - F(x_0)| < |F_n(x) - F_n(x_0)| + 2\epsilon, \quad x \in S. \tag{4.1.41}$$

Since F_n is continuous at x_0, there is a $\delta > 0$ such that

$$|F_n(x) - F_n(x_0)| < \epsilon \quad \text{if} \quad |x - x_0| < \delta,$$

so, from ,

$$|F(x) - F(x_0)| < 3\epsilon, \quad \text{if} \quad |x - x_0| < \delta.$$

Therefore, F is continuous at x_0. Similar arguments apply to the assertions on continuity from the right and left.

Now we consider the question of integrability of the uniform limit of integrable functions.

Since

$$\left|\int_a^b F_n(x)\,dx-\int_a^b F(x)\,dx\right| \ar\le \int_a^b |F_n(x)-F(x)|\,dx$$
$$\ar\le (b-a)\|F_n-F\|_S$$

and $\lim_{n\to\infty}\|F_n-F\|_S=0$, the conclusion follows.

In particular, this theorem implies that holds if each F_n is continuous on $[a,b]$, because then F is continuous (Corollary~) and therefore integrable on $[a,b]$.

The hypotheses of Theorem~ are stronger than necessary. We state the next theorem so that you will be better informed on this subject. We omit the proof, which is inaccessible if you skipped Section~3.5, and quite involved in any case.

Part of this theorem shows that it is not necessary to assume in Theorem~ that F is integrable on $[a,b]$, since this follows from the uniform convergence. Part is known as the {}. Neither of the assumptions of

can be omitted. Thus, in Example~, where $\{\|F_n\|_{[0,1]}\}$ is unbounded while F is integrable on $[0,1]$,

$$\int_0^1 F_n(x)\,dx=1,\quad n\ge 1,\quad \text{but}\quad \int_0^1 F(x)\,dx=0.$$

In Example~, where $\|F_n\|_{[a,b]}=1$ for every finite interval $[a,b]$, F_n is integrable for all $n\ge 1$, and F is nonintegrable on every interval (Exercise~).

After Theorems~ and , it may seem reasonable to expect that if a sequence $\{F_n\}$ of differentiable functions converges uniformly to F on S, then $F'=\lim_{n\to\infty}F_n'$ on S. The next example shows that this is not true in general.

Since F_n' is continuous on $[a,b]$, we can write

$$F_n(x)=F_n(x_0)+\int_{x_0}^x F_n'(t)\,dt,\quad a\le x\le b \tag{4.1.42}$$

(Theorem~). Now let

$$L\ar=\lim_{n\to\infty}F_n(x_0)$$
$$\arraytextand$$
$$G(x)\ar=\lim_{n\to\infty}F_n'(x). \tag{4.1.43}$$

Since F_n' is continuous and $\{F_n'\}$ converges uniformly to G on $[a,b]$, G is continuous on $[a,b]$ (Corollary~); therefore, and Theorem~ (with F and F_n replaced by G and F_n') imply that $\{F_n\}$ converges pointwise on $[a,b]$ to the limit function

$$F(x)=L+\int_{x_0}^x G(t)\,dt. \tag{4.1.44}$$

The convergence is actually uniform on $[a,b]$, since subtracting from yields

$$|F(x)-F_n(x)|\ar\le |L-F_n(x_0)|+\left|\int_{x_0}^x |G(t)-F_n'(t)|\,dt\right|$$
$$\ar\le |L-F_n(x_0)|+|x-x_0|\,\|G-F_n'\|_{[a,b]},$$

so

$$\|F-F_n\|_{[a,b]}\le |L-F_n(x_0)|+(b-a)\|G-F_n'\|_{[a,b]},$$

where the right side approaches zero as $n\to\infty$.

Since G is continuous on $[a,b]$, , , Definition~, and Theorem~ imply and .

In Section~4.3 we defined the sum of an infinite series of constants as the limit of the sequence of partial sums. The same definition can be applied to series of functions, as follows.

As for series of constants, the convergence, pointwise or uniform, of a series of functions is not changed by altering or omitting finitely many terms. This justifies adopting the convention that we used for series of constants: when we are interested only in whether a series of functions converges, and not in its sum, we will omit the limits on the summation sign and write simply $\sum f_n$.

Theorem~ is easily converted to a theorem on uniform convergence of series, as follows.

5pt Apply Theorem~ to the partial sums of $\sum f_n$, observing that

$$f_n + f_{n+1} + \cdots + f_m = F_m - F_{n-1}.$$

-2em2em

Setting $m = n$ in yields the following necessary, but not sufficient, condition for uniform convergence of series. It is analogous to Corollary~.

5pt

Theorem~ leads immediately to the following important test for uniform convergence of series.

5pt

5pt

From Cauchy's convergence criterion for series of constants, there is for each $\epsilon > 0$ an integer N such that

$$M_n + M_{n+1} + \cdots + M_m < \epsilon \quad \text{if} \quad m \ge n \ge N,$$

which, because of , implies that

$$\|f_n\|_S + \|f_{n+1}\|_S + \cdots + \|f_m\|_S < \epsilon \quad \text{if} \quad m, n \ge N.$$

Lemma~ and Theorem~ imply that $\sum f_n$ converges uniformly on S.

Weierstrass's test is very important, but applicable only to series that actually exhibit a stronger kind of convergence than we have considered so far. We say that $\sum f_n$ {} S if $\sum |f_n|$ converges pointwise on S, and {} on S if $\sum |f_n|$ converges uniformly on S. We leave it to you (Exercise~) to verify that our proof of Weierstrass's test actually shows that $\sum f_n$ converges absolutely uniformly on S. We also leave it to you to show that if a series converges absolutely uniformly on S, then it converges uniformly on S (Exercise~).

The next theorem applies to series that converge uniformly, but perhaps not absolutely uniformly, on a set S.

The proof is similar to the proof of Theorem~. Let

$$G_n = g_k + g_{k+1} + \cdots + g_n,$$

and consider the partial sums of $\sum_{n=k}^{\infty} f_n g_n$:

$$H_n = f_k g_k + f_{k+1} g_{k+1} + \cdots + f_n g_n. \tag{4.1.45}$$

By substituting

$$g_k = G_k \quad \text{and} \quad g_n = G_n - G_{n-1}, \quad n \ge k+1,$$

into , we obtain

$$H_n = f_k G_k + f_{k+1}(G_{k+1} - G_k) + \cdots + f_n(G_n - G_{n-1}),$$

which we rewrite as

$$H_n = (f_k - f_{k+1})G_k + (f_{k+1} - f_{k+2})G_{k+1} + \cdots + (f_{n-1} - f_n)G_{n-1} + f_n G_n,$$

or

$$H_n = J_{n-1} + f_n G_n, \tag{4.1.46}$$

where

$$J_{n-1} = (f_k - f_{k+1})G_k + (f_{k+1} - f_{k+2})G_{k+1} + \cdots + (f_{n-1} - f_n)G_{n-1}. \tag{4.1.47}$$

That is, $\{J_n\}$ is the sequence of partial sums of the series

$$\sum_{j=k}^{\infty}(f_j-f_{j+1})G_j. \tag{4.1.48}$$

From and the definition of G_j,

$$\left|\sum_{j=n}^{m}[f_j(x)-f_{j+1}(x)]G_j(x)\right| \le M\sum_{j=n}^{m}|f_j(x)-f_{j+1}(x)|, \quad x\in S,$$

so

$$\left\|\sum_{j=n}^{m}(f_j-f_{j+1})G_j\right\|_S \le M\left\|\sum_{j=n}^{m}|f_j-f_{j+1}|\right\|_S.$$

Now suppose that $\epsilon>0$. Since $\sum(f_j-f_{j+1})$ converges absolutely uniformly on S, Theorem~ implies that there is an integer N such that the right side of the last inequality is less than ϵ if $m\ge n\ge N$. The same is then true of the left side, so Theorem~ implies that converges uniformly on~S.

We have now shown that $\{J_n\}$ as defined in converges uniformly to a limit function J on S. Returning to , we see that

$$H_n-J=J_{n-1}-J+f_nG_n.$$

Hence, from Lemma~ and ,

$$\begin{aligned}\|H_n-J\|_S \ar &\le \|J_{n-1}-J\|_S+\|f_n\|_S\|G_n\|_S\\ \ar &\le \|J_{n-1}-J\|_S+M\|f_n\|_S.\end{aligned}$$

Since $\{J_{n-1}-J\}$ and $\{f_n\}$ converge uniformly to zero on S, it now follows that $\lim_{n\to\infty}\|H_n-J\|_S=0$. Therefore, $\{H_n\}$ converges uniformly on~S.

The proof is similar to that of Corollary~. We leave it to you (Exercise~).

\begin{example}\rm Consider the series

$$\sum_{n=1}^{\infty}\frac{\sin nx}{n}$$

with $f_n=1/n$ (constant), $g_n(x)=\sin nx$, and

$$G_n(x)=\sin x+\sin 2x+\cdots+\sin nx.$$

We saw in Example~ that

$$|G_n(x)|\le\frac{1}{|\sin(x/2)|}, \quad n\ge1, \quad n\ne 2k\pi \qquad (k=\text{integer}).$$

Therefore, $\{\|G_n\|_S\}$ is bounded, and the series converges uniformly on any set S on which $\sin x/2$ is bounded away from zero. For example, if $0<\delta<\pi$, then

$$\left|\sin\frac{x}{2}\right|\ge\sin\frac{\delta}{2}$$

if x is at least δ away from any multiple of 2π; hence, the series converges uniformly on

$$S=\bigcup_{k=-\infty}^{\infty}[2k\pi+\delta,2(k+1)\pi-\delta].$$

Since

$$\sum\left|\frac{\sin nx}{n}\right|=\infty, \quad x\ne k\pi$$

(Exercise~

), this result cannot be obtained from Weierstrass's test. \end{example}

We can obtain results on the continuity, differentiability, and integrability of infinite series by applying Theorems~, , and to their partial sums. We will state the theorems and give some examples, leaving the proofs to you.

Theorem~ implies the following theorem (Exercise~).

The next theorem gives conditions that permit the interchange of summation and integration of infinite series. It follows from Theorem~ (Exercise~). We leave it to you to formulate an analog of Theorem~ for series (Exercise~).

We say in this case that $\sum_{n=k}^{\infty} f_n$ can be integrated {} over $[a,b]$.

The next theorem gives conditions that permit the interchange of summation and differentiation of infinite series. It follows from Theorem~ (Exercise~).

We say in this case that $\sum_{n=k}^{\infty} f_n$ can be differentiated {} on $[a,b]$. To apply Theorem~, we first verify that $\sum_{n=k}^{\infty} f_n(x_0)$ converges for some x_0 in $[a,b]$ and then differentiate $\sum_{n=k}^{\infty} f_n$ term by term. If the resulting series converges uniformly, then term by term differentiation was legitimate.

We now consider a class of series sufficiently general to be interesting, but sufficiently specialized to be easily understood.

The following theorem summarizes the convergence properties of power series.

In any case, the series converges to a_0 if $x = x_0$. If

$$\sum |a_n| r^n < \infty \tag{4.1.49}$$

for some $r > 0$, then $\sum a_n (x - x_0)^n$ converges absolutely uniformly in $[x_0 - r, x_0 + r]$, by Weierstrass's test (Theorem~) and Exercise~. From Cauchy's root test (Theorem~), holds if

$$\limsup_{n\to\infty} \left(|a_n| r^n\right)^{1/n} < 1,$$

which is equivalent to

$$r \limsup_{n\to\infty} |a_n|^{1/n} < 1$$

(Exercise~). From , this can be rewritten as $r < R$, which proves the assertions concerning convergence in and

.

If $0 \le R < \infty$ and $|x - x_0| > R$, then

$$\frac{1}{R} > \frac{1}{|x - x_0|},$$

so implies that

$$|a_n|^{1/n} \ge \frac{1}{|x - x_0|} \quad \text{and therefore} \quad |a_n (x - x_0)^n| \ge 1$$

for infinitely many values of n. Therefore, $\sum a_n (x - x_0)^n$ diverges (Corollary~) if $|x - x_0| > R$. In particular, the series diverges for all $x \ne x_0$ if $R = 0$.

To prove the assertions concerning the possibilities at $x = x_0 + R$ and $x = x_0 - R$ requires examples, which follow. (Also, see Exercise~.)

The number R defined by is the {} of $\sum a_n (x - x_0)^n$. If $R > 0$, the {} interval $(x_0 - R, x_0 + R)$, or $(-\infty, \infty)$ if $R = \infty$, is the {} of the series. Theorem~ says that a power series with a nonzero radius of convergence converges absolutely uniformly in every compact subset of its interval of convergence and diverges at every point in the exterior of this interval. On this last we can make a stronger statement: Not only does $\sum a_n (x - x_0)^n$ diverge if $|x - x_0| > R$, but the sequence $\{a_n (x - x_0)^n\}$ is unbounded in this case (Exercise~

).

The next theorem provides an expression for R that, if applicable, is usually easier to use than .

From Theorem~, it suffices to show that if

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \tag{4.1.50}$$

exists in the extended reals, then

$$L = \limsup_{n\to\infty} |a_n|^{1/n}. \tag{4.1.51}$$

We will show that this is so if $0 < L < \infty$ and leave the cases where $L = 0$ or $L = \infty$ to you (Exercise~).

If holds with $0 < L < \infty$ and $0 < \epsilon < L$, there is an integer N such that

$$L - \epsilon < \left|\frac{a_{m+1}}{a_m}\right| < L + \epsilon \quad \text{if} \quad m \ge N,$$

so

$$|a_m|(L-\epsilon) < |a_{m+1}| < |a_m|(L+\epsilon) \quad \text{if} \quad m \ge N.$$

By induction,

$$|a_N|(L-\epsilon)^{n-N} < |a_n| < |a_N|(L+\epsilon)^{n-N} \quad \text{if} \quad n > N.$$

Therefore, if

$$K_1 = |a_N|(L-\epsilon)^{-N} \quad \text{and} \quad K_2 = |a_N|(L+\epsilon)^{-N},$$

then

$$K_1^{1/n}(L-\epsilon) < |a_n|^{1/n} < K_2^{1/n}(L+\epsilon). \tag{4.1.52}$$

Since $\lim_{n\to\infty} K^{1/n} = 1$ if K is any positive number, implies that

$$L - \epsilon \le \liminf_{n\to\infty} |a_n|^{1/n} \le \limsup_{n\to\infty} |a_n|^{1/n} \le L + \epsilon.$$

Since ϵ is an arbitrary positive number, it follows that

$$\lim_{n\to\infty} |a_n|^{1/n} = L,$$

which implies .

We now study the properties of functions defined by power series. Henceforth, we consider only power series with nonzero radii of convergence.

First, the series in and are the same, since the latter is obtained by shifting the index of summation in the former. Since

$$\begin{aligned}
\limsup_{n\to\infty} ((n+1)|a_n|)^{1/n} \ar &= \limsup_{n\to\infty} (n+1)^{1/n}|a_n|^{1/n} \\
\ar &= \left(\lim_{n\to\infty} (n+1)^{1/n}\right)\left(\limsup_{n\to\infty} |a_n|^{1/n}\right) \quad \text{(Exercise ???\parta)} \\
\ar &= \left[\lim_{n\to\infty} \exp\left(\frac{\log(n+1)}{n}\right)\right]\left(\limsup_{n\to\infty} |a_n|^{1/n}\right) = \frac{e^0}{R} = \frac{1}{R},
\end{aligned}$$

the radius of convergence of the power series in is R (Theorem~). Therefore, the power series in converges uniformly in every interval $[x_0 - r, x_0 + r]$ such that $0 < r < R$, and Theorem~ now implies for all x in $(x_0 - R, x_0 + R)$.

Theorem~ can be strengthened as follows.

The proof is by induction. The assertion is true for $k = 1$, by Theorem~. Suppose that it is true for some $k \ge 1$. By shifting the index of summation, we can rewrite as

$$f^{(k)}(x) = \sum_{n=0}^{\infty} (n+k)(n+k-1)\cdots(n+1)a_{n+k}(x-x_0)^n, \quad |x - x_0| < R.$$

Defining

$$b_n = (n+k)(n+k-1)\cdots(n+1)a_{n+k}, \tag{4.1.53}$$

we rewrite this as

$$f^{(k)}(x) = \sum_{n=0}^{\infty} b_n(x-x_0)^n, \quad |x-x_0| < R.$$

By Theorem~, we can differentiate this series term by term to obtain

$$f^{(k+1)}(x) = \sum_{n=1}^{\infty} nb_n(x-x_0)^{n-1}, \quad |x-x_0| < R.$$

Substituting from for b_n yields

$$f^{(k+1)}(x) = \sum_{n=1}^{\infty} (n+k)(n+k-1)\cdots(n+1)na_{n+k}(x-x_0)^{n-1}, \quad |x-x_0| < R.$$

Shifting the summation index yields

$$f^{(k+1)}(x) = \sum_{n=k+1}^{\infty} n(n-1)\cdots(n-k)a_n(x-x_0)^{n-k-1}, \quad |x-x_0| < R,$$

which is with k replaced by $k+1$. This completes the induction.

Theorem~ has two important corollaries.

Setting $x = x_0$ in yields

$$f^{(k)}(x_0) = k!a_k.$$

-2em2em

Let

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} b_n(x-x_0)^n.$$

From Corollary~,

$$a_n = \frac{f^{(n)}(x_0)}{n!} \quad \text{and} \quad b_n = \frac{g^{(n)}(x_0)}{n!}. \tag{4.1.54}$$

From , $f = g$ in (x_0-r, x_0+r). Therefore,

$$f^{(n)}(x_0) = g^{(n)}(x_0), \quad n \geq 0.$$

This and imply .

Theorems~ and imply the following theorem. We leave the proof to you (Exercise~).

Example~ presents an application of this theorem.

So far we have asked for what values of x a given power series converges, and what are the properties of its sum. Now we ask a related question: What properties guarantee that a given function f can be represented as the sum of a convergent power series in $x - x_0$? A partial answer to this question is provided by what we already know: Theorem~ tells us that f must have derivatives of all orders in some neighborhood of x_0, and Corollary~ tells us that the only power series in $x - x_0$ that can possibly converge to f in such a neighborhood is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n. \tag{4.1.55}$$

This is called the {} (also, the {} of f, if $x_0 = 0$). The mth partial sum of is the Taylor polynomial

$$T_m(x) = \sum_{n=0}^{m} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

defined in Section~2.5.

The Taylor series of an infinitely differentiable function f may converge to a sum different from f. For example, the function

$$f(x) = \begin{cases} \casespace e^{-1/x^2}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

is infinitely differentiable on $(-\infty, \infty)$ and $f^{(n)}(0) = 0$ for $n \geq 0$ (Exercise~), so its Maclaurin series is identically zero.

The answer to our question is provided by Taylor's theorem (Theorem~), which says that if f is infinitely differentiable on (a, b) and x and x_0 are in (a, b) then, for every integer $n \geq 0$,

$$f(x) - T_n(x) = \frac{f^{(n+1)}(c_n)}{(n+1)!}(x - x_0)^{n-1}, \tag{4.1.56}$$

where c_n is between x and x_0. Therefore,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

for an x in (a, b) if and only if

$$\lim_{n\to\infty} \frac{f^{(n+1)}(c_n)}{(n+1)!}(x - x_0)^{n+1} = 0.$$

It is not always easy to check this condition, because the sequence $\{c_n\}$ is usually not precisely known, or even uniquely defined; however, the next theorem is sufficiently general to be useful.

From ,

$$\|f - T_n\|_{I_r} \leq \frac{r^{n+1}}{(n+1)!}\|f^{(n+1)}\|_{I_r} \leq \frac{r^{n+1}}{(n+1)!}\|f^{(n+1)}\|_I,$$

so implies the conclusion.

We cannot prove in this way that the binomial series converges to $(1 + x)^q$ on $(-1, 0)$. This requires a form of the remainder in Taylor's theorem that we have not considered, or a different kind of proof altogether (Exercise~). The complete result is that

$$(1 + x)^q = \sum_{n=0}^{\infty} \binom{q}{n} x^n, \quad -1 < x < 1, \tag{4.1.57}$$

for all q, and, as we said earlier, the identity holds for all x if q is a nonnegative integer.

We now consider addition and multiplication of power series, and division of one by another.

We leave the proof of the next theorem to you (Exercise~).

Suppose that $R_1 \leq R_2$. Since the series and converge absolutely to $f(x)$ and $g(x)$ if $|x - x_0| < R_1$, their Cauchy product converges to $f(x)g(x)$ if $|x - x_0| < R_1$, by Theorem~. The nth term of this product is

$$\sum_{r=0}^{n} a_r(x - x_0)^r b_{n-r}(x - x_0)^{n-r} = \left(\sum_{r=0}^{n} a_r b_{n-r}\right)(x - x_0)^n = c_n(x - x_0)^n.$$

-4em4em

The quotient

$$f(x) = \frac{h(x)}{g(x)} \tag{4.1.58}$$

of two power series

$$h(x)\ar=\sum_{n=0}^{\infty}c_n(x-x_0)^n,\quad |x-x_0|<R_1,$$
$$\noalignand g(x)\ar=\sum_{n=0}^{\infty}b_n(x-x_0)^n,\quad |x-x_0|<R_2,$$

can be represented as a power series

$$f(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n \tag{4.1.59}$$

with a positive radius of convergence, provided that

$$b_0=g(x_0)\neq 0.$$

This is surely plausible. Since $g(x_0)\neq 0$ and g is continuous near x_0, the denominator of differs from zero on an interval about x_0. Therefore, f has derivatives of all orders on this interval, because g and h do. However, the proof that the Taylor series of f about x_0 converges to f near x_0 requires the use of the theory of functions of a complex variable. Therefore, we omit it. However, it is straightforward to compute the coefficients in if we accept the validity of the expansion. Since

$$f(x)g(x)=h(x),$$

Theorem~ implies that

$$\sum_{r=0}^{n}a_rb_{n-r}=c_n,\quad n\geq 0.$$

Solving these equations successively yields

$$a_0\ar=\frac{c_0}{b_0},$$
$$a_n\ar=\frac{1}{b_0}\left(c_n-\sum_{r=0}^{n-1}b_{n-r}a_r\right),\quad n\geq 1.$$

It is not worthwhile to memorize these formulas. Rather, it is usually better to view the procedure as follows: Multiply the series f (with unknown coefficients) and g according to the procedure of Theorem~, equate the resulting coefficients with those of h, and solve the resulting equations successively for a_0, a_1, .

From Theorem~, we know that a function f defined by a convergent power series

$$f(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n,\quad |x-x_0|<R, \tag{4.1.60}$$

is continuous in the open interval (x_0-R,x_0+R). The next theorem concerns the behavior of f as x approaches an endpoint of the interval of convergence.

We consider a simpler problem first. Let

$$g(y)\ar=\sum_{n=0}^{\infty}b_ny^n$$
$$\arraytextand$$
$$\sum_{n=0}^{\infty}b_n\ar=s\quad (\text{finite}).$$

We will show that

$$\lim_{y\to 1-}g(y)=s. \tag{4.1.61}$$

From Example~,

$$g(y)=(1-y)\sum_{n=0}^{\infty}s_ny^n, \tag{4.1.62}$$

where

$$s_n = b_0 + b_1 + \cdots + b_n.$$

Since

$$\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n \quad \text{and therefore} \quad 1 = (1-y)\sum_{n=0}^{\infty} y^n, \quad |y| < 1, \tag{4.1.63}$$

we can multiply through by s and write

$$s = (1-y)\sum_{n=0}^{\infty} s y^n, \quad |y| < 1.$$

Subtracting this from yields

$$g(y) - s = (1-y)\sum_{n=0}^{\infty} (s_n - s) y^n, \quad |y| < 1.$$

If $\epsilon > 0$, choose N so that

$$|s_n - s| < \epsilon \quad \text{if} \quad n \geq N+1.$$

Then, if $0 < y < 1$,

$$\begin{aligned} |g(y) - s| \ar &\leq (1-y)\sum_{n=0}^{N} |s_n - s| y^n + (1-y)\sum_{n=N+1}^{\infty} |s_n - s| y^n \\ \ar &< (1-y)\sum_{n=0}^{N} |s_n - s| y^n + (1-y)\epsilon y^{N+1}\sum_{n=0}^{\infty} y^n \\ \ar &< (1-y)\sum_{n=0}^{N} |s_n - s| + \epsilon, \end{aligned}$$

because of the second equality in . Therefore,

$$|g(y) - s| < 2\epsilon$$

if

$$(1-y)\sum_{n=0}^{N} |s_n - s| < \epsilon.$$

This proves .

To obtain from this, let $b_n = a_n R^n$ and $g(y) = f(x_0 + Ry)$; to obtain

, let $b_n = (-1)^n a_n R^n$ and $g(y) = f(x_0 - Ry)$.

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4.2: Earlier Topics Revisited With Sequences

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CHAPTER OVERVIEW

5: Real-Valued Functions of Several Variables

IN THIS CHAPTER we consider real-valued function of n variables, where $n > 1$.

- SECTION 5.1 deals with the structure of $\mathbf{R}^n$, the space of ordered n-tuples of real numbers, which we call {}. We define the sum of two vectors, the product of a vector and a real number, the length of a vector, and the inner product of two vectors. We study the arithmetic properties of $\mathbf{R}^n$, including Schwarz's inequality and the triangle inequality. We define neighborhoods and open sets in $\mathbf{R}^n$, define convergence of a sequence of points in $\mathbf{R}^n$, and extend the Heine–Borel theorem to $\mathbf{R}^n$. The section concludes with a discussion of connected subsets of $\mathbf{R}^n$.
- SECTION 5.2 deals with boundedness, limits, continuity, and uniform continuity of a function of n variables; that is, a function defined on a subset of $\mathbf{R}^n$.
- SECTION 5.3 defines directional and partial derivatives of a real-valued function of n variables. This is followed by the definition of differentiablity of such functions. We define the differential of such a function and give a geometric interpretation of differentiablity.
- SECTION 5.4 deals with the chain rule and Taylor's theorem for a real-valued function of n variables.

5.1: Structure of Rn

5.2: Continuous Real-Valued Function of n Variables

5.3: Partial Derivatives and the Differential

5.4: The Chain Rule and Taylor's Theorem

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5.1: Structure of Rn

In this chapter we study functions defined on subsets of the real n-dimensional space $\R^n$, which consists of all ordered n-tuples $\X = (x_1, x_2, \dots, x_n)$ of real numbers, called the {} or {} of $\X$. This space is sometimes called {}.

In this section we introduce an algebraic structure for $\R^n$. We also consider its {} properties; that is, properties that can be described in terms of a special class of subsets, the neighborhoods in $\R^n$. In Section~1.3 we studied the topological properties of $\R^1$, which we will continue to denote simply as $\R$. Most of the definitions and proofs in Section~1.3 were stated in terms of neighborhoods in $\R$. We will see that they carry over to $\R^n$ if the concept of neighborhood in $\R^n$ is suitably defined.

Members of $\R$ have dual interpretations: geometric, as points on the real line, and algebraic, as real numbers. We assume that you are familiar with the geometric interpretation of members of $\R^2$ and $\R^3$ as the rectangular coordinates of points in a plane and three-dimensional space, respectively. Although $\R^n$ cannot be visualized geometrically if $n \geq 4$, geometric ideas from $\R$, $\R^2$, and $\R^3$ often help us to interpret the properties of $\R^n$ for arbitrary n.

As we said in Section~1.3, the idea of neighborhood is always associated with some definition of ``closeness'' of points. The following definition imposes an algebraic structure on $\R^n$, in terms of which the distance between two points can be defined in a natural way. In addition, this algebraic structure will be useful later for other purposes.

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Note that `$+$'' has two distinct meanings in \eqref{eq:5.1.1}: on the left,` $+$''stands for the newly defined addition of members of $\R^n$ and, on the right, for addition of real numbers. However, this can never lead to confusion, since the meaning of ``$+$'' can always be deduced from the symbols on either side of it. A similar comment applies to the use of juxtaposition to indicate scalar multiplication on the left of and multiplication of real numbers on the right.

We leave the proof of the following theorem to you (Exercise~).

Clearly, $\0 = (0, 0, \dots, 0)$ and, if $\X = (x_1, x_2, \dots, x_n)$, then

$$-\X = (-x_1, -x_2, \dots, -x_n).$$

We write $\X + (-\Y)$ as $\X - \Y$. The point $\0$ is called the {}.

A nonempty set $V = \{\X, \Y, \Z, \dots\}$, together with rules such as , associating a unique member of V with every ordered pair of its members, and , associating a unique member of V with every real number and member of V, is said to be a {} if it has the properties listed in Theorem~. The members of a vector space are called {}. When we wish to emphasize that we are regarding a member of $\R^n$ as part of this algebraic structure, we will speak of it as a vector; otherwise, we will speak of it as a point.

If $n = 1$, this definition of length reduces to the familiar absolute value, and the distance between two points is the length of the interval having them as endpoints; for $n = 2$ and $n = 3$, the length and distance of Definition~ reduce to the familiar definitions for the plane and three-dimensional space.

If $\Y = \0$, then both sides of are $\0$, so holds, with equality. In this case, $\Y = 0\X$. Now suppose that $\Y \neq \0$ and t is any real number. Then

$$\begin{aligned} 0 &\leq \sum_{i=1}^n (x_i - t y_i)^2 \\ &= \sum_{i=1}^n x_i^2 - 2t \sum_{i=1}^n x_i y_i + t^2 \sum_{i=1}^n y_i^2 \end{aligned} \tag{5.1.1}$$

$$= |\X|^2 - 2(\X \cdot \Y)t + t^2 |\Y|^2.$$

The last expression is a second-degree polynomial p in t. From the quadratic formula, the zeros of p are

$$t = \frac{(\X \cdot \Y) \pm \sqrt{(\X \cdot \Y)^2 - |\X|^2 |\Y|^2}}{|\Y|^2}.$$

Hence,

$$(\X \cdot \Y)^2 \leq |\X|^2 |\Y|^2, \tag{5.1.2}$$

because if not, then p would have two distinct real zeros and therefore be negative between them (Figure~), contradicting the inequality . Taking square roots in yields if $\mathbf{Y} \neq \mathbf{0}$.

If $\mathbf{X} = t\mathbf{Y}$, then $|\mathbf{X} \cdot \mathbf{Y}| = |\mathbf{X}||\mathbf{Y}| = |t||\mathbf{Y}|^2$ (verify), so equality holds in . Conversely, if equality holds in , then p has the real zero $t_0 = (\mathbf{X} \cdot \mathbf{Y})/|\mathbf{Y}\|^2$, and

$$\sum_{i=1}^{n} (x_i - t_0 y_i)^2 = 0$$

from ; therefore, $\mathbf{X} = t_0\mathbf{Y}$.

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By definition,

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Hence,

$$|\mathbf{X} + \mathbf{Y}|^2 \leq (|\mathbf{X}| + |\mathbf{Y}|)^2.$$

Taking square roots yields .

From the third line of , equality holds in if and only if $\mathbf{X} \cdot \mathbf{Y} = |\mathbf{X}||\mathbf{Y}|$, which is true if and only if one of the vectors $\mathbf{X}$ and $\mathbf{Y}$ is a nonnegative scalar multiple of the other (Lemma~).

Write

$$\mathbf{X} - \mathbf{Z} = (\mathbf{X} - \mathbf{Y}) + (\mathbf{Y} - \mathbf{Z}),$$

and apply Theorem~ with $\mathbf{X}$ and $\mathbf{Y}$ replaced by $\mathbf{X} - \mathbf{Y}$ and $\mathbf{Y} - \mathbf{Z}$.

Since

$$\mathbf{X} = \mathbf{Y} + (\mathbf{X} - \mathbf{Y}),$$

Theorem~ implies that

$$|\mathbf{X}| \leq |\mathbf{Y}| + |\mathbf{X} - \mathbf{Y}|,$$

which is equivalent to

$$|\mathbf{X}| - |\mathbf{Y}| \leq |\mathbf{X} - \mathbf{Y}|.$$

Interchanging $\mathbf{X}$ and $\mathbf{Y}$ yields

$$|\mathbf{Y}| - |\mathbf{X}| \leq |\mathbf{Y} - \mathbf{X}|.$$

Since $|\mathbf{X} - \mathbf{Y}| = |\mathbf{Y} - \mathbf{X}|$, the last two inequalities imply the stated conclusion.

The next theorem lists properties of length, distance, and inner product that follow directly from Definitions~ and . We leave the proof to you (Exercise~).

The equation of a line through a point $\mathbf{X}_0 = (x_0, y_0, z_0)$ in $\R^3$ can be written parametrically as

$$x = x_0 + u_1 t, \quad y = y_0 + u_2 t, \quad z = z_0 + u_3 t, \quad -\infty < t < \infty,$$

where u_1, u_2, and u_3 are not all zero. We write this in vector form as

$$\mathbf{X} = \mathbf{X}_0 + t\mathbf{U}, \quad -\infty < t < \infty, \tag{5.1.3}$$

with $\mathbf{U} = (u_1, u_2, u_3)$, and we say that the line is {}.

There are many ways to represent a given line parametrically. For example,

$$\mathbf{X} = \mathbf{X}_0 + s\mathbf{V}, \quad -\infty < s < \infty, \tag{5.1.4}$$

represents the same line as if and only if $\mathbf{V} = a\mathbf{U}$ for some nonzero real number a. Then the line is traversed in the same direction as s and t vary from $-\infty$ to ∞ if $a > 0$, or in opposite directions if $a < 0$.

To write the parametric equation of a line through two points $\mathbf{X}_0$ and $\mathbf{X}_1$ in $\R^3$, we take $\mathbf{U} = \mathbf{X}_1 - \mathbf{0}$ in , which yields

$$\mathbf{X} = \mathbf{X}_0 + t(\mathbf{X}_1 - \mathbf{X}_0) = t\mathbf{X}_1 + (1-t)\mathbf{X}_0, \quad -\infty < t < \infty.$$

The line segment from $\mathbf{X}_0$ to $\mathbf{X}_1$ consists of those points for which $0 \le t \le 1$.

These familiar notions can be generalized to $\R^n$, as follows:

Having defined distance in $\R^n$, we are now able to say what we mean by a neighborhood of a point in $\R^n$.

An ϵ-neighborhood of a point $\mathbf{X}_0$ in $\R^2$ is the inside, but not the circumference, of the circle of radius ϵ about $\mathbf{X}_0$. In $\R^3$ it is the inside, but not the surface, of the sphere of radius ϵ about $\mathbf{X}_0$.

In Section~1.3 we stated several other definitions in terms of ϵ-neighborhoods: {}, {}, {}, {}, {},{}, {}, {}, {}, {}, {}, and {}. Since these definitions are the same for $\R^n$ as for $\R$, we will not repeat them. We advise you to read them again in Section~1.3, substituting $\R^n$ for $\R$ and $\mathbf{X}_0$ for x_0.

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Open and closed n-balls are generalizations to $\R^n$ of open and closed intervals.

The following lemma will be useful later in this section, when we consider connected sets.

The line segment is given by

$$\mathbf{X} = t\mathbf{X}_2 + (1-t)\mathbf{X}_1, \quad 0 < t < 1.$$

Suppose that $r > 0$. If

$$|\mathbf{X}_1 - \mathbf{X}_0| < r, \quad |\mathbf{X}_2 - \mathbf{X}_0| < r,$$

and $0 < t < 1$, then

$$\begin{aligned} |\mathbf{X} - \mathbf{X}_0| \ar &= |t\mathbf{X}_2 + (1-t)\mathbf{X}_1 - t\mathbf{X}_0 - (1-t)\mathbf{X}_0| \\ \ar &= |t(\mathbf{X}_2 - \mathbf{X}_0) + (1-t)\mathbf{X}_1 - \mathbf{X}_0)| \\ \ar &\le t|\mathbf{X}_2 - \mathbf{X}_0| + (1-t)|\mathbf{X}_1 - \mathbf{X}_0| \\ \ar &< tr + (1-t)r = r. \end{aligned}$$

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The proofs in Section~1.3 of Theorem~ (the union of open sets is open, the intersection of closed sets is closed) and Theorem~ and its Corollary~ (a set is closed if and only if it contains all its limit points) are also valid in $\R^n$. You should reread them now.

The Heine–Borel theorem (Theorem~) also holds in $\R^n$, but the proof in Section~1.3 is valid only for $n = 1$. To prove the Heine–Borel theorem for general n, we need some preliminary definitions and results that are of interest in their own right.

The next two theorems follow from this, the definition of distance in $\R^n$, and what we already know about convergence in $\R$. We leave the proofs to you (Exercises~ and).

The next definition generalizes the definition of the diameter of a circle or sphere.

Let $\{\mathbf{X}_r\}$ be a sequence such that $\mathbf{X}_r \in S_r$ $(r \ge 1)$. Because of , $\mathbf{X}_r \in S_k$ if $r \ge k$, so

$$|\mathbf{X}_r - \mathbf{X}_s| < d(S_k) \quad \text{if} \quad r, s \ge k.$$

From and Theorem~, $\mathbf{X}_r$ converges to a limit $\overline{\mathbf{X}}$. Since $\overline{\mathbf{X}}$ is a limit point of every S_k and every S_k is closed, $\overline{\mathbf{X}}$ is in every S_k (Corollary~). Therefore, $\overline{\mathbf{X}} \in I$, so $I \ne \emptyset$. Moreover, $\overline{\mathbf{X}}$ is the only point in I, since if $\mathbf{Y} \in I$, then

$$|\overline{\mathbf{X}} - \mathbf{Y}| \le d(S_k), \quad k \ge 1,$$

and implies that $\mathbf{Y} = \overline{\mathbf{X}}$.

We can now prove the Heine–Borel theorem for $\mathbf{R}^n$. This theorem concerns {} sets. As in $\mathbf{R}$, a compact set in $\mathbf{R}^n$ is a closed and bounded set.

Recall that a collection $\mathcal{H}$ of open sets is an open covering of a set S if

$$S \subset \cup\{H\} H \in \mathcal{H}.$$

The proof is by contradiction. We first consider the case where $n = 2$, so that you can visualize the method. Suppose that there is a covering $\mathcal{H}$ for S from which it is impossible to select a finite subcovering. Since S is bounded, S is contained in a closed square

$$T = \{(x, y) | a_1 \leq x \leq a_1 + L, a_2 \leq x \leq a_2 + L\}$$

with sides of length L (Figure~).

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Bisecting the sides of T as shown by the dashed lines in Figure~ leads to four closed squares, $T^{(1)}, T^{(2)}$, $T^{(3)}$, and $T^{(4)}$, with sides of length $L/2$. Let

$$S^{(i)} = S \cap T^{(i)}, \quad 1 \leq i \leq 4.$$

Each $S^{(i)}$, being the intersection of closed sets, is closed, and

$$S = \bigcup_{i=1}^{4} S^{(i)}.$$

Moreover, $\mathcal{H}$ covers each $S^{(i)}$, but at least one $S^{(i)}$ cannot be covered by any finite subcollection of $\mathcal{H}$, since if all the $S^{(i)}$ could be, then so could S. Let S_1 be a set with this property, chosen from $S^{(1)}$, $S^{(2)}$, $S^{(3)}$, and $S^{(4)}$. We are now back to the situation we started from: a compact set S_1 covered by $\mathcal{H}$, but not by any finite subcollection of $\mathcal{H}$. However, S_1 is contained in a square T_1 with sides of length $L/2$ instead of L. Bisecting the sides of T_1 and repeating the argument, we obtain a subset S_2 of S_1 that has the same properties as S, except that it is contained in a square with sides of length $L/4$. Continuing in this way produces a sequence of nonempty closed sets S_0 $(= S)$, S_1, S_2, , such that $S_k \supset S_{k+1}$ and $d(S_k) \leq L/2^{k-1/2}$ $(k \geq 0)$. From Theorem~, there is a point $\overline{\mathbf{X}}$ in $\bigcap_{k=1}^{\infty} S_k$. Since $\overline{\mathbf{X}} \in S$, there is an open set H in $\mathcal{H}$ that contains $\overline{\mathbf{X}}$, and this H must also contain some ϵ-neighborhood of $\overline{\mathbf{X}}$. Since every $\mathbf{X}$ in S_k satisfies the inequality

$$|\mathbf{X} - \overline{\mathbf{X}}| \leq 2^{-k+1/2} L,$$

it follows that $S_k \subset H$ for k sufficiently large. This contradicts our assumption on $\mathcal{H}$, which led us to believe that no S_k could be covered by a finite number of sets from $\mathcal{H}$. Consequently, this assumption must be false: $\mathcal{H}$ must have a finite subcollection that covers S. This completes the proof for $n = 2$.

The idea of the proof is the same for $n > 2$. The counterpart of the square T is the {} with sides of length L:

$$T = \{(x_1, x_2, \ldots, x_n)\} a_i \leq x_i \leq a_i + L, i = 1, 2, \ldots, n.$$

Halving the intervals of variation of the n coordinates x_1, x_2, , x_n divides T into 2^n closed hypercubes with sides of length $L/2$:

$$T^{(i)} = \{(x_1, x_2, \ldots, x_n)\} b_i \leq x_i \leq b_i + L/2, 1 \leq i \leq n,$$

where $b_i = a_i$ or $b_i = a_i + L/2$. If no finite subcollection of $\mathcal{H}$ covers S, then at least one of these smaller hypercubes must contain a subset of S that is not covered by any finite subcollection of S. Now the proof proceeds as for $n = 2$.

The Bolzano–Weierstrass theorem is valid in $\mathbf{R}^n$; its proof is the same as in $\mathbf{R}$.

Although it is legitimate to consider functions defined on arbitrary domains, we restricted our study of functions of one variable mainly to functions defined on intervals. There are good reasons for this. If we wish to raise questions of continuity and differentiability at every point of the domain D of a function f, then every point of D must be a limit point of D^0. Intervals have this property. Moreover, the definition of $\int_a^b f(x)\, dx$ is obviously applicable only if f is defined on $[a, b]$.

It is not productive to consider questions of continuity and differentiability of functions defined on the union of disjoint intervals, since many important results simply do not hold for such domains. For example, the intermediate value theorem (Theorem~; see also Exercise~) says that if f is continuous on an interval I and $f(x_1) < \mu < f(x_2)$ for some x_1 and x_2 in I, then $f(\overline{x}) = \mu$ for some $\overline{x}$ in I. Theorem~ says that f is constant on an interval I if $f' \equiv 0$ on I. Neither of these results holds if I is the union of disjoint intervals rather than a single interval; thus, if f is defined on $I = (0,1) \cup (2,3)$ by

$$f(x) = \left\{ \casespace \begin{array}{ll} 1, & 0 < x < 1, \\ 0, & 2 < x < 3, \end{array} \right.$$

then f is continuous on I, but does not assume any value between 0 and 1, and $f' \equiv 0$ on I, but f is not constant.

It is not difficult to see why these results fail to hold for this function: the domain of f consists of two disconnected pieces. It would be more sensible to regard f as two entirely different functions, one defined on $(0,1)$ and the other on $(2,3)$. The two results mentioned are valid for each of these functions.

As we will see when we study functions defined on subsets of $\R^n$, considerations like those just cited as making it natural to consider functions defined on intervals in $\R$ lead us to single out a preferred class of subsets as domains of functions of n variables. These subsets are called {}. To define this term, we first need the following definition.

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If $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_k$ are points in $\R^n$ and L_i is the line segment from $\mathbf{X}_i$ to $\mathbf{X}_{i+1}$, $1 \le i \le k-1$, we say that L_1, L_2, , L_{k-1} form a {} from $\mathbf{X}_1$ to $\mathbf{X}_k$, and that $\mathbf{X}_1$ and $\mathbf{X}_k$ are {} by the polygonal path. For example, Figure~ shows a polygonal path in $\R^2$ connecting $(0,0)$ to $(3,3)$. A set S is {} if every pair of points in S can be connected by a polygonal path lying entirely in S.

For sufficiency, we will show that if S is disconnected, then S is not poly-gonally connected. Let $S = A \cup B$, where A and B satisfy . Suppose that $\mathbf{X}_1 \in A$ and $\mathbf{X}_2 \in B$, and assume that there is a polygonal path in S connecting $\mathbf{X}_1$ to $\mathbf{X}_2$. Then some line segment L in this path must contain a point $\mathbf{Y}_1$ in A and a point $\mathbf{Y}_2$ in B. The line segment

$$\mathbf{X} = t\mathbf{Y}_2 + (1-t)\mathbf{Y}_1, \quad 0 \le t \le 1,$$

is part of L and therefore in S. Now define

$$\rho = \sup\{\tau\} t Y_2 + (1-t)\mathbf{Y}_1 \in A, \ 0 \le t \le \tau \le 1,$$

and let

$$\mathbf{X}_\rho = \rho\mathbf{Y}_2 + (1-\rho)\mathbf{Y}_1.$$

Then $\mathbf{X}_\rho \in \overline{A} \cap \overline{B}$. However, since $_AB $ and $\overline{A} \cap B = A \cap \overline{B} = \emptyset$, this is impossible. Therefore, the assumption that there is a polygonal path in S from $\mathbf{X}_1$ to $\mathbf{X}_2$ must be false.

For necessity, suppose that S is a connected open set and $\mathbf{X}_0 \in S$. Let A be the set consisting of $\mathbf{X}_0$ and the points in S can be connected to $\mathbf{X}_0$ by polygonal paths in S. Let B be set of points in S that cannot be connected to $\mathbf{X}_0$ by polygonal paths. If $\mathbf{Y}_0 \in S$, then S contains an ϵ-neighborhood $N_\epsilon(\mathbf{Y}_0)$ of $\mathbf{Y}_0$, since S is open. Any point $\mathbf{Y}_1$ in $N_\epsilon(\mathbf{Y}_0$ can be connected to $\mathbf{Y}_0$ by the line segment

$$\mathbf{X} = t\mathbf{Y}_1 + (1-t)\mathbf{Y}_0, \quad 0 \le t \le 1,$$

which lies in $N_\epsilon(\mathbf{Y}_0)$ (Lemma~) and therefore in S. This implies that $\mathbf{Y}_0$ can be connected to $\mathbf{X}_0$ by a polygonal path in S if and only if every member of $N_\epsilon(\mathbf{Y}_0)$ can also. Thus, $N_\epsilon(\mathbf{Y}_0) \subset A$ if $\mathbf{Y}_0 \in A$, and $N_\epsilon(\mathbf{Y}_0) \in B$ if $\mathbf{Y}_0 \in B$. Therefore, A and B are open. Since $A \cap B = \emptyset$, this implies that $A \cap \overline{B} = \overline{A} \cap B = \emptyset$ (Exercise~). Since A is nonempty ($\mathbf{X}_0 \in A$), it now follows that $B = \emptyset$, since if $B \neq \emptyset$, S would be disconnected (Definition~). Therefore, $A = S$, which completes the proof of necessity.

We did not use the assumption that S is open in the proof of sufficiency. In fact, we actually proved that any polygonally connected set, open or not, is connected. The converse is false. A set (not open) may be connected but not polygonally connected (Exercise~).

Our study of functions on $\R^n$ will deal mostly with functions whose domains are regions, defined next.

From Definition~, a sequence $\{\mathbf{X}_r\}$ of points in $\R^n$ converges to a limit $\overline{\mathbf{X}}$ if and only if for every $\epsilon > 0$ there is an integer K such that

$$|\mathbf{X}_r - \overline{\mathbf{X}}| < \epsilon \quad \text{if} \quad r \geq K.$$

The $\R^n$ definitions of divergence, boundedness, subsequence, and sums, differences, and constant multiples of sequences are analogous to those given in Sections 4.1 and 4.2 for the case where $n = 1$. Since $\R^n$ is not ordered for $n > 1$, monotonicity, limits inferior and superior of sequences in $\R^n$, and divergence to $\pm\infty$ are undefined for $n > 1$. Products and quotients of members of $\R^n$ are also undefined if $n > 1$.

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Several theorems from Sections~4.1 and 4.2 remain valid for sequences in $\R^n$, with proofs unchanged, provided that ``$|\quad|$'' is interpreted as distance in $\R^n$. (A trivial change is required: the subscript n, used in Sections~4.1 and 4.2 to identify the terms of the sequence, must be replaced, since n here stands for the dimension of the space.) These include Theorems~ (uniqueness of the limit), (boundedness of a convergent sequence), parts of (concerning limits of sums, differences, and constant multiples of convergent sequences), and (every subsequence of a convergent sequence converges to the limit of the sequence).

{

), and ???.}

We now study real-valued functions of n variables. We denote the domain of a function f by D_f and the value of f at a point $\mathbf{X} = (x_1, x_2, \dots, x_n)$ by $f(\mathbf{X})$ or $f(x_1, x_2, \dots, x_n)$. We continue the convention adopted in Section~2.1 for functions of one variable: If a function is defined by a formula such as

$$f(\mathbf{X})\ar = \left(1 - x_1^2 - x_2^2 - \cdots - x_n^2\right)^{1/2} \tag{5.1.5}$$

\arraytextor

$$g(\mathbf{X})\ar = \left(1 - x_1^2 - x_2^2 - \cdots - x_n^2\right)^{-1} \tag{5.1.6}$$

without specification of its domain, it is to be understood that its domain is the largest subset of $\R^n$ for which the formula defines a unique real number. Thus, in the absence of any other stipulation, the domain of f in is the closed n-ball $\{\mathbf{X}\}|\mathbf{X}| \leq 1$, while the domain of g in is the set $\{\mathbf{X}\}|\mathbf{X}| \neq 1$.

The main objective of this section is to study limits and continuity of functions of n variables. The proofs of many of the theorems here are similar to the proofs of their counterparts in Sections~2.1 and . We leave most of them to you.

Definition~ does not require that f be defined at $\mathbf{X}_0$, or even on a deleted neighborhood of $\mathbf{X}_0$.

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The following theorem is analogous to Theorem~2.1.3. We leave its proof to you (Exercise~).

When investigating whether a function has a limit at a point $\mathbf{X}_0$, no restriction can be made on the way in which $\mathbf{X}$ approaches $\mathbf{X}_0$, except that $\mathbf{X}$ must be in D_f. The next example shows that incorrect restrictions can lead to incorrect conclusions.

The sum, difference, and product of functions of n variables are defined in the same way as they are for functions of one variable (Definition~), and the proof of the next theorem is the same as the proof of Theorem~.

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We leave it to you to define $\lim_{|\mathbf{X}|\to\infty} f(\mathbf{X}) = \infty$ and $\lim_{|\mathbf{X}|\to\infty} f(\mathbf{X}) = -\infty$ (Exercise~).

