MAT 206
PRECALCULUS



Borough of Manhattan Community College

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MAT 206 Precalculus

Borough of Manhattan Community College

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Licensing

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CHAPTER OVERVIEW

0: Review - Linear Equations in 2 Variables

Recall that a function is a relation that assigns to every element in the domain exactly one element in the range. Linear functions are a specific type of function that can be used to model many real-world applications, such as plant growth over time. In this chapter, we will explore linear functions, their graphs, and how to relate them to data.

0.0: Prelude to Linear Functions

0.1: Linear Functions

0.2: Graphs of Linear Functions

0.3: Modeling with Linear Functions

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0.0: Prelude to Linear Functions

Imagine placing a plant in the ground one day and finding that it has doubled its height just a few days later. Although it may seem incredible, this can happen with certain types of bamboo species. These members of the grass family are the fastest-growing plants in the world. One species of bamboo has been observed to grow nearly 1.5 inches every hour.1 In a twenty-four hour period, this bamboo plant grows about 36 inches, or an incredible 3 feet! A constant rate of change, such as the growth cycle of this bamboo plant, is a linear function.

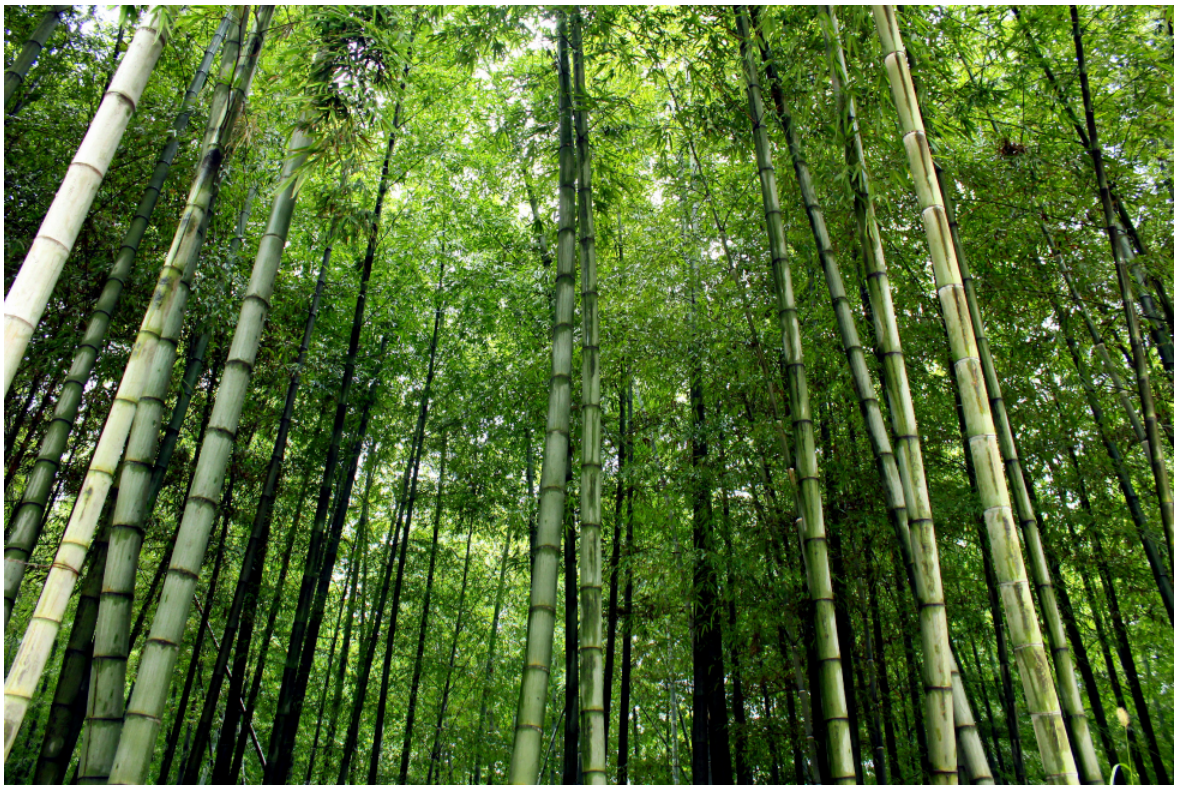


Figure* 0.0.1*: *A bamboo forest in China (credit: "JFXie"/Flickr)*

As we will see in Chapter 1, Section 1, a function is a relation that assigns to every element in the domain exactly one element in the range. Linear functions are a specific type of function that can be used to model many real-world applications, such as plant growth over time. In this chapter, we will explore linear functions, their graphs, and how to relate them to data.

Footnotes

1 http://www.guinnessworldrecords.com/...growing-plant/

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0.1: Linear Functions

Skills to Develop

- Represent a linear function.
- Determine whether a linear function is increasing, decreasing, or constant.
- Interpret slope as a rate of change.
- Write and interpret an equation for a linear function.
- Graph linear functions.
- Determine whether lines are parallel or perpendicular.
- Write the equation of a line parallel or perpendicular to a given line.

Just as with the growth of a bamboo plant, there are many situations that involve constant change over time. Consider, for example, the first commercial maglev train in the world, the Shanghai MagLev Train (maglev train in the world, the Shanghai MagLev Train (Figure 0.1.1). It carries passengers comfortably for a 30-kilometer trip from the airport to the subway station in only eight minutes.



Figure* 0.1.1*: *Shanghai MagLev Train (credit: "kanegen"/Flickr)*

Suppose a maglev train were to travel a long distance, and that the train maintains a constant speed of 83 meters per second for a period of time once it is 250 meters from the station. How can we analyze the train's distance from the station as a function of time? In this section, we will investigate a kind of function that is useful for this purpose, and use it to investigate real-world situations such as the train's distance from the station at a given point in time.maglev train were to travel a long distance, and that the train maintains a constant speed of 83 meters per second for a period of time once it is 250 meters from the station. How can we analyze the train's distance from the station as a function of time? In this section, we will investigate a kind of function that is useful for this purpose, and use it to investigate real-world situations such as the train's distance from the station at a given point in time.

Representing Linear Functions

The function describing the train's motion is a **linear function**, which is defined as a function with a constant rate of change; that is, a polynomial of degree 1. There are several ways to represent a linear function, including word form, function notation, tabular form, and graphical form. We will describe the train's motion as a function using each method.

Representing a Linear Function in Word Form

Let's begin by describing the linear function in words. For the train problem we just considered, the following word sentence may be used to describe the function relationship.

- *The train's distance from the station is a function of the time during which the train moves at a constant speed plus its original distance from the station when it began moving at constant speed.*

The speed is the rate of change. Recall that a rate of change is a measure of how quickly the dependent variable changes with respect to the independent variable. The rate of change for this example is constant, which means that it is the same for each input value. As the time (input) increases by 1 second, the corresponding distance (output) increases by 83 meters. The train began moving at this constant speed at a distance of 250 meters from the station.

Representing a Linear Function in Function Notation

Another approach to representing linear functions is by using function notation. One example of function notation is an equation written in the form known as the slope-intercept form of a line, where x is the input value, m is the rate of change, and b is the initial value of the dependent variable.

$$\begin{array}{ll} \textbf{Equation form} & y = mx + b \\ \textbf{Function notation} & f(x) = mx + b \end{array}$$

In the example of the train, we might use the notation $D(t)$ in which the total distance D is a function of the time t. The rate, m, is 83 meters per second. The initial value of the dependent variable b is the original distance from the station, 250 meters. We can write a function relationship to represent the motion of the train.

$$D(t) = 83t + 250$$

Representing a Linear Function in Tabular Form

A third method of representing a linear function is through the use of a table. The relationship between the distance from the station and the time is represented in Figure 0.1.2. From the table, we can see that the distance changes by 83 meters for every 1 second increase in time.

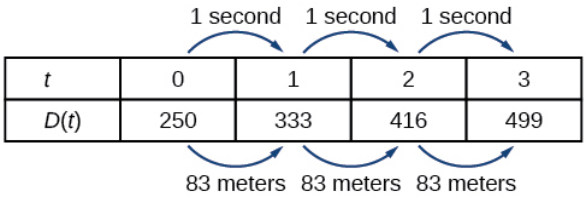


***Figure* 0.1.2:** *Tabular representation of the function D showing selected input and output values*

Can the input in the previous example be any real number?

No. The input represents time, so while nonnegative rational and irrational numbers are possible, negative real numbers are not possible for this example. The input consists of non-negative real numbers.

Representing a Linear Function in Graphical Form

Another way to represent linear functions is visually, using a graph. We can use the function relationship from above, $D(t) = 83t + 250$, to draw a graph, represented in Figure 0.1.3. Notice the graph is a line. When we plot a linear function, the graph is always a line.

The rate of change, which is constant, determines the slant, or steepness, or **slope** of the line. The point at which the input value is zero is the vertical intercept, or **y-intercept**, of the line. We can see from the graph in Figure 0.1.3 that the y-intercept in the train example we just saw is $(0, 250)$ and represents the distance of the train from the station when it began moving at a constant speed.

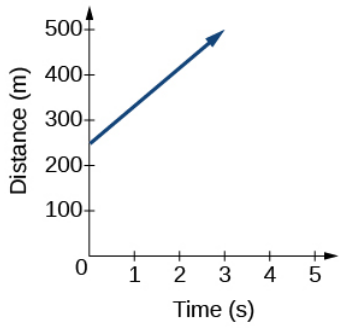


Figure* 0.1.3*: *The graph of $D(t) = 83t + 250$. Graphs of linear functions are lines because the rate of change is constant.*

Notice that the graph of the train example is restricted, but this is not always the case. Consider the graph of the line $f(x) = 2x + 1$. Ask yourself what numbers can be input to the function; that is, what is the domain of the function? The domain is comprised of all real numbers because any number may be doubled, and then have one added to the product.

definition: Linear Function

A **linear function** is a function whose graph is a line. Linear functions can be written in the slope-intercept form of a line

$$f(x) = mx + b$$

where b is the initial or starting value of the function (when the input $x = 0$), and m is the constant rate of change, or **slope** of the function. The **y-intercept** is at $(0, b)$.

Example 0.1.1: Using a Linear Function to Find the Pressure on a Diver

The pressure, P, in pounds per square inch (PSI) on the diver in **Figure 0.1.4** depends upon her depth below the water surface, d, in feet. This relationship may be modeled by the equation, $P(d) = 0.434d + 14.696$. Restate this function in words.



Figure* 0.1.4*: *(credit: Ilse Reijs and Jan-Noud Hutten)*

To restate the function in words, we need to describe each part of the equation. The pressure as a function of depth equals four hundred thirty-four thousandths times depth plus fourteen and six hundred ninety-six thousandths.

Analysis

The initial value, 14.696, is the pressure in PSI on the diver at a depth of 0 feet, which is the surface of the water. The rate of change, or slope, is 0.434 PSI per foot. This tells us that the pressure on the diver increases 0.434 PSI for each foot her depth increases.

Determining whether a Linear Function Is Increasing, Decreasing, or Constant

The linear functions we used in the two previous examples increased over time, but not every linear function does. A linear function may be increasing, decreasing, or constant. For an increasing function, as with the train example, the output values increase as the input values increase. The graph of an **increasing function** has a positive slope. A line with a positive slope slants upward from left to right as in Figure 0.1.5(a). For a **decreasing function**, the slope is negative. The output values decrease as the input values increase. A line with a negative slope slants downward from left to right as in Figure 0.1.5(b). If the function is constant, the output values are the same for all input values so the slope is zero. A line with a slope of zero is horizontal as in Figure 0.1.5(c).

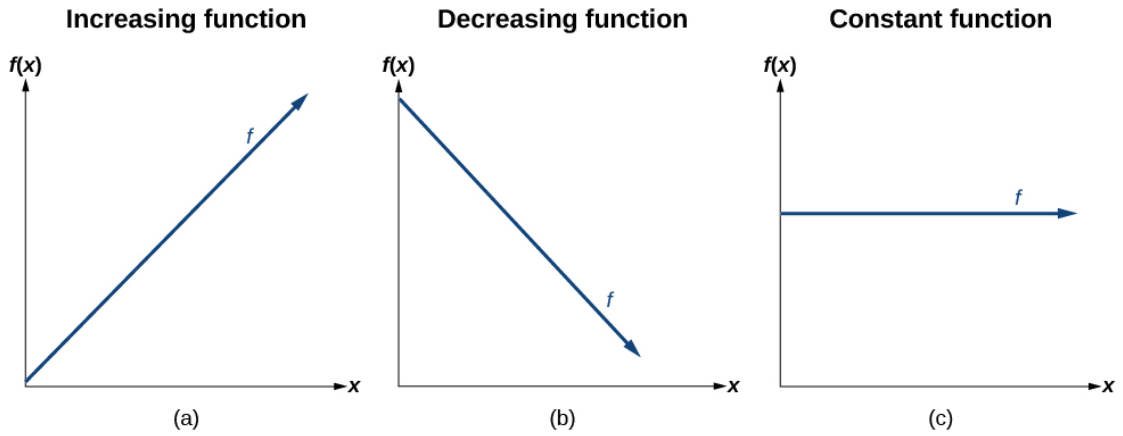


Figure* 0.1.5*: *Three graphs depicting an increasing function, a decreasing function, and a constant function.*

How To: determine whether a linear function is Increasing, Decreasing, or constant

The slope determines if the function is an **increasing linear function**, a **decreasing linear function**, or a **constant function**.

- $f(x) = mx + b$ is an increasing function if $m > 0$.
- $f(x) = mx + b$ is a decreasing function if $m < 0$.
- $f(x) = mx + b$ is a constant function if $m = 0$.

Example 0.1.2: Deciding whether a Function Is Increasing, Decreasing, or Constant

Some recent studies suggest that a teenager sends an average of 60 texts per day. For each of the following scenarios, find the linear function that describes the relationship between the input value and the output value. Then, determine whether the graph of the function is increasing, decreasing, or constant.

a. The total number of texts a teen sends is considered a function of time in days. The input is the number of days, and output is the total number of texts sent.
b. A teen has a limit of 500 texts per month in his or her data plan. The input is the number of days, and output is the total number of texts remaining for the month.
c. A teen has an unlimited number of texts in his or her data plan for a cost of $50 per month. The input is the number of days, and output is the total cost of texting each month.

Solution

Analyze each function.

a. The function can be represented as $f(x) = 60x$ where x is the number of days. The slope, 60, is positive so the function is increasing. This makes sense because the total number of texts increases with each day.
b. The function can be represented as $f(x) = 500 - 60x$ where x is the number of days. In this case, the slope is negative so the function is decreasing. This makes sense because the number of texts remaining decreases each day and this function represents the number of texts remaining in the data plan after x days.

c. The cost function can be represented as $f(x) = 0x + 50 = 50$ because the number of days does not affect the total cost. The slope is 0 so the function is constant.

Calculating and Interpreting Slope

In the examples we have seen so far, we have had the slope provided for us. However, we often need to calculate the slope, given input and output values.

Calculate Slope

Given two values for the input, x_1 and x_2, and two corresponding values for the output, y_1 and y_2 — which can be represented as points on the graph, (x_1, y_1) and (x_2, y_2) — we can calculate the slope m, as follows:

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

where Δy is the vertical displacement and Δx is the horizontal displacement.

Note that in function notation two corresponding values for the output y_1 and y_2 for the function f are $y_1 = f(x_1)$ and $y_2 = f(x_2)$, so we could equivalently write

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad \text{(this form will be important in calculus).}$$

Figure 0.1.6 indicates how the slope of the line between the points, (x_1, y_1) and (x_2, y_2), is calculated. Recall that the slope measures steepness. The greater the absolute value of the slope, the steeper the line is.

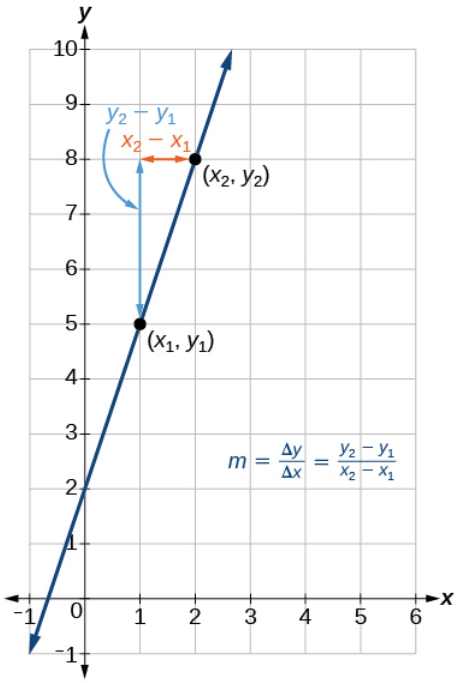


***Figure* 0.1.6:** *The slope of a function is calculated by the change in y divided by the change in x. It does not matter which point is used as the (x_2, y_2) and which is the (x_1, y_1), as long as each calculation is started with the elements from the same coordinate pair.*

Are the units for slope always $\frac{\text{units for the output}}{\text{units for the input}}$?

Yes. Think of the units as the change of output value for each unit of change in input value. An example of slope could be miles per hour or dollars per day. Notice the units appear as a ratio of units for the output per units for the input.

How To: Given two points from a linear function, calculate and interpret the slope.

1. Determine the units for output values and input values.
2. Calculate the change in output values (Δy) and change in input values (Δx).
3. Interpret the slope as the change in output values per unit of the input value.

Example 0.1.3: Finding the Slope of a Linear Function

If $f(x)$ is a linear function, and $(3,-2)$ and $(8,1)$ are points on the line, find the slope. Is this function increasing or decreasing?

Solution

The coordinate pairs are $(3,-2)$ and $(8,1)$. To find the rate of change, we divide the change in output by the change in input.

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{1-(-2)}{8-3} = \frac{3}{5} \tag{0.1.1}$$

We could also write the slope as $m = 0.6$. The function is increasing because $m > 0$.

Analysis

As noted earlier, the order in which we write the points does not matter when we compute the slope of the line as long as the first output value, or *y*-coordinate, used corresponds with the first input value, or *x*-coordinate, used.

Try It 0.1.1: If $f(x)$ is a linear function, and $(2,3)$ and $(0,4)$ are points on the line, find the slope. Is this function increasing or decreasing?

Solution

$m = \frac{4-3}{0-2} = \frac{1}{-2} = -\frac{1}{2}$; decreasing because $m < 0$.

Example 0.1.4: Finding the Population Change from a Linear Function

The population of a city increased from 23,400 to 27,800 between 2008 and 2012. Find the change of population per year if we assume the change was constant from 2008 to 2012.

The rate of change relates the change in population to the change in time. The population increased by $27,800 - 23,400 = 4400$ people over the four-year time interval. To find the rate of change, divide the change in the number of people by the number of years.

$$\frac{4,400 \text{ people}}{4 \text{ years}} = 1,100 \frac{\text{people}}{\text{year}} \tag{0.1.2}$$

So the population increased by 1,100 people per year.

Analysis

Because we are told that the population increased, we would expect the slope to be positive. This positive slope we calculated is therefore reasonable.

Try It 0.1.2: The population of a small town increased from 1,442 to 1,868 between 2009 and 2012. Find the change of population per year if we assume the change was constant from 2009 to 2012.

Solution

$m = \frac{1,868-1,442}{2,012-2,009} = \frac{426}{3} = 142$ people per year

Writing the Point-Slope Form of a Linear Equation

Up until now, we have been using the slope-intercept form of a linear equation to describe linear functions. Here, we will learn another way to write a linear function, the **point-slope form**.

$$y - y_1 = m(x - x_1) \tag{0.1.3}$$

The point-slope form is derived from the slope formula.

$$m = \frac{y - y_1}{x - x_1} \quad \text{assuming } x \neq x_1 \tag{0.1.4}$$

$$m(x - x_1) = \frac{y - y_1}{x - x_1}(x - x_1) \quad \text{Multiply both sides by } (x - x_1). \tag{0.1.5}$$

$$m(x - x_1) = y - y_1 \quad \text{Simplify} \tag{0.1.6}$$

$$y - y_1 = m(x - x_1) \quad \text{Rearrange} \tag{0.1.7}$$

Keep in mind that the slope-intercept form and the point-slope form can be used to describe the same function. We can move from one form to another using basic algebra. For example, suppose we are given an equation in point-slope form, $y - 4 = -\frac{1}{2}(x - 6)$. We can convert it to the slope-intercept form as shown.

$$y - 4 = -\frac{1}{2}(x - 6) \tag{0.1.8}$$

$$y - 4 = -\frac{1}{2}x + 3 \quad \text{Distribute the } -\frac{1}{2}. \tag{0.1.9}$$

$$y = -\frac{1}{2}x + 7 \quad \text{Add 4 to each side.} \tag{0.1.10}$$

Therefore, the same line can be described in slope-intercept form as $y = \frac{1}{2}x + 7$.

Point-Slope Form of a Linear Equation

The **point-slope** form of a linear equation takes the form

$$y - y_1 = m(x - x_1) \tag{0.1.11}$$

where m is the slope, x_1 and y_1 are the x and y coordinates of a specific point through which the line passes.

Writing the Equation of a Line Using a Point and the Slope

The point-slope form is particularly useful if we know one point and the slope of a line. Suppose, for example, we are told that a line has a slope of 2 and passes through the point $(4, 1)$. We know that $m = 2$ and that $x_1 = 4$ and $y_1 = 1$. We can substitute these values into the general point-slope equation.

$$y - y_1 = m(x - x_1) \tag{0.1.12}$$

$$y - 1 = 2(x - 4) \tag{0.1.13}$$

If we wanted to then rewrite the equation in slope-intercept form, we apply algebraic techniques.

$$y - 1 = 2(x - 4) \tag{0.1.14}$$

$$y - 1 = 2x - 8 \quad \text{Distribute the 2.} \tag{0.1.15}$$

$$y = 2x - 7 \quad \text{Add 1 to each side.} \tag{0.1.16}$$

Both equations, $y - 1 = 2(x - 4)$ and $y = 2x - 7$, describe the same line. See Figure 0.1.7.

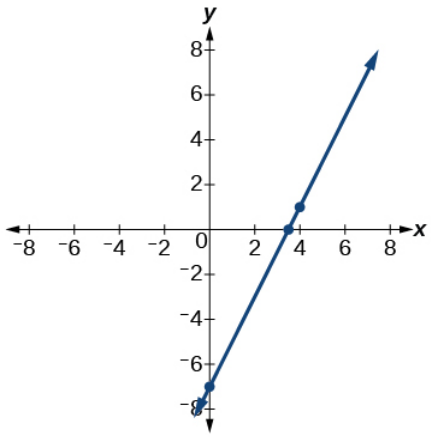


Figure 0.1.7

Example 0.1.5: Writing Linear Equations Using a Point and the Slope

Write the point-slope form of an equation of a line with a slope of 3 that passes through the point $(6,-1)$. Then rewrite it in the slope-intercept form.

Solution

Let's figure out what we know from the given information. The slope is 3, so $m = 3$. We also know one point, so we know $x_1 = 6$ and $y_1 = -1$. Now we can substitute these values into the general point-slope equation.

$$y - y_1 = m(x - x_1) \tag{0.1.17}$$
$$y - (-1) = 3(x - 6) \quad \text{Substitute known values.} \tag{0.1.18}$$
$$y + 1 = 3(x - 6) \quad \text{Distribute } -1 \text{ to find point-slope form.} \tag{0.1.19}$$

Then we use algebra to find the slope-intercept form.

$$y + 1 = 3(x - 6) \tag{0.1.20}$$
$$y + 1 = 3x - 18 \quad \text{Distribute 3.} \tag{0.1.21}$$
$$y = 3x - 19 \quad \text{Simplify to slope-intercept form.} \tag{0.1.22}$$

0.1.3: Write the point-slope form of an equation of a line with a slope of –2 that passes through the point $(-2,2)$. Then rewrite it in the slope-intercept form.

Solution

$y - 2 = -2(x + 2)$; $y = -2x - 2$

Writing the Equation of a Line Using Two Points

The point-slope form of an equation is also useful if we know any two points through which a line passes. Suppose, for example, we know that a line passes through the points $(0,1)$ and $(3,2)$. We can use the coordinates of the two points to find the slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{0.1.23}$$
$$= \frac{2-1}{3-0} \tag{0.1.24}$$
$$= \frac{1}{3} \tag{0.1.25}$$

Now we can use the slope we found and the coordinates of one of the points to find the equation for the line. Let use $(0,1)$ for our point.

$$y - y_1 = m(x - x_1) \tag{0.1.26}$$
$$y - 1 = \frac{1}{3}(x - 0) \tag{0.1.27}$$

As before, we can use algebra to rewrite the equation in the slope-intercept form.

$$y - 1 = \frac{1}{3}(x - 0) \tag{0.1.28}$$
$$y - 1 = \frac{1}{3}x \quad \text{Distribute the } \frac{1}{3}. \tag{0.1.29}$$
$$y = \frac{1}{3}x + 1 \quad \text{Add 1 to each side.} \tag{0.1.30}$$

Both equations describe the line shown in Figure 0.1.8.

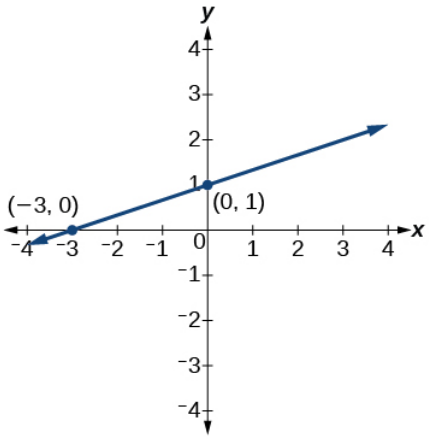


Figure 0.1.8

Example 0.1.6: Writing Linear Equations Using Two Points

Write the point-slope form of an equation of a line that passes through the points $(5,1)$ and $(8,7)$. Then rewrite it in the slope-intercept form.

Let's begin by finding the slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{0.1.31}$$
$$= \frac{7-1}{8-5} \tag{0.1.32}$$
$$= \frac{6}{3} \tag{0.1.33}$$
$$= 2 \tag{0.1.34}$$

So $m = 2$. Next, we substitute the slope and the coordinates for one of the points into the general point-slope equation. We can choose either point, but we will use $(5,1)$.

$$y - y_1 = m(x - x_1) \tag{0.1.35}$$
$$y - 1 = 2(x - 5) \tag{0.1.36}$$

The point-slope equation of the line is $y_2–1 = 2(x_2–5)$. To rewrite the equation in slope-intercept form, we use algebra.

$$y - 1 = 2(x - 5) \tag{0.1.37}$$
$$y - 1 = 2x - 10 \tag{0.1.38}$$
$$y = 2x - 9 \tag{0.1.39}$$

The slope-intercept equation of the line is $y = 2x–9$.

0.1.4: Write the point-slope form of an equation of a line that passes through the points $(–1,3)$ and $(0,0)$. Then rewrite it in the slope-intercept form.

Solution

$y – 0 = –3(x – 0)$; $y = –3x$

Writing and Interpreting an Equation for a Linear Function

Now that we have written equations for linear functions in both the slope-intercept form and the point-slope form, we can choose which method to use based on the information we are given. That information may be provided in the form of a graph, a point and a slope, two points, and so on. Look at the graph of the function f in Figure 0.1.9.

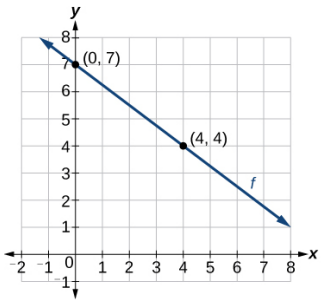


Figure* 0.1.9*: *Graph depicting how to calculate the slope of a line.*

We are not given the slope of the line, but we can choose any two points on the line to find the slope. Let's choose $(0,7)$ and $(4,4)$. We can use these points to calculate the slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{0.1.40}$$
$$= \frac{4-7}{4-0} \tag{0.1.41}$$
$$= -\frac{3}{4} \tag{0.1.42}$$

Now we can substitute the slope and the coordinates of one of the points into the point-slope form.

$$y - y_1 = m(x - x_1) \tag{0.1.43}$$
$$y - 4 = -\frac{3}{4}(x - 4) \tag{0.1.44}$$

If we want to rewrite the equation in the slope-intercept form, we would find

$$y - 4 = -\frac{3}{4}(x - 4) \tag{0.1.45}$$
$$y - 4 = -\frac{3}{4}x + 3 \tag{0.1.46}$$
$$y = -\frac{3}{4}x + 7 \tag{0.1.47}$$

If we wanted to find the slope-intercept form without first writing the point-slope form, we could have recognized that the line crosses the y-axis when the output value is 7. Therefore, $b = 7$. We now have the initial value b and the slope m so we can substitute m and b into the slope-intercept form of a line.

$$f(x) = mx + b$$
$$-\frac{3}{4} \qquad 7$$
$$f(x) = -\frac{3}{4}x + 7$$

***Figure* 0.1.10**

So the function is $f(x) = -\frac{3}{4}x + 7$, and the linear equation would be $y = -\frac{3}{4}x + 7$.

How To: Given the graph of a linear function, write an equation to represent the function.

1. Identify two points on the line.
2. Use the two points to calculate the slope.
3. Determine where the line crosses the y-axis to identify the y-intercept by visual inspection.
4. Substitute the slope and y-intercept into the slope-intercept form of a line equation.

Example 0.1.7: Writing an Equation for a Linear Function

Write an equation for a linear function given a graph of f shown in Figure **0.1.11**.

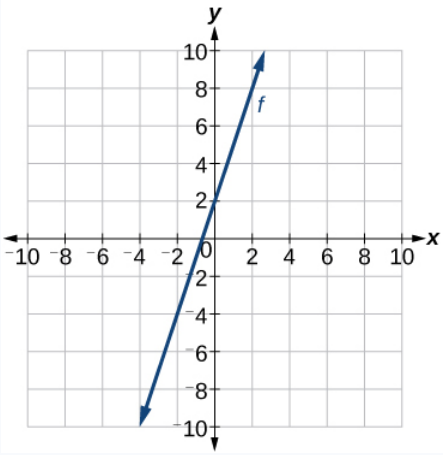


***Figure* 0.1.11**

Solution

Identify two points on the line, such as $(0, 2)$ and $(-2, -4)$. Use the points to calculate the slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{0.1.48}$$
$$= \frac{4 - 2}{-2 - 0} \tag{0.1.49}$$
$$= \frac{-6}{-2} \tag{0.1.50}$$
$$= 3 \tag{0.1.51}$$

Substitute the slope and the coordinates of one of the points into the point-slope form.

$$y - y_1 = m(x - x_1) \tag{0.1.52}$$
$$y - (-4) = 3(x - (-2)) \tag{0.1.53}$$
$$y + 4 = 3(x + 2) \tag{0.1.54}$$

We can use algebra to rewrite the equation in the slope-intercept form.

$$y + 4 = 3(x + 2) \tag{0.1.55}$$
$$y + 4 = 3x + 6 \tag{0.1.56}$$
$$y = 3x + 2 \tag{0.1.57}$$

Analysis

This makes sense because we can see from Figure **0.1.12** that the line crosses the y-axis at the point $(0, 2)$, which is the y-intercept, so $b = 2$.

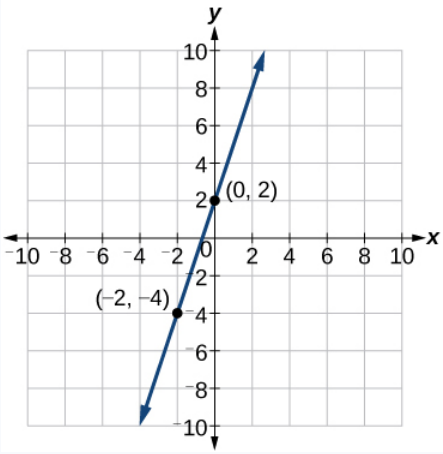


***Figure* 0.1.12:** *Graph of an increasing line with points at* $(0, 2)$ *and* $(-2, -4)$.

Example 0.1.8: Writing an Equation for a Linear Cost Function

Suppose Ben starts a company in which he incurs a fixed cost of $1,250 per month for the overhead, which includes his office rent. His production costs are $37.50 per item. Write a linear function C where $C(x)$ is the cost for x items produced in a given month.

Solution

The fixed cost is present every month, $1,250. The costs that can vary include the cost to produce each item, which is $37.50 for Ben. The variable cost, called the marginal cost, is represented by 37.5. The cost Ben incurs is the sum of these two costs, represented by $C(x) = 1250 + 37.5x$.

Analysis

If Ben produces 100 items in a month, his monthly cost is represented by

$$C(100) = 1250 + 37.5(100) \tag{0.1.58}$$
$$= 5000 \tag{0.1.59}$$

So his monthly cost would be $5,000.

Example 0.1.9: Writing an Equation for a Linear Function Given Two Points

If f is a linear function, with $f(3) = -2$, and $f(8) = 1$, find an equation for the function in slope-intercept form.

Solution

We can write the given points using coordinates.

$$f(3) = -2 \rightarrow (3, 2) \tag{0.1.60}$$
$$f(8) = 1 \rightarrow (8, 1) \tag{0.1.61}$$

We can then use the points to calculate the slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{0.1.62}$$
$$= \frac{1 - (-2)}{8 - 3} \tag{0.1.63}$$
$$= \frac{3}{5} \tag{0.1.64}$$

Substitute the slope and the coordinates of one of the points into the point-slope form.

$$y - y_1 = m(x - x_1) \tag{0.1.65}$$
$$y - (-2) = \frac{3}{5}(x - 3) \tag{0.1.66}$$

We can use algebra to rewrite the equation in the slope-intercept form.

$$y + 2 = \frac{3}{5}(x - 3) \tag{0.1.67}$$
$$y + 2 = \frac{3}{5}x - \frac{9}{5} \tag{0.1.68}$$
$$y = \frac{3}{5}x - \frac{19}{5} \tag{0.1.69}$$

0.1.5: If $f(x)$ is a linear function, with $f(2) = -11$, and $f(4) = -25$, find an equation for the function in slope-intercept form.

Solution

$y = -7x + 3$

Modeling Real-World Problems with Linear Functions

In the real world, problems are not always explicitly stated in terms of a function or represented with a graph. Fortunately, we can analyze the problem by first representing it as a linear function and then interpreting the components of the function. As long as we know, or can figure out, the initial value and the rate of change of a linear function, we can solve many different kinds of real-world problems.

How To: Given a linear function f and the initial value and rate of change, evaluate $f(c)$.

1. Determine the initial value and the rate of change (slope).
2. Substitute the values into $f(x) = mx + b$.
3. Evaluate the function at $x = c$.

Example 0.1.10: Using a Linear Function to Determine the Number of Songs in a Music Collection

Marcus currently has 200 songs in his music collection. Every month, he adds 15 new songs. Write a formula for the number of songs, N, in his collection as a function of time, t, the number of months. How many songs will he own in a year?

Solution

The initial value for this function is 200 because he currently owns 200 songs, so $N(0) = 200$, which means that $b = 200$.

The number of songs increases by 15 songs per month, so the rate of change is 15 songs per month. Therefore we know that $m = 15$. We can substitute the initial value and the rate of change into the slope-intercept form of a line.

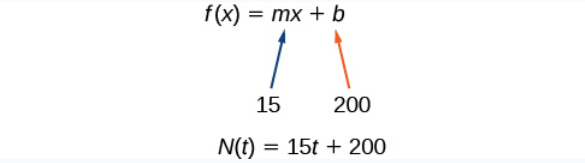


***Figure* 0.1.13**

We can write the formula $N(t) = 15t + 200$.

With this formula, we can then predict how many songs Marcus will have in 1 year (12 months). In other words, we can evaluate the function at $t = 12$.

$$\begin{aligned} N(12) &= 15(12) + 200 && (0.1.70)\\ &= 180 + 200 && (0.1.71)\\ &= 380 && (0.1.72) \end{aligned}$$

Marcus will have 380 songs in 12 months.

Analysis

Notice that N is an increasing linear function. As the input (the number of months) increases, the output (number of songs) increases as well.

Example 0.1.11: Using a Linear Function to Calculate Salary Plus Commission

Working as an insurance salesperson, Ilya earns a base salary plus a commission on each new policy. Therefore, Ilya's weekly income, I,depends on the number of new policies, n, he sells during the week. Last week he sold 3 new policies, and earned \$760 for the week. The week before, he sold 5 new policies and earned \$920. Find an equation for $I(n)$, and interpret the meaning of the components of the equation.

Solution

The given information gives us two input-output pairs: $(3, 760)$ and $(5, 920)$. We start by finding the rate of change.

$$m = \frac{920 - 760}{5 - 3} \tag{0.1.73}$$
$$= \frac{\$160}{2 \text{ policies}} \tag{0.1.74}$$
$$= \$80 \text{ per policy} \tag{0.1.75}$$

Keeping track of units can help us interpret this quantity. Income increased by $160 when the number of policies increased by 2, so the rate of change is $80 per policy. Therefore, Ilya earns a commission of $80 for each policy sold during the week.

We can then solve for the initial value.

$$I(n) = 80n + b \tag{0.1.76}$$
$$760 = 80(3) + b \text{ when } n = 3, I(3) = 760 \tag{0.1.77}$$
$$760 - 80(3) = b \tag{0.1.78}$$
$$520 = b \tag{0.1.79}$$

The value of b is the starting value for the function and represents Ilya's income when $n = 0$, or when no new policies are sold. We can interpret this as Ilya's base salary for the week, which does not depend upon the number of policies sold.

We can now write the final equation.

$$I(n) = 80n + 520 \tag{0.1.80}$$

Our final interpretation is that Ilya's base salary is $520 per week and he earns an additional $80 commission for each policy sold.

Example 0.1.12: Using Tabular Form to Write an Equation for a Linear Function

Table 0.1.1 relates the number of rats in a population to time, in weeks. Use the table to write a linear equation.

***Table* 0.1.1**

w, number of weeks	0	2	4	6
P(w), number of rats	1000	1080	1160	1240

Solution

We can see from the table that the initial value for the number of rats is 1000, so $b = 1000$.

Rather than solving for m, we can tell from looking at the table that the population increases by 80 for every 2 weeks that pass. This means that the rate of change is 80 rats per 2 weeks, which can be simplified to 40 rats per week.

$$P(w) = 40w + 1000 \tag{0.1.81}$$

If we did not notice the rate of change from the table we could still solve for the slope using any two points from the table. For example, using $(2, 1080)$ and $(6, 1240)$

$$m = \frac{1240 - 1080}{6 - 2} \tag{0.1.82}$$
$$= \frac{160}{4} \tag{0.1.83}$$
$$= 40 \tag{0.1.84}$$

Is the initial value always provided in a table of values like Table 0.1.1?

No. Sometimes the initial value is provided in a table of values, but sometimes it is not. If you see an input of 0, then the initial value would be the corresponding output. If the initial value is not provided because there is no value of input on the table equal to 0, find the slope, substitute one coordinate pair and the slope into $f(x) = mx + b$, and solve for b.

0.1.5: A new plant food was introduced to a young tree to test its effect on the height of the tree. Table **0.1.2** shows the height of the tree, in feet, x months since the measurements began. Write a linear function, $H(x)$, where x is the number of months since the start of the experiment.

***Table* 0.1.2**

x	1	2
$H(x)$	18	18.5

Solution

$H(x) = 0.5x + 12.5$

Key Equations

- slope-intercept form of a line: $f(x) = mx + b$
- slope: $m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$
- point-slope form of a line: $y - y_1 = m(x - x_1)$

Key Concepts

- The ordered pairs given by a linear function represent points on a line.
- Linear functions can be represented in words, function notation, tabular form, and graphical form.
- The rate of change of a linear function is also known as the slope.
- An equation in the slope-intercept form of a line includes the slope and the initial value of the function.
- The initial value, or y-intercept, is the output value when the input of a linear function is zero. It is the y-value of the point at which the line crosses the y-axis.
- An increasing linear function results in a graph that slants upward from left to right and has a positive slope.
- A decreasing linear function results in a graph that slants downward from left to right and has a negative slope.
- A constant linear function results in a graph that is a horizontal line.
- Analyzing the slope within the context of a problem indicates whether a linear function is increasing, decreasing, or constant.
- The slope of a linear function can be calculated by dividing the difference between y-values by the difference in corresponding x-values of any two points on the line.
- The slope and initial value can be determined given a graph or any two points on the line.
- One type of function notation is the slope-intercept form of an equation.
- The point-slope form is useful for finding a linear equation when given the slope of a line and one point.
- The point-slope form is also convenient for finding a linear equation when given two points through which a line passes.
- The equation for a linear function can be written if the slope m and initial value b are known.
- A linear function can be used to solve real-world problems.
- A linear function can be written from tabular form.

Section Exercises

Verbal

Terry is skiing down a steep hill. Terry's elevation, $E(t)$, in feet after t seconds is given by $E(t) = 3000 - 70t$. Write a complete sentence describing Terry's starting elevation and how it is changing over time.

Terry starts at an elevation of 3000 feet and descends 70 feet per second.

Maria is climbing a mountain. Maria's elevation, $E(t)$, in feet after t minutes is given by $E(t) = 1200 + 40t$. Write a complete sentence describing Maria's starting elevation and how it is changing over time.

Jessica is walking home from a friend's house. After 2 minutes she is 1.4 miles from home. Twelve minutes after leaving, she is 0.9 miles from home. What is her rate in miles per hour?

3 miles per hour

Sonya is currently 10 miles from home and is walking farther away at 2 miles per hour. Write an equation for her distance from home t hours from now.

A boat is 100 miles away from the marina, sailing directly toward it at 10 miles per hour. Write an equation for the distance of the boat from the marina after t hours.

$d(t) = 100 - 10t$

Timmy goes to the fair with $40. Each ride costs $2. How much money will he have left after riding n rides?

Algebraic

For the following exercises, determine whether the equation of the curve can be written as a linear function.

$y = \frac{1}{4}x + 6$

Yes.

$y = 3x - 5$

$y = 3x^2 - 2$

No.

$3x + 5y = 15$

$3x^2 + 5y = 15$

No.

$3x + 5y^2 = 15$

$-2x^2 + 3y^2 = 6$

No.

$-\frac{x-3}{5} = 2y$

For the following exercises, determine whether each function is increasing or decreasing.

$f(x) = 4x + 3$

Increasing.

$g(x) = 5x + 6$

$a(x) = 5 - 2x$

Decreasing.

$b(x) = 8 - 3x$

$h(x) = -2x + 4$

Decreasing.

$k(x) = -4x + 1$

$j(x) = \frac{1}{2}x - 3$

Increasing.

$p(x) = \frac{1}{4}x - 5$

$n(x) = -\frac{1}{3}x - 2$

Decreasing.

$m(x) = -\frac{3}{8}x + 3$

For the following exercises, find the slope of the line that passes through the two given points.

$(2, 4)$ and $(4, 10)$

3

$(1, 5)$ and $(4, 11)$

$(-1, 4)$ and $(5, 2)$

$-\frac{1}{3}$

$(8, -2)$ and $(4, 6)$

(6, 11) and (−4, 3)

$\frac{4}{5}$

For the following exercises, given each set of information, find a linear equation satisfying the conditions, if possible.

$f(-5) = -4$, and $f(5) = 2$

$f(-1) = 4$ and $f(5) = 1$

$f(x) = -\frac{1}{2}x + \frac{7}{2}$

$(2, 4)$ and $(4, 10)$

Passes through $(1, 5)$ and $(4, 11)$

$y = 2x + 3$

Passes through $(-1, 4)$ and $(5, 2)$

Passes through $(-2, 8)$ and $(4, 6)$

$y = -\frac{1}{3}x + \frac{22}{3}$

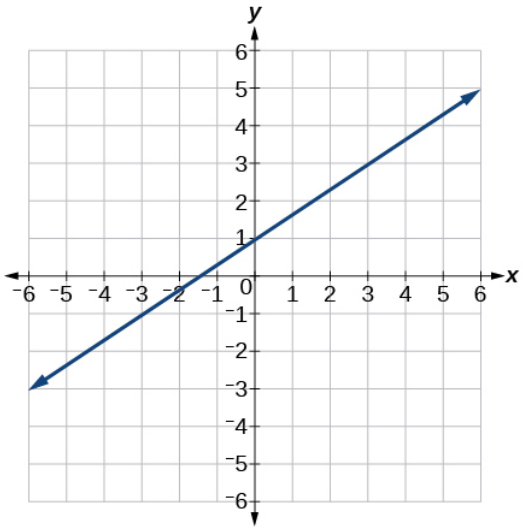
x intercept at (−2, 0) and y intercept at (0,−3)

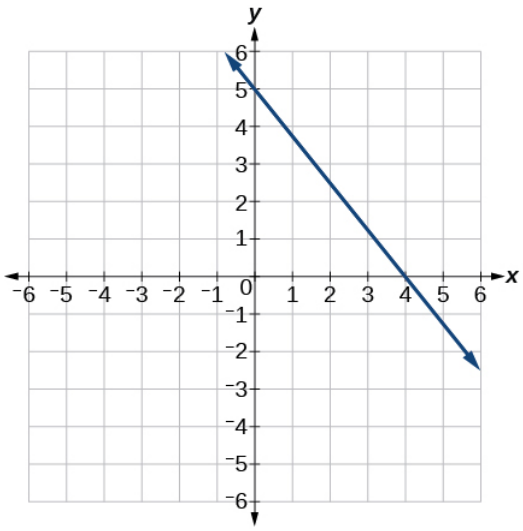
x intercept at (−5, 0) and y intercept at (0, 4)

$y = \frac{4}{5}x + 4$

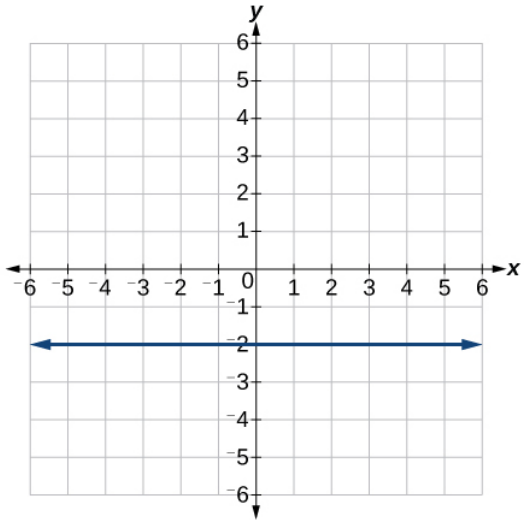
Graphical

For the following exercises, find the slope of the lines graphed.

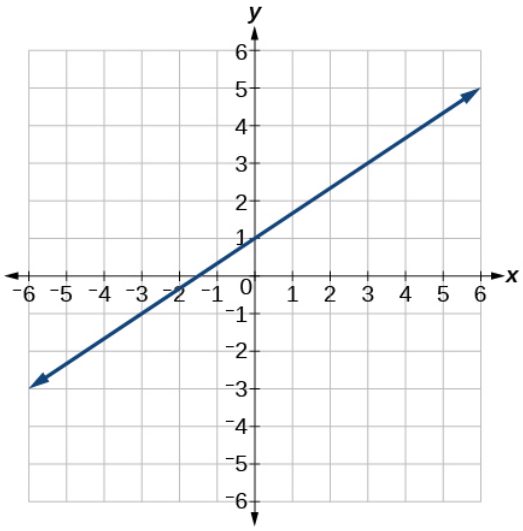




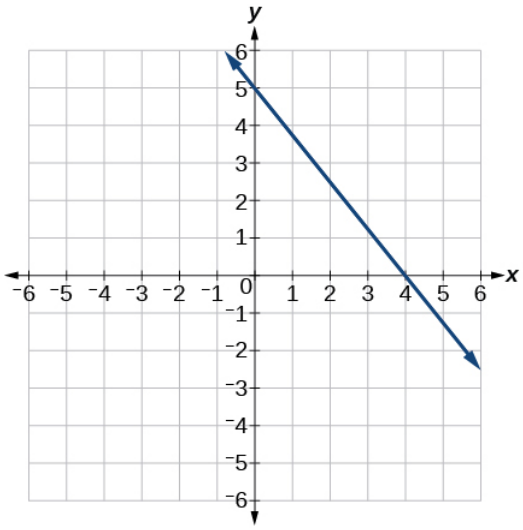
$-\frac{5}{4}$

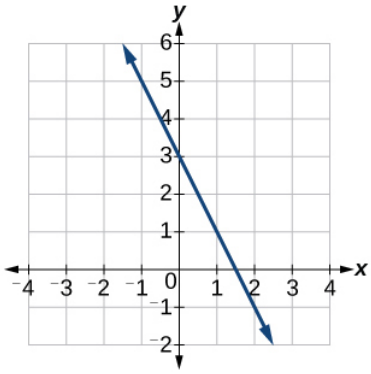


For the following exercises, write an equation for the lines graphed.

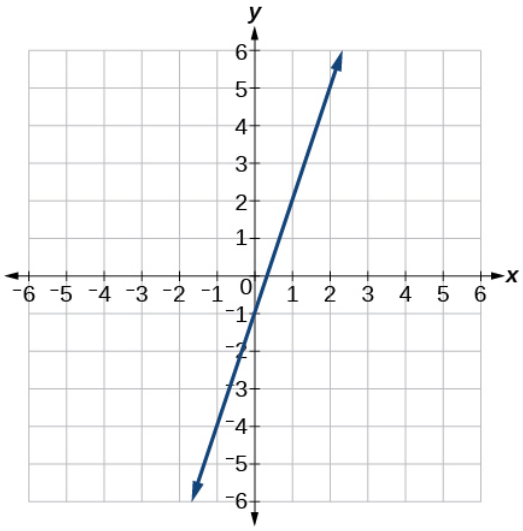


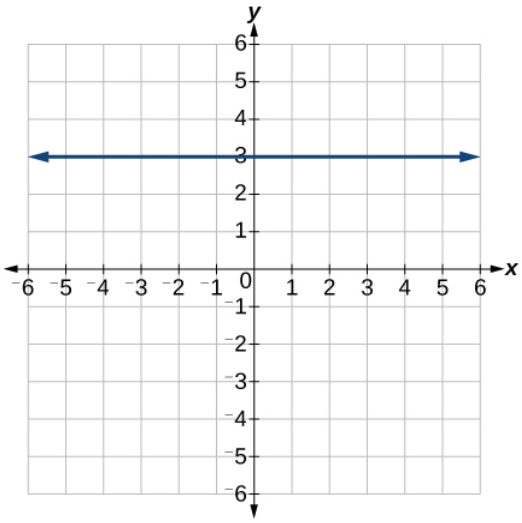
$y = \frac{2}{3}x + 1$



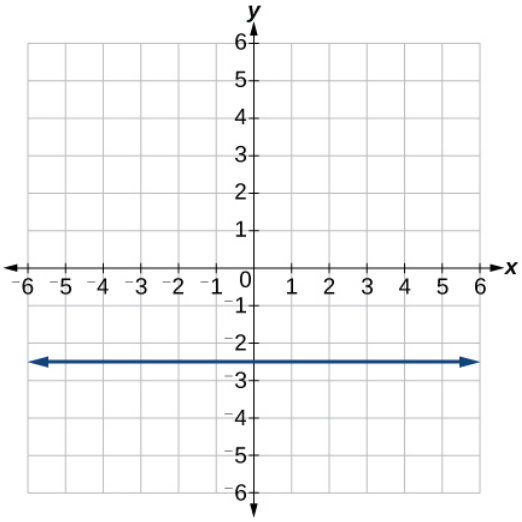


$y = -2x + 3$





$y = 3$



Numeric

For the following exercises, which of the tables could represent a linear function? For each that could be linear, find a linear equation that models the data.

x	0	5	10	15
$g(x)$	5	–10	–25	–40

Linear, $g(x) = -3x + 5$

x	0	5	10	15
$h(x)$	5	30	105	230

x	0	5	10	15
$f(x)$	–5	20	45	70

Linear, $f(x) = 5x - 5$

x	5	10	20	25
$k(x)$	28	13	58	73

x	0	2	4	6
$g(x)$	6	–19	–44	–69

Linear, $g(x) = -\frac{25}{2}x + 6$

x	2	4	6	8
$f(x)$	–4	16	36	56

x	2	4	6	8
$f(x)$	–4	16	36	56

Linear, $f(x) = 10x - 24$

x	0	2	6	8
$k(x)$	6	31	106	231

Technology

If f is a linear function, $f(0.1) = 11.5$, and $f(0.4) = -5.9$, find an equation for the function.

$f(x) = -58x + 17.3$

Graph the function f on a domain of $[-10, 10] : f(x) = 0.02x - 0.01$. Enter the function in a graphing utility. For the viewing window, set the minimum value of x to be −10 and the maximum value of x to be 10.

Graph the function f on a domain of $[-10, 10] : fx) = 2,500x + 4,000$

[link] shows the input, w, and output, k, for a linear function k. a. Fill in the missing values of the table. b. Write the linear function k, round to 3 decimal places.

w	–10	5.5	67.5	b
k	30	–26	a	–44

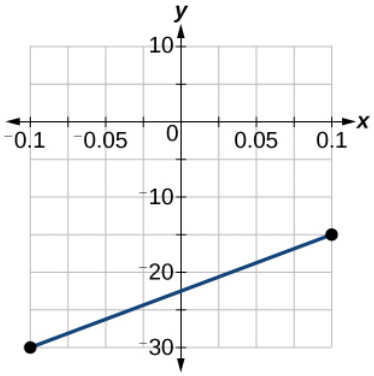
[link] shows the input,p,and output,q,for a linear function q. a. Fill in the missing values of the table. b. Write the linear function k.

p	0.5	0.8	12	b
q	400	700	a	1,000,000

a. $a = 11{,}900$; $b = 1001.1$ b. $q(p) = 1000p - 100$

Graph the linear function f on a domain of $[-10, 10]$ for the function whose slope is $\frac{1}{8}$ and y-intercept is $\frac{31}{16}$. Label the points for the input values of -10 and 10.

Graph the linear function f on a domain of $[-0.1, 0.1]$ for the function whose slope is 75 and y-intercept is −22.5. Label the points for the input values of −0.1 and 0.1.



Graph the linear function f where $f(x) = ax + b$ on the same set of axes on a domain of $[-4, 4]$ for the following values of a and b.

$a = 2; b = 3$
$a = 2; b = 4$
$a = 2; b = -4$
$a = 2; b = -5$

Extensions

Find the value of x if a linear function goes through the following points and has the following slope: $(x, 2), (-4, 6), m = 3$

$x = -\frac{16}{3}$

Find the value of y if a linear function goes through the following points and has the following slope: $(10, y), (25, 100), m = -5$

Find the equation of the line that passes through the following points: (a, b) and $(a, b + 1)$

$x = a$

Find the equation of the line that passes through the following points: $(2a, b)$ and (a, b+1)

Find the equation of the line that passes through the following points: $(a, 0)$ and (c, d)

$y = \frac{d}{c-a}x - \frac{ad}{c-a}$

Real-World Applications

At noon, a barista notices that she has $20 in her tip jar. If she makes an average of $0.50 from each customer, how much will she have in her tip jar if she serves n more customers during her shift?

A gym membership with two personal training sessions costs $125, while gym membership with five personal training sessions costs $260. What is cost per session?

$45 per training session.

A clothing business finds there is a linear relationship between the number of shirts, n, it can sell and the price, p, it can charge per shirt. In particular, historical data shows that 1,000 shirts can be sold at a price of $30, while 3,000 shirts can be sold at a price of $22. Find a linear equation in the form $p(n) = mn + b$ that gives the price p they can charge for n shirts.

A phone company charges for service according to the formula: $C(n) = 24 + 0.1$n, where n is the number of minutes talked, and $C(n)$ is the monthly charge, in dollars. Find and interpret the rate of change and initial value.

The rate of change is 0.1. For every additional minute talked, the monthly charge increases by $0.1 or 10 cents. The initial value is 24. When there are no minutes talked, initially the charge is $24.

A farmer finds there is a linear relationship between the number of bean stalks, n, she plants and the yield, y, each plant produces. When she plants 30 stalks, each plant yields 30 oz of beans. When she plants 34 stalks, each plant produces 28 oz of beans. Find a linear relationships in the form $y = mn + b$ that gives the yield when n stalks are planted.

A city's population in the year 1960 was 287,500. In 1989 the population was 275,900. Compute the rate of growth of the population and make a statement about the population rate of change in people per year.

The slope is −400. This means for every year between 1960 and 1989, the population dropped by 400 per year in the city.

A town's population has been growing linearly. In 2003, the population was 45,000, and the population has been growing by 1,700 people each year. Write an equation, $P(t)$, for the population t years after 2003.

Suppose that average annual income (in dollars) for the years 1990 through 1999 is given by the linear function: $I(x) = 1054x + 23,286$, where xis the number of years after 1990. Which of the following interprets the slope in the context of the problem?

As of 1990, average annual income was $23,286.
In the ten-year period from 1990–1999, average annual income increased by a total of $1,054.
Each year in the decade of the 1990s, average annual income increased by $1,054.
Average annual income rose to a level of $23,286 by the end of 1999.

c.

When temperature is 0 degrees Celsius, the Fahrenheit temperature is 32. When the Celsius temperature is 100, the corresponding Fahrenheit temperature is 212. Express the Fahrenheit temperature as a linear function of C, the Celsius temperature, $F(C)$.

Find the rate of change of Fahrenheit temperature for each unit change temperature of Celsius.
Find and interpret $F(28)$.
Find and interpret $F(-40)$.

Footnotes

1 http://www.chinahighlights.com/shang...glev-train.htm
2 http://www.cbsnews.com/8301-501465_1...ay-study-says/

Glossary

decreasing linear function

a function with a negative slope: If $f(x) = mx + b$, then $m < 0$.

increasing linear function

a function with a positive slope: If $f(x) = mx + b$, then $m > 0$.

linear function

a function with a constant rate of change that is a polynomial of degree 1, and whose graph is a straight line

point-slope form

the equation for a line that represents a linear function of the form \(y−y_1=m(x−x_1)

slope

the ratio of the change in output values to the change in input values; a measure of the steepness of a line

slope-intercept form

the equation for a line that represents a linear function in the form $f(x) = mx + b$

y-intercept

the value of a function when the input value is zero; also known as initial value

Contributors

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0.2: Graphs of Linear Functions

Skills to Develop

- Graph linear functions.
- Write the equation for a linear function from the graph of a line.
- Given the equations of two lines, determine whether their graphs are parallel or perpendicular.
- Write the equation of a line parallel or perpendicular to a given line.
- Solve a system of linear equations.

Two competing telephone companies offer different payment plans. The two plans charge the same rate per long distance minute, but charge a different monthly flat fee. A consumer wants to determine whether the two plans will ever cost the same amount for a given number of long distance minutes used. The total cost of each payment plan can be represented by a linear function. To solve the problem, we will need to compare the functions. In this section, we will consider methods of comparing functions using graphs.

Graphing Linear Functions

In Linear Functions, we saw that that the graph of a linear function is a straight line. We were also able to see the points of the function as well as the initial value from a graph. By graphing two functions, then, we can more easily compare their characteristics.

There are three basic methods of graphing linear functions. The first is by plotting points and then drawing a line through the points. The second is by using the y-intercept and slope. And the third is by using transformations of the identity function $f(x) = x$.

Graphing a Function by Plotting Points

To find points of a function, we can choose input values, evaluate the function at these input values, and calculate output values. The input values and corresponding output values form coordinate pairs. We then plot the coordinate pairs on a grid. In general, we should evaluate the function at a minimum of two inputs in order to find at least two points on the graph. For example, given the function, $f(x) = 2x$, we might use the input values 1 and 2. Evaluating the function for an input value of 1 yields an output value of 2, which is represented by the point $(1, 2)$. Evaluating the function for an input value of 2 yields an output value of 4, which is represented by the point $(2, 4)$. Choosing three points is often advisable because if all three points do not fall on the same line, we know we made an error.

HowTO: Given a linear function, graph by plotting points.

1. Choose a minimum of two input values.
2. Evaluate the function at each input value.
3. Use the resulting output values to identify coordinate pairs.
4. Plot the coordinate pairs on a grid.
5. Draw a line through the points.

Example 0.2.1: Graphing by Plotting Points

Graph $f(x) = -\frac{2}{3}x + 5$ by plotting points.

Solution

Begin by choosing input values. This function includes a fraction with a denominator of 3, so let's choose multiples of 3 as input values. We will choose 0, 3, and 6.

Evaluate the function at each input value, and use the output value to identify coordinate pairs.

$$x = 0 \qquad f(0) = -\frac{2}{3}(0) + 5 = 5 \rightarrow (0, 5) \tag{0.2.1}$$

$$x = 3 \qquad f(3) = -\frac{2}{3}(3) + 5 = 3 \rightarrow (3, 3) \tag{0.2.2}$$

$$x = 6 \qquad f(6) = -\frac{2}{3}(6) + 5 = 1 \rightarrow (6, 1) \tag{0.2.3}$$

Plot the coordinate pairs and draw a line through the points. Figure 0.2.1 represents the graph of the function $f(x) = -\frac{2}{3}x + 5$.

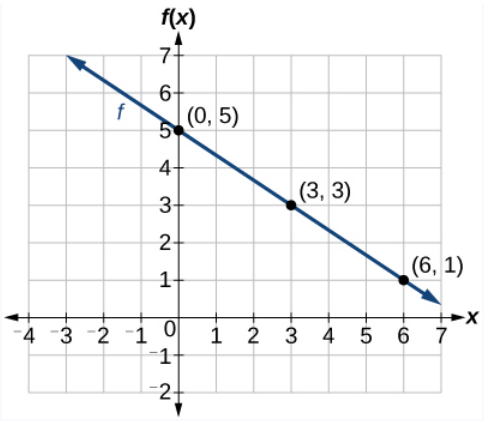


***Figure* 0.2.1**: *The graph of the linear function* $f(x) = -\frac{2}{3}x + 5$.

Analysis

The graph of the function is a line as expected for a linear function. In addition, the graph has a downward slant, which indicates a negative slope. This is also expected from the negative constant rate of change in the equation for the function.

Exercise 0.2.1

Graph $f(x) = -\frac{3}{4}x + 6$ by plotting points.

Answer

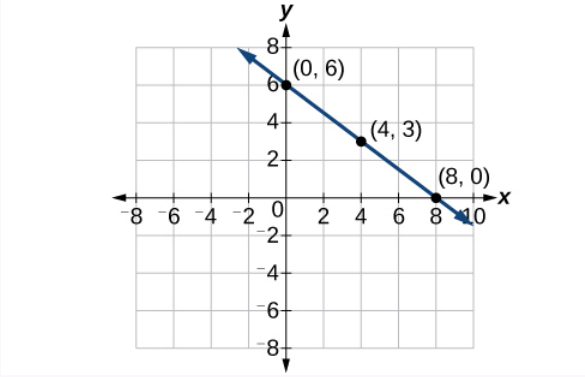


***Figure* 0.2.2**

Graphing a Function Using y-intercept and Slope

Another way to graph linear functions is by using specific characteristics of the function rather than plotting points. The first characteristic is its **y-intercept**, which is the point at which the input value is zero. To find the y-intercept, we can set $x = 0$ in the equation.

The other characteristic of the linear function is its slope m, which is a measure of its steepness. Recall that the slope is the rate of change of the function. The slope of a function is equal to the ratio of the change in outputs to the change in inputs. Another way to think about the slope is by dividing the vertical difference, or rise, by the horizontal difference, or run. We encountered both the y-intercept and the slope in Linear Functions.

Let's consider the following function.

$$f(x) = \frac{1}{2}x + 1 \tag{0.2.4}$$

The slope is $\frac{1}{2}$. Because the slope is positive, we know the graph will slant upward from left to right. The y-intercept is the point on the graph when $x = 0$. The graph crosses the y-axis at $(0, 1)$. Now we know the slope and the y-intercept. We can begin graphing by

plotting the point $(0,1)$ We know that the slope is rise over run, $m = \frac{\text{rise}}{\text{run}}$. From our example, we have $m = \frac{1}{2}$, which means that the rise is 1 and the run is 2. So starting from our y-intercept $(0,1)$, we can rise 1 and then run 2, or run 2 and then rise 1. We repeat until we have a few points, and then we draw a line through the points as shown in Figure 0.2.3.

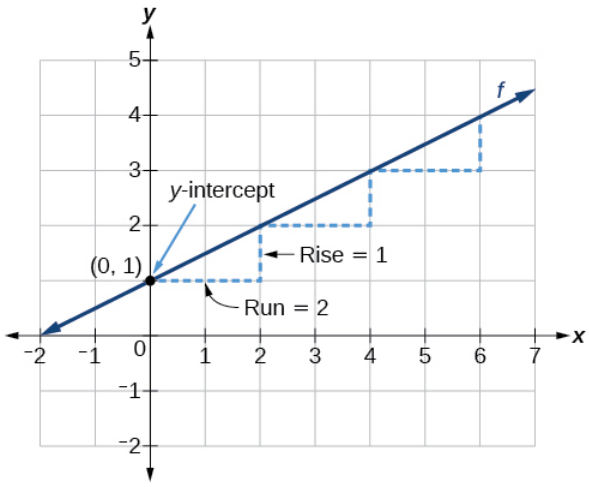


***Figure* 0.2.3:** *Graph of $f(x)$ showing slope and y-intercept*

Graphical Interpretation of a Linear Function

In the equation $f(x) = mx + b$

- b is the y-intercept of the graph and indicates the point $(0, b)$ at which the graph crosses the y-axis.
- m is the slope of the line and indicates the vertical displacement (rise) and horizontal displacement (run) between each successive pair of points. Recall the formula for the slope:

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \tag{0.2.5}$$

Exercise 0.2.1

Do all linear functions have y-intercepts?

Answer

Yes. All linear functions cross the y-axis and therefore have y-intercepts. (Note: A vertical line parallel to the y-axis does not have a y-intercept, but it is not a function.)

HowTO: Given the equation for a linear function, graph the function using the y-intercept and slope.

1. Evaluate the function at an input value of zero to find the y-intercept.
2. Identify the slope as the rate of change of the input value.
3. Plot the point represented by the y-intercept.
4. Use $\frac{\text{rise}}{\text{run}}$ to determine at least two more points on the line.
5. Sketch the line that passes through the points.

Example 0.2.2: Graphing by Using the y-intercept and Slope

Graph $f(x) = -\frac{2}{3}x + 5$ using the y-intercept and slope.

Solution

Evaluate the function at $x = 0$ to find the y-intercept. The output value when $x = 0$ is 5, so the graph will cross the y-axis at $(0, 5)$.

According to the equation for the function, the slope of the line is $-\frac{2}{3}$. This tells us that for each vertical decrease in the "rise" of –2 units, the "run" increases by 3 units in the horizontal direction. We can now graph the function by first plotting the y-intercept on the graph in Figure 0.2.4. From the initial value $(0, 5)$ we move down 2 units and to the right 3 units. We can extend the line to the left and right by repeating, and then draw a line through the points.

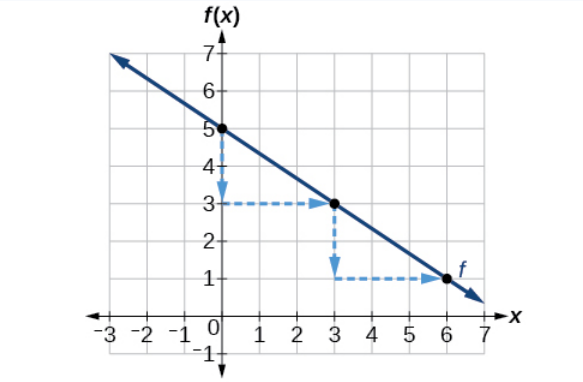


Figure* 0.2.4*: *Graph of* $f(x)$

Analysis

The graph slants downward from left to right, which means it has a negative slope as expected.

0.2.2: Find a point on the graph we drew in **Example** 0.2.2 that has a negative x-value.

Solution

Possible answers include $(-3, 7)$, $(-6, 9)$, or $(-9, 11)$.

Graphing a Function Using Transformations

Another option for graphing is to use **transformations** of the identity function $f(x) = x$. A function may be transformed by a shift up, down, left, or right. A function may also be transformed using a reflection, stretch, or compression.

Vertical Stretch or Compression

In the equation $f(x) = mx$, the m is acting as the **vertical stretch** or **compression** of the identity function. When m is negative, there is also a vertical reflection of the graph. Notice in Figure 0.2.5 that multiplying the equation of $f(x) = x$ by m stretches the graph of f by a factor of m units if $m > 1$ and compresses the graph of f by a factor of m units if $0 < m < 1$. This means the larger the absolute value of m, the steeper the slope.

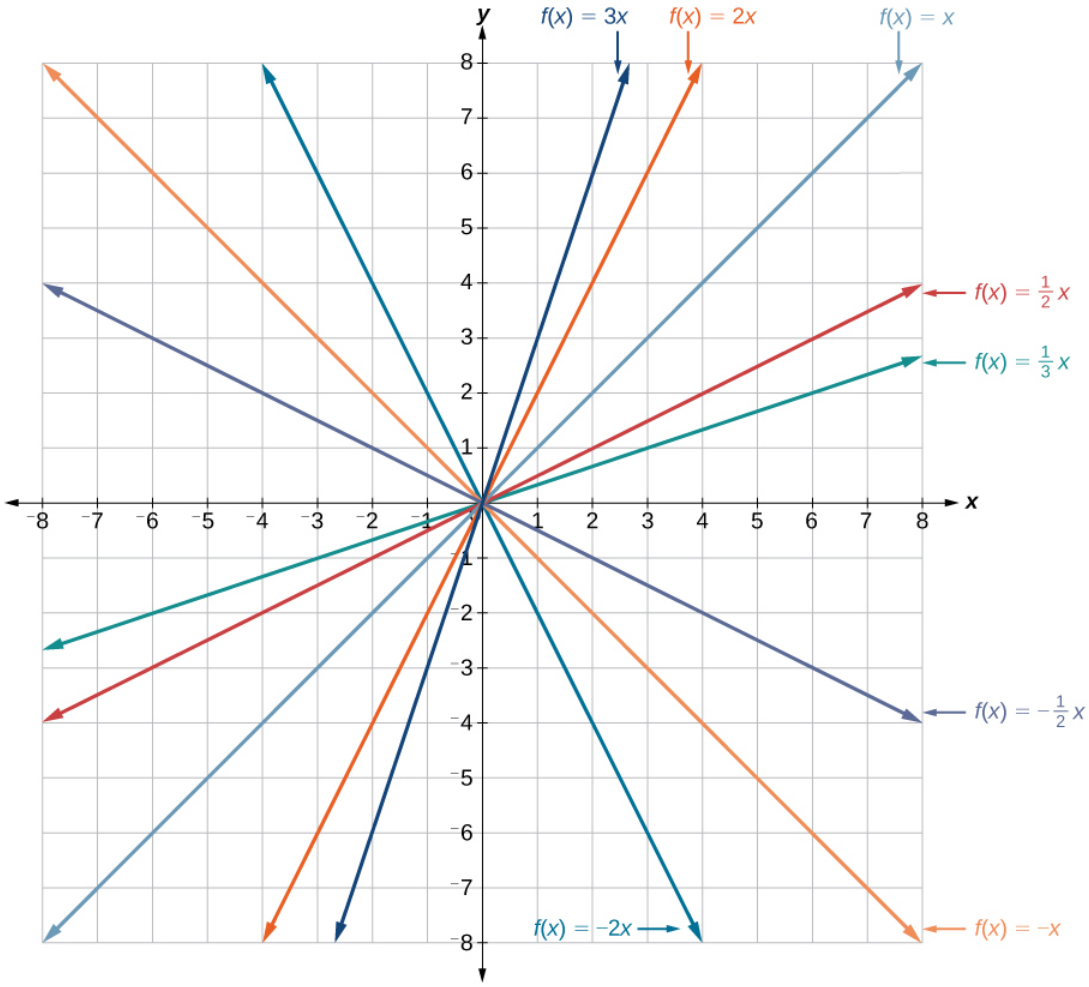


***Figure* 0.2.5:** *Vertical stretches and compressions and reflections on the function* $f(x) = x$.

Vertical Shift

In $f(x) = mx + b$, the b acts as the **vertical shift**, moving the graph up and down without affecting the slope of the line. Notice in Figure **0.2.6** that adding a value of b to the equation of $f(x) = x$ shifts the graph of f a total of b units up if b is positive and $|b|$ units down if b is negative.

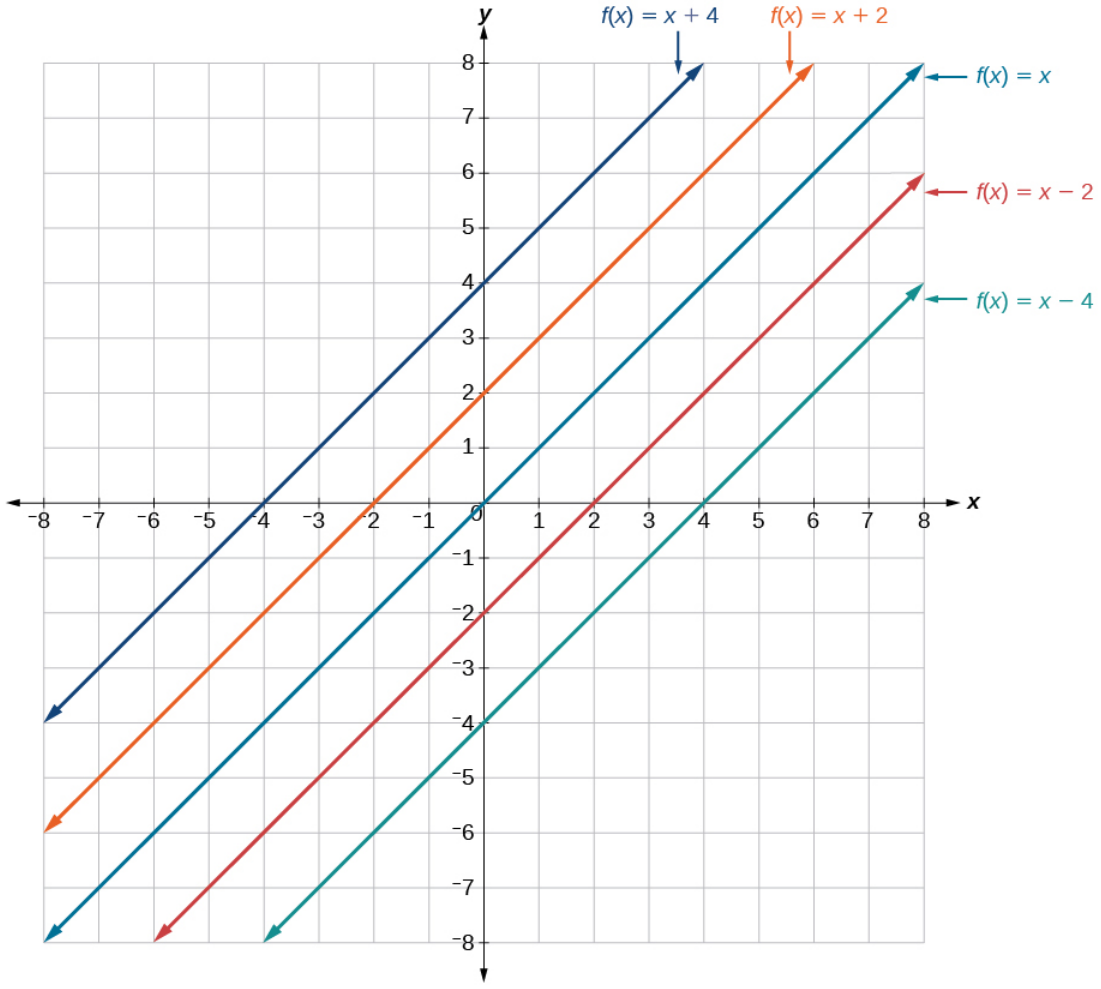


***Figure* 0.2.6:** *This graph illustrates vertical shifts of the function* $f(x) = x$.

Using vertical stretches or compressions along with vertical shifts is another way to look at identifying different types of linear functions. Although this may not be the easiest way to graph this type of function, it is still important to practice each method.

Given the equation of a linear function, use transformations to graph the linear function in the form $f(x) = mx + b$.

1. Graph $f(x) = x$.
2. Vertically stretch or compress the graph by a factor m.
3. Shift the graph up or down b units.

Example 0.2.3: Graphing by Using Transformations

Graph $f(x) = \frac{1}{2}x - 3$ using transformations.

Solution

The equation for the function shows that $m = \frac{1}{2}$ so the identity function is vertically compressed by $\frac{1}{2}$. The equation for the function also shows that $b = -3$ so the identity function is vertically shifted down 3 units. First, graph the identity function, and show the vertical compression as in Figure 0.2.7.

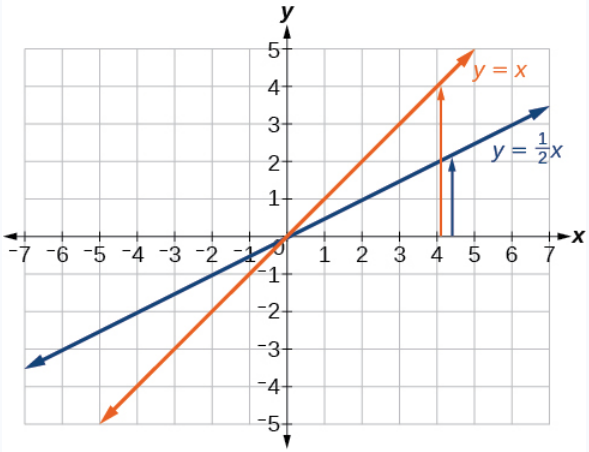


***Figure* 0.2.7**: *The function, $y = x$, compressed by a factor of $\frac{1}{2}$.*

Then show the vertical shift as in Figure 0.2.8.

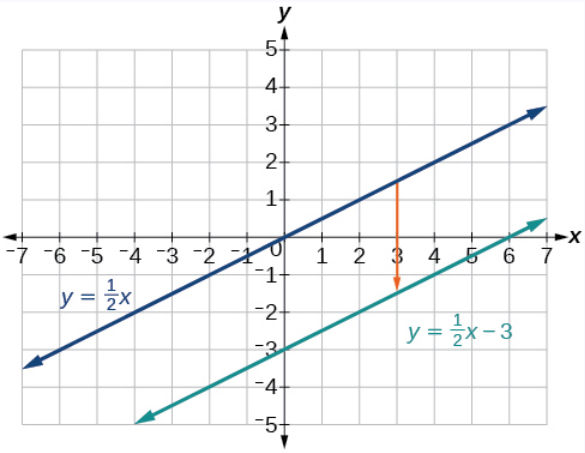


***Figure* 0.2.8:** *The function $y = \frac{1}{2}x$, shifted down 3 units.*

0.2.3: Graph $f(x) = 4 + 2x$, using transformations.

Solution

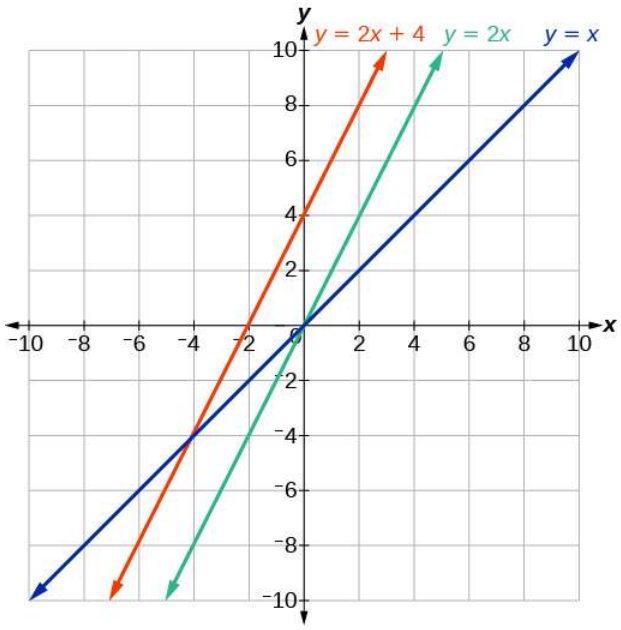


***Figure* 0.2.9**

In Example 2.2.3, could we have sketched the graph by reversing the order of the transformations?

No. The order of the transformations follows the order of operations. When the function is evaluated at a given input, the corresponding output is calculated by following the order of operations. This is why we performed the compression first. For example, following the order: Let the input be 2.

$$f(2) = \frac{1}{2}(2) - 3 \tag{0.2.6}$$
$$= 1 - 3 \tag{0.2.7}$$
$$= -2 \tag{0.2.8}$$

Writing the Equation for a Function from the Graph of a Line

Recall that in Linear Functions, we wrote the equation for a linear function from a graph. Now we can extend what we know about graphing linear functions to analyze graphs a little more closely. Begin by taking a look at Figure 0.2.10. We can see right away that the graph crosses the y-axis at the point $(0, 4)$ so this is the y-intercept.

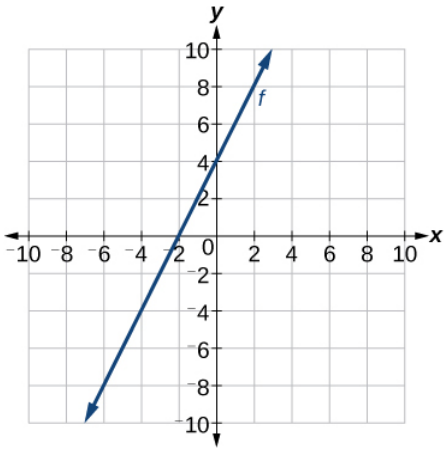


Figure* 0.2.10*:* *Graph of a linear function

Then we can calculate the slope by finding the rise and run. We can choose any two points, but let's look at the point (−2,0). To get from this point to the y-intercept, we must move up 4 units (rise) and to the right 2 units (run). So the slope must be

$$m = \frac{\text{rise}}{\text{run}} = \frac{4}{2} = 2 \tag{0.2.9}$$

Substituting the slope and y-intercept into the slope-intercept form of a line gives

$$y = 2x + 4 \tag{0.2.10}$$

How To: Given a graph of linear function, find the equation to describe the function.

1. Identify the y-intercept of an equation.
2. Choose two points to determine the slope.
3. Substitute the y-intercept and slope into the slope-intercept form of a line.

Example 0.2.4: Matching Linear Functions to Their Graphs

Match each equation of the linear functions with one of the lines in Figure 0.2.11.

a. $f(x) = 2x + 3$
b. $g(x) = 2x - 3$
c. $h(x) = -2x + 3$
d. $j(x) = \frac{1}{2}x + 3$

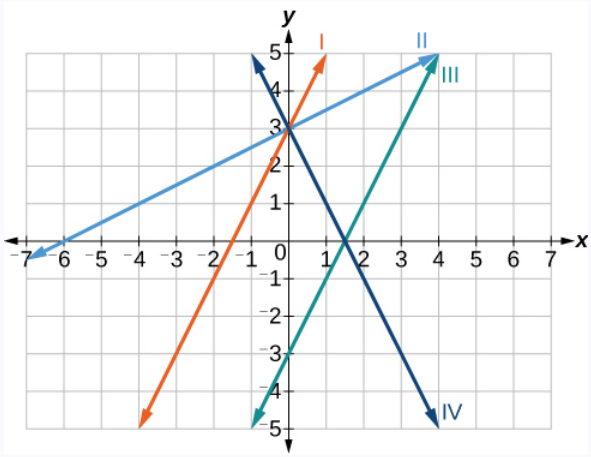


Figure 0.2.11: *Graph of four linear functions*

Solution

Analyze the information for each function.

a. This function has a slope of 2 and a y-intercept of 3. It must pass through the point $(0, 3)$ and slant upward from left to right. We can use two points to find the slope, or we can compare it with the other functions listed. Function g has the same slope, but a different y-intercept. Lines I and III have the same slant because they have the same slope. Line III does not pass through $(0, 3)$ so f must be represented by line I.
b. This function also has a slope of 2, but a y-intercept of −3. It must pass through the point $(0, -3)$ and slant upward from left to right. It must be represented by line III.
c. This function has a slope of –2 and a y-intercept of 3. This is the only function listed with a negative slope, so it must be represented by line IV because it slants downward from left to right.
d. This function has a slope of $\frac{1}{2}$ and a y-intercept of 3. It must pass through the point (0, 3) and slant upward from left to right. Lines I and II pass through $(0, 3)$, but the slope of j is less than the slope of f so the line for j must be flatter. This function is represented by Line II.

Now we can re-label the lines as in Figure 0.2.12.

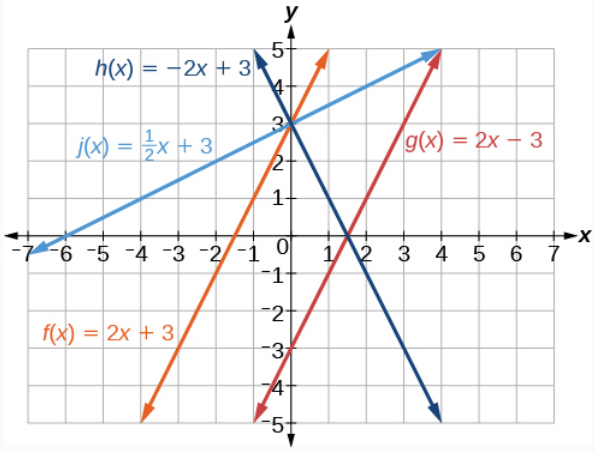


***Figure* 0.2.12**

Finding the x-intercept of a Line

So far, we have been finding the y-intercepts of a function: the point at which the graph of the function crosses the y-axis. A function may also have an x-intercept, which is the x-coordinate of the point where the graph of the function crosses the x-axis. In other words, it is the input value when the output value is zero.

To find the x-intercept, set a function $f(x)$ equal to zero and solve for the value of x. For example, consider the function shown.

$$f(x) = 3x - 6 \tag{0.2.11}$$

Set the function equal to 0 and solve for x.

$$0 = 3x - 6 \tag{0.2.12}$$
$$6 = 3x \tag{0.2.13}$$
$$2 = x \tag{0.2.14}$$
$$x = 2 \tag{0.2.15}$$

The graph of the function crosses the x-axis at the point $(2, 0)$.

Do all linear functions have x-intercepts?

No. However, linear functions of the form $y = c$, where c is a nonzero real number are the only examples of linear functions with no x-intercept. For example, $y = 5$ is a horizontal line 5 units above the x-axis. This function has no x-intercepts, as shown in Figure **0.2.13**.

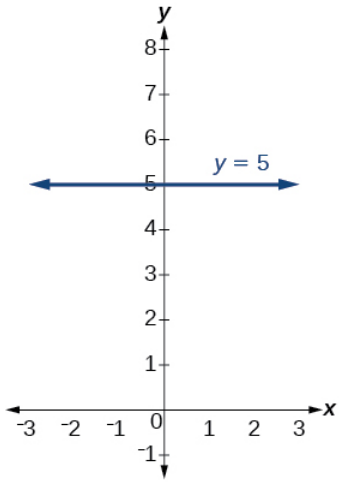


***Figure* 0.2.13**: Graph of $y = 5$.

Note: x-intercept

The **x-intercept** of the function is value of x when $f(x) = 0$. It can be solved by the equation $0 = mx + b$.

Example 0.2.5: Finding an x-intercept

Find the x-intercept of $f(x) = \frac{1}{2} - 3$.

Solution

Set the function equal to zero to solve for x.

$$0 = \frac{1}{2}x - 3$$
$$3 = \frac{1}{2}x$$
$$6 = x$$
$$x = 6$$

The graph crosses the x-axis at the point $(6, 0)$.

Analysis

A graph of the function is shown in Figure 0.2.14. We can see that the x-intercept is $(6, 0)$ as we expected.

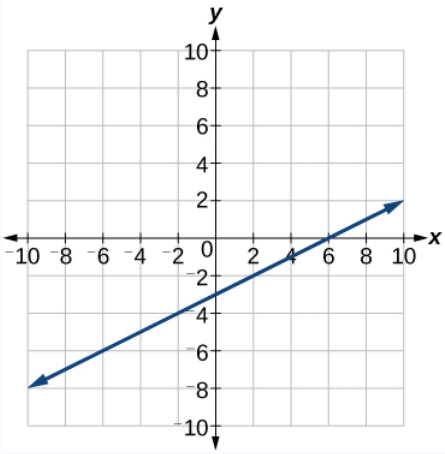


***Figure* 0.2.14:** *The graph of the linear function* $f(x) = \frac{1}{2}x - 3$.

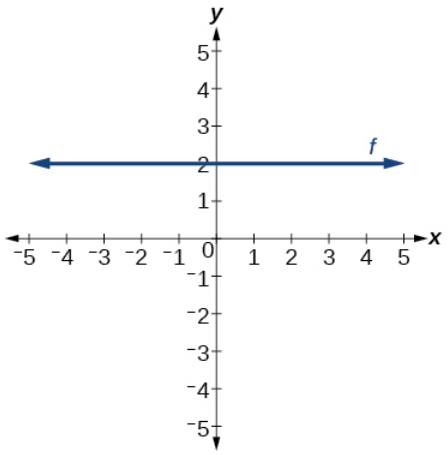
0.2.4: Find the x-intercept of $f(x) = \frac{1}{4}x - 4$.

Solution

$(16, 0)$

Describing Horizontal and Vertical Lines

There are two special cases of lines on a graph—horizontal and vertical lines. A **horizontal line** indicates a constant output, or y-value. In Figure 0.2.15, we see that the output has a value of 2 for every input value. The change in outputs between any two points, therefore, is 0. In the slope formula, the numerator is 0, so the slope is 0. If we use $m = 0$ in the equation $f(x) = mx + b$, the equation simplifies to $f(x) = b$. In other words, the value of the function is a constant. This graph represents the function $f(x) = 2$.



x	-4	-2	0	2	4
y	2	2	2	2	2

***Figure* 0.2.15:** *A horizontal line representing the function* $f(x) = 2$.

A **vertical line** indicates a constant input, or x-value. We can see that the input value for every point on the line is 2, but the output value varies. Because this input value is mapped to more than one output value, a vertical line does not represent a function. Notice that between any two points, the change in the input values is zero. In the slope formula, the denominator will be zero, so the slope of a vertical line is undefined.

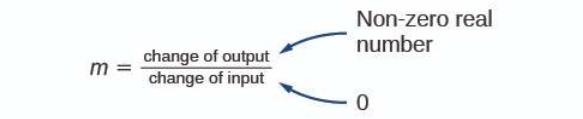
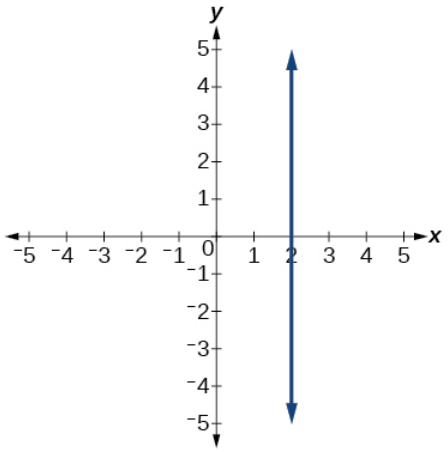


***Figure* 0.2.16**

Notice that a vertical line, such as the one in Figure 0.2.17, has an x-intercept, but no y-intercept unless it's the line $x = 0$. This graph represents the line $x = 2$.



x	2	2	2	2	2
y	-4	-2	0	2	4

***Figure* 0.2.17:** *The vertical line,* $x = 2$*, which does not represent a function.*

Note: Horizontal and Vertical Lines

Lines can be horizontal or vertical.

- A **horizontal line** is a line defined by an equation in the form $f(x) = b$.
- A **vertical line** is a line defined by an equation in the form $x = a$.

Example 0.2.6: Writing the Equation of a Horizontal Line

Write the equation of the line graphed in Figure 0.2.18.

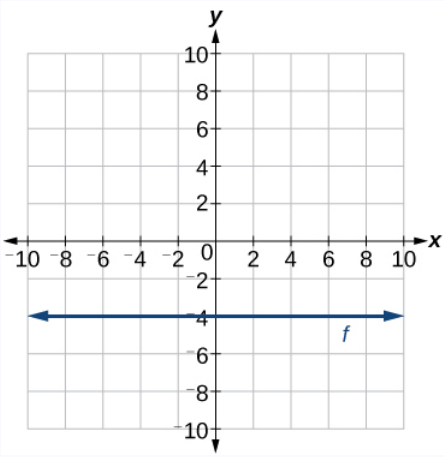


Figure 0.2.18: *Graph of $x = 7$.*

Solution

For any x-value, the y-value is −4, so the equation is $y = -4$.

Example 0.2.7: Writing the Equation of a Vertical Line

Write the equation of the line graphed in Figure 0.2.19.

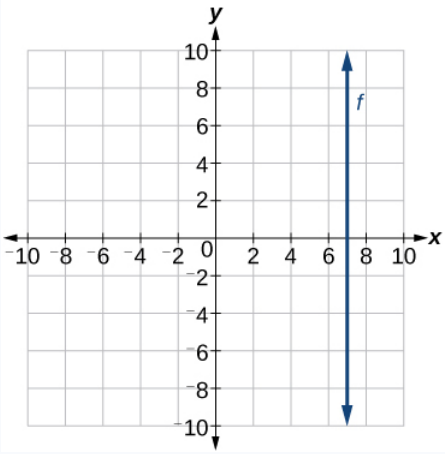


Figure 0.2.19: *Graph of a vertical line*

Solution

The constant x-value is 7, so the equation is $x = 7$.

Determining Whether Lines are Parallel or Perpendicular

The two lines in Figure 0.2.20 are **parallel lines**: they will never intersect. Notice that they have exactly the same steepness, which means their slopes are identical. The only difference between the two lines is the y-intercept. If we shifted one line vertically toward the y-intercept of the other, they would become the same line.

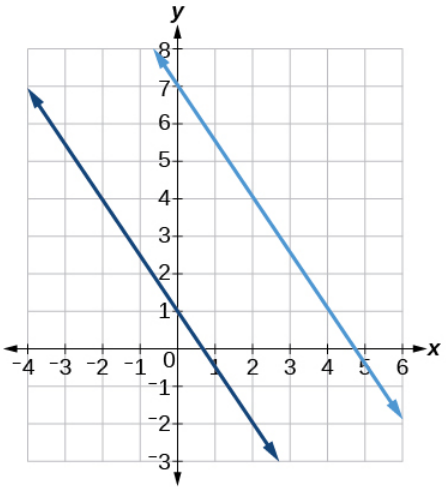


Figure* 0.2.20*: *Parallel lines.*

We can determine from their equations whether two lines are parallel by comparing their slopes. If the slopes are the same and the y-intercepts are different, the lines are parallel. If the slopes are different, the lines are not parallel.

$$\left.\begin{array}{l} f(x)=-2x+6 \\ f(x)=-2x-4 \end{array}\right\} \text{ parallel} \qquad \left.\begin{array}{l} f(x)=3x+2 \\ f(x)=2x+2 \end{array}\right\} \text{ not parallel}$$

Unlike parallel lines, **perpendicular lines** do intersect. Their intersection forms a right, or 90-degree, angle. The two lines in Figure 0.2.21 are perpendicular.

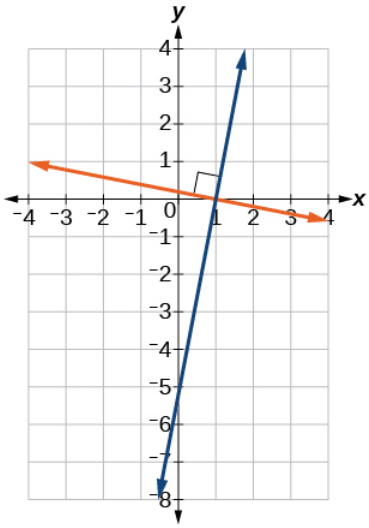


Figure* 0.2.21*: *Perpendicular lines.*

Perpendicular lines do not have the same slope. The slopes of perpendicular lines are different from one another in a specific way. The slope of one line is the negative reciprocal of the slope of the other line. The product of a number and its reciprocal is 1. So, if m_1 and m_2 are negative reciprocals of one another, they can be multiplied together to yield –1.

$$m_1 m_2 = -1 \tag{0.2.16}$$

To find the reciprocal of a number, divide 1 by the number. So the reciprocal of 8 is $\frac{1}{8}$, and the reciprocal of $\frac{1}{8}$ is 8. To find the negative reciprocal, first find the reciprocal and then change the sign.

As with parallel lines, we can determine whether two lines are perpendicular by comparing their slopes, assuming that the lines are neither horizontal nor perpendicular. The slope of each line below is the negative reciprocal of the other so the lines are perpendicular.

$$f(x) = \frac{1}{4}x + 2 \qquad \text{negative reciprocal of } \frac{1}{4} \text{ is -4} \tag{0.2.17}$$

$$f(x) = -4x + 3 \qquad \text{negative reciprocal of -4 is } \frac{1}{4} \tag{0.2.18}$$

The product of the slopes is –1.

$$-4\left(\frac{1}{4}\right) = -1 \tag{0.2.19}$$

Definition: Parallel and Perpendicular Lines

Two lines are **parallel lines** if they do not intersect. The slopes of the lines are the same.

$f(x) = m_1x + b_1$ and $g(x) = m_2x + b_2$ are parallel if $m_1 = m_2$.

If and only if $b_1 = b_2$ and $m_1 = m_2$, we say the lines coincide. Coincident lines are the same line.

Two lines are **perpendicular lines** if they intersect at right angles.

$f(x) = m_1x + b_1$ and $g(x) = m_2x + b_2$ are perpendicular if $m_1m_2 = -1$, and so $m_2 = \frac{-1}{m_1}$.

Example 0.2.8: Identifying Parallel and Perpendicular Lines

Given the functions below, identify the functions whose graphs are a pair of parallel lines and a pair of perpendicular lines.

$$f(x) = 2x + 3 \qquad h(x) = -2x + 2 \tag{0.2.20}$$

$$g(x) = \frac{1}{2}x - 4 \qquad f(x) = 2x - 6 \tag{0.2.21}$$

Solution

Parallel lines have the same slope. Because the functions $f(x) = 2x + 3$ and $j(x) = 2x - 6$ each have a slope of 2, they represent parallel lines. Perpendicular lines have negative reciprocal slopes. Because -2 and $\frac{1}{2}$ are negative reciprocals, the equations, $g(x) = \frac{1}{2}x - 4$ and $h(x) = -2x + 2$ represent perpendicular lines.

Analysis

A graph of the lines is shown in Figure 0.2.22.

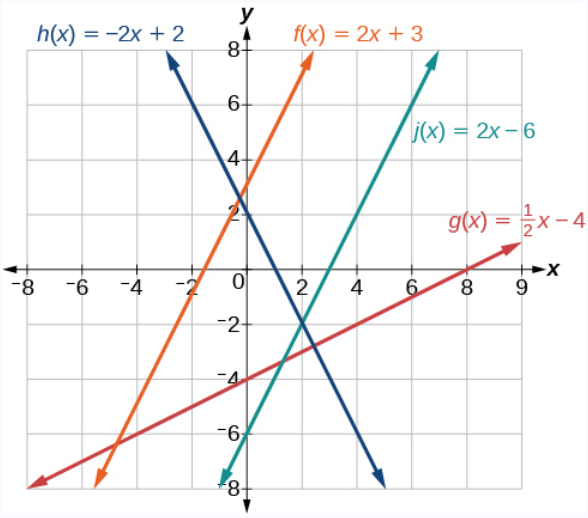


***Figure* 0.2.22**: *Graph of $h(x)$ (blue), $f(x)$ (orange), $j(x)$ (green), and $g(x)$ (red)*

The graph shows that the lines $f(x) = 2x + 3$ and $j(x) = 2x–6$ are parallel, and the lines $g(x) = \frac{1}{2}x–4$ and $h(x) = -2x + 2$ are perpendicular.

Writing the Equation of a Line Parallel or Perpendicular to a Given Line

If we know the equation of a line, we can use what we know about slope to write the equation of a line that is either parallel or perpendicular to the given line.

Writing Equations of Parallel Lines

Suppose for example, we are given the following equation.

$$f(x) = 3x + 1 \tag{0.2.22}$$

We know that the slope of the line formed by the function is 3. We also know that the y-intercept is $(0, 1)$. Any other line with a slope of 3 will be parallel to $f(x)$. So the lines formed by all of the following functions will be parallel to $f(x)$.

$$g(x) = 3x + 6 \tag{0.2.23}$$
$$h(x) = 3x + 1 \tag{0.2.24}$$
$$p(x) = 3x + \frac{2}{3} \tag{0.2.25}$$

Suppose then we want to write the equation of a line that is parallel to f and passes through the point $(1, 7)$. We already know that the slope is 3. We just need to determine which value for b will give the correct line. We can begin with the point-slope form of an equation for a line, and then rewrite it in the slope-intercept form.

$$y - y_1 = m(x - x_1) \tag{0.2.26}$$
$$y - 7 = 3(x - 1) \tag{0.2.27}$$
$$y - 7 = 3x - 3 \tag{0.2.28}$$
$$y = 3x + 4 \tag{0.2.29}$$

So $g(x) = 3x + 4$ is parallel to $f(x) = 3x + 1$ and passes through the point $(1, 7)$.

How To: Given the equation of a function and a point through which its graph passes, write the equation of a line parallel to the given line that passes through the given point.

1. Find the slope of the function.
2. Substitute the given values into either the general point-slope equation or the slope-intercept equation for a line.
3. Simplify.

Example 0.2.9: Finding a Line Parallel to a Given Line

Find a line parallel to the graph of $f(x) = 3x + 6$ that passes through the point $(3, 0)$.

Solution

The slope of the given line is 3. If we choose the slope-intercept form, we can substitute $m = 3$, $x = 3$, and $f(x) = 0$ into the slope-intercept form to find the y-intercept.

$$g(x) = 3x + b \tag{0.2.30}$$
$$0 = 3(3) + b \tag{0.2.31}$$
$$b = -9 \tag{0.2.32}$$

The line parallel to $f(x)$ that passes through $(3, 0)$ is $g(x) = 3x - 9$.

Analysis

We can confirm that the two lines are parallel by graphing them. Figure 0.2.23 shows that the two lines will never intersect.

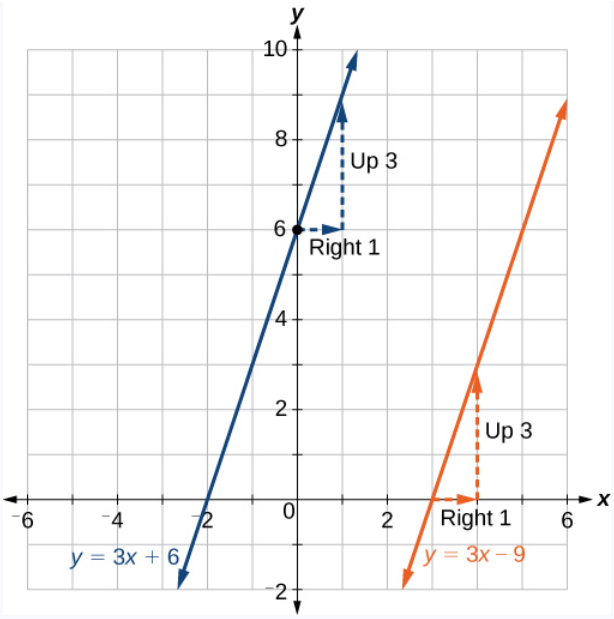


***Figure* 0.2.23:** *Graph of two functions where the blue line is* $y = 3x + 6$*, and the orange line is* $y = 3x - 9$.

Writing Equations of Perpendicular Lines

We can use a very similar process to write the equation for a line perpendicular to a given line. Instead of using the same slope, however, we use the negative reciprocal of the given slope. Suppose we are given the following function:

$$f(x) = 2x + 4 \tag{0.2.33}$$

The slope of the line is 2, and its negative reciprocal is $-\frac{1}{2}$. Any function with a slope of $-\frac{1}{2}$ will be perpendicular to $f(x)$. So the lines formed by all of the following functions will be perpendicular to $f(x)$.

$$g(x) = -\frac{1}{2}x + 4 \tag{0.2.34}$$

$$h(x) = -\frac{1}{2}x + 2 \tag{0.2.35}$$

$$p(x) = -\frac{1}{2}x - \frac{1}{2} \tag{0.2.36}$$

As before, we can narrow down our choices for a particular perpendicular line if we know that it passes through a given point. Suppose then we want to write the equation of a line that is perpendicular to $f(x)$ and passes through the point $(4, 0)$. We already know that the slope is $-\frac{1}{2}$. Now we can use the point to find the y-intercept by substituting the given values into the slope-intercept form of a line and solving for b.

$$g(x) = mx + b \tag{0.2.37}$$

$$0 = -\frac{1}{2}(4) + b \tag{0.2.38}$$

$$0 = -2 + b \tag{0.2.39}$$

$$2 = b \tag{0.2.40}$$

$$b = 2 \tag{0.2.41}$$

The equation for the function with a slope of $-\frac{1}{2}$ and a y-intercept of 2 is

$$g(x) = -\frac{1}{2}x + 2 \tag{0.2.42}$$

So $g(x) = -\frac{1}{2}x + 2$ is perpendicular to $f(x) = 2x + 4$ and passes through the point $(4, 0)$. Be aware that perpendicular lines may not look obviously perpendicular on a graphing calculator unless we use the square zoom feature.

A horizontal line has a slope of zero and a vertical line has an undefined slope. These two lines are perpendicular, but the product of their slopes is not –1. Doesn't this fact contradict the definition of perpendicular lines?

No. For two perpendicular linear functions, the product of their slopes is –1. However, a vertical line is not a function so the definition is not contradicted.

Given the equation of a function and a point through which its graph passes, write the equation of a line perpendicular to the given line.

1. Find the slope of the function.
2. Determine the negative reciprocal of the slope.
3. Substitute the new slope and the values for x and y from the coordinate pair provided into $g(x) = mx + b$.
4. Solve for b.
5. Write the equation for the line.

Example 0.2.10: Finding the Equation of a Perpendicular Line

Find the equation of a line perpendicular to $f(x) = 3x + 3$ that passes through the point $(3, 0)$.

Solution

The original line has slope $m = 3$, so the slope of the perpendicular line will be its negative reciprocal, or $-\frac{1}{3}$. Using this slope and the given point, we can find the equation for the line.

$$g(x) = -\frac{1}{3}x + b \tag{0.2.43}$$
$$0 = -\frac{1}{3}(3) + b \tag{0.2.44}$$
$$1 = b \tag{0.2.45}$$
$$b = 1 \tag{0.2.46}$$

The line perpendicular to $f(x)$ that passes through $(3, 0)$ is $g(x) = -\frac{1}{3}x + 1$.

Analysis

A graph of the two lines is shown in Figure 0.2.24 below.

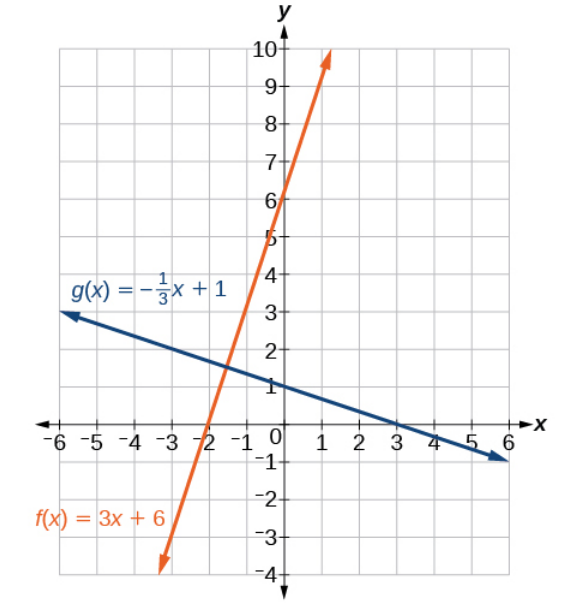


***Figure* 0.2.24:** *Graph of two functions where the blue line is $g(x) = -\frac{1}{3}x + 1$, and the orange line is $f(x) = 3x + 6$.*

Exercise 0.2.5

Given the function $h(x) = 2x - 4$, write an equation for the line passing through $(0, 0)$ that is

a. parallel to $h(x)$
b. perpendicular to $h(x)$

Answer

$f(x) = 2x$ $g(x) = -\frac{1}{2}x$

How to: Given two points on a line and a third point, write the equation of the perpendicular line that passes through the point.

1. Determine the slope of the line passing through the points.
2. Find the negative reciprocal of the slope.
3. Use the slope-intercept form or point-slope form to write the equation by substituting the known values.
4. Simplify.

Example 0.2.11: Finding the Equation of a Line Perpendicular to a Given Line Passing through a Point

A line passes through the points $(-2, 6)$ and $(4, 5)$. Find the equation of a perpendicular line that passes through the point $(4, 5)$.

From the two points of the given line, we can calculate the slope of that line.

$$m_1 = \frac{5-6}{4-(-2)} \tag{0.2.47}$$
$$= \frac{-1}{6} \tag{0.2.48}$$
$$= -\frac{1}{6} \tag{0.2.49}$$

Find the negative reciprocal of the slope.

$$m_2 = \frac{-1}{-\frac{1}{6}} \tag{0.2.50}$$
$$= -1\left(-\frac{6}{1}\right) \tag{0.2.51}$$
$$= 6 \tag{0.2.52}$$

We can then solve for the y-intercept of the line passing through the point $(4, 5)$.

$$g(x) = 6x + b \tag{0.2.53}$$
$$5 = 6(4) + b \tag{0.2.54}$$
$$5 = 24 + b \tag{0.2.55}$$
$$-19 = b \tag{0.2.56}$$
$$b = -19 \tag{0.2.57}$$

The equation for the line that is perpendicular to the line passing through the two given points and also passes through point $(4, 5)$ is

$$y = 6x - 19 \tag{0.2.58}$$

Try It 0.2.6: A line passes through the points, $(-2, -15)$ and $(2, -3)$. Find the equation of a perpendicular line that passes through the point, $(6, 4)$.

Solution

$y = -\frac{1}{3}x + 6$

Solving a System of Linear Equations Using a Graph

A system of linear equations includes two or more linear equations. The graphs of two lines will intersect at a single point if they are not parallel. Two parallel lines can also intersect if they are coincident, which means they are the same line and they intersect at every point. For two lines that are not parallel, the single point of intersection will satisfy both equations and therefore represent the solution to the system.

To find this point when the equations are given as functions, we can solve for an input value so that $f(x) = g(x)$. In other words, we can set the formulas for the lines equal to one another, and solve for the input that satisfies the equation.

Example 0.2.12: Finding a Point of Intersection Algebraically

Find the point of intersection of the lines $h(t) = 3t - 4$ and $j(t) = 5 - t$.

Solution

Set $h(t) = j(t)$.

$$3t - 4 = 5 - t \tag{0.2.59}$$
$$4t = 9 \tag{0.2.60}$$
$$t = \frac{9}{4} \tag{0.2.61}$$

This tells us the lines intersect when the input is $\frac{9}{4}$.

We can then find the output value of the intersection point by evaluating either function at this input.

$$j\left(\frac{9}{4}\right) = 5 - \frac{9}{4} \tag{0.2.62}$$
$$= \frac{11}{4} \tag{0.2.63}$$

These lines intersect at the point $\left(\frac{9}{4}, \frac{11}{4}\right)$.

Analysis

Looking at Figure 0.2.25, this result seems reasonable.

