

REVIEW FOR CALCULUS



Review for Calculus

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This text was compiled on 06/09/2025

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CHAPTER OVERVIEW

1: Algebra

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1.1: Dividing Polynomials

Learning Objectives

- Use synthetic division to divide polynomials.

Using Synthetic Division to Divide Polynomials

Synthetic division is a shorthand method of dividing polynomials for the special case of dividing by a linear factor whose leading coefficient is 1.

To illustrate the process, recall the example at the beginning of the section.

Divide $2x^3 - 3x^2 + 4x + 5$ by $x + 2$ using the long division algorithm.

The final form of the process looked like this:

$$\begin{array}{r}
 2x^2 + x + 18 \\
 x + 2 \overline{) 2x^3 - 3x^2 + 4x + 5} \\
 \underline{-(2x^3 + 4x^2)} \\
 -7x^2 + 4x \\
 \underline{-(-7x^2 - 14x)} \\
 18x + 5 \\
 \underline{-(18x + 36)} \\
 -31
 \end{array}$$

There is a lot of repetition in the table. If we don't write the variables but, instead, line up their coefficients in columns under the division sign and also eliminate the partial products, we already have a simpler version of the entire problem.

$$\begin{array}{r}
 2 \overline{) 2 \quad -3 \quad 4 \quad 5} \\
 \underline{-2 \quad -4} \\
 -7 \quad 14 \\
 \underline{18 \quad -36} \\
 -31
 \end{array}$$

Synthetic division carries this simplification even a few more steps. Collapse the table by moving each of the rows up to fill any vacant spots. Also, instead of dividing by 2, as we would in division of whole numbers, then multiplying and subtracting the middle product, we change the sign of the "divisor" to -2 , multiply and add. The process starts by bringing down the leading coefficient.

$$\begin{array}{r|rrrr}
 -2 & 2 & -3 & 4 & 5 \\
 & & -4 & 14 & -36 \\
 \hline
 & 2 & -7 & 18 & -31
 \end{array}$$

We then multiply it by the "divisor" and add, repeating this process column by column, until there are no entries left. The bottom row represents the coefficients of the quotient; the last entry of the bottom row is the remainder. In this case, the quotient is $2x^2 - 7x + 18$ and the remainder is -31 . The process will be made more clear in Example 1.1.3.

Synthetic Division

Synthetic division is a shortcut that can be used when the divisor is a binomial in the form $x - k$. In *synthetic division*, only the coefficients are used in the division process.

How To: Given two polynomials, use synthetic division to divide

1. Write k for the divisor.
2. Write the coefficients of the dividend.

3. Bring the lead coefficient down.
4. Multiply the lead coefficient by k . Write the product in the next column.
5. Add the terms of the second column.
6. Multiply the result by k . Write the product in the next column.
7. Repeat steps 5 and 6 for the remaining columns.
8. Use the bottom numbers to write the quotient. The number in the last column is the remainder and has degree 0, the next number from the right has degree 1, the next number from the right has degree 2, and so on.

Example 1.1.3: Using Synthetic Division to Divide a Second-Degree Polynomial

Use synthetic division to divide $5x^2 - 3x - 36$ by $x - 3$.

Solution

Begin by setting up the synthetic division. Write k and the coefficients.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \end{array}$$

Bring down the lead coefficient. Multiply the lead coefficient by k .

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ & & 15 & \\ \hline & 5 & & \end{array}$$

Continue by adding the numbers in the second column. Multiply the resulting number by k . Write the result in the next column. Then add the numbers in the third column.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ & & 15 & 36 \\ \hline & 5 & 12 & 0 \end{array}$$

The result is $5x + 12$. The remainder is 0. So $x - 3$ is a factor of the original polynomial.

Analysis

Just as with long division, we can check our work by multiplying the quotient by the divisor and adding the remainder.

$$(x - 3)(5x + 12) + 0 = 5x^2 - 3x - 36 \quad (1.1.1)$$

Example 1.1.4: Using Synthetic Division to Divide a Third-Degree Polynomial

Use synthetic division to divide $4x^3 + 10x^2 - 6x - 20$ by $x + 2$.

Solution

The binomial divisor is $x + 2$ so $k = -2$. Add each column, multiply the result by -2 , and repeat until the last column is reached.

$$\begin{array}{r|rrrr} -2 & 4 & 10 & -6 & -20 \\ & & -8 & -2 & 18 \\ \hline & 4 & 2 & -8 & -2 \end{array}$$

The result is $4x^2 + 2x - 10$.

The remainder is 0. Thus, $x + 2$ is a factor of $4x^3 + 10x^2 - 6x - 20$.

Analysis

The graph of the polynomial function $f(x) = 4x^3 + 10x^2 - 6x - 20$ in Figure 1.1.2 shows a zero at $x = k = -2$. This confirms that $x + 2$ is a factor of $4x^3 + 10x^2 - 6x - 20$.

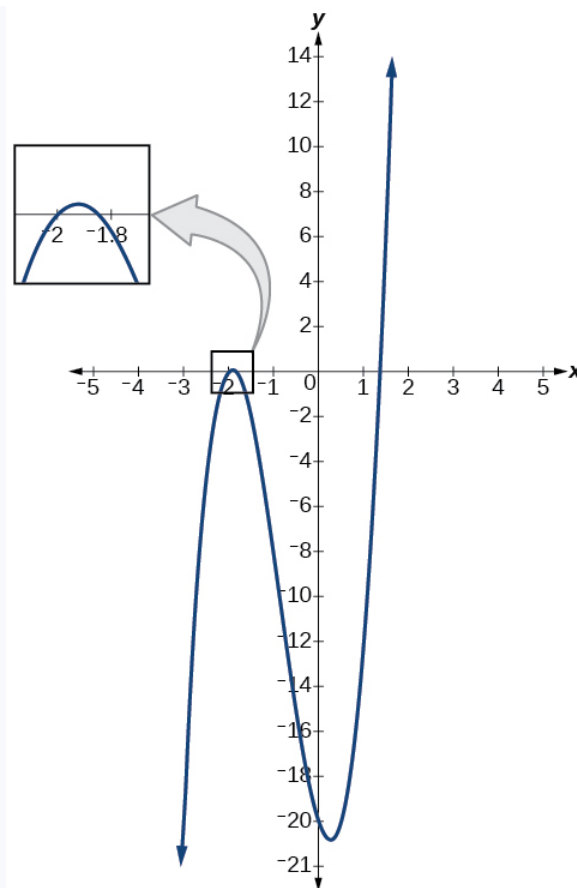


Figure 1.1.2

Example 1.1.5: Using Synthetic Division to Divide a Fourth-Degree Polynomial

Use synthetic division to divide $-9x^4 + 10x^3 + 7x^2 - 6$ by $x - 1$.

Solution

Notice there is no x -term. We will use a zero as the coefficient for that term.

$$\begin{array}{r|rrrrr} 1 & -9 & 10 & 7 & 0 & -6 \\ & & -9 & 1 & 8 & 8 \\ \hline & -9 & 1 & 8 & 8 & 2 \end{array}$$

The result is $-9x^3 + x^2 + 8x + 8 + \frac{2}{x-1}$.



1.1.5

Use synthetic division to divide $3x^4 + 18x^3 - 3x + 40$ by $x + 7$.

Solution

$$3x^3 - 3x^2 + 21x - 150 + \frac{1,090}{x+7}$$

Contributors

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1.2: Zeros of Polynomial Functions

Learning Objectives

- Evaluate a polynomial using the Remainder Theorem.
- Use the Factor Theorem to solve a polynomial equation.

A new bakery offers decorated sheet cakes for children's birthday parties and other special occasions. The bakery wants the volume of a small cake to be 351 cubic inches. The cake is in the shape of a rectangular solid. They want the length of the cake to be four inches longer than the width of the cake and the height of the cake to be one-third of the width. What should the dimensions of the cake pan be?

This problem can be solved by writing a cubic function and solving a cubic equation for the volume of the cake. In this section, we will discuss a variety of tools for writing polynomial functions and solving polynomial equations.

Evaluating a Polynomial Using the Remainder Theorem

In the last section, we learned how to divide polynomials. We can now use polynomial division to evaluate polynomials using the **Remainder Theorem**. If the polynomial is divided by $x - k$, the remainder may be found quickly by evaluating the polynomial function at k , that is, $f(k)$. Let's walk through the proof of the theorem.

Recall that the Division Algorithm states that, given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x)q(x) + r(x)$$

If the divisor, $d(x)$, is $x - k$, this takes the form

$$f(x) = (x - k)q(x) + r$$

Since the divisor $x - k$

is linear, the remainder will be a constant, r . And, if we evaluate this for $x = k$, we have

$$\begin{aligned} f(k) &= (k - k)q(k) + r \\ &= 0 \cdot q(k) + r \\ &= r \end{aligned}$$

In other words, $f(k)$ is the remainder obtained by dividing $f(x)$ by $x - k$.

The Remainder Theorem

If a polynomial $f(x)$ is divided by $x - k$, then the remainder is the value $f(k)$.



Given a polynomial function f , evaluate $f(x)$ at $x = k$ using the Remainder Theorem.

1. Use synthetic division to divide the polynomial by $x - k$.
2. The remainder is the value $f(k)$.

Example 1.2.1: Using the Remainder Theorem to Evaluate a Polynomial

Use the Remainder Theorem to evaluate $f(x) = 6x^4 - x^3 - 15x^2 + 2x - 7$ at $x = 2$.

Solution

To find the remainder using the Remainder Theorem, use synthetic division to divide the polynomial by $x - 2$.

$$\begin{array}{r|rrrrr} 2 & 6 & -1 & -15 & 2 & -7 \\ & & 12 & 22 & 14 & 32 \\ \hline & 6 & 11 & 7 & 16 & 25 \end{array} \quad (1.2.1)$$

The remainder is 25. Therefore, $f(2) = 25$.

Analysis

We can check our answer by evaluating $f(2)$.

$$\begin{aligned} f(x) &= 6x^4 - x^3 - 15x^2 + 2x - 7 \\ f(2) &= 6(2)^4 - (2)^3 - 15(2)^2 + 2(2) - 7 \\ &= 25 \end{aligned}$$

Exercise 1.2.1

Use the Remainder Theorem to evaluate $f(x) = 2x^5 - 3x^4 - 9x^3 + 8x^2 + 2$ at $x = -3$.

Answer

$$f(-3) = -412$$

Using the Factor Theorem to Solve a Polynomial Equation

The **Factor Theorem** is another theorem that helps us analyze polynomial equations. It tells us how the zeros of a polynomial are related to the factors. Recall that the Division Algorithm.

$$f(x) = (x - k)q(x) + r \quad (1.2.2)$$

If k is a zero, then the remainder r is $f(k) = 0$ and $f(x) = (x - k)q(x) + 0$ or $f(x) = (x - k)q(x)$.

Notice, written in this form, $x - k$ is a factor of $f(x)$. We can conclude if k is a zero of $f(x)$, then $x - k$ is a factor of $f(x)$.

Similarly, if $x - k$ is a factor of $f(x)$, then the remainder of the Division Algorithm $f(x) = (x - k)q(x) + r$ is 0. This tells us that k is a zero.

This pair of implications is the Factor Theorem. As we will soon see, a polynomial of degree n in the complex number system will have n zeros. We can use the Factor Theorem to completely factor a polynomial into the product of n factors. Once the polynomial has been completely factored, we can easily determine the zeros of the polynomial.

THE FACTOR THEOREM

According to the Factor Theorem, k is a zero of $f(x)$ if and only if $(x - k)$ is a factor of $f(x)$.

How to: Given a factor and a third-degree polynomial, use the Factor Theorem to factor the polynomial

1. Use synthetic division to divide the polynomial by $(x - k)$.
2. Confirm that the remainder is 0.
3. Write the polynomial as the product of $(x - k)$ and the quadratic quotient.
4. If possible, factor the quadratic.
5. Write the polynomial as the product of factors.

Example 1.2.2: Using the Factor Theorem to Solve a Polynomial Equation

Show that $(x + 2)$ is a factor of $x^3 - 6x^2 - x + 30$. Find the remaining factors. Use the factors to determine the zeros of the polynomial.

Solution

We can use synthetic division to show that $(x + 2)$ is a factor of the polynomial.

$$\begin{array}{r|rrrr} -2 & 1 & -6 & -1 & 30 \\ & & -2 & 16 & -30 \\ \hline & 1 & -8 & 15 & 0 \end{array} \quad (1.2.3)$$

The remainder is zero, so $(x + 2)$ is a factor of the polynomial. We can use the Division Algorithm to write the polynomial as the product of the divisor and the quotient:

$$(x + 2)(x^2 - 8x + 15) \quad (1.2.4)$$

We can factor the quadratic factor to write the polynomial as

$$(x + 2)(x - 3)(x - 5) \quad (1.2.5)$$

By the Factor Theorem, the zeros of $x^3 - 6x^2 - x + 30$ are -2 , 3 , and 5 .

Exercise 1.2.2

Use the Factor Theorem to find the zeros of $f(x) = x^3 + 4x^2 - 4x - 16$ given that $(x - 2)$ is a factor of the polynomial.

Answer

The zeros are 2 , -2 , and -4 .

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1.3: Rational Functions

Learning Objectives

- Use arrow notation.
- Solve applied problems involving rational functions.
- Find the domains of rational functions.
- Identify vertical asymptotes.
- Identify horizontal asymptotes.

Suppose we know that the cost of making a product is dependent on the number of items, x , produced. This is given by the equation $C(x) = 15,000x - 0.1x^2 + 1000$. If we want to know the average cost for producing x items, we would divide the cost function by the number of items, x . The average cost function, which yields the average cost per item for x items produced, is

$$f(x) = \frac{15,000x - 0.1x^2 + 1000}{x}$$

Many other application problems require finding an average value in a similar way, giving us variables in the denominator. Written without a variable in the denominator, this function will contain a negative integer power.

In the last few sections, we have worked with polynomial functions, which are functions with non-negative integers for exponents. In this section, we explore rational functions, which have variables in the denominator.

Using Arrow Notation

We have seen the graphs of the basic **reciprocal function** and the squared reciprocal function from our study of toolkit functions. Examine these graphs, as shown in Figure 1.3.1, and notice some of their features.

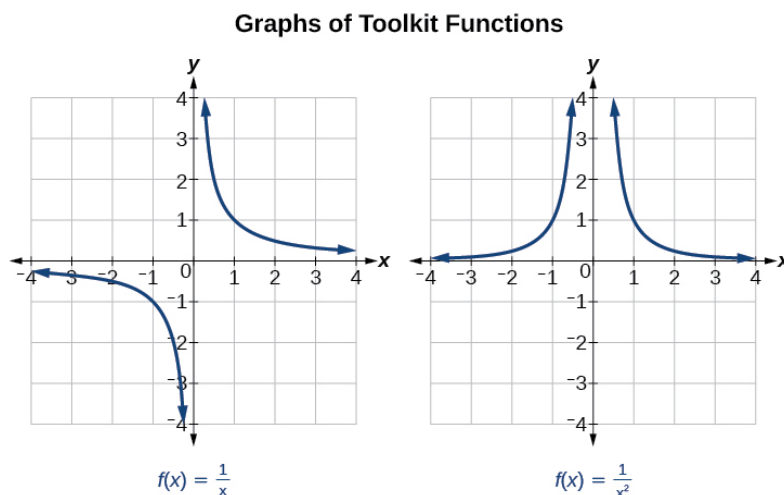


Figure 1.3.1

Several things are apparent if we examine the graph of $f(x) = \frac{1}{x}$.

- On the left branch of the graph, the curve approaches the x -axis ($y = 0$) as $x \rightarrow -\infty$.
- As the graph approaches $x = 0$ from the left, the curve drops, but as we approach zero from the right, the curve rises.
- Finally, on the right branch of the graph, the curves approaches the x -axis ($y = 0$) as $x \rightarrow \infty$.

To summarize, we use **arrow notation** to show that x or $f(x)$ is approaching a particular value (Table 1.3.1).

Table 1.3.1

Symbol	Meaning
$x \rightarrow a^-$	x approaches a from the left ($x < a$ but close to a)
$x \rightarrow a^+$	x approaches a from the right ($x > a$ but close to a)
$x \rightarrow \infty$	x approaches infinity (x increases without bound)
$x \rightarrow -\infty$	x approaches negative infinity (x decreases without bound)
$f(x) \rightarrow \infty$	the output approaches infinity (the output increases without bound)

Symbol	Meaning
$f(x) \rightarrow -\infty$	the output approaches negative infinity (the output decreases without bound)
$f(x) \rightarrow a$	the output approaches a

Local Behavior of $f(x) = \frac{1}{x}$

Let's begin by looking at the reciprocal function, $f(x) = \frac{1}{x}$. We cannot divide by zero, which means the function is undefined at $x = 0$; so zero is not in the domain. As the input values approach zero from the left side (becoming very small, negative values), the function values decrease without bound (in other words, they approach negative infinity). We can see this behavior in Table 1.3.2.

Table 1.3.2

x	-0.1	-0.01	-0.001	-0.0001
$f(x) = \frac{1}{x}$	-10	-100	-1000	-10,000

We write in arrow notation

$$\text{as } x \rightarrow 0^-, f(x) \rightarrow -\infty$$

As the input values approach zero from the right side (becoming very small, positive values), the function values increase without bound (approaching infinity). We can see this behavior in Table 1.3.3.

Table 1.3.3

x	0.1	0.01	0.001	0.0001
$f(x) = \frac{1}{x}$	10	100	1000	10,000

We write in arrow notation

$$\text{As } x \rightarrow 0^+, f(x) \rightarrow \infty.$$

See Figure 1.3.2.

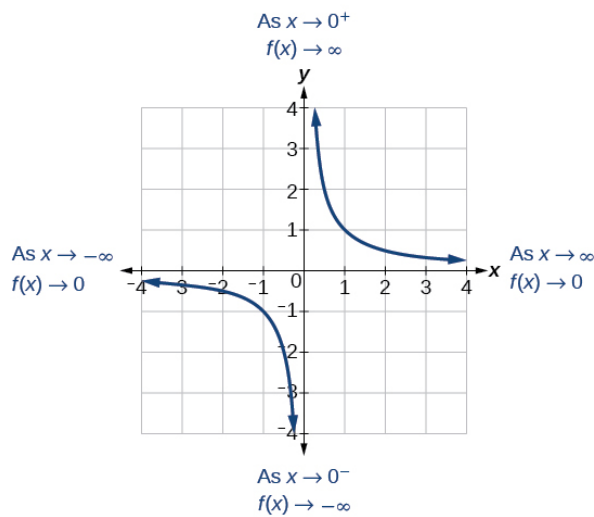


Figure 1.3.2.

This behavior creates a **vertical asymptote**, which is a vertical line that the graph approaches but never crosses. In this case, the graph is approaching the vertical line $x = 0$ as the input becomes close to zero (Figure 1.3.3).

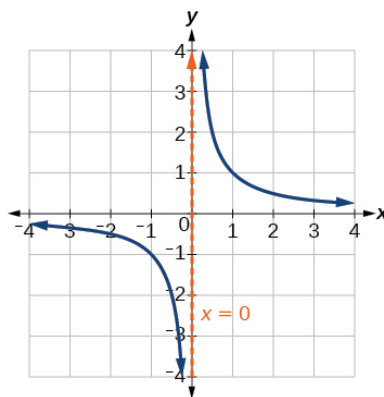


Figure 1.3.3.

Definition: VERTICAL ASYMPTOTE

A vertical asymptote of a graph is a vertical line $x = a$ where the graph tends toward positive or negative infinity as the inputs approach a . We write

As $x \rightarrow a$, $f(x) \rightarrow \infty$, or as $x \rightarrow a$, $f(x) \rightarrow -\infty$.

End Behavior of $f(x) = \frac{1}{x}$

As the values of x approach infinity, the function values approach 0. As the values of x approach negative infinity, the function values approach 0 (Figure 1.3.4). Symbolically, using arrow notation

As $x \rightarrow \infty$, $f(x) \rightarrow 0$, and as $x \rightarrow -\infty$, $f(x) \rightarrow 0$.

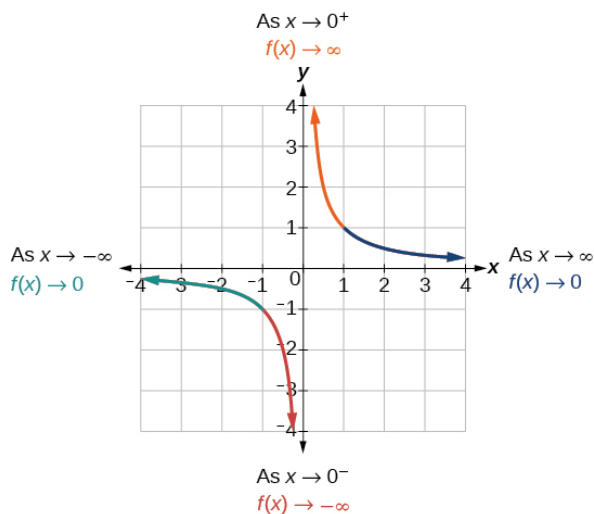


Figure 1.3.4.

Based on this overall behavior and the graph, we can see that the function approaches 0 but never actually reaches 0; it seems to level off as the inputs become large. This behavior creates a **horizontal asymptote**, a horizontal line that the graph approaches as the input increases or decreases without bound. In this case, the graph is approaching the horizontal line $y = 0$. See Figure 1.3.5.

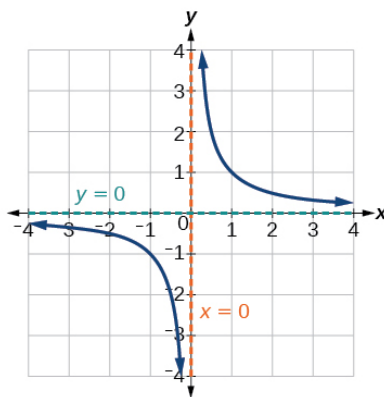


Figure 1.3.5.

Definition: HORIZONTAL ASYMPTOTE

A *horizontal asymptote* of a graph is a horizontal line $y = b$ where the graph approaches the line as the inputs increase or decrease without bound. We write

$$\text{As } x \rightarrow \infty \text{ or } x \rightarrow -\infty, f(x) \rightarrow b.$$

Example 1.3.1: Using Arrow Notation.

Use arrow notation to describe the end behavior and local behavior of the function graphed in Figure 1.3.6.

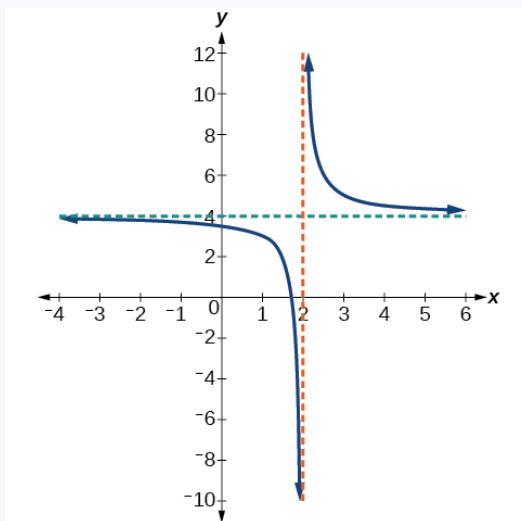


Figure 1.3.6.

Solution

Notice that the graph is showing a vertical asymptote at $x = 2$, which tells us that the function is undefined at $x = 2$.

$$\text{As } x \rightarrow 2^-, f(x) \rightarrow -\infty, \text{ and as } x \rightarrow 2^+, f(x) \rightarrow \infty.$$

And as the inputs decrease without bound, the graph appears to be leveling off at output values of 4, indicating a horizontal asymptote at $y = 4$. As the inputs increase without bound, the graph levels off at 4.

$$\text{As } x \rightarrow \infty, f(x) \rightarrow 4 \text{ and as } x \rightarrow -\infty, f(x) \rightarrow 4.$$

Exercises 1.3.1

Use arrow notation to describe the end behavior and local behavior for the reciprocal squared function.

Answer:

End behavior: as $x \rightarrow \pm\infty, f(x) \rightarrow 0$;

Local behavior: as $x \rightarrow 0, f(x) \rightarrow \infty$ (there are no x- or y-intercepts)

Example 1.3.2: Using Transformations to Graph a Rational Function.

Sketch a graph of the reciprocal function shifted two units to the left and up three units. Identify the horizontal and vertical asymptotes of the graph, if any.

Solution

Shifting the graph left 2 and up 3 would result in the function

$$f(x) = \frac{1}{x+2} + 3 \quad (1.3.1)$$

or equivalently, by giving the terms a common denominator,

$$f(x) = \frac{3x+7}{x+2} \quad (1.3.2)$$

The graph of the shifted function is displayed in Figure 1.3.7.

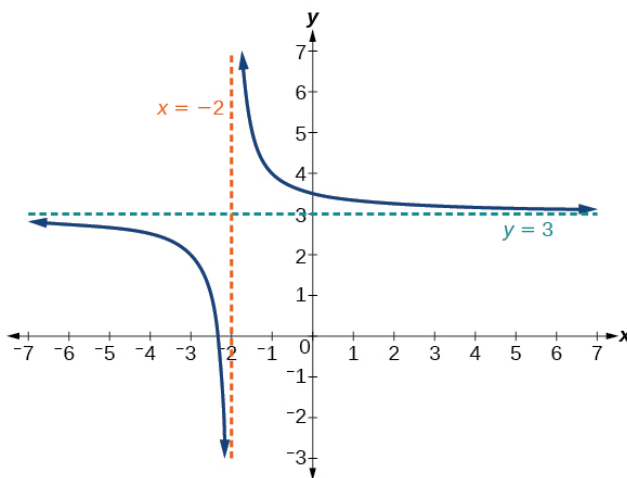


Figure 1.3.7.

Notice that this function is undefined at $x = -2$, and the graph also is showing a vertical asymptote at $x = -2$.

$$\text{As } x \rightarrow -2^-, f(x) \rightarrow -\infty, \text{ and as } x \rightarrow -2^+, f(x) \rightarrow \infty.$$

As the inputs increase and decrease without bound, the graph appears to be leveling off at output values of 3, indicating a horizontal asymptote at $y = 3$.

$$\text{As } x \rightarrow \pm\infty, f(x) \rightarrow 3.$$

Analysis

Notice that horizontal and vertical asymptotes are shifted left 2 and up 3 along with the function.

Exercise 1.3.2

Sketch the graph, and find the horizontal and vertical asymptotes of the reciprocal squared function that has been shifted right 3 units and down 4 units.

Solution

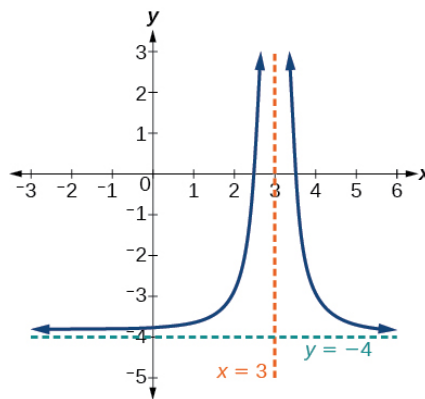


Figure 1.3.8.

The function and the asymptotes are shifted 3 units right and 4 units down. As $x \rightarrow 3$, $f(x) \rightarrow \infty$, and as $x \rightarrow \pm\infty$, $f(x) \rightarrow -4$.

The function is $f(x) = \frac{1}{(x-3)^2} - 4$.

Solving Applied Problems Involving Rational Functions

In Example 1.3.2, we shifted a toolkit function in a way that resulted in the function $f(x) = \frac{3x+7}{x+2}$. This is an example of a rational function. A *rational function* is a function that can be written as the quotient of two polynomial functions. Many real-world problems require us to find the ratio of two polynomial functions. Problems involving rates and concentrations often involve rational functions.

Definition: RATIONAL FUNCTION

A **rational function** is a function that can be written as the quotient of two polynomial functions $P(x)$ and $Q(x)$.

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_p x^p + a_{p-1} x^{p-1} + \dots + a_1 x + a_0}{b_q x^q + b_{q-1} x^{q-1} + \dots + b_1 x + b_0}, \quad Q(x) \neq 0 \quad (1.3.3)$$

Example 1.3.3: Solving an Applied Problem Involving a Rational Function

A large mixing tank currently contains 100 gallons of water into which 5 pounds of sugar have been mixed. A tap will open pouring 10 gallons per minute of water into the tank at the same time sugar is poured into the tank at a rate of 1 pound per minute. Find the concentration (pounds per gallon) of sugar in the tank after 12 minutes. Is that a greater concentration than at the beginning?

Solution

Let t be the number of minutes since the tap opened. Since the water increases at 10 gallons per minute, and the sugar increases at 1 pound per minute, these are constant rates of change. This tells us the amount of water in the tank is changing linearly, as is the amount of sugar in the tank. We can write an equation independently for each:

$$\begin{aligned} \text{water: } W(t) &= 100 + 10t \text{ in gallons} \\ \text{sugar: } S(t) &= 5 + 1t \text{ in pounds} \end{aligned}$$

The concentration, C , will be the ratio of pounds of sugar to gallons of water

$$C(t) = \frac{5 + t}{100 + 10t} \quad (1.3.4)$$

The concentration after 12 minutes is given by evaluating $C(t)$ at $t = 12$.

$$C(12) = \frac{5 + 12}{100 + 10(12)} \quad (1.3.5)$$

$$= \frac{17}{220} \quad (1.3.6)$$

This means the concentration is 17 pounds of sugar to 220 gallons of water.

At the beginning, the concentration is

$$C(0) = \frac{5 + 0}{100 + 10(0)} \quad (1.3.7)$$

$$= \frac{1}{20} \quad (1.3.8)$$

Since $\frac{17}{220} \approx 0.08 > \frac{1}{20} = 0.05$, the concentration is greater after 12 minutes than at the beginning.

Analysis

To find the horizontal asymptote, divide the leading coefficient in the numerator by the leading coefficient in the denominator:

$$\frac{1}{10} = 0.1 \quad (1.3.9)$$

Notice the horizontal asymptote is $y = 0.1$. This means the concentration, C , the ratio of pounds of sugar to gallons of water, will approach 0.1 in the long term.

Exercise 1.3.3

There are 1,200 freshmen and 1,500 sophomores at a prep rally at noon. After 12 p.m., 20 freshmen arrive at the rally every five minutes while 15 sophomores leave the rally. Find the ratio of freshmen to sophomores at 1 p.m.

Answer

$$\frac{12}{11}$$

Finding the Domains of Rational Functions

A *vertical asymptote* represents a value at which a rational function is undefined, so that value is not in the domain of the function. A reciprocal function cannot have values in its domain that cause the denominator to equal zero. In general, to find the domain of a rational function, we need to determine which inputs would cause division by zero.

Definition: DOMAIN OF A RATIONAL FUNCTION

The domain of a rational function includes all real numbers except those that cause the denominator to equal zero.

HowTO: Given a rational function, find the domain.

1. Set the denominator equal to zero.
2. Solve to find the x-values that cause the denominator to equal zero.
3. The domain is all real numbers except those found in Step 2.

Example 1.3.4: Finding the Domain of a Rational Function

Find the domain of $f(x) = \frac{x+3}{x^2-9}$.

Solution

Begin by setting the denominator equal to zero and solving.

$$x^2 - 9 = 0 \quad (1.3.10)$$

$$x^2 = 9 \quad (1.3.11)$$

$$x = \pm 3 \quad (1.3.12)$$

The denominator is equal to zero when $x = \pm 3$. The domain of the function is all real numbers except $x = \pm 3$.

Analysis

A graph of this function, as shown in Figure 1.3.9, confirms that the function is not defined when $x = \pm 3$.

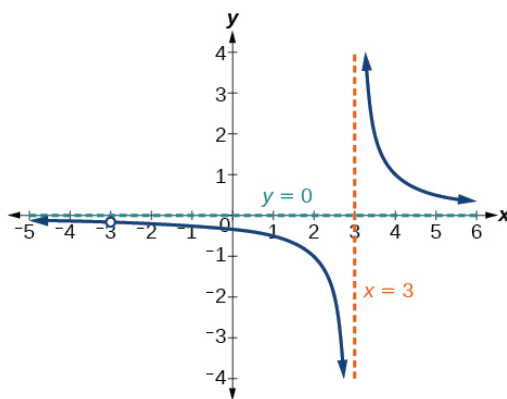


Figure 1.3.9.

There is a vertical asymptote at $x = 3$ and a hole in the graph at $x = -3$. We will discuss these types of holes in greater detail later in this section.

Exercise 1.3.4

Find the domain of $f(x) = \frac{4x}{5(x-1)(x-5)}$.

Answer

The domain is all real numbers except $x = 1$ and $x = 5$.

Identifying Vertical Asymptotes of Rational Functions

By looking at the graph of a rational function, we can investigate its local behavior and easily see whether there are asymptotes. We may even be able to approximate their location. Even without the graph, however, we can still determine whether a given rational function has any asymptotes, and calculate their location.