We will continue the convention adopted in Section~2.1: `$\lim_{\mathbf{X}\to \mathbf{X}_0} f(\mathbf{X})$ exists'' means that $\lim_{\mathbf{X}\to\mathbf{X}_0} f(\mathbf{X})=L$, where $L$ is finite; to leave open the possibility that $L=\pm\infty$, we will say that` $\lim_{\mathbf{X}\to\mathbf{X}_0} f(\mathbf{X})$ exists in the extended reals.'' A similar convention applies to limits as $|\mathbf{X}| \to \infty$.

Theorem~ remains valid if $\lim_{\mathbf{X}\to\mathbf{X}_0}$'' is replaced by $\lim_{|\mathbf{X}|\to\infty}$,'' provided that D is unbounded. Moreover, , , and are valid in either version of Theorem~ if either or both of L_1 and L_2 is infinite, provided that their right sides are not indeterminate, and remains valid if $L_2 \neq 0$ and L_1/L_2 is not indeterminate.

We now define continuity for functions of n variables. The definition is quite similar to the definition for functions of one variable.

The next theorem follows from this and Definition~.

In applying this theorem when $\mathbf{X}_0 \in D_f^0$, we will usually omit ``and $\mathbf{X} \in D_f$,'' it being understood that $S_\delta(\mathbf{X}_0) \subset D_f$.

We will say that f is {} S if f is continuous at every point of S.

Theorem~ implies the next theorem, which is analogous to Theorem~ and, like the latter, permits us to investigate continuity of a given function by regarding the function as the result of addition, subtraction, multiplication, and division of simpler functions.

Suppose that $g_1, g_2, , g_n$ are real-valued functions defined on a subset T of $\mathbb{R}^m$, and define the {} $\mathbf{G}$ on T by

$$\mathbf{G}(\mathbf{U}) = (g_1(\mathbf{U}), g_2(\mathbf{U}), \dots, g_n(\mathbf{U})), \quad \mathbf{U} \in T.$$

Then $g_1, g_2, , g_n$ are the {} of $\mathbf{G} = (g_1, g_2, \dots, g_n)$. We say that

$$\lim_{\mathbf{U}\to\mathbf{U}_0} \mathbf{G}(\mathbf{U}) = \mathbf{L} = (L_1, L_2, \dots, L_n)$$

if

$$\lim_{\mathbf{U}\to\mathbf{U}_0} g_i(\mathbf{U}) = L_i, \quad 1 \le i \le n,$$

and that $\mathbf{G}$ is {} at $\mathbf{U}_0$ if $g_1, g_2, , g_n$ are each continuous at $\mathbf{U}_0$.

The next theorem follows from Theorem~ and Definitions~ and . We omit the proof.

The following theorem on the continuity of a composite function is analogous to Theorem~.

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Suppose that $\epsilon > 0$. Since f is continuous at $\mathbf{X}_0 = \mathbf{G}(\mathbf{U}_0)$, there is an $\epsilon_1 > 0$ such that

$$|f(\mathbf{X}) - f(\mathbf{G}(\mathbf{U}_0))| < \epsilon \tag{5.1.7}$$

if

$$|\mathbf{X} - \mathbf{G}(\mathbf{U}_0)| < \epsilon_1 \quad \text{and} \quad \mathbf{X} \in D_f. \tag{5.1.8}$$

Since $\mathbf{G}$ is continuous at $\mathbf{U}_0$, there is a $\delta > 0$ such that

$$|\mathbf{G}(\mathbf{U}) - \mathbf{G}(\mathbf{U}_0)| < \epsilon_1 \quad \text{if} \quad |\mathbf{U} - \mathbf{U}_0| < \delta \quad \text{and} \quad \mathbf{U} \in D_{\mathbf{G}}.$$

By taking $\mathbf{X} = \mathbf{G}(\mathbf{U})$ in and , we see that

$$|h(\mathbf{U}) - h(\mathbf{U}_0)| = |f(\mathbf{G}(\mathbf{U}) - f(\mathbf{G}(\mathbf{U}_0))| < \epsilon$$

if

$$|\mathbf{U} - \mathbf{U}_0| < \delta \quad \text{and} \quad \mathbf{U} \in T.$$

-.4em The definitions of {}, and {} on a set S are the same for functions of n variables as for functions of one variable, as are the definitions of {} and {} of a function on a set S (Section 2.2). The proofs of the next two theorems are similar to those of Theorems~ and (Exercises~ and).

The next theorem is analogous to Theorem~.

If there is no such $\mathbf{C}$, then $S = R \cup T$, where

$$R\ar = \{\mathbf{X}\}\mathbf{X} \in S \text{ and } f(\mathbf{X}) < u$$
$$\arraytextand$$
$$T\ar = \{\mathbf{X}\}\mathbf{X} \in S \text{ and } f(\mathbf{X}) > u.$$

If $\mathbf{X}_0 \in R$, the continuity of f implies that there is a $\delta > 0$ such that $f(\mathbf{X}) < u$ if $|\mathbf{X} - \mathbf{X}_0| < \delta$ and $\mathbf{X} \in S$. This means that $\mathbf{X}_0 \notin \overline{T}$. Therefore, $R \cap \overline{T} = \emptyset$. Similarly, $\overline{R} \cap T = \emptyset$. Therefore, S is disconnected (Definition~), which contradicts the assumption that S is a region (Exercise~). Hence, we conclude that $f(\mathbf{C}) = u$ for some $\mathbf{C}$ in S.

-.4em The definition of uniform continuity for functions of n variables is the same as for functions of one variable; f is uniformly continuous on a subset S of its domain in $\R^n$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$|f(\mathbf{X}) - f(\mathbf{X}')| < \epsilon$$

whenever $|\mathbf{X} - \mathbf{X}'| < \delta$ and $\mathbf{X}, \mathbf{X}' \in S$. We emphasize again that δ must depend only on ϵ and S, and not on the particular points $\mathbf{X}$ and $\mathbf{X}'$.

The proof of the next theorem is analogous to that of Theorem~. We leave it to you (Exercise~).

{}

To say that a function of one variable has a derivative at x_0 is the same as to say that it is differentiable at x_0. The situation is not so simple for a function f of more than one variable. First, there is no specific number that can be called {} derivative of f at a point $\mathbf{X}_0$ in $\R^n$. In fact, there are infinitely many numbers, called the {} $\mathbf{X}_0$ (defined below), that are analogous to the derivative of a function of one variable. Second, we will see that the existence of directional derivatives at $\mathbf{X}_0$ does not imply that f is differentiable at $\mathbf{X}_0$, if differentiability at $\mathbf{X}_0$ is to imply (as it does for functions of one variable) that $f(\mathbf{X}) - f(\mathbf{X}_0)$ can be approximated well near $\mathbf{X}_0$ by a simple linear function, or even that f is continuous at $\mathbf{X}_0$.

We will now define directional derivatives and partial derivatives of functions of several variables. However, we will still have occasion to refer to derivatives of functions of one variable. We will call them {} derivatives when we wish to distinguish between them and the partial derivatives that we are about to define.

-2em2em

The directional derivatives that we are most interested in are those in the directions of the unit vectors

$$\mathbf{E}_1 = (1, 0, \dots, 0), \quad \mathbf{E}_2 = (0, 1, 0, \dots, 0), \dots, \quad \mathbf{E}_n = (0, \dots, 0, 1).$$

(All components of $\mathbf{E}_i$ are zero except for the ith, which is 1.) Since $\mathbf{X}$ and $\mathbf{X} + t\mathbf{E}_i$ differ only in the ith coordinate, $\partial f(\mathbf{X})/\partial \mathbf{E}_i$ is called the {}. It is also denoted by $\partial f(\mathbf{X})/\partial x_i$ or $f_{x_i}(\mathbf{X})$; thus,

$$\frac{\partial f(\mathbf{X})}{\partial x_1} = f_{x_1}(\mathbf{X}) = \lim_{t \to 0} \frac{f(x_1 + t, x_2, \dots, x_n) - f(x_1, x_2, \dots, x_n)}{t},$$

$$\frac{\partial f(\mathbf{X})}{\partial x_i} = f_{x_i}(\mathbf{X}) = \lim_{t \to 0} \frac{f(x_1, \dots, x_{i-1}, x_i + t, x_{i+1}, \dots, x_n) - f(x_1, x_2, \dots, x_n)}{t}$$

if $2 \le i \le n$, and

$$\frac{\partial f(\mathbf{X})}{\partial x_n} = f_{x_n}(\mathbf{X}) = \lim_{t \to 0} \frac{f(x_1, \dots, x_{n-1}, x_n + t) - f(x_1, \dots, x_{n-1}, x_n)}{t},$$

if the limits exist.

If we write $\mathbf{X} = (x, y)$, then we denote the partial derivatives accordingly; thus,

$$\frac{\partial f(x,y)}{\partial x} \ar = f_x(x,y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$
$$\arraytext{and}$$
$$\frac{\partial f(x,y)}{\partial y} \ar = f_y(x,y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}.$$

It can be seen from these definitions that to compute $f_{x_i}(\mathbf{X})$ we simply differentiate f with respect to x_i according to the rules for ordinary differentiation, while treating the other variables as constants.

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-3em3em

The next theorem follows from the rule just given for calculating partial derivatives.

If $f_{x_i}(\mathbf{X})$ exists at every point of a set D, then it defines a function f_{x_i} on D. If this function has a partial derivative with respect to x_j on a subset of D, we denote the partial derivative by

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)=\frac{\partial^2 f}{\partial x_j \partial x_i}=f_{x_i x_j}.$$

Similarly,

$$\frac{\partial}{\partial x_k}\left(\frac{\partial^2 f}{\partial x_j \partial x_i}\right)=\frac{\partial^3 f}{\partial x_k \partial x_j \partial x_i}=f_{x_i x_j x_k}.$$

The function obtained by differentiating f successively with respect to $x_{i_1}, x_{i_2}, \ldots, x_{i_r}$ is denoted by

$$\frac{\partial^r f}{\partial x_{i_r} \partial x_{i_{r-1}} \cdots \partial x_{i1}}=f_{x_{i_1}} \cdots x_{i_{r-1}} x_{i_r};$$

it is an {}.

This example shows that $f_{xy}(\mathbf{X}_0)$ and $f_{yx}(\mathbf{X}_0)$ may differ. However, the next theorem shows that they are equal if f satisfies a fairly mild condition.

Suppose that $\epsilon > 0$. Choose $\delta > 0$ so that the open square

$$S_\delta = \{(x,y)\} | x-x_0| < \delta, |y-y_0| < \delta$$

is in N and

$$|f_{xy}(\hat{x},\hat{y}) - f_{xy}(x_0,y_0)| < \epsilon \qquad \text{if} \quad (\hat{x},\hat{y}) \in S_\delta. \tag{5.1.9}$$

This is possible because of the continuity of f_{xy} at (x_0, y_0). The function

$$A(h,k) = f(x_0+h, y_0+k) - f(x_0+h, y_0) - f(x_0, y_0+k) + f(x_0, y_0) \tag{5.1.10}$$

is defined if $-\delta < h,\ k < \delta$; moreover,

$$A(h,k) = \phi(x_0+h) - \phi(x_0), \tag{5.1.11}$$

where

$$\phi(x) = f(x, y_0+k) - f(x, y_0).$$

Since

$$\phi'(x) = f_x(x, y_0+k) - f_x(x, y_0), \quad |x-x_0| < \delta,$$

and the mean value theorem imply that

$$A(h,k) = [f_x(\hat{x}, y_0+k) - f_x(\hat{x}, y_0)]\, h, \tag{5.1.12}$$

where $\hat{x}$ is between x_0 and x_0+h. The mean value theorem, applied to $f_x(\hat{x}, y)$ (where $\hat{x}$ is regarded as constant), also implies that

$$f_x(\hat{x}, y_0+k) - f_x(\hat{x}, y_0) = f_{xy}(\hat{x}, \hat{y})k,$$

where $\hat{y}$ is between y_0 and y_0+k. From this and ,

$$A(h,k) = f_{xy}(\hat{x},\hat{y})hk.$$

Now implies that

$$\left|\frac{A(h,k)}{hk} - f_{xy}(x_0,y_0)\right| = |f_{xy}(\hat{x},\hat{y}) - f_{xy}(x_0,y_0)| < \epsilon \quad \text{if} \quad 0 < |h|, |k| < \delta. \tag{5.1.13}$$

Since implies that

$$\lim_{k\to 0}\frac{A(h,k)}{hk} \ar = \lim_{k\to 0}\frac{f(x_0+h,y_0+k)-f(x_0+h,y_0)}{hk} \ar - \lim_{k\to 0}\frac{f(x_0,y_0+k)-f(x_0,y_0)}{hk} \ar = \frac{f_y(x_0+h,y_0)-f_y(x_0,y_0)}{h},$$

it follows from that

$$\left|\frac{f_y(x_0+h,y_0)-f_y(x_0,y_0)}{h}-f_{xy}(x_0,y_0)\right|\le\epsilon \quad \text{if} \quad 0<|h|<\delta.$$

Taking the limit as $h\to 0$ yields

$$|f_{yx}(x_0,y_0)-f_{xy}(x_0,y_0)|\le\epsilon.$$

Since ϵ is an arbitrary positive number, this proves .

Theorem~ implies the following theorem. We leave the proof to you (Exercises~ and).

For example, if f satisfies the hypotheses of Theorem~ with $k=4$ at a point $\mathbf{X}_0$ in $\R^n$ ($n\ge 2$), then

$$f_{xxyy}(\mathbf{X}_0)=f_{xyxy}(\mathbf{X}_0)=f_{xyyx}(\mathbf{X}_0)=f_{yyxx}(\mathbf{X}_0)=f_{yxyx}(\mathbf{X}_0)=f_{yxxy}(\mathbf{X}_0),$$

and their common value is denoted by

$$\frac{\partial^4 f(\mathbf{X}_0)}{\partial x^2\partial y^2}.$$

It can be shown (Exercise~) that if f is a function of $(x_1,x_2,\dots,x_n)$ and $(r_1,r_2,\dots,r_n)$ is a fixed ordered n-tuple that satisfies and , then the number of partial derivatives $f_{x_{i_1}x_{i_2}\cdots x_{i_r}}$ that involve differentiation r_i times with respect to x_i, $1\le i\le n$, equals the {}

$$\frac{r!}{r_1!r_2!\cdots r_n!}.$$

A function of several variables may have first-order partial derivatives at a point $\mathbf{X}_0$ but fail to be continuous at $\mathbf{X}_0$. For example, if

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then

$$f_x(0,0) \ar = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{0-0}{h}=0 \arraytextand f_y(0,0) \ar = \lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k}=\lim_{k\to 0}\frac{0-0}{k}=0,$$

but f is not continous at $(0,0)$. (See Examples~ and .) Therefore, if differentiability of a function of several variables is to be a stronger property than continuity, as it is for functions of one variable, the definition of differentiability must require more than the existence of first partial derivatives. Exercise~ characterizes differentiability of a function f of one variable in a way that suggests the proper generalization: f is differentiable at x_0 if and only if

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)-m(x-x_0)}{x-x_0}=0$$

for some constant m, in which case $m=f'(x_0)$.

The generalization to functions of n variables is as follows.

From , $m_1=f_x(x_0,y_0)$ and $m_2=f_y(x_0,y_0)$ in Example~. The next theorem shows that this is not a coincidence.

Let i be a given integer in $\{1,2,\dots,n\}$. Let $\mathbf{X}=\mathbf{X}_0+t\mathbf{E}_i$, so that $x_i=x_{i0}+t$, $x_j=x_{j0}$ if $j\ne i$, and $|\mathbf{X}-\mathbf{X}_0|=|t|$. Then and the differentiability of f at $\mathbf{X}_0$ imply that

$$\lim_{t\to 0}\frac{f(\mathbf{X}_0+t\mathbf{E}_i)-f(\mathbf{X}_0)-m_it}{t}=0.$$

Hence,

$$\lim_{t\to 0}\frac{f(\mathbf{X}_0+t\mathbf{E}_i)-f(\mathbf{X}_0)}{t}=m_i.$$

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This proves , since the limit on the left is $f_{x_i}(\mathbf{X}_0)$, by definition.

A {} is a function of the form

$$L(\mathbf{X})=m_1x_1+m_2x_2+\cdots+m_nx_n, \tag{5.1.14}$$

where m_1, m_2, , m_n are constants. From Definition~, f is differentiable at $\mathbf{X}_0$ if and only if there is a linear function L such that $f(\mathbf{X})-f(\mathbf{X}_0)$ can be approximated so well near $\mathbf{X}_0$ by

$$L(\mathbf{X})-L(\mathbf{X}_0)=L(\mathbf{X}-\mathbf{X}_0)$$

that

$$f(\mathbf{X})-f(\mathbf{X}_0)=L(\mathbf{X}-\mathbf{X}_0)+E(\mathbf{X})(|\mathbf{X}-\mathbf{X}_0|), \tag{5.1.15}$$

where

$$\lim_{\mathbf{X}\to\mathbf{X}_0}E(\mathbf{X})=0. \tag{5.1.16}$$

6pt

From and Schwarz's inequality,

$$|L(\mathbf{X}-\mathbf{X}_0)|\le M|\mathbf{X}-\mathbf{X}_0|,$$

where

$$M=(m_1^2+m_2^2+\cdots+m_n^2)^{1/2}.$$

This and imply that

$$|f(\mathbf{X})-f(\mathbf{X}_0)|\le(M+|E(\mathbf{X})|)|\mathbf{X}-\mathbf{X}_0|,$$

which, with , implies that f is continuous at $\mathbf{X}_0$.

Theorem~ implies that the function f defined by is not differentiable at $(0,0)$, since it is not continuous at $(0,0)$. However, $f_x(0,0)$ and $f_y(0,0)$ exist, so the converse of Theorem~ is false; that is, a function may have partial derivatives at a point without being differentiable at the point.

Theorem~ implies that if f is differentiable at $\mathbf{X}_0$, then there is exactly one linear function L that satisfies and :

$$L(\mathbf{X})=f_{x_1}(\mathbf{X}_0)x_1+f_{x_2}(\mathbf{X}_0)x_2+\cdots+f_{x_n}(\mathbf{X}_0)x_n.$$

This function is called the {}. We will denote it by $d_{\mathbf{X}_0}f$ and its value by $(d_{\mathbf{X}_0}f)(\mathbf{X})$; thus,

$$(d_{\mathbf{X}_0}f)(\mathbf{X})=f_{x_1}(\mathbf{X}_0)x_1+f_{x_2}(\mathbf{X}_0)x_2+\cdots+f_{x_n}(\mathbf{X}_0)x_n. \tag{5.1.17}$$

In terms of the differential, can be rewritten as

$$\lim_{\mathbf{X}\to\mathbf{X}_0}\frac{f(\mathbf{X})-f(\mathbf{X}_0)-(d_{\mathbf{X}_0}f)(\mathbf{X}-\mathbf{X}_0)}{|\mathbf{X}-\mathbf{X}_0|}=0.$$

For convenience in writing $d_{\mathbf{X}_0}f$, and to conform with standard notation, we introduce the function dx_i, defined by

$$dx_i(\mathbf{X})=x_i;$$

that is, dx_i is the function whose value at a point in $\R^n$ is the ith coordinate of the point. It is the differential of the function $g_i(\mathbf{X})=x_i$. From ,

$$d_{\mathbf{X}_0}f=f_{x_1}(\mathbf{X}_0)\,dx_1+f_{x_2}(\mathbf{X}_0\,dx_2+\cdots+f_{x_n}(\mathbf{X}_0)\,dx_n. \tag{5.1.18}$$

If we write $\mathbf{X} = (x, y, \dots,)$, then we write

$$d_{\mathbf{X}_0} f = f_x(\mathbf{X}_0)\, dx + f_y(\mathbf{X}_0)\, dy + \cdots,$$

where dx, dy, are the functions defined by

$$dx(\mathbf{X}) = x, \quad dy(\mathbf{X}) = y, \dots$$

When it is not necessary to emphasize the specific point $\mathbf{X}_0$, can be written more simply as

$$df = f_{x_1}\, dx_1 + f_{x_2}\, dx_2 + \cdots + f_{x_n}\, dx_n.$$

When dealing with a specific function at an arbitrary point of its domain, we may use the hybrid notation

$$df = f_{x_1}(\mathbf{X})\, dx_1 + f_{x_2}(\mathbf{X})\, dx_2 + \cdots + f_{x_n}(\mathbf{X})\, dx_n.$$

Unfortunately, the notation for the differential is so complicated that it obscures the simplicity of the concept. The peculiar symbols df, dx, dy, etc., were introduced in the early stages of the development of calculus to represent very small (``infinitesimal'') increments in the variables. However, in modern usage they are not quantities at all, but linear functions. This meaning of the symbol dx differs from its meaning in $\int_a^b f(x)\, dx$, where it serves merely to identify the variable of integration; indeed, some authors omit it in the latter context and write simply $\int_a^b f$.

Theorem~ implies the following lemma, which is analogous to Lemma~. We leave the proof to you (Exercise~).

Theorems~ and and the definition of the differential imply the following \theorem.

The next theorem provides a widely applicable sufficient condition for differentiability.

Let $\mathbf{X}_0 = (x_{10}, x_{20}, \dots, x_{n0})$ and suppose that $\epsilon > 0$. Our assumptions imply that there is a $\delta > 0$ such that $f_{x_1}, f_{x_2}, \dots, f_{x_n}$ are defined in the n-ball

$$S_\delta(\mathbf{X}_0) = \{\mathbf{X} | |\mathbf{X} - \mathbf{X}_0| < \delta$$

and

$$|f_{x_j}(\mathbf{X}) - f_{x_j}(\mathbf{X}_0)| < \epsilon \quad \text{if} \quad |\mathbf{X} - \mathbf{X}_0| < \delta, \quad 1 \le j \le n. \tag{5.1.19}$$

Let $\mathbf{X} = (x_1, x, \dots, x_n)$ be in $S_\delta(\mathbf{X}_0)$. Define

$$\mathbf{X}_j = (x_1, \dots, x_j, x_{j+1,0}, \dots, x_{n0}), \quad 1 \le j \le n-1,$$

and $\mathbf{X}_n = \mathbf{X}$. Thus, for $1 \le j \le n$, $\mathbf{X}_j$ differs from $\mathbf{X}_{j-1}$ in the jth component only, and the line segment from $\mathbf{X}_{j-1}$ to $\mathbf{X}_j$ is in $S_\delta(\mathbf{X}_0)$. Now write

$$f(\mathbf{X}) - f(\mathbf{X}_0) = f(\mathbf{X}_n) - f(\mathbf{X}_0) = \sum_{j=1}^{n} [f(\mathbf{X}_j) - f(\mathbf{X}_{j-1})], \tag{5.1.20}$$

and consider the auxiliary functions

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where, in each case, all variables except t are temporarily regarded as constants. Since

$$f(\mathbf{X}_j) - f(\mathbf{X}_{j-1}) = g_j(x_j) - g_j(x_{j0}),$$

the mean value theorem implies that

$$f(\mathbf{X}_j) - f(\mathbf{X}_{j-1}) = g_j'(\tau_j)(x_j - x_{j0}),$$

where τ_j is between x_j and x_{j0}. From ,

$$g_j'(\tau_j) = f_{x_j}(\widehat{\mathbf{X}}_j),$$

where $\widehat{\mathbf{X}}_j$ is on the line segment from $\mathbf{X}_{j-1}$ to $\mathbf{X}_j$. Therefore,

$$f(\mathbf{X}_j)-f(\mathbf{X}_{j-1})=f_{x_j}(\widehat{\mathbf{X}}_j)(x_j-x_{j0}),$$

and implies that

$$f(\mathbf{X})-f(\mathbf{X}_0)\ar=\sum_{j=1}^n f_{x_j}(\widehat{\mathbf{X}}_j)(x_j-x_{j0})$$
$$\ar=\sum_{j=1}^n f_{x_j}(\mathbf{X}_0)(x_j-x_{j0})+\sum_{j=1}^n [f_{x_j}(\widehat{\mathbf{X}}_j)-f_{x_j}(\mathbf{X}_0)](x_j-x_{j0}).$$

From this and ,

$$\left|f(\mathbf{X})-f(\mathbf{X}_0)-\sum_{j=1}^n f_{x_j}(\mathbf{X}_0)(x_j-x_{j0})\right|\le\epsilon\sum_{j=1}^n|x_j-x_{j0}|\le n\epsilon|\mathbf{X}-\mathbf{X}_0|,$$

which implies that f is differentiable at $\mathbf{X}_0$.

We say that f is {} on a subset S of $\R^n$ if S is contained in an open set on which f_{x_1}, f_{x_2}, , f_{x_n} are continuous. Theorem~ implies that such a function is differentiable at each $\mathbf{X}_0$ in S.

In Section~2.3 we saw that if a function f of one variable is differentiable at x_0, then the curve $y=f(x)$ has a tangent line

$$y=T(x)=f(x_0)+f'(x_0)(x-x_0)$$

that approximates it so well near x_0 that

$$\lim_{x\to x_0}\frac{f(x)-T(x)}{x-x_0}=0.$$

Moreover, the tangent line is the ``limit'' of the secant line through the points $(x_1,f(x_0))$ and $(x_0,f(x_0))$ as x_1 approaches x_0.

6pt

12pt Differentiability of a function of n variables has an analogous geometric interpretation. We will illustrate it for $n=2$. If f is defined in a region D in $\R^2$, then the set of points (x,y,z) such that

$$z=f(x,y),\quad (x,y)\in D, \tag{5.1.21}$$

is a {} in $\R^3$ (Figure~).

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If f is differentiable at $\mathbf{X}_0=(x_0,y_0)$, then the plane

$$z=T(x,y)=f(\mathbf{X}_0)+f_x(\mathbf{X}_0)(x-x_0)+f_y(\mathbf{X}_0)(y-y_0) \tag{5.1.22}$$

intersects the surface at $(x_0,y_0,f(x_0,y_0))$ and approximates the surface so well near (x_0,y_0) that

$$\lim_{(x,y)\to(x_0,y_0)}\frac{f(x,y)-T(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}}=0$$

(Figure~). Moreover, is the only plane in $\R^3$ with these properties (Exercise~). We say that this plane is {} $(x_0,y_0,f(x_0,y_0))$. We will now show that it is the `limit'' of` secant planes'' associated with the surface $z=f(x,y)$, just as a tangent line to a curve $y=f(x)$ in $\R^3$ is the limit of secant lines to the curve (Section~2.3).

Let $\mathbf{X}_i=(x_i,y_i)$ $(i=1,2,3)$. The equation of the ``secant plane'' through the points $(x_i,y_i,f(x_i,y_i))$ $(i=1,2,3)$ on the surface $z=f(x,y)$ (Figure~) is of the form

$$z=f(\mathbf{X}_0)+A(x-x_0)+B(y-y_0), \tag{5.1.23}$$

where A and B satisfy the system

$$f(\mathbf{X}_1)\ar=f(\mathbf{X}_0)+A(x_1-x_0)+B(y_1-y_0),$$
$$f(\mathbf{X}_2)\ar=f(\mathbf{X}_0)+A(x_2-x_0)+B(y_2-y_0).$$

Solving for A and B yields

$$A\ar = \frac{(f(\mathbf{X}_1)-f(\mathbf{X}_0))(y_2-y_0)-(f(\mathbf{X}_2)-f(\mathbf{X}_0))(y_1-y_0)}{(x_1-x_0)(y_2-y_0)-(x_2-x_0)(y_1-y_0)} \tag{5.1.24}$$

$$\arraytext{and}$$

$$B\ar = \frac{(f(\mathbf{X}_2)-f(\mathbf{X}_0))(x_1-x_0)-(f(\mathbf{X}_1)-f(\mathbf{X}_0))(x_2-x_0)}{(x_1-x_0)(y_2-y_0)-(x_2-x_0)(y_1-y_0)} \tag{5.1.25}$$

if

$$(x_1-x_0)(y_2-y_0)-(x_2-x_0)(y_1-y_0)\neq 0, \tag{5.1.26}$$

which is equivalent to the requirement that $\mathbf{X}_0$, $\mathbf{X}_1$, and $\mathbf{X}_2$ do not lie on a line (Exercise~). If we write

$$\mathbf{X}_1=\mathbf{X}_0+t\mathbf{U} \quad \text{and} \quad \mathbf{X}_2=\mathbf{X}_0+t\mathbf{V},$$

where $\mathbf{U}=(u_1,u_2)$ and $\mathbf{V}=(v_1,v_2)$ are fixed nonzero vectors (Figure~), then , , and take the more convenient forms

$$A\ar = \frac{\dst\frac{f(\mathbf{X}_0+t\mathbf{U})-f(\mathbf{X}_0)}{t}v_2-\frac{f(\mathbf{X}_0+t\mathbf{V})-f(\mathbf{X}_0)}{t}u_2}{u_1v_2-u_2v_1}, \tag{5.1.27}$$

$$B\ar = \frac{\dst\frac{f(\mathbf{X}_0+t\mathbf{V})-f(\mathbf{X}_0)}{t}u_1-\frac{f(\mathbf{X}_0+t\mathbf{U})-f(\mathbf{X}_0)}{t}v_1}{u_1v_2-u_2v_1}, \tag{5.1.28}$$

and

$$u_1v_2-u_2v_1\neq 0.$$

6pt

12pt

If f is differentiable at $\mathbf{X}_0$, then

$$f(\mathbf{X})-f(\mathbf{X}_0)=f_x(\mathbf{X}_0)(x-x_0)+f_y(\mathbf{X}_0)(y-y_0)+\epsilon(\mathbf{X})|\mathbf{X}-\mathbf{X}_0|, \tag{5.1.29}$$

where

$$\lim_{\mathbf{X}\to\mathbf{X}_0}\epsilon(\mathbf{X})=0. \tag{5.1.30}$$

Substituting first $\mathbf{X}=\mathbf{X}_0+t\mathbf{U}$ and then $\mathbf{X}=\mathbf{X}_0+t\mathbf{V}$ in and dividing by t yields

$$\frac{f(\mathbf{X}_0+t\mathbf{U})-f(\mathbf{X}_0)}{t}=f_x(\mathbf{X}_0)u_1+f_y(\mathbf{X}_0)u_2+E_1(t)|\mathbf{U}| \tag{5.1.31}$$

and

$$\frac{f(\mathbf{X}_0+t\mathbf{V})-f(\mathbf{X}_0)}{t}=f_x(\mathbf{X}_0)v_1+f_y(\mathbf{X}_0)v_2+E_2(t)|\mathbf{V}|, \tag{5.1.32}$$

where

$$E_1(t)=\epsilon(\mathbf{X}_0+t\mathbf{U})|t|/t \quad \text{and} \quad E_2(t)=\epsilon(\mathbf{X}_0+t\mathbf{V})|t|/t,$$

so

$$\lim_{t\to 0}E_i(t)=0, \quad i=1,2, \tag{5.1.33}$$

because of . Substituting and into and yields

$$A=f_x(\mathbf{X}_0)+\Delta_1(t), \quad B=f_y(\mathbf{X}_0)+\Delta_2(t), \tag{5.1.34}$$

where

$$\Delta_1(t)=\frac{v_2|\mathbf{U}|E_1(t)-u_2|\mathbf{V}|E_2(t)}{u_1v_2-u_2v_1}$$

and

$$\Delta_2(t) = \frac{u_1|\mathbf{V}|E_2(t) - v_1|\mathbf{U}|E_1(t)}{u_1v_2 - u_2v_1},$$

so

$$\lim_{t\to 0}\Delta_i(t) = 0, \quad i = 1, 2, \tag{5.1.35}$$

because of .

From and , the equation of the secant plane is

$$z = f(\mathbf{X}_0) + [f_x(\mathbf{X}_0) + \Delta_1(t)](x - x_0) + [f_y(\mathbf{X}_0) + \Delta_2(t)](y - y_0).$$

Therefore, because of , the secant plane ``approaches'' the tangent plane as t approaches zero.

We say that $\mathbf{X}_0$ is a {} of f if there is a $\delta > 0$ such that

$$f(\mathbf{X}) - f(\mathbf{X}_0)$$

does not change sign in $S_\delta(\mathbf{X}_0) \cap D_f$. More specifically, $\mathbf{X}_0$ is a {} if

$$f(\mathbf{X}) \le f(\mathbf{X}_0)$$

or a {} if

$$f(\mathbf{X}) \ge f(\mathbf{X}_0)$$

for all $\mathbf{X}$ in $S_\delta(\mathbf{X}_0) \cap D_f$.

The next theorem is analogous to Theorem~.

Let

$$\mathbf{E}_1 = (1, 0, \dots, 0), \quad \mathbf{E}_2 = (0, 1, 0, \dots, 0), \dots, \quad \mathbf{E}_n = (0, 0, \dots, 1),$$

and

$$g_i(t) = f(\mathbf{X}_0 + t\mathbf{E}_i), \quad 1 \le i \le n.$$

Then g_i is differentiable at $t = 0$, with

$$g_i'(0) = f_{x_i}(\mathbf{X}_0)$$

(Definition~). Since $\mathbf{X}_0$ is a local extreme point of f, $t_0 = 0$ is a local extreme point of g_i. Now Theorem~ implies that $g_i'(0) = 0$, and this implies .

The converse of Theorem~ is false, since may hold at a point $\mathbf{X}_0$ that is not a local extreme point of f. For example, let $\mathbf{X}_0 = (0, 0)$ and

$$f(x, y) = x^3 + y^3.$$

We say that a point $\mathbf{X}_0$ where holds is a {} of f. Thus, if f is defined in a neighborhood of a local extreme point $\mathbf{X}_0$, then $\mathbf{X}_0$ is a critical point of f; however, a critical point need not be a local extreme point of f.

The use of Theorem~ for finding local extreme points is covered in calculus, so we will not pursue it here.

We now consider the problem of differentiating a composite function

$$h(\mathbf{U}) = f(\mathbf{G}(\mathbf{U})),$$

where $\mathbf{G} = (g_1, g_2, \dots, g_n)$ is a vector-valued function, as defined in Section~5.2. We begin with the following definition.

We need the following lemma to prove the main result of the section.

Since g_1, g_2, , g_n are differentiable at $\mathbf{U}_0$, applying Lemma~ to g_i shows that

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where

$$\lim_{\mathbf{U}\to\mathbf{U}_0} E_i(\mathbf{U}) = 0, \quad 1 \le i \le n. \tag{5.1.36}$$

From Schwarz's inequality,

$$|g_i(\mathbf{U}) - g_i(\mathbf{U}_0)| \le (M_i + |E_i(\mathbf{U})|)|\mathbf{U} - \mathbf{U}_0|,$$

where

$$M_i = \left(\sum_{j=1}^{m} \left(\frac{\partial g_i(\mathbf{U}_0)}{\partial u_j}\right)^2\right)^{1/2}.$$

Therefore,

$$\frac{|\mathbf{G}(\mathbf{U}) - \mathbf{G}(\mathbf{U}_0)|}{|\mathbf{U} - \mathbf{U}_0|} \le \left(\sum_{i=1}^{n} (M_i + |E_i(\mathbf{U})|)^2\right)^{1/2}.$$

From ,

$$\lim_{\mathbf{U}\to\mathbf{U}_0} \left(\sum_{i=1}^{n} (M_i + |E_i(\mathbf{U})|)^2\right)^{1/2} = \left(\sum_{i=1}^{n} M_i^2\right)^{1/2} = M,$$

which implies the conclusion.

The following theorem is analogous to Theorem~.

We leave it to you to show that $\mathbf{U}_0$ is an interior point of the domain of h (Exercise~), so it is legitimate to ask if h is differentiable at $\mathbf{U}_0$.

Let $\mathbf{X}_0 = (x_{10}, x_{20}, \dots, x_{n0})$. Note that

$$x_{i0} = g_i(\mathbf{U}_0), \quad 1 \le i \le n,$$

by assumption. Since f is differentiable at $\mathbf{X}_0$, Lemma~ implies that

$$f(\mathbf{X}) - f(\mathbf{X}_0) = \sum_{i=1}^{n} f_{x_i}(\mathbf{X}_0)(x_i - x_{i0}) + E(\mathbf{X})|\mathbf{X} - \mathbf{X}_0|, \tag{5.1.37}$$

where

$$\lim_{\mathbf{X}\to\mathbf{X}_0} E(\mathbf{X}) = 0.$$

Substituting $\mathbf{X} = \mathbf{G}(\mathbf{U})$ and $\mathbf{X}_0 = \mathbf{G}(\mathbf{U}_0)$ in and recalling yields

$$h(\mathbf{U}) - h(\mathbf{U}_0) = \dst \sum_{i=1}^{n} f_{x_i}(\mathbf{X}_0)(g_i(\mathbf{U}) - g_i(\mathbf{U}_0)) + E(\mathbf{G}(\mathbf{U}))|\mathbf{G}(\mathbf{U}) - \mathbf{G}(\mathbf{U}_0)|. \tag{5.1.38}$$

Substituting into yields

$$h(\mathbf{U}) - h(\mathbf{U}_0) \ar = \dst \sum_{i=1}^{n} f_{x_i}(\mathbf{X}_0)(d_{\mathbf{U}_0} g_i)(\mathbf{U} - \mathbf{U}_0) + \dst\left(\sum_{i=1}^{n} f_{x_i}(\mathbf{X}_0) E_i(\mathbf{U})\right)|\mathbf{U} - \mathbf{U}_0|$$

$$\ar + E(\mathbf{G}(\mathbf{U}))|\mathbf{G}(\mathbf{U}) - \mathbf{G}(\mathbf{U}_0)|.$$

Since

$$\lim_{\mathbf{U}\to\mathbf{U}_0} E(\mathbf{G}(\mathbf{U})) = \lim_{\mathbf{X}\to\mathbf{X}_0} E(\mathbf{X}) = 0,$$

and Lemma~ imply that

$$\frac{h(\mathbf{U}) - h(\mathbf{U}_0) - \dst \sum_{i=1}^{n} f_{x_i}(\mathbf{X}_0 d_{\mathbf{U}_0} g_i(\mathbf{U} - \mathbf{U}_0)}{|\mathbf{U} - \mathbf{U}_0|} = 0.$$

Therefore, h is differentiable at $\mathbf{U}_0$, and $d_{\mathbf{U}_0}h$ is given by .

Substituting

$$d_{\mathbf{U}_0}g_i = \frac{\partial g_i(\mathbf{U}_0)}{\partial u_1}\,du_1 + \frac{\partial g_i(\mathbf{U}_0)}{\partial u_2}\,du_2 + \cdots + \frac{\partial g_i(\mathbf{U}_0)}{\partial u_m}\,du_m, \quad 1 \le i \le n,$$

into and collecting multipliers of du_1, du_2, , du_m yields

$$d_{\mathbf{U}_0}h = \sum_{i=1}^{m}\left(\sum_{j=1}^{n}\frac{\partial f(\mathbf{X}_0)}{\partial x_j}\frac{\partial g_j(\mathbf{U}_0)}{\partial u_i}\right)du_i.$$

However, from Theorem~,

$$d_{\mathbf{U}_0}h = \sum_{i=1}^{m}\frac{\partial h(\mathbf{U}_0)}{\partial u_i}\,du_i.$$

Comparing the last two equations yields .

When it is not important to emphasize the particular point $\mathbf{X}_0$, we write less formally as

$$\frac{\partial h}{\partial u_i} = \sum_{j=1}^{n}\frac{\partial f}{\partial x_j}\frac{\partial g_j}{\partial u_i}, \quad 1 \le i \le m, \tag{5.1.39}$$

with the understanding that in calculating $\partial h(\mathbf{U}_0)/\partial u_i$, $\partial g_j/\partial u_i$ is evaluated at $\mathbf{U}_0$ and $\partial f/\partial x_j$ at $\mathbf{X}_0 = \mathbf{G}(\mathbf{U}_0)$.

The formulas and can also be simplified by replacing the symbol $\mathbf{G}$ with $\mathbf{X} = \mathbf{X}(\mathbf{U})$; then we write

$$h(\mathbf{U}) = f(\mathbf{X}(\mathbf{U}))$$

and

$$\frac{\partial h(\mathbf{U}_0)}{\partial u_i} = \sum_{j=1}^{n}\frac{\partial f(\mathbf{X}_0)}{\partial x_j}\frac{\partial x_j(\mathbf{U}_0)}{\partial u_i},$$

or simply

$$\frac{\partial h}{\partial u_i} = \sum_{j=1}^{n}\frac{\partial f}{\partial x_j}\frac{\partial x_j}{\partial u_i}. \tag{5.1.40}$$

The proof of Corollary~ suggests a straightforward way to calculate the partial derivatives of a composite function without using explicitly. If $h(\mathbf{U}) = f(\mathbf{X}(\mathbf{U}))$, then Theorem~ , in the more casual notation introduced before Example~, implies that

$$dh = f_{x_1}dx_1 + f_{x_2}dx_2 + \cdots + f_{x_n}dx_n, \tag{5.1.41}$$

where dx_1, dx_2, , dx_n must be written in terms of the differentials du_1, du_2, , du_m of the independent variables; thus,

$$dx_i = \frac{\partial x_i}{\partial u_1}\,du_1 + \frac{\partial x_i}{\partial u_2}\,du_2 + \cdots + \frac{\partial x_i}{\partial u_m}\,du_m.$$

Substituting this into and collecting the multipliers of du_1, du_2, , du_m yields~.

Higher derivatives of composite functions can be computed by repeatedly applying the chain rule. For example, differentiating with respect to u_k yields

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We must be careful finding

$$\frac{\partial}{\partial u_k}\left(\frac{\partial f}{\partial x_j}\right),$$

which really stands here for

$$\frac{\partial}{\partial u_k}\left(\frac{\partial f(\mathbf{X}(\mathbf{U}))}{\partial x_j}\right). \tag{5.1.42}$$

The safest procedure is to write temporarily

$$g(\mathbf{X}) = \frac{\partial f(\mathbf{X})}{\partial x_j};$$

then becomes

$$\frac{\partial g(\mathbf{X}(\mathbf{U}))}{\partial u_k} = \sum_{s=1}^{n} \frac{\partial g(\mathbf{X}(\mathbf{U}))}{\partial x_s}\frac{\partial x_s(\mathbf{U})}{\partial u_k}.$$

Since

$$\frac{\partial g}{\partial x_s} = \frac{\partial^2 f}{\partial x_s\,\partial x_j},$$

this yields

$$\frac{\partial}{\partial u_k}\left(\frac{\partial f}{\partial x_k}\right) = \sum_{s=1}^{n} \frac{\partial^2 f}{\partial x_s\,\partial x_j}\frac{\partial x_s}{\partial u_k}.$$

Substituting this into yields

$$\frac{\partial^2 h}{\partial u_k\,\partial u_i} = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\frac{\partial^2 x_j}{\partial u_k\,\partial u_i} + \sum_{j=1}^{n}\frac{\partial x_j}{\partial u_i}\sum_{s=1}^{n}\frac{\partial^2 f}{\partial x_s\,\partial x_j}\frac{\partial x_s}{\partial u_k}\partial u_k. \tag{5.1.43}$$

To compute $h_{u_iu_k}(\mathbf{U}_0)$ from this formula, we evaluate the partial derivatives of x_1, x_2, , x_n at $\mathbf{U}_0$ and those of f at $\mathbf{X}_0 = \mathbf{X}(\mathbf{U}_0)$. The formula is valid if x_1, x_2, , x_n and their first partial derivatives are differentiable at $\mathbf{U}_0$ and f, f_{x_i}, f_{x_2}, , f_{x_n} and their first partial derivatives are differentiable at $\mathbf{X}_0$.

Instead of memorizing , you should understand how it is derived and use the method, rather than the formula, when calculating second partial derivatives of composite functions. The same method applies to the calculation of higher derivatives.

For a composite function of the form

$$h(t) = f(x_1(t), x_2(t), \dots, x_n(t))$$

where t is a real variable, x_1, x_2, , x_n are differentiable at t_0, and f is differentiable at $\mathbf{X}_0 = \mathbf{X}(t_0)$, takes the form

$$h'(t_0) = \sum_{j=1}^{n} f_{x_j}(\mathbf{X}(t_0))x_j'(t_0). \tag{5.1.44}$$

This will be useful in the proof of the following theorem.

An equation of L is

$$\mathbf{X} = \mathbf{X}(t) = t\mathbf{X}_2 + (1-t)\mathbf{X}_1, \quad 0 \le t \le 1.$$

Our hypotheses imply that the function

$$h(t) = f(\mathbf{X}(t))$$

is continuous on $[0,1]$ and differentiable on $(0,1)$. Since

$$x_i(t) = tx_{i2} + (1-t)x_{i1},$$

implies that

$$h'(t) = \sum_{i=1}^{n} f_{x_i}(\mathbf{X}(t))(x_{i2} - x_{i1}), \quad 0 < t < 1.$$

From the mean value theorem for functions of one variable (Theorem~),

$$h(1)-h(0)=h'(t_0)$$

for some $t_0\in(0,1)$. Since $h(1)=f(\mathbf{X}_2)$ and $h(0)=f(\mathbf{X}_1)$, this implies with $\mathbf{X}_0=\mathbf{X}(t_0)$.

We will show that if $\mathbf{X}_0$ and $\mathbf{X}$ are in S, then $f(\mathbf{X})=f(\mathbf{X}_0)$. Since S is an open region, S is polygonally connected (Theorem~). Therefore, there are points

$$\mathbf{X}_0,\mathbf{X}_1,\dots,\mathbf{X}_n=\mathbf{X}$$

such that the line segment L_i from $\mathbf{X}_{i-1}$ to $\mathbf{X}_i$ is in S, $1\le i\le n$. From Theorem~,

$$f(\mathbf{X}_i)-f(\mathbf{X}_{i-1})=\sum_{i=1}^{n}(d_{\widetilde{\mathbf{X}}_i}f)(\mathbf{X}_i-\mathbf{X}_{i-1}),$$

where $\widetilde{\mathbf{X}}$ is on L_i and therefore in S. Therefore,

$$f_{x_i}(\widetilde{\mathbf{X}}_i)=f_{x_2}(\widetilde{\mathbf{X}}_i)=\cdots=f_{x_n}(\widetilde{\mathbf{X}}_i)=0,$$

which means that $d_{\widetilde{\mathbf{X}}_i}f\equiv 0$. Hence,

$$f(\mathbf{X}_0)=f(\mathbf{X}_1)=\cdots=f(\mathbf{X}_n);$$

that is, $f(\mathbf{X})=f(\mathbf{X}_0)$ for every $\mathbf{X}$ in S.

Suppose that f is defined in an n-ball $B_\rho(\mathbf{X}_0)$, with $\rho>0$. If $\mathbf{X}\in B_\rho(\mathbf{X}_0)$, then

$$\mathbf{X}(t)=\mathbf{X}_0+t(\mathbf{X}-\mathbf{X}_0)\in B_\rho(\mathbf{X}),\quad 0\le t\le 1,$$

so the function

$$h(t)=f(\mathbf{X}(t))$$

is defined for $0\le t\le 1$. From Theorem~ (see also),

$$h'(t)=\sum_{i=1}^{n}f_{x_i}(\mathbf{X}(t)(x_i-x_{i0})$$

if f is differentiable in $B_\rho(\mathbf{X}_0)$, and

$$\begin{aligned}h''(t)\ar&=\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left(\sum_{i=1}^{n}\frac{\partial f(\mathbf{X}(t))}{\partial x_i}(x_i-x_{i0})\right)(x_j-x_{j0})\\ \ar&=\sum_{i,j=1}^{n}\frac{\partial^2 f(\mathbf{X}(t))}{\partial x_j\,\partial x_i}(x_i-x_{i0})(x_j-x_{j0})\end{aligned}$$

if f_{x_1}, f_{x_2}, , f_{x_n} are differentiable in $B_\rho(\mathbf{X}_0)$. Continuing in this way, we see that

$$h^{(r)}(t)=\sum_{i_1,i_2,\dots,i_r=1}^{n}\frac{\partial^r f(\mathbf{X}(t))}{\partial x_{i_r}\,\partial x_{i_{r-1}}\cdots\partial x_{i_1}}(x_{i_1}-x_{i_1,0})(x_{i_2}-x_{i_2,0})\cdots(x_{i_r}-x_{i_r,0}) \tag{5.1.45}$$

if all partial derivatives of f of order $\le r-1$ are differentiable in $B_\rho(\mathbf{X}_0)$.

This motivates the following definition.

Under the assumptions of Definition~, the value of

$$\frac{\partial^r f(\mathbf{X}_0)}{\partial x_{i_r}\partial x_{i_{r-1}}\cdots\partial x_{i_1}}$$

depends only on the number of times f is differentiated with respect to each variable, and not on the order in which the differentiations are performed (Exercise~). Hence, Exercise~ implies that can be rewritten as

$$d^{(r)}_{\mathbf{X}_0}f=\sum_{r}\frac{r!}{r_1!r_2!\cdots r_n!}\frac{\partial^r f(\mathbf{X}_0)}{\partial x_1^{r_1}\partial x_2^{r_2}\cdots\partial x_n^{r_n}}(dx_1)^{r_1}(dx_2)^{r_2}\cdots(dx_n)^{r_n}, \tag{5.1.46}$$

where $\sum_r$ indicates summation over all ordered n-tuples $(r_1,r_2,\dots,r_n)$ of nonnegative integers such that

$$r_1 + r_2 + \cdots + r_n = r$$

and $\partial x_i^{r_i}$ is omitted from the ``denominators'' of all terms in for which $r_i = 0$. In particular, if $n = 2$,

$$d_{\mathbf{X}_0}^{(r)} f = \sum_{j=0}^{r} \binom{r}{j} \frac{\partial^r f(x_0, y_0)}{\partial x^j \, \partial y^{r-j}} (dx)^j (dy)^{r-j}.$$

The next theorem is analogous to Taylor's theorem for functions of one variable (Theorem~).

Define

$$h(t) = f(\mathbf{X}_0 + t(\mathbf{X} - \mathbf{X}_0)). \tag{5.1.47}$$

With $\Phi = \mathbf{X} - \mathbf{X}_0$, our assumptions and the discussion preceding Definition~ imply that h, h', , $h^{(k+1)}$ exist on $[0, 1]$. From Taylor's theorem for functions of one variable,

$$h(1) = \sum_{r=0}^{k} \frac{h^{(r)}(0)}{r!} + \frac{h^{(k+1)}(\tau)}{(k+1)!}, \tag{5.1.48}$$

for some $\tau \in (0, 1)$. From ,

$$h(0) = f(\mathbf{X}_0) \quad \text{and} \quad h(1) = f(\mathbf{X}). \tag{5.1.49}$$

From and with $\Phi = \mathbf{X} - \mathbf{X}_0$,

$$h^{(r)}(0) \ar = (d_{\mathbf{X}_0}^{(r)} f)(\mathbf{X} - \mathbf{X}_0), \quad 1 \le r \le k, \tag{5.1.50}$$

\arraytext{and}

$$h^{(k+1)}(\tau) \ar = \left(d_{\widetilde{\mathbf{X}}}^{k+1} f\right)(\mathbf{X} - \mathbf{X}_0) \tag{5.1.51}$$

where

$$\widetilde{\mathbf{X}} = \mathbf{X}_0 + \tau(\mathbf{X} - \mathbf{X}_0)$$

is on L and distinct from $\mathbf{X}_0$ and $\mathbf{X}$. Substituting , , and into yields .