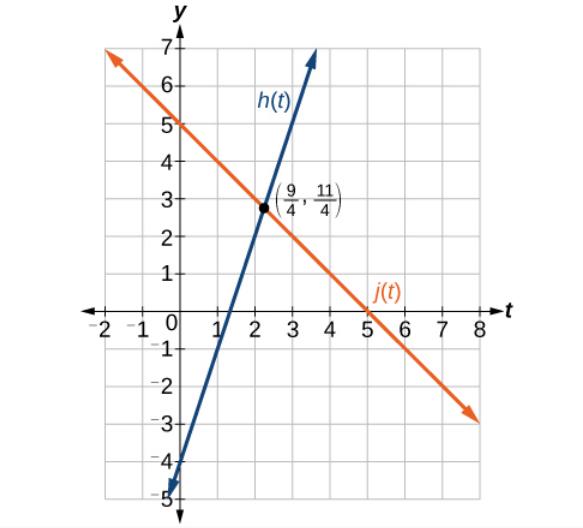


Figure 0.2.25: *Graph of two functions $h(t) = 3t - 4$ and $j(t) = t + 5$ and their intersection at $(\frac{9}{4}, \frac{11}{4})$.*

If we were asked to find the point of intersection of two distinct parallel lines, should something in the solution process alert us to the fact that there are no solutions?

Yes. After setting the two equations equal to one another, the result would be the contradiction "$0 =$ non-zero real number".

0.2.7: Look at the graph in Figure **0.2.23** and identify the following for the function $j(t)$:

a. y-intercept
b. x-intercept(s)
c. slope
d. Is $j(t)$ parallel or perpendicular to $h(t)$ (or neither)?
e. Is $j(t)$ an increasing or decreasing function (or neither)?
f. Write a transformation description for $j(t)$ from the identity toolkit function $f(x) = x$.

Solution

a. $(0, 5)$
b. (5, 0)
c. Slope -1
d. Neither parallel nor perpendicular
e. Decreasing function
f. Given the identity function, perform a vertical flip (over the t-axis) and shift up 5 units.

Example 0.2.13: Finding a Break-Even Point

A company sells sports helmets. The company incurs a one-time fixed cost for $250,000. Each helmet costs $120 to produce, and sells for $140.

a. Find the cost function, C, to produce x helmets, in dollars.
b. Find the revenue function, R, from the sales of x helmets, in dollars.
c. Find the break-even point, the point of intersection of the two graphs C and R.

Solution

a. The cost function in the sum of the fixed cost, $125,000, and the variable cost, $120 per helmet.

$$C(x) = 120x + 250,000$$

b. The revenue function is the total revenue from the sale of x helmets, $R(x) = 140x$.

c. The break-even point is the point of intersection of the graph of the cost and revenue functions. To find the x-coordinate of the coordinate pair of the point of intersection, set the two equations equal, and solve for x.

$$C(x) = R(x) \quad (0.2.64)$$
$$250,000 + 120x = 140x \quad (0.2.65)$$
$$250,000 = 20x \quad (0.2.66)$$
$$x = 12,500 \quad (0.2.67)$$

To find y, evaluate either the revenue or the cost function at 12,500.

$$R(20) = 140(12,500) \quad (0.2.68)$$
$$= \$1,750,000 \quad (0.2.69)$$

The break-even point is $(12,500, 1,750,000)$.

Analysis

This means if the company sells 12,500 helmets, they break even; both the sales and cost incurred equaled 1.75 million dollars. See Figure **0.2.26**.

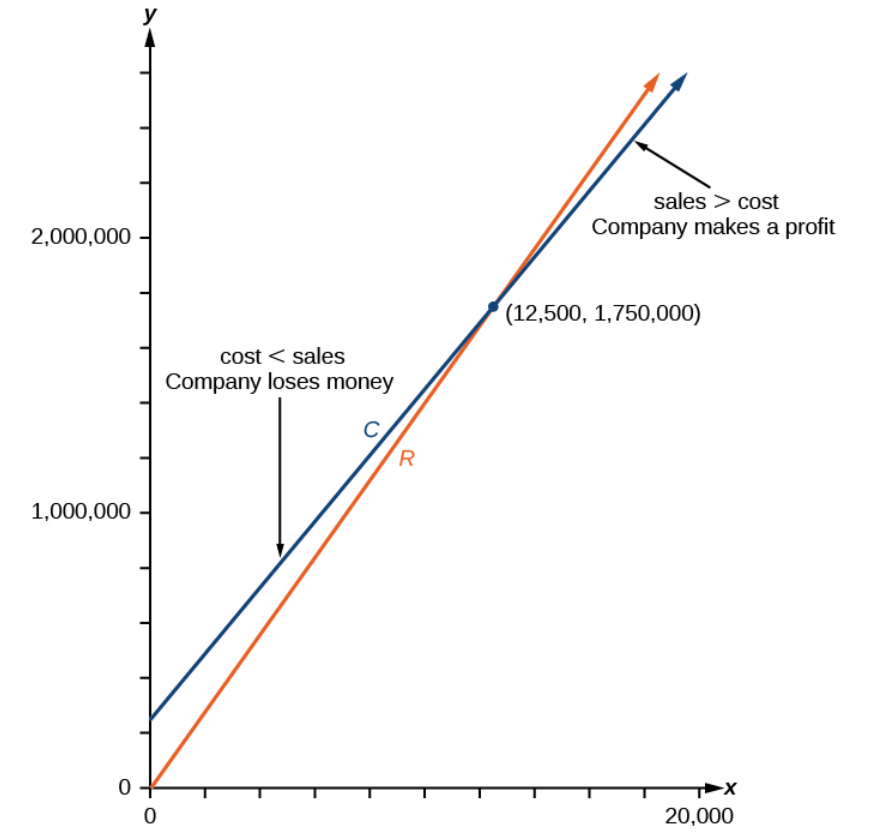


Figure 0.2.26: *Graph of the two functions, $C(x)$ and $R(x)$ where it shows that below $(12500, 1750000)$ the company loses money and above that point the company makes a profit.*

Key Concepts

- Linear functions may be graphed by plotting points or by using the y-intercept and slope.
- Graphs of linear functions may be transformed by using shifts up, down, left, or right, as well as through stretches, compressions, and reflections.
- The y-intercept and slope of a line may be used to write the equation of a line.
- The x-intercept is the point at which the graph of a linear function crosses the x-axis.
- Horizontal lines are written in the form, $f(x) = b$.
- Vertical lines are written in the form, $x = b$.
- Parallel lines have the same slope.
- Perpendicular lines have negative reciprocal slopes, assuming neither is vertical.
- A line parallel to another line, passing through a given point, may be found by substituting the slope value of the line and the x- and y-values of the given point into the equation, $f(x) = mx + b$, and using the b that results. Similarly, the point-slope form of an equation can also be used.
- A line perpendicular to another line, passing through a given point, may be found in the same manner, with the exception of using the negative reciprocal slope.
- A system of linear equations may be solved setting the two equations equal to one another and solving for x. The y-value may be found by evaluating either one of the original equations using this x-value.
- A system of linear equations may also be solved by finding the point of intersection on a graph.

Exercises

Verbal

If the graphs of two linear functions are parallel, describe the relationship between the slopes and the y-intercepts.

The slopes are equal; y-intercepts are not equal.

If the graphs of two linear functions are perpendicular, describe the relationship between the slopes and the y-intercepts.

If a horizontal line has the equation $f(x) = a$ and a vertical line has the equation $x = a$, what is the point of intersection? Explain why what you found is the point of intersection.

The point of intersection is (a, a). This is because for the horizontal line, all of the y coordinates are a and for the vertical line, all of the x coordinates are a. The point of intersection will have these two characteristics.

Explain how to find a line parallel to a linear function that passes through a given point.

Explain how to find a line perpendicular to a linear function that passes through a given point.

First, find the slope of the linear function. Then take the negative reciprocal of the slope; this is the slope of the perpendicular line. Substitute the slope of the perpendicular line and the coordinate of the given point into the equation $y = mx + b$ and solve for b. Then write the equation of the line in the form $y = mx + b$ by substituting in m and b.

Algebraic

For the following exercises, determine whether the lines given by the equations below are parallel, perpendicular, or neither parallel nor perpendicular:

$4x - 7y = 10$
$7x + 4y = 1$

$3y + x = 12$
$-y = 8x + 1$

neither parallel or perpendicular

$3y + 4x = 12$
$-6y = 8x + 1$

$6x - 9y = 10$
$3x + 2y = 1$

perpendicular

$y = \frac{2}{3}x + 1$
$3x + 2y = 1$

$y = \frac{3}{4}x + 1$
$-3x + 4y = 1$

parallel

For the following exercises, find the x- and y-intercepts of each equation

$f(x) = -x + 2$

$g(x) = 2x + 4$

$(-2, 0); (0, 4)$

$h(x) = 3x - 5$

$k(x) = -5x + 1$

$(\frac{1}{5}, 0); (0, 1)$

$-2x + 5y = 20$

$7x + 2y = 56$

$(8, 0); (0, 28)$

For the following exercises, use the descriptions of each pair of lines given below to find the slopes of Line 1 and Line 2. Is each pair of lines parallel, perpendicular, or neither?

Line 1: Passes through $(0, 6)$ and $(3, -24)$
Line 2: Passes through $(-1, 19)$ and $(8, -71)$

Line 1: Passes through $(-8, -55)$ and $(10, 89)$
Line 2: Passes through $(9, -44)$ and $(4, -14)$

Line 1: $m = 8$ Line 2: $m = -6$ Neither

Line 1: Passes through $(2, 3)$ and (4,−1)
Line 2: Passes through $(6, 3)$ and $(8, 5)$

Line 1: Passes through $(1, 7)$ and $(5, 5)$
Line 2: Passes through $(-1, -3)$ and $(1, 1)$

Line 1: $m = -12$ Line 2: $m = 2$ Perpendicular

Line 1: Passes through $(0, 5)$ and $(3, 3)$
Line 2: Passes through $(1, -5)$ and $(3, -2)$

Line 1: Passes through $(2, 5)$ and $(5, -1)$
Line 2: Passes through $(-3, 7)$ and $(3, -5)$

Line 1: $m = -2$ Line 2: $m = -2$ Parallel

Write an equation for a line parallel to $f(x) = -5x - 3$ and passing through the point $(2, -12)$.

Write an equation for a line parallel to $g(x) = 3x - 1$ and passing through the point $(4, 9)$.

$g(x) = 3x - 3$

Write an equation for a line perpendicular to $h(t) = -2t + 4$ and passing through the point $(-4, -1)$.

Write an equation for a line perpendicular to $p(t) = 3t + 4$ and passing through the point $(3, 1)$.

$p(t) = -\frac{1}{3}t + 2$

Find the point at which the line $f(x) = -2x - 1$ intersects the line $g(x) = -x$.

Find the point at which the line $f(x) = 2x + 5$ intersects the line $g(x) = -3x - 5$.

$(-2, 1)$

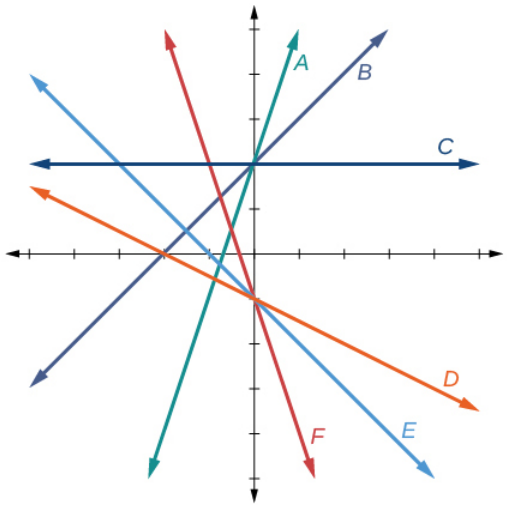
Use algebra to find the point at which the line $f(x) = -\frac{4}{5}x + \frac{274}{25}$ intersects the line $h(x) = \frac{9}{4}x + \frac{73}{10}$.

Use algebra to find the point at which the line $f(x) = \frac{7}{4}x + \frac{457}{60}$ intersects the line $g(x) = \frac{4}{3}x + \frac{31}{5}$.

$(-\frac{17}{5}, \frac{5}{3})$

Graphical

For the following exercises, the given linear equation with its graph in Figure.



$f(x) = -x - 1$

$f(x) = -2x - 1$

F

$f(x) = -\frac{1}{2}x - 1$

$f(x) = 2$

C

$f(x) = 2 + x$

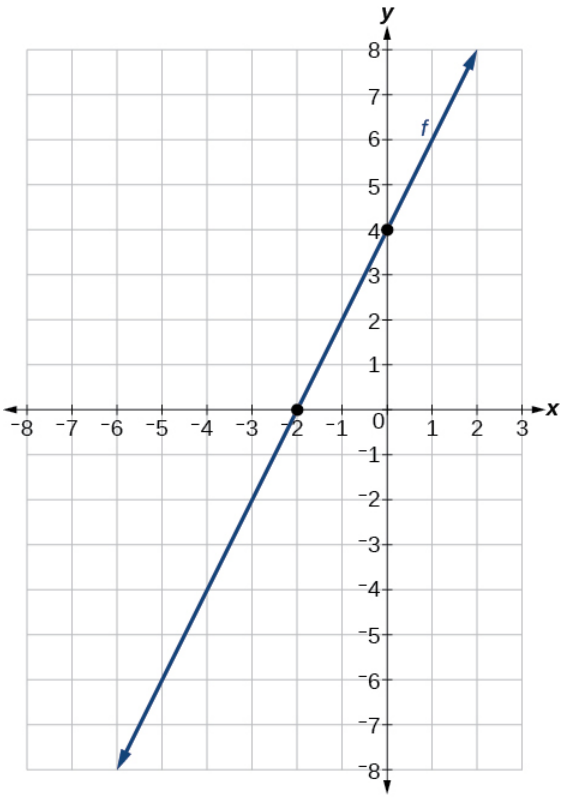
$f(x) = 3x + 2$

A

For the following exercises, sketch a line with the given features.

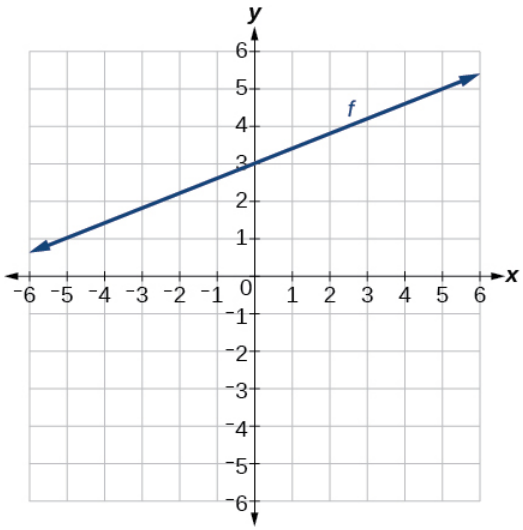
An x-intercept of $(-4, 0)$ and y-intercept of $(0, -2)$

An x-intercept of $(-2, 0)$ and y-intercept of $(0, 4)$



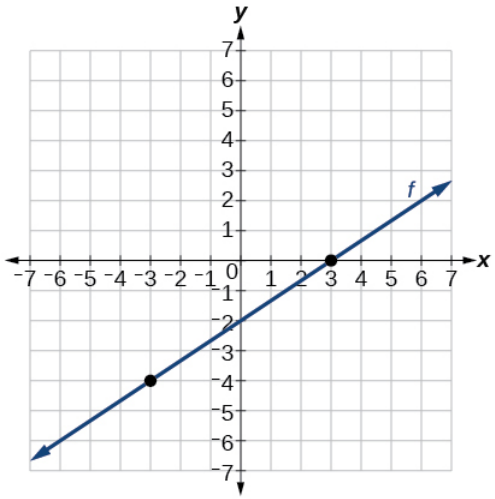
A y-intercept of $(0, 7)$ and slope $-\frac{3}{2}$

A y-intercept of $(0, 3)$ and slope $\frac{2}{5}$



Passing through the points $(-6, -2)$ and $(6, -6)$

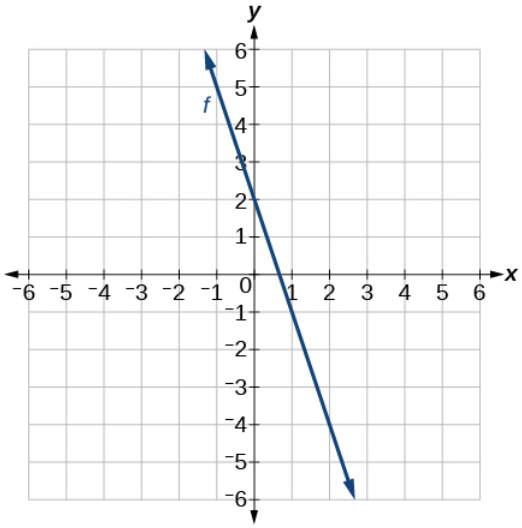
Passing through the points $(-3, -4)$ and $(3, 0)$



For the following exercises, sketch the graph of each equation.

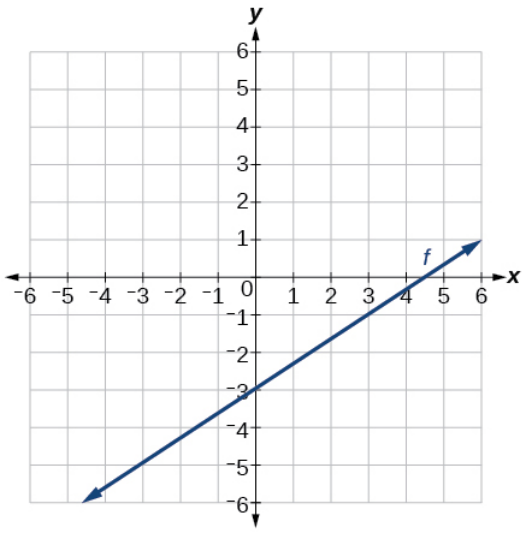
$f(x) = -2x - 1$

$g(x) = -3x + 2$



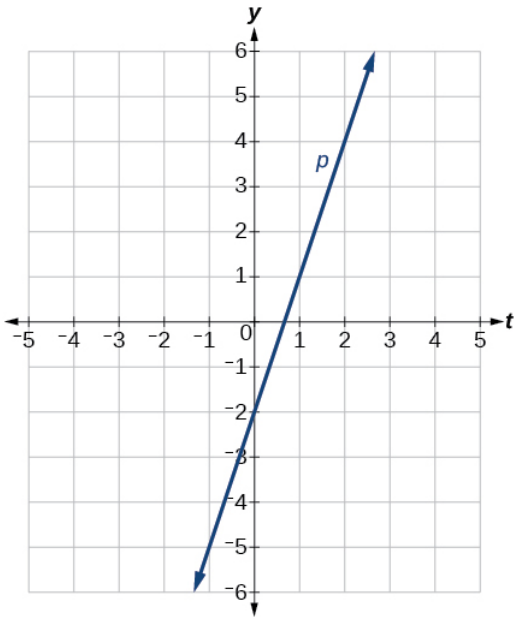
$h(x) = \frac{1}{3}x + 2$

$k(x) = \frac{2}{3}x - 3$



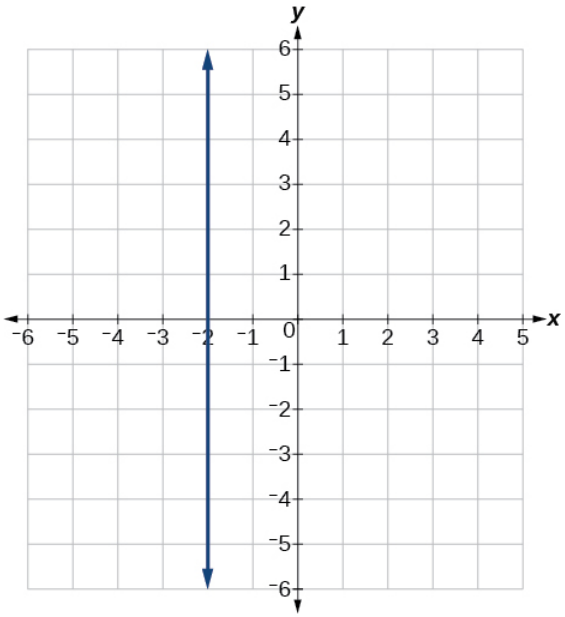
$f(t) = 3 + 2t$

$p(t) = -2 + 3t$



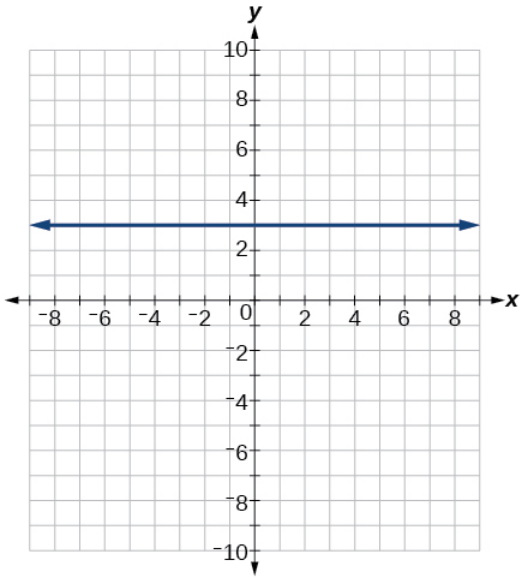
$x = 3$

$x = -2$



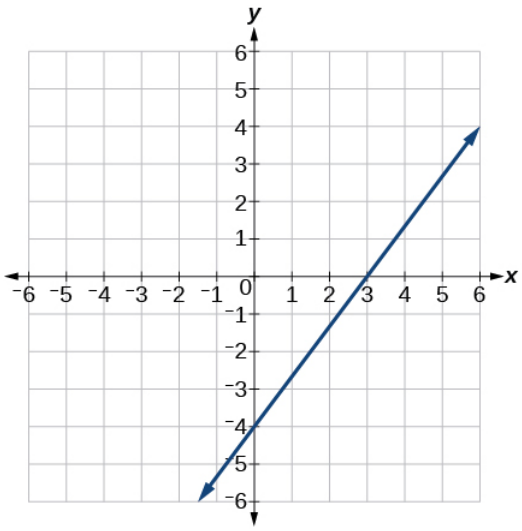
$r(x) = 4$

$q(x) = 3$



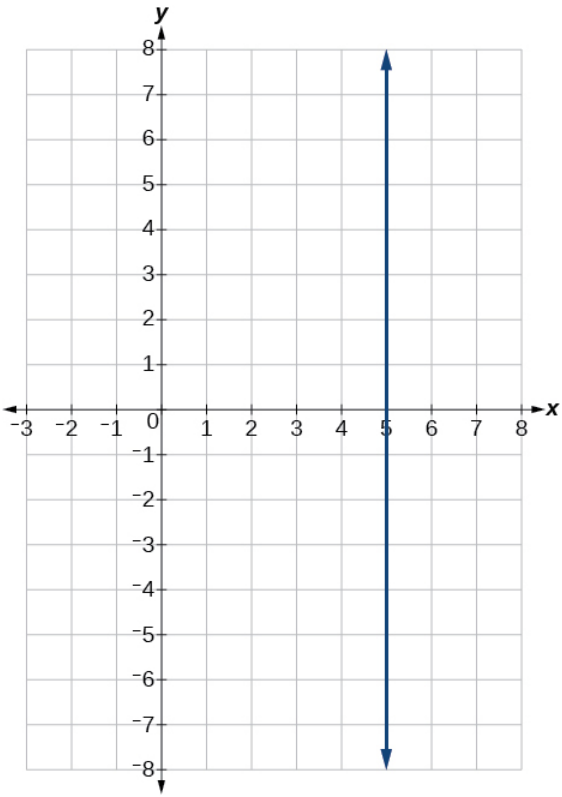
$4x = -9y + 36$

$\frac{x}{3} - \frac{y}{4} = 1$



$3x - 5y = 15$

$3x = 15$



$3y = 12$

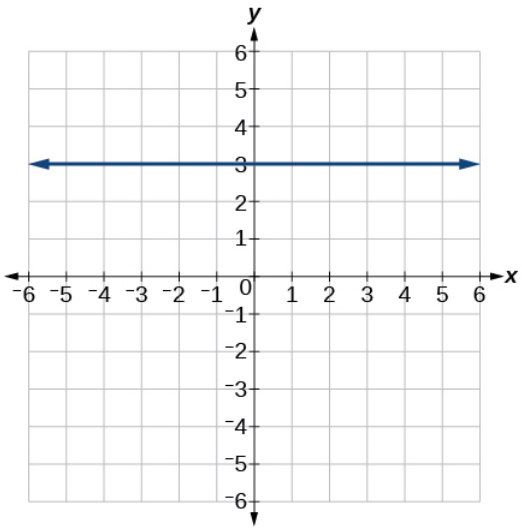
If $g(x)$ is the transformation of $f(x) = x$ after a vertical compression by $\frac{3}{4}$, a shift right by 2, and a shift down by 4

Write an equation for $g(x)$.
What is the slope of this line?
Find the y-intercept of this line.

$g(x) = 0.75x - 5.5$

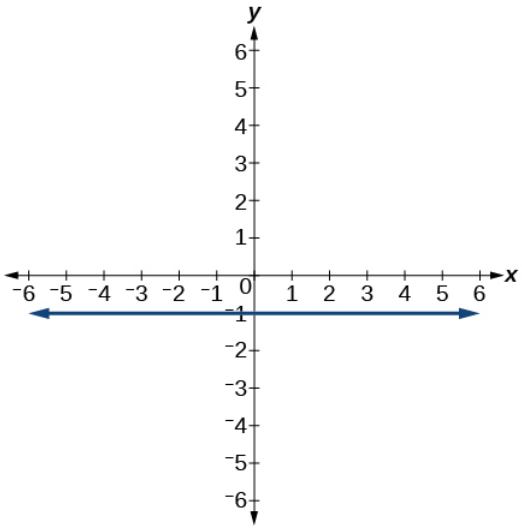
If $g(x)$ is the transformation of $f(x) = x$ after a vertical compression by $\frac{1}{3}$, a shift left by 1, and a shift up by 3

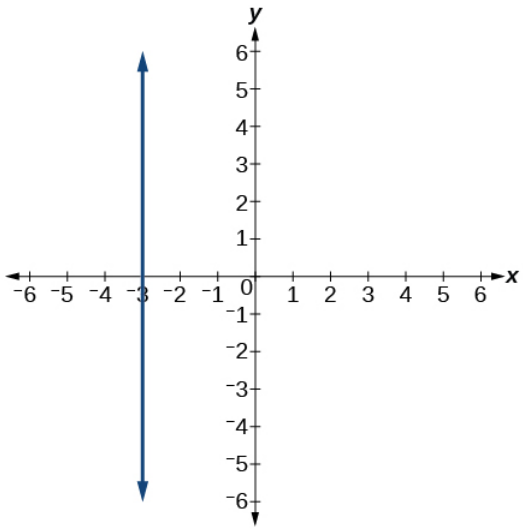
Write an equation for $g(x)$.
What is the slope of this line?
Find the y-intercept of this line.



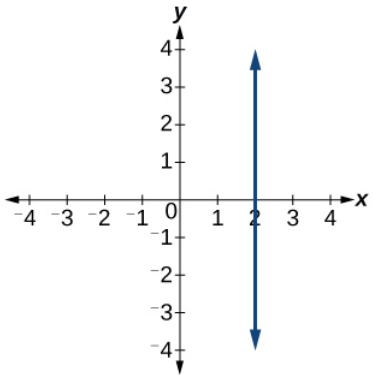
For the following exercises,, write the equation of the line shown in the graph.

$y = 3$





$x = -3$



For the following exercises, find the point of intersection of each pair of lines if it exists. If it does not exist, indicate that there is no point of intersection.

$y = \frac{3}{4}x + 1$
$-3x + 4y = 12$

no point of intersection

$2x - 3y = 12$
$5y + x = 30$

$2x = y - 3$
$y + 4x = 15$

$(2, 7)$

$x - 2y + 2 = 3$
$x - y = 3$

$5x + 3y = -65$
$x - y = -5$

$(-10, -5)$

Extensions

Find the equation of the line parallel to the line $g(x) = -0.01x + 2.01$ through the point $(1, 2)$.

Find the equation of the line perpendicular to the line $g(x) = -0.01x + 2.01$ through the point $(1, 2)$.

$y = 100x - 98$

For the following exercises, use the functions $f(x) = -0.1x + 200$ and $g(x) = 20x + 0.1$.

Find the point of intersection of the lines f and g.

Where is $f(x)$ greater than $g(x)$? Where is $g(x)$ greater than $f(x)$?

$x < \frac{1999}{201} x > \frac{1999}{201}$

Real-World Applications

A car rental company offers two plans for renting a car.

Plan A: $30 per day and $0.18 per mile
Plan B: $50 per day with free unlimited mileage

How many miles would you need to drive for plan B to save you money?

A cell phone company offers two plans for minutes.

Plan A: $20 per month and $1 for every one hundred texts.
Plan B: $50 per month with free unlimited texts.

How many texts would you need to send per month for plan B to save you money?

Less than 3000 texts

A cell phone company offers two plans for minutes.

Plan A: $15 per month and $2 for every 300 texts.
Plan B: $25 per month and $0.50 for every 100 texts.

How many texts would you need to send per month for plan B to save you money?

Glossary

horizontal line
a line defined by $f(x) = b$, where b is a real number. The slope of a horizontal line is 0.

parallel lines
two or more lines with the same slope

perpendicular lines
two lines that intersect at right angles and have slopes that are negative reciprocals of each other

vertical line
a line defined by $x = a$, where a is a real number. The slope of a vertical line is undefined.

x-intercept
the point on the graph of a linear function when the output value is 0; the point at which the graph crosses the horizontal axis

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0.3: Modeling with Linear Functions

Skills to Develop

- Build linear models from verbal descriptions.
- Model a set of data with a linear function.

Emily is a college student who plans to spend a summer in Seattle. She has saved $3,500 for her trip and anticipates spending $400 each week on rent, food, and activities. How can we write a linear model to represent her situation? What would be the x-intercept, and what can she learn from it? To answer these and related questions, we can create a model using a linear function. Models such as this one can be extremely useful for analyzing relationships and making predictions based on those relationships. In this section, we will explore examples of **linear function** models.



***Figure* 0.3.1:** *(credit: EEK Photography/Flickr)*

Identifying Steps to Model and Solve Problems

When **modeling** scenarios with linear functions and solving problems involving quantities with a **constant rate of change**, we typically follow the same problem strategies that we would use for any type of function. Let's briefly review them:

Identify changing quantities, and then define descriptive variables to represent those quantities. When appropriate, sketch a picture or define a coordinate system.
Carefully read the problem to identify important information. Look for information that provides values for the variables or values for parts of the functional model, such as slope and initial value.
Carefully read the problem to determine what we are trying to find, identify, solve, or interpret.
Identify a solution pathway from the provided information to what we are trying to find. Often this will involve checking and tracking units, building a table, or even finding a formula for the function being used to model the problem.
When needed, write a formula for the function.
Solve or evaluate the function using the formula.
Reflect on whether your answer is reasonable for the given situation and whether it makes sense mathematically.
Clearly convey your result using appropriate units, and answer in full sentences when necessary.

Building Linear Models

Now let's take a look at the student in Seattle. In her situation, there are two changing quantities: time and money. The amount of money she has remaining while on vacation depends on how long she stays. We can use this information to define our variables, including units.

Output: M, money remaining, in dollars
Input: t, time, in weeks

So, the amount of money remaining depends on the number of weeks: $M(t)$

We can also identify the initial value and the rate of change.

Initial Value: She saved $3,500, so $3,500 is the initial value for M.
Rate of Change: She anticipates spending $400 each week, so –$400 per week is the rate of change, or slope.

Notice that the unit of dollars per week matches the unit of our output variable divided by our input variable. Also, because the slope is negative, the linear function is decreasing. This should make sense because she is spending money each week.

The **rate of change is constant**, so we can start with the linear model $M(t) = mt + b$. Then we can substitute the intercept and slope provided.

$$M(t) = mt + b$$

$$M(t) = -400t + 3500$$

***Figure* 0.3.2**

To find the t-intercept, we set the output to zero, and solve for the input.

$$0 = -400t + 3500 \tag{0.3.1}$$

$$t = \frac{3500}{400} \tag{0.3.2}$$

$$= 8.75 \tag{0.3.3}$$

The t-intercept is 8.75 weeks. Because this represents the input value when the output will be zero, we could say that Emily will have no money left after 8.75 weeks.

When modeling any real-life scenario with functions, there is typically a limited domain over which that model will be valid—almost no trend continues indefinitely. Here the domain refers to the number of weeks. In this case, it doesn't make sense to talk about input values less than zero. A negative input value could refer to a number of weeks before she saved $3,500, but the scenario discussed poses the question once she saved $3,500 because this is when her trip and subsequent spending starts. It is also likely that this model is not valid after the x-intercept, unless Emily will use a credit card and goes into debt. The domain represents the set of input values, so the reasonable domain for this function is $0 \le t \le 8.75$.

In the above example, we were given a written description of the situation. We followed the steps of modeling a problem to analyze the information. However, the information provided may not always be the same. Sometimes we might be provided with an intercept. Other times we might be provided with an output value. We must be careful to analyze the information we are given, and use it appropriately to build a linear model.

Analysis:

Another way to view this situation would be to define a function whose input is again t, time, in weeks, but whose output is S, the amount of money already spent, in dollars. In this case, the Initial Value is $0, the Rate of Change is +$400 per week, and the function rule is $S(t) = 400t$. How much money will Emily have spent after 8 weeks?

Using a Given Intercept to Build a Model

Some real-world problems provide the y-intercept, which is the constant or initial value. Once the y-intercept is known, the x-intercept can be calculated. Suppose, for example, that Hannah plans to pay off a no-interest loan from her parents. Her loan balance is $1,000. She plans to pay $250 per month until her balance is $0. The y-intercept is the initial amount of her debt, or $1,000. The rate of change, or slope, is -$250 per month. We can then use the slope-intercept form and the given information to develop a linear model.

$$f(x) = mx + b \tag{0.3.4}$$
$$= -250x + 1000 \tag{0.3.5}$$

Now we can set the function equal to 0, and solve for x to find the x-intercept.

$$0 = -250 + 1000 \tag{0.3.6}$$
$$1000 = 250x \tag{0.3.7}$$
$$4 = x \tag{0.3.8}$$
$$x = 4 \tag{0.3.9}$$

The x-intercept is the number of months it takes her to reach a balance of $0. The x-intercept is 4 months, so it will take Hannah four months to pay off her loan.

Using a Given Input and Output to Build a Model

Many real-world applications are not as direct as the ones we just considered. Instead they require us to identify some aspect of a linear function. We might sometimes instead be asked to evaluate the linear model at a given input or set the equation of the linear model equal to a specified output.

Given a word problem that includes two pairs of input and output values, use the linear function to solve a problem.

1. Identify the input and output values.
2. Convert the data to two coordinate pairs.
3. Find the slope.
4. Write the linear model.
5. Use the model to make a prediction by evaluating the function at a given x-value.
6. Use the model to identify an x-value that results in a given y-value.
7. Answer the question posed.

Example 0.3.1: Using a Linear Model to Investigate a Town's Population

A town's population has been growing linearly. In 2004 the population was 6,200. By 2009 the population had grown to 8,100. Assume this trend continues.

a. Predict the population in 2013.
b. Identify the year in which the population will reach 15,000.

Solution

The two changing quantities are the population size and time. While we could use the actual year value as the input quantity, doing so tends to lead to very cumbersome equations because the y-intercept would correspond to the year 0, more than 2000 years ago!

To make computation a little nicer, we will define our input as the number of years since 2004:

- Input: t, years since 2004
- Output: $P(t)$, the town's population

To predict the population in 2013 ($t = 9$), we would first need an equation for the population. Likewise, to find when the population would reach 15,000, we would need to solve for the input that would provide an output of 15,000. To write an equation, we need the initial value and the rate of change, or slope.

To determine the rate of change, we will use the change in output per change in input.

$$m = \frac{\text{change in output}}{\text{change in input}} \tag{0.3.10}$$

The problem gives us two input-output pairs. Converting them to match our defined variables, the year 2004 would correspond to $t = 0$, giving the point $(0, 6200)$. Notice that through our clever choice of variable definition, we have "given" ourselves the y-intercept of the function. The year 2009 would correspond to $t = 5$, giving the point $(5, 8100)$.

The two coordinate pairs are $(0, 6200)$ and $(5, 8100)$. Recall that we encountered examples in which we were provided two points earlier in the chapter. We can use these values to calculate the slope.

$$m = \frac{8100 - 6200}{5 - 0} \tag{0.3.11}$$
$$= \frac{1900}{5} \tag{0.3.12}$$
$$= 380 \text{ people per year} \tag{0.3.13}$$

We already know the y-intercept of the line, so we can immediately write the equation:

$$P(t) = 380t + 6200 \tag{0.3.14}$$

To predict the population in 2013, we evaluate our function at $t = 9$.

$$P(9) = 380(9) + 6,200 \tag{0.3.15}$$
$$= 9,620 \tag{0.3.16}$$

If the trend continues, our model predicts a population of 9,620 in 2013.

To find when the population will reach 15,000, we can set $P(t) = 15000$ and solve for t.

$$15000 = 380t + 6200 \tag{0.3.17}$$
$$8800 = 380t \tag{0.3.18}$$
$$t \approx 23.158 \tag{0.3.19}$$

Our model predicts the population will reach 15,000 in a little more than 23 years after 2004, or somewhere around the year 2027.

0.3.1: A company sells doughnuts. They incur a fixed cost of $25,000 for rent, insurance, and other expenses. It costs $0.25 to produce each doughnut.

a. Write a linear model to represent the cost C of the company as a function of x, the number of doughnuts produced.
b. Find and interpret the y-intercept.

Solution

a. $y = C(x) = 0.25x + 25,000$ b. The y-intercept is $(0, 25,000)$. If the company does not produce a single doughnut, they still incur a cost of $25,000.

0.3.2: A city's population has been growing linearly. In 2008, the population was 28,200. By 2012, the population was 36,800. Assume this trend continues.

a. Predict the population in 2014.
b. Identify the year in which the population will reach 54,000.

Solution

a. 41,100 b. 2020

Using a Diagram to Model a Problem

It is useful for many real-world applications to draw a picture to gain a sense of how the variables representing the input and output may be used to answer a question. To draw the picture, first consider what the problem is asking for. Then, determine the input and the output. The diagram should relate the variables. Often, geometrical shapes or figures are drawn. Distances are often traced out.

If a right triangle is sketched, the Pythagorean Theorem relates the sides. If a rectangle is sketched, labeling width and height is helpful.

Example 0.3.2: Using a Diagram to Model Distance Walked

Anna and Emanuel start at the same intersection. Anna walks east at 4 miles per hour while Emanuel walks south at 3 miles per hour. They are communicating with a two-way radio that has a range of 2 miles. How long after they start walking will they fall out of radio contact?

Solution

In essence, we can partially answer this question by saying they will fall out of radio contact when they are 2 miles apart, which leads us to ask a new question:

"How long will it take them to be 2 miles apart?"

In this problem, our changing quantities are time and position, but ultimately we need to know how long will it take for them to be 2 miles apart. We can see that time will be our input variable, so we'll define our input and output variables.

Input: t, time in hours.
Output: $A(t)$, distance in miles, and $E(t)$, distance in miles

Because it is not obvious how to define our output variable, we'll start by drawing a picture such as Figure **0.3.3**.

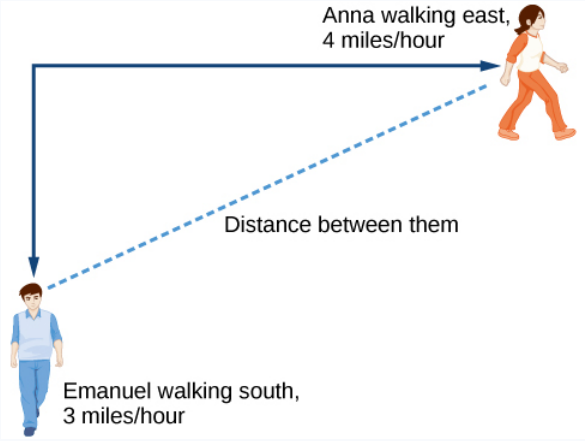


***Figure* 0.3.3**

Initial Value: They both start at the same intersection so when $t = 0$, the distance traveled by each person should also be 0. Thus the initial value for each is 0.

Rate of Change: Anna is walking 4 miles per hour and Emanuel is walking 3 miles per hour, which are both rates of change. The slope for A is 4 and the slope for E is 3.

Using those values, we can write formulas for the distance each person has walked.

$$A(t) = 4t \tag{0.3.20}$$

$$E(t) = 3t \tag{0.3.21}$$

For this problem, the distances from the starting point are important. To notate these, we can define a coordinate system, identifying the "starting point" at the intersection where they both started. Then we can use the variable, A, which we introduced above, to represent Anna's position, and define it to be a measurement from the starting point in the eastward direction. Likewise, can use the variable, E, to represent Emanuel's position, measured from the starting point in the southward direction. Note that in defining the coordinate system, we specified both the starting point of the measurement and the direction of measure.

We can then define a third variable, D, to be the measurement of the distance between Anna and Emanuel. Showing the variables on the diagram is often helpful, as we can see from Figure **0.3.4**.

Recall that we need to know how long it takes for D, the distance between them, to equal 2 miles. Notice that for any given input t, the outputs $A(t)$, $E(t)$, and $D(t)$ represent distances.

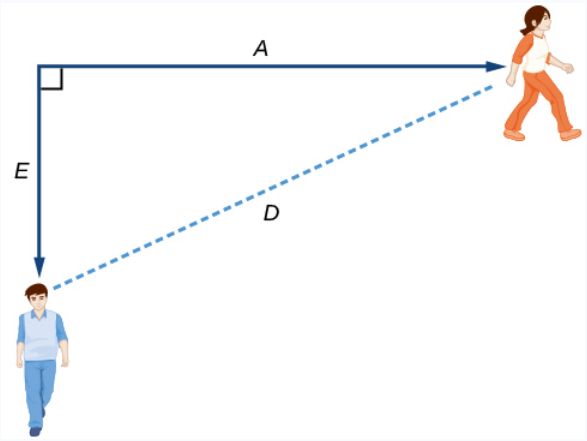


***Figure* 0.3.4:** *We can use the Pythagorean Theorem because we have drawn a right angle.*

Using the Pythagorean Theorem, we get:

$$d(t)^2 = A(t)^2 + E(t)^2 \tag{0.3.22}$$
$$= (4t)^2 + (3t)^2 \tag{0.3.23}$$
$$= 16t^2 + 9t^2 \tag{0.3.24}$$
$$= 25t^2 \tag{0.3.25}$$
$$D(t) = \pm\sqrt{25t^2} \quad \text{Solve for } D(t) \text{ using the square root} \tag{0.3.26}$$
$$= \pm 5|t| \tag{0.3.27}$$

In this scenario we are considering only positive values of t, so our distance $D(t)$ will always be positive. We can simplify this answer to $D(t) = 5t$. This means that the distance between Anna and Emanuel is also a linear function. Because D is a linear function, we can now answer the question of when the distance between them will reach 2 miles. We will set the output $D(t) = 2$ and solve for t.

$$D(t) = 2 \tag{0.3.28}$$
$$5t = 2 \tag{0.3.29}$$
$$t = \frac{2}{5} = 0.4 \tag{0.3.30}$$

They will fall out of radio contact in 0.4 hours, or 24 minutes.

Should I draw diagrams when given information based on a geometric shape?

Yes. Sketch the figure and label the quantities and unknowns on the sketch.

Example 0.3.3: Using a Diagram to Model Distance between Cities

There is a straight road leading from the town of Westborough to Agritown 30 miles east and 10 miles north. Partway down this road, it junctions with a second road, perpendicular to the first, leading to the town of Eastborough. If the town of Eastborough is located 20 miles directly east of the town of Westborough, how far is the road junction from Westborough?

Solution

It might help here to draw a picture of the situation. See Figure 0.3.5. It would then be helpful to introduce a coordinate system. While we could place the origin anywhere, placing it at Westborough seems convenient. This puts Agritown at coordinates $(30, 10)$, and Eastborough at $(20, 0)$.

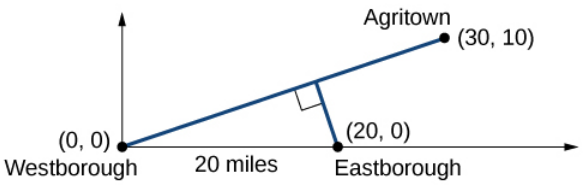


***Figure* 0.3.5**

Using this point along with the origin, we can find the slope of the line from Westborough to Agritown:

$$m = \frac{10-0}{30-0} = \frac{1}{3} \tag{0.3.31}$$

The equation of the road from Westborough to Agritown would be

$$W(x) = \frac{1}{3}x \tag{0.3.32}$$

From this, we can determine the perpendicular road to Eastborough will have slope $m = -3$. Because the town of Eastborough is at the point $(20, 0)$, we can find the equation:

$$E(x) = -3x + b \tag{0.3.33}$$
$$0 = -3(20) + b \quad \text{Substitute in } (20, 0) \tag{0.3.34}$$
$$b = 60 \tag{0.3.35}$$
$$E(x) = -3x + 60 \tag{0.3.36}$$

We can now find the coordinates of the junction of the roads by finding the intersection of these lines. Setting them equal,

$$\frac{1}{3}x = -3x + 60 \tag{0.3.37}$$
$$\frac{10}{3}x = 60 \tag{0.3.38}$$
$$10x = 180 \tag{0.3.39}$$
$$x = 18 \quad \text{Substituting this back into } W(x) \tag{0.3.40}$$
$$y = W(18) \tag{0.3.41}$$
$$= \frac{1}{3}(18) \tag{0.3.42}$$
$$= 6 \tag{0.3.43}$$

The roads intersect at the point $(18, 6)$. Using the distance formula, we can now find the distance from Westborough to the junction.

$$\text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{0.3.44}$$
$$= \sqrt{(18-0)^2 + (6-0)^2} \tag{0.3.45}$$
$$\approx 18.743 \text{miles} \tag{0.3.46}$$

Analysis

One nice use of linear models is to take advantage of the fact that the graphs of these functions are lines. This means real-world applications discussing maps need linear functions to model the distances between reference points.

0.3.3: There is a straight road leading from the town of Timpson to Ashburn 60 miles east and 12 miles north. Partway down the road, it junctions with a second road, perpendicular to the first, leading to the town of Garrison. If the town of Garrison is located 22 miles directly east of the town of Timpson, how far is the road junction from Timpson?

Solution

21.15 miles

Building Systems of Linear Models

Real-world situations including two or more linear functions may be modeled with a **system of linear equations**. Remember, when solving a system of linear equations, we are looking for points the two lines have in common. Typically, there are three types of answers possible, as shown in Figure **0.3.6**.

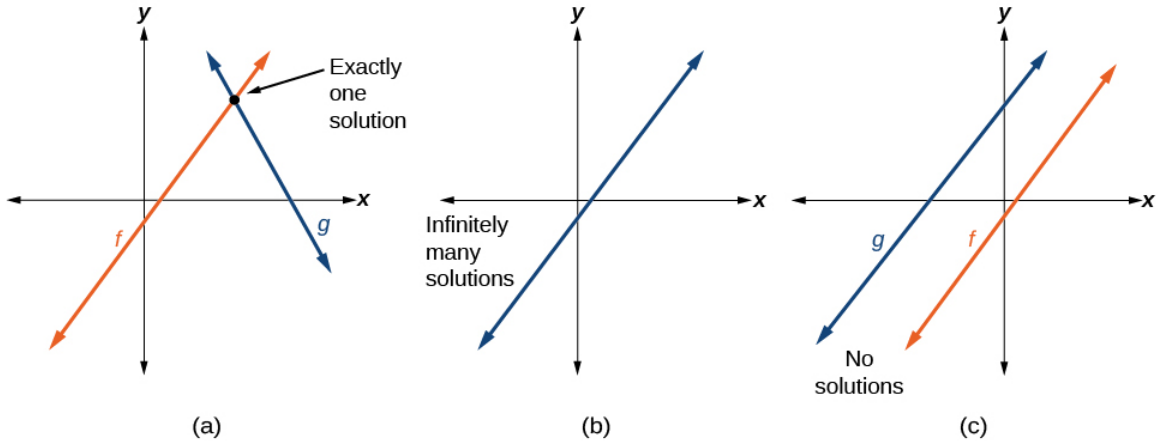


***Figure* 0.3.6**

Given a situation that represents a system of linear equations, write the system of equations and identify the solution.

a. Identify the input and output of each linear model.
b. Identify the slope and y-intercept of each linear model.
c. Find the solution by setting the two linear functions equal to another and solving for x,or find the point of intersection on a graph.

Example 0.3.4: Building a System of Linear Models to Choose a Truck Rental Company

Jamal is choosing between two truck-rental companies. The first, Keep on Trucking, Inc., charges an up-front fee of $20, then 59 cents a mile[1]. The second, Move It Your Way, charges an up-front fee of $16, then 63 cents a mile. When will Keep on Trucking, Inc. be the better choice for Jamal?

Solution

The two important quantities in this problem are the cost and the number of miles driven. Because we have two companies to consider, we will define two functions.

- Input: d, distance driven in miles
- Outputs: $K(d)$: cost, in dollars, for renting from Keep on Trucking

  $M(d)$: cost, in dollars, for renting from Move It Your Way
- Initial Value: Up-front fee: $K(0) = 20$ and $M(0) = 16$
- Rate of Change: $K(d) = \frac{\$0.59}{\text{mile}}$ and $P(d) = \frac{\$0.63}{\text{mile}}$

A linear function is of the form $f(x) = mx + b$. Using the rates of change and initial charges, we can write the equations

$$K(d) = 0.59d + 20 \tag{0.3.47}$$

$$M(d) = 0.63d + 16 \tag{0.3.48}$$

Using these equations, we can determine when Keep on Trucking, Inc., will be the better choice. Because all we have to make that decision from is the costs, we are looking for when Move It Your Way, will cost less, or when $K(d) < M(d)$. The solution pathway will lead us to find the equations for the two functions, find the intersection, and then see where the $K(d)$ function is smaller.

These graphs are sketched in Figure 0.3.7, with $K(d)$ in blue.

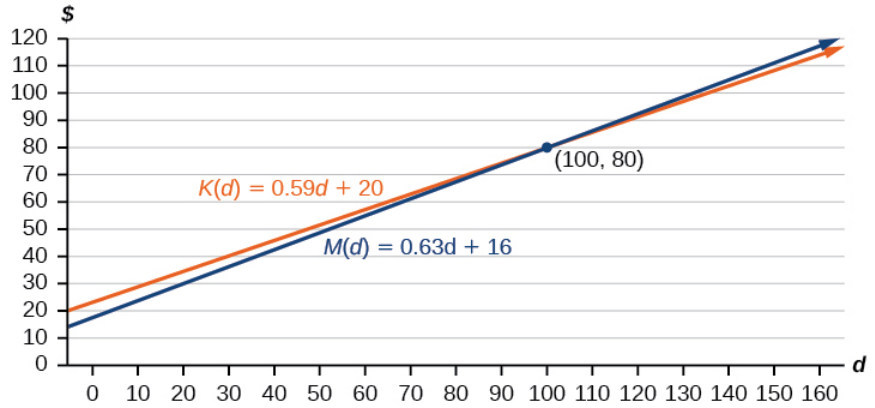


***Figure* 0.3.7**

To find the intersection, we set the equations equal and solve:

$$K(d) = M(d) \quad (0.3.49)$$
$$0.59d + 20 = 0.63d + 16 \quad (0.3.50)$$
$$4 = 0.04d \quad (0.3.51)$$
$$100 = d \quad (0.3.52)$$
$$d = 100 \quad (0.3.53)$$

This tells us that the cost from the two companies will be the same if 100 miles are driven. Either by looking at the graph, or noting that $K(d)$ is growing at a slower rate, we can conclude that Keep on Trucking, Inc. will be the cheaper price when more than 100 miles are driven; that is $d > 100$.

Key Concepts

- We can use the same problem strategies that we would use for any type of function.
- When modeling and solving a problem, identify the variables and look for key values, including the slope and y-intercept.
- Draw a diagram, where appropriate.
- Check for reasonableness of the answer.
- Linear models may be built by identifying or calculating the slope and using the y-intercept.
- The x-intercept may be found by setting $y = 0$, which is setting the expression $mx + b$ equal to 0.
- The point of intersection of a system of linear equations is the point where the x- and y-values are the same.
- A graph of the system may be used to identify the points where one line falls below (or above) the other line.

Section Excercises

Verbal

Explain how to find the input variable in a word problem that uses a linear function.

Determine the independent variable. This is the variable upon which the output depends.

Explain how to find the output variable in a word problem that uses a linear function.

Explain how to interpret the initial value in a word problem that uses a linear function.

To determine the initial value, find the output when the input is equal to zero.

Explain how to determine the slope in a word problem that uses a linear function.

Algebraic

Find the area of a parallelogram bounded by the y axis, the line $x = 3$, the line $f(x) = 1 + 2x$,and the line parallel to $f(x)$ passing through (2, 7).

6 square units

Find the area of a triangle bounded by the x-axis, the line $f(x) = 12 - \frac{1}{3}x$, and the line perpendicular to $f(x)$ that passes through the origin.

Find the area of a triangle bounded by the y-axis, the line $f(x) = 9 - \frac{6}{7}x$, and the line perpendicular to $f(x)$ that passes through the origin.

20.012 square units

Find the area of a parallelogram bounded by the x-axis, the line $g(x) = 2$, the line $f(x) = 3x$, and the line parallel to $f(x)$ passing through (6,1).

For the following exercises, consider this scenario: A town's population has been decreasing at a constant rate. In 2010 the population was 5,900. By 2012 the population had dropped 4,700. Assume this trend continues.

Predict the population in 2016.

2,300

Identify the year in which the population will reach 0.

For the following exercises, consider this scenario: A town's population has been increased at a constant rate. In 2010 the population was 46,020. By 2012 the population had increased to 52,070. Assume this trend continues.

Predict the population in 2016.

64,170

Identify the year in which the population will reach 75,000.

For the following exercises, consider this scenario: A town has an initial population of 75,000. It grows at a constant rate of 2,500 per year for 5 years.

Find the linear function that models the town's population P as a function of the year, t, where t is the number of years since the model began.

$P(t) = 75,000 + 2500t$

Find a reasonable domain and range for the function P.

If the function P is graphed, find and interpret the x- and y-intercepts.

(–30, 0) Thirty years before the start of this model, the town had no citizens. (0, 75,000) Initially, the town had a population of 75,000.

If the function P is graphed, find and interpret the slope of the function.

When will the output reached 100,000?

Ten years after the model began.

What is the output in the year 12 years from the onset of the model?

For the following exercises, consider this scenario: The weight of a newborn is 7.5 pounds. The baby gained one-half pound a month for its first year.

Find the linear function that models the baby's weight W as a function of the age of the baby, in months, t.

$W(t) = 7.5t + 0.5$

Find a reasonable domain and range for the function W.

If the function W is graphed, find and interpret the x- and y-intercepts.

$(-15, 0)$: The x-intercept is not a plausible set of data for this model because it means the baby weighed 0 pounds 15 months prior to birth. $(0, 7.5)$: The baby weighed 7.5 pounds at birth.

If the function W is graphed, find and interpret the slope of the function.

When did the baby weight 10.4 pounds?

At age 5.8 months.

What is the output when the input is 6.2? Interpret your answer.

For the following exercises, consider this scenario: The number of people afflicted with the common cold in the winter months steadily decreased by 205 each year from 2005 until 2010. In 2005, 12,025 people were afflicted.

Find the linear function that models the number of people inflicted with the common cold C as a function of the year, t.

$C(t) = 12,025 - 205t$

Find a reasonable domain and range for the function C.

If the function C is graphed, find and interpret the x- and y-intercepts.

$(58.7, 0)$: In roughly 59 years, the number of people inflicted with the common cold would be $0.(0, 12,025)$: Initially there were 12,025 people afflicted by the common cold.

If the function C is graphed, find and interpret the slope of the function.

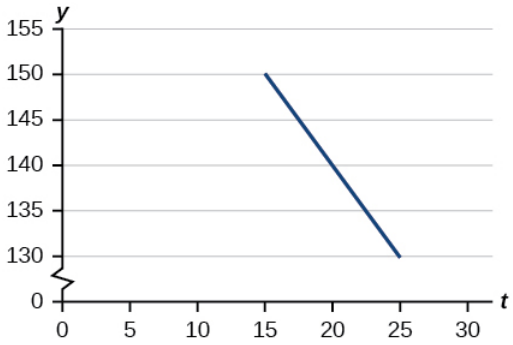
When will the output reach 0?

2064

In what year will the number of people be 9,700?

Graphical

For the following exercises, use the graph in Figure, which shows the profit, y, in thousands of dollars, of a company in a given year, t, where t represents the number of years since 1980.



Graph of a line from (15, 150) to (25, 130).

Find the linear function y, where y depends on t, the number of years since 1980.

$y = -2t + 180$

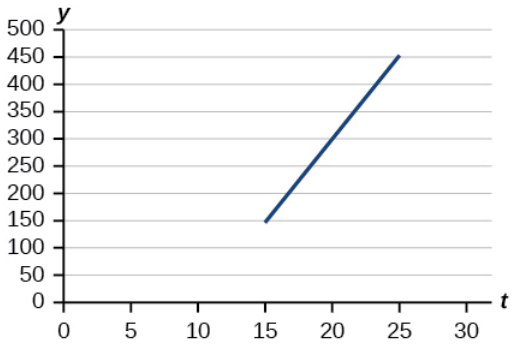
Find and interpret the y-intercept.

Find and interpret the x-intercept.

In 2070, the company's profit will be zero.

Find and interpret the slope.

For the following exercises, use the graph in Figure, which shows the profit, y, in thousands of dollars, of a company in a given year, t, where t represents the number of years since 1980.



Graph of a line from (15, 150) to (25, 450).

Find the linear function y, where y depends on t, the number of years since 1980.

$y = 30t - 300$

Find and interpret the y-intercept.

Find and interpret the x-intercept.

(10, 0) In 1990, the profit earned zero profit.

Find and interpret the slope.

Numeric

For the following exercises, use the median home values in Mississippi and Hawaii (adjusted for inflation) shown in Table. Assume that the house values are changing linearly.

Year	Mississippi	Hawaii
1950	$25,200	$74,400
2000	$71,400	$272,700

In which state have home values increased at a higher rate?

Hawaii

If these trends were to continue, what would be the median home value in Mississippi in 2010?

If we assume the linear trend existed before 1950 and continues after 2000, the two states' median house values will be (or were) equal in what year? (The answer might be absurd.)

During the year 1933

For the following exercises, use the median home values in Indiana and Alabama (adjusted for inflation) shown in Table. Assume that the house values are changing linearly.

Year	Indiana	Alabama
1950	$37,700	$27,100
2000	$94,300	$85,100

In which state have home values increased at a higher rate?

If these trends were to continue, what would be the median home value in Indiana in 2010?

$105,620

If we assume the linear trend existed before 1950 and continues after 2000, the two states' median house values will be (or were) equal in what year? (The answer might be absurd.)

Real-World Applications

In 2004, a school population was 1001. By 2008 the population had grown to 1697. Assume the population is changing linearly.

- a.How much did the population grow between the year 2004 and 2008?
- b.How long did it take the population to grow from 1001 students to 1697 students?
- c.What is the average population growth per year?
- d.What was the population in the year 2000?
- e.Find an equation for the population, P, of the school t years after 2000.
- f.Using your equation, predict the population of the school in 2011.

a. 696 people; b.4 years ;c.174 people per year; d. 305 people; e.$P(t) = 305 + 174t$; f. 2219 people

In 2003, a town's population was 1431. By 2007 the population had grown to 2134. Assume the population is changing linearly.

- a.How much did the population grow between the year 2003 and 2007?
- b.How long did it take the population to grow from 1431 people to 2134 people?
- c.What is the average population growth per year?
- d.What was the population in the year 2000?
- e.Find an equation for the population, P of the town t years after 2000.
- f.Using your equation, predict the population of the town in 2014.

A phone company has a monthly cellular plan where a customer pays a flat monthly fee and then a certain amount of money per minute used on the phone. If a customer uses 410 minutes, the monthly cost will be $71.50. If the customer uses 720 minutes, the monthly cost will be $118.

- a. Find a linear equation for the monthly cost of the cell plan as a function of x, the number of monthly minutes used.
- b. Interpret the slope and y-intercept of the equation.
- c. Use your equation to find the total monthly cost if 687 minutes are used.

a. $C(x) = 0.15x + 10$; b. The flat monthly fee is $10 and there is an additional $0.15 fee for each additional minute used; c. $113.05

A phone company has a monthly cellular data plan where a customer pays a flat monthly fee of $10 and then a certain amount of money per megabyte (MB) of data used on the phone. If a customer uses 20 MB, the monthly cost will be $11.20. If the customer uses 130 MB, the monthly cost will be $17.80.

- a. Find a linear equation for the monthly cost of the data plan as a function of x, the number of MB used.
- b. Interpret the slope and y-intercept of the equation.
- c. Use your equation to find the total monthly cost if 250 MB are used.

In 1991, the moose population in a park was measured to be 4,360. By 1999, the population was measured again to be 5,880. Assume the population continues to change linearly.

- a. Find a formula for the moose population, P since 1990.
- b. What does your model predict the moose population to be in 2003?

a. $P(t) = 190t + 4360$ b. 6640 moose

In 2003, the owl population in a park was measured to be 340. By 2007, the population was measured again to be 285. The population changes linearly. Let the input be years since 1990.

- a. Find a formula for the owl population, P. Let the input be years since 2003.
- b. What does your model predict the owl population to be in 2012?

The Federal Helium Reserve held about 16 billion cubic feet of helium in 2010 and is being depleted by about 2.1 billion cubic feet each year.

- a. Give a linear equation for the remaining federal helium reserves, R, in terms of t, the number of years since 2010.
- b. In 2015, what will the helium reserves be?
- c. If the rate of depletion doesn't change, in what year will the Federal Helium Reserve be depleted?

a. $R(t) = 16 - 2.1t$ b. 5.5 billion cubic feet c. During the year 2017

Suppose the world's oil reserves in 2014 are 1,820 billion barrels. If, on average, the total reserves are decreasing by 25 billion barrels of oil each year:

- a. Give a linear equation for the remaining oil reserves, R, in terms of t, the number of years since now.
- b. Seven years from now, what will the oil reserves be?
- c. If the rate at which the reserves are decreasing is constant, when will the world's oil reserves be depleted?

You are choosing between two different prepaid cell phone plans. The first plan charges a rate of 26 cents per minute. The second plan charges a monthly fee of $19.95 plus 11 cents per minute. How many minutes would you have to use in a month in order for the second plan to be preferable?

More than 133 minutes

You are choosing between two different window washing companies. The first charges $5 per window. The second charges a base fee of $40 plus $3 per window. How many windows would you need to have for the second company to be preferable?

When hired at a new job selling jewelry, you are given two pay options:
Option A: Base salary of $17,000 a year with a commission of 12% of your sales
Option B: Base salary of $20,000 a year with a commission of 5% of your sales

How much jewelry would you need to sell for option A to produce a larger income?

More than $42,857.14 worth of jewelry

When hired at a new job selling electronics, you are given two pay options:

- Option A: Base salary of $14,000 a year with a commission of 10% of your sales
- Option B: Base salary of $19,000 a year with a commission of 4% of your sales

How much electronics would you need to sell for option A to produce a larger income?

When hired at a new job selling electronics, you are given two pay options:

- Option A: Base salary of $20,000 a year with a commission of 12% of your sales
- Option B: Base salary of $26,000 a year with a commission of 3% of your sales

How much electronics would you need to sell for option A to produce a larger income?

$66,666.67

When hired at a new job selling electronics, you are given two pay options:

- Option A: Base salary of $10,000 a year with a commission of 9% of your sales
- Option B: Base salary of $20,000 a year with a commission of 4% of your sales

How much electronics would you need to sell for option A to produce a larger income?

Footnotes

1 Rates retrieved Aug 2, 2010 from http://www.budgettruck.com and http://www.uhaul.com/

Contributors

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CHAPTER OVERVIEW

1: Solving Linear Equations

In this chapter, you will explore linear equations, develop a strategy for solving them, and relate them to real-world situations.

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1: Solving Linear Equations

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2.1: Prelude to Solving Linear Equations

Imagine being a pilot, but not just any pilot—a drone pilot. Drones, or unmanned aerial vehicles, are devices that can be flown remotely. They contain sensors that can relay information to a command center where the pilot is located. Larger drones can also carry cargo. In the near future, several companies hope to use drones to deliver materials and piloting a drone will become an important career. Law enforcement and the military are using drones rather than send personnel into dangerous situations.



Figure **2.1.1**: This drone is flying high in the sky while its pilot remains safely on the ground. (credit: “Unsplash” / Pixabay)

Building and piloting a drone requires the ability to program a set of actions, including taking off, turning, and landing. This, in turn, requires the use of linear equations. In this chapter, you will explore linear equations, develop a strategy for solving them, and relate them to real-world situations.

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2.2: Use a General Strategy to Solve Linear Equations

Summary

By the end of this section, you will be able to:

- Use the commutative and associative properties
- Use the properties of identity, inverse, and zero
- Simplify expressions using the Distributive Property

Before you get started, take this readiness quiz.

1. Simplify: $\frac{3}{2}(12x+20)$.
2. Simplify: $5-2(n+1)$.
3. Find the LCD of $\frac{5}{6}$ and $\frac{1}{4}$.

Solve Linear Equations Using a General Strategy

Solving an equation is like discovering the answer to a puzzle. The purpose in solving an equation is to find the value or values of the variable that makes it a true statement. Any value of the variable that makes the equation true is called a **solution** to the equation. It is the answer to the puzzle!

Definition: SOLUTION OF AN EQUATION

A **solution** of an equation is a value of a variable that makes a true statement when substituted into the equation.

To determine whether a number is a solution to an equation, we substitute the value for the variable in the equation. If the resulting equation is a true statement, then the number is a solution of the equation.

How To: DETERMINE WHETHER A NUMBER IS A SOLUTION TO AN EQUATION

1. Substitute the number for the variable in the equation.
2. Simplify the expressions on both sides of the equation.
3. Determine whether the resulting equation is true.
   - If it is true, the number is a solution.
   - If it is not true, the number is not a solution.

Example 2.2.1

Determine whether the values are solutions to the equation: $5y+3=10y-4$.

a. $y=\frac{3}{5}$
b. $y=\frac{7}{5}$

Solution

Since a solution to an equation is a value of the variable that makes the equation true, begin by substituting the value of the solution for the variable.

a.

	$5y+3=10y-4$
Substitute $\frac{3}{5}$ for y	$5\left(\frac{3}{5}\right)+3\stackrel{?}{=}10\left(\frac{3}{5}\right)-4$
Multiply.	$3+3\stackrel{?}{=}6-4$
Simplify.	$6\neq 2$

Since $y=\frac{3}{5}$ does not result in a true equation, $y=\frac{3}{5}$ is not a solution to the equation $5y+3=10y-4$.

b.

	$5y+3=10y-4$
Substitute $\frac{7}{5}$ for y	$5\left(\frac{7}{5}\right)+3\stackrel{?}{=}10\left(\frac{7}{5}\right)-4$
Multiply.	$7+3\stackrel{?}{=}14-4$
Simplify.	$10=10\checkmark$

Since $y=\frac{7}{5}$ results in a true equation, $y=\frac{7}{5}$ is a solution to the equation $5y+3=10y-4$.

2.2.1A

Determine whether the values are solutions to the equation: $9y+2=6y+3$.

a. $y=\frac{4}{3}$
b. $y=\frac{1}{3}$

Answer a

no

Answer b

yes

2.2.1B

Determine whether the values are solutions to the equation: $4x-2=2x+1$.

a. $x=\frac{3}{2}$
b. $x=-\frac{1}{2}$

Answer a

yes

Answer b

no

There are many types of equations that we will learn to solve. In this section we will focus on a **linear equation.**

Definition: LINEAR EQUATION

A **linear equation** is an equation in one variable that can be written, where a and b are real numbers and $a\neq 0$, as:

$$ax+b=0$$

To solve a linear equation it is a good idea to have an overall strategy that can be used to solve any linear equation. In the next example, we will give the steps of a general strategy for solving any linear equation. Simplifying each side of the equation as much as possible first makes the rest of the steps easier.

EXAMPLE 2.2.2

Solve: $7(n-3)-8=-15$

Solution

Step 1. Simplify each side of the equation as much as possible.	Use the Distributive Property. Notice that each side of the equation is now simplified as much as possible.	$7(n-3)-8=-15$ $7n-21-8=-15$ $7n-29=-15$

Figure_02_01_003b_img.jpg" />

Step 3. Collect constant terms on the other side of the equation.	To get constants only on the right, add 29 to each side. Simplify.	$7n - 29 + 29 = -15 + 29$ $7n = 14$

Figure_02_01_003d_img.jpg" />

Step 5. Check the solution.	Let $n = 2$ Subtract.	**Check:** $7(n-3)-8=-15$ $7(2-3)-8 \stackrel{?}{=} -15$ $7(-1)-8 \stackrel{?}{=} -15$ $-7-8 \stackrel{?}{=} -15$ $-15 = -15$ ✓

2.2.2A

Solve: $2(m-4)+3=-1$.

Answer

$m = 2$

2.2.2B

Solve: $5(a-3)+5=-10$.

Answer

$a = 0$

These steps are summarized in the **General Strategy for Solving Linear Equations** below.

SOLVE LINEAR EQUATIONS USING A GENERAL STRATEGY

1. Simplify each side of the equation as much as possible.
   Use the Distributive Property to remove any parentheses.
   Combine like terms.
2. Collect all the variable terms on one side of the equation.
   Use the Addition or Subtraction Property of Equality.
3. Collect all the constant terms on the other side of the equation.
   Use the Addition or Subtraction Property of Equality.
4. Make the coefficient of the variable term equal to 1.
   Use the Multiplication or Division Property of Equality.
   State the solution to the equation.
5. Check the solution.
   Substitute the solution into the original equation to make sure the result is a true statement.

EXAMPLE 2.2.3

Solve: $\frac{2}{3}(3m-6)=5-m$.

Answer

	$\frac{2}{3}(3m-6)=5-m$
Distribute.	$2m-4=5-m$
Add m to both sides to get the variables only on the left.	$2m+m-4=5-m+m$
Simplify.	$3m-4=5$
Add 4 to both sides to get constants only on the right.	$3m-4+4=5+4$
Simplify.	$3m=9$
Divide both sides by three.	$\frac{3m}{3}=\frac{9}{3}$
Simplify.	$m=3$

Check:	$\frac{2}{3}(3m-6)=5-m$
Let m=3	$\frac{2}{3}(3\cdot 3-6)\stackrel{?}{=}5-3$
	$\frac{2}{3}(9-6)\stackrel{?}{=}2$
	$\frac{2}{3}(3)\stackrel{?}{=}2$
	$2=2\checkmark$

2.2.3A

Solve: $\frac{1}{3}(6u+3)=7-u$.

Answer

$u=2$

2.2.3B

Solve: $\frac{2}{3}(9x-12)=8+2x$.

Answer

$x=4$

We can solve equations by getting all the **variable** terms to either side of the **equal sign**. By collecting the variable terms on the side where the **coefficient** of the variable is larger, we avoid working with some negatives. This will be a good strategy when we solve inequalities later in this chapter. It also helps us prevent errors with negatives.

EXAMPLE 2.2.4

Solve: $4(x-1)-2=5(2x+3)+6$.

Answer

$4(x-1)-2=5(2x+3)+6$

Distribute.

$4x-4-2=10x+15+6$

Combine like terms.

$4x-6=10x+21$

Subtract $4x$ from each side.

$4x-4x-6=10x-4x+21$

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Divide both sides by 6.

$$\frac{-27}{6} = \frac{6x}{6}$$

Simplify $-\frac{9}{2} = x$ if y.

Check:

Let $x = -92$.

2.2.4A

Solve: $6(p-3)-7=5(4p+3)-12$.

Answer

$p=-2$

2.2.4B

Solve: $8(q+1)-5=3(2q-4)-1$.

Answer

$q=-8$

EXAMPLE 2.2.5

Solve: $10[3-8(2s-5)]=15(40-5s)$.

Answer

	$10[3-8(2s-5)]=15(40-5s)$
Simplify from the innermost parentheses first.	$10[3-16s+40]=15(40-5s)$
Combine like terms in the brackets.	$10[43-16s]=15(40-5s)$
Distribute.	$430-160s=600-75s$
Add $160s$ to both sides to get the variable to the right.	$430-160s+160s=600-75s+160s$
Simplify.	$430=600+85s$
Subtract 600 from both sides to get the constant to the left.	$430-600=600+85s-600$
Simplify.	$-170=85s$
Divide both sides by 85.	$\frac{-170}{85}=\frac{85s}{85}$
Simplify.	\(\-2=s)

Check:	$10[3-8(2s-5)]=15(40-5s)$
Let $s=-2$.	$10[3-8(2(-2)-5)] \stackrel{?}{=} 15(40-5(-2))$
	$10[3-8(-4-5)] \stackrel{?}{=} 15(40+10)$
	$10[3-8(-9)] \stackrel{?}{=} 15(50)$
	$10(3+72) \stackrel{?}{=} 750$
	$10(75) \stackrel{?}{=} 750$
	$750=750$ ✓

2.2.5A

Solve: $6[4-2(7y-1)]=8(13-8y)$.

Answer

$y = -\frac{17}{5}$

2.2.5B

Solve: $12[1 - 5(4z - 1)] = 3(24 + 11z)$.

Answer

$z = 0$

Classify Equations

Whether or not an equation is true depends on the value of the variable. The equation $7x + 8 = -13$ is true when we replace the variable, x, with the value -3, but not true when we replace x with any other value. An equation like this is called a **conditional equation**. All the equations we have solved so far are conditional equations.

Definition: CONDITIONAL EQUATION

An equation that is true for one or more values of the variable and false for all other values of the variable is a **conditional equation**.

Now let's consider the equation $7y + 14 = 7(y + 2)$. Do you recognize that the left side and the right side are equivalent? Let's see what happens when we solve for y.

Solve:

	$7y + 14 = 7(y + 2)$
Distribute.	$7y + 14 = 7y + 14$
Subtract $7y$ to each side to get the y's to one side.	$7y - 7y + 14$ $= 7y - 7y + 14$
Simplify—the y's are eliminated.	$14 = 14$
	But $14 = 14$ is true.

This means that the equation $7y + 14 = 7(y + 2)$ is true for any value of y. We say the solution to the equation is all of the real numbers. An equation that is true for any value of the variable is called an **identity**.

Definition: IDENTITY

An equation that is true for any value of the variable is called an **identity**.

The solution of an identity is valid for all real numbers.

What happens when we solve the equation $-8z = -8z + 9$?

Solve:

	$-8z = -8z + 9$
Add 8z to both sides to leave the constant alone on the right.	$-8z + 8z$ $= -8z + 8z + 9$
Simplify—the z's are eliminated.	$0 \neq 9$
	But $0 \neq 9$.

Solving the equation $-8z = -8z + 9$ led to the false statement $0 = 9$. The equation $-8z = -8z + 9$ will not be true for any value of z. It has no solution. An equation that has no solution, or that is false for all values of the variable, is called a contradiction.

definition: CONTRADICTION

An equation that is false for all values of the variable is called a **contradiction**.

A contradiction has no solution.

The next few examples will ask us to classify an equation as conditional, an identity, or as a contradiction.

EXAMPLE 2.2.6

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $6(2n-1)+3=2n-8+5(2n+1)$.

Answer

	$6(2n-1)+3=2n-8+5(2n+1)$
Distribute.	$12n-6+3=2n-8+10n+5$
Combine like terms.	$12n-3=12n-3$
Subtract 12n from each side to get the n's to one side.	$12n-12n-3=12n-12n-3$
Simplify.	$-3=-3$
This is a true statement.	The equation is an identity.
	The solution is all real numbers.

2.2.6A

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $4+9(3x-7)=-42x-13+23(3x-2)$.

Answer

identity; all real numbers

Exercise 2.2.6B

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $8(1-3x)+15(2x+7)=2(x+50)+4(x+3)+1$.

Answer

identity; all real numbers

EXAMPLE 2.2.7

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $8+3(a-4)=0$.

Answer

	$8 + 3(a - 4) = 0$
Distribute.	$8 + 3a - 12 = 0$
Combine like terms.	$3a - 4 = 0$
Add 4 to both sides.	$3a - 4 + 4 = 0 + 4$
Simplify.	$3a = 4$
Divide.	$\frac{3a}{3} = \frac{4}{3}$
Simplify.	$a = \frac{4}{3}$
The equation is true when $a = \frac{4}{3}$.	This is a conditional equation.
	The solution is $a = \frac{4}{3}$.

Exercise 2.2.7A

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $11(q + 3) - 5 = 19$.

Answer

conditional equation; $q = -\frac{9}{11}$

Exercise 2.2.7B

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $6 + 14(k - 8) = 95$.

Answer

conditional equation; $k = \frac{201}{14}$

EXAMPLE 2.2.8

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $5m + 3(9 + 3m) = 2(7m - 11)$.

Answer

	$5m + 3(9 + 3m) = 2(7m - 11)$
Distribute.	$5m + 27 + 9m = 14m - 22$
Combine like terms.	$14m + 27 = 14m - 22$
Subtract $14m$ from both sides.	$14m + 27 - 14m = 14m - 22 - 14m$
Simplify.	$27 \neq -22$
But $27 \neq -22$.	The equation is a contradiction.
	It has no solution.

Exercise 2.2.8A

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $12c + 5(5 + 3c) = 3(9c - 4)$.

Answer

contradiction; no solution

Exercise 2.2.8B

Classify the equation as a conditional equation, an identity, or a contradiction and then state the solution: $4(7d+18)=13(3d-2)-11d$.

Answer

contradiction; no solution

We summarize the methods for classifying equations in the table.

Type of equation	What happens when you solve it?	Solution
Conditional Equation	True for one or more values of the variables and false for all other values	One or more values
Identity	True for any value of the variable	All real numbers
Contradiction	False for all values of the variable	No solution

Solve Equations with Fraction or Decimal Coefficients

We could use the General Strategy to solve the next example. This method would work fine, but many students do not feel very confident when they see all those fractions. So, we are going to show an alternate method to solve equations with fractions. This alternate method eliminates the fractions.

We will apply the Multiplication Property of Equality and multiply both sides of an equation by the **least common denominator** (LCD) of all the fractions in the equation. The result of this operation will be a new equation, equivalent to the first, but without fractions. This process is called *clearing* the equation of fractions.

To clear an equation of decimals, we think of all the decimals in their fraction form and then find the LCD of those denominators.

EXAMPLE 2.2.9: How to Solve Equations with Fraction or Decimal Coefficients

Solve: $\frac{1}{12}x+\frac{5}{6}=\frac{3}{4}$.

Answer

Step 1. Find the least common denominator of *all* the fractions and decimals in the equation.	What is the LCD of $\frac{1}{12}$, $\frac{5}{6}$, and $\frac{3}{4}$?	$\frac{1}{12}x+\frac{5}{6}=\frac{3}{4}$ LCD = 12

Step 3. Solve using the General Strategy for Solving Linear Equations.	To isolate the variable term, subtract 10. Simplify.	$x+10-10=9-10$ $x=-1$ **Check:** $\frac{1}{12}x+\frac{5}{6}=\frac{3}{4}$ $\frac{1}{12}(-1)+\frac{5}{6}\stackrel{?}{=}\frac{3}{4}$ $-\frac{1}{12}+\frac{5}{6}\stackrel{?}{=}\frac{3}{4}$ $-\frac{1}{12}+\frac{10}{12}\stackrel{?}{=}\frac{9}{12}$ $\frac{9}{12}=\frac{9}{12}\checkmark$

Exercise 2.2.9A

Solve: $\frac{1}{4}x+\frac{1}{2}=\frac{5}{8}$.

Answer

$x=\frac{1}{2}$

Exercise 2.2.9B

Solve: $\frac{1}{8}x+\frac{1}{2}=\frac{1}{4}$.

Answer

$x=-2$

Notice in the previous example, once we cleared the equation of fractions, the equation was like those we solved earlier in this chapter. We changed the problem to one we already knew how to solve. We then used the **General Strategy for Solving Linear Equations**.

How To: SOLVE EQUATIONS WITH FRACTION OR DECIMAL COEFFICIENTS.

1. Find the least common denominator (LCD) of *all* the fractions and decimals (in fraction form) in the equation.
2. Multiply both sides of the equation by that LCD. This clears the fractions and decimals.
3. Solve using the General Strategy for Solving Linear Equations.

EXAMPLE 2.2.10

Solve: $5=\frac{1}{2}y+\frac{2}{3}y-\frac{3}{4}y$.

Answer

We want to clear the fractions by multiplying both sides of the equation by the LCD of all the fractions in the equation.

Find the LCD of all fractions in the equation.		$5=\frac{1}{2}y+\frac{2}{3}y-\frac{3}{4}y$
The LCD is 12.		
Multiply both sides of the equation by 12.		$12(5)=12\cdot\left(\frac{1}{2}y+\frac{2}{3}y-\frac{3}{4}y\right)$
Distribute.		$12(5)=12\cdot\frac{1}{2}y+12\cdot\frac{2}{3}y-12\cdot\frac{3}{4}y$
Simplify—notice, no more fractions.		$60=6y+8y-9y$
Combine like terms.		$60=5y$
Divide by five.		$\frac{60}{5}=\frac{5y}{5}$
Simplify.		$12=y$
Check:	$5=\frac{1}{2}y+\frac{2}{3}y-\frac{3}{4}y$	
Let $y=12$.	$5\stackrel{?}{=}\frac{1}{2}(12)+\frac{2}{3}(12)-\frac{3}{4}(12)$	
	$5\stackrel{?}{=}6+8-9$	
	$5=5\checkmark$	

Exercise 2.2.10A

Solve: $7=\frac{1}{2}x+\frac{3}{4}x-\frac{2}{3}x$.

Answer

$x=12$

Exercise 2.2.10B

Solve: $-1=\frac{1}{2}u+\frac{1}{4}u-\frac{2}{3}u$.

Answer

$u=-12$

In the next example, we'll distribute before we clear the fractions.

EXAMPLE 2.2.11

Solve: $\frac{1}{2}(y-5)=\frac{1}{4}(y-1)$.

Answer

	$\frac{1}{2}(y-5)=\frac{1}{4}(y-1)$
Distribute.	$\frac{1}{2}\cdot y-\frac{1}{2}\cdot 5=\frac{1}{4}\cdot y-\frac{1}{4}\cdot 1$
Simplify.	$\frac{1}{2}y-\frac{5}{2}=\frac{1}{4}y-\frac{1}{4}$
Multiply by the LCD, four.	$4\left(\frac{1}{2}y-\frac{5}{2}\right)=4\left(\frac{1}{4}y-\frac{1}{4}\right)$
Distribute.	$4\cdot\frac{1}{2}y-4\cdot\frac{5}{2}=4\cdot\frac{1}{4}y-4\cdot\frac{1}{4}$
Simplify.	$2y-10=y-1$
Collect the variables to the left.	$2y-y-10=y-y-1$
Simplify.	$y-10=-1$
Collect the constants to the right.	$y-10+10=-1+10$
Simplify.	$y=9$

An alternate way to solve this equation is to clear the fractions without distributing first. If you multiply the factors correctly, this method will be easier.

	$\frac{1}{2}(y-5)=\frac{1}{4}(y-1)$
Multiply by the LCD, 4.	$4\cdot\frac{1}{2}(y-5)=4\cdot\frac{1}{4}(y-1)$
Multiply four times the fractions.	$2(y-5)=1(y-1)$
Distribute.	$2y-10=y-1$
Collect the variables to the left.	$2y-y-10=y-y-1$
Simplify.	$y-10=-1$
Collect the constants to the right.	$y-10+10=-1+10$
Simplify.	$y=9$
Check:	$\frac{1}{2}(y-5)=\frac{1}{4}(y-1)$
Let $y=9$.	$\frac{1}{2}(9-5)\stackrel{?}{=}\frac{1}{4}(9-1)$

Finish the check on your own.

Exercise 2.2.11A

Solve: $\frac{1}{5}(n+3)=\frac{1}{4}(n+2)$.

Answer

$n=2$

Exercise 2.2.11B

Solve: $\frac{1}{2}(m-3)=\frac{1}{4}(m-7)$.

Answer

$m = -1$

When you multiply both sides of an equation by the LCD of the fractions, make sure you multiply each term by the LCD—even if it does not contain a fraction.

EXAMPLE 2.2.12

Solve: $\frac{4q+3}{2} + 6 = \frac{3q+5}{4}$

Answer

	$\frac{4q+3}{2} + 6 = \frac{3q+5}{4}$
Multiply both sides by the LCD, 4.	$4\left(\frac{4q+3}{2} + 6\right) = 4\left(\frac{3q+5}{4}\right)$
Distribute.	$4\left(\cdot \frac{4q+3}{2}\right) + 4 \cdot 6 = 4 \cdot \left(\frac{3q+5}{4}\right)$
Simplify.	\(\2(4 q+3)+24=3 q+5)
	$8q + 6 + 24 = 3q + 5$
	$8q + 30 = 3q + 5$
Collect the variables to the left.	$8q - 3q + 30 = 3q - 3q + 5$
Simplify.	$5q + 30 = 5$
Collect the constants to the right.	$5q + 30 - 30 = 5 - 30$
Simplify.	$5q = -25$
Divide both sides by five.	$\frac{5q}{5} = \frac{-25}{5}$
Simplify.	$q = -5$
Check:	$\frac{4q+3}{2} + 6 = \frac{3q+5}{4}$
Let $q = -5$.	$\frac{4(-5)+3}{2} + 6 \stackrel{?}{=} \frac{3(-5)+5}{4}$
Finish the check on your own.	

Exercise 2.2.12A

Solve: $\frac{3r+5}{6} + 1 = \frac{4r+3}{3}$.

Answer

$r = 3$

Exercise 2.2.12B

Solve: $\frac{2s+3}{2} + 1 = \frac{3s+2}{4}$.

Answer

$s = -8$

Some equations have decimals in them. This kind of equation may occur when we solve problems dealing with money or percentages. But decimals can also be expressed as fractions. For example, $0.7 = \frac{7}{10}$ and $0.29 = \frac{29}{100}$. So, with an equation with

decimals, we can use the same method we used to clear fractions—multiply both sides of the equation by the **least common denominator**.

The next example uses an equation that is typical of the ones we will see in the money applications in a later section. Notice that we will clear all decimals by multiplying by the LCD of their fraction form.

EXAMPLE 2.2.13

Solve: $0.25x + 0.05(x + 3) = 2.85$.

Answer

Look at the decimals and think of the equivalent fractions:

$$0.25 = \frac{25}{100}, \qquad 0.05 = \frac{5}{100}, \qquad 2.85 = 2\frac{85}{100}. \tag{2.2.1}$$

Notice, the LCD is 100. By multiplying by the LCD we will clear the decimals from the equation.

	$0.25x + 0.05(x + 3) = 2.85$
Distribute first.	$0.25x + 0.05x + 0.15 = 2.85$
Combine like terms.	$0.30x + 0.15 = 2.85$
To clear decimals, multiply by 100.	$100(0.30x + 0.15) = 100(2.85)$
Distribute.	$30x + 15 = 285$
Subtract 15 from both sides.	$30x + 15 - 15 = 285 - 15$
Simplify.	$30x = 270$
Divide by 30.	$\frac{30x}{30} = \frac{270}{30}$
Simplify.	$x = 9$
Check it yourself by substituting $x = 9$ into the original equation.	

Exercise 2.2.13A

Solve: $0.25n + 0.05(n + 5) = 2.95$.

Answer

$n = 9$

Exercise 2.2.13B

Solve: $0.10d + 0.05(d - 5) = 2.15$.

Answer

$d = 16$

Key Concepts

- **How to determine whether a number is a solution to an equation**
  1. Substitute the number in for the variable in the equation.
  2. Simplify the expressions on both sides of the equation.
  3. Determine whether the resulting equation is true.

     If it is true, the number is a solution.

If it is not true, the number is not a solution.

- **How to Solve Linear Equations Using a General Strategy**
  1. Simplify each side of the equation as much as possible.

     Use the Distributive Property to remove any parentheses.

     Combine like terms.
  2. Collect all the variable terms on one side of the equation.

     Use the Addition or Subtraction Property of Equality.
  3. Collect all the constant terms on the other side of the equation.

     Use the Addition or Subtraction Property of Equality.
  4. Make the coefficient of the variable term equal to 1.

     Use the Multiplication or Division Property of Equality.

     State the solution to the equation.
  5. Check the solution.

     Substitute the solution into the original equation to make sure the result is a true statement.
- **How to Solve Equations with Fraction or Decimal Coefficients**
  1. Find the least common denominator (LCD) of *all* the fractions and decimals (in fraction form) in the equation.
  2. Multiply both sides of the equation by that LCD. This clears the fractions and decimals.
  3. Solve using the General Strategy for Solving Linear Equations.

Glossary

conditional equation

An equation that is true for one or more values of the variable and false for all other values of the variable is a conditional equation.

contradiction

An equation that is false for all values of the variable is called a contradiction. A contradiction has no solution.

identity

An equation that is true for any value of the variable is called an Identity. The solution of an identity is all real numbers.

linear equation

A linear equation is an equation in one variable that can be written, where a and b are real numbers and $a \neq 0$, as $ax + b = 0$.

solution of an equation

A solution of an equation is a value of a variable that makes a true statement when substituted into the equation.

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2.2.1E: Use a General Strategy to Solve Linear Equations (Exercises)

Practice Makes Perfect

Solve Equations Using the General Strategy

In the following exercises, determine whether the given values are solutions to the equation.

$6y+10=12y$

ⓐ $y=\frac{5}{3}$

ⓑ $y=-\frac{1}{2}$

Answer

ⓐ yes ⓑ no

$4x+9=8x$

ⓐ $x=-\frac{7}{8}$

ⓑ $x=\frac{9}{4}$

$8u-1=6u$

ⓐ $u=-\frac{1}{2}$

ⓑ $u=\frac{1}{2}$

Answer

ⓐ no ⓑ yes

$9v-2=3v$

ⓐ $v=-\frac{1}{3}$

ⓑ $v=\frac{1}{3}$

In the following exercises, solve each linear equation.

$15(y-9)=-60$

Answer

$y=5$

$-16(3n+4)=32$

$-(w-12)=30$

Answer

$w=-18$

$-(t-19)=28$

$51+5(4-q)=56$

Answer

$q=3$

$-6+6(5-k)=15$

$3(10-2x)+54=0$

Answer

$x=14$

$-2(11-7x)+54=4$

$\frac{2}{3}(9c-3)=22$

Answer

$c=4$

$\frac{3}{5}(10x-5)=27$

$\frac{1}{5}(15c+10)=c+7$

Answer

$c=\frac{5}{2}$

$\frac{1}{4}(20d+12)=d+7$

$3(4n-1)-2=8n+3$

Answer

$n=2$

$9(2m-3)-8=4m+7$

$12+2(5-3y)=-9(y-1)-2$

Answer

$y=-5$

$-15+4(2-5y)=-7(y-4)+4$

$5+6(3s-5)=-3+2(8s-1)$

Answer

$s=10$

$-12+8(x-5)=-4+3(5x-2)$

$4(p-4)-(p+7)=5(p-3)$

Answer

$p=-4$

$3(a-2)-(a+6)=4(a-1)$

$4[5-8(4c-3)]=12(1-13c)-8$

Answer

$c=-4$

$5[9-2(6d-1)]=11(4-10d)-139$

$3[-9+8(4h-3)]=2(5-12h)-19$

Answer

$h=\frac{3}{4}$

$3[-14+2(15k-6)]=8(3-5k)-24$

$5[2(m+4)+8(m-7)]=2[3(5+m)-(21-3m)]$

Answer

$m=6$

$10[5(n+1)+4(n-1)]=11[7(5+n)-(25-3n)]$

Classify Equations

In the following exercises, classify each equation as a conditional equation, an identity, or a contradiction and then state the solution.

$23z+19=3(5z-9)+8z+46$

Answer

identity; all real numbers

$15y+32=2(10y-7)-5y+46$

$18(5j-1)+29=47$

Answer

conditional equation;$j=\frac{2}{5}$frac{2}{5}\)

$24(3d-4)+100=52$

$22(3m-4)=8(2m+9)$

Answer

conditional equation; $m=165$

$30(2n-1)=5(10n+8)$10n+8)\)

$7v+42=11(3v+8)-2(13v-1)$

Answer

contradiction; no solution

$18u-51=9(4u+5)-6(3u-10)$

$45(3y-2)=9(15y-6)$

Answer

contradiction; no solution

$60(2x-1)=15(8x+5)$

$9(14d+9)+4d=13(10d+6)+3$

Answer

identity; all real numbers

$11(8c+5)-8c=2(40c+25)+5$

Solve Equations with Fraction or Decimal Coefficients

In the following exercises, solve each equation with fraction coefficients.

$\frac{1}{4}x-\frac{1}{2}=-\frac{3}{4}$

Answer

$x=-1$

$\frac{3}{4}x-\frac{1}{2}=\frac{1}{4}$

$\frac{5}{6}y-\frac{2}{3}=-\frac{3}{2}$

Answer

$y=-1$

$\frac{5}{6}y-\frac{1}{3}=-\frac{7}{6}$

$\frac{1}{2}a+\frac{3}{8}=\frac{3}{4}$12a+38=34

Answer

$a=\frac{3}{4}$

$\frac{5}{8}b+\frac{1}{2}=-\frac{3}{4}$

$2=\frac{1}{3}x-\frac{1}{2}x+\frac{2}{3}x$

Answer

$x=4$

$2=\frac{3}{5}x-\frac{1}{3}x+\frac{2}{5}x$

$\frac{1}{3}w+\frac{5}{4}=w-\frac{1}{4}$

Answer

$w=\frac{9}{4}$

$\frac{1}{2}a-\frac{1}{4}=\frac{1}{6}a+\frac{1}{12}$

$\frac{1}{3}b+\frac{1}{5}=\frac{2}{5}b-\frac{3}{5}$

Answer

$b=12$

$\frac{1}{3}x+\frac{2}{5}=\frac{1}{5}x-\frac{2}{5}$

$\frac{1}{4}(p-7)=\frac{1}{3}(p+5)$

Answer

$p=-41$

$\frac{1}{5}(q+3)=\frac{1}{2}(q-3)$

$\frac{1}{2}(x+4)=\frac{3}{4}$

Answer

$x=-\frac{5}{2}$

$\frac{1}{3}(x+5)=\frac{5}{6}$

$\frac{4n+8}{4}$=\frac{n}{3}\)

Answer

$n=-3$

$\frac{3p+6}{3} = \frac{p}{2}$

$\frac{3x+4}{2} + 1 = \frac{5x+10}{8}$

Answer

$x = -2$

$\frac{10y-2}{3} + 3 = \frac{10y+1}{9}$

$\frac{7u-1}{4} - 1 = \frac{4u+8}{5}$

Answer

$u = 3$

$\frac{3v-6}{2} + 5 = \frac{11v-4}{5}$

In the following exercises, solve each equation with decimal coefficients.

$0.4x + 0.6 = 0.5x - 1.2$

Answer

$x = 18$

$0.7x + 0.4 = 0.6x + 2.4$

$0.9x - 1.25 = 0.75x + 1.75$

Answer

$x = 20$

$1.2x - 0.91 = 0.8x + 2.29$

$0.05n + 0.10(n + 8) = 2.15$

Answer

$n = 9$

$0.05n + 0.10(n + 7) = 3.55$

$0.10d + 0.25(d + 5) = 4.05$

Answer

$d = 8$

$0.10d + 0.25(d + 7) = 5.25$

Everyday Math

Fencing Micah has 74 feet of fencing to make a dog run in his yard. He wants the length to be 2.5 feet more than the width. Find the length, L, by solving the equation $2L + 2(L - 2.5) = 74$.

Answer

$L = 19.75$ feet

Stamps Paula bought $22.82 worth of 49-cent stamps and 21-cent stamps. The number of 21-cent stamps was eight less than the number of 49-cent stamps. Solve the equation $0.49s + 0.21(s - 8) = 22.82$ for s, to find the number of 49-cent stamps Paula bought.

Writing Exercises

Using your own words, list the steps in the general strategy for solving linear equations.

Answer

Answers will vary.

Explain why you should simplify both sides of an equation as much as possible before collecting the variable terms to one side and the constant terms to the other side.

What is the first step you take when solving the equation $3 - 7(y - 4) = 38$? Why is this your first step?

Answer

Answers will vary.

If an equation has several fractions, how does multiplying both sides by the LCD make it easier to solve?

If an equation has fractions only on one side, why do you have to multiply both sides of the equation by the LCD?

Answer

Answers will vary.

For the equation $0.35x + 2.1 = 3.85$, how do you clear the decimal?

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve linear equations using a general strategy.			
classify equations.			
solve equations with fraction or decimal coefficients.			

ⓑ If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

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2.3: Use a Problem Solving Strategy

Learning Objectives

By the end of this section, you will be able to:

- Use a problem solving strategy for word problems
- Solve number word problems
- Solve percent applications
- Solve simple interest applications

Before you get started, take this readiness quiz.

1. Translate “six less than twice x” into an algebraic expression.
   If you missed this problem, review [link].
2. Convert 4.5% to a decimal.
   If you missed this problem, review [link].
3. Convert 0.6 to a percent.
   If you missed this problem, review [link].

Have you ever had any negative experiences in the past with word problems? When we feel we have no control, and continue repeating negative thoughts, we set up barriers to success. Realize that your negative experiences with word problems are in your past. To move forward you need to calm your fears and change your negative feelings.

Start with a fresh slate and begin to think positive thoughts. Repeating some of the following statements may be helpful to turn your thoughts positive. Thinking positive thoughts is a first step towards success.

- I think I can! I think I can!
- While word problems were hard in the past, I think I can try them now.
- I am better prepared now—I think I will begin to understand word problems.
- I am able to solve equations because I practiced many problems and I got help when I needed it—I can try that with word problems.
- It may take time, but I can begin to solve word problems.
- You are now well prepared and you are ready to succeed. If you take control and believe you can be successful, you will be able to master word problems.

Use a Problem Solving Strategy for Word Problems

Now that we can solve equations, we are ready to apply our new skills to word problems. We will develop a strategy we can use to solve any word problem successfully.

EXAMPLE 2.3.1

Normal yearly snowfall at the local ski resort is 12 inches more than twice the amount it received last season. The normal yearly snowfall is 62 inches. What was the snowfall last season at the ski resort?

Answer

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	What was the snowfall last season?
Step 3. Name what we are looking for and choose a variable to represent it.	Let $s =$ the snowfall last season.
Step 4. Translate. Restate the problem in one sentence with all the important information.	The normal snowfall was twelve more than twice the amount last year
Translate into an equation.	$62 = 2s + 12$
Step 5. Solve the equation.	$62 = 2s + 12$
Subtract 12 from each side.	$62 - 12 = 2s + 12 - 12$
Simplify.	$50 = 2s$
Divide each side by two.	$\frac{50}{2} = \frac{2s}{2}$
Simplify.	$25 = s$
Step 6. Check: First, is our answer reasonable? Yes, having 25 inches of snow seems OK. The problem says the normal snowfall is twelve inches more than twice the number of last season. Twice 25 is 50 and 12 more than that is 62.	
Step 7. Answer the question.	The snowfall last season was 25 inches.

EXAMPLE 2.3.2

Guillermo bought textbooks and notebooks at the bookstore. The number of textbooks was three more than twice the number of notebooks. He bought seven textbooks. How many notebooks did he buy?

Answer

He bought two notebooks

EXAMPLE 2.3.3

Gerry worked Sudoku puzzles and crossword puzzles this week. The number of Sudoku puzzles he completed is eight more than twice the number of crossword puzzles. He completed 22 Sudoku puzzles. How many crossword puzzles did he do?

Answer

He did seven crosswords puzzles

We summarize an effective strategy for problem solving.

PROBLEM SOLVING STRATEGY FOR WORD PROBLEMS

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what you are looking for.
3. **Name** what you are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
5. **Solve** the equation using proper algebra techniques.
6. **Check** the answer in the problem to make sure it makes sense.
7. **Answer** the question with a complete sentence.

Solve Number Word Problems

We will now apply the problem solving strategy to "number word problems." Number word problems give some clues about one or more numbers and we use these clues to write an equation. Number word problems provide good practice for using the Problem Solving Strategy.

EXAMPLE 2.3.4

The sum of seven times a number and eight is thirty-six. Find the number.

Answer

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the number
Step 3. Name what you are looking for and choose a variable to represent it.	Let n = the number.
Step 4. Translate: Restate the problem as one sentence.	The sum of seven times a number and 8 is 36
Translate into an equation.	$7n + 8 = 36$
Step 5. Solve the equation.	$7n + 8 = 36$
Subtract eight from each side and simplify.	$7n = 28$
Divide each side by seven and simplify.	$n = 4$
Step 6. Check. Is the sum of seven times four plus eight equal to 36?	$7 \cdot 4 + 8 \stackrel{?}{=} 36$ (2.3.1) $28 + 8 \stackrel{?}{=} 36$ (2.3.2) $36 = 36\checkmark$ (2.3.3)
Step 7. Answer the question.	The number is 4.

Did you notice that we left out some of the steps as we solved this equation? If you're not yet ready to leave out these steps, write down as many as you need.

EXAMPLE 2.3.5

The sum of four times a number and two is fourteen. Find the number.

Answer

3

EXAMPLE 2.3.6

The sum of three times a number and seven is twenty-five. Find the number.

Answer

6

Some number word problems ask us to find two or more numbers. It may be tempting to name them all with different variables, but so far, we have only solved equations with one variable. In order to avoid using more than one variable, we will define the numbers in terms of the same variable. Be sure to read the problem carefully to discover how all the numbers relate to each other.

EXAMPLE 2.3.7

The sum of two numbers is negative fifteen. One number is nine less than the other. Find the numbers.

Answer

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	two numbers
Step 3. Name what you are looking for by choosing a variable to represent the first number. "One number is nine less than the other."	Let $n = 1^{st}$ number. $n-9 = 2^{nd}$ number
Step 4. Translate. Write as one sentence. Translate into an equation.	The sum of two numbers is negative fifteen. 1st number + 2nd number is negative fifteen $n + n-9 = -15$
Step 5. Solve the equation. Combine like terms. Add nine to each side and simplify. Simplify.	$n+n-9=-15$ $2n=-6$ $-3-9$ -12
Step 6. Check. Is -12 nine less than -3? $-3-9 \stackrel{?}{=} -12$ (2.3.4) $-12=-12\checkmark$ (2.3.5) Is their sum -15? $-3+(-12) \stackrel{?}{=} -15$ (2.3.6) $-15=-15\checkmark$ (2.3.7)	
Step 7. Answer the question.	The numbers are -3 and -12.

EXAMPLE 2.3.8

The sum of two numbers is negative twenty-three. One number is seven less than the other. Find the numbers.

Answer

$-15, -8$

EXAMPLE 2.3.9

The sum of two numbers is negative eighteen. One number is forty more than the other. Find the numbers.

Answer

$-29, 11$

Some number problems involve **consecutive integers**. Consecutive integers are integers that immediately follow each other. Examples of consecutive integers are:

$$\begin{array}{rrrr} 1, & 2, & 3, & 4 \\ -10, & -9, & -8, & -7 \\ 150, & 151, & 152, & 153 \end{array} \tag{2.3.8}$$

Notice that each number is one more than the number preceding it. Therefore, if we define the first integer as n, the next consecutive integer is $n+1$. The one after that is one more than $n+1$, so it is $n+1+1$, which is $n+2$.

$$\begin{array}{ll} n & 1^{\text{st}}\text{integer} \\ n+1 & 2^{\text{nd}}\text{consecutive integer} \\ n+2 & 3^{\text{rd}}\text{consecutive integer} \quad \text{etc.} \end{array} \tag{2.3.9}$$

We will use this notation to represent consecutive integers in the next example.

EXAMPLE 2.3.10

Find three consecutive integers whose sum is -54.

Answer

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	three consecutive integers
Step 3. Name each of the three numbers	Let $n = 1^{\text{st}}\text{integer}$. $n+1 = 2^{\text{nd}}\text{consecutive integer}$ $n+2 = 3^{\text{rd}}\text{consecutive integer}$
Step 4. Translate. Restate as one sentence. Translate into an equation.	The sum of the three integers is -54. $n+n+1+n+2=-54$
Step 5. Solve the equation. Combine like terms. Subtract three from each side. Divide each side by three.	$n+n+1+n+2=-54$ $3n+3=-54$ $3n=-57$ $n=-19$ 1st integer $n+1$ 2nd integer -18 $-19+2$ -17
Step 6. Check. $-19+(-18)+(-17)=-54$ (2.3.10) $-54=-54\checkmark$ (2.3.11)	
Step 7. Answer the question.	The three consecutive integers are $-17, -18$, and -19.

EXAMPLE 2.3.11

Find three consecutive integers whose sum is -96.

Answer

$-33, -32, -31$

EXAMPLE 2.3.12

Find three consecutive integers whose sum is -36.

Answer

$-13, -12, -11$

Now that we have worked with consecutive integers, we will expand our work to include **consecutive even integers** and **consecutive odd integers**. Consecutive even integers are even integers that immediately follow one another. Examples of

consecutive even integers are:

$$24, 26, 28 \tag{2.3.12}$$

$$-12, -10, -8 \tag{2.3.13}$$

Notice each integer is two more than the number preceding it. If we call the first one n, then the next one is $n+2$. The one after that would be $n+2+2$ or $n+4$.

$$\begin{array}{ll} n & 1^{\text{st}}\text{integer} \\ n+1 & 2^{\text{nd}}\text{consecutive integer} \\ n+2 & 3^{\text{rd}}\text{consecutive integer} \quad \text{etc.} \end{array} \tag{2.3.14}$$

Consecutive odd integers are odd integers that immediately follow one another. Consider the consecutive odd integers 63, 65, and 67.

$$63, 65, 67 \tag{2.3.15}$$

$$n, n+2, n+4 \tag{2.3.16}$$

$$\begin{array}{ll} n & 1^{\text{st}}\text{integer} \\ n+1 & 2^{\text{nd}}\text{consecutive integer} \\ n+2 & 3^{\text{rd}}\text{consecutive integer} \quad \text{etc.} \end{array} \tag{2.3.17}$$

Does it seem strange to have to add two (an even number) to get the next odd number? Do we get an odd number or an even number when we add 2 to 3? to 11? to 47?

Whether the problem asks for consecutive even numbers or odd numbers, you do not have to do anything different. The pattern is still the same—to get to the next odd or the next even integer, add two.

EXAMPLE 2.3.13

Find three consecutive even integers whose sum is 120.

Answer

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	three consecutive even integers
Step 3. Name each of the three numbers	Let $n = 1^{\text{st}}\text{consecutive even integer}$. $n+2 = 2^{\text{nd}}\text{consecutive even integer}$. $n+4 = 3^{\text{rd}}\text{consecutive even integer}$.
Step 4. Translate. Restate as one sentence. Translate into an equation.	The sum of the three even integers is 120 $n+n+2+n+4=120$
Step 5. Solve the equation. Combine like terms. Subtract three from each side. Divide each side by three.	$n+n+2+n+4=120$ $3n+6=120$ $3n=114$ $n=38$ $1^{\text{st}}\text{integer}$ $n+2$ $2^{\text{nd}}\text{integer}$ $38+2$ 40 $n+2$ $3^{\text{rd}}\text{integer}$ $38+4$ 42
Step 6. Check. $38+40+42 \stackrel{?}{=} 120$ $120 = 120\checkmark$	
Step 7. Answer the question.	The three consecutive integers are 38, 40, and 42.

EXAMPLE 2.3.14

Find three consecutive even integers whose sum is 102.

Answer

$32, 34, 36$

EXAMPLE 2.3.15

Find three consecutive even integers whose sum is -24.

Answer

$-10, -8, -6$

When a number problem is in a real life context, we still use the same strategies that we used for the previous examples.

EXAMPLE 2.3.16

A married couple together earns $110,000 a year. The wife earns $16,000 less than twice what her husband earns. What does the husband earn?

Answer

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	How much does the husband earn?
Step 3. Name each of the three numbers	Let $h =$ **the amount the husband earns.**
Step 4. Translate. Restate the problem in one sentence with all the important information. Translate into an equation.	$2h - 16,000 =$ **the amount the wife earns.** Together the husband and wife earn $110,000. $h + 2h - 16,000 = 110,000$
Step 5. Solve the equation. Combine like terms. Add 16,000 to both sides and simplify. Divide each side by three.	$h + 2h - 16,000 = 110,000$ $3h - 16,000 = 110,000$ $3h = 126,000$ $h = 42,000$ amount husband earns $2h - 16,000$ amount wife earns $2(42,000) - 16,000$ $84,000 - 16,000$ $68,000$
Step 6. Check. If the wife earns $68,000 and the husband earns $42,000, is that $110,000? Yes!	
Step 7. Answer the question.	The husband earns $42,000 a year.

EXAMPLE 2.3.17

According to the National Automobile Dealers Association, the average cost of a car in 2014 was $28,400. This was $1,600 less than six times the cost in 1975. What was the average cost of a car in 1975?

Answer

The average cost was $5,000.

EXAMPLE 2.3.18

US Census data shows that the median price of new home in the U.S. in November 2014 was $280,900. This was $10,700 more than 14 times the price in November 1964. What was the median price of a new home in November 1964?

Answer

The median price was $19,300.

Solve Percent Applications

There are several methods to solve percent equations. In algebra, it is easiest if we just translate English sentences into algebraic equations and then solve the equations. Be sure to change the given percent to a decimal before you use it in the equation.

EXAMPLE 2.3.19

Translate and solve:

a. What number is 45% of 84?
b. 8.5% of what amount is $4.76?
c. 168 is what percent of 112?

Answer

ⓐ

	What number is 45% of 84?
Translate into algebra. Let n = the number.	$n = 0.45 \cdot 84$
Multiply.	$n = 37.8$
	37.8 is 45% of 84.

ⓑ

	8.5% of what amount is $4.76?
Translate. Let n = the amount.	$0.085 \cdot n = 4.76$
Multiply.	$0.085n = 4.76$
Divide both sides by 0.085 and simplify.	$n = 56$
	8.5% of $56 is $4.76

ⓒ

We are asked to find percent, so we must have our result in percent form.	168 is what percent of 112?
Translate into algebra. Let p = the percent.	$168 = p \cdot 112$
Multiply.	$168 = 112p$
Divide both sides by 112 and simplify.	$1.5 = p$
Convert to percent.	$150\% = p$
	168 is 150% of 112.

EXAMPLE 2.3.20

Translate and solve:

a. What number is 45% of 80?
b. 7.5% of what amount is $1.95?

c. 110 is what percent of 88?

Answer

ⓐ 36 ⓑ $26 ⓒ 125%

EXAMPLE 2.3.21

Translate and solve:

a. What number is 55% of 60?
b. 8.5% of what amount is $3.06?
c. 126 is what percent of 72?

Answer

ⓐ 33 ⓑ $36 ⓐ 175%

Now that we have a problem solving strategy to refer to, and have practiced solving basic percent equations, we are ready to solve percent applications. Be sure to ask yourself if your final answer makes sense—since many of the applications we will solve involve everyday situations, you can rely on your own experience.

EXAMPLE 2.3.22

The label on Audrey's yogurt said that one serving provided 12 grams of protein, which is 24% of the recommended daily amount. What is the total recommended daily amount of protein?

Answer

What are you asked to find?	What total amount of protein is recommended?
Choose a variable to represent it.	Let a=a= total amount of protein.
Write a sentence that gives the information to find it.	12 g is 24% of the total amount
Translate into an equation.	$12 = 0.24 \cdot a$
Solve.	$50 = a$
Check: Does this make sense? Yes, 24% is about $\frac{1}{4}$ of the total and 12 is about $\frac{1}{4}$ of 50.	
Write a complete sentence to answer the question.	The amount of protein that is recommended is 50 g.

EXAMPLE 2.3.23

One serving of wheat square cereal has 7 grams of fiber, which is 28% of the recommended daily amount. What is the total recommended daily amount of fiber?