Vertical Asymptotes

The vertical asymptotes of a rational function may be found by examining the factors of the denominator that are not common to the factors in the numerator. Vertical asymptotes occur at the zeros of such factors.

HowTO: Given a rational function, identify any vertical asymptotes of its graph

1. Factor the numerator and denominator.
2. Note any restrictions in the domain of the function.
3. Reduce the expression by canceling common factors in the numerator and the denominator.
4. Note any values that cause the denominator to be zero in this simplified version. These are where the vertical asymptotes occur.
5. Note any restrictions in the domain where asymptotes do not occur. These are removable discontinuities, or “holes.”

Example 1.3.5: Identifying Vertical Asymptotes

Find the vertical asymptotes of the graph of $k(x) = \frac{5+2x^2}{2-x-x^2}$.

Solution

First, factor the numerator and denominator.

$$k(x) = \frac{5+2x^2}{2-x-x^2} \quad (1.3.13)$$

$$= \frac{5+2x^2}{(2+x)(1-x)} \quad (1.3.14)$$

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$(2+x)(1-x) = 0 \quad (1.3.15)$$

$$x = -2, 1 \quad (1.3.16)$$

Neither $x = -2$ nor $x = 1$ are zeros of the numerator, so the two values indicate two vertical asymptotes. The graph in Figure 1.3.10 confirms the location of the two vertical asymptotes.

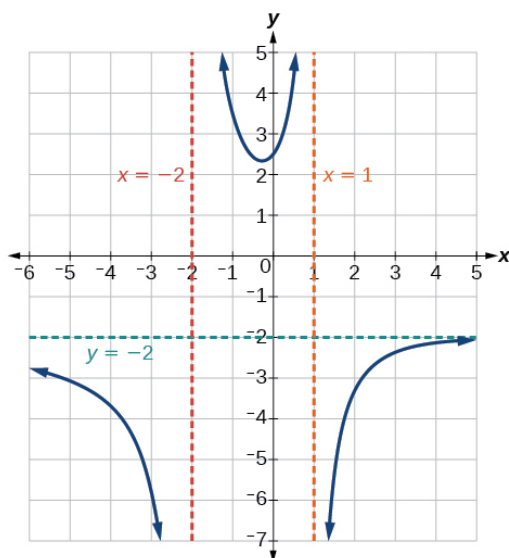


Figure 1.3.10.

Removable Discontinuities

Occasionally, a graph will contain a hole: a single point where the graph is not defined, indicated by an open circle. We call such a hole a *removable discontinuity*. For example, the function $f(x) = \frac{x^2-1}{x^2-2x-3}$ may be re-written by factoring the numerator and the denominator.

$$f(x) = \frac{(x+1)(x-1)}{(x+1)(x-3)} \quad (1.3.17)$$

Notice that $x+1$ is a common factor to the numerator and the denominator. The zero of this factor, $x = -1$, is the location of the removable discontinuity. Notice also that $x-3$ is not a factor in both the numerator and denominator. The zero of this factor, $x = 3$, is the vertical asymptote. See Figure 1.3.11. [Note that removable discontinuities may not be visible when we use a graphing calculator, depending upon the window selected.]

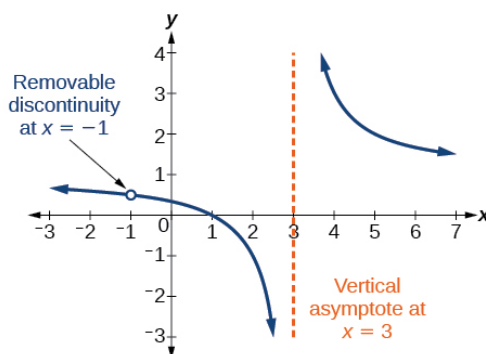


Figure 1.3.11.

REMOVABLE DISCONTINUITIES OF RATIONAL FUNCTIONS

A **removable discontinuity** occurs in the graph of a rational function at $x = a$ if a is a zero for a factor in the denominator that is common with a factor in the numerator. We factor the numerator and denominator and check for common factors. If we find any, we set the common factor equal to 0 and solve. This is the location of the removable discontinuity. This is true if the multiplicity of this factor is greater than or equal to that in the denominator. If the multiplicity of this factor is greater in the denominator, then there is still an asymptote at that value.

Example 1.3.6: Identifying Vertical Asymptotes and Removable Discontinuities for a Graph

Find the vertical asymptotes and removable discontinuities of the graph of $k(x) = \frac{x-2}{x^2-4}$.

Solution

Factor the numerator and the denominator.

$$k(x) = \frac{x-2}{(x-2)(x+2)} \quad (1.3.18)$$

Notice that there is a common factor in the numerator and the denominator, $x-2$. The zero for this factor is $x=2$. This is the location of the removable discontinuity.

Notice that there is a factor in the denominator that is not in the numerator, $x+2$. The zero for this factor is $x=-2$. The vertical asymptote is $x=-2$. See Figure 1.3.12

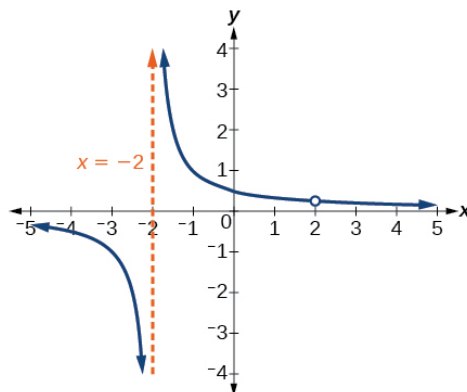


Figure 1.3.12.

The graph of this function will have the vertical asymptote at $x=-2$, but at $x=2$ the graph will have a hole.

Exercise 1.3.5

Find the vertical asymptotes and removable discontinuities of the graph of $f(x) = \frac{x^2-25}{x^3-6x^2+5x}$.

Answer

Removable discontinuity at $x=5$.

Vertical asymptotes: $x=0$, $x=1$.

Identifying Horizontal Asymptotes of Rational Functions

While vertical asymptotes describe the behavior of a graph as the *output* gets very large or very small, horizontal asymptotes help describe the behavior of a graph as the *input* gets very large or very small. Recall that a polynomial's end behavior will mirror that of the leading term. Likewise, a rational function's end behavior will mirror that of the ratio of the function that is the ratio of the leading terms.

There are two distinct outcomes when checking for horizontal asymptotes:

Case 1: If the degree of the denominator $>$ degree of the numerator, there is a **horizontal asymptote** at $y=0$.

$$\text{Example: } f(x) = \frac{4x+2}{x^2+4x-5}$$

In this case, the end behavior is $f(x) \approx \frac{4x}{x^2} = \frac{4}{x}$. This tells us that, as the inputs increase or decrease without bound, this function will behave similarly to the function $g(x) = \frac{4}{x}$, and the outputs will approach zero, resulting in a horizontal asymptote at $y=0$. See **Figure 1.3.13** Note that this graph crosses the horizontal asymptote.

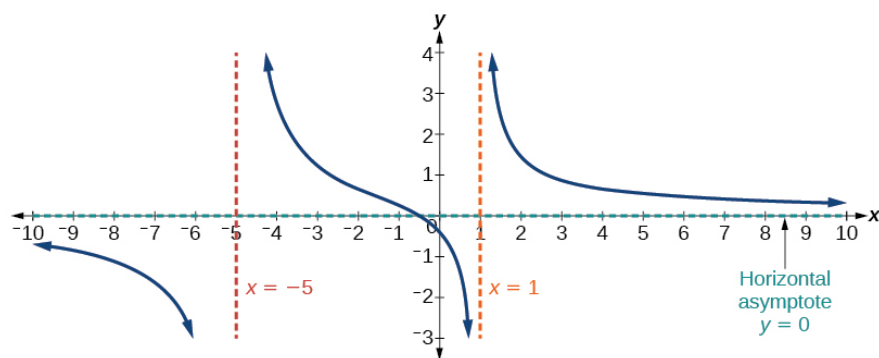


Figure 1.3.13: Horizontal asymptote $y = 0$ when $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$ where degree of $p <$ degree of q .

Case 2: If the degree of the denominator = degree of the numerator, there is a horizontal asymptote at $y = \frac{a_n}{b_n}$, where a_n and b_n are respectively the leading coefficients of $p(x)$ and $q(x)$ for $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$.

$$\text{Example: } f(x) = \frac{3x^2 + 2}{x^2 + 4x - 5}$$

In this case, the end behavior is $f(x) \approx \frac{3x^2}{x^2} = 3$. This tells us that as the inputs grow large, this function will behave like the function $g(x) = 3$, which is a horizontal line. As $x \rightarrow \pm\infty$, $f(x) \rightarrow 3$, resulting in a horizontal asymptote at $y = 3$. See Figure 1.3.15. Note that this graph crosses the horizontal asymptote.

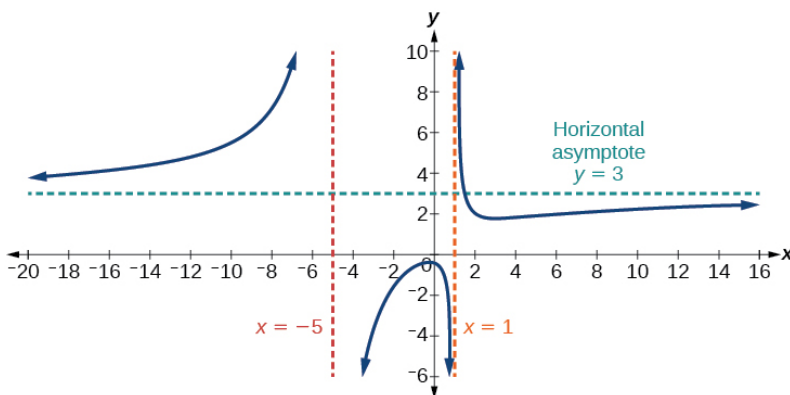


Figure 1.3.15: Horizontal asymptote when $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$ where degree of $p =$ degree of q .

Notice that, while the graph of a rational function will never cross a **vertical asymptote**, the graph may or may not cross a horizontal or slant asymptote. Also, although the graph of a rational function may have many vertical asymptotes, the graph will have at most one horizontal (or slant) asymptote.

It should be noted that, if the degree of the numerator is larger than the degree of the denominator by more than one, the **end behavior** of the graph will mimic the behavior of the reduced end behavior fraction. For instance, if we had the function

$$f(x) = \frac{3x^5 - x^2}{x + 3} \quad (1.3.19)$$

with end behavior

$$f(x) \approx \frac{3x^5}{x} = 3x^4 \quad (1.3.20)$$

the end behavior of the graph would look similar to that of an even polynomial with a positive leading coefficient.

$$x \rightarrow \pm\infty, f(x) \rightarrow \infty$$

HORIZONTAL ASYMPTOTES OF RATIONAL FUNCTIONS

The **horizontal asymptote** of a rational function can be determined by looking at the degrees of the numerator and denominator.

- Degree of numerator is less than degree of denominator: horizontal asymptote at $y = 0$.
- Degree of numerator is greater than degree of denominator by one: no horizontal asymptote; slant asymptote.
- Degree of numerator is equal to degree of denominator: horizontal asymptote at ratio of leading coefficients.

Example 1.3.7: Identifying Horizontal and Slant Asymptotes

For the functions listed, identify the horizontal or slant asymptote.

1. $g(x) = \frac{6x^3 - 10x}{2x^3 + 5x^2}$
2. $h(x) = \frac{x^2 - 4x + 1}{x + 2}$
3. $k(x) = \frac{x^2 + 4x}{x^3 - 8}$

Solution

For these solutions, we will use $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$.

1. $g(x) = \frac{6x^3 - 10x}{2x^3 + 5x^2}$: The degree of p = degree of $q = 3$, so we can find the horizontal asymptote by taking the ratio of the leading terms. There is a horizontal asymptote at $y = \frac{6}{2}$ or $y = 3$.

2. $h(x) = \frac{x^2 - 4x + 1}{x + 2}$: The degree of $p = 2$ and degree of $q = 1$. Since $p > q$ by 1, there is a slant asymptote found at $\frac{x^2 - 4x + 1}{x + 2}$.

$$\begin{array}{r|rr} -2 & 1 & -4 & 1 \\ & -2 & 12 & \\ \hline & 1 & -6 & 13 \end{array}$$

3. $k(x) = \frac{x^2 + 4x}{x^3 - 8}$: The degree of $p = 2 <$ degree of $q = 3$, so there is a horizontal asymptote $y = 0$.

Example 1.3.8 Identifying Horizontal Asymptotes

In the sugar concentration problem earlier, we created the equation $C(t) = \frac{5+t}{100+10t}$.

Find the horizontal asymptote and interpret it in context of the problem.

Solution

Both the numerator and denominator are linear (degree 1). Because the degrees are equal, there will be a horizontal asymptote at the ratio of the leading coefficients. In the numerator, the leading term is t , with coefficient 1. In the denominator, the leading term is $10t$, with coefficient 10. The horizontal asymptote will be at the ratio of these values:

$$t \rightarrow \infty, C(t) \rightarrow \frac{1}{10}$$

This function will have a horizontal asymptote at $y = \frac{1}{10}$.

This tells us that as the values of t increase, the values of C will approach $\frac{1}{10}$. In context, this means that, as more time goes by, the concentration of sugar in the tank will approach one-tenth of a pound of sugar per gallon of water or $\frac{1}{10}$ pounds per gallon.

Example 1.3.9: Identifying Horizontal and Vertical Asymptotes

Find the horizontal and vertical asymptotes of the function

$$f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$$

Solution

First, note that this function has no common factors, so there are no potential removable discontinuities.

The function will have vertical asymptotes when the denominator is zero, causing the function to be undefined. The denominator will be zero at $x = 1, -2$, and 5 , indicating vertical asymptotes at these values.

The numerator has degree 2, while the denominator has degree 3. Since the degree of the denominator is greater than the degree of the numerator, the denominator will grow faster than the numerator, causing the outputs to tend towards zero as the inputs get large, and so as $x \rightarrow \pm\infty$, $f(x) \rightarrow 0$. This function will have a horizontal asymptote at $y = 0$. See Figure 1.3.16.

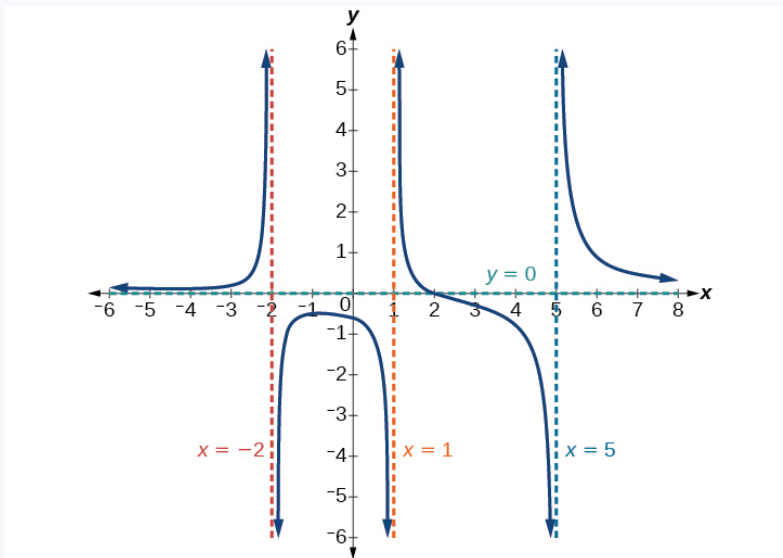


Figure 1.3.16.

Exercise 1.3.6

Find the vertical and horizontal asymptotes of the function:

$$f(x) = \frac{(2x - 1)(2x + 1)}{(x - 2)(x + 3)}$$

Answer

Vertical asymptotes at $x = 2$ and $x = -3$

horizontal asymptote at $y = 4$.

Contributor

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1.4: Logarithmic Properties

Learning Objectives

- Use the product rule for logarithms.
- Use the quotient rule for logarithms.
- Use the power rule for logarithms.
- Expand logarithmic expressions.
- Condense logarithmic expressions.
- Use the change-of-base formula for logarithms.

Using the Product Rule for Logarithms

Recall that the logarithmic and exponential functions “undo” each other. This means that logarithms have similar properties to exponents. Some important properties of logarithms are given here. First, the following properties are easy to prove.

$$\log_b 1 = 0$$

$$\log_b b = 1$$

For example, $\log_5 1 = 0$ since $5^0 = 1$. And $\log_5 5 = 1$ since $5^1 = 5$.

Next, we have the inverse property.

$$\log_b(b^x) = x$$

$$b^{\log_b x} = x, x > 0$$

For example, to evaluate $\log(100)$, we can rewrite the logarithm as $\log_{10}(10^2)$, and then apply the inverse property $\log_b(b^x) = x$ to get $\log_{10}(10^2) = 2$.

To evaluate $e^{\ln(7)}$, we can rewrite the logarithm as $e^{\log_e 7}$, and then apply the inverse property $b^{\log_b x} = x$ to get $e^{\log_e 7} = 7$.

Finally, we have the one-to-one property.

$$\log_b M = \log_b N \text{ if and only if } M = N \quad (1.4.1)$$

We can use the one-to-one property to solve the equation $\log_3(3x) = \log_3(2x + 5)$ for x . Since the bases are the same, we can apply the one-to-one property by setting the arguments equal and solving for x :

$$3x = 2x + 5 \text{ Set the arguments equal.}$$

$$x = 5 \text{ Subtract } 2x.$$

But what about the equation $\log_3(3x) + \log_3(2x + 5) = 2$? The one-to-one property does not help us in this instance. Before we can solve an equation like this, we need a method for combining terms on the left side of the equation.

Recall that we use the *product rule of exponents* to combine the product of exponents by adding: $x^a x^b = x^{a+b}$. We have a similar property for logarithms, called the **product rule for logarithms**, which says that the logarithm of a product is equal to a sum of logarithms. Because logs are exponents, and we multiply like bases, we can add the exponents. We will use the inverse property to derive the product rule below.

Given any real number x and positive real numbers M , N , and b , where $b \neq 1$, we will show

$$\log_b(MN) = \log_b(M) + \log_b(N).$$

Let $m = \log_b M$ and $n = \log_b N$. In exponential form, these equations are $b^m = M$ and $b^n = N$. It follows that

$$\begin{aligned} \log_b(MN) &= \log_b(b^m b^n) && \text{Substitute for M and N} \\ &= \log_b(b^{m+n}) && \text{Apply the product rule for exponents} \\ &= m + n && \text{Apply the inverse property of logs} \\ &= \log_b(M) + \log_b(N) && \text{Substitute for m and n} \end{aligned}$$

Note that repeated applications of the product rule for logarithms allow us to simplify the logarithm of the product of any number of factors. For example, consider $\log_b(wxyz)$. Using the product rule for logarithms, we can rewrite this logarithm of a product as the sum of logarithms of its factors:

$$\log_b(wxyz) = \log_b w + \log_b x + \log_b y + \log_b z$$

THE PRODUCT RULE FOR LOGARITHMS

The product rule for logarithms can be used to simplify a logarithm of a product by rewriting it as a sum of individual logarithms.

$$\log_b(MN) = \log_b(M) + \log_b(N) \text{ for } b > 0 \quad (1.4.2)$$

How to: Given the logarithm of a product, use the product rule of logarithms to write an equivalent sum of logarithms

1. Factor the argument completely, expressing each whole number factor as a product of primes.
2. Write the equivalent expression by summing the logarithms of each factor.

Example 1.4.1: Using the Product Rule for Logarithms

Expand $\log_3(30x(3x+4))$.

Solution

We begin by factoring the argument completely, expressing 30 as a product of primes.

$$\log_3(30x(3x+4)) = \log_3(2 \cdot 3 \cdot 5 \cdot x \cdot (3x+4))$$

Next we write the equivalent equation by summing the logarithms of each factor.

$$\log_3(30x(3x+4)) = \log_3(2) + \log_3(3) + \log_3(5) + \log_3(x) + \log_3(3x+4)$$

Exercise 1.4.1

Expand $\log_b(8k)$.

Answer

$$\log_b 2 + \log_b 2 + \log_b 2 + \log_b k = 3\log_b 2 + \log_b k$$

Using the Quotient Rule for Logarithms

For quotients, we have a similar rule for logarithms. Recall that we use the *quotient rule of exponents* to combine the quotient of exponents by subtracting: $x^a = x^{a-b}$. The **quotient rule for logarithms** says that the logarithm of a quotient is equal to a difference of logarithms.

THE QUOTIENT RULE FOR LOGARITHMS

The quotient rule for logarithms can be used to simplify a logarithm of a quotient by rewriting it as the difference of individual logarithms.

$$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N \quad (1.4.3)$$

Just as with the product rule, we can use the inverse property to derive the quotient rule.

Proof

Given any real number x and positive real numbers M , N , and b , where $b \neq 1$, we will show

$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N).$$

Let $m = \log_b M$ and $n = \log_b N$. In exponential form, these equations are $b^m = M$ and $b^n = N$. It follows that

$$\begin{aligned}
 \log_b \left(\frac{M}{N} \right) &= \log_b \left(\frac{b^m}{b^n} \right) && \text{Substitute for M and N} \\
 &= \log_b (b^{m-n}) && \text{Apply the quotient rule for exponents} \\
 &= m - n && \text{Apply the inverse property of logs} \\
 &= \log_b (M) - \log_b (N) && \text{Substitute for m and n}
 \end{aligned}$$

For example, to expand $\log \left(\frac{2x^2 + 6x}{3x + 9} \right)$, we must first express the quotient in lowest terms. Factoring and canceling we get,

$$\begin{aligned}
 \log \left(\frac{2x^2 + 6x}{3x + 9} \right) &= \log \left(\frac{2x(x + 3)}{3(x + 3)} \right) && \text{Factor the numerator and denominator} \\
 &= \log \left(\frac{2x}{3} \right) && \text{Cancel the common factors}
 \end{aligned}$$

Next we apply the quotient rule by subtracting the logarithm of the denominator from the logarithm of the numerator. Then we apply the product rule.

$$\begin{aligned}
 \log \left(\frac{2x}{3} \right) &= \log(2x) - \log(3) \\
 &= \log(2) + \log(x) - \log(3)
 \end{aligned}$$

How to: Given the logarithm of a quotient, use the quotient rule of logarithms to write an equivalent difference of logarithms

1. Express the argument in lowest terms by factoring the numerator and denominator and canceling common terms.
2. Write the equivalent expression by subtracting the logarithm of the denominator from the logarithm of the numerator.
3. Check to see that each term is fully expanded. If not, apply the product rule for logarithms to expand completely.

Example 1.4.2: Using the Quotient Rule for Logarithms

Expand $\log_2 \left(\frac{15x(x - 1)}{(3x + 4)(2 - x)} \right)$.

Solution

First we note that the quotient is factored and in lowest terms, so we apply the quotient rule.

$$\log_2 \left(\frac{15x(x - 1)}{(3x + 4)(2 - x)} \right) = \log_2(15x(x - 1)) - \log_2((3x + 4)(2 - x))$$

Notice that the resulting terms are logarithms of products. To expand completely, we apply the product rule, noting that the prime factors of the factor 15 are 3 and 5.

$$\begin{aligned}
 \log_2(15x(x - 1)) - \log_2((3x + 4)(2 - x)) &= [\log_2(3) + \log_2(5) + \log_2(x) + \log_2(x - 1)] - [\log_2(3x + 4) + \log_2(2 - x)] \\
 &= \log_2(3) + \log_2(5) + \log_2(x) + \log_2(x - 1) - \log_2(3x + 4) - \log_2(2 - x)
 \end{aligned}$$

Analysis

There are exceptions to consider in this and later examples. First, because denominators must never be zero, this expression is not defined for $x = -\frac{4}{3}$ and $x = 2$. Also, since the argument of a logarithm must be positive, we note as we observe the expanded logarithm, that $x > 0$, $x > 1$, $x > -\frac{4}{3}$, and $x < 2$. Combining these conditions is beyond the scope of this section, and we will not consider them here or in subsequent exercises.

Exercise 1.4.2

Expand $\log_3 \left(\frac{7x^2 + 21x}{7x(x - 1)(x - 2)} \right)$.

Answer

$$\log_3(x+3) - \log_3(x-1) - \log_3(x-2)$$

Using the Power Rule for Logarithms

We've explored the product rule and the quotient rule, but how can we take the logarithm of a power, such as x^2 ? One method is as follows:

$$\begin{aligned}\log_b(x^2) &= \log_b(x \cdot x) \\ &= \log_b x + \log_b x \\ &= 2\log_b x\end{aligned}$$

Notice that we used the **product rule for logarithms** to find a solution for the example above. By doing so, we have derived the **power rule for logarithms**, which says that the log of a power is equal to the exponent times the log of the base. Keep in mind that, although the input to a logarithm may not be written as a power, we may be able to change it to a power. For example,

$$100 = 10^2 \quad \sqrt{3} = 3^{\frac{1}{2}} \quad \frac{1}{e} = e^{-1}$$

THE POWER RULE FOR LOGARITHMS

The power rule for logarithms can be used to simplify the logarithm of a power by rewriting it as the product of the exponent times the logarithm of the base.

$$\log_b(M^n) = n\log_b M \quad (1.4.4)$$

How to: Given the logarithm of a power, use the power rule of logarithms to write an equivalent product of a factor and a logarithm

1. Express the argument as a power, if needed.
2. Write the equivalent expression by multiplying the exponent times the logarithm of the base.

Example 1.4.3: Expanding a Logarithm with Powers

Expand $\log_2 x^5$.

Solution

The argument is already written as a power, so we identify the exponent, 5, and the base, x , and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.

$$\log_2(x^5) = 5\log_2 x$$

Exercise 1.4.3

Expand $\ln x^2$.

Answer

$$2 \ln x$$

Example 1.4.4: Rewriting an Expression as a Power before Using the Power Rule

Expand $\log_3(25)$ using the power rule for logs.

Solution

Expressing the argument as a power, we get $\log_3(25) = \log_3(5^2)$.

Next we identify the exponent, 2, and the base, 5, and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.

$$\log_3(52) = 2\log_3(5)$$

Exercise 1.4.4

Expand $\ln\left(\frac{1}{x^2}\right)$.

Answer

$$-2 \ln(x)$$

Example 1.4.5: Using the Power Rule in Reverse

Rewrite $4 \ln(x)$ using the power rule for logs to a single logarithm with a leading coefficient of 1.

Solution

Because the logarithm of a power is the product of the exponent times the logarithm of the base, it follows that the product of a number and a logarithm can be written as a power. For the expression $4 \ln(x)$, we identify the factor, 4, as the exponent and the argument, x , as the base, and rewrite the product as a logarithm of a power: $4 \ln(x) = \ln(x^4)$.

Exercise 1.4.5

Rewrite $2 \log_3 4$ using the power rule for logs to a single logarithm with a leading coefficient of 1.

Answer

$$\log_3 16$$

Expanding Logarithmic Expressions

Taken together, the product rule, quotient rule, and power rule are often called “laws of logs.” Sometimes we apply more than one rule in order to simplify an expression. For example:

$$\begin{aligned}\log_b\left(\frac{6x}{y}\right) &= \log_b(6x) - \log_b y \\ &= \log_b 6 + \log_b x - \log_b y\end{aligned}$$

We can use the power rule to expand logarithmic expressions involving negative and fractional exponents. Here is an alternate proof of the quotient rule for logarithms using the fact that a reciprocal is a negative power:

$$\begin{aligned}\log_b\left(\frac{A}{C}\right) &= \log_b(AC^{-1}) \\ &= \log_b(A) + \log_b(C^{-1}) \\ &= \log_b A + (-1)\log_b C \\ &= \log_b A - \log_b C\end{aligned}$$

We can also apply the product rule to express a sum or difference of logarithms as the logarithm of a product.

With practice, we can look at a logarithmic expression and expand it mentally, writing the final answer. Remember, however, that we can only do this with products, quotients, powers, and roots—never with addition or subtraction inside the argument of the logarithm.

Example 1.4.6: Expanding Logarithms Using Product, Quotient, and Power Rules

Rewrite $\ln\left(\frac{x^4 y}{7}\right)$ as a sum or difference of logs.

Solution

First, because we have a quotient of two expressions, we can use the quotient rule:

$$\ln\left(\frac{x^4y}{7}\right) = \ln(x^4y) - \ln(7)$$

Then seeing the product in the first term, we use the product rule:

$$\ln(x^4y) - \ln(7) = \ln(x^4) + \ln(y) - \ln(7)$$

Finally, we use the power rule on the first term:

$$\ln(x^4) + \ln(y) - \ln(7) = 4\ln(x) + \ln(y) - \ln(7)$$

Exercise 1.4.6

Expand $\log\left(\frac{x^2y^3}{z^4}\right)$.

Answer

$$2\log x + 3\log y - 4\log z$$

Example 1.4.7: Using the Power Rule for Logarithms to Simplify the Logarithm of a Radical Expression

Expand $\log(x)$.

Solution

$$\begin{aligned}\log(\sqrt{x}) &= \log x^{\left(\frac{1}{2}\right)} \\ &= \frac{1}{2}\log x\end{aligned}$$

Exercise 1.4.7

Expand $\ln(\sqrt[3]{x^2})$.

Answer

$$\frac{2}{3}\ln x$$

Q&A: Can we expand $\ln(x^2 + y^2)$?

No. There is no way to expand the logarithm of a sum or difference inside the argument of the logarithm.

Example 1.4.8: Expanding Complex Logarithmic Expressions

Expand $\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right)$.

Solution

We can expand by applying the Product and Quotient Rules.

$$\begin{aligned}\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right) &= \log_6 64 + \log_6 x^3 + \log_6(4x+1) - \log_6(2x-1) && \text{Apply the Quotient Rule} \\ &= \log_6 26 + \log_6 x^3 + \log_6(4x+1) - \log_6(2x-1) && \text{Simplify by writing 64 as } 2^6 \\ &= 6\log_6 2 + 3\log_6 x + \log_6(4x+1) - \log_6(2x-1) && \text{Apply the Power Rule}\end{aligned}$$

Exercise 1.4.8

Expand $\ln \left(\frac{\sqrt{(x-1)(2x+1)^2}}{(x^2-9)} \right)$.

Answer

$$\frac{1}{2} \ln(x-1) + \ln(2x+1) - \ln(x+3) - \ln(x-3)$$

Condensing Logarithmic Expressions

We can use the rules of logarithms we just learned to condense sums, differences, and products with the same base as a single logarithm. It is important to remember that the logarithms must have the same base to be combined. We will learn later how to change the base of any logarithm before condensing.

How to: Given a sum, difference, or product of logarithms with the same base, write an equivalent expression as a single logarithm

1. Apply the power property first. Identify terms that are products of factors and a logarithm, and rewrite each as the logarithm of a power.
2. Next apply the product property. Rewrite sums of logarithms as the logarithm of a product.
3. Apply the quotient property last. Rewrite differences of logarithms as the logarithm of a quotient.

Example 1.4.9: Using the Product and Quotient Rules to Combine Logarithms

Write $\log_3(5) + \log_3(8) - \log_3(2)$ as a single logarithm.

Solution

Using the product and quotient rules

$$\log_3(5) + \log_3(8) = \log_3(5 \cdot 8) = \log_3(40)$$

This reduces our original expression to

$$\log_3(40) - \log_3(2)$$

Then, using the quotient rule

$$\log_3(40) - \log_3(2) = \log_3\left(\frac{40}{2}\right) = \log_3(20)$$

Exercise 1.4.9

Condense $\log_3 - \log_4 + \log_5 - \log_6$.

Answer

$$\log\left(\frac{3 \cdot 5}{4 \cdot 6}\right); \text{ can also be written } \log\left(\frac{5}{8}\right) \text{ by reducing the fraction to lowest terms.}$$

Example 1.4.10: Condensing Complex Logarithmic Expressions

Condense $\log_2(x^2) + \frac{1}{2}\log_2(x-1) - 3\log_2((x+3)^2)$.

Solution

We apply the power rule first:

$$\log_2(x^2) + \frac{1}{2}\log_2(x-1) - 3\log_2((x+3)^2) = \log_2(x^2) + \log_2(\sqrt{x-1}) - \log_2((x+3)^6)$$

Next we apply the product rule to the sum:

$$\log_2(x^2) + \log_2(\sqrt{x-1}) - \log_2((x+3)^6) = \log_2(x^2\sqrt{x-1}) - \log_2((x+3)^6)$$

Finally, we apply the quotient rule to the difference:

$$\log_2(x^2\sqrt{x-1}) - \log_2((x+3)^6) = \log_2 \frac{x^2\sqrt{x-1}}{(x+3)^6}$$

Example 1.4.11: Rewriting as a Single Logarithm

Rewrite $2 \log x - 4 \log(x+5) + \frac{1}{x} \log(3x+5)$ as a single logarithm.