By analogy with the situation for functions of one variable, we define the kth {} $\mathbf{X}_0$ by

$$T_k(\mathbf{X}) = \sum_{r=0}^{k} \frac{1}{r!} (d_{\mathbf{X}_0}^{(r)} f)(\mathbf{X} - \mathbf{X}_0)$$

if the differentials exist; then can be rewritten as

$$f(\mathbf{X}) = T_k(\mathbf{X}) + \frac{1}{(k+1)!} (d_{\widetilde{\mathbf{X}}}^{(k+1)} f)(\mathbf{X} - \mathbf{X}_0).$$

The next theorem leads to a useful sufficient condition for local maxima and minima. It is related to Theorem~. Strictly speaking, however, it is not a generalization of Theorem (Exercise).

If $\epsilon > 0$, there is a $\delta > 0$ such that $B_\delta(\mathbf{X}_0) \subset N$ and all kth-order partial derivatives of f satisfy the inequality

$$\left| \frac{\partial^k f(\widetilde{\mathbf{X}})}{\partial x_{i_k} \partial x_{i_{k-1}} \cdots \partial x_{i_1}} - \frac{\partial^k f(\mathbf{X}_0)}{\partial x_{i_k} \partial x_{i_{k-1}} \cdots \partial x_{i_1}} \right| < \epsilon, \quad \widetilde{\mathbf{X}} \in B_\delta(\mathbf{X}_0). \tag{5.1.52}$$

Now suppose that $\mathbf{X} \in B_\delta(\mathbf{X}_0)$. From Theorem~ with k replaced by $k - 1$,

$$f(\mathbf{X}) = T_{k-1}(\mathbf{X}) + \frac{1}{k!} (d_{\widetilde{\mathbf{X}}}^{(k)} f)(\mathbf{X} - \mathbf{X}_0), \tag{5.1.53}$$

where $\widetilde{\mathbf{X}}$ is some point on the line segment from $\mathbf{X}_0$ to $\mathbf{X}$ and is therefore in $B_\delta(\mathbf{X}_0)$. We can rewrite as

$$f(\mathbf{X}) = T_k(\mathbf{X}) + \frac{1}{k!} \left[(d_{\widetilde{\mathbf{X}}}^{(k)} f)(\mathbf{X} - \mathbf{X}_0) - (d_{\mathbf{X}_0}^{(k)} f)(\mathbf{X} - \mathbf{X}_0) \right]. \tag{5.1.54}$$

But and imply that

$$\left|(d^{(k)}_{\overline{\mathbf{X}}}f)(\mathbf{X}-\mathbf{X}_0)-(d^{(k)}_{\mathbf{X}_0}f)(\mathbf{X}-\mathbf{X}_0)\right|<n^k\epsilon|\mathbf{X}-\mathbf{X}_0|^k \tag{5.1.55}$$

(Exercise~), which implies that

$$\frac{|f(\mathbf{X})-T_k(\mathbf{X})|}{|\mathbf{X}-\mathbf{X}_0|^k}<\frac{n^k\epsilon}{k!},\quad \mathbf{X}\in B_\delta(\mathbf{X}_0),$$

from . This implies .

Let r be a positive integer and $\mathbf{X}_0=(x_{10},x_{20},\dots,x_{n0})$. A function of the form

$$p(\mathbf{X})=\sum_r a_{r_1r_2\dots r_n}(x_1-x_{10})^{r_1}(x_2-x_{20})^{r_2}\cdots(x_n-x_{n0})^{r_n}, \tag{5.1.56}$$

where the coefficients $\{a_{r_1r_2\dots r_n}\}$ are constants and the summation is over all n-tuples of nonnegative integers $(r_1,r_2,\dots,r_n)$ such that

$$r_1+r_2+\cdots+r_n=r,$$

is a {}, provided that at least one of the coefficients is nonzero. For example, if f satisfies the conditions of Definition~, then the function

$$p(\mathbf{X})=(d^{(r)}_{\mathbf{X}_0}f)(\mathbf{X}-\mathbf{X}_0)$$

is such a polynomial if at least one of the rth-order mixed partial derivatives of f at $\mathbf{X}_0$ is nonzero.

Clearly, $p(\mathbf{X}_0)=0$ if p is a homogeneous polynomial of degree $r\ge1$ in $\mathbf{X}-\mathbf{X}_0$. If $p(\mathbf{X})\ge0$ for all $\mathbf{X}$, we say that p is {}; if $p(\mathbf{X})>0$ except when $\mathbf{X}=\mathbf{X}_0$, p is {}.

Similarly, p is {} if $p(\mathbf{X})\le0$ or {} if $p(\mathbf{X})<0$ for all $\mathbf{X}\ne\mathbf{X}_0$. In all these cases, p is {}.

With p as in ,

$$p(-\mathbf{X}+2\mathbf{X}_0)=(-1)^rp(\mathbf{X}),$$

so p cannot be semidefinite if r is odd.

From Theorem~, if f is differentiable and attains a local extreme value at $\mathbf{X}_0$, then

$$d_{\mathbf{X}_0}f=0, \tag{5.1.57}$$

since $f_{x_1}(\mathbf{X}_0)=f_{x_2}(\mathbf{X}_0)=\cdots=f_{x_n}(\mathbf{X}_0)=0$. However, the converse is false. The next theorem provides a method for deciding whether a point satisfying is an extreme point. It is related to Theorem~.

From and Theorem~,

$$\lim_{\mathbf{X}\to\mathbf{X}_0}\frac{f(\mathbf{X})-f(\mathbf{X}_0)-\dst\frac{1}{k!}(d^{(k)}_{\mathbf{X}_0})(\mathbf{X}-\mathbf{X}_0)}{|\mathbf{X}-\mathbf{X}_0|^k}=0. \tag{5.1.58}$$

If $\mathbf{X}=\mathbf{X}_0+t\mathbf{U}$, where $\mathbf{U}$ is a constant vector, then

$$(d^{(k)}_{\mathbf{X}_0}f)(\mathbf{X}-\mathbf{X}_0)=t^k(d^{(k)}_{\mathbf{X}_0}f)(\mathbf{U}),$$

so implies that

$$\lim_{t\to0}\frac{f(\mathbf{X}_0+t\mathbf{U})-f(\mathbf{X}_0)-\dst\frac{t^k}{k!}(d^{(k)}_{\mathbf{X}_0}f)(\mathbf{U})}{t^k}=0,$$

or, equivalently,

$$\lim_{t\to0}\frac{f(\mathbf{X}_0+t\mathbf{U})-f(\mathbf{X}_0)}{t^k}=\frac{1}{k!}(d^{(k)}_{\mathbf{X}_0}f)(\mathbf{U}) \tag{5.1.59}$$

for any constant vector $\mathbf{U}$.

To prove

, suppose that $d^{(k)}_{\mathbf{X}_0}f$ is not semidefinite. Then there are vectors $\mathbf{U}_1$ and $\mathbf{U}_2$ such that

$$(d^{(k)}_{\mathbf{X}_0}f)(\mathbf{U}_1) > 0 \quad \text{and} \quad (d^{(k)}_{\mathbf{X}_0}f)(\mathbf{U}_2) < 0.$$

This and imply that

$$f(\mathbf{X}_0 + t\mathbf{U}_1) > f(\mathbf{X}_0) \quad \text{and} \quad f(\mathbf{X}_0 + t\mathbf{U}_2) < f(\mathbf{X}_0)$$

for t sufficiently small. Hence, $\mathbf{X}_0$ is not a local extreme point of f.

To prove

, first assume that $d^{(k)}_{\mathbf{X}_0}f$ is positive definite. Then it can be shown that there is a $\rho > 0$ such that

$$\frac{(d^{(k)}_{\mathbf{X}_0}f)(\mathbf{X}-\mathbf{X}_0)}{k!} \ge \rho|\mathbf{X}-\mathbf{X}_0|^k \tag{5.1.60}$$

for all $\mathbf{X}$ (Exercise~). From , there is a $\delta > 0$ such that

$$\frac{f(\mathbf{X}) - f(\mathbf{X}_0) - \dst\frac{1}{k!}(d^{(k)}_{\mathbf{X}_0}f)(\mathbf{X}-\mathbf{X}_0)}{|\mathbf{X}-\mathbf{X}_0|^k} > -\frac{\rho}{2} \quad \text{if} \quad |\mathbf{X}-\mathbf{X}_0| < \delta.$$

Therefore,

$$f(\mathbf{X}) - f(\mathbf{X}_0) > \frac{1}{k!}(d^{(k)}_{\mathbf{X}_0})(\mathbf{X}-\mathbf{X}_0) - \frac{\rho}{2}|\mathbf{X}-\mathbf{X}_0|^k \quad \text{if} \quad |\mathbf{X}-\mathbf{X}_0| < \delta.$$

This and imply that

$$f(\mathbf{X}) - f(\mathbf{X}_0) > \frac{\rho}{2}|\mathbf{X}-\mathbf{X}_0|^k \quad \text{if} \quad |\mathbf{X}-\mathbf{X}_0| < \delta,$$

which implies that $\mathbf{X}_0$ is a local minimum point of f. This proves half of

. We leave the other half to you (Exercise~).

To prove

merely requires examples; see Exercise~.

Write $(x-x_0, y-y_0) = (u, v)$ and

$$p(u,v) = (d^{(2)}_{\mathbf{X}_0}f)(u,v) = Au^2 + 2Buv + Cv^2,$$

where $A = f_{xx}(x_0,y_0)$, $B = f_{xy}(x_0,y_0)$, and $C = f_{yy}(x_0,y_0)$, so

$$D = AC - B^2.$$

If $D > 0$, then $A \neq 0$, and we can write

$$\begin{aligned} p(u,v)\ar &= A\left(u^2 + \frac{2B}{A}uv + \frac{B^2}{A^2}v^2\right) + \left(C - \frac{B^2}{A}\right)v^2 \\ \ar &= A\left(u + \frac{B}{A}v\right)^2 + \frac{D}{A}v^2. \end{aligned}$$

This cannot vanish unless $u = v = 0$. Hence, $d^{(2)}_{\mathbf{X}_0}f$ is positive definite if $A > 0$ or negative definite if $A < 0$, and Theorem~ implies

.

If $D < 0$, there are three possibilities:

In each case the two given values of p differ in sign, so $\mathbf{X}_0$ is not a local extreme point of f, from Theorem~

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5.2: Continuous Real-Valued Function of n Variables

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5.3: Partial Derivatives and the Differential

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5.4: The Chain Rule and Taylor's Theorem

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CHAPTER OVERVIEW

6: Vector-Valued Functions of Several Variables

IN THIS CHAPTER we study the differential calculus of vector-valued functions of several variables.

- SECTION 6.1 reviews matrices, determinants, and linear transformations, which are integral parts of the differential calculus as presented here.
- SECTION 6.2 defines continuity and differentiability of vector-valued functions of several variables. The differential of a vector-valued function $\mathbf{F}$ is defined as a certain linear transformation. The matrix of this linear transformation is called the differential matrix of $\mathbf{F}$, denoted by $\mathbf{F}'$. The chain rule is extended to compositions of differentiable vector-valued functions.
- SECTION 6.3 presents a complete proof of the inverse function theorem.
- SECTION 6.4. uses the inverse function theorem to prove the implicit function theorem.

6.1: Linear Transformations and Matrices

6.2: Continuity and Differentiability of Transformations

6.3: The Inverse Function Theorem

6.4: The Implicit Function Theorem

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6.1: Linear Transformations and Matrices

In this and subsequent sections it will often be convenient to write vectors vertically; thus, instead of $\mathbf{X} = (x_1, x_2, \dots, x_n)$ we will write

$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

when dealing with matrix operations. Although we assume that you have completed a course in linear algebra, we will review the pertinent matrix operations.

We have defined vector-valued functions as ordered n-tuples of real-valued functions, in connection with composite functions $h = f \circ \mathbf{G}$, where f is real-valued and $\mathbf{G}$ is vector-valued. We now consider vector-valued functions as objects of interest on their own.

If f_1, f_2, , f_m are real-valued functions defined on a set D in $\R^n$, then

$$\mathbf{F} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix}$$

assigns to every $\mathbf{X}$ in D an m-vector

$$\mathbf{F}(\mathbf{X}) = \begin{bmatrix} f_1(\mathbf{X}) \\ f_2(\mathbf{X}) \\ \vdots \\ f_m(\mathbf{X}) \end{bmatrix}.$$

Recall that f_1, f_2, , f_m are the {}, or simply {}, of $\mathbf{F}$. We write

$$\mathbf{F} : \R^n \to \R^m$$

to indicate that the domain of $\mathbf{F}$ is in $\R^n$ and the range of $\mathbf{F}$ is in $\R^m$. We also say that $\mathbf{F}$ is a {} $\R^n$ {} $\R^m$. If $m = 1$, we identify $\mathbf{F}$ with its single component function f_1 and regard it as a real-valued function.

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The simplest interesting transformations from $\R^n$ to $\R^m$ are the {}, defined as follows

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If can be seen by induction (Exercise~) that if $\mathbf{L}$ is linear, then

$$\mathbf{L}(a_1\mathbf{X}_1 + a_2\mathbf{X}_2 + \cdots + a_k\mathbf{X}_k) = a_1\mathbf{L}(\mathbf{X}_1) + a_2\mathbf{L}(\mathbf{X}_2) + \cdots + a_k\mathbf{L}(\mathbf{X}_k) \tag{6.1.1}$$

for any vectors $\mathbf{X}_1$, $\mathbf{X}_2$, , $\mathbf{X}_k$ and real numbers a_1, a_2, , a_k. Any $\mathbf{X}$ in $\R^n$ can be written as

$$\mathbf{X}\ar = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$
$$\ar = x_1\mathbf{E}_1 + x_2\mathbf{E}_2 + \cdots + x_n\mathbf{E}_n.$$

Applying with $k = n$, $\mathbf{X}_i = \mathbf{E}_i$, and $a_i = x_i$ yields

$$\mathbf{L}(\mathbf{X}) = x_1\mathbf{L}(\mathbf{E}_1) + x_2\mathbf{L}(\mathbf{E}_2) + \cdots + x_n\mathbf{L}(\mathbf{E}_n). \tag{6.1.2}$$

Now denote

$$\mathbf{L}(\mathbf{E}_j) = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix},$$

so becomes

$$\mathbf{L}(\mathbf{X}) = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix},$$

which is equivalent to . This proves that if $\mathbf{L}$ is linear, then $\mathbf{L}$ has the form . We leave the proof of the converse to you (Exercise~).

We call the rectangular array

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{21} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{6.1.3}$$

the {} of the linear transformation . The number a_{ij} in the ith row and jth column of $\mathbf{A}$ is called the {} $\mathbf{A}$. We say that $\mathbf{A}$ is an $m \times n$ matrix, since $\mathbf{A}$ has m rows and n columns. We will sometimes abbreviate as

$$\mathbf{A} = [a_{ij}].$$

We will now recall the matrix operations that we need to study the differential calculus of transformations.

We leave the proofs of next three theorems to you (Exercises~–)

The next theorem shows why Definition~ is appropriate. We leave the proof to you (Exercise~).

If a real-valued function $f : \R^n \to \R$ is differentiable at $\mathbf{X}_0$, then

$$d_{\mathbf{X}_0} f = f_{x_1}(\mathbf{X}_0)\, dx_1 + f_{x_2}(\mathbf{X}_0)\, dx_2 + \cdots + f_{x_n}(\mathbf{X}_0)\, dx_n.$$

This can be written as a matrix product

$$d_{\mathbf{X}_0} f = \begin{bmatrix} f_{x_1}(\mathbf{X}_0) & f_{x_2}(\mathbf{X}_0) & \cdots & f_{x_n}(\mathbf{X}_0) \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{bmatrix}. \tag{6.1.4}$$

We define the {} by

$$f'(\mathbf{X}_0) = \begin{bmatrix} f_{x_1}(\mathbf{X}_0) & f_{x_2}(\mathbf{X}_0) & \cdots & f_{x_n}(\mathbf{X}_0) \end{bmatrix} \tag{6.1.5}$$

and the {} by

$$d\mathbf{X} = \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{bmatrix}.$$

Then can be rewritten as

$$d_{\mathbf{X}_0} f = f'(\mathbf{X}_0)\, d\mathbf{X}. \tag{6.1.6}$$

This is analogous to the corresponding formula for functions of one variable (Example~), and shows that the differential matrix $f'(\mathbf{X}_0)$ is a natural generalization of the derivative. With this new notation we can express the defining property of the differential in a way similar to the form that applies for $n = 1$:

$$\lim_{\mathbf{X}\to\mathbf{X}_0}\frac{f(\mathbf{X})-f(\mathbf{X}_0)-f'(\mathbf{X}_0)(\mathbf{X}-\mathbf{X}_0)}{|\mathbf{X}-\mathbf{X}_0|}=0,$$

where $\mathbf{X}_0=(x_{10},x_{20},\dots,x_{n0})$ and $f'(\mathbf{X}_0)(\mathbf{X}-\mathbf{X}_0)$ is the matrix product

$$[f_{x_1}(\mathbf{X}_0)\quad f_{x_2}(\mathbf{X}_0)\quad\cdots\quad f_{x_n}(\mathbf{X}_0)]\begin{bmatrix}x_1-x_{10}\\x_2-x_{20}\\\vdots\\x_n-x_{n0}\end{bmatrix}.$$

As before, we omit the $\mathbf{X}_0$ in and when it is not necessary to emphasize the specific point; thus, we write

$$f'=[f_{x_1}\quad f_{x_2}\quad\cdots\quad f_{x_n}]\quad\text{and}\quad df=f'd\mathbf{X}.$$

We will need the following definition in the next section.

To justify this definition, we must show that $\|\mathbf{A}\|$ exists. The components of $\mathbf{Y}=\mathbf{AX}$ are

$$y_i=a_{i1}x_1+a_{i2}x_2+\cdots+a_{in}x_n,\quad 1\le i\le m.$$

By Schwarz's inequality,

$$y_i^2\le(a_{i1}^2+a_{i2}^2+\cdots+a_{in}^2)|\mathbf{X}|^2.$$

Summing this over $1\le i\le m$ yields

$$|\mathbf{Y}|^2\le\left(\sum_{i=1}^m\sum_{j=1}^n a_{ij}^2\right)|\mathbf{X}|^2.$$

Therefore, the set

$$B=\{K\}|\mathbf{AX}|\le K|\mathbf{X}|\quad\text{for all }\mathbf{X}\text{ in }\R^n$$

is nonempty. Since B is bounded below by zero, B has an infimum α. If $\epsilon>0$, then $\alpha+\epsilon$ is in B because if not, then no number less than $\alpha+\epsilon$ could be in B. Then $\alpha+\epsilon$ would be a lower bound for B, contradicting the definition of α. Hence,

$$|\mathbf{AX}|\le(\alpha+\epsilon)|\mathbf{X}|,\quad\mathbf{X}\in\R^n.$$

Since ϵ is an arbitrary positive number, this implies that

$$|\mathbf{AX}|\le\alpha|\mathbf{X}|,\quad\mathbf{X}\in\R^n,$$

so $\alpha\in B$. Since no smaller number is in B, we conclude that $\|\mathbf{A}\|=\alpha$.

In our applications we will not have to actually compute the norm of a matrix $\mathbf{A}$; rather, it will be sufficient to know that the norm exists (finite).

Linear transformations from $\R^n$ to $\R^n$ will be important when we discuss the inverse function theorem in Section~6.3 and change of variables in multiple integrals in Section~7.3. The matrix of such a transformation is $\{\}$; that is, it has the same number of rows and columns.

We assume that you know the definition of the determinant

$$\det(\mathbf{A})=\begin{vmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&\vdots&\ddots&\vdots\\a_{n1}&a_{n2}&\cdots&a_{nn}\end{vmatrix}$$

of an $n\times n$ matrix

$$\mathbf{A}=\begin{bmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&\vdots&\ddots&\vdots\\a_{n1}&a_{n2}&\cdots&a_{nn}\end{bmatrix}.$$

The {}, $\mathbf{A}^t$, of a matrix $\mathbf{A}$ (square or not) is the matrix obtained by interchanging the rows and columns of $\mathbf{A}$; thus, if

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 4 \\ 0 & 1 & -2 \end{bmatrix}, \quad \text{then} \quad \mathbf{A}^t = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 1 \\ 3 & 4 & -2 \end{bmatrix}.$$

A square matrix and its transpose have the same determinant; thus,

$$\det(\mathbf{A}^t) = \det(\mathbf{A}).$$

We take the next theorem from linear algebra as given.

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The entries a_{ii}, $1 \le i \le n$, of an $n \times n$ matrix $\mathbf{A}$ are on the {} of $\mathbf{A}$. The $n \times n$ matrix with ones on the main diagonal and zeros elsewhere is called the {} and is denoted by $\mathbf{I}$; thus, if $n = 3$, .4pc

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

.4pc We call $\mathbf{I}$ the identity matrix because $\mathbf{AI} = \mathbf{A}$ and $\mathbf{IA} = \mathbf{A}$ if $\mathbf{A}$ is any $n \times n$ matrix. We say that an $n \times n$ matrix $\mathbf{A}$ is {} if there is an $n \times n$ matrix $\mathbf{A}^{-1}$, the {} $\mathbf{A}$, such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$. Otherwise, we say that $\mathbf{A}$ is {}

Our main objective is to show that an $n \times n$ matrix $\mathbf{A}$ is nonsingular if and only if $\det(\mathbf{A}) \ne 0$. We will also find a formula for the inverse.

.1pc

For a proof of the following theorem, see any elementary linear algebra text.

If we compute $\det(\mathbf{A})$ from the formula

$$\det(\mathbf{A}) = \sum_{k=1}^{n} a_{ik}c_{ik},$$

we say that we are {}. Since we can choose i arbitrarily from $\{1, \dots, n\}$, there are n ways to do this. If we compute $\det(\mathbf{A})$ from the formula

$$\det(\mathbf{A}) = \sum_{k=1}^{n} a_{kj}c_{kj},$$

we say that we are {}. There are also n ways to do this.

In particular, we note that $\det(\mathbf{I}) = 1$ for all $n \ge 1$.

If $\det(\mathbf{A}) = 0$, then $\det(\mathbf{AB}) = 0$ for any $n \times n$ matrix, by Theorem~. Therefore, since $\det(\mathbf{I}) = 1$, there is no matrix $n \times n$ matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{I}$; that is, $\mathbf{A}$ is singular if $\det(\mathbf{A}) = 0$. Now suppose that $\det(\mathbf{A}) \ne 0$. Since implies that

$$\mathbf{A}\backslash\mathrm{adj}(\mathbf{A}) = \det(\mathbf{A})\mathbf{I}$$

and implies that

$$\backslash\mathrm{adj}(\mathbf{A})\mathbf{A} = \det(\mathbf{A})\mathbf{I},$$

dividing both sides of these two equations by $\det(\mathbf{A})$ shows that if $\mathbf{A}^{-1}$ is as defined in , then $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$. Therefore, $\mathbf{A}^{-1}$ is an inverse of $\mathbf{A}$. To see that it is the only inverse, suppose that $\mathbf{B}$ is an $n \times n$ matrix such that $\mathbf{AB} = \mathbf{I}$. Then $\mathbf{A}^{-1}(\mathbf{AB}) = \mathbf{A}^{-1}$, so $(\mathbf{A}^{-1}\mathbf{A})\mathbf{B} = \mathbf{A}^{-1}$. Since $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$ and $\mathbf{IB} = \mathbf{B}$, it follows that $\mathbf{B} = \mathbf{A}^{-1}$.

Now consider the equation

$$\mathbf{AX} = \mathbf{Y} \tag{6.1.7}$$

with

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \text{and} \quad \mathbf{Y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

Here $\mathbf{A}$ and $\mathbf{Y}$ are given, and the problem is to find $\mathbf{X}$.

Suppose that $\mathbf{A}$ is nonsingular, and let $\mathbf{X} = \mathbf{A}^{-1}\mathbf{Y}$. Then

$$\mathbf{AX} = \mathbf{A}(\mathbf{A}^{-1}\mathbf{Y}) = (\mathbf{A}\mathbf{A}^{-1})\mathbf{Y} = \mathbf{IY} = \mathbf{Y};$$

that is, $\mathbf{X}$ is a solution of . To see that $\mathbf{X}$ is the only solution of , suppose that $\mathbf{AX}_1 = \mathbf{Y}$. Then $\mathbf{AX}_1 = \mathbf{AX}$, so

$$\begin{aligned} \mathbf{A}^{-1}(\mathbf{AX}) &\ar = \mathbf{A}^{-1}(\mathbf{AX}_1) \\ &\arraytext{and} \\ (\mathbf{A}^{-1}\mathbf{A})\mathbf{X} &\ar = (\mathbf{A}^{-1}\mathbf{A})\mathbf{X}_1, \end{aligned}$$

which is equivalent to $\mathbf{IX} = \mathbf{IX}_1$, or $\mathbf{X} = \mathbf{X}_1$.

Conversely, suppose that has a solution for every $\mathbf{Y}$, and let $\mathbf{X}_i$ satisfy $\mathbf{AX}_i = \mathbf{E}_i$, $1 \le i \le n$. Let

$$\mathbf{B} = [\mathbf{X}_1\, \mathbf{X}_2\, \cdots\, \mathbf{X}_n];$$

that is, $\mathbf{X}_1$, $\mathbf{X}_2$, , $\mathbf{X}_n$ are the columns of $\mathbf{B}$. Then

$$\mathbf{AB} = [\mathbf{AX}_1\, \mathbf{AX}_2\, \cdots\, \mathbf{AX}_n] = [\mathbf{E}_1\, \mathbf{E}_2\, \cdots\, \mathbf{E}_n] = \mathbf{I}.$$

To show that $\mathbf{B} = \mathbf{A}^{-1}$, we must still show that $\mathbf{BA} = \mathbf{I}$. We first note that, since $\mathbf{AB} = \mathbf{I}$ and $\det(\mathbf{BA}) = \det(\mathbf{AB}) = 1$ (Theorem~), $\mathbf{BA}$ is nonsingular (Theorem~). Now note that

$$(\mathbf{BA})(\mathbf{BA}) = \mathbf{B}(\mathbf{AB})\mathbf{A}) = \mathbf{BIA};$$

that is,

$$(\mathbf{BA})(\mathbf{BA}) = (\mathbf{BA}).$$

Multiplying both sides of this equation on the left by $\mathbf{BA})^{-1}$ yields $\mathbf{BA} = \mathbf{I}$.

The following theorem gives a useful formula for the components of the solution of .

From Theorems~ and , the solution of $\mathbf{AX} = \mathbf{Y}$ is

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{A}^{-1}\mathbf{Y} &\ar = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} c_{11} & c_{21} & \cdots & c_{n1} \\ c_{12} & c_{22} & \cdots & c_{n2} \\ \cdots & \cdots & \ddots & \cdots \\ c_{1n} & c_{2n} & \cdots & c_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \\ &\ar = \begin{bmatrix} c_{11}y_1 + c_{21}y_2 + \cdots + c_{n1}y_n \\ c_{12}y_1 + c_{22}y_2 + \cdots + c_{n2}y_n \\ \vdots \\ c_{1n}y_1 + c_{2n}y_2 + \cdots + c_{nn}y_n \end{bmatrix}. \end{aligned}$$

But

$$c_{11}y_1 + c_{21}y_2 + \cdots + c_{n1}y_n = \begin{vmatrix} y_1 & a_{12} & \cdots & a_{1n} \\ y_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_n & a_{n2} & \cdots & a_{nn} \end{vmatrix},$$

as can be seen by expanding the determinant on the right in cofactors of its first column. Similarly,

$$c_{12}y_1 + c_{22}y_2 + \cdots + c_{n2}y_n = \begin{vmatrix} a_{11} & y_1 & a_{13} & \cdots & a_{1n} \\ a_{21} & y_2 & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & y_n & a_{n3} & \cdots & a_{nn} \end{vmatrix},$$

as can be seen by expanding the determinant on the right in cofactors of its second column. Continuing in this way completes the proof.

A system of n equations in n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \ar &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \ar &= 0 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \ar &= 0 \end{aligned} \tag{6.1.8}$$

(or, in matrix form, $\mathbf{AX} = \mathbf{0}$) is {}. It is obvious that $\mathbf{X}_0 = \mathbf{0}$ satisfies this system. We call this the {} of . Any other solutions of , if they exist, are {}.

We will need the following theorems. The proofs may be found in any linear algebra text.

Throughout the rest of this chapter, transformations $\mathbf{F}$ and points $\mathbf{X}$ should be considered as written in vertical form when they occur in connection with matrix operations. However, we will write $\mathbf{X} = (x_1, x_2, \ldots, x_n)$ when $\mathbf{X}$ is the argument of a function.

-.2em In Section~5.2 we defined a vector-valued function (transformation) to be continuous at $\mathbf{X}_0$ if each of its component functions is continuous at $\mathbf{X}_0$. We leave it to you to show that this implies the following theorem (Exercise~1).

This theorem is the same as Theorem~ except that the ``absolute value'' in now stands for distance in $\R^m$ rather than $\R$.

If $\mathbf{C}$ is a constant vector, then ``$\lim_{\mathbf{X}\to\mathbf{X}_0}\mathbf{F}(\mathbf{X}) = \mathbf{C}$'' means that

$$\lim_{\mathbf{X}\to\mathbf{X}_0} |\mathbf{F}(\mathbf{X}) - \mathbf{C}| = 0.$$

Theorem~ implies that $\mathbf{F}$ is continuous at $\mathbf{X}_0$ if and only if

$$\lim_{\mathbf{X}\to\mathbf{X}_0} \mathbf{F}(\mathbf{X}) = \mathbf{F}(\mathbf{X}_0).$$

In Section~5.4 we defined a vector-valued function (transformation) to be differentiable at $\mathbf{X}_0$ if each of its components is differentiable at $\mathbf{X}_0$ (Definition~). The next theorem characterizes this property in a useful way.

Let $\mathbf{X}_0 = (x_{10}, x_{20}, \ldots, x_{n0})$. If $\mathbf{F}$ is differentiable at $\mathbf{X}_0$, then so are f_1, f_2, , f_m (Definition~). Hence,

$$\lim_{\mathbf{X}\to\mathbf{X}_0} \frac{\dst f_i(\mathbf{X}) - f_i(\mathbf{X}_0) - \sum_{j=1}^n \frac{\partial f_i(\mathbf{X}_0)}{\partial x_j}(x_j - x_{j0})}{|\mathbf{X} - \mathbf{X}_0|} = 0, \quad 1 \le i \le m,$$

which implies with $\mathbf{A}$ as in .

Now suppose that holds with $\mathbf{A} = [a_{ij}]$. Since each component of the vector in approaches zero as $\mathbf{X}$ approaches $\mathbf{X}_0$, it follows that

$$\lim_{\mathbf{X}\to\mathbf{X}_0} \frac{\dst f_i(\mathbf{X}) - f_i(\mathbf{X}_0) - \dst\sum_{j=1}^n a_{ij}(x_j - x_{j0})}{|\mathbf{X} - \mathbf{X}_0|} = 0, \quad 1 \le i \le m,$$

so each f_i is differentiable at $\mathbf{X}_0$, and therefore so is $\mathbf{F}$ (Definition~). By Theorem~,

$$a_{ij} = \frac{\partial f_i(\mathbf{X}_0)}{\partial x_j}, \quad 1 \le i \le m, \quad 1 \le j \le n,$$

which implies .

A transformation $\mathbf{T} : \R^n \to \R^m$ of the form

$$\mathbf{T}(\mathbf{X}) = \mathbf{U} + \mathbf{A}(\mathbf{X} - \mathbf{X}_0),$$

where $\mathbf{U}$ is a constant vector in $\R^m$, $\mathbf{X}_0$ is a constant vector in $\R^n$, and $\mathbf{A}$ is a constant $m \times n$ matrix, is said to be {}. Theorem~ says that if $\mathbf{F}$ is differentiable at $\mathbf{X}_0$, then $\mathbf{F}$ can be well approximated by an affine transformation.

If $\mathbf{F} = (f_1, f_2, \ldots, f_m)$ is differentiable at $\mathbf{X}_0$, we define the {} $\mathbf{X}_0$ to be the linear transformation

$$d_{\mathbf{X}_0}\mathbf{F} = \begin{bmatrix} d_{\mathbf{X}_0} f_1 \\ d_{\mathbf{X}_0} f_2 \\ \vdots \\ d_{\mathbf{X}_0} f_m \end{bmatrix}. \tag{6.1.9}$$

We call the matrix $\mathbf{A}$ in and denote it by $\mathbf{F}'(\mathbf{X}_0)$; thus,

$$\mathbf{F}'(\mathbf{X}_0) = \begin{bmatrix} \dst\frac{\partial f_1(\mathbf{X}_0)}{\partial x_1} & \dst\frac{\partial f_1(\mathbf{X}_0)}{\partial x_2} & \cdots & \dst\frac{\partial f_1(\mathbf{X}_0)}{\partial x_n} \\ [3\jot] \dst\frac{\partial f_2(\mathbf{X}_0)}{\partial x_1} & \dst\frac{\partial f_2(\mathbf{X}_0)}{\partial x_2} & \cdots & \dst\frac{\partial f_2(\mathbf{X}_0)}{\partial x_n} \\ [3\jot] \vdots & \vdots & \ddots & \vdots \\ [3\jot] \dst\frac{\partial f_m(\mathbf{X}_0)}{\partial x_1} & \dst\frac{\partial f_m(\mathbf{X}_0)}{\partial x_2} & \cdots & \dst\frac{\partial f_m(\mathbf{X}_0)}{\partial x_n} \end{bmatrix}. \tag{6.1.10}$$

(It is important to bear in mind that while $\mathbf{F}$ is a function from $\R^n$ to $\R^m$, $\mathbf{F}'$ is not such a function; $\mathbf{F}'$ is an $m \times n$ matrix.) From Theorem~, the differential can be written in terms of the differential matrix as

$$d_{\mathbf{X}_0}\mathbf{F} = \mathbf{F}'(\mathbf{X}_0) \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{bmatrix} \tag{6.1.11}$$

or, more succinctly, as

$$d_{\mathbf{X}_0}\mathbf{F} = \mathbf{F}'(\mathbf{X}_0)\, d\mathbf{X},$$

where

$$d\mathbf{X} = \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{bmatrix},$$

as defined earlier.

When it is not necessary to emphasize the particular point $\mathbf{X}_0$, we write as

$$d\mathbf{F} = \begin{bmatrix} df_1 \\ df_2 \\ \vdots \\ df_m \end{bmatrix},$$

as .5pc

$$\mathbf{F}' = \begin{bmatrix} \dst\frac{\partial f_1}{\partial x_1} & \dst\frac{\partial f_1}{\partial x_2} & \cdots & \dst\frac{\partial f_1}{\partial x_n} \\ [3\jot] \dst\frac{\partial f_2}{\partial x_1} & \dst\frac{\partial f_2}{\partial x_2} & \cdots & \dst\frac{\partial f_2}{\partial x_n} \\ [3\jot] \vdots & \vdots & \ddots & \vdots \\ [3\jot] \dst\frac{\partial f_m}{\partial x_1} & \dst\frac{\partial f_m}{\partial x_2} & \cdots & \dst\frac{\partial f_m}{\partial x_n} \end{bmatrix},$$

.5pc and as

$$d\mathbf{F} = \mathbf{F}'\, d\mathbf{X}.$$

With the differential notation we can rewrite as

$$\lim_{\mathbf{X}\to\mathbf{X}_0} \frac{\mathbf{F}(\mathbf{X}) - \mathbf{F}(\mathbf{X}_0) - \mathbf{F}'(\mathbf{X}_0)(\mathbf{X}-\mathbf{X}_0)}{|\mathbf{X}-\mathbf{X}_0|} = \mathbf{0}.$$

If $m = n$, the differential matrix is square and its determinant is called the {}. The standard notation for this determinant is

$$\frac{\partial(f_1,f_2,\dots,f_n)}{\partial(x_1,x_2,\dots,x_n)}=\begin{vmatrix}\dst\frac{\partial f_1}{\partial x_1} & \dst\frac{\partial f_1}{\partial x_2} & \cdots & \dst\frac{\partial f_1}{\partial x_n}\\ [3\jot]\dst\frac{\partial f_2}{\partial x_1} & \dst\frac{\partial f_2}{\partial x_2} & \cdots & \dst\frac{\partial f_2}{\partial x_n}\\ [3\jot]\vdots & \vdots & \ddots & \vdots\\ [3\jot]\dst\frac{\partial f_n}{\partial x_1} & \dst\frac{\partial f_n}{\partial x_2} & \cdots & \dst\frac{\partial f_n}{\partial x_n}\end{vmatrix}.$$

We will often write the Jacobian of $\mathbf{F}$ more simply as $J(\mathbf{F})$, and its value at $\mathbf{X}_0$ as $J\mathbf{F}(\mathbf{X}_0)$.

Since an $n\times n$ matrix is nonsingular if and only if its determinant is nonzero, it follows that if $\mathbf{F}:\R^n\to\R^n$ is differentiable at $\mathbf{X}_0$, then $\mathbf{F}'(\mathbf{X}_0)$ is nonsingular if and only if $J\mathbf{F}(\mathbf{X}_0)\neq 0$. We will soon use this important fact.

We leave the proof of the following theorem to you (Exercise~).

Theorem~ and Definition~ imply the following theorem.

We say that $\mathbf{F}$ is {} on a set S if S is contained in an open set on which the partial derivatives in are continuous. The next three lemmas give properties of continuously differentiable transformations that we will need later.

Consider the auxiliary function

$$\mathbf{G}(\mathbf{X})=\mathbf{F}(\mathbf{X})-\mathbf{F}'(\mathbf{X}_0)\mathbf{X}. \tag{6.1.12}$$

The components of $\mathbf{G}$ are

$$g_i(\mathbf{X})=f_i(\mathbf{X})-\sum_{j=1}^n\frac{\partial f_i(\mathbf{X}_0)\partial x_j}{x}{}_j,$$

so

$$\frac{\partial g_i(\mathbf{X})}{\partial x_j}=\frac{\partial f_i(\mathbf{X})}{\partial x_j}-\frac{\partial f_i(\mathbf{X}_0)}{\partial x_j}.$$

Thus, $\partial g_i/\partial x_j$ is continuous on N and zero at $\mathbf{X}_0$. Therefore, there is a $\delta>0$ such that

$$\left|\frac{\partial g_i(\mathbf{X})}{\partial x_j}\right|<\frac{\epsilon}{\sqrt{mn}}\quad\text{for}\quad 1\le i\le m,\quad 1\le j\le n,\quad\text{if}\quad|\mathbf{X}-\mathbf{X}_0|<\delta. \tag{6.1.13}$$

Now suppose that $\mathbf{X}$, $\mathbf{Y}\in B_\delta(\mathbf{X}_0)$. By Theorem~,

$$g_i(\mathbf{X})-g_i(\mathbf{Y})=\sum_{j=1}^n\frac{\partial g_i(\mathbf{X}_i)}{\partial x_j}(x_j-y_j), \tag{6.1.14}$$

where $\mathbf{X}_i$ is on the line segment from $\mathbf{X}$ to $\mathbf{Y}$, so $\mathbf{X}_i\in B_\delta(\mathbf{X}_0)$. From , , and Schwarz's inequality,

$$(g_i(\mathbf{X})-g_i(\mathbf{Y}))^2\le\left(\sum_{j=1}^n\left[\frac{\partial g_i(\mathbf{X}_i)}{\partial x_j}\right]^2\right)|\mathbf{X}-\mathbf{Y}|^2<\frac{\epsilon^2}{m}|\mathbf{X}-\mathbf{Y}|^2.$$

Summing this from $i=1$ to $i=m$ and taking square roots yields

$$|\mathbf{G}(\mathbf{X})-\mathbf{G}(\mathbf{Y})|<\epsilon|\mathbf{X}-\mathbf{Y}|\quad\text{if}\quad\mathbf{X},\mathbf{Y}\in B_\delta(\mathbf{X}_0). \tag{6.1.15}$$

To complete the proof, we note that

$$\mathbf{F}(\mathbf{X})-\mathbf{F}(\mathbf{Y})=\mathbf{G}(\mathbf{X})-\mathbf{G}(\mathbf{Y})+\mathbf{F}'(\mathbf{X}_0)(\mathbf{X}-\mathbf{Y}), \tag{6.1.16}$$

so and the triangle inequality imply .

Let $\mathbf{X}$ and $\mathbf{Y}$ be arbitrary points in $D_{\mathbf{F}}$ and let $\mathbf{G}$ be as in . From ,

$$|\mathbf{F}(\mathbf{X})-\mathbf{F}(\mathbf{Y})|\ge\big||\mathbf{F}'(\mathbf{X}_0)(\mathbf{X}-\mathbf{Y})|-|\mathbf{G}(\mathbf{X})-\mathbf{G}(\mathbf{Y})|\big|, \tag{6.1.17}$$

Since

$$\mathbf{X}-\mathbf{Y}=[\mathbf{F}'(\mathbf{X}_0)]^{-1}\mathbf{F}'(\mathbf{X}_0)(\mathbf{X}-\mathbf{Y}),$$

implies that

$$|\mathbf{X}-\mathbf{Y}| \le \frac{1}{r}|\mathbf{F}'(\mathbf{X}_0)(\mathbf{X}-\mathbf{Y}|,$$

so

$$|\mathbf{F}'(\mathbf{X}_0)(\mathbf{X}-\mathbf{Y})| \ge r|\mathbf{X}-\mathbf{Y}|. \tag{6.1.18}$$

Now choose $\delta > 0$ so that holds. Then and imply .

See Exercise~ for a stronger conclusion in the case where $\mathbf{F}$ is linear.

On

$$S = \{(\mathbf{X},\mathbf{Y})\}\mathbf{X},\mathbf{Y} \in D \subset \R^{2n}$$

define

Bracket argument to \\ must be a dimension

Then g is continuous for all $(\mathbf{X},\mathbf{Y})$ in S such that $\mathbf{X} \ne \mathbf{Y}$. We now show that if $\mathbf{X}_0 \in D$, then

$$\lim_{(\mathbf{X},\mathbf{Y})\to(\mathbf{X}_0,\mathbf{X}_0)} g(\mathbf{X},\mathbf{Y}) = 0 = g(\mathbf{X}_0,\mathbf{X}_0); \tag{6.1.19}$$

that is, g is also continuous at points $(\mathbf{X}_0,\mathbf{X}_0)$ in S.

Suppose that $\epsilon > 0$ and $\mathbf{X}_0 \in D$. Since the partial derivatives of f_1, f_2, , f_m are continuous on an open set containing D, there is a $\delta > 0$ such that

$$\left|\frac{\partial f_i(\mathbf{Y})}{\partial x_j} - \frac{\partial f_i(\mathbf{X})}{\partial x_j}\right| < \frac{\epsilon}{\sqrt{mn}} \quad \text{if} \quad \mathbf{X},\mathbf{Y} \in B_\delta(\mathbf{X}_0),\ 1 \le i \le m,\ 1 \le j \le n. \tag{6.1.20}$$

(Note that $\partial f_i/\partial x_j$ is uniformly continuous on $\overline{B_\delta(\mathbf{X}_0)}$ for δ sufficiently small, from Theorem~.) Applying Theorem~ to f_1, f_2, , f_m, we find that if $\mathbf{X}$, $\mathbf{Y} \in B_\delta(\mathbf{X}_0)$, then

$$f_i(\mathbf{Y}) - f_i(\mathbf{X}) = \sum_{j=1}^n \frac{\partial f_i(\mathbf{X}_i)}{\partial x_j}(y_j - x_j),$$

where $\mathbf{X}_i$ is on the line segment from $\mathbf{X}$ to $\mathbf{Y}$. From this,

$$\begin{aligned}\left[f_i(\mathbf{Y}) - f_i(\mathbf{X}) - \dst\sum_{j=1}^n \frac{\partial f_i(\mathbf{X})}{\partial x_j}(y_j - x_j)\right]^2 \ar= \left[\sum_{j=1}^n \left[\frac{\partial f_i(\mathbf{X}_i)}{\partial x_j} - \frac{\partial f_i(\mathbf{X})}{\partial x_j}\right](y_j - x_j)\right]^2 \\ \ar\le |\mathbf{Y}-\mathbf{X}|^2 \sum_{j=1}^n \left[\frac{\partial f_i(\mathbf{X}_i)}{\partial x_j} - \frac{\partial f_i(\mathbf{X})}{\partial x_j}\right]^2 \\ \ar\text{(by Schwarz's inequality)} \\ \ar< \frac{\epsilon^2}{m}|\mathbf{Y}-\mathbf{X}|^2 \qquad \text{(by (6.1.20))} \quad .\end{aligned}$$

Summing from $i = 1$ to $i = m$ and taking square roots yields

$$|\mathbf{F}(\mathbf{Y}) - \mathbf{F}(\mathbf{X}) - \mathbf{F}'(\mathbf{X})(\mathbf{Y}-\mathbf{X})| < \epsilon|\mathbf{Y}-\mathbf{X}| \quad \text{if} \quad \mathbf{X},\mathbf{Y} \in B_\delta(\mathbf{X}_0).$$

This implies and completes the proof that g is continuous on S.

Since D is compact, so is S (Exercise~). Therefore, g is bounded on S (Theorem~); thus, for some M_1,

$$|\mathbf{F}(\mathbf{Y}) - \mathbf{F}(\mathbf{X}) - \mathbf{F}'(\mathbf{X})(\mathbf{Y}-\mathbf{X})| \le M_1|\mathbf{X}-\mathbf{Y}| \quad \text{if} \quad \mathbf{X},\mathbf{Y} \in D.$$

But

$$\begin{aligned}|\mathbf{F}(\mathbf{Y}) - \mathbf{F}(\mathbf{X})|\ar\le |\mathbf{F}(\mathbf{Y}) - \mathbf{F}(\mathbf{X}) - \mathbf{F}'(\mathbf{X})(\mathbf{Y}-\mathbf{X})| + |\mathbf{F}'(\mathbf{X})(\mathbf{Y}-\mathbf{X})| \\ \ar\le (M_1 + \|\mathbf{F}'(\mathbf{X})\|)|(\mathbf{Y}-\mathbf{X}|.\end{aligned} \tag{6.1.21}$$

Since

$$\|\mathbf{F}'(\mathbf{X})\| \le \left(\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial f_i(\mathbf{X})}{\partial x_j}\right]^2\right)^{1/2}$$

and the partial derivatives $\{\partial f_i/\partial x_j\}$ are bounded on D, it follows that $\|\mathbf{F}'(\mathbf{X})\|$ is bounded on D; that is, there is a constant M_2 such that

$$\|\mathbf{F}'(\mathbf{X})\| \le M_2, \quad \mathbf{X} \in D.$$

Now implies with $M = M_1 + M_2$.

By using differential matrices, we can write the chain rule for transformations in a form analogous to the form of the chain rule for real-valued functions of one variable (Theorem~).

The components of $\mathbf{H}$ are $h_1, h_2, , h_m$, where

$$h_i(\mathbf{U}) = f_i(\mathbf{G}(\mathbf{U})).$$

Applying Theorem~ to h_i yields

$$d_{\mathbf{U}_0}h_i = \sum_{j=1}^{n}\frac{\partial f_i(\mathbf{X}_0)}{\partial x_j}d_{\mathbf{U}_0}g_j, \quad 1 \le i \le m. \tag{6.1.22}$$

Since

$$d_{\mathbf{U}_0}\mathbf{H} = \begin{bmatrix} d_{\mathbf{U}_0}h_1 \\ d_{\mathbf{U}_0}h_2 \\ \vdots \\ d_{\mathbf{U}_0}h_m \end{bmatrix} \quad \text{and} \quad d_{\mathbf{U}_0}\mathbf{G} = \begin{bmatrix} d_{\mathbf{U}_0}g_1 \\ d_{\mathbf{U}_0}g_2 \\ \vdots \\ d_{\mathbf{U}_0}g_n \end{bmatrix},$$

5pt the m equations in can be written in matrix form as

$$d_{\mathbf{U}_0}\mathbf{H} = \mathbf{F}'(\mathbf{X}_0)d_{\mathbf{U}_0}\mathbf{G} = \mathbf{F}'(\mathbf{G}(\mathbf{U}_0))d_{\mathbf{U}_0}\mathbf{G}. \tag{6.1.23}$$

But

$$d_{\mathbf{U}_0}\mathbf{G} = \mathbf{G}'(\mathbf{U}_0)\,d\mathbf{U},$$

where

$$d\mathbf{U} = \begin{bmatrix} du_1 \\ du_2 \\ \vdots \\ du_k \end{bmatrix},$$

so can be rewritten as

$$d_{\mathbf{U}_0}\mathbf{H} = \mathbf{F}'(\mathbf{G}(\mathbf{U}_0))\mathbf{G}'(\mathbf{U}_0)\,d\mathbf{U}.$$

On the other hand,

$$d_{\mathbf{U}_0}\mathbf{H} = \mathbf{H}'(\mathbf{U}_0)\,d\mathbf{U}.$$

Comparing the last two equations yields . Since $\mathbf{G}'(\mathbf{U}_0)$ is the matrix of $d_{\mathbf{U}_0}\mathbf{G}$ and $\mathbf{F}'(\mathbf{G}(\mathbf{U}_0)) = \mathbf{F}'(\mathbf{X}_0)$ is the matrix of $d_{\mathbf{X}_0}\mathbf{F}$, Theorem~

and imply~.

So far our discussion of transformations has dealt mainly with properties that could just as well be defined and studied by considering the component functions individually. Now we turn to questions involving a transformation as a whole, that cannot be studied by regarding it as a collection of independent component functions.