Answer

25 grams

EXAMPLE 2.3.24

One serving of rice cereal has 190 mg of sodium, which is 8% of the recommended daily amount. What is the total recommended daily amount of sodium?

Answer

2,375 mg

Remember to put the answer in the form requested. In the next example we are looking for the percent.

EXAMPLE 2.3.25

Veronica is planning to make muffins from a mix. The package says each muffin will be 240 calories and 60 calories will be from fat. What percent of the total calories is from fat?

Answer

What are you asked to find?	What percent of the total calories is fat?
Choose a variable to represent it.	Let p=percent of fat.
Write a sentence that gives the information to find it.	What percent of 240 is 60?
Translate the sentence into an equation.	$p \cdot 240 = 60$
Multiply.	$240p = 60$
Divide both sides by 240.	$p = 0.25$
Put in percent form.	$p = 25\%$
Check: does this make sense? Yes, **25%** is one-fourth; 60 is one-fourth of 240. So, **25%** makes sense.	
Write a complete sentence to answer the question.	Of the total calories in each muffin, **25%** is fat.

EXAMPLE 2.3.26

Mitzi received some gourmet brownies as a gift. The wrapper said each 28% brownie was 480 calories, and had 240 calories of fat. What percent of the total calories in each brownie comes from fat? Round the answer to the nearest whole percent.

Answer

50%

EXAMPLE 2.3.27

The mix Ricardo plans to use to make brownies says that each brownie will be 190 calories, and 76 calories are from fat. What percent of the total calories are from fat? Round the answer to the nearest whole percent.

Answer

40%

It is often important in many fields—business, sciences, pop culture—to talk about how much an amount has increased or decreased over a certain period of time. This increase or decrease is generally expressed as a percent and called the **percent change**.

To find the percent change, first we find the amount of change, by finding the difference of the new amount and the original amount. Then we find what percent the amount of change is of the original amount.

FIND PERCENT CHANGE

1. Find the amount of change.

$$\text{change} = \text{new amount} - \text{original amount} \tag{2.3.18}$$

2. Find what percent the amount of change is of the original amount.

   change is what percent of the original amount?

EXAMPLE 2.3.28

Recently, the California governor proposed raising community college fees from $36 a unit to $46 a unit. Find the percent change. (Round to the nearest tenth of a percent.)

Answer

Find the amount of change.	$46 - 36 = 10$
Find the percent.	Change is what percent of the original amount?
Let p=p= the percent.	10 is what percent of 36?
Translate to an equation.	$10 = p \cdot 36$
Simplify.	$10 = 36p$
Divide both sides by 36.	$0.278 \approx p$
Change to percent form; round to the nearest tenth	$27.8\% \approx p$
Write a complete sentence to answer the question.	The new fees are approximately a 27.8% increase over the old fees.
Remember to round the division to the nearest thousandth in order to round the percent to the nearest tenth.	

EXAMPLE 2.3.29

Find the percent change. (Round to the nearest tenth of a percent.) In 2011, the IRS increased the deductible mileage cost to 55.5 cents from 51 cents.

Answer

8.8%

EXAMPLE 2.3.30

Find the percent change. (Round to the nearest tenth of a percent.) In 1995, the standard bus fare in Chicago was $1.50. In 2008, the standard bus fare was 2.25.

Answer

50

Applications of discount and mark-up are very common in retail settings.

When you buy an item on sale, the original price has been discounted by some dollar amount. The **discount rate**, usually given as a percent, is used to determine the **amount of the discount**. To determine the amount of discount, we multiply the discount rate by the original price.

The price a retailer pays for an item is called the **original cost**. The retailer then adds a **mark-up** to the original cost to get the **list price**, the price he sells the item for. The mark-up is usually calculated as a percent of the original cost. To determine the amount of mark-up, multiply the mark-up rate by the original cost.

DISCOUNT

$$\text{amount of discount} = \text{discount rate} \cdot \text{original price} \quad (2.3.19)$$

$$\text{sale price} = \text{original amount–discount price} \quad (2.3.20)$$

The sale price should always be less than the original price.

MARK-UP

$$\text{amount of mark-up} = \text{mark-up rate} \cdot \text{original price} \quad (2.3.21)$$

$$\text{list price} = \text{original cost–mark-up} \quad (2.3.22)$$

The list price should always be more than the original cost.

EXAMPLE 2.3.31

Liam's art gallery bought a painting at an original cost of $750. Liam marked the price up 40%. Find

a. the amount of mark-up and
b. the list price of the painting.

Answer

ⓐ

Identify what you are asked to find, and choose a variable to represent it.	What is the amount of mark-up? Let m=m= the amount of mark-up.
Write a sentence that gives the information to find it.	The mark-up is 40% of the $750 original cost
Translate into an equation.	$m = 0.40 \times 750$
Solve the equation.	$m = 300$
Write a complete sentence.	The mark-up on the painting was $300.

ⓑ

Identify what you are asked to find, and choose a variable to represent it.	What is the list price? Let $p =$ the list price.
Write a sentence that gives the information to find it.	The list price is original cost plus the mark-up
Translate into an equation.	$p = 750 + 300$
Solve the equation.	$p = 1{,}050$
Check.	Is the list price more than the original cost? Is $1,050 more than $750? Yes.
Write a complete sentence.	The list price of the painting was $1,050.

EXAMPLE 2.3.32

Find ⓐ the amount of mark-up and ⓑ the list price: Jim's music store bought a guitar at original cost $1,200. Jim marked the price up 50%.

Answer

ⓐ $600 ⓑ $1,800

EXAMPLE 2.3.33

Find ⓐ the amount of mark-up and ⓑ the list price: The Auto Resale Store bought Pablo's Toyota for $8,500. They marked the price up 35%.

Answer

ⓐ $2,975 ⓑ $11,475

Solve Simple Interest Applications

Interest is a part of our daily lives. From the interest earned on our savings to the interest we pay on a car loan or credit card debt, we all have some experience with interest in our lives.

The amount of money you initially deposit into a bank is called the **principal**, P, and the bank pays you interest, I. When you take out a loan, you pay interest on the amount you borrow, also called the principal.

In either case, the interest is computed as a certain percent of the principal, called the **rate of interest**, r. The rate of interest is usually expressed as a percent per year, and is calculated by using the decimal equivalent of the percent. The variable t, (for time) represents the number of years the money is saved or borrowed.

Interest is calculated as simple interest or compound interest. Here we will use simple interest.

SIMPLE INTEREST

If an amount of money, P, called the principal, is invested or borrowed for a period of t years at an annual interest rate r, the amount of interest, I, earned or paid is

Erroneous nesting of equation structures

Interest earned or paid according to this formula is called **simple interest**.

The formula we use to calculate interest is $I = Prt$. To use the formula we substitute in the values for variables that are given, and then solve for the unknown variable. It may be helpful to organize the information in a chart.

EXAMPLE 2.3.34

Areli invested a principal of $950 in her bank account that earned simple interest at an interest rate of 3%. How much interest did she earn in five years?

Answer

$I = ?$
$P = \$950$
$r = 3\%$
$t = 5$ years

Identify what you are asked to find, and choose a variable to represent it.	What is the simple interest? Let $I =$ interest.
Write the formula.	$I = Prt$
Substitute in the given information.	$I = (950)(0.03)(5)$
Simplify.	$I = 142.5$
Check.	
Is $142.50 a reasonable amount of interest on $ 950?	
Yes.	
Write a complete sentence.	The interest is $142.50.

EXAMPLE 2.3.35

Nathaly deposited $12,500 in her bank account where it will earn 4% simple interest. How much interest will Nathaly earn in five years?

Answer

He will earn $2,500.

EXAMPLE 2.3.36

Susana invested a principal of $36,000 in her bank account that earned simple interest at an interest rate of 6.5%.6.5%. How much interest did she earn in three years?

Answer

She earned $7,020.

There may be times when we know the amount of interest earned on a given principal over a certain length of time, but we do not know the rate.

EXAMPLE 2.3.37

Hang borrowed $7,500 from her parents to pay her tuition. In five years, she paid them $1,500 interest in addition to the $7,500 she borrowed. What was the rate of simple interest?

Answer

$I = \$1500$
$P = \$7500$
$r = ?$
$t = 5$ years

Identify what you are asked to find, and choose a variable to represent it.	What is the rate of simple interest? Let r = rate of interest.
Write the formula.	$I = Prt$
Substitute in the given information.	$1{,}500 = (7{,}500)r(5)$
Multiply.	$1{,}500 = 37{,}500r$
Divide.	$0.04 = r$
Change to percent form.	$4\% = r$
Check.	$I = Prt$ $1{,}500 \stackrel{?}{=} (7{,}500)(0.04)(5)$ $1{,}500 = 1{,}500$ ✓
Write a complete sentence.	The rate of interest was 4%.

chooseWhat is the rate of simple interest?a variable to represent it.Write the formula.Substitute in the given information.Multiply.Divide.Change to percent form.Letr=rate of interest.I=Prt1,500=(7,500)r(5)1,500=37,500r0.04=r4%=rCheck.I=Prt1,500=?(7,500)(0.04)(5)1,500=1,500✓Write a complete sentence.The rate of interest was 4%.

EXAMPLE 2.3.38

Jim lent his sister $5,000 to help her buy a house. In three years, she paid him the $5,000, plus $900 interest. What was the rate of simple interest?

Answer

The rate of simple interest was 6%.

EXAMPLE 2.3.39

Loren lent his brother $3,000 to help him buy a car. In four years, his brother paid him back the $3,000 plus $660 in interest. What was the rate of simple interest?

Answer

The rate of simple interest was 5.5%.

In the next example, we are asked to find the principal—the amount borrowed.

EXAMPLE 2.3.40

Sean's new car loan statement said he would pay $4,866,25 in interest from a simple interest rate of 8.5% over five years. How much did he borrow to buy his new car?

Answer

$I = 4,866.25$
$P = ?$
$r = 8.5\%$
$t = 5$ years

Identify what you are asked to find, and choose a variable to represent it.	**What is the amount borrowed (the principal)?** Let P = principal borrowed.
Write the formula.	$I = Prt$
Substitute in the given information.	$4,866.25 = P(0.085)(5)$
Multiply.	$4,866.25 = 0.425P$
Divide.	$11,450 = P$
Check.	$I = Prt$
	$4,866.25 \stackrel{?}{=} (11,450)(0.085)(5)$
	$4,866.25 = 4,866.25$ ✓
Write a complete sentence.	**The principal was $11,450.**

Prt4,866.25=P(0.085)(5)4,866.25=0.425P11,450=PCheck.I=Prt4,866.25=?(11,450)(0.085)(5)4,866.25=4,866.25 ✓ Write a complete sentence.The principal was $11,450.

EXAMPLE 2.3.41

Eduardo noticed that his new car loan papers stated that with a 7.5% simple interest rate, he would pay $6,596.25 in interest over five years. How much did he borrow to pay for his car?

Answer

He paid $17,590.

EXAMPLE 2.3.42

In five years, Gloria's bank account earned $2,400 interest at 5% simple interest. How much had she deposited in the account?

Answer

She deposited $9,600.

Access this online resource for additional instruction and practice with using a problem solving strategy.

- Begining Arithmetic Problems

Key Concepts

- **How To Use a Problem Solving Strategy for Word Problems**
  1. **Read** the problem. Make sure all the words and ideas are understood.
  2. **Identify** what you are looking for.
  3. **Name** what you are looking for. Choose a variable to represent that quantity.
  4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
  5. **Solve** the equation using proper algebra techniques.
  6. **Check** the answer in the problem to make sure it makes sense.
  7. **Answer** the question with a complete sentence.
- **How To Find Percent Change**
  1. Find the amount of change

     change = new amount − original amount

  2. Find what percent the amount of change is of the original amount.

     change is what percent of the original amount?

- **Discount**

$$\text{amount of discount} = \text{discount rate} \cdot \text{original price} \tag{2.3.23}$$
$$\text{sale price} = \text{original amount–discount price} \tag{2.3.24}$$

- **Mark-up**

$$\text{amount of mark-up} = \text{mark-up rate} \cdot \text{original price} \tag{2.3.25}$$
$$\text{list price} = \text{original cost–mark-up} \tag{2.3.26}$$

- **Simple Interest**

  If an amount of money, P, called the principal, is invested or borrowed for a period of t years at an annual interest rate r, the amount of interest, I, earned or paid is:

$$I = Prt \quad \begin{array}{l} I = interest \\ \text{where } P = principal \\ r = rate \\ t = time \end{array} \tag{2.3.27}$$

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2.3E: Exercises

Practice Makes Perfect

Use a Problem Solving Strategy for Word Problems

List five positive thoughts you can say to yourself that will help you approach word problems with a positive attitude. You may want to copy them on a sheet of paper and put it in the front of your notebook, where you can read them often.

Answer

Answers will vary.

List five negative thoughts that you have said to yourself in the past that will hinder your progress on word problems. You may want to write each one on a small piece of paper and rip it up to symbolically destroy the negative thoughts.

In the following exercises, solve using the problem solving strategy for word problems. Remember to write a complete sentence to answer each question.

There are 16 girls in a school club. The number of girls is four more than twice the number of boys. Find the number of boys.

Answer

six boys

There are 18 Cub Scouts in Troop 645. The number of scouts is three more than five times the number of adult leaders. Find the number of adult leaders.

Huong is organizing paperback and hardback books for her club's used book sale. The number of paperbacks is 12 less than three times the number of hardbacks. Huong had 162 paperbacks. How many hardback books were there?

Answer

58 hardback books

Jeff is lining up children's and adult bicycles at the bike shop where he works. The number of children's bicycles is nine less than three times the number of adult bicycles. There are 42 adult bicycles. How many children's bicycles are there?

Solve Number Word Problems

In the following exercises, solve each number word problem.

The difference of a number and 12 is three. Find the number.

Answer

15

The difference of a number and eight is four. Find the number.

The sum of three times a number and eight is 23. Find the number.

Answer

5

The sum of twice a number and six is 14. Find the number.

The difference of twice a number and seven is 17. Find the number.

Answer

12

The difference of four times a number and seven is 21. Find the number.

Three times the sum of a number and nine is 12. Find the number.

Answer

-5

Six times the sum of a number and eight is 30. Find the number.

One number is six more than the other. Their sum is 42. Find the numbers.

Answer

18, 24

One number is five more than the other. Their sum is 33. Find the numbers.

The sum of two numbers is 20. One number is four less than the other. Find the numbers.

Answer

8, 12

The sum of two numbers is 27. One number is seven less than the other. Find the numbers.

One number is 14 less than another. If their sum is increased by seven, the result is 85. Find the numbers.

Answer

32, 46

One number is 11 less than another. If their sum is increased by eight, the result is 71. Find the numbers.

The sum of two numbers is 14. One number is two less than three times the other. Find the numbers.

Answer

4, 10

The sum of two numbers is zero. One number is nine less than twice the other. Find the numbers.

The sum of two consecutive integers is 77. Find the integers.

Answer

$38, 39$

The sum of two consecutive integers is 89. Find the integers.

The sum of three consecutive integers is 78. Find the integers.

Answer

$25, 26, 27$

The sum of three consecutive integers is 60. Find the integers.

Find three consecutive integers whose sum is -36.

Answer

$-11, -12, -13$

Find three consecutive integers whose sum is -3.

Find three consecutive even integers whose sum is 258.

Answer

$84, 86, 88$

Find three consecutive even integers whose sum is 222.

Find three consecutive odd integers whose sum is -213.

Answer

$-69, -71, -73$

Find three consecutive odd integers whose sum is -267.

Philip pays $1,620 in rent every month. This amount is $120 more than twice what his brother Paul pays for rent. How much does Paul pay for rent?

Answer

$750

Marc just bought an SUV for $54,000. This is $7,400 less than twice what his wife paid for her car last year. How much did his wife pay for her car?

Laurie has $46,000 invested in stocks and bonds. The amount invested in stocks is $8,000 less than three times the amount invested in bonds. How much does Laurie have invested in bonds?

Answer

$13,500

Erica earned a total of $50,450 last year from her two jobs. The amount she earned from her job at the store was $1,250 more than three times the amount she earned from her job at the college. How much did she earn from her job at the college?

Solve Percent Applications

In the following exercises, translate and solve.

ⓐ What number is 45% of 120? ⓑ 81 is 75% of what number? ⓐ What percent of 260 is 78?

Answer

ⓐ 54 ⓑ 108 ⓐ 30%

ⓐ What number is 65% of 100? ⓑ 93 is 75% of what number? ⓐ What percent of 215 is 86?

ⓐ 250% of 65 is what number? ⓑ 8.2% of what amount is $2.87? ⓐ 30 is what percent of 20?

Answer

ⓐ162.5 ⓑ $35 ⓐ 150%

ⓐ 150% of 90 is what number? ⓑ 6.4% of what amount is $2.88? ⓐ 50 is what percent of 40?

In the following exercises, solve.

Geneva treated her parents to dinner at their favorite restaurant. The bill was $74.25. Geneva wants to leave 16% of the total bill as a tip. How much should the tip be?

Answer

$11.88

When Hiro and his co-workers had lunch at a restaurant near their work, the bill was $90.50. They want to leave 18% of the total bill as a tip. How much should the tip be?

One serving of oatmeal has 8 grams of fiber, which is 33% of the recommended daily amount. What is the total recommended daily amount of fiber?

Answer

24.2 g

One serving of trail mix has 67 grams of carbohydrates, which is 22% of the recommended daily amount. What is the total recommended daily amount of carbohydrates?

A bacon cheeseburger at a popular fast food restaurant contains 2070 milligrams (mg) of sodium, which is 86% of the recommended daily amount. What is the total recommended daily amount of sodium?

Answer

2407 mg

A grilled chicken salad at a popular fast food restaurant contains 650 milligrams (mg) of sodium, which is 27% of the recommended daily amount. What is the total recommended daily amount of sodium?

The nutrition fact sheet at a fast food restaurant says the fish sandwich has 380 calories, and 171 calories are from fat. What percent of the total calories is from fat?

Answer

45%

The nutrition fact sheet at a fast food restaurant says a small portion of chicken nuggets has 190 calories, and 114 calories are from fat. What percent of the total calories is from fat?

Emma gets paid $3,000 per month. She pays $750 a month for rent. What percent of her monthly pay goes to rent?

Answer

25%

Dimple gets paid $3,200 per month. She pays $960 a month for rent. What percent of her monthly pay goes to rent?

In the following exercises, solve.

Tamanika received a raise in her hourly pay, from $15.50 to $17.36. Find the percent change.

Answer

12%

Ayodele received a raise in her hourly pay, from $24.50 to $25.48. Find the percent change.

Annual student fees at the University of California rose from about $4,000 in 2000 to about $12,000 in 2010. Find the percent change.

Answer

200%

The price of a share of one stock rose from $12.50 to $50. Find the percent change.

A grocery store reduced the price of a loaf of bread from $2.80 to $2.73. Find the percent change.

Answer

−2.5%

The price of a share of one stock fell from $8.75 to $8.54. Find the percent change.

Hernando’s salary was $49,500 last year. This year his salary was cut to $44,055. Find the percent change.

Answer

−11%

In ten years, the population of Detroit fell from 950,000 to about 712,500. Find the percent change.

In the following exercises, find ⓐ the amount of discount and ⓑ the sale price.

Janelle bought a beach chair on sale at 60% off. The original price was $44.95.

Answer

ⓐ $26.97 ⓑ $17.98

Errol bought a skateboard helmet on sale at 40% off. The original price was $49.95.

In the following exercises, find ⓐ the amount of discount and ⓑ the discount rate (Round to the nearest tenth of a percent if needed.)

Larry and Donna bought a sofa at the sale price of $1,344. The original price of the sofa was $1,920.

Answer

ⓐ $576 ⓑ 30%

Hiroshi bought a lawnmower at the sale price of $240. The original price of the lawnmower is $300.

In the following exercises, find ⓐ the amount of the mark-up and ⓑ the list price.

Daria bought a bracelet at original cost $16 to sell in her handicraft store. She marked the price up 45%. What was the list price of the bracelet?

Answer

ⓐ $7.20 ⓑ $23.20

Regina bought a handmade quilt at original cost $120 to sell in her quilt store. She marked the price up 55%. What was the list price of the quilt?

Tom paid $0.60 a pound for tomatoes to sell at his produce store. He added a 33% mark-up. What price did he charge his customers for the tomatoes?

Answer

ⓐ $0.20 ⓑ $0.80

Flora paid her supplier $0.74 a stem for roses to sell at her flower shop. She added an 85% mark-up. What price did she charge her customers for the roses?

Solve Simple Interest Applications

In the following exercises, solve.

Casey deposited $1,450 in a bank account that earned simple interest at an interest rate of 4%. How much interest was earned in two years?

Answer

$116

Terrence deposited $5,720 in a bank account that earned simple interest at an interest rate of 6%. How much interest was earned in four years?

Robin deposited $31,000 in a bank account that earned simple interest at an interest rate of 5.2%. How much interest was earned in three years?

Answer

$4836

Carleen deposited $16,400 in a bank account that earned simple interest at an interest rate of 3.9% How much interest was earned in eight years?

Hilaria borrowed $8,000 from her grandfather to pay for college. Five years later, she paid him back the $8,000, plus $1,200 interest. What was the rate of simple interest?

Answer

3%

Kenneth lent his niece $1,200 to buy a computer. Two years later, she paid him back the $1,200, plus $96 interest. What was the rate of simple interest?

Lebron lent his daughter $20,000 to help her buy a condominium. When she sold the condominium four years later, she paid him the $20,000, plus $3,000 interest. What was the rate of simple interest?

Answer

3.75%

Pablo borrowed $50,000 to start a business. Three years later, he repaid the $50,000, plus $9,375 interest. What was the rate of simple interest?

In 10 years, a bank account that paid 5.25% simple interest earned $18,375 interest. What was the principal of the account?

Answer

$35,000

In 25 years, a bond that paid 4.75% simple interest earned $2,375 interest. What was the principal of the bond?

Joshua’s computer loan statement said he would pay $1,244.34 in simple interest for a three-year loan at 12.4%. How much did Joshua borrow to buy the computer?

Answer

$3345

Margaret's car loan statement said she would pay $7,683.20 in simple interest for a five-year loan at 9.8%. How much did Margaret borrow to buy the car?

Everyday Math

Tipping At the campus coffee cart, a medium coffee costs $1.65. MaryAnne brings $2.00 with her when she buys a cup of coffee and leaves the change as a tip. What percent tip does she leave?

Answer

17.5%

Tipping Four friends went out to lunch and the bill came to $53.75 They decided to add enough tip to make a total of $64, so that they could easily split the bill evenly among themselves. What percent tip did they leave?

Writing Exercises

What has been your past experience solving word problems? Where do you see yourself moving forward?

Answer

Answers will vary.

Without solving the problem "44 is 80% of what number" think about what the solution might be. Should it be a number that is greater than 44 or less than 44? Explain your reasoning.

After returning from vacation, Alex said he should have packed 50% fewer shorts and 200% more shirts. Explain what Alex meant.

Answer

Answers will vary.

Because of road construction in one city, commuters were advised to plan that their Monday morning commute would take 150% of their usual commuting time. Explain what this means.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objective of this section.

I can...	Confidently	With some help	No-I don't get it!
use a problem-solving strategy for word problems.			
solve number problems.			
solve percent applications.			
solve simple interest applications.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

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2.4: Solve a Formula for a Specific Variable

Learning Objectives

By the end of this section, you will be able to:

- Solve a formula for a specific variable
- Use formulas to solve geometry applications

Before you get started, take this readiness quiz.

1. Evaluate $2(x+3)$ when $x=5$.
   If you missed this problem, review [link].
2. The length of a rectangle is three less than the width. Let w represent the width. Write an expression for the length of the rectangle.
   If you missed this problem, review [link].
3. Evaluate $\frac{1}{2}bh$ when $b=14$ and $h=9$.
   If you missed this problem, review [link].

Solve a Formula for a Specific Variable

We have all probably worked with some geometric formulas in our study of mathematics. Formulas are used in so many fields, it is important to recognize formulas and be able to manipulate them easily.

It is often helpful to solve a formula for a specific variable. If you need to put a formula in a spreadsheet, it is not unusual to have to solve it for a specific variable first. We isolate that variable on one side of the equals sign with a coefficient of one and all other variables and constants are on the other side of the equal sign.

Geometric formulas often need to be solved for another variable, too. The formula $V=\frac{1}{3}\pi r^2 h$ is used to find the **volume** of a right circular cone when given the radius of the base and height. In the next example, we will solve this formula for the height.

EXAMPLE 2.4.1

Solve the formula $V=\frac{1}{3}\pi r^2 h$ for h.

Answer

Write the formula. $V = \frac{1}{3}\pi r^2 h$

Remove the fraction on the right. $3 \cdot V = 3 \cdot \frac{1}{3}\pi r^2 h$

Simplify on the right. $3V = \pi r^2 h$

Simplify.

f
y
.

D
i
v
i
d
e
b
o
t
$\frac{3V}{\pi r^2} = h$
s
i
d
e
s
b
y
πr^2
.

We could now use this formula to find the height of a right circular cone when we know the volume and the radius of the base, by using the formula $h = \frac{3V}{\pi r^2}$.

EXAMPLE 2.4.2:

Use the formula $A = \frac{1}{2}bh$ to solve for b.

Answer

$b = \frac{2A}{h}$

EXAMPLE 2.4.3:

Use the formula $A = \frac{1}{2}bh$ to solve for h.

Answer

$h = \frac{2A}{b}$

In the sciences, we often need to change **temperature** from Fahrenheit to Celsius or vice versa. If you travel in a foreign country, you may want to change the Celsius temperature to the more familiar Fahrenheit temperature.

EXAMPLE 2.4.4

Solve the formula $C = \frac{5}{9}(F - 32)$ for F.

Answer

Write the formula.			$C=\frac{5}{9}(F-32)$
Remove the fraction on the right.			$\frac{9}{5}C=\frac{9}{5}\cdot\frac{5}{9}(F-32)$
Simplify.			$\frac{9}{5}C=(F-32)$
Add 32 to both sides.			$\frac{9}{5}C+32=F$

We can now use the formula $F=\frac{9}{5}C+32$ to find the Fahrenheit temperature when we know the Celsius temperature.

EXAMPLE 2.4.5:

Solve the formula $F=\frac{9}{5}C+32$ for C.

Answer

$C=\frac{5}{9}(F-32)$

EXAMPLE 2.4.6:

Solve the formula $A=\frac{1}{2}h(b+B)$ for b.

Answer

$b=\frac{2A-Bh}{h}$

The next example uses the formula for the **surface area** of a right cylinder.

EXAMPLE 2.4.7

Solve the formula $S=2\pi r^2+2\pi rh$ for 2πr^2+2πrh\) for h.

Answer

Write the formula.	$S=2\pi r^2+2\pi r h$
Isolate the h term by subtracting $2\pi r^2$ from each side.2πr^2\) from each side.	$S-2\pi r^2=2\pi r^2-2\pi r^2+2\pi r h$
Simplify.	$S-2\pi r^2=2\pi r h$
Solve for h by dividing both sides by $2\pi r$.2πr.\)	$\frac{S-2\pi r^2}{2\pi r}=\frac{2\pi r h}{2\pi r}$
Simplify.	$\frac{S-2\pi r^2}{2\pi r}=h$

EXAMPLE 2.4.8:

Solve the formula $A=P+Prt$ for t.

Answer

$t=\frac{A-P}{Pr}$

EXAMPLE 2.4.9:

Solve the formula $A=P+Prt$ for r.

Answer

$r = \frac{A-P}{Pt}$

Sometimes we might be given an equation that is solved for y and need to solve it for x, or vice versa. In the following example, we're given an equation with both x and y on the same side and we'll solve it for y.

EXAMPLE 2.4.10

Solve the formula $8x + 7y = 15$ for y.

Answer

We will isolate y on one side of the equation.	$8x + 7y = 15$
Subtract 6x6x from both sides to isolate the term with y.	$8x - 8x + 7y = 15 - 8x$
Simplify.	$7y = 15 - 8x$
Divide both sides by 7 to make the coefficient of y one.	$\frac{7y}{7} = \frac{15-8x}{7}$
Simplify.	$y = \frac{15-8x}{7}$

EXAMPLE 2.4.11

Solve the formula $4x + 7y = 9$ for y.

Answer

$y = \frac{9-4x}{7}$

EXAMPLE 2.4.12

Solve the formula $5x + 8y = 1$ for y.

Answer

$y = \frac{1-5x}{8}$

Use Formulas to Solve Geometry Applications

In this objective we will use some common geometry formulas. We will adapt our problem solving strategy so that we can solve geometry applications. The geometry formula will name the variables and give us the equation to solve.

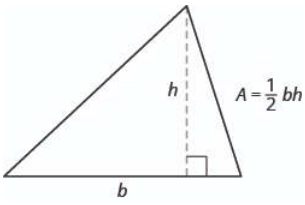
In addition, since these applications will all involve shapes of some sort, most people find it helpful to draw a figure and label it with the given information. We will include this in the first step of the problem solving strategy for geometry applications.

SOLVE GEOMETRY APPLICATIONS.

1. **Read** the problem and make sure all the words and ideas are understood.
2. **Identify** what you are looking for.
3. **Name** what we are looking for by choosing a variable to represent it. Draw the figure and label it with the given information.
4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

When we solve geometry applications, we often have to use some of the properties of the figures. We will review those properties as needed.

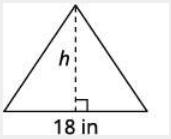
The next example involves the area of a triangle. The **area** of a triangle is one-half the base times the height. We can write this as $A = \frac{1}{2}bh$, where b = length of the base and h = height.



EXAMPLE 2.4.13

The area of a triangular painting is 126 square inches. The base is 18 inches. What is the height?

Answer

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	height of a triangle
Step 3. Name.	
Choose a variable to represent it.	Let $h =$ the height.
Draw the figure and label it with the given information.	Area = 126 sq. in.
	
Step 4. Translate.	
Write the appropriate formula.	$A = \frac{1}{2}bh$
Substitute in the given information.	$126 = \frac{1}{2} \cdot 18 \cdot h$
Step 5. Solve the equation.	$126 = 9h$
Divide both sides by 9.	$14 = h$
Step 6. Check. $A = \frac{1}{2}bh$ (2.4.1) $126 \stackrel{?}{=} 12 \cdot 18 \cdot 14$ (2.4.2) $126 = 126\checkmark$ (2.4.3)	
Step 7. Answer the question.	The height of the triangle is 14 inches.

EXAMPLE 2.4.14:

The area of a triangular church window is 90 square meters. The base of the window is 15 meters. What is the window's height?

Answer

The window's height is 12 meters.

EXAMPLE 2.4.15:

A triangular tent door has area 15 square feet. The height is five feet. What is the length of the base?

Answer

The length of the base is 6 feet.

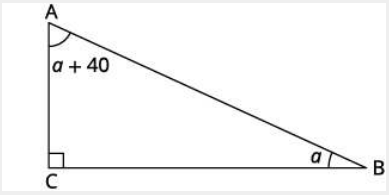
In the next example, we will work with a **right triangle**. To solve for the measure of each angle, we need to use two triangle properties. In any triangle, the sum of the measures of the angles is $180°$. We can write this as a formula: $(m\angle A+m\angle B+m\angle C=180\backslash)$. Also, since the triangle is a right triangle, we remember that a right triangle has one $90°$ angle.

Here, we will have to define one angle in terms of another. We will wait to draw the figure until we write expressions for all the angles we are looking for.

EXAMPLE 2.4.16:

The measure of one angle of a right triangle is 40 degrees more than the measure of the smallest angle. Find the measures of all three angles.

Answer

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the measures of all three angles
Step 3. Name. Choose a variable to represent it.	Let a = 1st angle. (2.4.4) $a+40$ = 2nd angle (2.4.5) 90 = 3rd angle (the right angle) (2.4.6)
Draw the figure and label it with the given information.	
Step 4. Translate.	
Write the appropriate formula.	$m\angle A+m\angle B+m\angle C=180$
Substitute into the formula.	$a+(a+40)+90=180$
Step 5. Solve the equation.	$2a+130=180$ $2a=50$ $a=25$ first angle $a+40$ second angle $25+40$ 65 90 third angle
Step 6. Check. $25+65+90\stackrel{?}{=}180$ (2.4.7) $180=180\checkmark$ (2.4.8)	
Step 7. Answer the question.	The three angles measure $25°, 65°$, and $90°$.

EXAMPLE 2.4.17:

The measure of one angle of a right triangle is 50 more than the measure of the smallest angle. Find the measures of all three angles.

Answer

The measures of the angles are 20°, 70°, and 90°.

EXAMPLE 2.4.18:

The measure of one angle of a right triangle is 30 more than the measure of the smallest angle. Find the measures of all three angles.

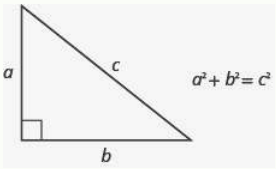
Answer

The measures of the angles are 30°, 60°, and 90°.

The next example uses another important geometry formula. The **Pythagorean Theorem** tells how the lengths of the three sides of a right triangle relate to each other. Writing the formula in every exercise and saying it aloud as you write it may help you memorize the Pythagorean Theorem.

THE PYTHAGOREAN THEOREM

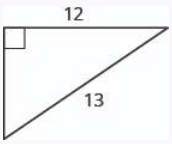
In any right triangle, where a and b are the lengths of the legs, and c is the length of the hypotenuse, the sum of the squares of the lengths of the two legs equals the square of the length of the hypotenuse.



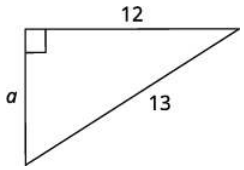
We will use the Pythagorean Theorem in the next example.

EXAMPLE 2.4.19:

Use the Pythagorean Theorem to find the length of the other leg in

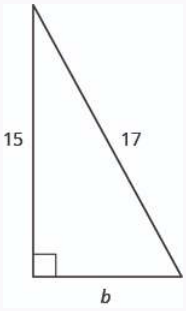


Answer

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the length of the leg of the triangle
Step 3. Name.	
Choose a variable to represent it.	Let a = the leg of the triangle.
Label side a.	
Step 4. Translate.	
Write the appropriate formula. Substitute.	$a^2+b^2=c^2$ (2.4.9) $a^2+12^2=13^2$ (2.4.10)
Step 5. Solve the equation. Isolate the variable term. Use the definition of square root. Simplify.	$a^2+144=169$ (2.4.11) $a^2=25$ (2.4.12) $a=\sqrt{25}$ (2.4.13) $a=5$ (2.4.14)
Step 6. Check. $5^2+12^2\stackrel{?}{=}13^2$ $25+144\stackrel{?}{=}169$ $169=169$ ✓	
Step 7. Answer the question.	The length of the leg is 5.

EXAMPLE 2.4.20:

Use the Pythagorean Theorem to find the length of the leg in the figure.

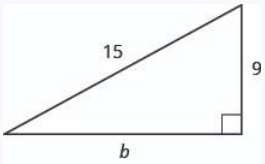


Answer

The length of the leg is 8.

EXAMPLE 2.4.21:

Use the Pythagorean Theorem to find the length of the leg in the figure.



Answer

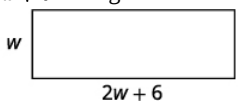
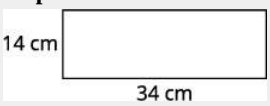
The length of the leg is 12.

The next example is about the **perimeter** of a rectangle. Since the perimeter is just the distance around the rectangle, we find the sum of the lengths of its four sides—the sum of two lengths and two widths. We can write is as $P = 2L + 2W$ where L is the length and W is the width. To solve the example, we will need to define the length in terms of the width.

EXAMPLE 2.4.22:

The length of a rectangle is six centimeters more than twice the width. The perimeter is 96 centimeters. Find the length and width.

Answer

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the length and the width
Step 3. Name. Choose a variable to represent the width. The length is six more than twice the width.	Let $w =$ width. $2w + 6 =$ length  $P = 96$ cm
Step 4. Translate.	
Write the appropriate formula.	$P = 2L + 2W$
Substitute in the given information.	$96 = 2(2w + 6) + 2w$
Step 5. Solve the equation.	$96 = 4w + 12 + 2w$ $96 = 6w + 12$ $84 = 6w$ $14 = w$ (width) $2w + 6$ (length) $2(14) + 6$ 34 The length is 34 cm.
Step 6. Check.  $P = 2L + 2W$ (2.4.15) $96 \stackrel{?}{=} 2 \cdot 34 + 2 \cdot 14$ (2.4.16) $96 = 96$✓ (2.4.17) 96✓ \end{align}\)	
Step 7. Answer the question.	The length is 34 cm and the width is 14 cm

EXAMPLE 2.4.23:

The length of a rectangle is seven more than twice the width. The perimeter is 110 inches. Find the length and width.

Answer

The length is 16 inches and the width is 39 inches.

EXAMPLE 2.4.24:

The width of a rectangle is eight yards less than twice the length. The perimeter is 86 yards. Find the length and width.

Answer

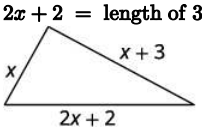
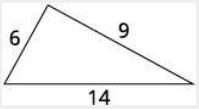
The length is 17 yards and the width is 26 yards.

The next example is about the **perimeter** of a triangle. Since the perimeter is just the distance around the triangle, we find the sum of the lengths of its three sides. We can write this as $P = a + b + c$, where a, b, and c are the lengths of the sides.

EXAMPLE 2.4.25:

One side of a triangle is three inches more than the first side. The third side is two inches more than twice the first. The perimeter is 29 inches. Find the length of the three sides of the triangle.

Answer

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the lengths of the three sides of a triangle
Step 3. Name. Choose a variable to represent the length of the first side.	Let $x =$ length of 1st side. (2.4.18) $x + 3 =$ length of 2nd side (2.4.19) $2x + 2 =$ length of 3rd side (2.4.20)  Perimeter is 29 in.
Step 4. Translate. Write the appropriate formula. Substitute in the given information.	$P = a + b + c$ $29 = x + (x + 3) + (2x + 2)$
Step 5. Solve the equation.	$29 = 4x + 5$ $24 = 4x$ $6 = x$ length of first side $x + 3$ length of second side $6 + 3$ 9 $2x + 2$ length of second side $2 \cdot 6 + 2$ 14
Step 6. Check.  $29 \stackrel{?}{=} 6 + 9 + 14$ (2.4.21) $29 = 29\checkmark$ (2.4.22)	
Step 7. Answer the question.	The lengths of the sides of the triangle are 6, 9, and 14 inches.

EXAMPLE 2.4.26:

One side of a triangle is seven inches more than the first side. The third side is four inches less than three times the first. The perimeter is 28 inches. Find the length of the three sides of the triangle.

Answer

The lengths of the sides of the triangle are 5, 11 and 12 inches.

EXAMPLE 2.4.27:

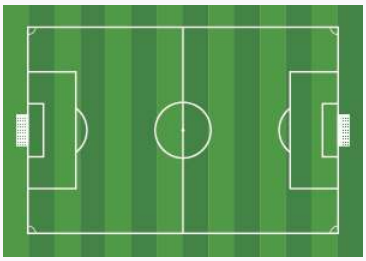
One side of a triangle is three feet less than the first side. The third side is five feet less than twice the first. The perimeter is 20 feet. Find the length of the three sides of the triangle.

Answer

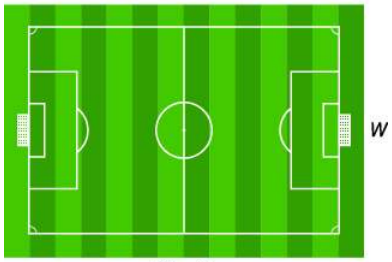
The lengths of the sides of the triangle are 4, 7 and 9 feet.

EXAMPLE 2.4.28:

The perimeter of a rectangular soccer field is 360 feet. The length is 40 feet more than the width. Find the length and width.



Answer

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the length and width of the soccer field
Step 3. Name. Choose a variable to represent it. The length is 40 feet more than the width. Draw the figure and label it with the given information.	Let w = width. $w + 40 =$ length Perimeter = 360 feet 
Step 4. Translate. Write the appropriate formula and substitute.	$P = 2L + 2W$ $360 = 2(w + 40) + 2w$
Step 5. Solve the equation.	
Step 6. Check. Extra close brace or missing open brace	$360 = 2w + 80 + 2w$ $360 = 4w + 80$ $280 = 4w$ $70 = w$ the width of the field $w + 40$ the length of the field $70 + 40$ 110
Step 7. Answer the question.	The length of the soccer field is 110 feet and the width is 70 feet.

EXAMPLE 2.4.29:

The perimeter of a rectangular swimming pool is 200 feet. The length is 40 feet more than the width. Find the length and width.

Answer

The length of the swimming pool is 70 feet and the width is 30 feet.

EXAMPLE 2.4.30:

The length of a rectangular garden is 30 yards more than the width. The perimeter is 300 yards. Find the length and width.

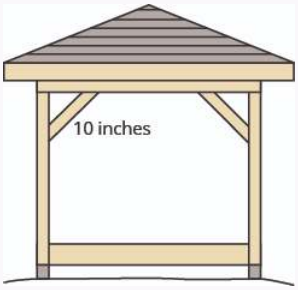
Answer

The length of the garden is 90 yards and the width is 60 yards.

Applications of these geometric properties can be found in many everyday situations as shown in the next example.

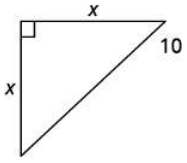
EXAMPLE 2.4.31:

Kelvin is building a gazebo and wants to brace each corner by placing a 10” piece of wood diagonally as shown.



How far from the corner should he fasten the wood if wants the distances from the corner to be equal? Approximate to the nearest tenth of an inch.

Answer

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	the distance from the corner that the bracket should be attached
Step 3. Name. Choose a variable to represent it. Draw the figure and label it with the given information.	Let $x =$ the distance from the corner. 
Step 4. Translate. Write the appropriate formula and substitute.	$a^2+b^2=c^2$ $x^2+x^2=10^2$
Step 5. Solve the equation. Isolate the variable. Use the definition of square root. Simplify. Approximate to the nearest tenth.	$2x^2=100$ (2.4.23) (2.4.24) $x^2=50$ (2.4.25) (2.4.26) $x=\sqrt{50}$ (2.4.27) (2.4.28) $x\approx 7.1$ (2.4.29)
Step 6. Check. $a^2+b^2=c^2$ (2.4.30) $(7.1)^2+(7.1)^2\approx 10^2$ Yes. (2.4.31)	
Step 7. Answer the question.	Kelvin should fasten each piece of wood approximately 7.1” from the corner.

EXAMPLE 2.4.32:

John puts the base of a 13-foot ladder five feet from the wall of his house as shown in the figure. How far up the wall does the ladder reach?

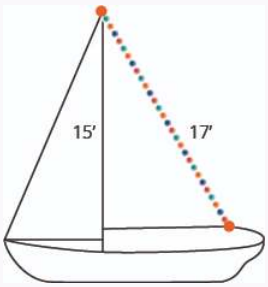


Answer

The ladder reaches 12 feet.

EXAMPLE 2.4.33:

Randy wants to attach a 17-foot string of lights to the top of the 15 foot mast of his sailboat, as shown in the figure. How far from the base of the mast should he attach the end of the light string?



Answer

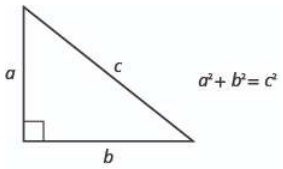
He should attach the lights 8 feet from the base of the mast.

Access this online resource for additional instruction and practice with solving for a variable in literal equations.

- Solving Literal Equations

Key Concepts

- **How To Solve Geometry Applications**
  1. **Read** the problem and make sure all the words and ideas are understood.
  2. **Identify** what you are looking for.
  3. **Name** what you are looking for by choosing a variable to represent it. Draw the figure and label it with the given information.
  4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
  5. **Solve** the equation using good algebra techniques.
  6. **Check** the answer in the problem and make sure it makes sense.
  7. **Answer** the question with a complete sentence.
- **The Pythagorean Theorem**
  - In any right triangle, where a and b are the lengths of the legs, and c is the length of the hypotenuse, the sum of the squares of the lengths of the two legs equals the square of the length of the hypotenuse.



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2.4E: Exercises

Practice Makes Perfect

Solve a Formula for a Specific Variable

In the following exercises, solve the given formula for the specified variable.

Solve the formula $C = \pi d$ for d.

Answer

$d = \frac{C}{\pi}$

Solve the formula $C = \pi d$ for π.

Solve the formula $V = LWH$ for L.

Answer

$L = \frac{V}{WH}$

Solve the formula $V = LWH$ for H.

Solve the formula $A = \frac{1}{2}bh$ for b.

Answer

$b = \frac{2A}{h}$

Solve the formula $A = \frac{1}{2}bh$ for h.

Solve the formula

$A = \frac{1}{2}d_1d_2$ for d_1.

Answer

$d_1 = \frac{2A}{d_2}$

Solve the formula

$A = \frac{1}{2}d_1d_2$ for d_2.

Solve the formula

$A = \frac{1}{2}h(b_1 + b_2)$ for b_1.

Answer

$b_1 = \frac{2A}{h} - b_2$

Solve the formula

$A = \frac{1}{2}h(b_1 + b_2)$ for b_2.

Solve the formula

$h = 54t + \frac{1}{2}at^2$ for a.

Answer

$a = \frac{2h-108t}{t^2}$

Solve the formula

$h = 48t + \frac{1}{2}at^2$ for a.

Solve $180 = a + b + c$ for a.

Answer

$a = 180 - b - c$

Solve $180 = a + b + c$ for c.

Solve the formula

$A = \frac{1}{2}pl + B$ for p.

Answer

$p = \frac{2A-2B}{l}$

Solve the formula

$A = \frac{1}{2}pl + B$ for l.

Solve the formula

$P = 2L + 2W$ for L.

Answer

$L = \frac{P-2W}{2}$

Solve the formula

$P = 2L + 2W$ for W.

In the following exercises, solve for the formula for y.

Solve the formula

$8x + y = 15$ for y.

Answer

$y = 15 - 8x$

Solve the formula

$9x + y = 13$ for y.

Solve the formula

$-4x + y = -6$ for y.

Answer

$y = -6 + 4x$

Solve the formula

$-5x + y = -1$ for y.

Solve the formula

$x - y = -4$ for y.

Answer

$y = 4 + x$

Solve the formula

$x - y = -3$ for y.

Solve the formula

$4x + 3y = 7$ for y.

Answer

$y = \frac{7-4x}{3}$

Solve the formula

$3x + 2y = 11$ for y.

Solve the formula

$2x + 3y = 12$ for y

Answer

$y = \frac{12-2x}{3}$

Solve the formula

$5x + 2y = 10$ for y.

Solve the formula

$3x - 2y = 18$ for y.

Answer

$y = \frac{18-3x}{-2}$

Solve the formula

$4x - 3y = 12$ for y.

Use Formulas to Solve Geometry Applications

In the following exercises, solve using a geometry formula.

A triangular flag has area 0.75 square feet and height 1.5 foot. What is its base?

Answer

1 foot

A triangular window has area 24 square feet and height six feet. What is its base?

What is the base of a triangle with area 207 square inches and height 18 inches?

Answer

23 inches

What is the height of a triangle with area 893 square inches and base 38 inches?

The two smaller angles of a right triangle have equal measures. Find the measures of all three angles.

Answer

$45°, 45°, 90°$

The measure of the smallest angle of a right triangle is $20°$ less than the measure of the next larger angle. Find the measures of all three angles.

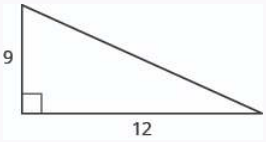
The angles in a triangle are such that one angle is twice the smallest angle, while the third angle is three times as large as the smallest angle. Find the measures of all three angles.

Answer

$30°, 60°, 90°$

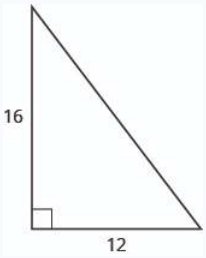
The angles in a triangle are such that one angle is 20 more than the smallest angle, while the third angle is three times as large as the smallest angle. Find the measures of all three angles.

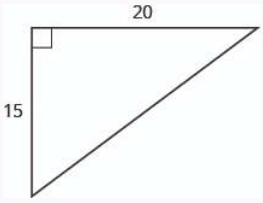
In the following exercises, use the Pythagorean Theorem to find the length of the hypotenuse.



Answer

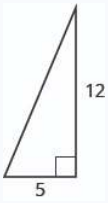
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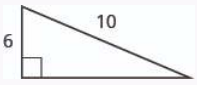


Answer

25

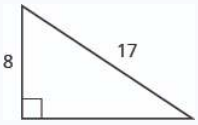


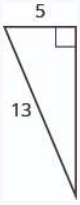
In the following exercises, use the Pythagorean Theorem to find the length of the leg. Round to the nearest tenth if necessary.



Answer

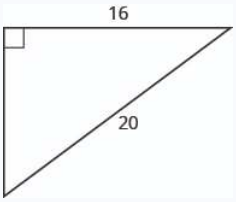
8

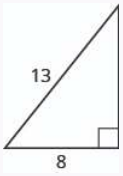




Answer

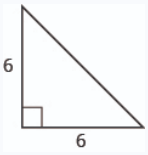
12

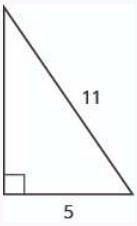




Answer

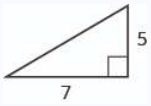
10.2





Answer

9.8



In the following exercises, solve using a geometry formula.

The width of a rectangle is seven meters less than the length. The perimeter is 58 meters. Find the length and width.

Answer

18 meters, 11 meters

The length of a rectangle is eight feet more than the width. The perimeter is 60 feet. Find the length and width.

The width of the rectangle is 0.7 meters less than the length. The perimeter of a rectangle is 52.6 meters. Find the dimensions of the rectangle.

Answer

13.5 m, 12.8 m

The length of the rectangle is 1.1 meters less than the width. The perimeter of a rectangle is 49.4 meters. Find the dimensions of the rectangle.

The perimeter of a rectangle of 150 feet. The length of the rectangle is twice the width. Find the length and width of the rectangle.

Answer

25 ft, 50 ft

The length of the rectangle is three times the width. The perimeter of a rectangle is 72 feet. Find the length and width of the rectangle.

The length of the rectangle is three meters less than twice the width. The perimeter of a rectangle is 36 meters. Find the dimensions of the rectangle.

Answer

7 m, 11 m

The length of a rectangle is five inches more than twice the width. The perimeter is 34 inches. Find the length and width.

The perimeter of a triangle is 39 feet. One side of the triangle is one foot longer than the second side. The third side is two feet longer than the second side. Find the length of each side.

Answer

12 ft, 13 ft, 14 ft

The perimeter of a triangle is 35 feet. One side of the triangle is five feet longer than the second side. The third side is three feet longer than the second side. Find the length of each side.

One side of a triangle is twice the smallest side. The third side is five feet more than the shortest side. The perimeter is 17 feet. Find the lengths of all three sides.

Answer

3 ft, 6 ft, 8 ft

One side of a triangle is three times the smallest side. The third side is three feet more than the shortest side. The perimeter is 13 feet. Find the lengths of all three sides.

The perimeter of a rectangular field is 560 yards. The length is 40 yards more than the width. Find the length and width of the field.

Answer

120 yd, 160 yd

The perimeter of a rectangular atrium is 160 feet. The length is 16 feet more than the width. Find the length and width of the atrium.

A rectangular parking lot has perimeter 250 feet. The length is five feet more than twice the width. Find the length and width of the parking lot.

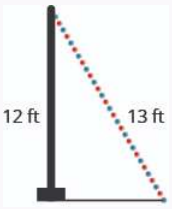
Answer

40 ft, 85 ft

A rectangular rug has perimeter 240 inches. The length is 12 inches more than twice the width. Find the length and width of the rug.

In the following exercises, solve. Approximate answers to the nearest tenth, if necessary.

A 13-foot string of lights will be attached to the top of a 12-foot pole for a holiday display as shown. How far from the base of the pole should the end of the string of lights be anchored?



Answer

5 feet

Pam wants to put a banner across her garage door diagonally, as shown, to congratulate her son for his college graduation. The garage door is 12 feet high and 16 feet wide. How long should the banner be to fit the garage door?



Chi is planning to put a diagonal path of paving stones through her flower garden as shown. The flower garden is a square with side 10 feet. What will the length of the path be?



Answer

14.1 feet

Brian borrowed a 20-foot extension ladder to use when he paints his house. If he sets the base of the ladder six feet from the house as shown, how far up will the top of the ladder reach?



Everyday Math

Converting temperature While on a tour in Greece, Tatyana saw that the temperature was 40° Celsius. Solve for F in the formula $C = \frac{5}{9}(F - 32)$ to find the Fahrenheit temperature.

Answer

104°F

Converting temperature Yon was visiting the United States and he saw that the temperature in Seattle one day was 50° Fahrenheit. Solve for C in the formula $F = \frac{9}{5}C + 32$ to find the Celsius temperature

Christa wants to put a fence around her triangular flowerbed. The sides of the flowerbed are six feet, eight feet and 10 feet. How many feet of fencing will she need to enclose her flowerbed?

Answer

24 ft

Jose just removed the children's play set from his back yard to make room for a rectangular garden. He wants to put a fence around the garden to keep the dog out. He has a 50-foot roll of fence in his garage that he plans to use. To fit in the backyard, the width of the garden must be 10 feet. How long can he make the other side?

Writing Exercises

If you need to put tile on your kitchen floor, do you need to know the perimeter or the area of the kitchen? Explain your reasoning.

Answer

Answers will vary.

If you need to put a fence around your backyard, do you need to know the perimeter or the area of the backyard? Explain your reasoning.

Look at the two figures below.



ⓐ Which figure looks like it has the larger area? Which looks like it has the larger perimeter?

ⓑ Now calculate the area and perimeter of each figure. Which has the larger area? Which has the larger perimeter?

ⓒ Were the results of part (b) the same as your answers in part (a)? Is that surprising to you?

Answer

ⓐ Answers will vary. ⓑ The areas are the same. The 2×8 rectangle has a larger perimeter than the 4×4 square.

ⓒ Answers will vary.

Write a geometry word problem that relates to your life experience, then solve it and explain all your steps.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve a formula for a specific variable.			
use formulas to solve geometry applications.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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2.5: Solve Mixture and Uniform Motion Applications

Learning Objectives

By the end of this section, you will be able to:

- Solve coin word problems
- Solve ticket and stamp word problems
- Solve mixture word problems
- Solve uniform motion applications

Before you get started, take this readiness quiz.

1. Simplify: $0.25x + 0.10(x + 4)$.
   If you missed this problem, review [link].
2. The number of adult tickets is three more than twice the number of children tickets. Let *c*represent the number of children tickets. Write an expression for the number of adult tickets.
   If you missed this problem, review [link].
3. Convert 4.2% to a decimal.
   If you missed this problem, review [link].

Solve Coin Word Problems

Using algebra to find the number of nickels and pennies in a piggy bank may seem silly. You may wonder why we just don't open the bank and count them. But this type of problem introduces us to some techniques that will be useful as we move forward in our study of mathematics.



If we have a pile of dimes, how would we determine its value? If we count the number of dimes, we'll know how many we have—the *number* of dimes. But this does not tell us the *value* of all the dimes. Say we counted 23 dimes, how much are they worth? Each dime is worth \$0.10—that is the *value* of one dime. To find the total value of the pile of 23 dimes, multiply 23 by \$0.10 to get \$2.30.

The number of dimes times the value of each dime equals the total value of the dimes.

$$\begin{aligned} number \cdot value &= total\ value \\ 23 \cdot \$0.10 &= \$2.30 \end{aligned}$$

This method leads to the following model.

TOTAL VALUE OF COINS

For the same type of coin, the total value of a number of coins is found by using the model

$$number \cdot value = total\ value$$

- *number* is the number of coins
- *value* is the value of each coin
- *total value* is the total value of all the coins

If we had several types of coins, we could continue this process for each type of coin, and then we would know the total value of each type of coin. To get the total value of *all* the coins, add the total value of each type of coin.

Example 2.5.1

Jesse has $3.02 worth of pennies and nickels in his piggy bank. The number of nickels is three more than eight times the number of pennies. How many nickels and how many pennies does Jesse have?

Answer

Step 1. Read the problem. Determine the types of coins involved. Create a table. Write in the value of each type of coin.	pennies and nickels Pennies are worth $0.10. Nickels are worth $0.05.
Step 2. Identify what we are looking for.	the number of pennies and nickels
Step 3. Name. Represent the number of each type of coin using variables. The number of nickels is defined in terms of the number of pennies, so start with pennies. The number of nickels is three more than eight times the number of pennies.	Let $p =$ number of pennies. $8p + 3 =$ number of nickels
In the chart, multiply the number and the value to get the total value of each type of coin.	

Type	Number • Value ($) = Total Value ($)		
pennies	p	0.01	$0.01p$
nickels	$8p + 3$	0.05	$0.05(8p + 3)$
			$3.02

Step 4. Translate. Write the equation by adding the total value of all the types of coins.	$0.01p + 0.05(8p + 3) = 3.02$
Step 5. Solve the equation.	$0.01p + 0.40p + 0.15 = 3.02$ $0.41p + 0.15 = 3.02$ $0.41p = 2.87$ $p = 7$ pennies
How many nickels?	$8p + 3$ $8(7) + 3$ 59 nickels
Step 6. Check the answer in the problem and make sure it makes sense. Jesse has 7 pennies and 59 nickels. Is the total value **$3.02**? $7(0.01) + 59(0.05) \stackrel{?}{=} 3.02$ $3.02 = 3.02$✓	

Example 2.5.2

Jesse has $6.55 worth of quarters and nickels in his pocket. The number of nickels is five more than two times the number of quarters. How many nickels and how many quarters does Jesse have?

Answer

Jess has 41 nickels and 18 quarters.

Example 2.5.3

Elane has $7.00 total in dimes and nickels in her coin jar. The number of dimes that Elane has is seven less than three times the number of nickels. How many of each coin does Elane have?

Answer

Elane has 22 nickels and 59 dimes.

The steps for solving a coin word problem are summarized below.

SOLVE COIN WORD PROBLEMS.

1. **Read** the problem. Make sure all the words and ideas are understood.
   - Determine the types of coins involved.
   - Create a table to organize the information.
     - Label the columns “type,” “number,” “value,” and “total value.”
     - List the types of coins.
     - Write in the value of each type of coin.
     - Write in the total value of all the coins.
     -

Type	Number	• Value ($) =	Total Value ($)

2. **Identify** what you are looking for.
3. **Name** what you are looking for. Choose a variable to represent that quantity.
   - Use variable expressions to represent the number of each type of coin and write them in the table.
   - Multiply the number times the value to get the total value of each type of coin.
4. **Translate** into an equation.
   - It may be helpful to restate the problem in one sentence with all the important information. Then, translate the sentence into an equation.
   - Write the equation by adding the total values of all the types of coins.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

Solve Ticket and Stamp Word Problems

Problems involving tickets or stamps are very much like coin problems. Each type of ticket and stamp has a value, just like each type of coin does. So to solve these problems, we will follow the same steps we used to solve coin problems.

Example 2.5.4

Danny paid $15.75 for stamps. The number of 49-cent stamps was five less than three times the number of 35-cent stamps. How many 49-cent stamps and how many 35-cent stamps did Danny buy?

Answer

Step 1. Determine the types of stamps involved.	49-cent stamps and 35-cent stamps
Step 2. Identify we are looking for.	the number of 49-cent stamps and the number of 35-cent stamps
Step 3. Write variable expressions to represent the number of each type of stamp.	Let x = number of 35-cent stamps.
"The number of 49-cent stamps was five less than three times the number of 35-cent stamps."	$3x-5=3x-5=$ number of 49-cent stamps

Type	Number	• Value($)	= Total Value ($)
49 cent stamps	$3x-5$	0.49	$0.49(3x-5)$
35 cent stamps	x	0.35	$0.35x$
			15.75

Step 4. Write the equation from the total values.	$0.49(3x-5)+0.35x=15.75$
Step 5. Solve the equation.	$1.47x-2.45+0.35x=15.75$ $1.82x-2.45=15.75$ $1.82x=18.2$
How many 49-cent stamps?	$x=10$ 35-cent stamps $3x-5$ $3(10)-5$ 25 49-cent stamps
Step 6. Check. $10(0.35)+25(0.49)3.50+12.2515.75=?=?=15.7515.7515.75\checkmark 10(0.35)+25(0.49)=?15.753.50+12.25=?15.7515.75=15.75\checkmark$	
Step 7. Answer the question with a complete sentence.	Danny bought ten 35-cent stamps and twenty-five 49-cent stamps.

Example 2.5.5

Eric paid $19.88 for stamps. The number of 49-cent stamps was eight more than twice the number of 35-cent stamps. How many 49-cent stamps and how many 35-cent stamps did Eric buy?

Answer

Eric bought thirty-two 49-cent stamps and twelve 35-cent stamps.

Example 2.5.6

Kailee paid $14.74 for stamps. The number of 49-cent stamps was four less than three times the number of 20-cent stamps. How many 49-cent stamps and how many 20-cent stamps did Kailee buy?

Answer

Kailee bought twenty-six 49-cent stamps and ten 20-cent stamps.

In most of our examples so far, we have been told that one quantity is four more than twice the other, or something similar. In our next example, we have to relate the quantities in a different way.

Suppose Aniket sold a total of 100 tickets. Each ticket was either an adult ticket or a child ticket. If he sold 20 child tickets, how many adult tickets did he sell?

Did you say "80"? How did you figure that out? Did you subtract 20 from 100?

If he sold 45 child tickets, how many adult tickets did he sell?

Did you say "55"? How did you find it? By subtracting 45 from 100?

Now, suppose Aniket sold x child tickets. Then how many adult tickets did he sell? To find out, we would follow the same logic we used above. In each case, we subtracted the number of child tickets from 100 to get the number of adult tickets. We now do the same with x.

We have summarized this in the table.

Child tickets	Adult tickets
20	80
45	55
75	25
x	$100 - x$

We will apply this technique in the next example.

Example 2.5.7

A whale-watching ship had 40 paying passengers on board. The total revenue collected from tickets was $1,196. Full-fare passengers paid $32 each and reduced-fare passengers paid $26 each. How many full-fare passengers and how many reduced-fare passengers were on the ship?

Answer

Step 1. Determine the types of tickets involved.	full-fare tickets and reduced-fare tickets
Step 2. Identify what we are looking for.	the number of full-fare tickets and reduced-fare tickets
Step 3. Name. Represent the number of each type of ticket using variables.	Let f = the number of full-fare tickets. $40-f=40-f=$ the number of reduced-fare tickets
We know the total number of tickets sold was 40. This means the number of reduced-fare tickets is 40 less the number of full-fare tickets. Multiply the number times the value to get the total value of each type of ticket.	
	Type / **Number** · **Value ($)** = **Total Value ($)** Full-fare: f · 32 = $32f$ Reduced-fare: $40-f$ · 26 = $26(40-f)$ $1{,}196$
Step 4. Translate. Write the equation by adding the total values of each type of ticket.	$32f+26(40-f)=1{,}196$
Step 5. Solve the equation.	$32f+1{,}040-26f=1{,}196$ $6f=156$ $f=26$ full-fare tickets
How many reduced-fare?	$40-f$ $40-26$ 14 reduced-fare tickets
Step 6. Check the answer. There were 26 full-fare tickets at $32 each and 14 reduced-fare tickets at $26 each. Is the total value $116? 26·3214·26==832364——1,196✓26·32=83214·26=364——1,196✓	
Step 7. Answer the question.	They sold 26 full-fare and 14 reduced-fare tickets.

Example 2.5.8

During her shift at the museum ticket booth, Leah sold 115 tickets for a total of $1,163. Adult tickets cost $12 and student tickets cost $5. How many adult tickets and how many student tickets did Leah sell?

Answer

84 adult tickets, 31 student tickets

Example 2.5.9

Galen sold 810 tickets for his church's carnival for a total revenue of $2,820. Children's tickets cost $3 each and adult tickets cost $5 each. How many children's tickets and how many adult tickets did he sell?

Answer

615 children's tickets and 195 adult tickets

Solve Mixture Word Problems

Now we'll solve some more general applications of the mixture model. In mixture problems, we are often mixing two quantities, such as raisins and nuts, to create a mixture, such as trail mix. In our tables we will have a row for each item to be mixed as well as one for the final mixture.

Example 2.5.10

Henning is mixing raisins and nuts to make 25 pounds of trail mix. Raisins cost $4.50 a pound and nuts cost $8 a pound. If Henning wants his cost for the trail mix to be $6.60 a pound, how many pounds of raisins and how many pounds of nuts should he use?

Answer

Step 1. Determine what is being mixed.	The 25 pounds of trail mix will come from mixing raisins and nuts.
Step 2. Identify what we are looking for.	the number of pounds of raisins and nuts
Step 3. Represent the number of each type of ticket using variables. As before, we fill in a chart to organize our information. We enter the price per pound for each item. We multiply the number times the value to get the total value.	Let x=x= number of pounds of raisins. 25−x=25−x= number of pounds of nuts

Type	Number of pounds	· Price per Pound ($)	= Total Value ($)
Raisins	x	4.50	$4.5x$
Nuts	$25 - x$	8	$8(25 - x)$
Trail mix	25	6.60	25(6.60)

Notice that the last column in the table gives the information for the total amount of the mixture.	
Step 4. Translate into an equation.	The value of the raisins plus the value of the nuts will be the value of the trail mix.
Step 5. Solve the equation.	$4.5x + 8(25 - x) = 25(6.60)$
	$4.5x + 200 - 8x = 165$ $-3.5x = -35$ $x = 10$ pounds of raisins
Find the number of pounds of nuts.	$25 - x$ $25 - 10$ 15 pounds of nuts
Step 6. Check. 4.5(10)+8(15)45+120165=?=? =25(6.60)165165✓4.5(10)+8(15)=?25(6.60)45+120=? 165165=165✓	
Step 7. Answer the question.	Henning mixed ten pounds of raisins with 15 pounds of nuts.

Example 2.5.11

Orlando is mixing nuts and cereal squares to make a party mix. Nuts sell for $7 a pound and cereal squares sell for $4 a pound. Orlando wants to make 30 pounds of party mix at a cost of $6.50 a pound, how many pounds of nuts and how many pounds of cereal squares should he use?

Answer

Orlando mixed five pounds of cereal squares and 25 pounds of nuts.

Example 2.5.12

Becca wants to mix fruit juice and soda to make a punch. She can buy fruit juice for $3 a gallon and soda for $4 a gallon. If she wants to make 28 gallons of punch at a cost of $3.25 a gallon, how many gallons of fruit juice and how many gallons of soda should she buy?

Answer

Becca mixed 21 gallons of fruit punch and seven gallons of soda.

Solve Uniform Motion Applications

When you are driving down the interstate using your cruise control, the speed of your car stays the same—it is uniform. We call a problem in which the speed of an object is constant a **uniform motion** application. We will use the distance, rate, and time formula, $D = rt$, to compare two scenarios, such as two vehicles traveling at different rates or in opposite directions.

Our problem solving strategies will still apply here, but we will add to the first step. The first step will include drawing a diagram that shows what is happening in the example. Drawing the diagram helps us understand what is happening so that we will write an appropriate equation. Then we will make a table to organize the information, like we did for the coin, ticket, and stamp applications.

The steps are listed here for easy reference:

SOLVE A UNIFORM MOTION APPLICATION.

1. **Read** the problem. Make sure all the words and ideas are understood.
   - Draw a diagram to illustrate what is happening.
   - Create a table to organize the information.
     - Label the columns rate, time, distance.
     - List the two scenarios.
     - Write in the information you know.

	Rate •	Time =	Distance

2. **Identify** what you are looking for.
3. **Name** what you are looking for. Choose a variable to represent that quantity.
   - Complete the chart.
   - Use variable expressions to represent that quantity in each row.
   - Multiply the rate times the time to get the distance.
4. **Translate** into an equation.
   - Restate the problem in one sentence with all the important information.
   - Then, translate the sentence into an equation.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

Example 2.5.13

Wayne and Dennis like to ride the bike path from Riverside Park to the beach. Dennis's speed is seven miles per hour faster than Wayne's speed, so it takes Wayne two hours to ride to the beach while it takes Dennis 1.5 hours for the ride. Find the speed of both bikers.

Answer

Step 1. Read the problem. Make sure all the words and ideas are understood.

Draw a diagram to illustrate what it happening. Shown below is a sketch of what is happening in the example.



Create a table to organize the information.

- Label the columns “Rate,” “Time,” and “Distance.”
- List the two scenarios.
- Write in the information you know.

	Rate (mph)	•	Time (hrs)	=	Distance (miles)
Dennis			1.5		
Wayne			2		

Step 2. Identify what you are looking for.

You are asked to find the speed of both bikers.

Notice that the distance formula uses the word “rate,” but it is more common to use “speed”

when we talk about vehicles in everyday English.

Step 3. Name what we are looking for. Choose a variable to represent that quantity.

Complete the chart Use variable expressions to represent that quantity in each row.
We are looking for the speed of the bikers. Let’s let r represent Wayne’s speed. Since Dennis’ speed is 7 mph faster, we represent that as $r+7$

$r+7=$ Dennis’ speed

$r=$ Wayne’s speed

Fill in the speeds into the chart.

	Rate (mph)	•	Time (hrs)	=	Distance (miles)
Dennis	$r+7$		1.5		
Wayne	r		2		

Multiply the rate times the time to get the distance.

	Rate (mph)	•	Time (hrs)	=	Distance (miles)
Dennis	$r+7$		1.5		$1.5(r+7)$
Wayne	r		2		$2r$

Step 4. Translate into an equation.

Restate the problem in one sentence with all the important information.

Then, translate the sentence into an equation.
The equation to model this situation will come from the relation between the distances. Look at the diagram we drew above.
How is the distance travelled by Dennis related to the distance travelled by Wayne?
Since both bikers leave from Riverside and travel to the beach, they travel the same distance. So we write:

	distance traveled by Dennis	=	distance traveled by Wayne
Translate to an equation.	$1.5(r+7)$	=	$2r$

Step 5. Solve the equation using algebra techniques.

Now solve this equation.	$1.5(r+7)=2r$ $1.5r+10.5=2r$ $10.5=0.5r$ $21=r$ So Wayne's speed is 21 mph.
Find Dennis' speed.	$r+7$ $21+7$ 28 Dennis' speed 28 mph.

Step 6. Check the answer in the problem and make sure it makes sense.

$28 \text{ mph}(1.5 \text{ hours}) = 42 \text{ miles}\checkmark$
$21 \text{ mph}(2 \text{ hours}) = 42 \text{ miles}\checkmark$

Step 7. Answer the question with a complete sentence.

Wayne rode at 21 mph and Dennis rode at 28 mph.

Restate the problem in one sentence with all the important information.

Then, translate the sentence into an equation.
The equation to model this situation will come from the relation between the distances. Look at the diagram we drew above. How is the distance travelled by Dennis related to the distance travelled by Wayne?
Since both bikers leave from Riverside and travel to the beach, they travel the same distance. So we write:

	distance traveled by Dennis	=	distance traveled by Wayne
Translate to an equation.	$1.5(r+7)$	=	$2r$

Example 2.5.14

An express train and a local train leave Pittsburgh to travel to Washington, D.C. The express train can make the trip in four hours and the local train takes five hours for the trip. The speed of the express train is 12 miles per hour faster than the speed of the local train. Find the speed of both trains.

Answer

The speed of the local train is 48 mph and the speed of the express train is 60 mph.

Example 2.5.15

Jeromy can drive from his house in Cleveland to his college in Chicago in 4.5 hours. It takes his mother six hours to make the same drive. Jeromy drives 20 miles per hour faster than his mother. Find Jeromy's speed and his mother's speed.

Answer

Jeromy drove at a speed of 80 mph and his mother drove 60 mph.

In Example, we had two bikers traveling the same distance. In the next example, two people drive toward each other until they meet.

Example 2.5.16

Carina is driving from her home in Anaheim to Berkeley on the same day her brother is driving from Berkeley to Anaheim, so they decide to meet for lunch along the way in Buttonwillow. The distance from Anaheim to Berkeley is 395 miles. It takes Carina three hours to get to Buttonwillow, while her brother drives four hours to get there. Carina's average speed is 15 miles per hour faster than her brother's average speed. Find Carina's and her brother's average speeds.

Answer

Step 1. Read the problem. Make sure all the words and ideas are understood.

Draw a diagram to illustrate what it happening. Below shows a sketch of what is happening in the example.



Create a table to organize the information.

- Label the columns rate, time, distance.
- List the two scenarios.
- Write in the information you know.

	Rate (mph)	•	Time (hrs)	=	Distance (miles)
Carina			3		
Brother			4		
					395

Step 2. Identify what we are looking for.

We are asked to find the average speeds of Carina and her brother.

Step 3. Name what we are looking for. Choose a variable to represent that quantity.

Complete the chart. Use variable expressions to represent that quantity in each row.
We are looking for their average speeds. Let's let r represent the average speed of Carina. Since the brother's speed is 15 mph faster, we represent that as $r + 15$.
Fill in the speeds into the chart. Multiply the rate times the time to get the distance.

	Rate (mph)	•	Time (hrs)	=	Distance (miles)
Carina	$r + 15$		3		$3(r + 15)$
Brother	r		4		$4r$
					395

Step 4. Translate into an equation.

Restate the problem in one sentence with all the important information. Then, translate the sentence into an equation.
Again, we need to identify a relationship between the distances in order to write an equation. Look at the diagram we created above and notice the relationship between the distance Carina traveled and the distance her brother traveled.
The distance Carina traveled plus the distance her brother travel must add up to 410 miles. So we write:

$$\underbrace{\text{distance traveled by Carina}}_{3(r+15)} + \underbrace{\text{distance traveled by her brother}}_{4r} = 395$$

Translate to an equation. $3(r + 15) + 4r = 395$

Step 5. Solve the equation using algebra techniques.

Now solve this equation.	$3(r+15)+4r=395$ $3r+45+4r=395$ $7r+45=395$ $7r=350$ $r=50$ So Carina's brother's speed was 50 mph.
Carina's speed is r+15.r+15.	$r+15$ $50+15$ 65 Her brother's speed was 65 mph.

Step 6. Check the answer in the problem and make sure it makes sense.

Carina drove	**65 mph(3 hours)**	**=**	**195 miles**
Her brother drove	**50 mph(4 hours)**	**=**	**200 miles**
			395 miles✓

Step 7. Answer the question with a complete sentence.

Carina drove 65 mph and her brother 50 mph.

my drove at a speed of 80 mph and his mother drove 60 mph.

Example 2.5.17

Christopher and his parents live 115 miles apart. They met at a restaurant between their homes to celebrate his mother's birthday. Christopher drove one and a half hours while his parents drove one hour to get to the restaurant. Christopher's average speed was ten miles per hour faster than his parents' average speed. What were the average speeds of Christopher and of his parents as they drove to the restaurant?

Answer

Christopher's speed was 50 mph and his parents' speed was 40 mph.

Example 2.5.18

Ashley goes to college in Minneapolis, 234 miles from her home in Sioux Falls. She wants her parents to bring her more winter clothes, so they decide to meet at a restaurant on the road between Minneapolis and Sioux Falls. Ashley and her parents both drove two hours to the restaurant. Ashley's average speed was seven miles per hour faster than her parents' average speed. Find Ashley's and her parents' average speed.

Answer

Ashley's parents drove 55 mph and Ashley drove 62 mph.

As you read the next example, think about the relationship of the distances traveled. Which of the previous two examples is more similar to this situation?

Example 2.5.19

Two truck drivers leave a rest area on the interstate at the same time. One truck travels east and the other one travels west. The truck traveling west travels at 70 mph and the truck traveling east has an average speed of 60 mph. How long will they travel before they are 325 miles apart?

Answer

Step 1. Read the problem. Make all the words and ideas are understood.

Draw a diagram to illustrate what it happening.



Create a table to organize the information.

- Label the columns rate, time, distance.
- List the two scenarios.
- Write in the information you know.

	Rate (mph)	• Time (hrs)	= Distance (miles)
West	70		
East	60		
			325

Step 2. Identify what we are looking for.

We are asked to find the amount of time the trucks will travel until they are 325 miles apart.

Step 3. Name what we are looking for. Choose a variable to represent that quantity.

Complete the chart. Use variable expressions to represent that quantity in each row.
We are looking for the time travelled. Both trucks will travel the same amount of time.
Let's call the time t. Since their speeds are different, they will travel different distances.

Multiply the rate times the time to get the distance.

	Rate (mph)	• Time (hrs)	= Distance (miles)
West	70	t	$70t$
East	60	t	$60t$
			325

Step 4. Translate into an equation.

Restate the problem in one sentence with all the important information. Then, translate the sentence into an equation.
We need to find a relation between the distances in order to write an equation. Looking at the diagram, what is the relationship between the distances each of the trucks will travel?
The distance travelled by the truck going west plus the distance travelled by the truck going east must add up to 325 miles. So we write:

	distance traveled by westbound truck	+	distance traveled by eastbound truck	= 325
Translate to an equation.	$70t$	+	$60t$	= 325

Step 5. Solve the equation using algebra techniques.

Erroneous nesting of equation structures

So it will take the trucks **2.5** hours to be 325 miles apart.

Step 6. Check the answer in the problem and make sure it makes sense.

Truck going West	70 mph(2.5 hours)	=	175 miles
Truck going East	60 mph(2.5 hours)	=	150 miles
			325 miles✓

Step 7. Answer the question with a complete sentence.
It will take the trucks 2.5 hours to be 325 miles apart.

Example 2.5.20

Pierre and Monique leave their home in Portland at the same time. Pierre drives north on the turnpike at a speed of 75 miles per hour while Monique drives south at a speed of 68 miles per hour. How long will it take them to be 429 miles apart?

Answer

Pierre and Monique will be 429 miles apart in 3 hours.

Example 2.5.21

Thanh and Nhat leave their office in Sacramento at the same time. Thanh drives north on I-5 at a speed of 72 miles per hour. Nhat drives south on I-5 at a speed of 76 miles per hour. How long will it take them to be 330 miles apart?

Answer

Thanh and Nhat will be 330 miles apart in 2.2 hours.

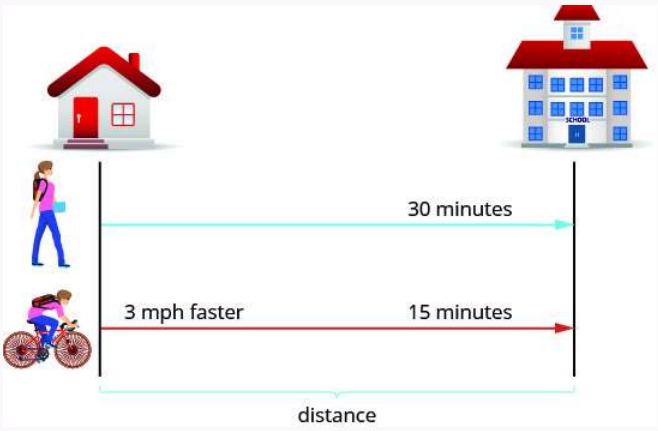
It is important to make sure that the units match when we use the distance rate and time formula. For instance, if the rate is in miles per hour, then the time must be in hours.

Example 2.5.22

When Naoko walks to school, it takes her 30 minutes. If she rides her bike, it takes her 15 minutes. Her speed is three miles per hour faster when she rides her bike than when she walks. What is her speed walking and her speed riding her bike?

Answer

First, we draw a diagram that represents the situation to help us see what is happening.



We are asked to find her speed walking and riding her bike. Let's call her walking speed r. Since her biking speed is three miles per hour faster, we will call that speed $r+3$. We write the speeds in the chart.

The speed is in miles per hour, so we need to express the times in hours, too, in order for the units to be the same. Remember, 1 hour is 60 minutes. So:

$$30 \text{ minutes is } \frac{30}{60} \text{ or } \frac{1}{2} \text{ hour}$$
$$15 \text{ minutes is } \frac{15}{60} \text{ or } \frac{1}{4} \text{ hour}$$

We write the times in the chart.

Next, we multiply rate times time to fill in the distance column.

	Rate (mph)	•	Time (hrs)	=	Distance (miles)
Walk	r		$\frac{1}{2}$		$\frac{1}{2}r$
Bike	$r+3$		$\frac{1}{4}$		$\frac{1}{4}(r+3)$

The equation will come from the fact that the distance from Naoko's home to her school is the same whether she is walking or riding her bike.
So we say:

	distance walked = distance covered by bike
Translate to an equation.	
Solve this equation.	
Clear the fractions by multiplying by the LCD of all the fractions in the equation.	$8 \cdot \frac{1}{2}r = 8 \cdot \frac{1}{4}(r+3)$
Simplify.	$4r = 2(r+3)$ $4r = 2r+6$ $2r = 6$ $r = 3$ mph walking speed $r+3$ biking speed $3+3$
	6

Let's check if this works.
Walk 3 mph (0.5 hour) = 1.5 miles
Bike 6 mph (0.25 hour) = 1.5 miles

Yes, either way Naoko travels 1.5 miles to school.

Naoko's walking speed is 3 mph and her speed riding her bike is 6 mph.

Example 2.5.23

Suzy takes 50 minutes to hike uphill from the parking lot to the lookout tower. It takes her 30 minutes to hike back down to the parking lot. Her speed going downhill is 1.2 miles per hour faster than her speed going uphill. Find Suzy's uphill and downhill speeds.

Answer

Suzy's speed uphill is 1.81.8 mph and downhill is three mph.

Example 2.5.24

Llewyn takes 45 minutes to drive his boat upstream from the dock to his favorite fishing spot. It takes him 30 minutes to drive the boat back downstream to the dock. The boat's speed going downstream is four miles per hour faster than its speed going upstream. Find the boat's upstream and downstream speeds.

Answer

The boat's speed upstream is eight mph and downstream is 12 mph.

In the distance, rate and time formula, time represents the actual amount of elapsed time (in hours, minutes, etc.). If a problem gives us starting and ending times as clock times, we must find the elapsed time in order to use the formula.

Example 2.5.25

Cruz is training to compete in a triathlon. He left his house at 6:00 and ran until 7:30. Then he rode his bike until 9:45. He covered a total distance of 51 miles. His speed when biking was 1.6 times his speed when running. Find Cruz's biking and running speeds.

Answer

A diagram will help us model this trip.



Next, we create a table to organize the information. We know the total distance is 51 miles. We are looking for the rate of speed for each part of the trip. The rate while biking is 1.6 times the rate of running. If we let r = the rate running, then the rate biking is $1.6r$.

The times here are given as clock times. Cruz started from home at 6:00 a.m. and started biking at 7:30 a.m. So he spent 1.5 hours running. Then he biked from 7:30 a.m until 9:45 a.m. So he spent 2.25 hours biking.

Now, we multiply the rates by the times.

	Rate (mph)	• Time (hrs)	= Distance (miles)
run	r	1.5	$1.5r$
bike	$1.6r$	2.25	$2.25(1.6r)$
			51

By looking at the diagram, we can see that the sum of the distance running and the distance biking is 255 miles.

	distance running + distance biking = 51
Translate to an equation.	
Solve this equation.	
Check. **Run 10 mph (1.5 hour) = 15 mi** **Bike 16 mph (2.25 hour) = 36 mi** **51 mi**	

Example 2.5.26

Hamilton loves to travel to Las Vegas, 255 miles from his home in Orange County. On his last trip, he left his house at 2:00 p.m. The first part of his trip was on congested city freeways. At 4:00 pm, the traffic cleared and he was able to drive through the desert at a speed 1.75 times faster than when he drove in the congested area. He arrived in Las Vegas at 6:30 p.m. How fast was he driving during each part of his trip?

Answer

Hamilton drove 40 mph in the city and 70 mph in the desert.

Example 2.5.27

Phuong left home on his bicycle at 10:00. He rode on the flat street until 11:15, then rode uphill until 11:45. He rode a total of 31 miles. His speed riding uphill was 0.6 times his speed on the flat street. Find his speed biking uphill and on the flat street.

Answer

Phuong rode uphill at a speed of 12 mph and on the flat street at 20 mph.

Key Concepts

- **Total Value of Coins**
  For the same type of coin, the total value of a number of coins is found by using the model
  number·value=totalvaluenumber·value=totalvalue
  - *number* is the number of coins
  - *value* is the value of each coin
  - *total value* is the total value of all the coins
- **How to solve coin word problems.**
  1. **Read** the problem. Make sure all the words and ideas are understood.
     Determine the types of coins involved.
     Create a table to organize the information.
     Label the columns “type,” “number,” “value,” “total value.”
     List the types of coins.
     Write in the value of each type of coin.
     Write in the total value of all the coins.

Type	Number •	Value ($) =	Total Value ($)

  2. **Identify** what you are looking for.
  3. **Name** what you are looking for. Choose a variable to represent that quantity.
     Use variable expressions to represent the number of each type of coin and write them in the table.
     Multiply the number times the value to get the total value of each type of coin.
  4. **Translate** into an equation.
     It may be helpful to restate the problem in one sentence with all the important information. Then, translate the sentence into an equation.
     Write the equation by adding the total values of all the types of coins.
  5. **Solve** the equation using good algebra techniques.
  6. **Check** the answer in the problem and make sure it makes sense.
  7. **Answer** the question with a complete sentence.
- **How To Solve a Uniform Motion Application**
  1. **Read** the problem. Make sure all the words and ideas are understood.
     Draw a diagram to illustrate what it happening.
     Create a table to organize the information.
     Label the columns rate, time, distance.
     List the two scenarios.

Write in the information you know.

	Rate •	Time =	Distance

2. **Identify** what you are looking for.
3. **Name** what you are looking for. Choose a variable to represent that quantity.
   Complete the chart.
   Use variable expressions to represent that quantity in each row.
   Multiply the rate times the time to get the distance.
4. **Translate** into an equation.
   Restate the problem in one sentence with all the important information.
   Then, translate the sentence into an equation.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

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2.5E: Exercises

Practice Makes Perfect

Solve Coin Word Problems

In the following exercises, solve each coin word problem.

Michaela has $2.05 in dimes and nickels in her change purse. She has seven more dimes than nickels. How many coins of each type does she have?

Answer

nine nickels, 16 dimes

Liliana has $2.10 in nickels and quarters in her backpack. She has 12 more nickels than quarters. How many coins of each type does she have?

In a cash drawer there is $125 in $5 and $10 bills. The number of $10 bills is twice the number of $5 bills. How many of each type of bill is in the drawer?

Answer

ten $10 bills, five $5 bills

Sumanta has $175 in $5 and $10 bills in his drawer. The number of $5 bills is three times the number of $10 bills. How many of each are in the drawer?

Chi has $11.30 in dimes and quarters. The number of dimes is three more than three times the number of quarters. How many of each are there?

Answer

63 dimes, 20 quarters

Alison has $9.70 in dimes and quarters. The number of quarters is eight more than four times the number of dimes. How many of each coin does she have?

Mukul has $3.75 in quarters, dimes and nickels in his pocket. He has five more dimes than quarters and nine more nickels than quarters. How many of each coin are in his pocket?

Answer

16 nickels, 12 dimes, seven quarters

Vina has $4.70 in quarters, dimes and nickels in her purse. She has eight more dimes than quarters and six more nickels than quarters. How many of each coin are in her purse?

Solve Ticket and Stamp Word Problems

In the following exercises, solve each ticket or stamp word problem.

The first day of a water polo tournament the total value of tickets sold was $17,610. One-day passes sold for $20 and tournament passes sold for $30. The number of tournament passes sold was 37 more than the number of day passes sold. How

many day passes and how many tournament passes were sold?

Answer

330 day passes, 367 tournament passes

At the movie theater, the total value of tickets sold was $2,612.50. Adult tickets sold for $10 each and senior/child tickets sold for $7.50 each. The number of senior/child tickets sold was 25 less than twice the number of adult tickets sold. How many senior/child tickets and how many adult tickets were sold?

Julie went to the post office and bought both $0.41 stamps and $0.26 postcards. She spent $51.40. The number of stamps was 20 more than twice the number of postcards. How many of each did she buy?

Answer

40 postcards, 100 stamps

Jason went to the post office and bought both $0.41 stamps and $0.26 postcards and spent $10.28 The number of stamps was four more than twice the number of postcards. How many of each did he buy?

Hilda has $210 worth of $10 and $12 stock shares. The number of $10 shares is five more than twice the number of $12 shares. How many of each type of share does she have?

Answer

15 $10 shares, five $12 shares

Mario invested $475 in $45 and $25 stock shares. The number of $25 shares was five less than three times the number of $45 shares. How many of each type of share did he buy?

The ice rink sold 95 tickets for the afternoon skating session, for a total of $828. General admission tickets cost $10 each and youth tickets cost $8 each. How many general admission tickets and how many youth tickets were sold?

Answer

34 general, 61 youth

For the 7:30 show time, 140 movie tickets were sold. Receipts from the $13 adult tickets and the $10 senior tickets totaled $1,664. How many adult tickets and how many senior tickets were sold?

The box office sold 360 tickets to a concert at the college. The total receipts were $4,170. General admission tickets cost $15 and student tickets cost $10. How many of each kind of ticket was sold?

Answer

114 general, 246 student

Last Saturday, the museum box office sold 281 tickets for a total of $3,954. Adult tickets cost $15 and student tickets cost $12. How many of each kind of ticket was sold

Solve Mixture Word Problems

In the following exercises, solve each mixture word problem.

Macario is making 12 pounds of nut mixture with macadamia nuts and almonds. Macadamia nuts cost $9 per pound and almonds cost $5.25 per pound. How many pounds of macadamia nuts and how many pounds of almonds should Macario use for the mixture to cost $6.50 per pound to make?

Answer

Four pounds of macadamia nuts, eight pounds almonds

Carmen wants to tile the floor of his house. He will need 1,000 square feet of tile. He will do most of the floor with a tile that costs $1.50 per square foot,

Riley is planning to plant a lawn in his yard. He will need nine pounds of grass seed. He wants to mix Bermuda seed that costs $4.80 per pound with Fescue seed that costs $3.50 per pound. How much of each seed should he buy so that the overall cost will be $4.02 per pound?

Answer

3.63.6 lbs Bermuda seed, 5.45.4 lbs Fescue seed

Vartan was paid $25,000 for a cell phone app that he wrote and wants to invest it to save for his son's education. He wants to put some of the money into a bond that pays 4% annual interest and the rest into stocks that pay 9% annual interest. If he wants to earn 7.4% annual interest on the total amount, how much money should he invest in each account?

Vern sold his 1964 Ford Mustang for $55,000 and wants to invest the money to earn him 5.8% interest per year. He will put some of the money into Fund A that earns 3% per year and the rest in Fund B that earns 10% per year. How much should he invest into each fund if he wants to earn 5.8% interest per year on the total amount?

Answer

$33,000 in Fund A, $22,000 in Fund B

Dominic pays 7% interest on his $15,000 college loan and 12% interest on his $11,000 car loan. What average interest rate does he pay on the total $26,000 he owes? (Round your answer to the nearest tenth of a percent.)

Liam borrowed a total of $35,000 to pay for college. He pays his parents 3% interest on the $8,000 he borrowed from them and pays the bank 6.8% on the rest. What average interest rate does he pay on the total $35,000? (Round your answer to the nearest tenth of a percent.)

Answer

5.9

Solve Uniform Motion Applications

In the following exercises, solve.

Lilah is moving from Portland to Seattle. It takes her three hours to go by train. Mason leaves the train station in Portland and drives to the train station in Seattle with all Lilah's boxes in his car. It takes him 2.4 hours to get to Seattle, driving at 15 miles per hour faster than the speed of the train. Find Mason's speed and the speed of the train.

Kathy and Cheryl are walking in a fundraiser. Kathy completes the course in 4.8 hours and Cheryl completes the course in eight hours. Kathy walks two miles per hour faster than Cheryl. Find Kathy's speed and Cheryl's speed.

Answer

Kathy 5 mph, Cheryl 3 mph

Two busses go from Sacramento to San Diego. The express bus makes the trip in 6.8 hours and the local bus takes 10.2 hours for the trip. The speed of the express bus is 25 mph faster than the speed of the local bus. Find the speed of both busses.

A commercial jet and a private airplane fly from Denver to Phoenix. It takes the commercial jet 1.6 hours for the flight, and it takes the private airplane 2.6 hours. The speed of the commercial jet is 210 miles per hour faster than the speed of the private airplane. Find the speed of both airplanes to the nearest 10 mph.

Answer

commercial 540 mph, private plane 330 mph

Saul drove his truck three hours from Dallas towards Kansas City and stopped at a truck stop to get dinner. At the truck stop he met Erwin, who had driven four hours from Kansas City towards Dallas. The distance between Dallas and Kansas City is 542 miles, and Erwin's speed was eight miles per hour slower than Saul's speed. Find the speed of the two truckers.

Charlie and Violet met for lunch at a restaurant between Memphis and New Orleans. Charlie had left Memphis and drove 4.8 hours towards New Orleans. Violet had left New Orleans and drove two hours towards Memphis, at a speed 10 miles per hour faster than Charlie's speed. The distance between Memphis and New Orleans is 394 miles. Find the speed of the two drivers.

Answer

Violet 65 mph, Charlie 55 mph

Sisters Helen and Anne live 332 miles apart. For Thanksgiving, they met at their other sister's house partway between their homes. Helen drove 3.2 hours and Anne drove 2.8 hours. Helen's average speed was four miles per hour faster than Anne's. Find Helen's average speed and Anne's average speed.

Ethan and Leo start riding their bikes at the opposite ends of a 65-mile bike path. After Ethan has ridden 1.5 hours and Leo has ridden two hours, they meet on the path. Ethan's speed is six miles per hour faster than Leo's speed. Find the speed of the two bikers.

Answer

Ethan 22 mph, Leo 16 mph

Elvira and Aletheia live 3.1 miles apart on the same street. They are in a study group that meets at a coffee shop between their houses. It took Elvira half an hour and Aletheia two-thirds of an hour to walk to the coffee shop. Aletheia's speed is 0.6 miles per hour slower than Elvira's speed. Find both women's walking speeds.

DaMarcus and Fabian live 23 miles apart and play soccer at a park between their homes. DaMarcus rode his bike for three-quarters of an hour and Fabian rode his bike for half an hour to get to the park. Fabian's speed was six miles per hour faster than DaMarcus' speed. Find the speed of both soccer players.

Answer

DaMarcus 16 mph, Fabian 22 mph

Cindy and Richard leave their dorm in Charleston at the same time. Cindy rides her bicycle north at a speed of 18 miles per hour. Richard rides his bicycle south at a speed of 14 miles per hour. How long will it take them to be 96 miles apart?

Matt and Chris leave their uncle's house in Phoenix at the same time. Matt drives west on I-60 at a speed of 76 miles per hour. Chris drives east on I-60 at a speed of 82 miles per hour. How many hours will it take them to be 632 miles apart?

Answer

four hours

Two busses leave Billings at the same time. The Seattle bus heads west on I-90 at a speed of 73 miles per hour while the Chicago bus heads east at a speed of 79 miles an hour. How many hours will it take them to be 532 miles apart?

Two boats leave the same dock in Cairo at the same time. One heads north on the Mississippi River while the other heads south. The northbound boat travels four miles per hour. The southbound boat goes eight miles per hour. How long will it take them to be 54 miles apart?

Answer

4.5 hours

Lorena walks the path around the park in 30 minutes. If she jogs, it takes her 20 minutes. Her jogging speed is 1.5 miles per hour faster than her walking speed. Find Lorena's walking speed and jogging speed

Julian rides his bike uphill for 45 minutes, then turns around and rides back downhill. It takes him 15 minutes to get back to where he started. His uphill speed is 3.2 miles per hour slower than his downhill speed. Find Julian's uphill and downhill speed.

Answer

uphill **1.6** mph, downhill **4.8** mph

Cassius drives his boat upstream for 45 minutes. It takes him 30 minutes to return downstream. His speed going upstream is three miles per hour slower than his speed going downstream. Find his upstream and downstream speeds.

It takes Darline 20 minutes to drive to work in light traffic. To come home, when there is heavy traffic, it takes her 36 minutes. Her speed in light traffic is 24 miles per hour faster than her speed in heavy traffic. Find her speed in light traffic and in heavy traffic.

Answer

light traffic 54 mph, heavy traffic 30 mph

At 1:30, Marlon left his house to go to the beach, a distance of 7.6 miles. He rode his skateboard until 2:15, and then walked the rest of the way. He arrived at the beach at 3:00. Marlon's speed on his skateboard is 2.5 times his walking speed. Find his speed when skateboarding and when walking.

Aaron left at 9:15 to drive to his mountain cabin 108 miles away. He drove on the freeway until 10:45 and then drove on a mountain road. He arrived at 11:05. His speed on the freeway was three times his speed on the mountain road. Find Aaron's speed on the freeway and on the mountain road.

Answer

freeway 72 mph, mountain road 24 mph

Marisol left Los Angeles at 2:30 to drive to Santa Barbara, a distance of 95 miles. The traffic was heavy until 3:20. She drove the rest of the way in very light traffic and arrived at 4:20. Her speed in heavy traffic was 40 miles per hour slower than her speed in light traffic. Find her speed in heavy traffic and in light traffic.

Lizette is training for a marathon. At 7:00 she left her house and ran until 8:15 then she walked until 11:15. She covered a total distance of 19 miles. Her running speed was five miles per hour faster than her walking speed. Find her running and walking speeds.

Answer

running eight mph, walking three mph

Everyday Math

John left his house in Irvine at 8:35 a.m. to drive to a meeting in Los Angeles, 45 miles away. He arrived at the meeting at 9:50 a.m.. At 5:30 p.m. he left the meeting and drove home. He arrived home at 7:18 p.m.

ⓐ What was his average speed on the drive from Irvine to Los Angeles?

ⓑ What was his average speed on the drive from Los Angeles to Irvine?

ⓒ What was the total time he spent driving to and from this meeting?

Sarah wants to arrive at her friend's wedding at 3:00. The distance from Sarah's house to the wedding is 95 miles. Based on usual traffic patterns, Sarah predicts she can drive the first 15 miles at 60 miles per hour, the next 10 miles at 30 miles per hour, and the remainder of the drive at 70 miles per hour.

ⓐ How long will it take Sarah to drive the first 15 miles?

ⓑ How long will it take Sarah to drive the next 10 miles?

ⓒ How long will it take Sarah to drive the rest of the trip?

ⓓ What time should Sarah leave her house?

Answer

ⓐ 15 minutes ⓑ 20 minutes ⓒ one hour (d) 1:25

Writing Exercises

Suppose you have six quarters, nine dimes, and four pennies. Explain how you find the total value of all the coins.

Do you find it helpful to use a table when solving coin problems? Why or why not?

Answer

Answers will vary.

In the table used to solve coin problems, one column is labeled "number" and another column is labeled "value." What is the difference between the "number" and the "value"?

When solving a uniform motion problem, how does drawing a diagram of the situation help you?

Answer

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve coin word problems.			
solve ticket and stamp word problems.			
solve mixture word problems.			
solve uniform motion applications.			

ⓑ On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

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2.6: Solve Linear Inequalities

Learning Objectives

By the end of this section, you will be able to:

- Graph inequalities on the number line
- Solve linear inequalities
- Translate words to an inequality and solve
- Solve applications with linear inequalities

Before you get started, take this readiness quiz.

1. Translate from algebra to English: $15 > x$.
   If you missed this problem, review [link].
2. Translate to an algebraic expression: 15 is less than x.
   If you missed this problem, review [link].

Graph Inequalities on the Number Line

What number would make the inequality $x > 3$ true? Are you thinking, "x could be four"? That's correct, but x could be 6, too, or 37, or even 3.001. Any number greater than three is a solution to the inequality $x > 3$. We show all the solutions to the inequality $x > 3$ on the number line by shading in all the numbers to the right of three, to show that all numbers greater than three are solutions. Because the number three itself is not a solution, we put an open parenthesis at three.

We can also represent inequalities using **interval notation**. There is no upper end to the solution to this inequality. In interval notation, we express $x > 3$ as $(3, \infty)$. The symbol ∞ is read as "**infinity**." It is not an actual number. Figure 2.6.1 shows both the number line and the interval notation.

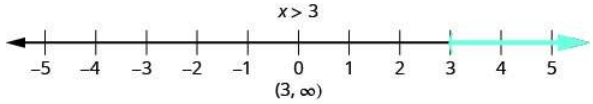


Figure 2.6.1: The inequality $x > 3$ is graphed on this number line and written in interval notation.

We use the left parenthesis symbol, (, to show that the endpoint of the inequality is not included. The left bracket symbol, [, shows that the endpoint is included.

The inequality $x \leq 1$ means all numbers less than or equal to one. Here we need to show that one is a solution, too. We do that by putting a bracket at $x = 1$. We then shade in all the numbers to the left of one, to show that all numbers less than one are solutions (Figure 2.6.2). There is no lower end to those numbers. We write $x \leq 1 x \leq 1$ in interval notation as $(-\infty, 1]$. The symbol $-\infty$ is read as "negative infinity."

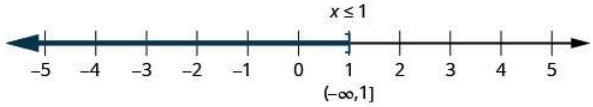
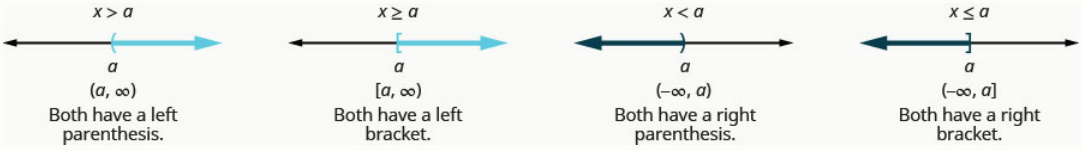


Figure 2.6.2: The inequality $x \leq 1$ is graphed on this number line and written in interval notation.

Figure 2.6.3 shows both the number line and interval notation.

INEQUALITIES, NUMBER LINES, AND INTERVAL NOTATION



The notation for inequalities on a number line and in interval notation use the same symbols to express the endpoints of intervals.

Example 2.6.1

Graph each inequality on the number line and write in interval notation.

a. $x \geq -3$
b. $x < 2.5$
c. $x \leq -\frac{3}{5}$

Answer

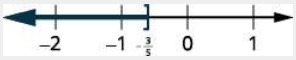
ⓐ

	$x \geq -3$
Shade to the right of -3, and put a bracket at -3.	
Write in interval notation.	$[-3, \infty)$

ⓑ

	$x < 2.5$
Shade to the left of 2.5 and put a parenthesis at 2.5.	
Write in interval notation.	$(-\infty, 2.5)$

ⓒ

	$x \leq -\frac{3}{5}$
Shade to the left of $-\frac{3}{5}$, and put a bracket at $-\frac{3}{5}$.	
Write in interval notation.	$\left(-\infty, \frac{3}{5}\right]$

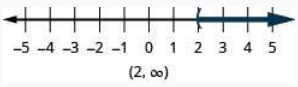
Example 2.6.2

Graph each inequality on the number line and write in interval notation:

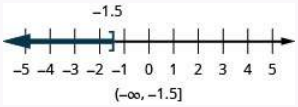
a. $x > 2$
b. $x \leq -1.5$
c. $x \geq \frac{3}{4}$.

Answer

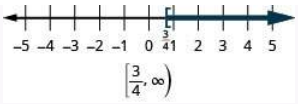
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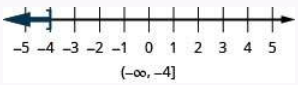
Example 2.6.3

Graph each inequality on the number line and write in interval notation:

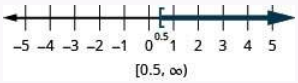
a. $x \leq -4$
b. $x \geq 0.5$
c. $x < -\frac{2}{3}$.

Answer

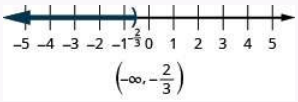
ⓐ



ⓑ



ⓒ



What numbers are greater than two but less than five? Are you thinking say, $2.5,\ 3,\ 3\frac{2}{3},\ 4,\ 4,\ 99$? We can represent all the numbers between two and five with the inequality $2 < x < 5$. We can show $2 < x < 5$ on the number line by shading all the numbers between two and five. Again, we use the parentheses to show the numbers two and five are not included. See Figure.

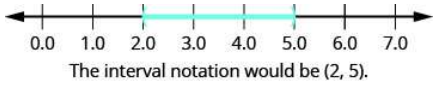


Figure 2.6.3

Example 2.6.4

Graph each inequality on the number line and write in interval notation.

ⓐ $-3 < x < 4$ ⓑ $-6 \leq x < -1$ ⓒ $0 \leq x \leq 2.5$

Answer

ⓐ

		$-3 < x < 4$
Shade between -3 and 4. Put a parentheses at -3 and 4.		
Write in interval notation.		$(-3, 4)$

ⓑ

			$-6 \le x < -1$
Shade between -6 and -1. Put a bracket at -6, and a parenthesis at -1.			
Write in interval notation.			$[-6, 1)$

ⓒ

			$0 \le x \le 2.5$
Shade between 0 and 2.5. Put a bracket at 0 and at 2.5.			
Write in interval notation.			$[0, 2.5]$

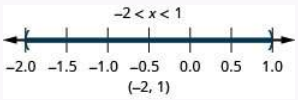
Example 2.6.5

Graph each inequality on the number line and write in interval notation:

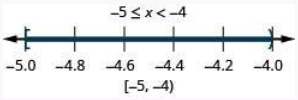
ⓐ $-2 < x < 1$ ⓑ $-5 \le x < -4$ ⓒ $1 \le x \le 4.25$

Answer

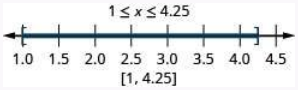
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ⓑ



ⓒ



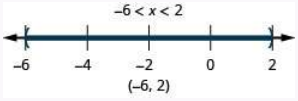
Example 2.6.6

Graph each inequality on the number line and write in interval notation:

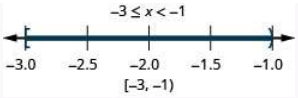
ⓐ $-6 < x < 2$ ⓑ $-3 \le x < -1$ ⓒ $2.5 \le x \le 6$

Answer

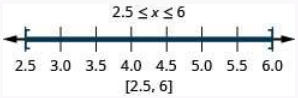
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ⓑ



ⓒ



Solve Linear Inequalities

A linear inequality is much like a linear equation—but the equal sign is replaced with an inequality sign. A **linear inequality** is an inequality in one variable that can be written in one of the forms, $ax + b < c$, $ax + b \le c$, $ax + b > c$, or $ax + b \ge c$.

LINEAR INEQUALITY

A linear inequality is an inequality in one variable that can be written in one of the following forms where a, b, and c are real numbers and $a \ne 0$:

$$ax + b < c, \quad ax + b \le c, \quad ax + b > c, \quad ax + b \ge c.$$

When we solved linear equations, we were able to use the properties of equality to add, subtract, multiply, or divide both sides and still keep the equality. Similar properties hold true for inequalities.

We can add or subtract the same quantity from both sides of an inequality and still keep the inequality. For example:

$$\begin{array}{ll} -4 < 2 & -4 < 2 \\ -4 - 5 < 2 - 5 & -4 + 7 < 2 + 7 \\ -9 < -3 \text{ True} & 3 < 9 \text{ True} \end{array}$$

Notice that the inequality sign stayed the same.

This leads us to the Addition and Subtraction Properties of Inequality.

ADDITION AND SUBTRACTION PROPERTY OF INEQUALITY

For any numbers a, b, and c, if $a < b$, then

$$\begin{array}{ll} a + c < b + c & a - c < b - c \\ a + c > b + c & a - c > b - c \end{array}$$

We can add or subtract the same quantity from both sides of an inequality and still keep the inequality

What happens to an inequality when we divide or multiply both sides by a constant?

Let's first multiply and divide both sides by a positive number.

$$\begin{array}{ll} 10 < 15 & 10 < 15 \\ 10(5) < 15(5) & \frac{10}{5} < \frac{15}{5} \\ 50 < 75 \text{ True} & 2 < 3 \text{ True} \end{array}$$

The inequality signs stayed the same.

Does the inequality stay the same when we divide or multiply by a negative number?

$$\begin{array}{ll} 10 < 15 & 10 < 15 \\ 10(-5) \;?\; 15(-5) & \frac{10}{-5} \;?\; \frac{15}{-5} \\ -50 \;?\; -75 & -2 \;?\; -3 \\ -50 > -75 & -2 > -3 \end{array}$$

Notice that when we filled in the inequality signs, the inequality signs reversed their direction.

When we divide or multiply an inequality by a positive number, the inequality sign stays the same. When we divide or multiply an inequality by a negative number, the inequality sign reverses.

This gives us the Multiplication and Division Property of Inequality.

MULTIPLICATION AND DIVISION PROPERTY OF INEQUALITY

For any numbers a, b, and c,

multiply or divide by a positive

if $a < b$ and $c > 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.
if $a > b$ and $c > 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.

multiply or divide by a negative

if $a < b$ and $c < 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.
if $a > b$ and $c < 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.

When we divide or multiply an inequality by a:

- positive number, the inequality stays the same.
- negative number, the inequality reverses.

Sometimes when solving an inequality, as in the next example, the variable ends upon the right. We can rewrite the inequality in reverse to get the variable to the left.

$x > a$ has the same meaning as $a < x$

Think about it as “If Xander is taller than Andy, then Andy is shorter than Xander.”

Example 2.6.7

Solve each inequality. Graph the solution on the number line, and write the solution in interval notation.

ⓐ $x - \frac{3}{8} \leq \frac{3}{4}$ ⓑ $9y < 54$ ⓒ $-15 < \frac{3}{5}z$

Answer

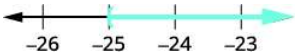
ⓐ

	$x - \frac{3}{8} \leq \frac{3}{4}$
Add 3838 to both sides of the inequality.	
Simplify.	
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, \frac{9}{8}]$

ⓑ

	$9y < 54$
Divide both sides of the inequality by 9; since 9 is positive, the inequality stays the same.	
Simplify.	$y < 6$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, 6)$

ⓒ

	$-15 < \frac{3}{5}z$
Multiply both sides of the inequality by $\frac{5}{3}$. Since $\frac{5}{3}$ is positive, the inequality stays the same.	
Simplify.	
Rewrite with the variable on the left.	
Graph the solution on the number line.	
Write the solution in interval notation.	

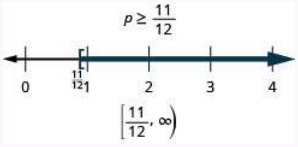
Example 2.6.8

Solve each inequality, graph the solution on the number line, and write the solution in interval notation:

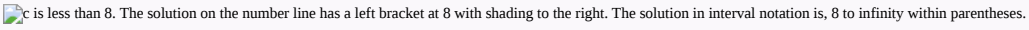
ⓐ $p - \frac{3}{4} \geq \frac{1}{6}$ ⓑ $9c > 72$ ⓒ $24 \leq \frac{3}{8}m$

Answer

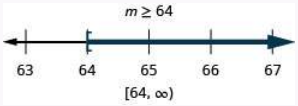
ⓐ



ⓑ


c is less than 8. The solution on the number line has a left bracket at 8 with shading to the right. The solution in interval notation is, 8 to infinity within parentheses.

ⓒ



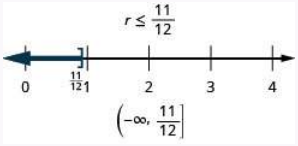
Example 2.6.9

Solve each inequality, graph the solution on the number line, and write the solution in interval notation:

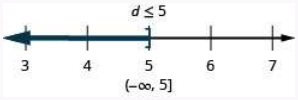
ⓐ $r - \frac{1}{3} \leq \frac{7}{12}$ ⓑ $12d \leq 60$ ⓒ $-24 < \frac{4}{3}n$

Answer

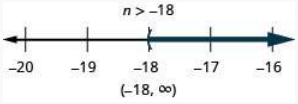
ⓐ



ⓑ



ⓒ



Be careful when you multiply or divide by a negative number—remember to reverse the inequality sign.