Solution

We apply the power rule first:

$$2 \log x - 4 \log(x+5) + \frac{1}{x} \log(3x+5) = \log(x^2) - \log(x+5)^4 + \log((3x+5)^{x^{-1}})$$

Next we rearrange and apply the product rule to the sum:

$$\begin{aligned} \log(x^2) - \log(x+5)^4 + \log((3x+5)^{x^{-1}}) &= \log(x^2) + \log((3x+5)^{x^{-1}}) - \log(x+5)^4 \\ &= \log(x^2(3x+5)^{x^{-1}}) - \log(x+5)^4 \\ &= \log \frac{x^2(3x+5)^{x^{-1}}}{(x+5)^4} \quad \text{Apply the quotient rule to the difference} \end{aligned}$$

Exercise 1.4.10

Rewrite $\log(5) + 0.5 \log(x) - \log(7x-1) + 3 \log(x-1)$ as a single logarithm.

Answer

$$\log \frac{5(x-1)^3 \sqrt{x}}{(7x-1)}$$

Exercise 1.4.11

Condense $4(3 \log(x) + \log(x+5) - \log(2x+3))$.

Answer

$$\log \frac{x^{12}(x+5)^4}{(2x+3)^4}; \text{ this answer could also be written } \log \left(\frac{x^3(x+5)}{(2x+3)} \right)^4$$

Example 1.4.12: Applying of the Laws of Logs

Recall that, in chemistry, $pH = -\log[H^+]$. If the concentration of hydrogen ions in a liquid is doubled, what is the effect on pH?

Solution

Suppose C is the original concentration of hydrogen ions, and P is the original pH of the liquid. Then $P = -\log(C)$. If the concentration is doubled, the new concentration is $2C$. Then the pH of the new liquid is

$$pH = -\log(2C)$$

Using the product rule of logs

$$pH = -\log(2C) = -(\log(2) + \log(C)) = -\log(2) - \log(C)$$

Since $P = -\log(C)$, the new pH is

$$pH = P - \log(2) \approx P - 0.301$$

Exercise 1.4.12

When the concentration of hydrogen ions is doubled, the pH decreases by about 0.301.

How does the pH change when the concentration of positive hydrogen ions is decreased by half?

Answer

The pH increases by about 0.301.

Using the Change-of-Base Formula for Logarithms

Most calculators can evaluate only common and natural logs. In order to evaluate logarithms with a base other than 10 or e , we use the **change-of-base formula** to rewrite the logarithm as the quotient of logarithms of any other base; when using a calculator, we would change them to common or natural logs.

To derive the change-of-base formula, we use the **one-to-one** property and **power rule for logarithms**.

Given any positive real numbers M , b , and n , where $n \neq 1$ and $b \neq 1$, we show

$$\log_b M = \frac{\log_n M}{\log_n b}$$

Let $y = \log_b M$. By taking the log base n of both sides of the equation, we arrive at an exponential form, namely $b^y = M$. It follows that

$$\begin{aligned} \log_n (b^y) &= \log_n M && \text{Apply the one-to-one property} \\ y \log_n b &= \log_n M && \text{Apply the power rule for logarithms} \\ y &= \frac{\log_n M}{\log_n b} && \text{Isolate } y \\ \log_b M &= \frac{\log_n M}{\log_n b} && \text{Substitute for } y \end{aligned}$$

For example, to evaluate $\log_5 36$ using a calculator, we must first rewrite the expression as a quotient of common or natural logs. We will use the common log.

$$\begin{aligned} \log_5 36 &= \frac{\log(36)}{\log(5)} && \text{Apply the change of base formula using base 10} \\ &\approx 2.2266 && \text{Use a calculator to evaluate to 4 decimal places} \end{aligned}$$

THE CHANGE-OF-BASE FORMULA

The change-of-base formula can be used to evaluate a logarithm with any base.

For any positive real numbers M , b , and n , where $n \neq 1$ and $b \neq 1$,

$$\log_b M = \frac{\log_n M}{\log_n b} \quad (1.4.5)$$

It follows that the change-of-base formula can be used to rewrite a logarithm with any base as the quotient of common or natural logs.

$$\log_b M = \frac{\ln M}{\ln b} \quad (1.4.6)$$

and

$$\log_b M = \frac{\log M}{\log b} \quad (1.4.7)$$

How to: Given a logarithm with the form $\log_b M$, use the change-of-base formula to rewrite it as a quotient of logs with any positive base n , where $n \neq 1$

1. Determine the new base n , remembering that the common log, $\log(x)$, has base 10, and the natural log, $\ln(x)$, has base e .
2. Rewrite the log as a quotient using the change-of-base formula
 - The numerator of the quotient will be a logarithm with base n and argument M .
 - The denominator of the quotient will be a logarithm with base n and argument b .

Example 1.4.13: Changing Logarithmic Expressions to Expressions Involving Only Natural Logs

Change $\log_5 3$ to a quotient of natural logarithms.

Solution

Because we will be expressing $\log_5 3$ as a quotient of natural logarithms, the new base, $n = e$.

We rewrite the log as a quotient using the change-of-base formula. The numerator of the quotient will be the natural log with argument 3. The denominator of the quotient will be the natural log with argument 5.

$$\log_b M = \frac{\ln M}{\ln b}$$
$$\log_5 3 = \frac{\ln 3}{\ln 5}$$

Exercise 1.4.13

Change $\log 0.58$ to a quotient of natural logarithms.

Answer

$$\frac{\ln 8}{\ln 0.5}$$

Q&A: Can we change common logarithms to natural logarithms?

Yes. Remember that $\log 9$ means $\log_{10} 9$. So, $\log 9 = \frac{\ln 9}{\ln 10}$.

Example 1.4.14: Using the Change-of-Base Formula with a Calculator

Evaluate $\log_2(10)$ using the change-of-base formula with a calculator.

Solution

According to the change-of-base formula, we can rewrite the log base 2 as a logarithm of any other base. Since our calculators can evaluate the natural log, we might choose to use the natural logarithm, which is the log base e .

$$\log_2 10 = \frac{\ln 10}{\ln 2} \quad \text{Apply the change of base formula using base } e$$
$$\approx 3.3219 \quad \text{Use a calculator to evaluate to 4 decimal places}$$

Exercise 1.4.14

Evaluate $\log_5(100)$ using the change-of-base formula.

Answer

$$\frac{\ln 100}{\ln 5} \approx \frac{4.6051}{1.6094} = 2.861$$

Media

Access these online resources for additional instruction and practice with laws of logarithms.

- [The Properties of Logarithms](#)
- [Expand Logarithmic Expressions](#)
- [Evaluate a Natural Logarithmic Expression](#)

Key Equations

The Product Rule for Logarithms	$\log_b(MN) = \log_b(M) + \log_b(N)$
The Quotient Rule for Logarithms	$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$
The Power Rule for Logarithms	$\log_b(M^n) = n\log_b M$
The Change-of-Base Formula	$\log_b M = \frac{\log_n M}{\log_n b} \quad n > 0, n \neq 1, b \neq 1$

Contributors

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CHAPTER OVERVIEW

2: Trigonometry

- 2.1: Unit Circle - Sine and Cosine Functions
- 2.2: The Other Trigonometric Functions
- 2.3: Right Triangle Trigonometry
- 2.4: Solving Trigonometric Equations with Identities
- 2.5: Sum and Difference Identities
- 2.6: Double-Angle, Half-Angle, and Reduction Formulas

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2.1: Unit Circle - Sine and Cosine Functions

Learning Objectives

- Find function values for the sine and cosine of 30° or $(\frac{\pi}{6})$, 45° or $(\frac{\pi}{4})$, and 60° or $(\frac{\pi}{3})$.
- Identify the domain and range of sine and cosine functions.
- Find reference angles.
- Use reference angles to evaluate trigonometric functions.

Looking for a thrill? Then consider a ride on the Singapore Flyer, the world's tallest Ferris wheel. Located in Singapore, the Ferris wheel soars to a height of 541 feet—a little more than a tenth of a mile! Described as an observation wheel, riders enjoy spectacular views as they travel from the ground to the peak and down again in a repeating pattern. In this section, we will examine this type of revolving motion around a circle. To do so, we need to define the type of circle first, and then place that circle on a coordinate system. Then we can discuss circular motion in terms of the coordinate pairs.



Figure 2.1.1: The Singapore Flyer is the world's tallest Ferris wheel. (credit: "Vibin JK"/Flickr)

Finding Function Values for the Sine and Cosine

To define our trigonometric functions, we begin by drawing a unit circle, a circle centered at the origin with radius 1, as shown in Figure 2.1.2. The angle (in radians) that t intercepts forms an arc of length s . Using the formula $s = rt$, and knowing that $r = 1$, we see that for a **unit circle**, $s = t$.

Recall that the x - and y -axes divide the coordinate plane into four quarters called quadrants. We label these quadrants to mimic the direction a positive angle would sweep. The four quadrants are labeled I, II, III, and IV.

For any angle t , we can label the intersection of the terminal side and the unit circle as by its coordinates, (x, y) . The coordinates x and y will be the outputs of the trigonometric functions $f(t) = \cos t$ and $f(t) = \sin t$, respectively. This means $x = \cos t$ and $y = \sin t$.

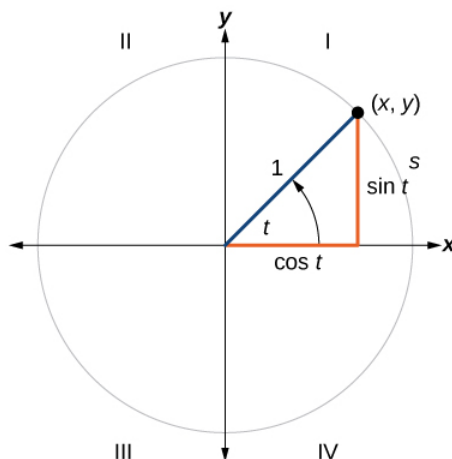


Figure 2.1.2: Unit circle where the central angle is t radians

UNIT CIRCLE

A unit circle has a center at $(0, 0)$ and radius 1. The length of the intercepted arc is equal to the radian measure of the central angle t .

Let (x, y) be the endpoint on the unit circle of an arc of arc length s . The (x, y) coordinates of this point can be described as functions of the angle.

Defining Sine and Cosine Functions

Now that we have our unit circle labeled, we can learn how the (x, y) coordinates relate to the **arc length** and **angle**. The **sine function** relates a real number t to the y -coordinate of the point where the corresponding angle intercepts the unit circle. More precisely, the sine of an angle t equals the y -value of the endpoint on the unit circle of an arc of length t . In Figure 2.1.3, the sine is equal to y . Like all functions, the sine function has an input and an output. Its input is the measure of the angle; its output is the y -coordinate of the corresponding point on the unit circle.

The **cosine function** of an angle t equals the x -value of the endpoint on the unit circle of an arc of length t . In Figure 2.1.1, the cosine is equal to x .

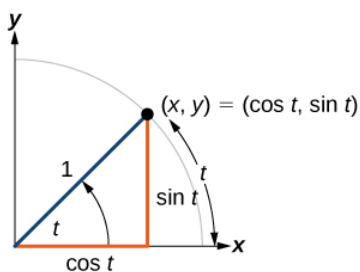


Figure 2.1.3

Because it is understood that sine and cosine are functions, we do not always need to write them with parentheses: $\sin t$ is the same as $\sin(t)$ and $\cos t$ is the same as $\cos(t)$. Likewise, $\cos^2 t$ is a commonly used shorthand notation for $(\cos(t))^2$. Be aware that many calculators and computers do not recognize the shorthand notation. When in doubt, use the extra parentheses when entering calculations into a calculator or computer.

SINE AND COSINE FUNCTIONS

If t is a real number and a point (x, y) on the unit circle corresponds to an angle of t , then

$$\cos t = x \quad (2.1.1)$$

$$\sin t = y \quad (2.1.2)$$

How To: Given a point $P(x, y)$ on the unit circle corresponding to an angle of t , find the sine and cosine

1. The sine of t is equal to the y -coordinate of point P : $\sin t = y$.
2. The cosine of t is equal to the x -coordinate of point P : $\cos t = x$.

Example 2.1.1: Finding Function Values for Sine and Cosine

Point P is a point on the unit circle corresponding to an angle of t , as shown in Figure 2.1.4. Find $\cos(t)$ and $\sin(t)$.

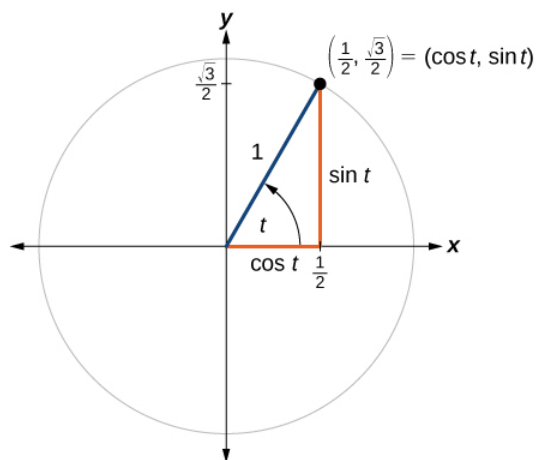


Figure 2.1.4

Solution

We know that $\cos t$ is the x -coordinate of the corresponding point on the unit circle and $\sin t$ is the y -coordinate of the corresponding point on the unit circle. So:

$$x = \cos t = \frac{1}{2}$$

$$y = \sin t = \frac{\sqrt{3}}{2}$$

Try It 2.1.1

A certain angle t corresponds to a point on the unit circle at $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ as shown in Figure 2.1.5. Find $\cos t$ and $\sin t$.

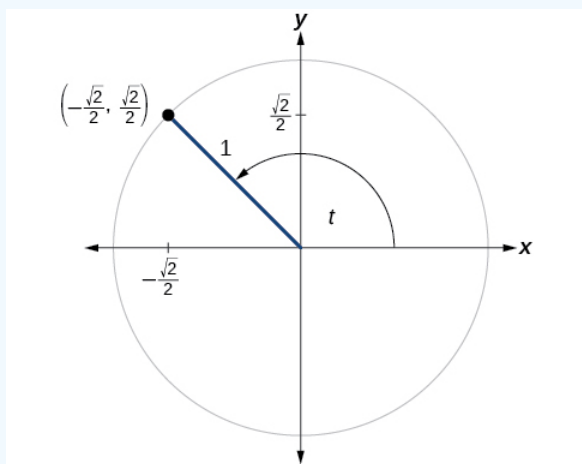


Figure 2.1.5

Answer

$$\cos(t) = -\frac{\sqrt{2}}{2}, \sin(t) = \frac{\sqrt{2}}{2}$$

Finding Sines and Cosines of Angles on an Axis

For quadrantal angles, the corresponding point on the unit circle falls on the x - or y -axis. In that case, we can easily calculate cosine and sine from the values of x and y .

Example 2.1.2: Calculating Sines and Cosines along an Axis

Find $\cos(90^\circ)$ and $\sin(90^\circ)$.

Solution

Moving 90° counterclockwise around the unit circle from the positive x -axis brings us to the top of the circle, where the (x, y) coordinates are $(0, 1)$, as shown in Figure 2.1.6.

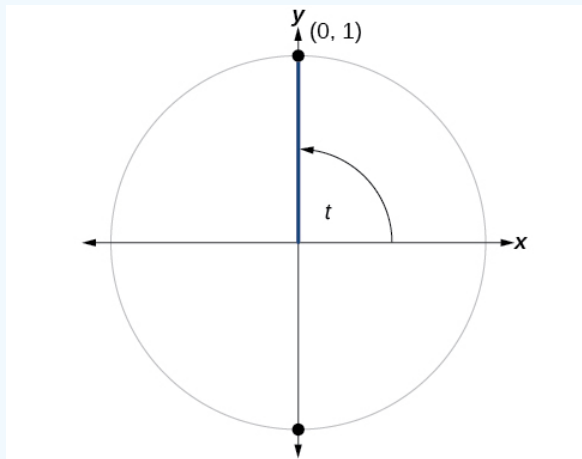


Figure 2.1.6

Using our definitions of cosine and sine,

$$\begin{aligned} x &= \cos t = \cos(90^\circ) = 0 \\ y &= \sin t = \sin(90^\circ) = 1 \end{aligned}$$

The cosine of 90° is 0; the sine of 90° is 1.

Try It 2.1.2

Find cosine and sine of the angle π .

Answer

$$\cos(\pi) = -1, \sin(\pi) = 0$$

The Pythagorean Identity

Now that we can define sine and cosine, we will learn how they relate to each other and the unit circle. Recall that the equation for the unit circle is $x^2 + y^2 = 1$. Because $x = \cos t$ and $y = \sin t$, we can substitute for x and y to get $\cos^2 t + \sin^2 t = 1$. This equation, $\cos^2 t + \sin^2 t = 1$, is known as the **Pythagorean Identity**. See Figure 2.1.7.

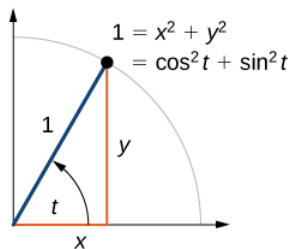


Figure 2.1.7

We can use the Pythagorean Identity to find the cosine of an angle if we know the sine, or vice versa. However, because the equation yields two solutions, we need additional knowledge of the angle to choose the solution with the correct sign. If we know the quadrant where the angle is, we can easily choose the correct solution.

PYTHAGOREAN IDENTITY

The **Pythagorean Identity** states that, for any real number t ,

$$\cos^2 t + \sin^2 t = 1 \quad (2.1.3)$$

How To: Given the sine of some angle t and its quadrant location, find the cosine of t

1. Substitute the known value of $\sin(t)$ into the Pythagorean Identity.
2. Solve for $\cos(t)$.
3. Choose the solution with the appropriate sign for the x -values in the quadrant where t is located.

Example 2.1.3: Finding a Cosine from a Sine or a Sine from a Cosine

If $\sin(t) = \frac{3}{7}$ and t is in the second quadrant, find $\cos(t)$.

Solution

If we drop a vertical line from the point on the unit circle corresponding to t , we create a right triangle, from which we can see that the Pythagorean Identity is simply one case of the Pythagorean Theorem. See Figure 2.1.8.

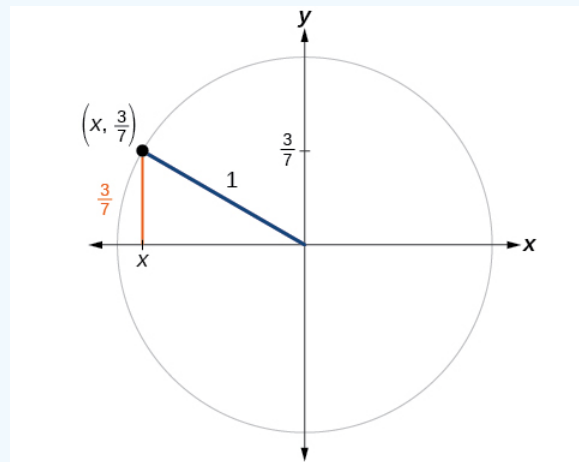


Figure 2.1.8

Substituting the known value for sine into the Pythagorean Identity,

$$\begin{aligned} \cos^2(t) + \sin^2(t) &= 1 \\ \cos^2(t) + \frac{9}{49} &= 1 \\ \cos^2(t) &= \frac{40}{49} \\ \cos(t) &= \pm \sqrt{\frac{40}{49}} = \pm \frac{\sqrt{40}}{7} = \pm \frac{2\sqrt{10}}{7} \end{aligned}$$

Because the angle is in the second quadrant, we know the x -value is a negative real number, so the cosine is also negative. So

$$\cos(t) = -\frac{2\sqrt{10}}{7}$$

Try It 2.1.3

If $\cos(t) = \frac{24}{25}$ and t is in the fourth quadrant, find $\sin(t)$.

Answer

$$\sin(t) = -\frac{7}{25}$$

Finding Sines and Cosines of Special Angles

We have already learned some properties of the special angles, such as the conversion from radians to degrees. We can also calculate sines and cosines of the special angles using the **Pythagorean Identity** and our knowledge of triangles.

Finding Sines and Cosines of 45° Angles

First, we will look at angles of 45° or $\frac{\pi}{4}$, as shown in Figure 2.1.9. A $45^\circ - 45^\circ - 90^\circ$ triangle is an isosceles triangle, so the x - and y -coordinates of the corresponding point on the circle are the same. Because the x - and y -values are the same, the sine and cosine values will also be equal.

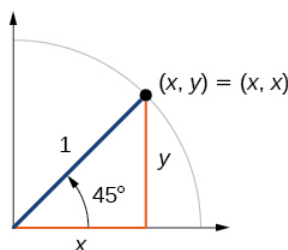


Figure 2.1.9

At $t = \frac{\pi}{4}$, which is 45 degrees, the radius of the unit circle bisects the first **quadrantal angle**. This means the radius lies along the line $y = x$. A unit circle has a radius equal to 1. So, the right triangle formed below the line $y = x$ has sides x and y (with $y = x$), and a radius = 1. See Figure 2.1.10.

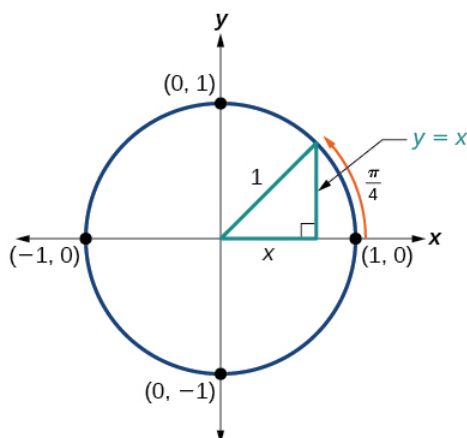


Figure 2.1.10

From the Pythagorean Theorem we get

$$x^2 + y^2 = 1$$

Substituting $y = x$, we get

$$x^2 + x^2 = 1$$

Combining like terms we get

$$2x^2 = 1$$

And solving for x , we get

$$x^2 = \frac{1}{2}$$

$$x = \pm \frac{1}{\sqrt{2}}$$

In quadrant I, $x = \frac{1}{\sqrt{2}}$.

At $t = \frac{\pi}{4}$ or 45 degrees,

$$(x, y) = (x, x) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$x = \frac{1}{\sqrt{2}}, y = \frac{1}{\sqrt{2}}$$

$$\cos t = \frac{1}{\sqrt{2}}, \sin t = \frac{1}{\sqrt{2}}$$

If we then rationalize the denominators, we get

$$\cos t = \frac{1}{\sqrt{2}} \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{2}}{2}$$

$$\sin t = \frac{1}{\sqrt{2}} \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{2}}{2}$$

Therefore, the (x, y) coordinates of a point on a circle of radius 1 at an angle of 45° are $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right)$.

Finding Sines and Cosines of 30° and 60° Angles

Next, we will find the cosine and sine at an angle of 30° , or $\frac{\pi}{6}$. First, we will draw a triangle inside a circle with one side at an angle of 30° , and another at an angle of -30° , as shown in Figure 2.1.11. If the resulting two right triangles are combined into one large triangle, notice that all three angles of this larger triangle will be 60° , as shown in Figure 2.1.12

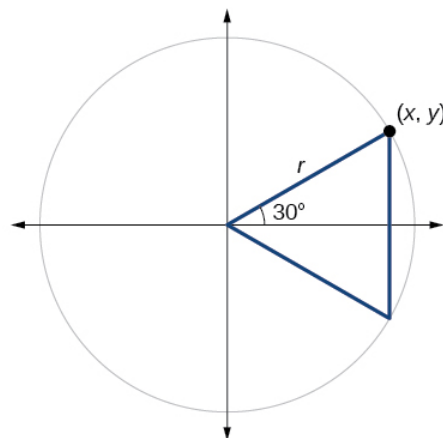


Figure 2.1.11

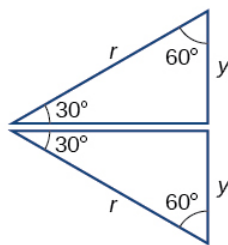


Figure 2.1.12

Because all the angles are equal, the sides are also equal. The vertical line has length $2y$, and since the sides are all equal, we can also conclude that $r = 2y$ or $y = \frac{1}{2}r$. Since $\sin t = y$,

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

And since $r = 1$ in our **unit circle**,

$$\begin{aligned}\sin\left(\frac{\pi}{6}\right) &= \frac{1}{2}(1) \\ &= \frac{1}{2}\end{aligned}$$

Using the Pythagorean Identity, we can find the cosine value.

$$\cos^2\left(\frac{\pi}{6}\right) + \sin^2\left(\frac{\pi}{6}\right) = 1$$

$$\cos^2\left(\frac{\pi}{6}\right) + \left(\frac{1}{2}\right)^2 = 1$$

$$\cos^2\left(\frac{\pi}{6}\right) = \frac{3}{4}$$

$$\cos\left(\frac{\pi}{6}\right) = \pm\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

Use the square root property.

Since y is positive, choose the positive root.

The (x, y) coordinates for the point on a circle of radius 1 at an angle of 30° are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. At $t = \frac{\pi}{3}$ (60°), the radius of the unit circle, 1, serves as the hypotenuse of a 30-60-90 degree right triangle, BAD , as shown in Figure 2.1.13. Angle A has measure 60° . At point B , we draw an angle ABC with measure of 60° . We know the angles in a triangle sum to 180° , so the measure of angle C is also 60° . Now we have an equilateral triangle. Because each side of the equilateral triangle ABC is the same length, and we know one side is the radius of the unit circle, all sides must be of length 1.

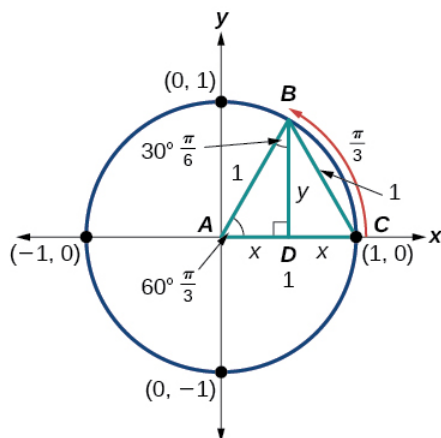


Figure 2.1.13

The measure of angle ABD is 30° . So, if double, angle ABC is 60° . BD is the perpendicular bisector of AC , so it cuts AC in half. This means that AD is $\frac{1}{2}$ the radius, or $\frac{1}{2}$. Notice that AD is the x -coordinate of point B , which is at the intersection of the 60°

angle and the unit circle. This gives us a triangle BAD with hypotenuse of 1 and side x of length $\frac{1}{2}$.

From the Pythagorean Theorem, we get

$$x^2 + y^2 = 1$$

Substituting $x = \frac{1}{2}$, we get

$$\left(\frac{1}{2}\right)^2 + y^2 = 1$$

Solving for y , we get

$$\begin{aligned}\frac{1}{4} + y^2 &= 1 \\ y^2 &= 1 - \frac{1}{4} \\ y^2 &= \frac{3}{4} \\ y &= \pm \frac{\sqrt{3}}{2}\end{aligned}$$

Since $t = \frac{\pi}{3}$ has the terminal side in quadrant I where the y -coordinate is positive, we choose $y = \frac{\sqrt{3}}{2}$, the positive value.

At $t = \frac{\pi}{3}$ (60°), the (x, y) coordinates for the point on a circle of radius 1 at an angle of 60° are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, so we can find the sine and cosine.

$$\begin{aligned}(x, y) &= \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \\ x &= \frac{1}{2}, y = \frac{\sqrt{3}}{2} \\ \cos t &= \frac{1}{2}, \sin t = \frac{\sqrt{3}}{2}\end{aligned}$$

We have now found the cosine and sine values for all of the most commonly encountered angles in the first quadrant of the unit circle. Table 2.1.1 summarizes these values.

Table 2.1.1

Angle	0	$\frac{\pi}{6}$, or 30°	$\frac{\pi}{4}$, or 45°	$\frac{\pi}{3}$, or 60°	$\frac{\pi}{2}$, or 90°
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Figure 2.1.14 shows the common angles in the first quadrant of the unit circle.

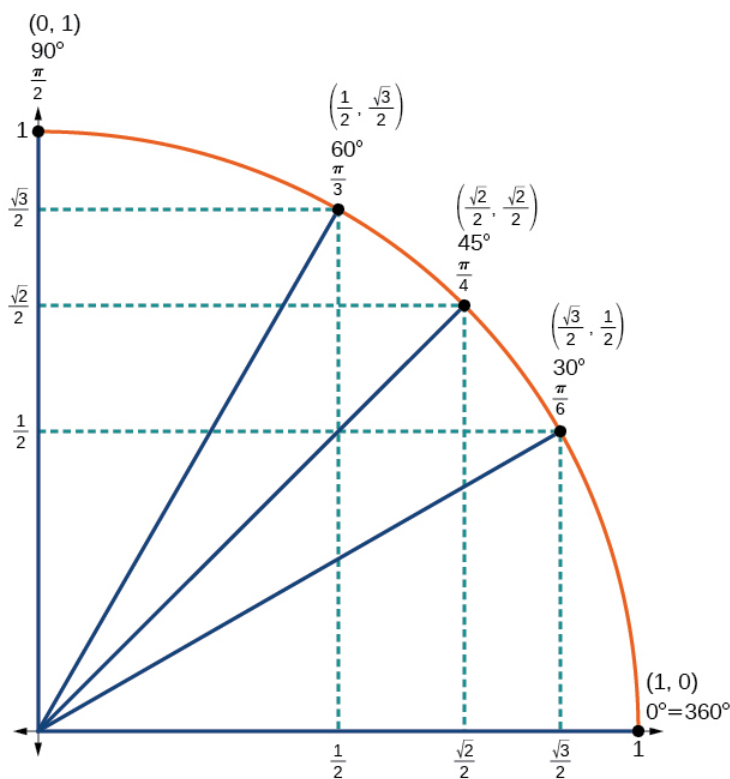


Figure 2.1.14

Using a Calculator to Find Sine and Cosine

To find the cosine and sine of angles other than the **special angles**, we turn to a computer or calculator. **Be aware:** Most calculators can be set into “degree” or “radian” mode, which tells the calculator the units for the input value. When we evaluate $\cos(30)$ on our calculator, it will evaluate it as the cosine of 30 degrees if the calculator is in degree mode, or the cosine of 30 radians if the calculator is in radian mode.

How To: Given an angle in radians, use a graphing calculator to find the cosine

1. If the calculator has degree mode and radian mode, set it to radian mode.
2. Press the COS key.
3. Enter the radian value of the angle and press the close-parentheses key “)”.
4. Press ENTER.

Example 2.1.4: Using a Graphing Calculator to Find Sine and Cosine

Evaluate $\cos\left(\frac{5\pi}{3}\right)$ using a graphing calculator or computer.

Solution

Enter the following keystrokes:

$\text{COS}(5 \times \pi \div 3) \text{ ENTER}$

$$\cos\left(\frac{5\pi}{3}\right) = 0.5$$

Analysis

We can find the cosine or sine of an angle in degrees directly on a calculator with degree mode. For calculators or software that use only radian mode, we can find the sign of 20° , for example, by including the conversion factor to radians as part of the input:

Try It 2.1.4

Evaluate $\sin\left(\frac{\pi}{3}\right)$.

Answer

approximately 0.866025403

Identifying the Domain and Range of Sine and Cosine Functions

Now that we can find the sine and cosine of an angle, we need to discuss their domains and ranges. What are the domains of the sine and cosine functions? That is, what are the smallest and largest numbers that can be inputs of the functions? Because angles smaller than 0 and angles larger than 2π can still be graphed on the unit circle and have real values of x , y , and r , there is no lower or upper limit to the angles that can be inputs to the sine and cosine functions. The input to the sine and cosine functions is the rotation from the positive x -axis, and that may be any real number.

What are the ranges of the sine and cosine functions? What are the least and greatest possible values for their output? We can see the answers by examining the **unit circle**, as shown in Figure 2.1.15. The bounds of the x -coordinate are $[-1, 1]$. The bounds of the y -coordinate are also $[-1, 1]$. Therefore, the range of both the sine and cosine functions is $[-1, 1]$.

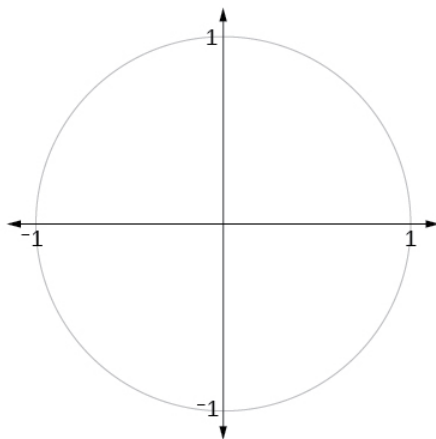


Figure 2.1.15

Finding Reference Angles

We have discussed finding the sine and cosine for angles in the first quadrant, but what if our angle is in another quadrant? For any given angle in the first quadrant, there is an angle in the second quadrant with the same sine value. Because the sine value is the y -coordinate on the unit circle, the other angle with the same sine will share the same y -value, but have the opposite x -value. Therefore, its cosine value will be the opposite of the first angle's cosine value.