In this section we restrict our attention to transformations from $\R^n$ to itself. It is useful to interpret such transformations geometrically. If $\mathbf{F} = (f_1, f_2, \dots, f_n)$, we can think of the components of

$$\mathbf{F}(\mathbf{X}) = (f_1(\mathbf{X}), f_2(\mathbf{X}), \dots, f_n(\mathbf{X}))$$

as the coordinates of a point $\mathbf{U} = \mathbf{F}(\mathbf{X})$ in another ``copy'' of \R^n. Thus, $\mathbf{U} = (u_1, u_2, \dots, u_n)$, with

$$u_1 = f_1(\mathbf{X}), \quad u_2 = f_2(\mathbf{X}), \quad \dots, \quad u_n = f_n(\mathbf{X}).$$

We say that $\mathbf{F}$ {}, and that $\mathbf{U}$ {} $\mathbf{F}$. Occasionally we will also write $\partial u_i/\partial x_j$ to mean $\partial f_i/\partial x_j$. If $S \subset D_{\mathbf{F}}$, then the set

$$\mathbf{F}(S) = \{\mathbf{U} \} \mathbf{U} = \mathbf{F}(\mathbf{X}),\ \mathbf{X} \in S$$

is the {}.

We will often denote the components of $\mathbf{X}$ by x, y, , and the components of $\mathbf{U}$ by u, v, .

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-.15em A transformation $\mathbf{F}$ is {}, or {}, if $\mathbf{F}(\mathbf{X}_1)$ and $\mathbf{F}(\mathbf{X}_2)$ are distinct whenever $\mathbf{X}_1$ and $\mathbf{X}_2$ are distinct points of $D_{\mathbf{F}}$. In this case, we can define a function $\mathbf{G}$ on the range

$$R(\mathbf{F}) = \{\mathbf{U} \} \mathbf{U} = \mathbf{F}(\mathbf{X}) \text{ for some } \mathbf{X} \in D_{\mathbf{F}}$$

of $\mathbf{F}$ by defining $\mathbf{G}(\mathbf{U})$ to be the unique point in $D_{\mathbf{F}}$ such that $\mathbf{F}(\mathbf{U}) = \mathbf{U}$. Then

$$D_{\mathbf{G}} = R(\mathbf{F}) \quad \text{and} \quad R(\mathbf{G}) = D_{\mathbf{F}}.$$

Moreover, $\mathbf{G}$ is one-to-one,

$$\mathbf{G}(\mathbf{F}(\mathbf{X})) = \mathbf{X}, \quad \mathbf{X} \in D_{\mathbf{F}},$$

and

$$\mathbf{F}(\mathbf{G}(\mathbf{U})) = \mathbf{U}, \quad \mathbf{U} \in D_{\mathbf{G}}.$$

We say that $\mathbf{G}$ is the {} of $\mathbf{F}$, and write $\mathbf{G} = \mathbf{F}^{-1}$. The relation between $\mathbf{F}$ and $\mathbf{G}$ is symmetric; that is, $\mathbf{F}$ is also the inverse of $\mathbf{G}$, and we write $\mathbf{F} = \mathbf{G}^{-1}$.

The crucial difference between the transformations of Examples~ and is that the matrix of $\mathbf{L}$ is nonsingular while the matrix of $\mathbf{L}_1$ is singular. Thus, $\mathbf{L}$ (see) can be written as

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \tag{6.1.24}$$

where the matrix has the inverse

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ [3\jot] -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

(Verify.) Multiplying both sides of by this matrix yields

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ [3\jot] -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix},$$

which is equivalent to .

Since the matrix

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

of $\mathbf{L}_1$ is singular, cannot be solved uniquely for (x, y) in terms of (u, v). In fact, it cannot be solved at all unless $v = 2u$.

The following theorem settles the question of invertibility of linear transformations from \R^n to \R^n. We leave the proof to you (Exercise~).

-.5em We will now briefly review polar coordinates, which we will use in some of the following examples.

The coordinates of any point (x,y) can be written in infinitely many ways as

$$x = r\cos\theta, \quad y = r\sin\theta, \tag{6.1.25}$$

where

$$r^2 = x^2 + y^2$$

and, if $r > 0$, θ is the angle from the x-axis to the line segment from $(0,0)$ to (x,y), measured counterclockwise (Figure~).

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For each $(x,y) \neq (0,0)$ there are infinitely many values of θ, differing by integral multiples of 2π, that satisfy . If θ is any of these values, we say that θ is an {} of (x,y), and write

$$\theta = \arg(x,y).$$

By itself, this does not define a function. However, if ϕ is an arbitrary fixed number, then

$$\theta = \arg(x,y), \quad \phi \le \theta < \phi + 2\pi,$$

does define a function, since every half-open interval $[\phi, \phi + 2\pi)$ contains exactly one argument of (x,y).

We do not define $\arg(0,0)$, since places no restriction on θ if $(x,y) = (0,0)$ and therefore $r = 0$.

The transformation

$$\begin{bmatrix} r \\ \theta \end{bmatrix} = \mathbf{G}(x,y) = \begin{bmatrix} \sqrt{x^2+y^2} \\ [3\jot] \arg(x,y) \end{bmatrix}, \quad \phi \le \arg(x,y) < \phi + 2\pi,$$

is defined and one-to-one on

$$D_{\mathbf{G}} = \{(x,y)\}(x,y) \neq (0,0),$$

and its range is

$$R(\mathbf{G}) = \{(r,\theta)\}r > 0,\ \phi \le \theta < \phi + 2\pi.$$

For example, if $\phi = 0$, then

$$\mathbf{G}(1,1) = \begin{bmatrix} \sqrt{2} \\ [3\jot] \dst \frac{\pi}{4} \end{bmatrix},$$

since $\pi/4$ is the unique argument of $(1,1)$ in $[0, 2\pi)$. If $\phi = \pi$, then

$$\mathbf{G}(1,1) = \begin{bmatrix} \sqrt{2} \\ [3\jot] \dst \frac{9\pi}{4} \end{bmatrix},$$

since $9\pi/4$ is the unique argument of $(1,1)$ in $[\pi, 3\pi)$.

If $\arg(x_0,y_0) = \phi$, then (x_0,y_0) is on the half-line shown in Figure~ and $\mathbf{G}$ is not continuous at (x_0,y_0), since every neighborhood of (x_0,y_0) contains points (x,y) for which the second component of $\mathbf{G}(x,y)$ is arbitrarily close to $\phi + 2\pi$, while the second component of $\mathbf{G}(x_0,y_0)$ is ϕ. We will show later, however, that $\mathbf{G}$ is continuous, in fact, continuously differentiable, on the plane with this half-line deleted.

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A transformation $\mathbf{F}$ may fail to be one-to-one, but be one-to-one on a subset S of $D_{\mathbf{F}}$. By this we mean that $\mathbf{F}(\mathbf{X}_1)$ and $\mathbf{F}(\mathbf{X}_2)$ are distinct whenever $\mathbf{X}_1$ and $\mathbf{X}_2$ are distinct points of S. In this case, $\mathbf{F}$ is not invertible, but if $\mathbf{F}_S$ is defined on S by

$$\mathbf{F}_S(\mathbf{X}) = \mathbf{F}(\mathbf{X}), \quad \mathbf{X} \in S,$$

and left undefined for $\mathbf{X} \notin S$, then $\mathbf{F}_S$ is invertible. We say that $\mathbf{F}_S$ is the {}, and that $\mathbf{F}_S^{-1}$ is the {}. The domain of $\mathbf{F}_S^{-1}$ is $\mathbf{F}(S)$.

If $\mathbf{F}$ is one-to-one on a neighborhood of $\mathbf{X}_0$, we say that $\mathbf{F}$ is {}. If this is true for every $\mathbf{X}_0$ in a set S, then $\mathbf{F}$ is {}.

The question of invertibility of an arbitrary transformation $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is too general to have a useful answer. However, there is a useful and easily applicable sufficient condition which implies that one-to-one restrictions of continuously differentiable transformations have continuously differentiable inverses.

To motivate our study of this question, let us first consider the linear transformation

$$\mathbf{F}(\mathbf{X}) = \mathbf{A}\mathbf{X} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

From Theorem~, $\mathbf{F}$ is invertible if and only if $\mathbf{A}$ is nonsingular, in which case $R(\mathbf{F}) = \mathbb{R}^n$ and

$$\mathbf{F}^{-1}(\mathbf{U}) = \mathbf{A}^{-1}\mathbf{U}.$$

Since $\mathbf{A}$ and $\mathbf{A}^{-1}$ are the differential matrices of $\mathbf{F}$ and $\mathbf{F}^{-1}$, respectively, we can say that a linear transformation is invertible if and only if its differential matrix $\mathbf{F}'$ is nonsingular, in which case the differential matrix of $\mathbf{F}^{-1}$ is given by

$$(\mathbf{F}^{-1})' = (\mathbf{F}')^{-1}.$$

Because of this, it is tempting to conjecture that if $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable and $\mathbf{A}'(\mathbf{X})$ is nonsingular, or, equivalently, $J\mathbf{F}(\mathbf{X}) \neq 0$, for $\mathbf{X}$ in a set S, then $\mathbf{F}$ is one-to-one on S. However, this is false. For example, if

$$\mathbf{F}(x, y) = \begin{bmatrix} e^x \cos y \\ e^x \sin y \end{bmatrix},$$

then

$$J\mathbf{F}(x, y) = \begin{vmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{vmatrix} = e^{2x} \neq 0, \tag{6.1.26}$$

but $\mathbf{F}$ is not one-to-one on $\mathbb{R}^2$ (Example~). The best that can be said in general is that if $\mathbf{F}$ is continuously differentiable and $J\mathbf{F}(\mathbf{X}) \neq 0$ in an open set S, then $\mathbf{F}$ is locally invertible on S, and the local inverses are continuously differentiable. This is part of the inverse function theorem, which we will prove presently. First, we need the following definition.

We first show that if $\mathbf{X}_0 \in S$, then a neighborhood of $\mathbf{F}(\mathbf{X}_0)$ is in $\mathbf{F}(S)$. This implies that $\mathbf{F}(S)$ is open.

Since S is open, there is a $\rho > 0$ such that $\overline{B_\rho(\mathbf{X}_0)} \subset S$. Let B be the boundary of $B_\rho(\mathbf{X}_0)$; thus,

$$B = \{\mathbf{X}\} |\mathbf{X} - \mathbf{X}_0| = \rho. \tag{6.1.27}$$

The function

$$\sigma(\mathbf{X}) = |\mathbf{F}(\mathbf{X}) - \mathbf{F}(\mathbf{X}_0)|$$

is continuous on S and therefore on B, which is compact. Hence, by Theorem~, there is a point $\mathbf{X}_1$ in B where $\sigma(\mathbf{X})$ attains its minimum value, say m, on B. Moreover, $m > 0$, since $\mathbf{X}_1 \neq \mathbf{X}_0$ and $\mathbf{F}$ is one-to-one on S. Therefore,

$$|\mathbf{F}(\mathbf{X}) - \mathbf{F}(\mathbf{X}_0)| \geq m > 0 \quad \text{if} \quad |\mathbf{X} - \mathbf{X}_0| = \rho. \tag{6.1.28}$$

The set

$$\{\mathbf{U}\} |\mathbf{U} - \mathbf{F}(\mathbf{X}_0)| < m/2$$

is a neighborhood of $\mathbf{F}(\mathbf{X}_0)$. We will show that it is a subset of $\mathbf{F}(S)$. To see this, let $\mathbf{U}$ be a fixed point in this set; thus,

$$|\mathbf{U} - F(\mathbf{X}_0)| < m/2. \tag{6.1.29}$$

Consider the function

$$\sigma_1(\mathbf{X}) = |\mathbf{U} - \mathbf{F}(\mathbf{X})|^2,$$

which is continuous on S. Note that

$$\sigma_1(\mathbf{X}) \geq \frac{m^2}{4} \quad \text{if} \quad |\mathbf{X} - \mathbf{X}_0| = \rho, \tag{6.1.30}$$

since if $|\mathbf{X} - \mathbf{X}_0| = \rho$, then

$$\begin{aligned} |\mathbf{U} - \mathbf{F}(\mathbf{X})| \ar &= |(\mathbf{U} - \mathbf{F}(\mathbf{X}_0)) + (\mathbf{F}(\mathbf{X}_0) - \mathbf{F}(\mathbf{X}))| \\ \ar &\geq \big||\mathbf{F}(\mathbf{X}_0) - \mathbf{F}(\mathbf{X})| - |\mathbf{U} - \mathbf{F}(\mathbf{X}_0)|\big| \\ \ar &\geq m - \frac{m}{2} = \frac{m}{2}, \end{aligned}$$

from and .

Since σ_1 is continuous on S, σ_1 attains a minimum value μ on the compact set $\overline{B_\rho(\mathbf{X}_0)}$ (Theorem~); that is, there is an $\overline{\mathbf{X}}$ in $\overline{B_\rho(\mathbf{X}_0)}$ such that

$$\sigma_1(\mathbf{X}) \geq \sigma_1(\overline{\mathbf{X}}) = \mu, \quad \mathbf{X} \in \overline{B_\rho(\mathbf{X}_0)}.$$

Setting $\mathbf{X} = \mathbf{X}_0$, we conclude from this and that

$$\sigma_1(\overline{\mathbf{X}}) = \mu \leq \sigma_1(\mathbf{X}_0) < \frac{m^2}{4}.$$

Because of and , this rules out the possibility that $\overline{\mathbf{X}} \in B$, so $\overline{\mathbf{X}} \in B_\rho(\mathbf{X}_0)$.

Now we want to show that $\mu = 0$; that is, $\mathbf{U} = \mathbf{F}(\overline{\mathbf{X}})$. To this end, we note that $\sigma_1(\mathbf{X})$ can be written as

$$\sigma_1(\mathbf{X}) = \sum_{j=1}^{n} (u_j - f_j(\mathbf{X}))^2,$$

-.4em so σ_1 is differentiable on $B_\rho(\mathbf{X}_0)$. Therefore, the first partial derivatives of σ_1 are all zero at the local minimum point $\overline{\mathbf{X}}$ (Theorem~), so 3pt

$$\sum_{j=1}^{n} \frac{\partial f_j(\overline{\mathbf{X}})}{\partial x_i}(u_j - f_j(\overline{\mathbf{X}})) = 0, \quad 1 \leq i \leq n,$$

or, in matrix form,

$$\mathbf{F}'(\overline{\mathbf{X}})(\mathbf{U} - \mathbf{F}(\overline{\mathbf{X}})) = \mathbf{0}.$$

Since $\mathbf{F}'(\overline{\mathbf{X}})$ is nonsingular this implies that $\mathbf{U} = \mathbf{F}(\overline{\mathbf{X}})$ (Theorem~). Thus, we have shown that every $\mathbf{U}$ that satisfies is in $\mathbf{F}(S)$. Therefore, since $\mathbf{X}_0$ is an arbitrary point of S, $\mathbf{F}(S)$ is open.

Next, we show that $\mathbf{G}$ is continuous on $\mathbf{F}(S)$. Suppose that $\mathbf{U}_0 \in \mathbf{F}(S)$ and $\mathbf{X}_0$ is the unique point in S such that $\mathbf{F}(\mathbf{X}_0) = \mathbf{U}_0$. Since $\mathbf{F}'(\mathbf{X}_0)$ is invertible, Lemma~ implies that there is a $\lambda > 0$ and an open neighborhood N of $\mathbf{X}_0$ such that $N \subset S$ and

$$|\mathbf{F}(\mathbf{X}) - \mathbf{F}(\mathbf{X}_0)| \geq \lambda |\mathbf{X} - \mathbf{X}_0| \quad \text{if} \quad \mathbf{X} \in N. \tag{6.1.31}$$

(Exercise~ also implies this.) Since $\mathbf{F}$ satisfies the hypotheses of the present theorem on N, the first part of this proof shows that $\mathbf{F}(N)$ is an open set containing $\mathbf{U}_0 = \mathbf{F}(\mathbf{X}_0)$. Therefore, there is a $\delta > 0$ such that $\mathbf{X} = \mathbf{G}(\mathbf{U})$ is in N if $\mathbf{U} \in B_\delta(\mathbf{U}_0)$. Setting $\mathbf{X} = \mathbf{G}(\mathbf{U})$ and $\mathbf{X}_0 = \mathbf{G}(\mathbf{U}_0)$ in yields

$$|\mathbf{F}(\mathbf{G}(\mathbf{U})) - \mathbf{F}(\mathbf{G}(\mathbf{U}_0))| \geq \lambda |\mathbf{G}(\mathbf{U}) - \mathbf{G}(\mathbf{U}_0)| \quad \text{if} \quad \mathbf{U} \in B_\delta(\mathbf{U}_0).$$

Since $\mathbf{F}(\mathbf{G}(\mathbf{U})) = \mathbf{U}$, this can be rewritten as

$$|\mathbf{G}(\mathbf{U}) - \mathbf{G}(\mathbf{U}_0)| \leq \frac{1}{\lambda} |\mathbf{U} - \mathbf{U}_0| \quad \text{if} \quad \mathbf{U} \in B_\delta(\mathbf{U}_0), \tag{6.1.32}$$

which means that $\mathbf{G}$ is continuous at $\mathbf{U}_0$. Since $\mathbf{U}_0$ is an arbitrary point in $\mathbf{F}(S)$, it follows that $\mathbf{G}$ is continous on $\mathbf{F}(S)$.

We will now show that $\mathbf{G}$ is differentiable at $\mathbf{U}_0$. Since

$$\mathbf{G}(\mathbf{F}(\mathbf{X}))=\mathbf{X},\quad \mathbf{X}\in S,$$

the chain rule (Theorem~) implies that {} $\mathbf{G}$ is differentiable at $\mathbf{U}_0$, then

$$\mathbf{G}'(\mathbf{U}_0)\mathbf{F}'(\mathbf{X}_0)=\mathbf{I}$$

(Example~). Therefore, if $\mathbf{G}$ is differentiable at $\mathbf{U}_0$, the differential matrix of $\mathbf{G}$ must be

$$\mathbf{G}'(\mathbf{U}_0)=[\mathbf{F}'(\mathbf{X}_0)]^{-1},$$

so to show that $\mathbf{G}$ is differentiable at $\mathbf{U}_0$, we must show that if

$$\mathbf{H}(\mathbf{U})=\frac{\mathbf{G}(\mathbf{U})-\mathbf{G}(\mathbf{U}_0)-[\mathbf{F}'(\mathbf{X}_0)]^{-1}(\mathbf{U}-\mathbf{U}_0)}{|\mathbf{U}-\mathbf{U}_0|}\quad(\mathbf{U}\neq\mathbf{U}_0), \tag{6.1.33}$$

then

$$\lim_{\mathbf{U}\to\mathbf{U}_0}\mathbf{H}(\mathbf{U})=\mathbf{0}. \tag{6.1.34}$$

Since $\mathbf{F}$ is one-to-one on S and $\mathbf{F}(\mathbf{G}(\mathbf{U}))=\mathbf{U}$, it follows that if $\mathbf{U}\neq\mathbf{U}_0$, then $\mathbf{G}(\mathbf{U})\neq\mathbf{G}(\mathbf{U}_0)$. Therefore, we can multiply the numerator and denominator of by $|\mathbf{G}(\mathbf{U})-\mathbf{G}(\mathbf{U}_0)|$ to obtain

$$\mathbf{H}(\mathbf{U})\ar=\dst\frac{|\mathbf{G}(\mathbf{U})-\mathbf{G}(\mathbf{U}_0|}{|\mathbf{U}-\mathbf{U}_0|}\left(\frac{\mathbf{G}(\mathbf{U})-\mathbf{G}(\mathbf{U}_0)-[\mathbf{F}'(\mathbf{X}_0)]^{-1}(\mathbf{U}-\mathbf{U}_0)}{|\mathbf{G}(\mathbf{U})-\mathbf{G}(\mathbf{U}_0)|}\right)$$

$$\ar=-\dst\frac{|\mathbf{G}(\mathbf{U})-\mathbf{G}(\mathbf{U}_0)|}{|\mathbf{U}-\mathbf{U}_0|}[\mathbf{F}'(\mathbf{X}_0)]^{-1}\left(\frac{\mathbf{U}-\mathbf{U}_0-\mathbf{F}'(\mathbf{X}_0)(\mathbf{G}(\mathbf{U})-\mathbf{G}(\mathbf{U}_0))}{|\mathbf{G}(\mathbf{U})-\mathbf{G}(\mathbf{U}_0)|}\right)$$

if $0<|\mathbf{U}-\mathbf{U}_0|<\delta$. Because of , this implies that

$$|\mathbf{H}(\mathbf{U})|\le\frac{1}{\lambda}\|[\mathbf{F}'(\mathbf{X}_0)]^{-1}\|\left|\frac{\mathbf{U}-\mathbf{U}_0-\mathbf{F}'(\mathbf{X}_0)(\mathbf{G}(\mathbf{U})-\mathbf{G}(\mathbf{U}_0))}{|\mathbf{G}(\mathbf{U})-\mathbf{G}(\mathbf{U}_0)|}\right|$$

if $0<|\mathbf{U}-\mathbf{U}_0|<\delta$. Now let

$$\mathbf{H}_1(\mathbf{U})=\frac{\mathbf{U}-\mathbf{U}_0-\mathbf{F}'(\mathbf{X}_0)(\mathbf{G}(\mathbf{U})-\mathbf{G}(\mathbf{U}_0))}{|\mathbf{G}(\mathbf{U})-\mathbf{G}(\mathbf{U}_0)|}$$

To complete the proof of , we must show that

$$\lim_{\mathbf{U}\to\mathbf{U}_0}\mathbf{H}_1(\mathbf{U})=\mathbf{0}. \tag{6.1.35}$$

Since $\mathbf{F}$ is differentiable at $\mathbf{X}_0$, we know that if

$$\mathbf{H}_2(\mathbf{X})=\lim_{\mathbf{X}\to\mathbf{X}_0}\frac{\mathbf{F}(\mathbf{X})-\mathbf{F}(\mathbf{X}_0)-\mathbf{F}'(\mathbf{X}_0)(\mathbf{X}-\mathbf{X}_0)}{|\mathbf{X}-\mathbf{X}_0|},$$

then

$$\lim_{\mathbf{X}\to\mathbf{X}_0}\mathbf{H}_2(\mathbf{X})=\mathbf{0}. \tag{6.1.36}$$

Since $\mathbf{F}(\mathbf{G}(\mathbf{U}))=\mathbf{U}$ and $\mathbf{X}_0=\mathbf{G}(\mathbf{U}_0)$,

$$\mathbf{H}_1(\mathbf{U})=\mathbf{H}_2(\mathbf{G}(\mathbf{U})).$$

Now suppose that $\epsilon>0$. From , there is a $\delta_1>0$ such that

$$|\mathbf{H}_2(\mathbf{X})|<\epsilon\quad\text{if}\quad 0<|\mathbf{X}-\mathbf{X}_0|=|\mathbf{X}-\mathbf{G}(\mathbf{U}_0)|<\delta_1. \tag{6.1.37}$$

Since $\mathbf{G}$ is continuous at $\mathbf{U}_0$, there is a $\delta_2\in(0,\delta)$ such that

$$|\mathbf{G}(\mathbf{U})-\mathbf{G}(\mathbf{U}_0)|<\delta_1\quad\text{if}\quad 0<|\mathbf{U}-\mathbf{U}_0|<\delta_2.$$

This and imply that

$$|\mathbf{H}_1(\mathbf{U})|=|\mathbf{H}_2(\mathbf{G}(\mathbf{U}))|<\epsilon\quad\text{if}\quad 0<|\mathbf{U}-\mathbf{U}_0|<\delta_2.$$

Since this implies , $\mathbf{G}$ is differentiable at $\mathbf{X}_0$.

Since $\mathbf{U}_0$ is an arbitrary member of $\mathbf{F}(N)$, we can now drop the zero subscript and conclude that $\mathbf{G}$ is continuous and differentiable on $\mathbf{F}(N)$, and

$$\mathbf{G}'(\mathbf{U}) = [\mathbf{F}'(\mathbf{X})]^{-1}, \quad \mathbf{U} \in \mathbf{F}(N).$$

To see that $\mathbf{G}$ is on $\mathbf{F}(N)$, we observe that by Theorem~, each entry of $\mathbf{G}'(\mathbf{U})$ (that is, each partial derivative $\partial g_i(\mathbf{U})/\partial u_j$, $1 \le i, j \le n$) can be written as the ratio, with nonzero denominator, of determinants with entries of the form

$$\frac{\partial f_r(\mathbf{G}(\mathbf{U}))}{\partial x_s}. \tag{6.1.38}$$

Since $\partial f_r/\partial x_s$ is continuous on N and $\mathbf{G}$ is continuous on $\mathbf{F}(N)$, Theorem~ implies that is continuous on $\mathbf{F}(N)$. Since a determinant is a continuous function of its entries, it now follows that the entries of $\mathbf{G}'(\mathbf{U})$ are continuous on $\mathbf{F}(N)$.

-.4em If $\mathbf{F}$ is regular on an open set S, we say that $\mathbf{F}_S^{-1}$ is a {} $\mathbf{F}^{-1}$. (This is a convenient terminology but is not meant to imply that $\mathbf{F}$ actually has an inverse.) From this definition, it is possible to define a branch of $\mathbf{F}^{-1}$ on a set $T \subset R(\mathbf{F})$ if and only if $T = \mathbf{F}(S)$, where $\mathbf{F}$ is regular on S. There may be open subsets of $R(\mathbf{F})$ that do not have this property, and therefore no branch of $\mathbf{F}^{-1}$ can be defined on them. It is also possible that $T = \mathbf{F}(S_1) = \mathbf{F}(S_2)$, where S_1 and S_2 are distinct subsets of $D_{\mathbf{F}}$. In this case, more than one branch of $\mathbf{F}^{-1}$ is defined on T. Thus, we saw in Example~ that two branches of $\mathbf{F}^{-1}$ may be defined on a set T. In Example~ infinitely many branches of $\mathbf{F}^{-1}$ are defined on the same set.

It is useful to define branches of the argument To do this, we think of the relationship between polar and rectangular coordinates in terms of the transformation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{F}(r, \theta) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}, \tag{6.1.39}$$

where for the moment we regard r and θ as rectangular coordinates of a point in an $r\theta$-plane. Let S be an open subset of the right half of this plane (that is, $S \subset \{(r, \theta)\} r > 0$)

that does not contain any pair of points (r, θ) and $(r, \theta + 2k\pi)$, where k is a nonzero integer. Then $\mathbf{F}$ is one-to-one and continuously differentiable on S, with

$$\mathbf{F}'(r, \theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \tag{6.1.40}$$

and

$$J\mathbf{F}(r, \theta) = r > 0, \quad (r, \theta) \in S. \tag{6.1.41}$$

Hence, $\mathbf{F}$ is regular on S. Now let $T = \mathbf{F}(S)$, the set of points in the xy-plane with polar coordinates in S. Theorem~ states that T is open and $\mathbf{F}_S$ has a continuously differentiable inverse (which we denote by $\mathbf{G}$, rather than $\mathbf{F}_S^{-1}$, for typographical reasons)

$$\begin{bmatrix} r \\ \theta \end{bmatrix} = \mathbf{G}(x, y) = \begin{bmatrix} \sqrt{x^2 + y^2} \\ [3\jot] \arg_S(x, y) \end{bmatrix}, \quad (x, y) \in T,$$

where $\arg_S(x, y)$ is the unique value of $\arg(x, y)$ such that

$$(r, \theta) = \left(\sqrt{x^2 + y^2}, \arg_S(x, y)\right) \in S.$$

We say that $\arg_S(x, y)$ is a {}. Theorem~ also implies that

$$\begin{aligned} \mathbf{G}'(x, y) = [\mathbf{F}'(r, \theta)]^{-1} \ar &= \begin{bmatrix} \cos\theta & \sin\theta \\ \dst -\frac{\sin\theta}{r} & \dst \frac{\cos\theta}{r} \end{bmatrix} \quad \text{(see (6.1.40))} \\ \ar &= \begin{bmatrix} \dst \frac{x}{\sqrt{x^2+y^2}} & \dst \frac{y}{\sqrt{x^2+y^2}} \\ [3\jot] \dst -\frac{y}{x^2+y^2} & \dst \frac{x}{x^2+y^2} \end{bmatrix} \quad \text{(see (6.1.39))}. \end{aligned}$$

1pc Therefore,

$$\frac{\partial \arg_S(x,y)}{\partial x} = -\frac{y}{x^2+y^2}, \quad \frac{\partial \arg_S(x,y)}{\partial y} = \frac{x}{x^2+y^2}. \tag{6.1.42}$$

A branch of $\arg(x,y)$ can be defined on an open set T of the xy-plane if and only if the polar coordinates of the points in T form an open subset of the $r\theta$-plane that does not intersect the θ-axis or contain any two points of the form (r,θ) and $(r,\theta+2k\pi)$, where k is a nonzero integer. No subset containing the origin $(x,y)=(0,0)$ has this property, nor does any deleted neighborhood of the origin (Exercise~), so there are open sets on which no branch of the argument can be defined. However, if one branch can be defined on T, then so can infinitely many others. (Why?) All branches of $\arg(x,y)$ have the same partial derivatives, given in .

We leave it to you (Exercise~) to verify that and can also be obtained by differentiating directly.

\begin{example}\rm If

$$\left[\begin{array}{c} u \\ v \end{array}\right] = \mathbf{F}(x,y) = \left[\begin{array}{c} e^x\cos y \\ e^x \sin y \end{array}\right]$$

(Example~), we can also define a branch $\mathbf{G}$ of $\mathbf{F}^{-1}$ on any subset T of the uv-plane on which a branch of $\arg(u,v)$ can be defined, and $\mathbf{G}$ has the form

$$\left[\begin{array}{c} x \\ y \end{array}\right] = \mathbf{G}(u,v) = \left[\begin{array}{c} \log\,(u^2+v^2)^{1/2} \\ \arg(u,v) \end{array}\right]. \tag{6.1.43}$$

Since the branches of the argument differ by integral multiples of 2π, implies that if $\mathbf{G}_1$ and $\mathbf{G}_2$ are branches of $\mathbf{F}^{-1}$, both defined on T, then

$$\mathbf{G}_1(u,v) - \mathbf{G}_2(u,v) = \left[\begin{array}{c} 0 \\ 2k\pi \end{array}\right] \quad (k = \text{integer}).$$

From Theorem~,

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Substituting for x and y in terms of u and v from , we find that

$$\frac{\partial x}{\partial u}\ar = \frac{\partial y}{\partial v} = e^{-x}\cos y = e^{-2x}u = \frac{u}{u^2+v^2}$$
$$\arraytextand$$
$$\frac{\partial x}{\partial v}\ar = -\frac{\partial y}{\partial u} = e^{-x}\sin y = e^{-2x}v = \frac{v}{u^2+v^2}.$$

\end{example}

Examples~ and show that a continuously differentiable function $\mathbf{F}$ may fail to have an inverse on a set S even if $J\mathbf{F}(\mathbf{X}) \neq 0$ on S. However, the next theorem shows that in this case $\mathbf{F}$ is locally invertible on S.

Unknown environment 'theorem'

Lemma~ implies that there is an open neighborhood N of $\mathbf{X}_0$ on which $\mathbf{F}$ is one-to-one. The rest of the conclusions then follow from applying Theorem~ to $\mathbf{F}$ on N.

By continuity, since $J\mathbf{F}'(\mathbf{X}_0) \neq 0$, $J\mathbf{F}'(\mathbf{X})$ is nonzero for all $\mathbf{X}$ in some open neighborhood S of $\mathbf{X}_0$. Now apply Theorem~.

Theorem~ and imply that the transformation is locally invertible on $S = \{(r,\theta)\}r > 0$, which means that it is possible to define a branch of $\arg(x,y)$ in a neighborhood of any point $(x_0,y_0) \neq (0,0)$. It also implies, as we have already seen, that

the transformation of Example~ is locally invertible everywhere except at $(0,0)$, where its Jacobian equals zero, and the transformation of Example~ is locally invertible everywhere.

In this section we consider transformations from $\R^{n+m}$ to $\R^m$. It will be convenient to denote points in $\R^{n+m}$ by

$$(\mathbf{X},\mathbf{U}) = (x_1,x_2,\dots,x_n,u_1,u_2,\dots,u_m).$$

We will often denote the components of $\mathbf{X}$ by x, y, , and the components of $\mathbf{U}$ by u, v, .

To motivate the problem we are interested in, we first ask whether the linear system of m equations in $m+n$ variables

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determines $u_1, u_2, , u_m$ uniquely in terms of $x_1, x_2, , x_n$. By rewriting the system in matrix form as

$$\mathbf{AX}+\mathbf{BU}=\mathbf{0},$$

where

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{B}=\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mm} \end{bmatrix},$$

$$\mathbf{X}=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \text{and} \quad \mathbf{U}=\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix},$$

we see that can be solved uniquely for $\mathbf{U}$ in terms of $\mathbf{X}$ if the square matrix $\mathbf{B}$ is nonsingular. In this case the solution is

$$\mathbf{U}=-\mathbf{B}^{-1}\mathbf{AX}.$$

For our purposes it is convenient to restate this: If

$$\mathbf{F}(\mathbf{X},\mathbf{U})=\mathbf{AX}+\mathbf{BU}, \tag{6.1.44}$$

where $\mathbf{B}$ is nonsingular, then the system

$$\mathbf{F}(\mathbf{X},\mathbf{U})=\mathbf{0}$$

determines $\mathbf{U}$ as a function of $\mathbf{X}$, for all $\mathbf{X}$ in $\R^n$.

Notice that $\mathbf{F}$ in is a linear transformation. If $\mathbf{F}$ is a more general transformation from $\R^{n+m}$ to $\R^m$, we can still ask whether the system

$$\mathbf{F}(\mathbf{X},\mathbf{U})=\mathbf{0},$$

or, in terms of components,

$$\begin{aligned} f_1(x_1,x_2,\dots,x_n,u_1,u_2,\dots,u_m)\ar &= 0 \\ f_2(x_1,x_2,\dots,x_n,u_1,u_2,\dots,u_m)\ar &= 0 \\ &\vdots \\ f_m(x_1,x_2,\dots,x_n,u_1,u_2,\dots,u_m)\ar &= 0, \end{aligned}$$

can be solved for $\mathbf{U}$ in terms of $\mathbf{X}$. However, the situation is now more complicated, even if $m=1$. For example, suppose that $m=1$ and

$$f(x,y,u)=1-x^2-y^2-u^2.$$

If $x^2+y^2>1$, then no value of u satisfies

$$f(x,y,u)=0. \tag{6.1.45}$$

However, infinitely many functions $u=u(x,y)$ satisfy on the set

$$S=\{(x,y)\}x^2+y^2\le 1.$$

They are of the form

$$u(x,y)=\epsilon(x,y)\sqrt{1-x^2-y^2},$$

where $\epsilon(x,y)$ can be chosen arbitrarily, for each (x,y) in S, to be 1 or -1. We can narrow the choice of functions to two by requiring that u be continuous on S; then

$$u(x,y)\ar=\sqrt{1-x^2-y^2}\arraytextor u(x,y)\ar=-\sqrt{1-x^2-y^2}. \tag{6.1.46}$$

We can define a unique continuous solution u of by specifying its value at a single interior point of S. For example, if we require that

$$u\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)=\frac{1}{\sqrt{3}},$$

then u must be as defined by .

The question of whether an arbitrary system

$$\mathbf{F}(\mathbf{X},\mathbf{U})=\mathbf{0}$$

determines $\mathbf{U}$ as a function of $\mathbf{X}$ is too general to have a useful answer. However, there is a theorem, the implicit function theorem, that answers this question affirmatively in an important special case. To facilitate the statement of this theorem, we partition the differential matrix of $\mathbf{F}:\R^{n+m}\to\R^m$:

$$\mathbf{F}'=\left[\begin{array}{cccc|cccc} \dst\frac{\partial f_1}{\partial x_1} & \dst\frac{\partial f_1}{\partial x_2} & \cdots & \dst\frac{\partial f_1}{\partial x_n} & \dst\frac{\partial f_1}{\partial u_1} & \dst\frac{\partial f_1}{\partial u_2} & \cdots & \dst\frac{\partial f_1}{\partial u_m} \\ [3\jot] \dst\frac{\partial f_2}{\partial x_1} & \dst\frac{\partial f_2}{\partial x_2} & \cdots & \dst\frac{\partial f_2}{\partial x_n} & \dst\frac{\partial f_2}{\partial u_1} & \dst\frac{\partial f_2}{\partial u_2} & \cdots & \dst\frac{\partial f_2}{\partial u_m} \\ [3\jot] \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \dst\frac{\partial f_m}{\partial x_1} & \dst\frac{\partial f_m}{\partial x_2} & \cdots & \dst\frac{\partial f_m}{\partial x_n} & \dst\frac{\partial f_m}{\partial u_1} & \dst\frac{\partial f_m}{\partial u_2} & \cdots & \dst\frac{\partial f_m}{\partial u_m} \end{array}\right] \tag{6.1.47}$$

or

$$\mathbf{F}'=[\mathbf{F}_\mathbf{X},\mathbf{F}_\mathbf{U}],$$

where $\mathbf{F}_\mathbf{X}$ is the submatrix to the left of the dashed line in and $\mathbf{F}_\mathbf{U}$ is to the right.

For the linear transformation , $\mathbf{F}_\mathbf{X}=\mathbf{A}$ and $\mathbf{F}_\mathbf{U}=\mathbf{B}$, and we have seen that the system $\mathbf{F}(\mathbf{X},\mathbf{U})=\mathbf{0}$ defines $\mathbf{U}$ as a function of $\mathbf{X}$ for all $\mathbf{X}$ in $\R^n$ if $\mathbf{F}_\mathbf{U}$ is nonsingular. The next theorem shows that a related result holds for more general transformations.

Define $\mathbf{\Phi}:\R^{n+m}\to\R^{n+m}$ by

$$\mathbf{\Phi}(\mathbf{X},\mathbf{U})=\left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \\ f_1(\mathbf{X},\mathbf{U}) \\ [3\jot] f_2(\mathbf{X},\mathbf{U}) \\ \vdots \\ f_m(\mathbf{X},\mathbf{U}) \end{array}\right] \tag{6.1.48}$$

or, in ``horizontal''notation by

$$\mathbf{\Phi}(\mathbf{X},\mathbf{U})=(\mathbf{X},\mathbf{F}(\mathbf{X},\mathbf{U})). \tag{6.1.49}$$

Then $\mathbf{\Phi}$ is continuously differentiable on S and, since $\mathbf{F}(\mathbf{X}_0,\mathbf{U}_0)=\mathbf{0}$,

$$\mathbf{\Phi}(\mathbf{X}_0,\mathbf{U}_0)=(\mathbf{X}_0,\mathbf{0}). \tag{6.1.50}$$

The differential matrix of $\mathbf{\Phi}$ is

$$\Phi' = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ [3\jot]0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ [3\jot]\dst\frac{\partial f_1}{\partial x_1} & \dst\frac{\partial f_1}{\partial x_2} & \cdots & \dst\frac{\partial f_1}{\partial x_n} & \dst\frac{\partial f_1}{\partial u_1} & \dst\frac{\partial f_1}{\partial u_2} & \cdots & \dst\frac{\partial f_1}{\partial u_m} \\ [3\jot]\dst\frac{\partial f_2}{\partial x_1} & \dst\frac{\partial f_2}{\partial x_2} & \cdots & \dst\frac{\partial f_2}{\partial x_n} & \dst\frac{\partial f_2}{\partial u_1} & \dst\frac{\partial f_2}{\partial u_2} & \cdots & \dst\frac{\partial f_2}{\partial u_m} \\ [3\jot]\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ [3\jot]\dst\frac{\partial f_m}{\partial x_1} & \dst\frac{\partial f_m}{\partial x_2} & \cdots & \dst\frac{\partial f_m}{\partial x_n} & \dst\frac{\partial f_m}{\partial u_1} & \dst\frac{\partial f_m}{\partial u_2} & \cdots & \dst\frac{\partial f_m}{\partial u_m} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{F_X} & \mathbf{F_U} \end{bmatrix},$$

where $\mathbf{I}$ is the $n \times n$ identity matrix, $\mathbf{0}$ is the $n \times m$ matrix with all zero entries, and $\mathbf{F_X}$ and $\mathbf{F_U}$ are as in . By expanding $\det(\Phi')$ and the determinants that evolve from it in terms of the cofactors of their first rows, it can be shown in n steps that .5pc

$$J\Phi = \det(\Phi') = \begin{vmatrix} \dst\frac{\partial f_1}{\partial u_1} & \dst\frac{\partial f_1}{\partial u_2} & \cdots & \dst\frac{\partial f_1}{\partial u_m} \\ [3\jot]\dst\frac{\partial f_2}{\partial u_1} & \dst\frac{\partial f_2}{\partial u_2} & \cdots & \dst\frac{\partial f_2}{\partial u_m} \\ [3\jot]\vdots & \vdots & \ddots & \vdots \\ \dst\frac{\partial f_m}{\partial u_1} & \dst\frac{\partial f_m}{\partial u_2} & \cdots & \dst\frac{\partial f_m}{\partial u_m} \end{vmatrix} = \det(\mathbf{F_U}).$$

.5pc In particular,

$$J\Phi(\mathbf{X}_0, \mathbf{U}_0) = \det(\mathbf{F_U}(\mathbf{X}_0, \mathbf{U}_0) \neq 0.$$

Since Φ is continuously differentiable on S, Corollary~ implies that Φ is regular on some open neighborhood M of $(\mathbf{X}_0, \mathbf{U}_0)$ and that $\widehat{M} = \Phi(M)$ is open.

Because of the form of Φ (see or), we can write points of $\widehat{M}$ as $(\mathbf{X}, \mathbf{V})$, where $\mathbf{V} \in \R^m$. Corollary~ also implies that Φ has a a continuously differentiable inverse $\Gamma(\mathbf{X}, \mathbf{V})$ defined on $\widehat{M}$ with values in M. Since Φ leaves the $\mathbf{X}$ part" of $(\mathbf{X},\mathbf{U})$ fixed, a local inverse of $\boldsymbol{\Phi}$ must also have this property. Therefore, $\boldsymbol{\Gamma}$ must have the form \vskip.5pc \[\boldsymbol{\Gamma} (\mathbf{X},\mathbf{V})=\left[\begin{array}{c} x_1\\ x_2\\\vdots\\ x_n\\[3\jot] h_1(\mathbf{X},\mathbf{V})\\[3\jot] h_2(\mathbf{X},\mathbf{V})\\ \vdots\\ [3\jot] h_m(\mathbf{X},\mathbf{V})\end{array}\right] $$ \vskip1pc or, in horizontal'' notation,

$$\Gamma(\mathbf{X}, \mathbf{V}) = (\mathbf{X}, \mathbf{H}(\mathbf{X}, \mathbf{V})),$$

where $\mathbf{H} : \R^{n+m} \to \R^m$ is continuously differentiable on $\widehat{M}$. We will show that $\mathbf{G}(\mathbf{X}) = \mathbf{H}(\mathbf{X}, \mathbf{0})$ has the stated properties.

From , $(\mathbf{X}_0, \mathbf{0}) \in \widehat{M}$ and, since $\widehat{M}$ is open, there is a neighborhood N of $\mathbf{X}_0$ in $\R^n$ such that $(\mathbf{X}, \mathbf{0}) \in \widehat{M}$ if $\mathbf{X} \in N$ (Exercise~). Therefore, $(\mathbf{X}, \mathbf{G}(\mathbf{X})) = \Gamma(\mathbf{X}, \mathbf{0}) \in M$ if $\mathbf{X} \in N$. Since $\Gamma = \Phi^{-1}$, $(\mathbf{X}, \mathbf{0}) = \Phi(\mathbf{X}, \mathbf{G}(\mathbf{X}))$. Setting $\mathbf{X} = \mathbf{X}_0$ and recalling shows that $\mathbf{G}(\mathbf{X}_0) = \mathbf{U}_0$, since Φ is one-to-one on M.

Henceforth we assume that $\mathbf{X} \in N$. Now,

$$\begin{aligned} (\mathbf{X}, \mathbf{0}) &= \Phi(\Gamma(\mathbf{X}, \mathbf{0})) && (\text{since } \Phi = \Gamma^{-1}) \\ &= \Phi(\mathbf{X}, \mathbf{G}(\mathbf{X})) && (\text{since } \Gamma(\mathbf{X}, \mathbf{0}) = (\mathbf{X}, \mathbf{G}(\mathbf{X}))) \\ &= (\mathbf{X}, \mathbf{F}(\mathbf{X}, \mathbf{G}(\mathbf{X}))) && (\text{since } \Phi(\mathbf{X}, \mathbf{U}) = (\mathbf{X}, \mathbf{F}(\mathbf{X}, \mathbf{U}))). \end{aligned}$$

Therefore, $\mathbf{F}(\mathbf{X}, \mathbf{G}(\mathbf{X})) = \mathbf{0}$; that is, $\mathbf{G}$ satisfies . To see that $\mathbf{G}$ is unique, suppose that $\mathbf{G}_1 : \R^n \to \R^m$ also satisfies . Then

$$\Phi(\mathbf{X}, \mathbf{G}(\mathbf{X})) = (\mathbf{X}, \mathbf{F}(\mathbf{X}, \mathbf{G}(\mathbf{X}))) = (\mathbf{X}, \mathbf{0})$$

and

$$\Phi(\mathbf{X}, \mathbf{G}_1(\mathbf{X})) = (\mathbf{X}, \mathbf{F}(\mathbf{X}, \mathbf{G}_1(\mathbf{X}))) = (\mathbf{X}, \mathbf{0})$$

for all $\mathbf{X}$ in N. Since Φ is one-to-one on M, this implies that $\mathbf{G}(\mathbf{X}) = \mathbf{G}_1(\mathbf{X})$.

Since the partial derivatives

$$\frac{\partial h_i}{\partial x_j}, \quad 1 \le i \le m, \quad 1 \le j \le n,$$

are continuous functions of $(\mathbf{X}, \mathbf{V})$ on $\widehat{M}$, they are continuous with respect to $\mathbf{X}$ on the subset $\{(\mathbf{X}, \mathbf{0})\}\mathbf{X} \in N$ of $\widehat{M}$. Therefore, $\mathbf{G}$ is continuously differentiable on N. To verify , we write $\mathbf{F}(\mathbf{X}, \mathbf{G}(\mathbf{X})) = \mathbf{0}$ in terms of components; thus,

$$f_i(x_1, x_2, \dots, x_n, g_1(\mathbf{X}), g_2(\mathbf{X}), \dots, g_m(\mathbf{X})) = 0, \quad 1 \le i \le m, \quad \mathbf{X} \in N.$$

Since f_i and g_1, g_2, , g_m are continuously differentiable on their respective domains, the chain rule (Theorem~) implies that

$$\frac{\partial f_i(\mathbf{X}, \mathbf{G}(\mathbf{X}))}{\partial x_j} + \sum_{r=1}^{m} \frac{\partial f_i(\mathbf{X}, \mathbf{G}(\mathbf{X}))}{\partial u_r} \frac{\partial g_r(\mathbf{X})}{\partial x_j} = 0, \quad 1 \le i \le m, \ 1 \le j \le n, \tag{6.1.51}$$

or, in matrix form,

$$\mathbf{F}_{\mathbf{X}}(\mathbf{X}, \mathbf{G}(\mathbf{X})) + \mathbf{F}_{\mathbf{U}}(\mathbf{X}, \mathbf{G}(\mathbf{X}))\mathbf{G}'(\mathbf{X}) = \mathbf{0}. \tag{6.1.52}$$

Since $(\mathbf{X}, \mathbf{G}(\mathbf{X})) \in M$ for all $\mathbf{X}$ in N and $\mathbf{F}_{\mathbf{U}}(\mathbf{X}, \mathbf{U})$ is nonsingular when $(\mathbf{X}, \mathbf{U}) \in M$, we can multiply on the left by $\mathbf{F}_{\mathbf{U}}^{-1}(\mathbf{X}, \mathbf{G}(\mathbf{X}))$ to obtain . This completes the proof.

In Theorem~ we denoted the implicitly defined transformation by $\mathbf{G}$ for reasons of clarity in the proof. However, in applying the theorem it is convenient to denote the transformation more informally by $\mathbf{U} = \mathbf{U}(\mathbf{X})$; thus, $\mathbf{U}(\mathbf{X}_0) = \mathbf{U}_0$, and we replace and by

$$(\mathbf{X}, \mathbf{U}(\mathbf{X})) \in M \quad \text{and} \quad \mathbf{X}(\mathbf{X}, \mathbf{U}(\mathbf{X})) = 0 \quad \text{if} \quad \mathbf{X} \in N,$$

and

$$\mathbf{U}'(\mathbf{X}) = -[\mathbf{F}_{\mathbf{U}}(\mathbf{X}, \mathbf{U}(\mathbf{X}))]^{-1}\mathbf{F}_{\mathbf{X}}(\mathbf{X}, \mathbf{U}(\mathbf{X})), \quad \mathbf{X} \in N,$$

while becomes

$$\frac{\partial f_i}{\partial x_j} + \sum_{r=1}^{m} \frac{\partial f_i}{\partial u_r} \frac{\partial u_r}{\partial x_j} = 0, \quad 1 \le i \le m, \ 1 \le j \le n, \tag{6.1.53}$$

it being understood that the partial derivatives of u_r and f_i are evaluated at $\mathbf{X}$ and $(\mathbf{X}, \mathbf{U}(\mathbf{X}))$, respectively.

The following corollary is the implicit function theorem for $m = 1$.

-2em2em

It is not necessary to memorize formulas like and . Since we know that f and u are differentiable, we can obtain and by applying the chain rule to the identity

$$f(x, y, u(x, y)) = 0.$$

If we try to solve for u, we see very clearly that Theorem~ and Corollary~ are {} theorems; that is, they tell us that there is a function $u = u(x, y)$ that satisfies , but not how to find it. In this case there is no convenient formula for the function, although its partial derivatives can be expressed conveniently in terms of x, y, and $u(x, y)$:

$$u_x(x, y) = -\frac{f_x(x, y, u(x, y))}{f_u(x, y, u(x, y))}, \quad u_y(x, y) = -\frac{f_y(x, y, u(x, y))}{f_u(x, y, u(x, y))}.$$

In particular, since $u(1, -1) = 1$,

$$u_x(1, -1) = -\frac{0}{-7} = 0, \quad u_y(1, -1) = -\frac{4}{-7} = \frac{4}{7}.$$

Again, it is not necessary to memorize , since the partial derivatives of an implicitly defined function can be obtained from the chain rule and Cramer's rule, as in the next example.