Example 2.6.10

Solve each inequality, graph the solution on the number line, and write the solution in interval notation.

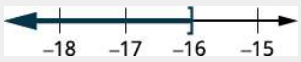
ⓐ $-\frac{1}{3}m \ge \frac{6}{5}$ ⓑ $\frac{n}{-2} \ge 8$

Answer

ⓐ

	$-13m \ge 65$
Divide both sides of the inequality by $-\frac{1}{3}$. Since $-\frac{1}{3}$ is a negative, the inequality reverses.	$\frac{-13m}{-13} \le \frac{65}{-13}$
Simplify.	$m \le -5$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, -5]$

ⓑ

	$\frac{n}{-2} \ge 8$
Multiply both sides of the inequality by -2. Since -2 is a negative, the inequality reverses.	$-2\left(\frac{n}{-2}\right) \le -2(8)$
Simplify.	$n \le -16$
Graph the solution on the number line.	
Write the solution in interval notation.	$(-\infty, -16]$

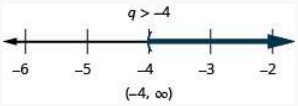
Example 2.6.11

Solve each inequality, graph the solution on the number line, and write the solution in interval notation:

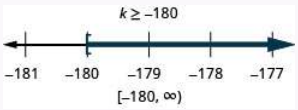
ⓐ $-8q < 32$ ⓑ $\frac{k}{-12} \le 15$.

Answer

ⓐ



ⓑ



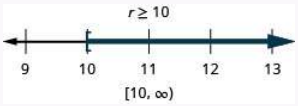
Example 2.6.12

Solve each inequality, graph the solution on the number line, and write the solution in interval notation:

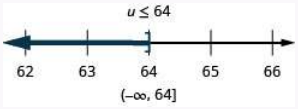
ⓐ $-7r \leq -70$ ⓑ $\frac{u}{-4} \geq -16$.

Answer

ⓐ



ⓑ

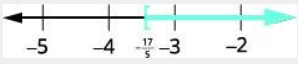


Most inequalities will take more than one step to solve. We follow the same steps we used in the general strategy for solving linear equations, but make sure to pay close attention when we multiply or divide to isolate the variable.

Example 2.6.13

Solve the inequality $6y \leq 11y + 17$, graph the solution on the number line, and write the solution in interval notation.

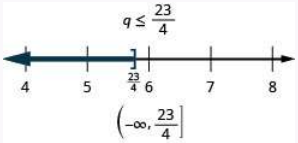
Answer

	$6y \leq 11y + 17$
Subtract 11y11y from both sides to collect the variables on the left.	$6y - 11y \leq 11y - 11y + 17$
Simplify.	$-5y \leq 17$
Divide both sides of the inequality by -5,-5, and reverse the inequality.	
Simplify.	$y \geq \frac{17}{-5}$
Graph the solution on the number line.	
Write the solution in interval notation.	$\left[-\frac{17}{5}, \infty\right)$

Example 2.6.14

Solve the inequality, graph the solution on the number line, and write the solution in interval notation: $3q \geq 7q - 23$.

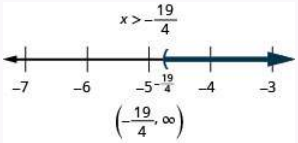
Answer



Example 2.6.15

Solve the inequality, graph the solution on the number line, and write the solution in interval notation: $6x < 10x + 19$.

Answer



When solving inequalities, it is usually easiest to collect the variables on the side where the coefficient of the variable is largest. This eliminates negative coefficients and so we don't have to multiply or divide by a negative—which means we don't have to remember to reverse the inequality sign.

Example 2.6.16

Solve the inequality $8p + 3(p - 12) > 7p - 28$, graph the solution on the number line, and write the solution in interval notation.

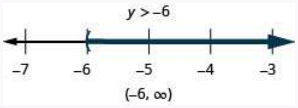
Answer

	$8p + 3(p - 12) > 7p - 28$
Simplify each side as much as possible.	
Distribute.	$8p + 3p - 36 > 7p - 28$
Combine like terms.	$11p - 36 > 7p - 28$
Subtract $7p$ from both sides to collect the variables on the left, since $11 > 7$.	$11p - 36 - 7p > 7p - 28 - 7p$
Simplify.	$4p - 36 > -28$
Add 36 to both sides to collect the constants on the right.	$4p - 36 + 36 > -28 + 36$
Simplify.	$4p > 8$
Divide both sides of the inequality by 4; the inequality stays the same.	$\frac{4p}{4} > \frac{8}{4}$
Simplify.	$p > 2$
Graph the solution on the number line.	
Write the solution in interval notation.	$(2, \infty)$

Example 2.6.17

Solve the inequality $9y + 2(y + 6) > 5y - 24$, graph the solution on the number line, and write the solution in interval notation.

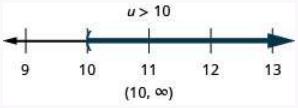
Answer



Example 2.6.18

Solve the inequality $6u + 8(u - 1) > 10u + 32$, graph the solution on the number line, and write the solution in interval notation.

Answer



Just like some equations are identities and some are contradictions, inequalities may be identities or contradictions, too. We recognize these forms when we are left with only constants as we solve the inequality. If the result is a true statement, we have an identity. If the result is a false statement, we have a contradiction.

Example 2.6.19

Solve the inequality $8x - 2(5 - x) < 4(x + 9) + 6x$, graph the solution on the number line, and write the solution in interval notation.

Answer

Simplify each side as much as possible.		$8x - 2(5 - x) < 4(x + 9) + 6x$
Distribute.		$8x - 10 + 2x < 4x + 36 + 6x$
Combine like terms.		$10x - 10 < 10x + 36$
Subtract $10x$ from both sides to collect the variables on the left.		$10x - 10 - 10x < 10x + 36 - 10x$
Simplify.		$-10 < 36$
The x's are gone, and we have a true statement.		The inequality is an identity. The solution is all real numbers.
Graph the solution on the number line.		
Write the solution in interval notation.		$(-\infty, \infty)$

Example 2.6.20

Solve the inequality $4b - 3(3 - b) > 5(b - 6) + 2b$, graph the solution on the number line, and write the solution in interval notation.

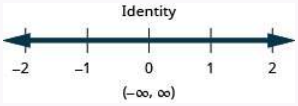
Answer

The inequality is an identity. Its solution on the number line is shaded for all values. The solution in interval notation is negative infinity to infinity within parentheses.

Example 2.6.21

Solve the inequality $9h - 7(2 - h) < 8(h + 11) + 8h$, graph the solution on the number line, and write the solution in interval notation.

Answer



We can clear fractions in inequalities much as we did in equations. Again, be careful with the signs when multiplying or dividing by a negative.

Example 2.6.22

Solve the inequality $\frac{1}{3}a - \frac{1}{8}a > \frac{5}{24}a + \frac{3}{4}$, graph the solution on the number line, and write the solution in interval notation.

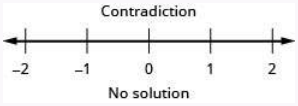
Answer

	$\frac{1}{3}a - \frac{1}{8}a > \frac{5}{24}a + \frac{3}{4}$
Multiply both sides by the LCD, 24, to clear the fractions.	$24\left(\frac{1}{3}a - \frac{1}{8}a\right) > 24\left(\frac{5}{24}a + \frac{3}{4}\right)$
Simplify.	$8a - 3a > 5a + 18$
Combine like terms.	$5a > 5a + 18$
Subtract $5a$ from both sides to collect the variables on the left.	$5a - 5a > 5a - 5a + 18$
Simplify.	$0 > 18$
The statement is false.	The inequality is a contradiction. There is no solution.
Graph the solution on the number line.	
Write the solution in interval notation.	There is no solution.

Example 2.6.23

Solve the inequality $\frac{1}{4}x - \frac{1}{12}x > \frac{1}{6}x + \frac{7}{8}$, graph the solution on the number line, and write the solution in interval notation.

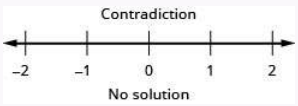
Answer



Example 2.6.24

Solve the inequality $\frac{2}{5}z - \frac{1}{3}z < \frac{1}{15}z - \frac{3}{5}$, graph the solution on the number line, and write the solution in interval notation.

Answer



Translate to an Inequality and Solve

To translate English sentences into inequalities, we need to recognize the phrases that indicate the inequality. Some words are easy, like “more than” and “less than.” But others are not as obvious. Table shows some common phrases that indicate inequalities.

$>$	$\geq$	$<$	$\leq$
is greater than	is greater than or equal to	is less than	is less than or equal to
is more than	is at least	is smaller than	is at most
is larger than	is no less than	has fewer than	is no more than
exceeds	is the minimum	is lower than	is the maximum

Example 2.6.25

Translate and solve. Then graph the solution on the number line, and write the solution in interval notation.

Twenty-seven less than x is at least 48.

Answer

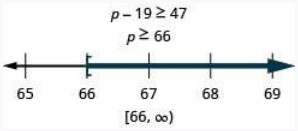
	Twenty-seven less than x is at least 48.
Translate.	$x - 27 \geq 48$
Solve—add 27 to both sides.	$x - 27 + 27 \geq 48 + 27$
Simplify.	$x \geq 75$
Graph on the number line.	
Write in interval notation.	$[75, \infty)$

Example 2.6.26

Translate and solve. Then graph the solution on the number line, and write the solution in interval notation.

Nineteen less than p is no less than 47.

Answer

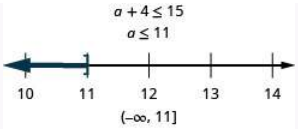


Example 2.6.27

Translate and solve. Then graph the solution on the number line, and write the solution in interval notation.

Four more than a is at most 15.

Answer



Solve Applications with Linear Inequalities

Many real-life situations require us to solve inequalities. The method we will use to solve applications with linear inequalities is very much like the one we used when we solved applications with equations.

We will read the problem and make sure all the words are understood. Next, we will identify what we are looking for and assign a variable to represent it. We will restate the problem in one sentence to make it easy to translate into an inequality. Then, we will solve the inequality.

Sometimes an application requires the solution to be a whole number, but the algebraic solution to the inequality is not a whole number. In that case, we must round the algebraic solution to a whole number. The context of the application will determine whether we round up or down.

Example 2.6.28

Dawn won a mini-grant of $4,000 to buy tablet computers for her classroom. The tablets she would like to buy cost $254.12 each, including tax and delivery. What is the maximum number of tablets Dawn can buy?

Answer

Step 1. Read the problem.	
Step 2. Identify what you are looking for.	the maximum number of tablets Dawn can buy
Step 3. Name what you are looking for.	
Choose a variable to represent that quantity.	Let $n =$ the number of tablets.
Step 4. Translate. Write a sentence that gives the information to find it.	$254.12 times the number of tablets is no more than $4,000.
Translate into an inequality.	$254.12n \leq 4000$
Step 5. Solve the inequality.	$n \leq 15.74$
But n must be a whole number of tablets, so round to 15.	$n \leq 15$
Step 6. Check the answer in the problem and make sure it makes sense.	Rounding down the price to $250, 15 tablets would cost $3,750, while 16 tablets would be $4,000. So a maximum of 15 tablets at $254.12 seems reasonable.
Step 7. Answer the question with a complete sentence.	Dawn can buy a maximum of 15 tablets.

Example 2.6.29

Angie has $20 to spend on juice boxes for her son's preschool picnic. Each pack of juice boxes costs $2.63. What is the maximum number of packs she can buy?

Answer

Angie can buy 7 packs of juice.

Example 2.6.30

Daniel wants to surprise his girlfriend with a birthday party at her favorite restaurant. It will cost $42.75 per person for dinner, including tip and tax. His budget for the party is $500. What is the maximum number of people Daniel can have at the party?

Answer

Daniel can have 11 people at the party.

Example 2.6.31

Taleisha's phone plan costs her $28.80 a month plus $0.20 per text message. How many text messages can she send/receive and keep her monthly phone bill no more than $50?

Answer

Step 1. Read the problem.

Step 2. Identify what you are looking for. — the number of text messages Taleisha can make

Step 3. Name what you are looking for.

Choose a variable to represent that quantity. — Let t = the number of text messages.

Step 4. Translate Write a sentence that gives the information to find it. — $28.80 plus $0.20 times the number of text messages is less than or equal to $50.

Translate into an inequality. — $28.80 + 0.20t \leq 50$

Step 5. Solve the inequality.

$$0.2t \leq 21.2$$
$$t \leq 106 \text{ text messages}$$

Step 6. Check the answer in the problem and make sure it makes sense.

Yes, $28.80 + 0.20(106) = 50$.

Step 7. Write a sentence that answers the question. — Taleisha can send/receive no more than 106 text messages to keep her bill no more than $50.

Example 2.6.32

Sergio and Lizeth have a very tight vacation budget. They plan to rent a car from a company that charges $75 a week plus $0.25 a mile. How many miles can they travel during the week and still keep within their $200 budget?

Answer

Sergio and Lizeth can travel no more than 500 miles.

Example 2.6.33

Rameen's heating bill is $5.42 per month plus $1.08 per therm. How many therms can Rameen use if he wants his heating bill to be a maximum of $87.50.

Answer

Rameen's heating bill is $5.42 per month plus $1.08 per therm. How many therms can Rameen use if he wants his heating bill to be a maximum of $87.50.

Profit is the money that remains when the costs have been subtracted from the revenue. In the next example, we will find the number of jobs a small businesswoman needs to do every month in order to make a certain amount of profit.

Example 2.6.34

Felicity has a calligraphy business. She charges $2.50 per wedding invitation. Her monthly expenses are $650. How many invitations must she write to earn a profit of at least $2,800 per month?

Answer

Step 1. Read the problem.

Step 2. Identify what you are looking for. — the number of invitations Felicity needs to write

Step 3. Name what you are looking for. — Let $j =$ the number of invitations.

Choose a variable to represent it.

Step 4. Translate. Write a sentence that gives the information to find it. — $2.50 times the number of invitations minus $650 is at least $2,800.

$$2.50j - 650 \geq 2,800$$

Translate into an inequality.

Step 5. Solve the inequality.

$$2.5j \geq 3,450$$
$$j \geq 1,380 \text{ invitations}$$

Step 6. Check the answer in the problem and make sure it makes sense.

If Felicity wrote 1400 invitations, her profit would be $2.50(1400) - 650$, or $2,850. This is more than $2800.

Step 7. Write a sentence that answers the question. — Felicity must write at least 1,380 invitations.

Example 2.6.35

Caleb has a pet sitting business. He charges $32 per hour. His monthly expenses are $2,272. How many hours must he work in order to earn a profit of at least $800 per month?

Answer

Caleb must work at least 96 hours.

Example 2.6.36

Elliot has a landscape maintenance business. His monthly expenses are $1,100. If he charges $60 per job, how many jobs must he do to earn a profit of at least $4,000 a month?

Answer

Elliot must work at least 85 jobs.

There are many situations in which several quantities contribute to the total expense. We must make sure to account for all the individual expenses when we solve problems like this.

Example 2.6.37

Malik is planning a six-day summer vacation trip. He has $840 in savings, and he earns $45 per hour for tutoring. The trip will cost him $525 for airfare, $780 for food and sightseeing, and $95 per night for the hotel. How many hours must he tutor to have enough money to pay for the trip?

Answer

Step 1. Read the problem.

Step 2. Identify what you are looking for. — the number of hours Malik must tutor

Step 3. Name what you are looking for.

Choose a variable to represent that quantity. — Let $h =$ the number of hours.

Step 4. Translate. Write a sentence that gives the information to find it. — The expenses must be less than or equal to the income. The cost of airfare plus the cost of food and sightseeing and the hotel bill must be less than the savings plus the amount earned tutoring.

Translate into an inequality. — $525 + 780 + 95(6) \leq 840 + 45h$

Step 5. Solve the inequality.

$$\begin{aligned} 1,875 &\leq 840 + 45h \\ 1,035 &\leq 45h \\ 23 &\leq h \\ h &\geq 23 \end{aligned}$$

Step 6. Check the answer in the problem and make sure it makes sense.
We substitute 23 into the inequality.

$$\begin{aligned} 1,875 &\leq 840 + 45h \\ 1,875 &\leq 840 + 45(23) \\ 1,875 &\leq 1875 \end{aligned}$$

Step 7. Write a sentence that answers the question. — Malik must tutor at least 23 hours.

Example 2.6.38

Brenda's best friend is having a destination wedding and the event will last three days. Brenda has $500 in savings and can earn $15 an hour babysitting. She expects to pay $350 airfare, $375 for food and entertainment and $60 a night for her share of a hotel room. How many hours must she babysit to have enough money to pay for the trip?

Answer

Brenda must babysit at least 27 hours.

Example 2.6.39

Josue wants to go on a 10-night road trip with friends next spring. It will cost him $180 for gas, $450 for food, and $49 per night to share a motel room. He has $520 in savings and can earn $30 per driveway shoveling snow. How many driveways must he shovel to have enough money to pay for the trip?

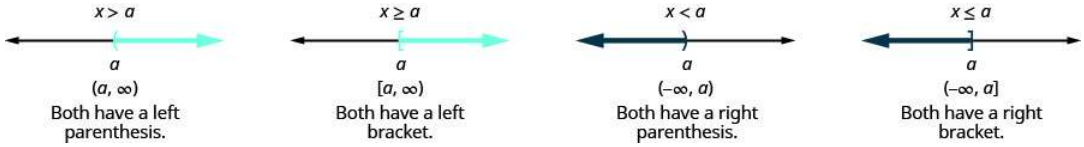
Answer

Josue must shovel at least 20 driveways.

Key Concepts

- **Inequalities, Number Lines, and Interval Notation**

$x > a \quad x \geq a \quad x < a \quad x \leq a$



- **Linear Inequality**
  - A **linear inequality** is an inequality in one variable that can be written in one of the following forms where a, b, and $c$are real numbers and $a \neq 0$:

$$ax + b < c, \qquad ax + b \leq c, \qquad ax + b > c, \qquad ax + b \geq c.$$

- **Addition and Subtraction Property of Inequality**
  - For any numbers a, b, and c, if a<b,thena<b,then

$$a + c < b + c \quad a - c < b - c$$
$$a + c > b + c \quad a - c > b - c$$

  - We can add or subtract the same quantity from both sides of an inequality and still keep the inequality.
- **Multiplication and Division Property of Inequality**
  - For any numbers a, b, and c,

multiply or divide by a positive

$$\text{if } a < b \text{ and } c > 0, \text{ then } ac < bc \text{ and } \frac{a}{c} < \frac{b}{c}.$$
$$\text{if } a > b \text{ and } c > 0, \text{ then } ac > bc \text{ and } \frac{a}{c} > \frac{b}{c}.$$

multiply or divide by a negative

$$\text{if } a < b \text{ and } c < 0, \text{ then } ac > bc \text{ and } \frac{a}{c} > \frac{b}{c}.$$
$$\text{if } a > b \text{ and } c < 0, \text{ then } ac < bc \text{ and } \frac{a}{c} < \frac{b}{c}.$$

- **Phrases that indicate inequalities**

>	≥	<	≤
is greater than	is greater than or equal to	is less than	is less than or equal to
is more than	is at least	is smaller than	is at most
is larger than	is no less than	has fewer than	is no more than
exceeds	is the minimum	is lower than	is the maximum

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2.6E: Exercises

Practice Makes Perfect

Graph Inequalities on the Number Line

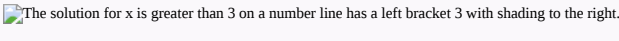
In the following exercises, graph each inequality on the number line and write in interval notation.

ⓐ $x < -2$
ⓑ $x \geq -3.5$
ⓒ $x \leq \frac{2}{3}$

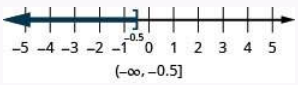
ⓐ $x > 3$
ⓑ $x \leq -0.5$
ⓒ $x \geq \frac{1}{3}$

Answer

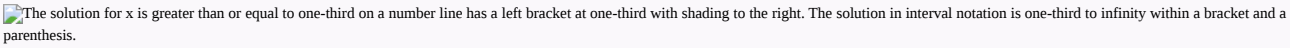
ⓐ


The solution for x is greater than 3 on a number line has a left bracket 3 with shading to the right. The solution in interval notation is 3 to infinity within parentheses.

ⓑ



ⓒ

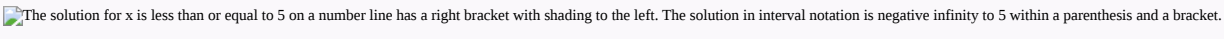

The solution for x is greater than or equal to one-third on a number line has a left bracket at one-third with shading to the right. The solution in interval notation is one-third to infinity within a bracket and a parenthesis.

ⓐ $x \geq -4$
ⓑ $x < 2.5$
ⓒ $x > -\frac{3}{2}$

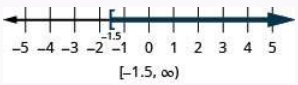
ⓐ $x \leq 5$
ⓑ $x \geq -1.5$
ⓒ $x < -\frac{7}{3}$

Answer

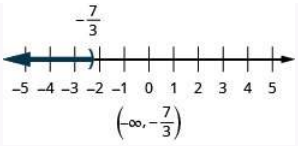
ⓐ


The solution for x is less than or equal to 5 on a number line has a right bracket with shading to the left. The solution in interval notation is negative infinity to 5 within a parenthesis and a bracket.

ⓑ



ⓒ



ⓐ $-5 < x < 2$
ⓑ $-3 \leq x < -1$
ⓒ $0 \leq x \leq 1.5$

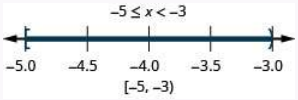
ⓐ $-2 < x < 0$
ⓑ $-5 \leq x < -3$
ⓒ $0 \leq x \leq 3.5$

Answer

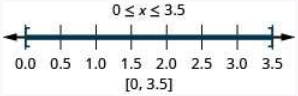
ⓐ

Negative 2 is less than x which is less than 0. There is an open circle at negative 2 and an open circle at 0 and shading between negative 2 and 0 on the number line. The interval notation is negative 2 and 0 within parentheses.

ⓑ



ⓒ



ⓐ $-1 < x < 3$
ⓑ $-3 < x \leq -2$
ⓒ $-1.25 \leq x \leq 0$

ⓐ $-4 < x < 2$
ⓑ $-5 < x \leq -2$
ⓒ $-3.75 \leq x \leq 0$

Answer

ⓐ

Negative 4 is less than x which is less than 2. There is a open circle at negative 4 and an open circle at 2 and shading between negative 4 and 2 on the number line. The interval notation is negative 4 and 2 within parentheses.

ⓑ

Negative 5 is less than x which is less than or equal to 2. There is a open circle at negative 5 and a closed circle at negative 2 and shading between negative 5 and negative 2 on the number line. The interval notation is negative 5 and negative 2 within a parenthesis and a bracket.

ⓒ


Negative 3.75 is less than or equal to x which is less than or equal to 0. There is a closed circle at negative 3.75 and a closed circle at 0 and shading between negative 3.75 and 0 on the number line. The interval notation is negative 3.75 and 0 within brackets.

Solve Linear Inequalities

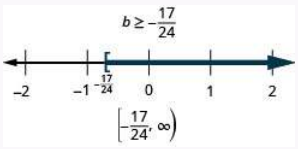
In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

ⓐ $a+\frac{3}{4} \geq \frac{7}{10}$
ⓑ $8x > 72$
ⓒ $20 > \frac{2}{5}h$

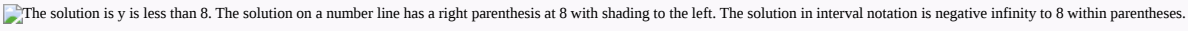
ⓐ $b+\frac{7}{8} \geq \frac{1}{6}$
ⓑ $6y < 48$
ⓒ $40 < \frac{5}{8}k$

Answer

ⓐ



ⓑ


The solution is y is less than 8. The solution on a number line has a right parenthesis at 8 with shading to the left. The solution in interval notation is negative infinity to 8 within parentheses.

ⓒ

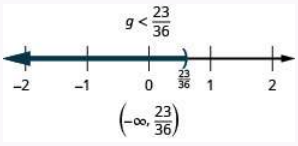

The solution is k is greater than 64. The solution on a number line has a left parenthesis at 64 with shading to the right. The solution in interval notation is 64 to infinity within parentheses.

ⓐ $f-\frac{13}{20} < -\frac{5}{12}$
ⓑ $9t \geq -27$
ⓒ $\frac{7}{6}j \geq 42$

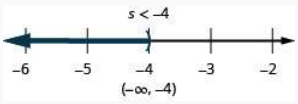
ⓐ $g-\frac{11}{12} < -\frac{5}{18}$
ⓑ $7s < -28$
ⓒ $\frac{9}{4}g \leq 36$

Answer

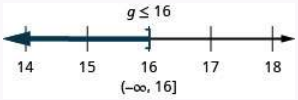
ⓐ



ⓑ



ⓒ

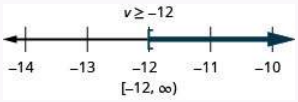


ⓐ $-5u \geq 65$
ⓑ $\frac{a}{-3} \leq 9$

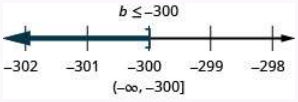
ⓐ $-8v \leq 96$
ⓑ $\frac{b}{-10} \geq 30$

Answer

ⓐ



ⓑ

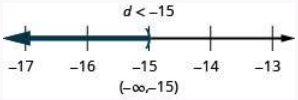


ⓐ $-9c < 126$
ⓑ $-25 < \frac{p}{-5}$

ⓐ $-7d > 105$
ⓑ $-18 > \frac{q}{-6}$

Answer

ⓐ



ⓑ

The solution is q is greater than 108. The solution on a number line has a left parentheses at 108 with shading to the right. The solution in interval notation is 108 to infinity within parentheses.

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

$4v \geq 9v - 40$

$5u \leq 8u - 21$

Answer

The solution is u is greater than or equal to 7. The solution on a number line has a left bracket at 7 with shading to the right. The solution in interval notation is 7 to infinity within a a bracket and a parenthesis.

$13q < 7q - 29$

$9p > 14p - 18$

Answer

The solution is p is less than eighteen fifths. The solution on a number line has a right parenthesis at eighteen fifths with shading to the left. The solution in interval notation negative infinity to eighteen fifths within parentheses.

$12x + 3(x + 7) > 10x - 24$

$9y + 5(y + 3) < 4y - 35$

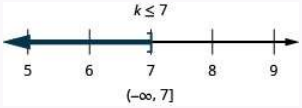
Answer

The solution is y is less than negative 5. The solution on a number line has a right parenthesis at negative 5 with shading to the left. The solution in interval notation is negative infinity to negative 5 within parentheses.

$6h - 4(h - 1) \leq 7h - 11$

$4k - (k - 2) \geq 7k - 26$

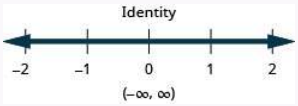
Answer



$8m - 2(14 - m) \geq 7(m - 4) + 3m$

$6n - 12(3 - n) \leq 9(n - 4) + 9n$

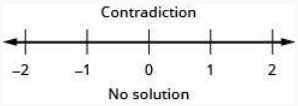
Answer



$\frac{3}{4}b - \frac{1}{3}b < \frac{5}{12}b - \frac{1}{2}$

$9u + 5(2u - 5) \geq 12(u - 1) + 7u$

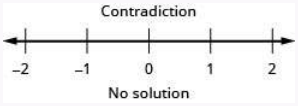
Answer



$\frac{2}{3}g - \frac{1}{2}(g - 14) \leq \frac{1}{6}(g + 42)$

$\frac{4}{5}h - \frac{2}{3}(h-9) \geq \frac{1}{15}(2h+90)$

Answer



$\frac{5}{6}a - \frac{1}{4}a > \frac{7}{12}a + \frac{2}{3}$

$12v + 3(4v-1) \leq 19(v-2) + 5v$

Answer

The inequality is a contradiction. So, there is no solution. As a result, there is no graph on the number line or interval notation.

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

$15k \leq -40$

$35k \geq -77$

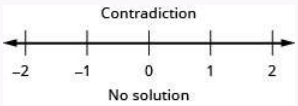
Answer

The solution is k is greater than or equal to negative eleven fifthss. The solution on a number line has a left bracket at negative eleven fifths with shading to the right. The solution in interval notation is negative eleven fifths to negative infinity within a bracket and a parenthesis.

$23p - 2(6-5p) > 3(11p-4)$

$18q - 4(10-3q) < 5(6q-8)$

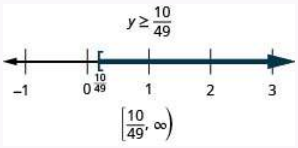
Answer



$-\frac{9}{4}x \geq -\frac{5}{12}$

$-\frac{21}{8}y \leq -\frac{15}{28}$

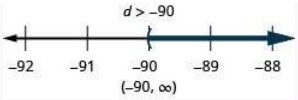
Answer



$c + 34 < -99$

$d + 29 > -61$

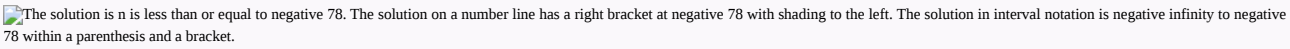
Answer



$\frac{m}{18} \geq -4$

$\frac{n}{13} \leq -6$

Answer


The solution is n is less than or equal to negative 78. The solution on a number line has a right bracket at negative 78 with shading to the left. The solution in interval notation is negative infinity to negative 78 within a parenthesis and a bracket.

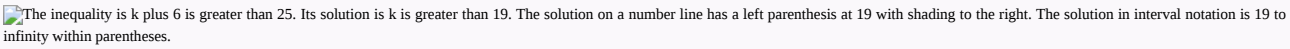
Translate to an Inequality and Solve

In the following exercises, translate and solve. Then graph the solution on the number line and write the solution in interval notation.

Three more than h is no less than 25.

Six more than k exceeds 25.

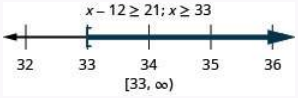
Answer


The inequality is k plus 6 is greater than 25. Its solution is k is greater than 19. The solution on a number line has a left parenthesis at 19 with shading to the right. The solution in interval notation is 19 to infinity within parentheses.

Ten less than w is at least 39.

Twelve less than x is no less than 21.

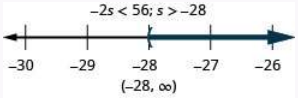
Answer



Negative five times r is no more than 95.

Negative two times s is lower than 56.

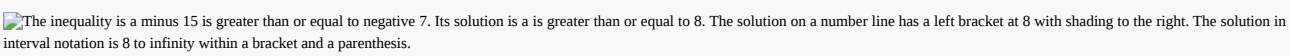
Answer



Nineteen less than b is at most -22.

Fifteen less than a is at least -7.

Answer


The inequality is a minus 15 is greater than or equal to negative 7. Its solution is a is greater than or equal to 8. The solution on a number line has a left bracket at 8 with shading to the right. The solution in interval notation is 8 to infinity within a bracket and a parenthesis.

Solve Applications with Linear Inequalities

In the following exercises, solve.

Alan is loading a pallet with boxes that each weighs 45 pounds. The pallet can safely support no more than 900 pounds. How many boxes can he safely load onto the pallet?

The elevator in Yehire's apartment building has a sign that says the maximum weight is 2100 pounds. If the average weight of one person is 150 pounds, how many people can safely ride the elevator?

Answer

A maximum of 14 people can safely ride in the elevator.

Andre is looking at apartments with three of his friends. They want the monthly rent to be no more than $2,360. If the roommates split the rent evenly among the four of them, what is the maximum rent each will pay?

Arleen got a $20 gift card for the coffee shop. Her favorite iced drink costs $3.79. What is the maximum number of drinks she can buy with the gift card?

Answer

five drinks

Teegan likes to play golf. He has budgeted $60 next month for the driving range. It costs him $10.55 for a bucket of balls each time he goes. What is the maximum number of times he can go to the driving range next month?

Ryan charges his neighbors $17.50 to wash their car. How many cars must he wash next summer if his goal is to earn at least $1,500?

Answer

86 cars

Keshad gets paid $2,400 per month plus 6% of his sales. His brother earns $3,300 per month. For what amount of total sales will Keshad's monthly pay be higher than his brother's monthly pay?

Kimuyen needs to earn $4,150 per month in order to pay all her expenses. Her job pays her $3,475 per month plus 4% of her total sales. What is the minimum Kimuyen's total sales must be in order for her to pay all her expenses?

Answer

$16,875

Andre has been offered an entry-level job. The company offered him $48,000 per year plus 3.5% of his total sales. Andre knows that the average pay for this job is $62,000. What would Andre's total sales need to be for his pay to be at least as high as the average pay for this job?

Nataly is considering two job offers. The first job would pay her $83,000 per year. The second would pay her $66,500 plus 15% of her total sales. What would her total sales need to be for her salary on the second offer be higher than the first?

Answer

$110,000

Jake’s water bill is $24.80 per month plus $2.20 per ccf (hundred cubic feet) of water. What is the maximum number of ccf Jake can use if he wants his bill to be no more than $60?

Kiyoshi’s phone plan costs $17.50 per month plus $0.15 per text message. What is the maximum number of text messages Kiyoshi can use so the phone bill is no more than $56.60?

Answer

260 messages

Marlon’s TV plan costs $49.99 per month plus $5.49 per first-run movie. How many first-run movies can he watch if he wants to keep his monthly bill to be a maximum of $100?

Kellen wants to rent a banquet room in a restaurant for her cousin’s baby shower. The restaurant charges $350 for the banquet room plus $32.50 per person for lunch. How many people can Kellen have at the shower if she wants the maximum cost to be $1,500?

Answer

35 people

Moshde runs a hairstyling business from her house. She charges $45 for a haircut and style. Her monthly expenses are $960. She wants to be able to put at least $1,200 per month into her savings account order to open her own salon. How many “cut & styles” must she do to save at least $1,200 per month?

Noe installs and configures software on home computers. He charges $125 per job. His monthly expenses are $1,600. How many jobs must he work in order to make a profit of at least $2,400?

Answer

32 jobs

Katherine is a personal chef. She charges $115 per four-person meal. Her monthly expenses are $3,150. How many four-person meals must she sell in order to make a profit of at least $1,900?

Melissa makes necklaces and sells them online. She charges $88 per necklace. Her monthly expenses are $3,745. How many necklaces must she sell if she wants to make a profit of at least $1,650?

Answer

62 necklaces

Five student government officers want to go to the state convention. It will cost them $110 for registration, $375 for transportation and food, and $42 per person for the hotel. There is $450 budgeted for the convention in the student government savings account. They can earn the rest of the money they need by having a car wash. If they charge $5 per car, how many cars must they wash in order to have enough money to pay for the trip?

Cesar is planning a four-day trip to visit his friend at a college in another state. It will cost him $198 for airfare, $56 for local transportation, and $45 per day for food. He has $189 in savings and can earn $35 for each lawn he mows. How many lawns

must he mow to have enough money to pay for the trip?

Answer

seven lawns

Alonzo works as a car detailer. He charges $175 per car. He is planning to move out of his parents' house and rent his first apartment. He will need to pay $120 for application fees, $950 for security deposit, and first and last months' rent at $1,140 per month. He has $1,810 in savings. How many cars must he detail to have enough money to rent the apartment?

Eun-Kyung works as a tutor and earns $60 per hour. She has $792 in savings. She is planning an anniversary party for her parents. She would like to invite 40 guests. The party will cost her $1,520 for food and drinks and $150 for the photographer. She will also have a favor for each of the guests, and each favor will cost $7.50. How many hours must she tutor to have enough money for the party?

Answer

20 hours

Everyday Math

Maximum load on a stage In 2014, a high school stage collapsed in Fullerton, California, when 250 students got on stage for the finale of a musical production. Two dozen students were injured. The stage could support a maximum of 12,750 pounds. If the average weight of a student is assumed to be 140 pounds, what is the maximum number of students who could safely be on the stage?

Maximum weight on a boat In 2004, a water taxi sank in Baltimore harbor and five people drowned. The water taxi had a maximum capacity of 3,500 pounds (25 people with average weight 140 pounds). The average weight of the 25 people on the water taxi when it sank was 168 pounds per person. What should the maximum number of people of this weight have been?

Answer

20 people

Wedding budget Adele and Walter found the perfect venue for their wedding reception. The cost is $9850 for up to 100 guests, plus $38 for each additional guest. How many guests can attend if Adele and Walter want the total cost to be no more than $12,500?

Shower budget Penny is planning a baby shower for her daughter-in-law. The restaurant charges $950 for up to 25 guests, plus $31.95 for each additional guest. How many guests can attend if Penny wants the total cost to be no more than $1,500?

Answer

42 guests

Writing Exercises

Explain why it is necessary to reverse the inequality when solving $-5x > 10$.

Explain why it is necessary to reverse the inequality when solving $n - 3 < 12$.

Answer

Answers will vary.

Find your last month's phone bill and the hourly salary you are paid at your job. Calculate the number of hours of work it would take you to earn at least enough money to pay your phone bill by writing an appropriate inequality and then solving it. Do you feel this is an appropriate number of hours? Is this the appropriate phone plan for you?

Find out how many units you have left, after this term, to achieve your college goal and estimate the number of units you can take each term in college. Calculate the number of terms it will take you to achieve your college goal by writing an appropriate inequality and then solving it. Is this an acceptable number of terms until you meet your goal? What are some ways you could accelerate this process?

Answer

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
graph inequalities on the number line.			
solve linear inequalities.			
translate words to an inequality and solve.			
solve applications with linear inequalities.			

ⓑ After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

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2.7: Solve Compound Inequalities

Learning Objectives

By the end of this section, you will be able to:

- Solve compound inequalities with “and”
- Solve compound inequalities with “or”
- Solve applications with compound inequalities

Before you get started, take this readiness quiz.

1. Simplify: $\frac{2}{5}(x+10)$.
   If you missed this problem, review [link].
2. Simplify: $-(x-4)$.
   If you missed this problem, review [link].

Solve Compound Inequalities with “and”

Now that we know how to solve linear inequalities, the next step is to look at compound inequalities. A **compound inequality** is made up of two inequalities connected by the word “and” or the word “or.” For example, the following are compound inequalities.

$$\begin{array}{lll} x+3>-4 & \textbf{and} & 4x-5\leq 3 \\ 2(y+1)<0 & \textbf{or} & y-5\geq -2 \end{array}$$

COMPOUND INEQUALITY

A **compound inequality** is made up of two inequalities connected by the word “and” or the word “or.”

To solve a compound inequality means to find all values of the variable that make the compound inequality a true statement. We solve compound inequalities using the same techniques we used to solve linear inequalities. We solve each inequality separately and then consider the two solutions.

To solve a compound inequality with the word “and,” we look for all numbers that make *both* inequalities true. To solve a compound inequality with the word “or,” we look for all numbers that make *either* inequality true.

Let’s start with the compound inequalities with “and.” Our solution will be the numbers that are solutions to *both* inequalities known as the intersection of the two inequalities. Consider the intersection of two streets—the part where the streets overlap—belongs to both streets.



To find the solution of the compound inequality, we look at the graphs of each inequality and then find the numbers that belong to both graphs—where the graphs overlap.

For the compound inequality $x>-3$ and $x\leq 2$, we graph each inequality. We then look for where the graphs “overlap”. The numbers that are shaded on both graphs, will be shaded on the graph of the solution of the compound inequality. See Figure **2.7.1**.

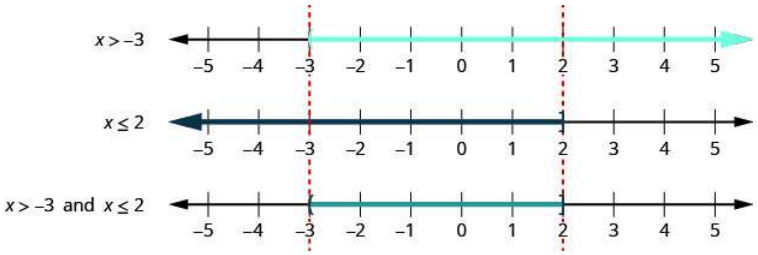


Figure 2.7.1

We can see that the numbers between -3 and 2 are shaded on both of the first two graphs. They will then be shaded on the solution graph.

The number -3 is not shaded on the first graph and so since it is not shaded on both graphs, it is not included on the solution graph.

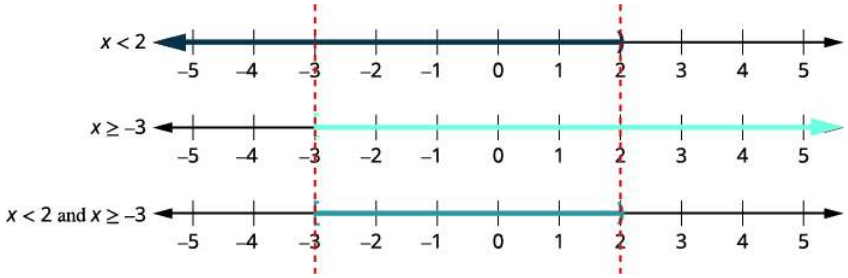
The number two is shaded on both the first and second graphs. Therefore, it is be shaded on the solution graph.

This is how we will show our solution in the next examples.

Example 2.7.1

Solve $6x-3<9$ and $2x+7\geq 3$. Graph the solution and write the solution in interval notation.

Answer

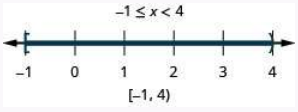
	$6x-3<9$	and	$2x+9\geq 3$
Step 1. Solve each inequality.	$6x-3<9$		$2x+9\geq 3$
	$6x<12$		$2x\geq -6$
	$x<2$	and	$x\geq -3$
Step 2. Graph each solution. Then graph the numbers that make both inequalities true. The final graph will show all the numbers that make both inequalities true—the numbers shaded on *both* of the first two graphs.			
Step 3. Write the solution in interval notation.	$[-3,2)$		

All the numbers that make both inequalities true are the solution to the compound inequality.

Example 2.7.2

Solve the compound inequality. Graph the solution and write the solution in interval notation: $4x-7<9$ and $5x+8\geq 3$.

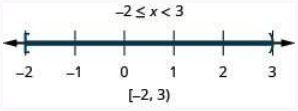
Answer



Example 2.7.3

Solve the compound inequality. Graph the solution and write the solution in interval notation: $3x-4<5$ and $4x+9\geq 1$.

Answer



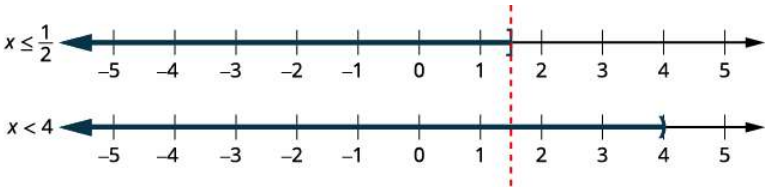
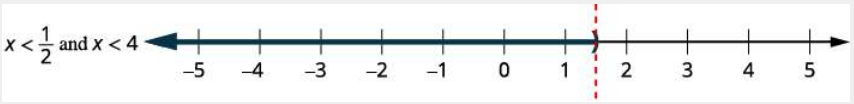
SOLVE A COMPOUND INEQUALITY WITH "AND."

1. Solve each inequality.
2. Graph each solution. Then graph the numbers that make *both* inequalities true.
   This graph shows the solution to the compound inequality.
3. Write the solution in interval notation.

Example 2.7.4

Solve $3(2x+5) \le 18$ and $2(x-7) < -6$. Graph the solution and write the solution in interval notation.

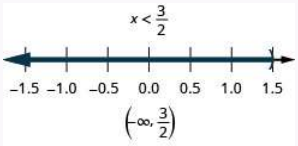
Answer

	$3(2x+5) \le 18$	and	$2(x-7) < -6$
Solve each inequality.	$6x+15 \le 18$		$2x-14 < -6$
	$6x \le 3$		$2x < 8$
	$x \le \frac{1}{2}$	and	$x < 4$
Graph each solution.			
Graph the numbers that make both inequalities true.			
Write the solution in interval notation.	$(-\infty, \frac{1}{2}]$		

Example 2.7.5

Solve the compound inequality. Graph the solution and write the solution in interval notation: $2(3x+1) \le 20$ and $4(x-1) < 2$.

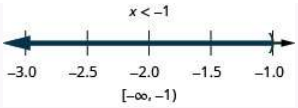
Answer



Example 2.7.6

Solve the compound inequality. Graph the solution and write the solution in interval notation: $5(3x-1) \le 10$ and $4(x+3) < 8$.

Answer



Example 2.7.7

Solve $\frac{1}{3}x - 4 \geq -2$ and $-2(x - 3) \geq 4$. Graph the solution and write the solution in interval notation.

Answer

$\frac{1}{3}x$
$-2(x-3) \geq 4$
$x \geq -2$

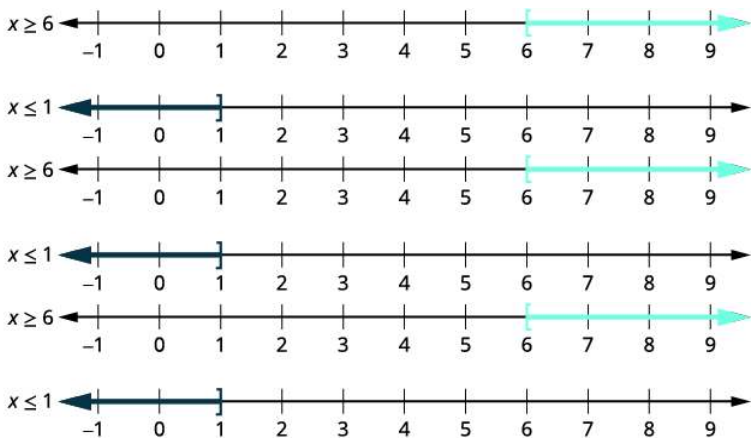
Solve each inequality.

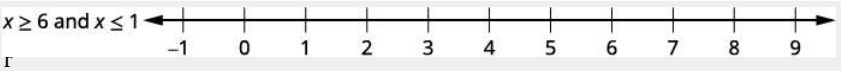
$\frac{1}{3}x$
$-2x+6 \geq 4$
$x \geq -2$

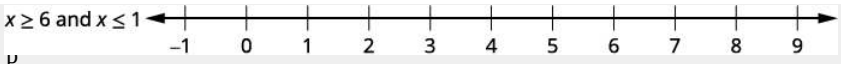
$\frac{1}{3}x$
$-2x \geq -2$
$x \geq 2$

x
$x \leq 1$
$x \geq 6$

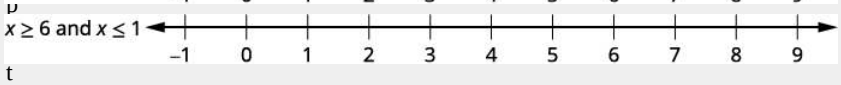
Graph each solution.







b



the number that make both inequalities true.

There are no numbers that make both inequalities true.

This is a contradiction so there is no solution.There are no numbers that make both inequalities true.

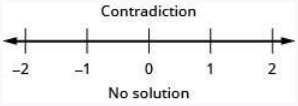
This is a contradiction so there is no solution.There are no numbers that make both inequalities true.

This is a contradiction so there is no solution.

Example 2.7.8

Solve the compound inequality. Graph the solution and write the solution in interval notation: $\frac{1}{4}x - 3 \geq -1$ and $-3(x-2) \geq 2$.

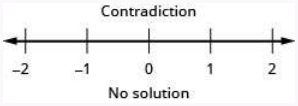
Answer



Example 2.7.9

Solve the compound inequality. Graph the solution and write the solution in interval notation: $\frac{1}{5}x - 5 \geq -3$ and $-4(x-1) \geq -2$.

Answer



Sometimes we have a compound inequality that can be written more concisely. For example, $a < x$ and $x < b$ can be written simply as $a < x < b$ and then we call it a **double inequality**. The two forms are equivalent.

DOUBLE INEQUALITY

A double inequality is a compound inequality such as $a < x < b$. It is equivalent to $a < x$ and $x < b$.

Other forms:	$a < x < b$	is equivalent to	$a < x$	and	$x < b$
	$a \leq x \leq b$	is equivalent to	$a \leq x$	and	$x \leq b$
	$a > x > b$	is equivalent to	$a > x$	and	$x > b$
	$a \geq x \geq b$	is equivalent to	$a \geq x$	and	$x \geq b$

To solve a double inequality we perform the same operation on all three "parts" of the double inequality with the goal of isolating the variable in the center.

Example 2.7.10

Solve $-4 \leq 3x - 7 < 8$. Graph the solution and write the solution in interval notation.

Answer

	$-4 \leq 3x - 7 < 8$
Add 7 to all three parts.	$-4+7 \leq 3x - 7+7 < 8+7$
Simplify.	$3 \leq 3x < 15$
Divide each part by three.	$\frac{3}{3} \leq \frac{3x}{3} < \frac{15}{3}$
Simplify.	$1 \leq x < 5$
Graph the solution.	
Write the solution in interval notation.	$[1, 5)$

When written as a double inequality, $1 \leq x < 5$, it is easy to see that the solutions are the numbers caught between one and five, including one, but not five. We can then graph the solution immediately as we did above.

Another way to graph the solution of $1 \leq x < 5$ is to graph both the solution of $x \geq 1$ and the solution of $x < 5$. We would then find the numbers that make both inequalities true as we did in previous examples.

Example 2.7.11

Solve the compound inequality. Graph the solution and write the solution in interval notation: $-5 \leq 4x - 1 < 7$.

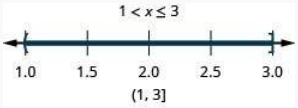
Answer



Example 2.7.12

Solve the compound inequality. Graph the solution and write the solution in interval notation: $-3 < 2x - 5 \leq 1$.

Answer



Solve Compound Inequalities with “or”

To solve a **compound inequality** with “or”, we start out just as we did with the compound inequalities with “and”—we solve the two inequalities. Then we find all the numbers that make *either* inequality true.

Just as the United States is the union of all of the 50 states, the solution will be the union of all the numbers that make either inequality true. To find the solution of the compound inequality, we look at the graphs of each inequality, find the numbers that belong to either graph and put all those numbers together.

To write the solution in **interval notation**, we will often use the **union symbol**, ∪, to show the union of the solutions shown in the graphs.

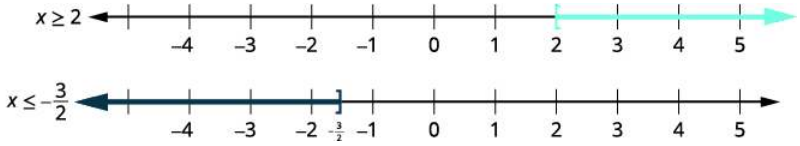
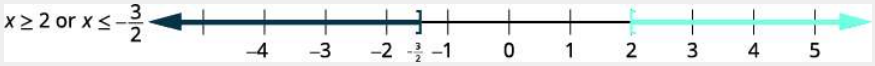
SOLVE A COMPOUND INEQUALITY WITH “OR.”

1. Solve each inequality.
2. Graph each solution. Then graph the numbers that make either inequality true.
3. Write the solution in interval notation.

Example 2.7.13

Solve $5 - 3x \leq -1$ or $8 + 2x \leq 5$. Graph the solution and write the solution in interval notation.

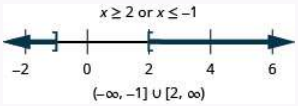
Answer

	$5-3x \le -1$	or	$8+2x \le 5$
Solve each inequality.	$5-3x \le -1$		$8+2x \le 5$
	$-3x \le -6$		$2x \le -3$
	$x \ge 2$	or	$x \le -\frac{3}{2}$
Graph each solution.			
Graph numbers that make either inequality true.			
	$(-\infty, -32] \cup [2, \infty)$		

Example 2.7.14

Solve the compound inequality. Graph the solution and write the solution in interval notation: $1-2x \le -3$ or $7+3x \le 4$.

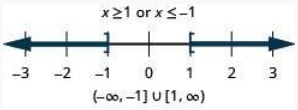
Answer



Example 2.7.15

Solve the compound inequality. Graph the solution and write the solution in interval notation: $2-5x \le -3$ or $5+2x \le 3$.

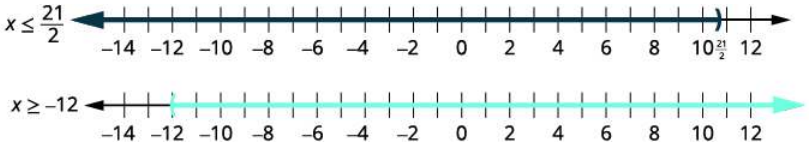
Answer



Example 2.7.16

Solve $\frac{2}{3}x-4 \le 3$ or $\frac{1}{4}(x+8) \ge -1$. Graph the solution and write the solution in interval notation.

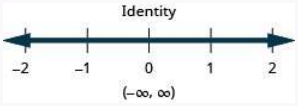
Answer

	$\frac{2}{3}x-4\le 3$	or	$\frac{1}{4}(x+8)\ge -1$
Solve each inequality.	$3(\frac{2}{3}x-4)\le 3(3)$		$4\cdot\frac{1}{4}(x+8)\ge 4\cdot(-1)$
	$2x-12\le 9$		$x+8\ge -4$
	$2x\le 21$		$x\ge -12$
	$x\le \frac{21}{2}$		
	$x\le \frac{21}{2}$	or	$x\ge -12$
Graph each solution.			
Graph numbers that make either inequality true.			
	The solution covers all real numbers.		
	$(-\infty,\infty)$		

Example 2.7.17

Solve the compound inequality. Graph the solution and write the solution in interval notation: $\frac{3}{5}x-7\le -1$ or $\frac{1}{3}(x+6)\ge -2$.

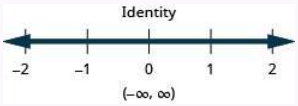
Answer



Example 2.7.18

Solve the compound inequality. Graph the solution and write the solution in interval notation: $\frac{3}{4}x-3\le 3$ or $\frac{2}{5}(x+10)\ge 0$.

Answer



Solve Applications with Compound Inequalities

Situations in the real world also involve compound inequalities. We will use the same problem solving strategy that we used to solve linear equation and inequality applications.

Recall the problem solving strategies are to first read the problem and make sure all the words are understood. Then, identify what we are looking for and assign a variable to represent it. Next, restate the problem in one sentence to make it easy to translate into a **compound inequality**. Last, we will solve the compound inequality.

Example 2.7.19

Due to the drought in California, many communities have tiered water rates. There are different rates for Conservation Usage, Normal Usage and Excessive Usage. The usage is measured in the number of hundred cubic feet (hcf) the property owner uses.

During the summer, a property owner will pay $24.72 plus $1.54 per hcf for Normal Usage. The bill for Normal Usage would be between or equal to $57.06 and $171.02. How many hcf can the owner use if he wants his usage to stay in the normal range?

Answer

Identify what we are looking for.	The number of hcf he can use and stay in the "normal usage" billing range.
Name what we are looking for.	Let x=x= the number of hcf he can use.
Translate to an inequality.	Bill is $24.72 plus $1.54 times the number of hcf he uses or $24.72+1.54x$.
	$\underbrace{\text{His bill will be between or equal to \$57.06 and \$171.02.}}$ $57.06 \le 24.74+1.54x \le 171.02$
Solve the inequality.	$57.06 \le 24.74+1.54x \le 171.02$ $57.06-24.72 \le 24.74-24.72+1.54x \le 171.02-24.72$ $32.34 \le 1.54x \le 146.3$ $\frac{32.34}{1.54} \le \frac{1.54x}{1.54} \le \frac{146.3}{1.54}$ $21 \le x \le 95$
Answer the question.	The property owner can use 21–95 hcf and still fall within the "normal usage" billing range.

Example 2.7.20

Due to the drought in California, many communities now have tiered water rates. There are different rates for Conservation Usage, Normal Usage and Excessive Usage. The usage is measured in the number of hundred cubic feet (hcf) the property owner uses.

During the summer, a property owner will pay $24.72 plus $1.32 per hcf for Conservation Usage. The bill for Conservation Usage would be between or equal to $31.32 and $52.12. How many hcf can the owner use if she wants her usage to stay in the conservation range?

Answer

The homeowner can use 5–20 hcf and still fall within the "conservation usage" billing range.

Example 2.7.21

Due to the drought in California, many communities have tiered water rates. There are different rates for Conservation Usage, Normal Usage and Excessive Usage. The usage is measured in the number of hundred cubic feet (hcf) the property owner uses.

During the winter, a property owner will pay $24.72 plus $1.54 per hcf for Normal Usage. The bill for Normal Usage would be between or equal to $49.36 and $86.32. How many hcf will he be allowed to use if he wants his usage to stay in the normal range?

Answer

The homeowner can use 16–40 hcf and still fall within the "normal usage" billing range.

Access this online resource for additional instruction and practice with solving compound inequalities.

- Compound inequalities

Key Concepts

- **How to solve a compound inequality with "and"**
  1. Solve each inequality.
  2. Graph each solution. Then graph the numbers that make *both* inequalities true. This graph shows the solution to the compound inequality.
  3. Write the solution in interval notation.
- **Double Inequality**
  - A **double inequality** is a compound inequality such as $a < x < b$. It is equivalent to $a < x$ and $x < b$.

    Other forms:

$$\begin{array}{lll} a < x < b & \text{is equivalent to} & a < x \text{ and } x < b \\ a \leq x \leq b & \text{is equivalent to} & a \leq x \text{ and } x \leq b \\ a > x > b & \text{is equivalent to} & a > x \text{ and } x > b \\ a \geq x \geq b & \text{is equivalent to} & a \geq x \text{ and } x \geq b \end{array}$$

- **How to solve a compound inequality with "or"**
  1. Solve each inequality.
  2. Graph each solution. Then graph the numbers that make either inequality true.
  3. Write the solution in interval notation.

Glossary

compound inequality

A compound inequality is made up of two inequalities connected by the word "and" or the word "or."

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2.7E: Exercises

Practice Makes Perfect

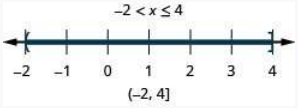
Solve Compound Inequalities with "and"

In the following exercises, solve each inequality, graph the solution, and write the solution in interval notation.

$x < 3$ and $x \geq 1$

$x \leq 4$ and $x > -2$

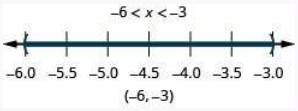
Answer



$x \geq -4$ and $x \leq -1$

$x > -6$ and $x < -3$

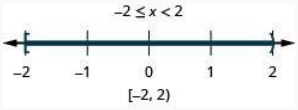
Answer



$5x - 2 < 8$ and $6x + 9 \geq 3$

$4x - 1 < 7$ and $2x + 8 \geq 4$

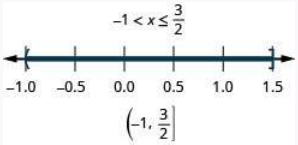
Answer



$4x + 6 \leq 2$ and
$2x + 1 \geq -5$

$4x - 2 \leq 4$ and
$7x - 1 > -8$

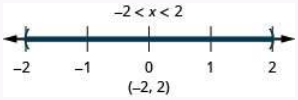
Answer



$2x - 11 < 5$ and
$3x - 8 > -5$

$7x - 8 < 6$ and
$5x + 7 > -3$

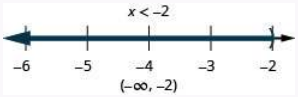
Answer



$4(2x - 1) \leq 12$ and
$2(x + 1) < 4$

$5(3x - 2) \leq 5$ and
$3(x + 3) < 3$

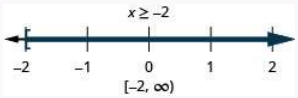
Answer



$3(2x - 3) > 3$ and
$4(x + 5) \geq 4$

$-3(x + 4) < 0$ and
$-1(3x - 1) \leq 7$

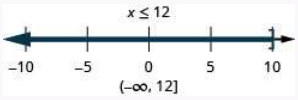
Answer



$\frac{1}{2}(3x - 4) \leq 1$ and
$\frac{1}{3}(x + 6) \leq 4$

$\frac{3}{4}(x - 8) \leq 3$ and
$\frac{1}{5}(x - 5) \leq 3$

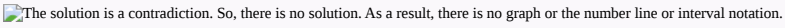
Answer



$5x - 2 \leq 3x + 4$ and
$3x - 4 \geq 2x + 1$

$\frac{3}{4}x - 5 \geq -2$ and
$-3(x + 1) \geq 6$

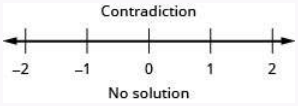
Answer



$\frac{2}{3}x - 6 \geq -4$ and
$-4(x + 2) \geq 0$

$\frac{1}{2}(x - 6) + 2 < -5$ and
$4 - \frac{2}{3}x < 6$

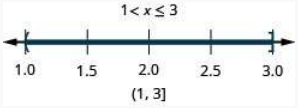
Answer



$-5 \leq 4x - 1 < 7$

$-3 < 2x - 5 \leq 1$

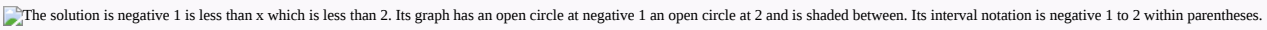
Answer



$5 < 4x + 1 < 9$

$-1 < 3x + 2 < 8$

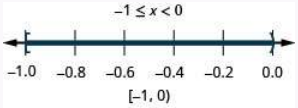
Answer



$-8 < 5x + 2 \leq -3$

$-6 \leq 4x - 2 < -2$

Answer



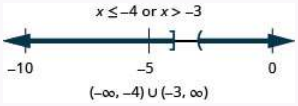
Solve Compound Inequalities with "or"

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

$x \leq -2$ or $x > 3$

$x \leq -4$ or $x > -3$

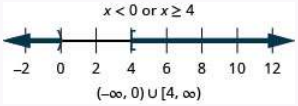
Answer



$x < 2$ or $x \geq 5$

$x < 0$ or $x \geq 4$

Answer



$2 + 3x \leq 4$ or
$5 - 2x \leq -1$

$4 - 3x \leq -2$ or
$2x - 1 \leq -5$

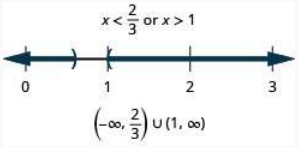
Answer

The solution is x is less than or equal to negative 2 or x is greater than or equal to 2. The graph of the solutions on a number line has a closed circle at negative 2 and shading to the left and a closed circle at 2 with shading to the right. The interval notation is the union of negative infinity to negative 2 within a parenthesis and a bracket and 2 to infinity within a bracket and a parenthesis.

$2(3x - 1) < 4$ or
$3x - 5 > 1$

$3(2x - 3) < -5$ or
$4x - 1 > 3$

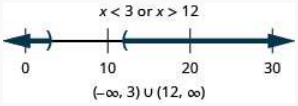
Answer



$\frac{3}{4}x - 2 > 4$ or $4(2 - x) > 0$

$\frac{2}{3}x - 3 > 5$ or $3(5 - x) > 6$

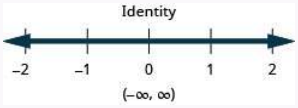
Answer



$3x - 2 > 4$ or $5x - 3 \leq 7$

$2(x + 3) \geq 0$ or
$3(x + 4) \leq 6$

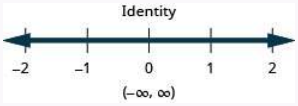
Answer



$\frac{1}{2}x - 3 \leq 4$ or
$\frac{1}{3}(x - 6) \geq -2$

$\frac{3}{4}x + 2 \leq -1$ or
$\frac{1}{2}(x + 8) \geq -3$

Answer



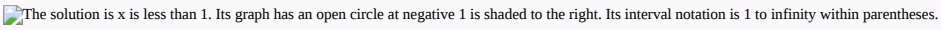
Mixed practice

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

$3x + 7 \leq 1$ and
$2x + 3 \geq -5$

$6(2x - 1) > 6$ and
$5(x + 2) \geq 0$

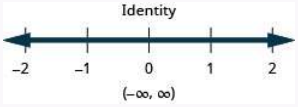
Answer


The solution is x is less than 1. Its graph has an open circle at negative 1 is shaded to the right. Its interval notation is 1 to infinity within parentheses.

$4 - 7x \geq -3$ or
$5(x - 3) + 8 > 3$

$\frac{1}{2}x - 5 \leq 3$ or
$\frac{1}{4}(x - 8) \geq -3$

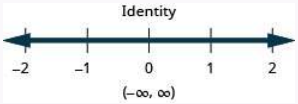
Answer



$-5 \leq 2x - 1 < 7$

$\frac{1}{5}(x-5)+6<4$ and
$3-\frac{2}{3}x<5$

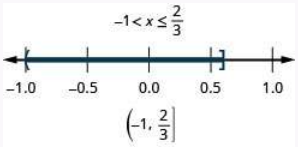
Answer



$4x-2>6$ or
$3x-1\leq -2$

$6x-3\leq 1$ and
$5x-1>-6$

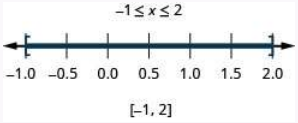
Answer



$-2(3x-4)\leq 2$ and
$-4(x-1)<2$

$-5\leq 3x-2\leq 4$

Answer



Solve Applications with Compound Inequalities

In the following exercises, solve.

Penelope is playing a number game with her sister June. Penelope is thinking of a number and wants June to guess it. Five more than three times her number is between 2 and 32. Write a compound inequality that shows the range of numbers that Penelope might be thinking of.

Gregory is thinking of a number and he wants his sister Lauren to guess the number. His first clue is that six less than twice his number is between four and forty-two. Write a compound inequality that shows the range of numbers that Gregory might be thinking of.

Answer

$5\leq n\leq 24$

Andrew is creating a rectangular dog run in his back yard. The length of the dog run is 18 feet. The perimeter of the dog run must be at least 42 feet and no more than 72 feet. Use a compound inequality to find the range of values for the width of the dog

run.

Elouise is creating a rectangular garden in her back yard. The length of the garden is 12 feet. The perimeter of the garden must be at least 36 feet and no more than 48 feet. Use a compound inequality to find the range of values for the width of the garden.

Answer

$6 \leq w \leq 12$

Everyday Math

Blood Pressure A person's blood pressure is measured with two numbers. The systolic blood pressure measures the pressure of the blood on the arteries as the heart beats. The diastolic blood pressure measures the pressure while the heart is resting.

ⓐ Let x be your systolic blood pressure. Research and then write the compound inequality that shows you what a normal systolic blood pressure should be for someone your age.

ⓑ Let y be your diastolic blood pressure. Research and then write the compound inequality that shows you what a normal diastolic blood pressure should be for someone your age.

Body Mass Index (BMI) is a measure of body fat is determined using your height and weight.

ⓐ Let x be your BMI. Research and then write the compound inequality to show the BMI range for you to be considered normal weight.

ⓑ Research a BMI calculator and determine your BMI. Is it a solution to the inequality in part (a)?

Answer

ⓐ answers vary ⓑ answers vary

Writing Exercises

In your own words, explain the difference between the properties of equality and the properties of inequality.

Explain the steps for solving the compound inequality $2 - 7x \geq -5$ or $4(x - 3) + 7 > 3$.

Answer

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve compound inequalities with "and".			
solve compound inequalities with "or".			
solve applications with compound inequalities.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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2.8: Solve Absolute Value Inequalities

Learning Objectives

By the end of this section, you will be able to:

- Solve absolute value equations
- Solve absolute value inequalities with “less than”
- Solve absolute value inequalities with “greater than”
- Solve applications with absolute value

Before you get started, take this readiness quiz.

1. Evaluate: $-|7|$.
   If you missed this problem, review [link].
2. Fill in $<, >, <, >$, or $=$ for each of the following pairs of numbers.
   ⓐ '_' allowed only in math mode ⓑ '_' allowed only in math mode ⓒ '_' allowed only in math mode ⓓ '_' allowed only in math mode
   If you missed this problem, review [link].
3. Simplify: $14 - 2|8 - 3(4 - 1)|$.
   If you missed this problem, review [link].

Solve Absolute Value Equations

As we prepare to solve absolute value equations, we review our definition of **absolute value**.

ABSOLUTE VALUE

The absolute value of a number is its distance from zero on the number line.

The absolute value of a number n is written as $|n|$ and $|n| \geq 0$ for all numbers.

Absolute values are always greater than or equal to zero.

We learned that both a number and its opposite are the same distance from zero on the number line. Since they have the same distance from zero, they have the same absolute value. For example:

- -5 is 5 units away from 0, so $|-5| = 5$.
- 5 is 5 units away from 0, so $|5| = 5$.

Figure 2.8.1 illustrates this idea.

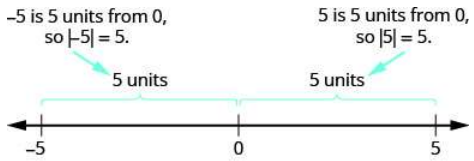


Figure 2.8.1: The numbers 5 and -5 are both five units away from zero.

For the equation |x|=5,|x|=5, we are looking for all numbers that make this a true statement. We are looking for the numbers whose distance from zero is 5. We just saw that both 5 and −5−5 are five units from zero on the number line. They are the solutions to the equation.

If $|x| = 5$
then $x = -5$ **or** $x = 5$

The solution can be simplified to a single statement by writing $x = \pm 5$. This is read, “x is equal to positive or negative 5”.

We can generalize this to the following property for absolute value equations.

ABSOLUTE VALUE EQUATIONS

For any algebraic expression, u, and any positive real number, a,

$$\begin{array}{ll} \textbf{if} & |u| = a \\ \textbf{then} & u = -a \text{ or } u = a \end{array}$$

Remember that an absolute value cannot be a negative number.

Example 2.8.1

Solve:

a. $|x| = 8$
b. $|y| = -6$
c. $|z| = 0$

Answer a

$$\begin{array}{ll} & |x| = 8 \\ \textbf{Write the equivalent equations.} & x = -8 \text{ or } x = 8 \\ & x = \pm 8 \end{array}$$

Answer b

$|y| = -6$
No solution
Since an absolute value is always positive, there are no solutions to this equation.

Answer c

$$\begin{array}{ll} & |z| = 0 \\ \textbf{Write the equivalent equations.} & z = -0 \text{ or } z = 0 \\ \textbf{Since } -0 = 0, & z = 0 \end{array}$$

Both equations tell us that z=0z=0 and so there is only one solution.

Example 2.8.2

Solve:

a. $|x| = 2$
b. $|y| = -4$
c. $|z| = 0$

Answer a

± 2

Answer b

no solution

Answer c

0

Example 2.8.3

Solve:

a. $|x| = 11$
b. $|y| = -5$
c. $|z| = 0$

Answer a

± 11

Answer b

no solution

Answer c

0

To solve an **absolute value equation**, we first isolate the absolute value expression using the same procedures we used to solve linear equations. Once we isolate the absolute value expression we rewrite it as the two equivalent equations.

How to Solve Absolute Value Equations

Example 2.8.4

Solve $|5x-4|-3=8$.

Answer

Step 1. Isolate the absolute value expression.	Add 3 to both sides.	$\|5x-4\|-3=8$ $\|5x-4\|=11$
Step 3. Solve each equation.	Add 4 to each side. Divide each side by 5.	$5x=-7$ $\quad 5x=15$ $x=-\frac{7}{5}$ or $x=3$
Step 4. Check each solution.	Substitute 3 and $-\frac{7}{5}$ into the original equation.	$\|5x-4\|-3=8$ $x=3$ $\|5\cdot 3-4\|-3\stackrel{?}{=}8$ $\|15-4\|-3\stackrel{?}{=}8$ $\|11\|-3\stackrel{?}{=}8$ $11-3\stackrel{?}{=}8$ $8=8\checkmark$ $\|5x-4\|-3=8$ $x=-\frac{7}{5}$ $\left\|5\left(-\frac{7}{5}\right)-4\right\|-3\stackrel{?}{=}8$ $\|-7-4\|-3\stackrel{?}{=}8$ $\|-11\|-3\stackrel{?}{=}8$ $11-3\stackrel{?}{=}8$ $8=8\checkmark$

Example 2.8.5

Solve: $|3x-5|-1=6$.

Answer

$x=4,\ x=-\frac{2}{3}$

Example 2.8.6

Solve: $|4x-3|-5=2$.

Answer

$x=-1,\ x=\frac{5}{2}$

The steps for solving an absolute value equation are summarized here.

SOLVE ABSOLUTE VALUE EQUATIONS.

1. Isolate the absolute value expression.
2. Write the equivalent equations.
3. Solve each equation.
4. Check each solution.

Example 2.8.7

Solve $2|x-7|+5=9$.

Answer

	$2\|x-7\|+5=9$
Isolate the absolute value expression.	$2\|x-7\|=4$
	$\|x-7\|=2$
Write the equivalent equations.	$x-7=-2$ or $x-7=2$
Solve each equation.	$x=5$ or $x=9$
Check: $2\|x-7\|+5=9$ $2\|5-7\|+5\stackrel{?}{=}9$ $2\|-2\|+5\stackrel{?}{=}9$ $2\cdot 2+5\stackrel{?}{=}9$ $4+5\stackrel{?}{=}9$ $9=9$ ✓ $2\|x-7\|+5=9$ $2\|9-7\|+5\stackrel{?}{=}9$ $2\|2\|+5\stackrel{?}{=}9$ $2\cdot 2+5\stackrel{?}{=}9$ $4+5\stackrel{?}{=}9$ $9=9$ ✓	

Example 2.8.8

Solve: $3|x-4|-4=8$.

Answer

$x=8,\ x=0$

Example 2.8.9

Solve: $2|x-5|+3=9$.

Answer

$x=8,\ x=2$

Remember, an absolute value is always positive!

Example 2.8.10

Solve: $|\frac{2}{3}x-4|+11=3$.

Answer

	$\|\frac{2}{3}x-4\|=-8$
Isolate the absolute value term.	$\|\frac{2}{3}x-4\|=-8$
An absolute value cannot be negative.	No solution

Example 2.8.11

Solve: $|\frac{3}{4}x-5|+9=4$.

Answer

No solution

Example 2.8.12

Solve: $|\frac{5}{6}x+3|+8=6$.

Answer

No solution

Some of our absolute value equations could be of the form $|u|=|v|$ where u and v are algebraic expressions. For example, $|x-3|=|2x+1|$.

How would we solve them? If two algebraic expressions are equal in absolute value, then they are either equal to each other or negatives of each other. The property for absolute value equations says that for any algebraic expression, u, and a positive real number, a, if $|u|=a$, then $u=-a$ or $u=a$.

This tell us that

\(\begin{array} {llllll} {\text{if}} &{|u|=|v|} &{} &{} &{} &{} \ {\text{then]} &{u=−v} &{\text{or}} &{u=v} &{\text{or}} &{u=v} \ \end{array}\)

This leads us to the following property for equations with two absolute values.

EQUATIONS WITH TWO ABSOLUTE VALUES

For any algebraic expressions, u and v,

$$\begin{array}{ll} \textbf{if} & |u|=|v| \\ \textbf{then} & u=-v \text{ or } u=v \end{array}$$

When we take the opposite of a quantity, we must be careful with the signs and to add parentheses where needed.

Example 2.8.13

Solve: $|5x-1|=|2x+3|$.

Answer

$$|5x-1|=|2x+3|$$

Write the equivalent equations.	$5x-1=-(2x+3)$	or	$5x-1=2x+3$
	$5x-1=-2x-3$	or	$3x-1=3$
Solve each equation.	$7x-1=-3$		$3x=4$
	$7x=-2$		$x=43$
	$x=-27$	or	$x=43$

Check.
We leave the check to you.

Example 2.8.14

Solve: $|7x-3|=|3x+7|$.

Answer

$x=-\frac{2}{5},\ x=\frac{5}{2}$

Example 2.8.15

Solve: $|6x-5|=|3x+4|$.

Answer

$x=3, x=19$

Solve Absolute Value Inequalities with "Less Than"

Let's look now at what happens when we have an **absolute value inequality**. Everything we've learned about solving inequalities still holds, but we must consider how the absolute value impacts our work. Again we will look at our definition of absolute value. The absolute value of a number is its distance from zero on the number line. For the equation $|x|=5$, we saw that both 5 and -5 are five units from zero on the number line. They are the solutions to the equation.

$$|x|=5$$
$$x=-5 \quad \text{or} \quad x=5$$

What about the inequality $|x|\leq 5$? Where are the numbers whose distance is less than or equal to 5? We know -5 and 5 are both five units from zero. All the numbers between -5 and 5 are less than five units from zero (Figure 2.8.2).

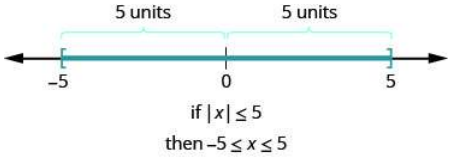


Figure 2.8.2.

In a more general way, we can see that if $|u|\leq a$, then $-a\leq u\leq a$ (Figure 2.8.3).

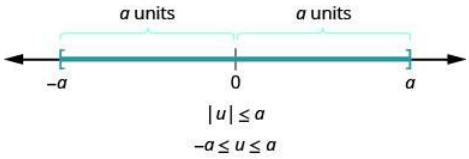


Figure 2.8.3.

This result is summarized here.

ABSOLUTE VALUE INEQUALITIES WITH $<$ OR $\leq$

For any algebraic expression, u, and any positive real number, a,

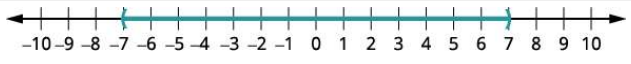
$$\text{if} \quad |u|<a, \quad \text{then} \quad -a<u<a$$
$$\text{if} \quad |u|\leq a, \quad \text{then} \quad -a\leq u\leq a$$

After solving an inequality, it is often helpful to check some points to see if the solution makes sense. The graph of the solution divides the number line into three sections. Choose a value in each section and substitute it in the original inequality to see if it makes the inequality true or not. While this is not a complete check, it often helps verify the solution.

Example 2.8.16

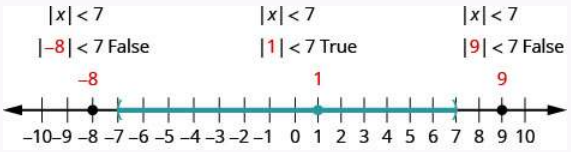
Solve $|x|<7$. Graph the solution and write the solution in interval notation.

Answer

	$\lvert x\rvert < 7$
Write the equivalent inequality.	$-7 < x < 7$
Graph the solution.	
Write the solution using interval notation.	$(-7, 7)$

Check:

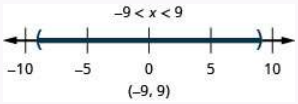
To verify, check a value in each section of the number line showing the solution. Choose numbers such as −8,−8, 1, and 9.



Example 2.8.17

Graph the solution and write the solution in interval notation: $|x| < 9$.

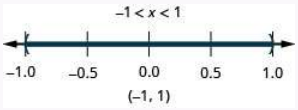
Answer



Example 2.8.18

Graph the solution and write the solution in interval notation: $|x| < 1$.

Answer



Example 2.8.19

Solve $|5x - 6| \leq 4$. Graph the solution and write the solution in interval notation.

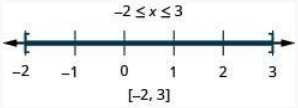
Answer

Step 1. Isolate the absolute value expression. It is isolated.	$\lvert 5x - 6\rvert \leq 4$
Step 2. Write the equivalent compound inequality.	$-4 \leq 5x - 6 \leq 4$
Step 3. Solve the compound inequality.	$2 \leq 5x \leq 10$ $\frac{2}{5} \leq x \leq 2$
Step 4. Graph the solution.	
Step 5. Write the solution using interval notation.	$[\frac{2}{5}, 2]$
Check: The check is left to you.	

Example 2.8.20

Solve $|2x-1| \leq 5$. Graph the solution and write the solution in interval notation:

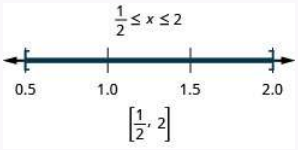
Answer



Example 2.8.21

Solve $|4x-5| \leq 3$. Graph the solution and write the solution in interval notation:

Answer



SOLVE ABSOLUTE VALUE INEQUALITIES WITH < OR ≤

1. Isolate the absolute value expression.
2. Write the equivalent compound inequality.

$$|u| < a \quad \text{is equivalent to} \quad -a < u < a$$
$$|u| \leq a \quad \text{is equivalent to} \quad -a \leq u \leq a$$

3. Solve the compound inequality.
4. Graph the solution
5. Write the solution using interval notation.

Solve Absolute Value Inequalities with "Greater Than"

What happens for absolute value inequalities that have "greater than"? Again we will look at our definition of absolute value. The absolute value of a number is its distance from zero on the number line.

We started with the inequality $|x| \leq 5$. We saw that the numbers whose distance is less than or equal to five from zero on the number line were -5 and 5 and all the numbers between -5 and 5 (Figure 2.8.4).

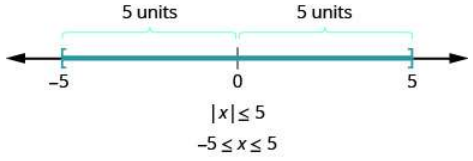


Figure 2.8.4.

Now we want to look at the inequality $|x| \geq 5$. Where are the numbers whose distance from zero is greater than or equal to five?

Again both -5 and 5 are five units from zero and so are included in the solution. Numbers whose distance from zero is greater than five units would be less than -5 and greater than 5 on the number line (Figure 2.8.5).

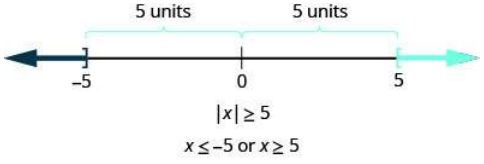


Figure 2.8.5.

In a more general way, we can see that if $|u| \geq a$, then $u \leq -a$ or $u \leq a$. See Figure.

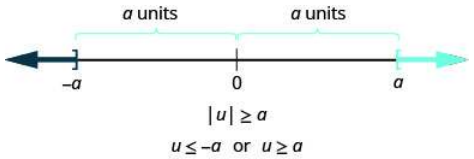


Figure 2.8.6.

This result is summarized here.

ABSOLUTE VALUE INEQUALITIES WITH > OR ≥

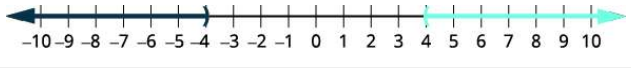
For any algebraic expression, u, and any positive real number, a,

$$\begin{array}{lll} \text{if} & |u| > a, & \text{then } u < -a \text{ or } u > a \\ \text{if} & |u| \geq a, & \text{then } u \leq -a \text{ or } u \geq a \end{array}$$

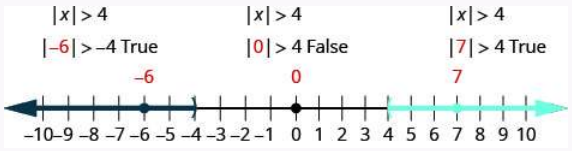
Example 2.8.22

Solve $|x| > 4$. Graph the solution and write the solution in interval notation.

Answer

	$\|x\| > 4$
Write the equivalent inequality.	$x < -4$ or $x > 4$
Graph the solution.	
Write the solution using interval notation.	$(-\text{inf}, -4) \cup (4, \text{inf})$
Check:	

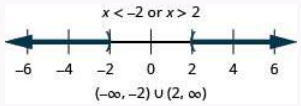
To verify, check a value in each section of the number line showing the solution. Choose numbers such as −6,−6, 0, and 7.



Example 2.8.23

Solve $|x| > 2$. Graph the solution and write the solution in interval notation.

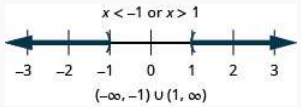
Answer



Example 2.8.24

Solve $|x| > 1$. Graph the solution and write the solution in interval notation.

Answer



Example 2.8.25

Solve $|2x-3| \geq 5$. Graph the solution and write the solution in interval notation.

Answer

	$\|2x-3\| \geq 5$
Step 1. Isolate the absolute value expression. It is isolated.	
Step 2. Write the equivalent compound inequality.	$2x-3 \leq -5$ or $2x-3 \geq 5$
Step 3. Solve the compound inequality.	$2x \leq -2$ or $2x \geq 8$ $x \leq -1$ or $x \geq 4$
Step 4. Graph the solution.	
Step 5. Write the solution using interval notation.	$(-\inf, -1] \cup [4, \inf)$
Check: The check is left to you.	

Example 2.8.26

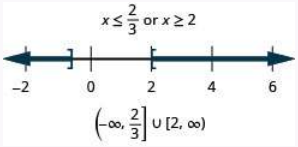
Solve $|4x-3| \geq 5$. Graph the solution and write the solution in interval notation.

Answer

Example 2.8.27

Solve $|3x-4| \geq 2$. Graph the solution and write the solution in interval notation.

Answer



SOLVE ABSOLUTE VALUE INEQUALITIES WITH > OR ≥.

1. Isolate the absolute value expression.
2. Write the equivalent compound inequality.

   |u|>ais equivalent tou<−aoru>a|u|\geq ais equivalent tou\leq −aoru\geq a|u|>ais equivalent tou<−aoru>a|u|\geq ais equivalent tou\leq −aoru\geq a

3. Solve the compound inequality.
4. Graph the solution
5. Write the solution using interval notation.

Solve Applications with Absolute Value

Absolute value inequalities are often used in the manufacturing process. An item must be made with near perfect specifications. Usually there is a certain *tolerance* of the difference from the specifications that is allowed. If the difference from the specifications exceeds the tolerance, the item is rejected.

$$|\text{actual-ideal}| \leq \text{tolerance}$$

Example 2.8.28

The ideal diameter of a rod needed for a machine is 60 mm. The actual diameter can vary from the ideal diameter by **0.075** mm. What range of diameters will be acceptable to the customer without causing the rod to be rejected?

Answer

	Let $x =$ the actual measurement
Use an absolute value inequality to express this situation.	$\lvert\text{actual-ideal}\rvert \leq \text{tolerance}$
	$\lvert x - 60\rvert \leq 0.075$
Rewrite as a compound inequality.	$-0.075 \leq x - 60 \leq 0.075$
Solve the inequality.	$59.925 \leq x \leq 60.075$
Answer the question.	The diameter of the rod can be between $59.925mm$ and $60.075mm$.

Example 2.8.29

The ideal diameter of a rod needed for a machine is 80 mm. The actual diameter can vary from the ideal diameter by 0.009 mm. What range of diameters will be acceptable to the customer without causing the rod to be rejected?

Answer

The diameter of the rod can be between 79.991 and 80.009 mm.

Example 2.8.30

The ideal diameter of a rod needed for a machine is 75 mm. The actual diameter can vary from the ideal diameter by 0.05 mm. What range of diameters will be acceptable to the customer without causing the rod to be rejected?

Answer

The diameter of the rod can be between 74.95 and 75.05 mm.

Access this online resource for additional instruction and practice with solving linear absolute value equations and inequalities.

- Solving Linear Absolute Value Equations and Inequalities

Key Concepts

- **Absolute Value**
  The absolute value of a number is its distance from 0 on the number line.
  The absolute value of a number n is written as $|n|$ and $|n| \geq 0$ for all numbers.
  Absolute values are always greater than or equal to zero.
- **Absolute Value Equations**
  For any algebraic expression, u, and any positive real number, a,
  if $|u| = a$
  then $u = -a$ or $u = a$
  Remember that an absolute value cannot be a negative number.
- **How to Solve Absolute Value Equations**
  1. Isolate the absolute value expression.
  2. Write the equivalent equations.
  3. Solve each equation.
  4. Check each solution.
- **Equations with Two Absolute Values**
  For any algebraic expressions, u and v,
  if $|u| = |v|$
  then $u = -v$ or $u = v$

- **Absolute Value Inequalities with $<$ or $\leq$**
  For any algebraic expression, u, and any positive real number, a,
  if $|u| = a$ **then** $-a < u < a$
  if $|u| \leq a$ **then** $-a \leq u \leq a$
- **How To Solve Absolute Value Inequalities with $<$ or $\leq$**
  1. Isolate the absolute value expression.
  2. Write the equivalent compound inequality.
     $|u| < a$ **is equivalent to** $-a < u < a$
     $|u| \leq a$ **is equivalent to** $-a \leq u \leq a$
  3. Solve the compound inequality.
  4. Graph the solution
  5. Write the solution using interval notation
- **Absolute Value Inequalities with $>$ or $\geq$**
  For any algebraic expression, u, and any positive real number, a,
  if $|u| > a$, **then** $u < -a$ **or** $u > a$
  if $|u| \geq a$, **then** $u \leq -a$ **or** $u \geq a$
- **How To Solve Absolute Value Inequalities with $>$ or $\geq$**
  1. Isolate the absolute value expression.
  2. Write the equivalent compound inequality.
     $|u| > a$ **is equivalent to** $u < -a$ **or** $u > a$
     $|u| \geq a$ **is equivalent to** $u \leq -a$ **or** $u \geq a$
  3. Solve the compound inequality.
  4. Graph the solution
  5. Write the solution using interval notation

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2.8E: Exercises

Practice Makes Perfect

Solve Absolute Value Equations

In the following exercises, solve.

ⓐ $|x| = 6$ ⓑ $|y| = -3$
ⓒ $|z| = 0$

ⓐ $|x| = 4$ ⓑ $|y| = -5$
ⓒ $|z| = 0$

Answer

ⓐ $x = 4, x = -4$ ⓑ no solution ⓒ $z = 0$

ⓐ $|x| = 7$ⓑ\(|y| = -11$
ⓒ $|z| = 0$

ⓐ $|x| = 3$ ⓑ $|y| = -1$
ⓒ $|z| = 0$

Answer

ⓐ $x = 3, x = -3$ ⓑ no solution ⓒ $z = 0$

$|2x - 3| - 4 = 1$

$|4x - 1| - 3 = 0$

Answer

$x = 1, x = -\frac{1}{2}$

$|3x - 4| + 5 = 7$

$|4x + 7| + 2 = 5$

Answer

$x = -1, x = -\frac{5}{2}$

$4|x - 1| + 2 = 10$

$3|x - 4| + 2 = 11$

Answer

$x = 7, x = 1$

$3|4x - 5| - 4 = 11$

$3|x+2|-5=4$

Answer

$x=1, x=-5$

$-2|x-3|+8=-4$

$-3|x-4|+4=-5$

Answer

$x=7, x=1$

$|34x-3|+7=2$

$|35x-2|+5=2$

Answer

no solution

$|12x+5|+4=1$

$|14x+3|+3=1$

Answer

no solution

$|3x-2|=|2x-3|$

$|4x+3|=|2x+1|$

Answer

$x=-1, x=-\frac{2}{3}$

$|6x-5|=|2x+3|$

$|6-x|=|3-2x|$

Answer

$x=-3, x=3$

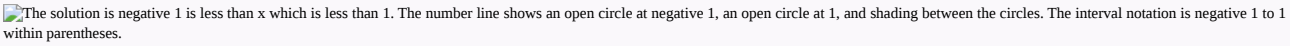
Solve Absolute Value Inequalities with "less than"

In the following exercises, solve each inequality. Graph the solution and write the solution in interval notation.

$|x|<5$

$|x|<1$

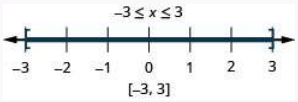
Answer


The solution is negative 1 is less than x which is less than 1. The number line shows an open circle at negative 1, an open circle at 1, and shading between the circles. The interval notation is negative 1 to 1 within parentheses.

$|x| \leq 8$

$|x| \leq 3$

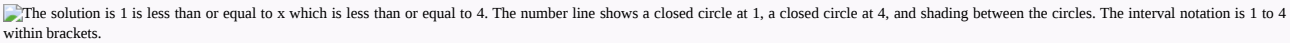
Answer



$|3x - 3| \leq 6$

$|2x - 5| \leq 3$

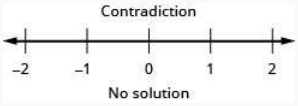
Answer


The solution is 1 is less than or equal to x which is less than or equal to 4. The number line shows a closed circle at 1, a closed circle at 4, and shading between the circles. The interval notation is 1 to 4 within brackets.

$|2x + 3| + 5 < 4$

$|3x - 7| + 3 < 1$

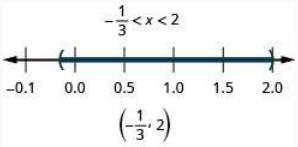
Answer



$|4x - 3| < 1$

$|6x - 5| < 7$

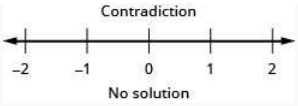
Answer



$|x - 4| \leq -1$

$|5x + 1| \leq -2$

Answer



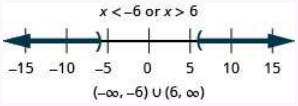
Solve Absolute Value Inequalities with "greater than"

In the following exercises, solve each inequality. Graph the solution and write the solution in interval notation.

$|x| > 3$

$|x| > 6$

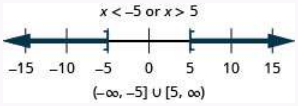
Answer



$|x| \geq 2$

$|x| \geq 5$

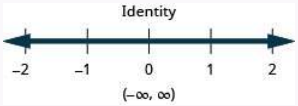
Answer



$|3x - 8| > -1$

$|x - 5| > -2$

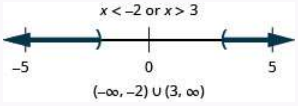
Answer



$|3x - 2| > 4$

$|2x - 1| > 5$

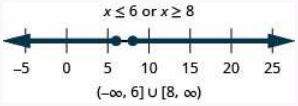
Answer



$|x + 3| \geq 5$

$|x - 7| \geq 1$

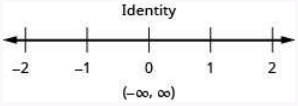
Answer



$3|x|+4 \geq 1$

$5|x|+6 \geq 1$

Answer



In the following exercises, solve. For each inequality, also graph the solution and write the solution in interval notation.

$2|x+6|+4=8$

$|3x-4| \geq 2$

Answer

$x=4, x=27$

$|6x-5|=|2x+3|$

$|4x-3|<5$

Answer

$x=3, x=2$

$|2x-5|+2=3$

$|3x+1|-3=7$

Answer

$x=3, x=-\frac{11}{3}$

$|7x+2|+8<4$

$5|2x-1|-3=7$

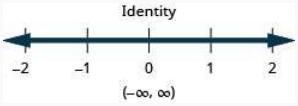
Answer

$x=\frac{3}{2}, x=-\frac{1}{2}$

$|x-7|>-3$

$|8-x|=|4-3x|$

Answer



Solve Applications with Absolute Value

In the following exercises, solve.

A chicken farm ideally produces 200,000 eggs per day. But this total can vary by as much as 25,000 eggs. What is the maximum and minimum expected production at the farm?

An organic juice bottler ideally produces 215,000 bottle per day. But this total can vary by as much as 7,500 bottles. What is the maximum and minimum expected production at the bottling company?

Answer

The minimum to maximum expected production is 207,500 to 2,225,000 bottles

In order to insure compliance with the law, Miguel routinely overshoots the weight of his tortillas by 0.5 gram. He just received a report that told him that he could be losing as much as $100,000 per year using this practice. He now plans to buy new equipment that guarantees the thickness of the tortilla within 0.005 inches. If the ideal thickness of the tortilla is 0.04 inches, what thickness of tortillas will be guaranteed?

At Lilly's Bakery, the ideal weight of a loaf of bread is 24 ounces. By law, the actual weight can vary from the ideal by 1.5 ounces. What range of weight will be acceptable to the inspector without causing the bakery being fined?

Answer

The acceptable weight is 22.5 to 25.5 ounces.

Writing Exercises

Write a graphical description of the absolute value of a number

In your own words, explain how to solve the absolute value inequality, $|3x-2| \geq 4$.

Answer

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve absolute value equations.			
solve absolute value inequalities with "less than".			
solve absolute value inequalities with "greater than".			
solve applications with absolute value.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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Chapter 2 Review Exercises

Chapter Review Exercises

Use a General Strategy to Solve Linear Equations

Solve Equations Using the General Strategy for Solving Linear Equations

In the following exercises, determine whether each number is a solution to the equation.

$10x - 1 = 5x, x = \frac{1}{5}$

$-12n + 5 = 8n, n = -\frac{5}{4}$

Answer

no

In the following exercises, solve each linear equation.

$6(x + 6) = 24$

$-(s + 4) = 18$

Answer

$s = -22$

$23 - 3(y - 7) = 8$

$\frac{1}{3}(6m + 21) = m - 7$

Answer

$m = -14$

$4(3.5y + 0.25) = 365$

$0.25(q - 8) = 0.1(q + 7)$

Answer

$q = 18$

$8(r - 2) = 6(r + 10)$

$5 + 7(2 - 5x) = 2(9x + 1) - (13x - 57)$

Answer

$x = -1$

$(9n + 5) - (3n - 7) = 20 - (4n - 2)$

$2[-16 + 5(8k - 6)] = 8(3 - 4k) - 32$

Answer

$k = \frac{3}{4}$

Classify Equations

In the following exercises, classify each equation as a conditional equation, an identity, or a contradiction and then state the solution.

$17y - 3(4 - 2y) = 11(y - 1) + 12y - 1$

$9u + 32 = 15(u - 4) - 3(2u + 21)$

Answer

$contradiction; no solution$

$-8(7m + 4) = -6(8m + 9)$

Solve Equations with Fraction or Decimal Coefficients

In the following exercises, solve each equation.

$\frac{2}{5}n - \frac{1}{10} = \frac{7}{10}$

Answer

$n = 2$

$\frac{3}{4}a - \frac{1}{3} = \frac{1}{2}a + \frac{5}{6}$

$\frac{1}{2}(k + 3) = \frac{1}{3}(k + 16)$

Answer

$k = 23$

$\frac{5y-1}{3} + 4 = \frac{-8y+4}{6}$

$0.8x - 0.3 = 0.7x + 0.2$

Answer

$x = 5$

$0.10d + 0.05(d - 4) = 2.05$

Use a Problem-Solving Strategy

Use a Problem Solving Strategy for Word Problems

In the following exercises, solve using the problem solving strategy for word problems.

Three-fourths of the people at a concert are children. If there are 87 children, what is the total number of people at the concert?

Answer

There are 116 people.

There are nine saxophone players in the band. The number of saxophone players is one less than twice the number of tuba players. Find the number of tuba players.

Solve Number Word Problems

In the following exercises, solve each number word problem.

The sum of a number and three is forty-one. Find the number.

Answer

38

One number is nine less than another. Their sum is negative twenty-seven. Find the numbers.

One number is two more than four times another. Their sum is negative thirteen. Find the numbers.

Answer

$-3, -10$

The sum of two consecutive integers is -135. Find the numbers.

Find three consecutive even integers whose sum is 234.

Answer

76, 78, 80

Find three consecutive odd integers whose sum is 51.

Koji has $5,502 in his savings account. This is $30 less than six times the amount in his checking account. How much money does Koji have in his checking account?

Answer

$922

Solve Percent Applications

In the following exercises, translate and solve.

What number is 67% of 250?

12.5% of what number is 20?

Answer

160

What percent of 125 is 150?

In the following exercises, solve.

The bill for Dino's lunch was $19.45. He wanted to leave 20% of the total bill as a tip. How much should the tip be?

Answer

$3.89

Dolores bought a crib on sale for $350. The sale price was 40% of the original price. What was the original price of the crib?

Jaden earns $2,680 per month. He pays $938 a month for rent. What percent of his monthly pay goes to rent?

Answer

35

Angel received a raise in his annual salary from $55,400 to $56,785. Find the percent change.

Rowena's monthly gasoline bill dropped from $83.75 last month to $56.95 this month. Find the percent change.

Answer

32

Emmett bought a pair of shoes on sale at 40% off from an original price of $138. Find ⓐ the amount of discount and ⓑthe sale price.

Lacey bought a pair of boots on sale for $95. The original price of the boots was $200. Find ⓐ the amount of discount and ⓑ the discount rate. (Round to the nearest tenth of a percent, if needed.)

Answer

ⓐ $105 ⓑ 52.5

Nga and Lauren bought a chest at a flea market for $50. They re-finished it and then added a 350% mark-up. Find ⓐthe amount of the mark-up and ⓑ the list price.

Solve Simple Interest Applications

In the following exercises, solve.

Winston deposited $3,294 in a bank account with interest rate 2.6% How much interest was earned in five years?

Answer

$428.22

Moira borrowed $4,500 from her grandfather to pay for her first year of college. Three years later, she repaid the $4,500 plus $243 interest. What was the rate of interest?

Jaime's refrigerator loan statement said he would pay $1,026 in interest for a four-year loan at 13.5%. How much did Jaime borrow to buy the refrigerator?

Answer

$1,900

Solve a formula for a Specific Variable

Solve a Formula for a Specific Variable

In the following exercises, solve the formula for the specified variable.

Solve the formula
$V = LWH$ for L.

Solve the formula
$A = \frac{1}{2}d_1d_2$ for d_2.

Answer

$d_2 = \frac{2A}{d_1}$

Solve the formula
$h = 48t + \frac{1}{2}at^2$ for t.

Solve the formula
$4x-3y=12$ for y.

Answer

$y = \frac{4x}{3} - 4$

Use Formulas to Solve Geometry Applications

In the following exercises, solve using a geometry formula.

What is the height of a triangle with area 67.567.5 square meters and base 9 meters?

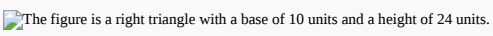
The measure of the smallest angle in a right triangle is $45°45°$ less than the measure of the next larger angle. Find the measures of all three angles.

Answer

$22.5°, 67.5°, 90°$

The perimeter of a triangle is 97 feet. One side of the triangle is eleven feet more than the smallest side. The third side is six feet more than twice the smallest side. Find the lengths of all sides.

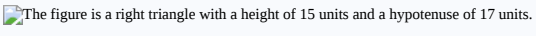
Find the length of the hypotenuse.


The figure is a right triangle with a base of 10 units and a height of 24 units.

Answer

26

Find the length of the missing side. Round to the nearest tenth, if necessary.


The figure is a right triangle with a height of 15 units and a hypotenuse of 17 units.

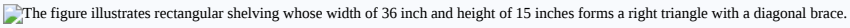
Sergio needs to attach a wire to hold the antenna to the roof of his house, as shown in the figure. The antenna is eight feet tall and Sergio has 10 feet of wire. How far from the base of the antenna can he attach the wire? Approximate to the nearest tenth, if necessary.


The figure is a right triangle with a height of 8 feet and a hypotenuse of 10 feet.

Answer

6 feet

Seong is building shelving in his garage. The shelves are 36 inches wide and 15 inches tall. He wants to put a diagonal brace across the back to stabilize the shelves, as shown. How long should the brace be?


The figure illustrates rectangular shelving whose width of 36 inch and height of 15 inches forms a right triangle with a diagonal brace.

The length of a rectangle is 12 cm more than the width. The perimeter is 74 cm. Find the length and the width.

Answer

24.5 cm, 12.5 cm

The width of a rectangle is three more than twice the length. The perimeter is 96 inches. Find the length and the width.

The perimeter of a triangle is 35 feet. One side of the triangle is five feet longer than the second side. The third side is three feet longer than the second side. Find the length of each side.

Answer

9 ft, 14 ft, 12 ft

Solve Mixture and Uniform Motion Applications

Solve Coin Word Problems

In the following exercises, solve.

Paulette has $140 in $5 and $10 bills. The number of $10 bills is one less than twice the number of $5 bills. How many of each does she have?

Lenny has $3.69 in pennies, dimes, and quarters. The number of pennies is three more than the number of dimes. The number of quarters is twice the number of dimes. How many of each coin does he have?

Answer

nine pennies, six dimes, 12 quarters

Solve Ticket and Stamp Word Problems

In the following exercises, solve each ticket or stamp word problem.

Tickets for a basketball game cost $2 for students and $5 for adults. The number of students was three less than 10 times the number of adults. The total amount of money from ticket sales was $619. How many of each ticket were sold?

125 tickets were sold for the jazz band concert for a total of $1,022. Student tickets cost $6 each and general admission tickets cost $10 each. How many of each kind of ticket were sold?

Answer

57 students, 68 adults

Yumi spent $34.15 buying stamps. The number of $0.56 stamps she bought was 10 less than four times the number of $0.41 stamps. How many of each did she buy?

Solve Mixture Word Problems

In the following exercises, solve.

Marquese is making 10 pounds of trail mix from raisins and nuts. Raisins cost $3.45 per pound and nuts cost $7.95 per pound. How many pounds of raisins and how many pounds of nuts should Marquese use for the trail mix to cost him $6.96 per pound?

Answer

2.2 lbs of raisins, 7.8 lbs of nuts

Amber wants to put tiles on the backsplash of her kitchen counters. She will need 36 square feet of tile. She will use basic tiles that cost $8 per square foot and decorator tiles that cost $20 per square foot. How many square feet of each tile should she use so that the overall cost of the backsplash will be $10 per square foot?

Enrique borrowed $23,500 to buy a car. He pays his uncle 2% interest on the $4,500 he borrowed from him, and he pays the bank 11.5% interest on the rest. What average interest rate does he pay on the total $23,500? (Round your answer to the nearest tenth of a percent.)

Answer

9.7

Solve Uniform Motion Applications

In the following exercises, solve.

When Gabe drives from Sacramento to Redding it takes him 2.2 hours. It takes Elsa two hours to drive the same distance. Elsa's speed is seven miles per hour faster than Gabe's speed. Find Gabe's speed and Elsa's speed.

Louellen and Tracy met at a restaurant on the road between Chicago and Nashville. Louellen had left Chicago and drove 3.2 hours towards Nashville. Tracy had left Nashville and drove 4 hours towards Chicago, at a speed one mile per hour faster than Louellen's speed. The distance between Chicago and Nashville is 472 miles. Find Louellen's speed and Tracy's speed.

Answer

Louellen 65 mph, Tracy 66 mph

Two busses leave Amarillo at the same time. The Albuquerque bus heads west on the I-40 at a speed of 72 miles per hour, and the Oklahoma City bus heads east on the I-40 at a speed of 78 miles per hour. How many hours will it take them to be 375 miles apart?

Kyle rowed his boat upstream for 50 minutes. It took him 30 minutes to row back downstream. His speed going upstream is two miles per hour slower than his speed going downstream. Find Kyle's upstream and downstream speeds.

Answer

upstream 3 mph, downstream 5 mph

At 6:30, Devon left her house and rode her bike on the flat road until 7:30. Then she started riding uphill and rode until 8:00. She rode a total of 15 miles. Her speed on the flat road was three miles per hour faster than her speed going uphill. Find Devon's speed on the flat road and riding uphill.

Anthony drove from New York City to Baltimore, which is a distance of 192 miles. He left at 3:45 and had heavy traffic until 5:30. Traffic was light for the rest of the drive, and he arrived at 7:30. His speed in light traffic was four miles per hour more than twice his speed in heavy traffic. Find Anthony's driving speed in heavy traffic and light traffic.

Answer

heavy traffic 32 mph, light traffic 66 mph

Solve Linear Inequalities

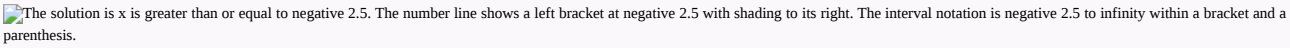
Graph Inequalities on the Number Line

In the following exercises, graph the inequality on the number line and write in interval notation.

$x < -1$

$x \geq -2.5$

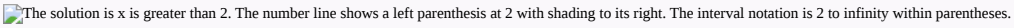
Answer


The solution is x is greater than or equal to negative 2.5. The number line shows a left bracket at negative 2.5 with shading to its right. The interval notation is negative 2.5 to infinity within a bracket and a parenthesis.

$x \leq \frac{5}{4}$

$x > 2$

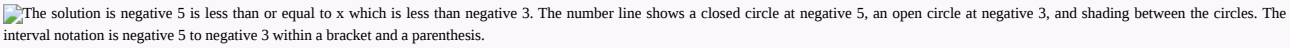
Answer


The solution is x is greater than 2. The number line shows a left parenthesis at 2 with shading to its right. The interval notation is 2 to infinity within parentheses.

$-2 < x < 0$

$-5 \leq x < -3$

Answer


The solution is negative 5 is less than or equal to x which is less than negative 3. The number line shows a closed circle at negative 5, an open circle at negative 3, and shading between the circles. The interval notation is negative 5 to negative 3 within a bracket and a parenthesis.

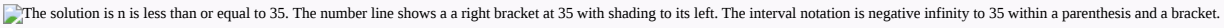
$0 \leq x \leq 3.5$

Solve Linear Inequalities

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

$n - 12 \leq 23$

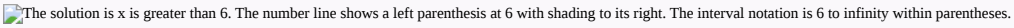
Answer


The solution is n is less than or equal to 35. The number line shows a a right bracket at 35 with shading to its left. The interval notation is negative infinity to 35 within a parenthesis and a bracket.

$a + \frac{2}{3} \geq \frac{7}{12}$

$9x > 54$

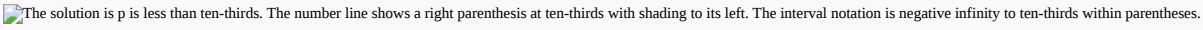
Answer


The solution is x is greater than 6. The number line shows a left parenthesis at 6 with shading to its right. The interval notation is 6 to infinity within parentheses.

$\frac{q}{-2} \geq -24$

$6p > 15p - 30$

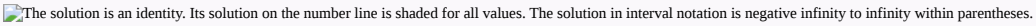
Answer


The solution is p is less than ten-thirds. The number line shows a right parenthesis at ten-thirds with shading to its left. The interval notation is negative infinity to ten-thirds within parentheses.

$9h - 7(h - 1) \leq 4h - 23$

$5n - 15(4 - n) < 10(n - 6) + 10n$

Answer


The solution is an identity. Its solution on the number line is shaded for all values. The solution in interval notation is negative infinity to infinity within parentheses.

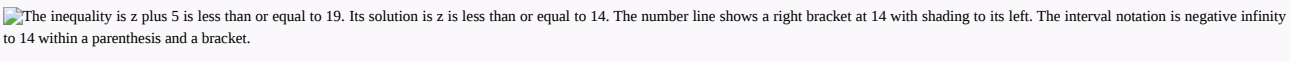
$\frac{3}{8}a - \frac{1}{12}a > \frac{5}{12}a + \frac{3}{4}$

Translate Words to an Inequality and Solve

In the following exercises, translate and solve. Then write the solution in interval notation and graph on the number line.

Five more than z is at most 19.

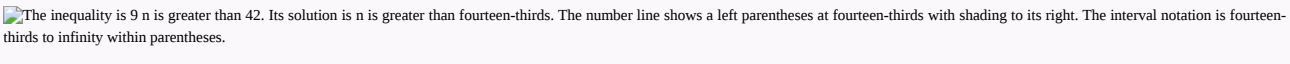
Answer


The inequality is z plus 5 is less than or equal to 19. Its solution is z is less than or equal to 14. The number line shows a right bracket at 14 with shading to its left. The interval notation is negative infinity to 14 within a parenthesis and a bracket.

Three less than c is at least 360.

Nine times n exceeds 42.

Answer


The inequality is 9 n is greater than 42. Its solution is n is greater than fourteen-thirds. The number line shows a left parentheses at fourteen-thirds with shading to its right. The interval notation is fourteen-thirds to infinity within parentheses.