Likewise, there will be an angle in the fourth quadrant with the same cosine as the original angle. The angle with the same cosine will share the same x -value but will have the opposite y -value. Therefore, its sine value will be the opposite of the original angle's sine value.

As shown in Figure 2.1.16, angle α has the same sine value as angle t ; the cosine values are opposites. Angle β has the same cosine value as angle t ; the sine values are opposites.

$$\begin{array}{ll} \sin(t) = \sin(\alpha) & \text{and} \quad \cos(t) = -\cos(\alpha) \\ \sin(t) = -\sin(\beta) & \text{and} \quad \cos(t) = \cos(\beta) \end{array}$$

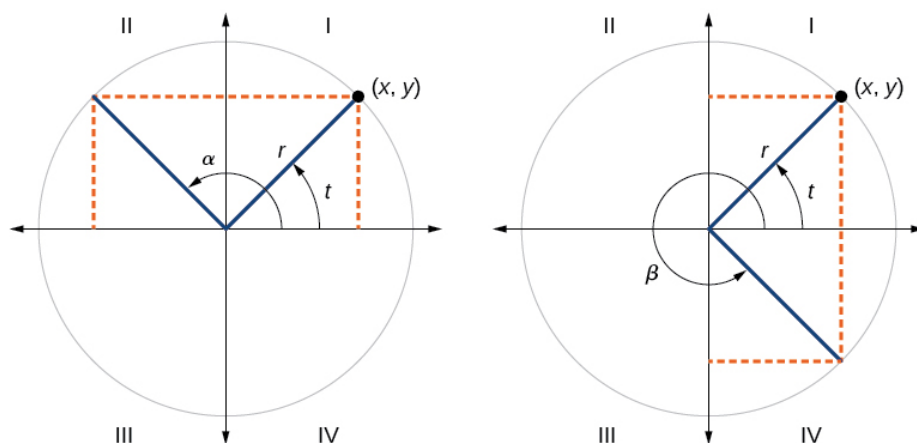


Figure 2.1.16

Recall that an angle's reference angle is the acute angle, t , formed by the terminal side of the angle t and the horizontal axis. A reference angle is always an angle between 0 and 90° , or 0 and $\frac{\pi}{2}$ radians. As we can see from Figure 2.1.17, for any angle in quadrants II, III, or IV, there is a reference angle in quadrant I.

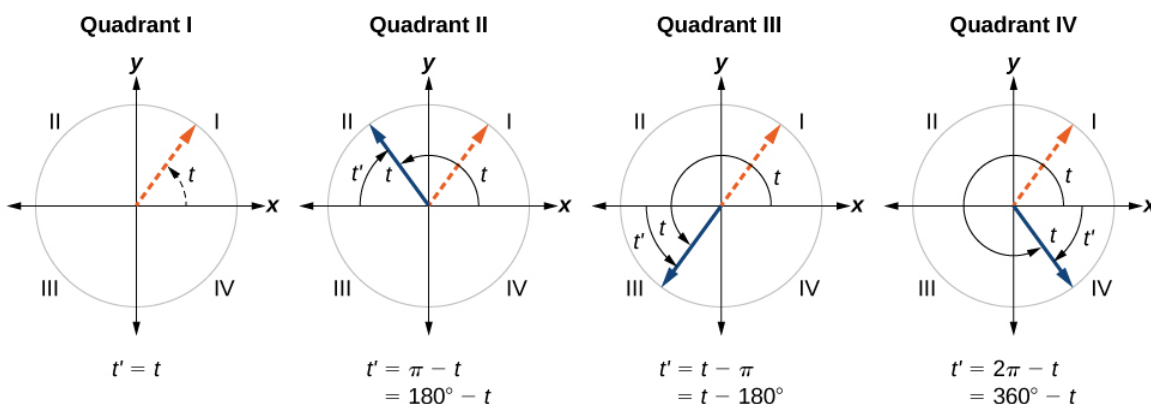


Figure 2.1.17

How To: Given an angle between 0 and 2π , find its reference angle

1. An angle in the first quadrant is its own reference angle.
2. For an angle in the second or third quadrant, the reference angle is $|\pi - t|$ or $|180^\circ - t|$.
3. For an angle in the fourth quadrant, the reference angle is $2\pi - t$ or $360^\circ - t$.
4. If an angle is less than 0 or greater than 2π , add or subtract 2π as many times as needed to find an equivalent angle between 0 and 2π .

Example 2.1.5: Finding a Reference Angle

Find the reference angle of 225° as shown in Figure 2.1.18

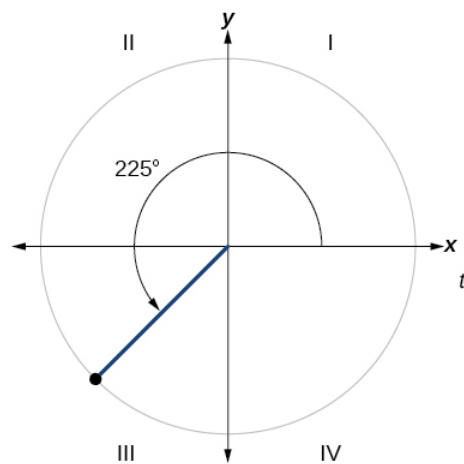


Figure 2.1.18

Solution

Because 225° is in the third quadrant, the reference angle is

$$|(180^\circ - 225^\circ)| = |-45^\circ| = 45^\circ$$

Try It 2.1.5

Find the reference angle of $\frac{5\pi}{3}$.

Answer

$$\left(\frac{\pi}{3}\right)$$

Using Reference Angles

Now let's take a moment to reconsider the Ferris wheel introduced at the beginning of this section. Suppose a rider snaps a photograph while stopped twenty feet above ground level. The rider then rotates three-quarters of the way around the circle. What is the rider's new elevation? To answer questions such as this one, we need to evaluate the sine or cosine functions for angles that are greater than 90 degrees or at a **negative angle**. Reference angles make it possible to evaluate trigonometric functions for angles outside the first quadrant. They can also be used to find (x, y) coordinates for those angles. We will use the **reference angle** of the angle of rotation combined with the quadrant in which the terminal side of the angle lies.

Using Reference Angles to Evaluate Trigonometric Functions

We can find the cosine and sine of any angle in any quadrant if we know the cosine or sine of its reference angle. The absolute values of the cosine and sine of an angle are the same as those of the reference angle. The sign depends on the quadrant of the original angle. The cosine will be positive or negative depending on the sign of the x -values in that quadrant. The sine will be positive or negative depending on the sign of the y -values in that quadrant.

USING REFERENCE ANGLES TO FIND COSINE AND SINE

Angles have cosines and sines with the same absolute value as cosines and sines of their reference angles. The sign (positive or negative) can be determined from the quadrant of the angle.

How To: Given an angle in standard position, find the reference angle, and the cosine and sine of the original angle

1. Measure the angle between the terminal side of the given angle and the horizontal axis. That is the reference angle.
2. Determine the values of the cosine and sine of the reference angle.
3. Give the cosine the same sign as the x -values in the quadrant of the original angle.
4. Give the sine the same sign as the y -values in the quadrant of the original angle.

Example 2.1.6: Using Reference Angles to Find Sine and Cosine

1. Using a reference angle, find the exact value of $\cos(150^\circ)$ and $\sin(150^\circ)$.
2. Using the reference angle, find $\cos \frac{5\pi}{4}$ and $\sin \frac{5\pi}{4}$.

Solution

This tells us that 150° has the same sine and cosine values as 30° , except for the sign. We know that

Since 150° is in the second quadrant, the x -coordinate of the point on the circle is negative, so the cosine value is negative. The y -coordinate is positive, so the sine value is positive.

$\frac{5\pi}{4}$ is in the third quadrant. Its reference angle is $\left| \frac{5\pi}{4} - \pi \right| = \frac{\pi}{4}$. The cosine and sine of $\frac{\pi}{4}$ are both $\frac{\sqrt{2}}{2}$. In the third quadrant, both x and y are negative, so:

$$\cos \frac{5\pi}{4} = -\frac{\sqrt{2}}{2} \text{ and } \sin \frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$$

Try It 2.1.6

1. Use the reference angle of 315° to find $\cos(315^\circ)$ and $\sin(315^\circ)$.
2. Use the reference angle of $-\frac{\pi}{6}$ to find $\cos\left(-\frac{\pi}{6}\right)$ and $\sin\left(-\frac{\pi}{6}\right)$.

Answer

1. $\cos(315^\circ) = \frac{\sqrt{2}}{2}$, $\sin(315^\circ) = -\frac{\sqrt{2}}{2}$
2. $\cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$, $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$

Using Reference Angles to Find Coordinates

Now that we have learned how to find the cosine and sine values for special angles in the first quadrant, we can use symmetry and reference angles to fill in cosine and sine values for the rest of the special angles on the unit circle. They are shown in Figure 2.1.19. Take time to learn the (x, y) coordinates of all of the major angles in the first quadrant.

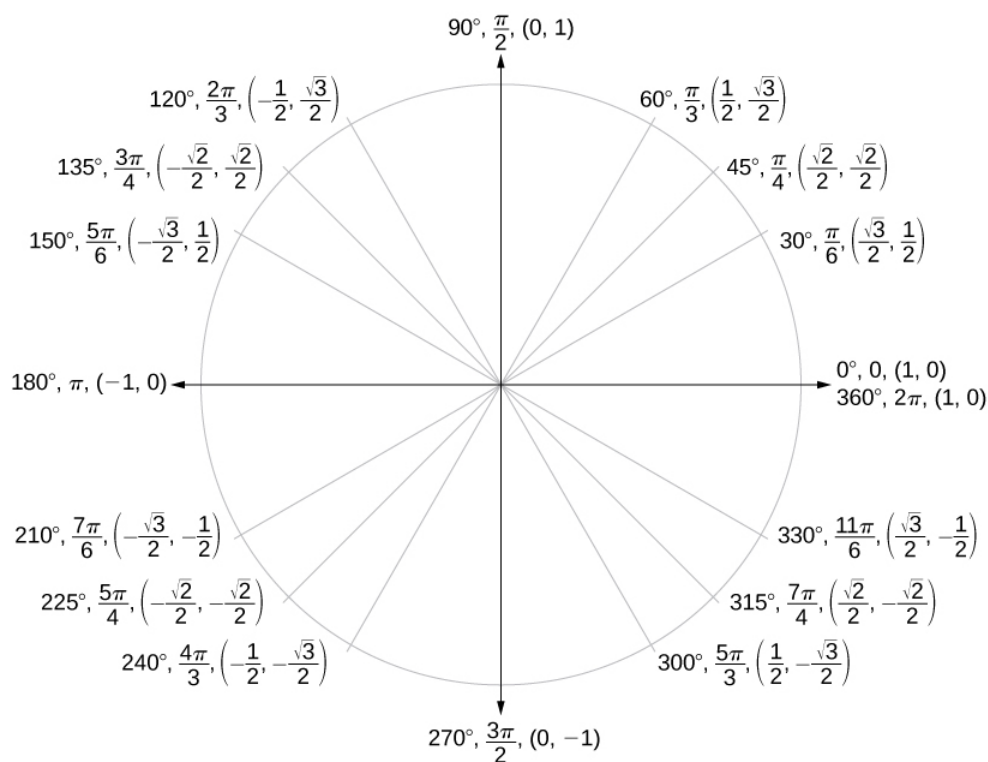


Figure 2.1.19: Special angles and coordinates of corresponding points on the unit circle

In addition to learning the values for special angles, we can use reference angles to find (x, y) coordinates of any point on the unit circle, using what we know of reference angles along with the **identities**

$$x = \cos t$$

$$y = \sin t$$

First we find the reference angle corresponding to the given angle. Then we take the sine and cosine values of the **reference angle**, and give them the signs corresponding to the y - and x -values of the quadrant.

How To: Given the angle of a point on a circle and the radius of the circle, find the (x, y) coordinates of the point

1. Find the reference angle by measuring the smallest angle to the x -axis.
2. Find the cosine and sine of the reference angle.
3. Determine the appropriate signs for x and y in the given quadrant.

Example 2.1.7: Using the Unit Circle to Find Coordinates

Find the coordinates of the point on the unit circle at an angle of $\frac{7\pi}{6}$.

Solution

We know that the angle $\frac{7\pi}{6}$ is in the third quadrant.

First, let's find the reference angle by measuring the angle to the x -axis. To find the reference angle of an angle whose terminal side is in quadrant III, we find the difference of the angle and π .

$$\frac{7\pi}{6} - \pi = \frac{\pi}{6}$$

Next, we will find the cosine and sine of the reference angle:

$$\cos\left(\frac{\pi}{6}\right) = \frac{3}{2} \quad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

We must determine the appropriate signs for x and y in the given quadrant. Because our original angle is in the third quadrant, where both x and y are negative, both cosine and sine are negative.

$$\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$$

Now we can calculate the (x, y) coordinates using the identities $x = \cos \theta$ and $y = \sin \theta$.

The coordinates of the point are $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ on the unit circle.

Try It 2.1.7

Find the coordinates of the point on the unit circle at an angle of $\frac{5\pi}{3}$.

Answer

$$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

media

Access these online resources for additional instruction and practice with sine and cosine functions.

- [Trigonometric Functions Using the Unit Circle](#)
- [Sine and Cosine from the Unit Circle](#)
- [Sine and Cosine from the Unit Circle and Multiples of Pi Divided by Six](#)
- [Sine and Cosine from the Unit Circle and Multiples of Pi Divided by Four](#)
- [Trigonometric Functions Using Reference Angles](#)

Key Equations

Cosine	$\cos t = x$
Sine	$\sin t = y$
Pythagorean Identity	$\cos^2 t + \sin^2 t = 1$

Key Concepts

- Finding the function values for the sine and cosine begins with drawing a unit circle, which is centered at the origin and has a radius of 1 unit.
- Using the unit circle, the sine of an angle t equals the y -value of the endpoint on the unit circle of an arc of length t whereas the cosine of an angle t equals the x -value of the endpoint. See [Example](#).
- The sine and cosine values are most directly determined when the corresponding point on the unit circle falls on an axis. See [Example](#).
- When the sine or cosine is known, we can use the Pythagorean Identity to find the other. The Pythagorean Identity is also useful for determining the sines and cosines of special angles. See [Example](#).
- Calculators and graphing software are helpful for finding sines and cosines if the proper procedure for entering information is known. See [Example](#).
- The domain of the sine and cosine functions is all real numbers.
- The range of both the sine and cosine functions is $[-1, 1]$.
- The sine and cosine of an angle have the same absolute value as the sine and cosine of its reference angle.
- The signs of the sine and cosine are determined from the x - and y -values in the quadrant of the original angle.
- An angle's reference angle is the size angle, t , formed by the terminal side of the angle t and the horizontal axis. See [Example](#).

- Reference angles can be used to find the sine and cosine of the original angle. See [Example](#).
- Reference angles can also be used to find the coordinates of a point on a circle. See [Example](#).

Glossary

cosine function

the x -value of the point on a unit circle corresponding to a given angle

Pythagorean Identity

a corollary of the Pythagorean Theorem stating that the square of the cosine of a given angle plus the square of the sine of that angle equals 1

sine function

the y -value of the point on a unit circle corresponding to a given angle

unit circle

a circle with a center at $(0, 0)$ and radius 1.

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2.2: The Other Trigonometric Functions

Learning Objectives

- Find exact values of the trigonometric functions secant, cosecant, tangent, and cotangent of $\frac{\pi}{3}$, $\frac{\pi}{4}$, and $\frac{\pi}{6}$.
- Use reference angles to evaluate the trigonometric functions secant, tangent, and cotangent.
- Use properties of even and odd trigonometric functions.
- Recognize and use fundamental identities.
- Evaluate trigonometric functions with a calculator.

A wheelchair ramp that meets the standards of the Americans with Disabilities Act must make an angle with the ground whose tangent is $\frac{1}{12}$ or less, regardless of its length. A tangent represents a ratio, so this means that for every 1 inch of rise, the ramp must have 12 inches of run. Trigonometric functions allow us to specify the shapes and proportions of objects independent of exact dimensions. We have already defined the sine and cosine functions of an angle. Though sine and cosine are the trigonometric functions most often used, there are four others. Together they make up the set of six trigonometric functions. In this section, we will investigate the remaining functions.

Finding Exact Values of the Trigonometric Functions Secant, Cosecant, Tangent, and Cotangent

To define the remaining functions, we will once again draw a unit circle with a point (x, y) corresponding to an angle of t , as shown in Figure 2.2.1. As with the sine and cosine, we can use the (x, y) coordinates to find the other functions.

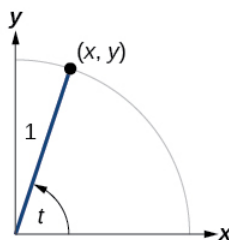


Figure 2.2.1

The first function we will define is the tangent. The **tangent** of an angle is the ratio of the y-value to the x-value of the corresponding point on the unit circle. In Figure 2.2.1, the tangent of angle t is equal to $\frac{y}{x}$, $x \neq 0$. Because the y-value is equal to the sine of t , and the x-value is equal to the cosine of t , the tangent of angle t can also be defined as $\frac{\sin t}{\cos t}$, $\cos t \neq 0$. The tangent function is abbreviated as $\tan$. The remaining three functions can all be expressed as reciprocals of functions we have already defined.

- The **secant** function is the reciprocal of the cosine function. In Figure 2.2.1, the secant of angle t is equal to $\frac{1}{\cos t} = \frac{1}{x}$, $x \neq 0$. The secant function is abbreviated as $\sec$.
- The **cotangent** function is the reciprocal of the tangent function. In Figure 2.2.1, the cotangent of angle t is equal to $\frac{\cos t}{\sin t} = \frac{x}{y}$, $y \neq 0$. The cotangent function is abbreviated as $\cot$.
- The **cosecant** function is the reciprocal of the sine function. In Figure 2.2.1, the cosecant of angle t is equal to $\frac{1}{\sin t} = \frac{1}{y}$, $y \neq 0$. The cosecant function is abbreviated as $\csc$.

TANGENT, SECANT, COSECANT, AND COTANGENT FUNCTIONS

If t is a real number and (x, y) is a point where the terminal side of an angle of t radians intercepts the unit circle, then

$$\tan t = \frac{y}{x}, x \neq 0 \quad (2.2.1)$$

$$\sec t = \frac{1}{x}, x \neq 0 \quad (2.2.2)$$

$$\csc t = \frac{1}{y}, y \neq 0 \quad (2.2.3)$$

$$\cot t = \frac{x}{y}, y \neq 0 \quad (2.2.4)$$

Example 2.2.1: Finding Trigonometric Functions from a Point on the Unit Circle

The point $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$ is on the unit circle, as shown in Figure 2.2.2. Find $\sin t$, $\cos t$, $\tan t$, $\sec t$, $\csc t$, and $\cot t$.

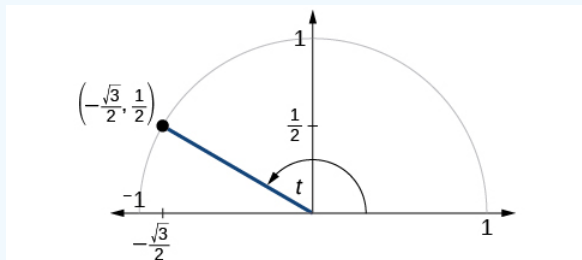


Figure 2.2.2

Solution

Because we know the (x, y) coordinates of the point on the unit circle indicated by angle t , we can use those coordinates to find the six functions:

$$\begin{aligned}\sin t &= y = \frac{1}{2} \\ \cos t &= x = -\frac{\sqrt{3}}{2} \\ \tan t &= \frac{y}{x} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{2} \left(-\frac{2}{\sqrt{3}} \right) = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3} \\ \sec t &= \frac{1}{x} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3} \\ \csc t &= \frac{1}{y} = \frac{1}{\frac{1}{2}} = 2 \\ \cot t &= \frac{x}{y} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\frac{\sqrt{3}}{2} \left(\frac{2}{1} \right) = -\sqrt{3}\end{aligned}$$

Exercise 2.2.1:

The point $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ is on the unit circle, as shown in Figure 2.2.3. Find $\sin t$, $\cos t$, $\tan t$, $\sec t$, $\csc t$, and $\cot t$.

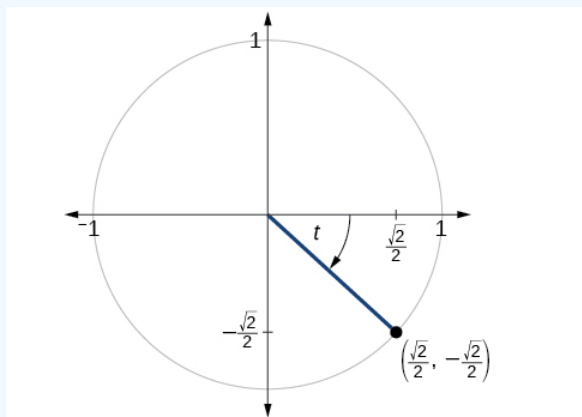


Figure 2.2.3

Solution

$$\sin t = -\frac{\sqrt{2}}{2}, \cos t = \frac{\sqrt{2}}{2}, \tan t = -1, \sec t = \sqrt{2}, \csc t = -\sqrt{2}, \cot t = -1$$

Example 2.2.2: Finding the Trigonometric Functions of an Angle

Find $\sin t$, $\cos t$, $\tan t$, $\sec t$, $\csc t$, and $\cot t$ when $t = \frac{\pi}{6}$.

Solution

We have previously used the properties of equilateral triangles to demonstrate that $\sin \frac{\pi}{6} = \frac{1}{2}$ and $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$. We can use these values and the definitions of tangent, secant, cosecant, and cotangent as functions of sine and cosine to find the remaining function values.

$$\begin{aligned}\tan \frac{\pi}{6} &= \frac{\sin \frac{\pi}{6}}{\cos \frac{\pi}{6}} \\ &= \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \\ \sec \frac{\pi}{6} &= \frac{1}{\cos \frac{\pi}{6}} \\ &= \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \\ \csc \frac{\pi}{6} &= \frac{1}{\sin \frac{\pi}{6}} = \frac{1}{\frac{1}{2}} = 2 \\ \cot \frac{\pi}{6} &= \frac{\cos \frac{\pi}{6}}{\sin \frac{\pi}{6}} \\ &= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}\end{aligned}$$

Exercise 2.2.2:

Find $\sin t$, $\cos t$, $\tan t$, $\sec t$, $\csc t$, and $\cot t$ when $t = \frac{\pi}{3}$.

Solution

$$\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} \quad (2.2.5)$$

$$\cos \frac{\pi}{3} = \frac{1}{2} \quad (2.2.6)$$

$$\tan \frac{\pi}{3} = \sqrt{3} \quad (2.2.7)$$

$$\sec \frac{\pi}{3} = 2 \quad (2.2.8)$$

$$\csc \frac{\pi}{3} = \frac{2\sqrt{3}}{3} \quad (2.2.9)$$

$$\cot \frac{\pi}{3} = \frac{\sqrt{3}}{3} \quad (2.2.10)$$

Because we know the sine and cosine values for the common first-quadrant angles, we can find the other function values for those angles as well by setting x equal to the cosine and y equal to the sine and then using the definitions of tangent, secant, cosecant, and cotangent. The results are shown in Table 2.2.1.

Table 2.2.1

Angle	0	$\frac{\pi}{6}$, or 30°	$\frac{\pi}{4}$, or 45°	$\frac{\pi}{3}$, or 60°	$\frac{\pi}{2}$, or 90°
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Angle	0	$\frac{\pi}{6}$, or 30°	$\frac{\pi}{4}$, or 45°	$\frac{\pi}{3}$, or 60°	$\frac{\pi}{2}$, or 90°
Tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined
Secant	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	Undefined
Cosecant	Undefined	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
Cotangent	Undefined	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

Using Reference Angles to Evaluate Tangent, Secant, Cosecant, and Cotangent

We can evaluate trigonometric functions of angles outside the first quadrant using reference angles as we have already done with the sine and cosine functions. The procedure is the same: Find the **reference angle** formed by the terminal side of the given angle with the horizontal axis. The trigonometric function values for the original angle will be the same as those for the reference angle, except for the positive or negative sign, which is determined by x - and y -values in the original quadrant. Figure 2.2.4 shows which functions are positive in which quadrant.

To help us remember which of the six trigonometric functions are positive in each quadrant, we can use the mnemonic phrase “A Smart Trig Class.” Each of the four words in the phrase corresponds to one of the four quadrants, starting with quadrant I and rotating counterclockwise. In quadrant I, which is “A,” all of the six trigonometric functions are positive. In quadrant II, “Smart,” only sine and its reciprocal function, cosecant, are positive. In quadrant III, “Trig,” only tangent and its reciprocal function, cotangent, are positive. Finally, in quadrant IV, “Class,” only cosine and its reciprocal function, secant, are positive.

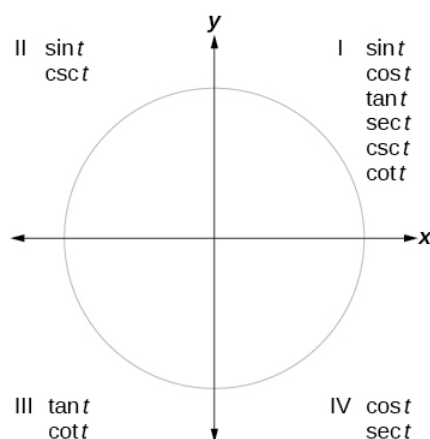


Figure 2.2.4

HOWTO: Given an angle not in the first quadrant, use reference angles to find all six trigonometric functions

1. Measure the angle formed by the terminal side of the given angle and the horizontal axis. This is the reference angle.
2. Evaluate the function at the reference angle.
3. Observe the quadrant where the terminal side of the original angle is located. Based on the quadrant, determine whether the output is positive or negative.

Example 2.2.3: Using Reference Angles to Find Trigonometric Functions

Use reference angles to find all six trigonometric functions of $-\frac{5\pi}{6}$.

Solution

The angle between this angle's terminal side and the x -axis is $\frac{\pi}{6}$, so that is the reference angle. Since $-\frac{5\pi}{6}$ is in the third quadrant, where both x and y are negative, cosine, sine, secant, and cosecant will be negative, while tangent and cotangent will be positive.

$$\cos\left(-\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}, \sin\left(-\frac{5\pi}{6}\right) = -\frac{1}{2}, \tan\left(-\frac{5\pi}{6}\right) = \frac{\sqrt{3}}{3} \quad (2.2.11)$$

$$\sec\left(-\frac{5\pi}{6}\right) = -\frac{2\sqrt{3}}{3}, \csc\left(-\frac{5\pi}{6}\right) = -2, \cot\left(-\frac{5\pi}{6}\right) = \sqrt{3} \quad (2.2.12)$$

Exercise 2.2.3

Use reference angles to find all six trigonometric functions of $-\frac{7\pi}{4}$.

Solution

$$\sin\left(-\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2}, \cos\left(-\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2}, \tan\left(-\frac{7\pi}{4}\right) = 1,$$

$$\sec\left(-\frac{7\pi}{4}\right) = \sqrt{2}, \csc\left(-\frac{7\pi}{4}\right) = \sqrt{2}, \cot\left(-\frac{7\pi}{4}\right) = 1$$

Using Even and Odd Trigonometric Functions

To be able to use our six trigonometric functions freely with both positive and negative angle inputs, we should examine how each function treats a negative input. As it turns out, there is an important difference among the functions in this regard.

Consider the function $f(x) = x^2$, shown in Figure 2.2.5. The graph of the function is symmetrical about the y-axis. All along the curve, any two points with opposite x-values have the same function value. This matches the result of calculation: $(4)^2 = (-4)^2$, $(-5)^2 = (5)^2$, and so on. So $f(x) = x^2$ is an **even function**, a function such that two inputs that are opposites have the same output. That means $f(-x) = f(x)$.

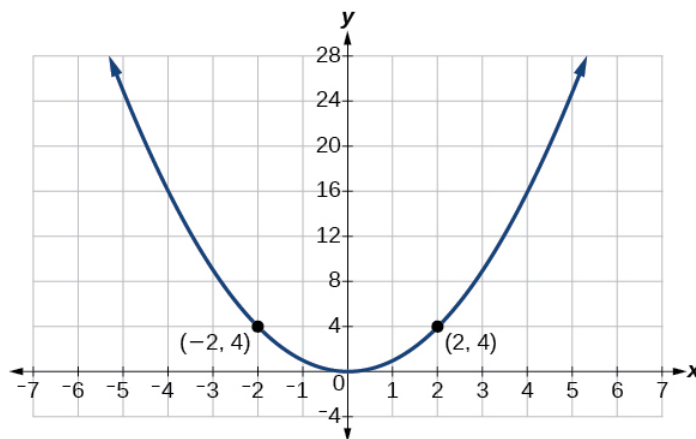


Figure 2.2.5: The function $f(x) = x^2$ is an even function.

Now consider the function $f(x) = x^3$, shown in Figure 2.2.6. The graph is not symmetrical about the y-axis. All along the graph, any two points with opposite x-values also have opposite y-values. So $f(x) = x^3$ is an **odd function**, one such that two inputs that are opposites have outputs that are also opposites. That means $f(-x) = -f(x)$.

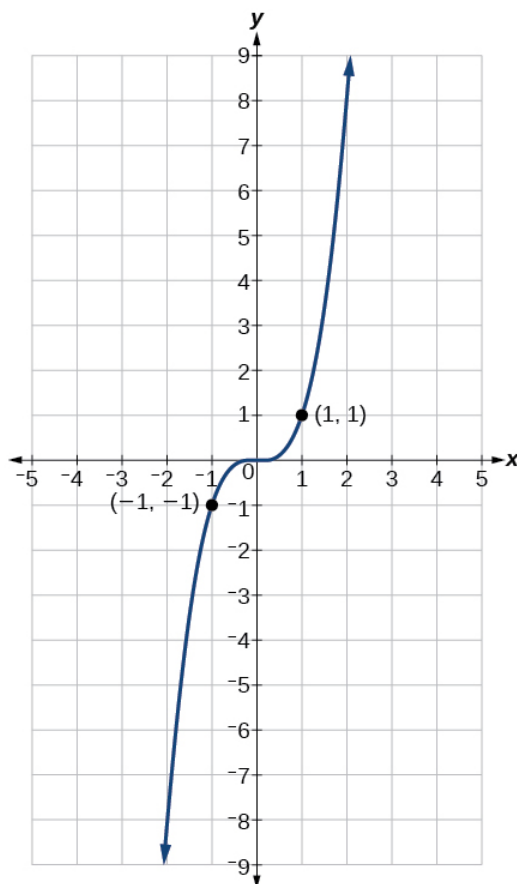


Figure 2.2.6: The function $f(x) = x^3$ is an odd function.

We can test whether a trigonometric function is even or odd by drawing a **unit circle** with a positive and a negative angle, as in Figure 2.2.7. The sine of the positive angle is y . The sine of the negative angle is $-y$. The **sine function**, then, is an odd function. We can test each of the six trigonometric functions in this fashion. The results are shown in Table 2.2.2.

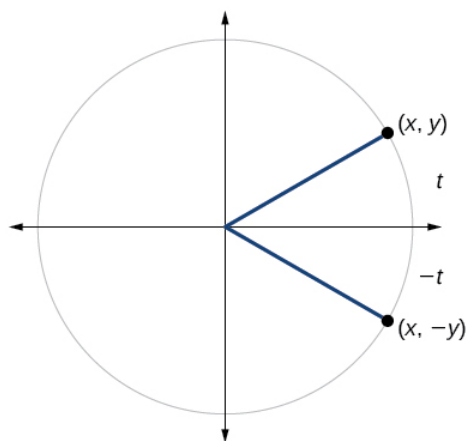


Figure 2.2.7
Table 2.2.2

$\sin t = y$	(2.2.13)	$\cos t = x$	(2.2.16)	$\tan(t) = \frac{y}{x}$	(2.2.19)
$\sin(-t) = -y$	(2.2.14)	$\cos(-t) = x$	(2.2.17)	$\tan(-t) = -\frac{y}{x}$	(2.2.20)
$\sin t \neq \sin(-t)$	(2.2.15)	$\cos t = \cos(-t)$	(2.2.18)	$\tan t \neq \tan(-t)$	(2.2.21)

$$\sec t = \frac{1}{x} \quad (2.2.22)$$

$$\sec(-t) = \frac{1}{x} \quad (2.2.23)$$

$$\sec t = \sec(-t) \quad (2.2.24)$$

$$\csc t = \frac{1}{y} \quad (2.2.25)$$

$$\csc(-t) = \frac{1}{-y} \quad (2.2.26)$$

$$\csc t \neq \csc(-t) \quad (2.2.27)$$

$$\cot t = \frac{x}{y} \quad (2.2.28)$$

$$\cot(-t) = \frac{x}{-y} \quad (2.2.29)$$

$$\cot t \neq \cot(-t) \quad (2.2.30)$$

EVEN AND ODD TRIGONOMETRIC FUNCTIONS

- An **even function** is one in which $f(-x) = f(x)$.
- An **odd function** is one in which $f(-x) = -f(x)$.