-.4em It is convenient to extend the notation introduced in Section 6.2 for the Jacobian of a transformation $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^m$. If f_1, f_2, , f_m are real-valued functions of k variables, $k \ge m$, and ξ_1, ξ_2, , ξ_m are any m of the variables, then we call the determinant

$$\begin{vmatrix} \dst\frac{\partial f_1}{\partial \xi_1} & \dst\frac{\partial f_1}{\partial \xi_2} & \cdots & \dst\frac{\partial f_1}{\partial \xi_m} \\ [3\jot] \dst\frac{\partial f_2}{\partial \xi_1} & \dst\frac{\partial f_2}{\partial \xi_2} & \cdots & \dst\frac{\partial f_2}{\partial \xi_m} \\ [3\jot] \vdots & \vdots & \ddots & \vdots \\ [3\jot] \dst\frac{\partial f_m}{\partial \xi_1} & \dst\frac{\partial f_m}{\partial \xi_2} & \cdots & \dst\frac{\partial f_m}{\partial \xi_m} \end{vmatrix},$$

the {}. We denote this Jacobian by

$$\frac{\partial(f_1, f_2, \cdots, f_m)}{\partial(\xi_1, \xi_2, \cdots, \xi_m)},$$

and we denote the value of the Jacobian at a point $\mathbf{P}$ by

$$\left.\frac{\partial(f_1, f_2, \cdots, f_m)}{\partial(\xi_1, \xi_2, \cdots, \xi_m)}\right|_{\mathbf{P}}.$$

The requirement in Theorem~ that $\mathbf{F_U}(\mathbf{X}_0, \mathbf{U}_0)$ be nonsingular is equivalent to

$$\left.\frac{\partial(f_1, f_2, \cdots, f_m)}{\partial(u_1, u_2, \cdots, u_m)}\right|_{(\mathbf{X}_0, \mathbf{U}_0)} \neq 0.$$

If this is so then, for a fixed j, Cramer's rule allows us to write the solution of as

$$\frac{\partial u_i}{\partial x_j} = -\frac{\dst\frac{\partial(f_1, f_2, \ldots, f_i, \ldots, f_m)}{\partial(u_1, u_2, \ldots, x_j, \ldots, u_m)}}{\dst\frac{\partial(f_1, f_2, \ldots, f_i, \ldots, f_m)}{\partial(u_1, u_2, \ldots, u_i, \ldots, u_m)}}, \quad 1 \leq i \leq m,$$

Notice that the determinant in the numerator on the right is obtained by replacing the ith column of the determinant in the denominator, which is

$$\begin{bmatrix} \dst\frac{\partial f_1}{\partial u_i} \\ [3\jot] \dst\frac{\partial f_2}{\partial u_i} \\ \vdots \\ \dst\frac{\partial f_m}{\partial u_i} \end{bmatrix}, \quad \text{by} \quad \begin{bmatrix} \dst\frac{\partial f_1}{\partial x_j} \\ [3\jot] \dst\frac{\partial f_2}{\partial x_j} \\ \vdots \\ \dst\frac{\partial f_m}{\partial x_j} \end{bmatrix}.$$

So far we have considered only the problem of solving a continuously differentiable system

$$\mathbf{F}(\mathbf{X}, \mathbf{U}) = \mathbf{0} \quad (\mathbf{F} : \R^{n+m} \to \R^m) \tag{6.1.54}$$

for the last m variables, u_1, u_2, , u_m, in terms of the first n, x_1, x_2, , x_n. This was merely for convenience; can be solved near $(\mathbf{X}_0, \mathbf{U}_0)$ for any m of the variables in terms of the other n, provided only that the Jacobian of f_1, f_2, , f_m with respect to the chosen m variables is nonzero at $(\mathbf{X}_0, \mathbf{U}_0)$. This can be seen by renaming the variables and applying Theorem~.

We close this section by observing that the functions u_1, u_2, , u_m defined in Theorem~ have higher derivatives if $f_1, f_2, \ldots, f_m$ do, and they may be obtained by differentiating , using the chain rule. (Exercise~).

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6.2: Continuity and Differentiability of Transformations

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6.3: The Inverse Function Theorem

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6.4: The Implicit Function Theorem

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CHAPTER OVERVIEW

7: Integrals of Functions of Several Variables

IN THIS CHAPTER we study the integral calculus of real-valued functions of several variables.

- SECTION 7.1 defines multiple integrals, first over rectangular parallelepipeds in $\R^n$ and then over more general sets. The discussion deals with the multiple integral of a function whose discontinuities form a set of Jordan content zero, over a set whose boundary has Jordan content zero.
- SECTION 7.2 deals with evaluation of multiple integrals by means of iterated integrals.
- SECTION 7.3 begins with the definition of Jordan measurability, followed by a derivation of the rule for change of content under a linear transformation, an intuitive formulation of the rule for change of variables in multiple integrals, and finally a careful statement and proof of the rule. This is a complicated proof.

7.1: Definition and Existence of the Multiple Integral
7.2: Iterated Integrals and Multiple Integrals
7.3: Change of Variables in Multiple Integrals

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7.1: Definition and Existence of the Multiple Integral

We now consider the Riemann integral of a real-valued function f defined on a subset of $\R^n$, where $n \geq 2$. Much of this development will be analogous to the development in Sections~3.1–3 for $n = 1$, but there is an important difference: for $n = 1$, we considered integrals over closed intervals only, but for $n > 1$ we must consider more complicated regions of integration. To defer complications due to geometry, we first consider integrals over rectangles in $\R^n$, which we now define.

-.4em The

$$S_1 \times S_2 \times \cdots \times S_n$$

of subsets S_1, S_2, , S_n of $\R$ is the set of points $(x_1, x_2, \ldots, x_n)$ in $\R^n$ such that $x_1 \in S_1, x_2 \in S_2, \ldots, x_n \in S_n$. For example, the Cartesian product of the two closed intervals

$$[a_1, b_1] \times [a_2, b_2] = \{(x, y)\} a_1 \leq x \leq b_1,\ a_2 \leq y \leq b_2$$

is a rectangle in $\R^2$ with sides parallel to the x- and y-axes (Figure~).

6pt

6pt

The Cartesian product of three closed intervals

$$[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] = \{(x, y, z)\} a_1 \leq x \leq b_1,\ a_2 \leq y \leq b_2,\ a_3 \leq z \leq b_3$$

is a rectangular parallelepiped in $\R^3$ with faces parallel to the coordinate axes (Figure~).

6pt

6pt

If $n = 1$, 2, or 3, then $V(R)$ is, respectively, the length of an interval, the area of a rectangle, or the volume of a rectangular parallelepiped. Henceforth, `rectangle'' or` cube''will always mean `coordinate rectangle'' or` coordinate cube'' unless it is stated otherwise.

If

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$$

and

$$P_r\colon\ a_r = a_{r0} < a_{r1} < \cdots < a_{rm_r} = b_r$$

is a partition of $[a_r, b_r]$, $1 \leq r \leq n$, then the set of all rectangles in $\R^n$ that can be written as

$$[a_{1,j_1-1}, a_{1j_1}] \times [a_{2,j_2-1}, a_{2j_2}] \times \cdots \times [a_{n,j_n-1}, a_{nj_n}], \quad 1 \leq j_r \leq m_r, \quad 1 \leq r \leq n,$$

is a {} of R. We denote this partition by

$$\mathbf{P} = P_1 \times P_2 \times \cdots \times P_n \tag{7.1.1}$$

and define its {} to be the maximum of the norms of P_1, P_2, , P_n, as defined in Section~3.1; thus,

$$\|\mathbf{P}\| = \max\{\|P_1\|, \|P_2\|, \ldots, \|P_n\|\}.$$

Put another way, $\|\mathbf{P}\|$ is the largest of the edge lengths of all the subrectangles in~$\mathbf{P}$.

Geometrically, a rectangle in $\R^2$ is partitioned by drawing horizontal and vertical lines through it (Figure~); in $\R^3$, by drawing planes through it parallel to the coordinate axes. Partitioning divides a rectangle R into finitely many subrectangles that we can number in arbitrary order as R_1, R_2, , R_k. Sometimes it is convenient to write

$$\mathbf{P} = \{R_1, R_2, \ldots, R_k\}$$

rather than .

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If $\mathbf{P} = P_1 \times P_2 \times \cdots \times P_n$ and $\mathbf{P}' = P_1' \times P_2' \times \cdots \times P_n'$ are partitions of the same rectangle, then $\mathbf{P}'$ is a {} of $\mathbf{P}$ if P_i' is a refinement of P_i, $1 \le i \le n$, as defined in Section~3.1.

Suppose that f is a real-valued function defined on a rectangle R in $\R^n$, $\mathbf{P} = \{R_1, R_2, \ldots, R_k\}$ is a partition of R, and $\mathbf{X}_j$ is an arbitrary point in R_j, $1 \le j \le k$. Then

$$\sigma = \sum_{j=1}^{k} f(\mathbf{X}_j) V(R_j)$$

is a {}. Since $\mathbf{X}_j$ can be chosen arbitrarily in R_j, there are infinitely many Riemann sums for a given function f over any partition $\mathbf{P}$ of R.

The following definition is similar to Definition~.

If R is degenerate, then Definition~ implies that $\int_R f(\mathbf{X})\, d\mathbf{X} = 0$ for any function f defined on R (Exercise~). Therefore, it should be understood henceforth that whenever we speak of a rectangle in $\R^n$ we mean a nondegenerate rectangle, unless it is stated to the contrary.

The integral $\int_R f(\mathbf{X}) d\mathbf{X}$ is also written as

$$\int_R f(x,y)\, d(x,y) \quad (n=2), \qquad \int_R f(x,y,z)\, d(x,y,z) \quad (n=3),$$

or

$$\int_R f(x_1, x_2, \ldots, x_n)\, d(x_1, x_2, \ldots, x_n) \quad (n \text{ arbitrary}).$$

Here $d\mathbf{X}$ does not stand for the differential of $\mathbf{X}$, as defined in Section~6.2. It merely identifies x_1, x_2, , x_n, the components of $\mathbf{X}$, as the variables of integration. To avoid this minor inconsistency, some authors write simply $\int_R f$ rather than $\int_R f(\mathbf{X})\, d\mathbf{X}$.

As in the case where $n = 1$, we will say simply integrable'' or integral'‘when we mean Riemann integrable'' or Riemann integral.'' If $n \ge 2$, we call the integral of Definition~ a {}; for $n = 2$ and $n = 3$ we also call them {} and {}, respectively. When we wish to distinguish between multiple integrals and the integral we studied in Chapter~ $(n = 1)$, we will call the latter an {} integral.

Let P_1 and P_2 be partitions of $[a,b]$ and $[c,d]$; thus,

$$P_1 : a = x_0 < x_1 < \cdots < x_r = b \quad \text{and} \quad P_2 : c = y_0 < y_1 < \cdots < y_s = d.$$

A typical Riemann sum of f over $\mathbf{P} = P_1 \times P_2$ is given by

$$\sigma = \sum_{i=1}^{r} \sum_{j=1}^{s} (\xi_{ij} + \eta_{ij})(x_i - x_{i-1})(y_j - y_{j-1}), \tag{7.1.2}$$

where

$$x_{i-1} \le \xi_{ij} \le x_i \quad \text{and} \quad y_{j-1} \le \eta_{ij} \le y_j. \tag{7.1.3}$$

The midpoints of $[x_{i-1}, x_i]$ and $[y_{j-1}, y_j]$ are

$$\overline{x}_i = \frac{x_i + x_{i-1}}{2} \quad \text{and} \quad \overline{y}_j = \frac{y_j + y_{j-1}}{2}, \tag{7.1.4}$$

and implies that

$$|\xi_{ij} - \overline{x}_i| \ar \le \frac{x_i - x_{i-1}}{2} \le \frac{\|P_1\|}{2} \le \frac{\|\mathbf{P}\|}{2} \tag{7.1.5}$$

$$\arraytext{and}$$

$$|\eta_{ij} - \overline{y}_j| \ar \le \frac{y_j - y_{j-1}}{2} \le \frac{\|P_2\|}{2} \le \frac{\|\mathbf{P}\|}{2}. \tag{7.1.6}$$

Now we rewrite as

Bracket argument to \\ must be a dimension

To find $\int_R f(x,y)\,d(x,y)$ from , we recall that

$$\sum_{i=1}^{r}(x_i-x_{i-1})=b-a,\quad \sum_{j=1}^{s}(y_j-y_{j-1})=d-c \tag{7.1.7}$$

(Example~), and

$$\sum_{i=1}^{r}(x_i^2-x_{i-1}^2)=b^2-a^2,\quad \sum_{j=1}^{s}(y_j^2-y_{j-1}^2)=d^2-c^2 \tag{7.1.8}$$

(Example~).

Because of and the absolute value of the second sum in does not exceed

$$\|\mathbf{P}\|\sum_{j=1}^{r}\sum_{j=1}^{s}(x_i-x_{i-1})(y_j-y_{j-1})\ar=\|\mathbf{P}\|\left[\sum_{i=1}^{r}(x_i-x_{i-1})\right]\left[\sum_{j=1}^{s}(y_j-y_{j-1})\right]$$
$$\ar=\|\mathbf{P}\|(b-a)(d-c)$$

(see), so implies that

$$\left|\sigma-\sum_{i=1}^{r}\sum_{j=1}^{s}(\overline{x}_i+\overline{y}_j)(x_i-x_{i-1})(y_j-y_{j-1})\right|\le\|\mathbf{P}\|(b-a)(d-c). \tag{7.1.9}$$

It now follows that

Missing dimension or its units for \hspace

Similarly,

$$\sum_{i=1}^{r}\sum_{j=1}^{s}\overline{y}_j(x_i-x_{i-1})(y_j-y_{j-1})=\frac{b-a}{2}(d^2-c^2).$$

Therefore, can be written as

$$\left|\sigma-\frac{d-c}{2}(b^2-a^2)-\frac{b-a}{2}(d^2-c^2)\right|\le\|\mathbf{P}\|(b-a)(d-c).$$

Since the right side can be made as small as we wish by choosing $\|\mathbf{P}\|$ sufficiently small,

$$\int_R(x+y)\,d(x,y)=\frac{1}{2}\left[(d-c)(b^2-a^2)+(b-a)(d^2-c^2)\right].$$

The following theorem is analogous to Theorem~.

We will show that if f is unbounded on R, $\mathbf{P}=\{R_1,R_2,\dots,R_k\}$ is any partition of R, and $M>0$, then there are Riemann sums σ and σ' of f over $\mathbf{P}$ such that

$$|\sigma-\sigma'|\ge M. \tag{7.1.10}$$

This implies that f cannot satisfy Definition~. (Why?)

Let

$$\sigma=\sum_{j=1}^{k}f(\mathbf{X}_j)V(R_j)$$

be a Riemann sum of f over $\mathbf{P}$. There must be an integer i in $\{1,2,\dots,k\}$ such that

$$|f(\mathbf{X})-f(\mathbf{X}_i)|\ge\frac{M}{V(R_i)} \tag{7.1.11}$$

for some $\mathbf{X}$ in R_i, because if this were not so, we would have

$$|f(\mathbf{X})-f(\mathbf{X}_j)|<\frac{M}{V(R_j)},\quad \mathbf{X}\in R_j,\qquad 1\le j\le k.$$

If this is so, then

$$\begin{aligned}|f(\mathbf{X})|\ar &= |f(\mathbf{X}_j)+f(\mathbf{X})-f(\mathbf{X}_j)|\le |f(\mathbf{X}_j)|+|f(\mathbf{X})-f(\mathbf{X}_j)|\\ \ar &\le |f(\mathbf{X}_j)|+\frac{M}{V(R_j)},\quad \mathbf{X}\in R_j,\quad 1\le j\le k.\end{aligned}$$

However, this implies that

$$|f(\mathbf{X})|\le \max\{|f(\mathbf{X}_j)|+\frac{M}{V(R_j)}\}1\le j\le k,\quad \mathbf{X}\in R,$$

which contradicts the assumption that f is unbounded on R.

Now suppose that $\mathbf{X}$ satisfies , and consider the Riemann sum

$$\sigma'=\sum_{j=1}^{n}f(\mathbf{X}_j')V(R_j)$$

over the same partition $\mathbf{P}$, where

$$\mathbf{X}_j'=\begin{cases}\casespace \mathbf{X}_j, & j\ne i,\\ \mathbf{X}, & j=i.\end{cases}$$

Since

$$|\sigma-\sigma'|=|f(\mathbf{X})-f(\mathbf{X}_i)|V(R_i),$$

implies .

Because of Theorem~, we need consider only bounded functions in connection with Definition~. As in the case where $n=1$, it is now convenient to define the upper and lower integrals of a bounded function over a rectangle. The following definition is analogous to Definition~.

The following theorem is analogous to Theorem~.

Exercise~.

If

$$m\le f(\mathbf{X})\le M\quad \text{for } \mathbf{X} \text{ in } R,$$

then

$$mV(R)\le s(\mathbf{P})\le S(\mathbf{P})\le MV(R);$$

therefore, $\overline{\int_R} f(\mathbf{X})\,d\mathbf{X}$ and $\underline{\int_R} f(\mathbf{X})\,d\mathbf{X}$ exist, are unique, and satisfy the inequalities

$$mV(R)\le \overline{\int_R} f(\mathbf{X})\,d\mathbf{X}\le MV(R)$$

and

$$mV(R)\le \underline{\int_R} f(\mathbf{X})\,d\mathbf{X}\le MV(R).$$

The upper and lower integrals are also written as

$$\overline{\int_R} f(x,y)\,d(x,y)\quad \text{and}\quad \underline{\int_R} f(x,y)\,d(x,y)\quad (n=2),$$

$$\overline{\int_R} f(x,y,z)\,d(x,y,z)\quad \text{and}\quad \underline{\int_R} f(x,y,z)\,d(x,y,z)\quad (n=3),$$

or

$$\overline{\int_R} f(x_1, x_2, \dots, x_n)\, d(x_1, x_2, \dots, x_n)$$

and

$$\underline{\int_R} f(x_1, x_2, \dots, x_n)\, d(x_1, x_2, \dots, x_n) \qquad (n \text{ arbitrary}).$$

Let P_1 and P_2 be partitions of $[a, b]$ and $[c, d]$; thus,

$$P_1 : a = x_0 < x_1 < \cdots < x_r = b \quad \text{and} \quad P_2 : c = y_0 < y_1 < \cdots < y_s = d.$$

The maximum and minimum values of f on the rectangle $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$ are $x_i + y_j$ and $x_{i-1} + y_{j-1}$, respectively. Therefore,

$$S(\mathbf{P})\ar = \sum_{i=1}^{r} \sum_{j=1}^{s} (x_i + y_j)(x_i - x_{i-1})(y_j - y_{j-1}) \tag{7.1.12}$$

$$\arraytextand$$

$$s(\mathbf{P})\ar = \sum_{i=1}^{r} \sum_{j=1}^{s} (x_{i-1} + y_{j-1})(x_i - x_{i-1})(y_j - y_{j-1}). \tag{7.1.13}$$

By substituting

$$x_i + y_j = \frac{1}{2}[(x_i + x_{i-1}) + (y_j + y_{j-1}) + (x_i - x_{i-1}) + (y_j - y_{j-1})]$$

into , we find that

$$S(\mathbf{P}) = \frac{1}{2}(\Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4), \tag{7.1.14}$$

where

Bracket argument to \\ must be a dimension

Substituting these four results into shows that

$$I < S(\mathbf{P}) < I + \|\mathbf{P}\|(b - a)(d - c),$$

where

$$I = \frac{(d - c)(b^2 - a^2) + (b - a)(d^2 - c^2)}{2}.$$

From this, we see that

$$\overline{\int_R} (x + y)\, d(x, y) = I.$$

After substituting

$$x_{i-1} + y_{j-1} = \frac{1}{2}[(x_i + x_{i-1}) + (y_j + y_{j-1}) - (x_i - x_{i-1}) - (y_j - y_{j-1})]$$

into , a similar argument shows that

$$I - \|\mathbf{P}\|(b - a)(d - c) < s(\mathbf{P}) < I,$$

so

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We now prove an analog of Lemma~.

We will prove and leave the proof of to you (Exercise~). First suppose that P_1' is obtained by adding one point to P_1, and $P_j' = P_j$ for $2 \le j \le n$. If P_r is defined by

$$P_r : a_r = a_{r0} < a_{r1} < \cdots < a_{rm_r} = b_r, \quad 1 \le r \le n,$$

then a typical subrectangle of $\mathbf{P}$ is of the form

$$R_{j_1 j_2 \cdots j_n} = [a_{1,j_1-1}, a_{1j_1}] \times [a_{2,j_2-1}, a_{2j_2}] \times \cdots \times [a_{n,j_n-1}, a_{nj_n}].$$

Let c be the additional point introduced into P_1 to obtain P_1', and suppose that

$$a_{1,k-1} < c < a_{1k}.$$

If $j_1 \ne k$, then $R_{j_1 j_2 \cdots j_n}$ is common to $\mathbf{P}$ and $\mathbf{P}'$, so the terms associated with it in $S(\mathbf{P}')$ and $S(\mathbf{P})$ cancel in the difference $S(\mathbf{P}) - S(\mathbf{P}')$. To analyze the terms that do not cancel, define

Bracket argument to \\ must be a dimension

$$M_{kj_2 \cdots j_n} = \sup\{f(\mathbf{X})\}\mathbf{X} \in R_{kj_2 \cdots j_n} \tag{7.1.15}$$

and

$$M^{(i)}_{kj_2 \cdots j_n} = \sup\{f(\mathbf{X})\}\mathbf{X} \in R^{(i)}_{kj_2 \cdots j_n}, \quad i = 1, 2. \tag{7.1.16}$$

Then $S(\mathbf{P}) - S(\mathbf{P}')$ is the sum of terms of the form

$$\begin{aligned}\left[M_{kj_2 \cdots j_n}(a_{1k} - a_{1,k-1}) - M^{(1)}_{kj_2 \cdots j_n}(c - a_{1,k-1}) - M^{(2)}_{kj_2 \cdots j_n}(a_{1k} - c)\right] \\ \times (a_{2j_2} - a_{2,j_2-1}) \cdots (a_{nj_n} - a_{n,j_n-1}).\end{aligned} \tag{7.1.17}$$

The terms within the brackets can be rewritten as

$$(M_{kj_2 \cdots j_n} - M^{(1)}_{kj_2 \cdots j_n})(c - a_{1,k-1}) + (M_{kj_2 \cdots j_n} - M^{(2)}_{kj_2 \cdots j_n})(a_{1k} - c), \tag{7.1.18}$$

which is nonnegative, because of and . Therefore,

$$S(\mathbf{P}') \le S(\mathbf{P}). \tag{7.1.19}$$

Moreover, the quantity in is not greater than $2M(a_{1k} - a_{1,k-1})$, so implies that the general surviving term in $S(\mathbf{P}) - S(\mathbf{P}')$ is not greater than

$$2M\|\mathbf{P}\|(a_{2j_2} - a_{2,j_2-1}) \cdots (a_{nj_n} - a_{n,j_n-1}).$$

The sum of these terms as j_2, , j_n assume all possible values $1 \le j_i \le m_i$, $2 \le i \le n$, is

$$2M\|\mathbf{P}\|(b_2 - a_2) \cdots (b_n - a_n) = \frac{2M\|\mathbf{P}\|V(R)}{b_1 - a_1}.$$

This implies that

$$S(\mathbf{P}) \le S(\mathbf{P}') + \frac{2M\|\mathbf{P}\|V(R)}{b_1 - a_1}.$$

This and imply for $r_1 = 1$ and $r_2 = \cdots = r_n = 0$.

Similarly, if $r_i = 1$ for some i in $\{1, \ldots, n\}$ and $r_j = 0$ if $j \ne i$, then

$$S(\mathbf{P}) \le S(\mathbf{P}') + \frac{2M\|\mathbf{P}\|V(R)}{b_i - a_i}.$$

To obtain in the general case, repeat this argument $r_1 + r_2 + \cdots + r_n$ times, as in the proof of Lemma~.

Lemma~ implies the following theorems and lemma, with proofs analogous to the proofs of their counterparts in Section~3.2.

Exercise~.

The next theorem is analogous to Theorem~3.2.3.

Exercise~.

Exercise~.

The next theorem is analogous to Theorem~3.2.5.

Exercise~.

Theorems~ and imply the following theorem, which is analogous to Theorem~.

The next theorem translates this into a test that can be conveniently applied. It is analogous to Theorem~.

Exercise~.

Theorem~ provides a useful criterion for integrability. The next theorem is an important application. It is analogous to Theorem~.

Let $\epsilon > 0$. Since f is uniformly continuous on R (Theorem~), there is a $\delta > 0$ such that

$$|f(\mathbf{X}) - f(\mathbf{X}')| < \frac{\epsilon}{V(\mathbf{R})} \tag{7.1.20}$$

if $\mathbf{X}$ and $\mathbf{X}'$ are in R and $|\mathbf{X} - \mathbf{X}'| < \delta$. Let $\mathbf{P} = \{R_1, R_2, \dots, R_k\}$ be a partition of R with $\|P\| < \delta/\sqrt{n}$. Since f is continuous on R, there are points $\mathbf{X}_j$ and $\mathbf{X}'_j$ in R_j such that

$$f(\mathbf{X}_j) = M_j = \sup_{\mathbf{X} \in R_j} f(\mathbf{X}) \quad \text{and} \quad f(\mathbf{X}'_j) = m_j = \inf_{\mathbf{X} \in R_j} f(\mathbf{X})$$

(Theorem~). Therefore,

$$S(\mathbf{P}) - s(\mathbf{P}) = \sum_{j=1}^{n} (f(\mathbf{X}_j) - f(\mathbf{X}'_j)) V(R_j).$$

Since $\|\mathbf{P}\| < \delta/\sqrt{n}$, $|\mathbf{X}_j - \mathbf{X}'_j| < \delta$, and, from with $\mathbf{X} = \mathbf{X}_j$ and $\mathbf{X}' = \mathbf{X}'_j$,

$$S(\mathbf{P}) - s(\mathbf{P}) < \frac{\epsilon}{V(R)} \sum_{j=1}^{k} V(R_j) = \epsilon.$$

Hence, f is integrable on R, by Theorem~.

The next definition will enable us to establish the existence of $\int_R f(\mathbf{X})\, d\mathbf{X}$ in cases where f is bounded on the rectangle R, but is not necessarily continuous for all $\mathbf{X}$ in R.

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The next lemma follows immediately from Definition~.

The following theorem will enable us to define multiple integrals over more general subsets of $\mathbb{R}^n$.

Suppose that $\epsilon > 0$. Since E has zero content, there are rectangles T_1, T_2, , T_m such that

$$E \subset \bigcup_{j=1}^{m} T_j \tag{7.1.21}$$

and

$$\sum_{j=1}^{m} V(T_j) < \epsilon. \tag{7.1.22}$$

We may assume that T_1, T_2, , T_m are contained in R, since, if not, their intersections with R would be contained in R, and still satisfy and . We may also assume that if T is any rectangle such that

$$T \bigcap \left(\bigcup_{j=1}^{m} T_j^0 \right) = \emptyset, \quad \text{then} \quad T \cap E = \emptyset \tag{7.1.23}$$

since if this were not so, we could make it so by enlarging T_1, T_2, , T_m slightly while maintaining . Now suppose that

$$T_j = [a_{1j}, b_{1j}] \times [a_{2j}, b_{2j}] \times \cdots \times [a_{nj}, b_{nj}], \quad 1 \le j \le m,$$

let P_{i0} be the partition of $[a_i, b_i]$ (see) with partition points

$$a_i, b_i, a_{i1}, b_{i1}, a_{i2}, b_{i2}, \ldots, a_{im}, b_{im} \backslash \text{vspace} * 1pt$$

(these are not in increasing order), $1 \le i \le n$, and let

$$\mathbf{P}_0 = P_{10} \times P_{20} \times \cdots \times P_{n0}.$$

-.3em Then $\mathbf{P}_0$ consists of rectangles whose union equals $\cup_{j=1}^m T_j$ and other rectangles T_1', T_2', , T_k' that do not intersect E. (We need to be sure that $T_i' \cap E = \emptyset, 1 \le i \le k$.) If we let

$$B = \bigcup_{j=1}^m T_j \quad \text{and} \quad C = \bigcup_{i=1}^k T_i',$$

then $R = B \cup C$ and f is continuous on the compact set C. If $\mathbf{P} = \{R_1, R_2, \ldots, R_k\}$ is a refinement of $\mathbf{P}_0$, then every subrectangle R_j of $\mathbf{P}$ is contained entirely in B or entirely in C. Therefore, we can write

$$S(\mathbf{P}) - s(\mathbf{P}) = \Sigma_1 (M_j - m_j) V(R_j) + \Sigma_2 (M_j - m_j) V(R_j), \tag{7.1.24}$$

-.3em where Σ_1 and Σ_2 are summations over values of j for which $R_j \subset B$ and $R_j \subset C$, respectively. Now suppose that

$$|f(\mathbf{X})| \le M \quad \text{for } \mathbf{X} \text{ in } R.$$

Then

$$\Sigma_1 (M_j - m_j) V(R_j) \le 2M \, \Sigma_1 V(R_j) = 2M \sum_{j=1}^m V(T_j) < 2M\epsilon, \tag{7.1.25}$$

from . Since f is uniformly continuous on the compact set C (Theorem~), there is a $\delta > 0$ such that $M_j - m_j < \epsilon$ if $\|\mathbf{P}\| < \delta$ and $R_j \subset C$; hence,

$$\Sigma_2 (M_j - m_j) V(R_j) < \epsilon \Sigma_2 \, V(R_j) \le \epsilon V(R).$$

This, , and imply that

$$S(\mathbf{P}) - s(\mathbf{P}) < [2M + V(R)]\epsilon$$

if $\|\mathbf{P}\| < \delta$ and $\mathbf{P}$ is a refinement of $\mathbf{P}_0$. Therefore, Theorem~ implies that f is integrable on R.

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We can now define the integral of a bounded function over more general subsets of~$\mathbb{R}^n$.

To see that this definition makes sense, we must show that if R_1 and R_2 are two rectangles containing S and $\int_{R_1} f_S(\mathbf{X})\, d\mathbf{X}$ exists, then so does $\int_{R_2} f_S(\mathbf{X})\, d\mathbf{X}$, and the two integrals are equal. The proof of this is sketched in Exercise~.

Let f_S be as in . Since a discontinuity of f_S is either a discontinuity of f or a point of ∂S, the set of discontinuities of f_S is the union of two sets of zero content and therefore is of zero content (Lemma~). Therefore, f_S is integrable on any rectangle containing S (from Theorem~), and consequently on S (Definition~).

, defined as follows, form an important class of sets of zero content in $\mathbb{R}^n$.

Let S, D, and $\mathbf{G}$ be as in Definition~. From Lemma~, there is a constant M such that

$$|\mathbf{G}(\mathbf{X}) - \mathbf{G}(\mathbf{Y})| \le M|\mathbf{X} - \mathbf{Y}| \quad \text{if} \quad \mathbf{X}, \mathbf{Y} \in D. \tag{7.1.26}$$

Since D is bounded, D is contained in a cube

$$C = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_m, b_m],$$

where

$$b_i - a_i = L, \quad 1 \le i \le m.$$

Suppose that we partition C into N^m smaller cubes by partitioning each of the intervals $[a_i, b_i]$ into N equal subintervals. Let R_1, R_2, , R_k be the smaller cubes so produced that contain points of D, and select points $\mathbf{X}_1$, $\mathbf{X}_2$, , $\mathbf{X}_k$ such that $\mathbf{X}_i \in D \cap R_i$, $1 \le i \le k$. If $\mathbf{Y} \in D \cap R_i$, then implies that

$$|\mathbf{G}(\mathbf{X}_i) - \mathbf{G}(\mathbf{Y})| \le M|\mathbf{X}_i - \mathbf{Y}|. \tag{7.1.27}$$

Since $\mathbf{X}_i$ and $\mathbf{Y}$ are both in the cube R_i with edge length L/N,

$$|\mathbf{X}_i - \mathbf{Y}| \le \frac{L\sqrt{m}}{N}.$$

This and imply that

$$|\mathbf{G}(\mathbf{X}_i) - \mathbf{G}(\mathbf{Y})| \le \frac{ML\sqrt{m}}{N},$$

which in turn implies that $\mathbf{G}(\mathbf{Y})$ lies in a cube $\widetilde{R}_i$ in $\R^n$ centered at $\mathbf{G}(\mathbf{X}_i)$, with sides of length $2ML\sqrt{m}/N$. Now

$$\sum_{i=1}^{k} V(\widetilde{R}_i) = k\left(\frac{2ML\sqrt{m}}{N}\right)^n \le N^m\left(\frac{2ML\sqrt{m}}{N}\right)^n = (2ML\sqrt{m})^n N^{m-n}.$$

Since $n > m$, we can make the sum on the left arbitrarily small by taking N sufficiently large. Therefore, S has zero content.

Theorems~ and imply the following theorem.

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We now list some theorems on properties of multiple integrals. The proofs are similar to those of the analogous theorems in Section~3.3.

Note: Because of Definition~, if we say that a function f is integrable on a set S, then S is necessarily bounded.

Exercise~.

Exercise~.

Exercise~.

Exercise~.

Exercise~.

Exercise~.

From Definition~ with f and S replaced by f_S and T,

$$(f_S)_T(\mathbf{X}) = \begin{cases} \casespace f_S(\mathbf{X}), & \mathbf{X} \in T, \\ 0, & \mathbf{X} \notin T. \end{cases}$$

Since $S \subset T$, $(f_S)_T = f_S$. (Verify.) Now suppose that R is a rectangle containing T. Then R also contains S (Figure~), 6pt

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which completes the proof.

For $i = 1, 2$, let

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From Lemma~ with $S = S_i$ and $T = S_1 \cup S_2$, f_{S_i} is integrable on $S_1 \cup S_2$, and

$$\int_{S_1\cup S_2} f_{S_i}(\mathbf{X})\,d\mathbf{X} = \int_{S_i} f(\mathbf{X})\,d\mathbf{X}, \quad i = 1, 2.$$

Theorem~ now implies that $f_{S_1} + f_{S_2}$ is integrable on $S_1 \cup S_2$ and

$$\int_{S_1\cup S_2} (f_{S_1} + f_{S_2})(\mathbf{X})\,d\mathbf{X} = \int_{S_1} f(\mathbf{X})\,d\mathbf{X} + \int_{S_2} f(\mathbf{X})\,d\mathbf{X}. \tag{7.1.28}$$

Since $S_1 \cap S_2 = \emptyset$,

$$(f_{S_1} + f_{S_2})(\mathbf{X}) = f_{S_1}(\mathbf{X}) + f_{S_2}(\mathbf{X}) = f(\mathbf{X}), \quad \mathbf{X} \in S_1 \cup S_2.$$

Therefore, implies .

We leave it to you to prove the following extension of Theorem. (Exercise

).

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We will discuss this example further in the next section.

\begin{exerciselist}

Prove: If R is degenerate, then Definition~ implies that $\int_R f(\mathbf{X})\,d\mathbf{X} = 0$ if f is bounded on R.

Evaluate directly from Definition~.

Suppose that $\int_a^b f(x)\,dx$ and $\int_c^d g(y)\,dy$ exist, and let $R = [a, b] \times [c, d]$. Criticize the following ``proof'' that $\int_R f(x)g(y)\,d(x, y)$ exists and equals

$$\left(\int_a^b f(x)\,dx\right)\left(\int_c^d g(y)\,dy\right).$$

(See Exercise~ for a correct proof of this assertion.)

``Proof.'' Let

$$P_1 : a = x_0 < x_1 < \cdots < x_r = b \quad \text{and} \quad P_2 : c = y_0 < y_1 < \cdots < y_s = d$$

be partitions of $[a, b]$ and $[c, d]$, and $\mathbf{P} = P_1 \times P_2$. Then a typical Riemann sum of fg over $\mathbf{P}$ is of the form

$$\sigma = \sum_{i=1}^{r}\sum_{j=1}^{s} f(\xi_i)g(\eta_j)(x_i - x_{i-1})(y_j - y_{j-1}) = \sigma_1\sigma_2,$$

where

$$\sigma_1 = \sum_{i=1}^{r} f(\xi_i)(x_i - x_{i-1}) \quad \text{and} \quad \sigma_2 = \sum_{j=1}^{s} g(\eta_j)(y_j - y_{j-1})$$

are typical Riemann sums of f over $[a, b]$ and g over $[c, d]$. Since f and g are integrable on these intervals,

$$\left|\sigma_1 - \int_a^b f(x)\,dx\right| \quad \text{and} \quad \left|\sigma_2 - \int_c^d g(y)\,dy\right|$$

can be made arbitrarily small by taking $\|P_1\|$ and $\|P_2\|$ sufficiently small. From this, it is straightforward to show that

$$\left|\sigma - \left(\int_a^b f(x)\,dx\right)\left(\int_c^d g(y)\,dy\right)\right|$$

can be made arbitrarily small by taking $\|\mathbf{P}\|$ sufficiently small. This implies the stated result.

Suppose that $f(x, y) \geq 0$ on $R = [a, b] \times [c, d]$. Justify the interpretation of $\int_R f(x, y)\,d(x, y)$, if it exists, as the volume of the region in $\R^3$ bounded by the surfaces $z = f(x, y)$ and the planes $z = 0$, $x = a$, $x = b$, $y = c$, and $y = d$.

Prove Theorem~.

Suppose that

$$f(x,y) = \begin{cases} \casespace & \begin{array}{ll} 0 & \text{if } x \text{ and } y \text{ are rational,} \\ 1 & \text{if } x \text{ is rational and } y \text{ is irrational,} \\ 2 & \text{if } x \text{ is irrational and } y \text{ is rational,} \\ 3 & \text{if } x \text{ and } y \text{ are irrational.} \end{array} \end{cases}$$

Find

$$\overline{\int_R} f(x,y)\,d(x,y) \quad \text{and} \quad \underline{\int_R} f(x,y)\,d(x,y) \quad \text{if} \quad R=[a,b]\times[c,d].$$

Prove Eqn.~ of Lemma~.

Prove Theorem~

Prove Theorem~

Prove Lemma~

Prove Theorem~

Prove Theorem~

Give an example of a denumerable set in $\R^2$ that does not have zero content.

%:14 Prove:

Show that a degenerate rectangle has zero content.

Suppose that f is continuous on a compact set S in $\R^n$. Show that the surface $z=f(\mathbf{X})$, $\mathbf{X}\in S$, has zero content in $\R^{n+1}$.

Let S be a bounded set such that $S\cap \partial S$ does not have zero content.

Suppose that f is integrable on a set S and S_0 is a subset of S such that ∂S_0 has zero content. Show that f is integrable on S_0.

Prove Theorem~

Prove Theorem~.

Prove Theorem~

Prove Theorem~

Prove Theorem~

Prove Theorem~

Prove: If f is integrable on a rectangle R, then f is integrable on any subrectangle of R.

Suppose that R and $\widetilde{R}$ are rectangles, $R\subset\widetilde{R}$, g is bounded on $\widetilde{R}$, and $g(\mathbf{X})=0$ if $\mathbf{X}\notin R$.

Suppose that f is continuously differentiable on a rectangle R. Show that there is a constant M such that

$$\left|\sigma-\int_R f(\mathbf{X})\,d\mathbf{X}\right|\le M\|\mathbf{P}\|$$

if σ is any Riemann sum of f over a partition $\mathbf{P}$ of R.

Suppose that $\int_a^b f(x)\,dx$ and $\int_c^d g(y)\,dy$ exist, and let $R=[a,b]\times[c,d]$.

\end{exerciselist}

Except for very simple examples, it is impractical to evaluate multiple integrals directly from Definitions~ and . Fortunately, this can usually be accomplished by evaluating n successive ordinary integrals. To motivate the method, let us first assume that f is continuous on $R=[a,b]\times[c,d]$. Then, for each y in $[c,d]$, $f(x,y)$ is continuous with respect to x on $[a,b]$, so the integral

$$F(y) = \int_a^b f(x,y)\,dx$$

exists. Moreover, the uniform continuity of f on R implies that F is continuous (Exercise~) and therefore integrable on $[c,d]$. We say that

$$I_1 = \int_c^d F(y)\,dy = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

is an {} of f over R. We will usually write it as

$$I_1 = \int_c^d dy \int_a^b f(x,y)\,dx.$$

Another iterated integral can be defined by writing

$$G(x) = \int_c^d f(x,y)\,dy, \quad a \le x \le b,$$

and defining

$$I_2 = \int_a^b G(x)\,dx = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx,$$

which we usually write as

$$I_2 = \int_a^b dx \int_c^d f(x,y)\,dy.$$

In this example, $I_1 = I_2$; moreover, on setting $a = 0$, $b = 1$, $c = 1$, and $d = 2$ in Example~, we see that

$$\int_R (x+y)\,d(x,y) = 2,$$

so the common value of the iterated integrals equals the multiple integral. The following theorem shows that this is not an accident.

Let

$$P_1 : a = x_0 < x_1 < \cdots < x_r = b \quad \text{and} \quad P_2 : c = y_0 < y_1 < \cdots < y_s = d$$

be partitions of $[a,b]$ and $[c,d]$, and $\mathbf{P} = P_1 \times P_2$. Suppose that

$$y_{j-1} \le \eta_j \le y_j, \quad 1 \le j \le s, \tag{7.1.29}$$

so

$$\sigma = \sum_{j=1}^{s} F(\eta_j)(y_j - y_{j-1}) \tag{7.1.30}$$

is a typical Riemann sum of F over P_2. Since

$$F(\eta_j) = \int_a^b f(x,\eta_j)\,dx = \sum_{i=1}^{r} \int_{x_{i-1}}^{x} f(x,\eta_j)\,dx,$$

implies that if

$$\begin{aligned} m_{ij} \ar &= \inf\{f(x,y)\} x_{i-1} \le x \le x_i,\ y_{j-1} \le y \le y_j \\ &\arraytext{and} \\ M_{ij} \ar &= \sup\{f(x,y)\} x_{i-1} \le x \le x_i,\ y_{j-1} \le y \le y_j, \end{aligned}$$

then

$$\sum_{i=1}^{r} m_{ij}(x_i - x_{i-1}) \le F(\eta_j) \le \sum_{i=1}^{r} M_{ij}(x_i - x_{i-1}).$$

Multiplying this by $y_j - y_{j-1}$ and summing from $j = 1$ to $j = s$ yields

$$\sum_{j=1}^{s}\sum_{i=1}^{r} m_{ij}(x_i - x_{i-1})(y_j - y_{j-1})\ar \le \sum_{j=1}^{s} F(\eta_j)(y_j - y_{j-1})$$

$$\ar \le \sum_{j=1}^{s}\sum_{i=1}^{r} M_{ij}(x_i - x_{i-1})(y_j - y_{j-1}),$$

which, from , can be rewritten as

$$s_f(\mathbf{P}) \le \sigma \le S_f(\mathbf{P}), \tag{7.1.31}$$

where $s_f(\mathbf{P})$ and $S_f(\mathbf{P})$ are the lower and upper sums of f over $\mathbf{P}$. Now let $s_F(P_2)$ and $S_F(P_2)$ be the lower and upper sums of F over P_2; since they are respectively the infimum and supremum of the Riemann sums of F over P_2 (Theorem~), implies that

$$s_f(\mathbf{P}) \le s_F(P_2) \le S_F(P_2) \le S_f(\mathbf{P}). \tag{7.1.32}$$

Since f is integrable on R, there is for each $\epsilon > 0$ a partition $\mathbf{P}$ of R such that $S_f(\mathbf{P}) - s_f(\mathbf{P}) < \epsilon$, from Theorem~. Consequently, from , there is a partition P_2 of $[c,d]$ such that $S_F(P_2) - s_F(P_2) < \epsilon$, so F is integrable on $[c,d]$, from Theorem~.

It remains to verify . From and the definition of $\int_c^d F(y)\,dy$, there is for each $\epsilon > 0$ a $\delta > 0$ such that

$$\left|\int_c^d F(y)\,dy - \sigma\right| < \epsilon \quad \text{if} \quad \|P_2\| < \delta;$$

that is,

$$\sigma - \epsilon < \int_c^d F(y)\,dy < \sigma + \epsilon \quad \text{if} \quad \|P_2\| < \delta.$$

This and imply that

$$s_f(\mathbf{P}) - \epsilon < \int_c^d F(y)\,dy < S_f(\mathbf{P}) + \epsilon \quad \text{if} \quad \|\mathbf{P}\| < \delta,$$

and this implies that

$$\underline{\int_R} f(x,y)\,d(x,y) - \epsilon \le \int_c^d F(y)\,dy \le \overline{\int_R} f(x,y)\,d(x,y) + \epsilon \tag{7.1.33}$$

(Definition). Since

$$\underline{\int_R} f(x,y)\,d(x,y) = \overline{\int_R} f(x,y)\,d(x,y)$$

(Theorem) and ϵ can be made arbitrarily small, implies .

If f is continuous on R, then f satisfies the hypotheses of Theorem~ (Exercise~), so is valid in this case.

If $\int_R f(x,y)\,d(x,y)$ and

$$\int_c^d f(x,y)\,dy, \quad a \le x \le b,$$

exist, then by interchanging x and y in Theorem~, we see that

$$\int_a^b dx \int_c^d f(x,y)\,dy = \int_R f(x,y)\,d(x,y).$$

This and yield the following corollary of Theorem~.

-2em2em

A plausible partial converse of Theorem~ would be that if $\int_c^d dy \int_a^b f(x,y)\,dx$ exists then so does $\int_R f(x,y)\,d(x,y)$; however, the next example shows that this need not be so. \begin{example}\rm If f is defined on $R = [0,1] \times [0,1]$ by

$$f(x,y)=\left\{\casespace \begin{array}{ll} 2xy & \text{if } y \text{ is rational,} \\ y & \text{if } y \text{ is irrational,} \end{array}\right.$$

then

$$\int_0^1 f(x,y)\,dx = y, \quad 0 \le y \le 1,$$

and

$$\int_0^1 dy \int_0^1 f(x,y)\,dx = \int_0^1 y\,dy = \frac{1}{2}.$$

However, f is not integrable on R (Exercise~). \end{example}

The next theorem generalizes Theorem~ to $\R^n$.

For convenience, denote $(x_{p+1}, x_{p+2}, \dots, x_n)$ by $\mathbf{Y}$. Denote $\widehat{R} = I_1 \times I_2 \times \cdots \times I_p$ and $T = I_{p+1} \times I_{p+2} \times \cdots \times I_n$. Let $\widehat{\mathbf{P}} = \{\widehat{R}_1, \widehat{R}_2, \dots, \widehat{R}_k\}$ and $\mathbf{Q} = \{T_1, T_2, \dots, T_s\}$ be partitions of $\widehat{R}$ and T, respectively. Then the collection of rectangles of the form $\widehat{R}_i \times T_j$ $(1 \le i \le k,\ 1 \le j \le s)$ is a partition $\mathbf{P}$ of R; moreover, every partition $\mathbf{P}$ of R is of this form.

Suppose that

$$\mathbf{Y}_j \in T_j, \quad 1 \le j \le s, \tag{7.1.34}$$

so

$$\sigma = \sum_{j=1}^{s} F_p(\mathbf{Y}_j) V(T_j) \tag{7.1.35}$$

is a typical Riemann sum of F_p over $\mathbf{Q}$. Since

$$\begin{aligned} F_p(\mathbf{Y_j}) \ar &= \int_{\widehat{R}} f(x_1, x_2, \dots, x_p, \mathbf{Y}_j)\, d(x_1, x_2, \dots, x_p) \\ \ar &= \sum_{j=1}^{k} \int_{\widehat{R}_j} f(x_1, x_2, \dots, x_p, \mathbf{Y}_j)\, d(x_1, x_2, \dots, x_p), \end{aligned}$$

implies that if

$$\begin{aligned} m_{ij} \ar = \inf\{f(x_1, x_2, \dots, x_p, \mathbf{Y})\}(x_1, x_2, \dots, x_p) \in \widehat{R}_i,\ \mathbf{Y} \in T_j \\ &\arraytext{and} \\ M_{ij} \ar = \sup\{f(x_1, x_2, \dots, x_p, \mathbf{Y})\}(x_1, x_2, \dots, x_p) \in \widehat{R}_i,\ \mathbf{Y} \in T_j, \end{aligned}$$

then

$$\sum_{i=1}^{k} m_{ij} V(\widehat{R}_i) \le F_p(\mathbf{Y_j}) \le \sum_{i=1}^{k} M_{ij} V(\widehat{R}_i).$$

Multiplying this by $V(T_j)$ and summing from $j = 1$ to $j = s$ yields

$$\sum_{j=1}^{s} \sum_{i=1}^{k} m_{ij} V(\widehat{R}_i) V(T_j) \le \sum_{j=1}^{s} F_p(\mathbf{Y}_j) V(T_j) \le \sum_{j=1}^{s} \sum_{i=1}^{k} M_{ij} V(\widehat{R}_i) V(T_j),$$

which, from , can be rewritten as

$$s_f(\mathbf{P}) \le \sigma \le S_f(\mathbf{P}), \tag{7.1.36}$$

where $s_f(\mathbf{P})$ and $S_f(\mathbf{P})$ are the lower and upper sums of f over $\mathbf{P}$. Now let $s_{F_p}(\mathbf{Q})$ and $S_{F_p}(\mathbf{Q})$ be the lower and upper sums of F_p over $\mathbf{Q}$; since they are respectively the infimum and supremum of the Riemann sums of F_p over $\mathbf{Q}$ (Theorem~), implies that

$$s_f(\mathbf{P}) \le s_{F_p}(\mathbf{Q}) \le S_{F_p}(\mathbf{Q}) \le S_f(\mathbf{P}). \tag{7.1.37}$$

Since f is integrable on R, there is for each $\epsilon > 0$ a partition $\mathbf{P}$ of R such that $S_f(\mathbf{P}) - s_f(\mathbf{P}) < \epsilon$, from Theorem~. Consequently, from , there is a partition $\mathbf{Q}$ of T such that $S_{F_p}(\mathbf{Q}) - s_{F_p}(\mathbf{Q}) < \epsilon$, so F_p is integrable on T, from Theorem~.