Negative two times a is no more than eight.

Solve Applications with Linear Inequalities

In the following exercises, solve.

Julianne has a weekly food budget of $231 for her family. If she plans to budget the same amount for each of the seven days of the week, what is the maximum amount she can spend on food each day?

Answer

$33 per day

Rogelio paints watercolors. He got a $100 gift card to the art supply store and wants to use it to buy 12″ × 16″ canvases. Each canvas costs $10.99. What is the maximum number of canvases he can buy with his gift card?

Briana has been offered a sales job in another city. The offer was for $42,500 plus 8% of her total sales. In order to make it worth the move, Briana needs to have an annual salary of at least $66,500. What would her total sales need to be for her to move?

Answer

at least $300,000

Renee's car costs her $195 per month plus $0.09 per mile. How many miles can Renee drive so that her monthly car expenses are no more than $250?

Costa is an accountant. During tax season, he charges $125 to do a simple tax return. His expenses for buying software, renting an office, and advertising are $6,000. How many tax returns must he do if he wants to make a profit of at least $8,000?

Answer

at least 112 jobs

Jenna is planning a five-day resort vacation with three of her friends. It will cost her $279 for airfare, $300 for food and entertainment, and $65 per day for her share of the hotel. She has $550 saved towards her vacation and can earn $25 per hour as an assistant in her uncle's photography studio. How many hours must she work in order to have enough money for her vacation?

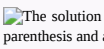
Solve Compound Inequalities

Solve Compound Inequalities with "and"

In each of the following exercises, solve each inequality, graph the solution, and write the solution in interval notation.

$x \leq 5$ and $x > -3$

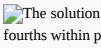
Answer


The solution is negative 3 is less than x which is less than or equal to 5. The number line shows an open circle at negative 3 and a closed circle at 5. The interval notation is negative 3 to 5 within a parenthesis and a bracket.

$4x - 2 \leq 4$ and
$7x - 1 > -8$

$5(3x - 2) \leq 5$ and
$4(x + 2) < 3$

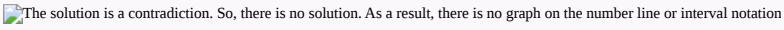
Answer


The solution is negative x is less than negative five-fourths. The number line shows an open circle at negative five-fourths with shading to its left. The interval notation is negative infinity to negative five-fourths within parentheses.

$34(x-8) \leq 3$ and
$15(x-5) \leq 3$

$34x-5 \geq -2$ and
$-3(x+1) \geq 6$

Answer


The solution is a contradiction. So, there is no solution. As a result, there is no graph on the number line or interval notation

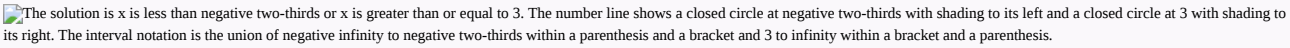
$-5 \leq 4x-1 < 7$

Solve Compound Inequalities with "or"

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

$5-2x \leq -1$ or
$6+3x \leq 4$

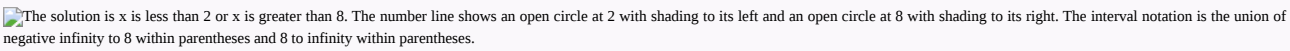
Answer


The solution is x is less than negative two-thirds or x is greater than or equal to 3. The number line shows a closed circle at negative two-thirds with shading to its left and a closed circle at 3 with shading to its right. The interval notation is the union of negative infinity to negative two-thirds within a parenthesis and a bracket and 3 to infinity within a bracket and a parenthesis.

$3(2x-3) < -5$ or
$4x-1 > 3$

$34x-2 > 4$ or $4(2-x) > 0$

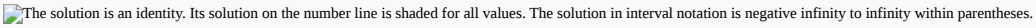
Answer


The solution is x is less than 2 or x is greater than 8. The number line shows an open circle at 2 with shading to its left and an open circle at 8 with shading to its right. The interval notation is the union of negative infinity to 8 within parentheses and 8 to infinity within parentheses.

$2(x+3) \geq 0$ or
$3(x+4) \leq 6$

$12x-3 \leq 4$ or
$13(x-6) \geq -2$

Answer


The solution is an identity. Its solution on the number line is shaded for all values. The solution in interval notation is negative infinity to infinity within parentheses.

Solve Applications with Compound Inequalities

In the following exercises, solve.

Liam is playing a number game with his sister Audry. Liam is thinking of a number and wants Audry to guess it. Five more than three times her number is between 2 and 32. Write a compound inequality that shows the range of numbers that Liam might be thinking of.

Elouise is creating a rectangular garden in her back yard. The length of the garden is 12 feet. The perimeter of the garden must be at least 36 feet and no more than 48 feet. Use a compound inequality to find the range of values for the width of the garden.

Answer

$6 \leq w \leq 12$

Solve Absolute Value Inequalities

Solve Absolute Value Equations

In the following exercises, solve.

$|x| = 8$

$|y| = -14$

Answer

no solution

$|z| = 0$

$|3x - 4| + 5 = 7$

Answer

$x = 2, x = \frac{2}{3}$

$4|x - 1| + 2 = 10$

$-2|x - 3| + 8 = -4$

Answer

$x = 9, x = -3$

$|12x + 5| + 4 = 1$

$|6x - 5| = |2x + 3|$

Answer

$x = 2, x = 14$

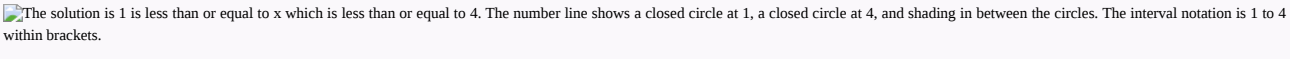
Solve Absolute Value Inequalities with "less than"

In the following exercises, solve each inequality. Graph the solution and write the solution in interval notation.

$|x| \leq 8$

$|2x - 5| \leq 3$

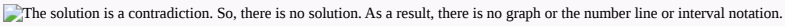
Answer


The solution is 1 is less than or equal to x which is less than or equal to 4. The number line shows a closed circle at 1, a closed circle at 4, and shading in between the circles. The interval notation is 1 to 4 within brackets.

$|6x - 5| < 7$

$|5x+1| \leq -2$

Answer


The solution is a contradiction. So, there is no solution. As a result, there is no graph or the number line or interval notation.

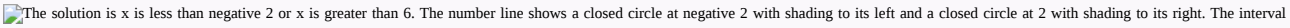
Solve Absolute Value Inequalities with "greater than"

In the following exercises, solve. Graph the solution and write the solution in interval notation.

$|x| > 6$

$|x| \geq 2$

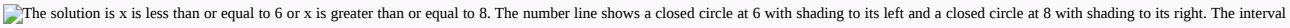
Answer


The solution is x is less than negative 2 or x is greater than 6. The number line shows a closed circle at negative 2 with shading to its left and a closed circle at 2 with shading to its right. The interval notation is the union of negative infinity to negative 2 within a parenthesis and a bracket and 2 to infinity within a bracket and a parenthesis.

$|x-5| > -2$

$|x-7| \geq 1$

Answer


The solution is x is less than or equal to 6 or x is greater than or equal to 8. The number line shows a closed circle at 6 with shading to its left and a closed circle at 8 with shading to its right. The interval notation is the union of negative infinity to negative 6 within a parenthesis and a bracket and 8 to infinity within a bracket and a parenthesis.

$3|x|+4 \geq 1$

Solve Applications with Absolute Value

In the following exercises, solve.

A craft beer brewer needs 215,000 bottle per day. But this total can vary by as much as 5,000 bottles. What is the maximum and minimum expected usage at the bottling company?

Answer

The minimum to maximum expected usage is 210,000 to 220,000 bottles

At Fancy Grocery, the ideal weight of a loaf of bread is 16 ounces. By law, the actual weight can vary from the ideal by 1.5 ounces. What range of weight will be acceptable to the inspector without causing the bakery being fined?

Practice Test

In the following exercises, solve each equation.

$-5(2x+1) = 45$

Answer

$x = -5$

$\frac{1}{4}(12m+28) = 6+2(3m+1)$

$8(3a+5)-7(4a-3)=20-3a$

Answer

$a=41$

$0.1d+0.25(d+8)=4.1$

$14n-3(4n+5)=-9+2(n-8)$

Answer

contradiction; no solution

$3(3u+2)+4[6-8(u-1)]=3(u-2)$

$\frac{3}{4}x-\frac{2}{3}=\frac{1}{2}x+\frac{5}{6}$

Answer

$x=6$

$|3x-4|=8$

$|2x-1|=|4x+3|$

Answer

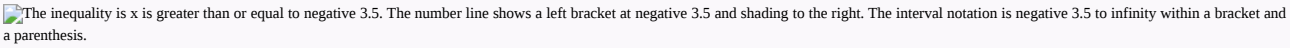
$x=-2, x=-13$

Solve the formula
$x+2y=5$ for y.

In the following exercises, graph the inequality on the number line and write in interval notation.

$x\geq -3.5$

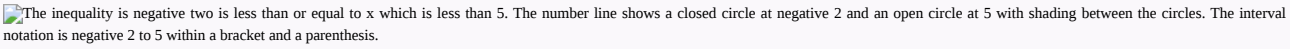
Answer


The inequality is x is greater than or equal to negative 3.5. The number line shows a left bracket at negative 3.5 and shading to the right. The interval notation is negative 3.5 to infinity within a bracket and a parenthesis.

$x<\frac{11}{4}$

$-2\leq x<5$

Answer

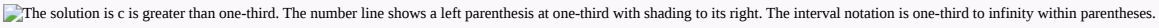

The inequality is negative two is less than or equal to x which is less than 5. The number line shows a closed circle at negative 2 and an open circle at 5 with shading between the circles. The interval notation is negative 2 to 5 within a bracket and a parenthesis.

In the following exercises, solve each inequality, graph the solution on the number line, and write the solution in interval notation.

$8k\geq 5k-120$

$3c-10(c-2)<5c+16$

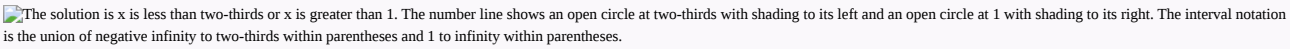
Answer


The solution is c is greater than one-third. The number line shows a left parenthesis at one-third with shading to its right. The interval notation is one-third to infinity within parentheses.

$\frac{3}{4}x-5\geq -2$ and
$-3(x+1)\geq 6$

$3(2x-3)<-5$ or
$4x-1>3$

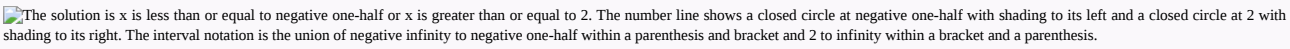
Answer


The solution is x is less than two-thirds or x is greater than 1. The number line shows an open circle at two-thirds with shading to its left and an open circle at 1 with shading to its right. The interval notation is the union of negative infinity to two-thirds within parentheses and 1 to infinity within parentheses.

$\frac{1}{2}x-3\leq 4$ or
$\frac{1}{3}(x-6)\geq -2$

$|4x-3|\geq 5$

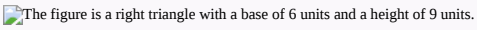
Answer


The solution is x is less than or equal to negative one-half or x is greater than or equal to 2. The number line shows a closed circle at negative one-half with shading to its left and a closed circle at 2 with shading to its right. The interval notation is the union of negative infinity to negative one-half within a parenthesis and bracket and 2 to infinity within a bracket and a parenthesis.

In the following exercises, translate to an equation or inequality and solve.

Four less than twice x is 16.

Find the length of the missing side.


The figure is a right triangle with a base of 6 units and a height of 9 units.

Answer

10.8

One number is four more than twice another. Their sum is -47. Find the numbers.

The sum of two consecutive odd integers is -112.. Find the numbers.

Answer

$-57, -55$

Marcus bought a television on sale for $626.50 The original price of the television was $895. Find ⓐ the amount of discount and ⓑ the discount rate.

Bonita has $2.95 in dimes and quarters in her pocket. If she has five more dimes than quarters, how many of each coin does she have?

Answer

12 dimes, seven quarters

Kim is making eight gallons of punch from fruit juice and soda. The fruit juice costs $6.04 per gallon and the soda costs $4.28 per gallon. How much fruit juice and how much soda should she use so that the punch costs $5.71 per gallon?

The measure of one angle of a triangle is twice the measure of the smallest angle. The measure of the third angle is three times the measure of the smallest angle. Find the measures of all three angles.

Answer

$30°, 60°, 90°$

The length of a rectangle is five feet more than four times the width. The perimeter is 60 feet. Find the dimensions of the rectangle.

Two planes leave Dallas at the same time. One heads east at a speed of 428 miles per hour. The other plane heads west at a speed of 382 miles per hour. How many hours will it take them to be 2,025 miles apart?

Answer

2.5 hours

Leon drove from his house in Cincinnati to his sister's house in Cleveland, a distance of 252 miles. It took him $4\frac{1}{2}$ hours. For the first half hour, he had heavy traffic, and the rest of the time his speed was five miles per hour less than twice his speed in heavy traffic. What was his speed in heavy traffic?

Sara has a budget of $1,000 for costumes for the 18 members of her musical theater group. What is the maximum she can spend for each costume?

Answer

At most $55.56 per costume.

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1.3: Logic

Two of the most subtle words in the English language are the words “and” and “or.” One has only three letters, the other two, but it is absolutely amazing how much confusion these two tiny words can cause. Our intent in this section is to clear the mystery surrounding these words and prepare you for the mathematics that depends upon a thorough understanding of the words “and” and “or.”

Set Notation

We begin with the definition of a set.

Definition 1

A set is a collection of objects.

The objects in the set could be anything at all: numbers, letters, first names, cities, you name it. In this section we will focus on sets of numbers, but it is important to understand that the objects in a set can be whatever you choose them to be.

If the number of objects in a set is finite and small enough, we can describe the set simply by listing the elements (objects) in the set. This is usually done by enclosing the list of objects in the set with curly braces. For example, let

$$A = \{1, 3, 5, 7, 9, 11\} \tag{1.3.1}$$

Now, when we refer to the set A in the narrative, everyone should know we’re talking about the set of numbers 1, 3, 5, 7, 9, and 11.

It is also possible to describe the set A with words. Although there are many ways to do this, one possible description might be “Let A be the set of odd natural numbers between 1 and 11, inclusive.” This descriptive technique is particularly efficient when the set you are describing is either infinite or too large to enumerate in a list.

For example, we might say “let A be the set of all real numbers that are greater than 4.” This is much better than trying to list each of the numbers in the set A, which would be futile in this case. Another possibility is to combine the curly brace notation with a textual description and write something like

$$A = \{real numbers that are greater than 4\} \tag{1.3.2}$$

If we’re called upon to read this notation aloud, we would say “A is the set of all real numbers that are greater than 4,” or something similar.

There are a number of more sophisticated methods we can use to describe a set. One description that we will often employ is called set-builder notation and has the following appearance.

$$A = \{x : some statement describing x\} \tag{1.3.3}$$

It is standard to read the notation $\{x : \quad\}$ aloud as follows: “The set of all x such that.” That is, the colon is pronounced “such that.” Then you would read the description that follows the colon. For example, the set

$$A = \{x : x < 3\} \tag{1.3.4}$$

is read aloud “A is the set of all x such that x is less than 3.” Some people prefer to use a “bar” instead of a colon and they write

$$A = \{x | \text{ some statement describing } x\} \tag{1.3.5}$$

This is also pronounced “A is the set of all x such that,” and then you would read the text description that follows the “bar.” Thus, the notation

$$A = \{x | x < 3\} \tag{1.3.6}$$

is identical to the notation $A = \{x : x < 3\}$ used above and is read in exactly the same manner, “A is the set of all x such that x is less than 3.” We prefer the colon notation, but feel free to use the “bar” if you like it better. It means the same thing.

A moment’s thought will reveal the fact that the notation $A = \{x : x < 3\}$ is not quite descriptive enough. It’s probably safe to say, since the description of x is “x < 3,” that this notation is referring to numbers that are less than 3, but what kind of numbers?

Natural numbers? Integers? Rational numbers? Irrational numbers? Real numbers? The notation $A = \{x : x < 3\}$ doesn't really tell the whole story.

We'll fix this deficiency in a moment, but first recall that in our preliminary chapter, we used specific symbols to represent certain sets of numbers. Indeed, we used the following:

$$\begin{aligned} \mathbb{N} &= \{ \text{ natural numbers } \} \\ \mathbb{Z} &= \{ \text{ integers } \} \\ \mathbb{Q} &= \{ \text{ rational numbers } \} \\ \mathbb{R} &= \{ \text{ real numbers } \} \end{aligned} \tag{1.3.7}$$

We can use these symbols to help denote the type of number described with our setbuilder notation. For example, if we write

$$A = \{x \in \mathbb{N} : x < 3\} \tag{1.3.8}$$

then we say "A is the set of all x in the natural numbers such that x is less than 3," or more simply, "the set of all natural numbers that are less than 3." The symbol $\in$ is the Greek letter "epsilon," and when used in set-builder notation, it is pronounced "is an element of," or "is in." Of course, the only natural numbers $\mathbb{N} = \{1, 2, 3, ...\}$ that are less than 3 are the natural numbers 1 and 2. Thus, $A = 1, 2$, the "set whose members are 1 and 2."

On the other hand, if we write

$$A = \{x \in \mathbb{Z} : x < 3\} \tag{1.3.9}$$

then we say that "A is the set of x in the set of integers such that x is less than 3," or more informally, "A is the set of all integers less than 3." Of course, the integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, ...\}$[2] less than 3 are infinite in number. We cannot list all of them unless we appeal to the imagination with something like

$$A = \{..., -3, -2, -1, 0, 1, 2\} \tag{1.3.10}$$

The ellipsis . . . means "etc." We've listed enough of the numbers to establish a pattern, so we're permitted to say "and so on." The reader intuits that the earlier numbers in the list are −4, −5, etc.

Let's look at another example. Suppose that we write

$$A = \{x \in \mathbb{R} : x < 3\} \tag{1.3.11}$$

Then we say "A is the set of all x in the set of real numbers such that x is less than 3," or more informally, "A is the set of all real numbers less than 3." Of course, this is another infinite set and it's not hard to imagine that the notation $\{x \in \mathbb{R} : x < 3\}$ used above is already optimal for describing this set of real numbers.

In this text, we will mostly deal with sets of real numbers. Thus, from this point forward, if we write

$$A = \{x : x < 3\} \tag{1.3.12}$$

we will assume that we mean to say that "A is the set of all real numbers less than 3." That is, if we write $A = \{x : x < 3\}$, we understand this to mean $A = \{x \in \mathbb{R} : x < 3\}$. In the case when we want to use a specific set of numbers, we will indicate that as we did above, for example, in $A = \{x \in \mathbb{N} : x < 3\}$.

The Real Line and Interval Notation

Suppose that we draw a line (affectionately known as the "real line"), then plot a point anywhere on that line, then map the number zero to that point (called the "origin"), as shown in Figure **1.3.1**. Secondly, decide on a unit distance and map the number 1 to that point, again shown in Figure **1.3.1**.

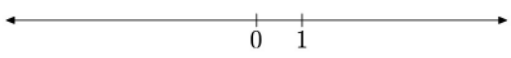


Figure **1.3.1** Establishing the origin and a unit length on the real line.

Now that we've established a unit distance, every real number corresponds to a point on the real line. Vice-versa, every point on the real line corresponds to a real number. This defines a one-to-one correspondence between the real numbers in $\mathbb{R}$ and the points on

the real line. In this manner, the point on the line and the real number can be thought of as synonymous. Figure **1.3.2** shows several real numbers plotted on the real line.

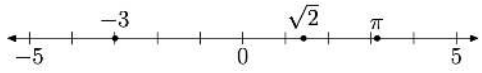


Figure **1.3.2** Sample numbers on the real line.

Now, suppose that we're asked to shade all real numbers in the set $\{x : x > 3\}$. Because this requires that we shade every real number that is greater than 3 (to the right of 3), we use the shading shown in Figure **1.3.3** to represent the set $\{x : x > 3\}$.

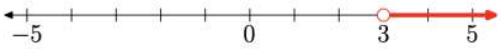


Figure **1.3.3**. Shading all real numbers greater than 3.

Although technically correct, the image in Figure 3 contains more information than is really needed. The picture is acceptable, but crowded. The really important information is the fact that the shading starts at 3, then moves to the right. Also, because 3 is not in the set $x : x > 3$, that is, 3 is not greater than 3, we do not shade the point corresponding to the real number 3. Note that we've indicated this fact with an "empty" circle at 3 on the real line.

Thus, when shading the set $x : x > 3$ on the real line, we need only label the endpoint at 3, use an "empty" circle at 3, and shade all the real numbers to the right of 3, as shown in Figure **1.3.4**.

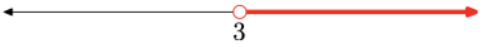


Figure **1.3.4**. Shading all real numbers greater than 3. The endpoint is the only information that needs to be labeled. It is not necessary to show any other tickpoints and/or labels.

Because we're shading all numbers from 3 to positive infinity in Figure **1.3.4**, we'll use the following interval notation to represent this "interval" of numbers (everything between 3 and positive infinity).

$$(3, \infty) = \{x : x > 3\} \tag{1.3.13}$$

Similarly, Table **1.3.1** lists the set-builder and interval notations, as well as shading of the sets on the real line, for several situations, including the one just discussed.

There are several points of emphasis regarding the intervals in Table **1.3.1**.

1. When we want to emphasize that we are not including a point on the real line, we use an "empty circle." Conversely, a "filled circle" means that we are including the point on the real line. Thus, the real lines in the first two rows of Table **1.3.1** do not include the number 3, but the real lines in the last two rows in Table **1.3.1** do include the number 3.

2. The use of a parenthesis in interval notation means that we are not including that endpoint in the interval. Thus, the parenthesis use in $(-\infty, 3)$ in the second row of Table **1.3.1** means that we are not including the number 3 in the interval.

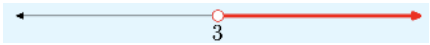
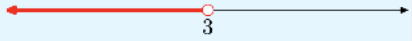
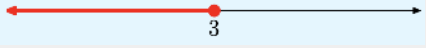
Number line	Set-builder notation	Interval notation
	{x : x > 3}	$(3, \infty)$
	{x : x < 3}	$(-\infty, 3)$
	$\{x : x \geq 3\}$	$[3, \infty)$
	$\{x : x \leq 3\}$	$(-\infty, 3]$

Table 1.3.1 Number lines, set-builder notation, and interval notation.

3. The use of a bracket in interval notation means that we are including the bracketed number in the interval. Thus, the bracket used in $[3, \infty)$, as seen in the third row of Table 1.3.1, means that we are including the number 3 in the interval.

4. The use of ∞ in $(3, \infty)$ in row one of Table 1.3.1 means that we are including every real number greater than 3. The use of $-\infty$ in $(-\infty, 3]$ means that we are including every real number less than or equal to 3. As $-\infty$ and ∞ are not actual numbers, it makes no sense to include them with a bracket. Consequently, you must always use a parenthesis with $-\infty$ or ∞.

Union and Intersection

The intersection of two sets A and B is defined as follows.

Definition 3

The intersection of the sets A and B is the set of all objects that are in A and in B. In symbols, we write

$$A \cap B = \{x : x \in A \text{ and } x \in B\} \tag{1.3.14}$$

In order to understand this definition, it's absolutely crucial that we understand the meaning of the word "and." The word "and" is a conjunction, used between statements P and Q, as in "It is raining today and my best friend is the Lone Ranger." In order to determine the truth or falsehood of this statement, you must first examine the truth or falsehood of the statements P and Q on each side of the word "and."

The only way that the speaker is telling the truth is if both statements P and Q are true. In other words, the statement "It is raining today and my best friend is the Lone Ranger" is true if and only if the statement "It is raining today" is true and the statement "my best friend is the Lone Ranger" is also true. Logicians like to make up a construct called a truth table, like the one shown in Table 1.3.2.

Points in Table 1.3.2 to consider:

P	Q	P **and** Q
T	T	T
T	F	F
F	T	F
F	F	F

Table 1.3.2 Truth table for the conjunction "and."

- In the first row (after the header row) of Table 1.3.2, if statements P and Q are both true (indicated with a T), then the statement "P and Q" is also true.
- In the remaining rows of Table 1.3.2, one or the other of statements P or Q are false (indicated with an F), so the statement "P and Q" is also false.

Therefore, the statement "P and Q" is true if and only if P is true and Q is true.

Example 1.3.1

If A = {1, 3, 5, 7, 9} and B = {2, 5, 7, 8, 11}, find the intersection of A and B.

Solution

As a reminder, the *intersection* of A and B is

$$A \cap B = \{x : x \in A \text{ and } x \in B\} \tag{1.3.15}$$

Thus, we are looking for the objects that are in A and in B. The only objects that are in A and in B (remember, both statements "in A" and "in B" must be true) are 5 and 7, so we write:

$$A \cap B = \{5, 7\} \tag{1.3.16}$$

Mathematicians and logicians both use a visual aid called a Venn Diagram to represent sets. John Venn was an English mathematician who devised this visualization of logical relationships. Consider the ellipse A in Figure **1.3.5**. Everything inside the boundary of this ellipse constitutes the set A = {1, 3, 5, 7, 9}. That's why you see these numbers inside the boundary of this ellipse.

Consider the ellipse B in Figure **1.3.5**. Everything inside the boundary of this ellipse constitutes the set B = {2, 5, 7, 8, 11}. That's why you see these numbers inside the boundary of this ellipse.

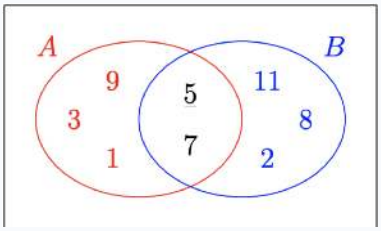


Figure **1.3.5** Venn Diagram

Now, note that only two numbers, 5 and 7, are contained within the boundaries of both A and B. These are the numbers that are in the intersection of the sets A and B.

The shaded region in Figure **1.3.6** is the area that belongs to both of the sets A and B. Note how this shaded region is aptly named "the intersection of the sets A and B." This is the region that is in common to the sets A and B, the region where the sets A and B overlap or "intersect."

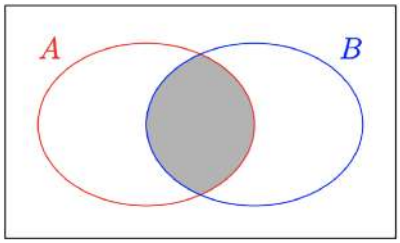


Figure **1.3.6** The shaded region is the intersection of the sets A and B. That is, the shaded region is $A \cap B$.

This leads to the following important piece of advice.

Note

When asked to find the intersection of two sets A and B, look to see where the sets intersect or overlap. That is, look to see the elements that are in both sets A and B.

Let's move on to the definition of the union of two sets A and B.

Definition

The union of the sets A and B is the set of all objects that are in A or in B. In symbols, we write

$$A \cup B = \{x : x \in A \text{ or } x \in B\} \tag{1.3.17}$$

In order to understand this definition, it's critical that we understand the meaning of the word "or." The word "or" is a disjunction, used between statements P and Q, as in "It is raining today or my best friend is the Lone Ranger." In order to determine the truth or falsehood of this statement, you must first examine the truth or falsehood of the statements P and Q on each side of the word "or."

The speaker is telling the truth if either statement P is true or statement Q is true. In other words, the statement "It is raining today or my best friend is the Lone Ranger" is true if and only if the statement "It is raining today" is true or the statement "my best friend is the Lone Ranger" is true. Logicians like to make up a construct called a truth table, like the one shown in Table **1.3.3**.

P	Q	P **or** Q
T	T	T
T	F	T
F	T	T
F	F	F

Table **1.3.3** Truth table for the conjunction “or.”

Points in Table **1.3.3** to consider:

- In the last row of Table **1.3.3** , both statements P and Q are false (indicated with an F), so the statement P or Q is also false.
- In the first three rows (after the header row) of Table **1.3.3** , either statement P is true or statement Q is true (indicated with a T), so the statement P or Q is also true.

Therefore, the statement “P or Q” is true if and only if either statement, P or Q, is true.

Example 1.3.2

If A = {1, 3, 5, 7, 9} and B = {2, 5, 7, 8, 11}, find the union of A and B.

Solution

As a reminder, the union of A and B is

$$A \cup B = \{x : x \in A \text{ or } x \in B\} \tag{1.3.18}$$

Thus, an object is in the union of A and B if and only if it is in either set. The numbers that are in either set are the numbers

$$A \cup B = \{1, 2, 3, 5, 7, 8, 9, 11\} \tag{1.3.19}$$

If we look again at the Venn Diagram in Figure **1.3.5**, we see that this union $A \cup B$ = {1, 2, 3, 5, 7, 8, 9, 11} lists every number that is in either set in Figure **1.3.5**.

Thus, the shaded region in Figure **1.3.7** is the union of sets A and B. Note how this region is well-named, as that’s what you’re actually doing, taking the “union” of the two sets A and B. That is, the union contains all elements that belong to either A or B. Less formally, the union is a way of combining everything that occurs in either set.

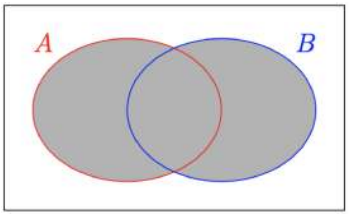


Figure **1.3.7**. The shaded region is the union of sets A and B. That is, the shaded region is $A \cup B$.

This leads to the following important piece of advice.

Note

When asked to find the union of two sets A and B, in your answer, include everything from both sets.

Simple Compound Inequalities

Let’s apply what we’ve learned to find the unions and/or intersections of intervals of real numbers. The easiest approach is through a series of examples. Let’s begin.

Example 1.3.3

On the real line, sketch the set of real numbers in the set {x : x < 3 or x < 5}. Use interval notation to describe your final answer.

Solution

First, let's sketch two sets, $\{x : x < 3\}$ and $\{x : x < 5\}$, on separate real lines, one atop the other as shown in Figure **1.3.8**.

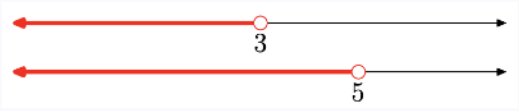


Figure **1.3.8** Sketch each set separately.

Now, to sketch the solution, note the word "or" in the set $\{x : x < 3 \text{ or } x < 5\}$. Thus, we need to take the union of the two shaded real lines in Figure **1.3.8**. That is, we need to shade everything that is shaded on either of the two number lines. Of course, this would be everything less than 5, as shown in Figure **1.3.9**.



Figure **1.3.9** The final solution is the union of the two shaded sets in Figure **1.3.8**.

Thus, the final solution is $\{x : x < 5\}$, which in interval notation, is $(-\infty, 5)$.

Let's look at another example.

Example 1.3.4

On the real line, sketch the set of real numbers in the set $\{x : x < 3 \text{ and } x < 5\}$. Use interval notation to describe your final answer.

Solution

In Example **1.3.3**, you were asked to shade the set $\{x : x < 3 \text{ or } x < 5\}$ on the real line. In this example, we're asked to sketch the set $\{x : x < 3 \text{ and } x < 5\}$. Note that the set-builder notations are identical except for one change, the "or" of Example **1.3.3** has been replaced with the word "and."

Again, sketch two sets, $\{x : x < 3\}$ and $\{x : x < 5\}$, on separate real lines, one atop the other as shown in Figure **1.3.10**.

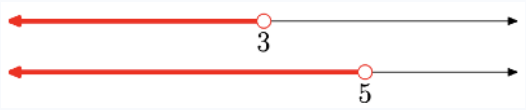


Figure **1.3.10**. Sketch each set separately.

Now, to sketch the solution, note the word "and" in the set $\{x : x < 3 \text{ and } x < 5\}$. Thus, we need to take the intersection of the two shaded real lines in Figure **1.3.10**.. That is, we need to shade everything that is common to the two number lines. Of course, this would be everything less than 3, as shown in Figure **1.3.11**.



Figure **1.3.11**. The final solution is the intersection of the two shaded sets in Figure **1.3.10**.

Thus, the final solution is $\{x : x < 3\}$, which in interval notation, is $(-\infty, 3)$.

Note

If you answer "or" when the answer requires "and," or vice-versa, you have not made a minor mistake. Indeed, this is a huge error, as demonstrated in Example **1.3.3** and Example **1.3.4**.

Before attempting another example, we pause to define a bit of notation that will be extremely important in our upcoming work.

Definition

The notation

$$a < x < b \tag{1.3.20}$$

is interpreted to mean

$$x > a \text{ and } x < b \tag{1.3.21}$$

Alternatively, we could have said that a < x < b is identical to saying “a < x and x < b,” but saying “a < x” is the same as saying “x > a.” We prefer to say “x > a and x < b,” and will use this order throughout our work, but the form “a < x and x < b” is equally valid.

The really key point to make here is the fact that the statement a < x < b is an “and” statement. If it is used properly, it’s a good way to describe the numbers that lie between a and b.

Let’s look at an example.

Example 1.3.5

On the real line, sketch the set of real numbers in the set {x : 3 < x < 5}. Use interval notation to describe your answer.

Solution

First, let’s write what’s meant by the notation {x : 3 < x < 5}. By definition, this set is the same as the set

$$\{x : x > 3 \text{ and } x < 5\} \tag{1.3.22}$$

Thus, the first step is to sketch the sets {x : x > 3} and {x : x < 5} on individual real lines, stacked one atop the other, as shown in Figure 1.3.12.



Figure 1.3.12. Sketch each set separately.

Now, to sketch the solution, note the word “and” in the set {x : x > 3 and x < 5}. Thus, we need to take the intersection of the two lines in Figure 1.3.12. That is, we need to shade the numbers on the real line that are in common to the two lines shown in Figure 1.3.12. The numbers 3 and 5 are not shaded in both sets in Figure 1.3.12, so they will not be shaded in our final solution. However, all real numbers between 3 and 5 are shaded in both sets in Figure 1.3.12, so these numbers will be shaded in the final solution shown in Figure 1.3.13.



Figure 1.3.13. The final solution is the intersection of the two shaded sets in Figure 1.3.12.

In a most natural way, the interval notation for the shaded solution in Figure 1.3.13 is (3, 5). That is,

$$(3, 5) = \{x : 3 < x < 5\} \tag{1.3.23}$$

Similarly, here are the set-builder and interval notations, as well as shading of the sets on the real line, for several situations, including the one just discussed.

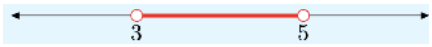
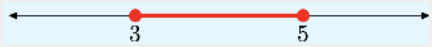
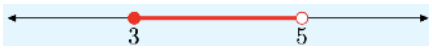
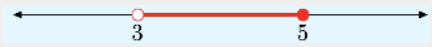
Number line	Set-builder notation	Interval notation
	$\{x : 3 < x < 5\}$	$(3, 5)$
	$\{x : 3 \leq x \leq 5\}$	$[3, 5]$
	$\{x : 3 \leq x < 5\}$	$[3, 5)$
	$\{x : 3 < x \leq 5\}$	$(3, 5]$

Table 1.3.4. Number lines, set-builder notation, and interval notation.

There are several points of emphasis regarding the intervals in Table 1.3.4.

1. When we want to emphasize that we are not including a point on the real line, we use an "empty circle." Conversely, a "filled circle" means that we are including the point on the real line. Thus, the interval in the first row of Table 1.3.4 does not include the endpoints at 3 and 5, but the interval in the second row of Table 1.3.4 does include the endpoints at 3 and 5.
2. The use of a parenthesis in interval notation means that we are not including that endpoint in the interval. Thus, the parentheses used in (3, 5) in the first row of Table 1.3.4 means that we are not including the numbers 3 and 5 in that interval.
3. The use of a bracket in interval notation means that we are including the bracketed number in the interval. Thus, the brackets used in [3, 5], as seen in the second row of Table 1.3.4, means that we are including the numbers 3 and 5 in the interval.
4. Finally, note that some of our intervals are "open" on one end but "closed" (filled) on the other end, such as those in rows 3 and 4 of Table 1.3.4.

Definition

Some terminology:

- The interval (3, 5) is open at each end. Therefore, we call the interval (3, 5) an open interval.
- The interval [3, 5] is closed (filled) at each end. Therefore, we call the interval [3, 5] a closed interval.
- The intervals (3, 5] and [3, 5) are neither open nor closed.

Let's look at another example.

Example 1.3.6

On the real line, sketch the set of all real numbers in the set $\{x : x > 3 or x < 5\}$. Use interval notation to describe your answer.

Solution

Note that the only difference between this current example and the set shaded in Example 1.3.4 is the fact that we've replaced the word "and" in $\{x : x > 3 and x < 5\}$ with the word "or" in $\{x : x > 3 or x < 5\}$. But, as we've seen before, this can make a world of difference.

Thus, the first step is to sketch the sets $\{x : x > 3\}$ and $\{x : x < 5\}$ on individual real lines, stacked one atop the other, as shown in Figure 1.3.14.



Figure 1.3.14. Sketch each set separately

Now, to sketch the solution, note the word "or" in the set $\{x : x > 3 or x < 5\}$. Thus, we need to take the union of the two lines in Figure 1.3.14. That is, we need to shade the numbers on the real line that are shaded on either of the two lines shown in Figure 1.3.14. However, this means that we will have to shade every number on the line, as shown in Figure 1.3.15. You'll note no labels for 3 and 5 on the real line in Figure 1.3.15, as there are no endpoints in this solution. The endpoints, if you will, are at negative and positive infinity.



Figure 1.3.15. The final solution is the union of the two shaded sets in Figure 1.3.14.

Thus, in a most natural way, the interval notation for the shaded solution in Figure 1.3.15 is $(-\infty, \infty)$.

Let's look at another example.

Example 1.3.7

On the real line, sketch the set of all real numbers in the set $\{x : x < -1 or x > 3\}$. Use interval notation to describe your answer.

Solution

The first step is to sketch the sets $\{x : x < -1\}$ and $\{x : x > 3\}$ on separate real lines, stacked one atop the other, as shown in Figure 1.3.16.



Figure 1.3.16. Sketch each set separately.

To sketch the solution, note the word "or" in the set $\{x : x < -1 or x > 3\}$. Thus, we need to take the union of the two shaded real lines in Figure 1.3.16. That is, we need to shade the numbers on the real line that are shaded on either real line in Figure 1.3.16. Thus, every number smaller than −1 is shaded, as well as every number greater than 3. The result is shown in Figure 1.3.17.



Figure 1.3.17. The final solution is the union of the shaded real lines in Figure 1.3.16.

Here is an important tip.

Note

If you wish to use interval notation correctly, follow one simple rule: Always sweep your eyes from left to right describing what you see shaded on the real line.

If we follow this advice, as we sweep our eyes from left to right across the real line shaded in Figure 1.3.17, we see that numbers are shaded from negative infinity to −1, and from 3 to positive infinity. Thus, in a most natural way, the interval notation for the shaded solution set in Figure 1.3.17 is

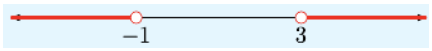
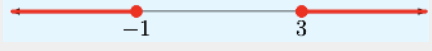
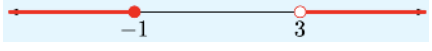
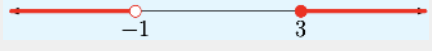
$$(-\infty, -1) \cup (3, \infty) \tag{1.3.24}$$

There are several important points to make here:

Note how we used the union symbol ∪ to join the two intervals in $(-\infty, -1) \cup (3, \infty)$ in a natural manner.

The union symbol is used between sets of numbers, while the word "or" is used between statements about numbers. It is incorrect to exchange the roles of the union symbol and the word "or." Thus, writing $\{x : x < -1 \cup x > 3\}$ is incorrect, as it would also be to write $(-\infty, -1)$ or $(3, \infty)$.

We reinforce earlier discussion about the difference between "filled" and "open" circles, brackets, and parentheses in Table 5, where we include several comparisons of interval and set-builder notation, including the current solution to Example 1.3.7.

Number line	Set-builder notation	Interval notation
	$\{x : x < -1 \text{ or } x > 3\}$	$(-\infty, -1) \cup (3, \infty)$
	$\{x : x \leq -1 \text{ or } x \geq 3\}$	$(-\infty, -1] \cup [3, \infty)$
	$\{x : x \leq -1 \text{ or } x > 3\}$	$(-\infty, -1] \cup (3, \infty)$
	$\{x : x < -1 \text{ or } x \geq 3\}$	$(-\infty, -1) \cup [3, \infty)$

Again, we reinforce the following points.

- Note how sweeping your eyes from left to right, describing what is shaded on the real line, insures that you write the interval notation in the correct order.
- A bracket is equivalent to a filled dot and includes the endpoint, while a parenthesis is equivalent to an open dot and does not include the endpoint.

Let's do one last example that should forever cement the notion that there is a huge difference between the words "and" and "or."

Example 1.3.8

On the real line, sketch the set of all real numbers in the set $\{x : x < -1 \text{ and } x > 3\}$. Describe your solution.

Solution

First and foremost, note that the only difference between this example and Example **1.3.6** is the fact that we changed the "or" in $\{x : x < -1 \text{ or } x > 3\}$ to an "and" in $\{x : x < -1 \text{ and } x > 3\}$. The preliminary sketches are identical to those in Figure 16.



Figure **1.3.18**. Sketch $\{x : x < -1\}$ and $\{x : x > 3\}$ on separate real lines.

Now, note the word "and" in $\{x : x < -1 and x > 3\}$. Thus, we need to take the intersection of the shaded real lines in Figure **1.3.18**. That is, we need to shade on a single real line all of the numbers that are shaded on both real lines in Figure **1.3.18**. However, there are no points shaded in common on the real lines in Figure 18, so the solution set is empty, as shown in Figure **1.3.19**.

Figure **1.3.19**. The solution is empty so we leave the real line blank.

Pretty impressive! The last two examples clearly demonstrate that if you interchange the roles of "and" and "or," you have not made a minor mistake. Indeed, you've changed the whole meaning of the problem. So, be careful with your "ands" and "ors."

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CHAPTER OVERVIEW

3: Polynomial and Rational Functions

In this chapter, we will learn about these concepts and discover how mathematics can be used in such applications.

3.0: Prelude to Polynomial and Rational Functions
3.0E: Exercises
3.1: Complex Numbers
3.1E: Exercises
3.2: Quadratic Functions
3.2E: Exercises
3.3: Power Functions and Polynomial Functions
3.3E: Exercises
3.4: Graphs of Polynomial Functions
3.4E: Exercises
3.5: Dividing Polynomials
3.6: Zeros of Polynomial Functions
3.6E: Exercises
3.7: Rational Functions
3.7E: Exercises
3.8: Inverses and Radical Functions
3.8E: Exercises

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3.0: Prelude to Polynomial and Rational Functions

Digital photography has dramatically changed the nature of photography. No longer is an image etched in the emulsion on a roll of film. Instead, nearly every aspect of recording and manipulating images is now governed by mathematics. An image becomes a series of numbers, representing the characteristics of light striking an image sensor. When we open an image file, software on a camera or computer interprets the numbers and converts them to a visual image. Photo editing software uses complex polynomials to transform images, allowing us to manipulate the image in order to crop details, change the color palette, and add special effects. Inverse functions make it possible to convert from one file format to another. In this chapter, we will learn about these concepts and discover how mathematics can be used in such applications.

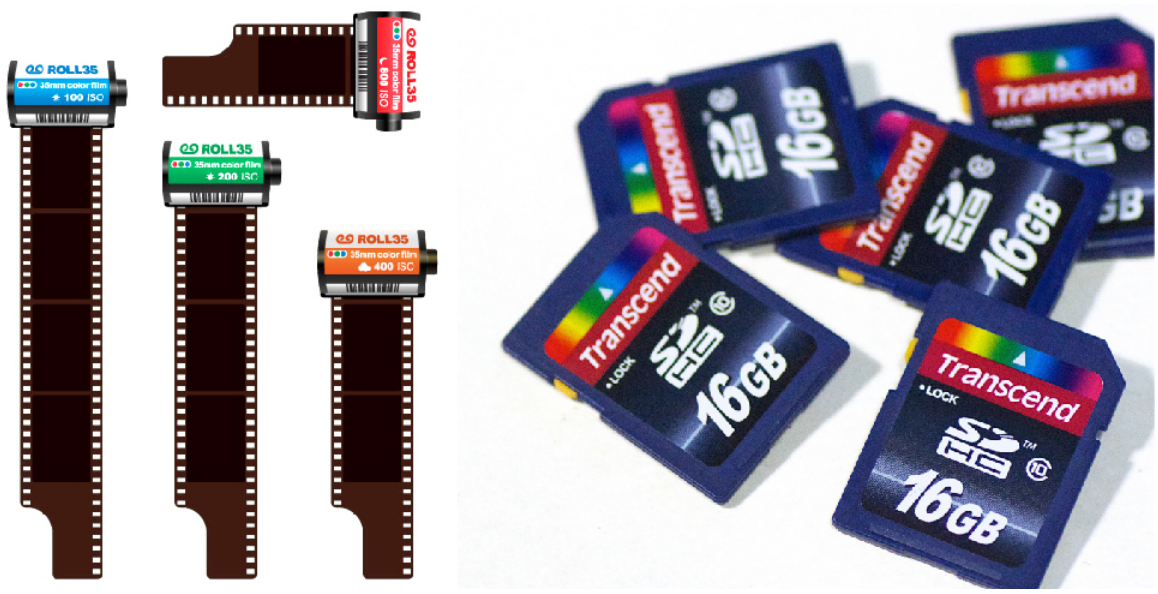


Figure* 3.0.1*: *35-mm film, once the standard for capturing photographic images, has been made largely obsolete by digital photography. (credit "film": modification of work by Horia Varlan; credit "memory cards": modification of work by Paul Hudson)*

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3.0E: Exercises

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3.1: Complex Numbers

Skills to Develop

- Express square roots of negative numbers as multiples of i.
- Plot complex numbers on the complex plane.
- Add and subtract complex numbers.
- Multiply and divide complex numbers.

The study of mathematics continuously builds upon itself. Negative integers, for example, fill a void left by the set of positive integers. The set of rational numbers, in turn, fills a void left by the set of integers. The set of real numbers fills a void left by the set of rational numbers. Not surprisingly, the set of real numbers has voids as well. For example, we still have no solution to equations such as

$$x^2 + 4 = 0$$

Our best guesses might be $x = +2$ or $x = -2$. But if we test +2 in this equation, it does not work. If we test –2, it does not work. If we want to have a solution for this equation, we will have to go farther than we have so far. After all, to this point we have described the square root of a negative number as undefined. Fortunately, there is another system of numbers that provides solutions to problems such as these. In this section, we will explore this number system and how to work within it.

Expressing Square Roots of Negative Numbers as Multiples of i

We know how to find the square root of any positive real number. In a similar way, we can find the square root of a negative number. The difference is that the root is not real. If the value in the radicand is negative, the root is said to be an imaginary number. The imaginary number i is defined as the square root of -1.

$$\sqrt{-1} = i$$

So, using properties of radicals,

$$i^2 = (\sqrt{-1})^2 = -1$$

We can write the square root of any negative number as a multiple of i. Consider the square root of –25.

$$\begin{aligned}\sqrt{-25} &= \sqrt{25 \cdot (-1)} \\ &= \sqrt{25}\sqrt{-1} \\ &= 5i\end{aligned}$$

We use $5i$ and not $-5i$ because the principal root of 25 is the positive root.

A **complex number** is the sum of a real number and an imaginary number. A complex number is expressed in standard form when written $a + bi$ (with a, b real numbers) where a is the real part and bi is the imaginary part. For example, $5 + 2i$ is a complex number. So, too, is $3 + 4\sqrt{3}i$.

Note: since it is so easy to make the mistake of thinking that the i at the end of the number sits under the radical sign, it is customary to write this number as $3 + 4i\sqrt{3}$. Also, as usual, if a term is 0, or a coefficient is 1, we often omit it; so $0 + 1i$ (correct standard form) is often written simply as i.

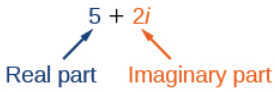


***Figure* 3.1.1**

Imaginary numbers are distinguished from real numbers because a squared imaginary number produces a negative real number. Recall, when a positive real number is squared, the result is a positive real number and when a negative real number is squared, again, the result is a positive real number. Complex numbers are a combination of real and imaginary numbers.

Definition: Imaginary and Complex Numbers

A complex number is a number of the form $a+bi$ where

- a is the real part of the complex number.
- bi is the imaginary part of the complex number.

If $b=0$, then $a+bi$ is a real number. If $a=0$ and b is not equal to 0, the complex number is called an **imaginary number**. An imaginary number is an even root of a negative number.

How To: Given an imaginary number, express it in standard form

1. Write $\sqrt{-a}$ as $\sqrt{a}\sqrt{-1}$.
2. Express $\sqrt{-1}$ as i.
3. Write $\sqrt{a}\cdot i$ in simplest form.

Example 3.1.1: Expressing an Imaginary Number in Standard Form

Express $\sqrt{-9}$ in standard form.

Solution

$$\sqrt{-9}=\sqrt{9}\sqrt{-1}=3i$$

In standard form, this is $0+3i$.

Try It 3.1.1

Express $\sqrt{-24}$ in standard form.

Answer

$\sqrt{-24}=0+2i\sqrt{6}$

Plotting a Complex Number on the Complex Plane

We cannot plot complex numbers on a number line as we might real numbers. However, we can still represent them graphically. To represent a complex number we need to address the two components of the number. We use the complex plane, which is a coordinate system in which the horizontal axis represents the real component and the vertical axis represents the imaginary component. Complex numbers are the points on the plane, expressed as ordered pairs (a,b), where a represents the coordinate for the horizontal axis and b represents the coordinate for the vertical axis.

Let's consider the number $-2+3i$. The real part of the complex number is -2 and the imaginary part is $3i$. We plot the ordered pair $(-2,3)$ to represent the complex number $-2+3i$ as shown in Figure 3.1.2

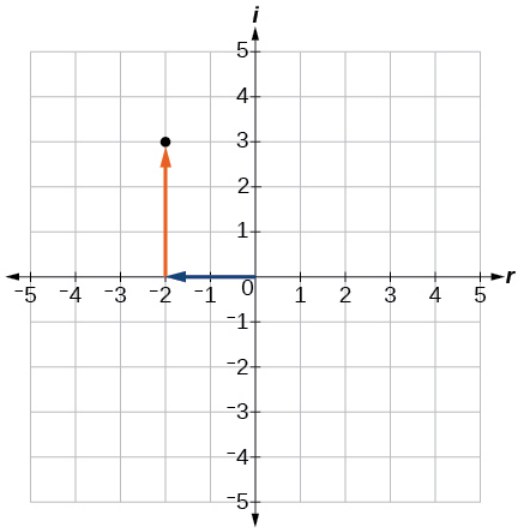


***Figure* 3.1.2:** *Plot of a complex number, $-2+3i$. Note that the real part (-2) is plotted on the x-axis and the imaginary part $(3i)$ is plotted on the y-axis.*

Complex Plane

In the complex plane, the horizontal axis is the real axis, and the vertical axis is the imaginary axis as shown in Figure 3.1.3.

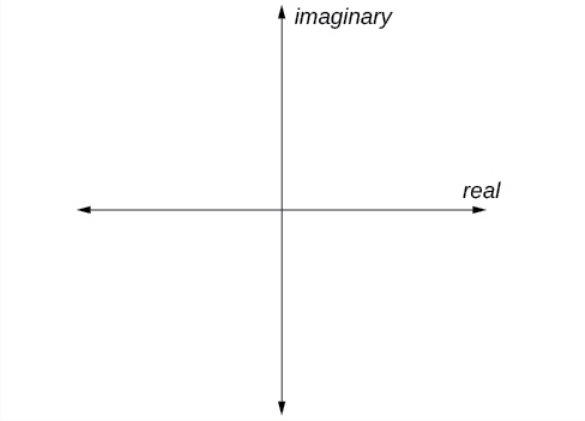


***Figure* 3.1.3:** *The complex plane showing that the horizontal axis (in the real plane, the x-axis) is known as the real axis and the vertical axis (in the real plane, the y-axis) is known as the imaginary axis.*

Given a complex number, represent its components on the complex plane.

1. Determine the real part and the imaginary part of the complex number.
2. Move along the horizontal axis to show the real part of the number.
3. Move parallel to the vertical axis to show the imaginary part of the number.
4. Plot the point.

Example 3.1.2: Plotting a Complex Number on the Complex Plane

Plot the complex number $3-4i$ on the complex plane.

Solution

The real part of the complex number is 3, and the imaginary part is $-4i$. We plot the ordered pair $(3,-4)$ as shown in Figure 3.1.4.

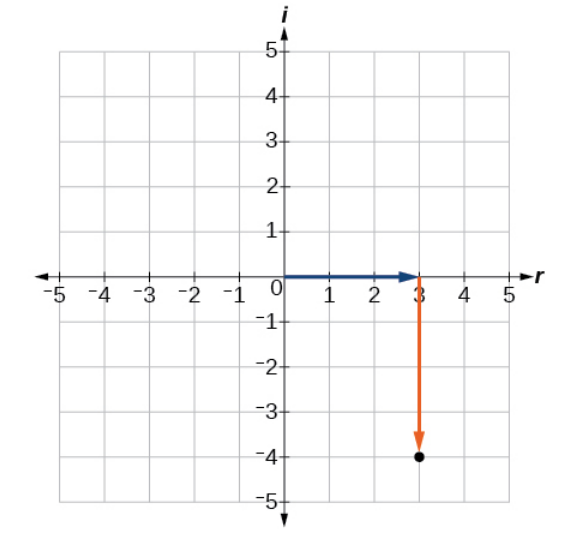


***Figure* 3.1.4:** *Plot of a complex number, $3-4i$. Note that the real part (3) is plotted on the x-axis and the imaginary part $(-4i)$ is plotted on the y-axis.*

3.1.2

Plot the complex number $-4-i$ on the complex plane.

Answer

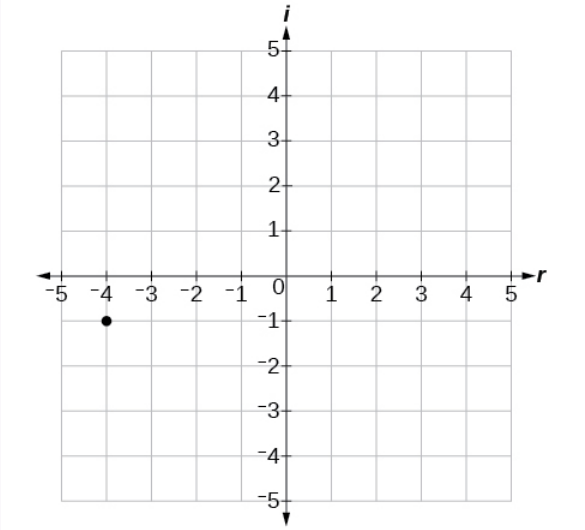


***Figure* 3.1.5**

Adding and Subtracting Complex Numbers

Just as with real numbers, we can perform arithmetic operations on complex numbers. To add or subtract complex numbers, we combine the real parts and combine the imaginary parts.

Complex Numbers: Addition and Subtraction

Adding complex numbers:

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$

Subtracting complex numbers:

$$(a+bi)-(c+di)=(a-c)+(b-d)i$$

How To: Given two complex numbers, find the sum or difference.

1. Identify the real and imaginary parts of each number.
2. Add or subtract the real parts.
3. Add or subtract the imaginary parts.

Example 3.1.3: Adding Complex Numbers

Add $3-4i$ and $2+5i$.

Solution

We add the real parts and add the imaginary parts.

$$\begin{aligned}(a+bi)+(c+di)&=(a+c)+(b+d)i\\(3-4i)+(2+5i)&=(3+2)+(-4+5)i\\&=5+i\end{aligned}$$

Try It 3.1.3

Subtract $2+5i$ from $3–4i$.

Answer

$$(3-4i)-(2+5i)=1-9i$$

Multiplying Complex Numbers

Multiplying complex numbers is much like multiplying binomials. The "twist" is that whenever we have an i^2 term, we rewrite i^2 as -1.

Multiplying a Complex Numbers by a Real Number

Let's begin by multiplying a complex number by a real number. We distribute the real number just as we would with a binomial. So, for example,

$$\begin{aligned}3(6+2i)&=(3\cdot 6)+(3\cdot 2i) && \text{Distribute.}\\&=18+6i && \text{Simplify.}\end{aligned}$$

***Figure* 3.1.6**

How To: Given a complex number and a real number, multiply to find the product.

1. Use the distributive property.
2. Simplify.

Example 3.1.4: Multiplying a Complex Number by a Real Number

Find the product $4(2+5i)$.

Solution

Distribute the 4.

$$\begin{aligned}4(2+5i)&=(4\cdot 2)+(4\cdot 5i)\\&=8+20i\end{aligned}$$

Try It 3.1.4

Find the product $-4(2+6i)$.

Answer

$-8-24i$

Multiplying Complex Numbers Together

Now, let's multiply two complex numbers. We can use either the distributive property or the FOIL method. Recall that FOIL is an acronym for multiplying First, Outer, Inner, and Last terms together. Using either the distributive property or the FOIL method, we get

$$(a+bi)(c+di)=ac+adi+bci+bdi^2$$

Here is the "twist": because $i^2=-1$, we have

$$(a+bi)(c+di)=ac+adi+bci-bd$$

To simplify, we combine the real parts, and we combine the imaginary parts to get standard form.

$$(a+bi)(c+di)=(ac-bd)+(ad+bc)i$$

How To: Given two complex numbers, multiply to find the product.

1. Use the distributive property or the FOIL method.
2. If there is a term containing i^2, rewrite i^2 as -1.
3. Simplify.
4. Write the answer in standard form.

Example 3.1.5: Multiplying a Complex Number by a Complex Number

Multiply $(4+3i)(2-5i)$.

Solution

Use $(a+bi)(c+di)=(ac-bd)+(ad+bc)i$

$$\begin{aligned}(4+3i)(2-5i)&=(4\cdot 2-4(5i)+3i(2)-15i^2\\&=(8+15)+(-20+6)i\\&=23-14i\end{aligned}$$

Try It 3.1.5

Multiply $(3-4i)(2+3i)$.

Answer

$18+i$

Dividing Complex Numbers

Division of two complex numbers is more complicated than addition, subtraction, and multiplication because standard form of a complex number does not allow an imaginary number in the denominator. "Division" of complex numbers becomes a task of rewriting a fraction so that the result is in standard form. We need a term by which we can multiply the numerator and the denominator that will eliminate the imaginary portion of the denominator. This term is called the **complex conjugate** of the denominator, which is found by changing the sign of the imaginary part of the complex number. In other words, the complex conjugate of $a+bi$ is $a-bi$.

Definition: The Complex Conjugate

The *complex conjugate* of a complex number $a+bi$ is $a-bi$. It is found by changing the sign of the imaginary part of the complex number. The real part of the number is left unchanged.

- When a complex number is multiplied by its complex conjugate, the result is a real number: $(a+bi)(a-bi)=a^2+b^2$.

- When a complex number is added to its complex conjugate, the result is a real number: $(a+bi)+(a-bi)=2a$.

Note that complex conjugates have a reciprocal relationship: The complex conjugate of $a+bi$ is $a-bi$, and the complex conjugate of $a-bi$ is $a+bi$. Note: as we will see in Section 3.2, when a quadratic equation with real coefficients has complex solutions, the solutions are always complex conjugates of one another.

Suppose we want to divide $c+di$ by $a+bi$, where neither a nor b equals zero. We first write the division as a fraction, then find the complex conjugate of the denominator, and multiply.

$$\frac{c+di}{a+bi} \text{ where } a \text{ and } b \text{ are not both } 0$$

Multiply the numerator and denominator by the complex conjugate of the denominator.

$$\frac{(c+di)}{(a+bi)}\cdot\frac{(a-bi)}{(a-bi)}=\frac{(c+di)(a-bi)}{(a+bi)(a-bi)}$$

Apply the distributive property.

$$=\frac{ca-cbi+adi-bdi^2}{a^2-abi+abi-b^2i^2}$$

Simplify, remembering that $i^2=-1$.

$$=\frac{ca-cbi+adi-bd(-1)}{a^2-abi+abi-b^2(-1)}$$
$$=\frac{(ca+bd)+(ad-cbi)}{a^2+b^2}$$

Example 3.1.6: Finding Complex Conjugates

Find the complex conjugate of each number.

a. $2+i\sqrt{5}$
b. $-\frac{1}{2}i$

Solution

a. The number is already in the form $a+bi$. The complex conjugate is $a-bi$, or $2-i\sqrt{5}$.
b. Rewrite this number in the form $a+bi$ to get $0-\frac{1}{2}i$. The complex conjugate is $a-bi$, or $0-(-\frac{1}{2}i)$, which simplifies to $0+\frac{1}{2}i$. This is usually written simply as $\frac{1}{2}i$.

How To: Given two complex numbers, divide one by the other.

1. Write the division problem as a fraction.
2. Determine the complex conjugate of the denominator.
3. Multiply the numerator and denominator of the fraction by the complex conjugate of the denominator.
4. Simplify.

Example 3.1.7: Dividing Complex Numbers

Divide $(2+5i)$ by $(4-i)$.

Solution

We begin by writing the problem as a fraction.

$$\frac{(2+5i)}{(4-i)}$$

We multiply the numerator and denominator by the complex conjugate of the denominator.

$$\frac{(2+5i)}{(4-i)}\cdot\frac{(4+i)}{(4+i)}$$

To multiply two complex numbers, we expand the product as we would with binomials (the process commonly called FOIL).

$$\begin{aligned}\frac{(2+5i)}{(4-i)}\cdot\frac{(4+i)}{(4+i)} &= \frac{8+2i+20i+5i^2}{16+4i-4i-i^2} \\ &= \frac{8+2i+20i+5(-1)}{16+4i-4i-(-1)} && \text{because } i^2=-1 \\ &= \frac{3+22i}{17} \\ &= \frac{3}{17}+\frac{22}{17}i && \text{Separate real and imaginary parts.}\end{aligned}$$

Note that this expresses the quotient in standard form.

Extra Credit: We multipliy the original fraction by $\frac{4+i}{4+i}$. **(a)** What is the value of $\frac{4+i}{4+i}$? **(b)** Explain the importance of this value in the process of dividing complex numbers. Write this up and give it to your instructor for Extra Credit.

Example 3.1.8: Substituting a Complex Number into a Polynomial Function

Let $f(x)=x^2-5x+2$. Evaluate $f(3+i)$.

Solution

Substitute $x=3+i$ into the function $f(x)=x^2-5x+2$ and simplify.

$$\begin{aligned}f(3+i) &= (3+i)^2-5(3+i)+2 && \text{Substitute } 3+i \text{ for } x \\ &= (3+6i+i^2)-(15+5i)+2 && \text{Multiply} \\ &= 9+6i+(-1)-15-5i+2 && \text{Subsitute -1 for } i^2 \\ &= -5+i && \text{Combine like terms}\end{aligned}$$

Analysis

We write $f(3+i)=-5+i$. Notice that the input is $3+i$ and the output is $-5+i$.

3.1.6

Let $f(x)=2x^2-3x$. Evaluate $f(8-i)$.

Answer

$102-29i$

Example 3.1.9: Substituting an Imaginary Number in a Rational Function

Let $f(x)=\frac{2+x}{x+3}$. Evaluate $f(10i)$.

Solution

Substitute $x=10i$ and simplify.

$\frac{2+10i}{10i+3}$	Substitute $10i$ for x.
$\frac{2+10i}{3+10i}$	Rewrite the denominator in standard form.
$\frac{2+10i}{3+10i}\cdot\frac{3-10i}{3-10i}$	Multiply the numerator and denominator by the complex conjugate of the denominator.
$\frac{6-20i+30i-100i^2}{9-30i+30i-100i^2}$	Multiply using the distributive property or the FOIL method.
$\frac{6-20i+30i-100(-1)}{9-30i+30i-100(-1)}$	Substitute -1 for i^2.
$\frac{106+10i}{109}$	Simplify.
$\frac{106}{109}+\frac{10}{109}$	Separate the real and imaginary parts.

3.1.7

Let $f(x)=\frac{x+1}{x-4}$. Evaluate $f(-i)$.

Answer

$-\frac{3}{17}+\frac{5i}{17}$

Simplifying Powers of i

The powers of i are cyclic. Let's look at what happens when we raise i to increasing powers.

$$i^1=i$$
$$i^2=-1$$
$$i^3=i^2\cdot i=-1\cdot i=-i$$
$$i^4=i^3\cdot i=-i\cdot i=-i^2=-(-1)=1$$
$$i^5=i^4\cdot i=1\cdot i=i$$

We can see that when we get to the fifth power of i, it is equal to the first power. As we continue to multiply i by itself for increasing powers, we will see a cycle of length 4. Let's examine the next 4 powers of i.

$$i^6=i^5\cdot i=i\cdot i=i^2=-1$$
$$i^7=i^6\cdot i=i^2\cdot i=i^3=-i$$
$$i^8=i^7\cdot i=i^3\cdot i=i^4=1$$
$$i^9=i^8\cdot i=i^4\cdot i=i^5=i$$

Example 3.1.10: Simplifying Powers of i

Evaluate i^{35}.

Solution

Since $i^4=1$, we can simplify the problem by factoring out as many factors of i^4 as possible. To do so, first determine how many times 4 goes into 35: $35=4\cdot 8+3$.

$$i^{35}=i^{4\cdot 8+3}=i^{4\cdot 8}\cdot i^3=(i^4)^8\cdot i^3=1^8\cdot i^3=i^3=-i$$

Can we write i^{35} in other helpful ways?

As we saw in Example **3.1.10**, *we reduced* i^{35} *to* i^3 *by dividing the exponent by 4 and using the remainder to find the simplified form. But perhaps another factorization of* i^{35} *may be more useful. Table* **3.1.1** *shows some other possible factorizations.*

Table 3.1.1*: Each of these will eventually result in the answer we obtained above but may require several more steps than our earlier method.*

Factorization of i^{35}	$i^{34}\cdot i$	$i^{33}\cdot i^2$	$i^{31}\cdot i^4$	$i^{19}\cdot i^{16}$
Reduced form	$\left(i^2\right)^{17}\cdot i$	$i^{33}\cdot(-1)$	$i^{31}\cdot 1$	$i^{19}\cdot\left(i^4\right)^4$
Simplified form	$(-1)^{17}\cdot i$	$-i^{33}$	i^{31}	i^{19}

Key Concepts

- The square root of any negative number can be written as a multiple of i.
- To plot a complex number, we use two number lines, crossed to form the complex plane. The horizontal axis is the real axis, and the vertical axis is the imaginary axis.
- Complex numbers can be added and subtracted by combining the real parts and combining the imaginary parts.
- Complex numbers can be multiplied and divided.
- To multiply complex numbers, distribute just as with polynomials.
- To divide complex numbers, multiply both the numerator and denominator by the complex conjugate of the denominator to eliminate the complex number from the denominator.
- The powers of i are cyclic, repeating every fourth one.

Glossary

complex conjugate
the complex number in which the sign of the imaginary part is changed and the real part of the number is left unchanged; when added to or multiplied by the original complex number, the result is a real number

complex number
the sum of a real number and an imaginary number, written in the standard form
$a+bi\ (a,b\in\mathbb{R})$,
where a is the real part, and bi is the imaginary part

complex plane
a coordinate system in which the horizontal axis is used to represent the real part of a complex number and the vertical axis is used to represent the imaginary part of a complex number

imaginary number
a number in the form bi where $i=\sqrt{-1}$

Contributors

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3.1E: Exercises

Verbal

Exercise 3.1E.1

Explain how to add complex numbers.

Answer

Add the real parts together and the imaginary parts together.

Exercise 3.1E.2

What is the basic principle in multiplication of complex numbers?

Exercise 3.1E.3

Give an example to show the product of two imaginary numbers is not always imaginary.

Answer

i times i equals –1, which is not imaginary. (answers will vary)

Exercise 3.1E.4

What is a characteristic of the plot of a real number in the complex plane?

Algebraic

For the following exercises, evaluate the algebraic expressions.

Exercise 3.1E.5

If $f(x) = x^2 + x - 4$, evaluate $f(2i)$.

Answer

$-8 + 2i$

Exercise 3.1E.6

If $f(x) = x^3 - 2$, evaluate $f(i)$.

Exercise 3.1E.7

If $f(x) = x^2 + 3x + 5$,evaluate $f(2 + i)$.

Answer

$14 + 7i$

Exercise 3.1E.8

If $f(x) = 2x^2 + x - 3$, evaluate $f(2 - 3i)$.

Exercise 3.1E.9

If $f(x) = \frac{x+1}{2-x}$, evaluate $f(5i)$.

Answer

$-\frac{23}{29} + \frac{15}{29}i$

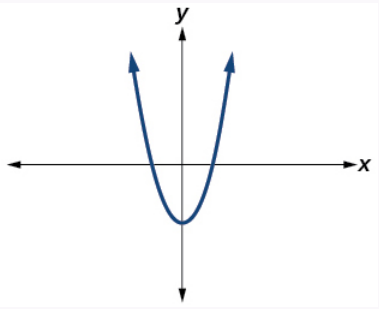
Exercise 3.1E.10

If $f(x) = \frac{1+2x}{x+3}$, evaluate $f(4i)$.

Graphical

For the following exercises, determine the number of real and nonreal solutions for each quadratic function shown.

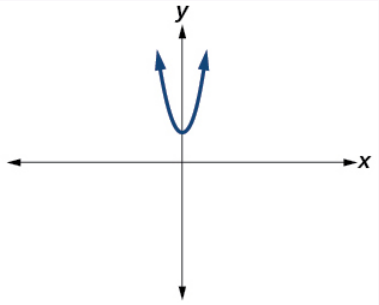
Exercise 3.1E.11



Answer

2 real and 0 nonreal

Exercise 3.1E.12

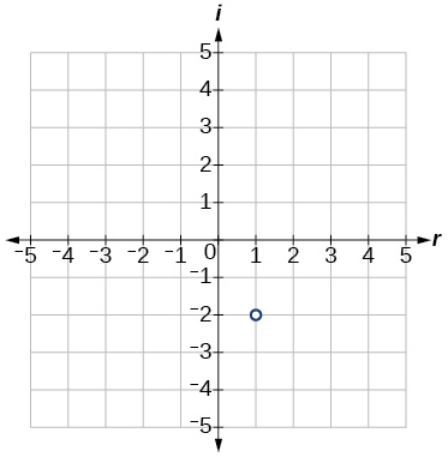


For the following exercises, plot the complex numbers on the complex plane.

Exercise 3.1E.13

$1 - 2i$

Answer



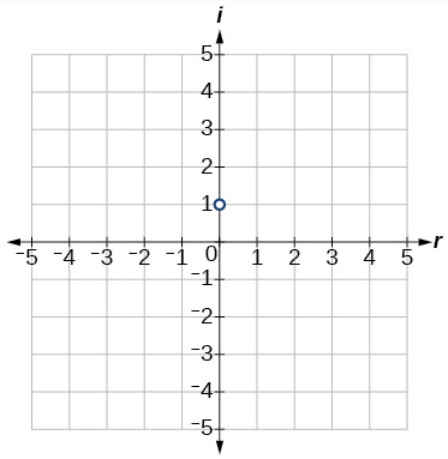
Exercise 3.1E.14

$-2+3i$

Exercise 3.1E.15

i

Answer



Exercise 3.1E.16

$-3-4i$

Numeric

For the following exercises, perform the indicated operation and express the result as a simplified complex number.

Exercise 3.1E.17

$(3+2i)+(5-3i)$

Answer

$8-i$

Exercise 3.1E.18

$(-2-4i)+(1+6i)$

Exercise 3.1E.19

$(-5+3i)-(6-i)$

Answer

$-11+4i$

Exercise 3.1E.20

$(2-3i)-(3+2i)$

Exercise 3.1E.21

$(-4+4i)-(-6+9i)$

Answer

$2-5i$

Exercise 3.1E.22

$(2+3i)(4i)$

Exercise 3.1E.23

$(5-2i)(3i)$

Answer

$6+15i$

Exercise 3.1E.24

$(6-2i)(5)$

Exercise 3.1E.25

$(-2+4i)(8)$

Answer

$-16+32i$

Exercise 3.1E.26

$(2+3i)(4-i)$

Exercise 3.1E.27

$(-1+2i)(-2+3i)$

Answer

$-4-7i$

Exercise 3.1E.28

$(4-2i)(4+2i)$

Exercise 3.1E.29

$(3+4i)(3-4i)$

Answer

25

Exercise 3.1E.30

$\frac{3+4i}{2}$

Exercise 3.1E.31

$\frac{6-2i}{3}$

Answer

$2-\frac{2}{3}i$

Exercise 3.1E.32

$\frac{-5+3i}{2i}$

Exercise 3.1E.33

$\frac{6+4i}{i}$

Answer

$4-6i$

Exercise 3.1E.34

$\frac{2-3i}{4+3i}$

Exercise 3.1E.35

$\frac{3+4i}{2-i}$

Answer

$\frac{2}{5}+\frac{11}{5}i$

Exercise 3.1E.36

$\frac{2+3i}{2-3i}$

Exercise 3.1E.37

$\sqrt{-9}+3\sqrt{-16}$

Answer

$15i$

Exercise 3.1E.38

$-\sqrt{-4}-4\sqrt{-25}$

Exercise 3.1E.39

$\frac{2+\sqrt{-12}}{2}$

Answer

$1+i\sqrt{3}$

Exercise 3.1E.40

$\frac{4+\sqrt{-20}}{2}$

Exercise 3.1E.41

i^8

Answer

1

Exercise 3.1E.42

i^{15}

Exercise 3.1E.43

i^{22}

Answer

−1

Technology

For the following exercises, use a calculator to help answer the questions.

44. Evaluate $(1+i)^k$ for $k=4, 8,$ and 12.Predict the value if $k=16$.

45. Evaluate $(1-i)^k$ for $k=2, 6,$ and 10.Predict the value if $k=14$.

Answer: 128i

46. Evaluate (1+i)k−(1−i)k for $k=4$, 8, and 12. Predict the value for $k=16$.

47. Show that a solution of $x^6+1=0$ is $\frac{\sqrt{3}}{2}+\frac{1}{2}i$.

Answer: $(\frac{\sqrt{3}}{2}+\frac{1}{2}i)^6=-1$

48. Show that a solution of $x^8-1=0$ is $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$.

Extensions

For the following exercises, evaluate the expressions, writing the result as a simplified complex number.

49. $\frac{1}{i}+\frac{4}{i^3}$

Answer: $3i$

50. $\frac{1}{i^{11}}-\frac{1}{i^{21}}$

51. $i^7(1+i^2)$

Answer: 0

52. $i^{-3}+5i^7$

53. $\frac{(2+i)(4-2i)}{(1+i)}$

Answer: $5-5i$

54. $\frac{(1+3i)(2-4i)}{(1+2i)}$

55. $\frac{(3+i)^2}{(1+2i)^2}$

Answer: $-2i$

56. $\frac{3+2i}{2+i}+(4+3i)$

https://math.libretexts.org/@go/page/31079

57. $\frac{4+i}{i}+\frac{3-4i}{1-i}$

Answer: $\frac{9}{2}-\frac{9}{2}i$

58. $\frac{3+2i}{1+2i}-\frac{2-3i}{3+i}$

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3.2: Quadratic Functions

Skills to Develop

- Recognize characteristics of parabolas.
- Understand how the graph of a parabola is related to its quadratic function.
- Determine a quadratic function's minimum or maximum value.
- Solve problems involving a quadratic function's minimum or maximum value.

Curved antennas, such as the ones shown in Figure 3.2.1, are commonly used to focus microwaves and radio waves to transmit television and telephone signals, as well as satellite and spacecraft communication. The cross-section of the antenna is in the shape of a parabola, which can be described by a quadratic function.



Figure* 3.2.1*: *An array of satellite dishes. (credit: Matthew Colvin de Valle, Flickr)*

In this section, we will investigate quadratic functions, which frequently model problems involving area and projectile motion. Working with quadratic functions can be less complex than working with higher degree functions, so they provide a good opportunity for a detailed study of function behavior.

Recognizing Characteristics of Parabolas

The graph of a quadratic function is a U-shaped curve called a parabola. One important feature of the graph is that it has an extreme point, called the **vertex**. If the parabola opens up, the vertex represents the lowest point on the graph, or the **minimum value** of the quadratic function. If the parabola opens down, the vertex represents the highest point on the graph, or the **maximum value**. In either case, the vertex is a turning point on the graph. The graph is also symmetric with a vertical line drawn through the vertex, called the **axis of symmetry**. These features are illustrated in Figure 3.2.2.

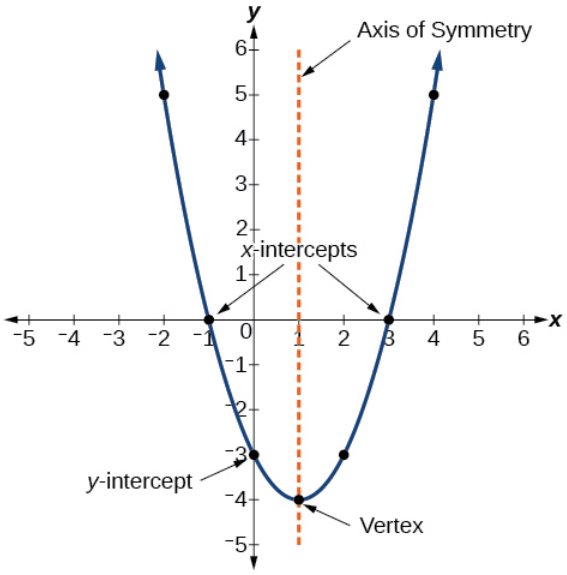


***Figure* 3.2.2:** *Graph of a parabola showing where the x and y intercepts, vertex, and axis of symmetry are.*

The y-intercept is the point at which the parabola crosses the y-axis. The x-intercepts are the points at which the parabola crosses the x-axis. If they exist, the x-intercepts represent the **zeros**, or **roots**, of the quadratic function, the values of x at which $y = 0$.

Example 3.2.1: Identifying the Characteristics of a Parabola

Determine the vertex, axis of symmetry, zeros, and y-intercept of the parabola shown in Figure 3.2.3.

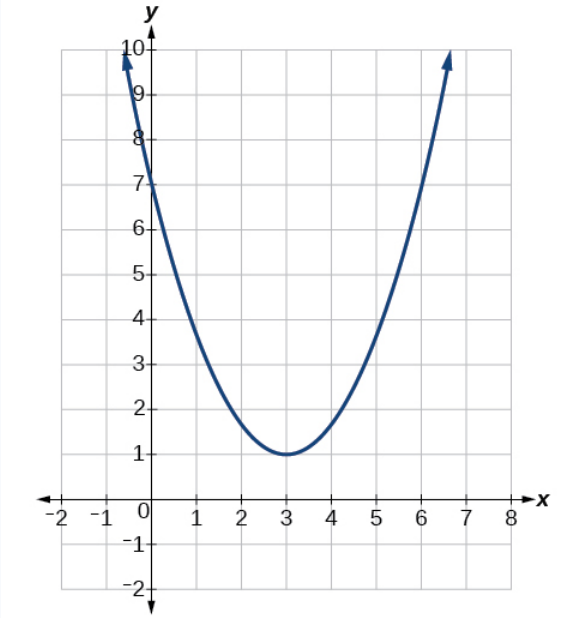


***Figure* 3.2.3.**

Solution

The vertex is the turning point of the graph. We can see that the vertex is at $(3, 1)$. Because this parabola opens upward, the axis of symmetry is the vertical line that intersects the parabola at the vertex. So the axis of symmetry is $x = 3$. This parabola does not cross the x-axis, so it has no zeros. It crosses the y-axis at $(0, 7)$ so this is the y-intercept.

Understanding How the Graphs of Parabolas are Related to Their Quadratic Functions

The **general form of a quadratic function** presents the function in the form

$$f(x) = ax^2 + bx + c$$

where a, b, and c are real numbers and $a \neq 0$. If $a > 0$, the parabola opens upward. If $a < 0$, the parabola opens downward. We can use the general form of a parabola to find the equation for the axis of symmetry.

The axis of symmetry is defined by $x = -\frac{b}{2a}$. If we use the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, to solve $ax^2 + bx + c = 0$ for the x-intercepts, or zeros, we find that the value of x halfway between them is always $x = -\frac{b}{2a}$, the equation for the axis of symmetry.

Figure 3.2.4 represents the graph of the quadratic function written in general form as $y = x^2 + 4x + 3$. In this form, $a = 1$, $b = 4$, and $c = 3$. Because $a > 0$, the parabola opens upward. The axis of symmetry is $x = -\frac{4}{2(1)} = -2$. This also makes sense because we can see from the graph that the vertical line $x = -2$ divides the graph in half. The vertex always occurs along the axis of symmetry. For a parabola that opens upward, the vertex occurs at the lowest point on the graph, in this instance, $(-2, -1)$. The x-intercepts, those points where the parabola crosses the x-axis, occur at $(-3, 0)$ and $(-1, 0)$.

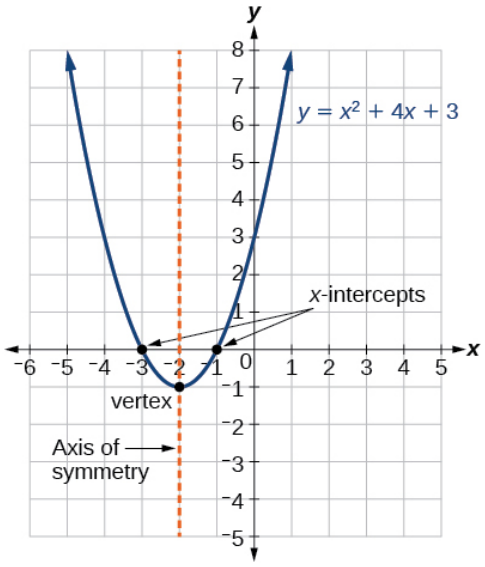


Figure 3.2.4: *Graph of a parabola showing where the x-intercepts, vertex, and axis of symmetry are for the function $y = x^2 + 4x + 3$.*

The **standard form of a quadratic function** presents the function in the form

$$f(x) = a(x - h)^2 + k$$

where (h, k) is the vertex. Because the vertex can be seen in the standard form of the quadratic function, this form is also known as the **vertex form of a quadratic function**.

As with the general form, if $a > 0$, the parabola opens upward and the vertex is a minimum. If $a < 0$, the parabola opens downward, and the vertex is a maximum. Figure 3.2.5 represents the graph of the quadratic function written in standard form as $y = -3(x + 2)^2 + 4$. Since $x - h = x + 2$ in this example, $h = -2$. In this form, $a = -3$, $h = -2$, and $k = 4$. Because $a < 0$, the parabola opens downward. The vertex is at $(-2, 4)$.

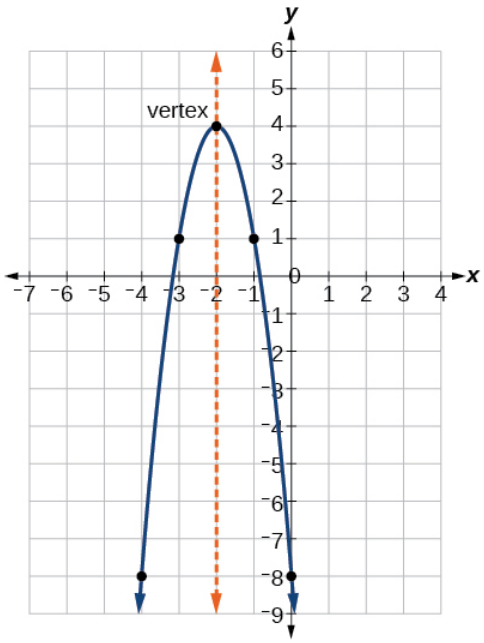


Figure 3.2.5: *Graph of a parabola showing where the vertex and axis of symmetry are for the function* $y = -3(x + 2)^2 + 4$.

The standard form is useful for determining how the graph is transformed from the graph of $y = x^2$. Figure 3.2.6 is the graph of this basic function.

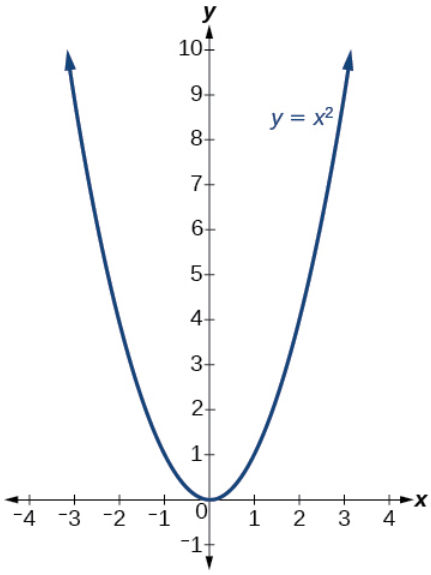


Figure 3.2.6: *Graph of* $y = x^2$.

If $h > 0$, the graph shifts toward the right and if $h < 0$, the graph shifts to the left. In Figure 3.2.5, $h < 0$, so the graph is shifted 2 units to the left. The magnitude of a indicates the stretch of the graph. If $|a| > 1$, the point associated with a particular x-value shifts farther from the x-axis, so the graph appears to become narrower, and there is a vertical stretch. But if $|a| < 1$, the point associated with a particular x-value shifts closer to the x-axis. The graph appears to become wider, but in fact there is a vertical compression. In Figure 3.2.5, $|a| > 1$, so the graph becomes narrower. Since $a < 0$, the graph is reflected across the x-axis. If $k > 0$, the graph shifts upward, whereas if $k < 0$, the graph shifts downward. In Figure 3.2.5, $k > 0$, so the graph is shifted 4 units upward.

The standard form and the general form are equivalent methods of describing the same function. If we wish to solve for h and k in terms of $a, b, |)and\(x$, we can expand the general form and set it equal to the standard form. In both forms, a represents the same value.

$$a(x-h)^2+k=ax^2+bx+c$$
$$ax^2-2ahx+(ah^2+k)=ax^2+bx+c$$

For the linear terms to be equal, the coefficients must be equal.

$$-2ah=b, \text{ so } h=-\frac{b}{2a}.$$

This is the **axis of symmetry** we defined earlier.

Setting the constant terms equal, we get:

$$\begin{aligned} ah^2+k&=c\\ k&=c-ah^2\\ &=c-a\left(\left(\frac{b}{2a}\right)^2\right)\\ &=c-\frac{b^2}{4a}. \end{aligned}$$

In practice, though, it is usually easier to remember that k is the output value of the function when the input is h, so $f(h)=k$.

Definitions: Forms of Quadratic Functions

A *quadratic function* is a function of degree two. The graph of a quadratic function is a *parabola*.

- The *general form* of a quadratic function is $f(x)=ax^2+bx+c$ where a, b, and c are real numbers and $a\neq 0$.
- The *standard form* or *vertex form* of a quadratic function is $f(x)=a(x-h)^2+k$.
- The *vertex* (h,k) is located at

$$h=-\frac{b}{2a},\ k=f(h)=f\left(\frac{-b}{2a}\right).$$

How To: Write a quadratic function in vertex form, then general form

Given a graph of a quadratic function, write the equation of the function in general form.

1. Identify the horizontal shift of the parabola; this value is h. Identify the vertical shift of the parabola; this value is k.
2. Substitute the values of the horizontal and vertical shift for h and k. in the function $f(x)=a(x-h)^2+k$.
3. Substitute the values of any point, other than the vertex, on the graph of the parabola for x and $f(x)$.
4. Solve for the coefficient, a.
5. If the parabola opens up, $a>0$. If the parabola opens down, $a<0$ since this means the graph was reflected across the x-axis.
6. Expand and simplify to write in general form.

Example 3.2.2: Writing the Equation of a Quadratic Function from the Graph

Write an equation for the quadratic function g in Figure **3.2.7** as a transformation of $f(x)=x^2$, and then expand the formula, and simplify terms to write the equation in general form.

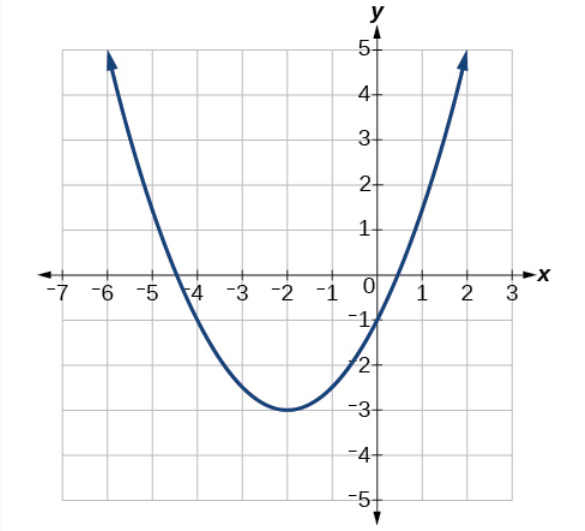


***Figure* 3.2.7:** *Graph of a parabola with its vertex at* $(-2,-3)$.

Solution

We can see the graph of g is the graph of $f(x)=x^2$ shifted to the left 2 and down 3, giving a formula in the form $g(x)=a(x+2)^2-3$.

Substituting the coordinates of a point on the curve, such as $(0,-1)$, we can solve for the stretch factor.

$$\begin{aligned} -1 &= a(0+2)^2-3 \\ 2 &= 4a \\ a &= \frac{1}{2} \end{aligned}$$

In standard form, the algebraic model for this graph is $g(x)=\frac{1}{2}(x+2)^2-3$.

To write this in general polynomial form, we can expand the formula and simplify terms.

$$\begin{aligned} g(x) &= \frac{1}{2}(x+2)^2-3 \\ &= \frac{1}{2}(x+2)(x+2)-3 \\ &= \frac{1}{2}(x^2+4x+4)-3 \\ &= \frac{1}{2}x^2+2x+2-3 \\ &= \frac{1}{2}x^2+2x-1 \end{aligned}$$

Notice that the horizontal and vertical shifts of the basic graph of the quadratic function determine the location of the vertex of the parabola; the vertex is unaffected by stretches and compressions.

3.2.1

A coordinate grid has been superimposed over the quadratic path of a basketball in Figure 3.2.8. Find an equation for the path of the ball. Does the shooter make the basket?

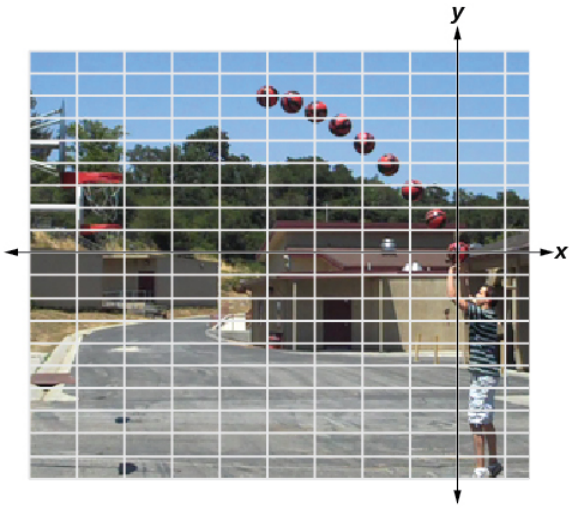


***Figure* 3.2.8:** *Stop motioned picture of a boy throwing a basketball into a hoop to show the parabolic curve it makes. (credit: modification of work by Dan Meyer)*

Answer

The path passes through the origin and has vertex at $(-4, 7)$, so $h(x) = -\frac{7}{16}(x+4)^2 + 7$. To make the shot, $h(-7.5)$ would need to be about 4 but $h(-7.5) \approx 1.64$, below the basket; he doesn't make it.

How To: Given a quadratic function in general form, find the vertex of the parabola.

1. Identify a, b, and c.
2. Find h, the x-coordinate of the vertex, by substituting a and b into $h = -\frac{b}{2a}$.
3. Find k, the y-coordinate of the vertex, by evaluating $k = f(h) = f\left(-\frac{b}{2a}\right)$.

Example 3.2.3: Finding the Vertex of a Quadratic Function

Find the vertex of the quadratic function $f(x) = 2x^2 - 6x + 7$. Rewrite the quadratic in standard form (vertex form).

Solution

The horizontal coordinate of the vertex will be at

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{-6}{2(2)} \\ &= \frac{6}{4} \\ &= \frac{3}{2}. \end{aligned}$$

The vertical coordinate of the vertex will be at

$$\begin{aligned} k &= f(h) \\ &= f\left(\frac{3}{2}\right) \\ &= 2\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) + 7 \\ &= \frac{5}{2}. \end{aligned}$$

Rewriting into standard form, the stretch factor will be the same as the a in the original quadratic:

$$f(x) = ax^2 + bx + c$$
$$f(x) = 2x^2 - 6x + 7$$

Using the h and k values of the vertex, rewrite as

$$f(x) = 2\left(x - \frac{3}{2}\right)^2 + \frac{5}{2}.$$

Analysis

One reason we may want to identify the vertex of the parabola is that this point will inform us what the maximum or minimum value of the function is (k), and where in the domain it occurs (h).

3.2.2

Given the equation $g(x) = 13 + x^2 - 6x$, write the equation in general form and then in standard form.

Answer

$g(x) = x^2 - 6x + 13$ in general form; $g(x) = (x-3)^2 + 4$ in standard form.

Finding the Domain and Range of a Quadratic Function

Any number can be the input value of a quadratic function. Therefore, the domain of any quadratic function is all real numbers. Because parabolas have a maximum or a minimum point, the range is restricted. Since the vertex of a parabola will be either a maximum or a minimum, the range will consist of all y-values greater than or equal to the y-coordinate at the turning point or less than or equal to the y-coordinate at the turning point, depending on whether the parabola opens up or down.

Domain and Range of a Quadratic Function

The domain of any **quadratic function** is all real numbers.

The range of a quadratic function written in general form $f(x) = ax^2 + bx + c$ with a positive a value is $f(x) \geq f\left(-\frac{b}{2a}\right)$, or $[f(-\frac{b}{2a}), \infty)$; the range of a quadratic function written in general form with a negative a value is $f(x) \leq f(-\frac{b}{2a})$, or $(-\infty, f(-\frac{b}{2a})]$.

The range of a quadratic function written in standard form $f(x) = a(x-h)^2 + k$ with a positive a value is $f(x) \geq k$; the range of a quadratic function written in standard form with a negative a value is $f(x) \leq k$.

How To: Given a quadratic function, find the domain and range.

1. Identify the domain of any quadratic function as all real numbers.
2. Determine whether a is positive or negative. If a is positive, the parabola has a minimum. If a is negative, the parabola has a maximum.
3. Determine the maximum or minimum value of the parabola, k.
4. If the parabola has a minimum, the range is given by $f(x) \geq k$, or $[k, \infty)$. If the parabola has a maximum, the range is given by $f(x) \leq k$, or $(-\infty, k]$.

Example 3.2.4: Finding the Domain and Range of a Quadratic Function

Find the domain and range of $f(x) = -5x^2 + 9x - 1$.

Solution

As with any quadratic function, the domain is all real numbers.

Because a is negative, the parabola opens downward and has a maximum value. We need to determine the maximum value. We can begin by finding the x-value of the vertex.

$$\begin{aligned} h &= -\frac{b}{2a} \\ &= -\frac{9}{2(-5)} \\ &= \frac{9}{10} \end{aligned}$$

The maximum value is given by $f(h)$.

$$\begin{aligned} f\left(\frac{9}{10}\right) &= -5(\frac{9}{10})^2 + 9(\frac{9}{10}) - 1 \\ &= \frac{61}{20} \end{aligned}$$

The range is $f(x) \leq \frac{61}{20}$, or $(-\infty, \frac{61}{20}]$.

3.2.3

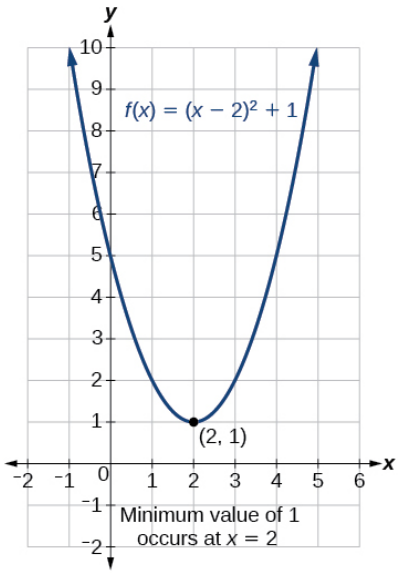
Find the domain and range of $f(x) = 2\left(x - \frac{4}{7}\right)^2 + \frac{8}{11}$.

Answer

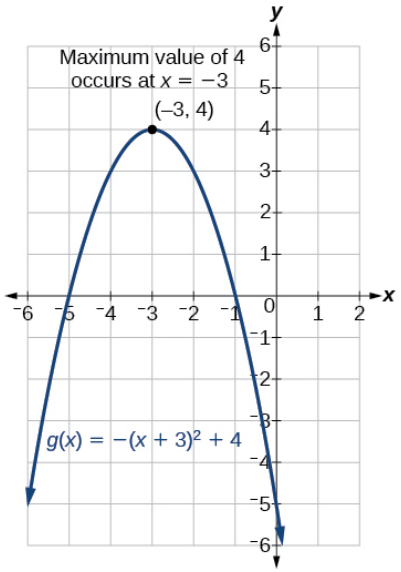
The domain is all real numbers. The range is $f(x) \geq \frac{8}{11}$, or $[\frac{8}{11}, \infty)$.

Determining the Maximum and Minimum Values of Quadratic Functions

The output of the quadratic function at the vertex is the maximum or minimum value of the function, depending on the orientation of the **parabola**. We can see the maximum and minimum values in Figure **3.2.9**.



(a)



(b)

***Figure* 3.2.9:** *Minimum and maximum of two quadratic functions.*

There are many real-world scenarios that involve finding the maximum or minimum value of a quadratic function, such as applications involving area and revenue.

Example 3.2.5: Finding the Maximum Value of a Quadratic Function

A backyard farmer wants to enclose a rectangular space for a new garden within her fenced backyard. She has purchased 80 feet of wire fencing to enclose three sides, and she will use a section of the backyard fence as the fourth side.

a. Find a formula for the area enclosed by the fence if the sides of fencing perpendicular to the existing fence have length L.
b. What dimensions should she make her garden to maximize the enclosed area?

Solution

Let's use a diagram such as Figure 3.2.10 to record the given information. It is also helpful to introduce a temporary variable, W, to represent the width of the garden and the length of the fence section parallel to the backyard fence.

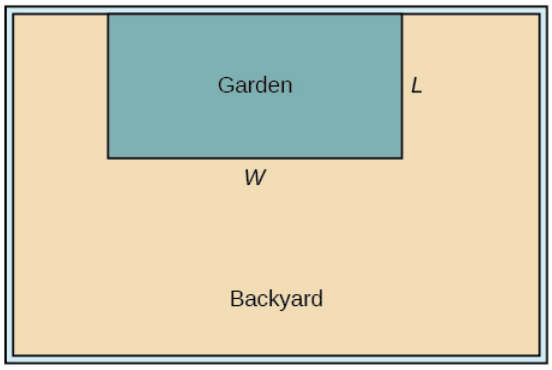


***Figure* 3.2.10:** *Diagram of the garden and the backyard.*

a. We know we have only 80 feet of fence available, and $L + W + L = 80$, or more simply, $2L + W = 80$. This allows us to represent the width W in terms of L.

$$W = 80 - 2L$$

Now we are ready to write an equation for the area the fence encloses. We know the area of a rectangle is length multiplied by width, so

$$A = LW = L(80 - 2L)$$
$$A(L) = 80L - 2L^2$$

This formula represents the area of the fence in terms of the variable length L. The function, written in general form, is

$$A(L) = -2L^2 + 80L$$

.

b. The quadratic has a negative leading coefficient, so the graph will open downward, and the vertex will be the maximum value for the area. In finding the vertex, we must be careful because the equation is not written in standard polynomial form with decreasing powers. This is why we rewrote the function in general form above. Since a is the coefficient of the squared term, $a = -2$, $b = 80$, and $c = 0$.

To find the vertex:

$$h = -\frac{80}{2(-2)} \quad \text{and} \quad k = A(20)$$
$$= 20 \qquad\qquad = 80(20) - 2(20)^2$$
$$= 800$$

The maximum value of the function is an area of 800 square feet, which occurs when $L = 20$ feet. When the shorter sides are 20 feet, there is 40 feet of fencing left for the longer side. To maximize the area, she should enclose the garden so the two shorter sides have length 20 feet and the longer side parallel to the existing fence has length 40 feet.

Analysis

This problem also could be solved by graphing the quadratic function. We can see where the maximum area occurs on a graph of the quadratic function in Figure 3.2.11.

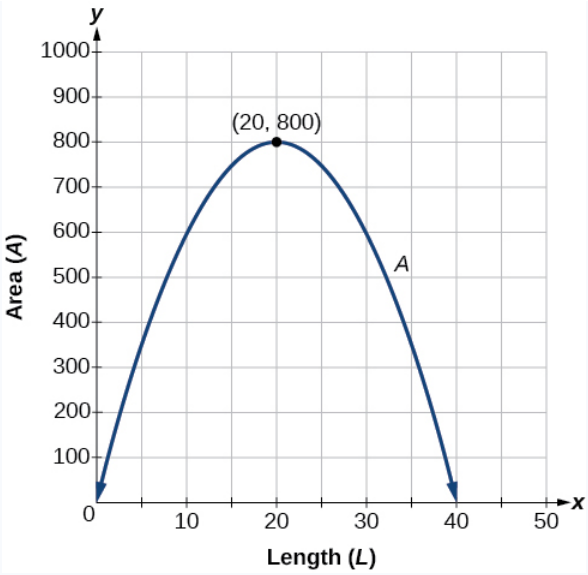


***Figure* 3.2.11:** *Graph of the parabolic function* $A(L) = -2L^2 + 80L$

How To: Given an application involving revenue, use a quadratic equation to find the maximum

1. Write a quadratic equation for revenue.
2. Find the vertex of the quadratic equation.
3. Determine the y-value of the vertex.

Example 3.2.6: Finding Maximum Revenue

The unit price of an item affects its supply and demand. That is, if the unit price goes up, the demand for the item will usually decrease. For example, a local newspaper currently has 84,000 subscribers at a quarterly charge of $30. Market research has suggested that if the owners raise the price to $32, they would lose 5,000 subscribers. Assuming that subscriptions are linearly related to the price, what price should the newspaper charge for a quarterly subscription to maximize their revenue?

Solution

Revenue is the amount of money a company brings in. In this case, the revenue can be found by multiplying the *price per subscription* times the number of subscribers, or *quantity*. We can introduce variables, p for price per subscription and Q for quantity, giving us the equation $\text{Revenue} = pQ$.

Because the number of subscribers changes with the price, we need to find a relationship between the variables. We know that currently $p = 30$ and $Q = 84,000$. We also know that if the price rises to $32, the newspaper would lose 5,000 subscribers, giving a second pair of values, $p = 32$ and $Q = 79{,}000$. From this we can find a linear equation relating the two quantities. The slope will be

$$\begin{aligned} m &= \frac{79,000 - 84,000}{32 - 30} \\ &= -\frac{5,000}{2} \\ &= -2,500 \end{aligned}$$

This tells us the paper will lose 2,500 subscribers for each dollar they raise the price. We can then solve for the y-intercept.

$$\begin{aligned} Q &= -2500p + b && \text{Substitute in the point } Q = 84,000 \text{ and } p = 30 \\ 84,000 &= -2500(30) + b && \text{Solve for } b \\ b &= 159,000 \end{aligned}$$

This gives us the linear equation $Q = -2,500p + 159,000$ relating cost and subscribers. We now return to our revenue equation.

$$\text{Revenue} = pQ$$
$$\text{Revenue} = p(-2,500p + 159,000)$$
$$\text{Revenue} = -2,500p^2 + 159,000p$$

We have a quadratic function for revenue as a function of the subscription charge. To find the price that will maximize revenue for the newspaper, we can find the vertex.

$$\begin{aligned} h &= -\frac{159,000}{2(-2,500)} \\ &= 31.8 \end{aligned}$$

The model tells us that the maximum revenue will occur if the newspaper charges $31.80 for a subscription. To find what the maximum revenue is, we evaluate the revenue function.

$$\begin{aligned} \text{maximum revenue} &= -2,500(31.8)^2 + 159,000(31.8) \\ &= 2,528,100 \end{aligned}$$

Analysis

This could also be solved by graphing the quadratic as in Figure 3.2.12. We can see the maximum revenue on a graph of the quadratic function.

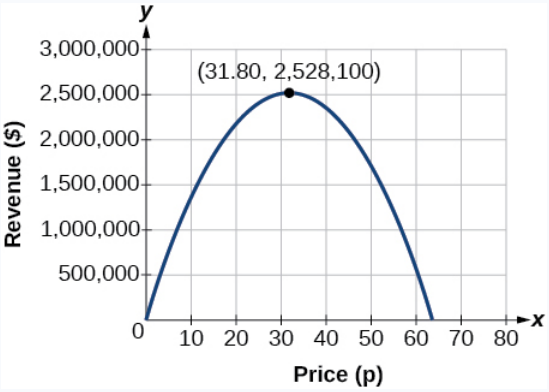


Figure 3.2.12: *Graph of the parabolic function*

Finding the x- and y-Intercepts of a Quadratic Function

As a tool to help us graph parabolas, we need to find intercepts of quadratic equations. Recall that we find the y-intercept of a quadratic by evaluating the function at an input of zero, and we find the x-intercepts at locations where the output is zero. Notice in Figure 3.2.13 that the number of x-intercepts can vary depending upon the location of the graph.

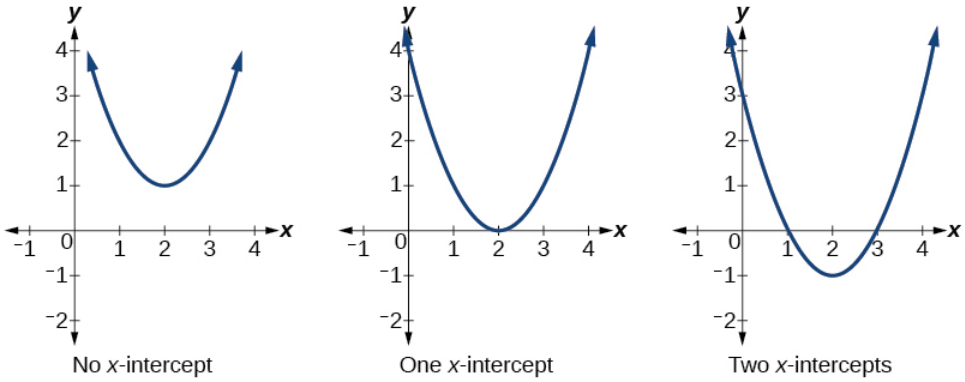


Figure 3.2.13: *Number of x-intercepts of a parabola.*

How To: Given a quadratic function $f(x)$, find the y- and x-intercepts

1. Evaluate $f(0)$ to find the y-intercept.
2. Solve the quadratic equation $f(x) = 0$ to find the x-intercepts.

Example 3.2.7: Finding the y- and x-Intercepts of a Parabola

Find the y- and x-intercepts of the quadratic $f(x) = 3x^2 + 5x - 2$.

Solution

We find the y-intercept by evaluating $f(0)$.

$$\begin{aligned} f(0) &= 3(0)^2 + 5(0) - 2 \\ &= -2 \end{aligned}$$

The y-intercept is at $(0, -2)$.

For the x-intercepts, we find all solutions of $f(x) = 0$.

$$0 = 3x^2 + 5x - 2$$

In this case, the quadratic can be factored, providing the simplest method for solution.

$$0 = (3x - 1)(x + 2)$$

$$\begin{aligned} 0 &= 3x - 1 \qquad\qquad 0 = x + 2 \\ x &= \frac{1}{3} \qquad \text{or} \qquad x = -2 \end{aligned}$$

The x-intercepts are at $(\frac{1}{3}, 0)$ and $(-2, 0)$.

Analysis

By graphing the function, we can confirm that the graph crosses the y-axis at $(0, -2)$. We can also confirm that the graph crosses the x-axis at $\left(\frac{1}{3}, 0\right)$ and $(-2, 0)$. See Figure 3.2.14.

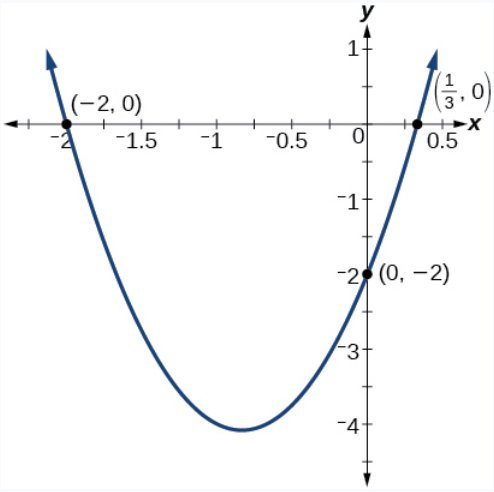


***Figure* 3.2.14:** *Graph of a parabola.*

Rewriting Quadratics in Standard Form

In Example 3.2.7, the quadratic equation was fairly easily solved by factoring. However, there are many quadratics that cannot be factored using rational numbers. Another method for solving quadratic equations is by first rewriting the quadratic in standard form. This method is also known as solving by *completing the square*.

How To: Given a quadratic function, find the x-intercepts by rewriting in standard form.

1. Substitute a and b into $h = -\frac{b}{2a}$.
2. Substitute $x = h$ into the general form of the quadratic function to find k.
3. Rewrite the quadratic in standard form using h and k.
4. Solve for when the output of the function will be zero to find the x-intercepts.

Example 3.2.8: Finding the x-Intercepts of a Parabola

Find the x-intercepts of the quadratic function $f(x) = 2x^2 + 4x - 4$.

Solution

We need to solve for when the output will be zero:

$$0 = 2x^2 + 4x - 4.$$

Because the quadratic is not easily factorable in this case, we solve for the intercepts by first rewriting the quadratic in standard form.

$$f(x) = a(x-h)^2 + k$$

We know that $a = 2$. Then we solve for h and k.

$$\begin{aligned} h &= -\frac{b}{2a} & k &= f(-1) \\ &= -\frac{4}{2(2)} & &= 2(-1)^2 + 4(-1) - 4 \\ &= -1 & &= -6 \end{aligned}$$

Now we can rewrite in standard form.

$$f(x) = 2(x+1)^2 - 6$$

We are ready to solve for when the output will be zero.

$$\begin{aligned} 0 &= 2(x+1)^2 - 6 \\ 6 &= 2(x+1)^2 \\ 3 &= (x+1)^2 \\ x+1 &= \pm\sqrt{3} \\ x &= -1 \pm \sqrt{3} \approx 0.73 \text{ or } -2.73 \end{aligned}$$

The graph has x-intercepts at $(-1-\sqrt{3}, 0)$ and $(-1+\sqrt{3}, 0)$. Note that the x-values are irrational numbers.

Analysis

We can check our work by graphing the given function on a graphing utility and observing an approximation of the x-intercepts. See Figure 3.2.15.