Cosine and secant are even:

$$\cos(-t) = \cos t \quad (2.2.31)$$

$$\sec(-t) = \sec t \quad (2.2.32)$$

Sine, tangent, cosecant, and cotangent are odd:

$$\sin(-t) = -\sin t \quad (2.2.33)$$

$$\tan(-t) = -\tan t \quad (2.2.34)$$

$$\csc(-t) = -\csc t \quad (2.2.35)$$

$$\cot(-t) = -\cot t \quad (2.2.36)$$

Example 2.2.4: Using Even and Odd Properties of Trigonometric Functions

If the secant of angle t is 2, what is the secant of $-t$?

Solution

Secant is an even function. The secant of an angle is the same as the secant of its opposite. So if the secant of angle t is 2, the secant of $-t$ is also 2.

Exercise 2.2.4:

If the cotangent of angle t is $\sqrt{3}$, what is the cotangent of $-t$?

Solution

$$-\sqrt{3}$$

Recognizing and Using Fundamental Identities

We have explored a number of properties of trigonometric functions. Now, we can take the relationships a step further, and derive some fundamental identities. Identities are statements that are true for all values of the input on which they are defined. Usually, identities can be derived from definitions and relationships we already know. For example, the Pythagorean Identity we learned earlier was derived from the Pythagorean Theorem and the definitions of sine and cosine.

FUNDAMENTAL IDENTITIES

We can derive some useful **identities** from the six trigonometric functions. The other four trigonometric functions can be related back to the sine and cosine functions using these basic relationships:

$$\tan t = \frac{\sin t}{\cos t} \quad (2.2.37)$$

$$\sec t = \frac{1}{\cos t} \quad (2.2.38)$$

$$\csc t = \frac{1}{\sin t} \quad (2.2.39)$$

$$\cot t = \frac{1}{\tan t} = \frac{\cos t}{\sin t} \quad (2.2.40)$$

Example 2.2.5: Using Identities to Evaluate Trigonometric Functions

- a. Given $\sin(45^\circ) = \frac{\sqrt{2}}{2}$, $\cos(45^\circ) = \frac{\sqrt{2}}{2}$, evaluate $\tan(45^\circ)$.
 b. Given $\sin(\frac{5\pi}{6}) = \frac{1}{2}$, $\cos(\frac{5\pi}{6}) = -\frac{\sqrt{3}}{2}$, evaluate $\sec(\frac{5\pi}{6})$.

Solution

Because we know the sine and cosine values for these angles, we can use identities to evaluate the other functions.

a.

$$\begin{aligned} \tan(45^\circ) &= \frac{\sin(45^\circ)}{\cos(45^\circ)} \\ &= \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} \\ &= 1 \end{aligned}$$

b.

$$\begin{aligned} \sec(\frac{5\pi}{6}) &= \frac{1}{\cos(\frac{5\pi}{6})} \\ &= \frac{1}{-\frac{\sqrt{3}}{2}} \\ &= \frac{-2\sqrt{3}}{1} \\ &= -2\sqrt{3} \end{aligned}$$

Exercise 2.2.5

Evaluate $\csc(\frac{7\pi}{6})$.

Solution

−2

Example 2.2.6: Using Identities to Simplify Trigonometric Expressions

Simplify $\frac{\sec t}{\tan t}$.

Solution

We can simplify this by rewriting both functions in terms of sine and cosine.

$$\begin{aligned} \frac{\sec t}{\tan t} &= \frac{1/\cos t}{\sin t/\cos t} && \text{To divide the functions, we multiply by the reciprocal.} \\ &= \frac{1}{\cos t} \cdot \frac{\cos t}{\sin t} && \text{Divide out the cosines.} \\ &= \frac{1}{\sin t} && \text{Simplify and use the identity.} \\ &= \csc t \end{aligned} \quad (2.2.41)$$

By showing that $\frac{\sec t}{\tan t}$ can be simplified to $\csc t$, we have, in fact, established a new identity.

$$\frac{\sec t}{\tan t} = \csc t$$

Exercise 2.2.6

Simplify $(\tan t)(\cos t)$.

Solution

$\sin t$

Alternate Forms of the Pythagorean Identity

We can use these fundamental identities to derive alternative forms of the **Pythagorean Identity**, $\cos^2 t + \sin^2 t = 1$. One form is obtained by dividing both sides by $\cos^2 t$:

$$\frac{\cos^2 t}{\cos^2 t} + \frac{\sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t} \quad (2.2.42)$$

$$1 + \tan^2 t = \sec^2 t \quad (2.2.43)$$

The other form is obtained by dividing both sides by $\sin^2 t$:

$$\frac{\cos^2 t}{\sin^2 t} + \frac{\sin^2 t}{\sin^2 t} = \frac{1}{\sin^2 t} \quad (2.2.44)$$

$$\cot^2 t + 1 = \csc^2 t \quad (2.2.45)$$

ALTERNATE FORMS OF THE PYTHAGOREAN IDENTITY

$$1 + \tan^2 t = \sec^2 t \quad (2.2.46)$$

$$\cot^2 t + 1 = \csc^2 t \quad (2.2.47)$$

Example 2.2.7: Using Identities to Relate Trigonometric Functions

If $\cos(t) = \frac{12}{13}$ and t is in quadrant IV, as shown in Figure 2.2.8, find the values of the other five trigonometric functions.

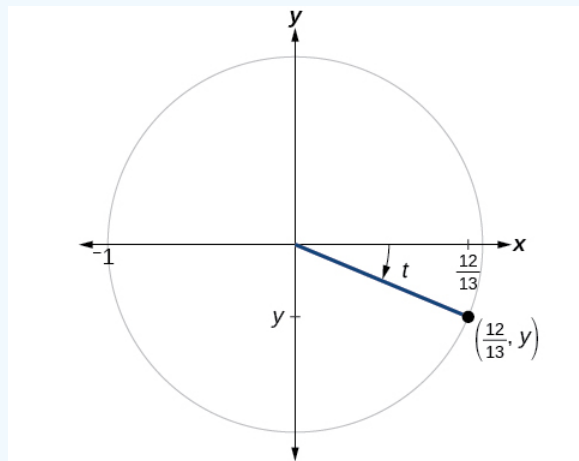


Figure 2.2.8

Solution

We can find the sine using the Pythagorean Identity, $\cos^2 t + \sin^2 t = 1$, and the remaining functions by relating them to sine and cosine.

$$\left(\frac{12}{13}\right)^2 + \sin^2 t = 1 \quad (2.2.48)$$

$$\sin^2 t = 1 - \left(\frac{12}{13}\right)^2 \quad (2.2.49)$$

$$\sin^2 t = 1 - \frac{144}{169} \quad (2.2.50)$$

$$\sin^2 t = \frac{25}{169} \quad (2.2.51)$$

$$\sin t = \pm \sqrt{\frac{25}{169}} \quad (2.2.52)$$

$$\sin t = \pm \frac{\sqrt{25}}{\sqrt{169}} \quad (2.2.53)$$

$$\sin t = \pm \frac{5}{13} \quad (2.2.54)$$

The sign of the sine depends on the y-values in the quadrant where the angle is located. Since the angle is in quadrant IV, where the y-values are negative, its sine is negative, $-\frac{5}{13}$.

The remaining functions can be calculated using identities relating them to sine and cosine.

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{5}{13}}{\frac{12}{13}} = -\frac{5}{12} \quad (2.2.55)$$

$$\sec t = \frac{1}{\cos t} = \frac{1}{\frac{12}{13}} = \frac{13}{12} \quad (2.2.56)$$

$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{5}{13}} = -\frac{13}{5} \quad (2.2.57)$$

$$\cot t = \frac{1}{\tan t} = \frac{1}{-\frac{5}{12}} = -\frac{12}{5} \quad (2.2.58)$$

Exercise 2.2.7:

If $\sec(t) = -\frac{17}{8}$ and $0 < t < \pi$, find the values of the other five functions.

Solution

$$\cos t = -\frac{8}{17}, \sin t = \frac{15}{17}, \tan t = -\frac{15}{8}$$

$$\csc t = \frac{17}{15}, \cot t = -\frac{8}{15}$$

As we discussed in the chapter opening, a function that repeats its values in regular intervals is known as a **periodic function**. The trigonometric functions are periodic. For the four trigonometric functions, sine, cosine, cosecant and secant, a revolution of one circle, or 2π , will result in the same outputs for these functions. And for tangent and cotangent, only a half a revolution will result in the same outputs.

Other functions can also be periodic. For example, the lengths of months repeat every four years. If x represents the length time, measured in years, and $f(x)$ represents the number of days in February, then $f(x+4) = f(x)$. This pattern repeats over and over through time. In other words, every four years, February is guaranteed to have the same number of days as it did 4 years earlier. The positive number 4 is the smallest positive number that satisfies this condition and is called the period. A **period** is the shortest interval over which a function completes one full cycle—in this example, the period is 4 and represents the time it takes for us to be certain February has the same number of days.

PERIOD OF A FUNCTION

The **period** P of a repeating function f is the number representing the interval such that $f(x+P) = f(x)$ for any value of x .

The period of the cosine, sine, secant, and cosecant functions is 2π .

The period of the tangent and cotangent functions is π .

Example 2.2.8: Finding the Values of Trigonometric Functions

Find the values of the six trigonometric functions of angle t based on Figure 2.2.9.

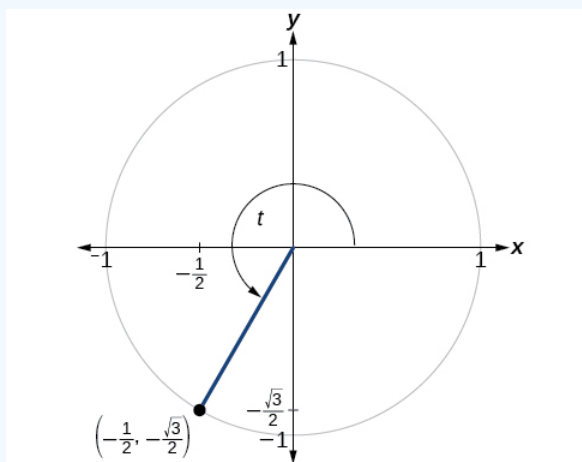


Figure 2.2.9

Solution

$$\sin t = y = -\frac{\sqrt{3}}{2}$$

$$\cos t = x = -\frac{1}{2}$$

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \sqrt{3}$$

$$\sec t = \frac{1}{\cos t} = \frac{1}{-\frac{1}{2}} = -2$$

$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2\sqrt{3}}{3}$$

$$\cot t = \frac{1}{\tan t} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Exercise 2.2.8

Find the values of the six trigonometric functions of angle t based on Figure 2.2.10.

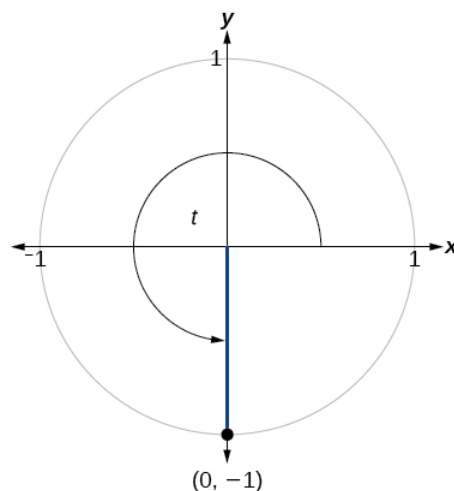


Figure 2.2.10

Solution

$$\sin t = -1, \cos t = 0, \tan t = \text{Undefined} \quad (2.2.59)$$

$$(2.2.60)$$

$$\sec t = \text{Undefined}, \csc t = -1, \cot t = 0 \quad (2.2.61)$$

Example 2.2.9: Finding the Value of Trigonometric Functions

If $\sin(t) = -\frac{\sqrt{3}}{2}$ and $\cos(t) = \frac{1}{2}$, find $\sec(t)$, $\csc(t)$, $\tan(t)$, $\cot(t)$.

Solution

$$\sec t = \frac{1}{\cos t} = \frac{1}{\frac{1}{2}} = 2 \quad (2.2.62)$$

$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2\sqrt{3}}{3} \quad (2.2.63)$$

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3} \quad (2.2.64)$$

$$\cot t = \frac{1}{\tan t} = \frac{1}{-\sqrt{3}} = -\frac{\sqrt{3}}{3} \quad (2.2.65)$$

Exercise 2.2.9:

If $\sin(t) = \frac{\sqrt{2}}{2}$ and $\cos(t) = \frac{\sqrt{2}}{2}$, find $\sec(t)$, $\csc(t)$, $\tan(t)$, and $\cot(t)$.

Solution

$$\sec t = \sqrt{2}, \csc t = \sqrt{2}, \tan t = 1, \cot t = 1$$

Evaluating Trigonometric Functions with a Calculator

We have learned how to evaluate the six trigonometric functions for the common first-quadrant angles and to use them as reference angles for angles in other quadrants. To evaluate trigonometric functions of other angles, we use a scientific or graphing calculator or computer software. If the calculator has a degree mode and a radian mode, confirm the correct mode is chosen before making a calculation.

Evaluating a tangent function with a scientific calculator as opposed to a graphing calculator or computer algebra system is like evaluating a sine or cosine: Enter the value and press the TAN key. For the reciprocal functions, there may not be any dedicated keys that say CSC, SEC, or COT. In that case, the function must be evaluated as the reciprocal of a sine, cosine, or tangent.

If we need to work with degrees and our calculator or software does not have a degree mode, we can enter the degrees multiplied by the conversion factor $\frac{\pi}{180}$ to convert the degrees to radians. To find the secant of 30° , we could press

$$\text{(for a scientific calculator)} : \frac{1}{30 \times \frac{\pi}{180}} \text{COS} \quad (2.2.66)$$

or

$$\text{(for a graphing calculator)} : \frac{1}{\cos(\frac{30\pi}{180})} \quad (2.2.67)$$

how to: Given an angle measure in radians, use a scientific calculator to find the cosecant

1. If the calculator has degree mode and radian mode, set it to radian mode.
2. Enter: 1 /
3. Enter the value of the angle inside parentheses.
4. Press the SIN key.
5. Press the = key.

how to: Given an angle measure in radians, use a graphing utility/calculator to find the cosecant

1. If the graphing utility has degree mode and radian mode, set it to radian mode.
2. Enter: 1 /
3. Press the SIN key.
4. Enter the value of the angle inside parentheses.
5. Press the ENTER key.

Example 2.2.10: Evaluating the Cosecant Using Technology

Evaluate the cosecant of $\frac{5\pi}{7}$.

Solution

For a scientific calculator, enter information as follows:

$$1 / (5 \times \pi / 7) \text{SIN} = \quad (2.2.68)$$

$$\csc\left(\frac{5\pi}{7}\right) \approx 1.279 \quad (2.2.69)$$

Exercise 2.2.10:

Evaluate the cotangent of $-\frac{\pi}{8}$.

≈ -2.414

media

Access these online resources for additional instruction and practice with other trigonometric functions.

- [Determining Trig Function Values](#)
- [More Examples of Determining Trig Functions](#)
- [Pythagorean Identities](#)
- [Trig Functions on a Calculator](#)

Key Equations

Tangent function	$\tan t = \frac{\sin t}{\cos t}$
Secant function	$\sec t = \frac{1}{\cos t}$

Cosecant function	$\csc t = \frac{1}{\sin t}$
Cotangent function	$\cot t = \frac{1}{\tan t} = \frac{\cos t}{\sin t}$

Key Concepts

- The tangent of an angle is the ratio of the y-value to the x-value of the corresponding point on the unit circle.
- The secant, cotangent, and cosecant are all reciprocals of other functions. The secant is the reciprocal of the cosine function, the cotangent is the reciprocal of the tangent function, and the cosecant is the reciprocal of the sine function.
- The six trigonometric functions can be found from a point on the unit circle. See [Example](#).
- Trigonometric functions can also be found from an angle. See [Example](#).
- Trigonometric functions of angles outside the first quadrant can be determined using reference angles. See [Example](#).
- A function is said to be even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$.
- Cosine and secant are even; sine, tangent, cosecant, and cotangent are odd.
- Even and odd properties can be used to evaluate trigonometric functions. See [Example](#).
- The Pythagorean Identity makes it possible to find a cosine from a sine or a sine from a cosine.
- Identities can be used to evaluate trigonometric functions. See [Example](#) and [Example](#).
- Fundamental identities such as the Pythagorean Identity can be manipulated algebraically to produce new identities. See [Example](#).
- The trigonometric functions repeat at regular intervals.
- The period P of a repeating function f is the smallest interval such that $f(x + P) = f(x)$ for any value of x .
- The values of trigonometric functions of special angles can be found by mathematical analysis.
- To evaluate trigonometric functions of other angles, we can use a calculator or computer software. See [Example](#).

Glossary

cosecant

the reciprocal of the sine function: on the unit circle, $\csc t = \frac{1}{y}, y \neq 0$

cotangent

the reciprocal of the tangent function: on the unit circle, $\cot t = \frac{x}{y}, y \neq 0$

identities

statements that are true for all values of the input on which they are defined

period

the smallest interval P of a repeating function f such that $f(x + P) = f(x)$

secant

the reciprocal of the cosine function: on the unit circle, $\sec t = \frac{1}{x}, x \neq 0$

tangent

the quotient of the sine and cosine: on the unit circle, $\tan t = \frac{y}{x}, x \neq 0$

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2.3: Right Triangle Trigonometry

Learning Objectives

- Use right triangles to evaluate trigonometric functions.
- Find function values for $30^\circ(\frac{\pi}{6})$, $45^\circ(\frac{\pi}{4})$, and $60^\circ(\frac{\pi}{3})$.
- Use equal cofunctions of complementary angles.
- Use the definitions of trigonometric functions of any angle.
- Use right-triangle trigonometry to solve applied problems.

Mt. Everest, which straddles the border between China and Nepal, is the tallest mountain in the world. Measuring its height is no easy task and, in fact, the actual measurement has been a source of controversy for hundreds of years. The measurement process involves the use of triangles and a branch of mathematics known as trigonometry. In this section, we will define a new group of functions known as trigonometric functions, and find out how they can be used to measure heights, such as those of the tallest mountains.

We have previously defined the sine and cosine of an angle in terms of the coordinates of a point on the unit circle intersected by the terminal side of the angle:

$$\begin{aligned}\cos t &= x \\ \sin t &= y\end{aligned}$$

In this section, we will see another way to define trigonometric functions using properties of *right triangles*.

Using Right Triangles to Evaluate Trigonometric Functions

In earlier sections, we used a unit circle to define the *trigonometric functions*. In this section, we will extend those definitions so that we can apply them to right triangles. The value of the sine or cosine function of t is its value at t radians. First, we need to create our right triangle. Figure 2.3.1 shows a point on a unit circle of radius 1. If we drop a vertical line segment from the point (x, y) to the x -axis, we have a right triangle whose vertical side has length y and whose horizontal side has length x . We can use this right triangle to redefine sine, cosine, and the other trigonometric functions as ratios of the sides of a right triangle.

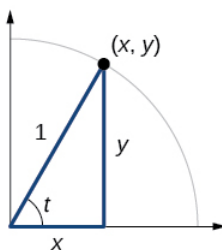


Figure 2.3.1

We know

$$\cos t = \frac{x}{1} = x \quad (2.3.1)$$

Likewise, we know

$$\sin t = \frac{y}{1} = y \quad (2.3.2)$$

These ratios still apply to the sides of a right triangle when no unit circle is involved and when the triangle is not in standard position and is not being graphed using (x, y) coordinates. To be able to use these ratios freely, we will give the sides more general names: Instead of x , we will call the side between the given angle and the right angle the **adjacent side** to angle t . (Adjacent means “next to.”) Instead of y , we will call the side most distant from the given angle the **opposite side** from angle t . And instead of 1, we will call the side of a right triangle opposite the right angle the **hypotenuse**. These sides are labeled in Figure 2.3.2.

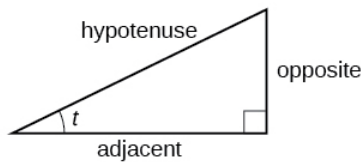


Figure 2.3.2: The sides of a right triangle in relation to angle t .

Understanding Right Triangle Relationships

Given a right triangle with an acute angle of t ,

$$\sin(t) = \frac{\text{opposite}}{\text{hypotenuse}} \quad (2.3.3)$$

$$\cos(t) = \frac{\text{adjacent}}{\text{hypotenuse}} \quad (2.3.4)$$

$$\tan(t) = \frac{\text{opposite}}{\text{adjacent}} \quad (2.3.5)$$

A common mnemonic for remembering these relationships is SohCahToa, formed from the first letters of “Sine is opposite over hypotenuse, Cosine is adjacent over hypotenuse, Tangent is opposite over adjacent.”

how to: Given the side lengths of a right triangle and one of the acute angles, find the sine, cosine, and tangent of that angle

1. Find the sine as the ratio of the opposite side to the hypotenuse.
2. Find the cosine as the ratio of the adjacent side to the hypotenuse.
3. Find the tangent is the ratio of the opposite side to the adjacent side.

Example 2.3.1: Evaluating a Trigonometric Function of a Right Triangle

Given the triangle shown in Figure 2.3.3, find the value of $\cos \alpha$.

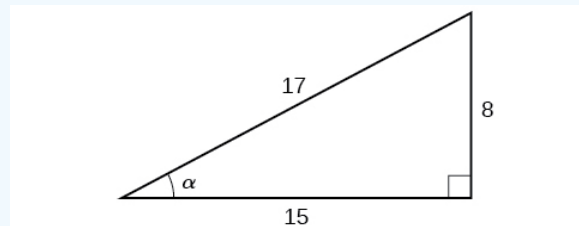


Figure 2.3.3

Solution

The side adjacent to the angle is 15, and the hypotenuse of the triangle is 17, so via Equation 2.3.4:

$$\begin{aligned} \cos(\alpha) &= \frac{\text{adjacent}}{\text{hypotenuse}} \\ &= \frac{15}{17} \end{aligned}$$

Exercise 2.3.1

Given the triangle shown in Figure 2.3.4, find the value of $\sin t$.

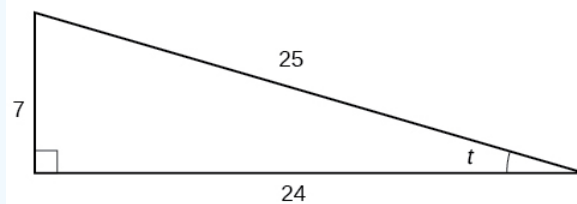


Figure 2.3.4

Answer

$$\frac{7}{25}$$

Relating Angles and Their Functions

When working with right triangles, the same rules apply regardless of the orientation of the triangle. In fact, we can evaluate the six trigonometric functions of either of the two acute angles in the triangle in Figure 2.3.5. The side opposite one acute angle is the side adjacent to the other acute angle, and vice versa.

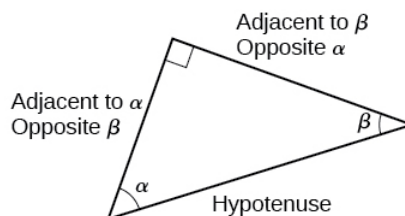


Figure 2.3.5: The side adjacent to one angle is opposite the other.

We will be asked to find all six trigonometric functions for a given angle in a triangle. Our strategy is to find the sine, cosine, and tangent of the angles first. Then, we can find the other trigonometric functions easily because we know that the reciprocal of sine is cosecant, the reciprocal of cosine is secant, and the reciprocal of tangent is cotangent.

how to: Given the side lengths of a right triangle, evaluate the six trigonometric functions of one of the acute angles

1. If needed, draw the right triangle and label the angle provided.
2. Identify the angle, the adjacent side, the side opposite the angle, and the hypotenuse of the right triangle.
3. Find the required function:
 - sine as the ratio of the opposite side to the hypotenuse
 - cosine as the ratio of the adjacent side to the hypotenuse
 - tangent as the ratio of the opposite side to the adjacent side
 - secant as the ratio of the hypotenuse to the adjacent side
 - cosecant as the ratio of the hypotenuse to the opposite side
 - cotangent as the ratio of the adjacent side to the opposite side

Example 2.3.2: Evaluating Trigonometric Functions of Angles Not in Standard Position

Using the triangle shown in Figure 2.3.6, evaluate $\sin \alpha$, $\cos \alpha$, $\tan \alpha$, $\sec \alpha$, $\csc \alpha$, and $\cot \alpha$.

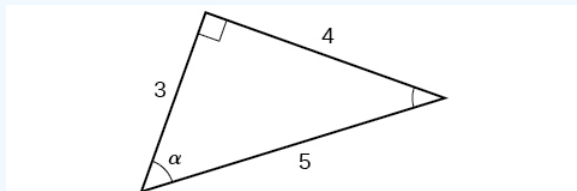


Figure 2.3.6

Solution

$$\begin{aligned}\sin \alpha &= \frac{\text{opposite } \alpha}{\text{hypotenuse}} = \frac{4}{5} \\ \cos \alpha &= \frac{\text{adjacent to } \alpha}{\text{hypotenuse}} = \frac{3}{5} \\ \tan \alpha &= \frac{\text{opposite } \alpha}{\text{adjacent to } \alpha} = \frac{4}{3} \\ \sec \alpha &= \frac{\text{hypotenuse}}{\text{adjacent to } \alpha} = \frac{5}{3} \\ \csc \alpha &= \frac{\text{hypotenuse}}{\text{opposite } \alpha} = \frac{5}{4} \\ \cot \alpha &= \frac{\text{adjacent to } \alpha}{\text{opposite } \alpha} = \frac{3}{4}\end{aligned}$$

Exercise 2.3.2

Using the triangle shown in Figure 2.3.7, evaluate $\sin t$, $\cos t$, $\tan t$, $\sec t$, $\csc t$, and $\cot t$.

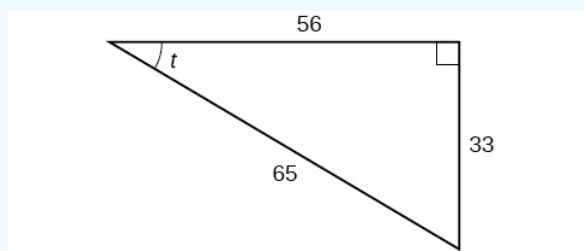


Figure 2.3.7

Answer

$$\begin{aligned}\sin t &= \frac{33}{65}, \cos t = \frac{56}{65}, \tan t = \frac{33}{56}, \\ \sec t &= \frac{65}{56}, \csc t = \frac{65}{33}, \cot t = \frac{56}{33}\end{aligned}$$

Finding Trigonometric Functions of Special Angles Using Side Lengths

We have already discussed the trigonometric functions as they relate to the **special angles** on the unit circle. Now, we can use those relationships to evaluate triangles that contain those special angles. We do this because when we evaluate the special angles in trigonometric functions, they have relatively friendly values, values that contain either no or just one square root in the ratio. Therefore, these are the angles often used in math and science problems. We will use multiples of 30° , 60° , and 45° , however, remember that when dealing with right triangles, we are limited to angles between 0° and 90° .

Suppose we have a 30° , 60° , 90° triangle, which can also be described as a $\frac{\pi}{6}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$ triangle. The sides have lengths in the relation s , $\sqrt{3}s$, $2s$. The sides of a 45° , 45° , 90° triangle, which can also be described as a $\frac{\pi}{4}$, $\frac{\pi}{4}$, $\frac{\pi}{2}$ triangle, have lengths in the relation s , s , $\sqrt{2}s$. These relations are shown in Figure 2.3.8.

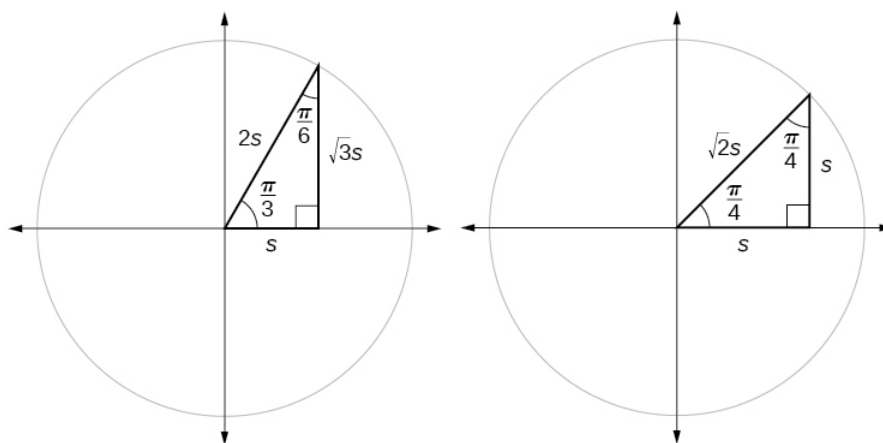


Figure 2.3.8: Side lengths of special triangles

We can then use the ratios of the side lengths to evaluate trigonometric functions of special angles.

Given trigonometric functions of a special angle, evaluate using side lengths.

1. Use the side lengths shown in Figure 2.3.8 for the special angle you wish to evaluate.
2. Use the ratio of side lengths appropriate to the function you wish to evaluate.

Example 2.3.3: Evaluating Trigonometric Functions of Special Angles Using Side Lengths

Find the exact value of the trigonometric functions of $\frac{\pi}{3}$, using side lengths.

Solution

$$\begin{aligned}\sin\left(\frac{\pi}{3}\right) &= \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}s}{2s} = \frac{\sqrt{3}}{2} \\ \cos\left(\frac{\pi}{3}\right) &= \frac{\text{adj}}{\text{hyp}} = \frac{s}{2s} = \frac{1}{2} \\ \tan\left(\frac{\pi}{3}\right) &= \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{3}s}{s} = \sqrt{3} \\ \sec\left(\frac{\pi}{3}\right) &= \frac{\text{hyp}}{\text{adj}} = \frac{2s}{s} = 2 \\ \csc\left(\frac{\pi}{3}\right) &= \frac{\text{hyp}}{\text{opp}} = \frac{2s}{\sqrt{3}s} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \\ \cot\left(\frac{\pi}{3}\right) &= \frac{\text{adj}}{\text{opp}} = \frac{s}{\sqrt{3}s} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}\end{aligned}$$

Exercise 2.3.3

Find the exact value of the trigonometric functions of $\frac{\pi}{4}$ using side lengths.

Answer

$$\begin{aligned}\sin\left(\frac{\pi}{4}\right) &= \frac{\sqrt{2}}{2}, \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \tan\left(\frac{\pi}{4}\right) = 1, \\ \sec\left(\frac{\pi}{4}\right) &= \sqrt{2}, \csc\left(\frac{\pi}{4}\right) = \sqrt{2}, \cot\left(\frac{\pi}{4}\right) = 1\end{aligned}$$

Using Equal Cofunction of Complements

If we look more closely at the relationship between the sine and cosine of the special angles relative to the unit circle, we will notice a pattern. In a right triangle with angles of $\frac{\pi}{6}$ and $\frac{\pi}{3}$, we see that the sine of $\frac{\pi}{3}$, namely $\frac{\sqrt{3}}{2}$, is also the cosine of $\frac{\pi}{6}$, while the

sine of $\frac{\pi}{6}$, namely $\frac{1}{2}$, is also the cosine of $\frac{\pi}{3}$ (Figure 2.3.9).

$$\sin \frac{\pi}{3} = \cos \frac{\pi}{6} = \frac{\sqrt{3}s}{2s} = \frac{\sqrt{3}}{2}$$

$$\sin \frac{\pi}{6} = \cos \frac{\pi}{3} = \frac{s}{2s} = \frac{1}{2}$$

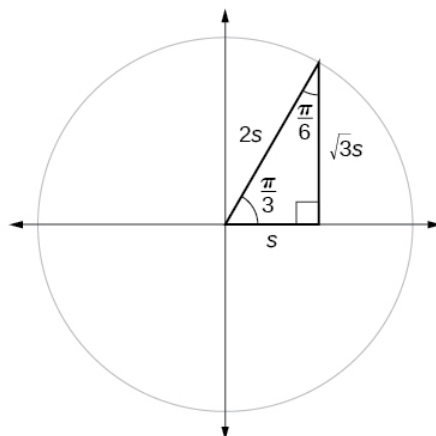
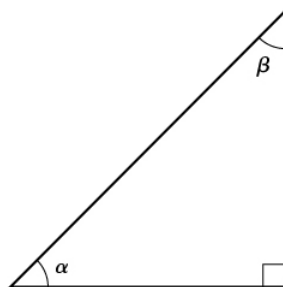


Figure 2.3.9: The sine of $\frac{\pi}{3}$ equals the cosine of $\frac{\pi}{6}$ and vice versa.