It remains to verify that

$$\int_R f(\mathbf{X})\,d\mathbf{X} = \int_T F_p(\mathbf{Y})\,d\mathbf{Y}. \tag{7.1.38}$$

From and the definition of $\int_T F_p(\mathbf{Y})\,d\mathbf{Y}$, there is for each $\epsilon > 0$ a $\delta > 0$ such that

$$\left|\int_T F_p(\mathbf{Y})\,d\mathbf{Y} - \sigma\right| < \epsilon \quad \text{if} \quad \|\mathbf{Q}\| < \delta;$$

that is,

$$\sigma - \epsilon < \int_T F_p(\mathbf{Y})\,d\mathbf{Y} < \sigma + \epsilon \quad \text{if} \quad \|\mathbf{Q}\| < \delta.$$

This and imply that

$$s_f(\mathbf{P}) - \epsilon < \int_T F_p(\mathbf{Y})\,d\mathbf{Y} < S_f(\mathbf{P}) + \epsilon \quad \text{if} \quad \|\mathbf{P}\| < \delta,$$

and this implies that

$$\underline{\int_R} f(\mathbf{X})\,d\mathbf{X} - \epsilon \le \int_T F_p(\mathbf{Y})\,d\mathbf{Y} \le \overline{\int_R} f(\mathbf{X})\,d\mathbf{X} + \epsilon. \tag{7.1.39}$$

Since \dst $\underline{\int_R}\, f(\mathbf{X})\,d\mathbf{X} = \overline{\int_R} f(\mathbf{X})\,d\mathbf{X}$ (Theorem~) and ϵ can be made arbitrarily small, implies .

The proof is by induction. From Theorem~, the proposition is true for $n = 2$. Now assume $n > 2$ and the proposition is true with n replaced by $n-1$. Holding x_n fixed and applying this assumption yields

$$F_n(x_n) = \int_{a_{n-1}}^{b_{n-1}} dx_{n-1} \int_{a_{n-2}}^{b_{n-2}} dx_{n-2} \cdots \int_{a_2}^{b_2} dx_2 \int_{a_1}^{b_1} f(\mathbf{X})\,dx_1.$$

Now Theorem~ with $p = n-1$ completes the induction.

The hypotheses of Theorems~ and ~ are stated so as to justify successive integrations with respect to x_1, then x_2, then x_3, and so forth. It is legitimate to use other orders of integration if the hypotheses are adjusted accordingly. For example, suppose that

$\{i_1, i_2, \dots, i_n\}$ is a permutation of $\{1, 2, \dots, n\}$ and $\int_R f(\mathbf{X})\,d\mathbf{X}$ exists, along with

$$\int_{I_{i_1} \times I_{i_2} \times \cdots \times I_{i_j}} f(\mathbf{X})\,d(x_{i_1}, x_{i_2}, \dots, x_{i_j}), \quad 1 \le j \le n-1, \tag{7.1.40}$$

for each

$$(x_{i_{j+1}}, x_{i_{j+2}}, \dots, x_{i_n}) \quad \text{in} \quad I_{i_{j+1}} \times I_{i_{j+2}} \times \cdots \times I_{i_n}. \tag{7.1.41}$$

Then, by renaming the variables, we infer from Theorem~ that

$$\int_R f(\mathbf{X})\,d\mathbf{X} = \int_{a_{i_n}}^{b_{i_n}} dx_{i_n} \int_{a_{i_{n-1}}}^{b_{i_{n-1}}} dx_{i_{n-1}} \cdots \int_{a_{i_2}}^{b_{i_2}} dx_{i_2} \int_{a_{i_1}}^{b_{i_1}} f(\mathbf{X})\,dx_{i_1}. \tag{7.1.42}$$

Since there are $n!$ permutations of $\{1, 2, \dots, n\}$, there are $n!$ ways of evaluating a multiple integral over a rectangle in $\R^n$, provided that the integrand satisfies appropriate hypotheses. In particular, if f is continuous on R and $\{i_1, i_2, \dots, i_n\}$ is any permutation of $\{1, 2, \dots, n\}$, then f is continuous with respect to $(x_{i_1}, x_{i_2}, \dots, x_{i_j})$ on $I_{i_1} \times I_{i_2} \times \cdots \times I_{i_j}$ for each fixed $(x_{i_{j+1}}, x_{i_{j+2}}, \dots, x_{i_n})$ satisfying . Therefore, the integrals exist for every permutation of $\{1, 2, \dots, n\}$ (Theorem~). We summarize this in the next theorem, which now follows from Theorem~.

We now consider the problem of evaluating multiple integrals over more general sets. First, suppose that f is integrable on a set of the form

$$S = \{(x,y)\} u(y) \le x \le v(y),\ c \le y \le d \tag{7.1.43}$$

(Figure~).

If $u(y) \ge a$ and $v(y) \le b$ for $c \le y \le d$, and

Bracket argument to \\ must be a dimension

then

$$\int_S f(x,y)\,d(x,y) = \int_R f_S(x,y)\,d(x,y),$$

where $R = [a,b] \times [c,d]$.. From Theorem~,

$$\int_R f_S(x,y)\,d(x,y) = \int_c^d dy \int_a^b f_S(x,y)\,dx$$

provided that $\int_a^b f_S(x,y)\,dx$ exists for each y in $[c,d]$. From and , this integral can be written as

$$\int_{u(y)}^{v(y)} f(x,y)\,dx. \tag{7.1.44}$$

Thus, we have proved the following theorem.

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From Theorem~, the assumptions of Theorem~ are satisfied if f is continuous on S and u and v are continuously differentiable on $[c,d]$.

Interchanging x and y in Theorem~ shows that if f is integrable on

$$S = \{(x,y)\} u(x) \le y \le v(x),\ a \le x \le b \tag{7.1.45}$$

(Figure~) and

$$\int_{u(x)}^{v(x)} f(x,y)\,dy$$

exists for $a \le x \le b$, then

$$\int_S f(x,y)\,d(x,y) = \int_a^b dx \int_{u(x)}^{v(x)} f(x,y)\,dy. \tag{7.1.46}$$

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12pt Theorem~ has an analog for $n > 2$. Suppose that f is integrable on a set S of points $\mathbf{X} = (x_1, x_2, \dots, x_n)$ satisfying the inequalities

$$u_j(x_{j+1}, \dots, x_n) \le x_j \le v_j(x_{j+1}, \dots, x_n), \quad 1 \le j \le n-1,$$

and

$$a_n \le x_n \le b_n.$$

Then, under appropriate additional assumptions, it can be shown by an argument analogous to the one that led to Theorem~ that

$$\int_S f(\mathbf{X})\,d\mathbf{X} = \int_{a_n}^{b_n} dx_n \int_{u_n(x_n)}^{v_n(x_n)} dx_{n-1} \cdots \int_{u_2(x_3,\ldots,x_n)}^{v_2(x_3,\ldots,x_n)} dx_2 \int_{u_1(x_2,\ldots,x_n)}^{v_1(x_2,\ldots,x_n)} f(\mathbf{X})\,dx_1.$$

These additional assumptions are tedious to state for general n. The following theorem contains a complete statement for $n = 3$.

2em

Thus far we have viewed the iterated integral as a tool for evaluating multiple integrals. In some problems the iterated integral is itself the object of interest. In this case a result

like Theorem~ can be used to evaluate the iterated integral. The procedure is as follows.

This procedure is called {} of an iterated integral.

6pt

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\begin{exerciselist}

Evaluate

Let $I_j = [a_j, b_j]$, $1 \le j \le 3$, and suppose that f is integrable on $R = I_1 \times I_2 \times I_3$. Prove:

Prove: If f is continuous on $[a,b] \times [c,d]$, then the function

$$F(y) = \int_a^b f(x,y)\,dx$$

is continuous on $[c,d]$.

Suppose that

$$f(x',y') \ge f(x,y) \quad \text{if} \quad a \le x \le x' \le b,\ c \le y \le y' \le d.$$

Show that f satisfies the hypotheses of Theorem~ on $R = [a,b] \times [c,d]$.

Evaluate by means of iterated integrals:

Let A be the set of points of the form $(2^{-m}p, 2^{-m}q)$, where p and q are odd integers and m is a nonnegative integer. Let

Bracket argument to \\ must be a dimension

Show that f is not integrable on any rectangle $R = [a,b] \times [c,d]$, but

$$\int_a^b dx \int_c^d f(x,y)\,dy = \int_c^d dy \int_a^b f(x,y)\,dx = (b-a)(d-c).\eqno(A)$$

Let

$$f(x,y) = \begin{cases} \casespace 2xy & \text{if } y \text{ is rational,} \\ y & \text{if } y \text{ is irrational,} \end{cases}$$

and $R = [0,1] \times [0,1]$ (Example~).

Let $R = [0,1] \times [0,1] \times [0,1]$, $\widetilde{R} = [0,1] \times [0,1]$, and

$$f(x,y,z) = \begin{cases} \casespace 2xy + 2xz & \text{if } y \text{ and } z \text{ are rational,} \\ y + 2xz & \text{if } y \text{ is irrational and } z \text{ is rational,} \\ 2xy + z & \text{if } y \text{ is rational and } z \text{ is irrational,} \\ y + z & \text{if } y \text{ and } z \text{ are irrational.} \end{cases}$$

Calculate Suppose that f is bounded on $R = [a,b] \times [c,d]$. Prove:

Use Exercise~ to prove the following generalization of Theorem~: If f is integrable on $R = [a,b] \times [c,d]$, then

$$\overline{\int_a^b} f(x,y)\,dy \quad \text{and} \quad \underline{\int_c^d} f(x,y)\,dy$$

are integrable on $[a,b]$, and

$$\int_a^b \left(\overline{\int_c^d} f(x,y)\,dy\right) dx = \int_a^b \left(\underline{\int_c^d} f(x,y)\,dy\right) dx = \int_R f(x,y)\,d(x,y).$$

Evaluate Evaluate

Evaluate $\int_S (x+y)\,d(x,y)$, where S is bounded by $y=x^2$ and $y=2x$, using iterated integrals of both possible types.

Find the area of the set bounded by the given curves.

In Example~, verify the last five representations of $\int_S f(x,y,z)\,d(x,y,z)$ as iterated integrals.

Let S be the region in $\R^3$ bounded by the coordinate planes and the plane $x+2y+3z=1$. Let f be continuous on S. Set up six iterated integrals that equal $\int_S f(x,y,z)\,d(x,y,z)$.

Evaluate Find the volume of S.

Let $R=[a_1,b_2]\times[a_2,b_2]\times\cdots\times[a_n,b_n]$. Evaluate

Assuming that f is continuous, express

$$\int_{1/2}^1 dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx$$

as an iterated integral with the order of integration reversed.

Evaluate $\int_S (x+y)\,d(x,y)$ of Example~ by means of iterated integrals in which the first integration is with respect to x.

Evaluate $\dst \int_0^1 x\,dx \int_0^{\sqrt{1-x^2}} \frac{dy}{\sqrt{x^2+y^2}}$.

Suppose that f is continuous on $[a,\infty)$,

$$y^{(n)}(x) = f(x), \quad t \ge a,$$

and $y(a)=y'(a)=\cdots=y^{(n-1)}(a)=0$.

Let $T_\rho=[0,\rho]\times[0,\rho]$, $\rho>0$. By calculating

$$I(a)=\lim_{\rho\to\infty}\int_{T_\rho} e^{-xy}\sin ax\,d(x,y)$$

in two different ways, show that

$$\int_0^\infty \frac{\sin ax}{x}dx = \frac{\pi}{2} \quad \text{if} \quad a>0.$$

\end{exerciselist}

In Section~3.3 we saw that a change of variables may simplify the evaluation of an ordinary integral. We now consider change of variables in multiple integrals.

Prior to formulating the rule for change of variables, we must deal with some rather involved preliminary considerations.

-.4em In Section~ we defined the content of a set S to be

$$V(S)=\int_S d\mathbf{X} \tag{7.1.47}$$

if the integral exists. If R is a rectangle containing S, then can be rewritten as

$$V(S)=\int_R \psi_S(\mathbf{X})\,d\mathbf{X},$$

where ψ_S is the characteristic function of S, defined by

$$\psi_S(\mathbf{X}) = \left\{\casespace \begin{array}{ll} 1, & \mathbf{X}\in S, \\ 0, & \mathbf{X}\notin S. \end{array}\right.$$

From Exercise~, the existence and value of $V(S)$ do not depend on the particular choice of the enclosing rectangle R. We say that S is {} if $V(S)$ exists. Then $V(S)$ is the {}S.

We leave it to you (Exercise~) to show that S has zero content according to Definition~ if and only if S has Jordan content zero.

Let R be a rectangle containing S. Suppose that $V(\partial S) = 0$. Since ψ_S is bounded on R and discontinuous only on ∂S (Exercise~), Theorem~ implies that $\int_R \psi_S(\mathbf{X})\,d\mathbf{X}$ exists. For the converse, suppose that ∂S does not have zero content and let $\mathbf{P} = \{R_1, R_2, \ldots, R_k\}$ be a partition of R. For each j in $\{1, 2, \ldots, k\}$ there are three possibilities:

Let

$$\mathcal{U}_1 = \{j\} R_j \subset S \quad \text{and} \quad \mathcal{U}_2 = \{j\} R_j \cap S \neq \emptyset \text{ and } R_j \cap S^c \neq \emptyset. \tag{7.1.48}$$

Then the upper and lower sums of ψ_S over $\mathbf{P}$ are

Bracket argument to \\ must be a dimension

and

$$\begin{aligned} s(\mathbf{P}) \ar &= \dst \sum_{j\in\mathcal{U}_1} V(R_j) \\ \ar &= \text{total content of the subrectangles in } \mathbf{P} \text{ contained in } S. \end{aligned} \tag{7.1.49}$$

Therefore,

$$S(\mathbf{P}) - s(\mathbf{P}) = \sum_{j\in\mathcal{U}_2} V(R_j),$$

which is the total content of the subrectangles in $\mathbf{P}$ that intersect both S and S^c. Since these subrectangles contain ∂S, which does not have zero content, there is an $\epsilon_0 > 0$ such that

$$S(\mathbf{P}) - s(\mathbf{P}) \geq \epsilon_0$$

for every partition $\mathbf{P}$ of R. By Theorem~, this implies that ψ_S is not integrable on R, so S is not Jordan measurable.

Theorems~ and imply the following corollary.

Since $V(K) = 0$,

$$\int_C \psi_K(\mathbf{X})\,d\mathbf{X} = 0$$

if C is any cube containing K. From this and the definition of the integral, there is a $\delta > 0$ such that if $\mathbf{P}$ is any partition of C with $\|\mathbf{P}\| \leq \delta$ and σ is any Riemann sum of ψ_K over $\mathbf{P}$, then

$$0 \leq \sigma \leq \epsilon. \tag{7.1.50}$$

Now suppose that $\mathbf{P} = \{C_1, C_2, \ldots, C_k\}$ is a partition of C into cubes with

$$\|\mathbf{P}\| < \min(\rho, \delta), \tag{7.1.51}$$

and let $C_1, C_2, , C_k$ be numbered so that $C_j \cap K \neq \emptyset$ if $1 \leq j \leq r$ and $C_j \cap K = \emptyset$ if $r+1 \leq j \leq k$. Then holds, and a typical Riemann sum of ψ_K over $\mathbf{P}$ is of the form

$$\sigma = \sum_{j=1}^{r} \psi_K(\mathbf{X}_j) V(C_j)$$

with $\mathbf{X}_j \in C_j$, $1 \leq j \leq r$. In particular, we can choose $\mathbf{X}_j$ from K, so that $\psi_K(\mathbf{X}_j) = 1$, and

$$\sigma = \sum_{j=1}^{r} V(C_j).$$

Now and imply that $C_1, C_2, , C_r$ have the required properties.

To formulate the theorem on change of variables in multiple integrals, we must first consider the question of preservation of Jordan measurability under a regular transformation.

Since K is a compact subset of the open set S, there is a $\rho_1 > 0$ such that the compact set

$$K_{\rho_1} = \{\mathbf{X}\}\backslash\mathrm{dist}(\mathbf{X}, K) \le \rho_1$$

is contained in S (Exercise$_{5.1.26)}$. From Lemma, there is a constant M such that

$$|\mathbf{G}(\mathbf{Y}) - \mathbf{G}(\mathbf{X})| \le M|\mathbf{Y} - \mathbf{X}| \quad \text{if} \quad \mathbf{X}, \mathbf{Y} \in K_{\rho_1}. \tag{7.1.52}$$

Now suppose that $\epsilon > 0$. Since $V(K) = 0$, there are cubes C_1, C_2, , C_r with edge lengths s_1, s_2, , $s_r < \rho_1/\sqrt{n}$ such that $C_j \cap K \neq \emptyset$, $1 \le j \le r$,

$$K \subset \bigcup_{j=1}^{r} C_j,$$

and

$$\sum_{j=1}^{r} V(C_j) < \epsilon \tag{7.1.53}$$

(Lemma~). For $1 \le j \le r$, let $\mathbf{X}_j \in C_j \cap K$. If $\mathbf{X} \in C_j$, then

$$|\mathbf{X} - \mathbf{X}_j| \le s_j\sqrt{n} < \rho_1,$$

so $\mathbf{X} \in K$ and $|\mathbf{G}(\mathbf{X}) - \mathbf{G}(\mathbf{X}_j)| \le M|\mathbf{X} - \mathbf{X}_j| \le M\sqrt{n}\, s_j$, from . Therefore, $\mathbf{G}(C_j)$ is contained in a cube $\widetilde{C}_j$ with edge length $2M\sqrt{n}$ s_j, centered at $\mathbf{G}(\mathbf{X}_j)$. Since

$$V(\widetilde{C}_j) = (2M\sqrt{n})^n s_j^n = (2M\sqrt{n})^n V(C_j),$$

we now see that

$$\mathbf{G}(K) \subset \bigcup_{j=1}^{r} \widetilde{C}_j$$

and

$$\sum_{j=1}^{r} V(\widetilde{C}_j) \le (2M\sqrt{n})^n \sum_{j=1}^{r} V(C_j) < (2M\sqrt{n})^n \epsilon,$$

where the last inequality follows from . Since $(2M\sqrt{n})^n$ does not depend on ϵ, it follows that $V(\mathbf{G}(K)) = 0$.

We leave it to you to prove that $\mathbf{G}(S)$ is compact (Exercise~6.2.23). Since S is Jordan measurable, $V(\partial S) = 0$, by Theorem~. Therefore, $V(\mathbf{G}(\partial S)) = 0$, by Lemma~. But $\mathbf{G}(\partial S) = \partial(\mathbf{G}(S))$ (Exercise~), so $V(\partial(\mathbf{G}(S))) = 0$, which implies that $\mathbf{G}(S)$ is Jordan measurable, again by Theorem~.

-.4em To motivate and prove the rule for change of variables in multiple integrals, we must know how $V(\mathbf{L}(S))$ is related to $V(S)$ if S is a compact Jordan measurable set and $\mathbf{L}$ is a nonsingular linear transformation. (From Theorem~, $\mathbf{L}(S)$ is compact and Jordan measurable in this case.) The next lemma from linear algebra will help to establish this relationship. We omit the proof.

Matrices of the kind described in this lemma are called {} matrices. The key to the proof of the lemma is that if $\mathbf{E}$ is an elementary $n \times n$ matrix and $\mathbf{A}$ is any $n \times n$ matrix, then $\mathbf{EA}$ is the matrix obtained by applying to $\mathbf{A}$ the same operation that must be applied to $\mathbf{I}$ to produce $\mathbf{E}$ (Exercise~). Also, the inverse of an elementary matrix of type , , or

is an elementary matrix of the same type (Exercise~).

The next example illustrates the procedure for finding the factorization .

Lemma~ and Theorem~

imply that an arbitrary invertible linear transformation $\mathbf{L} : \mathrm{R}^n \to \mathrm{R}^n$, defined by

$$\mathbf{X} = \mathbf{L}(\mathbf{Y}) = \mathbf{A}\mathbf{Y}, \tag{7.1.54}$$

can be written as a composition

$$\mathbf{L} = \mathbf{L}_k \circ \mathbf{L}_{k-1} \circ \cdots \circ \mathbf{L}_1, \tag{7.1.55}$$

where

$$\mathbf{L}_i(\mathbf{Y}) = \mathbf{E}_i\mathbf{Y}, \quad 1 \le i \le k.$$

Theorem~ implies that $\mathbf{L}(S)$ is Jordan measurable. If

$$V(\mathbf{L}(R)) = |\det(\mathbf{A})|V(R) \tag{7.1.56}$$

whenever R is a rectangle, then holds if S is any compact Jordan measurable set. To see this, suppose that $\epsilon > 0$, let R be a rectangle containing S, and let $\mathbf{P} = \{R_1, R_2, \ldots, R_k\}$ be a partition of R such that the upper and lower sums of ψ_S over $\mathbf{P}$ satisfy the inequality

$$S(\mathbf{P}) - s(\mathbf{P}) < \epsilon. \tag{7.1.57}$$

Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be as in . From and ,

$$s(\mathbf{P}) = \sum_{j\in\mathcal{U}_1} V(R_j) \le V(S) \le \sum_{j\in\mathcal{U}_1} V(R_j) + \sum_{j\in\mathcal{U}_2} V(R_j) = S(\mathbf{P}). \tag{7.1.58}$$

Theorem~ implies that $\mathbf{L}(R_1)$, $\mathbf{L}(R_2)$, , $\mathbf{L}(R_k)$ and $\mathbf{L}(S)$ are all Jordan measurable. Since

$$\bigcup_{j\in\mathcal{U}_1} R_j \subset S \subset \bigcup_{j\in\mathcal{S}_1\cup\mathcal{S}_2} R_j,$$

it follows that

$$L\left(\bigcup_{j\in\mathcal{U}_1} R_j\right) \subset L(S) \subset L\left(\bigcup_{j\in\mathcal{S}_1\cup\mathcal{S}_2} R_j\right).$$

Since L is one-to-one on $\R^n$, this implies that

$$\sum_{j\in\mathcal{U}_1} V(\mathbf{L}(R_j)) \le V(\mathbf{L}(S)) \le \sum_{j\in\mathcal{U}_1} V(\mathbf{L}(R_j)) + \sum_{j\in\mathcal{U}_2} V(\mathbf{L}(R_j)). \tag{7.1.59}$$

If we assume that holds whenever R is a rectangle, then

$$V(\mathbf{L}(R_j)) = |\det(\mathbf{A})|V(R_j), \quad 1 \le j \le k,$$

so implies that

$$s(\mathbf{P}) \le \frac{V(\mathbf{L}(S))}{|\det(\mathbf{A})|} \le S(\mathbf{P}).$$

This, and imply that

$$\left|V(S) - \frac{V(\mathbf{L}(S))}{|\det(\mathbf{A})|}\right| < \epsilon;$$

hence, since ϵ can be made arbitrarily small, follows for any Jordan measurable set.

To complete the proof, we must verify for every rectangle

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] = I_1 \times I_2 \times \cdots \times I_n.$$

Suppose that $\mathbf{A}$ in is an elementary matrix; that is, let

$$\mathbf{X} = \mathbf{L}(\mathbf{Y}) = \mathbf{E}\mathbf{Y}.$$

{Case 1}. If $\mathbf{E}$ is obtained by interchanging the ith and jth rows of $\mathbf{I}$, then

$$x_r = \begin{cases} y_r & \text{if } r \ne i \text{ and } r \ne j; \\ y_j & \text{if } r = i; \\ y_i & \text{if } r = j. \end{cases}$$

Then $\mathbf{L}(R)$ is the Cartesian product of I_1, I_2, , I_n with I_i and I_j interchanged, so

$$V(\mathbf{L}(R)) = V(R) = |\det(\mathbf{E})|V(R)$$

since $\det(\mathbf{E}) = -1$ in this case (Exercise~

).

{Case 2}. If $\mathbf{E}$ is obtained by multiplying the rth row of $\mathbf{I}$ by a, then

$$x_r = \left\{\casespace \begin{array}{ll} y_r & \text{if } r \neq i, \\ ay_i & \text{if } r = i. \end{array}\right.$$

Then

$$\mathbf{L}(R) = I_1 \times \cdots \times I_{i-1} \times I_i' \times I_{i+1} \times \cdots \times I_n,$$

where I_i' is an interval with length equal to $|a|$ times the length of I_i, so

$$V(\mathbf{L}(R)) = |a|V(R) = |\det(\mathbf{E})|V(R)$$

since $\det(\mathbf{E}) = a$ in this case (Exercise~

).

{Case 3}. If $\mathbf{E}$ is obtained by adding a times the jth row of $\mathbf{I}$ to its ith row ($j \neq i$), then

$$x_r = \left\{\casespace \begin{array}{ll} y_r & \text{if } r \neq i; \\ y_i + ay_j & \text{if } r = i. \end{array}\right.$$

Then

$$\mathbf{L}(R) = \{(x_1, x_2, \ldots, x_n)\}a_i + ax_j \leq x_i \leq b_i + ax_j \text{ and } a_r \leq x_r \leq b_r \text{if } r \neq i,$$

which is a parallelogram if $n = 2$ and a parallelepiped if $n = 3$ (Figure~). Now

$$V(\mathbf{L}(R)) = \int_{\mathbf{L}(R)} d\mathbf{X},$$

which we can evaluate as an iterated integral in which the first integration is with respect to x_i. For example, if $i = 1$, then

$$V(\mathbf{L}(R)) = \int_{a_n}^{b_n} dx_n \int_{a_{n-1}}^{b_{n-1}} dx_{n-1} \cdots \int_{a_2}^{b_2} dx_2 \int_{a_1+ax_j}^{b_1+ax_j} dx_1. \tag{7.1.60}$$

Since

$$\int_{a_1+ax_j}^{b_1+ax_j} dy_1 = \int_{a_1}^{b_1} dy_1,$$

can be rewritten as

$$\begin{aligned} V(\mathbf{L}(R))\ar &= \int_{a_n}^{b_n} dx_n \int_{a_{n-1}}^{b_{n-1}} dx_{n-1} \cdots \int_{a_2}^{b_2} dx_2 \int_{a_1}^{b_1} dx_1 \\ \ar &= (b_n - a_n)(b_{n-1} - a_{n-1}) \cdots (b_1 - a_1) = V(R). \end{aligned}$$

Hence, $V(\mathbf{L}(R)) = |\det(\mathbf{E})|V(R)$, since $\det(\mathbf{E}) = 1$ in this case (Exercise~

).

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From what we have shown so far, holds if $\mathbf{A}$ is an elementary matrix and S is any compact Jordan measurable set. If $\mathbf{A}$ is an arbitrary nonsingular matrix,

-.0em then we can write $\mathbf{A}$ as a product of elementary matrices and apply our known result successively to $\mathbf{L}_1$, $\mathbf{L}_2$, , $\mathbf{L}_k$ (see). This yields

$$V(\mathbf{L}(S)) = |\det(\mathbf{E}_k)|\,|\det(\mathbf{E}_{k-1})| \cdots |\det \mathbf{E}_1|V(S) = |\det(\mathbf{A})|V(S),$$

by Theorem~ and induction.

We now formulate the rule for change of variables in a multiple integral. Since we are for the present interested only in ``discovering'' the rule, we will make any assumptions that ease this task, deferring questions of rigor until the proof.

Throughout the rest of this section it will be convenient to think of the range and domain of a transformation $\mathbf{G}:\R^n\to\R^n$ as subsets of distinct copies of $\R^n$. We will denote the copy containing $D_{\mathbf{G}}$ as $\E^n$, and write $\mathbf{G}:\E^n\to\R^n$ and $\mathbf{X}=\mathbf{G}(\mathbf{Y})$, reversing the usual roles of $\mathbf{X}$ and $\mathbf{Y}$.

If $\mathbf{G}$ is regular on a subset S of $\E^n$, then each $\mathbf{X}$ in $\mathbf{G}(S)$ can be identified by specifying the unique point $\mathbf{Y}$ in S such that $\mathbf{X}=\mathbf{G}(\mathbf{Y})$.

Suppose that we wish to evaluate $\int_T f(\mathbf{X})\,d\mathbf{X}$, where T is the image of a compact Jordan measurable set S under the regular transformation $\mathbf{X}=\mathbf{G}(\mathbf{Y})$. For simplicity, we take S to be a rectangle and assume that f is continuous on $T=\mathbf{G}(S)$.

Now suppose that $\mathbf{P}=\{R_1,R_2,\dots,R_k\}$ is a partition of S and $T_j=\mathbf{G}(R_j)$ (Figure~).

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Then

$$\int_T f(\mathbf{X})\,d\mathbf{X}=\sum_{j=1}^k\int_{T_j}f(\mathbf{X})\,d\mathbf{X} \tag{7.1.61}$$

(Corollary~ and induction). Since f is continuous, there is a point $\mathbf{X}_j$ in T_j such that

$$\int_{T_j}f(\mathbf{X})\,d\mathbf{X}=f(\mathbf{X}_j)\int_{T_j}d\mathbf{X}=f(\mathbf{X}_j)V(T_j)$$

(Theorem~), so can be rewritten as

$$\int_T f(\mathbf{X})\,d\mathbf{X}=\sum_{j=1}^k f(\mathbf{X}_j)V(T_j). \tag{7.1.62}$$

Now we approximate $V(T_j)$. If

$$\mathbf{X}_j=\mathbf{G}(\mathbf{Y}_j), \tag{7.1.63}$$

then $\mathbf{Y}_j\in R_j$ and, since $\mathbf{G}$ is differentiable at $\mathbf{Y}_j$,

$$\mathbf{G}(\mathbf{Y})\approx\mathbf{G}(\mathbf{Y}_j)+\mathbf{G}'(\mathbf{Y}_j)(\mathbf{Y}-\mathbf{Y}_j). \tag{7.1.64}$$

Here $\mathbf{G}$ and $\mathbf{Y}-\mathbf{Y}_j$ are written as column matrices, $\mathbf{G}'$ is a differential matrix, and `$\approx$'' means` approximately equal'' in a sense that we could make precise if we wished (Theorem~).

It is reasonable to expect that the Jordan content of $\mathbf{G}(R_j)$ is approximately equal to the Jordan content of $\mathbf{A}(R_j)$, where $\mathbf{A}$ is the affine transformation

$$\mathbf{A}(\mathbf{Y})=\mathbf{G}(\mathbf{Y}_j)+\mathbf{G}'(\mathbf{Y}_j)(\mathbf{Y}-\mathbf{Y}_j)$$

on the right side of ; that is,

$$V(\mathbf{G}(R_j))\approx V(\mathbf{A}(R_j)). \tag{7.1.65}$$

We can think of the affine transformation $\mathbf{A}$ as a composition $\mathbf{A}=\mathbf{A}_3\circ\mathbf{A}_2\circ\mathbf{A}_1$, where

$$\begin{aligned}\mathbf{A}_1(\mathbf{Y})\ar&=\mathbf{Y}-\mathbf{Y}_j,\\ \mathbf{A}_2(\mathbf{Y})\ar&=\mathbf{G}'(\mathbf{Y}_j)\mathbf{Y},\\ &\arraytext{and}\\ \mathbf{A}_3(\mathbf{Y})\ar&=\mathbf{G}(\mathbf{Y}_j)+\mathbf{Y}.\end{aligned}$$

Let $R_j'=\mathbf{A}_1(R_j)$. Since $\mathbf{A}_1$ merely shifts R_j to a different location, R_j' is also a rectangle, and

$$V(R_j') = V(R_j). \tag{7.1.66}$$

Now let $R_j'' = \mathbf{A}_2(R_j')$. (In general, R_j'' is not a rectangle.) Since $\mathbf{A}_2$ is the linear transformation with nonsingular matrix $\mathbf{G}'(\mathbf{Y}_j)$, Theorem~ implies that

$$V(R_j'')) = |\det \mathbf{G}'(\mathbf{Y}_j)|V(R_j') = |J\mathbf{G}(\mathbf{Y}_j)|V(R_j), \tag{7.1.67}$$

where $J\mathbf{G}$ is the Jacobian of $\mathbf{G}$. Now let $R_j''' = \mathbf{A}_3(R_j'')$. Since $\mathbf{A}_3$ merely shifts all points in the same way,

$$V(R_j''') = V(R_j''). \tag{7.1.68}$$

Now – suggest that

$$V(T_j) \approx |J\mathbf{G}(\mathbf{Y}_j)|V(R_j).$$

(Recall that $T_j = \mathbf{G}(R_j)$.) Substituting this and into yields

$$\int_T f(\mathbf{X})\,d\mathbf{X} \approx \sum_{j=1}^{k} f(\mathbf{G}(\mathbf{Y}_j))|J\mathbf{G}(\mathbf{Y}_j)|V(R_j).$$

But the sum on the right is a Riemann sum for the integral

$$\int_S f(\mathbf{G}(\mathbf{Y}))|J\mathbf{G}(\mathbf{Y})|\,d\mathbf{Y},$$

which suggests that

$$\int_T f(\mathbf{X})\,d\mathbf{X} = \int_S f(\mathbf{G}(\mathbf{Y}))|J\mathbf{G}(\mathbf{Y})|\,d\mathbf{Y}.$$

We will prove this by an argument that was published in the {} [Vol. 61 (1954), pp. 81-85] by J. Schwartz.

We now prove the following form of the rule for change of variable in a multiple integral.

Since the proof is complicated, we break it down to a series of lemmas. We first observe that both integrals in exist, by Corollary~, since their integrands are continuous. (Note that S is compact and Jordan measurable by assumption, and $\mathbf{G}(S)$ is compact and Jordan measurable by Theorem~.) Also, the result is trivial if $V(S) = 0$, since then $V(\mathbf{G}(S)) = 0$ by Lemma~, and both integrals in vanish. Hence, we assume that $V(S) > 0$. We need the following definition.

Let s be the edge length of C. Let $\mathbf{Y}_0 = (c_1, c_2, \ldots, c_n)$ be the center of C, and suppose that $\mathbf{H} = (y_1, y_2, \ldots, y_n) \in C$. If $\mathbf{H} = (h_1, h_2, \ldots, h_n)$ is continuously differentiable on C, then applying the mean value theorem (Theorem~) to the components of $\mathbf{H}$ yields

$$h_i(\mathbf{Y}) - h_i(\mathbf{Y}_0) = \sum_{j=1}^{n} \frac{\partial h_i(\mathbf{Y}_i)}{\partial y_j}(y_j - c_j), \quad 1 \le i \le n,$$

where $\mathbf{Y}_i \in C$. Hence, recalling that

$$\mathbf{H}'(\mathbf{Y}) = \left[\frac{\partial h_i}{\partial y_j}\right]_{i,j=1}^{n},$$

applying Definition~, and noting that $|y_j - c_j| \le s/2$, $1 \le j \le n$, we infer that

$$|h_i(\mathbf{Y}) - h_i(\mathbf{Y}_0)| \le \frac{s}{2} \max\{\|\mathbf{H}'(\mathbf{Y})\|_\infty\}\mathbf{Y} \in C, \quad 1 \le i \le n.$$

This means that $\mathbf{H}(C)$ is contained in a cube with center $\mathbf{X}_0 = \mathbf{H}(\mathbf{Y}_0)$ and edge length

$$s \max\{\|\mathbf{H}'(\mathbf{Y})\|_\infty\}\mathbf{Y} \in C.$$

Therefore,

Missing \left or extra \right

Now let

$$\mathbf{L}(\mathbf{X}) = \mathbf{A}^{-1}\mathbf{X}$$

and set $\mathbf{H} = \mathbf{L} \circ \mathbf{G}$; then

$$\mathbf{H}(C) = \mathbf{L}(\mathbf{G}(C)) \quad \text{and} \quad \mathbf{H}' = \mathbf{A}^{-1}\mathbf{G}',$$

so implies that

$$V(\mathbf{L}(\mathbf{G}(C))) \le \left[\max\{\|\mathbf{A}^{-1}\mathbf{G}'(\mathbf{Y})\|_\infty\}\mathbf{Y} \in C\right]^n V(C). \tag{7.1.69}$$

Since $\mathbf{L}$ is linear, Theorem~ with $\mathbf{A}$ replaced by $\mathbf{A}^{-1}$ implies that

$$V(\mathbf{L}(\mathbf{G}(C))) = |\det(\mathbf{A})^{-1}|V(\mathbf{G}(C)).$$

This and imply that

$$|\det(\mathbf{A}^{-1})|V(\mathbf{G}(C)) \le \left[\max\{\|\mathbf{A}^{-1}\mathbf{G}'(\mathbf{Y})\|_\infty\}\mathbf{Y} \in C\right]^n V(C).$$

Since $\det(\mathbf{A}^{-1}) = 1/\det(\mathbf{A})$, this implies .

Let $\mathbf{P}$ be a partition of C into subcubes C_1, C_2, , C_k with centers $\mathbf{Y}_1$, $\mathbf{Y}_2$, , $\mathbf{Y}_k$. Then

$$V(\mathbf{G}(C)) = \sum_{j=1}^{k} V(\mathbf{G}(C_j)). \tag{7.1.70}$$

Applying Lemma~ to C_j with $\mathbf{A} = \mathbf{G}'(\mathbf{A}_j)$ yields

$$V(\mathbf{G}(C_j)) \le |J\mathbf{G}(\mathbf{Y}_j)|\left[\max\{\|(\mathbf{G}'(\mathbf{Y}_j))^{-1}\mathbf{G}'(\mathbf{Y})\|_\infty\}\mathbf{Y} \in C_j\right]^n V(C_j). \tag{7.1.71}$$

Exercise~ implies that if $\epsilon > 0$, there is a $\delta > 0$ such that

$$\max\{\|(\mathbf{G}'(\mathbf{Y}_j))^{-1}\mathbf{G}'(\mathbf{Y})\|_\infty\}\mathbf{Y} \in C_j < 1 + \epsilon, \quad 1 \le j \le k, \quad \text{if} \quad \|\mathbf{P}\| < \delta.$$

Therefore, from ,

$$V(\mathbf{G}(C_j)) \le (1+\epsilon)^n |J\mathbf{G}(\mathbf{Y}_j)|V(C_j),$$

so implies that

$$V(\mathbf{G}(C)) \le (1+\epsilon)^n \sum_{j=1}^{k} |J\mathbf{G}(\mathbf{Y}_j)|V(C_j) \quad \text{if} \quad \|\mathbf{P}\| < \delta.$$

Since the sum on the right is a Riemann sum for $\int_C |J\mathbf{G}(\mathbf{Y})|\,d\mathbf{Y}$ and ϵ can be taken arbitrarily small, this implies .

Since S is Jordan measurable,

$$\int_C \psi_S(\mathbf{X})\,d\mathbf{X} = V(S)$$

if C is any cube containing S. From this and the definition of the integral, there is a $\delta > 0$ such that if $\mathbf{P}$ is any partition of C with $\|\mathbf{P}\| < \delta$ and σ is any Riemann sum of ψ_S over $\mathbf{P}$, then $\sigma > V(S) - \epsilon/2$. Therefore, if $s(P)$ is the lower sum of ψ_S over $\mathbf{P}$, then

$$s(\mathbf{P}) > V(S) - \epsilon \quad \text{if} \quad \|\mathbf{P}\| < \delta. \tag{7.1.72}$$

Now suppose that $\mathbf{P} = \{C_1, C_2, \dots, C_k\}$ is a partition of C into cubes with $\|\mathbf{P}\| < \min(\rho, \delta)$, and let C_1, C_2, , C_k be numbered so that $C_j \subset S$ if $1 \le j \le r$ and $C_j \cap S^c \ne \emptyset$ if $j > r$. From , $s(\mathbf{P}) = \sum_{j=1}^{r} V(C_k)$. This and imply . Clearly, $C_i^0 \cap C_j^0 = \emptyset$ if $i \ne j$.

From the continuity of $J\mathbf{G}$ and f on the compact sets S and $\mathbf{G}(S)$, there are constants M_1 and M_2 such that

$$|J\mathbf{G}(\mathbf{Y})| \le M_1 \quad \text{if} \quad \mathbf{Y} \in S \tag{7.1.73}$$

and

$$|f(\mathbf{X})| \le M_2 \quad \text{if} \quad \mathbf{X} \in \mathbf{G}(S) \tag{7.1.74}$$

(Theorem~). Now suppose that $\epsilon > 0$. Since $f \circ \mathbf{G}$ is uniformly continuous on S (Theorem~), there is a $\delta > 0$ such that

$$|f(\mathbf{G}(\mathbf{Y}))-f(\mathbf{G}(\mathbf{Y}'))|<\epsilon \quad \text{if} \quad |\mathbf{Y}-\mathbf{Y}'|<\delta \text{ and } \mathbf{Y},\mathbf{Y}'\in S. \tag{7.1.75}$$

Now let $C_1, C_2, , C_r$ be chosen as described in Lemma~, with $\rho=\delta/\sqrt{n}$. Let

$$S_1=\{\mathbf{Y}\in S\}\mathbf{Y}\notin\bigcup_{j=1}^r C_j.$$

Then $V(S_1)<\epsilon$ and

$$S=\left(\bigcup_{j=1}^r C_j\right)\cup S_1. \tag{7.1.76}$$

Suppose that $\mathbf{Y}_1, \mathbf{Y}_2, , \mathbf{Y}_r$ are points in $C_1, C_2, , C_r$ and $\mathbf{X}_j=\mathbf{G}(\mathbf{Y}_j)$, $1\le j\le r$. From and Theorem~,

$$\begin{aligned}
Q(S)\ar &= \int_{\mathbf{G}(S_1)} f(\mathbf{X})\,d\mathbf{X}-\int_{S_1} f(\mathbf{G}(\mathbf{Y}))|J\mathbf{G}(\mathbf{Y})|\,d\mathbf{Y}\\
\ar &+\sum_{j=1}^r\int_{\mathbf{G}(C_j)} f(\mathbf{X})\,d\mathbf{X}-\sum_{j=1}^r\int_{C_j} f(\mathbf{G}(\mathbf{Y}))|J\mathbf{G}(\mathbf{Y})|\,d\mathbf{Y}\\
\ar &= \int_{\mathbf{G}(S_1)} f(\mathbf{X})\,d\mathbf{X}-\int_{S_1} f(\mathbf{G}(\mathbf{Y}))|J\mathbf{G}(\mathbf{Y})|\,d\mathbf{Y}\\
\ar &+\sum_{j=1}^r\int_{\mathbf{G}(C_j)} (f(\mathbf{X})-f(\mathbf{A}_j))\,d\mathbf{X}\\
\ar &+\sum_{j=1}^r\int_{C_j} ((f(\mathbf{G}(\mathbf{Y}_j))-f(\mathbf{G}(\mathbf{Y})))|J(\mathbf{G}(\mathbf{Y})|\,d\mathbf{Y}\\
\ar &+\sum_{j=1}^r f(\mathbf{X}_j)\left(V(\mathbf{G}(C_j))-\int_{C_j}|J\mathbf{G}(\mathbf{Y})|\,d\mathbf{Y}\right).
\end{aligned}$$

Since $f(\mathbf{X})\ge 0$,

$$\int_{S_1} f(\mathbf{G}(\mathbf{Y}))|J\mathbf{G}(\mathbf{Y})|\,d\mathbf{Y}\ge 0,$$

and Lemma~ implies that the last sum is nonpositive. Therefore,

$$Q(S)\le I_1+I_2+I_3, \tag{7.1.77}$$

where

$$I_1=\int_{\mathbf{G}(S_1)} f(\mathbf{X})\,d\mathbf{X},\quad I_2=\sum_{j=1}^r\int_{\mathbf{G}(C_j)}|f(\mathbf{X})-f(\mathbf{X}_j)|\,d\mathbf{X},$$

and

$$I_3=\sum_{j=1}^r\int_{C_j}|f(\mathbf{G})(\mathbf{Y}_j))-f(\mathbf{G}(\mathbf{Y}))||J\mathbf{G}(\mathbf{Y})|\,d\mathbf{Y}.$$

We will now estimate these three terms. Suppose that $\epsilon>0$.

To estimate I_1, we first remind you that since $\mathbf{G}$ is regular on the compact set S, $\mathbf{G}$ is also regular on some open set $\mathcal{O}$ containing S (Definition~). Therefore, since $S_1\subset S$ and $V(S_1)<\epsilon$, S_1 can be covered by cubes $T_1, T_2, , T_m$ such that

$$\sum_{j=1}^r V(T_j)<\epsilon \tag{7.1.78}$$

and $\mathbf{G}$ is regular on $\bigcup_{j=1}^m T_j$. Now,

Bracket argument to \\ must be a dimension

To estimate I_2, we note that if $\mathbf{X}$ and $\mathbf{X}_j$ are in $\mathbf{G}(C_j)$ then $\mathbf{X}=\mathbf{G}(\mathbf{Y})$ and $\mathbf{X}_j=\mathbf{G}(\mathbf{Y}_j)$ for some $\mathbf{Y}$ and $\mathbf{Y}_j$ in C_j. Since the edge length of C_j is less than $\delta/\sqrt{n}$, it follows that $|\mathbf{Y}-\mathbf{Y}_j|<\delta$, so $|f(\mathbf{X})-f(\mathbf{X}_j)|<\epsilon$, by . Therefore,

Bracket argument to \\ must be a dimension

To estimate I_3, we note again from that $|f(\mathbf{G}(\mathbf{Y}_j)) - f(\mathbf{G}(\mathbf{Y}))| < \epsilon$ if $\mathbf{Y}$ and $\mathbf{Y}_j$ are in C_j. Hence,

$$\begin{aligned} I_3 \ar &< \epsilon \sum_{j=1}^{r} \int_{C_j} |J\mathbf{G}(\mathbf{Y})|\,d\mathbf{Y} \\ \ar &\le M_1\epsilon \sum_{j=1}^{r} V(C_j) \quad \text{(from (7.1.73)} \\ \ar &\le M_1 V(S)\epsilon \end{aligned}$$

because $\bigcup_{j=1}^{r} C_j \subset S$ and $C_i^0 \cap C_j^0 = \emptyset$ if $i \ne j$.

From these inequalities on I_1, I_2, and I_3, now implies that

$$Q(S) < M_1(M_2 + 2V(S))\epsilon.$$

Since ϵ is an arbitrary positive number, it now follows that $Q(S) \le 0$.

Let

$$\mathbf{G}_1 = \mathbf{G}^{-1}, \quad S_1 = \mathbf{G}(S), \quad f_1 = (|J\mathbf{G}|)f \circ \mathbf{G}, \tag{7.1.79}$$

and

$$Q_1(S_1) \ar = \dst \int_{\mathbf{G}_1(S_1)} f_1(\mathbf{Y})\,d\mathbf{Y} - \int_{S_1} f_1(\mathbf{G}_1(\mathbf{X}))|J\mathbf{G}_1(\mathbf{X})|\,d\mathbf{X}. \tag{7.1.80}$$

Since $\mathbf{G}_1$ is regular on S_1 (Theorem~) and f_1 is continuous and nonnegative on $\mathbf{G}_1(S_1) = S$, Lemma~ implies that $Q_1(S_1) \le 0$. However, substituting from into and again noting that $\mathbf{G}_1(S_1) = S$ yields

$$\begin{aligned} Q_1(S_1) \ar = \dst \int_S f(\mathbf{G}(\mathbf{Y}))|J\mathbf{G}(\mathbf{Y})|\,d\mathbf{Y} \\ \ar - \dst \int_{\mathbf{G}(S)} f(\mathbf{G}(\mathbf{G}^{-1}(\mathbf{X})))|J\mathbf{G}(\mathbf{G}^{-1}(\mathbf{X}))||J\mathbf{G}^{-1}(\mathbf{X})|\,d\mathbf{X}. \end{aligned} \tag{7.1.81}$$

Since $\mathbf{G}(\mathbf{G}^{-1}(\mathbf{X})) = \mathbf{X}$, $f(\mathbf{G}(\mathbf{G}^{-1}(\mathbf{X}))) = f(\mathbf{X})$. However, it is important to interpret the symbol $J\mathbf{G}(\mathbf{G}^{-1}(\mathbf{X}))$ properly. We are not substituting $\mathbf{G}^{-1}(\mathbf{X})$ into $\mathbf{G}$ here; rather, we are evaluating the determinant of the differential matrix of $\mathbf{G}$ at the point $\mathbf{Y} = \mathbf{G}^{-1}(\mathbf{X})$. From Theorems~ and ~,

$$|J\mathbf{G}(\mathbf{G}^{-1}(\mathbf{X}))||J\mathbf{G}^{-1}(\mathbf{X})| = 1,$$

so can be rewritten as

$$Q_1(S_1) = \int_S f(\mathbf{G}(\mathbf{Y}))|J\mathbf{G}(\mathbf{Y})|\,d\mathbf{Y} - \int_{\mathbf{G}(S)} f(\mathbf{X})\,d\mathbf{X} = -Q(S).$$

Since $Q_1(S_1) \le 0$, it now follows that $Q(S) \ge 0$.

We can now complete the proof of Theorem. Lemmas and imply if f is nonnegative on S. Now suppose that

$$m = \min\{f(\mathbf{X})\}\mathbf{X} \in \mathbf{G}(S) < 0.$$

Then $f - m$ is nonnegative on $\mathbf{G}(S)$, so with f replaced by $f - m$ implies that

$$\int_{\mathbf{G}(S)} (f(\mathbf{X}) - m)\,d\mathbf{X} = \int_S (f(\mathbf{G}(\mathbf{Y}) - m)|J\mathbf{G}(\mathbf{Y})|\,d\mathbf{Y}. \tag{7.1.82}$$

However, setting $f = 1$ in yields

$$\int_{\mathbf{G}(S)} d\mathbf{X} = \int_S |J\mathbf{G}(\mathbf{Y})|\,d\mathbf{Y},$$

so implies .

The assumptions of Theorem~ are too stringent for many applications. For example, to find the area of the disc

$$\{(x, y)\}x^2 + y^2 \le 1,$$

it is convenient to use polar coordinates and regard the circle as $\mathbf{G}(S)$, where

$$\mathbf{G}(r,\theta)=\begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} \tag{7.1.83}$$

and S is the compact set

$$S=\{(r,\theta)\}0\le r\le 1,\ 0\le\theta\le 2\pi \tag{7.1.84}$$

(Figure~).

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Since

$$\mathbf{G}'(r,\theta)=\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix},$$

it follows that $J\mathbf{G}(r,\theta)=r$. Therefore, formally applying Theorem~ with $f\equiv 1$ yields

$$A=\int_{\mathbf{G}(S)}d\mathbf{X}=\int_S r\,d(r,\theta)=\int_0^1 r\,dr\int_0^{2\pi}d\theta=\pi.$$

Although this is a familiar result, Theorem~ does not really apply here, since $\mathbf{G}(r,0)=\mathbf{G}(r,2\pi)$, \$0r1 \$, so $\mathbf{G}$ is not one-to-one on S, and therefore not regular on S.

The next theorem shows that the assumptions of Theorem~ can be relaxed so as to include this example.