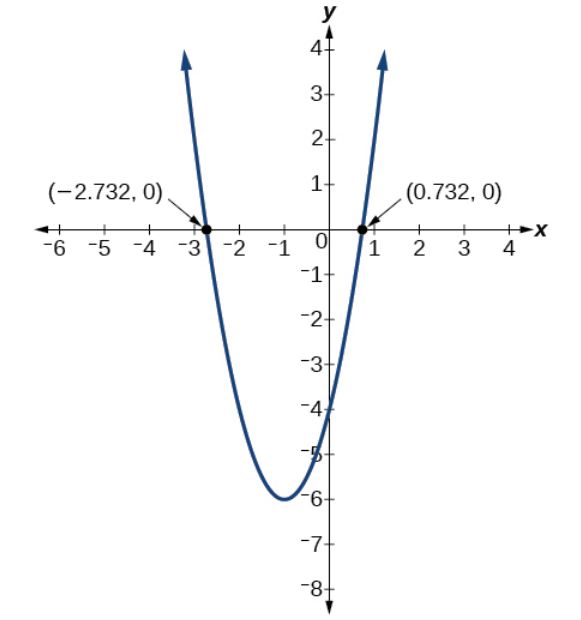


***Figure* 3.2.15:** *Graph of a parabola which has the following x-intercepts:* $(-2.73, 0)$ *and* $(0.73, 0)$.

Try it 3.2.4

In Try it 3.2.2, we found the standard and general form for the function $g(x) = 13 + x^2 - 6x$. Now find the y- and x-intercepts (if any).

Answer

y-intercept at $(0, 13)$, no x-intercepts

Example 3.2.9: Solving a Quadratic Equation with the Quadratic Formula

Find the zeros of $f(x) = x^2 + x + 2$. We set $f(x) = 0$, and solve: $x^2 + x + 2 = 0$.

Solution

Let's begin by writing the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

When applying the quadratic formula, we identify the coefficients a, b and c. For the equation $x^2 + x + 2 = 0$, we have $a = 1$, $b = 1$, and $c = 2$. Substituting these values into the formula we have:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (2)}}{2 \cdot 1} \\ &= \frac{-1 \pm \sqrt{1 - 8}}{2} \\ &= \frac{-1 \pm \sqrt{-7}}{2} \\ &= \frac{-1 \pm i\sqrt{7}}{2} \end{aligned}$$

The solutions to the equation are $x = \frac{-1 + i\sqrt{7}}{2}$ and $x = \frac{-1 - i\sqrt{7}}{2}$ or $x = -\frac{1}{2} + \frac{i\sqrt{7}}{2}$ and $x = \frac{-1}{2} - \frac{i\sqrt{7}}{2}$. Since there are no real-valued solutions to the equation, there are no real zeros. The graph of this parabola will not cross the x-axis.

Example 3.2.10: Applying the Vertex and x-Intercepts of a Parabola

A ball is thrown upward from the top of a 40 foot high building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation $H(t) = -16t^2 + 80t + 40$.

When does the ball reach the maximum height?
What is the maximum height of the ball?
When does the ball hit the ground?

The ball reaches the maximum height at the vertex of the parabola.

$$\begin{aligned} h &= -\frac{80}{2(-16)} \\ &= \frac{80}{32} \\ &= \frac{5}{2} \\ &= 2.5 \end{aligned}$$

The ball reaches a maximum height after 2.5 seconds.

To find the maximum height, find the y-coordinate of the vertex of the parabola.

$$\begin{aligned} k &= H\left(-\frac{b}{2a}\right) \\ &= H(2.5) \\ &= -16(2.5)^2 + 80(2.5) + 40 \\ &= 140 \end{aligned}$$

The ball reaches a maximum height of 140 feet.

To find when the ball hits the ground, we need to determine when the height is zero, $H(t) = 0$.

We use the quadratic formula.

$$\begin{aligned} t &= \frac{-80 \pm \sqrt{80^2 - 4(-16)(40)}}{2(-16)} \\ &= \frac{-80 \pm \sqrt{8960}}{-32} \end{aligned}$$

Because the square root does not simplify nicely, we can use a calculator to approximate the values of the solutions.

$$t = \frac{-80 - \sqrt{8960}}{-32} \approx 5.458 \text{ or } t = \frac{-80 + \sqrt{8960}}{-32} \approx -0.458 \tag{3.2.1}$$

The second answer is outside the reasonable domain of our model, so we conclude the ball will hit the ground after about 5.458 seconds. See Figure 3.2.16.

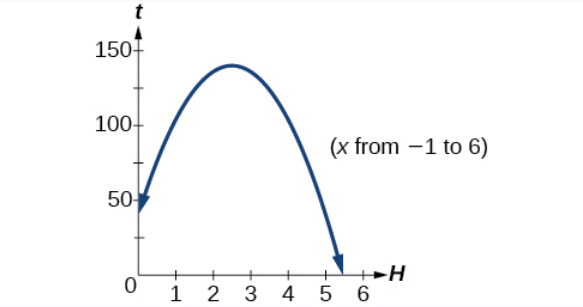


Figure 3.2.16

3.2.5

A rock is thrown upward from the top of a 112-foot high cliff overlooking the ocean at a speed of 96 feet per second. The rock's height above ocean can be modeled by the equation $H(t) = -16t^2 + 96t + 112$.

a. When does the rock reach the maximum height?
b. What is the maximum height of the rock?
c. When does the rock hit the ocean?

Answer

a. 3 seconds
b. 256 feet
c. 7 seconds

Key Equations

- general form of a quadratic function: $f(x) = ax^2 + bx + c$
- the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- standard, or vertex, form of a quadratic function: $f(x) = a(x - h)^2 + k$

Key Concepts

- A polynomial function of degree two is called a quadratic function.
- The graph of a quadratic function is a parabola. A parabola is a U-shaped curve that can open either up or down.
- The axis of symmetry is the vertical line passing through the vertex. The zeros, or x-intercepts, are the points at which the parabola crosses the x-axis. The y-intercept is the point at which the parabola crosses the y-axis.
- Quadratic functions are often written in general form. Standard or vertex form is useful to easily identify the vertex of a parabola. Either form can be written from a graph.
- The vertex can be found from either the general form or the standard form of a quadratic function.
- The domain of a quadratic function is all real numbers. The range varies with the function.
- A quadratic function's minimum or maximum value is given by the y-value of the vertex.
- The minimum or maximum value of a quadratic function can be used to determine the range of the function and to solve many kinds of real-world problems, including problems involving area and revenue.
- Some quadratic equations must be solved by using the quadratic formula, or by completing the square.
- The vertex and the intercepts can be identified and interpreted to solve real-world problems.

Glossary

axis of symmetry
a vertical line drawn through the vertex of a parabola around which the parabola is symmetric; defined by $x = -\frac{b}{2a}$

general form of a quadratic function
the function that describes a parabola, written in the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$

standard form of a quadratic function
the function that describes a parabola, written in the form $f(x) = a(x - h)^2 + k$, where (h, k) is the vertex

vertex
the point at which a parabola changes direction, corresponding to the minimum or maximum value of the quadratic function

vertex form of a quadratic function
another name for the standard form of a quadratic function

zeros
in a given function, the values of x at which $y = 0$, also called roots

Contributors

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3.2E: Exercises

Verbal

1. Explain the advantage of writing a quadratic function in standard form.

Answer: When written in that form, the vertex can be easily identified.

2. How can the vertex of a parabola be used in solving real world problems?

3. Explain why the condition of $a \neq 0$ is imposed in the definition of the quadratic function.

Answer: If $a = 0$ then the function becomes a linear function.

4. What is another name for the standard form of a quadratic function?

5. What algebraic methods can be used to find the horizontal intercepts of a quadratic function?

Answer: If possible, we can use factoring. Otherwise, we can use the quadratic formula. A third possibility is completing the square to solve.

Algebraic

For the following exercises, rewrite the quadratic functions in standard form and give the vertex.

6. $f(x) = x^2 - 12x + 32$

Exercise 3.2E.7

$g(x) = x^2 + 2x - 3$

Answer

$g(x) = (x+1)^2 - 4$, Vertex $(-1, -4)$

8. $f(x) = x^2 - x$

9. $f(x) = x^2 + 5x - 2$

Answer: $f(x) = (x + \frac{5}{2})^2 - \frac{33}{4}$, Vertex $(-\frac{5}{2}, -\frac{33}{4})$

10. $h(x) = 2x^2 + 8x - 10$

Exercise 3.2E.11

$k(x) = 3x^2 - 6x - 9$

Answer

$k(x) = 3(x-1)^2 - 12$, Vertex $(1, -12)$

12. $f(x) = 2x^2 - 6x$

13. $f(x) = 3x^2 - 5x - 1$

Answer: $f(x) = 3(x - \frac{5}{6})^2 - \frac{37}{12}$, Vertex $(\frac{5}{6}, -\frac{37}{12})$

For the following exercises, determine whether there is a minimum or maximum value to each quadratic function. Find the value and the axis of symmetry.

14. $y(x) = 2x^2 + 10x + 12$

Exercise 3.2E.15

$f(x) = 2x^2 - 10x + 4$

Answer

Minimum is $-\frac{17}{2}$ and occurs at $x=\frac{5}{2}$. Axis of symmetry is $x=\frac{5}{2}$.

16. $f(x)=-x^2+4x+3$

17. $f(x)=4x^2+x-1$

Answer: Minimum is $-\frac{17}{16}$ and occurs at $x=-\frac{1}{8}$. Axis of symmetry is $x=-\frac{1}{8}$.

18. $h(t)=-4t^2+6t-1$

19. $f(x)=-\frac{1}{3}x^2-2x+3$

Answer: Minimum is 6 and occurs at $x=-3$. Axis of symmetry is $x=-3$.

20. $f(x)=\frac{1}{2}x^2+3x+1$

For the following exercises, determine the domain and range of the quadratic function.

---

21. $f(x)=(x-3)^2+2$

Answer: Domain is $(-\infty,\infty)$. Range is $[2,\infty)$.

22. $f(x)=-2(x+3)^2-6$

Exercise $3.2E.23$

$f(x)=x^2+10x+4$

Answer

Domain is $(-\infty,\infty)$. Range is $[-5,\infty)$.

24. $f(x)=2x^2-4x+2$

Exercise $3.2E.25$

$k(x)=-3x^2-6x-9$

Answer

Domain is $(-\infty,\infty)$. Range is $(-\infty,12]$.

For the following exercises, solve the equations over the complex numbers.

---

26. $x^2=-25$

27. $x^2=-8$

Answer: $2i\sqrt{2},-2i\sqrt{2}$

28. $x^2+36=0$

29. $x^2+27=0$

Answer: $3i\sqrt{3},-3i\sqrt{3}$

30. $x^2+2x+5=0$

31. $x^2-4x+5=0$

Answer: $2+i,2-i$

32. $x^2+8x+25=0$

33. $x^2-4x+13=0$

Answer: $2+3i,2-3i$

34. $x^2+6x+25=0$

35. $x^2-10x+26=0$

Answer: $5+i, 5-i$

36. $x^2-6x+10=0$

37. $x(x-4)=20$

Answer: $2+2\sqrt{6}, 2-2\sqrt{6}$

38. $x(x-2)=10$

39. $2x^2+2x+5=0$

Answer: $-\frac{1}{2}+\frac{3}{2}i, -\frac{1}{2}-\frac{3}{2}i$

40. $5x^2-8x+5=0$

41. $5x^2+6x+2=0$

Answer: $-\frac{3}{5}+\frac{1}{5}i, -\frac{3}{5}-\frac{1}{5}i$

42. $2x^2-6x+5=0$

43. $x^2+x+2=0$

Answer: $-\frac{1}{2}+\frac{1}{2}i\sqrt{7}, -\frac{1}{2}-\frac{1}{2}i\sqrt{7}$

44. $x^2-2x+4=0$

For the following exercises, use the vertex (h,k) and a point on the graph (x,y) to find the general form of the equation of the quadratic function.

Exercise 3.2E.45

$(h,k)=(2,0),(x,y)=(4,4)$

Answer

$f(x)=x^2-4x+4$

46. $(h,k)=(-2,-1),(x,y)=(-4,3)$

47. $(h,k)=(0,1),(x,y)=(2,5)$

Answer: $f(x)=x^2+1$

48. $(h,k)=(2,3),(x,y)=(5,12)$

49. $(h,k)=(-5,3),(x,y)=(2,9)$

Answer: $f(x)=\frac{6}{49}x^2+\frac{60}{49}x+\frac{297}{49}$

50. $(h,k)=(3,2),(x,y)=(10,1)$

51. $(h,k)=(0,1),(x,y)=(1,0)$

Answer: $f(x)=-x^2+1$

52. $(h,k)=(1,0),(x,y)=(0,1)$

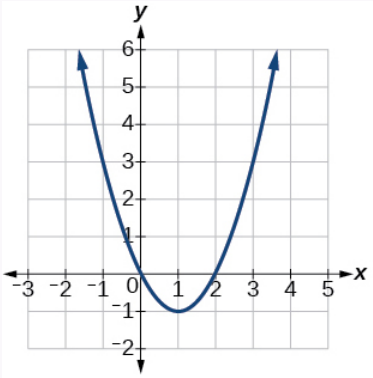
Graphical

For the following exercises, sketch a graph of the quadratic function and give the vertex, axis of symmetry, and intercepts.

Exercise 3.2E.53

$f(x)=x^2-2x$

Answer

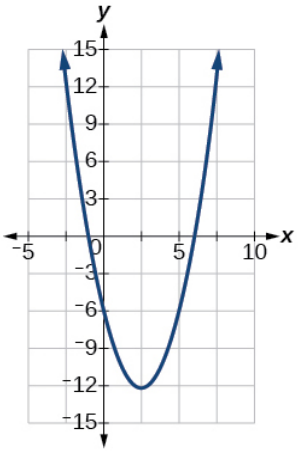


Vertex $(1,-1)$, Axis of symmetry is $x=1$. Intercepts are $(0,0),(2,0)$.

54. $f(x)=x^2-6x-1$

55. $f(x)=x^2-5x-6$

Answer:



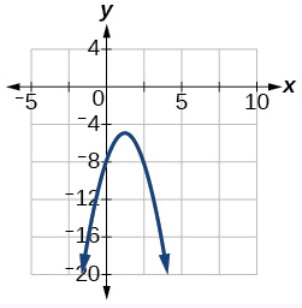
Vertex $(\frac{5}{2},\frac{-49}{4})$, Axis of symmetry is $x=\frac{5}{2}$. Intercepts are $(0,-6),(-1,0),(6,0)$.

56. $f(x)=x^2-7x+3$

Exercise $3.2E.57$

$f(x)=-2x^2+5x-8$

Answer



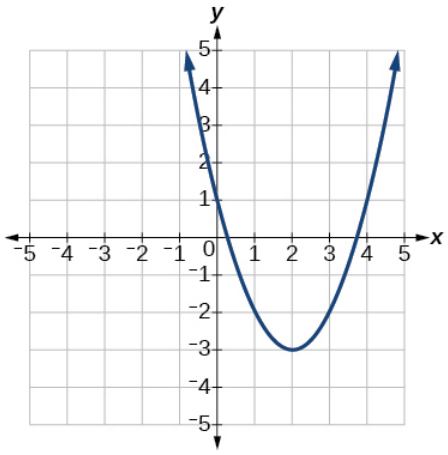
Vertex $(\frac{5}{4},-\frac{39}{8})$, Axis of symmetry is $x=\frac{5}{4}$. Intercepts are $(0,-8)$.

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58. $f(x) = 4x^2 - 12x - 3$

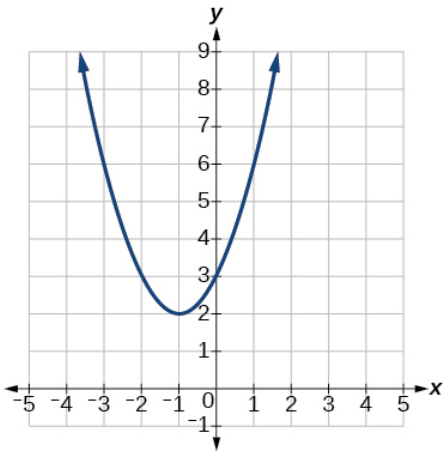
For the following exercises, write the equation for the graphed function.

59.

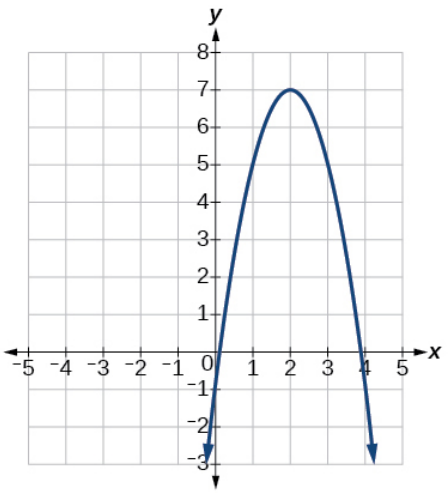


Answer: $f(x) = x^2 - 4x + 1$

60.

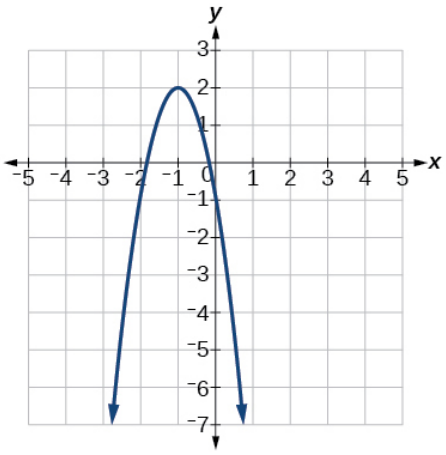


61.

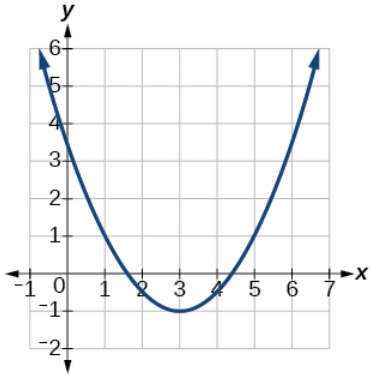


Answer: $f(x) = -2x^2 + 8x - 1$

62.

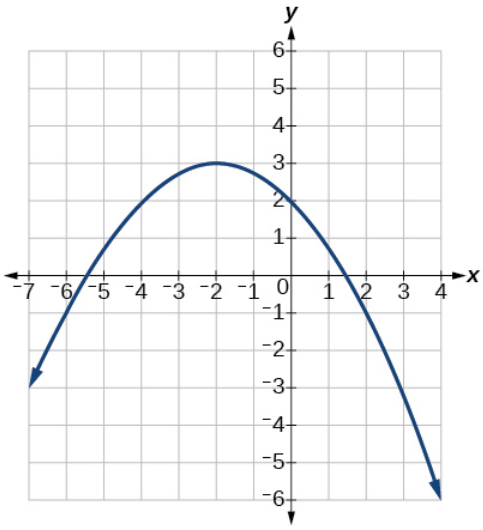


63.



Answer: $f(x) = \frac{1}{2}x^2 - 3x + \frac{7}{2}$

64.



Numeric

For the following exercises, use the table of values that represent points on the graph of a quadratic function. By determining the vertex and axis of symmetry, find the general form of the equation of the quadratic function.

65.

x	–2	–1	0	1	2
y	5	2	1	2	5

Answer: $f(x) = x^2 + 1$

66.

x	–2	–1	0	1	2
y	1	0	1	4	9

67.

x	–2	–1	0	1	2
y	–2	1	2	1	–2

Answer: $f(x) = 2 - x^2$

68.

x	–2	–1	0	1	2
y	–8	–3	0	1	0

69.

x	–2	–1	0	1	2
y	8	2	0	2	8

Answer: $f(x) = 2x^2$

Technology

For the following exercises, use a calculator to find the answer.

70. Graph on the same set of axes the functions $f(x) = x^2, f(x) = 2x^2$, and $f(x) = \frac{1}{3}x^2$.

What appears to be the effect of changing the coefficient?

71. Graph on the same set of axes $f(x) = x^2, f(x) = x^2 + 2$ and $f(x) = x^2, f(x) = x^2 + 5$ and $f(x) = x^2 - 3$. What appears to be the effect of adding a constant?

Answer: The graph is shifted up or down (a vertical shift).

72. Graph on the same set of axes $f(x) = x^2, f(x) = (x - 2)^2, f(x - 3)2$, and $f(x) = (x + 4)2$.

What appears to be the effect of adding or subtracting those numbers?

73. The path of an object projected at a 45 degree angle with initial velocity of 80 feet per second is given by the function $h(x) = -\frac{32}{(80)^2}x^2 + x$ where x is the horizontal distance traveled and $h(x)$ is the height in feet. Use the TRACE feature of your calculator to determine the height of the object when it has traveled 100 feet away horizontally.

Answer: 50 feet

74. A suspension bridge can be modeled by the quadratic function $h(x) = .0001x^2$ with $-2000 \le x \le 2000$ where $|x|$ is the number of feet from the center and $h(x)$ is height in feet. Use the TRACE feature of your calculator to estimate how far from the center does the bridge have a height of 100 feet.

Extensions

For the following exercises, use the vertex of the graph of the quadratic function and the direction the graph opens to find the domain and range of the function.

75. Vertex $(1, -2)$, opens up.

Answer: Domain is $(-\infty, \infty)$. Range is $[-2, \infty)$.

76. Vertex $(-1, 2)$ opens down.

77. Vertex $(-5, 11)$, opens down.

Answer: Domain is $(-\infty, \infty)$ Range is $(-\infty, 11]$.

78. Vertex $(-100, 100)$, opens up.

For the following exercises, write the equation of the quadratic function that contains the given point and has the same shape as the given function.

---

79. Contains $(1, 1)$ and has shape of $f(x) = 2x^2$. Vertex is on the y-axis.

Answer: $f(x) = 2x^2 - 1$

80. Contains $(-1, 4)$ and has the shape of $f(x) = 2x^2$. Vertex is on the y-axis.

81. Contains $(2, 3)$ and has the shape of $f(x) = 3x^2$. Vertex is on the y-axis.

Answer: $f(x) = 3x^2 - 9$

82. Contains $(1, -3)$ and has the shape of $f(x) = -x^2$. Vertex is on the y-axis.

83. Contains $(4, 3)$ and has the shape of $f(x) = 5x^2$. Vertex is on the y-axis.

Answer: $f(x) = 5x^2 - 77$

84. Contains $(1, -6)$ has the shape of $f(x) = 3x^2$. Vertex has x-coordinate of -1.

Real-World Applications

85. Find the dimensions of the rectangular corral producing the greatest enclosed area given 200 feet of fencing.

Answer: 50 feet by 50 feet. Maximize $f(x) = -x^2 + 100x$.

86. Find the dimensions of the rectangular corral split into 2 pens of the same size producing the greatest possible enclosed area given 300 feet of fencing.

87. Find the dimensions of the rectangular corral producing the greatest enclosed area split into 3 pens of the same size given 500 feet of fencing.

Answer: 125 feet by 62.5 feet. Maximize $f(x) = -2x^2 + 250x$.

88. Among all of the pairs of numbers whose sum is 6, find the pair with the largest product. What is the product?

Exercise 3.2E.89

Among all of the pairs of numbers whose difference is 12, find the pair with the smallest product. What is the product?

Answer

6 and -6; product is –36; maximize $f(x) = x^2 + 12x$.

90. Suppose that the price per unit in dollars of a cell phone production is modeled by $p = \$45 - 0.0125x$, where x is in thousands of phones produced, and the revenue represented by thousands of dollars is $R = x \cdot p$. Find the production level that will maximize revenue.

91. A rocket is launched in the air. Its height, in meters above sea level, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 229t + 234$. Find the maximum height the rocket attains.

Answer: 2909.56 meters

92. A ball is thrown in the air from the top of a building. Its height, in meters above ground, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 24t + 8$. How long does it take to reach maximum height?

Exercise $3.2E.93$

A soccer stadium holds 62,000 spectators. With a ticket price of $11, the average attendance has been 26,000. When the price dropped to $9, the average attendance rose to 31,000. Assuming that attendance is linearly related to ticket price, what ticket price would maximize revenue?

Answer

$10.70

94. A farmer finds that if she plants 75 trees per acre, each tree will yield 20 bushels of fruit. She estimates that for each additional tree planted per acre, the yield of each tree will decrease by 3 bushels. How many trees should she plant per acre to maximize her harvest?

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3.3: Power Functions and Polynomial Functions

Skills to Develop

- Identify power functions.
- Identify end behavior of power functions.
- Identify polynomial functions.
- Identify the degree and leading coefficient of polynomial functions.

Suppose a certain species of bird thrives on a small island. Its population over the last few years is shown in Table 3.3.1.

***Table* 3.3.1**

Year	2009	2010	2011	2012	2013
Bird Population	800	897	992	1,083	1,169

The population can be estimated using the function $P(t) = -0.3t^3 + 97t + 800$, where $P(t)$ represents the bird population on the island t years after 2009. We can use this model to estimate the maximum bird population and when it will occur. We can also use this model to predict when the bird population will disappear from the island. In this section, we will examine functions that we can use to estimate and predict these types of changes.



Figure* 3.3.1*: *(credit: Jason Bay, Flickr)*

Identifying Power Functions

In order to better understand the bird problem, we need to understand a specific type of function. A power function is a function with a single term that is the product of a real number, a coefficient, and a variable raised to a fixed real number. (A number that multiplies a variable raised to an exponent is known as a coefficient.)

As an example, consider functions for area or volume. The function for the area of a circle with radius r is

$$A(r) = \pi r^2$$

and the function for the volume of a sphere with radius r is

$$V(r) = \frac{4}{3}\pi r^3$$

Both of these are examples of power functions because they consist of a coefficient, π or $\frac{4}{3}\pi$, multiplied by a variable r raised to a power.

Definition: Power Function

A *power function* is a function that can be represented in the form

$$f(x) = kx^p \tag{3.3.1}$$

where k and p are real numbers, and k is known as the *coefficient*.

Is $f(x) = 2^x$ a power function?

No. A power function contains a variable base raised to a fixed power (Equation 3.3.1). This function has a constant base raised to a variable power. This is called an exponential function, not a power function. This function will be discussed later.

Example 3.3.1: Identifying Power Functions

Which of the following functions are power functions?

$f(x) = 1$	Constant function
$f(x) = x$	Identify function
$f(x) = x^2$	Quadratic function
$f(x) = x^3$	Cubic function
$f(x) = \frac{1}{x}$	Reciprocal function
$f(x) = \frac{1}{x^2}$	Reciprocal squared function
$f(x) = \sqrt{x}$	Square root function
$f(x) = \sqrt[3]{x}$	Cube root function

Solution

All of the listed functions are power functions.

The constant and identity functions are power functions because they can be written as $f(x) = x^0$ and $f(x) = x^1$ respectively.

The quadratic and cubic functions are power functions with whole number powers $f(x) = x^2$ and $f(x) = x^3$.

The **reciprocal** and reciprocal squared functions are power functions with negative whole number powers because they can be written as $f(x) = x^{-1}$ and $f(x) = x^{-2}$.

The square and **cube root** functions are power functions with fractional powers because they can be written as $f(x) = x^{1/2}$ or $f(x) = x^{1/3}$.

3.3.1

Which functions are power functions?

- $f(x) = 2x^2 \cdot 4x^3$
- $g(x) = -x^5 + 5x^3 - 4x$
- $h(x) = \frac{2x^5-1}{3x^2+4}$

Answer

$f(x)$ is a power function because it can be written as $f(x) = 8x^5$. The other functions are not power functions.

Identifying End Behavior of Power Functions

Figure 3.3.2 shows the graphs of $f(x) = x^2$, $g(x) = x^4$ and $h(x) = x^6$, which are all power functions with even, whole number powers. (**Note**: In this section we will be mostly concerned with whole number powers, unless otherwise specified.) Notice that these graphs have similar shapes, very much like that of the quadratic function in the toolkit. However, as the power increases, the graphs flatten somewhat near the origin and become steeper away from the origin.

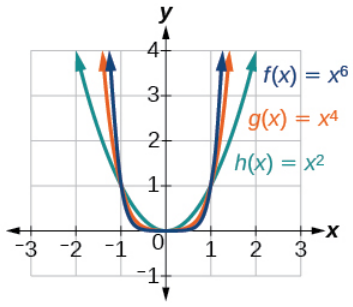


***Figure* 3.3.2:** *Even-power functions*

Note that the graph of $f(x) = x^6$ actually appears to be constant near the origin; however, this is not the case. The output values are changing very slightly. If we use a graphing tool (for example, Desmos), we could zoom in near the origin to see these changes.

To describe the behavior as numbers become larger and larger, we use the idea of infinity. We use the symbol ∞ for positive infinity and $-\infty$ for negative infinity. When we say that "x approaches infinity," which can be symbolically written as $x \to \infty$, we are describing a behavior; we are saying that x is increasing without bound.

With the even-power function, as the input increases or decreases without bound, the output values become very large, positive numbers. Equivalently, we could describe this behavior by saying that as x approaches positive or negative infinity, the $f(x)$ values increase without bound. In symbolic form, we could write

$$\text{as } x \to \pm\infty, f(x) \to \infty$$

Figure **3.3.3** shows the graphs of $f(x) = x^3$, $g(x) = x^5$, and $h(x) = x^7$, which are all power functions with odd, whole number powers. Notice that these graphs look similar to the cubic function in the toolkit. Again, as the power increases, the graphs flatten near the origin and become steeper away from the origin.

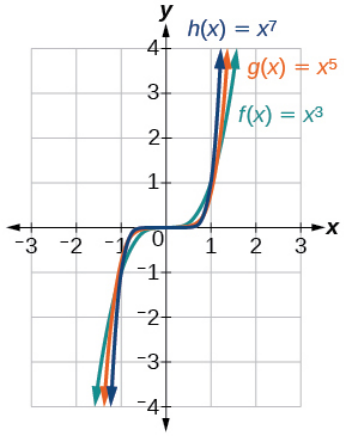


***Figure* 3.3.3:** *Odd-power function*

These examples illustrate that functions of the form $f(x) = x^n$, where n is a non-negative integer, reveal symmetry of one kind or another. First, in Figure **3.3.2** we see that even functions of the form $f(x) = x^n$, n even, are symmetric about the y-axis. In Figure **3.3.3** we see that odd functions of the form $f(x) = x^n$, n odd, are symmetric about the origin. **Note** that x^0 is a power function whose exponent is a non-negative integer. If you can explain how it fits, write your explanation up carefully and clearly, and give it to your instructor for ***Extra Credit***.

For these odd power functions, as x approaches negative infinity, $f(x)$ decreases without bound. As x approaches positive infinity, $f(x)$ increases without bound. In symbolic form we write

$$\text{as } x \to -\infty,\ f(x) \to -\infty$$
$$\text{as } x \to \infty,\ f(x) \to \infty$$

The behavior of the graph of a function as the input values move to the left on the number line ($x \to -\infty$) and move to the right on the number line ($x \to \infty$) is referred to as the *end behavior* of the function. We can use words or symbols to describe end behavior. **Note**: some texts may talk about x getting "small", when x is moving to the left on the number line; however, since x is getting very **large** in magnitude as it moves away from the origin "small" can be confusing, and we try to avoid this terminology.

Figure **3.3.4** shows the end behavior of power functions in the form $f(x) = kx^n$, where n is a whole number, depending on the power and the constant.

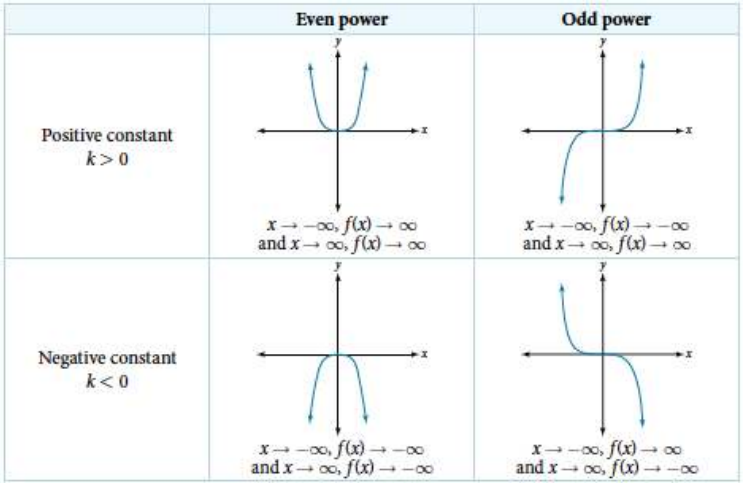


***Figure* 3.3.4**

How To: Given a power function $f(x) = kx^n$ where n is a whole number, identify the end behavior.

1. Determine whether the power is even or odd.
2. Determine whether the constant is positive or negative.
3. Use Figure **3.3.4** to identify the end behavior.

Example 3.3.2: Identifying the End Behavior of a Power Function where the power is an even number

Describe the end behavior of the graph of $f(x) = x^8$.

Solution

The coefficient is 1 (positive) and the exponent of the power function is 8 (an even number). As x approaches infinity, the output (value of $f(x)$) increases without bound. We write as $x \to \infty, f(x) \to \infty$. As x approaches negative infinity, the output increases without bound. In symbolic form, as $x \to -\infty, f(x) \to \infty$. We can graphically represent the function as shown in Figure **3.3.5**.

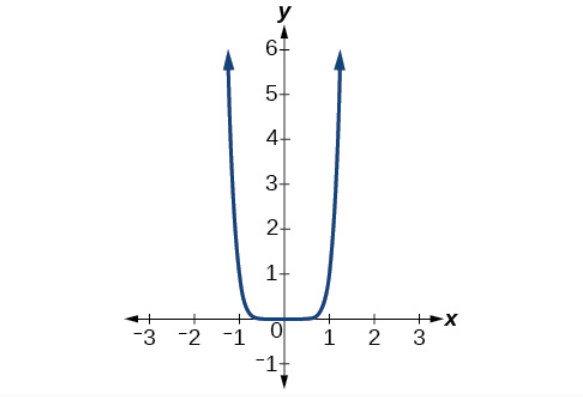


***Figure* 3.3.5:** *Graph of* $f(x) = x^8$.

Example 3.3.3: Identifying the End Behavior of a Power Function where the power is an odd number

Describe the end behavior of the graph of $f(x) = -x^9$.

Solution

The exponent of the power function is 9 (an odd number). Because the coefficient is –1 (negative), the graph is the reflection about the x-axis of the graph of $f(x) = x^9$. Figure 3.3.6 shows that as x approaches infinity, the output decreases without bound. As x approaches negative infinity, the output increases without bound. In symbolic form, we would write

$$\text{as } x \to -\infty,\ f(x) \to \infty$$
$$\text{as } x \to \infty,\ f(x) \to -\infty$$

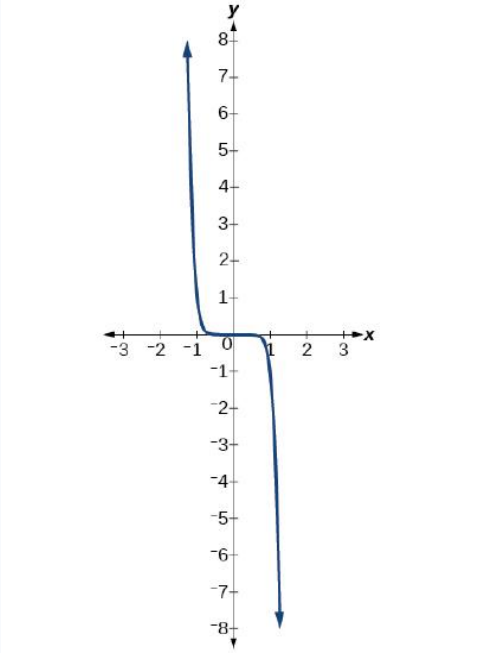


Figure 3.3.6: *Graph of* $f(x) = -x^9$.

Analysis

We can check our work by using the table feature on a graphing utility.

Table 3.3.2

x	$f(x)$
-10	1,000,000,000
-5	1,953,125
0	0
5	-1,953,125
10	-1,000,000,000

We can see from Table 3.3.2 that, when we substitute very negative values for x that are large in magnitude, the output is very large, and when we substitute very large positive values for x, the output is negative with very large magnitude.

3.3.2

Describe in words and symbols the end behavior of $f(x) = -5x^4$.

Answer

As x approaches positive or negative infinity, $f(x)$ decreases without bound: as $x \to \pm\infty$, $f(x) \to -\infty$ because of the negative coefficient.

Identifying Polynomial Functions

An oil pipeline bursts in the Gulf of Mexico, causing an oil slick in a roughly circular shape. The slick is currently 24 miles in radius, but that radius is increasing by 8 miles each week. We want to write a formula for the area covered by the oil slick by combining two functions. The radius r of the spill depends on the number of weeks w that have passed. This relationship is linear.

$$r(w) = 24 + 8w$$

We can combine this with the formula for the area A of a circle.

$$A(r) = \pi r^2$$

Composing these functions gives a formula for the area in terms of weeks.

$$\begin{aligned} A(w) &= A(r(w)) \\ &= A(24 + 8w) \\ &= \pi(24 + 8w)^2 \end{aligned}$$

Multiplying gives the formula.

$$A(w) = 576\pi + 384\pi w + 64\pi w^2$$

This formula is an example of a polynomial function. A polynomial function consists of either zero or the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power.

Definition: Polynomial Functions

Let n be a non-negative integer. A *polynomial function* is a function that can be written in the form

$$f(x) = a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0 \tag{3.3.2}$$

This is called the general form of a polynomial function. Each a_i is a coefficient and can be any real number. Each product $a_i x^i$ is a *term of a polynomial function*.

3.3.3

Which of the following are polynomial functions?

- $f(x) = 2x^3 \cdot 3x + 4$
- $g(x) = -x(x^2 - 4)$
- $h(x) = 5\sqrt{x} + 2$

Answer

The first two functions are examples of polynomial functions because they can be written in the form of Equation 3.3.2, where the powers are non-negative integers and the coefficients are real numbers.

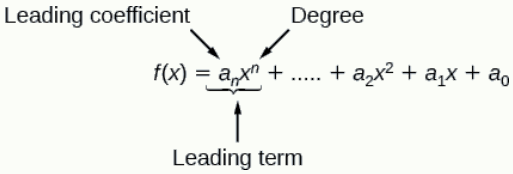
$f(x)$ can be written as $f(x) = 6x^4 + 4$. $g(x)$ can be written as $g(x) = -x^3 + 4x$. $h(x)$ cannot be written in this form and is therefore not a polynomial function.

Identifying the Degree and Leading Coefficient of a Polynomial Function

Because of the form of a polynomial function, we can see an infinite variety in the number of terms and the power of the variable. Although the order of the terms in the polynomial function is not important for performing operations, we typically arrange the terms in descending order of power, or in general form. The degree of the polynomial is the highest power of the variable that occurs in the polynomial; it is the power of the first variable if the function is in general form. The leading term is the term containing the highest power of the variable, or the term with the highest degree. The leading coefficient is the coefficient of the leading term.

Terminology of Polynomial Functions

We often rearrange polynomials so that the powers are descending.



When a polynomial is written in this way, we say that it is in *general form*.

How To: Given a polynomial function, identify the degree, leading term, and leading coefficient

1. Find the highest power of x to determine the degree of the function.
2. Identify the term containing the highest power of x to find the leading term.
3. Identify the coefficient of the leading term.

Example 3.3.5: Identifying the Degree, leading term, and Leading Coefficient of a Polynomial Function

Identify the degree, leading term, and leading coefficient of the following polynomial functions.

$$f(x) = 3 + 2x^2 - 4x^3$$

$$g(t) = 5t^5 - 2t^3 + 7t$$

$$h(p) = 6p - p^3 - 2$$

Solution

For the function $f(x)$, the highest power of x is 3, so the degree is 3. The leading term is the term containing that degree, $-4x^3$. The leading coefficient is the coefficient of that term, -4.

For the function $g(t)$, the highest power of t is 5, so the degree is 5. The leading term is the term containing that degree, $5t^5$. The leading coefficient is the coefficient of that term, 5.

For the function $h(p)$, the highest power of p is 3, so the degree is 3. The leading term is the term containing that degree, $-p^3$; the leading coefficient is the coefficient of that term, -1.

Try It 3.3.4

Identify the degree, leading term, and leading coefficient of the polynomial $f(x) = 4x^2 - x^6 + 2x - 6$.

Answer

AThe degree is 6. The leading term is $-x^6$. The leading coefficient is -1.

Example 3.3.6: Identifying the Degree, leading term, and Leading Coefficient of a Polynomial Function Given in factored form

Identify the degree, leading term, and leading coefficient of the polynomial function $f(x) = 5(x + 2)(x^2 + 1)$.

Solution

To see the leading term of $f(x)$, we will multiply the factors and rearrange terms to get the general form.

$$f(x) = 5(x^3 + x + 2x^2 + 2) = 5x^3 + 10x^2 + 5x + 10$$

The degree is 3, the leading term is $5x^3$, and the leading coefficient is 5.

Note: When a polynomial function is given in factored form, it is harder to identify the leading term, the degree, and the leading coefficient. However, we shall see in Example **3.3.9** how factored form can be quite useful!

Identifying End Behavior of Polynomial Functions

Knowing the degree of a polynomial function is useful in helping us predict its end behavior. To determine its end behavior, look at the leading term of the polynomial function. Because the power of the leading term is the highest, that term will grow significantly faster than the other terms as x gets very large or very small, so its behavior will dominate the graph. For any polynomial, the end behavior of the polynomial will match the end behavior of the term of highest degree (Table **3.3.3**).

***Table* 3.3.3**

Polynomial Function	Leading Term
$f(x) = -2x^6 - x^5 + 3x^4 + x^3$	$-2x^6$
$f(x) = -6x^3 + 7x^2 + 3x + 1$	$-6x^3$
$f(x) = 3x^5 - 4x^4 + 2x^2 + 1$	$3x^5$
$f(x) = 5x^4 + 2x^3 - x - 4$	$5x^4$

Example 3.3.7: Identifying End Behavior and Degree of a Polynomial Function

Describe the end behavior and determine a possible degree of the polynomial function in Figure **3.3.7**.

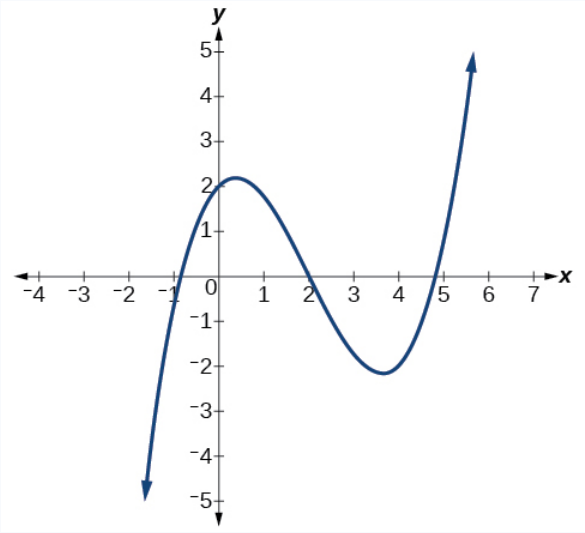


***Figure* 3.3.7.**

Solution

As the input values x increase, the output values $f(x)$ increase without bound. As the input values x decrease, the output values $f(x)$ decrease without bound. We can describe the end behavior symbolically by writing

$$\text{as } x \to -\infty, f(x) \to -\infty$$

$$\text{as } x \to \infty, f(x) \to \infty$$

In words, we could say that as x values approach infinity, the function values approach infinity, and as x values approach negative infinity, the function values approach negative infinity.

We can tell this graph has the shape of an odd degree power function that has not been reflected, so the degree of the polynomial creating this graph must be odd and the leading coefficient must be positive.

3.3.5

Describe the end behavior, and determine a possible degree of the polynomial function in Figure **3.3.8**.

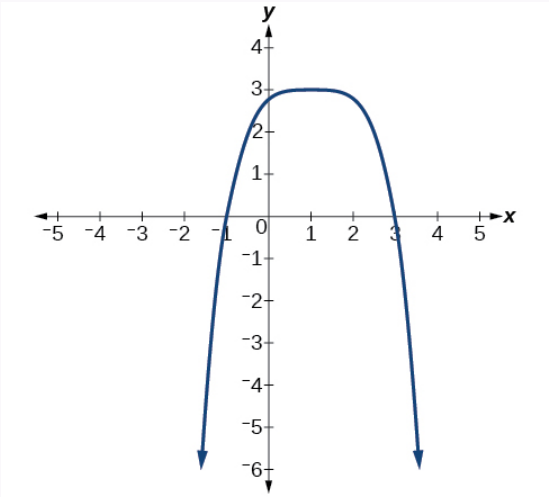


***Figure* 3.3.8**

Answer

As $x \to \infty$, $f(x) \to -\infty$; as $x \to -\infty$, $f(x) \to -\infty$. It has the shape of an even degree power function with a negative coefficient.

Example 3.3.8: Identifying End Behavior and Degree of a Polynomial Function

Given the function $f(x) = -3x^2(x-1)(x+4)$, express the function as a polynomial in general form, and determine the leading term, degree, and end behavior of the function.

Solution

Obtain the general form by expanding the given expression for $f(x)$.

$$\begin{aligned} f(x) &= -3x^2(x-1)(x+4) \\ &= -3x^2(x^2+3x-4) \\ &= -3x^4 - 9x^3 + 12x^2 \end{aligned}$$

The general form is $f(x) = -3x^4 - 9x^3 + 12x^2$. The leading term is $-3x^4$; therefore, the degree of the polynomial is 4. The degree is even (4) and the leading coefficient is negative (–3), so the end behavior is

$$\text{as } x \to -\infty, f(x) \to -\infty$$

$$\text{as } x \to \infty, f(x) \to -\infty$$

3.3.6

Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$, express the function as a polynomial in general form and determine the leading term, degree, and end behavior of the function.

Answer

The general form is $f(x) = 0.2x^3 - 1.2x^2 + 0.6x + 2$. The leading term is $0.2x^3$, so it is a degree 3 polynomial. As x approaches positive infinity, $f(x)$ increases without bound; as x approaches negative infinity, $f(x)$ decreases without bound.

Identifying Local Behavior of Polynomial Functions

In addition to the end behavior of polynomial functions, we are also interested in what happens in the "middle" of the function. In particular, we are interested in locations where graph behavior changes. A **turning point** is a point at which the function values change from increasing to decreasing or decreasing to increasing.

We are also interested in the intercepts. As with all functions, the y-intercept is the point at which the graph intersects the vertical axis. The point corresponds to the coordinate pair in which the input value is zero. Because a polynomial is a function, only one output value corresponds to each input value so there can be only one y-intercept $(0, a_0)$. The x-intercepts occur at the input values that correspond to an output value of zero. It is possible to have more than one x-intercept. See Figure 3.3.9.

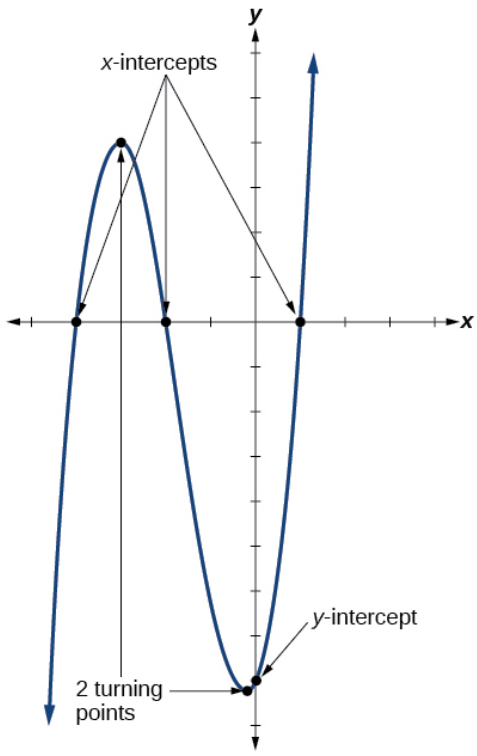


***Figure* 3.3.9**

DefinItion: Intercepts and Turning Points of Polynomial Functions

A **turning point** of a graph is a point at which the graph changes direction from increasing to decreasing or decreasing to increasing. The y-intercept is the point at which the function has an input value of zero. The x-intercepts are the points at which the output value is zero.

How to: Given a polynomial function, determine the intercepts

1. Determine the y-intercept by setting $x = 0$ and finding the corresponding output value.
2. Determine the x-intercepts by solving (if possible) for the input values that yield an output value of zero. **Note**: this is not always easy to do!

Example 3.3.9: Determining the Intercepts of a Polynomial Function

Given the polynomial function $f(x) = (x-2)(x+1)(x-4)$, written in factored form rather than general form for your convenience, determine the y-and x-intercepts.

Solution

The y-intercept occurs when the input is zero, so substitute 0 for x.

$$\begin{aligned} f(0) &= (0-2)(0+1)(0-4) \\ &= (-2)(1)(-4) \\ &= 8 \end{aligned}$$

The y-intercept is $(0, 8)$.

The x-intercepts occur when the output is zero.

$$0 = (x-2)(x+1)(x-4)$$

$$\begin{array}{ccccccc} x-2=0 & \text{or} & x+1=0 & \text{or} & x-4=0 \\ x=2 & \text{or} & x=-1 & \text{or} & x=4 \end{array}$$

The x-intercepts are $(2,0)$,$(-1,0)$, and $(4,0)$.

We can see these intercepts on the graph of the function shown in Figure 3.3.10.

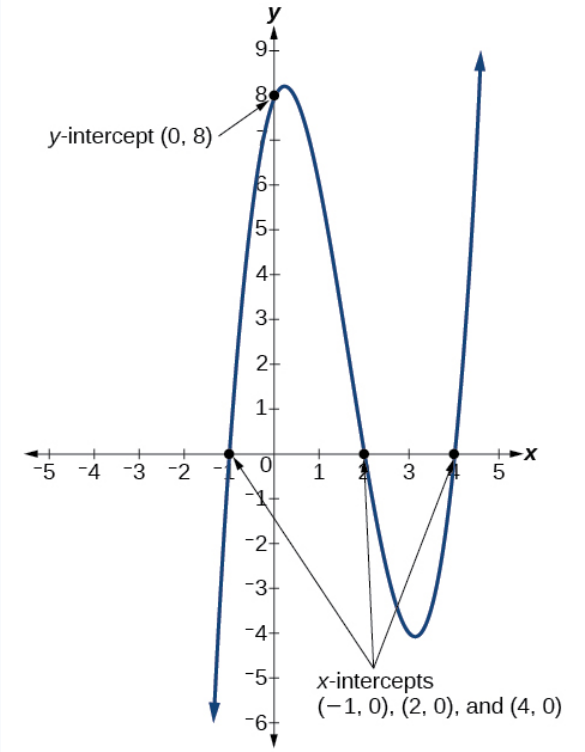


***Figure* 3.3.10:** *Graph of* $f(x)=(x-2)(x+1)(x-4)$.

Example 3.3.10: Determining the Intercepts of a Polynomial Function with Factoring

Given the polynomial function $f(x)=x^4-4x^2-45$, determine the y-and x-intercepts.

Solution

The y-intercept occurs when the input is zero.

$$\begin{aligned} f(0) &= (0)^4-4(0)^2-45 \\ &= -45 \end{aligned}$$

The y-intercept is $(0,-45)$.

The x-intercepts occur when the output is zero. To determine when the output is zero, we will need to factor the polynomial and set it equal to zero.

$$\begin{aligned} f(x) &= x^4-4x^2-45 \\ &= (x^2-9)(x^2+5) \\ &= (x-3)(x+3)(x^2+5) \end{aligned}$$

$$0=(x-3)(x+3)(x^2+5)$$

$$\begin{array}{ccccccc} x-3=0 & \text{or} & x+3=0 & \text{or} & x^2+5=0 \\ x=3 & \text{or} & x=-3 & \text{or} & \text{(no real solution)} \end{array}$$

The x-intercepts are $(3,0)$ and $(-3,0)$.

We can see these intercepts on the graph of the function shown in Figure 3.3.11. The function is even because $f(x)=f(-x)$.

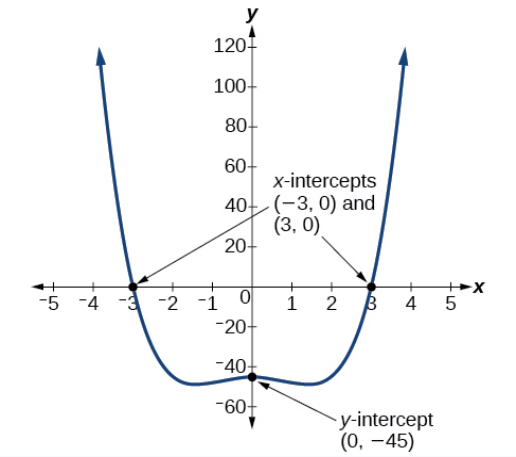


***Figure* 3.3.11:** *Graph of* $f(x) = x^4 - 4x^2 - 45$

Note: Although $f(x)$ cannot be factored any further if we stay within a domain of real numbers, it **can** be factored using complex numbers:

$$f(x) = (x-3)(x+3)(x-i\sqrt{5})(x+i\sqrt{5}).$$

There is an important theorem in mathematics called the *Fundamental Theorem of Algebra*, which says that every polynomial of degree n can be factored into n linear factors using complex numbers. However, here we are only considering a domain of real numbers, so we say $x^2 + 5$ cannot be factored any further.

3.3.7

Given the polynomial function $f(x) = 2x^3 - 6x^2 - 20x$, determine the y-and x-intercepts.

Answer

y-intercept $(0, 0)$; x-intercepts $(0, 0)$,$(-2, 0)$, and $(5, 0)$

Comparing Smooth and Continuous Graphs

The degree of a polynomial function helps us to determine the number of x-intercepts and the number of turning points. The graph of the polynomial function of degree n will have at most $n-1$ turning points. This means the graph has at most one fewer turning point than the degree of the polynomial.

A **continuous function** has no breaks in its graph: the graph can be drawn without lifting the pen from the paper. A smooth curve is a graph that has no sharp corners. The turning points of a smooth graph must always occur at rounded curves. The graphs of polynomial functions are both continuous and smooth.

Intercepts and Turning Points of Polynomials

A polynomial of degree n will have (a) **at most** n x-intercepts and (b) $n - 1$ turning points.

Note: Fact (a) follows from the **Fundamental Theorem of Algebra**, which tells us that a polynomial function of degree n is the product of n linear factors (possibly using complex numbers), so it will have at most n real roots or zeros, or x-intercepts. (We will say more about the Fundamental Theorem in Section 3.6.) Fact (b) can be shown using tools from calculus.

Example 3.3.11: Determining the Number of Intercepts and Turning Points of a Polynomial

Without graphing the function, determine the local behavior of the function by finding the maximum possible number of x-intercepts and turning points for $f(x) = -3x^{10} + 4x^7 - x^4 + 2x^3$.

Solution

The polynomial has a degree of 10, so there are at most 10 x-intercepts and at most 9 turning points.

3.3.8

Without graphing the function, determine the maximum possible number of x-intercepts and turning points for $f(x) = 108 - 13x^9 - 8x^4 + 14x^{12} + 2x^3$.

Answer

There are at most 12 x-intercepts and at most 11 turning points.

Example 3.3.12: Drawing Conclusions about a Polynomial Function from the Graph

What can we conclude about the polynomial represented by the graph shown in Figure 3.3.12?

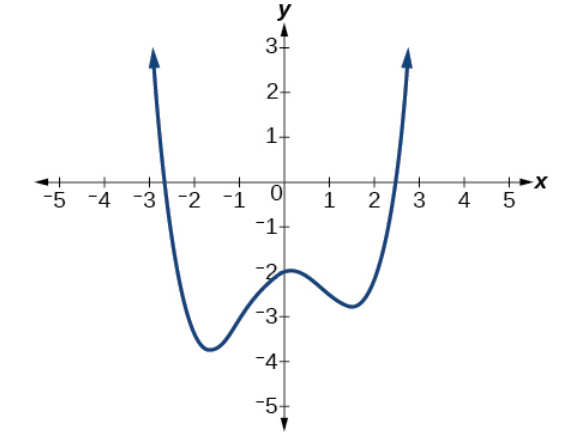


***Figure* 3.3.12.**

Solution

The end behavior of the graph tells us this is the graph of an even-degree polynomial. Now look at Figure 3.3.13.

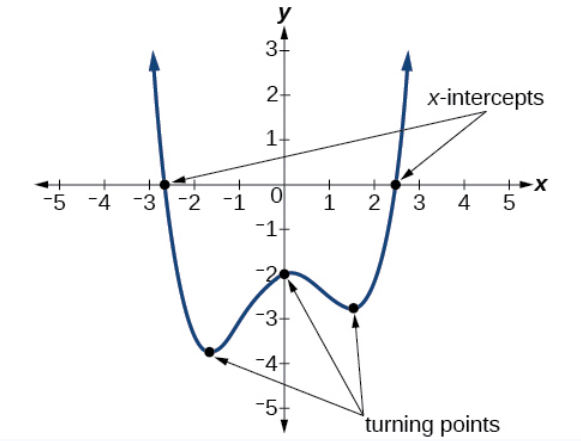


***Figure* 3.3.13:** *Graph of an even-degree polynomial.*

The graph has 2 x-intercepts, indicating a degree of 2 or greater, and 3 turning points, indicating a degree of 4 or greater. Based on this, we can conclude that the degree is even and at least 4.

3.3.9

What can we conclude about the polynomial represented by the graph shown in Figure 3.3.14?

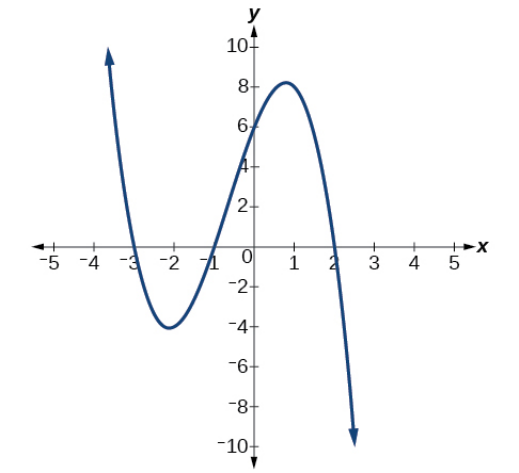


***Figure* 3.3.14**

Answer

The end behavior indicates an odd-degree polynomial function; there are 3 x-intercepts and 2 turning points, so the degree is odd and at least 3. Because of the end behavior, we know that the leading coefficient must be negative.

Example 3.3.13: Drawing Conclusions about a Polynomial Function from the Factors

Given the function $f(x) = -4x(x+3)(x-4)$, determine the local behavior.

Solution

The y-intercept is found by evaluating $f(0)$.

$$\begin{aligned} f(0) &= -4(0)(0+3)(0-4) \\ &= 0 \end{aligned}$$

The y-intercept is $(0, 0)$.

The x-intercepts are found by determining the zeros of the function.

$$\begin{aligned} 0 &= -4x(x+3)(x-4) \\ x = 0 \quad &\text{or} \quad x+3=0 \quad \text{or} \quad x-4=0 \\ x = 0 \quad &\text{or} \quad x=-3 \quad \text{or} \quad x=4 \end{aligned}$$

The x-intercepts are $(0, 0)$,$(-3, 0)$, and $(4, 0)$.

The degree is 3 so the graph has at most 2 turning points.

Try It 3.3.10

Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$, determine the local behavior.

Answer

The x-intercepts are $(2, 0)$, $(-1, 0)$, and $(5, 0)$, the y-intercept is $(0, 2)$, and the graph has at most 2 turning points.

Key Equations

- general form of a polynomial function: $f(x) = a_n x^n + a_{n-1} x^{n-1} \ldots + a_2 x^2 + a_1 x + a_0$

Key Concepts

- A power function is a variable base raised to a number power.
- The behavior of a graph as the input decreases beyond bound and increases beyond bound is called the end behavior.
- The end behavior depends on whether the power is even or odd.
- A polynomial function is the sum of terms, each of which consists of a transformed power function with positive whole number power.
- The degree of a polynomial function is the highest power of the variable that occurs in a polynomial. The term containing the highest power of the variable is called the leading term. The coefficient of the leading term is called the leading coefficient.
- The end behavior of a polynomial function is the same as the end behavior of the power function represented by the leading term of the function.
- A polynomial of degree n will have at most n x-intercepts and at most $n-1$ turning points.

Glossary

coefficient

a nonzero real number multiplied by a variable raised to an exponent

continuous function

a function whose graph can be drawn without lifting the pen from the paper because there are no breaks in the graph

degree

the highest power of the variable that occurs in a polynomial

end behavior

the behavior of the graph of a function as the input decreases without bound and increases without bound

leading coefficient

the coefficient of the leading term

leading term

the term containing the highest power of the variable

polynomial function

a function that consists of either zero or the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power

power function

a function that can be represented in the form $f(x)=kx^p$ where k is a constant, the base is a variable, and the exponent, p, is a constant

smooth curve

a graph with no sharp corners

term of a polynomial function

any a_ix^i of a polynomial function in the form $f(x)=a_nx^n+a_{n-1}x^{n-1}...+a_2x^2+a_1x+a_0$

turning point

the location at which the graph of a function changes direction

Contributors

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3.3E: Exercises

Section Exercises

Verbal

1. Explain the difference between the coefficient of a power function and its degree.

Answer: The coefficient of the power function is the real number that is multiplied by the variable raised to a power. The degree is the highest power appearing in the function.

2. If a polynomial function is in factored form, what would be a good first step in order to determine the degree of the function?

3. In general, explain the end behavior of a power function with odd degree if the leading coefficient is positive.

Answer: As x decreases without bound, so does $f(x)$. As x increases without bound, so does $f(x)$.

4. What is the relationship between the degree of a polynomial function and the maximum number of turning points in its graph?

5. What can we conclude if, in general, the graph of a polynomial function exhibits the following end behavior? As $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to -\infty$.

Answer: The polynomial function is of even degree and leading coefficient is negative.

Algebraic

For the following exercises, identify the function as a power function, a polynomial function, or neither.

---

6. $f(x) = x^5$

7. $f(x) = (x^2)^3$

Answer: Power function

8. $f(x) = x - x^4$

Exercise 3.3E.9

$f(x) = \frac{x^2}{x^2-1}$

Answer

Neither

10. $f(x) = 2x(x+2)(x-1)^2$

11. $f(x) = 3^{x+1}$

Answer: Neither

For the following exercises, find the degree and leading coefficient for the given polynomial.

---

12. $-3x^4$

13. $7 - 2x^2$

Answer: Degree = 2, Coefficient = –2

14. $-2x^2 - 3x^5 + x - 6$

15. $x(4 - x^2)(2x + 1)$

Answer: Degree =4, Coefficient = –2

16. $x^2(2x - 3)^2$

For the following exercises, determine the end behavior of the functions.

---

17. $f(x) = x^4$

Answer: As $x \to \infty$, $f(x) \to \infty$, as $x \to -\infty$, $f(x) \to \infty$

18. $f(x) = x^3$

19. $f(x) = -x^4$

Answer: As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to -\infty$

20. $f(x) = -x^9$

21. $f(x) = -2x^4 - 3x^2 + x - 1$

Answer: As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to -\infty$

22. $f(x) = 3x^2 + x - 2$

23. $f(x) = x^2(2x^3 - x + 1)$

Answer: As $x \to \infty$, $f(x) \to \infty$, as $x \to -\infty$, $f(x) \to -\infty$

24. $f(x) = (2 - x)^7$

For the following exercises, find the intercepts of the functions.

---

25. $f(t) = 2(t-1)(t+2)(t-3)$

Answer: y-intercept is $(0, 12)$, t-intercepts are $(1, 0)$; $(-2, 0)$; and $(3, 0)$.

26. $g(n) = -2(3n-1)(2n+1)$

Exercise 3.3E.27

$f(x) = x^4 - 16$

Answer

y-intercept is $(0, -16)$. x-intercepts are $(2, 0)$ and $(-2, 0)$.

28. $f(x) = x^3 + 27$

29. $f(x) = x(x^2 - 2x - 8)$

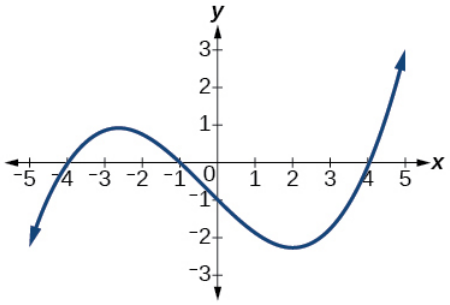
Answer: y-intercept is $(0, 0)$. x-intercepts are $(0, 0)$, $(4, 0)$, and $(-2, 0)$.

30. $f(x) = (x + 3)(4x^2 - 1)$

Graphical

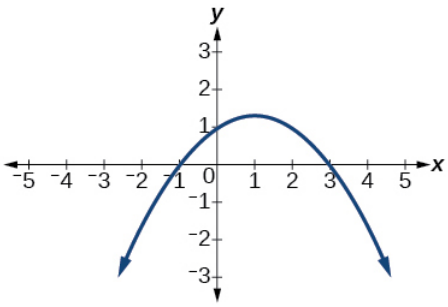
For the following exercises, determine the least possible degree of the polynomial function shown.

31.

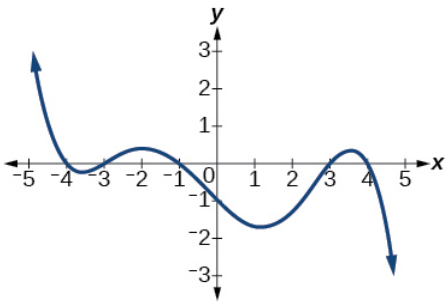


Answer: 3

32.

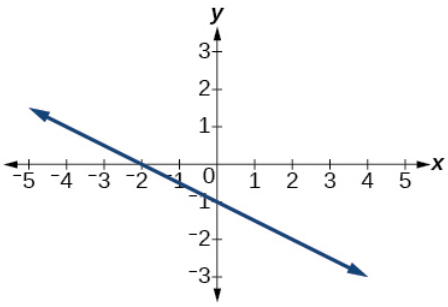


33.

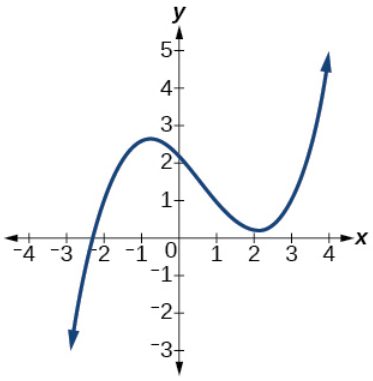


Answer: 5

34.



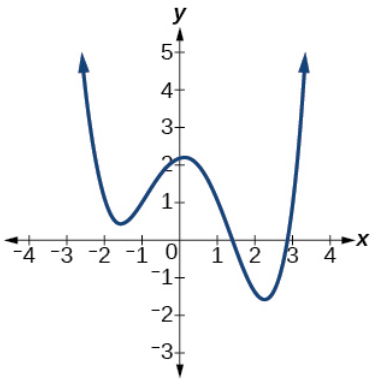
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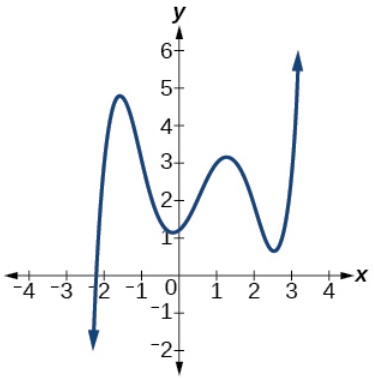
Answer: 3

36.

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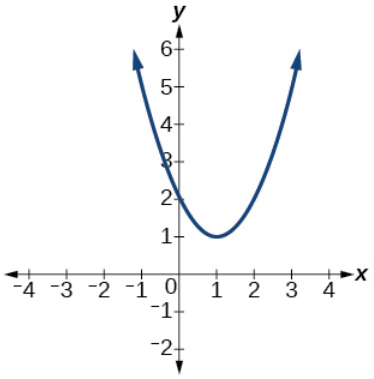


37.



Answer: 5

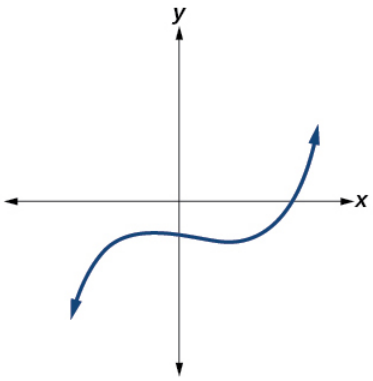
38.



For the following exercises, determine whether the graph of the function provided is a graph of a polynomial function. If so, determine the number of turning points and the least possible degree for the function.

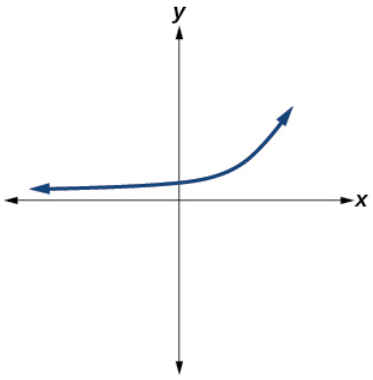
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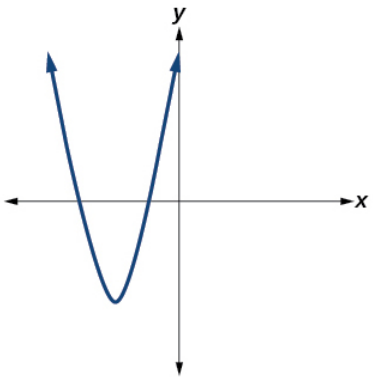


Answer: Yes. Number of turning points is 2. Least possible degree is 3.

40.

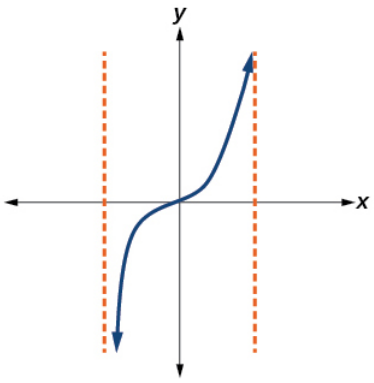


41.

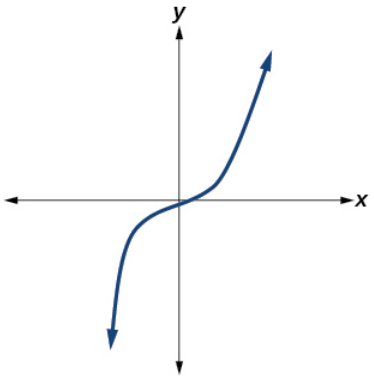


Answer: Yes. Number of turning points is 1. Least possible degree is 2.

42.

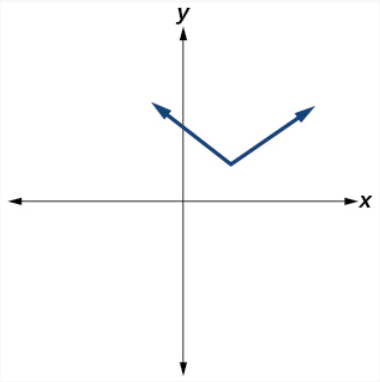


43.



Answer: Yes. Number of turning points is 0. Least possible degree is 3.

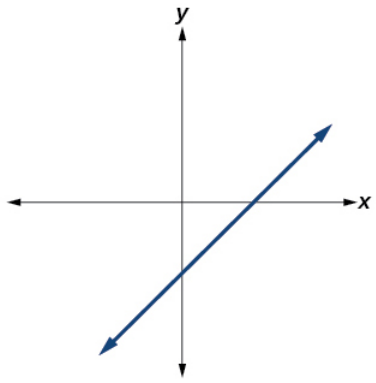
Exercise 3.3E.44



Answer

No (the graph is not smooth)

45.



Answer: Yes. Number of turning points is 0. Least possible degree is 1.

Numeric

For the following exercises, make a table to confirm the end behavior of the function.

---

46. $f(x) = -x^3$

47. $f(x) = x^4 - 5x^2$

Answer:

x	$f(x)$
10	9,500
100	99,950,000
–10	9,500
–100	99,950,000

as $x \to -\infty$, $f(x) \to \infty$, as $x \to \infty$, $f(x) \to \infty$

48. $f(x) = x^2(1-x)^2$

49. $f(x) = (x-1)(x-2)(3-x)$

Answer:

x	$f(x)$
10	9,500
100	99,950,000
–10	9,500
–100	99,950,000

as $x \to -\infty$, $f(x) \to \infty$, as $x \to \infty$, $f(x) \to -\infty$

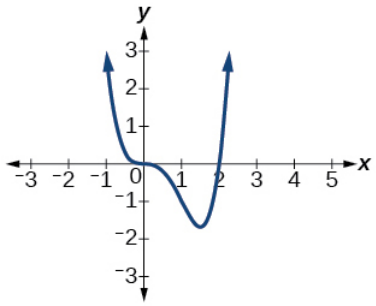
50. $f(x) = \frac{x^5}{10} - x^4$

Technology

For the following exercises, graph the polynomial functions using a calculator. Based on the graph, determine the intercepts and the end behavior.

51. $f(x) = x^3(x-2)$

Answer:

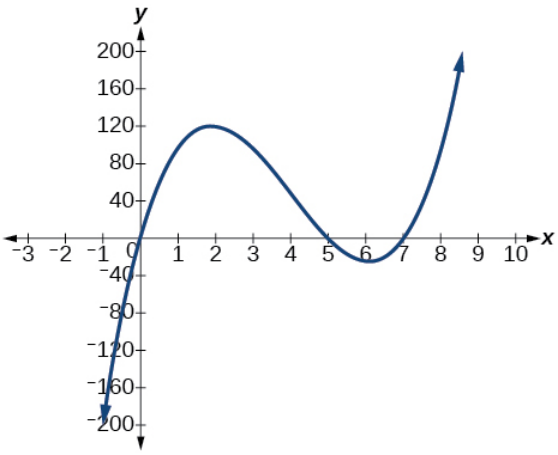


The y-intercept is $(0,0)$. The x-intercepts are $(0,0),(2,0)$. As $x \to -\infty$, $f(x) \to \infty$, as $x \to \infty$, $f(x) \to \infty$

52. $f(x) = x(x-3)(x+3)$

53. $f(x) = x(14-2x)(10-2x)$

Answer:

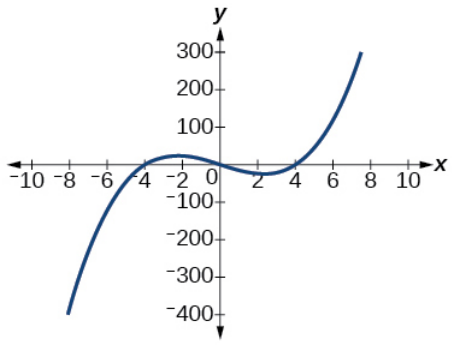


The y-intercept is $(0,0)$. The x-intercepts are $(0,0),(5,0),(7,0)$. As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to \infty$

54. $f(x) = x(14-2x)(10-2x)^2$

55. $f(x) = x^3 - 16x$

Answer:

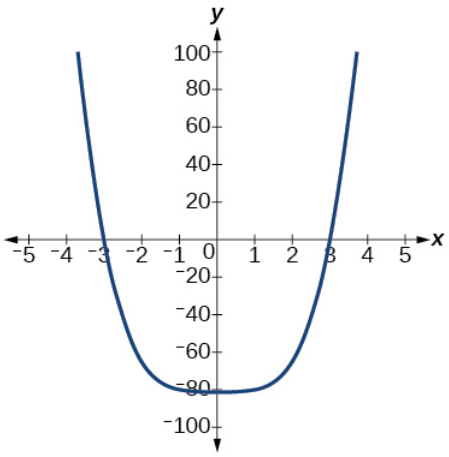


The y-intercept is (0, 0). The x-intercept is $(-4,0),(0,0),(4,0)$. As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to \infty$

56. $f(x) = x^3 - 27$

57. $f(x) = x^4 - 81$

Answer:

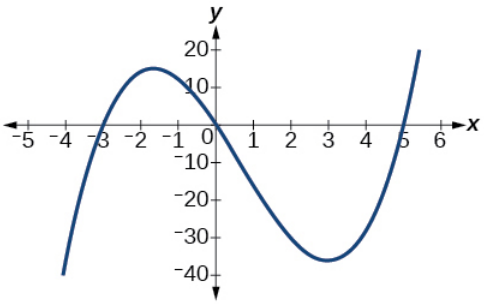


The y-intercept is (0, −81). The x-intercept are $(3,0),(-3,0)$. As $x \to -\infty,\ f(x) \to \infty$, as $x \to \infty,\ f(x) \to \infty$

58. $f(x) = -x^3 + x^2 + 2x$

59. $f(x) = x^3 - 2x^2 - 15x$

Answer:



The y-intercept is $(0,0)$. The x-intercepts are $(-3,0),(0,0),(5,0)$. As $x \to -\infty,\ f(x) \to -\infty$, as $x \to \infty,\ f(x) \to \infty$

60. $f(x) = x^3 - 0.01x$

Extensions

For the following exercises, use the information about the graph of a polynomial function to determine the function. Assume the leading coefficient is 1 or –1. There may be more than one correct answer.

61. The y-intercept is $(0,-4)$. The x-intercepts are $(-2,0),(2,0)$. Degree is 2.

End behavior: as $x \to -\infty,\ f(x) \to \infty$, as $x \to \infty,\ f(x) \to \infty$.

Answer: $f(x) = x^2 - 4$

62. The y-intercept is $(0,9)$. The x-intercepts are $(-3,0),(3,0)$. Degree is 2.

End behavior: as $x \to -\infty,\ f(x) \to -\infty$, as $x \to \infty,\ f(x) \to -\infty$.

63. The y-intercept is $(0,0)$. The x-intercepts are $(0,0),(2,0)$. Degree is 3.

End behavior: as $x \to -\infty,\ f(x) \to -\infty$, as $x \to \infty,\ f(x) \to \infty$.

Answer: $f(x) = x^3 - 4x^2 + 4x$

64. The y-intercept is $(0,1)$. The x-intercept is $(1,0)$. Degree is 3.

End behavior: as $x \to -\infty,\ f(x) \to \infty$, as $x \to \infty,\ f(x) \to -\infty$.

65. The y-intercept is $(0,1)$. There is no x-intercept. Degree is 4.

End behavior: as $x \to -\infty$, $f(x) \to \infty$, as $x \to \infty$, $f(x) \to \infty$.

Answer: $f(x) = x^4 + 1$

Real-World Applications

For the following exercises, use the written statements to construct a polynomial function that represents the required information.

66. An oil slick is expanding as a circle. The radius of the circle is increasing at the rate of 20 meters per day. Express the area of the circle as a function of d, the number of days elapsed.

67. A cube has an edge of 3 feet. The edge is increasing at the rate of 2 feet per minute. Express the volume of the cube as a function of m, the number of minutes elapsed.

Answer: $V(m) = 8m^3 + 36m^2 + 54m + 27$

68. A rectangle has a length of 10 inches and a width of 6 inches. If the length is increased by x inches and the width increased by twice that amount, express the area of the rectangle as a function of x.

Exercise $3.3E.69$

An open box is to be constructed by cutting out square corners of x-inch sides from a piece of cardboard 8 inches by 8 inches and then folding up the sides. Express the volume of the box as a function of x.

Answer

$V(x) = 4x^3 - 32x^2 + 64x$

70. A rectangle is twice as long as it is wide. Squares of side 2 feet are cut out from each corner. Then the sides are folded up to make an open box. Express the volume of the box as a function of the width (x).

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3.4: Graphs of Polynomial Functions

Skills to Develop

- Recognize characteristics of graphs of polynomial functions.
- Use factoring to find zeros of polynomial functions.
- Identify zeros and their multiplicities.
- Determine end behavior.
- Understand the relationship between degree and turning points.
- Graph polynomial functions.
- Use the Intermediate Value Theorem.

The revenue in millions of dollars for a fictional cable company from 2006 through 2013 is shown in Table **3.4.1**.

***Table* 3.4.1**

Year	2006	2007	2008	2009	2010	2011	2012	2013
Revenues	52.4	52.8	51.2	49.5	48.6	48.6	48.7	47.1

The revenue can be modeled by the polynomial function

$$R(t) = -0.037t^4 + 1.414t^3 - 19.777t^2 + 118.696t - 205.332$$

where R represents the revenue in millions of dollars and t represents the year, with $t = 6$ corresponding to 2006. Over which intervals is the revenue for the company increasing? Over which intervals is the revenue for the company decreasing? These questions, along with many others, can be answered by examining the graph of the polynomial function. We have already explored the local behavior of quadratics, a special case of polynomials. In this section we will explore the local behavior of polynomials in general.

Recognizing Characteristics of Graphs of Polynomial Functions

Polynomial functions of degree 2 or more have graphs that do not have sharp corners; recall that these types of graphs are called smooth curves. Polynomial functions also display graphs that have no breaks. Curves with no breaks are called continuous. Figure **3.4.1** shows a graph that represents a **polynomial function** and a graph that represents a function that is not a polynomial.

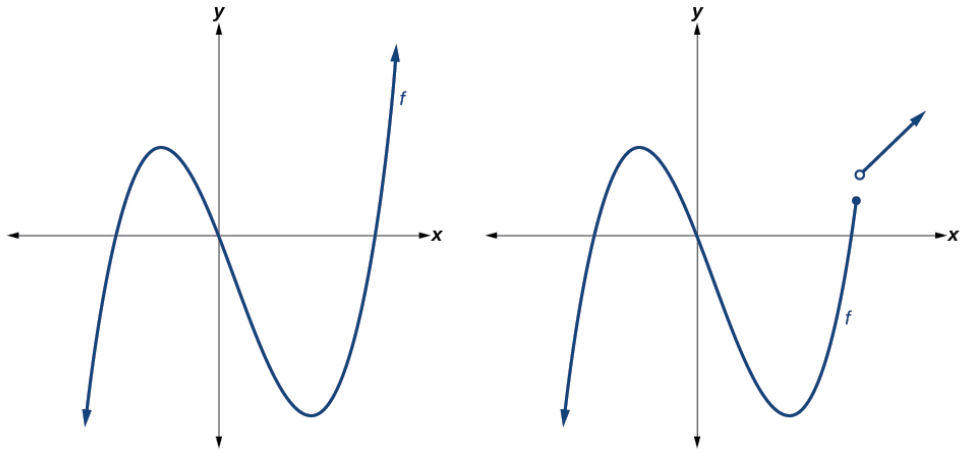


***Figure* 3.4.1**

Example 3.4.1: Recognizing Polynomial Functions

Which of the graphs in Figure **3.4.2** represents a polynomial function?

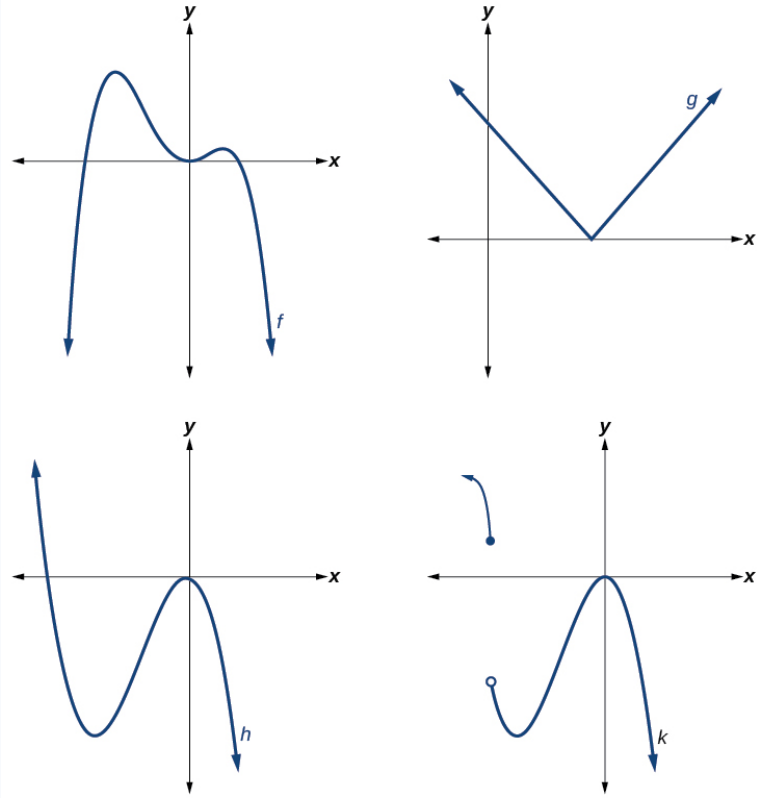


***Figure* 3.4.2**

Solution

The graphs of f and h are graphs of polynomial functions. They are smooth and **continuous**.

The graphs of g and k are graphs of functions that are not polynomials. The graph of function g has a sharp corner. The graph of function k is not continuous.

Do all polynomial functions have as their domain all real numbers?

Yes. Any real number is a valid input for a polynomial function.

Using Factoring to Find Zeros of Polynomial Functions

Recall that if f is a polynomial function, the values of x for which $f(x) = 0$ are called **zeros** of f.

If the equation of the polynomial function can be factored, we can set each factor equal to zero and solve for the zeros. (Also, any value $x = a$ that is a zero of a polynomial function yields a factor of the polynomial, of the form $(x - a)$.

Given the equation of a polynomial function, we can use this method to find x-intercepts because at the x-intercepts we find the input values whose output value is zero. For general polynomials, this can be a challenging prospect. While quadratics can be solved using the relatively simple quadratic formula, the corresponding formulas for cubic and fourth-degree polynomials are not simple enough to remember, and formulas do not exist for general higher-degree polynomials. Consequently, we will limit ourselves to three cases in this section:

- The polynomial can be factored using known methods: greatest common factor, factor by grouping, and trinomial factoring.
- The polynomial is given in factored form.
- Technology is used to determine the intercepts.

How To: given a polynomial function f, find the x-intercepts by factoring

1. Set $f(x) = 0$.
2. If the polynomial function is not given in factored form:
    a. Factor out any common monomial factors.
    b. Factor any factorable binomials or trinomials.
3. Set each factor equal to zero and solve to find the x-intercepts.

Example 3.4.2: Finding the x-Intercepts of a Polynomial Function by Factoring

Find the x-intercepts of $f(x) = x^6 - 3x^4 + 2x^2$.

Solution

We can factor this polynomial to find solutions for $f(x) = 0$.

$$\begin{aligned} x^6 - 3x^4 + 2x^2 &= 0 && \text{Factor out the greatest common factor.} \\ x^2(x^4 - 3x^2 + 2) &= 0 && \text{Factor the trinomial, which is in quadratic form.} \\ x^2(x^2 - 1)(x^2 - 2) &= 0 && \text{Set each factor equal to zero.} \end{aligned}$$

$$\begin{aligned} x^2 = 0 \quad &\text{or} \quad (x^2 - 1) = 0 \quad \text{or} \quad (x^2 - 2) = 0 \\ &\qquad\quad x^2 = 1 \qquad\quad x^2 = 2 \\ x = 0 \quad &\qquad\quad x = \pm 1 \qquad\quad x = \pm\sqrt{2} \approx \pm 1.41 \end{aligned}$$

.

This gives us five x-intercepts: $(0, 0)$, $(1, 0)$, $(-1, 0)$, $(1.41, 0)$,and $(-1.41, 0)$.

See Figure 3.4.3. We can see that this is an even function. To confirm algebraically, we have

$$\begin{aligned} f(-x) &= (-x)^6 - 3(-x)^4 + 2(-x)^2 \\ &= x^6 - 3x^4 + 2x^2 \\ &= f(x). \end{aligned}$$

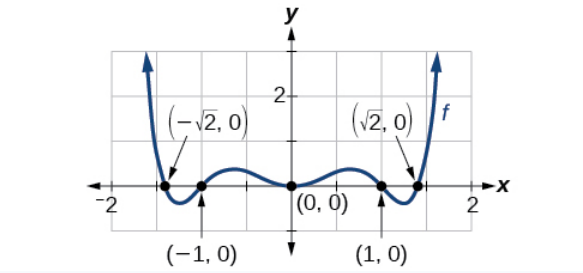


***Figure* 3.4.3.**

Example 3.4.3: Finding the x-Intercepts of a Polynomial Function by Factoring

Find the x-intercepts of $f(x) = x^3 - 5x^2 - x + 5$.

Solution

Find solutions for $f(x) = 0$ by factoring.

$$\begin{aligned} x^3 - 5x^2 - x + 5 &= 0 && \text{Factor by grouping.} \\ x^2(x-5) - (x-5) &= 0 && \text{Factor out the common factor.} \\ (x^2-1)(x-5) &= 0 && \text{Factor the difference of squares.} \\ (x+1)(x-1)(x-5) &= 0 && \text{Set each factor equal to zero.} \end{aligned}$$

$$\begin{aligned} x+1 &= 0 \quad \text{or} \quad x-1 = 0 \quad \text{or} \quad x-5 = 0 \\ x &= -1 \qquad\quad x = 1 \qquad\quad x = 5 \end{aligned}$$

There are three x-intercepts: $(-1, 0)$, $(1, 0)$, and $(5, 0)$. See Figure 3.4.4.

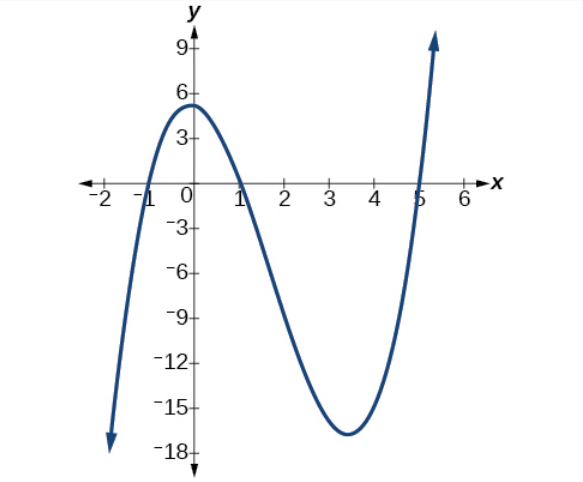


***Figure* 3.4.4:** *Graph of $f(x)$.*

Example 3.4.4: Finding the y- and x-Intercepts of a Polynomial in Factored Form

Find the y- and x-intercepts of $g(x) = (x-2)^2(2x+3)$.

Solution

The y-intercept can be found by evaluating $g(0)$.

$$\begin{aligned} g(0) &= (0-2)^2(2(0)+3) \\ &= 12 \end{aligned}$$

The y-intercept is $(0, 12)$.

The x-intercepts can be found by solving $g(x) = 0$.

$$(x-2)^2(2x+3)=0$$

$$\begin{aligned}(x-2)^2&=0 \quad \text{or} \quad (2x+3)=0\\ x-2&=0 \quad\quad\quad x=-\frac{3}{2}\\ x&=2\end{aligned}$$

The x-intercepts are $(2,0)$ and $\left(-\frac{3}{2},0\right)$.

Analysis

We can always check that our answers are reasonable by using a graphing utility such as Desmos to graph the polynomial as shown in Figure 3.4.5.

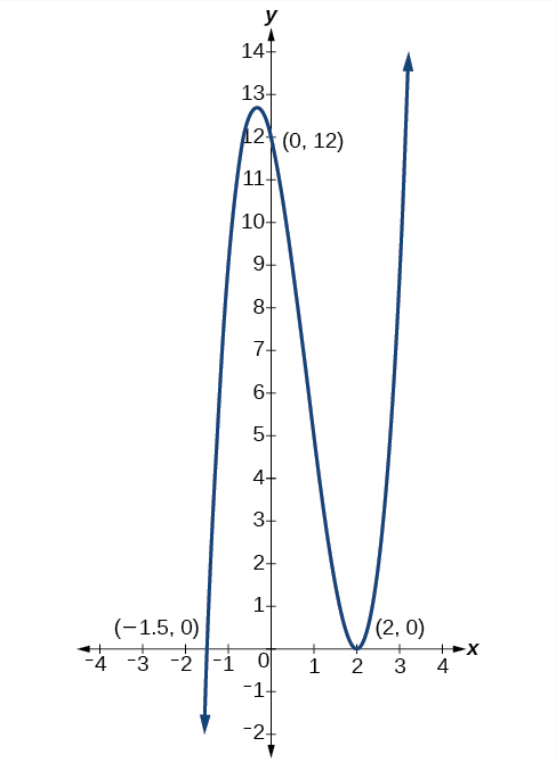


Figure 3.4.5: *Graph of $g(x)$.*

Example 3.4.5: Finding the x-Intercepts of a Polynomial Function Using a Graph

Find the x-intercepts of $h(x)=x^3+4x^2+x-6$.

Solution

This polynomial is not in factored form, has no common factors, and does not appear to be factorable using techniques previously discussed. **Note**: If you are familiar with the **Rational Root Theorem** and **synthetic division**, use these techniques to factor this polynomial, show your work, and hand it in for **Extra Credit**. Fortunately, we can use technology to find the intercepts. Keep in mind that some values make graphing difficult by hand. In these cases, we can take advantage of graphing utilities.

Looking at the graph of this function, as shown in Figure 3.4.6, it appears that there are x-intercepts at $x=-3,-2,$ and 1.

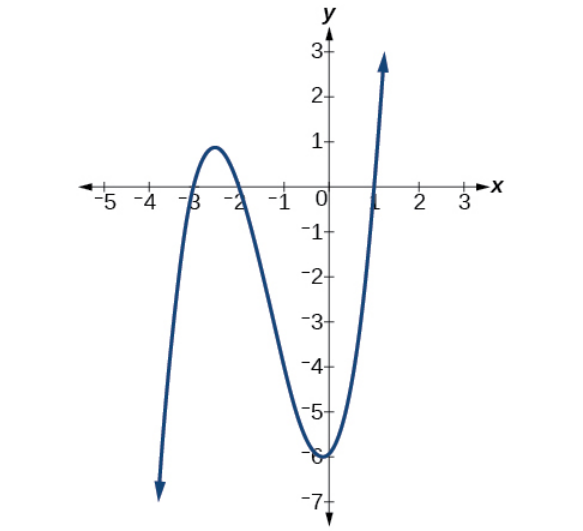


Figure 3.4.6: *Graph of* $h(x)$.

We can check whether these are correct by substituting these values for x and verifying that

$$h(-3) = h(-2) = h(1) = 0.$$

Since $h(x) = x^3 + 4x^2 + x - 6$, we have:

$$h(-3) = (-3)^3 + 4(-3)^2 + (-3) - 6 = -27 + 36 - 3 - 6 = 0$$
$$h(-2) = (-2)^3 + 4(-2)^2 + (-2) - 6 = -8 + 16 - 2 - 6 = 0$$
$$h(1) = (1)^3 + 4(1)^2 + (1) - 6 = 1 + 4 + 1 - 6 = 0$$

Each x-intercept corresponds to a zero of the polynomial function and each zero yields a factor, so we can now write the polynomial in factored form.

$$\begin{aligned} h(x) &= x^3 + 4x^2 + x - 6 \\ &= (x+3)(x+2)(x-1) \end{aligned}$$

3.4.1

Use a graphing utility (like Desmos) to find the y-and x-intercepts of the function $f(x) = x^4 - 19x^2 + 30x$.

Answer

- y-intercept $(0, 0)$;
- x-intercepts $(0, 0)$, $(-5, 0)$, $(2, 0)$, and $(3, 0)$

Identifying Zeros and Their Multiplicities

Graphs behave differently at various x-intercepts. Sometimes, the graph will cross over the horizontal axis at an intercept. Other times, the graph will touch the horizontal axis and "bounce off." Suppose, for example, we graph the function

$$f(x) = (x+3)(x-2)^2(x+1)^3.$$

Notice in Figure **3.4.7** that the behavior of the function at each of the x-intercepts is different.

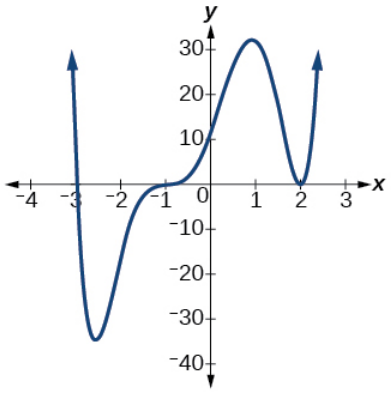


***Figure* 3.4.7:** *Identifying the behavior of the graph at an x-intercept by examining the multiplicity of the zero.*

The x-intercept −3 is the solution of the equation $(x+3)=0$. The graph passes directly through the x-intercept at $x=-3$. The factor is linear (has a degree of 1), so the behavior near the intercept is like that of a line — it passes directly through the intercept. We call this a single zero because the zero corresponds to a single factor of the function.

The x-intercept 2 is the repeated solution of the equation $(x-2)^2=0$. The graph touches the axis at the intercept and changes direction. The factor is quadratic (degree 2), so the behavior near the intercept is like that of a quadratic — it bounces off of the horizontal axis at the intercept.

$$(x-2)^2=(x-2)(x-2)$$

The factor is repeated, that is, the factor $(x-2)$ appears twice. The number of times a given factor appears in the factored form of the equation of a polynomial is called the **multiplicity**. The zero associated with this factor, $x=2$, has multiplicity 2 because the factor $(x-2)$ occurs twice.

The x-intercept −1 is the repeated solution of the equation $(x+1)^3=0$.The graph passes through the axis at the intercept, but flattens out a bit first. This factor is cubic (degree 3), so the behavior near the intercept is like that of a cubic — with the same S-shape near the intercept as the toolkit function $f(x)=x^3$. We call this a triple zero, or a zero with multiplicity 3.

For **zeros** with even multiplicities, the graphs touch or are **tangent** to the x-axis. For zeros with odd multiplicities, the graphs cross, or **intersect**, the x-axis. See Figure 3.4.8 for examples of graphs of polynomial functions with multiplicity $p=1, p=2$, and $p=3$.

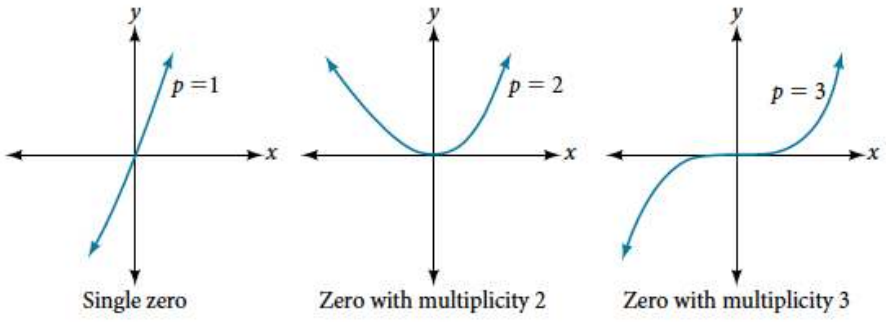


***Figure* 3.4.8:** *Three graphs showing three different polynomial functions with multiplicity 1, 2, and 3.*

For higher even powers, such as 4, 6, and 8, the graph will still touch and bounce off of the horizontal axis but, for each increasing even power, the graph will appear flatter as it approaches and leaves the x-axis.

For higher odd powers, such as 5, 7, and 9, the graph will still cross through the horizontal axis, but for each increasing odd power, the graph will appear flatter as it approaches and leaves the x-axis.

Graphical Behavior of Polynomials at x-Intercepts

If a polynomial contains a factor of the form $(x-h)^p$, the behavior near the x-intercept h is determined by the power p. We say that $x=h$ is a zero of **multiplicity** p.

The graph of a polynomial function will touch but not cross the x-axis at zeros with even multiplicities. The graph will cross the x-axis at zeros with odd multiplicities.

The sum of the multiplicities is no greater than the degree of the polynomial function.

How To: Given a graph of a polynomial function of degree n, identify the zeros and their multiplicities

1. If the graph crosses the x-axis and appears almost linear at the intercept, it is a single zero.
2. If the graph touches the x-axis and bounces off of the axis, it is a zero with even multiplicity.
3. If the graph crosses the x-axis, it is a zero with odd multiplicity.
4. The sum of all of the multiplicities is no greater than n.

Example 3.4.6: Identifying Zeros and Their Multiplicities

Use the graph of the function of degree 6 in Figure **3.4.9** to identify the zeros of the function and their possible multiplicities.

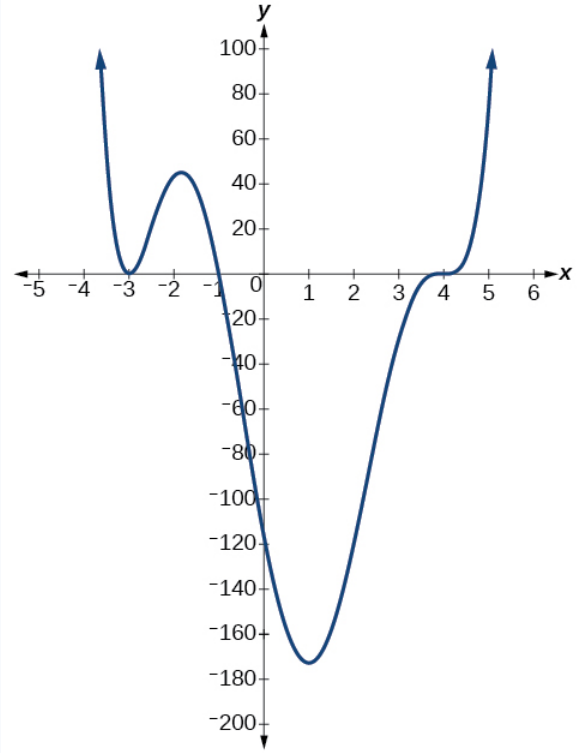


Figure* 3.4.9*: *Graph of a polynomial function with degree 6.*

Solution

The polynomial function is of degree **6**. The sum of the multiplicities cannot be greater than **6**.

Starting from the left, the first zero occurs at $x = -3$. The graph touches the x-axis, so the multiplicity of the zero must be even. The zero of $x = -3$ has multiplicity 2 or 4. It cannot have multiplicity 6 since there are other zeros.

The next zero occurs at $x = -1$. The graph looks almost linear at this point. This is probably a single zero of multiplicity 1.

The last zero occurs at $x = 4$.The graph crosses the x-axis, so the multiplicity of the zero must be odd, but is probably not 1 since the graph does not seem to cross in a linear fashion. The multiplicity is probably 3, which means the multiplicity of $x = -3$ must be 2, and that the sum of the multiplicities is 6.

3.4.2

Use the graph of the function of degree 5 in Figure **3.4.10** to identify the zeros of the function and their multiplicities.

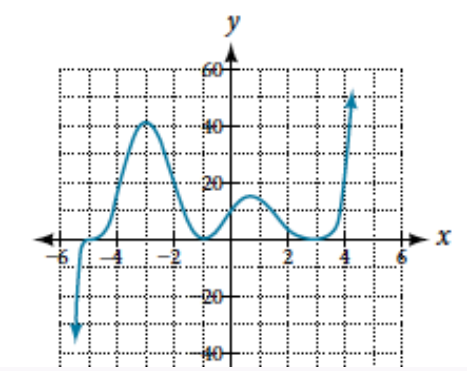


Figure* 3.4.10*: *Graph of a polynomial function with degree 5.*

Answer

The graph has a zero of –5 with multiplicity 1, a zero of –1 with multiplicity 2, and a zero of 3 with multiplicity 2.

Determining End Behavior

As we have already learned, the behavior of a graph of a polynomial function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

will either ultimately rise or fall as x increases without bound and will either rise or fall as x decreases without bound. This is because for very large inputs, say 100 or 1,000, the leading term dominates the size of the output. The same is true for very small inputs, say –100 or –1,000.

Recall that we call this behavior the end behavior of a function. As we pointed out when discussing quadratic equations, when the leading term of a polynomial function, $a_n x^n$, is an even power function and $a_n > 0$, as x increases or decreases without bound, $f(x)$ increases without bound. When the leading term is an odd power function, as x decreases without bound, $f(x)$ also decreases without bound; as x increases without bound, $f(x)$ also increases without bound. If the leading term is negative, it will change the direction of the end behavior. Figure **3.4.11** summarizes all four cases.

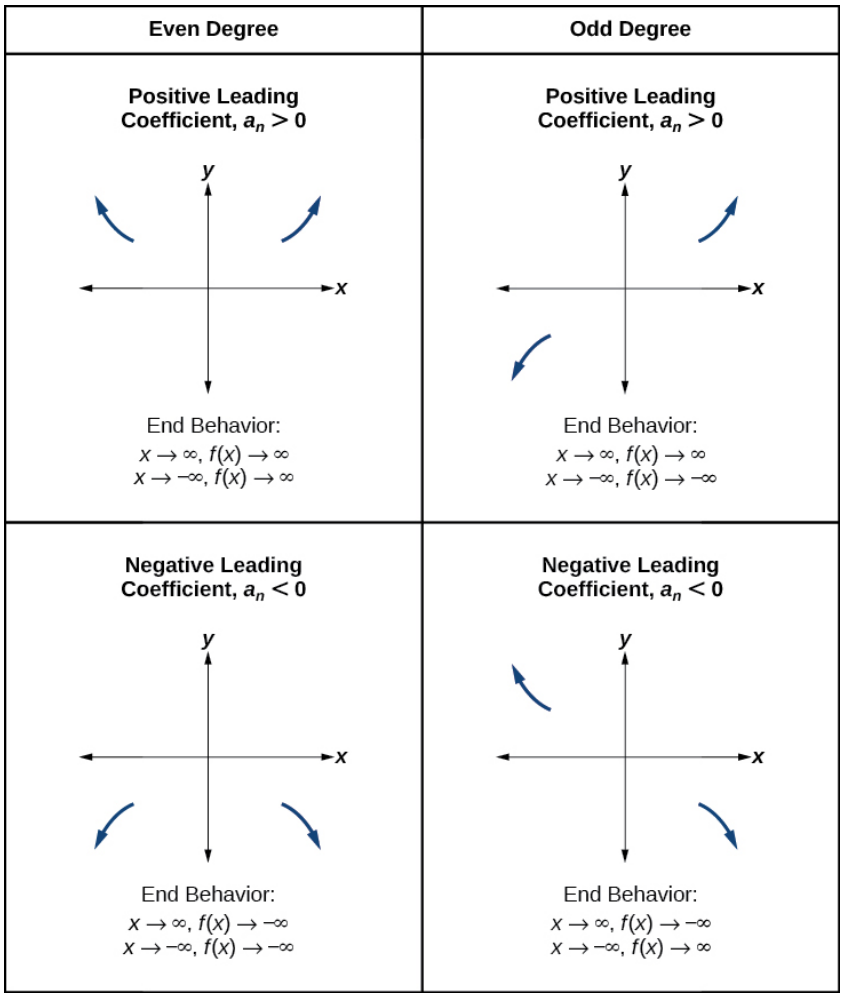


***Figure* 3.4.11.**

Understanding the Relationship between Degree and Turning Points

In addition to the end behavior, recall that we can analyze a polynomial function's local behavior. It may have a turning point where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising). Look at the graph of the polynomial function $f(x) = x^4 - x^3 - 4x^2 + 4x$ in Figure **3.4.12**. The graph has three turning points.

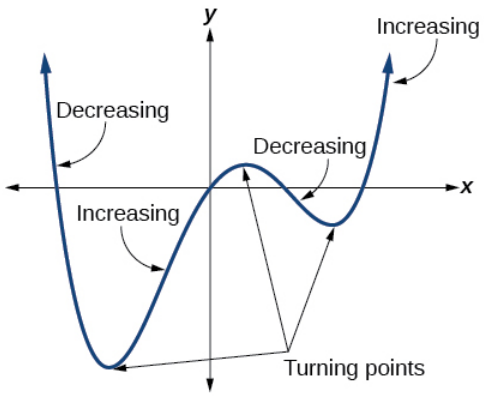


***Figure* 3.4.12:** *Graph of* $f(x) = x^4 - x^3 - 4x^2 + 4x$

This function f is a 4th degree polynomial function and has 3 turning points. The maximum number of turning points of a polynomial function is always one less than the degree of the function.

Interpreting Turning Points

A turning point is a point of the graph where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising). A polynomial of degree n will have at most $n-1$ turning points.

Example 3.4.7: Finding the Maximum possible Number of Turning Points Using the Degree of a Polynomial Function

Find the maximum possible number of turning points of each polynomial function.

a. $f(x) = -x^3 + 4x^5 - 3x^2 + 1$
b. $f(x) = -(x-1)^2(1+2x^2)$

Solution

a. $f(x) = -x^3 + 4x^5 - 3x^2 + 1$

First, rewrite the polynomial function in descending order: $f(x) = 4x^5 - x^3 - 3x^2 + 1$

Identify the degree of the polynomial function. This polynomial function is of degree 5.

The maximum possible number of turning points is $5-1=4$.

b. $f(x) = -(x-1)^2(1+2x^2)$

First, calculate the leading term of the polynomial function if the function were expanded.

$$f(x) = -(x-1)^2(1+2x^2)$$
$$a_n x^n = -(x^2)(2x^2) = -2x^4.$$

Then, identify the degree of the polynomial function. This polynomial function is of degree 4.

The maximum possible number of turning points is $4-1=3$.

Graphing Polynomial Functions

We can use what we have learned about multiplicities, end behavior, and turning points to sketch possible graphs of polynomial functions. Let us put this all together and look at the steps required. Note: we will need tools from calculus in order to find the exact location of any turning points; so while our sketch will be as correct as we can make it, it may not be entirely accurate.

How To: Given a polynomial function, sketch a possible graph

1. Find the intercepts, if possible.
2. Check for symmetry. If the function is an even function, its graph is symmetrical about the y-axis, that is, $f(-x) = f(x)$. If a function is an odd function, its graph is symmetrical about the origin, that is, $f(-x) = -f(x)$.
3. Use the multiplicities of the zeros to determine the behavior of the polynomial at the x-intercepts.
4. Determine the end behavior by examining the leading term.
5. Use the end behavior and the behavior at the intercepts to sketch a graph.
6. Ensure that the number of turning points does not exceed one less than the degree of the polynomial.
7. Optionally, use technology to check the graph.

Example 3.4.8: Sketching the Graph of a Polynomial Function

Sketch a graph of $f(x) = -2(x+3)^2(x-5)$.

Solution

This graph has two x-intercepts. At $x=-3$, the factor is squared, indicating a multiplicity of 2. The graph will bounce at this x-intercept. At $x=5$,the function has a multiplicity of one, indicating the graph will cross through the axis at this intercept.

The y-intercept is found by evaluating $f(0)$.

$$\begin{aligned} f(0) &= -2(0+3)^2(0-5) \\ &= -2 \cdot 9 \cdot (-5) \\ &= 90 \end{aligned}$$

The y-intercept is $(0, 90)$.

Additionally, we can see the leading term is $-2(x^2)(x) = -2x^3$, so the end behavior is that of a vertically reflected cubic, with the outputs decreasing as the inputs approach positive infinity, and the outputs increasing as the inputs approach negative infinity. See Figure 3.4.13.

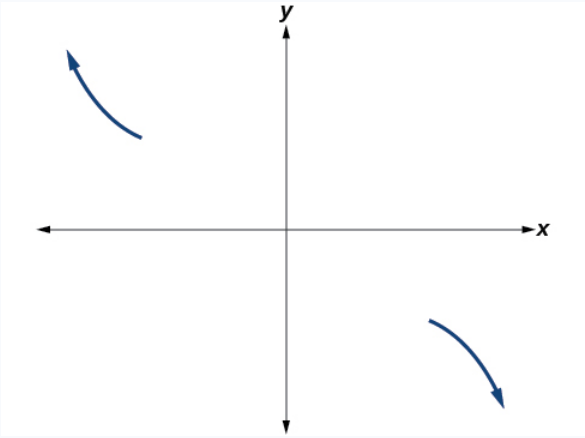


Figure 3.4.13: *Showing the distribution for the leading term.*

To sketch this, we consider that:

- As $x \to -\infty$ the function $f(x) \to \infty$, so we know the graph starts in the second quadrant and is decreasing toward the x-axis.
- Since $f(-x) = -2(-x+3)^2(-x-5)$ is not equal to $f(x)$, nor $-f(x)$, the graph does not display symmetry.
- At $(-3, 0)$, the graph bounces off of the x-axis, so the function must start increasing.
- At $(0, 90)$, the graph crosses the y-axis at the y-intercept. See Figure 3.4.14.