This result should not be surprising because, as we see from Figure 2.3.9, the side opposite the angle of $\frac{\pi}{3}$ is also the side adjacent to $\frac{\pi}{6}$, so $\sin(\frac{\pi}{3})$ and $\cos(\frac{\pi}{6})$ are exactly the same ratio of the same two sides, $\sqrt{3}s$ and $2s$. Similarly, $\cos(\frac{\pi}{3})$ and $\sin(\frac{\pi}{6})$ are also the same ratio using the same two sides, s and $2s$.

The interrelationship between the sines and cosines of $\frac{\pi}{6}$ and $\frac{\pi}{3}$ also holds for the two acute angles in any right triangle, since in every case, the ratio of the same two sides would constitute the sine of one angle and the cosine of the other. Since the three angles of a triangle add to π , and the right angle is $\frac{\pi}{2}$, the remaining two angles must also add up to $\frac{\pi}{2}$. That means that a right triangle can be formed with any two angles that add to $\frac{\pi}{2}$ —in other words, any two complementary angles. So we may state a *cofunction identity*: If any two angles are complementary, the sine of one is the cosine of the other, and vice versa. This identity is illustrated in Figure 2.3.10



$$\sin \alpha = \cos \beta$$

$$\sin \beta = \cos \alpha$$

Figure 2.3.10: Cofunction identity of sine and cosine of complementary angles

Using this identity, we can state without calculating, for instance, that the sine of $\frac{\pi}{12}$ equals the cosine of $\frac{5\pi}{12}$, and that the sine of $\frac{5\pi}{12}$ equals the cosine of $\frac{\pi}{12}$. We can also state that if, for a certain angle t , $\cos t = \frac{5}{13}$, then $\sin(\frac{\pi}{2} - t) = \frac{5}{13}$ as well.

COFUNCTION IDENTITIES

The **cofunction identities** in radians are listed in Table 2.3.1.

Table 2.3.1

$$\cos t = \sin(\frac{\pi}{2} - t)$$

$$\sin t = \cos(\frac{\pi}{2} - t)$$

$$\tan t = \cot(\frac{\pi}{2} - t)$$

$$\cot t = \tan(\frac{\pi}{2} - t)$$

$$\sec t = \csc\left(\frac{\pi}{2} - t\right)$$

$$\csc t = \sec\left(\frac{\pi}{2} - t\right)$$

how to: Given the sine and cosine of an angle, find the sine or cosine of its complement.

1. To find the sine of the complementary angle, find the cosine of the original angle.
2. To find the cosine of the complementary angle, find the sine of the original angle.

Example 2.3.4: Using Cofunction Identities

If $\sin t = \frac{5}{12}$, find $(\cos \frac{\pi}{2} - t)$.

Solution

According to the cofunction identities for sine and cosine,

$$\sin t = \cos\left(\frac{\pi}{2} - t\right).$$

So

$$\cos\left(\frac{\pi}{2} - t\right) = \frac{5}{12}.$$

Exercise 2.3.4

If $\csc\left(\frac{\pi}{6}\right) = 2$, find $\sec\left(\frac{\pi}{3}\right)$.

Solution

2

Using Trigonometric Functions

In previous examples, we evaluated the sine and cosine in triangles where we knew all three sides. But the real power of right-triangle trigonometry emerges when we look at triangles in which we know an angle but do not know all the sides.

how to: Given a right triangle, the length of one side, and the measure of one acute angle, find the remaining sides

1. For each side, select the trigonometric function that has the unknown side as either the numerator or the denominator. The known side will in turn be the denominator or the numerator.
2. Write an equation setting the function value of the known angle equal to the ratio of the corresponding sides.
3. Using the value of the trigonometric function and the known side length, solve for the missing side length.

Example 2.3.5: Finding Missing Side Lengths Using Trigonometric Ratios

Find the unknown sides of the triangle in Figure 2.3.11.

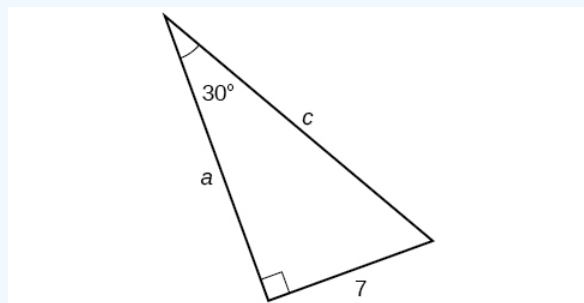


Figure 2.3.11

Solution

We know the angle and the opposite side, so we can use the tangent to find the adjacent side.

$$\tan(30^\circ) = \frac{7}{a}$$

We rearrange to solve for a .

$$a = \frac{7}{\tan(30^\circ)} \quad (2.3.6)$$

$$= 12.1 \quad (2.3.7)$$

We can use the sine to find the hypotenuse.

$$\sin(30^\circ) = \frac{7}{c}$$

Again, we rearrange to solve for c .

$$c = \frac{7}{\sin(30^\circ)} = 14$$

Exercise 2.3.5:

A right triangle has one angle of $\frac{\pi}{3}$ and a hypotenuse of 20. Find the unknown sides and angle of the triangle.

Answer

adjacent = 10; opposite = $10\sqrt{3}$; missing angle is $\frac{\pi}{6}$

Using Right Triangle Trigonometry to Solve Applied Problems

Right-triangle trigonometry has many practical applications. For example, the ability to compute the lengths of sides of a triangle makes it possible to find the height of a tall object without climbing to the top or having to extend a tape measure along its height. We do so by measuring a distance from the base of the object to a point on the ground some distance away, where we can look up to the top of the tall object at an angle. The **angle of elevation** of an object above an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. The right triangle this position creates has sides that represent the unknown height, the measured distance from the base, and the angled line of sight from the ground to the top of the object. Knowing the measured distance to the base of the object and the angle of the line of sight, we can use trigonometric functions to calculate the unknown height. Similarly, we can form a triangle from the top of a tall object by looking downward. The **angle of depression** of an object below an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. See Figure 2.3.12

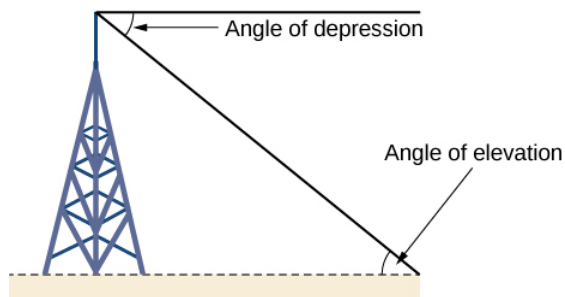


Figure 2.3.12

how to: Given a tall object, measure its height indirectly

1. Make a sketch of the problem situation to keep track of known and unknown information.
2. Lay out a measured distance from the base of the object to a point where the top of the object is clearly visible.
3. At the other end of the measured distance, look up to the top of the object. Measure the angle the line of sight makes with the horizontal.

4. Write an equation relating the unknown height, the measured distance, and the tangent of the angle of the line of sight.
5. Solve the equation for the unknown height.

Example 2.3.6: Measuring a Distance Indirectly

To find the height of a tree, a person walks to a point 30 feet from the base of the tree. She measures an angle of 57° between a line of sight to the top of the tree and the ground, as shown in Figure 2.3.13. Find the height of the tree.

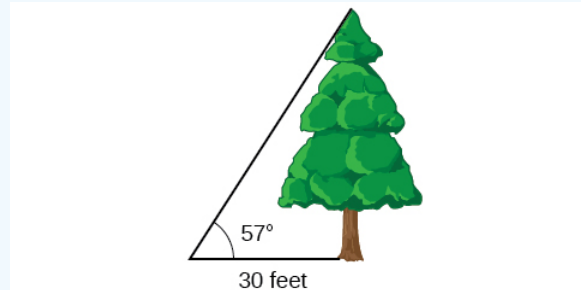


Figure 2.3.13

Solution

We know that the angle of elevation is 57° and the adjacent side is 30 ft long. The opposite side is the unknown height.

The trigonometric function relating the side opposite to an angle and the side adjacent to the angle is the tangent. So we will state our information in terms of the tangent of 57° , letting h be the unknown height.

$$\begin{aligned}\tan \theta &= \frac{\text{opposite}}{\text{adjacent}} \\ \tan(57^\circ) &= \frac{h}{30} && \text{Solve for } h. \\ h &= 30 \tan(57^\circ) && \text{Multiply.} \\ h &\approx 46.2 && \text{Use a calculator.}\end{aligned}\tag{2.3.8}$$

The tree is approximately 46 feet tall.

Exercise 2.3.6:

How long a ladder is needed to reach a windowsill 50 feet above the ground if the ladder rests against the building making an angle of $\frac{5\pi}{12}$ with the ground? Round to the nearest foot.

Answer

About 52 ft

media:

Access these online resources for additional instruction and practice with right triangle trigonometry.

- [Finding Trig Functions on Calculator](#)
- [Finding Trig Functions Using a Right Triangle](#)
- [Relate Trig Functions to Sides of a Right Triangle](#)
- [Determine Six Trig Functions from a Triangle](#)
- [Determine Length of Right Triangle Side](#)

Visit this website for additional practice questions from Learningpod.

Key Equations

Cofunction Identities

$$\cos t = \sin\left(\frac{\pi}{2} - t\right)$$

$$\sin t = \cos\left(\frac{\pi}{2} - t\right)$$

$$\tan t = \cot\left(\frac{\pi}{2} - t\right)$$

$$\cot t = \tan\left(\frac{\pi}{2} - t\right)$$

$$\sec t = \csc\left(\frac{\pi}{2} - t\right)$$

$$\csc t = \sec\left(\frac{\pi}{2} - t\right)$$

Key Concepts

- We can define trigonometric functions as ratios of the side lengths of a right triangle. See [Example](#).
- The same side lengths can be used to evaluate the trigonometric functions of either acute angle in a right triangle. See [Example](#).
- We can evaluate the trigonometric functions of special angles, knowing the side lengths of the triangles in which they occur. See [Example](#).
- Any two complementary angles could be the two acute angles of a right triangle.
- If two angles are complementary, the cofunction identities state that the sine of one equals the cosine of the other and vice versa. See [Example](#).
- We can use trigonometric functions of an angle to find unknown side lengths.
- Select the trigonometric function representing the ratio of the unknown side to the known side. See [Example](#).
- Right-triangle trigonometry permits the measurement of inaccessible heights and distances.
- The unknown height or distance can be found by creating a right triangle in which the unknown height or distance is one of the sides, and another side and angle are known. See [Example](#).

Glossary

adjacent side

in a right triangle, the side between a given angle and the right angle

angle of depression

the angle between the horizontal and the line from the object to the observer's eye, assuming the object is positioned lower than the observer

angle of elevation

the angle between the horizontal and the line from the object to the observer's eye, assuming the object is positioned higher than the observer

opposite side

in a right triangle, the side most distant from a given angle

hypotenuse

the side of a right triangle opposite the right angle

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2.4: Solving Trigonometric Equations with Identities

Learning Objectives

- Verify the fundamental trigonometric identities.
- Simplify trigonometric expressions using algebra and the identities.

In espionage movies, we see international spies with multiple passports, each claiming a different identity. However, we know that each of those passports represents the same person. The trigonometric identities act in a similar manner to multiple passports—there are many ways to represent the same trigonometric expression. Just as a spy will choose an Italian passport when traveling to Italy, we choose the identity that applies to the given scenario when solving a trigonometric equation.



Figure 2.4.1: International passports and travel documents

In this section, we will begin an examination of the fundamental trigonometric identities, including how we can verify them and how we can use them to simplify trigonometric expressions.

Verifying the Fundamental Trigonometric Identities

Identities enable us to simplify complicated expressions. They are the basic tools of trigonometry used in solving trigonometric equations, just as factoring, finding common denominators, and using special formulas are the basic tools of solving algebraic equations. In fact, we use algebraic techniques constantly to simplify trigonometric expressions. Basic properties and formulas of algebra, such as the difference of squares formula and the perfect squares formula, will simplify the work involved with trigonometric expressions and equations. We already know that all of the trigonometric functions are related because they all are defined in terms of the unit circle. Consequently, any trigonometric identity can be written in many ways.

To verify the trigonometric identities, we usually start with the more complicated side of the equation and essentially rewrite the expression until it has been transformed into the same expression as the other side of the equation. Sometimes we have to factor expressions, expand expressions, find common denominators, or use other algebraic strategies to obtain the desired result. In this first section, we will work with the fundamental identities: the **Pythagorean identities**, the even-odd identities, the reciprocal identities, and the quotient identities.

We will begin with the **Pythagorean identities** (Table 2.4.1), which are equations involving trigonometric functions based on the properties of a right triangle. We have already seen and used the first of these identities, but now we will also use additional identities.

Table 2.4.1: Pythagorean Identities

$\sin^2\theta + \cos^2\theta = 1$	$1 + \cot^2\theta = \csc^2\theta$	$1 + \tan^2\theta = \sec^2\theta$
-----------------------------------	-----------------------------------	-----------------------------------

The second and third identities can be obtained by manipulating the first. The identity $1 + \cot^2\theta = \csc^2\theta$ is found by rewriting the left side of the equation in terms of sine and cosine.

Prove: $1 + \cot^2\theta = \csc^2\theta$

$$\begin{aligned}
 1 + \cot^2 \theta &= \left(1 + \frac{\cos^2}{\sin^2}\right) && \text{Rewrite the left side} \\
 &= \left(\frac{\sin^2}{\sin^2}\right) + \left(\frac{\cos^2}{\sin^2}\right) && \text{Write both terms with the common denominator} \\
 &= \frac{\sin^2 + \cos^2}{\sin^2} \\
 &= \frac{1}{\sin^2} \\
 &= \csc^2
 \end{aligned}$$

Similarly, $1 + \tan^2 \theta = \sec^2 \theta$ can be obtained by rewriting the left side of this identity in terms of sine and cosine. This gives

$$\begin{aligned}
 1 + \tan^2 \theta &= 1 + \left(\frac{\sin \theta}{\cos \theta}\right)^2 && \text{Rewrite left side} \\
 &= \left(\frac{\cos \theta}{\cos \theta}\right)^2 + \left(\frac{\sin \theta}{\cos \theta}\right)^2 && \text{Write both terms with the common denominator} \\
 &= \frac{\cos^2 \theta + \sin^2 \theta}{\cos^2 \theta} \\
 &= \frac{1}{\cos^2 \theta} \\
 &= \sec^2 \theta
 \end{aligned}$$

Recall that we determined which trigonometric functions are odd and which are even. The next set of fundamental identities is the set of **even-odd identities**. The **even-odd identities** relate the value of a trigonometric function at a given angle to the value of the function at the opposite angle (Table 2.4.2).

Table 2.4.2: Even-Odd Identities

$\tan(-\theta) = -\tan \theta$	$\sin(-\theta) = -\sin \theta$	$\cos(-\theta) = \cos \theta$
$\cot(-\theta) = -\cot \theta$	$\csc(-\theta) = -\csc \theta$	$\sec(-\theta) = \sec \theta$

Recall that an **odd function** is one in which $f(-x) = -f(x)$ for all x in the domain of f . The sine function is an odd function because $\sin(-\theta) = -\sin \theta$. The graph of an odd function is symmetric about the origin. For example, consider corresponding inputs of $\frac{\pi}{2}$ and $-\frac{\pi}{2}$. The output of $\sin\left(\frac{\pi}{2}\right)$ is opposite the output of $\sin\left(-\frac{\pi}{2}\right)$. Thus,

$$\begin{aligned}
 \sin\left(\frac{\pi}{2}\right) &= 1 \\
 \sin\left(-\frac{\pi}{2}\right) &= -\sin\left(\frac{\pi}{2}\right) \\
 &= -1
 \end{aligned}$$

This is shown in Figure 2.4.2.

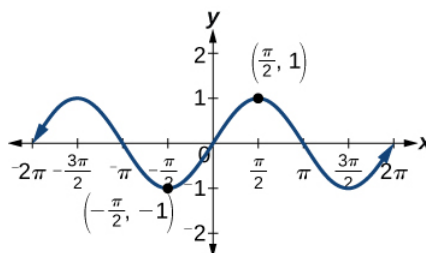


Figure 2.4.2: Graph of $y = \sin \theta$

Recall that an **even function** is one in which

$$f(-x) = f(x) \text{ for all } x \text{ in the domain of } f$$

The graph of an even function is symmetric about the y-axis. The cosine function is an even function because $\cos(-\theta) = \cos \theta$. For example, consider corresponding inputs $\frac{\pi}{4}$ and $-\frac{\pi}{4}$. The output of $\cos\left(\frac{\pi}{4}\right)$ is the same as the output of $\cos\left(-\frac{\pi}{4}\right)$. Thus,

$$\begin{aligned}\cos\left(-\frac{\pi}{4}\right) &= \cos\left(\frac{\pi}{4}\right) \\ &\approx 0.707\end{aligned}$$

See Figure 2.4.3.

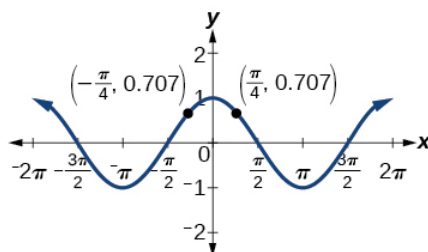


Figure 2.4.3: Graph of $y = \cos \theta$

For all θ in the domain of the sine and cosine functions, respectively, we can state the following:

- Since $\sin(-\theta) = -\sin \theta$, sine is an odd function.
- Since $\cos(-\theta) = \cos \theta$, cosine is an even function.

The other even-odd identities follow from the even and odd nature of the sine and cosine functions. For example, consider the tangent identity, $\tan(-\theta) = -\tan \theta$. We can interpret the tangent of a negative angle as

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin \theta}{\cos \theta} = -\tan \theta.$$

Tangent is therefore an odd function, which means that $\tan(-\theta) = -\tan(\theta)$ for all θ in the domain of the **tangent function**.

The cotangent identity, $\cot(-\theta) = -\cot \theta$, also follows from the sine and cosine identities. We can interpret the cotangent of a negative angle as

$$\cot(-\theta) = \frac{\cos(-\theta)}{\sin(-\theta)} = \frac{\cos \theta}{-\sin \theta} = -\cot \theta.$$

Cotangent is therefore an odd function, which means that $\cot(-\theta) = -\cot(\theta)$ for all θ in the domain of the **cotangent function**.

The **cosecant function** is the reciprocal of the sine function, which means that the cosecant of a negative angle will be interpreted as

$$\csc(-\theta) = \frac{1}{\sin(-\theta)} = \frac{1}{-\sin \theta} = -\csc \theta.$$

The cosecant function is therefore odd.

Finally, the secant function is the reciprocal of the cosine function, and the secant of a negative angle is interpreted as

$$\sec(-\theta) = \frac{1}{\cos(-\theta)} = \frac{1}{\cos \theta} = \sec \theta.$$

The secant function is therefore even.

To sum up, only two of the trigonometric functions, cosine and secant, are even. The other four functions are odd, verifying the even-odd identities.

The next set of fundamental identities is the set of reciprocal identities, which, as their name implies, relate trigonometric functions that are reciprocals of each other. (Table 2.4.3). Recall that we first encountered these identities when defining trigonometric functions from right angles in [Right Angle Trigonometry](#).

Table 2.4.3: Reciprocal Identities

$\sin \theta = \frac{1}{\csc \theta}$	$\csc \theta = \frac{1}{\sin \theta}$
$\cos \theta = \frac{1}{\sec \theta}$	$\sec \theta = \frac{1}{\cos \theta}$
$\tan \theta = \frac{1}{\cot \theta}$	$\cot \theta = \frac{1}{\tan \theta}$

The final set of identities is the set of **quotient identities**, which define relationships among certain trigonometric functions and can be very helpful in verifying other identities (Table 2.4.4).

Table 2.4.4: Quotient Identities

$\tan \theta = \frac{\sin \theta}{\cos \theta}$	$\cot \theta = \frac{\cos \theta}{\sin \theta}$
---	---

The reciprocal and quotient identities are derived from the definitions of the basic trigonometric functions.

SUMMARIZING TRIGONOMETRIC IDENTITIES

The Pythagorean identities are based on the properties of a right triangle.

$$\cos^2 \theta + \sin^2 \theta = 1 \quad (2.4.1)$$

$$1 + \cot^2 \theta = \csc^2 \theta \quad (2.4.2)$$

$$1 + \tan^2 \theta = \sec^2 \theta \quad (2.4.3)$$

The even-odd identities relate the value of a trigonometric function at a given angle to the value of the function at the opposite angle.

$$\tan(-\theta) = -\tan \theta \quad (2.4.4)$$

$$\cot(-\theta) = -\cot \theta \quad (2.4.5)$$

$$\sin(-\theta) = -\sin \theta \quad (2.4.6)$$

$$\csc(-\theta) = -\csc \theta \quad (2.4.7)$$

$$\cos(-\theta) = \cos \theta \quad (2.4.8)$$

$$\sec(-\theta) = \sec \theta \quad (2.4.9)$$

The reciprocal identities define reciprocals of the trigonometric functions.

$$\sin \theta = \frac{1}{\csc \theta} \quad (2.4.10)$$

$$\cos \theta = \frac{1}{\sec \theta} \quad (2.4.11)$$

$$\tan \theta = \frac{1}{\cot \theta} \quad (2.4.12)$$

$$\csc \theta = \frac{1}{\sin \theta} \quad (2.4.13)$$

$$\sec \theta = \frac{1}{\cos \theta} \quad (2.4.14)$$

$$\cot \theta = \frac{1}{\tan \theta} \quad (2.4.15)$$

The **quotient identities** define the relationship among the trigonometric functions.

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \quad (2.4.16)$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta} \quad (2.4.17)$$

✓ Example 2.4.1: Graphing the Equations of an Identity

Graph both sides of the identity $\cot \theta = \frac{1}{\tan \theta}$. In other words, on the graphing calculator, graph $y = \cot \theta$ and $y = \frac{1}{\tan \theta}$.

Solution

See Figure 2.4.4.

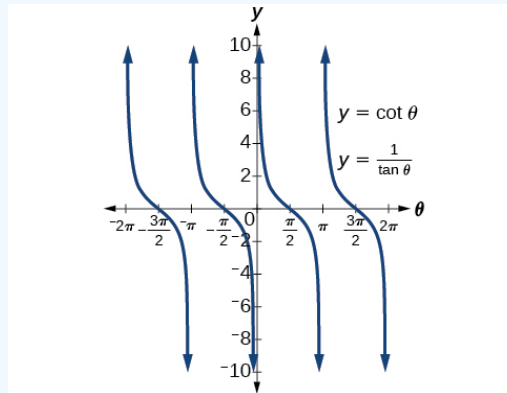


Figure 2.4.4

Analysis

We see only one graph because both expressions generate the same image. One is on top of the other. This is a good way to prove any identity. If both expressions give the same graph, then they must be identities.

📌 How to: Given a trigonometric identity, verify that it is true.

1. Work on one side of the equation. It is usually better to start with the more complex side, as it is easier to simplify than to build.
2. Look for opportunities to factor expressions, square a binomial, or add fractions.
3. Noting which functions are in the final expression, look for opportunities to use the identities and make the proper substitutions.
4. If these steps do not yield the desired result, try converting all terms to sines and cosines.

✓ Example 2.4.2: Verifying a Trigonometric Identity

Verify $\tan \theta \cos \theta = \sin \theta$.

Solution

We will start on the left side, as it is the more complicated side:

$$\begin{aligned} \tan \theta \cos \theta &= \left(\frac{\sin \theta}{\cos \theta} \right) \cos \theta \\ &= \sin \theta \end{aligned}$$

Analysis

This identity was fairly simple to verify, as it only required writing $\tan \theta$ in terms of $\sin \theta$ and $\cos \theta$.

? Exercise 2.4.1

Verify the identity $\csc \theta \cos \theta \tan \theta = 1$.

Answer

$$\begin{aligned}\csc \theta \cos \theta \tan \theta &= \left(\frac{1}{\sin \theta} \right) \cos \theta \left(\frac{\sin \theta}{\cos \theta} \right) \\ &= \frac{\cos \theta}{\sin \theta} \left(\frac{\sin \theta}{\cos \theta} \right) \\ &= \frac{\sin \theta \cos \theta}{\sin \theta \cos \theta} \\ &= 1\end{aligned}$$

✓ Example 2.4.3A: Verifying the Equivalency Using the Even-Odd Identities

Verify the following equivalency using the even-odd identities:

$$(1 + \sin x)[1 + \sin(-x)] = \cos^2 x$$

Solution

Working on the left side of the equation, we have

$$(1 + \sin x)[1 + \sin(-x)] = (1 + \sin x)(1 - \sin x)$$

Since

$$\sin(-x) = -\sin x$$

$$[5pt] = 1 - \sin^2 x \quad \text{Difference of squares}$$

$$[5pt] = \cos^2 x$$

$$\cos^2 x = 1 - \sin^2 x$$

✓ Example 2.4.3B: Verifying a Trigonometric Identity Involving $\sec^2 \theta$

Verify the identity $\frac{\sec^2 \theta - 1}{\sec^2 \theta} = \sin^2 \theta$

Solution

As the left side is more complicated, let's begin there.

$$\begin{aligned}\frac{\sec^2 \theta - 1}{\sec^2 \theta} &= \frac{(\tan^2 \theta + 1) - 1}{\sec^2 \theta} \\ \sec^2 \theta &= \tan^2 \theta + 1 \\ &= \frac{\tan^2 \theta}{\sec^2 \theta} \\ &= \tan^2 \theta \left(\frac{1}{\sec^2 \theta} \right) \\ &= \tan^2 \theta (\cos^2 \theta) \\ \cos^2 \theta &= \frac{1}{\sec^2 \theta} \\ &= \left(\frac{\sin^2 \theta}{\cos^2 \theta} \right) \\ \tan^2 \theta &= \frac{\sin^2 \theta}{\cos^2 \theta} \\ &= \sin^2 \theta\end{aligned}$$

There is more than one way to verify an identity. Here is another possibility. Again, we can start with the left side.

$$\begin{aligned}\frac{\sec^2\theta - 1}{\sec^2\theta} &= \frac{\sec^2\theta}{\sec^2\theta} - \frac{1}{\sec^2\theta} \\ &= 1 - \cos^2\theta \\ &= \sin^2\theta\end{aligned}$$

Analysis

In the first method, we used the identity $\sec^2\theta = \tan^2\theta + 1$ and continued to simplify. In the second method, we split the fraction, putting both terms in the numerator over the common denominator. This problem illustrates that there are multiple ways we can verify an identity. Employing some creativity can sometimes simplify a procedure. As long as the substitutions are correct, the answer will be the same.

? Exercise 2.4.2

Show that $\frac{\cot\theta}{\csc\theta} = \cos\theta$.

Answer

$$\begin{aligned}\frac{\cot\theta}{\csc\theta} &= \frac{\frac{\cos\theta}{\sin\theta}}{\frac{1}{\sin\theta}} \\ &= \frac{\cos\theta}{\sin\theta} \cdot \frac{\sin\theta}{1} \\ &= \cos\theta\end{aligned}$$

✓ Example 2.4.4: Creating and Verifying an Identity

Create an identity for the expression $2 \tan\theta \sec\theta$ by rewriting strictly in terms of sine.

Solution

There are a number of ways to begin, but here we will use the quotient and reciprocal identities to rewrite the expression:

$$\begin{aligned}2 \tan\theta \sec\theta &= 2 \left(\frac{\sin\theta}{\cos\theta} \right) \left(\frac{1}{\cos\theta} \right) \\ &= \frac{2 \sin\theta}{\cos^2\theta} \\ &= \frac{2 \sin\theta}{1 - \sin^2\theta} \quad \text{Substitute } 1 - \sin^2\theta \text{ for } \cos^2\theta\end{aligned}$$

Thus,

$$2 \tan\theta \sec\theta = \frac{2 \sin\theta}{1 - \sin^2\theta}$$

✓ Example 2.4.5: Verifying an Identity Using Algebra and Even/Odd Identities

Verify the identity:

$$\frac{\sin^2(-\theta) - \cos^2(-\theta)}{\sin(-\theta) - \cos(-\theta)} = \cos\theta - \sin\theta$$

Solution

Let's start with the left side and simplify:

$$\begin{aligned}
 \frac{\sin^2(-\theta) - \cos^2(-\theta)}{\sin(-\theta) - \cos(-\theta)} &= \frac{[\sin(-\theta)]^2 - [\cos(-\theta)]^2}{\sin(-\theta) - \cos(-\theta)} \\
 &= \frac{(-\sin\theta)^2 - (\cos\theta)^2}{-\sin\theta - \cos\theta}, \sin(-x) = -\sin x \text{ and } \cos(-x) = \cos x \\
 &= \frac{(\sin\theta)^2 - (\cos\theta)^2}{-\sin\theta - \cos\theta} \quad \text{Difference of squares} \\
 &= \frac{(\sin\theta - \cos\theta)(\sin\theta + \cos\theta)}{-(\sin\theta + \cos\theta)} \\
 &= \cos\theta - \sin\theta
 \end{aligned}$$

? Exercise 2.4.3

Verify the identity $\frac{\sin^2\theta - 1}{\tan\theta\sin\theta - \tan\theta} = \frac{\sin\theta + 1}{\tan\theta}$.

Answer

$$\begin{aligned}
 \frac{\sin^2\theta - 1}{\tan\theta\sin\theta - \tan\theta} &= \frac{(\sin\theta + 1)(\sin\theta - 1)}{\tan\theta(\sin\theta - 1)} \\
 &= \frac{\sin\theta + 1}{\tan\theta}
 \end{aligned}$$

✓ Example 2.4.6: Verifying an Identity Involving Cosines and Cotangents

Verify the identity: $(1 - \cos^2x)(1 + \cot^2x) = 1$.

Solution

$$\begin{aligned}
 (1 - \cos^2x)(1 + \cot^2x) &= (1 - \cos^2x) \left(1 + \frac{\cos^2x}{\sin^2x} \right) \\
 &= (1 - \cos^2x) \left(\frac{\sin^2x}{\sin^2x} + \frac{\cos^2x}{\sin^2x} \right) \quad \text{Find the common denominator} \\
 &= (1 - \cos^2x) \left(\frac{\sin^2x + \cos^2x}{\sin^2x} \right) \\
 &= (\sin^2x) \left(\frac{1}{\sin^2x} \right) \\
 &= 1
 \end{aligned}$$

Using Algebra to Simplify Trigonometric Expressions

We have seen that algebra is very important in verifying trigonometric identities, but it is just as critical in simplifying trigonometric expressions before solving. Being familiar with the basic properties and formulas of algebra, such as the difference of squares formula, the perfect square formula, or substitution, will simplify the work involved with trigonometric expressions and equations.

For example, the equation $(\sin x + 1)(\sin x - 1) = 0$ resembles the equation $(x + 1)(x - 1) = 0$, which uses the factored form of the difference of squares. Using algebra makes finding a solution straightforward and familiar. We can set each factor equal to zero and solve. This is one example of recognizing algebraic patterns in trigonometric expressions or equations.

Another example is the difference of squares formula, $a^2 - b^2 = (a - b)(a + b)$, which is widely used in many areas other than mathematics, such as engineering, architecture, and physics. We can also create our own identities by continually expanding an expression and making the appropriate substitutions. Using algebraic properties and formulas makes many trigonometric equations easier to understand and solve.

✓ Example 2.4.7A: Writing the Trigonometric Expression as an Algebraic Expression

Write the following trigonometric expression as an algebraic expression: $2\cos^2\theta + \cos\theta - 1$.

Solution

Notice that the pattern displayed has the same form as a standard quadratic expression, $ax^2 + bx + c$. Letting $\cos\theta = x$, we can rewrite the expression as follows:

$$2x^2 + x - 1$$

This expression can be factored as $(2x + 1)(x - 1)$. If it were set equal to zero and we wanted to solve the equation, we would use the zero factor property and solve each factor for x . At this point, we would replace x with $\cos\theta$ and solve for θ .

✓ Example 2.4.7B: Rewriting a Trigonometric Expression Using the Difference of Squares

Rewrite the trigonometric expression using the difference of squares: $4\cos^2\theta - 1$.

Solution

Notice that both the coefficient and the trigonometric expression in the first term are squared, and the square of the number 1 is 1. This is the difference of squares.

$$\begin{aligned} 4\cos^2\theta - 1 &= (2\cos\theta)^2 - 1 \\ &= (2\cos\theta - 1)(2\cos\theta + 1) \end{aligned}$$

Analysis

If this expression were written in the form of an equation set equal to zero, we could solve each factor using the zero factor property. We could also use substitution like we did in the previous problem and let $\cos\theta = x$, rewrite the expression as $4x^2 - 1$, and factor $(2x - 1)(2x + 1)$. Then replace x with $\cos\theta$ and solve for the angle.

? Exercise 2.4.4

Rewrite the trigonometric expression using the difference of squares: $25 - 9\sin^2\theta$.