Since f is continuous on $\mathbf{G}(S)$ and $(|J\mathbf{G}|)f\circ\mathbf{G}$ is continuous on S, the integrals in both exist, by Corollary~. Now let

$$\rho=\backslash\mathrm{dist}\ (\partial S,N^c)$$

(Exercise~5.1.25), and

$$P=\{\mathbf{Y}\}\backslash\mathrm{dist}(\mathbf{Y},\partial S)\le\frac{\rho}{2}.$$

Then P is a compact subset of N (Exercise~5.1.26) and $\partial S\subset P^0$ (Figure~).

Since S is Jordan measurable, $V(\partial S)=0$, by Theorem~. Therefore, if $\epsilon>0$, we can choose cubes C_1, C_2, , C_k in P^0 such that

$$\partial S\subset\bigcup_{j=1}^k C_j^0 \tag{7.1.85}$$

and

$$\sum_{j=1}^k V(C_j)<\epsilon \tag{7.1.86}$$

Now let S_1 be the closure of the set of points in S that are not in any of the cubes C_1, C_2, , C_k; thus,

$$S_1=\overline{S\cap\left(\cup_{j=1}^k C_j\right)^c}.$$

Because of , $S_1\cap\partial S=\emptyset$, so S_1 is a compact Jordan measurable subset of S^0. Therefore, $\mathbf{G}$ is regular on S_1, and f is continuous on $\mathbf{G}(S_1)$. Consequently, if Q is as defined in , then $Q(S_1)=0$ by Theorem~.

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Now

$$Q(S)=Q(S_1)+Q(S\cap S_1^c)=Q(S\cap S_1^c) \tag{7.1.87}$$

(Exercise~) and

$$|Q(S\cap S_1^c)|\le\left|\int_{\mathbf{G}(S\cap S_1^c)}f(\mathbf{X})\,d\mathbf{X}\right|+\left|\int_{S\cap S_1^c}f(\mathbf{G}(\mathbf{Y}))|J\mathbf{G}(\mathbf{Y})|\,d\mathbf{Y}|\right|.$$

But

$$\left|\int_{S\cap S_1^c}f(\mathbf{G}(\mathbf{Y}))|J\mathbf{G}(\mathbf{Y})|\,d\mathbf{Y}|\right|\le M_1M_2V(S\cap S_1^c),\tag{7.1.88}$$

where M_1 and M_2 are as defined in and . Since $S\cap S_1^c\subset\cup_{j=1}^k C_j$, implies that $V(S\cap S_1^k)<\epsilon$; therefore,

$$\left|\int_{S\cap S_1^c}f(\mathbf{G}(\mathbf{Y}))|J\mathbf{G}(\mathbf{Y})|\,d\mathbf{Y}|\right|\le M_1M_2\epsilon,\tag{7.1.89}$$

from . Also

$$\left|\int_{\mathbf{G}(S\cap S_1^c)}f(\mathbf{X})\,d\mathbf{X}\right|\le M_2V(\mathbf{G}(S\cap S_1^c))\le M_2\sum_{j=1}^k V(\mathbf{G}(C_j)).\tag{7.1.90}$$

By the argument that led to with $\mathbf{H}=\mathbf{G}$ and $C=C_j$,

$$V(\mathbf{G}(C_j))\le[\max\{\|\mathbf{G}'(\mathbf{Y})\|_\infty\}\mathbf{Y}\in C_j]^n\,V(C_j),$$

so can be rewritten as

$$\left|\int_{\mathbf{G}(S\cap S_1^c)}f(\mathbf{X})\,d\mathbf{X}\right|\le M_2\,[\max\{\|\mathbf{G}'(\mathbf{Y})\|_\infty\}\mathbf{Y}\in P]^n\,\epsilon,$$

because of . Since ϵ can be made arbitrarily small, this and imply that $Q(S\cap S_1^c)=0$. Now $Q(S)=0$, from .

The transformation to polar coordinates to compute the area of the disc is now justified, since $\mathbf{G}$ and S as defined by and satisfy the assumptions of Theorem~.

If $\mathbf{G}$ is the transformation from polar to rectangle coordinates

$$\begin{bmatrix}x\\y\end{bmatrix}=\mathbf{G}(r,\theta)=\begin{bmatrix}r\cos\theta\\r\sin\theta\end{bmatrix},\tag{7.1.91}$$

then $J\mathbf{G}(r,\theta)=r$ and becomes

$$\int_{\mathbf{G}(S)}f(x,y)\,d(x,y)=\int_S f(r\cos\theta,r\sin\theta)r\,d(r,\theta)$$

if we assume, as is conventional, that S is in the closed right half of the $r\theta$-plane. This transformation is especially useful when the boundaries of S can be expressed conveniently in terms of polar coordinates, as in the example preceding Theorem~. Two more examples follow.

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If $\mathbf{G}$ is the transformation from spherical to rectangular coordinates,

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\mathbf{G}(r,\theta,\phi)=\begin{bmatrix}r\cos\theta\cos\phi\\r\sin\theta\cos\phi\\r\sin\phi\end{bmatrix},\tag{7.1.92}$$

then

$$\mathbf{G}'(r,\theta,\phi)=\begin{bmatrix}\cos\theta\cos\phi&-r\sin\theta\cos\phi&-r\cos\theta\sin\phi\\\sin\theta\cos\phi&r\cos\theta\cos\phi&-r\sin\theta\sin\phi\\\sin\phi&0&r\cos\phi\end{bmatrix}$$

and $J_G(r,\theta,\phi) = r^2\cos\phi$, so becomes

$$\begin{aligned}&\dst\int_{G(S)} f(x,y,z)\,d(x,y,z)\\ &= \dst\int_S f(r\cos\theta\cos\phi, r\sin\theta\cos\phi, r\sin\phi)r^2\cos\phi\,d(r,\theta,\phi)\end{aligned} \tag{7.1.93}$$

if we make the conventional assumption that $|\phi| \le \pi/2$ and $r \ge 0$.

We now consider other applications of Theorem~.

12pt 6pt

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7.2: Iterated Integrals and Multiple Integrals

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7.3: Change of Variables in Multiple Integrals

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CHAPTER OVERVIEW

8: Metric Spaces

IN THIS CHAPTER we study metric spaces.

- SECTION 8.1 defines the concept and basic properties of a metric space. Several examples of metric spaces are considered.
- SECTION 8.2 defines and discusses compactness in a metric space.
- SECTION 8.3 deals with continuous functions on metric spaces.

8.1: Introduction to Metric Spaces
8.2: Compact Sets in a Metric Space
8.3: Continuous Functions on Metric Spaces

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8.1: Introduction to Metric Spaces

If $n \geq 2$ and $u_1, u_2, , u_n$ are arbitrary members of A, then

and induction yield the inequality

$$\rho(u_1, u_n) \leq \sum_{i=1}^{n-1} \rho(u_i, u_{i+1}).$$

Motivated by this example, in an arbitrary metric space we call $\rho(u, v)$ {}, and we call Definition~

{}.

Example~ shows that it is possible to define a metric on any nonempty set A. In fact, it is possible to define infinitely many metrics on any set with more than one member (Exercise~). Therefore, to specify a metric space completely, we must specify the couple (A, ρ), where A is the set and ρ is the metric. (In some cases we will not be so precise; for example, we will always refer to the real numbers with the metric $\rho(u, v) = |u - v|$ simply as $\R$.)

There is an important kind of metric space that arises when a definition of length is imposed on a vector space. Although we assume that you are familiar with the definition of a vector space, we restate it here for convenience. We confine the definition to vector spaces over the real numbers.

We say that A is {} if (1) is true, and that A is {} if (6) is true. It can be shown that if B is any nonempty subset of A that is closed under vector addition and scalar multiplication, then B together with these operations is itself a vector space. (See any linear algebra text for the proof.) We say that B is a {} of A.

From with $u = x - y$, $\rho(x, y) = N(x - y) \geq 0$, with equality if and only if $x = y$. From with $u = x - y$ and $a = -1$,

$$\rho(y, x) = N(y - x) = N(-(x - y)) = N(x - y) = \rho(x, y).$$

From

with $u = x - z$ and $v = z - y$,

$$\rho(x, y) = N(x - y) \leq N(x - z) + N(z - y) = \rho(x, z) + \rho(z, y).$$

-2em2em

We will say that the metric in is {} N. Whenever we speak of a normed vector space (A, N), it is to be understood that we are regarding it as a metric space (A, ρ), where ρ is the metric induced by N.

We will often write $N(u)$ as $\|u\|$. In this case we will denote the normed vector space as $(A, \|\cdot\|)$.

Since

$$x = y + (x - y),$$

Definition~ with $u = y$ and $v = x - y$ implies that

$$N(x) \leq N(y) + N(x - y),$$

or

$$N(x) - N(y) \leq N(x - y).$$

Interchanging x and y yields

$$N(y) - N(x) \leq N(y - x).$$

Since $N(x - y) = N(y - x)$ (Definition~

with $u = x - y$ and $a = -1$), the last two inequalities imply .

In Section~5.1 we defined the norm of a vector $\mathbf{X} = (x_1, x_2, \ldots, x_n)$ in $\R^n$ as

$$\|\mathbf{X}\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}.$$

The metric induced by this norm is

$$\rho(\mathbf{X},\mathbf{Y}) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2}.$$

Whenever we write $\R^n$ without identifying the norm or metric specifically, we are referring to $\R^n$ with this norm and this induced metric.

The following definition provides infinitely many norms and metrics on $\R^n$.

To justify this definition, we must verify that actually defines a norm. Since it is clear that $\|\mathbf{X}\|_p \ge 0$ with equality if and only if $\mathbf{X} = \mathbf{0}$, and $\|a\mathbf{X}\|_p = |a|\|\mathbf{X}\|_p$ if a is any real number and $\mathbf{X} \in \R^n$, this reduces to showing that

$$\|\mathbf{X}+\mathbf{Y}\|_p \le \|\mathbf{X}\|_p + \|\mathbf{Y}\|_p \tag{8.1.1}$$

for every $\mathbf{X}$ and $\mathbf{Y}$ in $\R^n$. Since

$$|x_i + y_i| \le |x_i| + |y_i|,$$

summing both sides of this equation from $i = 1$ to n yields with $p = 1$. To handle the case where $p > 1$, we need the following lemmas. The inequality established in the first lemma is known as {}.

Let α and β be any two positive numbers, and consider the function

$$f(\beta) = \frac{\alpha^p}{p} + \frac{\beta^q}{q} - \alpha\beta,$$

where we regard α as a constant. Since $f'(\beta) = \beta^{q-1} - \alpha$ and $f''(\beta) = (q-1)\beta^{q-2} > 0$ for $\beta > 0$, f assumes its minimum value on $[0,\infty)$ at $\beta = \alpha^{1/(q-1)} = \alpha^{p-1}$. But

$$f(\alpha^{p-1}) = \frac{\alpha^p}{p} + \frac{\alpha^{(p-1)q}}{q} - \alpha^p = \alpha^p\left(\frac{1}{p} + \frac{1}{q} - 1\right) = 0.$$

Therefore,

$$\alpha\beta \le \frac{\alpha^p}{p} + \frac{\beta^q}{q} \quad \text{if} \quad \alpha,\beta \ge 0. \tag{8.1.2}$$

Now let

$$\alpha_i = \mu_i\left(\sum_{j=1}^{n} \mu_j^p\right)^{-1/p} \quad \text{and} \quad \beta_i = \nu_i\left(\sum_{j=1}^{n} \nu_j^q\right)^{-1/q}.$$

From ,

$$\alpha_i\beta_i \le \frac{\mu_i^p}{p}\left(\sum_{j=1}^{n} \mu_j^p\right)^{-1} + \frac{\nu_i^q}{q}\left(\sum_{j=1}^{n} \nu_j^q\right)^{-1}.$$

From , summing this from $i = 1$ to n yields $\sum_{i=1}^{n} \alpha_i\beta_i \le 1$, which implies .

Again, let $q = p/(p-1)$. We write

$$\sum_{i=1}^{n} (u_i + v_i)^p = \sum_{i=1}^{n} u_i(u_i + v_i)^{p-1} + \sum_{i=1}^{n} v_i(u_i + v_i)^{p-1}. \tag{8.1.3}$$

From H"older's inequality with $\mu_i = u_i$ and $\nu_i = (u_i + v_i)^{p-1}$,

$$\sum_{i=1}^{n} u_i(u_i + v_i)^{p-1} \le \left(\sum_{i=1}^{n} u_i^p\right)^{1/p}\left(\sum_{i=1}^{n} (u_i + v_i)^p\right)^{1/q}, \tag{8.1.4}$$

since $q(p-1)=p$. Similarly,

$$\sum_{i=1}^{n} v_i(u_i+v_i)^{p-1} \le \left(\sum_{i=1}^{n} v_i^p\right)^{1/p} \left(\sum_{i=1}^{n} (u_i+v_i)^p\right)^{1/q}.$$

This, , and imply that

$$\sum_{i=1}^{n} (u_i+v_i)^p \le \left[\left(\sum_{i=1}^{n} u_i^p\right)^{1/p} + \left(\sum_{i=1}^{n} v_i^p\right)^{1/p}\right] \left(\sum_{i=1}^{n} (u_i+v_i)^p\right)^{1/q}.$$

Since $1-1/q=1/p$, this implies , which is known as {}.

We leave it to you to verify that Minkowski's inequality implies if $p>1$.

We now define the ∞-{} on $\R^n$ by

$$\|\mathbf{X}\|_\infty = \max\{|x_i|\}1 \le i \le n. \tag{8.1.5}$$

We leave it to you to verify (Exercise~) that $\|\cdot\|_\infty$ is a norm on $\R^n$. The associated metric is

$$\rho_\infty(\mathbf{X},\mathbf{Y}) = \max\{|x_i-y_i|\}1 \le i \le n.$$

The following theorem justifies the notation in .

Let $u_1, u_2, , u_n$ be nonnegative and $M=\max\{u_i\}1 \le i \le n$. Define

$$\sigma(p) = \left(\sum_{i=1}^{n} u_i^p\right)^{1/p}.$$

Since $u_i/\sigma(p) \le 1$ and $p_2>p_1$,

$$\left(\frac{u_i}{\sigma(p_2)}\right)^{p_1} \ge \left(\frac{u_i}{\sigma(p_2)}\right)^{p_2};$$

therefore,

$$\frac{\sigma(p_1)}{\sigma(p_2)} = \left(\sum_{i=1}^{n} \left(\frac{u_i}{\sigma(p_2)}\right)^{p_1}\right)^{1/p_1} \ge \left(\sum_{i=1}^{n} \left(\frac{u_i}{\sigma(p_2)}\right)^{p_2}\right)^{1/p_1} = 1,$$

so $\sigma(p_1) \ge \sigma(p_2)$. Since $M \le \sigma(p) \le Mn^{1/p}$, $\lim_{p\to\infty}\sigma(p)=M$. Letting $u_i=|x_i|$ yields and .

Since Minkowski's inequality is false if $p<1$ (Exercise~), is not a norm in this case. However, if $0<p<1$, then

$$\|\mathbf{X}\|_p = \sum_{i=1}^{n} |x_i|^p$$

is a norm on $\R^n$ (Exercise~).

In this section and in the exercises we will consider subsets of the vector space $\R^\infty$ consisting of sequences $\mathbf{X}=\{x_i\}_{i=1}^\infty$, with vector addition and scalar multiplication defined by

$$\mathbf{X}+\mathbf{Y} = \{x_i+y_i\}_{i=1}^\infty \quad \text{and} \quad r\mathbf{X} = \{rx_i\}_{i=1}^\infty.$$

The metric induced by $\|\cdot\|_p$ is

$$\rho_p(\mathbf{X},\mathbf{Y}) = \left(\sum_{i=1}^{\infty} |x_i-y_i|^p\right)^{1/p}.$$

Henceforth, we will denote $(\ell_p, \|\cdot\|_p)$ simply by ℓ_p.

The metric induced by $\|\cdot\|_\infty$ is

$$\rho_\infty(\mathbf{X},\mathbf{Y}) = \sup\{|x_i-y_i|\}i \ge 1.$$

Henceforth, we will denote $(\ell_\infty, \|\cdot\|_\infty)$ simply by ℓ_∞.

At this point you may want to review Definition~ and Exercises~ and , which apply equally well to subsets of a metric space (A, ρ).

We will now state some definitions and theorems for a general metric space (A, ρ) that are analogous to definitions and theorems presented in Section~1.3 for the real numbers. To avoid repetition, it is to be understood in all these definitions that we are discussing a given metric space (A, ρ).

We must show that if $u_1 \in S_r(u_0)$, then there is an $\epsilon > 0$ such that

$$S_\epsilon(u_1) \subset S_r(u_0). \tag{8.1.6}$$

If $u_1 \in S_r(u_0)$, then $\rho(u_1, u_0) < r$. Since

$$\rho(u, u_0) \le \rho(u, u_1) + \rho(u_1, u_0)$$

for any u in A, $\rho(u, u_0) < r$ if $\rho(u, u_1) < r - \rho(u_1, u_0)$. Therefore, holds if $\epsilon < r - \rho(u_1, u_0)$.

The entire space A is open and therefore $\emptyset\ (= A^c)$ is closed. However, $\emptyset$ is also open, for to deny this is to say that it contains a point that is not an interior point, which is absurd because $\emptyset$ contains no points. Since $\emptyset$ is open, $A\ (= \emptyset^c)$ is closed. If $A = \R$, these are the only sets that are both open and closed, but this is not so in all metric spaces. For example, if ρ is the discrete metric, then every subset of A is both open and closed. (Verify!)

A {} of a point u_0 is a set that contains every point of some neighborhood of u_0 except u_0 itself. (If ρ is the discrete metric then the empty set is a deleted neighborhood of every member of A!)

The proof of the following theorem is identical to the proof Theorem~.

Although this definition is identical to Definition~, you should not assume that conclusions valid for the real numbers are necessarily valid in all metric spaces. For example, if $A = \R$ and $\rho(u, v) = |u - v|$, then

$$\overline{S}_r(u_0) = \{u \} \rho(u, u_0) \le r.$$

This is not true in every metric space (Exercise~).

For the proof of the following theorem, see the proofs of Theorem~ and Corollary~.

Since metric spaces are not ordered, concepts and results concerning the real numbers that depend on order for their definitions must be redefined and reexamined in the context of metric spaces. The first example of this kind is completeness. To discuss this concept, we begin by defining an {} (more briefly, a {}) in a metric space (A, ρ) as a function defined on the integers $n \ge k$ with values in A. As we did for real sequences, we denote a sequence in A by, for example, $\{u_n\} = \{u_n\}_{n=k}^\infty$. A subsequence of a sequence in A is defined in exactly the same way as a subsequence of a sequence of real numbers (Definition~).

We leave the proof of the following theorem to you. (See the proofs of Theorems~ and .)

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We note that if ρ is the metric induced by a norm $\|\cdot\|$ on A, then and can be replaced by

$$\lim_{n\to\infty} \|u_n - u\| = 0$$

and

$$\|u_n - u_m\| < \epsilon \quad \text{and} \quad m, n > N,$$

respectively.

Suppose that $\lim_{n\to\infty} u_n = u$. If $\epsilon > 0$, there is an integer N such that $\rho(u_n, u) < \epsilon/2$ if $n > N$. Therefore, if m, $n > N$, then

$$\rho(u_n, u_m) \le \rho(u_n, u) + \rho(u, u_m) < \epsilon.$$

-2em2em

This example raises a question that we should resolve before going further. In Section~1.1 we defined completeness to mean that the real numbers have the following property:

Axiom {}. Every nonempty set of real numbers that is bounded above has a supremum.

Here we are saying that the real numbers are complete because every Cauchy sequence of real numbers has a limit. We will now show that these two usages of ``complete'' are consistent.

The proof of Theorem~ requires the existence of the (finite) limits inferior and superior of a bounded sequence of real numbers, a consequence of Axiom~{}. However, the assertion in Axiom~{} can be deduced as a theorem if Axiom~{} is replaced by the assumption that every Cauchy sequence of real numbers has a limit. To see this, let T be a nonempty set of real numbers that is bounded above. We first show that there are sequences $\{u_i\}_{i=1}^{\infty}$ and $\{v_i\}_{i=1}^{\infty}$ with the following properties for all $i \geq 1$:

Since T is nonempty and bounded above, u_1 and v_1 can be chosen to satisfy with $i = 1$. Clearly, holds with $i = 1$. Let $w_1 = (u_1 + v_1)/2$, and let

$$(u_2, v_2) = \begin{cases} (w_1, v_1) & \text{if } w_1 \leq t \text{ for some } t \in T, \\ (u_1, w_1) & \text{if } w_1 \geq t \text{ for all } t \in T. \end{cases}$$

In either case, and hold with $i = 2$ and holds with $i = 1$. Now suppose that $n > 1$ and $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ have been chosen so that and hold for $1 \leq i \leq n$ and holds for $1 \leq i \leq n-1$. Let $w_n = (u_n + v_n)/2$ and let

$$(u_{n+1}, v_{n+1}) = \begin{cases} (w_n, v_n) & \text{if } w_n \leq t \text{ for some } t \in T, \\ (u_n, w_n) & \text{if } w_n \geq t \text{ for all } t \in T. \end{cases}$$

Then and hold for $1 \leq i \leq n+1$ and

holds for $1 \leq i \leq n$. This completes the induction.

Now and imply that

$$0 \leq u_{i+1} - u_i \leq 2^{i-1}(v_1 - u_1) \quad \text{and} \quad 0 \leq v_i - v_{i+1} \leq 2^{i-1}(v_1 - u_1), \quad i \geq 1.$$

By an argument similar to the one used in Example~, this implies that $\{u_i\}_{i=1}^{\infty}$ and $\{v_i\}_{i=1}^{\infty}$ are Cauchy sequences. Therefore the sequences both converge (because of our assumption), and

implies that they have the same limit. Let

$$\lim_{i\to\infty} u_i = \lim_{i\to\infty} v_i = \beta.$$

If $t \in T$, then $v_i \geq t$ for all i, so $\beta = \lim_{i\to\infty} v_i \geq t$; therefore, β is an upper bound of T. Now suppose that $\epsilon > 0$. Then there is an integer N such that $u_N > \beta - \epsilon$. From the definition of u_N, there is a t_N in T such that $t_N \geq u_N > \beta - \epsilon$. Therefore, $\beta = \sup T$.

-2em

We say that a sequence $\{T_n\}$ of sets is {} if $T_{n+1} \subset T_n$ for all n.

Suppose that (A, ρ) is complete and $\{T_n\}$ is a nested sequence of nonempty closed subsets of A such that $\lim_{n\to\infty} d(T_n) = 0$. For each n, choose $t_n \in T_n$. If $m \geq n$, then $t_m, t_n \in T_n$, so $\rho(t_n, t_m) < d(T_n)$. Since $\lim_{n\to\infty} d(T_n) = 0$, $\{t_n\}$ is a Cauchy sequence. Therefore, $\lim_{n\to\infty} t_n = \bar{t}$ exists. Since $\bar{t}$ is a limit point of T_n and T_n is closed for all n, $\bar{t} \in T_n$ for all n. Therefore, $\bar{t} \in \cap_{n=1}^{\infty} T_n$; in fact, $\cap_{n=1}^{\infty} T_n = \{\bar{t}\}$. (Why?)

Now suppose that (A, ρ) is not complete, and let $\{t_n\}$ be a Cauchy sequence in A that does not have a limit. Choose n_1 so that $\rho(t_n, t_{n_1}) < 1/2$ if $n \geq n_1$, and let $T_1 = \{t \rho(t, t_{n_1}) \leq 1$. Now suppose that $j > 1$ and we have specified $n_1, n_2, , n_{j-1}$ and $T_1, T_2, , T_{j-1}$. Choose $n_j > n_{j-1}$ so that $\rho(t_n, t_{n_j}) < 2^{-j}$ if $n \geq n_j$, and let $T_j = \{t \rho(t, t_{n_j}) \leq 2^{-j+1}$. Then T_j is closed and nonempty, $T_{j+1} \subset T_j$ for all j, and $\lim_{j\to\infty} d(T_j) = 0$. Moreover, $t_n \in T_j$ if $n \geq n_j$. Therefore, if $\bar{t} \in \cap_{j=1}^{\infty} T_j$, then $\rho(t_n, \bar{t}) < 2^{-j}$, $n \geq n_j$, so $\lim_{n\to\infty} t_n = \bar{t}$, contrary to our assumption. Hence, $\cap_{j=1}^{\infty} T_j = \emptyset$.

When considering more than one metric on a given set A we must be careful, for example, in saying that a set is open, or that a sequence converges, etc., since the truth or falsity of the statement will in general depend on the metric as well as the set on which it is imposed. In this situation we will alway refer to the metric space by its ``full name;'' that is, (A, ρ) rather than just A.

Suppose that holds. Let S be an open set in (A, ρ) and let $x_0 \in S$. Then there is an $\epsilon > 0$ such that $x \in S$ if $\rho(x, x_0) < \epsilon$, so the second inequality in implies that $x_0 \in S$ if $\sigma(x, x_0) \leq \epsilon/\beta$. Therefore, S is open in (A, σ).

Conversely, suppose that S is open in (A,σ) and let $x_0 \in S$. Then there is an $\epsilon > 0$ such that $x \in S$ if $\sigma(x,x_0) < \epsilon$, so the first inequality in implies that $x_0 \in S$ if $\rho(x,x_0) \le \epsilon\alpha$. Therefore, S is open in (A,ρ).

It suffices to show that there are positive constants α and β such

$$\alpha \le \frac{N_1(\mathbf{X})}{N_2(\mathbf{X})} \le \beta \quad \text{if} \quad \mathbf{X} \neq \mathbf{0}. \tag{8.1.7}$$

We will show that if N is any norm on $\mathbb{R}^n$, there are positive constants a_N and b_N such that

$$a_N\|\mathbf{X}\|_2 \le N(\mathbf{X}) \le b_N\|\mathbf{X}\|_2 \quad \text{if} \quad \mathbf{X} \neq \mathbf{0} \tag{8.1.8}$$

and leave it to you to verify that this implies with $\alpha = a_{N_1}/b_{N_2}$ and $\beta = b_{N_1}/a_{N_2}$.

We write $\mathbf{X}-\mathbf{Y} = (x_1, x_2, \dots, x_n)$ as

$$\mathbf{X}-\mathbf{Y} = \sum_{i=1}^{n} (x_i - y_i)\mathbf{E}_i,$$

where $\mathbf{E}_i$ is the vector with ith component equal to 1 and all other components equal to 0. From Definition~ , , and induction,

$$N(\mathbf{X}-\mathbf{Y}) \le \sum_{i=1}^{n} |x_i - y_i| N(\mathbf{E}_i);$$

therefore, by Schwarz's inequality,

$$N(\mathbf{X}-\mathbf{Y}) \le K\|\mathbf{X}-\mathbf{Y}\|_2, \tag{8.1.9}$$

where

$$K = \left(\sum_{i=1}^{n} N^2(\mathbf{E}_i)\right)^{1/2}.$$

From and Theorem~,

$$|N(\mathbf{X}) - N(\mathbf{Y})| \le K\|\mathbf{X}-\mathbf{Y}\|_2,$$

so N is continuous on $\mathbb{R}_2^n = \mathbb{R}^n$. By Theorem~, there are vectors $\mathbf{U}_1$ and $\mathbf{U}_2$ such that $\|\mathbf{U}_1\|_2 = \|\mathbf{U}_2\|_2 = 1$,

$$N(\mathbf{U}_1) = \min\{N(\mathbf{U}) \| \|\mathbf{U}\|_2 = 1, \quad \text{and} \quad N(\mathbf{U}_2) = \max\{N(\mathbf{U}) \| \|\mathbf{U}\|_2 = 1.$$

If $a_N = N(\mathbf{U}_1)$ and $b_N = N(\mathbf{U}_2)$, then a_N and b_N are positive (Definition~), and

$$a_N \le N\left(\frac{\mathbf{X}}{\|\mathbf{X}\|_2}\right) \le b_N \quad \text{if} \quad \mathbf{X} \neq \mathbf{0}.$$

This and Definition~

imply .

We leave the proof of the following theorem to you.

Throughout this section it is to be understood that (A,ρ) is a metric space and that the sets under consideration are subsets of A.

We say that a collection $\mathcal{H}$ of open subsets of A is an {} of T if $T \subset \cup\{H\} H \in \mathcal{H}$. We say that T {} if every open covering $\mathcal{H}$ of T contains a finite collection $\widehat{\mathcal{H}}$ such that

$$T \subset \cup\{H\} H \in \widehat{\mathcal{H}}.$$

From Theorem~, every nonempty closed and bounded subset of the real numbers has the Heine–Borel property. Moreover, from Exercise~, any nonempty set of reals that has the Heine–Borel property is closed and bounded. Given these results, we defined a compact set of reals to be a closed and bounded set, and we now draw the following conclusion:

{}.

The definition of boundedness of a set of real numbers is based on the ordering of the real numbers: if a and b are distinct real numbers then either $a < b$ or $b < a$. Since there is no such ordering in a general metric space, we introduce the following definition.

As we will see below, a closed and bounded subset of a general metric space may fail to have the Heine–Borel property. Since we want compact" and has the Heine–Borel property" to be synonymous in connection with a general metric space, we simply make the following definition.

Suppose that T has an infinite subset E with no limit point in T. Then, if $t \in T$, there is an open set H_t such that $t \in H_t$ and H_t contains at most one member of E. Then $\mathcal{H} = \cup\{H_t\} t \in T$ is an open covering of T, but no finite collection $\{H_{t_1}, H_{t_2}, \ldots, H_{t_k}\}$ of sets from $\mathcal{H}$ can cover E, since E is infinite. Therefore, no such collection can cover T; that is, T is not compact.

Now suppose that every infinite subset of T has a limit point in T, and let $\mathcal{H}$ be an open covering of T. We first show that there is a sequence $\{H_i\}_{i=1}^{\infty}$ of sets from $\mathcal{H}$ that covers T.

If $\epsilon > 0$, then T can be covered by ϵ-neighborhoods of finitely many points of T. We prove this by contradiction. Let $t_1 \in T$. If $N_\epsilon(t_1)$ does not cover T, there is a $t_2 \in T$ such that $\rho(t_1, t_2) \geq \epsilon$. Now suppose that $n \geq 2$ and we have chosen t_1, t_2, , t_n such that $\rho(t_i, t_j) \geq \epsilon$, $1 \leq i < j \leq n$. If $\cup_{i=1}^{n} N_\epsilon(t_i)$ does not cover T, there is a $t_{n+1} \in T$ such that $\rho(t_i, t_{n+1}) \geq \epsilon$, $1 \leq i \leq n$. Therefore, $\rho(t_i, t_j) \geq \epsilon$, $1 \leq i < j \leq n+1$. Hence, by induction, if no finite collection of ϵ-neighborhoods of points in T covers T, there is an infinite sequence $\{t_n\}_{n=1}^{\infty}$ in T such that $\rho(t_i, t_j) \geq \epsilon$, $i \neq j$. Such a sequence could not have a limit point, contrary to our assumption.

By taking ϵ successively equal to 1, $1/2$, , $1/n$, , we can now conclude that, for each n, there are points t_{1n}, t_{2n}, , $t_{k_n,n}$ such that

$$T \subset \bigcup_{i=1}^{k_n} N_{1/n}(t_{in}).$$

Denote $B_{in} = N_{1/n}(t_{in})$, $1 \leq i \leq n$, $n \geq 1$, and define

$$\{G_1, G_2, G_3, \ldots\} = \{B_{11}, \ldots, B_{k_1,1}, B_{12}, \ldots, B_{k_2,2}, B_{13}, \ldots, B_{k_3,3}, \ldots\}.$$

If $t \in T$, there is an H in $\mathcal{H}$ such that $t \in H$. Since H is open, there is an $\epsilon > 0$ such that $N_\epsilon(t) \subset H$. Since $t \in G_j$ for infinitely many values of j and $\lim_{j\to\infty} d(G_j) = 0$,

$$G_j \subset N_\epsilon(t) \subset H$$

for some j. Therefore, if $\{G_{j_i}\}_{i=1}^{\infty}$ is the subsequence of $\{G_j\}$ such that G_{j_i} is a subset of some H_i in $\mathcal{H}$ (the $\{H_i\}$ are not necessarily distinct), then

$$T \subset \bigcup_{i=1}^{\infty} H_i. \tag{8.1.10}$$

We will now show that

$$T \subset \bigcup_{i=1}^{N} H_i. \tag{8.1.11}$$

for some integer N. If this is not so, there is an infinite sequence $\{t_n\}_{n=1}^{\infty}$ in T such that

$$t_n \notin \bigcup_{i=1}^{n} H_i, \quad n \geq 1. \tag{8.1.12}$$

From our assumption, $\{t_n\}_{n=1}^{\infty}$ has a limit $\bar{t}$ in T. From , $\bar{t} \in H_k$ for some k, so $N_\epsilon(\bar{t}) \subset H_k$ for some $\epsilon > 0$. Since $\lim_{n\to\infty} t_n = \bar{t}$, there is an integer N such that

$$t_n \in N_\epsilon(\bar{t}) \subset H_k \subset \bigcup_{i=1}^{n} H_i, \quad n > k,$$

which contradicts . This verifies , so T is compact.

Any finite subset of a metric space obviously has the Heine–Borel property and is therefore compact. Since Theorem~ does not deal with finite sets, it is often more convenient to work with the following criterion for compactness, which is also applicable to finite sets.

Suppose that T is compact and $\{t_n\} \subset T$. If $\{t_n\}$ has only finitely many distinct terms, there is a $\bar{t}$ in T such that $t_n = \bar{t}$ for infinitely many values of n; if this is so for $n_1 < n_2 < \cdots$, then $\lim_{j\to\infty} t_{n_j} = \bar{t}$. If $\{t_n\}$ has infinitely many distinct terms, then $\{t_n\}$ has a limit point $\bar{t}$ in T, so there are integers $n_1 < n_2 < \cdots$ such that $\rho(t_{n_j}, \bar{t}) < 1/j$; therefore, $\lim_{j\to\infty} t_{n_j} = \bar{t}$.

Conversely, suppose that every sequence in T has a subsequence that converges to a limit in T. If S is an infinite subset of T, we can choose a sequence $\{t_n\}$ of distinct points in S. By assumption, $\{t_n\}$ has a subsequence that converges to a member $\bar{t}$ of T. Since $\bar{t}$ is a limit point of $\{t_n\}$, and therefore of T, T is compact.

By Theorem~, $\{t_n\}$ has a subsequence $\{t_{n_j}\}$ such that

$$\lim_{j\to\infty} t_{n_j} = \bar{t} \in T. \tag{8.1.13}$$

We will show that $\lim_{n\to\infty} t_n = \bar{t}$.

Suppose that $\epsilon > 0$. Since $\{t_n\}$ is a Cauchy sequence, there is an integer N such that $\rho(t_n, t_m) < \epsilon$, $n > m \geq N$. From , there is an $m = n_j \geq N$ such that $\rho(t_m, \bar{t}) < \epsilon$. Therefore,

$$\rho(t_n, \bar{t}) \leq \rho(t_n, t_m) + \rho(t_m, \bar{t}) < 2\epsilon, \quad n \geq m.$$

-2em2em

Suppose that $\bar{t}$ is a limit point of T. For each n, choose $t_n \neq \bar{t} \in B_{1/n}(\bar{t}) \cap T$. Then $\lim_{n\to\infty} t_n = \bar{t}$. Since every subsequence of $\{t_n\}$ also converges to $\bar{t}$, $\bar{t} \in T$, by Theorem~. Therefore, T is closed.

The family of unit open balls $\mathcal{H} = \{B_1(t)\}t \in T$ is an open covering of T. Since T is compact, there are finitely many members t_1, t_2, , t_n of T such that $S \subset \cup_{j=1}^{n} B_1(t_j)$. If u and v are arbitrary members of T, then $u \in B_1(t_r)$ and $v \in B_1(t_s)$ for some r and s in $\{1, 2, \ldots, n\}$, so

$$\begin{aligned} \rho(u, v) \ar &\leq \rho(u, t_r) + \rho(t_r, t_s) + \rho(t_s, v) \\ \ar &\leq 2 + \rho(t_r, t_s) \leq 2 + \max\{\rho(t_i, t_j)\}1 \leq i < j \leq n. \end{aligned}$$

Therefore, T is bounded.

The converse of Theorem~ is false; for example, if A is any infinite set equipped with the discrete metric (Example~.), then every subset of A is bounded and closed. However, if T is an infinite subset of A, then $\mathcal{H} = \{\{t\}\}t \in T$ is an open covering of T, but no finite subfamily of $\mathcal{H}$ covers T.

We leave it to you (Exercise~) to show that every totally bounded set is bounded and that the converse is false.

We will prove that if T is not totally bounded, then T is not compact. If T is not totally bounded, there is an $\epsilon > 0$ such that there is no finite ϵ-net for T. Let $t_1 \in T$. Then there must be a t_2 in T such that $\rho(t_1, t_2) > \epsilon$. (If not, the singleton set $\{t_1\}$ would be a finite ϵ-net for T.) Now suppose that $n \geq 2$ and we have chosen t_1, t_2, , t_n such that $\rho(t_i, t_j) \geq \epsilon$, $1 \leq i < j \leq n$. Then there must be a $t_{n+1} \in T$ such that $\rho(t_i, t_{n+1}) \geq \epsilon$, $1 \leq i \leq n$. (If not, $\{t_1, t_2, \ldots, t_n\}$ would be a finite ϵ-net for T.) Therefore, $\rho(t_i, t_j) \geq \epsilon$, $1 \leq i < j \leq n+1$. Hence, by induction, there is an infinite sequence $\{t_n\}_{n=1}^{\infty}$ in T such that $\rho(t_i, t_j) \geq \epsilon$, $i \neq j$. Since such a sequence has no limit point, T is not compact, by Theorem~.

Let S be an infinite subset of T, and let $\{s_i\}_{i=1}^{\infty}$ be a sequence of distinct members of S. We will show that $\{s_i\}_{i=1}^{\infty}$ has a convergent subsequence. Since T is closed, the limit of this subsequence is in T, which implies that T is compact, by Theorem~.

For $n \geq 1$, let $T_{1/n}$ be a finite $1/n$-net for T. Let $\{s_{i0}\}_{i=1}^{\infty} = \{s_i\}_{i=1}^{\infty}$. Since T_1 is finite and $\{s_{i0}\}_{i=1}^{\infty}$ is infinite, there must be a member t_1 of T_1 such that $\rho(s_{i0}, t_1) \leq 1$ for infinitely many values of i. Let $\{s_{i1}\}_{i=1}^{\infty}$ be the subsequence of $\{s_{i0}\}_{i=1}^{\infty}$ such that $\rho(s_{i1}, t_1) \leq 1$.

We continue by induction. Suppose that $n > 1$ and we have chosen an infinite subsequence $\{s_{i,n-1}\}_{i=1}^{\infty}$ of $\{s_{i,n-2}\}_{i=1}^{\infty}$. Since $T_{1/n}$ is finite and $\{s_{i,n-1}\}_{i=1}^{\infty}$ is infinite, there must be member t_n of $T_{1/n}$ such that $\rho(s_{i,n-1}, t_n) \leq 1/n$ for infinitely many values of i. Let $\{s_{in}\}_{i=1}^{\infty}$ be the subsequence of $\{s_{i,n-1}\}_{i=1}^{\infty}$ such that $\rho(s_{in}, t_n) \leq 1/n$. From the triangle inequality,

$$\rho(s_{in}, s_{jn}) \leq 2/n, \quad i, j \geq 1, \quad n \geq 1. \tag{8.1.14}$$

Now let $\hat{s}_i = s_{ii}$, $i \geq 1$. Then $\{\hat{s}_i\}_{i=1}^{\infty}$ is an infinite sequence of members of T. Moroever, if $i, j \geq n$, then $\hat{s}_i$ and $\hat{s}_j$ are both included in $\{s_{in}\}_{i=1}^{\infty}$, so implies that $\rho(\hat{s}_i, \hat{s}_j) \leq 2/n$; that is, $\{\hat{s}_i\}_{i=1}^{\infty}$ is a Cauchy sequence and therefore has a limit, since (A, ρ) is complete.

We will show that T is totally bounded in ℓ_∞. Since ℓ_∞ is complete (Exercise$_{\text{), Theorem}}$ will then imply that T is compact.

Let $\epsilon > 0$. Choose N so that $\mu_i \le \epsilon$ if $i > N$. Let $\mu = \max\{\mu_i\} 1 \le i \le n$ and let p be an integer such that $p\epsilon > \mu$. Let $Q_\epsilon = \{r_i \epsilon\} r_i = \text{integer in} [-p, p]$. Then the subset of ℓ_∞ such that $x_i \in Q_\epsilon$, $1 \le i \le N$, and $x_i = 0$, $i > N$, is a finite ϵ-net for T.

In Example~ we showed that $C[a, b]$ is a complete metric space under the metric

$$\rho(f, g) = \|f - g\| = \max\{|f(x) - g(x)|\} a \le x \le b.$$

We will now give necessary and sufficient conditions for a subset of $C[a, b]$ to be compact.

Theorem~ implies that for each f in $C[a, b]$ there is a constant M_f {} f, such that

$$|f(x)| \le M_f \quad \text{if} \quad a \le x \le b,$$

and Theorem~ implies that there is a constant δ_f {} such that

$$|f(x_1) - f(x_2)| \le \epsilon \quad \text{if} \quad x_1, x_2 \in [a, b] \quad \text{and} \quad |x_1 - x_2| < \delta_f.$$

The difference in Definition~ is that the {} M and δ apply to {} f in T.

For necessity, suppose that T is compact. Then T is closed (Theorem~) and totally bounded (Theorem~). Therefore, if $\epsilon > 0$, there is a finite subset $T_\epsilon = \{g_1, g_2, \dots, g_k\}$ of $C[a, b]$ such that if $f \in T$, then $\|f - g_i\| \le \epsilon$ for some i in $\{1, 2, \dots, k\}$. If we temporarily let $\epsilon = 1$, this implies that

$$\|f\| = \|(f - g_i) + g_i\| \le \|f - g_i\| + \|g_i\| \le 1 + \|g_i\|,$$

which implies with

$$M = 1 + \max\{\|g_i\|\} 1 \le i \le k.$$

For , we again let ϵ be arbitary, and write

$$\begin{aligned} |f(x_1) - f(x_2)| \text{ar} &\le |f(x_1) - g_i(x_1)| + |g_i(x_1) - g_i(x_2)| + |g_i(x_2) - f(x_2)| \\ \text{ar} &\le |g_i(x_1) - g_i(x_2)| + 2\|f - g_i\| \\ \text{ar} &< |g_i(x_1) - g_i(x_2)| + 2\epsilon. \end{aligned} \tag{8.1.15}$$

Since each of the finitely many functions g_1, g_2, , g_k is uniformly continuous on $[a, b]$ (Theorem~), there is a $\delta > 0$ such that

$$|g_i(x_1) - g_i(x_2)| < \epsilon \quad \text{if} \quad |x_1 - x_2| < \delta, \quad 1 \le i \le k.$$

This and imply with ϵ replaced by 3ϵ. Since this replacement is of no consequence, this proves necessity.

For sufficiency, we will show that T is totally bounded. Since T is closed by assumption and $C[a, b]$ is complete, Theorem~ will then imply that T is compact.

Let m and n be positive integers and let

$$\xi_r = a + \frac{r}{m}(b - a), \quad 0 \le r \le m, \quad \text{and} \quad \eta_s = \frac{sM}{n}, \quad -n \le s \le n;$$

that is, $a = \xi_0 < \xi_1 < \cdots < \xi_m = b$ is a partition of $[a, b]$ into subintervals of length $(b - a)/m$, and $-M = \eta_{-n} < \eta_{-n+1} < \cdots < \eta_{n-1} < \eta_n = M$ is a partition of the

segment of the y-axis between $y = -M$ and $y = M$ into subsegments of length M/n. Let S_{mn} be the subset of $C[a, b]$ consisting of functions g such that

$$\{g(\xi_0), g(\xi_1), \dots, g(\xi_m)\} \subset \{\eta_{-n}, \eta_{-n+1} \cdots, \eta_{n-1}, \eta_n\}$$

and g is linear on $[\xi_{i-1}, \xi_i]$, $1 \le i \le m$. Since there are only $(m + 1)(2n + 1)$ points of the form (ξ_r, η_s), S_{mn} is a finite subset of $C[a, b]$.

Now suppose that $\epsilon > 0$, and choose $\delta > 0$ to satisfy . Choose m and n so that $(b - a)/m < \delta$ and $2M/n < \epsilon$. If f is an arbitrary member of T, there is a g in S_{mn} such that

$$|g(\xi_i) - f(\xi_i)| < \epsilon, \quad 0 \le i \le m. \tag{8.1.16}$$

If $0 \le i \le m - 1$,

$$|g(\xi_i) - g(\xi_{i+1})| = |g(\xi_i) - f(\xi_i)| + |f(\xi_i) - f(\xi_{i+1})| + |f(\xi_{i+1}) - g(\xi_{i+1})|. \tag{8.1.17}$$

Since $\xi_{i+1} - \xi_i < \delta$, , , and imply that

$$|g(\xi_i) - g(\xi_{i+1})| < 3\epsilon.$$

Therefore,

$$|g(\xi_i) - g(x)| < 3\epsilon, \quad \xi_i \le x \le \xi_{i+1}, \tag{8.1.18}$$

since g is linear on $[\xi_i, \xi_{i+1}]$.

Now let x be an arbitrary point in $[a, b]$, and choose i so that $x \in [\xi_i, \xi_{i+1}]$. Then

$$|f(x) - g(x)| \le |f(x) - f(\xi_i)| + |f(\xi_i) - g(\xi_i)| + |g(\xi_i) - g(x)|,$$

so , , and imply that $|f(x) - g(x)| < 5\epsilon$, $a \le x \le b$. Therefore, S_{mn} is a finite 5ϵ-net for T, so T is totally bounded.

Let T be the closure of $\mathcal{F}$; that is, $f \in T$ if and only if either $f \in T$ or f is the uniform limit of a sequence of members of $\mathcal{F}$. Then T is also uniformly bounded and equicontinuous (verify), and T is closed. Hence, T is compact, by Theorem~. Therefore, $\mathcal{F}$ has a limit point in T. (In this context, the limit point is a function f in T.) Since f is a limit point of $\mathcal{F}$, there is for each integer n a function f_n in $\mathcal{F}$ such that $\|f_n - f\| < 1/n$; that is $\{f_n\}$ converges uniformly to f on $[a, b]$.

In Chapter~ we studied real-valued functions defined on subsets of $\R^n$, and in Chapter~6.4. we studied functions defined on subsets of $\R^n$ with values in $\R^m$. These are examples of functions defined on one metric space with values in another metric space.(Of course, the two spaces are the same if $n = m$.)

In this section we briefly consider functions defined on subsets of a metric space (A, ρ) with values in a metric space (B, σ). We indicate that f is such a function by writing

$$f : (A, \rho) \to (B, \sigma).$$

The {} and {} of f are the sets

$$D_f = \{u \in A \} f(u) \text{ is defined}$$

and

$$R_f = \{v \in B \} v = f(u) \text{ for some } u \text{ in } D_f.$$

Note that can be written as

$$f(D_f \cap N_\delta(\hat{u})) \subset N_\epsilon(f(\hat{u})).$$

Also, f is automatically continuous at every isolated point of D_f. (Why?)

Suppose that is true, and let $\{u_n\}$ be a sequence in D_f that satisfies . Let $\epsilon > 0$ and choose $\delta > 0$ to satisfy . From , there is an integer N such that $\rho(u_n, \hat{u}) < \delta$ if $n \ge N$. Therefore, $\sigma(f(u_n), \hat{v}) < \epsilon$ if $n \ge N$, which implies .

For the converse, suppose that is false. Then there is an $\epsilon_0 > 0$ and a sequence $\{u_n\}$ in D_f such that $\rho(u_n, \hat{u}) < 1/n$ and $\sigma(f(u_n), \hat{v}) \ge \epsilon_0$, so is false.

We leave the proof of the next two theorems to you.

Let $\{v_n\}$ be an infinite sequence in $f(T)$. For each n, $v_n = f(u_n)$ for some $u_n \in T$. Since T is compact, $\{u_n\}$ has a subsequence $\{u_{n_j}\}$ such that $\lim_{j\to\infty} u_{n_j} = \hat{u} \in T$ (Theorem~). From Theorem~, $\lim_{j\to\infty} f(u_{n_j}) = f(\hat{u})$; that is, $\lim_{j\to\infty} v_{n_j} = f(\hat{u})$. Therefore, $f(T)$ is compact, again by Theorem~.

If f is not uniformly continuous on T, then for some $\epsilon_0 > 0$ there are sequences $\{u_n\}$ and $\{v_n\}$ in T such that $\rho(u_n, v_n) < 1/n$ and

$$\sigma(f(u_n), f(v_n)) \ge \epsilon_0. \tag{8.1.19}$$

Since T is compact, $\{u_n\}$ has a subsequence $\{u_{n_k}\}$ that converges to a limit $\hat{u}$ in T (Theorem~). Since $\rho(u_{n_k}, v_{n_k}) < 1/n_k$, $\lim_{k\to\infty} v_{n_k} = \hat{u}$ also. Then

$$\lim_{k\to\infty} f(u_{n_k}) = \dst \lim_{k\to\infty} f(v_{n_k}) = f(\hat{u})$$

(Theorem~~), which contradicts .

We note that a contraction of (A,ρ) is uniformly continuous on A.

To see that cannot have more than one solution, suppose that $u = f(u)$ and $v = f(v)$. Then

$$\rho(u,v) = \rho(f(u),f(v)). \tag{8.1.20}$$

However, implies that

$$\rho(f(u),f(v)) \le \alpha\rho(u,v). \tag{8.1.21}$$

Since and imply that

$$\rho(u,v) \le \alpha\rho(u,v)$$

and $\alpha < 1$, it follows that $\rho(u,v) = 0$. Hence $u = v$.

We will now show that has a solution. With u_0 arbitrary, define

$$u_n = f(u_{n-1}), \quad n \ge 1. \tag{8.1.22}$$

We will show that $\{u_n\}$ converges. From and ,

$$\rho(u_{n+1},u_n) = \rho(f(u_n),f(u_{n-1})) \le \alpha\rho(u_n,u_{n-1}). \tag{8.1.23}$$

The inequality

$$\rho(u_{n+1},u_n) \le \alpha^n\rho(u_1,u_0), \quad n \ge 0, \tag{8.1.24}$$

follows by induction from . If $n > m$, repeated application of the triangle inequality yields

$$\rho(u_n,u_m) \le \rho(u_n,u_{n-1}) + \rho(u_{n-1},u_{n-2}) + \cdots + \rho(u_{m+1},u_m),$$

and yields

$$\rho(u_n,u_m) \le \rho(u_1,u_0)\alpha^m(1+\alpha+\cdots+\alpha^{n-m-1}) < \frac{\alpha^m}{1-\alpha}.$$

Now it follows that

$$\rho(u_n,u_m) < \frac{\rho(u_1,u_0)}{1-\alpha}\alpha^N \quad \text{if} \quad n,m > N,$$

and, since $\lim_{N\to\infty}\alpha^N = 0$, $\{u_n\}$ is a Cauchy sequence. Since A is complete, $\{u_n\}$ has a limit $\hat{u}$. Since f is continuous at $\hat{u}$,

$$f(\hat{u}) = \lim_{n\to\infty} f(u_{n-1}) = \lim_{n\to\infty} u_n = \hat{u},$$

where Theorem~~ implies the first equality and implies the second.