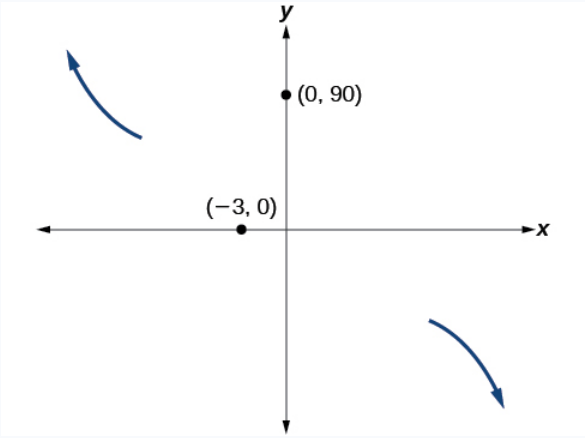


Figure 3.4.14: *Graph of the end behavior and intercepts, $(-3, 0)$ and $(0, 90)$, for the function $f(x) = -2(x+3)^2(x-5)$.*

Somewhere before or after this point, the graph must turn back down or start decreasing toward the horizontal axis because the graph has another x-intercept at $(5, 0)$. See Figure 3.4.15.

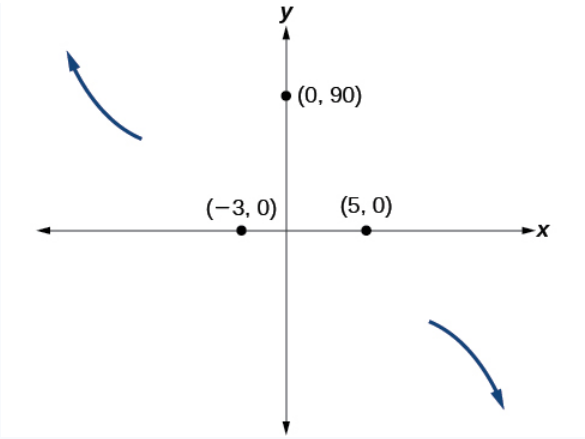


***Figure* 3.4.15:** *Graph of the end behavior and intercepts,* $(-3,0)$, $(0,90)$ *and* $(5,0)$, *for the function* $f(x)=-2(x+3)^2(x-5)$.

As $x \to \infty$ the function $f(x) \to -\infty$,

so we know the graph continues to decrease, and we can stop drawing the graph in the fourth quadrant.

Using technology, we can create the graph for the polynomial function, shown in Figure **3.4.16**, and verify that the resulting graph looks like our sketch in Figure **3.4.15**.

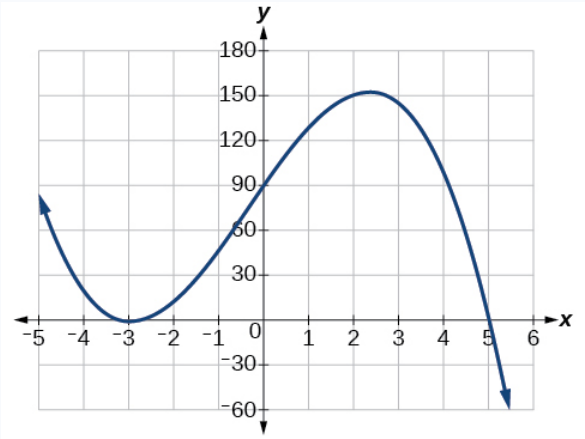


***Figure* 3.4.16:** *The complete graph of the polynomial function* $f(x)=-2(x+3)^2(x-5)$.

3.4.3

Sketch a graph of $f(x)=\frac{1}{6}(x-1)^3(x+2)(x+3)$.

Answer

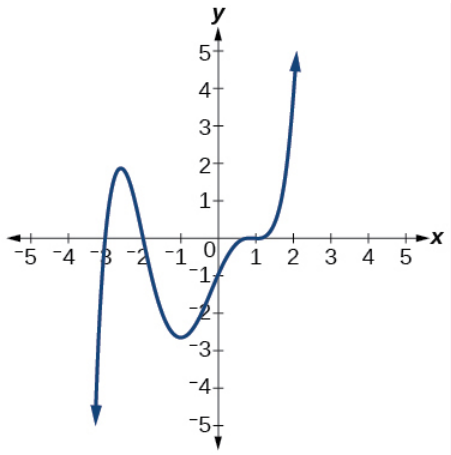


***Figure* 3.4.17:** *Graph of* $f(x) = \frac{1}{6}(x-1)^3(x+2)(x+3)$

Using the Intermediate Value Theorem

In some situations, we may know two points on a graph but not the zeros. If those two points are on opposite sides of the x-axis, we can confirm that there is a zero between them. Consider a polynomial function f whose graph is smooth and continuous. The **Intermediate Value Theorem** states that for two numbers a and b in the domain of f, if $a < b$ and $f(a) \neq f(b)$,then the function f takes on every value between $f(a)$ and $f(b)$. We can apply this theorem to a special case that is useful in graphing polynomial functions. If a point on the graph of a continuous function f at $x = a$ lies above the x-axis and another point at $x = b$ lies below the x-axis, there must exist a third point between $x = a$ and $x = b$ where the graph crosses the x-axis. Call this point $(c, f(c))$.This means that we are assured there is a solution c where $f(c) = 0$.

In other words, the **Intermediate Value Theorem** tells us that when a polynomial function changes from a negative value to a positive value, the function must cross the x-axis. Figure 3.4.18 shows that there is a zero between a and b.

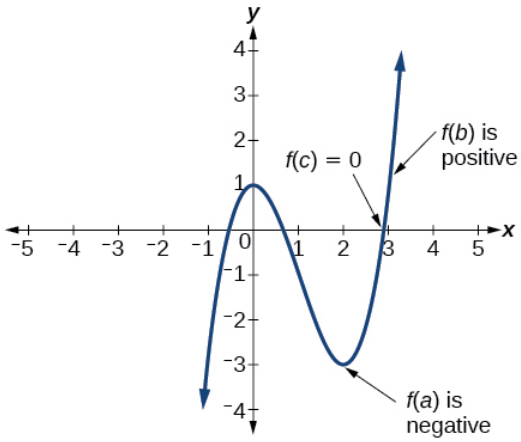


***Figure* 3.4.18:** *Using the Intermediate Value Theorem to show there exists a zero.*

Intermediate Value Theorem

Let f be a polynomial function. The **Intermediate Value Theorem** states that if $f(a)$ and $f(b)$ have opposite signs, then there exists at least one value c between a and b for which $f(c) = 0$.

Example 3.4.9: Using the Intermediate Value Theorem

Show that the function $f(x) = x^3 - 5x^2 + 3x + 6$ has at least two real zeros between $x = 1$ and $x = 4$.

Solution

As a start, evaluate $f(x)$ at the integer values $x = 1,\ 2,\ 3,$ and 4.

See Table 3.4.2.

Table 3.4.2

x	4
$f(x)$	3

We see that one zero occurs at $x = 2$. Also, since $f(3)$ is negative and $f(4)$ is positive, by the Intermediate Value Theorem, there must be at least one real zero between 3 and 4.

We have shown that there are at least two real zeros between $x = 1$ and $x = 4$.

Analysis

We can also see on the graph of the function in Figure 3.4.19 that there are two real zeros between $x = 1$ and $x = 4$.

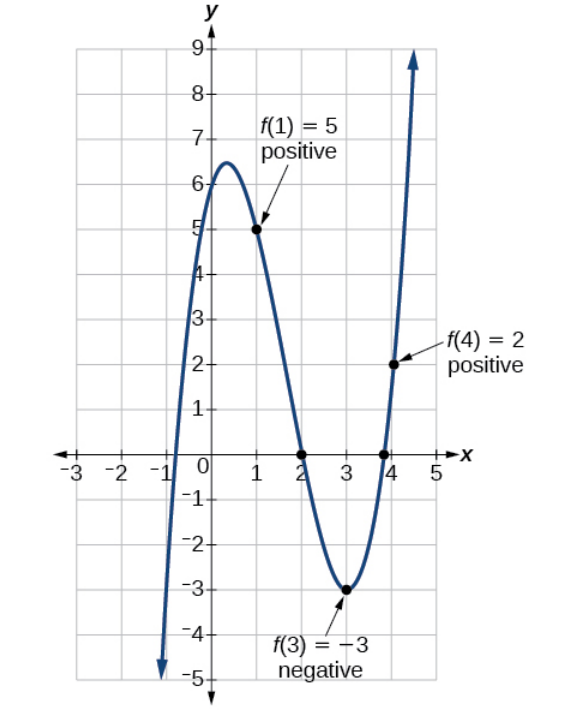


Figure 3.4.19.

3.4.4: Show that the function $f(x) = 7x^5 - 9x^4 - x^2$ has at least one real zero between $x = 1$ and $x = 2$.

Answer

Because f is a polynomial function and since $f(1)$ is negative and $f(2)$ is positive, there is at least one real zero between $x = 1$ and $x = 2$.

Writing Formulas for Polynomial Functions

Now that we know how to find zeros of polynomial functions, we can use them to write formulas based on graphs. Because a **polynomial function** written in factored form will have an x-intercept where each factor is equal to zero, we can form a function that will pass through a set of x-intercepts by introducing a corresponding set of factors.

Factored Form of Polynomials

The polynomial of lowest degree p that has horizontal intercepts at $x = x_1, x_2, \ldots, x_n$ can be written in the factored form: $f(x) = a(x - x_1)^{p_1}(x - x_2)^{p_2} \cdots (x - x_n)^{p_n}$ where the powers p_i on each factor can be determined by the behavior of the graph at the

corresponding intercept, and the stretch factor a can be determined given a value of the function other than an x-intercept.

How To: Given a graph of a polynomial function, write a possible formula for the function.

1. Identify the x-intercepts of the graph to find the factors of the polynomial.
2. Examine the behavior of the graph at the x-intercepts to determine the multiplicity of each factor.
3. Find the polynomial of least degree containing all the factors found in the previous step.
4. Use any other point on the graph (the y-intercept may be easiest) to determine the stretch factor.

Example 3.4.10: Writing a Formula for a Polynomial Function from the Graph

Write a formula for the polynomial function shown in Figure 3.4.20.

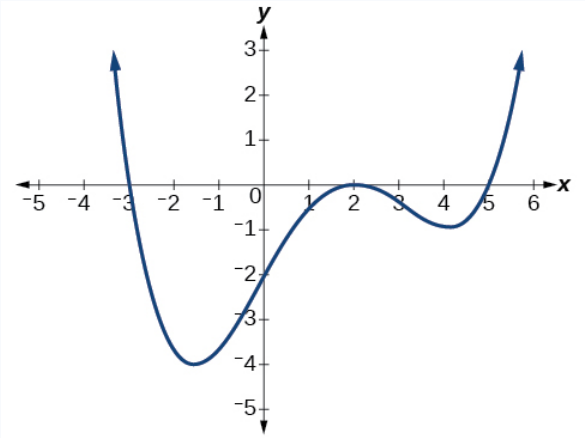


***Figure* 3.4.20.**

Solution

This graph has three x-intercepts: $x = -3,\ 2,$ and 5 and three turning points. The y-intercept is located at $(0, -2)$. At $x = -3$ and $x = 5$, the graph passes through the axis linearly, suggesting the corresponding factors of the polynomial will be linear. At $x = 2$, the graph bounces at the intercept, suggesting the corresponding factor of the polynomial could be second degree (quadratic). Thus, this is the graph of a polynomial of degree at least 5. Together, this gives us the possibility that

$$f(x) = a(x+3)(x-2)^2(x-5) \tag{3.4.1}$$

To determine the stretch factor, we utilize another point on the graph. We will use the y-intercept $(0, -2)$, to solve for a.

$$f(0) = a(0+3)(0-2)^2(0-5) \tag{3.4.2}$$
$$-2 = a(0+3)(0-2)^2(0-5) \tag{3.4.3}$$
$$-2 = -60a \tag{3.4.4}$$
$$a = \frac{1}{30} \tag{3.4.5}$$

The graphed polynomial appears to represent the function $f(x) = \frac{1}{30}(x+3)(x-2)^2(x-5)$.

Try It 3.4.5: Given the graph shown in Figure 3.4.21, write a formula for the function shown.

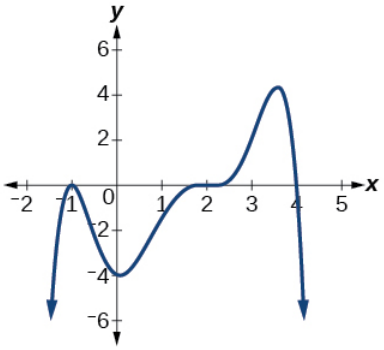


***Figure* 3.4.21.**

Answer

$f(x) = -\frac{1}{8}(x-2)^3(x+1)^2(x-4)$

Using Local and Global Extrema

With quadratics, we were able to algebraically find the maximum or minimum value of the function by finding the vertex. For general polynomials, finding these turning points is not possible without more advanced techniques from calculus. Even then, finding where extrema occur can still be algebraically challenging. For now, we will estimate the locations of turning points using technology to generate a graph.

Each turning point represents a local minimum or maximum. Sometimes, a turning point is the highest or lowest point on the entire graph. In these cases, we say that the turning point is a **global maximum** or a **global minimum**. These are also referred to as the absolute maximum and absolute minimum values of the function.

Local and Global Extrema

A **local maximum** or **local minimum** at $x = a$ (sometimes called the relative maximum or minimum, respectively) is the output at the highest or lowest point on the graph in an open interval around $x = a$.If a function has a local maximum at a, then $f(a) \geq f(x)$for all x in an open interval around $x = a$. If a function has a local minimum at a, then $f(a) \leq f(x)$for all x in an open interval around $x = a$.

A **global maximum** or **global minimum** is the output at the highest or lowest point of the function. If a function has a global maximum at a, then $f(a) \geq f(x)$ for all x. If a function has a global minimum at a, then $f(a) \leq f(x)$ for all x.

We can see the difference between local and global extrema in Figure **3.4.22**.

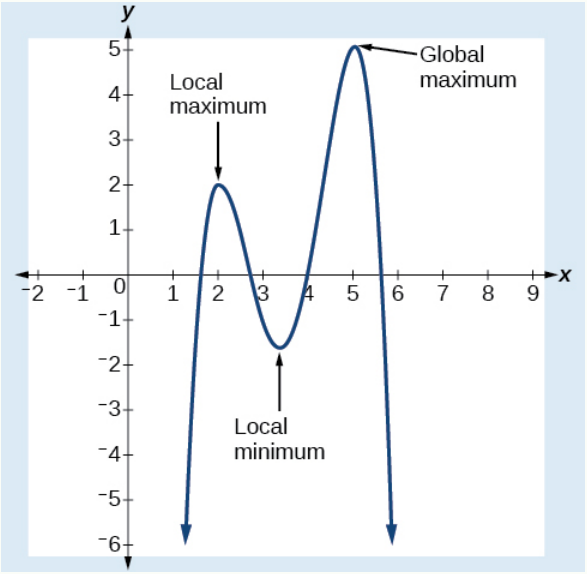


***Figure* 3.4.22:** *Graph of an even-degree polynomial that denotes the local maximum and minimum and the global maximum.*

Do all polynomial functions have a global minimum or maximum?

No. Only polynomial functions of even degree have a global minimum or maximum. For example, $f(x) = x$ has neither a global maximum nor a global minimum.

Example 3.4.11: Using Local Extrema to Solve Applications

An open-top box is to be constructed by cutting out squares from each corner of a 14 cm by 20 cm sheet of plastic then folding up the sides. Find the size of squares that should be cut out to maximize the volume enclosed by the box.

Solution

We will start this problem by drawing a picture like that in Figure 3.4.23, labeling the width of the cut-out squares with a variable,w.

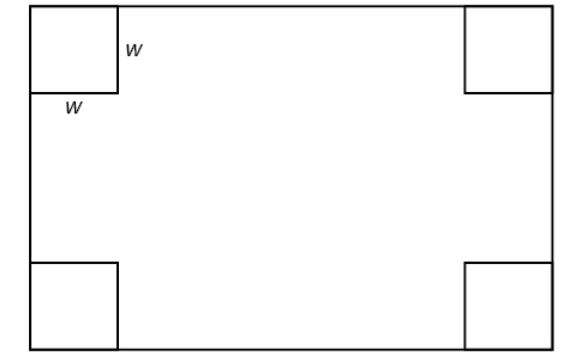


***Figure* 3.4.23:** *Diagram of a rectangle with four squares at the corners.*

Notice that after a square is cut out from each end, it leaves a $(14 - 2w)$ cm by $(20 - 2w)$ cm rectangle for the base of the box, and the box will be w cm tall. This gives the volume

$$V(w) = (20 - 2w)(14 - 2w)w \tag{3.4.6}$$
$$= 280w - 68w^2 + 4w^3 \tag{3.4.7}$$

Notice, since the factors are w, $20–2w$ and $14–2w$, the three zeros are $x = 10, 7$, and 0, respectively. Because a height of 0 cm is not reasonable, we consider only the zeros 10 and 7. The shortest side is 14 and we are cutting off two squares, so values w may take on are greater than zero or less than 7. This means we will restrict the domain of this function to $0 < w < 7$.Using technology to sketch the graph of $V(w)$ on this reasonable domain, we get the graph shown in Figure 3.4.24. We can use this graph to estimate the maximum value for the volume, restricted to values for w that are reasonable for this problem—values from 0 to 7.

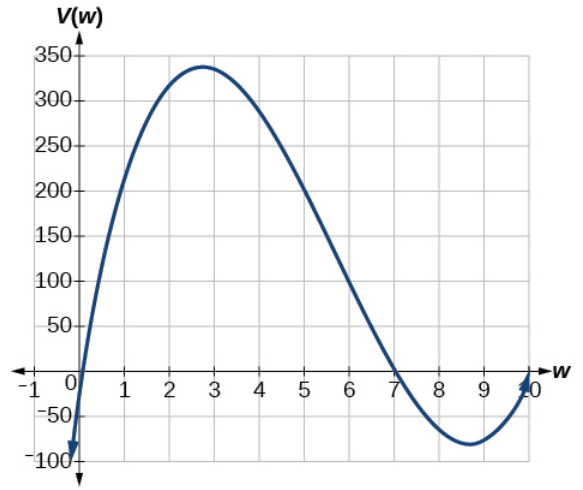


Figure 3.4.24: *Graph of* $V(w) = (20 - 2w)(14 - 2w)w$

From this graph, we turn our focus to only the portion on the reasonable domain, $[0, 7]$. We can estimate the maximum value to be around 340 cubic cm, which occurs when the squares are about 2.75 cm on each side. To improve this estimate, we could use advanced features of our technology, if available, or simply change our window to zoom in on our graph to produce Figure **3.4.25**.

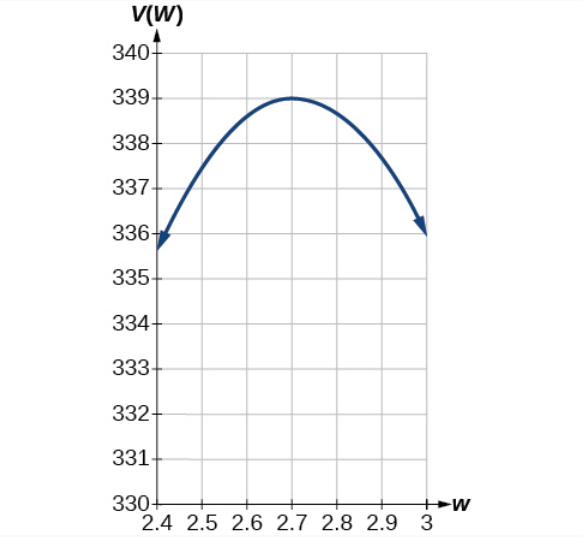


Figure 3.4.25: *Graph of* $V(w) = (20 - 2w)(14 - 2w)w$.

From this zoomed-in view, we can refine our estimate for the maximum volume to about 339 cubic cm, when the squares measure approximately 2.7 cm on each side.

3.4.6: Use technology to find the maximum and minimum values on the interval $[-1, 4]$ of the function $f(x) = -0.2(x - 2)^3(x + 1)^2(x - 4)$.

Answer

The minimum occurs at approximately the point $(0, -6.5)$,
and the maximum occurs at approximately the point $(3.5, 7)$.

Key Concepts

- Polynomial functions of degree 2 or more are smooth, continuous functions.
- To find the zeros of a polynomial function, if it can be factored, factor the function and set each factor equal to zero.

- Another way to find the x-intercepts of a polynomial function is to graph the function and identify the points at which the graph crosses the x-axis.
- The multiplicity of a zero determines how the graph behaves at the x-intercepts.
- The graph of a polynomial will cross the horizontal axis at a zero with odd multiplicity.
- The graph of a polynomial will touch the horizontal axis at a zero with even multiplicity.
- The end behavior of a polynomial function depends on the leading term.
- The graph of a polynomial function changes direction at its turning points.
- A polynomial function of degree n has at most $n-1$ turning points.
- To graph polynomial functions, find the zeros and their multiplicities, determine the end behavior, and ensure that the final graph has at most $n-1$ turning points.
- Graphing a polynomial function helps to estimate local and global extremas.
- The Intermediate Value Theorem tells us that if $f(a)$ and $f(b)$ have opposite signs, then there exists at least one value c between a and b for which $f(c)=0$.

Glossary

global maximum
highest turning point on a graph; $f(a)$ where $f(a) \geq f(x)$ for all x.

global minimum
lowest turning point on a graph; $f(a)$ where $f(a) \leq f(x)$ for all x.

Intermediate Value Theorem
for two numbers a and b in the domain of f, if $a<b$ and $f(a) \neq f(b)$, then the function f takes on every value between $f(a)$ and $f(b)$; specifically, when a polynomial function changes from a negative value to a positive value, the function must cross the x-axis

multiplicity
the number of times a given factor appears in the factored form of the equation of a polynomial; if a polynomial contains a factor of the form $(x-h)^p$, $x=h$ is a zero of multiplicity p.

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3.4E: Exercises

Section Exercises

Verbal

1. What is the difference between an x-intercept and a zero of a polynomial function f?

Answer: The x-intercept is where the graph of the function crosses thex-axis, and the zero of the function is the input value for which $f(x) = 0$.

2. If a polynomial function of degree n has n distinct zeros, what do you know about the graph of the function?

3. Explain how the Intermediate Value Theorem can assist us in finding a zero of a function.

Answer: If we evaluate the function at a and at b and the sign of the function value changes, then we know a zero exists between a and b.

4. Explain how the factored form of the polynomial helps us in graphing it.

5. If the graph of a polynomial just touches the x-axis and then changes direction, what can we conclude about the factored form of the polynomial?

Answer: There will be a factor raised to an even power.

Algebraic

For the following exercises, find the x-or t-intercepts of the polynomial functions.

6. $C(t) = 2(t-4)(t+1)(t-6)$

7. $C(t) = 3(t+2)(t-3)(t+5)$

Answer: $(-2,0), (3,0), (-5,0)$

8. $C(t) = 4t(t-2)^2(t+1)$

9. $C(t) = 2t(t-3)(t+1)^2$

Answer: $(3,0), (-1,0), (0,0)$

10. $C(t) = 2t^4 - 8t^3 + 6t^2$

11. $C(t) = 4t^4 + 12t^3 - 40t^2$

Answer: $(0,0), (-5,0), (2,0)$

12. $f(x) = x^4 - x^2$

13. $f(x) = x^3 + x^2 - 20x$

Answer: $(0,0), (-5,0), (4,0)$

14. $f(x) = x^3 + 6x^2 - 7x$

Exercise 3.4E.15

$f(x) = x^3 + x^2 - 4x - 4$

Answer

$(2,0), (-2,0), (-1,0)$

16. $f(x) = x^3 + 2x^2 - 9x - 18$

17. $f(x) = 2x^3 - x^2 - 8x + 4$

Answer: $(-2,0), (2,0), (\frac{1}{2},0)$

18. $f(x) = x^6 - 7x^3 - 8$

19. $f(x) = 2x^4 + 6x^2 - 8$

Answer: $(1, 0), (-1, 0)$

20. $f(x) = x^3 - 3x^2 - x + 3$

21. $f(x) = x^6 - 2x^4 - 3x^2$

Answer: $(0, 0), (\sqrt{3}, 0), (-\sqrt{3}, 0)$

22. $f(x) = x^6 - 3x^4 - 4x^2$

23. $f(x) = x^5 - 5x^3 + 4x$

Answer: $(0, 0), (1, 0), (-1, 0), (2, 0), (-2, 0)$

For the following exercises, use the Intermediate Value Theorem to confirm that the given polynomial has at least one zero within the given interval.

---

24. $f(x) = x^3 - 9x$, between $x = -4$ and $x = -2$.

25. $f(x) = x^3 - 9x$, between $x = 2$ and $x = 4$.

Answer: $f(2) = -10$ and $f(4) = 28$. Sign change confirms.

26. $f(x) = x^5 - 2x$, between $x = 1$ and $x = 2$.

27. $f(x) = -x^4 + 4$, between $x = 1$ and $x = 3$.

Answer: $f(1) = 3$ and $f(3) = -77$. Sign change confirms.

28. $f(x) = -2x^3 - x$, between $x = -1$ and $x = 1$.

Exercise 3.4E.29

$f(x) = x^3 - 100x + 2$, between $x = 0.01$ and $x = 0.1$

Answer

$f(0.01) = 1.000001$ and $f(0.1) = -7.999$. Sign change confirms.

For the following exercises, find the zeros and give the multiplicity of each.

---

30. $f(x) = (x + 2)^3(x - 3)^2$

31. $f(x) = x^2(2x + 3)^5(x - 4)^2$

Answer: 0 with multiplicity 2,$-\frac{3}{2}$ with multiplicity 5, 4 with multiplicity 2

32. $f(x) = x^3(x - 1)^3(x + 2)$

33. $f(x) = x^2(x^2 + 4x + 4)$

Answer: 0 with multiplicity 2, –2 with multiplicity 2

34. $f(x) = (2x + 1)^3(9x^2 - 6x + 1)$

Exercise 3.4E.35

$f(x) = (3x + 2)^5(x^2 - 10x + 25)$

Answer

$-\frac{2}{3}$ with multiplicity 5,5 with multiplicity 2

36. $f(x) = x(4x^2 - 12x + 9)(x^2 + 8x + 16)$

37. $f(x) = x^6 - x^5 - 2x^4$

Answer: 0 with multiplicity 4,2 with multiplicity 1,–1 with multiplicity 1

38. $f(x) = 3x^4 + 6x^3 + 3x^2$

39. $f(x) = 4x^5 - 12x^4 + 9x^3$

Answer: $\frac{3}{2}$ with multiplicity 2, 0 with multiplicity 3

40. $f(x) = 2x^4(x^3 - 4x^2 + 4x)$

41. $f(x) = 4x^4(9x^4 - 12x^3 + 4x^2)$

Answer: 0 with multiplicity 6, \frac{2}{3}\) with multiplicity 2

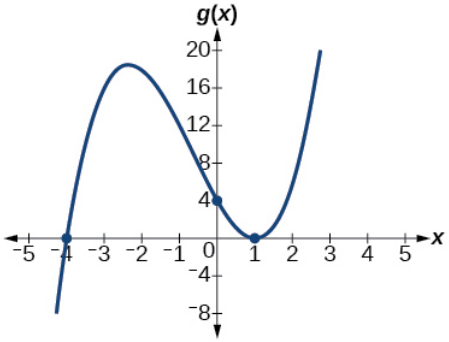
Graphical

For the following exercises, graph the polynomial functions. Note x- and y-intercepts, multiplicity, and end behavior.

42. $f(x) = (x + 3)^2(x - 2)$

43. $g(x) = (x + 4)(x - 1)^2$

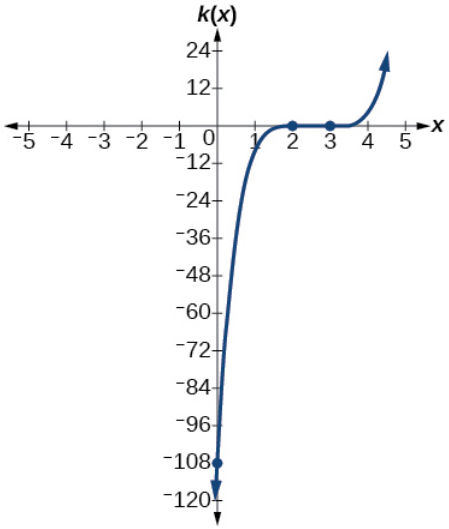
Answer: x-intercepts, $(1, 0)$ with multiplicity 2, $(-4, 0)$ with multiplicity 1, y- intercept $(0, 4)$. As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to \infty$.



44. $h(x) = (x - 1)^3(x + 3)^2$

45. $k(x) = (x - 3)^3(x - 2)^2$

Answer: x-intercepts $(3, 0)$ with multiplicity 3, $(2, 0)$ with multiplicity 2, y- intercept $(0, -108)$. As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to \infty$.



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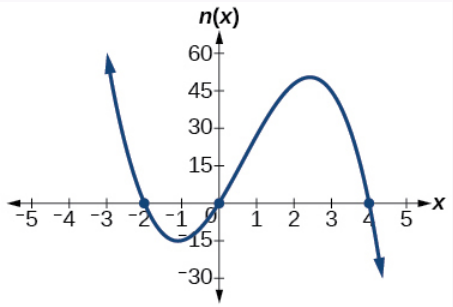
46. $m(x) = -2x(x-1)(x+3)$

Exercise 3.4E.47

$n(x) = -3x(x+2)(x-4)$

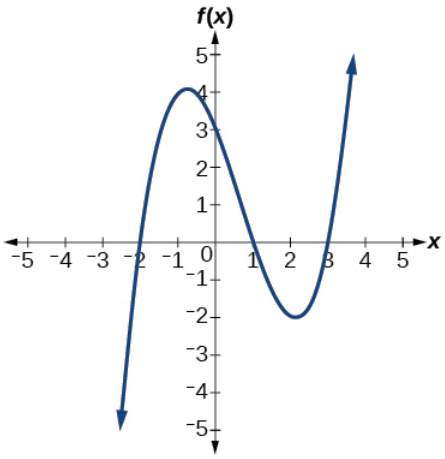
Answer

x-intercepts $(0,0), (-2,0), (4,0)$ with multiplicity 1, y-intercept (0, 0). As $x \to -\infty$, $f(x) \to \infty$, as $x \to \infty$, $f(x) \to -\infty$.

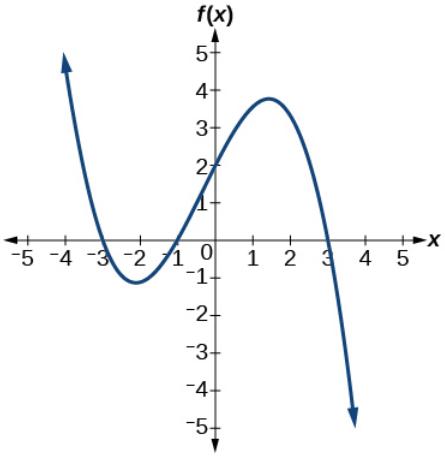


For the following exercises, use the graphs to write the formula for a polynomial function of least degree.

48.

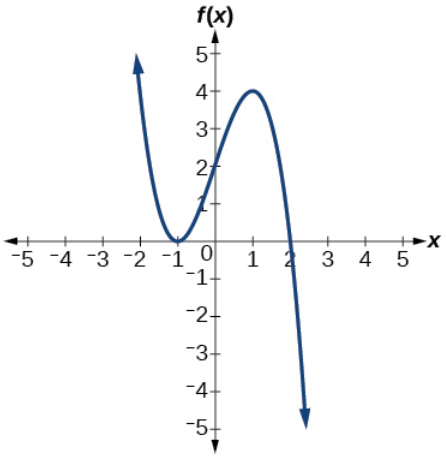


49.

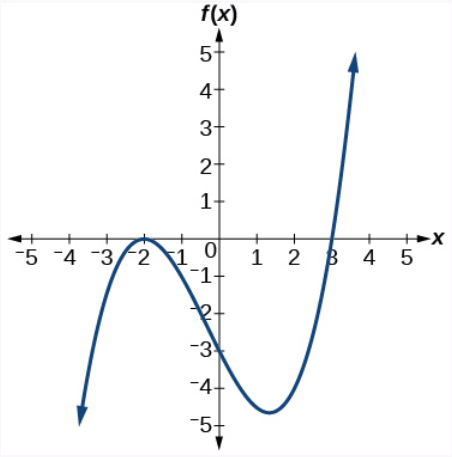


Answer: $f(x) = -\frac{2}{9}(x-3)(x+1)(x+3)$

50.



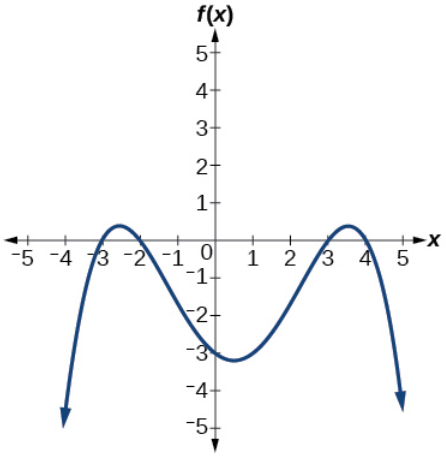
Exercise 3.4E.51



Answer

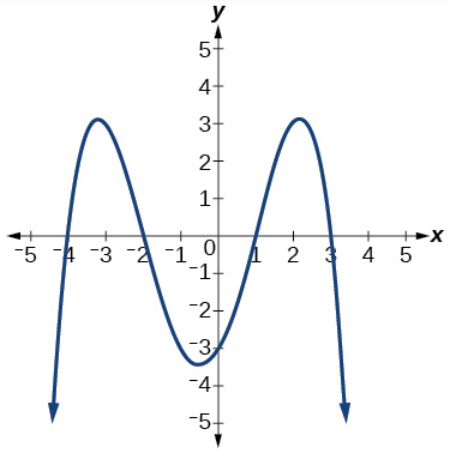
$f(x) = \frac{1}{4}(x+2)^2(x-3)$

52.



For the following exercises, use the graph to identify zeros and multiplicity.

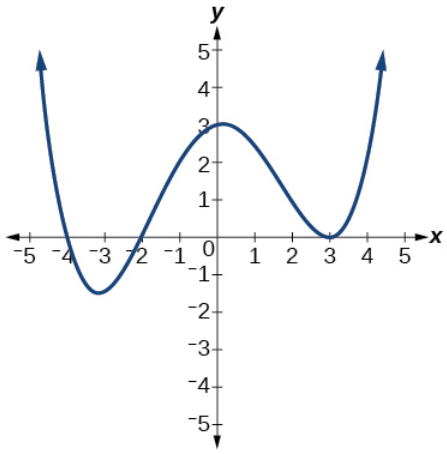
Exercise 3.4E.53



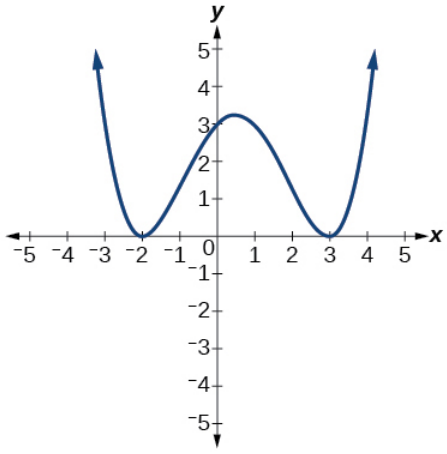
Answer

–4, –2, 1, 3 with multiplicity 1

54.

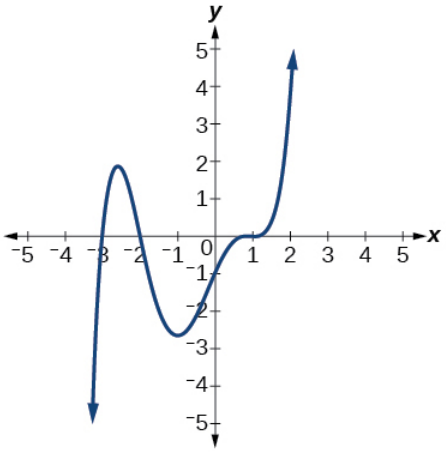


55.



Answer: –2, 3 each with multiplicity 2

56.



For the following exercises, use the given information about the polynomial graph to write the equation.

---

57. Degree 3. Zeros at $x = -2, x = 1$, and $x = 3$. y-intercept at $(0, -4)$.

Answer: $f(x) = -\frac{2}{3}(x+2)(x-1)(x-3)$

58. Degree 3. Zeros at $x = -5$, $x = -2$, and $x = 1$. y-intercept at $(0, 6)$

Exercise 3.4E.59

Degree 5. Roots of multiplicity 2 at $x = 3$ and $x = 1$, and a root of multiplicity 1 at $x = -3$. y-intercept at $(0, 9)$

Answer

$f(x) = \frac{1}{3}(x-3)^2(x-1)^2(x+3)$

60. Degree 4. Root of multiplicity 2 at $x = 4$, and a roots of multiplicity 1 at $x = 1$ and $x = -2$. y-intercept at $(0, -3)$.

61. Degree 5. Double zero at $x = 1$, and triple zero at $x = 3$. Passes through the point $(2, 15)$.

Answer: $f(x) = -15(x-1)^2(x-3)^3$

62. Degree 3. Zeros at $x = 4$, $x = 3$, and $x = 2$. y-intercept at $(0, -24)$.

63. Degree 3. Zeros at $x = -3$, $x = -2$ and $x = 1$. y-intercept at $(0, 12)$.

Answer: $f(x) = -2(x+3)(x+2)(x-1)$

64. Degree 5. Roots of multiplicity 2 at $x = -3$ and $x = 2$ and a root of multiplicity 1 at $x = -2$. y-intercept at $(0, 4)$.

65. Degree 4. Roots of multiplicity 2 at $x = \frac{1}{2}$
and roots of multiplicity 1 at $x = 6$ and $x = -2$. y-intercept at $(0, 18)$.

Answer: $f(x) = -\frac{3}{2}(2x-1)^2(x-6)(x+2)$

66. Double zero at $x = -3$ and triple zero at $x = 0$. Passes through the point $(1, 32)$.

Technology

For the following exercises, use a calculator to approximate local minima and maxima or the global minimum and maximum.

67. $f(x) = x^3 - x - 1$

Answer: local max $(-.58, -.62)$, local min $(.58, -1.38)$

68. $f(x) = 2x^3 - 3x - 1$

69. $f(x) = x^4 + x$

Answer: global min $(-.63, -.47)$

70. $f(x) = -x^4 + 3x - 2$

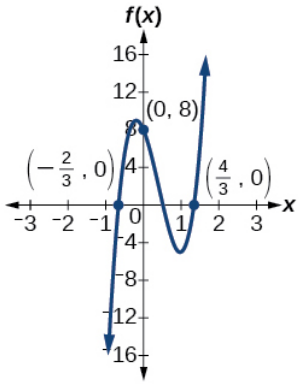
71. $f(x) = x^4 - x^3 + 1$

Answer: global min $(.75, .89)$

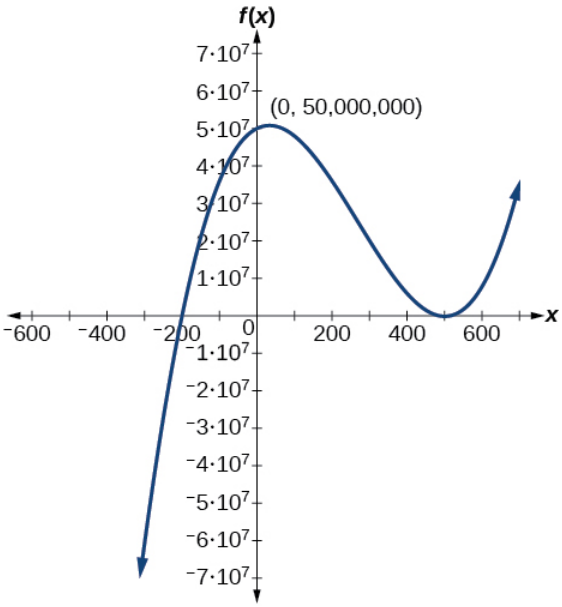
Extensions

For the following exercises, use the graphs to write a polynomial function of least degree.

72.



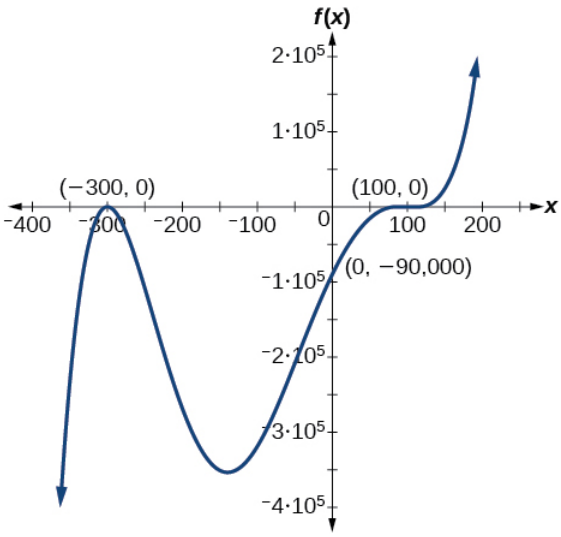
73.



Answer: $f(x) = (x - 500)^2(x + 200)$

74.

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Real-World Applications

For the following exercises, write the polynomial function that models the given situation.

Exercise 3.4E.75

A rectangle has a length of 10 units and a width of 8 units. Squares of x by x units are cut out of each corner, and then the sides are folded up to create an open box. Express the volume of the box as a polynomial function in terms of x.

Answer

$f(x) = 4x^3 - 36x^2 + 80x$

76. Consider the same rectangle of the preceding problem. Squares of $2x$ by $2x$ units are cut out of each corner. Express the volume of the box as a polynomial in terms of x.

77. A square has sides of 12 units. Squares $x+1$ by $x+1$ units are cut out of each corner, and then the sides are folded up to create an open box. Express the volume of the box as a function in terms of x.

Answer: $f(x) = 4x^3 - 36x^2 + 60x + 100$

78. A cylinder has a radius of $x+2$ units and a height of 3 units greater. Express the volume of the cylinder as a polynomial function.

79. A right circular cone has a radius of $3x+6$ and a height 3 units less. Express the volume of the cone as a polynomial function. The volume of a cone is $V = \frac{1}{3}\pi r^2 h$ for radius r and height h.

Answer: $f(x) = \pi(9x^3 + 45x^2 + 72x + 36)$

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3.5: Dividing Polynomials

Learning Objectives

- Use long division to divide polynomials.
- Use synthetic division to divide polynomials.

Work in Progesss - do not read yet! (maybe take out all worked examples and hand out new ones in class) The exterior of the Lincoln Memorial in Washington, D.C., is a large rectangular solid with length 61.5 meters (m), width 40 m, and height 30 m.1 We can easily find the volume using elementary geometry.

$$\begin{aligned} V &= l \cdot w \cdot h \\ &= 61.5 \cdot 40 \cdot 30 \\ &= 73,800 \end{aligned}$$

So the volume is 73,800 cubic meters (m^3).



Figure **3.5.1**: Lincoln Memorial, Washington, D.C. (credit: Ron Cogswell, Flickr)

Suppose we knew the volume, length, and width. We could divide to find the height.

$$\begin{aligned} h &= \frac{V}{l \cdot w} \\ &= \frac{73,800}{61.5 \cdot 40} \\ &= 30 \end{aligned}$$

As we can confirm from the dimensions above, the height is 30 m. We can use similar methods to find any of the missing dimensions. We can also use the same method if any or all of the measurements contain variable expressions. For example, suppose the volume of a rectangular solid is given by the polynomial $3x^4 - 3x^3 - 33x^2 + 54x$. The length of the solid is given by $3x$; the width is given by $x - 2$.

To find the height of the solid, we can use polynomial division, which is the focus of this section.

Using Long Division to Divide Polynomials

We are familiar with the **long division** algorithm for ordinary arithmetic. We begin by dividing into the digits of the dividend that have the greatest place value. We divide, multiply, subtract, include the digit in the next place value position, and repeat. For example, let's divide 178 by 3 using long division. **Needs Work - replace!**

Long Division

$$\begin{array}{r} 59 \\ 3\overline{)178} \\ \underline{-15\downarrow} \\ 28 \\ \underline{-27} \\ 1 \end{array}$$

Step 1: $5 \times 3 = 15$ and $17 - 15 = 2$
Step 2: Bring down the 8
Step 3: $9 \times 3 = 27$ and $28 - 27 = 1$

Answer: $59\,R\,1$ or $59\frac{1}{3}$

Another way to look at the solution is as a sum of parts. This should look familiar, since it is the same method used to check division in elementary arithmetic.

$$\begin{aligned}\text{dividend} &= (\text{divisor}\cdot\text{quotient}) + \text{remainder}\\ 178 &= (3\cdot 59) + 1\\ &= 177 + 1\\ &= 178\end{aligned}$$

We call this the **Division Algorithm** and will discuss it more formally after looking at an example.

Division of polynomials that contain more than one term has similarities to long division of whole numbers. We can write a polynomial dividend as the product of the divisor and the quotient added to the remainder. The terms of the polynomial division correspond to the digits (and place values) of the whole number division. This method allows us to divide two polynomials. For example, if we were to divide $2x^3 - 3x^2 + 4x + 5$ by $x + 2$ using the long division algorithm, it would look like this: **Needs work - replace!**

$x+2\overline{)2x^3-3x^2+4x+5}$	Set up the division problem.
$\quad 2x^2$	
$x+2\overline{)2x^3-3x^2+4x+5}$	$2x^3$ divided by x is $2x^2$.
$\quad 2x^2$	
$x+2\overline{)2x^3-3x^2+4x+5}$	Multiply $x+2$ by $2x^2$.
$\quad -(2x^3+4x^2)$	
$\quad\quad -7x^2+4x$	Subtract.
$\quad\quad\ 2x^2-7x$	Bring down the next term.
$x+2\overline{)2x^3-3x^2+4x+5}$	$-7x^2$ divided by x is $-7x$.
$\quad -(2x^3+4x^2)$	
$\quad\quad -7x^2+4x$	
$\quad\quad -(-7x^2+14x)$	Multiply $x+2$ by $-7x$.
$\quad\quad\quad\ 18x+5$	Subtract. Bring down the next term.
$\quad\quad\quad 2x^2-7x+18$	
$x+2\overline{)2x^3-3x^2+4x+5}$	$18x$ divided by x is 18.
$\quad -(2x^3+4x^2)$	
$\quad\quad -7x^2+4x$	
$\quad\quad -(-7x^2+14x)$	
$\quad\quad\quad\ 18x+5$	
$\quad\quad\quad -18x+36$	Multiply $x+2$ by 18.
$\quad\quad\quad\quad -31$	Subtract.

We have found

$$\frac{2x^3-3x^2+4x+5}{x+2} = 2x^2-7x+18-\frac{31}{x+2} \tag{3.5.1}$$

or

$$2x^3-3x^2+4x+5 = (x+2)(2x^2-7x+18)-31 \tag{3.5.2}$$

We can identify the dividend, the **divisor**, the **quotient**, and the **remainder**.

Writing the result in this manner illustrates the Division Algorithm.

The Division Algorithm

The Division Algorithm states that, given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x)q(x) + r(x) \tag{3.5.3}$$

$q(x)$ is the quotient and $r(x)$ is the remainder. The remainder is either equal to zero or has degree strictly less than $d(x)$.

If $r(x) = 0$, then $d(x)$ divides evenly into $f(x)$. This means that, in this case, both $d(x)$ and $q(x)$ are factors of $f(x)$.

How To: Given a polynomial and a binomial, use long division to divide the polynomial by the binomial

1. Set up the division problem.
2. Determine the first term of the quotient by dividing the leading term of the dividend by the leading term of the divisor.
3. Multiply the answer by the divisor and write it below the like terms of the dividend.
4. Subtract the bottom **binomial** from the top binomial.
5. Bring down the next term of the dividend.
6. Repeat steps 2–5 until reaching the last term of the dividend.
7. If the remainder is non-zero, express as a fraction using the divisor as the denominator.

Example 3.5.1: Using Long Division to Divide a Second-Degree Polynomial

Divide $5x^2 + 3x - 2$ by $x + 1$. **Needs Work - more space between steps! (replace!)**

Solution

$$
\begin{array}{rl}
x+1\,\overline{)\,5x^2+3x-2} & \text{Set up division problem.} \\
\quad 5x & \\
x+1\,\overline{)\,5x^2+3x-2} & 5x^2 \text{ divided by } x \text{ is } 5x. \\
\quad 5x & \\
x+1\,\overline{)\,5x^2+3x-2} & \text{Multiply } x+1 \text{ by } 5x. \\
\quad -(5x^2+5x) & \\
\quad\quad -2x-2 & \text{Subtract.} \\
\quad\quad\quad 5x-2 & \text{Bring down the next term.} \\
x+1\,\overline{)\,5x^2+3x-2} & -2x \text{ divided by } x \text{ is } -2. \\
\quad -(5x^2+5x) & \\
\quad\quad -2x-2 & \\
\quad\quad -(-2x-2) & \text{Multiply } x+1 \text{ by } -2. \\
\quad\quad\quad 0 & \text{Subtract.}
\end{array}
$$

The quotient is $5x - 2$. The remainder is 0. We write the result as

$$\frac{5x^2 + 3x - 2}{x + 1} = 5x - 2 \tag{3.5.4}$$

or

$$5x^2 + 3x - 2 = (x + 1)(5x - 2) \tag{3.5.5}$$

Analysis

This division problem had a remainder of 0. This tells us that the dividend is divided evenly by the divisor, and that the divisor is a factor of the dividend.

Example 3.5.2: Using Long Division to Divide a Third-Degree Polynomial

Divide $6x^3 + 11x^2 - 31x + 15$ by $3x - 2$. **Needs work - + sign in line 5 should be a minus! 21x in next line should be -21x! constant term of dividend should be 15! (screen shot)**

Solution

$$
\begin{array}{rl}
\quad\quad 2x^2 + 5x - 7 & \\
3x - 2\overline{)\,6x^3 + 11x^2 - 31x + 1} & \\
\quad -(6x^3 - 4x^2) & \\
\hline
\quad\quad 15x^2 - 31x & \\
\quad -(15x^2 + 10x) & \\
\hline
\quad\quad\quad 21x + 15 & \\
\quad\quad -(-21x + 14) & \\
\hline
\quad\quad\quad\quad 1 &
\end{array}
$$

$6x^3$ divided by $3x$ is $2x^2$.

Multiply $x + 2$ by $2x^2$.
Subtract. Bring down the next term. $15x^2$ divided by $3x$ is $5x$.
Multiply $3x - 2$ by $5x$.
Subtract. Bring down the next term. $-21x$ divided by $3x$ is -7.
Multiply $3x - 2$ by -7.
Subtract. The remainder is 1.

There is a remainder of 1. We can express the result as:

$$\frac{6x^3 + 11x^2 - 31x + 15}{3x - 2} = 2x^2 + 5x - 7 + \frac{1}{3x - 2} \tag{3.5.6}$$

Analysis

We can check our work by using the Division Algorithm to rewrite the solution. Then multiply.

$$(3x - 2)(2x^2 + 5x - 7) + 1 = 6x^3 + 11x^2 - 31x + 15 \tag{3.5.7}$$

Notice, as we write our result,

- the dividend is $6x^3 + 11x^2 - 31x + 15$
- the divisor is $3x - 2$
- the quotient is $2x^2 + 5x - 7$
- the remainder is 1

3.5.2

Divide $16x^3 - 12x^2 + 20x - 3$ by $4x + 5$.

Solution

$$4x^2 - 8x + 15 - \frac{78}{4x + 5}$$

Using Synthetic Division to Divide Polynomials

As we've seen, long division of polynomials can involve many steps and be quite cumbersome. **Synthetic division** is a shorthand method of dividing polynomials for the special case of dividing by a linear factor whose leading coefficient is 1.

To illustrate the process, recall the example at the beginning of the section.

Divide $2x^3 - 3x^2 + 4x + 5$ by $x + 2$ using the long division algorithm.

The final form of the process looked like this: **Make sure this matches example at beginning of section! (quotient is wrong here)**

$$
\begin{array}{r}
2x^2 + x + 18 \\
x+2\overline{)2x^3 - 3x^2 + 4x + 5} \\
\underline{-(2x^3 + 4x^2)} \\
-7x^2 + 4x \\
\underline{-(-7x^2 - 14x)} \\
18x + 5 \\
\underline{-(18x + 36)} \\
-31
\end{array}
$$

There is a lot of repetition in the table. If we don't write the variables but, instead, line up their coefficients in columns under the division sign and also eliminate the partial products, we already have a simpler version of the entire problem.

$$
\begin{array}{r|llll}
2 & 2 & -3 & 4 & 5 \\
 & -2 & -4 & & \\
\hline
 & & -7 & 14 & \\
 & & & 18 & -36 \\
\hline
 & & & & -31
\end{array}
$$

Synthetic division carries this simplification even a few more steps. Collapse the table by moving each of the rows up to fill any vacant spots. Also, instead of dividing by 2, as we would in division of whole numbers, then multiplying and subtracting the middle product, we change the sign of the "divisor" to –2, multiply and add. The process starts by bringing down the leading coefficient.

$$
\begin{array}{r|llll}
-2 & 2 & -3 & 4 & 5 \\
 & & -4 & 14 & -36 \\
\hline
 & 2 & -7 & 18 & -31
\end{array}
$$

We then multiply it by the "divisor" and add, repeating this process column by column, until there are no entries left. The bottom row represents the coefficients of the quotient; the last entry of the bottom row is the remainder. In this case, the quotient is $2x^2-7x + 18$ and the remainder is –31.The process will be made more clear in Example 3.5.3.

Synthetic Division

Synthetic division is a shortcut that can be used when the divisor is a binomial in the form $x - k$. In *synthetic division*, only the coefficients are used in the division process.

How To: Given two polynomials, use synthetic division to divide

1. Write k for the divisor.
2. Write the coefficients of the dividend.
3. Bring the lead coefficient down.
4. Multiply the lead coefficient by k. Write the product in the next column.
5. Add the terms of the second column.
6. Multiply the result by k. Write the product in the next column.
7. Repeat steps 5 and 6 for the remaining columns.
8. Use the bottom numbers to write the quotient. The number in the last column is the remainder and has degree 0, the next number from the right has degree 1, the next number from the right has degree 2, and so on.

Example 3.5.3: Using Synthetic Division to Divide a Second-Degree Polynomial

Use synthetic division to divide $5x^2 - 3x - 36$ by $x - 3$.

Solution

Begin by setting up the synthetic division. Write k and the coefficients.

3	5	−3	−36

Bring down the lead coefficient. Multiply the lead coefficient by k.

3	5	−3	−36
		15	
	5		

Continue by adding the numbers in the second column. Multiply the resulting number by k.Write the result in the next column. Then add the numbers in the third column.

3	5	−3	−36
		15	36
	5	12	0

The result is $5x + 12$. The remainder is 0. So $x - 3$ is a factor of the original polynomial.

Analysis

Just as with long division, we can check our work by multiplying the quotient by the divisor and adding the remainder.

$$(x - 3)(5x + 12) + 0 = 5x^2 - 3x - 36 \tag{3.5.8}$$

Example 3.5.4: Using Synthetic Division to Divide a Third-Degree Polynomial

Use synthetic division to divide $4x^3 + 10x^2 - 6x - 20$ by $x + 2$.

Solution

The binomial divisor is $x + 2$ so $k = -2$. Add each column, multiply the result by –2, and repeat until the last column is reached.

−2	1	-1	-11	18
		2	2	-18
	1	1	-9	0

The result is $4x^2 + 2x - 10$.

The remainder is 0. Thus, $x + 2$ is a factor of $4x^3 + 10x^2 - 6x - 20$.

Analysis

The graph of the polynomial function $f(x) = 4x^3 + 10x^2 - 6x - 20$ in Figure 3.5.2 shows a zero at $x = k = -2$. This confirms that $x + 2$ is a factor of $4x^3 + 10x^2 - 6x - 20$.

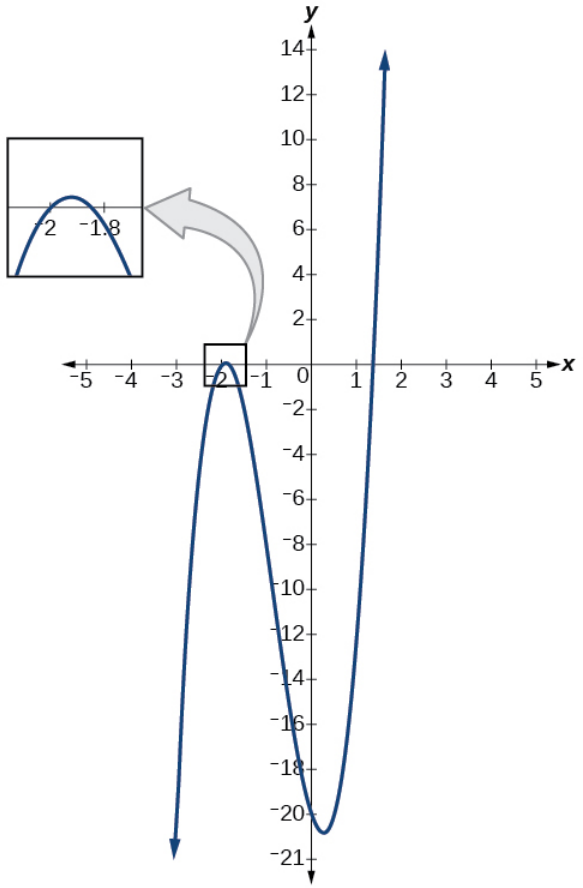


Figure 3.5.2

Example 3.5.5: Using Synthetic Division to Divide a Fourth-Degree Polynomial

Use synthetic division to divide $-9x^4 + 10x^3 + 7x^2 - 6$ by $x - 1$.

Solution

Notice there is no x-term. We will use a zero as the coefficient for that term.

$$\begin{array}{r|rrrrr} 1 & -9 & 10 & 7 & 0 & -6 \\ & & -9 & 1 & 8 & 8 \\ \hline & -9 & 1 & 8 & 8 & 2 \end{array}$$

The result is $-9x^3 + x^2 + 8x + 8 + \frac{2}{x-1}$.

3.5.5

Use synthetic division to divide $3x^4 + 18x^3 - 3x + 40$ by $x + 7$.

Solution

$3x^3 - 3x^2 + 21x - 150 + \frac{1,090}{x+7}$

Using Polynomial Division to Solve Application Problems

Polynomial division can be used to solve a variety of application problems involving expressions for area and volume. We looked at an application at the beginning of this section. Now we will solve that problem in the following example.

Example 3.5.6: Using Polynomial Division in an Application Problem

The volume of a rectangular solid is given by the polynomial $3x^4 - 3x^3 - 33x^2 + 54x$. The length of the solid is given by $3x$ and the width is given by $x - 2$. Find the height of the solid.

Solution

There are a few ways to approach this problem. We need to divide the expression for the volume of the solid by the expressions for the length and width. Let us create a sketch as in Figure 3.5.3.

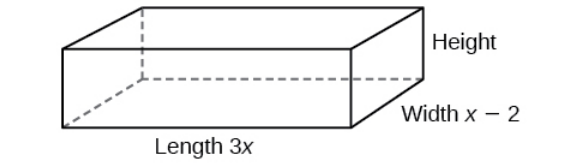


Figure 3.5.3

We can now write an equation by substituting the known values into the formula for the volume of a rectangular solid.

$$V = l \cdot w \cdot h$$
$$3x^4 - 3x^3 - 33x^2 + 54x = 3x \cdot (x - 2) \cdot h$$

To solve for h, first divide both sides by $3x$.

$$\frac{3x \cdot (x - 2) \cdot h}{3x} = \frac{3x^4 - 3x^3 - 33x^2 + 54x}{3x} \tag{3.5.9}$$

$$(x - 2)h = \frac{x^3 - x^2 - 11x + 18}{x - 2} \tag{3.5.10}$$

Now solve for h using synthetic division.

$$h = \frac{x^3 - x^2 - 11x + 18}{x - 2} \tag{3.5.11}$$

−2	1	-1	-11	18
		2	2	-18
	1	1	-9	0

The quotient is $x^2 + x - 9$ and the remainder is 0. The height of the solid is $x^2 + x - 9$.

3.5.6

The area of a rectangle is given by $3x^3 + 14x^2 - 23x + 6$. The width of the rectangle is given by $x + 6$. Find an expression for the length of the rectangle.

Solution

$3x^2 - 4x + 1$

Key Equations

Division Algorithm $f(x) = d(x)q(x) + r(x)$ where $q(x) \neq 0$

Key Concepts

- Polynomial long division can be used to divide a polynomial by any polynomial with equal or lower degree.
- The Division Algorithm tells us that a polynomial dividend can be written as the product of the divisor and the quotient added to the remainder.
- Synthetic division is a shortcut that can be used to divide a polynomial by a binomial in the form x−k.
- Polynomial division can be used to solve application problems, including area and volume.

Footnotes

1 National Park Service. "Lincoln Memorial Building Statistics." http://www.nps.gov/linc/historycultu...statistics.htm. Accessed 4/3/2014

Glossary

Division Algorithm

given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that $f(x) = d(x)q(x) + r(x)$ where $q(x)$ is the quotient and $r(x)$ is the remainder. The remainder is either equal to zero or has degree strictly less than $d(x)$.

synthetic division

a shortcut method that can be used to divide a polynomial by a binomial of the form $x - k$

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3.6: Zeros of Polynomial Functions

Skills to Develop

- Evaluate a polynomial using the Remainder Theorem.
- Use the Factor Theorem to solve a polynomial equation.
- Use the Rational Zero Theorem to find rational zeros.
- Find zeros of a polynomial function.
- Use the Linear Factorization Theorem to find polynomials with given zeros.
- Use Descartes' Rule of Signs.
- Solve real-world applications of polynomial equations

A new bakery offers decorated sheet cakes for children's birthday parties and other special occasions. The bakery wants the volume of a small cake to be 351 cubic inches. The cake is in the shape of a rectangular solid. They want the length of the cake to be four inches longer than the width of the cake and the height of the cake to be one-third of the width. What should the dimensions of the cake pan be?

This problem can be solved by writing a cubic function and solving a cubic equation for the volume of the cake. In this section, we will discuss a variety of tools for writing polynomial functions and solving polynomial equations.

Evaluating a Polynomial Using the Remainder Theorem

In the last section, we learned how to divide polynomials. We can now use polynomial division to evaluate polynomials using the **Remainder Theorem**. If the polynomial is divided by $x–k$, the remainder may be found quickly by evaluating the polynomial function at k, that is, $f(k)$. Let's walk through the proof of the theorem.

Recall that the Division Algorithm states that, given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$,there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x)q(x) + r(x) \tag{3.6.1}$$

If the divisor, $d(x)$, is $x - k$, this takes the form

$$f(x) = (x - k)q(x) + r \tag{3.6.2}$$

Since the divisor $x - k$

is linear, the remainder will be a constant, r. And, if we evaluate this for $x = k$, we have

$$\begin{aligned} f(k) &= (k - k)q(k) + r && (3.6.3)\\ &= 0 \cdot q(k) + r && (3.6.4)\\ &= r && (3.6.5) \end{aligned}$$

In other words, $f(k)$ is the remainder obtained by dividing $f(x)$by $x - k$.

The Remainder Theorem:

If a polynomial $f(x)$ is divided by $x - k$,then the remainder is the value $f(k)$.

How To: Given a polynomial function f, evaluate $f(x)$ at $x = k$ using the Remainder Theorem.

1. Use synthetic division to divide the polynomial by $x - k$.
2. The remainder is the value $f(k)$.

Example 3.6.1: Using the Remainder Theorem to Evaluate a Polynomial

Use the Remainder Theorem to evaluate $f(x) = 6x^4 - x^3 - 15x^2 + 2x - 7$ at $x = 2$.

Solution

To find the remainder using the Remainder Theorem, use synthetic division to divide the polynomial by $x - 2$.

$$\begin{array}{r|rrrrr} 2 & 6 & -1 & -15 & 2 & -7 \\ & & 12 & 22 & 14 & 32 \\ \hline & 6 & 11 & 7 & 16 & 25 \end{array} \tag{3.6.6}$$

The remainder is 25. Therefore, $f(2) = 25$.

Analysis

We can check our answer by evaluating $f(2)$.

$$f(x) = 6x^4 - x^3 - 15x^2 + 2x - 7 \tag{3.6.7}$$
$$f(2) = 6(2)^4 - (2)^3 - 15(2)^2 + 2(2) - 7 \tag{3.6.8}$$
$$= 25 \tag{3.6.9}$$

3.6.1: Use the Remainder Theorem to evaluate $f(x) = 2x^5 - 3x^4 - 9x^3 + 8x^2 + 2$ at $x = -3$.

Answer

$f(-3) = -412$

Using the Factor Theorem to Solve a Polynomial Equation

The **Factor Theorem** is another theorem that helps us analyze polynomial equations. It tells us how the zeros of a polynomial are related to the factors. Recall that the Division Algorithm.

$$f(x) = (x - k)q(x) + r \tag{3.6.10}$$

If k is a zero, then the remainder r is $f(k) = 0$ and $f(x) = (x - k)q(x) + 0$ or $f(x) = (x - k)q(x)$.

Notice, written in this form, $x - k$ is a factor of $f(x)$. We can conclude if k is a zero of $f(x)$, then $x - k$ is a factor of $f(x)$.

Similarly, if $x - k$ is a factor of $f(x)$, then the remainder of the Division Algorithm $f(x) = (x - k)q(x) + r$ is 0. This tells us that k is a zero.

This pair of implications is the Factor Theorem. As we will soon see, a polynomial of degree n in the complex number system will have n zeros. We can use the Factor Theorem to completely factor a polynomial into the product of n factors. Once the polynomial has been completely factored, we can easily determine the zeros of the polynomial.

THE FACTOR THEOREM

According to the Factor Theorem, k is a zero of $f(x)$ if and only if $(x - k)$ is a factor of $f(x)$.

How To: Given a factor k and a third-degree polynomial, use the Factor Theorem to factor the polynomial

1. Use synthetic division to divide the polynomial by $(x - k)$.
2. Confirm that the remainder is 0.
3. Write the polynomial as the product of $(x - k)$ and the quadratic quotient.
4. If possible, factor the quadratic.
5. Write the polynomial as the product of factors.

Example 3.6.2: Using the Factor Theorem to Solve a Polynomial Equation

Show that $(x + 2)$ is a factor of $x^3 - 6x^2 - x + 30$. Find the remaining factors. Use the factors to determine the zeros of the **polynomial**.

Solution:

We can use synthetic division to show that $(x + 2)$ is a factor of the polynomial.

$$\begin{array}{r|rrrr} -2 & 1 & -6 & -1 & 30 \\ & & -2 & 16 & -30 \\ \hline & 1 & -8 & 15 & 0 \end{array} \tag{3.6.11}$$

The remainder is zero, so $(x+2)$ is a factor of the polynomial. We can use the Division Algorithm to write the polynomial as the product of the divisor and the quotient:

$$(x+2)(x^2-8x+15) \tag{3.6.12}$$

We can factor the quadratic factor to write the polynomial as

$$(x+2)(x-3)(x-5) \tag{3.6.13}$$

By the Factor Theorem, the zeros of x^3-6x^2-x+30 are –2, 3, and 5.

Try It 3.6.2

Use the Factor Theorem to find the zeros of $f(x)=x^3+4x^2-4x-16$ given that $(x-2)$ is a factor of the polynomial.

Answer

The zeros are 2, –2, and –4.

Using the Rational Zero Theorem to Find Rational Zeros

Another use for the Remainder Theorem is to test whether a rational number is a zero for a given polynomial. But first we need a pool of rational numbers to test. The **Rational Zero Theorem** helps us to narrow down the number of possible rational zeros using the ratio of the factors of the constant term and factors of the leading **coefficient** of the polynomial

Consider a quadratic function with two zeros, $x=\frac{2}{5}$ and $x=\frac{3}{4}$. By the Factor Theorem, these zeros have factors associated with them. Let us set each factor equal to 0, and then construct the original quadratic function absent its stretching factor.

$x-\frac{2}{5}=0$ or $x-\frac{3}{4}=0$	Set each factor equal to 0.
$5x-2=0$ or $4x-3=0$	Multiply both sides of the equation to eliminate fractions.
$f(x)=(5x-2)(4x-3)$	Create the quadratic function, multiplying the factors.
$f(x)=20x^2-23x+6$	Expand the polynomial.
$f(x)=(5\cdot4)x^2-23x+(2\cdot3)$	

Notice that two of the factors of the constant term, 6, are the two numerators from the original rational roots: 2 and 3. Similarly, two of the factors from the leading coefficient, 20, are the two denominators from the original rational roots: 5 and 4.

We can infer that the numerators of the rational roots will always be factors of the constant term and the denominators will be factors of the leading coefficient. This is the essence of the Rational Zero Theorem; it is a means to give us a pool of possible rational zeros.

THE RATIONAL ZERO THEOREM

The Rational Zero Theorem states that, if the polynomial $f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$ has integer coefficients, then every rational zero of $f(x)$ has the form $\frac{p}{q}$ where p is a factor of the constant term a_0 and q is a factor of the leading coefficient a_n.

When the leading coefficient is 1, the possible rational zeros are the factors of the constant term.

How To: Given a polynomial function $f(x)$, use the Rational Zero Theorem to find rational zeros.

1. Determine all factors of the constant term and all factors of the leading coefficient.
2. Determine all possible values of $\frac{p}{q}$, where p is a factor of the constant term and q is a factor of the leading coefficient. Be sure to include both positive and negative candidates.
3. Determine which possible zeros are actual zeros by evaluating each case of $f\left(\frac{p}{q}\right)$.

Example 3.6.3: Listing All Possible Rational Zeros

List all possible rational zeros of $f(x)=2x^4-5x^3+x^2-4$.

Solution:

The only possible rational zeros of $f(x)$ are the quotients of the factors of the last term, –4, and the factors of the leading coefficient, 2.

The constant term is –4; the factors of –4 are $p = \pm 1, \pm 2, \pm 4$.

The leading coefficient is 2; the factors of 2 are $q = \pm 1, \pm 2$.

If any of the four real zeros are rational zeros, then they will be of one of the following factors of –4 divided by one of the factors of 2.

$$\frac{p}{q} = \pm\frac{1}{1}, \pm\frac{1}{2} \quad \frac{p}{q} = \pm\frac{2}{1}, \pm\frac{2}{2} \quad \frac{p}{q} = \pm\frac{4}{1}, \pm\frac{4}{2} \tag{3.6.14}$$

Note that $\frac{2}{2} = 1$ and $\frac{4}{2} = 2$, which have already been listed. So we can shorten our list.

$$\frac{p}{q} = \frac{\text{Factors of the last}}{\text{Factors of the first}} = \pm 1, \pm 2, \pm 4, \pm\frac{1}{2} \tag{3.6.15}$$

Example 3.6.4: Using the Rational Zero Theorem to Find Rational Zeros

Use the Rational Zero Theorem to find the rational zeros of $f(x) = 2x^3 + x^2 - 4x + 1$.

Solution:

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of 1 and q is a factor of 2.

$$\frac{p}{q} = \frac{factor\ of\ constant\ term}{factor\ of\ leading\ coefficient} \tag{3.6.16}$$

$$= \frac{factor\ of\ 1}{factor\ of\ 2} \tag{3.6.17}$$

The factors of 1 are ±1 and the factors of 2 are ±1 and ±2. The possible values for $\frac{p}{q}$ are ±1 and $\pm\frac{1}{2}$. These are the possible rational zeros for the function. We can determine which of the possible zeros are actual zeros by substituting these values for x in $f(x)$.

$$f(-1) = 2(-1)^3 + (-1)^2 - 4(-1) + 1 = 4 \tag{3.6.18}$$

$$f(1) = 1(1)^3 + (1)^2 - 4(1) + 1 = 0 \tag{3.6.19}$$

$$f(-\frac{1}{2}) = 2(-\frac{1}{2})^3 + (-\frac{1}{2})^2 - 4(-\frac{1}{2}) + 1 = 3 \tag{3.6.20}$$

$$f(\frac{1}{2}) = 2(\frac{1}{2})^3 + (\frac{1}{2})^2 - 4(\frac{1}{2}) + 1 = -\frac{1}{2} \tag{3.6.21}$$

Of those, $-1, -\frac{1}{2}$, and $\frac{1}{2}$ are not zeros of $f(x)$. 1 is the only rational zero of $f(x)$.

Try It 3.6.3

Use the Rational Zero Theorem to find the rational zeros of $f(x) = x^3 - 5x^2 + 2x + 1$.

Answer

There are no rational zeros.

Finding the Zeros of Polynomial Functions

The Rational Zero Theorem helps us to narrow down the list of possible rational zeros for a polynomial function. Once we have done this, we can use **synthetic division** repeatedly to determine all of the **zeros** of a polynomial function.

How To: Given a polynomial function f, use synthetic division to find its zeros.

1. Use the Rational Zero Theorem to list all possible rational zeros of the function.

2. Use synthetic division to evaluate a given possible zero by synthetically dividing the candidate into the polynomial. If the remainder is 0, the candidate is a zero. If the remainder is not zero, discard the candidate.
3. Repeat step two using the quotient found with synthetic division. If possible, continue until the quotient is a quadratic.
4. Find the zeros of the quadratic function. Three possible methods for solving quadratics are factoring, completing the square, and using the quadratic formula.

Example 3.6.5: Finding the Zeros of a Polynomial Function with Repeated Real Zeros

Find the zeros of $f(x) = 4x^3 - 3x - 1$.

Solution:

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of –1 and q is a factor of 4.

$$\frac{p}{q} = \frac{factor\ of\ constant\ term}{factor\ of\ leading\ coefficient} \tag{3.6.22}$$

$$= \frac{factor\ of\ -1}{factor\ of\ 4} \tag{3.6.23}$$

The factors of –1 are ±1 and the factors of 4 are ±1,±2, and ±4. The possible values for $\frac{p}{q}$ are $\pm 1, \pm\frac{1}{2}$, and $\pm\frac{1}{4}$. These are the possible rational zeros for the function. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. It is easy to determine whether 1 is a zero: just add the coefficients of each term. If the result is 0, $x = 1$ is a zero. Let's begin with 1.

1	4	0	−3	−1
		4	4	1
	4	4	1	0

Dividing by $(x - 1)$ gives a remainder of 0, so 1 is a zero of the function. The polynomial can be written as

$$(x-1)(4x^2 + 4x + 1) = (x-1)(9(2x)^2 + 4x + 1^2) \tag{3.6.24}$$

The quadratic is a perfect square. $f(x)$ can be written as

$$(x-1)(2x+1)^2 \tag{3.6.25}$$

We already know that 1 is a zero. The other zero will have a multiplicity of 2 because the factor is squared. To find the other zero, we can set the factor equal to 0.

$$2x + 1 = 0 \tag{3.6.26}$$

$$x = -\frac{1}{2} \tag{3.6.27}$$

The zeros of the function are 1 and $-\frac{1}{2}$ with multiplicity 2.

Analysis

Look at the graph of the function f in Figure 3.6.1. Notice, at $x = -0.5$, the graph bounces off the x-axis, indicating the even multiplicity (2,4,6…) for the zero −0.5. At $x = 1$, the graph crosses the x-axis, indicating the odd multiplicity (1,3,5…) for the zero $x = 1$.

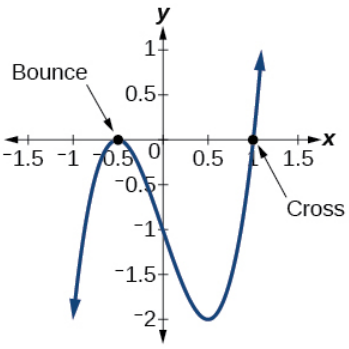


***Figure* 3.6.1.**

Using the Fundamental Theorem of Algebra

Now that we can find rational zeros for a polynomial function, we will look at a theorem that discusses the number of complex zeros of a polynomial function. The **Fundamental Theorem of Algebra** tells us that every polynomial function has at least one complex zero. This theorem forms the foundation for solving polynomial equations.

Suppose f is a polynomial function of degree four, and $f(x) = 0$. The Fundamental Theorem of Algebra states that there is at least one complex solution, call it c_1. By the Factor Theorem, we can write $f(x)$ as a product of $x - c_1$ and a polynomial quotient. Since $x - c_1$ is linear, the polynomial quotient will be of degree three. Now we apply the Fundamental Theorem of Algebra to the third-degree polynomial quotient. It will have at least one complex zero, call it c_2. So we can write the polynomial quotient as a product of $x - c_2$ and a new polynomial quotient of degree two. Continue to apply the Fundamental Theorem of Algebra until all of the zeros are found. There will be four of them and each one will yield a factor of $f(x)$.

THE FUNDAMENTAL THEOREM OF ALGEBRA

The *Fundamental Theorem of Algebra* states that, if $f(x)$ is a polynomial of degree $n > 0$, then $f(x)$ has at least one complex zero.

We can use this theorem to argue that, if $f(x)$ is a polynomial of degree $n > 0$, and a is a non-zero real number, then $f(x)$ has exactly n linear factors

$$f(x) = a(x - c_1)(x - c_2)...(x - c_n) \tag{3.6.28}$$

where $c_1, c_2,...,c_n$ are complex numbers. Therefore, $f(x)$ has n conplex roots if we allow for multiplicities.

Does every polynomial have at least one imaginary zero?

No. Real numbers are a subset of complex numbers, but not the other way around. A complex number is not necessarily imaginary. Real numbers are also complex numbers.

Example 3.6.6: Finding the Zeros of a Polynomial Function with Complex Zeros

Find the zeros of $f(x) = 3x^3 + 9x^2 + x + 3$.

Solution:

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of 3 and q is a factor of 3.

$$\frac{p}{q} = \frac{factor\ of\ constant\ term}{factor\ of\ leading\ coefficient} \tag{3.6.29}$$

$$= \frac{factor\ of\ 3}{factor\ of\ 3} \tag{3.6.30}$$

The factors of 3 are ±1 and ±3. The possible values for $\frac{p}{q}$, and therefore the possible rational zeros for the function, are ±3,±1, and $\pm\frac{1}{3}$. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with –3.

$$\begin{array}{r|cccc} -3 & 3 & 9 & 1 & 3 \\ & & -9 & 0 & -3 \\ \hline & 3 & 0 & 1 & 0 \end{array}$$

Dividing by $(x+3)$ gives a remainder of 0, so –3 is a zero of the function. The polynomial can be written as

$$(x+3)(3x^2+1). \tag{3.6.31}$$

(Note: this polynomial could also have been factored using the technique of grouping.)

We can then set the quadratic equal to 0 and solve to find the other zeros of the function.

$$3x^2+1=0 \tag{3.6.32}$$

$$x^2=-\frac{1}{3} \tag{3.6.33}$$

$$x=\pm-\sqrt{\frac{1}{3}}=\pm\frac{i\sqrt{3}}{3} \tag{3.6.34}$$

The zeros of $f(x)$ are -3 and $\pm\frac{i\sqrt{3}}{3}$.

Analysis

Suppose we do not choose to find the zeros by factoring, but we have a graph of the function. Look at the graph of the function $f(x)=3x^3+9x^2+x+3$ in Figure 3.6.2. Notice that, at $x=-3$, the graph crosses the x-axis, indicating an odd multiplicity (1) for the zero $x=-3$. Also note the presence of the two turning points. This means that, since there is a 3rd degree polynomial, we are looking at the maximum number of turning points. So, the end behavior of increasing without bound to the right and decreasing without bound to the left will continue. Thus, all the x-intercepts for the function are shown. So either the multiplicity of $x=-3$ is 1 and there are two complex solutions, which is what we found, or the multiplicity at $x=-3$ is three. Either way, our result is correct.

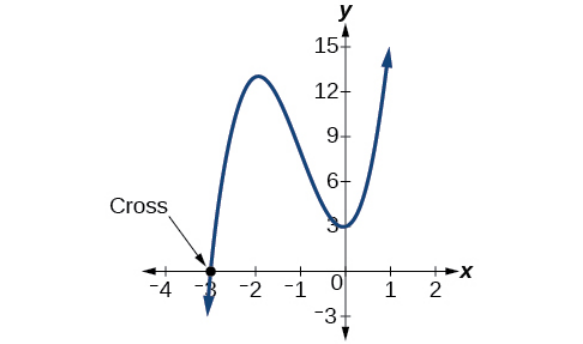


***Figure* 3.6.2.**

3.6.4

Find the zeros of $f(x)=2x^3+5x^2-11x+4$.

Answer

The zeros are -4, $\frac{1}{2}$, and 1.

Using the **Linear Factorization Theorem** to Find Polynomials with Given Zeros

A vital implication of the **Fundamental Theorem of Algebra**, as we stated above, is that a polynomial function of degree n will have n zeros in the set of complex numbers, if we allow for multiplicities. This means that we can factor the polynomial function into n factors. The **Linear Factorization Theorem** states that a polynomial function will have the same number of factors as its degree, and that each factor will be in the form $(x-c)$, where c is a complex number.

Let f be a polynomial function with real coefficients, and suppose $a+bi$, $b \neq 0$, is a zero of $f(x)$. Then, by the Factor Theorem, $x-(a+bi)$ is a factor of $f(x)$. For f to have real coefficients, $x-(a-bi)$ must also be a factor of $f(x)$. This is true because any factor other than $x-(a-bi)$, when multiplied by $x-(a+bi)$, will leave imaginary components in the product. Only multiplication with conjugate pairs will eliminate the imaginary parts and result in real coefficients. In other words, if a polynomial function f with real coefficients has a complex zero $a+bi$, then the complex conjugate $a-bi$ must also be a zero of $f(x)$. This is called the **Complex Conjugate Theorem**.

COMPLEX CONJUGATE THEOREM

According to the Linear Factorization Theorem, a polynomial function will have the same number of factors as its degree, and each factor will be in the form $(x-c)$, where c is a complex number.

If the polynomial function f has real coefficients and a complex zero in the form $a+bi$, then the complex conjugate of the zero, $a-bi$, is also a zero.

How To: Given the zeros of a polynomial function f and a point $(c, f(c))$ on the graph of f, use the Linear Factorization Theorem to find the polynomial function.

1. Use the zeros to construct the linear factors of the polynomial.
2. Multiply the linear factors to expand the polynomial.
3. Substitute $(c, f(c))$ into the function to determine the leading coefficient.
4. Simplify.

Example 3.6.7: Using the Linear Factorization Theorem to Find a Polynomial with Given Zeros

Find a fourth degree polynomial with real coefficients that has zeros of –3, 2, i, such that $f(-2)=100$.

Solution:

Because $x=i$ is a zero, by the Complex Conjugate Theorem $x=-i$ is also a zero. The polynomial must have factors of $(x+3),(x-2),(x-i)$, and $(x+i)$. Since we are looking for a degree 4 polynomial, and now have four zeros, we have all four factors. Let's begin by multiplying these factors.

$$f(x)=a(x+3)(x-2)(x-i)(x+i) \tag{3.6.35}$$
$$f(x)=a(x^2+x-6)(x^2+1) \tag{3.6.36}$$
$$f(x)=a(x^4+x^3-5x^2+x-6) \tag{3.6.37}$$

We need to find a to ensure $f(-2)=100$. Substitute $x=-2$ and $f(-2)=100$ into $f(x)$.

$$100=a((-2)^4+(-2)^3-5(-2)^2+(-2)-6) \tag{3.6.38}$$
$$100=a(-20) \tag{3.6.39}$$
$$-5=a \tag{3.6.40}$$

So the polynomial function is

$$f(x)=-5(x^4+x^3-5x^2+x-6) \tag{3.6.41}$$

or

$$f(x)=-5x^4-5x^3+25x^2-5x+30 \tag{3.6.42}$$

Analysis

We found that both i and $-i$ were zeros, but only one of these zeros needed to be given. If i is a zero of a polynomial with real coefficients, then $-i$ must also be a zero of the polynomial because $-i$ is the complex conjugate of i .

Q A If $2+3i$ were given as a zero of a polynomial with real coefficients, would $2-3i$ also need to be a zero?

Yes. When any complex number with an imaginary component is given as a zero of a polynomial with real coefficients, the conjugate must also be a zero of the polynomial.

3.6.5

Find a third degree polynomial with real coefficients that has zeros of 5 and $-2i$ such that $f(1) = 10$.

Answer

$f(x) = -\frac{1}{2}x^3 + \frac{5}{2}x^2 - 2x + 10$

Using Descartes' Rule of Signs

There is a straightforward way to determine the possible numbers of positive and negative real zeros for any polynomial function. If the polynomial is written in descending order, **Descartes' Rule of Signs** tells us of a relationship between the number of sign changes in $f(x)$ and the number of positive real zeros. For example, the polynomial function below has one sign change.

$$f(x) = x^4 + x^3 + x^2 + x - 1$$

This tells us that the function must have 1 positive real zero.

There is a similar relationship between the number of sign changes in $f(-x)$ and the number of negative real zeros.

$$f(-x) = (-x)^4 + (-x)^3 + (-x)^2 + (-x) - 1$$
$$f(-x) = +x^4 - x^3 + x^2 - x - 1$$

In this case, $f(-x)$ has 3 sign changes. This tells us (see below) that $f(x)$ could have 3 or 1 negative real zeros.

DESCARTES' RULE OF SIGNS

According to *Descartes' Rule of Signs*, if we let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial function with real coefficients:

- The number of positive real zeros is either equal to the number of sign changes of $f(x)$ or is less than the number of sign changes by an even integer.
- The number of negative real zeros is either equal to the number of sign changes of $f(-x)$ or is less than the number of sign changes by an even integer.

Example 3.6.8: Using Descartes' Rule of Signs

Use Descartes' Rule of Signs to determine the possible numbers of positive and negative real zeros for $f(x) = -x^4 - 3x^3 + 6x^2 - 4x - 12$.

Solution:

Begin by determining the number of sign changes.

$$f(x) = -x^4 - 3x^3 + 6x^2 - 4x - 12$$

There are two sign changes, so there are either 2 or 0 positive real roots. Next, we examine $f(-x)$ to determine the number of negative real roots.

$$f(-x) = -(-x)^4 - 3(-x)^3 + 6(-x)^2 - 4(-x) - 12 \quad (3.6.43)$$
$$f(-x) = -x^4 + 3x^3 + 6x^2 + 4x - 12 \quad (3.6.44)$$

$$f(-x) = -x^4 + 3x^3 + 6x^2 + 4x - 12$$

Again, there are two sign changes, so there are either 2 or 0 negative real roots.

There are four possibilities, as we can see in Table **3.6.1**.

Table **3.6.1**

Positive Real Zeros	Negative Real Zeros	Complex Zeros	Total Zeros
2	2	0	4
2	0	2	4
0	2	2	4
0	0	4	4

Analysis

We can confirm the numbers of positive and negative real roots by examining a graph of the function. See Figure **3.6.3**. We can see from the graph that the function has 0 positive real roots and 2 negative real roots.

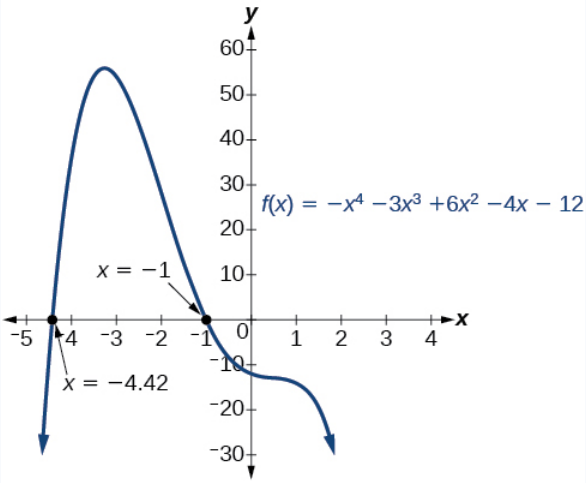


***Figure* 3.6.3.**

3.6.6

Use Descartes' Rule of Signs to determine the maximum possible numbers of positive and negative real zeros for $f(x) = 2x^4 - 10x^3 + 11x^2 - 15x + 12$. Use a graph to verify the numbers of positive and negative real zeros for the function.

Answer

There must be 4, 2, or 0 positive real roots and 0 negative real roots. The graph shows that there are 2 positive real zeros and 0 negative real zeros.

Solving Real-World Applications

We have now introduced a variety of tools for solving polynomial equations. Let's use these tools to solve the bakery problem from the beginning of the section.

Example 3.6.9

Solving Polynomial Equations

A new bakery offers decorated sheet cakes for children's birthday parties and other special occasions. The bakery wants the volume of a small cake to be 351 cubic inches. The cake is in the shape of a rectangular solid. They want the length of the cake to be four inches longer than the width of the cake and the height of the cake to be one-third of the width. What should the dimensions of the cake pan be?

Solution:

Begin by writing an equation for the volume of the cake. The volume of a rectangular solid is given by $V = lwh$. We were given that the length must be four inches longer than the width, so we can express the length of the cake as $l = w + 4$. We were given

that the height of the cake is one-third of the width, so we can express the height of the cake as $h = \frac{1}{3}w$. Let's write the volume of the cake in terms of width of the cake.

$$V = (w+4)(w)(\frac{1}{3}w) \tag{3.6.45}$$

$$V = \frac{1}{3}w^3 + \frac{4}{3}w^2 \tag{3.6.46}$$

Substitute the given volume into this equation.

$351 = 13w^3 + 43w^2$ Substitute 351 for V.
$1053 = w^3 + 4w^2$ Multiply both sides by 3.
$0 = w^3 + 7w^2 - 1053$ Subtract 1053 from both sides.

Descartes' rule of signs tells us there is one positive solution. The Rational Zero Theorem tells us that the possible rational zeros are $\pm 1, \pm 3, \pm 9, \pm 13, \pm 27, \pm 39, \pm 81, \pm 117, \pm 351,$ and ± 1053. We can use synthetic division to test these possible zeros. Only positive numbers make sense as dimensions for a cake, so we need not test any negative values. Let's begin by testing values that make the most sense as dimensions for a small sheet cake. A 1-inch wide cake is not likely, and by adding the coefficients, we can see that 1 is not a solution.

We will use synthetic division to check $x = 3$.

3	1	4	0	−1053
		3	21	63
	1	7	21	−990

Since 3 is not a solution either, we will test $x = 9$.

9	1	4	0	−1053
		9	117	1053
	1	13	117	0

Synthetic division gives a remainder of 0, so 9 is a solution to the equation. We can use the relationships between the width and the other dimensions to determine the length and height of the sheet cake pan.

$l = w + 4 = 9 + 4 = 13$ and $h = \frac{1}{3}w = \frac{1}{3}(9) = 3$

The sheet cake pan should have dimensions 13 inches by 9 inches by 3 inches

3.6.7

A shipping container in the shape of a rectangular solid must have a volume of 84 cubic meters. The client tells the manufacturer that, because of the contents, the length of the container must be one meter longer than the width, and the height must be one meter greater than twice the width. What should the dimensions of the container be?

Answer

3 meters by 4 meters by 7 meters

Media

Access these online resources for additional instruction and practice with zeros of polynomial functions.

- Real Zeros, Factors, and Graphs of Polynomial Functions
- Complex Factorization Theorem
- Find the Zeros of a Polynomial Function
- Find the Zeros of a Polynomial Function 2
- Find the Zeros of a Polynomial Function 3

Key Concepts

- To find $f(k)$, determine the remainder of the polynomial $f(x)$ when it is divided by $x-k$. This is known as the Remainder Theorem. See Example 3.6.1.
- According to the Factor Theorem, k is a zero of $f(x)$ if and only if $(x-k)$ is a factor of $f(x)$. See Example3.6.2.
- According to the Rational Zero Theorem, each rational zero of a polynomial function with integer coefficients will be equal to a factor of the constant term divided by a factor of the leading coefficient. See Example 3.6.3 and Example 3.6.4.
- When the leading coefficient is 1, the possible rational zeros are the factors of the constant term.
- Synthetic division can be used to find the zeros of a polynomial function. See Example 3.6.5.
- According to the Fundamental Theorem, every polynomial function with degree greater than 0 has at least one complex zero (keep in mind that real numbers are complex numbers). See Example 3.6.6., as well as Examples 3.6.2, 3.6.4, and 3.6.5.
- Allowing for multiplicities, a polynomial function will have the same number of factors as its degree. Each factor will be in the form $(x-c)$, where c is a complex number. See Example 3.6.7.
- The number of positive real zeros of a polynomial function is either the number of sign changes of the function or less than the number of sign changes by an even integer.
- The number of negative real zeros of a polynomial function is either the number of sign changes of $f(-x)$ or less than the number of sign changes by an even integer. See Example 3.6.8.
- Polynomial equations model many real-world scenarios. Synthetic division can often be used to solve the equation. See Example 3.6.9.

Glossary

Descartes' Rule of Signs

a rule that determines the maximum possible numbers of positive and negative real zeros based on the number of sign changes of $f(x)$ and $f(-x)$

Factor Theorem

k is a zero of polynomial function $f(x)$ if and only if $(x-k)$ is a factor of $f(x)$

Fundamental Theorem of Algebra

A polynomial function with degree greater than 0 has at least one complex zero.

Linear Factorization Theorem

Allowing for multiplicities, a polynomial function will have the same number of factors as its degree, and each factor will be in the form $(x-c)$, where c is a complex number.

Rational Zero Theorem

the possible rational zeros of a polynomial function have the form $\frac{p}{q}$ where p is a factor of the constant term and q is a factor of the leading coefficient.

Remainder Theorem

if a polynomial $f(x)$ is divided by $x-k$,then the remainder is equal to the value $f(k)$

Contributor

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3.6E: Exercises

Section Exercises

Verbal

1. Describe a use for the Remainder Theorem.

Answer: The theorem can be used to evaluate a polynomial.

2. Explain why the Rational Zero Theorem does not guarantee finding zeros of a polynomial function.

3. What is the difference between rational and real zeros?

Answer: Rational zeros can be expressed as fractions whereas real zeros include irrational numbers.

4. If Descartes' Rule of Signs reveals a no change of signs or one sign of changes, what specific conclusion can be drawn?

5. If synthetic division reveals a zero, why should we try that value again as a possible solution?

Answer: Polynomial functions can have repeated zeros, so the fact that number is a zero doesn't preclude it being a zero again.

Algebraic

For the following exercises, use the Remainder Theorem to find the remainder.

---

6. $(x^4 - 9x^2 + 14) \div (x - 2)$

7. $(3x^3 - 2x^2 + x - 4) \div (x + 3)$

Answer: -106

8. $(x^4 + 5x^3 - 4x - 17) \div (x + 1)$

9. $(-3x^2 + 6x + 24) \div (x - 4)$

Answer: 0

10. $(5x^5 - 4x^4 + 3x^3 - 2x^2 + x - 1) \div (x + 6)$

11. $(x^4 - 1) \div (x - 4)$

Answer: 255

12. $(3x^3 + 4x^2 - 8x + 2) \div (x - 3)$

13. $(4x^3 + 5x^2 - 2x + 7) \div (x + 2)$

Answer: -1

For the following exercises, use the Factor Theorem to find all real zeros for the given polynomial function and one factor.

---

14. $f(x) = 2x^3 - 9x^2 + 13x - 6; x - 1$

15. $f(x) = 2x^3 + x^2 - 5x + 2; x + 2$

Answer: $-2, 1, \frac{1}{2}$

16. $f(x) = 3x^3 + x^2 - 20x + 12; x + 3$

17. $f(x) = 2x^3 + 3x^2 + x + 6; x + 2$

Answer: -2

18. $f(x) = -5x^3 + 16x^2 - 9; x - 3$

19. $x^3 + 3x^2 + 4x + 12; x + 3$

Answer: -3

20. $4x^3 - 7x + 3; x - 1$

21. $2x^3 + 5x^2 - 12x - 30, 2x + 5$

Answer: $-\frac{5}{2}, \sqrt{6}, -\sqrt{6}$

For the following exercises, use the Rational Zero Theorem to find all real zeros.

---

22. $x^3 - 3x^2 - 10x + 24 = 0$

23.$2x^3 + 7x^2 - 10x - 24 = 0$

Answer: $2, -4, -\frac{3}{2}$

24. $x^3 + 2x^2 - 9x - 18 = 0$

25. $x^3 + 5x^2 - 16x - 80 = 0$

Answer: $4, -4, -5$

26. $x^3 - 3x^2 - 25x + 75 = 0$

27. $2x^3 - 3x^2 - 32x - 15 = 0$

Answer: $5, -3, -\frac{1}{2}$

28. $2x^3 + x^2 - 7x - 6 = 0$

29. $2x^3 - 3x^2 - x + 1 = 0$

Answer: $\frac{1}{2}, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$

30. $3x^3 - x^2 - 11x - 6 = 0$

31. $2x^3 - 5x^2 + 9x - 9 = 0$

Answer: $\frac{3}{2}$

32. $2x^3 - 3x^2 + 4x + 3 = 0$

33. $x^4 - 2x^3 - 7x^2 + 8x + 12 = 0$

Answer: $2, 3, -1, -2$

34. $x^4 + 2x^3 - 9x^2 - 2x + 8 = 0$

35. $4x^4 + 4x^3 - 25x^2 - x + 6 = 0$

Answer: $\frac{1}{2}, -\frac{1}{2}, 2, -3$

36. $2x^4 - 3x^3 - 15x^2 + 32x - 12 = 0$

37. $x^4 + 2x^3 - 4x^2 - 10x - 5 = 0$

Answer: $-1, -1, \sqrt{5}, -\sqrt{5}$

38. $4x^3 - 3x + 1 = 0$

39. $8x^4 + 26x^3 + 39x^2 + 26x + 6$

Answer: $-\frac{3}{4}, -\frac{1}{2}$

For the following exercises, find all complex solutions (real and non-real).

---

40. $x^3 + x^2 + x + 1 = 0$

41. $x^3 - 8x^2 + 25x - 26 = 0$

Answer: $2, 3 + 2i, 3 - 2i$

42. $x^3 + 13x^2 + 57x + 85 = 0$

43. $3x^3 - 4x^2 + 11x + 10 = 0$

Answer: $-\frac{2}{3}, 1 + 2i, 1 - 2i$

44. $x^4 + 2x^3 + 22x^2 + 50x - 75 = 0$

45. $2x^3 - 3x^2 + 32x + 17 = 0$

Answer: $-\frac{1}{2}, 1 + 4i, 1 - 4i$

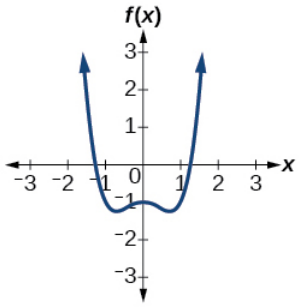
Graphical

For the following exercises, use Descartes' Rule to determine the possible number of positive and negative solutions. Confirm with the given graph.

46. $f(x) = x^3 - 1$

47. $f(x) = x^4 - x^2 - 1$

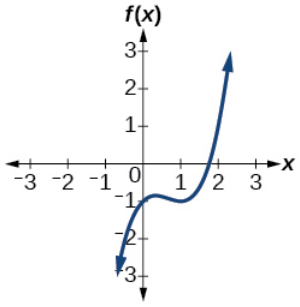
Answer: 1 positive, 1 negative



48. $f(x) = x^3 - 2x^2 - 5x + 6$

49. $f(x) = x^3 - 2x^2 + x - 1$

Answer: 3 or 1 positive, 0 negative

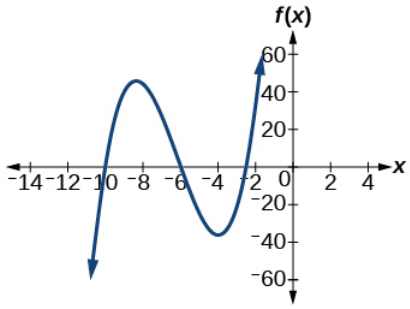


50. $f(x) = x^4 + 2x^3 - 12x^2 + 14x - 5$

51. $f(x) = 2x^3 + 37x^2 + 200x + 300$

Answer: 0 positive, 3 or 1 negative

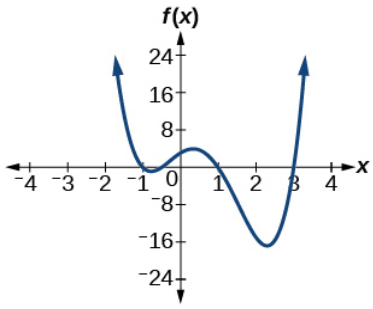
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52. $f(x) = x^3 - 2x^2 - 16x + 32$

53. $f(x) = 2x^4 - 5x^3 - 5x^2 + 5x + 3$

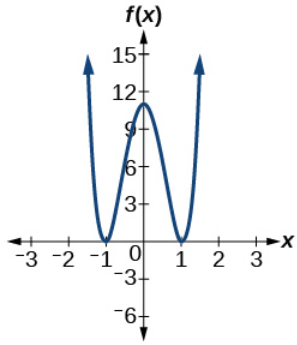
Answer: 2 or 0 positive, 2 or 0 negative



54. $f(x) = 2x^4 - 5x^3 - 14x^2 + 20x + 8$

55. $f(x) = 10x^4 - 21x^2 + 11$

Answer: 2 or 0 positive, 2 or 0 negative



Numeric

For the following exercises, list all possible rational zeros for the functions.

$f(x) = x^4 + 3x^3 - 4x + 4$

$f(x) = 2x^3 + 3x^2 - 8x + 5$

$\pm 5, \pm 1, \pm \frac{5}{2}$

$f(x) = 3x^3 + 5x^2 - 5x + 4$

$f(x) = 6x^4 - 10x^2 + 13x + 1$

$\pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{6}$

$f(x) = 4x^5 - 10x^4 + 8x^3 + x^2 - 8$

Technology

For the following exercises, use your calculator to graph the polynomial function. Based on the graph, find the rational zeros. All real solutions are rational.

$f(x) = 6x^3 - 7x^2 + 1$

$1, \frac{1}{2}, -\frac{1}{3}$

$f(x) = 4x^3 - 4x^2 - 13x - 5$

$f(x) = 8x^3 - 6x^2 - 23x + 6$

$2, \frac{1}{4}, -\frac{3}{2}$

$f(x) = 12x^4 + 55x^3 + 12x^2 - 117x + 54$

$f(x) = 16x^4 - 24x^3 + x^2 - 15x + 25$

$\frac{5}{4}$

Extensions

For the following exercises, construct a polynomial function of least degree possible using the given information.

Real roots: –1, 1, 3 and $(2, f(2)) = (2, 4)$

Real roots: –1 (with multiplicity 2 and 1) and $(2, f(2)) = (2, 4)$

$f(x) = \frac{4}{9}(x^3 + x^2 - x - 1)$

Real roots: $-2, \frac{1}{2}$

(with multiplicity 2) and $(-3, f(-3)) = (-3, 5)$

Real roots:$-\frac{1}{2}, 0, \frac{1}{2}$ and $(-2, f(-2)) = (-2, 6)$

$f(x) = -\frac{1}{5}(4x^3 - x)$

Real roots: –4, –1, 1, 4 and $(-2, f(-2)) = (-2, 10)$

Real-World Applications

For the following exercises, find the dimensions of the box described.

The length is twice as long as the width. The height is 2 inches greater than the width. The volume is 192 cubic inches.

8 by 4 by 6 inches

The length, width, and height are consecutive whole numbers. The volume is 120 cubic inches.

The length is one inch more than the width, which is one inch more than the height. The volume is 86.625 cubic inches.

5.5 by 4.5 by 3.5 inches

The length is three times the height and the height is one inch less than the width. The volume is 108 cubic inches.

The length is 3 inches more than the width. The width is 2 inches more than the height. The volume is 120 cubic inches.

8 by 5 by 3 inches

For the following exercises, find the dimensions of the right circular cylinder described.

The radius is 3 inches more than the height. The volume is 16π cubic meters.

The height is one less than one half the radius. The volume is 72π cubic meters.

Radius = 6 meters, Height = 2 meters

The radius and height differ by one meter. The radius is larger and the volume is 48π cubic meters.

The radius and height differ by two meters. The height is greater and the volume is 28.125π cubic meters.

Radius = 2.5 meters, Height = 4.5 meters

80. The radius is $\frac{1}{3}$ meter greater than the height. The volume is $\frac{98}{9}\pi$ cubic meters.

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3.7: Rational Functions

Skills to Develop

- Use arrow notation.
- Solve applied problems involving rational functions.
- Find the domains of rational functions.
- Identify vertical asymptotes.
- Identify horizontal asymptotes.
- Graph rational functions.

Suppose we know that the cost of making a product is dependent on the number of items, x, produced, given by the equation $C(x) = 15,000x - 0.1x^2 + 1000$. If we want to know the average cost for producing x items, we would divide the cost function by the number of items, x.

The average cost function, which yields the average cost per item for x items produced, is

$$f(x) = \frac{15,000x - 0.1x^2 + 1000}{x}$$

Many other application problems require finding an average value in a similar way, giving us variables in the denominator. Written without a variable in the denominator, this function will contain a negative integer power.

In the last few sections, we have worked with polynomial functions, which are functions with non-negative integers for exponents. In this section, we explore rational functions, which have variables in the denominator.

Using Arrow Notation

We have seen the graphs of the basic **reciprocal function** and the squared reciprocal function from our study of toolkit functions. Examine these graphs, as shown in Figure 3.7.1, and notice some of their features.

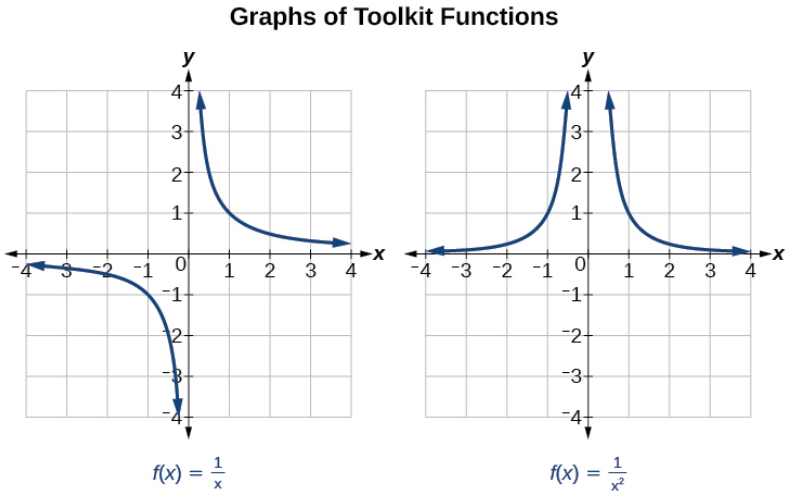


***Figure* 3.7.1.**

Several things are apparent if we examine the graph of $f(x) = \frac{1}{x}$.

On the left branch of the graph, the curve approaches the x-axis $(y = 0)$ as $x \to -\infty$.

As the graph approaches $x = 0$ from the left, the curve drops, but as we approach zero from the right, the curve rises.

Finally, on the right branch of the graph, the curves approaches the x-axis $(y = 0)$ as $x \to \infty$.

To summarize, we use **arrow notation** to show that x or $f(x)$ is approaching a particular value. See Table 3.7.1.

***Table* 3.7.1.**

Symbol	Meaning
$x \to a^-$	x approaches a from the left ($x < a$ but close to a)
$x \to a^+$	x approaches a from the right ($x > a$ but close to a)
$x \to a$	x approaches a from both left and right ($x \neq a$ but close to a)
$x \to \infty$	x approaches infinity (x increases without bound)
$x \to -\infty$	x approaches negative infinity (x decreases without bound)
$f(x) \to \infty$	the output approaches infinity (the output increases without bound)
$f(x) \to -\infty$	the output approaches negative infinity (the output decreases without bound)
$f(x) \to b$	the output approaches b

Local Behavior of $f(x) = \frac{1}{x}$

Let's begin by looking at the reciprocal function, $f(x) = \frac{1}{x}$. We cannot divide by zero, which means the function is undefined at $x = 0$; so zero is not in the domain. As the input values approach zero from the left side (negative values that are very small in magnitude), the function values decrease without bound (in other words, they approach negative infinity). We can see this behavior in Table **3.7.2**.

***Table* 3.7.2.**

x	–0.1	–0.01	–0.001	–0.0001
$f(x) = \frac{1}{x}$	–10	–100	–1000	–10,000

We write in arrow notation

$$\text{as } x \to 0^-, f(x) \to -\infty$$

As the input values approach zero from the right side (positive values that are very small in magnitude), the function values increase without bound (approaching infinity). We can see this behavior in Table **3.7.3**.

***Table* 3.7.3**

x	0.1	0.01	0.001	0.0001
$f(x) = \frac{1}{x}$	10	100	1000	10,000

We write in arrow notation

$$\text{As } x \to 0^+, f(x) \to \infty.$$

See Figure **3.7.2**.

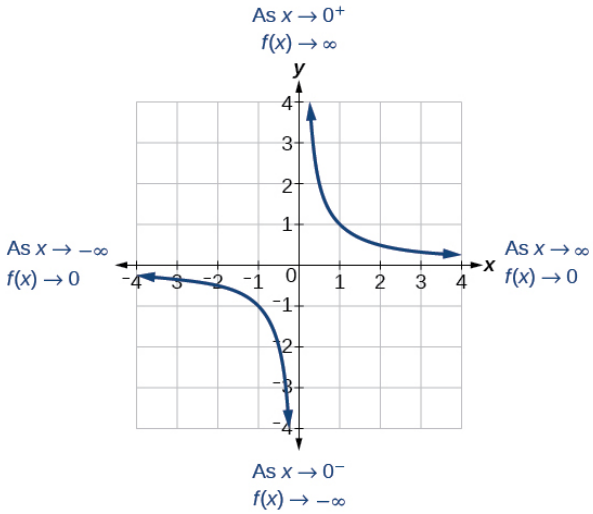


Figure 3.7.2.

This behavior creates a **vertical asymptote**, which is a vertical line that the graph approaches but never crosses. In this case, the graph is approaching the vertical line $x = 0$ as the input becomes close to zero. See Figure 3.7.3.

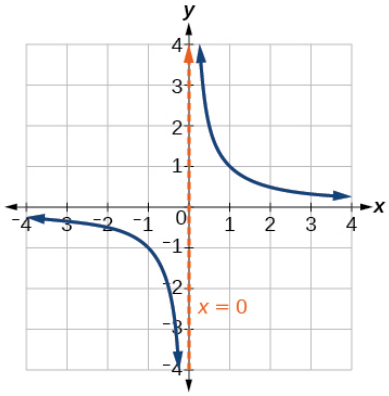


Figure 3.7.3.

VERTICAL ASYMPTOTE

A vertical asymptote of a graph is a vertical line $x = a$ where the graph tends toward positive or negative infinity as the inputs approach a from either or both sides of a. In the case when the graph heads in the same direction from both sides, we write

As $x \to a$, $f(x) \to \infty$, or as $x \to a$, $f(x) \to -\infty$.

End Behavior of $f(x) = \frac{1}{x}$

As the values of x approach infinity, the function values approach 0. As the values of x approach negative infinity, the function values approach 0. See Figure 3.7.4. Symbolically, using arrow notation

As $x \to \infty$, $f(x) \to 0$,and as $x \to -\infty$, $f(x) \to 0$.