Answer

This is a difference of squares formula: $25 - 9\sin^2\theta = (5 - 3\sin\theta)(5 + 3\sin\theta)$.

✓ Example 2.4.8: Simplify by Rewriting and Using Substitution

Simplify the expression by rewriting and using identities:

$$\csc^2\theta - \cot^2\theta$$

Solution

We can start with the Pythagorean identity.

$$\begin{aligned} 1 + \cot^2\theta &= \csc^2\theta \\ \text{Now we can simplify by substituting } 1 + \cot^2\theta &\text{ for } \csc^2\theta \\ \csc^2\theta - \cot^2\theta &= 1 + \cot^2\theta - \cot^2\theta \\ &= 1 \end{aligned}$$

? Exercise 2.4.5

Use algebraic techniques to verify the identity: $\frac{\cos\theta}{1 + \sin\theta} = \frac{1 - \sin\theta}{\cos\theta}$.

(Hint: Multiply the numerator and denominator on the left side by $1 - \sin\theta$.)

Answer

$$\begin{aligned}\frac{\cos \theta}{1 + \sin \theta} \left(\frac{1 - \sin \theta}{1 - \sin \theta} \right) &= \frac{\cos \theta (1 - \sin \theta)}{1 - \sin^2 \theta} \\ &= \frac{\cos \theta (1 - \sin \theta)}{\cos^2 \theta} \\ &= \frac{1 - \sin \theta}{\cos \theta}\end{aligned}$$

Media

Access these online resources for additional instruction and practice with the fundamental trigonometric identities.

- Fundamental Trigonometric Identities
- Verifying Trigonometric Identities

Key Equations

Pythagorean identities	$\cos^2 \theta + \sin^2 \theta = 1$ $1 + \cot^2 \theta = \csc^2 \theta$ $1 + \tan^2 \theta = \sec^2 \theta$
Even-odd identities	$\tan(-\theta) = -\tan \theta$ $\cot(-\theta) = -\cot \theta$ $\sin(-\theta) = -\sin \theta$ $\csc(-\theta) = -\csc \theta$ $\cos(-\theta) = \cos \theta$ $\sec(-\theta) = \sec \theta$
Reciprocal identities	$\sin \theta = \frac{1}{\csc \theta}$ $\cos \theta = \frac{1}{\sec \theta}$ $\tan \theta = \frac{1}{\cot \theta}$ $\csc \theta = \frac{1}{\sin \theta}$ $\sec \theta = \frac{1}{\cos \theta}$ $\cot \theta = \frac{1}{\tan \theta}$
Quotient identities	$\tan \theta = \frac{\sin \theta}{\cos \theta}$ $\cot \theta = \frac{\cos \theta}{\sin \theta}$

Key Concepts

- There are multiple ways to represent a trigonometric expression. Verifying the identities illustrates how expressions can be rewritten to simplify a problem.
- Graphing both sides of an identity will verify it. See Example 2.4.1.
- Simplifying one side of the equation to equal the other side is another method for verifying an identity. See Example 2.4.2 and Example 2.4.3.
- The approach to verifying an identity depends on the nature of the identity. It is often useful to begin on the more complex side of the equation. See Example 2.4.4.
- We can create an identity and then verify it. See Example 2.4.5.
- Verifying an identity may involve algebra with the fundamental identities. See Example 2.4.6 and Example 2.4.7.

- Algebraic techniques can be used to simplify trigonometric expressions. We use algebraic techniques throughout this text, as they consist of the fundamental rules of mathematics. See Example 2.4.8, Example 2.4.9, and Example 2.4.10

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2.5: Sum and Difference Identities

Learning Objectives

- Use sum and difference formulas for cosine.
- Use sum and difference formulas for sine.
- Use sum and difference formulas for tangent.
- Use sum and difference formulas for cofunctions.
- Use sum and difference formulas to verify identities.

How can the height of a mountain be measured? What about the distance from Earth to the sun? Like many seemingly impossible problems, we rely on mathematical formulas to find the answers. The trigonometric identities, commonly used in mathematical proofs, have had real-world applications for centuries, including their use in calculating long distances.

The trigonometric identities we will examine in this section can be traced to a Persian astronomer who lived around 950 AD, but the ancient Greeks discovered these same formulas much earlier and stated them in terms of chords. These are special equations or postulates, true for all values input to the equations, and with innumerable applications.



Figure 2.5.1: Mount McKinley, in Denali National Park, Alaska, rises 20,237 feet (6,168 m) above sea level. It is the highest peak in North America. (credit: Daniel A. Leifheit, Flickr)

In this section, we will learn techniques that will enable us to solve problems such as the ones presented above. The formulas that follow will simplify many trigonometric expressions and equations. Keep in mind that, throughout this section, the term *formula* is used synonymously with the word *identity*.

Using the Sum and Difference Formulas for Cosine

Finding the exact value of the sine, cosine, or tangent of an angle is often easier if we can rewrite the given angle in terms of two angles that have known trigonometric values. We can use the special angles, which we can review in the unit circle shown in Figure 2.5.2.

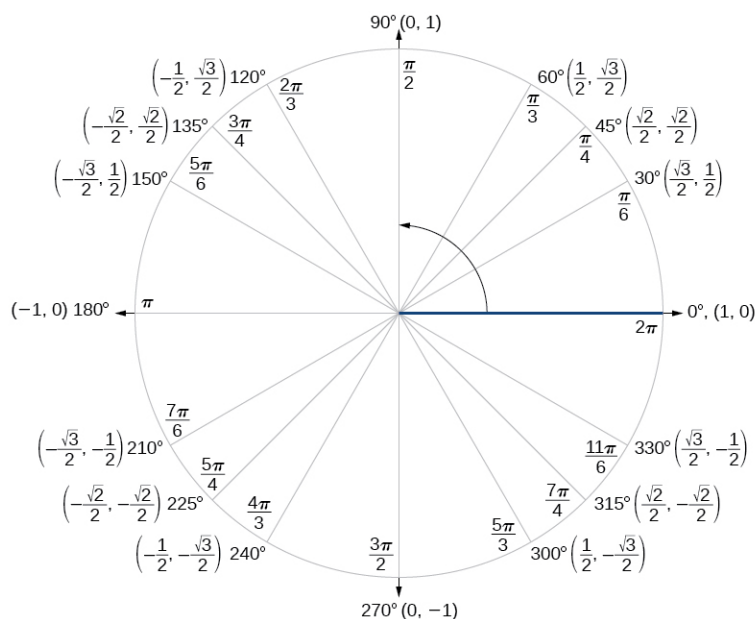


Figure 2.5.2: The Unit Circle

We will begin with the sum and difference formulas for cosine, so that we can find the cosine of a given angle if we can break it up into the sum or difference of two of the special angles (Table 2.5.1).

Table 2.5.1

Sum formula for cosine	$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$
Difference formula for cosine	$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$

First, we will prove the difference formula for cosines. Let's consider two points on the unit circle (Figure 2.5.3). Point P is at an angle α from the positive x -axis with coordinates $(\cos \alpha, \sin \alpha)$ and point Q is at an angle of β from the positive x -axis with coordinates $(\cos \beta, \sin \beta)$. Note the measure of angle POQ is $\alpha - \beta$.

Label two more points: A at an angle of $(\alpha - \beta)$ from the positive x -axis with coordinates $(\cos(\alpha - \beta), \sin(\alpha - \beta))$; and point B with coordinates $(1, 0)$. Triangle POQ is a rotation of triangle AOB and thus the distance from P to Q is the same as the distance from A to B .

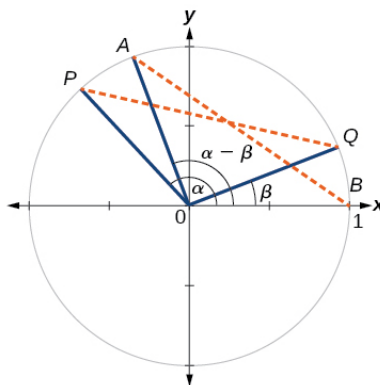


Figure 2.5.3

We can find the distance from P to Q using the distance formula.

$$\begin{aligned}
 d_{PQ} &= \sqrt{(\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2} \\
 &= \sqrt{\cos^2 \alpha - 2 \cos \alpha \cos \beta + \cos^2 \beta + \sin^2 \alpha - 2 \sin \alpha \sin \beta + \sin^2 \beta} && \text{Apply Pythagorean identity and simplify.} \\
 &= \sqrt{(\cos^2 \alpha + \sin^2 \alpha) + (\cos^2 \beta + \sin^2 \beta) - 2 \cos \alpha \cos \beta - 2 \sin \alpha \sin \beta} \\
 &= \sqrt{1 + 1 - 2 \cos \alpha \cos \beta - 2 \sin \alpha \sin \beta} \\
 &= \sqrt{2 - 2 \cos \alpha \cos \beta - 2 \sin \alpha \sin \beta}
 \end{aligned}$$

Similarly, using the distance formula we can find the distance from A to B .

$$\begin{aligned}
 d_{AB} &= \sqrt{(\cos(\alpha - \beta) - 1)^2 + (\sin(\alpha - \beta) - 0)^2} \\
 &= \sqrt{\cos^2(\alpha - \beta) - 2 \cos(\alpha - \beta) + 1 + \sin^2(\alpha - \beta)} && \text{Apply Pythagorean identity and simplify} \\
 &= \sqrt{(\cos^2(\alpha - \beta) + \sin^2(\alpha - \beta)) - 2 \cos(\alpha - \beta) + 1} \\
 &= \sqrt{1 - 2 \cos(\alpha - \beta) + 1} \\
 &= \sqrt{2 - 2 \cos(\alpha - \beta)} && \text{Subtract 2 from both sides and divide both sides by } -2.
 \end{aligned}$$

$$\cos \alpha \cos \beta + \sin \alpha \sin \beta = \cos(\alpha - \beta)$$

Thus, we have the difference formula for cosine. We can use similar methods to derive the cosine of the sum of two angles.

SUM AND DIFFERENCE FORMULAS FOR COSINE

These formulas can be used to calculate the cosine of sums and differences of angles.

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \quad (2.5.1)$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \quad (2.5.2)$$

How to: Given two angles, find the cosine of the difference between the angles

1. Write the difference formula for cosine.
2. Substitute the values of the given angles into the formula.
3. Simplify.

✓ Example 2.5.1: Finding the Exact Value Using the Formula for the Cosine of the Difference of Two Angles

Using the formula for the cosine of the difference of two angles, find the exact value of $\cos\left(\frac{5\pi}{4} - \frac{\pi}{6}\right)$.

Solution

Begin by writing the formula for the cosine of the difference of two angles. Then substitute the given values.

$$\begin{aligned}\cos(\alpha - \beta) &= \cos \alpha \cos \beta + \sin \alpha \sin \beta \\ \cos\left(\frac{5\pi}{4} - \frac{\pi}{6}\right) &= \cos\left(\frac{5\pi}{4}\right) \cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{5\pi}{4}\right) \sin\left(\frac{\pi}{6}\right) \\ &= \left(-\frac{\sqrt{2}}{2}\right) \left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right) \left(\frac{1}{2}\right) \\ &= -\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \\ &= \frac{-\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

Keep in mind that we can always check the answer using a graphing calculator in radian mode.

? Exercise 2.5.1:

Find the exact value of $\cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$.

Answer

$$\frac{\sqrt{2} + \sqrt{6}}{4}$$

✓ Example 2.5.2: Finding the Exact Value Using the Formula for the Sum of Two Angles for Cosine

Find the exact value of $\cos(75^\circ)$.

Solution

As $75^\circ = 45^\circ + 30^\circ$, we can evaluate $\cos(75^\circ)$ as $\cos(45^\circ + 30^\circ)$.

$$\begin{aligned}\cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \\ \cos(45^\circ + 30^\circ) &= \cos(45^\circ) \cos(30^\circ) - \sin(45^\circ) \sin(30^\circ) \\ &= \frac{\sqrt{2}}{2} \left(\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{2}}{2} \left(\frac{1}{2}\right) \\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

Keep in mind that we can always check the answer using a graphing calculator in degree mode.

Analysis

Note that we could have also solved this problem using the fact that $75^\circ = 135^\circ - 60^\circ$.

$$\begin{aligned}
 \cos(\alpha - \beta) &= \cos \alpha \cos \beta + \sin \alpha \sin \beta \\
 \cos(135^\circ - 60^\circ) &= \cos(135^\circ) \cos(60^\circ) + \sin(135^\circ) \sin(60^\circ) \\
 &= \left(-\frac{\sqrt{2}}{2}\right) \left(\frac{1}{2}\right) + \left(\frac{\sqrt{2}}{2}\right) \left(\frac{\sqrt{3}}{2}\right) \\
 &= -\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} \\
 &= \frac{\sqrt{6} - \sqrt{2}}{4}
 \end{aligned}$$

? Exercise 2.5.2

Find the exact value of $\cos(105^\circ)$.

Answer

$$\frac{\sqrt{2} - \sqrt{6}}{4}$$

Using the Sum and Difference Formulas for Sine

The sum and difference formulas for sine can be derived in the same manner as those for cosine, and they resemble the cosine formulas.

✚ SUM AND DIFFERENCE FORMULAS FOR SINE

These formulas can be used to calculate the sines of sums and differences of angles.

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \quad (2.5.3)$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta \quad (2.5.4)$$

✚ How to: Given two angles, find the sine of the difference between the angles

1. Write the difference formula for sine.
2. Substitute the given angles into the formula.
3. Simplify.

✓ Example 2.5.3: Using Sum and Difference Identities to Evaluate the Difference of Angles

Use the sum and difference identities to evaluate the difference of the angles and show that part *a* equals part *b*.

- a. $\sin(45^\circ - 30^\circ)$
- b. $\sin(135^\circ - 120^\circ)$

Solution

- a. Let's begin by writing the formula and substitute the given angles.

$$\begin{aligned}
 \sin(\alpha - \beta) &= \sin \alpha \cos \beta - \cos \alpha \sin \beta \\
 \sin(45^\circ - 30^\circ) &= \sin(45^\circ) \cos(30^\circ) - \cos(45^\circ) \sin(30^\circ)
 \end{aligned}$$

Next, we need to find the values of the trigonometric expressions.

$$\sin(45^\circ) = \frac{\sqrt{2}}{2}, \quad \cos(30^\circ) = \frac{\sqrt{3}}{2}, \quad \cos(45^\circ) = \frac{\sqrt{2}}{2}, \quad \sin(30^\circ) = \frac{1}{2}$$

Now we can substitute these values into the equation and simplify.

$$\begin{aligned}
 \sin(45^\circ - 30^\circ) &= \frac{\sqrt{2}}{2} \left(\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{2}}{2} \left(\frac{1}{2}\right) \\
 &= \frac{\sqrt{6} - \sqrt{2}}{4}
 \end{aligned}$$

- b. Again, we write the formula and substitute the given angles.

$$\begin{aligned}
 \sin(\alpha - \beta) &= \sin \alpha \cos \beta - \cos \alpha \sin \beta \\
 \sin(135^\circ - 120^\circ) &= \sin(135^\circ) \cos(120^\circ) - \cos(135^\circ) \sin(120^\circ)
 \end{aligned}$$

Next, we find the values of the trigonometric expressions.

$$\sin(135^\circ) = \frac{\sqrt{2}}{2}, \quad \cos(120^\circ) = -\frac{1}{2}, \quad \cos(135^\circ) = -\frac{\sqrt{2}}{2}, \quad \sin(120^\circ) = \frac{\sqrt{3}}{2}$$

Now we can substitute these values into the equation and simplify.

$$\begin{aligned} \sin(135^\circ - 120^\circ) &= \frac{\sqrt{2}}{2} \left(-\frac{1}{2} \right) - \left(-\frac{\sqrt{2}}{2} \right) \left(\frac{\sqrt{3}}{2} \right) \\ &= \frac{\sqrt{6} - \sqrt{2}}{4} \end{aligned}$$

✓ Example 2.5.4: Finding the Exact Value of an Expression Involving an Inverse Trigonometric Function

Find the exact value of $\sin(\cos^{-1} \frac{1}{2} + \sin^{-1} \frac{3}{5})$. Then check the answer with a graphing calculator.

Solution

The pattern displayed in this problem is $\sin(\alpha + \beta)$. Let $\alpha = \cos^{-1} \frac{1}{2}$ and $\beta = \sin^{-1} \frac{3}{5}$. Then we can write

$$\begin{aligned} \cos \alpha &= \frac{1}{2}, \quad 0 \leq \alpha \leq \pi \\ \sin \beta &= \frac{3}{5}, \quad -\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2} \end{aligned}$$

We will use the Pythagorean identities to find $\sin \alpha$ and $\cos \beta$

$$\begin{aligned} \sin \alpha &= \sqrt{1 - \cos^2 \alpha} \\ &= \sqrt{1 - \frac{1}{4}} \\ &= \sqrt{\frac{3}{4}} \\ &= \frac{\sqrt{3}}{2} \\ \cos \beta &= \sqrt{1 - \sin^2 \beta} \\ &= \sqrt{1 - \frac{9}{25}} \\ &= \sqrt{\frac{16}{25}} \\ &= \frac{4}{5} \end{aligned}$$

Using the sum formula for sine,

$$\begin{aligned} \sin(\cos^{-1} \frac{1}{2} + \sin^{-1} \frac{3}{5}) &= \sin(\alpha + \beta) \\ &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \\ &= \frac{\sqrt{3}}{2} \cdot \frac{4}{5} + \frac{1}{2} \cdot \frac{3}{5} \\ &= \frac{4\sqrt{3} + 3}{10} \end{aligned}$$

Using the Sum and Difference Formulas for Tangent

Finding exact values for the tangent of the sum or difference of two angles is a little more complicated, but again, it is a matter of recognizing the pattern.

Finding the sum of two angles formula for tangent involves taking quotient of the sum formulas for sine and cosine and simplifying. Recall,

$$\tan x = \frac{\sin x}{\cos x}, \text{ when } \cos x \neq 0.$$

Let's derive the sum formula for tangent.

$$\begin{aligned}
 \tan(\alpha + \beta) &= \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} \\
 &= \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta - \sin \alpha \sin \beta} \\
 &= \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta} \cdot \frac{\cos \alpha \cos \beta}{\cos \alpha \cos \beta - \sin \alpha \sin \beta} \\
 &= \frac{\frac{\sin \alpha \cos \beta}{\cos \alpha \cos \beta} + \frac{\cos \alpha \sin \beta}{\cos \alpha \cos \beta}}{1 - \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta}} \\
 &= \frac{\frac{\sin \alpha}{\cos \alpha} + \frac{\sin \beta}{\cos \beta}}{1 - \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta}} \\
 &= \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}
 \end{aligned}$$

We can derive the difference formula for tangent in a similar way.

📌 SUM AND DIFFERENCE FORMULAS FOR TANGENT

The sum and difference formulas for tangent are:

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \quad (2.5.5)$$

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} \quad (2.5.6)$$

📌 How to: Given two angles, find the tangent of the sum of the angles

1. Write the sum formula for tangent.
2. Substitute the given angles into the formula.
3. Simplify.

✓ Example 2.5.5: Finding the Exact Value of an Expression Involving Tangent

Find the exact value of $\tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right)$.

Solution

Let's first write the sum formula for tangent and then substitute the given angles into the formula.

$$\begin{aligned}
 \tan(\alpha + \beta) &= \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \\
 \tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right) &= \frac{\tan\left(\frac{\pi}{6}\right) + \tan\left(\frac{\pi}{4}\right)}{1 - \left(\tan\left(\frac{\pi}{6}\right)\right)\left(\tan\left(\frac{\pi}{4}\right)\right)}
 \end{aligned}$$

Next, we determine the individual function values within the formula:

$$\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}, \quad \text{and} \quad \tan\left(\frac{\pi}{4}\right) = 1 \quad (2.5.7)$$

So we have,

$$\begin{aligned}\tan\left(\frac{\pi}{6} + \frac{\pi}{4}\right) &= \frac{\frac{1}{\sqrt{3}} + 1}{1 - \left(\frac{1}{\sqrt{3}}\right)} \quad (1) \\ &= \frac{\frac{1 + \sqrt{3}}{\sqrt{3}}}{\frac{\sqrt{3} - 1}{\sqrt{3}}} \\ &= \frac{1 + \sqrt{3}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3} - 1} \\ &= \frac{\sqrt{3} + 1}{\sqrt{3} - 1}\end{aligned}$$

? Exercise 2.5.5:

Find the exact value of $\tan\left(\frac{2\pi}{3} + \frac{\pi}{4}\right)$.

Answer

$$\frac{1 - \sqrt{3}}{1 + \sqrt{3}}$$

✓ Example 2.5.6: Finding Multiple Sums and Differences of Angles

Given $\sin \alpha = \frac{3}{5}$, $0 < \alpha < \frac{\pi}{2}$, and $\cos \beta = -\frac{5}{13}$, $\pi < \beta < \frac{3\pi}{2}$,

find

- $\sin(\alpha + \beta)$
- $\cos(\alpha + \beta)$
- $\tan(\alpha + \beta)$
- $\tan(\alpha - \beta)$

Solution

We can use the sum and difference formulas to identify the sum or difference of angles when the ratio of sine, cosine, or tangent is provided for each of the individual angles. To do so, we construct what is called a reference triangle to help find each component of the sum and difference formulas.

- a. To find $\sin(\alpha + \beta)$, we begin with $\sin \alpha = \frac{3}{5}$ and $0 < \alpha < \frac{\pi}{2}$. The side opposite α has length 3, the hypotenuse has length 5, and α is in the first quadrant. See Figure 2.5.4. Using the Pythagorean Theorem, we can find the length of side a :

$$a^2 + 3^2 = 5^2$$

$$a^2 = 16$$

$$a = 4$$

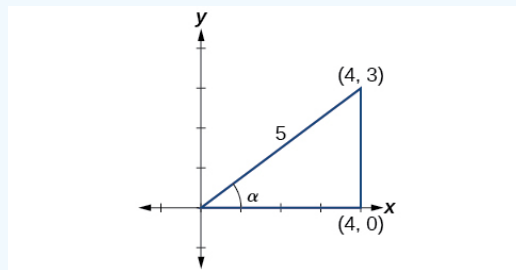


Figure 2.5.4

Since $\cos \beta = -\frac{5}{13}$ and $\pi < \beta < \frac{3\pi}{2}$, the side adjacent to β is -5 , the hypotenuse is 13, and β is in the third quadrant. See Figure 2.5.5. Again, using the Pythagorean Theorem, we have

$$\begin{aligned}(-5)^2 + a^2 &= 13^2 \\ 25 + a^2 &= 169 \\ a^2 &= 144 \\ a &= \pm 12\end{aligned}$$

Since β is in the third quadrant, $a = -12$.

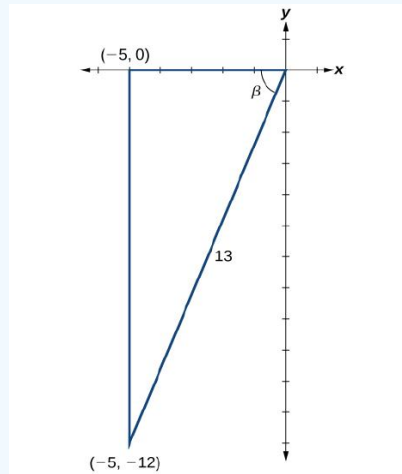


Figure 2.5.5

The next step is finding the cosine of α and the sine of β . The cosine of α is the adjacent side over the hypotenuse. We can find it from the triangle in Figure 2.5.5: $\cos \alpha = \frac{4}{5}$. We can also find the sine of β from the triangle in Figure 2.5.5, as opposite side over the hypotenuse: $\sin \beta = -\frac{12}{13}$. Now we are ready to evaluate $\sin(\alpha + \beta)$.

$$\begin{aligned}\sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \\ &= \left(\frac{3}{5}\right) \left(-\frac{5}{13}\right) + \left(\frac{4}{5}\right) \left(-\frac{12}{13}\right) \\ &= -\frac{15}{65} - \frac{48}{65} \\ &= -\frac{63}{65}\end{aligned}$$

b. We can find $\cos(\alpha + \beta)$ in a similar manner. We substitute the values according to the formula.

$$\begin{aligned}\cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \\ &= \left(\frac{4}{5}\right) \left(-\frac{5}{13}\right) - \left(\frac{3}{5}\right) \left(-\frac{12}{13}\right) \\ &= -\frac{20}{65} + \frac{36}{65} \\ &= \frac{16}{65}\end{aligned}$$

c. For $\tan(\alpha + \beta)$, if $\sin \alpha = \frac{3}{5}$ and $\cos \alpha = \frac{4}{5}$, then

$$\tan \alpha = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$$

If $\sin \beta = -\frac{12}{13}$ and $\cos \beta = -\frac{5}{13}$, then

$$\tan \beta = \frac{-\frac{12}{13}}{-\frac{5}{13}} = \frac{12}{5}$$

Then,

$$\begin{aligned}\tan(\alpha + \beta) &= \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \\ &= \frac{\frac{3}{4} + \frac{12}{5}}{1 - \frac{3}{4} \left(\frac{12}{5} \right)} \\ &= \frac{\frac{63}{20}}{-\frac{16}{20}} \\ &= -\frac{63}{16}\end{aligned}$$

d. To find $\tan(\alpha - \beta)$, we have the values we need. We can substitute them in and evaluate.

$$\begin{aligned}\tan(\alpha - \beta) &= \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} \\ &= \frac{\frac{3}{4} - \frac{12}{5}}{1 + \frac{3}{4} \left(\frac{12}{5} \right)} \\ &= \frac{-\frac{33}{20}}{\frac{56}{20}} \\ &= -\frac{33}{56}\end{aligned}$$

Analysis

A common mistake when addressing problems such as this one is that we may be tempted to think that α and β are angles in the same triangle, which of course, they are not. Also note that

$$\tan(\alpha + \beta) = \sin(\alpha + \beta) \cos(\alpha + \beta)$$

Using Sum and Difference Formulas for Cofunctions

Now that we can find the sine, cosine, and tangent functions for the sums and differences of angles, we can use them to do the same for their cofunctions. You may recall from [Right Triangle Trigonometry](#) that, if the sum of two positive angles is $\frac{\pi}{2}$, those two angles are complements, and the sum of the two acute angles in a right triangle is $\frac{\pi}{2}$, so they are also complements. In Figure 2.5.6, notice that if one of the acute angles is labeled as θ , then the other acute angle must be labeled $\frac{\pi}{2} - \theta$.

Notice also that $\sin \theta = \cos(\frac{\pi}{2} - \theta)$, which is opposite over hypotenuse. Thus, when two angles are complimentary, we can say that the sine of θ equals the cofunction of the complement of θ . Similarly, tangent and cotangent are cofunctions, and secant and cosecant are cofunctions.

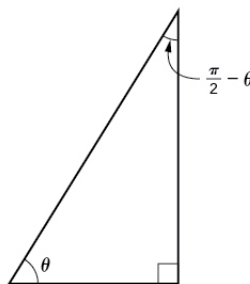


Figure 2.5.6

From these relationships, the cofunction identities are formed. Recall that you first encountered these identities in [The Unit Circle: Sine and Cosine Functions](#).

COFUNCTION IDENTITIES

The cofunction identities are summarized in Table 2.5.2.

Table 2.5.2

$\sin \theta = \cos \left(\frac{\pi}{2} - \theta \right)$	$\cos \theta = \sin \left(\frac{\pi}{2} - \theta \right)$
$\tan \theta = \cot \left(\frac{\pi}{2} - \theta \right)$	$\cot \theta = \tan \left(\frac{\pi}{2} - \theta \right)$
$\sec \theta = \csc \left(\frac{\pi}{2} - \theta \right)$	$\csc \theta = \sec \left(\frac{\pi}{2} - \theta \right)$

Notice that the formulas in the table may also be justified algebraically using the sum and difference formulas. For example, using

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

We can write

$$\begin{aligned} \cos \left(\frac{\pi}{2} - \theta \right) &= \cos \frac{\pi}{2} \cos \theta + \sin \frac{\pi}{2} \sin \theta \\ &= (0) \cos \theta + (1) \sin \theta \\ &= \sin \theta \end{aligned}$$

✓ Example 2.5.7: Finding a Cofunction with the Same Value as the Given Expression

Write $\tan \frac{\pi}{9}$ in terms of its cofunction.

Solution

The cofunction of $\tan \theta = \cot \left(\frac{\pi}{2} - \theta \right)$. Thus,

$$\begin{aligned} \tan \left(\frac{\pi}{9} \right) &= \cot \left(\frac{\pi}{2} - \frac{\pi}{9} \right) \\ &= \cot \left(\frac{9\pi}{18} - \frac{2\pi}{18} \right) \\ &= \cot \left(\frac{7\pi}{18} \right) \end{aligned}$$

? Exercise 2.5.7

Write $\sin \frac{\pi}{7}$ in terms of its cofunction.

Answer

$$\cos \left(\frac{5\pi}{14} \right)$$

Using the Sum and Difference Formulas to Verify Identities

Verifying an identity means demonstrating that the equation holds for all values of the variable. It helps to be very familiar with the identities or to have a list of them accessible while working the problems. Reviewing the general rules presented earlier may help simplify the process of verifying an identity.

✚ How to: Given an identity, verify using sum and difference formulas

1. Begin with the expression on the side of the equal sign that appears most complex. Rewrite that expression until it matches the other side of the equal sign. Occasionally, we might have to alter both sides, but working on only one side is the most efficient.
2. Look for opportunities to use the sum and difference formulas.
3. Rewrite sums or differences of quotients as single quotients.
4. If the process becomes cumbersome, rewrite the expression in terms of sines and cosines.

✓ Example 2.5.8A: Verifying an Identity Involving Sine

Verify the identity $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin \alpha \cos \beta$.

Solution

We see that the left side of the equation includes the sines of the sum and the difference of angles.

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

We can rewrite each using the sum and difference formulas.

$$\begin{aligned} \sin(\alpha + \beta) + \sin(\alpha - \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta + \sin \alpha \cos \beta - \cos \alpha \sin \beta \\ &= 2 \sin \alpha \cos \beta \end{aligned}$$

We see that the identity is verified.

✓ Example 2.5.8B: Verifying an Identity Involving Tangent

Verify the following identity.

$$\frac{\sin(\alpha - \beta)}{\cos \alpha \cos \beta} = \tan \alpha - \tan \beta$$

Solution

We can begin by rewriting the numerator on the left side of the equation.

$$\begin{aligned} \frac{\sin(\alpha - \beta)}{\cos \alpha \cos \beta} &= \frac{\sin \alpha \cos \beta - \cos \alpha \sin \beta}{\cos \alpha \cos \beta} \\ &= \frac{\sin \alpha \cos \beta}{\cos \alpha \cos \beta} - \frac{\cos \alpha \sin \beta}{\cos \alpha \cos \beta} && \text{Rewrite using a common denominator} \\ &= \frac{\sin \alpha}{\cos \alpha} - \frac{\sin \beta}{\cos \beta} && \text{Cancel} \\ &= \tan \alpha - \tan \beta && \text{Rewrite in terms of tangent} \end{aligned}$$

? Exercise 2.5.8:

Verify the identity: $\tan(\pi - \theta) = -\tan \theta$.

Answer

$$\begin{aligned} \tan(\pi - \theta) &= \frac{\tan(\pi) - \tan \theta}{1 + \tan(\pi) \tan \theta} \\ &= \frac{0 - \tan \theta}{1 + 0 \cdot \tan \theta} \\ &= -\tan \theta \end{aligned}$$

✓ Example 2.5.9A: Using Sum and Difference Formulas to Solve an Application Problem

Let L_1 and L_2 denote two non-vertical intersecting lines, and let θ denote the acute angle between L_1 and L_2 . See Figure 2.5.7. Show that

$$\tan \theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

where m_1 and m_2 are the slopes of L_1 and L_2 respectively. (**Hint:** Use the fact that $\tan \theta_1 = m_1$ and $\tan \theta_2 = m_2$.)

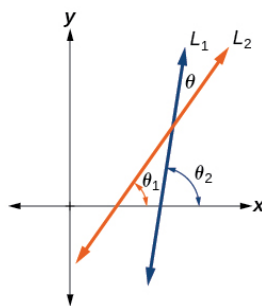


Figure 2.5.7

Solution

Using the difference formula for tangent, this problem does not seem as daunting as it might.

$$\begin{aligned}\tan \theta &= \tan(\theta_2 - \theta_1) \\ &= \frac{\tan \theta_2 - \tan \theta_1}{1 + \tan \theta_1 \tan \theta_2} \\ &= \frac{m_2 - m_1}{1 + m_1 m_2}\end{aligned}$$

✓ Example 2.5.9B: Investigating a Guy-wire Problem

For a climbing wall, a guy-wire R is attached 47 feet high on a vertical pole. Added support is provided by another guy-wire S attached 40 feet above ground on the same pole. If the wires are attached to the ground 50 feet from the pole, find the angle α between the wires. See Figure 2.5.8.