Let A be $C[a,b]$, which is complete. If $u \in C[a,b]$, let $f(u) = v$, where

$$v(x) = h(x) + \lambda\int_a^b K(x,y)u(y)\,dy, \quad a \le x \le b.$$

Since $v \in C[a,b]$, $f : C[a,b] \to C[a,b]$. If u_1, $u_2 \in C[a,b]$, then

$$|v_1(x) - v_2(x)| \le |\lambda|\int_a^b |K(x,y)||u_1(y) - v_1(y)|\,dy,$$

so

$$\|v_1 - v_2\| \le |\lambda|M(b-a)\|u_1 - u_2\|.$$

Since $|\lambda|M(b-a) < 1$, f is a contraction. Hence, there is a unique u in $C[a,b]$ such that $f(u) = u$. This u satisfies .

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  - 7: Integrals of Functions of Several Variables - *CC BY-NC-SA 3.0*
    - 7.1: Definition and Existence of the Multiple Integral - *CC BY-NC-SA 3.0*
    - 7.2: Iterated Integrals and Multiple Integrals - *CC BY-NC-SA 3.0*
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  - 8: Metric Spaces - *CC BY-NC-SA 3.0*
    - 8.1: Introduction to Metric Spaces - *CC BY-NC-SA 3.0*
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Answers to Selected Exercises

$2\max(a,b)$ $2\min(a,b)$ $4\max(a,b,c)$

$4\min(a,b,c)$

∞ (no); -1 (yes) 3 (no); -3 (no) $\sqrt{7}$ (yes); $-\sqrt{7}$ (yes)\ 2 (no); -3 (no) 1 (no); -1 (no)

$\sqrt{7}$ (no); $-\sqrt{7}$ (no)

$2^n/(2n)!$ $2\cdot 3^n/(2n+1)!$ $2^{-n}(2n)!/(n!)^2$

$n^n/n!$\

no

no

\dst$A_n = \frac{x^n}{n!}\left(\ln x - \sum_{j=1}^{n}\frac{1}{j}\right)$

$f_n(x_1,x_2,\ldots,x_n) = 2^{n-1}\max(x_1,x_2,\ldots,x_n)$, $g_n(x_1,x_2,\ldots,x_n) = 2^{n-1}\min(x_1,x_2,\ldots,x_n)$

$[\frac{1}{2},1)$; $(-\infty,\frac{1}{2})\cup[1,\infty)$; $(-\infty,0]\cup(\frac{3}{2},\infty)$; $(0,\frac{3}{2}]$; $(-\infty,0]\cup(\frac{3}{2},\infty)$; $(-\infty,\frac{1}{2}]\cup[1,\infty)$ $(-3,-2)\cup(2,3)$; $(-\infty,-3]\cup[-2,2]\cup[3,\infty)$; $\emptyset$; $(-\infty,\infty)$;$\emptyset$; $(-\infty,-3]\cup[-2,2]\cup[3,\infty)$

$\emptyset$; $(-\infty,\infty)$; $\emptyset$; $(-\infty,\infty)$; $\emptyset$; $(-\infty,\infty)$

$\emptyset$; $(-\infty,\infty)$; $[-1,1]$; $(-\infty,-1)\cup(1,\infty)$; $[-1,1]$; $(-\infty,\infty)$

$(0,3]$ $[0,2]$ $(-\infty,1)\cup(2,\infty)$

$(-\infty,0]\cup(3,\infty)$

$\frac{1}{4}$ $\frac{1}{6}$ 6

1

neither; $(-1,2)\cup(3,\infty)$; $(-\infty,-1)\cup(2,3)$; $(-\infty,-1]\cup(2,3)$; $(-\infty,-1]\cup[2,3]$ open; S; $(1,2)$; $[1,2]$ closed; $(-3,-2)\cup(7,8)$; $(-\infty,-3)\cup(-2,7)\cup(8,\infty)$; $(-\infty-3]\cup[-2,7]\cup[8,\infty)$

closed; $\emptyset$; $\bigcup\{(n,n+1)\}n=$ integer; $(-\infty,\infty)$

$\{x\}x = 1/n,\ n=1,2,\ldots$; $\emptyset$, $S_1 =$ rationals, $S_2 =$ irrationals any set whose supremum is an isolated point of the set , the rationals

$S_1 =$ rationals, $S_2 =$ irrationals

$D_f = [-2,1)\cup[3,\infty)$, $D_g = (-\infty,-3]\cup[3,7)\cup(7,\infty)$, $D_{f\pm g} = D_{fg} = [3,7)\cup(7,\infty)$, $D_{f/g} = (3,4)\cup(4,7)\cup(7,\infty)$

, $\{x\}x \neq (2k+1)\pi/2$ where $k=$ integer $\{x\}x\neq 0,1$ $\{x\}x\neq 0$

$[1,\infty)$

4 12 -1 2

-2

$\frac{11}{17}$ $-\frac{2}{3}$ $\frac{1}{3}$

2

$0, 2$ 0, none $-\frac{1}{3},\frac{1}{3}$

none, 0

0 0 none 0 none

0

0 0 none none none

0

∞ $-\infty$ ∞ ∞ ∞

$-\infty$

none ∞ ∞

none

∞ ∞ ∞ $-\infty$ none

∞

$\frac{3}{2}$ $\frac{3}{2}$ ∞ $-\infty$ ∞

$\frac{1}{2}$

$\lim_{x\to\infty} r(x) = \infty$ if $n > m$ and $a_n/b_m > 0$; $= -\infty$ if $n > m$ and $a_n/b_m < 0$; $= a_n/b_m$ if $n = m$; $= 0$ if $n < m$. $\lim_{x\to-\infty} r(x) = (-1)^{n-m} \lim_{x\to\infty} r(x)$

$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x)$

$\limsup_{x\to x_0-}(f-g)(x) \le \limsup_{x\to x_0-} f(x) - \liminf_{x\to x_0-} g(x)$; $\liminf_{x\to x_0-}(f-g)(x) \ge \liminf_{x\to x_0-} f(x) - \limsup_{x\to x_0-} g(x)$

from the right continuous none continuous none continuous

from the left

$[0,1)$, $(0,1)$, $[1,2)$, $(1,2)$, $(1,2]$, $[1,2]$ $[0,1)$, $(0,1)$, $(1,\infty)$

$\tanh x$ is continuous for all x, $\coth x$ for all $x \ne 0$

No $[-1,1]$, $[0,\infty)$ $\bigcup_{n=-\infty}^{\infty}(2n\pi,(2n+1)\pi)$, $(0,\infty)$ $\bigcup_{n=-\infty}^{\infty}(n\pi,(n+1)\pi)$, $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$

$\bigcup_{n=-\infty}^{\infty}[n\pi,(n+\frac{1}{2})\pi]$, $[0,\infty)$

$(-1,1)$ $(-\infty,\infty)$ $x_0 \ne (2k+\frac{3}{2}\pi)$, $k =$ integer $x \ne \frac{1}{2}$ $x \ne 1$ $x \ne (k+\frac{1}{2}\pi)$, $k =$ integer $x \ne (k+\frac{1}{2}\pi)$, $k =$ integer $x \ne 0$

$x \ne 0$

$p(c) = q(c)$ and $p'_-(c) = q'_+(c)$

$f^{(k)}(x) = n(n-1)\cdots(n-k-1)x^{n-k-1}|x|$ if $1 \le k \le n-1$; $f^{(n)}(x) = n!$ if $x > 0$; $f^{(n)}(x) = -n!$ if $x < 0$; $f^{(k)}(x) = 0$ if $k > n$ and $x \ne 0$; $f^{(k)}(0)$ does not exist if $k \ge n$.

$c' = ac - bs$, $s' = bc + as$

$c(x) = e^{ax}\cos bx$, $s(x) = e^{ax}\sin bx$

$f(x) = -1$ if $x \le 0$, $f(x) = 1$ if $x > 0$; then $f'(0+) = 0$, but $f'_+(0)$ does not exist.

continuous from the right

There is no such function (Theorem~2.3.9).

Counterexample: Let $x_0 = 0$, $f(x) = |x|^{3/2}\sin(1/x)$ if $x \ne 0$, and $f(0) = 0$.

Counterexample: Let $x_0 = 0$, $f(x) = x/|x|$ if $x \ne 0$, $f(0) = 0$.

$-\infty$ if $\alpha \le 0$, 0 if $\alpha > 0$

∞ if $\alpha > 0$, $-\infty$ if $\alpha \le 0$

$-\infty$ $-\infty$ if $\alpha \le 0$, 0 if $\alpha > 0$

0

Suppose that g' is continuous at x_0 and $f(x) = g(x)$ if $x \le x_0$, $f(x) = 1 + g(x)$ if $x > x_0$.

1 e

1 e^L

$f^{(n+1)}(x_0)/(n+1)!$.

Counterexample: Let $x_0 = 0$ and $f(x) = x|x|$.

Let $g(x) = 1 + |x - x_0|$, so $f(x) = (x - x_0)(1 + |x - x_0|)$.

Let $g(x) = 1 + |x - x_0|$, so $f(x) = (x - x_0)^2(1 + |x - x_0|)$.

$1, 2, 2, 0\ 0, -\pi, 3\pi/2, -4\pi + \pi^3/2\backslash -\pi^2/4, -2\pi, -6 + \pi^2/4, 4\pi$

$-2, 5, -16, 65$

$0, -1, 0, 5$

$0, 1, 0, 5\ -1, 0, 6, -24\ \sqrt{2}, 3\sqrt{2}, 11\sqrt{2}, 57\sqrt{2}\ -1, 3, -14, 88$ min neither min max min neither min

min

$f(x) = e^{-1/x^2}$ if $x \neq 0$, $f(0) = 0$ (Exercise~)

None if $b^2 - 4c < 0$; local min at $x_1 = (-b + \sqrt{b^2 - 4c})/2$ and local max at $x_1 = (-b - \sqrt{b^2 - 4c})/2$ if $b^2 - 4c > 0$; if $b^2 = 4c$ then $x = -b/2$ is a critical point, but not a local extreme point.

$\dst \frac{1}{6}\left(\frac{\pi}{20}\right)^3\ \dst \frac{1}{8^3}\ \dst \frac{\pi^2}{512\sqrt{2}}$

$\dst \frac{1}{4(63)^4}$

$M_3 h/3$, where $M_3 = \sup_{|x-c|\leq h} |f^{(3)}(c)|\backslash$

$M_4 h^2/12$ where $M_4 = \sup_{|x-c|\leq h} |f^{(4)}(c)|$

$k = -h/2$

monotonic functions

Let $[a, b] = [0, 1]$ and $P = \{0, 1\}$. Let $f(0) = f(1) = \frac{1}{2}$ and $f(x) = x$ if $0 < x < 1$. Then $s(P) = 0$ and $S(P) = 1$, but neither is a Riemann sum of f over P.

$\frac{1}{2}, -\frac{1}{2}$

$\frac{1}{2}, 1\ e^b - e^a\ 1 - \cos b \sin b$

$f(a)[g_1 - g(a)] + f(d)(g_2 - g_1) + f(b)[g(b) - g_2]$

$f(a)[g_1 - g(a)] + f(b)[g(b) - g_p] + \sum_{m=1}^{p-1} f(a_m)(g_{m+1} - g_m)$

If $g \equiv 1$ and f is arbitrary, then $\int_a^b f(x)\, dg(x) = 0$.

$\overline{u} = c = \frac{2}{3}\ \overline{u} = c = 0$

$\overline{u} = (e-2)/(e-1),\ c = \sqrt{\overline{u}}$

$n!\ \frac{1}{2}$ divergent $1\ -1$

0

divergent convergent divergent convergent convergent

divergent

$p < 2\ p < 1\ p > -1\ -1 < p < 2$ none none

$p < 1$

$p - q < 1\ p, q < 1\ -1 < p < 2q - 1\ q > -1, p + q > 1\ p + q > 1$

$q + 1 < p < 3q + 1$

$\deg g - \deg f \geq 2$

$2\ 1\ 0\ 1/2 \quad 1/2 \quad 1/2$

$1/2$

$\sqrt{A}\ 1\ 1\ 1\ -\infty$

0

If $s_n = 1$ and $t_n = -1/n$, then $(\lim_{n\to\infty} s_n)/(\lim_{n\to\infty} t_n) = 1/0 = \infty$, but $\lim_{n\to\infty} s_n/t_n = -\infty$.

$\infty, 0\ \infty, -\infty$ if $|r| > 1$; $2, -2$ if $r = -1$; $0, 0$ if $r = 1$; $1, -1$ if $|r| < 1\ \infty, -\infty$ if $r < -1$; $0, 0$ if $|r| < 1$; $\frac{1}{2}, \frac{1}{2}$ if $r = 1$; ∞,∞ if $r > 1 \setminus \infty, \infty$

$|t|, -|t|$

$1, -1\ 2, -2\ 3, -1$

$\sqrt{3}/2, -\sqrt{3}/2$

If $\{s_n\} = \{1, 0, 1, 0, \ldots\}$, then $\lim_{n\to\infty} t_n = \frac{1}{2}$

$\lim_{m\to\infty} s_{2m} = \infty$, $\lim_{m\to\infty} s_{2m+1} = -\infty$\ $\lim_{m\to\infty} s_{4m} = 1$, $\lim_{m\to\infty} s_{4m+2} = -1$, $\lim_{m\to\infty} s_{2m+1} = 0$\ $\lim_{m\to\infty} s_{2m} = 0$, $\lim_{m\to\infty} s_{4m+1} = 1$, $\lim_{m\to\infty} s_{4m+3} = -1$\ $\lim_{n\to\infty} s_n = 0$ $\lim_{m\to\infty} s_{2m} = \infty$, $\lim_{m\to\infty} s_{2m+1} = 0$\

$\lim_{m\to\infty} s_{8m} = \lim_{m\to\infty} s_{8m+2} = 1$, $\lim_{m\to\infty} s_{8m+1} = \sqrt{2}$,\ $\lim_{m\to\infty} s_{8m+3} = \lim_{m\to\infty} s_{8m+7} = 0$, $\lim_{m\to\infty} s_{8m+5} = -\sqrt{2}$,\ $\lim_{m\to\infty} s_{8m+4} = \lim_{m\to\infty} s_{8m+6} = -1$

$\{1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \ldots\}$

Let $\{t_n\}$ be any convergent sequence and $\{s_n\} = \{t_1, 1, t_2, 2, \ldots, t_n, n, \ldots\}$.

No; consider $\sum 1/n$

convergent convergent divergent divergent \ convergent convergent divergent

convergent

$p > 1\ p > 1$

$p > 1$

convergent convergent if $0 < r < 1$, divergent if $r \geq 1$\ divergent convergent divergent

convergent

convergent convergent convergent

convergent

divergent convergent if and only if $0 < r < 1$ or $r = 1$ and $p < -1$ convergent convergent

convergent

divergent convergent convergent

convergent if $\alpha < \beta - 1$, divergent if $\alpha \geq \beta - 1$

divergent convergent convergent

convergent

$\sum (-1)^n\ \sum (-1)^n/n$, $\sum$ \dst $\left[\frac{(-1)^n}{n} + \frac{1}{n \log n}\right]$\ $\sum (-1)^n 2^n$

$\sum (-1)^n$

conditionally convergent conditionally convergent absolutely convergent

absolutely convergent

%\begin{exercisepatch1}

Let k and s be the degrees of the numerator and denominator, respectively. If $|r| = 1$, the series converges absolutely if and only if $s \geq k + 2$. The series converges conditionally if $s = k + 1$ and $r = -1$, and diverges in all other cases, where $s \geq k + 1$ and $|r| = 1$.

$\sum (-1)^n/\sqrt{n}\ 0$

$2A - a_0$

$F(x) = 0,\ |x| \leq 1\ F(x) = 0,\ |x| \leq 1$ \ $F(x) = 0,\ -1 < x \leq 1\ F(x) = \sin x,\ -\infty < x < \infty$\ $F(x) = 1,\ -1 < x \leq 1;\ F(x) = 0,\ |x| > 1$ $F(x) = x,\ -\infty < x < \infty$\ $F(x) = x^2/2,\ -\infty < x < \infty\ F(x) = 0,\ -\infty < x < \infty$\

$F(x) = 1,\ -\infty < x < \infty$

$F(x) = 0\ F(x) = 1,\ |x| < 1;\ F(x) = 0,\ |x| > 1$ \

$F(x) = \sin x/x$

$F_n(x) = x^n$; $S_k = [-k/(k+1), k/(k+1)]$

$[-1,1]$ $[-r,r] \cup \{1\} \cup \{-1\}$, $0 < r < 1$ $[-r,r] \cup \{1\}$, $0 < r < 1$ $\setminus [-r,r]$, $r > 0$ $(-\infty, -1/r] \cup [-r,r] \cup [1/r, \infty) \cup \{1\}$, $0 < r < 1 \setminus [-r,r]$, $r > 0$ $[-r,r]$, $r > 0$ $(-\infty, -r] \cup [r, \infty) \cup \{0\}$, $r > 0$ \

$[-r,r]$, $r > 0$

Let $S = (0,1]$, $F_n(x) = \sin(x/n)$, $G_n(x) = 1/x^2$; then $F = 0$, $G = 1/x^2$, and the convergence is uniform, but $\|F_n G_n\|_S = \infty$.

3 1 $\frac{1}{2}$

$e - 1$

compact subsets of $(-\frac{1}{2}, \infty)$ $[-\frac{1}{2}, \infty)$ closed subsets of \dst $\left(\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right)$ $(-\infty, \infty)$ $[r, \infty)$, $r > 1$

compact subsets of $(-\infty, 0) \cup (0, \infty)$

Let $S = (-\infty, \infty)$, $f_n = a_n$ (constant), where $\sum a_n$ converges conditionally, and $g_n = |a_n|$.

``absolutely"

means that $\sum |f_n(x)|$ converges pointwise and $\sum f_n(x)$ converges uniformly on S, while

means that $\sum |f_n(x)|$ converges uniformly on S.

\dst $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!(2n+1)}$

\dst $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)(2n+1)!}$

$1/3e$ 1 $\frac{1}{3}$ 1

∞

1 $\frac{1}{2}$ $\frac{1}{4}$ 4 $1/e$

1

$x(1+x)/(1-x)^3$ e^{-x^2} \ \dst $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}(x-1)^n$; $R = 1$

$\mathrm{Tan}^{-1} x =$ \dst $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)}$; $f^{(2n)}(0) = 0$; $f^{(2n+1)}(0) = (-1)^2(2n)!$;

\dst $\frac{\pi}{6} = \mathrm{Tan}^{-1} \frac{1}{\sqrt{3}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^{n+1/2}}$

$\cosh x =$ \dst $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$, $\sinh x =$ \dst $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

$(1-x)\sum_{n=0}^{\infty} x^n = 1$ converges for all x

\dst $x + x^2 + \frac{x^3}{3} - \frac{3x^5}{40} + \cdots$ \dst $1 - x - \frac{x^2}{2} + \frac{5x^3}{6} + \cdots$ \dst $1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{721x^6}{720} + \cdots$

\dst $x^2 - \frac{x^3}{2} + \frac{x^4}{6} - \frac{x^5}{6} + \cdots$

\dst $1 + x + \frac{2x^2}{3} + \frac{x^3}{3} + \cdots$ \dst $1 - x - \frac{x^2}{2} + \frac{3x^3}{2} + \cdots$ \dst $1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots$ \dst $1 + \frac{x^2}{6} + \frac{7x^4}{360} + \frac{31x^6}{15120} + \cdots$

\dst $2 - x^2 + \frac{x^4}{12} - \frac{x^6}{360} + \cdots$

\dst $F(x) = \frac{5}{(1-3x)(1+2x)} = \frac{3}{1-3x} + \frac{2}{1+2x} = \sum_{n=0}^{\infty} [3^{n+1} - (-2)^{n+1}]x^n$ 1

$(3,0,3,3)$ $(-1,-1,4)$

$(\frac{1}{6}, \frac{11}{12}, \frac{23}{24}, \frac{5}{36})$

$\sqrt{15}$ $\sqrt{65}/12$ $\sqrt{31}$

$\sqrt{3}$

$\sqrt{89}$ $\sqrt{166}/12$ 3

$\sqrt{31}$

12 $\frac{1}{32}$

27

$\mathbf{X} = \mathbf{X}_0 + t\mathbf{U}$ $(-\infty < t < \infty)$ in all cases.

... $\mathbf{U}$ and $\mathbf{X}_1 - \mathbf{X}_0$ are scalar multiples of $\mathbf{V}$.

$\mathbf{X} = (1, -3, 4, 2) + t(1, 3, -5, 3)$

$\mathbf{X} = (3, 1, -2, 1, 4,) + t(-1, -1, 1, 3, -7)$

$\mathbf{X} = (1, 2, -1) + t(-1, -3, 0)$

5 2

$1/2\sqrt{5}$

$\{(x_1, x_2, x_3, x_4)\}|x_i| \leq 3$ $(i = 1, 2, 3)$ with at least one equality $\{(x_1, x_2, x_3, x_4)\}|x_i| \leq 3$ $(i = 1, 2, 3)$

S

$\{(x_1, x_2, x_3, x_4)\}|x_i| > 3$ for at least one of $i = 1, 2, 3$

S S $\emptyset$

$\{(x, y, z)\}z \neq 1$ or $x^2 + y^2 > 1$

open neither

closed

$(\pi, 1, 0)$

$(1, 0, e)$

6 6 $2\sqrt{5}$ $2L\sqrt{n}$

∞

$\{(x, y)\}x^2 + y^2 = 1$

... if for A there is an integer R such that $|\mathbf{X}_r| > A$ if $r \geq R$.

10 3 1 0 0

0

$a/(1 + a^2)$

∞ ∞ no $-\infty$

no

0 0 none 0

none

if D_f is unbounded and for each M there is an R such that $f(\mathbf{X}) > M$ if $\mathbf{X} \in D_f$ and $|\mathbf{X}| > R$. Replace $>M$'' by $<M$'' in

.

$\lim_{\mathbf{X}\to 0} f(\mathbf{X}) = 0$ if $a_1 + a_2 + \cdots + a_n > b$; no limit if $a_1 + a_2 + \cdots + a_n \leq b$ and $a_1^2 + a_2^2 + \cdots + a_n^2 \neq 0$; $\lim_{\mathbf{X}\to 0} f(\mathbf{X}) = \infty$ if $a_1 = a_2 = \cdots = a_n = 0$ and $b > 0$.

No; for example, $\lim_{x\to\infty} g(x, \sqrt{x}) = 0$.

$\mathbb{R}^3$ $\mathbb{R}^2$ $\mathbb{R}^3$ $\mathbb{R}^2$ $\{(x, y)\}x \geq y$

$\mathbb{R}^n$

$\mathbb{R}^3 - \{(0, 0, 0)\}$ $\mathbb{R}^2$ $\mathbb{R}^2$ $\mathbb{R}^2$

$\mathbb{R}^2$

$f(x, y) = xy/(x^2 + y^2)$ if $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$

$\dst\frac{2}{\sqrt{3}}(x + y\cos x - xy\sin x) - 2\sqrt{\frac{2}{3}}(x\cos x)$ $\dst\frac{1-2y}{\sqrt{3}}e^{-x+y^2+2z}$ $\dst\frac{2}{\sqrt{n}}(x_1 + x_2 + \cdots + x_n)$

$1/(1 + x + y + z)$

$\phi_1^2\phi_2$ $-5\pi/\sqrt{6}$ $-2e$ 0

0

$f_x = f_y = 1/(x+y+2z)$, $f_z = 2/(x+y+2z)$

$f_x = 2x+3yz+2y$, $f_y = 3xz+2x$, $f_z = 3xy$ $f_x = e^{yz}$, $f_y = xze^{yz}$, $f_z = xye^{yz}$

$f_x = 2xy\cos x^2y$, $f_y = x^2\cos x^2y$, $f_z = 1$

$f_{xx} = f_{yy} = f_{xy} = f_{yx} = -1/(x+y+2z)^2$, $f_{xz} = f_{zx} = f_{yz} = f_{zy} = -2/(x+y+2z)^2$, $f_{zz} = -4/(x+y+2z)^2$

$f_{xx} = 2$, $f_{yy} = f_{zz} = 0$, $f_{xy} = f_{yx} = 3z+2$, $f_{xz} = f_{zx} = 3y$, $f_{yz} = f_{zy} = 3x$

$f_{xx} = 0$, $f_{yy} = xz^2e^{yz}$, $f_{zz} = xy^2e^{yz}$, $f_{xy} = f_{yx} = ze^{yz}$, $f_{xz} = f_{zx} = ye^{yz}$, $f_{yz} = f_{zy} = xe^{yz}$

$f_{xx} = 2y\cos x^2y - 4x^2y^2\sin x^2y$, $f_{yy} = -x^4\sin x^2y$, $f_{zz} = 0$, $f_{xy} = f_{yx} = 2x\cos x^2y - 2x^3y\sin x^2y$, $f_{xz} = f_{zx} = f_{yz} = f_{zy} = 0$

$f_{xx}(0,0) = f_{yy}(0,0) = 0$, $f_{xy}(0,0) = -1$, $f_{yx}(0,0) = 1$\

$f_{xx}(0,0) = f_{yy}(0,0) = 0$, $f_{xy}(0,0) = -1$, $f_{yx}(0,0) = 1$

$f(x,y) = g(x,y) + h(y)$, where g_{xy} exists everywhere and h is nowhere differentiable.

$df = (3x^2+4y^2+2y\sin x + 2xy\cos x)\,dx + (8xy+2x\sin x)\,dy$,\ $d_{\mathbf{X}_0}f = 16\,dx$, $(d_{\mathbf{X}_0}f)(\mathbf{X}-\mathbf{X}_0) = 16x$ \ $df = -e^{-x-y-z}(dx+dy+dz)$, $d_{\mathbf{X}_0}f = -dx-dy-dz$, \ $(d_{\mathbf{X}_0}f)(\mathbf{X}-\mathbf{X}_0) = -x-y-z$ \ $df = (1+x_1+2x_2+\cdots+nx_n)^{-1}\sum_{j=1}^n j\,dx_j$, $d_{\mathbf{X}_0}f = \sum_{j=1}^n j\,dx_j$, \ $(d_{\mathbf{X}_0}f)(\mathbf{X}-\mathbf{X}_0) = \sum_{j=1}^n jx_j$,\

$df = 2r|\mathbf{X}|^{2r-2}\sum_{j=1}^n x_j\,dx_j$, $d_{\mathbf{X}_0}f = 2rn^{r-1}\sum_{j=1}^n dx_j$, \ $(d_{\mathbf{X}_0}f)(\mathbf{X}-\mathbf{X}_0) = 2rn^{r-1}\sum_{j=1}^n(x_j-1)$,

The unit vector in the direction of $(f_{x_1}(\mathbf{X}_0), f_{x_2}(\mathbf{X}_0), \ldots, f_{x_n}(\mathbf{X}_0))$ provided that this is not $\mathbf{0}$; if it is $\mathbf{0}$, then $\partial f(\mathbf{X}_0)/\partial\mathbf{\Phi} = 0$ for every $\mathbf{\Phi}$.

$z = 2x+4y-6$ $z = 2x+3y+1$ $z = (\pi x)/2 + y - \pi/2$

$z = x+10y+4$

$5\,du + 34\,dv$ 0 $6\,du - 18\,dv$

$8\,du$

$h_r = f_x\cos\theta + f_y\sin\theta$, $h_\theta = r(-f_x\sin\theta + f_y\cos\theta)$, $h_z = f_z$

$h_r = f_x\sin\phi\cos\theta + f_y\sin\phi\sin\theta + f_z\cos\phi$, $h_\theta = r\sin\phi(-f_x\sin\theta + f_y\cos\theta)$, $h_\phi = r(f_x\cos\phi\cos\theta + f_y\cos\phi\sin\theta - f_z\sin\phi)$

$h_y = g_xx_y + g_y + g_ww_y$, $h_z = g_xx_z + g_z + g_ww_z$

$h_{rr} = f_{xx}\sin^2\phi\cos^2\theta + f_{yy}\sin^2\phi\sin^2\theta + f_{zz}\cos^2\phi + f_{xy}\sin^2\phi\sin 2\theta + f_{yz}\sin 2\phi\sin\theta + f_{xz}\sin 2\phi\cos\theta$,

\dst$h_{r\theta} = (-f_x\sin\theta + f_y\cos\theta)\sin\phi + \frac{r}{2}(f_{yy}-f_{xx})\sin^2\phi\sin 2\theta + rf_{xy}\sin^2\phi\cos 2\theta + \frac{r}{2}(f_{zy}\cos\theta - f_{zx}\sin\theta)\sin 2\phi$

\dst$1 + x + \frac{x^2}{2} - \frac{y^2}{2} + \frac{x^3}{6} - \frac{xy^2}{2}$

\dst$1 - x - y + \frac{x^2}{2} + xy + \frac{y^2}{2} - \frac{x^3}{6} - \frac{x^2y}{2} - \frac{xy^2}{2} - \frac{y^3}{6}$\ 0

xyz

$(d^2_{(0,0)}p)(x,y) = (d^2_{(0,0)}q)(x,y) = 2(x-y)^2$

$$\begin{bmatrix} 3 & 4 & 6 \\ 2 & -4 & 2 \\ 7 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 \\ 3 & -2 \\ 7 & -4 \\ 6 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 8 & 16 & 24 \\ 0 & 0 & 4 & 12 \\ 12 & 16 & 28 & 44 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -6 & 0 \\ 0 & -2 & -4 \\ -2 & 2 & -6 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 & 6 \\ 6 & 7 & -3 \\ 0 & -2 & 6 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 7 \\ 3 & 5 \\ 5 & 14 \end{bmatrix}$$

$$\begin{bmatrix} 13 & 25 \\ 16 & 31 \\ 16 & 25 \end{bmatrix}$$

$$\begin{bmatrix} 29 \\ 50 \end{bmatrix}$$

A and **B** are square of the same order.

$$\begin{bmatrix} 7 & 3 & 3 \\ 4 & 7 & 7 \\ 6 & -9 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 14 & 10 \\ 6 & -2 \\ 14 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -7 & 6 & 4 \\ -9 & 7 & 13 \\ 5 & 0 & -14 \end{bmatrix}, \begin{bmatrix} -5 & 6 & 0 \\ 4 & -12 & 3 \\ 4 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 6xyz & 3xz^2 & 3x^2y \end{bmatrix};$$
$$\begin{bmatrix} -6 & 3 & -3 \end{bmatrix}$$

$\cos(x+y) \begin{bmatrix} 1 & 1 \end{bmatrix}; \begin{bmatrix} 0 & 0 \end{bmatrix}$

$\begin{bmatrix} (1-xz)ye^{-xz} & xe^{-xz} & -x^2ye^{-xz} \end{bmatrix}; \begin{bmatrix} 2 & 1 & -2 \end{bmatrix}\backslash$

$\sec^2(x+2y+z) \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}; \begin{bmatrix} 2 & 4 & 2 \end{bmatrix}$

$|\mathbf{X}|^{-1} \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}; \dst\frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}$

$(2,3,-2)\ (2,3,0)\ (-2,0,-1)$

$(3,1,3,2)$

$$\dst\frac{1}{10} \begin{bmatrix} 4 & 2 \\ -3 & 1 \end{bmatrix}$$

$$\dst\frac{1}{2} \begin{bmatrix} -1 & 1 & 2 \\ 3 & 1 & -4 \\ -1 & -1 & 2 \end{bmatrix}$$

$$\dst\frac{1}{25} \begin{bmatrix} 4 & 3 & -5 \\ 6 & -8 & 5 \\ -3 & 4 & 10 \end{bmatrix}$$

$$\dst\frac{1}{2} \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\dst\frac{1}{7} \begin{bmatrix} 3 & -2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\dst \frac{1}{10}\begin{bmatrix} -1 & -2 & 0 & 5 \\ -14 & -18 & 10 & 20 \\ 21 & 22 & -10 & -25 \\ 17 & 24 & -10 & -25 \end{bmatrix}$$

Bracket argument to \\ must be a dimension;\[2] $J\mathbf{F}(\mathbf{X}) = e^{xyz}\sin(x+y+z)[x(1-2x)(y-z) - z(x-y)];\[2]$ $\mathbf{G}(\mathbf{X}) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$$+ \begin{bmatrix} 2 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x-1 \\ y+1 \\ z \end{bmatrix}$$

$\mathbf{F}'(\mathbf{X}) = \begin{bmatrix} e^x\cos y & -e^x\sin y \\ e^x\sin y & e^x\cos y \end{bmatrix}$; $J\mathbf{F}(\mathbf{X}) = e^{2x};\[2]$ $\mathbf{G}(\mathbf{X}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y-\pi/2 \end{bmatrix}\[2]$

$\mathbf{F}'(\mathbf{X}) = \begin{bmatrix} 2x & -2y & 0 \\ 0 & 2y & -2z \\ -2x & 0 & 2z \end{bmatrix}$; $J\mathbf{F} = 0;\[2]$ $\mathbf{G}(\mathbf{X}) = \begin{bmatrix} 2 & -2 & 0 \\ 0 & 2 & -2 \\ -2 & 0 & 2 \end{bmatrix}\begin{bmatrix} x-1 \\ y-1 \\ z-1 \end{bmatrix}$

$\mathbf{F}'(\mathbf{X}) = \begin{bmatrix} (x+y+z+1)e^x & e^x & e^x \\ (2x-x^2-y^2)e^{-x} & 2ye^{-x} & 0 \end{bmatrix}$

$\mathbf{F}'(\mathbf{X}) = \begin{bmatrix} g_1'(x) \\ g_2'(x) \\ \vdots \\ g_n'(x) \end{bmatrix}$ \

$\mathbf{F}'(r,\theta) = \begin{bmatrix} e^x\sin yz & ze^x\cos yz & ye^x\cos yz \\ ze^y\cos xz & e^y\sin xz & xe^y\cos xz \\ ye^z\cos xy & xe^z\cos xy & e^z\sin xy \end{bmatrix}$

$\mathbf{F}'(r,\theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$; $J\mathbf{F}(r,\theta) = r\[2]$

$\mathbf{F}'(r,\theta,\phi) = \begin{bmatrix} \cos\theta\cos\phi & -r\sin\theta\cos\phi & -r\cos\theta\sin\phi \\ \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\phi & 0 & r\cos\phi \end{bmatrix}$;\ $J\mathbf{F}(r,\theta,\phi) = r^2\cos\phi\[2]$

$\mathbf{F}'(r,\theta,z) = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $J\mathbf{F}(r,\theta,z) = r$

$$\begin{bmatrix} 0 & 0 & 4 \\ 0 & -\frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} -18 & 0 \\ 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 9 & -3 \\ 3 & -8 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -3 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 10 \\ 9 & 18 \\ -4 & -8 \end{bmatrix}$$

$[1,\pi/2]$ $[1,2\pi]$ $[1,\pi]$ $[2\sqrt{2},9\pi/4]$

$[\sqrt{2},3\pi/4]$

$[1,-3\pi/2]$ $[1,-2\pi]$ $[1,-\pi]$ $[2\sqrt{2},-7\pi/4]$

$[\sqrt{2},-5\pi/4]$

Let $f(x) = x$ $(0 \le x \le \frac{1}{2})$, $f(x) = x - \frac{1}{2}$ $(\frac{1}{2} < x \le 1)$; then f is locally invertible but not invertible on $[0,1]$.

$\mathbf{F}(S) = \{(u,v)\} - \pi + 2\phi < \arg(u,v) < \pi + 2\phi$, where ϕ is an argument of (a,b);

Bracket argument to \\ must be a dimension

$$\begin{bmatrix} x \\ y \end{bmatrix} = \dst \frac{1}{10} \begin{bmatrix} u-2v \\ 3u+4v \end{bmatrix};\ (\mathbf{F}^{-1})' = \dst \frac{1}{10} \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \dst \frac{1}{2} \begin{bmatrix} u+2v+3w \\ u-w \\ u+v+2w \end{bmatrix};\ (\mathbf{F}^{-1})' = \dst \frac{1}{2} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\mathbf{G}_1(u,v) = \dst \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{u+v} \\ \sqrt{u-v} \end{bmatrix},\ \mathbf{G}_1'(u,v) = \dst \frac{1}{2\sqrt{2}} \begin{bmatrix} 1/\sqrt{u+v} & 1/\sqrt{u+v} \\ 1/\sqrt{u-v} & -1/\sqrt{u-v} \end{bmatrix}$$

$$\mathbf{G}_2(u,v) = \dst \frac{1}{\sqrt{2}} \begin{bmatrix} -\sqrt{u+v} \\ \sqrt{u-v} \end{bmatrix},\ \mathbf{G}_2'(u,v) = \dst \frac{1}{2\sqrt{2}} \begin{bmatrix} -1/\sqrt{u+v} & -1/\sqrt{u+v} \\ 1/\sqrt{u-v} & -1/\sqrt{u-v} \end{bmatrix}$$

$$\mathbf{G}_3(u,v) = \dst \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{u+v} \\ -\sqrt{u-v} \end{bmatrix},\ \mathbf{G}_3'(u,v) = \dst \frac{1}{2\sqrt{2}} \begin{bmatrix} 1/\sqrt{u+v} & 1/\sqrt{u+v} \\ -1/\sqrt{u-v} & 1/\sqrt{u-v} \end{bmatrix}$$

$$\mathbf{G}_4(u,v) = \dst \frac{1}{\sqrt{2}} \begin{bmatrix} -\sqrt{u+v} \\ -\sqrt{u-v} \end{bmatrix},\ \mathbf{G}_4'(u,v) = \dst \frac{1}{2\sqrt{2}} \begin{bmatrix} -1/\sqrt{u+v} & -1/\sqrt{u+v} \\ -1/\sqrt{u-v} & 1/\sqrt{u-v} \end{bmatrix}$$

From solving $x = r\cos\theta$, $y = r\sin\theta$ for $\theta = \arg(x,y)$. Each equation is satisfied by angles that are not arguments of (x,y), since none of the formulas identifies the quadrant of (x,y) uniquely. Moreover,

does not hold if $x = 0$.

Bracket argument to \\ must be a dimension,

where $\beta - \pi/2 < \arg(u,v) < \beta + \pi/2$ and β is an argument of (a,b);

$$\mathbf{G}'(u,v) = \dst \frac{1}{2(x^2+y^2)} \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$$

If $\mathbf{F}(x_1, x_2, \dots, x_n) = (x_1^3, x_2^3, \dots, x_n^3)$, then $\mathbf{F}$ is invertible, but\ $J\mathbf{F}(\mathbf{0}) = 0$.

$$\mathbf{A}(\mathbf{U}) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \dst \frac{1}{25} \begin{bmatrix} 5 & 5 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} u+5 \\ v-4 \end{bmatrix}$$

$$\mathbf{A}(\mathbf{U}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \dst \frac{1}{6} \begin{bmatrix} 4 & -2 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} u-2 \\ v-3 \end{bmatrix}$$

$$\mathbf{A}(\mathbf{U}) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u-1 \\ v-1 \\ w-2 \end{bmatrix}$$

$$\mathbf{A}(\mathbf{U}) = \begin{bmatrix} 1 \\ \pi/2 \\ \pi \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u \\ v+1 \\ w \end{bmatrix}$$

Bracket argument to \\ must be a dimension

Bracket argument to \\ must be a dimension

$$\begin{bmatrix} u \\ v \end{bmatrix} = \dst \frac{1}{2} \begin{bmatrix} -3 & 4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \dst -\frac{1}{2} \begin{bmatrix} 3 & 3 \\ -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \dst \frac{1}{5} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -y+\sin x \\ -x+\sin y \end{bmatrix}$$

$u = -x$, $v = -y$, $z = -w$

$f_i(\mathbf{X}, \mathbf{U}) = \dst \left(\sum_{j=1}^{n} a_{ij}(x_j - x_{j0})\right)^r - (u_i - u_{i0})^s$, $1 \le i \le m$, where r and s are positive integers and not all $a_{ij} = 0$. $r = s = 3$; $r = 1$, $s = 3$;

$r = s = 2$

$u_x(1,1) = -\frac{5}{8}$, $u_y(1,1) = -\frac{1}{2}$

$u_x(1,1,1) = \frac{5}{8}$, $u_y(1,1,1) = -\frac{9}{8}$, $u_z(1,1,1) = \frac{1}{2}$

$u(1,2) = 0$, $u_x(1,2) = u_y(1,2) = -4$

$u(-1,-2) = 2$, $u_x(-1,-2) = 1$, $u_y(-1,-2) = -\frac{1}{2}$

$u(\pi/2,\pi/2) = u_x(\pi/2,\pi/2) = u_y(\pi/2,\pi/2) = 0$

$u(1,1) = 1$, $u_x(1,1) = u_y(1,1) = -1$

$u_1(1,1) = 1$,$\dst\frac{\partial u_1(1,1)}{\partial x} = 5$, $\dst\frac{\partial u_1(1,1)}{\partial y} = 2$

$u_2(1,1) = 2$, $\dst\frac{\partial u_2(1,1)}{\partial x} = -14$; $\dst\frac{\partial u_2(1,1)}{\partial y} = -2$

$u_k(0,\pi) = (2k+1)\pi/2$, $\dst\frac{\partial u_k(0,\pi)}{\partial x} = 0$, $\dst\frac{\partial u_k(0,\pi)}{\partial y} = -1, k = \text{integer}$

$\dst\frac{1}{5}\begin{bmatrix} -1 & -2 & 1 \\ -1 & -2 & 1 \end{bmatrix}$ $u'(0) = 3$, $v'(0) = -1$

$\dst\frac{1}{6}\begin{bmatrix} 5 & 5 \\ -5 & -5 \\ 6 & 6 \end{bmatrix}$

$\mathbf{U}_1(1,1) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\mathbf{U}_1'(1,1) = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$;

$\mathbf{U}_2(1,1) = -\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\mathbf{U}_2'(1,1) = -\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$

$u_x(0,0,0) = 2$, $v_x(0,0,0) = w_x(0,0,0) = -2$

$y_x = -\dst\frac{\dst\frac{\partial(f,g,h)}{\partial(x,z,w)}}{\dst\frac{\partial(f,g,h)}{\partial(y,z,w)}}$, $y_v = -\dst\frac{\dst\frac{\partial(f,g,h)}{\partial(v,z,w)}}{\dst\frac{\partial(f,g,h)}{\partial(y,z,w)}}$, $z_x = -\dst\frac{\dst\frac{\partial(f,g,h)}{\partial(y,x,w)}}{\dst\frac{\partial(f,g,h)}{\partial(y,z,w)}}$,

$z_v = -\dst\frac{\dst\frac{\partial(f,g,h)}{\partial(y,v,w)}}{\dst\frac{\partial(f,g,h)}{\partial(y,z,w)}}$, $u_x = -\dst\frac{\dst\frac{\partial(f,g,h)}{\partial(y,z,x)}}{\dst\frac{\partial(f,g,h)}{\partial(y,z,w)}}$, $u_v = -\dst\frac{\dst\frac{\partial(f,g,h)}{\partial(y,z,v)}}{\dst\frac{\partial(f,g,h)}{\partial(y,z,w)}}$

$x = -2y - u$, $z = -2v$; $x = -2y - u$, $v = -\dst\frac{z}{2}$; $y = -\dst\frac{x}{2} - \dst\frac{u}{2}$, $z = -2v$; $y = -\dst\frac{x}{2} - \dst\frac{u}{2}$, $v = -\dst\frac{z}{2}$; $z = -2v$, $u = -x - 2y$; $u = -x - 2y$, $v = -\dst\frac{z}{2}$

$y_x(1,-1,-2) = -\frac{1}{2}$, $v_u(1,-1,-2) = 1$

$u_w(0,-1) = \frac{5}{6}$, $u_y(0,-1) = 0$, $v_w(0,-1) = -\frac{5}{6}$, $v_y(0,-1) = 0$,\ $x_w(0,-1) = 1$, $x_y(0,-1) = -1$

$u_x(1,1) = 0$, $u_y(1,1) = 0$, $v_x(1,1) = -1$, $v_y(1,1) = -1$, $u_{xx}(1,1) = 2$,\ $u_{xy}(1,1) = 1$, $u_{yy}(1,1) = 2$, $v_{xx}(1,1) = -2$, $v_{xy}(1,1) = -1$, $v_{yy}(1,1) = -2$

$u_x(1,-1) = 0$, $u_y(1,-1) = \dst\frac{1}{2}$, $v_x(1,-1) = -\dst\frac{1}{2}$, $v_y(1,-1) = 0$,\ $u_{xx}(1,-1) = -\dst\frac{1}{8}$, $u_{xy}(1,-1) = \dst\frac{1}{8}$, $u_{yy}(1,-1) = \dst\frac{1}{8}$, $v_{xx}(1,-1) = -\dst\frac{1}{8}$,\ $v_{xy}(1,-1) = -\dst\frac{1}{8}$, $v_{yy}(1,-1) = \dst\frac{1}{8}$

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$\frac{1}{4}$ $3(b-a)(d-c)$, 0 $\{(m,n)\}m,n = \text{integers}$

12 $\frac{79}{20}$ -1

$(1 - \log 2)/2$

$\frac{7}{4}$ 17 $\frac{2}{3}(\sqrt{2}-1)$

$1/4\pi$

$\frac{3}{8}$, $\frac{5}{8}$ $\frac{3}{8}$, $\frac{5}{8}$ $\frac{3}{4}$, $\frac{5}{4}$ $\frac{3}{4}$ $\left(z+\frac{1}{2}\right)$, $\frac{5}{4}$ $\left(z+\frac{1}{2}\right)$

$z+\frac{1}{2}$, 1

-285 0 0

$\frac{1}{4}\left(e - \frac{5}{2}\right)$

324 $\frac{1}{6}$

1 $\frac{52}{15}$

$36\ 1\ \frac{64}{3}$

$(e^6+17)/2$

$\frac{2}{27}\ \frac{1}{2}(e-\frac{5}{2})\ \frac{1}{24}$

$\frac{1}{36}$

$16\pi\ \frac{1}{6}\ \frac{128}{21}$

$\frac{\pi}{2}$

$\frac{1}{2}(b_1-a_1)\cdots(b_n-a_n)\sum_{j=1}^{n}(a_j+b_j)$

$\frac{1}{3}(b_1-a_1)\cdots(b_n-a_n)\sum_{j=1}^{n}(a_j^2+a_jb_j+b_j^2)$

$2^{-n}(b_1^2-a_1^2)\cdots(b_n^2-a_n^2)$

$\int_{-\sqrt{3}/2}^{\sqrt{3}/2}dx\int_{1/2}^{\sqrt{1-x^2}}f(x,y)\,dy\ \frac{1}{2}$

Let S_1 and S_2 be dense subsets of $\mathbb{R}$ such that $S_1\cup S_2=\mathbb{R}$.

-1; c (constant); $1\ (u_2-u_1)(v_2-v_1)/|ad-bc|$

$\frac{5}{6}\ \frac{4}{9}$

$\log\frac{5}{2}\ 3\ \frac{1}{2}\ \frac{5}{4}e(e-1)$

$\frac{4}{3}\pi abc\ 2\pi(e^{25}-e^9)\ 16\pi/3\ 21/64$

$(\pi/8)\log 5\ (\pi/4)(e^4-1)$

$2\pi/15$

$\pi^2a^4/2$

$(\beta_1-\alpha_1)\cdots(\beta_n-\alpha_n)/|\det(\mathbf{A})|\ |a_1a_2\cdots a_n|V_n$

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{}

Suppose $f\in C[a,b]$. Acording to the , if $\epsilon>0$, there is a polynonmial p such that $|f(x)-p(x)|<\epsilon$, $a\le x\le b$. Now suppose that

$$a\le x_{1n}<x_{2n}<\cdots<x_{nn}\le b,\quad a\le y_{1n}<y_{2n}<\cdots<y_{nn}\le b,\quad n\ge 1.$$

and

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}|x_{in}-y_{in}|=0.$$

Show that if $f\in C[a,b]$, then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}|f(x_{in})-f(y_{in})|=0.$$

By the triangle inequality,

$$|f(x_{in})-f(y_{in})|\le|f(x_{in})-p(x_{in})|+|p(x_{in})-p(y_{in})|+|p(y_{in})-f(y_{in})|,$$

so

$$|f(x_{in})-f(y_{in})|<|p(x_{in})-p(y_{in})|+2\epsilon. \tag{A}$$

Let $M=\max_{a\le x\le b}|p'(x)|$. By the mean value theorem,

$$|p(x_{in})-p(y_{in})|\le M|x_{in}-y_{in}|.$$

This and (A) imply that

$$\frac{1}{n}\sum_{i=1}^n |f(x_{in}) - f(y_{in})| < 2\epsilon + \frac{M}{n}\sum_{i=1}^n |x_{in} - y_{in}|.$$

From this and (A),

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^n |f(x_{in}) - f(y_{in})| \le 2\epsilon.$$

Since ϵ is arbitrary, this implies the conclusion.

{} In the setting of Exercise~1, let

$$y_{in} = a + \frac{i}{n}\sum_{i=1}^n (b-a), \quad 1 \le i \le n, \quad n \ge 1.$$

Show that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n f(x_{in}) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

{Solution 2.} Since $f \in C[a,b]$, $\int_a^b f(x)\,dx$ exists (Theorem~3.2.8, p.~133).

Therefore, from Definition~3.1.1 (p.114),

$$\lim_{n\to\infty} \sum_{i=1}^n f(y_{in}) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Since

$$\frac{1}{n}\left|\sum_{i=1}^n f(x_{in}) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{1}{n}\sum_{i=1}^n |f(x_{in}) - f(y_{in})| + \left|\frac{1}{n}\sum_{i=1}^n f(y_{in}) - \frac{1}{b-a}\int_a^b f(x)\,dx\right|,$$

Exercise1 implies the conclusion.

Suppose g is continuous and nondecreasing on $[c,d]$. For

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