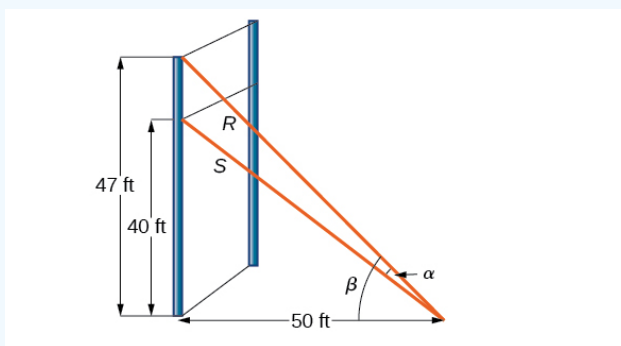


Figure 2.5.8

Solution

Let's first summarize the information we can gather from the diagram. As only the sides adjacent to the right angle are known, we can use the tangent function. Notice that $\tan \beta = \frac{47}{50}$, and $\tan(\beta - \alpha) = \frac{40}{50} = \frac{4}{5}$. We can then use difference formula for tangent.

$$\tan(\beta - \alpha) = \frac{\tan \beta - \tan \alpha}{1 + \tan \beta \tan \alpha} \quad (2.5.8)$$

Now, substituting the values we know into the formula, we have,

$$\begin{aligned}\frac{4}{5} &= \frac{\frac{47}{50} - \tan \alpha}{1 + \frac{47}{50} \tan \alpha} \\ 4 \left(1 + \frac{47}{50} \tan \alpha \right) &= 5 \left(\frac{47}{50} - \tan \alpha \right)\end{aligned}$$

Use the distributive property, and then simplify the functions.

$$\begin{aligned}
 4(1) + 4\left(\frac{47}{50}\right) \tan \alpha &= 5\left(\frac{47}{50}\right) - 5 \tan \alpha \\
 4 + 3.76 \tan \alpha &= 4.7 - 5 \tan \alpha \\
 5 \tan \alpha + 3.76 \tan \alpha &= 0.7 \\
 8.76 \tan \alpha &= 0.7 \\
 \tan \alpha &\approx 0.07991 \\
 \tan^{-1}(0.07991) &\approx .079741
 \end{aligned}$$

Now we can calculate the angle in degrees.

$$\begin{aligned}
 \alpha &\approx 0.079741 \left(\frac{180}{\pi} \right) \\
 &\approx 4.57^\circ
 \end{aligned}$$

Analysis

Occasionally, when an application appears that includes a right triangle, we may think that solving is a matter of applying the Pythagorean Theorem. That may be partially true, but it depends on what the problem is asking and what information is given.

Media

Access these online resources for additional instruction and practice with sum and difference identities.

- Sum and Difference Identities for Cosine
- Sum and Difference Identities for Sine
- Sum and Difference Identities for Tangent

Key Equations

Sum Formula for Cosine	$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$
Difference Formula for Cosine	$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$
Sum Formula for Sine	$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$
Difference Formula for Sine	$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$
Sum Formula for Tangent	$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$
Difference Formula for Tangent	$\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha) \tan(\beta)}$
Cofunction identities	$\sin \theta = \cos\left(\frac{\pi}{2} - \theta\right)$ $\cos \theta = \sin\left(\frac{\pi}{2} - \theta\right)$ $\tan \theta = \cot\left(\frac{\pi}{2} - \theta\right)$ $\cot \theta = \tan\left(\frac{\pi}{2} - \theta\right)$ $\sec \theta = \csc\left(\frac{\pi}{2} - \theta\right)$ $\csc \theta = \sec\left(\frac{\pi}{2} - \theta\right)$

Key Concepts

- The sum formula for cosines states that the cosine of the sum of two angles equals the product of the cosines of the angles minus the product of the sines of the angles. The difference formula for cosines states that the cosine of the difference of two angles equals the product of the cosines of the angles plus the product of the sines of the angles.
- The sum and difference formulas can be used to find the exact values of the sine, cosine, or tangent of an angle. See Example 2.5.1 and Example 2.5.2.
- The sum formula for sines states that the sine of the sum of two angles equals the product of the sine of the first angle and cosine of the second angle plus the product of the cosine of the first angle and the sine of the second angle. The difference formula for sines states that the sine of the difference of two angles equals the product of the sine of the first angle and cosine of the second angle minus the product of the cosine of the first angle and the sine of the second angle. See Example 2.5.3.
- The sum and difference formulas for sine and cosine can also be used for inverse trigonometric functions. See Example 2.5.4.

- The sum formula for tangent states that the tangent of the sum of two angles equals the sum of the tangents of the angles divided by 1 minus the product of the tangents of the angles. The difference formula for tangent states that the tangent of the difference of two angles equals the difference of the tangents of the angles divided by 1 plus the product of the tangents of the angles. See Example 2.5.5.
- The Pythagorean Theorem along with the sum and difference formulas can be used to find multiple sums and differences of angles. See Example 2.5.6.
- The cofunction identities apply to complementary angles and pairs of reciprocal functions. See Example 2.5.7.
- Sum and difference formulas are useful in verifying identities. See Example 2.5.8 and Example 2.5.9.
- Application problems are often easier to solve by using sum and difference formulas. See Example 2.5.10 and Example 2.5.11.

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2.6: Double-Angle, Half-Angle, and Reduction Formulas

Learning Objectives

- Use double-angle formulas to find exact values
- Use double-angle formulas to verify identities
- Use reduction formulas to simplify an expression
- Use half-angle formulas to find exact values

Bicycle ramps made for competition (see Figure 2.6.1) must vary in height depending on the skill level of the competitors. For advanced competitors, the angle formed by the ramp and the ground should be θ such that $\tan \theta = \frac{5}{3}$. The angle is divided in half for novices. What is the steepness of the ramp for novices? In this section, we will investigate three additional categories of identities that we can use to answer questions such as this one.

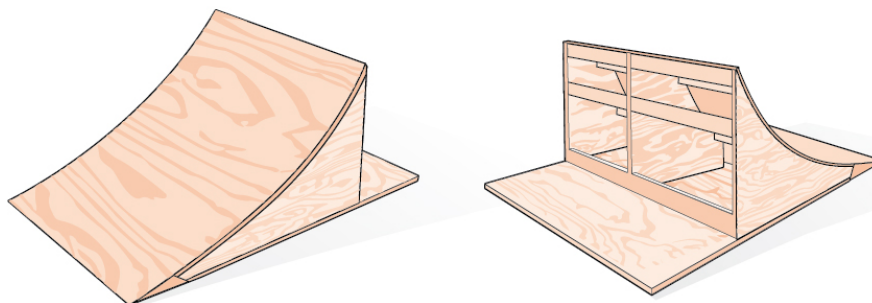


Figure 2.6.1: Bicycle ramps for advanced riders have a steeper incline than those designed for novices.

Using Double-Angle Formulas to Find Exact Values

In the previous section, we used addition and subtraction formulas for trigonometric functions. Now, we take another look at those same formulas. The **double-angle formulas** are a special case of the sum formulas, where $\alpha = \beta$. Deriving the double-angle formula for sine begins with the sum formula,

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \quad (2.6.1)$$

If we let $\alpha = \beta = \theta$, then we have

$$\begin{aligned} \sin(\theta + \theta) &= \sin \theta \cos \theta + \cos \theta \sin \theta \\ \sin(2\theta) &= 2 \sin \theta \cos \theta \end{aligned}$$

Deriving the double-angle for cosine gives us three options. First, starting from the sum formula, $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$, and letting $\alpha = \beta = \theta$, we have

$$\begin{aligned} \cos(\theta + \theta) &= \cos \theta \cos \theta - \sin \theta \sin \theta \\ \cos(2\theta) &= \cos^2 \theta - \sin^2 \theta \end{aligned}$$

Using the Pythagorean properties, we can expand this double-angle formula for cosine and get two more variations. The first variation is:

$$\begin{aligned} \cos(2\theta) &= \cos^2 \theta - \sin^2 \theta \\ &= (1 - \sin^2 \theta) - \sin^2 \theta \end{aligned}$$

The second variation is:

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= \cos^2\theta - (1 - \cos^2\theta) \\ &= 2\cos^2\theta - 1\end{aligned}$$

Similarly, to derive the double-angle formula for tangent, replacing $\alpha = \beta = \theta$ in the sum formula gives

$$\begin{aligned}\tan(\alpha + \beta) &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \\ \tan(\theta + \theta) &= \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta} \\ \tan(2\theta) &= \frac{2\tan\theta}{1 - \tan^2\theta}\end{aligned}$$

DOUBLE-ANGLE FORMULAS

The *double-angle formulas* are summarized as follows:

$$\sin(2\theta) = 2\sin\theta\cos\theta \quad (2.6.2)$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1 \quad (2.6.3)$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta} \quad (2.6.4)$$

How to: Given the tangent of an angle and the quadrant in which it is located, use the double-angle formulas to find the exact value

1. Draw a triangle to reflect the given information.
2. Determine the correct double-angle formula.
3. Substitute values into the formula based on the triangle.
4. Simplify.

✓ Example 2.6.1: Using a Double-Angle Formula to Find the Exact Value Involving Tangent

Given that $\tan\theta = -\frac{3}{4}$ and θ is in quadrant II, find the following:

- a. $\sin(2\theta)$
- b. $\cos(2\theta)$
- c. $\tan(2\theta)$

Solution

If we draw a triangle to reflect the information given, we can find the values needed to solve the problems on the image. We are given $\tan\theta = -\frac{3}{4}$, such that θ is in quadrant II. The tangent of an angle is equal to the opposite side over the adjacent side, and because θ is in the second quadrant, the adjacent side is on the x-axis and is negative. Use the **Pythagorean Theorem** to find the length of the hypotenuse:

$$\begin{aligned}(-4)^2 + (3)^2 &= c^2 \\ 16 + 9 &= c^2 \\ 25 &= c^2 \\ c &= 5\end{aligned}$$

Now we can draw a triangle similar to the one shown in Figure 2.6.2.

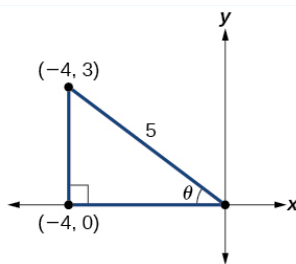


Figure 2.6.2

a. Let's begin by writing the double-angle formula for sine.

$$\sin(2\theta) = 2 \sin \theta \cos \theta$$

We see that we need to find $\sin \theta$ and $\cos \theta$. Based on Figure 2.6.2, we see that the hypotenuse equals 5, so $\sin \theta = \frac{3}{5}$, $\cos \theta = -\frac{4}{5}$. Substitute these values into the equation, and simplify.

Thus,

$$\begin{aligned} \sin(2\theta) &= 2 \left(\frac{3}{5} \right) \left(-\frac{4}{5} \right) \\ &= -\frac{24}{25} \end{aligned}$$

b. Write the double-angle formula for cosine.

$$\cos(2\theta) = \cos^2 \theta - \sin^2 \theta$$

Again, substitute the values of the sine and cosine into the equation, and simplify.

$$\begin{aligned} \cos(2\theta) &= \left(-\frac{4}{5} \right)^2 - \left(\frac{3}{5} \right)^2 \\ &= \frac{16}{25} - \frac{9}{25} \\ &= \frac{7}{25} \end{aligned}$$

c. Write the double-angle formula for tangent.

$$\tan(2\theta) = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

In this formula, we need the tangent, which we were given as $\tan \theta = -\frac{3}{4}$. Substitute this value into the equation, and simplify.

$$\begin{aligned} \tan(2\theta) &= \frac{2 \left(-\frac{3}{4} \right)}{1 - \left(-\frac{3}{4} \right)^2} \\ &= \frac{-\frac{3}{2}}{1 - \frac{9}{16}} \\ &= -\frac{3}{2} \left(\frac{16}{7} \right) \\ &= -\frac{24}{7} \end{aligned}$$

? Exercise 2.6.1

Given $\sin \alpha = \frac{5}{8}$, with θ in quadrant I, find $\cos(2\alpha)$.

Answer

$$\cos(2\alpha) = \frac{7}{32}$$

✓ Example 2.6.2: Using the Double-Angle Formula for Cosine without Exact Values

Use the double-angle formula for cosine to write $\cos(6x)$ in terms of $\cos(3x)$.

Solution

$$\begin{aligned}\cos(6x) &= \cos(3x + 3x) \\ &= \cos 3x \cos 3x - \sin 3x \sin 3x \\ &= \cos^2 3x - \sin^2 3x\end{aligned}$$

Analysis

This example illustrates that we can use the double-angle formula without having exact values. It emphasizes that the pattern is what we need to remember and that identities are true for all values in the domain of the trigonometric function.

Using Double-Angle Formulas to Verify Identities

Establishing identities using the double-angle formulas is performed using the same steps we used to derive the sum and difference formulas. Choose the more complicated side of the equation and rewrite it until it matches the other side.

✓ Example 2.6.3: Using the Double-Angle Formulas to Verify an Identity

Verify the following identity using double-angle formulas:

$$1 + \sin(2\theta) = (\sin \theta + \cos \theta)^2$$

Solution

We will work on the right side of the equal sign and rewrite the expression until it matches the left side.

$$\begin{aligned}(\sin \theta + \cos \theta)^2 &= \sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta \\ &= (\sin^2 \theta + \cos^2 \theta) + 2 \sin \theta \cos \theta \\ &= 1 + 2 \sin \theta \cos \theta \\ &= 1 + \sin(2\theta)\end{aligned}$$

Analysis

This process is not complicated, as long as we recall the perfect square formula from algebra:

$$(a \pm b)^2 = a^2 \pm 2ab + b^2$$

where $a = \sin \theta$ and $b = \cos \theta$. Part of being successful in mathematics is the ability to recognize patterns. While the terms or symbols may change, the algebra remains consistent.

? Exercise 2.6.2

Verify the identity: $\cos^4\theta - \sin^4\theta = \cos(2\theta)$.

Answer

$$\cos^4\theta - \sin^4\theta = (\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta) = \cos(2\theta)$$

✓ Example 2.6.4: Verifying a Double-Angle Identity for Tangent

Verify the identity: $\tan(2\theta) = 2 \cot \theta - \tan \theta$

Solution

In this case, we will work with the left side of the equation and simplify or rewrite until it equals the right side of the equation.

$$\begin{aligned} \tan(2\theta) &= \frac{2 \tan \theta}{1 - \tan^2 \theta} && \text{Double-angle formula} \\ &= \frac{2 \tan \theta \left(\frac{1}{\tan \theta} \right)}{(1 - \tan^2 \theta) \left(\frac{1}{\tan \theta} \right)} && \text{Multiply by a term that results in desired numerator} \\ &= \frac{2}{\frac{1}{\tan \theta} - \tan \theta} \\ &= \frac{2}{\cot \theta - \tan \theta} && \text{Use reciprocal identity for } \frac{1}{\tan \theta} \end{aligned}$$

Analysis

Here is a case where the more complicated side of the initial equation appeared on the right, but we chose to work the left side. However, if we had chosen the left side to rewrite, we would have been working backwards to arrive at the equivalency. For example, suppose that we wanted to show

$$\begin{aligned} \frac{2 \tan \theta}{1 - \tan^2 \theta} &= \frac{2}{\cot \theta - \tan \theta} \\ \text{Let's work on the right side} \\ \frac{2}{\cot \theta - \tan \theta} &= \frac{2}{\frac{1}{\tan \theta} - \tan \theta} \left(\frac{\tan \theta}{\tan \theta} \right) \\ &= \frac{2 \tan \theta}{\frac{1}{\tan \theta} (\tan \theta) - \tan \theta (\tan \theta)} \\ &= \frac{2 \tan \theta}{1 - \tan^2 \theta} \end{aligned}$$

When using the identities to simplify a trigonometric expression or solve a trigonometric equation, there are usually several paths to a desired result. There is no set rule as to what side should be manipulated. However, we should begin with the guidelines set forth earlier.

? Exercise 2.6.3

Verify the identity: $\cos(2\theta) \cos \theta = \cos^3 \theta - \cos \theta \sin^2 \theta$.

Answer

$$\cos(2\theta) \cos \theta = (\cos^2 \theta - \sin^2 \theta) \cos \theta = \cos^3 \theta - \cos \theta \sin^2 \theta$$

Use Reduction Formulas to Simplify an Expression

The double-angle formulas can be used to derive the **reduction formulas**, which are formulas we can use to reduce the power of a given expression involving even powers of sine or cosine. They allow us to rewrite the even powers of sine or cosine in terms of the first power of cosine. These formulas are especially important in higher-level math courses, calculus in particular. Also called the power-reducing formulas, three identities are included and are easily derived from the double-angle formulas.

We can use two of the three double-angle formulas for cosine to derive the reduction formulas for sine and cosine. Let's begin with $\cos(2\theta) = 1 - 2 \sin^2 \theta$. Solve for $\sin^2 \theta$:

$$\begin{aligned}\cos(2\theta) &= 1 - 2\sin^2 \theta \\ 2\sin^2 \theta &= 1 - \cos(2\theta) \\ \sin^2 \theta &= \frac{1 - \cos(2\theta)}{2}\end{aligned}$$

Next, we use the formula $\cos(2\theta) = 2 \cos^2 \theta - 1$. Solve for $\cos^2 \theta$:

$$\begin{aligned}\cos(2\theta) &= 2\cos^2 \theta - 1 \\ 1 + \cos(2\theta) &= 2\cos^2 \theta \\ \frac{1 + \cos(2\theta)}{2} &= \cos^2 \theta\end{aligned}$$

The last reduction formula is derived by writing tangent in terms of sine and cosine:

$$\begin{aligned}\tan^2 \theta &= \frac{\sin^2 \theta}{\cos^2 \theta} \\ &= \frac{\frac{1 - \cos(2\theta)}{2}}{\frac{1 + \cos(2\theta)}{2}} && \text{(Substitute the reduction formulas)} \\ &= \left(\frac{1 - \cos(2\theta)}{2} \right) \left(\frac{2}{1 + \cos(2\theta)} \right) \\ &= \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}\end{aligned}$$

REDUCTION FORMULAS

The reduction formulas are summarized as follows:

$$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2} \quad (2.6.5)$$

$$\cos^2 \theta = \frac{1 + \cos(2\theta)}{2} \quad (2.6.6)$$

$$\tan^2 \theta = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)} \quad (2.6.7)$$

✓ Example 2.6.5: Writing an Equivalent Expression Not Containing Powers Greater Than 1

Write an equivalent expression for $\cos^4 x$ that does not involve any powers of sine or cosine greater than 1.

Solution

We will apply the reduction formula for cosine twice.

$$\begin{aligned}
 \cos^4 x &= (\cos^2 x)^2 \\
 &= \left(\frac{1 + \cos(2x)}{2} \right)^2 && \text{Substitute reduction formula} \\
 &= \frac{1}{4} (1 + 2\cos(2x) + \cos^2(2x)) \\
 &= \frac{1}{4} + \frac{1}{2}\cos(2x) + \frac{1}{4} \left(\frac{1 + \cos^2(2x)}{2} \right) && \text{Substitute reduction formula for } \cos^2 x \\
 &= \frac{1}{4} + \frac{1}{2}\cos(2x) + \frac{1}{8} + \frac{1}{8}\cos(4x) \\
 &= \frac{3}{8} + \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)
 \end{aligned}$$

Analysis

The solution is found by using the reduction formula twice, as noted, and the perfect square formula from algebra.

✓ Example 2.6.6: Using the Power-Reducing Formulas to Prove an Identity

Use the power-reducing formulas to prove $\sin^3(2x) = \left[\frac{1}{2} \sin(2x) \right] [1 - \cos(4x)]$

Solution

We will work on simplifying the left side of the equation:

$$\begin{aligned}
 \sin^3(2x) &= [\sin(2x)][\sin^2(2x)] \\
 &= \sin(2x) \left[\frac{1 - \cos(4x)}{2} \right] && \text{Substitute the power-reduction formula.} \\
 &= \sin(2x) \left(\frac{1}{2} \right) [1 - \cos(4x)] \\
 &= \frac{1}{2} [\sin(2x)][1 - \cos(4x)]
 \end{aligned}$$

Analysis

Note that in this example, we substituted $\frac{1 - \cos(4x)}{2}$ for $\sin^2(2x)$. The formula states $\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}$

We let $\theta = 2x$, so $2\theta = 4x$.

? Exercise 2.6.4

Use the power-reducing formulas to prove that $10\cos^4 x = \frac{15}{4} + 5\cos(2x) + \frac{5}{4}\cos(4x)$.

Answer

$$\begin{aligned}
 10\cos^4 x &= 10(\cos^2 x)^2 \\
 &= 10\left[\frac{1 + \cos(2x)}{2}\right]^2 && \text{Substitute reduction formula for } \cos^2 x \\
 &= \frac{10}{4}[1 + 2\cos(2x) + \cos^2(2x)] \\
 &= \frac{10}{4} + \frac{10}{2}\cos(2x) + \frac{10}{4}\left(\frac{1 + \cos^2(2x)}{2}\right) && \text{Substitute reduction formula for } \cos^2 x \\
 &= \frac{10}{4} + \frac{10}{2}\cos(2x) + \frac{10}{8} + \frac{10}{8}\cos(4x) \\
 &= \frac{30}{8} + 5\cos(2x) + \frac{10}{8}\cos(4x) \\
 &= \frac{15}{4} + 5\cos(2x) + \frac{5}{4}\cos(4x)
 \end{aligned}$$

Using Half-Angle Formulas to Find Exact Values

The next set of identities is the set of **half-angle formulas**, which can be derived from the reduction formulas and we can use when we have an angle that is half the size of a special angle. If we replace θ with $\frac{\alpha}{2}$, the half-angle formula for sine is found by simplifying the equation and solving for $\sin\left(\frac{\alpha}{2}\right)$. Note that the half-angle formulas are preceded by a $\pm$ sign. This does not mean that both the positive and negative expressions are valid. Rather, it depends on the quadrant in which $\frac{\alpha}{2}$ terminates.

The half-angle formula for sine is derived as follows:

$$\begin{aligned}
 \sin^2 \theta &= \frac{1 - \cos(2\theta)}{2} \\
 \sin^2\left(\frac{\alpha}{2}\right) &= \frac{1 - \left(\cos 2 \cdot \frac{\alpha}{2}\right)}{2} \\
 &= \frac{1 - \cos \alpha}{2} \\
 \sin\left(\frac{\alpha}{2}\right) &= \pm \sqrt{\frac{1 - \cos \alpha}{2}}
 \end{aligned}$$

To derive the half-angle formula for cosine, we have

$$\begin{aligned}
 \cos^2 \theta &= \frac{1 + \cos(2\theta)}{2} \\
 \cos^2\left(\frac{\alpha}{2}\right) &= \frac{1 + \cos\left(2 \cdot \frac{\alpha}{2}\right)}{2} \\
 &= \frac{1 + \cos \alpha}{2} \\
 \cos\left(\frac{\alpha}{2}\right) &= \pm \sqrt{\frac{1 + \cos \alpha}{2}}
 \end{aligned}$$

For the tangent identity, we have

$$\tan^2 \theta = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}$$

$$\tan^2 \left(\frac{\alpha}{2} \right) = \frac{1 - \cos \left(2 \cdot \frac{\alpha}{2} \right)}{1 + \cos \left(2 \cdot \frac{\alpha}{2} \right)}$$

$$\tan \left(\frac{\alpha}{2} \right) = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}}$$

✚ HALF-ANGLE FORMULAS

The **half-angle formulas** are as follows:

$$\sin \left(\frac{\alpha}{2} \right) = \pm \sqrt{\frac{1 - \cos \alpha}{2}} \quad (2.6.8)$$

$$\cos \left(\frac{\alpha}{2} \right) = \pm \sqrt{\frac{1 + \cos \alpha}{2}} \quad (2.6.9)$$

$$\tan \left(\frac{\alpha}{2} \right) = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} = \frac{\sin \alpha}{1 + \cos \alpha} = \frac{1 - \cos \alpha}{\sin \alpha} \quad (2.6.10)$$

✓ Example 2.6.7

Using a Half-Angle Formula to Find the Exact Value of a Sine Function. Find $\sin(15^\circ)$ using a half-angle formula.

Solution

Since $15^\circ = \frac{30^\circ}{2}$, we use the half-angle formula for sine (Equation 2.6.8):

$$\begin{aligned} \sin \frac{30^\circ}{2} &= \sqrt{\frac{1 - \cos 30^\circ}{2}} \\ &= \sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{2}} \\ &= \sqrt{\frac{\frac{2 - \sqrt{3}}{2}}{2}} \\ &= \sqrt{\frac{2 - \sqrt{3}}{4}} \\ &= \frac{\sqrt{2 - \sqrt{3}}}{2} \end{aligned}$$

Remember that we can check the answer with a graphing calculator.

Analysis

Notice that we used only the positive root because $\sin(15^\circ)$ is positive.

✚ **Howto:** Given the tangent of an angle and the quadrant in which the angle lies, find the exact values of trigonometric functions of half of the angle.

1. Draw a triangle to represent the given information.
2. Determine the correct half-angle formula.
3. Substitute values into the formula based on the triangle.
4. Simplify.

✓ Example 2.6.8: Finding Exact Values Using Half-Angle Identities

Given that $\tan \alpha = \frac{8}{15}$ and α lies in quadrant III, find the exact value of the following:

- $\sin\left(\frac{\alpha}{2}\right)$
- $\cos\left(\frac{\alpha}{2}\right)$
- $\tan\left(\frac{\alpha}{2}\right)$

Solution

Using the given information, we can draw the triangle shown in Figure 2.6.3. Using the Pythagorean Theorem, we find the hypotenuse to be 17. Therefore, we can calculate $\sin \alpha = -\frac{8}{17}$ and $\cos \alpha = -\frac{15}{17}$.

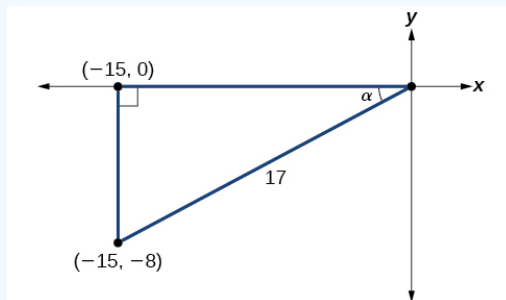


Figure 2.6.3

- Before we start, we must remember that if α is in quadrant III, then $180^\circ < \alpha < 270^\circ$, so $\frac{180^\circ}{2} < \frac{\alpha}{2} < \frac{270^\circ}{2}$. This means that the terminal side of $\frac{\alpha}{2}$ is in quadrant II, since $90^\circ < \frac{\alpha}{2} < 135^\circ$. To find $\sin \frac{\alpha}{2}$, we begin by writing the half-angle formula for sine. Then we substitute the value of the cosine we found from the triangle in Figure 2.6.3 and simplify.

$$\begin{aligned}\sin \frac{\alpha}{2} &= \pm \sqrt{\frac{1 - \cos \alpha}{2}} \\ &= \pm \sqrt{\frac{1 - \left(-\frac{15}{17}\right)}{2}} \\ &= \pm \sqrt{\frac{\frac{32}{17}}{2}} \\ &= \pm \sqrt{\frac{32}{17} \cdot \frac{1}{2}} \\ &= \pm \sqrt{\frac{16}{17}} \\ &= \pm \frac{4}{\sqrt{17}} \\ &= \frac{4\sqrt{17}}{17}\end{aligned}$$

We choose the positive value of $\sin \frac{\alpha}{2}$ because the angle terminates in quadrant II and sine is positive in quadrant II.

- To find $\cos \frac{\alpha}{2}$, we will write the half-angle formula for cosine, substitute the value of the cosine we found from the triangle in Figure 2.6.3, and simplify.

$$\begin{aligned}
 \cos \frac{\alpha}{2} &= \pm \sqrt{\frac{1 + \cos \alpha}{2}} \\
 &= \pm \sqrt{\frac{1 + \left(-\frac{15}{17}\right)}{2}} \\
 &= \pm \sqrt{\frac{\frac{2}{17}}{2}} \\
 &= \pm \sqrt{\frac{2}{17} \cdot \frac{1}{2}} \\
 &= \pm \sqrt{\frac{1}{17}} \\
 &= -\frac{\sqrt{17}}{17}
 \end{aligned}$$

We choose the negative value of $\cos \frac{\alpha}{2}$ because the angle is in quadrant II because cosine is negative in quadrant II.

c. To find $\tan \frac{\alpha}{2}$, we write the half-angle formula for tangent. Again, we substitute the value of the cosine we found from the triangle in Figure 2.6.3 and simplify.

$$\begin{aligned}
 \tan \frac{\alpha}{2} &= \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} \\
 &= \pm \sqrt{\frac{1 - \left(-\frac{15}{17}\right)}{1 + \left(-\frac{15}{17}\right)}} \\
 &= \pm \sqrt{\frac{\frac{32}{17}}{\frac{2}{17}}} \\
 &= \pm \sqrt{\frac{32}{2}} \\
 &= -\sqrt{16} \\
 &= -4
 \end{aligned}$$

We choose the negative value of $\tan \frac{\alpha}{2}$ because $\frac{\alpha}{2}$ lies in quadrant II, and tangent is negative in quadrant II.

? Exercise 2.6.5

Given that $\sin \alpha = -\frac{4}{5}$ and α lies in quadrant IV, find the exact value of $\cos \left(\frac{\alpha}{2}\right)$.

Answer

$$-\frac{2}{\sqrt{5}}$$

✓ Example 2.6.9: Finding the Measurement of a Half Angle

Now, we will return to the problem posed at the beginning of the section. A bicycle ramp is constructed for high-level competition with an angle of θ formed by the ramp and the ground. Another ramp is to be constructed half as steep for novice competition. If $\tan \theta = \frac{5}{3}$ for higher-level competition, what is the measurement of the angle for novice competition?

Solution

Since the angle for novice competition measures half the steepness of the angle for the high level competition, and $\tan \theta = \frac{5}{3}$ for high competition, we can find $\cos \theta$ from the right triangle and the Pythagorean theorem so that we can use the half-angle identities. See Figure 2.6.4.

$$3^2 + 5^2 = 34$$

$$c = \sqrt{34}$$

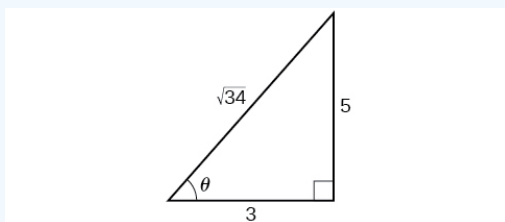


Figure 2.6.4

We see that $\cos \theta = \frac{3}{\sqrt{34}} = \frac{3\sqrt{34}}{34}$. We can use the half-angle formula for tangent: $\tan \frac{\theta}{2} = \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}}$. Since $\tan \theta$ is in the first quadrant, so is $\tan \frac{\theta}{2}$.

$$\begin{aligned} \tan \frac{\theta}{2} &= \sqrt{\frac{1 - \frac{3\sqrt{34}}{34}}{1 + \frac{3\sqrt{34}}{34}}} \\ &= \sqrt{\frac{\frac{34 - 3\sqrt{34}}{34}}{\frac{34 + 3\sqrt{34}}{34}}} \\ &= \sqrt{\frac{34 - 3\sqrt{34}}{34 + 3\sqrt{34}}} \\ &\approx 0.57 \end{aligned}$$

We can take the inverse tangent to find the angle: $\tan^{-1}(0.57) \approx 29.7^\circ$. So the angle of the ramp for novice competition is $\approx 29.7^\circ$.

📌 Media

Access these online resources for additional instruction and practice with double-angle, half-angle, and reduction formulas.

- Double-Angle Identities
- Half-Angle Identities

Key Equations

Double-angle formulas	$\sin(2\theta) = 2 \sin \theta \cos \theta$ $\cos(2\theta) = \cos^2 \theta - \sin^2 \theta$ $= 1 - 2\sin^2 \theta$ $= 2\cos^2 \theta - 1$ $\tan(2\theta) = \frac{2 \tan \theta}{1 - \tan^2 \theta}$
Reduction formulas	$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}$ $\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}$ $\tan^2 \theta = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}$
Half-angle formulas	$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}$ $\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$ $\tan \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}}$ $= \frac{\sin \alpha}{1 + \cos \alpha}$ $= \frac{1 - \cos \alpha}{\sin \alpha}$

Key Concepts

- Double-angle identities are derived from the sum formulas of the fundamental trigonometric functions: sine, cosine, and tangent. See Example 2.6.1, Example 2.6.2, Example 2.6.3, and Example 2.6.4.
- Reduction formulas are especially useful in calculus, as they allow us to reduce the power of the trigonometric term. See Example 2.6.5 and Example 2.6.6.
- Half-angle formulas allow us to find the value of trigonometric functions involving half-angles, whether the original angle is known or not. See Example 2.6.7, Example 2.6.8, and Example 2.6.9.

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