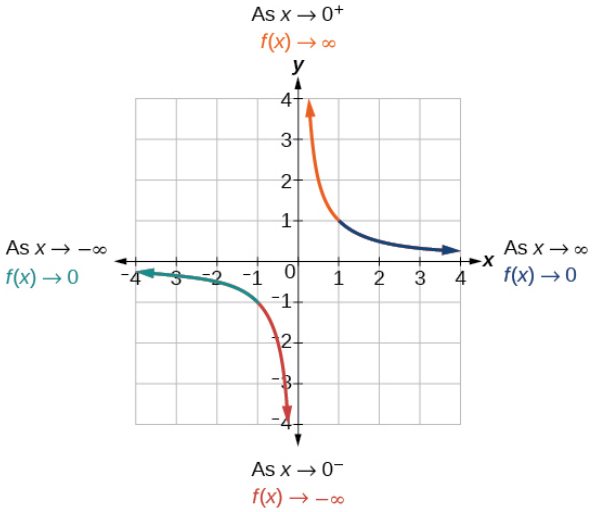


Figure 3.7.4.

Based on this overall behavior and the graph, we can see that the function approaches 0 but never actually reaches 0; it seems to level off as the inputs become large. This behavior creates a **horizontal asymptote**, a horizontal line that the graph approaches as the input increases or decreases without bound. In this case, the graph is approaching the horizontal line $y = 0$. See Figure 3.7.5.

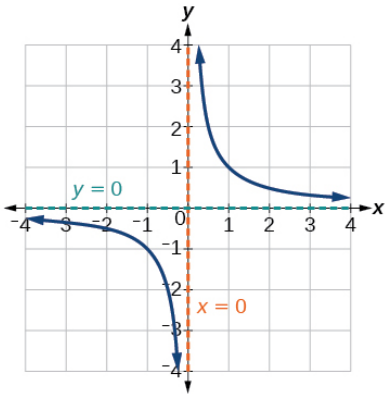


***Figure* 3.7.5.**

HORIZONTAL ASYMPTOTE

A **horizontal asymptote** of a graph is a horizontal line $y = b$ where the graph approaches the line as the inputs increase or decrease without bound. We write

$$\text{As } x \to \infty \text{ or } x \to -\infty,\ f(x) \to b.$$

Example 3.7.1: Using Arrow Notation.

Use arrow notation to describe the end behavior and local behavior of the function graphed in Figure **3.7.6**.

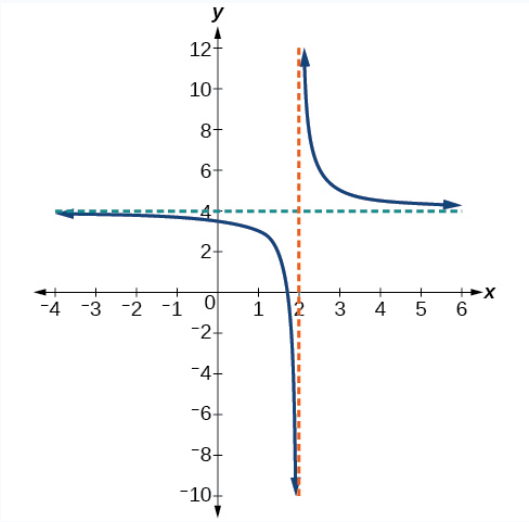


***Figure* 3.7.6.**

Solution:

Notice that the graph is showing a vertical asymptote at $x = 2$, which tells us that the function is undefined at $x = 2$.

$$\text{As } x \to 2^-,\ f(x) \to -\infty, \text{ and as } x \to 2^+,\ f(x) \to \infty.$$

And as the inputs decrease without bound, the graph appears to be leveling off at output values of 4, indicating a horizontal asymptote at $y = 4$. As the inputs increase without bound, the graph also levels off at 4.

$$\text{As } x \to \infty,\ f(x) \to 4 \text{ and as } x \to -\infty,\ f(x) \to 4.$$

Note that the lines representing the asymptotes do **not** represent points on the graph; however, it is essential to indicate any asymptote that belongs to a graph.

There is an x-intercept at about $(1.7, 0)$ and a y-intercept at about $(0, 3.7)$. There are no turning points.

Try it **3.7.1**: Use arrow notation to describe the end behavior and local behavior for the reciprocal squared function.

Answer

End behavior: as $x \to \pm\infty$, $f(x) \to 0$; Local behavior: as $x \to 0$, $f(x) \to \infty$ (There are no x- or y-intercepts, and no turning points.)

Example 3.7.2: Using Transformations to Graph a Rational Function.

Sketch a graph of the reciprocal function shifted two units to the left and up three units. Identify the horizontal and vertical asymptotes of the graph, if any.

Solution:

Shifting the graph left 2 and up 3 would result in the function

$$f(x) = \frac{1}{x+2} + 3$$

or equivalently, by giving the terms a common denominator,

$$f(x) = \frac{1}{x+2} + \frac{3(x+2)}{x+2} = \frac{3x+7}{x+2}$$

The graph of the shifted function is displayed in Figure **3.7.7.**

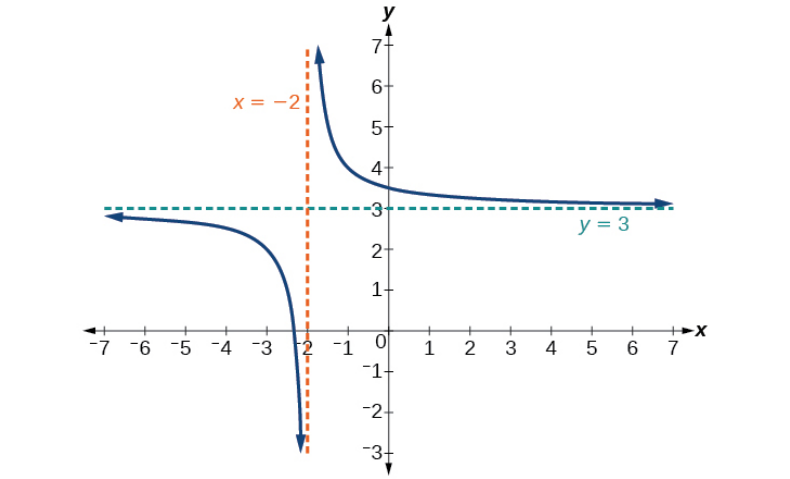


***Figure* 3.7.7.**

Notice that this function is undefined at $x = -2$, and the graph also is showing a vertical asymptote at $x = -2$.

$$\text{As } x \to -2^-,\ f(x) \to -\infty, \text{ and as } x \to -2^+,\ f(x) \to \infty.$$

As the inputs increase and decrease without bound, the graph appears to be leveling off at output values of 3, indicating a horizontal asymptote at $y = 3$.

$$\text{As } x \to \pm\infty,\ f(x) \to 3.$$

Analysis

Notice that horizontal and vertical asymptotes are shifted left 2 and up 3 along with the function.

Try It **3.7.2**: Sketch the graph, and find the horizontal and vertical asymptotes of the reciprocal squared function that has been shifted right 3 units and down 4 units.

Answer

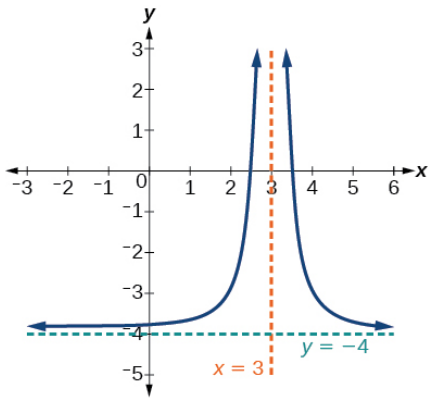


***Figure* 3.7.8.**

The function and the asymptotes are shifted 3 units right and 4 units down. As $x \to 3$, $f(x) \to \infty$, and as $x \to \pm\infty$, $f(x) \to -4$.

The function is $f(x) = \frac{1}{(x-3)^2} - 4$.

Solving Applied Problems Involving Rational Functions

In Example 3.7.2, we shifted a toolkit function in a way that resulted in the function $f(x) = \frac{3x+7}{x+2}$. This is an example of a rational function. A **rational function** is a function that can be written as the quotient of two polynomial functions. Many real-world problems require us to find the ratio of two polynomial functions. Problems involving rates and concentrations often involve rational functions.

RATIONAL FUNCTION

A **rational function** is a function that can be written as the quotient of two polynomial functions $P(x)$ and $Q(x)$.

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_p x^p + a_{p-1}x^{p-1} + \ldots + a_1 x + a_0}{b_q x^q + b_{q-1}x^{q-1} + \ldots + b_1 x + b_0}$$

Example 3.7.3: Solving an Applied Problem Involving a Rational Function

A large mixing tank currently contains 100 gallons of water into which 5 pounds of sugar have been mixed. A tap will open pouring 10 gallons per minute of water into the tank at the same time sugar is poured into the tank at a rate of 1 pound per minute. Find the concentration (pounds per gallon) of sugar in the tank after 12 minutes. Is that a greater concentration than at the beginning?

Solution:

Let t be the number of minutes since the tap opened. Since the water increases at 10 gallons per minute, and the sugar increases at 1 pound per minute, these are constant rates of change. This tells us the amount of water in the tank is changing linearly, as is the amount of sugar in the tank. We can write an equation independently for each:

water: $W(t) = 100 + 10t$ in gallons
sugar: $S(t) = 5 + 1t$ in pounds

The concentration, C, will be the ratio of pounds of sugar to gallons of water

$$C(t) = \frac{5+t}{100+10t}$$

The concentration after 12 minutes is given by evaluating $C(t)$ at $t = 12$.

$$\begin{aligned} C(12) &= \frac{5+12}{100+10(12)} \\ &= \frac{17}{220} \end{aligned}$$

This means the concentration is 17 pounds of sugar to 220 gallons of water.

At the beginning, the concentration is

$$\begin{aligned} C(0) &= \frac{5+0}{100+10(0)} \\ &= \frac{1}{20} \end{aligned}$$

Since $\frac{17}{220} \approx 0.08 > \frac{1}{20} = 0.05$, the concentration is greater after 12 minutes than at the beginning.

Analysis

To find the horizontal asymptote, divide the leading coefficient in the numerator by the leading coefficient in the denominator:

$$\frac{1}{10} = 0.1$$

In this problem, the horizontal asymptote is $y = 0.1$. This means the concentration C, the ratio of pounds of sugar to gallons of water, will approach 0.1 in the long term.

3.7.3: There are 1,200 freshmen and 1,500 sophomores at a prep rally at noon. After 12 p.m., 20 freshmen arrive at the rally every five minutes while 15 sophomores leave the rally. Find the ratio of freshmen to sophomores at 1 p.m.

Answer

$\frac{12}{11}$

Finding the Domains of Rational Functions

A **vertical asymptote** represents a value at which a rational function is undefined, so that value is not in the domain of the function. A reciprocal function cannot have values in its domain that cause the denominator to equal zero. In general, to find the domain of a rational function, we need to determine which inputs would cause division by zero.

DOMAIN OF A RATIONAL FUNCTION

The domain of a rational function includes all real numbers except those that cause the denominator to equal zero.

Given a rational function, find the domain.

1. Set the denominator equal to zero.
2. Solve to find the x-values that cause the denominator to equal zero.
3. The domain is all real numbers except those found in Step 2.

Example 3.7.4: Finding the Domain of a Rational Function

Find the domain of $f(x) = \frac{x+3}{x^2-9}$.

Solution:

Begin by setting the denominator equal to zero and solving.

$$x^2 - 9 = 0$$

$$x^2 = 9$$

$$x = \pm 3$$

The denominator is equal to zero when $x = \pm 3$. The domain of the function is all real numbers except $x = \pm 3$.

Analysis

A graph of this function, as shown in Figure **3.7.9**, confirms that the function is not defined when $x = \pm 3$.

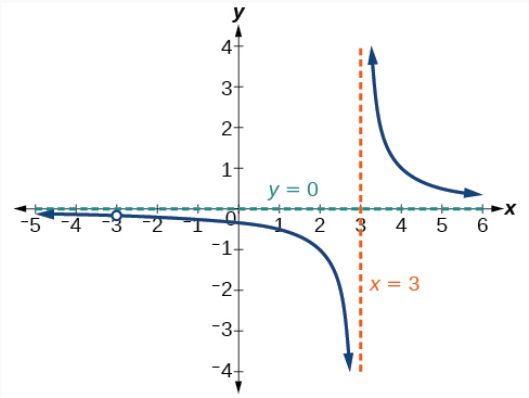


***Figure* 3.7.9.**

There is a vertical asymptote at $x = 3$ and a hole in the graph at $x = -3$. We will discuss these types of holes in greater detail later in this section.

3.7.4: Find the domain of $f(x) = \frac{4x}{5(x-1)(x-5)}$.

Answer

The domain is all real numbers except $x = 1$ and $x = 5$.

Identifying Vertical Asymptotes of Rational Functions

By looking at the graph of a rational function, we can investigate its local behavior and easily see whether there are asymptotes. We may even be able to approximate their location. Even without the graph, however, we can still determine whether a given rational function has any asymptotes, and calculate their location.

Vertical Asymptotes

The vertical asymptotes of a rational function may be found by examining the factors of the denominator that are not common to the factors in the numerator. Vertical asymptotes occur at the zeros of such factors.

Given a rational function, identify any vertical asymptotes of its graph

1. Factor the numerator and denominator.
2. Note any restrictions in the domain of the function.
3. Reduce the expression by canceling common factors in the numerator and the denominator.
4. Note any values that cause the denominator to be zero in this simplified version. These are where the vertical asymptotes occur.
5. Note any restrictions in the domain where asymptotes do not occur. These are **removable discontinuities**, or "holes."

Example 3.7.5: Identifying Vertical Asymptotes

Find the vertical asymptotes of the graph of $k(x) = \frac{5+2x^2}{2-x-x^2}$.

Solution:

First, factor the numerator and denominator.

$$k(x) = \frac{5 + 2x^2}{2 - x - x^2}$$

.

$$= \frac{5 + 2x^2}{(2 + x)(1 - x)}$$

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$(2 + x)(1 - x) = 0$$

$$x = -2, \text{ or } 1$$

Neither $x = -2$ nor $x = 1$ are zeros of the numerator, so the two values indicate two vertical asymptotes. The graph in Figure 3.7.10 confirms the location of the two vertical asymptotes.

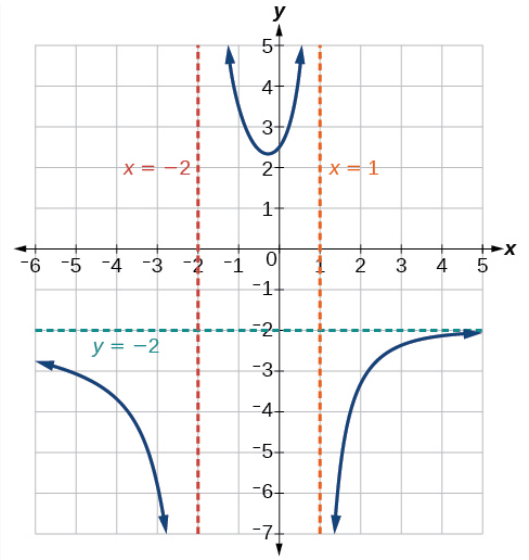


Figure 3.7.10.

Removable Discontinuities

Occasionally, a graph will contain a hole: a single point where the graph is not defined, indicated by an open circle. We call such a hole a **removable discontinuity**. For example, the function $f(x) = \frac{x^2-1}{x^2-2x-3}$ may be re-written by factoring the numerator and the denominator.

$$f(x) = \frac{(x+1)(x-1)}{(x+1)(x-3)}$$

Notice that $x+1$ is a common factor to the numerator and the denominator. The zero of this factor, $x = -1$, is the location of the removable discontinuity. Notice also that $x-3$ is not a factor in both the numerator and denominator. The zero of this factor, $x = 3$, is the vertical asymptote. See Figure 3.7.11. [Note that removable discontinuities may not be visible when we use a graphing calculator, depending upon the window selected.]

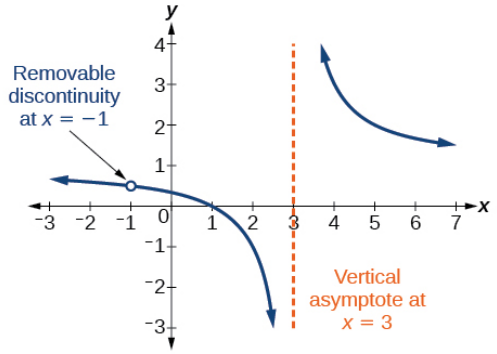


Figure 3.7.11.

REMOVABLE DISCONTINUITIES OF RATIONAL FUNCTIONS

A **removable discontinuity** may occur in the graph of a rational function at $x = a$ if a is a zero for a factor in the denominator that is common with a factor in the numerator. We factor the numerator and denominator and check for common factors. If we find any, we set the common factor equal to 0 and solve. If the multiplicity of this factor is greater than or equal to that in the denominator, this is the location of a removable discontinuity. If the multiplicity of the factor is greater in the denominator than in the numerator, then there is still an asymptote at $x = a$.

Example 3.7.6: Identifying Vertical Asymptotes and Removable Discontinuities for a Graph

Find the vertical asymptotes and removable discontinuities of the graph of $k(x) = \frac{x-2}{x^2-4}$.

Solution:

Factor the numerator and the denominator.

$$k(x) = \frac{x-2}{(x-2)(x+2)}$$

$(x-2)$ is a common factor in the numerator and the denominator. The zero for this factor is $x = 2$. Since the multiplicities of $(x-2)$ in both the numerator and denominator are equal, this is the location of a removable discontinuity.

The factor $(x+2)$ is in the denominator but is **not** in the numerator. The zero for this factor is $x = -2$. The vertical asymptote is $x = -2$. See Figure 3.7.12.

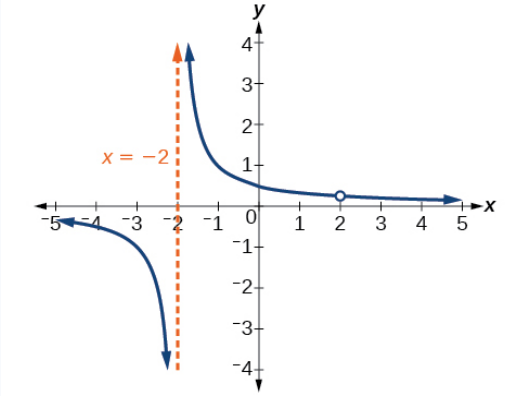


***Figure* 3.7.12.**

The graph of this function will have a vertical asymptote at $x = -2$, but at $x = 2$ the graph will have a hole.

Try it 3.7.5: Find the vertical asymptotes and removable discontinuities of the graph of $f(x) = \frac{x^2-25}{x^3-6x^2+5x}$.

Answer

Removable discontinuity at $x = 5$. Vertical asymptotes: $x = 0$, $x = 1$.

Identifying Horizontal Asymptotes of Rational Functions

While vertical asymptotes describe the behavior of a graph as the *output* increases or decreases without bound, horizontal asymptotes help describe the behavior of a graph as the *input* gets approaches infinity or negative infinity. Recall that a polynomial's end behavior will mirror that of the leading term. Likewise, a rational function's end behavior will mirror that of the ratio of the function that is the ratio of the leading terms.

There are four distinct outcomes when checking for horizontal asymptotes:

Case 1: If the degree of the denominator is greater than the degree of the numerator, there is a **horizontal asymptote** at $y = 0$.

$$\text{Example: } f(x) = \frac{4x+2}{x^2+4x-5}$$

In this case, the end behavior is $f(x) \approx \frac{4x}{x^2} = \frac{4}{x}$ (looking at the ratio of the leading term of the numerator to the leading term of the denominator). This tells us that, as the inputs increase or decrease without bound, this function will behave similarly to the function $g(x) = \frac{4}{x}$, and the outputs will approach zero, resulting in a horizontal asymptote at $y = 0$. See **Figure 3.7.13**. Note that this graph crosses the horizontal asymptote.

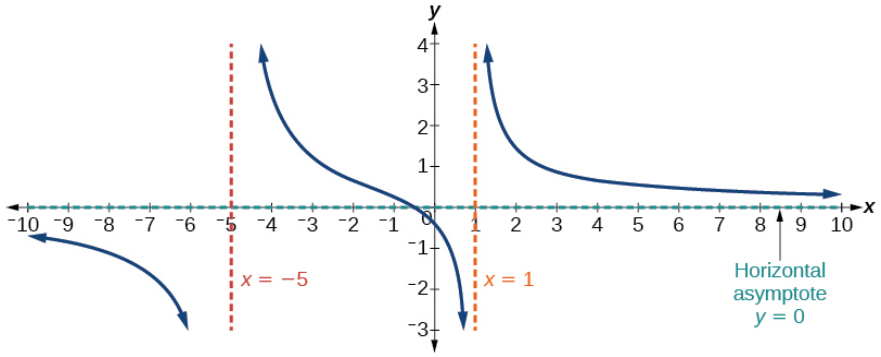


***Figure* 3.7.13:** *Horizontal asymptote* $y = 0$ *when* $f(x) = \frac{p(x)}{q(x)}$, *where degree of* $p <$ *degree of* q.

Case 2: If the degree of the denominator is less than the degree of the numerator by one, we get a **slant asymptote**.

Example: $f(x) = \dfrac{3x^2 - 2x + 1}{x - 1}$

In this case, the end behavior is $f(x) \approx \frac{3x^2}{x} = 3x$. This tells us that as the inputs increase or decrease without bound, this function will behave similarly to the function $g(x) = 3x$. As the inputs grow large, the outputs will grow and not level off, so this graph has no horizontal asymptote. However, the graph of $g(x) = 3x$ lis a line with a slope of 3, and since f will behave similarly to g, it will approach a line whose slope is 3. This line is a slant asymptote.

To find the equation of the slant asymptote, divide $\frac{3x^2-2x+1}{x-1}$. The quotient is $3x + 1$, with a remainder of 2. The slant asymptote is the graph of the line $g(x) = 3x + 1$. See Figure 3.7.14.

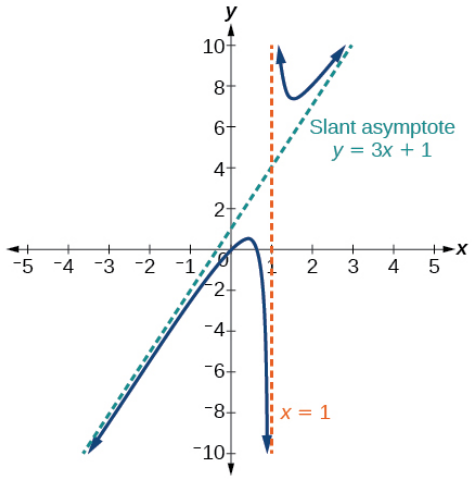


***Figure* 3.7.14:** *Slant asymptote when $f(x) = \frac{p(x)}{q(x)}$, where degree of p >degree of q by 1.*

Case 3: If the degree of the denominator equals the degree of the numerator, there is a horizontal asymptote at $y = \dfrac{a_n}{b_n}$, where a_n and b_n are respectively the leading coefficients of $p(x)$ and $q(x)$ for $f(x) = \dfrac{p(x)}{q(x)}$.

Example: $f(x) = \dfrac{3x^2 + 2}{x^2 + 4x - 5}$

In this case, the end behavior is $f(x) \approx \dfrac{3x^2}{x^2} = 3$. This tells us that as the inputs grow large in magnitude, this function will behave like the function $g(x) = 3$, which is a horizontal line. As $x \to \pm\infty$, $f(x) \to 3$, resulting in a horizontal asymptote at $y = 3$. See Figure 3.7.15. Note that this graph crosses the horizontal asymptote.

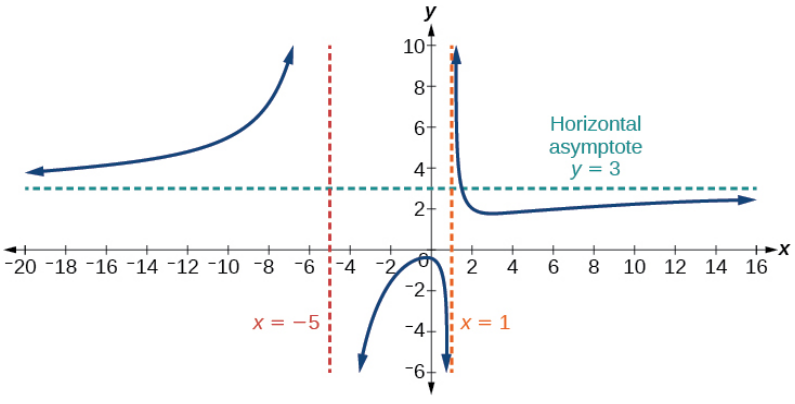


***Figure* 3.7.15:** *Horizontal asymptote when $f(x) = \frac{p(x)}{q(x)}$, where degree of p =degree of q.*

Notice that, while the graph of a rational function will never cross a **vertical asymptote**, the graph may or may not cross a horizontal or slant asymptote. Also, although the graph of a rational function may have many vertical asymptotes, the graph will have at most one horizontal (or slant) asymptote.

Case 4: If the degree of the numerator is larger than the degree of the denominator by more than one, the graph does not have a horizontal asymptote. The **end behavior** of the graph will mimic the behavior of the reduced end behavior fraction. For instance, if we had the function

$$f(x) = \frac{3x^5 - x^2}{x + 3}$$

with end behavior

$$f(x) \approx \frac{3x^5}{x} = 3x^4,$$

the end behavior of the graph would look similar to that of an even polynomial with a positive leading coefficient.

$$x \to \pm\infty, f(x) \to \infty$$

HORIZONTAL ASYMPTOTES OF RATIONAL FUNCTIONS

The **horizontal asymptote** of a rational function can be determined by looking at the degrees of the numerator and denominator.

- Degree of numerator is less than degree of denominator: horizontal asymptote at $y = 0$.
- Degree of numerator is greater than degree of denominator by one: no horizontal asymptote; slant asymptote.
- Degree of numerator is equal to degree of denominator: horizontal asymptote at ratio of leading coefficients, $y = \frac{a_n}{b_n}$.
- Degree of numerator is larger than degree of the denominator by more than one: no horizontal asymptote.

Example 3.7.7: Identifying Horizontal and Slant Asymptotes

For the functions listed, identify the horizontal or slant asymptote.

1. $g(x) = \frac{6x^3 - 10x}{2x^3 + 5x^2}$
2. $h(x) = \frac{x^2 - 4x + 1}{x + 2}$
3. $k(x) = \frac{x^2 + 4x}{x^3 - 8}$

Solution:

For these solutions, we will refer to the polynomial in the numerator as $p(x)$ and the polynomial in the denominator as $q(x)$.

1. $g(x) = \frac{6x^3 - 10x}{2x^3 + 5x^2}$: The degree of p =degree of $q = 3$, so we can find the horizontal asymptote by taking the ratio of the leading terms. There is a horizontal asymptote at $y = \frac{6}{2}$ or $y = 3$.
2. $h(x) = \frac{x^2 - 4x + 1}{x + 2}$: The degree of $p = 2$ and degree of $q = 1$. Since $p > q$ by 1, there is a slant asymptote $y = x - 6$.

   $$\begin{array}{r|rrr} -2 & 1 & -4 & 1 \\ & & -2 & 12 \\ \hline & 1 & -6 & 13 \end{array}$$
3. $k(x) = \frac{x^2 + 4x}{x^3 - 8}$: The degree of $p = 2$ < degree of $q = 3$, so there is a horizontal asymptote $y = 0$.

Example 3.7.8: Identifying Horizontal Asymptotes

In Example 3.7.3, which gives the concentration of sugar in a mixing tank at time t, we created the function $C(t) = \frac{5 + t}{100 + 10t}$.

Find the horizontal asymptote and interpret it in the context of the problem.

Solution:

Both the numerator and denominator are linear (degree 1). Because the degrees are equal, there will be a horizontal asymptote at the ratio of the leading coefficients. In the numerator, the leading term is t, with coefficient 1. In the denominator, the leading term is 10t, with coefficient 10. The horizontal asymptote will be at the ratio of these values:

$$t \to \infty,\ C(t) \to \frac{1}{10}$$

This function will have a horizontal asymptote at $y = \frac{1}{10}$.

This tells us that as the values of t increase, the values of C will approach $\frac{1}{10}$. In context, this means that, as more time goes by, the concentration of sugar in the tank will approach one-tenth of a pound of sugar per gallon of water or $\frac{1}{10}$ pounds per gallon.

Example 3.7.9: Identifying Horizontal and Vertical Asymptotes

Find the horizontal and vertical asymptotes of the function

$$f(x) = \frac{(x - 2)(x + 3)}{(x - 1)(x + 2)(x - 5)}$$

Solution:

First, note that this function has no common factors, so there are no potential removable discontinuities.

The function will have vertical asymptotes when the denominator is zero, causing the function to be undefined. The denominator will be zero at $x = 1, -2,$ and 5, indicating vertical asymptotes at these values.

The numerator has degree 2, while the denominator has degree 3. Since the degree of the denominator is greater than the degree of the numerator, the denominator will grow faster than the numerator, causing the outputs to tend towards zero as the inputs get large, and so as $x \to \pm\infty$, $f(x) \to 0$. This function will have a horizontal asymptote at $y = 0$. See Figure 3.7.16.

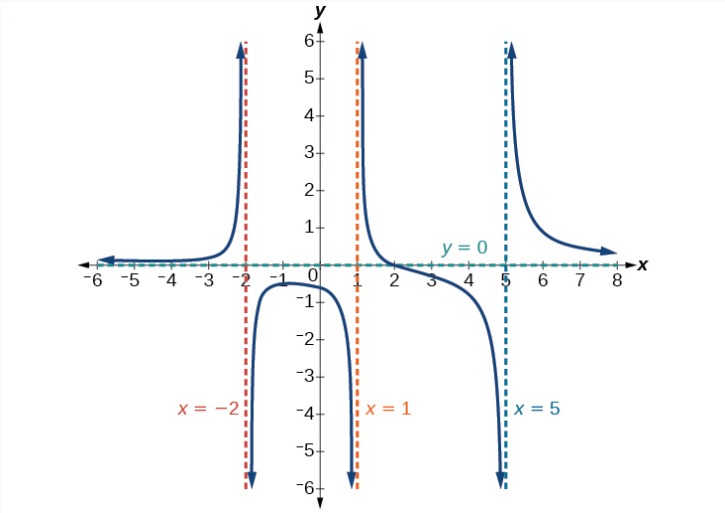


***Figure* 3.7.16.**

Try It **3.7.6**: Find the vertical and horizontal asymptotes of the function:

$$f(x) = \frac{(2x-1)(2x+1)}{(x-2)(x+3)}$$

Answer

Vertical asymptotes at $x = 2$ and $x = -3$; horizontal asymptote at $y = 4$.

INTERCEPTS OF RATIONAL FUNCTIONS

A **rational function** will have a y-intercept at $f(0)$, if the function is defined at zero. A rational function will not have a y-intercept if the function is not defined at zero.

Likewise, a rational function will have x-intercepts at the inputs that cause the output to be zero. A fraction is only equal to zero when the numerator is zero; thus, x-intercepts can only occur when the numerator of the rational function is equal to zero.

Example 3.7.10: Finding the Intercepts of a Rational Function

Find the intercepts of $f(x) = \dfrac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$.

Solution:

We can find the y-intercept by evaluating the function at zero

$$\begin{aligned} f(0) &= \frac{(0-2)(0+3)}{(0-1)(0+2)(0-5)} \\ &= -\frac{6}{10} \\ &= -\frac{3}{5} \\ &= -0.6 \end{aligned}$$

The x-intercepts will occur when the function is equal to zero:

$$0 = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)} \text{ This is zero when the numerator is zero.}$$

$$\begin{aligned} 0 &= (x-2)(x+3) \\ x &= 2, -3 \end{aligned}$$

The \(y\)-intercept is $(0, -0.6)$, the x-intercepts are $(2, 0)$ and $(-3, 0)$.See Figure 3.7.17.

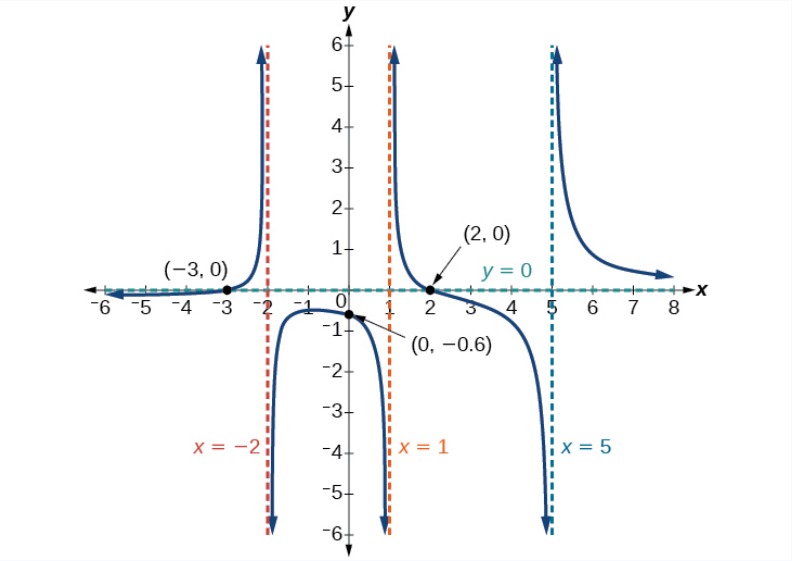


***Figure* 3.7.17.**

Try it 3.7.7: Given the reciprocal squared function that is shifted right 3 units and down 4 units, write this as a rational function. Then, find the x- and y-intercepts and the horizontal and vertical asymptotes.

Answer

For the transformed reciprocal squared function, we find the rational form.

$$f(x) = \frac{1}{(x-3)^2} - 4 = \frac{1-4(x-3)^2}{(x-3)^2} = \frac{1-4(x^2-6x+9)}{(x-3)(x-3)} = \frac{-4x^2+24x-35}{x^2-6x+9}$$

Because the numerator is the same degree as the denominator we know that as $x \to \pm\infty$, $f(x) \to -4$; so $y = -4$ is the horizontal asymptote. Next, we set the denominator equal to zero, and find that the vertical asymptote is $x = 3$, because as $x \to 3$, $f(x) \to \infty$. We then set the numerator equal to 0 and find the x-intercepts are at $(2.5, 0)$ and $(3.5, 0)$. Finally, we evaluate the function at 0 and find the y-intercept to be at $(0, -\frac{35}{9})$.

Graphing Rational Functions

In Example 3.7.10, we see that the factors of the numerator of a rational function reveal the x-intercepts of the graph, whereas the factors of the denominator reveal the vertical asymptotes of the graph. The factors of the numerator may have multiplicity greater than one. The effect on the shape of the graph at those x-intercepts is the same as we saw with polynomial functions.

The vertical asymptotes associated with the factors of the denominator will mirror one of the two toolkit reciprocal functions. When the degree of the factor in the denominator is odd, the distinguishing characteristic is that on one side of the vertical asymptote the graph heads towards positive infinity, and on the other side the graph heads towards negative infinity. See Figure **3.7.18**.

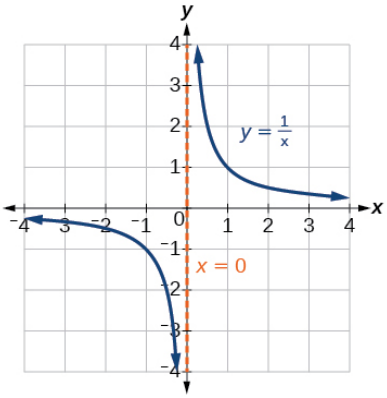


***Figure* 3.7.18.**

When the degree of the factor in the denominator is even, the distinguishing characteristic is that the graph either heads toward positive infinity on both sides of the vertical asymptote or heads toward negative infinity on both sides. See Figure **3.7.19**.

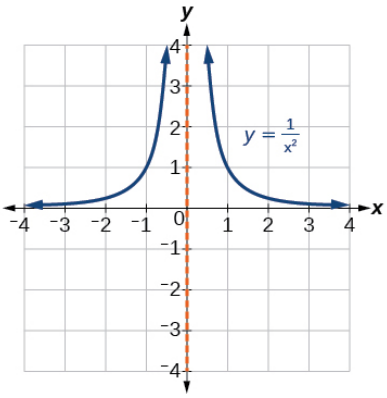


***Figure* 3.7.19.**

For example, the graph of $f(x) = \dfrac{(x+1)^2(x-3)}{(x+3)^2(x-2)}$ is shown in Figure **3.7.20**.

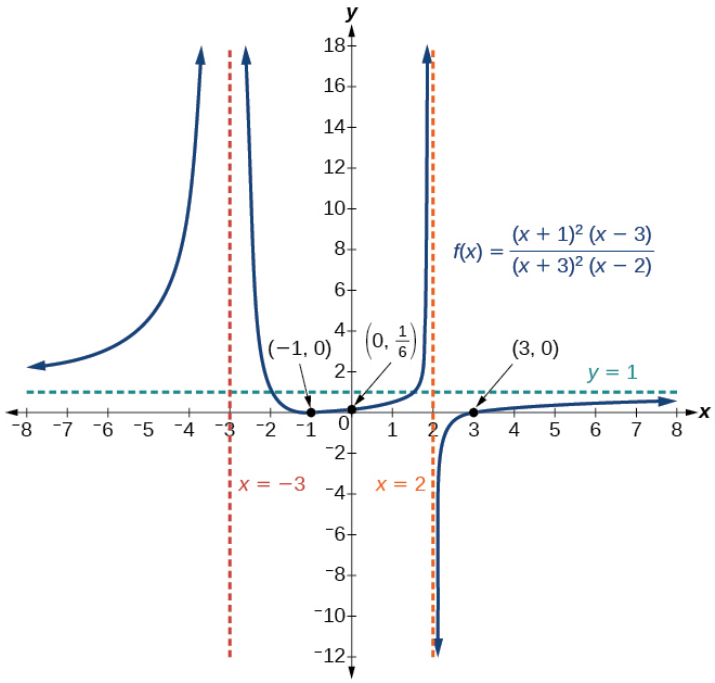


***Figure* 3.7.20.**

- At the x-intercept $x = -1$ corresponding to the $(x+1)^2$ factor of the numerator, the graph "bounces", consistent with the quadratic nature of the factor.
- At the x-intercept $x = 3$ corresponding to the $(x-3)$ factor of the numerator, the graph passes through the axis as we would expect from a linear factor.
- At the vertical asymptote $x = -3$ corresponding to the $(x+3)^2$ factor of the denominator, the graph heads towards positive infinity on both sides of the asymptote, consistent with the behavior of the function $f(x) = \frac{1}{x^2}$.
- At the vertical asymptote $x = 2$, corresponding to the $(x-2)$ factor of the denominator, the graph heads towards positive infinity on the left side of the asymptote and towards negative infinity on the right side, consistent with the behavior of the function $f(x) = \frac{1}{x}$.

How To: Given a rational function, sketch a possible graph.

1. Evaluate the function at $x = 0$ (if zero is in the domain) to find the y-intercept.
2. Factor the numerator and denominator.
3. For factors in the numerator not common to the denominator, determine where each factor of the numerator is zero to find the x-intercepts.
4. Find the multiplicities of the x-intercepts to determine the behavior of the graph at those points.
5. For factors in the denominator, note the multiplicities of the zeros to determine the local behavior. For those factors not common to the numerator, find the vertical asymptotes by setting those factors equal to zero and then solve.
6. For factors in the denominator common to factors in the numerator, find the removable discontinuities by setting those factors equal to zero and then solve.
7. Compare the degrees of the numerator and the denominator to determine the horizontal or slant asymptotes.
8. Sketch the graph.

Example 3.7.11: Graphing a Rational Function

Sketch a possible graph of $f(x) = \frac{(x+2)(x-3)}{(x+1)^2(x-2)}$.

Solution:

We can start by noting that the function is already factored, saving us a step.

Next, we find the intercepts. Evaluating the function at zero gives the y-intercept:

$$f(0) = \frac{(0+2)(0-3)}{(0+1)^2(0-2)} = 3$$

To find the x-intercepts, we determine when the numerator of the function is zero. Setting each factor equal to zero, we find x-intercepts at $x = -2$ and $x = 3$. At each, the behavior will be linear (multiplicity 1), with the graph passing through the intercept.

We have a y-intercept at $(0,3)$ and x-intercepts at $(-2,0)$ and $(3,0)$.

To find the vertical asymptotes, we determine when the denominator is equal to zero. This occurs when $x+1=0$ and when $x-2=0$, giving us vertical asymptotes at $x=-1$ and $x=2$.

There are no common factors in the numerator and denominator. This means there are no removable discontinuities.

Finally, the degree of denominator is larger than the degree of the numerator, telling us this graph has a horizontal asymptote at $y=0$.

To sketch the graph, we might start by plotting the three intercepts. Since the graph has no x-intercepts between the vertical asymptotes, and the y-intercept is positive, we know the function must remain positive between the asymptotes, letting us fill in the middle portion of the graph as shown in Figure 3.7.21. Note that we have no method to find the exact location of the local minimum.

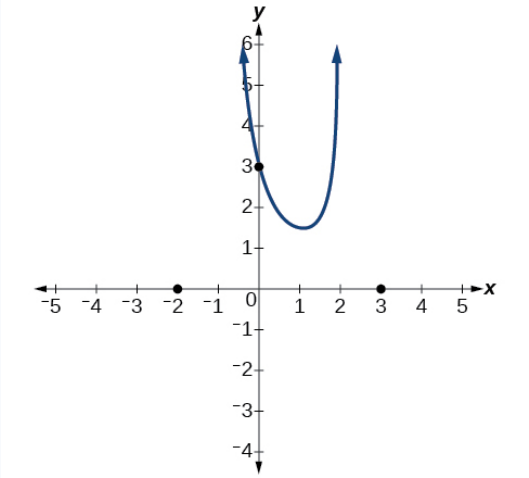


***Figure* 3.7.21.**

The factor associated with the vertical asymptote at $x=-1$ was squared, so we know the behavior will be the same on both sides of the asymptote. The graph heads toward positive infinity as the inputs approach the asymptote on the right, so the graph will head toward positive infinity on the left as well.

For the vertical asymptote at $x=2$, the factor was not squared, so the graph will have opposite behavior on either side of the asymptote. See Figure 3.7.22. After passing through the x-intercepts, the graph will then level off toward an output of zero, as indicated by the horizontal asymptote. Note: The scale of this graph does not allow us to see all of the details. Can you explain why there must be one more local minimum (in Quadrant III), and a local maximum (in Quadrant I)? (If you can, write it up and give it to your instructor for **extra credit**!)

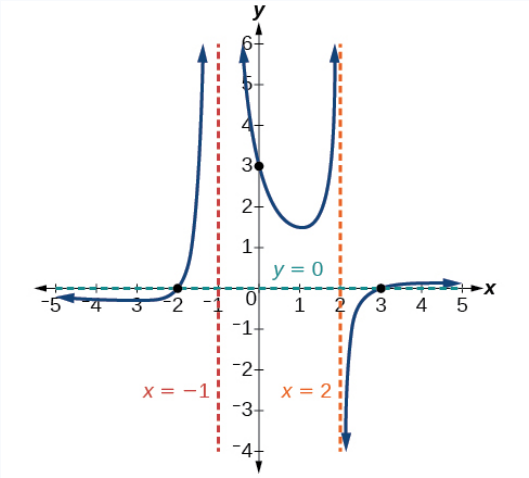


***Figure* 3.7.22.**

3.7.8: Given the function $f(x)=\frac{(x+2)^2(x-2)}{2(x-1)^2(x-3)}$, use the characteristics of polynomials and rational functions to describe its behavior and sketch a possible graph of the function.

Answer

Horizontal asymptote at $y = \frac{1}{2}$. Vertical asymptotes at $x = 1$ (multiplicity 2) and $x = 3$ (multiplicity 1). y-intercept at $(0, \frac{4}{3})$. x-intercepts at $(2, 0)$ and $(-2, 0)$. $(-2, 0)$ is a zero with multiplicity 2, and the graph bounces off the x-axis at this point. $(2, 0)$ is a single zero and the graph crosses the x-axis at this point.

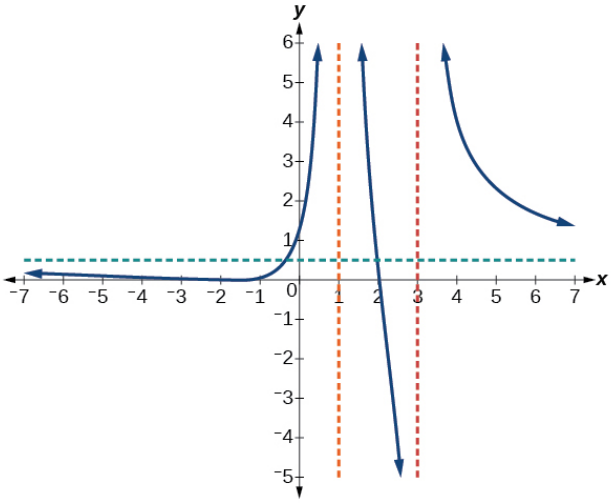


***Figure* 3.7.23.**

Writing Rational Functions

Now that we have analyzed the equations for rational functions and how they relate to a graph of the function, we can use information given by a graph to write a possible function. A rational function written in factored form will have an x-intercept where each factor of the numerator is equal to zero. (An exception occurs in the case of a removable discontinuity.) As a result, we can form a numerator of a function whose graph will pass through a set of x-intercepts by introducing a corresponding set of factors. Likewise, because the function will have a vertical asymptote where each factor of the denominator is equal to zero, we can form a denominator that will produce the vertical asymptotes by introducing a corresponding set of factors.

WRITING RATIONAL FUNCTIONS FROM INTERCEPTS AND ASYMPTOTES

If a **rational function** has x-intercepts at $x = x_1, x_2, \ldots, x_n$, vertical asymptotes at $x = v_1, v_2, \ldots, v_m$, and no $x_i =$ any v_j, then a possible function can be written in the form:

$$f(x) = a\frac{(x - x_1)^{p_1}(x - x_2)^{p_2} \cdots (x - x_n)^{p_n}}{(x - v_1)^{q_1}(x - v_2)^{q_2} \cdots (x - v_m)^{q_n}}$$

where the powers p_i or q_i on each factor can be determined by the behavior of the graph at the corresponding intercept or asymptote, and the stretch factor a can be determined given a value of the function other than the x-intercept **or** by the horizontal asymptote if it is nonzero.

How To: Given a graph of a rational function, write a possible function.

1. Determine the factors of the numerator. Examine the behavior of the graph at the x-intercepts to determine the zeroes and their multiplicities. (This is easy to do when finding the "simplest" function with small multiplicities—such as 1 or 3—but may be difficult for larger multiplicities—such as 5 or 7, for example.)
2. Determine the factors of the denominator. Examine the behavior on both sides of each vertical asymptote to determine the factors and their powers.
3. Use any clear point on the graph to find the stretch factor.

Example 3.7.12: Writing a Rational Function from Intercepts and Asymptotes

Write a possible equation for the rational function shown in Figure **3.7.24**.

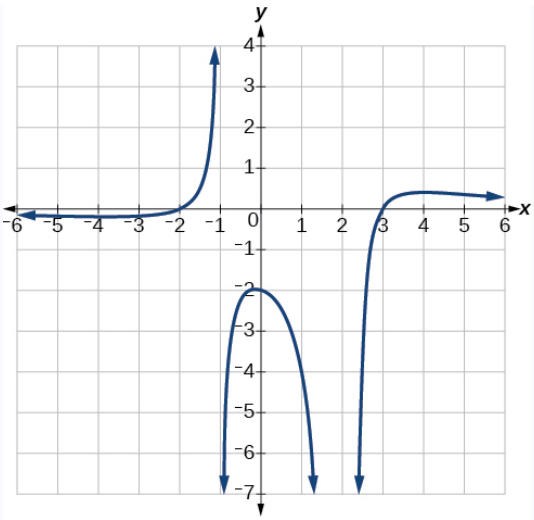


Figure 3.7.24.

Solution:

The graph appears to have x-intercepts at $x = -2$ and $x = 3$. At both, the graph passes through the intercept, suggesting linear factors. The graph has two vertical asymptotes. The one at $x = -1$ exhibits the basic behavior similar to $\frac{1}{x}$, with the graph heading toward positive infinity on one side and heading toward negative infinity on the other. The asymptote at $x = 2$ is exhibiting a behavior similar to $\frac{1}{x^2}$, with the graph heading toward negative infinity on both sides of the asymptote. See Figure 3.7.25.

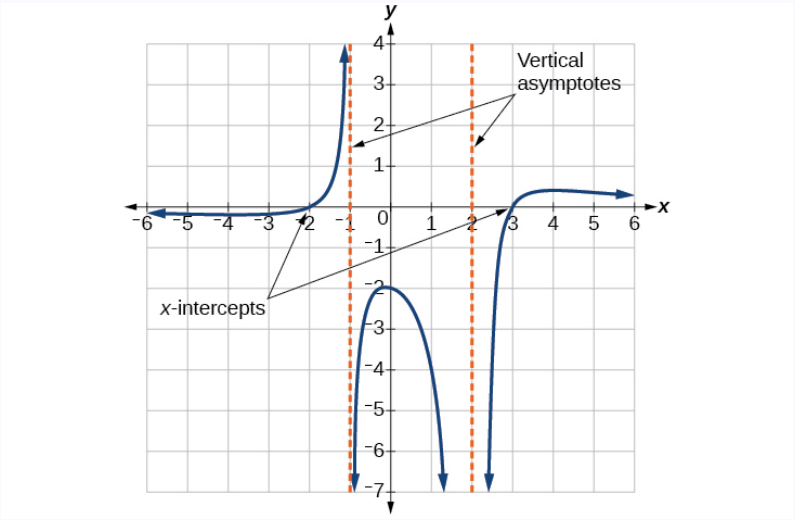


Figure 3.7.25.

We can use this information to write a function of the form

$$f(x) = a\frac{(x+2)(x-3)}{(x+1)(x-2)^2}$$

To find the stretch factor, we can use another clear point on the graph, such as the y-intercept $(0, -2)$.

$$-2 = a\frac{(0+2)(0-3)}{(0+1)(0-2)^2}$$
$$-2 = a\frac{-6}{4}$$
$$a = \frac{-8}{-6} = \frac{4}{3}$$

This gives us a final function of $f(x) = \frac{4(x+2)(x-3)}{3(x+1)(x-2)^2}$.

Media

Access these online resources for additional instruction and practice with rational functions.

- Graphing Rational Functions
- Find the Equation of a Rational Function
- Determining Vertical and Horizontal Asymptotes
- Find the Intercepts, Asymptotes, and Hole of a Rational Function

Key Equations

Rational Function	$f(x)=\frac{P(x)}{Q(x)}=\frac{a_p x^p+a_{p-1}x^{p-1}+\ldots+a_1x+a_0}{b_q x^q+b_{q-1}x^{q-1}+\ldots+b_1x+b_0}$, Domain $=\{x\|Q(x)\neq 0\}$

Key Concepts

- Arrow notation can be used to describe local behavior and end behavior of the toolkit functions $f(x)=\frac{1}{x}$ and $f(x)=\frac{1}{x^2}$.
- A function that tends toward positive or negative infinity as the input values approach an x-value that is not in the domain has a vertical asymptote.
- A function can have more than one vertical asymptote.
- A function that levels off at a horizontal value has a horizontal asymptote.
- The domain of a rational function includes all real numbers except those that cause the denominator to equal zero.
- The vertical asymptotes of a rational function will occur where the denominator of the function is equal to zero and the numerator is not zero.
- A rational functions behavior at a vertical asymptote will mirror $y=\frac{1}{x}$ if the degree of the factor in the denominator is odd, and will mirror $y=\frac{1}{x^2}$ if the degree of the factor in the denominator is even.
- A removable discontinuity might occur in the graph of a rational function if an input causes both numerator and denominator to be zero.
- A rational function's end behavior will mirror that of the ratio of the leading terms of the numerator and denominator functions.
- Graph rational functions by finding the intercepts, behavior at the intercepts and asymptotes, and end behavior.
- If a rational function has x-intercepts at $x=x_1,x_2,\ldots,x_n$, vertical asymptotes at $x=v_1,v_2,\ldots,v_m$, and no x_i equals any v_j, then the function can be written in the form

$$f(x)=a\frac{(x-x_1)^{p_1}(x-x_2)^{p_2}\cdots(x-x_n)^{p_n}}{(x-v_1)^{q_1}(x-v_2)^{q_2}\cdots(x-v_m)^{q_m}}.$$

Contributor

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3.7E: Exercises

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3.8: Inverses and Radical Functions

Skills to Develop

In this section, you will:

- Find the inverse of an invertible polynomial function.
- Restrict the domain to find the inverse of a polynomial function.

A mound of gravel is in the shape of a cone with the height equal to twice the radius.

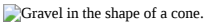


***Figure* 3.8.1**

The volume is found using a formula from elementary geometry.

$$\begin{aligned} V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi r^2 (2r) \\ &= \frac{2}{3}\pi r^3 \end{aligned}$$

We have written the volume V in terms of the radius r. However, in some cases, we may start out with the volume and want to find the radius. For example: A customer purchases 100 cubic feet of gravel to construct a cone shape mound with a height twice the radius. What are the radius and height of the new cone? To answer this question, we use the formula

$$r = \sqrt[3]{\frac{3V}{2\pi}}$$

This function is the inverse of the formula for V in terms of r.

In this section, we will explore the inverses of polynomial and rational functions, and the radical functions we encounter in the process.

Finding the Inverse of a Polynomial Function

Two functions f and g are inverse functions if for every coordinate pair (a, b) in f, there exists a corresponding coordinate pair (b, a) in the inverse function, g. In other words, the coordinate pairs of the inverse functions have the input and output interchanged. Only one-to-one functions have inverses. Recall that a one-to-one function has a unique output value for each input value and passes the horizontal line test.

For example, suppose a water runoff collector is built in the shape of a parabolic trough as shown in Figure **3.8.2**. We can use the information in the figure to find the surface area of the water in the trough as a function of the depth of the water.

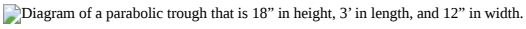


***Figure* 3.8.2**

Because it will be helpful to have an equation for the parabolic cross-sectional shape, we will impose a coordinate system at the cross section, with x measured horizontally and y measured vertically, with the origin at the vertex of the parabola (Figure **3.8.3**).

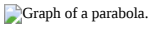


***Figure* 3.8.3**

From this we find an equation for the parabolic shape. We placed the origin at the vertex of the parabola, so we know the equation will have form $y(x) = ax^2$. Our equation will need to pass through the point $(6, 18)$, from which we can solve for the stretch factor a.

$$18 = a6^2$$
$$a = \frac{18}{36}$$
$$= \frac{1}{2}$$

Our parabolic cross section has the equation

$$y(x) = \frac{1}{2}x^2$$

We are interested in the **surface area** of the water, so we must determine the width at the top of the water as a function of the water depth. For any depth y, the width will be given by $2x$, so we need to solve the equation above for x and find the inverse function. However, notice that the original function is not one-to-one, and indeed, given any output there are two inputs that produce the same output, one positive and one negative.

To find an inverse, we can restrict our original function to a limited domain on which it *is* one-to-one. In this case, it makes sense to restrict ourselves to positive x values. On this domain, we can find an inverse by solving for the input variable:

$$y = \frac{1}{2}x^2$$
$$2y = x^2$$
$$x = \pm\sqrt{2y}$$

This is not a function as written. We are limiting ourselves to positive x values, so we eliminate the negative solution, giving us the inverse function we're looking for.

$$y = \frac{x^2}{2},\ x > 0$$

Because x is the distance from the center of the parabola to either side, the entire width of the water at the top will be $2x$. The trough is 3 feet (36 inches) long, so the surface area will then be:

$$Area = l \cdot w$$
$$= 36 \cdot 2x$$
$$= 72x$$
$$= 72\sqrt{2y}$$

This example illustrates two important points:

1. When finding the inverse of a quadratic, we have to limit ourselves to a domain on which the function is one-to-one.
2. The inverse of a quadratic function is a square root function. Both are toolkit functions and different types of power functions.

Functions involving roots are often called **radical functions**. While it is not possible to find an inverse of most polynomial functions, some basic polynomials do have inverses. Such functions are called **invertible functions**, and we use the notation $f^{-1}(x)$.

Warning: $f^{-1}(x)$ is not the same as the reciprocal of the function $f(x)$. This use of "–1" is reserved to denote inverse functions. To denote the reciprocal of a function $f(x)$, we would need to write $(f(x))^{-1} = \frac{1}{f(x)}$.

An important relationship between a function and its inverse is that they "undo" each other. If f^{-1} is the inverse of a function f, then f is the inverse of the function f^{-1}. In other words, whatever the function f does to x, f^{-1} undoes it—and vice-versa.

$$f^{-1}(f(x)) = x, \text{ for all } x \text{ in the domain of } f$$

and

$$f(f^{-1}(x)) = x, \text{ for all } x \text{ in the domain of } f^{-1}$$

Note that the inverse switches the domain and range of the original function.

VERIFYING TWO FUNCTIONS ARE INVERSES OF ONE ANOTHER

Two functions, f and g, are inverses of one another if for all x in the domain of f and g,

$$g(f(x)) = f(g(x)) = x$$

How To: Given a polynomial function, find the inverse of the function by restricting the domain in such a way that the new function is one-to-one

1. Replace $f(x)$ with y.
2. Interchange x and y.
3. Solve for y, and rename the function $f^{-1}(x)$.

Example 3.8.1: Verifying Inverse Functions

Show that $f(x) = \frac{1}{x+1}$ and $f^{-1}(x) = \frac{1}{x} - 1$ are inverses, for $x \neq 0, -1$.

Solution

We must show that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}\left(\frac{1}{x+1}\right) \\ &= \frac{1}{\frac{1}{x+1}} - 1 \\ &= (x+1) - 1 \\ &= x \end{aligned}$$

$$\begin{aligned} f(f^{-1}(x)) &= f\left(\frac{1}{x-1}\right) \\ &= \frac{1}{\left(\frac{1}{x-1}\right)+1} \\ &= \frac{1}{\frac{1}{x}} \\ &= x \end{aligned}$$

Therefore, $f(x) = \dfrac{1}{x+1}$ and $f^{-1}(x) = \dfrac{1}{x} - 1$ are inverses.

Try It 3.8.1: Show that $f(x) = \frac{x+5}{3}$ and $f^{-1}(x) = 3x - 5$ are inverses.

Answer

$f^{-1}(f(x)) = f^{-1}\left(\frac{x+5}{3}\right) = 3\left(\frac{x+5}{3}\right) - 5 = (x-5) + 5 = x$

$f(f^{-1}(x)) = f(3x-5) = \frac{(3x-5)+5}{3} = \frac{3x}{3} = x$

Example 3.8.2: Finding the Inverse of a Cubic Function

Find the inverse of the function $f(x) = 5x^3 + 1$.

Solution

This is a transformation of the basic cubic toolkit function, and based on our knowledge of that function, we know it is one-to-one. Solve for the inverse by interchanging x and y and solving for y.

$$\begin{aligned} y &= 5x^3 + 1 \\ x &= 5y^3 + 1 \\ x - 1 &= 5y^3 \end{aligned}$$

$$\frac{x-1}{5} = y^3$$

$$f^{-1}(x) = \sqrt[3]{\frac{x-1}{5}}$$

Analysis

Look at the graph of f and f^{-1}. Notice that one graph is the reflection of the other about the line $y = x$. This is always the case when graphing a function and its inverse function.

Also, since the method involved interchanging x and y, notice corresponding points. If (a, b) is on the graph of f,then (b, a) is on the graph of f^{-1}. Since $(0, 1)$ is on the graph of f, then $(1, 0)$ is on the graph of f^{-1}. Similarly, since $(1, 6)$ is on the graph of f,then $(6, 1)$ is on the graph of f^{-1} (Figure 3.8.4).

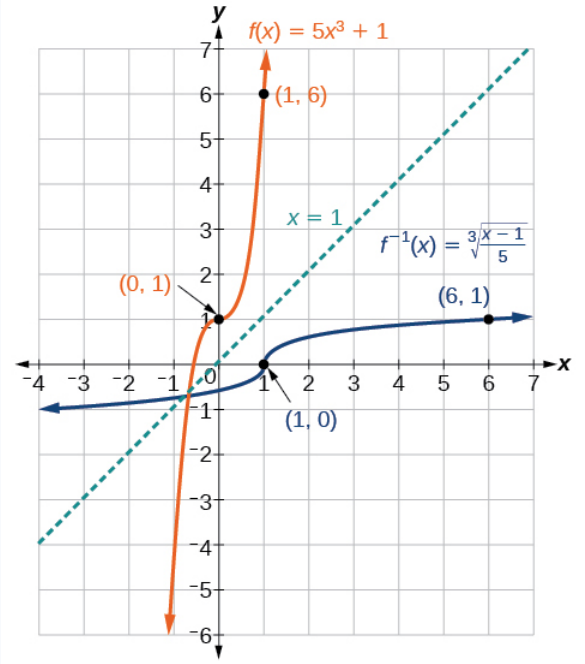


***Figure* 3.8.4**

Try It 3.8.2: Find the inverse function of $f(x) = \sqrt[3]{x+4}$.

Answer

$f^{-1}(x) = x^3 - 4$

Restricting the Domain to Find the Inverse of a Polynomial Function

So far, we have been able to find the inverse functions of **cubic functions** without having to restrict their domains. However, as we know, not all cubic polynomials are one-to-one. Some functions that are not one-to-one may have their domain restricted so that they are one-to-one, but only over that domain. The function over the restricted domain would then have an **inverse function**. Since quadratic functions are not one-to-one, we must restrict their domain in order to find their inverses.

RESTRICTING THE DOMAIN

If a function is not one-to-one, it cannot have an inverse. If we restrict the domain of the function so that it becomes one-to-one, thus creating a new function, this new function will have an inverse.

How To: Given a polynomial function, restrict the domain of a function that is not one-to-one and then find the inverse

1. Restrict the domain by determining a smaller domain on which the original function is one-to-one.

2. Replace $f(x)$ with y.
3. Interchange x and y.
4. Solve for y, and rename the function $f^{-1}(x)$.
5. Revise the formula for $f^{-1}(x)$ by ensuring that the outputs of the inverse function correspond to the restricted domain of the original function.

Example 3.8.3: Restricting the Domain to Find the Inverse of a Polynomial Function

Find the inverse function of f, for the following functions:

1. $f(x) = (x-4)^2$, $x \geq 4$
2. $f(x) = (x-4)^2$, $x \leq 4$

Note: Although the formula is the same in each case, these two functions are not considered the same, because their domains are different.

Solution

The original function $f(x) = (x-4)^2$ is not one-to-one, but the function is restricted to a domain of $x \geq 4$ (for the first case) or $x \leq 4$ (for the second case) on which it is one-to-one (Figure 3.8.5).



***Figure* 3.8.5**

To find the inverse, start by replacing $f(x)$ with the variable y.

$$y = (x-4)^2 \text{ Interchange } x \text{ and } y.$$

$$x = (y-4)^2 \text{ Take the square root.}$$

$$\pm\sqrt{x} = y - 4 \text{ Add } 4 \text{ to both sides.}$$

$$4 \pm \sqrt{x} = y$$

This is not a function as written. We need to examine the restrictions on the domain of the original function to determine the inverse.

1. The domain of the original function was restricted to $x \geq 4$, so the outputs of the inverse need to be the same, $f^{-1}(x) \geq 4$, and we must use the + case:

$$f^{-1}(x) = 4 + \sqrt{x}$$

2. The domain of the original function was restricted to $x \leq 4$, so the outputs of the inverse need to be the same, $f^{-1}(x) \leq 4$, and we must use the – case:

$$f^{-1}(x) = 4 - \sqrt{x}$$

Note: for both of the restrictions (case 1 and case 2) of the original function, the range is the same, $[0, \infty)$. Thus, the domain for the inverse in the first case is the same as the domain for the inverse in the second case: $D = [0, \infty)$.

Analysis

On the graphs in Figure 3.8.6, we see the original function graphed on the same set of axes as its inverse function. Notice that together the graphs show symmetry about the line $y = x$. The coordinate pair $(4, 0)$ is on the graph off f and the coordinate pair $(0, 4)$ is on the graph of f^{-1}. For any coordinate pair, if (a, b) is on the graph of f, then (b, a) is on the graph of f^{-1}. Finally, observe that the graph of f intersects the graph of f^{-1} on the line $y = x$. Points of intersection for the graphs of f and f^{-1} will always lie on the line $y = x$.



***Figure* 3.8.6**

Example 3.8.4: Finding the Inverse of a Quadratic Function When the Restriction Is Not Specified

Restrict the domain and then find the inverse of

$$f(x) = (x-2)^2 - 3.$$

Solution:

We can see this is a parabola with vertex at $(2, -3)$ that opens upward. Because the graph will be decreasing on one side of the vertex and increasing on the other side, we can restrict this function to a domain on which it will be one-to-one by limiting the domain to $x \geq 2$.

To find the inverse, we will use the vertex form of the quadratic. We start by replacing $f(x)$ with the variable, y, then solve for x.

$y = (x-2)^2 - 3$ Interchange x and y.

$x = (y-2)^2 - 3$ Add 3 to both sides.

$x + 3 = (y-2)^2$ Take the square root.

$\pm\sqrt{x+3} = y - 2$ Add 2 to both sides.

$2 \pm \sqrt{x+3} = y$ Rename the function.

$$f^{-1}(x) = 2 \pm \sqrt{x+3}$$

Now we need to determine which case to use. Because we restricted our original function to a domain of $x \geq 2$, the range (outputs) of the inverse should be the same, telling us to utilize the + case

$$f^{-1}(x) = 2 + \sqrt{x+3}$$

If the quadratic had not been given in vertex form, rewriting it into vertex form would be the first step. This way we may easily observe the coordinates of the vertex to help us restrict the domain.

Analysis

Notice that we arbitrarily decided to restrict the domain on $x \geq 2$. We could just as easily have opted to restrict the domain to $x \leq 2$, in which case $f^{-1}(x) = 2 - \sqrt{x+3}$. Observe the original function graphed on the same set of axes as its inverse function in Figure 3.8.7. Notice that both graphs show symmetry about the line $y = x$. The coordinate pair $(2, -3)$ is on the graph of f and the coordinate pair $(-3, 2)$ is on the graph of f^{-1}. Observe from the graph of both functions on the same set of axes that

$$\text{domain of } f = \text{range of } f^{-1} = [2, \infty)$$

and

$$\text{domain of } f^{-1} = \text{range of } f = [-3, \infty).$$

Finally, observe that the graph of f intersects the graph of f^{-1} along the line $y = x$.



***Figure* 3.8.7**

Try it 3.8.4: Find the inverse of the function $f(x) = x^2 + 1$, on the domain $x \geq 0$.

Answer

$f^{-1}(x) = \sqrt{x-1}$

Finding Inverses of Radical Functions

Notice that the functions from previous examples were all polynomials, and their inverses were radical functions. If we want to find the inverse of a radical function, we will need to restrict the domain of the answer because the range of the original function is limited.

How To: Given a radical function, find the inverse

1. Determine the range of the original function.
2. Replace $f(x)$ with y, then solve for x.
3. If necessary, restrict the domain of the inverse function to the range of the original function.

Example 3.8.5: Finding the Inverse of a Radical Function

Find the inverse of the function $f(x) = \sqrt{x-4}$ and then restrict its domain to the range of the original function.

Solution

Note that the original function has range $f(x) \geq 0$. Replace $f(x)$ with y:

$$y = \sqrt{x-4} \tag{3.8.1}$$

Interchange x and y:

$$x = \sqrt{y-4} \tag{3.8.2}$$

Square each side:

$$x^2 = y - 4 \tag{3.8.3}$$

Add 4:

$$x^2 + 4 = y \tag{3.8.4}$$

Rename the function $f^{-1}(x)$:

$$f^{-1}(x) = x^2 + 4 \tag{3.8.5}$$

Recall that the domain of this function must be limited to the range of the original function.

$$f^{-1}(x) = x^2 + 4,\ x \geq 0$$

Analysis

Notice in Figure 3.8.8 that the inverse is a reflection of the original function over the line $y = x$. Because the original function has only nonnegative outputs, the inverse function has only nonnegative inputs.

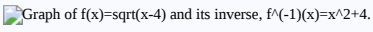


***Figure* 3.8.8**

Try It 3.8.5: Restrict the domain and then find the inverse of the function $f(x) = \sqrt{2x+3}$.

Answer

$f^{-1}(x) = \frac{x^2-3}{2},\ x \geq 0$

Solving Applications of Radical Functions

Radical functions are common in physical models, as we saw in the section opener. We now have enough tools to be able to solve the problem posed at the start of the section.

Example 3.8.6: Solving an Application with a Cubic Function

A mound of gravel is in the shape of a cone with the height equal to twice the radius. The volume of the cone in terms of the radius is given by

$$V = \frac{2}{3}\pi r^3$$

Find the inverse of the function $V = \frac{2}{3}\pi r^3$ that determines the volume V of a cone and is a function of the radius r. Then use the inverse function to calculate the radius of such a mound of gravel measuring 100.00 cubic feet. Use $\pi = 3.1416$.

Solution:

Start with the given function for V. Notice that the meaningful domain for the function is $r > 0$ since negative radii would not make sense in this context nor would a radius of 0. Also note the range of the function (hence, the domain of the inverse function) is $V > 0$. Solve for r in terms of V, using the method outlined previously. Note that in real-world applications, we do not swap the variables when finding inverses. Instead, we change which variable is considered to be the independent variable.

$$V = \frac{2}{3}\pi r^3$$

Solve for r^3:

$$r^3 = \frac{3V}{2\pi}$$

Solve for r:

$$r = \sqrt[3]{\frac{3V}{2\pi}}$$

This is the result stated in the section opener. Now evaluate this for $V = 100.00$ and $\pi = 3.1416$.

$$\begin{aligned} r &= \sqrt[3]{\frac{3V}{2\pi}} \\ &= \sqrt[3]{\frac{3 \cdot 100.00}{2 \cdot 3.1416}} \\ &\approx \sqrt[3]{47.7707} \\ &\approx 3.63 \end{aligned}$$

Therefore, the radius is about 3.63 ft. (Anybody who is a science major knows how many significant figures we should be using in the answer. Your professor guesses 3. If you can write down the rules for significant figures in calculated quantities, give them to your professor and make her give you extra credit.)

Determining the Domain of a Radical Function Composed with Other Functions

When radical functions are composed with other functions, determining domain can become more complicated.

Example 3.8.7: Finding the Domain of a Radical Function Composed with a Rational Function

Find the domain of the function:

$$f(x) = \sqrt{\frac{(x+2)(x-3)}{(x-1)}}.$$

Solution

Because a square root is only defined when the quantity under the radical is non-negative, we need to determine where

$$\frac{(x+2)(x-3)}{(x-1)} \geq 0.$$

The output of a rational function can change signs (change from positive to negative or vice versa) at *x*-intercepts and at vertical asymptotes. For this equation, the graph could change signs at $x = -2$, 1, and 3.

To determine the intervals on which the rational expression is positive, we could test some values in the expression or sketch a graph. While both approaches work equally well, for this example we will use a graph as shown in Figure 3.8.9.

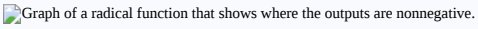


***Figure* 3.8.9**

This function has two *x*-intercepts, both of which exhibit linear behavior near the *x*-intercepts. There is one vertical asymptote, corresponding to a linear factor; this behavior is similar to the basic reciprocal toolkit function, and there is no horizontal

asymptote because the degree of the numerator is larger than the degree of the denominator. There is a y-intercept at $(0, \sqrt{6})$.

From the y-intercept and x-intercept at $x = -2$, we can sketch the left side of the graph. From the behavior at the asymptote, we can sketch the right side of the graph.

From the graph, we can now tell on which intervals the outputs will be non-negative, so that we can be sure that the original function $f(x)$ will be defined. $f(x)$ has domain $-2 \leq x < 1$ or $x \geq 3$, or in interval notation, $[-2, 1) \cup [3, \infty)$.

Finding Inverses of Rational Functions

As with finding inverses of quadratic functions, it is sometimes desirable to find the *inverse of a rational function*, particularly of rational functions that are the ratio of linear functions, such as in concentration applications.

Example 3.8.8: Finding the Inverse of a Rational Function

The function

$$C = \frac{20 + 0.4n}{100 + n} \tag{3.8.6}$$

represents the concentration C of an acid solution after n mL of 40% solution has been added to 100 mL of a 20% solution. First, find the inverse of the function; that is, find an expression for n in terms of C. Then use your result to determine how much of the 40% solution should be added so that the final mixture is a 35% solution.

Solution

We first want the inverse of the function in order to determine how many mL we need for a given concentration. We will solve for n in terms of C.

$$\begin{aligned} C &= \frac{20 + 0.4n}{100 + n} \\ C(100 + n) &= 20 + 0.4n \\ 100C + Cn &= 20 + 0.4n \\ 100C - 20 &= 0.4n - Cn \\ 100C - 20 &= (0.4 - C)n \\ n &= \frac{100C - 20}{0.4 - C} \end{aligned}$$

Now evaluate this function at 35%, which is $C = 0.35$.

$$\begin{aligned} n &= \frac{100(0.35) - 20}{0.4 - 0.35} \\ &= \frac{15}{0.05} \\ &= 300 \end{aligned}$$

We can conclude that 300 mL of the 40% solution should be added.

Try it 3.8.8: Find the inverse of the function $f(x) = \frac{x+3}{x-2}$.

Answer

$f^{-1}(x) = \frac{2x+3}{x-1}$

Media

Access these online resources for additional instruction and practice with inverses and radical functions.

- Graphing the Basic Square Root Function
- Find the Inverse of a Square Root Function
- Find the Inverse of a Rational Function

- Find the Inverse of a Rational Function and an Inverse Function Value
- Inverse Functions

Key Concepts

- The inverse of a quadratic function is a square root function.
- If f^{-1} is the inverse of a function f, then f is the inverse of the function f^{-1}.
- While it is not possible to find an inverse of most polynomial functions, some basic polynomials are invertible.
- To find the inverse of certain functions, we must restrict the function to a domain on which it will be one-to-one.
- When finding the inverse of a radical function, we need a restriction on the domain of the answer.
- Inverse and radical and functions can be used to solve application problems.

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3.8E: Exercises

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CHAPTER OVERVIEW

4: Exponential and Logarithmic Functions

In this chapter, we will explore exponential functions, which can be used for, among other things, modeling growth patterns such as those found in bacteria. We will also investigate logarithmic functions, which are closely related to exponential functions. Both types of functions have numerous real-world applications when it comes to modeling and interpreting data.

4.0: Prelude to Exponential and Logarithmic Functions
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4.1: Exponential Functions
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4.2: Graphs of Exponential Functions
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4.4: Graphs of Logarithmic Functions
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4.6: Exponential and Logarithmic Equations
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Thumbnail: The functions $y = e^x$ and $y = \ln(x)$ are inverses of each other, so their graphs are symmetric about the line $y = x$. Image used with permission (CC BY-SA; OpenStax).

Contributors

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4.0: Prelude to Exponential and Logarithmic Functions

Focus in on a square centimeter of your skin. Look closer. Closer still. If you could look closely enough, you would see hundreds of thousands of microscopic organisms. They are bacteria, and they are not only on your skin, but in your mouth, nose, and even your intestines. In fact, the bacterial cells in your body at any given moment outnumber your own cells. But that is no reason to feel bad about yourself. While some bacteria can cause illness, many are healthy and even essential to the body.

Bacteria commonly reproduce through a process called binary fission, during which one bacterial cell splits into two. When conditions are right, bacteria can reproduce very quickly. Unlike humans and other complex organisms, the time required to form a new generation of bacteria is often a matter of minutes or hours, as opposed to days or years. For simplicity's sake, suppose we begin with a culture of one bacterial cell that can divide every hour. Table **4.0.1** shows the number of bacterial cells at the end of each subsequent hour. We see that the single bacterial cell leads to over one thousand bacterial cells in just ten hours! And if we were to extrapolate the table to twenty-four hours, we would have over 16 million!

Table **4.0.1**

Hour	0	1	2	3	4	5	6	7	8	9	10
Bacteria	1	2	4	8	16	32	64	128	256	512	1024

In this chapter, we will explore exponential functions, which can be used for, among other things, modeling growth patterns such as those found in bacteria. We will also investigate logarithmic functions, which are closely related to exponential functions. Both types of functions have numerous real-world applications when it comes to modeling and interpreting data.

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4.0E: Exercises

No Exercises for this Section.

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4.1: Exponential Functions

Skills to Develop

- Evaluate exponential functions.
- Find the equation of an exponential function.
- Use compound interest formulas.
- Evaluate exponential functions with base e.

India is the second most populous country in the world with a population of about 1.25 billion people in 2013. The population is growing at a rate of about 0.2% each year. If this rate continues, the population of India will exceed China's population by the year 2031. When populations grow rapidly, we often say that the growth is "exponential," meaning that something is growing very rapidly. To a mathematician, however, the term *exponential growth* has a very specific meaning. In this section, we will take a look at *exponential functions*, which model this kind of rapid growth.

Identifying Exponential Functions

When exploring linear growth, we observed a constant rate of change—a constant number by which the output increased for each unit increase in input. For example, in the equation $f(x) = 3x + 4$, the slope tells us the output increases by 3 each time the input increases by 1. The scenario in the India population example is different because we have a *percent* change per unit time (rather than a constant change) in the number of people.

Defining an Exponential Function

A study found that the percent of the population who are vegans in the United States doubled from 2009 to 2011. In 2011, 2.5% of the population was vegan, adhering to a diet that does not include any animal products—no meat, poultry, fish, dairy, or eggs. If this rate continues, vegans will make up 10% of the U.S. population in 2015, 40% in 2019, and 80% in 2050.

What exactly does it mean to *grow exponentially*? What does the word *double* have in common with *percent increase*? People toss these words around errantly. Are these words used correctly? The words certainly appear frequently in the media.

- **Percent change** refers to a *change* based on a *percent* of the original amount.
- **Exponential growth** refers to an *increase* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* increase of the original amount over time.
- **Exponential decay** refers to a *decrease* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* decrease of the original amount over time.

For us to gain a clear understanding of **exponential growth**, let us contrast exponential growth with **linear growth**. We will construct two functions. The first function is exponential. We will start with an initial input of 0, and increase each input by 1. We will double the corresponding consecutive outputs. The second function is linear. We will start with an input of 0, and increase each input by 1. We will add 2 to the corresponding consecutive outputs (Table 4.1.1).

From Table 4.1.1 we can infer that for these two functions, exponential growth dwarfs linear growth.

- With **exponential growth**, the initial output value from the range increases by the *same percentage* over equal increments found in the domain.
- With **linear growth**, the initial output value from the range increases by the *same amount* over equal increments found in the domain.

Table 4.1.1

x	$f(x) = 2^x$	$g(x) = 2x$
0	1	0
1	2	2
2	4	4
3	8	6
4	16	8
5	32	10
6	64	12

Apparently, the difference between "the same percentage" and "the same amount" is quite significant. For exponential growth, over equal increments, the constant multiplicative rate of change resulted in doubling the output, or increasing the output by 100%, whenever the input increased by one. For linear growth, the constant additive rate of change over equal increments resulted in adding 2 to the output whenever the input was increased by one.

The general form of the **exponential function** is $f(x) = ab^x$, where a is any nonzero number, and b is a positive real number not equal to 1. The exponential function is unlike any we have studied thus far, and we will add it to our collection of **Toolkit functions**.

- If $b > 1$,the function grows at a rate proportional to its size.
- If $0 < b < 1$, the function decays at a rate proportional to its size.

Let's look at the function $f(x) = 2^x$ from our example. We will create a table (Table 4.1.2) to determine the corresponding outputs over an interval in the domain from -3 to 3.

Table 4.1.2

x	-3	-2	-1	0	1	2	3
$f(x) = 2^x$	$2^{-3} = \frac{1}{8}$	$2^{-2} = \frac{1}{4}$	$2^{-1} = \frac{1}{2}$	$2^0 = 1$	$2^1 = 2$	$2^2 = 4$	$2^3 = 8$

Let us plot the ordered pairs from Table 4.1.2 and then make a few observations (Figure 4.1.1).

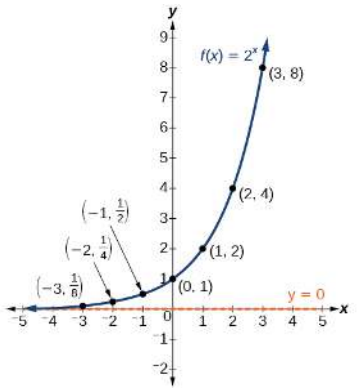


***Figure* 4.1.1**

Let's examine the behavior of the graph of the exponential function $f(x) = 2^x$ and highlight some of its key characteristics.

- the domain is $(-\infty, \infty)$,
- the range is $(0, \infty)$,
- as $x \to \infty$, $f(x) \to \infty$,
- as $x \to -\infty$, $f(x) \to 0$,
- $f(x)$ is always increasing,
- the graph of $f(x)$ will never touch the x-axis because base two raised to any exponent never has the result of zero.
- $y = 0$ is the horizontal asymptote.
- the y-intercept is 1.

Definition: EXPONENTIAL FUNCTION

For any real number x, an **exponential function** is a function with the form

$$f(x) = ab^x \tag{4.1.1}$$

where a is a non-zero real number called the **initial value** and b is any positive real number such that $b \neq 1$.

Characteristics of an exponential function:

- The domain of f is all real numbers.
- The range of f is all positive real numbers if $a > 0$.
- The range of f is all negative real numbers if $a < 0$.
- The y-intercept is $(0, a)$,and the horizontal asymptote is $y = 0$.

Example 4.1.1: Identifying Exponential Functions

Which of the following equations are *not* exponential functions?

- $f(x) = 4^{3(x-2)}$
- $g(x) = x^3$
- $h(x) = \left(\frac{1}{3}\right)^x$
- $j(x) = (-2)^x$

Solution

By definition, an exponential function has a constant as a base and an independent variable as an exponent. Thus, $g(x) = x^3$ does not represent an exponential function because the base is an independent variable. In fact, $g(x) = x^3$ is a power function.

Recall that the base b of an exponential function is always a positive constant, and $b \neq 1$. Thus, $j(x) = (-2)^x$ does not represent an exponential function because the base, -2, is less than 0.

$h(x)$ is clearly an exponential function. We can rewrite $f(x)$ to see that it is an exponential function: $f(x) = 4^{3(x-2)} = 4^{3x-6} = (4^3)^x \cdot 4^{-6} = 4^{-6}(64)^x$. Here, 4^{-6} is the initial value a and 64 is the base b.

Try It 4.1.1

Which of the following equations represent exponential functions?

- $f(x) = 2x^2 - 3x + 1$
- $g(x) = 0.875^x$
- $h(x) = 1.75x + 2$
- $j(x) = 1095.6^{-2x}$

Answer

$g(x) = 0.875^x$ and $j(x) = 1095.6^{-2x}$ represent exponential functions.

Evaluating Exponential Functions

Recall that the base of an exponential function must be a positive real number other than 1. Why do we limit the base b to positive values? To ensure that the outputs will be real numbers. Observe what happens if the base is not positive:

- Let $b = -9$ and $x = \frac{1}{2}$. Then $f(x) = f\left(\frac{1}{2}\right) = (-9)^{\frac{1}{2}} = \sqrt{-9}$,which is not a real number.

Why do we limit the base to positive values other than 1? Because base 1 results in the constant function. Observe what happens if the base is 1:

- Let $b = 1$. Then $f(x) = 1^x = 1$ for any value of x.

To evaluate an exponential function with the form $f(x) = b^x$,we simply substitute x with the given value, and calculate the resulting power. For example:

Let $f(x) = 2^x$. What is $f(3)$?

$$\begin{aligned} f(x) &= 2^x \\ f(3) &= 2^3 \quad \text{Substitute } x = 3 \\ &= 8 \quad \text{Evaluate the power} \end{aligned}$$

To evaluate an exponential function with a form other than the basic form, it is important to follow the order of operations. For example:

Let $f(x) = 30(2)^x$. What is $f(3)$?

$$\begin{aligned} f(x) &= 30(2)^x \\ f(3) &= 30(2)^3 \quad \text{Substitute } x = 3 \\ &= 30(8) \quad \text{Simplify the power first} \\ &= 240 \quad \text{Multiply} \end{aligned}$$

Note that if the order of operations were not followed, the result would be incorrect:

$$f(3) = 30(2)^3 \neq 60^3 = 216,000$$

Example 4.1.2: Evaluating Exponential Functions

Let $f(x) = 5(3)^{x+1}$. Evaluate $f(2)$ without using a calculator.

Solution

Follow the order of operations. Be sure to pay attention to the parentheses.

$$\begin{aligned} f(x) &= 5(3)^{x+1} \\ f(2) &= 5(3)^{2+1} \quad \text{Substitute } x = 2 \\ &= 5(3)^3 \quad \text{Add the exponents} \\ &= 5(27) \quad \text{Simplify the power} \\ &= 135 \quad \text{Multiply} \end{aligned}$$

4.1.2

Let $f(x) = 8(1.2)^{x-5}$. Evaluate $f(3)$ using a calculator. Round to four decimal places.

Answer

5.5556

Defining Exponential Growth

Because the output of exponential functions increases very rapidly, the term "exponential growth" is often used in everyday language to describe anything that grows or increases rapidly. However, exponential growth can be defined more precisely in a mathematical sense. If the growth rate is proportional to the amount present, the function models exponential growth.

EXPONENTIAL GROWTH

A function that models exponential growth grows by a rate proportional to the amount present. For any real number x and any positive real numbers a and b such that $b \neq 1$,an exponential growth function has the form

$$f(x) = ab^x \tag{4.1.2}$$

where

- a is the initial or starting value of the function.
- b is the growth factor or growth multiplier per unit x.

In more general terms, we have an *exponential function*, in which a constant base is raised to a variable exponent. To differentiate between linear and exponential functions, let's consider two companies, A and B. Company A has 100 stores and expands by opening 50 new stores a year, so its growth can be represented by the function $A(x) = 100 + 50x$. Company B has 100 stores and expands by increasing the number of stores by 50% each year, so its growth can be represented by the function $B(x) = 100(1 + 0.5)^x$.

A few years of growth for these companies are illustrated in Table 4.1.3.

Table 4.1.3

Year, x	Stores, Company A	Stores, Company B
0	$100 + 50(0) = 100$	$100(1 + 0.5)^0 = 100$
1	$100 + 50(1) = 150$	$100(1 + 0.5)^1 = 150$
2	$100 + 50(2) = 200$	$100(1 + 0.5)^2 = 225$
3	$100 + 50(3) = 250$	$100(1 + 0.5)^3 = 337.5$
x	$A(x) = 100 + 50x$	$B(x) = 100(1 + 0.5)^x$

The graphs comparing the number of stores for each company over a five-year period are shown in Figure 4.1.2. We can see that, with exponential growth, the number of stores increases much more rapidly than with linear growth.

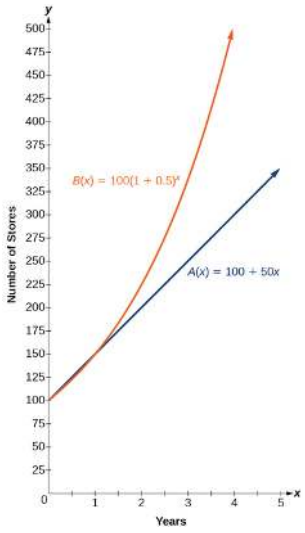


Figure 4.1.2: *The graph shows the numbers of stores Companies A and B opened over a five-year period.*

Notice that the domain for both functions is $[0, \infty)$,and the range for both functions is $[100, \infty)$. After year 1, Company B always has more stores than Company A. Although the domain is theoretically $[0, \infty)$, in reality neither company will be able to keep opening stores indefinitely. What do you think could be a reasonable domain for Company B's growth? Back up your claim mathematically, give it to your professor, and make her give you extra credit!

Now we will turn our attention to the function representing the number of stores for Company B, $B(x) = 100(1 + 0.5)^x$. In this exponential function, 100 represents the initial number of stores, 0.50 represents the growth rate, and $1 + 0.5 = 1.5$ represents the growth factor. Generalizing further, we can write this function as $B(x) = 100(1.5)^x$,where 100 is the initial value, 1.5 is called the *base*, and x is called the *exponent*.

Example 4.1.3: Evaluating a Real-World Exponential Model

At the beginning of this section, we learned that the population of India was about 1.25 billion in the year 2013, with an annual growth rate of about 1.2%. This situation is represented by the growth function $P(t) = 1.25(1.012)^t$, where t is the number of years since 2013. To the nearest thousandth, what will the population of India be in 2031?

Solution

To estimate the population in 2031, we evaluate the models for $t = 18$, because 2031 is 18 years after 2013. Rounding to the nearest thousandth,

$$P(18) = 1.25(1.012)^{18} \approx 1.549$$

There will be about 1.549 billion people in India in the year 2031.

4.1.3

The population of China was about 1.39 billion in the year 2013, with an annual growth rate of about 0.6%. This situation is represented by the growth function $P(t) = 1.39(1.006)^t$, where t is the number of years since 2013.To the nearest thousandth, what will the population of China be for the year 2031? How does this compare to the population prediction we made for India in Example 4.1.3?

Answer

About 1.548 billion people; by the year 2031, India's population will exceed China's by about 0.001 billion, or 1 million people.

Finding Equations of Exponential Functions

In the previous examples, we were given an exponential function, which we then evaluated for a given input. Sometimes we are given information about an exponential function without knowing the function explicitly. We must use the information to first write the form of the function, then determine the constants a and b,and evaluate the function.

Given two data points, write an exponential model

1. If one of the data points has the form $(0, a)$, then a is the initial value. Using a, substitute the second point into the equation $f(x) = a(b)^x$, and solve for b.
2. If neither of the data points have the form $(0, a)$, substitute both points into two equations with the form $f(x) = a(b)^x$. Solve the resulting system of two equations in two unknowns to find a and b.
3. Using the a and b found in the steps above, write the exponential function in the form $f(x) = a(b)^x$.

Example 4.1.4: Writing an Exponential Model When the Initial Value Is Known

In 2006, 80 deer were introduced into a wildlife refuge. By 2012, the population had grown to 180 deer. The population was growing exponentially. Write an algebraic function $N(t)$ representing the population (N) of deer over time t.

Solution

We let our independent variable t be the number of years after 2006. Thus, the information given in the problem can be written as input-output pairs: (0, 80) and (6, 180). Notice that by choosing our input variable to be measured as years after 2006, we have given ourselves the initial value for the function, $a = 80$. We can now substitute the second point into the equation $N(t) = 80b^t$ to find b:

$$\begin{aligned} N(t) &= 80b^t \\ 180 &= 80b^6 && \text{Substitute using point } (6, 180) \\ \frac{9}{4} &= b^6 && \text{Divide and write in lowest terms} \\ b &= \left(\frac{9}{4}\right)^{\frac{1}{6}} && \text{Isolate b using properties of exponents} \\ b &\approx 1.1447 && \text{Round to 4 decimal places} \end{aligned}$$

> *Unless otherwise stated, do not round any intermediate calculations. Then round the final answer to four places for the remainder of this section.*

The exponential model for the population of deer is $N(t) = 80(1.1447)^t$. (Note that this exponential function models short-term growth. As the inputs gets large, the output will get increasingly larger, so much so that the model may not be useful in the long term.)

We can graph our model to observe the population growth of deer in the refuge over time. Notice that the graph in Figure **4.1.3** passes through the initial points given in the problem, $(0, 80)$ and $(6, 180)$. We can also see that the domain for the function is $[0, \infty)$,and the range for the function is $[80, \infty)$.

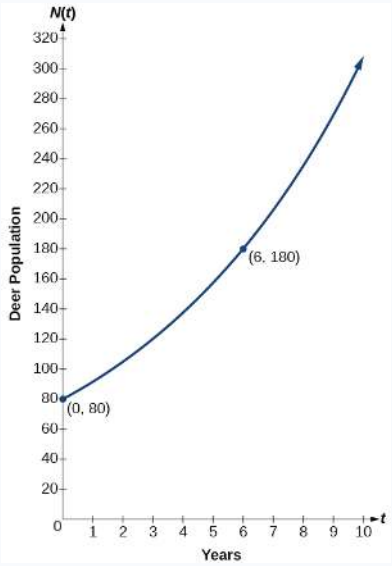


***Figure 4.1.3**: Graph showing the population of deer over time, $N(t) = 80(1.1447)^t$, t years after 2006*

4.1.4

A wolf population is growing exponentially. In 2011, **129** wolves were counted. By 2013, the population had reached **236** wolves. What two points can be used to derive an exponential equation modeling this situation? Write the equation representing the population N of wolves over time t.

Answer

$(0, 129)$ and $(2, 236)$; $N(t) = 129(1.3526)^t$

Example 4.1.5: Writing an Exponential Model When the Initial Value is Not Known

Find an exponential function that passes through the points $(-2, 6)$ and $(2, 1)$.

Solution

Because we don't have the initial value, we substitute both points into an equation of the form $f(x) = ab^x$, and then solve the system for a and b.

- Substituting $(-2, 6)$ gives $6 = ab^{-2}$
- Substituting $(2, 1)$ gives $1 = ab^2$

Use the first equation to solve for a in terms of b:

$$\begin{aligned} 6 &= ab^{-2} \\ \frac{6}{b^{-2}} &= a \quad \text{Divide} \\ a &= 6b^2 \quad \text{Use properties of exponents to rewrite the denominator} \end{aligned}$$

Substitute $6b^2$ for a in the second equation, and solve for b:

$$\begin{aligned} 1 &= ab^2 \\ 1 &= 6b^2b^2 \quad \text{Substitute for } a \\ &= 6b^4 \\ b &= \left(\frac{1}{6}\right)^{\frac{1}{4}} \quad \text{Round to 4 decimal places} \\ b &\approx 0.6389 \end{aligned}$$

Use the value of b in the first equation to solve for the value of a:

$$\begin{aligned} a &= 6b^2 \\ &\approx 6(0.6389)^2 \\ &\approx 2.4492 \end{aligned}$$

Thus, the equation is $f(x) = 2.4492(0.6389)^x$.

We can graph our model to check our work. Notice that the graph in Figure **4.1.4** passes through the initial points given in the problem, $(-2, 6)$ and $(2, 1)$. The graph is an example of an exponential decay function.

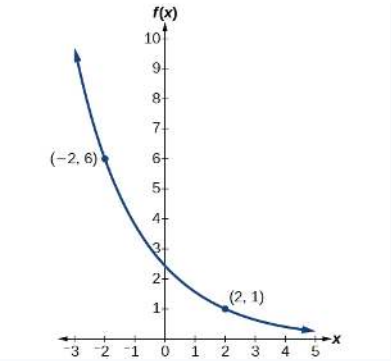


***Figure* 4.1.4:** *The graph of* $f(x) = 2.4492(0.6389)^x$ *models exponential decay.*

4.1.5

Given the two points $(1, 3)$ and $(2, 4.5)$,find the equation of the exponential function that passes through these two points.

Answer

$f(x) = 2(1.5)^x$

Q&A: Do two points always determine a unique exponential function?

Yes, provided the two points are either both above the x-axis or both below the x-axis and have different x-coordinates. But keep in mind that we also need to know that the graph is, in fact, an exponential function. Not every graph that looks exponential really is exponential. We need to know the graph is based on a model that shows the same percent growth with each unit increase in x*, which in many real world cases represents time.*

How to: Given the graph of an exponential function, write its equation

1. First, identify two points on the graph. Choose the y-intercept as one of the two points whenever possible. Try to choose points that are as far apart as possible to reduce round-off error.
2. If one of the data points is the y-intercept $(0, a)$, then a is the initial value. Using a, substitute the second point into the equation $f(x) = a(b)^x$, and solve for b
3. If neither of the data points have the form $(0, a)$, substitute both points into two equations with the form $f(x) = a(b)^x$. Solve the resulting system of two equations in two unknowns to find a and b.
4. Write the exponential function, $f(x) = a(b)^x$.

Example 4.1.6: Writing an Exponential Function Given Its Graph

Find an equation for the exponential function graphed in Figure 4.1.5.

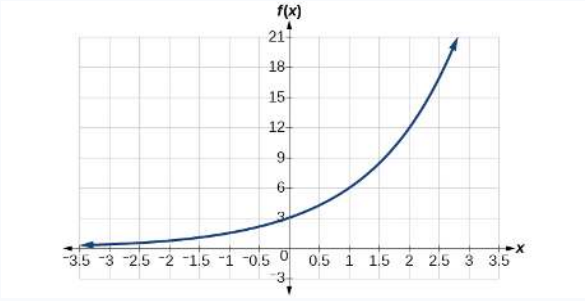


***Figure* 4.1.5**

Solution

We can choose the y-intercept of the graph, $(0, 3)$, as our first point. This gives us the initial value, $a = 3$. Next, choose a point on the curve some distance away from $(0, 3)$ that has integer coordinates. One such point is $(2, 12)$.

$y = ab^x$	Write the general form of an exponential equation
$y = 3b^x$	Substitute the initial value 3 for a
$12 = 3b^2$	Substitute in 12 for y and 2 for x
$4 = b^2$	Divide by 3
$b = \pm 2$	Take the square root

Because we restrict ourselves to positive values of b, we use $b = 2$. Substitute a and b into the standard form to yield the equation $f(x) = 3(2)^x$.

4.1.6

Find an equation for the exponential function graphed in Figure 4.1.6.

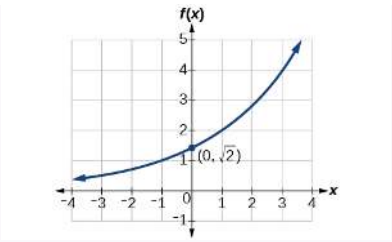


Figure 4.1.6

Answer

$f(x) = \sqrt{2}(\sqrt{2})^x$. Answers may vary due to round-off error. The answer should be very close to $1.4142(1.4142)^x$.

Given two points on the curve of an exponential function, use a graphing calculator to find the equation

1. Press **[STAT].**
2. Clear any existing entries in columns **L1** or **L2.**
3. In **L1**, enter the *x*-coordinates given.
4. In **L2**, enter the corresponding *y*-coordinates.
5. Press **[STAT]** again. Cursor right to **CALC**, scroll down to **ExpReg (Exponential Regression)**, and press **[ENTER].**
6. The screen displays the values of *a* and *b* in the exponential equation $y = a \cdot b^x$.

Example 4.1.7: Using a Graphing Calculator to Find an Exponential Function

Use a graphing calculator to find the exponential equation that includes the points $(2, 24.8)$ and $(5, 198.4)$.

Solution

Follow the guidelines above. First press **[STAT]**, **[EDIT]**, **[1: Edit…],** and clear the lists **L1** and **L2**. Next, in the **L1** column, enter the x-coordinates, **2** and **5**. Do the same in the **L2** column for the y-coordinates, **24.8** and **198.4**.

Now press **[STAT]**, **[CALC]**, **[0: ExpReg]** and press **[ENTER]**. The values $a = 6.2$ and $b = 2$ will be displayed. The exponential equation is $y = 6.2 \cdot 2^x$.

4.1.7

Use a graphing calculator to find the exponential equation that includes the points $(3, 75.98)$ and $(6, 481.07)$.

Answer

$y \approx 12 \cdot 1.85^x$

Applying the Compound-Interest Formula

Savings instruments in which earnings are continually reinvested, such as mutual funds and retirement accounts, use **compound interest**. The term *compounding* refers to interest earned not only on the original value, but on the accumulated value of the account.

The **annual percentage rate (APR)** of an account, also called the **nominal rate**, is the yearly interest rate earned by an investment account. The term *nominal* is used when the compounding occurs a number of times other than once per year. In fact, when interest is compounded more than once a year, the effective interest rate ends up being *greater* than the nominal rate! This is a powerful tool for investing.

We can calculate the compound interest using the compound interest formula, which is an exponential function of the variables time t, principal P, APR r, and number of compounding periods in a year n:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

For example, observe Table 4.1.4, which shows the result of investing $\$1,000$ at 10% for one year. Notice how the value of the account increases as the compounding frequency increases.

Table 4.1.4

Frequency	Value after 1 year
Annually	$1100
Semiannually	$1102.50
Quarterly	$1103.81
Monthly	$1104.71
Daily	$1105.16

Definition: THE COMPOUND INTEREST FORMULA

Compound interest can be calculated using the formula

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt} \tag{4.1.3}$$

where

- $A(t)$ is the account value,
- t is measured in years,
- P is the starting amount of the account, often called the principal, or more generally present value,
- r is the annual percentage rate (APR) expressed as a decimal, and
- n is the number of compounding periods in one year.

Example 4.1.8: Calculating Compound Interest

If we invest $3,000 in an investment account paying 3% interest compounded quarterly, how much will the account be worth in 10 years?

Solution

Because we are starting with $3,000, $P = 3000$. Our interest rate is 3%, so $r = 0.03$. Because we are compounding quarterly, we are compounding 4 times per year, so $n = 4$. We want to know the value of the account in 10 years, so we are looking for $A(10)$,the value when $t = 10$.

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt} \quad \text{Use the compound interest formula}$$
$$A(10) = 3000\left(1 + \frac{0.03}{4}\right)^{(4)\cdot(10)} \quad \text{Substitute using given values}$$
$$\approx \$4045.05 \quad \text{Round to two decimal places}$$

The account will be worth about $4,045.05 in 10 years.

4.1.8

An initial investment of $100,000 at 12% interest is compounded weekly (use 52 weeks in a year). What will the investment be worth in 30 years?

Answer

about $3,644,675.88

Example 4.1.9: Using the Compound Interest Formula to Solve for the Principal

A 529 Plan is a college-savings plan that allows relatives to invest money to pay for a child's future college tuition; the account grows tax-free. Lily wants to set up a 529 account for her new granddaughter and wants the account to grow to $40,000 over 18 years. She believes the account will earn 6% compounded semi-annually (twice a year). To the nearest dollar, how much will Lily need to invest in the account now?

Solution

The nominal interest rate is 6%, so $r = 0.06$. Interest is compounded twice a year, so $k = 2$.

We want to find the initial investment, P, needed so that the value of the account will be worth $40,000 in 18 years. Substitute the given values into the compound interest formula, and solve for P.

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt} \quad \text{Use the compound interest formula}$$
$$40,000 = P\left(1 + \frac{0.06}{2}\right)^{2(18)} \quad \text{Substitute using given values } A, r, n, t$$
$$40,000 = P(1.03)^{36} \quad \text{Simplify}$$
$$\frac{40,000}{(1.03)^{36}} = P \quad \text{Isolate } P$$
$$P \approx \$13,801 \quad \text{Divide and round to the nearest dollar}$$

Lily will need to invest $13,801 to have $40,000 in 18 years.

4.1.9

Refer to Example 4.1.9. To the nearest dollar, how much would Lily need to invest if the account is compounded quarterly?

Answer

$13,693

Evaluating Functions with Base e

As we saw earlier, the amount earned on an account increases as the compounding frequency increases. Table 4.1.5 shows that the increase from annual to semi-annual compounding is larger than the increase from monthly to daily compounding. This might lead us to ask whether this pattern will continue.

Examine the value of $1 invested at 100% interest for 1 year, compounded at various frequencies, listed in Table 4.1.5.

Table 4.1.5

Frequency	$A(t) = \left(1 + \frac{1}{n}\right)^n$	Value
Annually	$\left(1 + \frac{1}{1}\right)^1$	$2
Semiannually	$\left(1 + \frac{1}{2}\right)^2$	$2.25
Quarterly	$\left(1 + \frac{1}{4}\right)^4$	$2.441406
Monthly	$\left(1 + \frac{1}{12}\right)^{12}$	$2.613035
Daily	$\left(1 + \frac{1}{365}\right)^{365}$	$2.714567
Hourly	$\left(1 + \frac{1}{8760}\right)^{8760}$	$2.718127
Once per minute	$\left(1 + \frac{1}{525600}\right)^{525600}$	$2.718279
Once per second	$\left(1 + \frac{1}{31536000}\right)^{31536000}$	$2.718282

These values appear to be approaching a limit as n increases without bound. In fact, as n gets larger and larger, the expression $\left(1 + \frac{1}{n}\right)^n$ approaches a number used so frequently in mathematics that it has its own name: the letter e. This value is an irrational number, which means that its decimal expansion goes on forever without repeating. Its approximation to six decimal places is shown below.

Definition: THE NUMBER e

The letter e represents the irrational number

$$\left(1 + \frac{1}{n}\right)^n \tag{4.1.4}$$

as n increases without bound

The letter e is used as a base for many real-world exponential models. To work with base e, we use the approximation, $e \approx 2.718282$. The constant was named by the Swiss mathematician Leonhard Euler (1707–1783) who first investigated and discovered many of its properties.

Example 4.1.10: Using a Calculator to Find Powers of e

Calculate $e^{3.14}$. Round to five decimal places.

Solution

On a calculator, press the button labeled $[e^x]$. The window shows $[e^(]$. Type 3.14 and then close parenthesis, $[)]$. Press [ENTER]. Rounding to 5 decimal places, $e^{3.14} \approx 23.10387$. Caution: Many scientific calculators have an "Exp" button, which is used to enter numbers in scientific notation. It is not used to find powers of e.

Try It 4.1.10

Use a calculator to find $e^{-0.5}$. Round to five decimal places.

Answer

$e^{-0.5} \approx 0.60653$

Investigating Continuous Growth

So far we have worked with rational bases for exponential functions. For most real-world phenomena, however, e is used as the base for exponential functions. Exponential models that use e as the base are called continuous growth or decay models. We see these models in finance, computer science, and most of the sciences, such as physics, toxicology, and fluid dynamics.

Definition: THE CONTINUOUS GROWTH/DECAY FORMULA

For all real numbers t,and all positive numbers a and r,continuous growth or decay is represented by the formula

$$A(t) = ae^{rt} \tag{4.1.5}$$

where

- a is the initial value,
- r is the continuous growth rate per unit time,
- t is the elapsed time.

If $r > 0$, then the formula represents continuous growth. If $r < 0$, then the formula represents continuous decay.

For business applications, the continuous growth formula is called the continuous compounding formula and takes the form

$$A(t) = Pe^{rt} \tag{4.1.6}$$

where

- P is the principal or the initial invested,
- r is the growth or interest rate per unit time,
- t is the period or term of the investment.

How to: Given the initial value, rate of growth or decay, and time t, solve a continuous growth or decay function

1. Use the information in the problem to determine a, the initial value of the function.
2. Use the information in the problem to determine the growth rate r.
   - If the problem refers to continuous growth, then $r > 0$.
   - If the problem refers to continuous decay, then $r < 0$.
3. Use the information in the problem to determine the time t.
4. Substitute the given information into the continuous growth formula and solve for $A(t)$.

Example 4.1.11: Calculating Continuous Growth

A person invested $\$1,000$ in an account earning a nominal 10% per year compounded continuously. How much was in the account at the end of one year?

Solution

Since the account is growing in value, this is a continuous compounding problem with growth rate $r = 0.10$. The initial investment was $\$1,000$, so $P = 1000$. We use the continuous compounding formula to find the value after $t = 1$ year:

$$\begin{aligned} A(t) &= Pe^{rt} && \text{Use the continuous compounding formula} \\ &= 1000(e)^{0.1} && \text{Substitute known values for } P, r, t \\ &\approx 1105.17 && \text{Use a calculator to approximate} \end{aligned}$$

The account is worth $\$1,105.17$ after one year.

Try It 4.1.11

A person invests $\$100,000$ at a nominal 12% interest per year compounded continuously. What will be the value of the investment in 30 years?

Answer

$\$3,659,823.44$

Example 4.1.12: Calculating Continuous Decay

$Radon - 222$ decays at a continuous rate of 17.3% per day. How much will $100mg$ of $Radon - 222$ decay to in 3 days?

Solution

Since the substance is decaying, the rate, 17.3%, is negative. So, $r = -0.173$. The initial amount of $Radon - 222$ was 100 mg, so $a = 100$. We use the continuous decay formula to find the value after $t = 3$ days:

$$\begin{aligned} A(t) &= ae^{rt} && \text{Use the continuous growth formula} \\ &= 100e-0.173(3) && \text{Substitute known values for } a, r, t \\ &\approx 59.5115 && \text{Use a calculator to approximate} \end{aligned}$$

So 59.5115 mg of $Radon-222$ will remain.

4.1.12

Using the data in Example $4.1.12$, how much $Radon-222$ will remain after one year?

Answer

$3.77E-26$ (This is calculator notation for the number written as 3.77×10^{-26} in scientific notation. While the output of an exponential function is never zero, this number is so close to zero that for all practical purposes we can accept zero as the answer.)

Media

Access these online resources for additional instruction and practice with exponential functions.

- Exponential Growth Function
- Compound Interest

Key Equations

definition of the exponential function	$f(x)=b^x$, where $b>0$, $b\neq1$
definition of exponential growth	$f(x)=ab^x$, where $a>0$, $b>0$, $b\neq1$
compound interest formula	$A(t)=P(1+\frac{r}{n})^{nt}$, where $A(t)$ is the account value at time t t is the number of years P is the initial investment, often called the principal r is the annual percentage rate (APR), or nominal rate n is the number of compounding periods in one year
continuous growth formula	$A(t)=ae^{rt}$, where t is the number of unit time periods of growth a is the starting amount (in the continuous compounding formula a is replaced with P, the princip

Key Concepts

- An exponential function is defined as a function with a positive constant other than 1 raised to a variable exponent.
- A function is evaluated by solving at a specific value.
- An exponential model can be found when the growth rate and initial value are known.
- An exponential model can be found when the two data points from the model are known.
- An exponential model can be found using two data points from the graph of the model.
- An exponential model can be found using two data points from the graph and a calculator.
- The value of an account at any time t can be calculated using the compound interest formula when the principal, annual interest rate, and compounding periods are known.
- The initial investment of an account can be found using the compound interest formula when the value of the account, annual interest rate, compounding periods, and life span of the account are known.
- The number e is a mathematical constant often used as the base of real world exponential growth and decay models. Its decimal approximation is $e\approx2.718282$.
- Scientific and graphing calculators have the key $[e^x]$ or $[exp(x)]$ for calculating powers of e.
- Continuous growth or decay models are exponential models that use e as the base. Continuous growth and decay models can be found when the initial value and growth or decay rate are known.

Contributors

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4.1E: Exercises

Verbal

1) Explain why the values of an increasing exponential function will eventually overtake the values of an increasing linear function.

Answer:

Linear functions have a constant rate of change. Exponential functions increase based on a percent of the original.

2) Given a formula for an exponential function, is it possible to determine whether the function grows or decays exponentially just by looking at the formula? Explain.

3)The Oxford Dictionary defines the word nominal as a value that is "stated or expressed but not necessarily corresponding exactly to the real value." Develop a reasonable argument for why the term nominal rate is used to describe the annual percentage rate of an investment account that compounds interest.

Answer:

When interest is compounded, the percentage of interest earned to principal ends up being greater than the annual percentage rate for the investment account. Thus, the annual percentage rate does not necessarily correspond to the real interest earned, which is the very definition of nominal.

Algebraic

For the following exercises, identify whether the statement represents an exponential function. Explain.

4) The average annual population increase of a pack of wolves is 25.

5) A population of bacteria decreases by a factor of $\frac{1}{8}$ every 24 hours.

Answer:

exponential; the population decreases by a proportional rate.

6) The value of a coin collection has increased by 3.25% annually over the last 20 years.

7) For each training session, a personal trainer charges his clients $\$5$ less than the previous training session.

Answer:

not exponential; the charge decreases by a constant amount each visit, so the statement represents a linear function.

8) The height of a projectile at time t is represented by the function $h(t) = -4.9t^2 + 18t + 40$

For the following exercises, consider this scenario: For each year t, the population of a forest of trees is represented by the function $A(t) = 115(1.025)^t$. In a neighboring forest, the population of the same type of tree is represented by the function $B(t) = 82(1.029)^t$. (Round answers to the nearest whole number.)

9) Which forest's population is growing at a faster rate?

Answer:

The forest represented by the function $B(t) = 82(1.029)^t$.

10) Which forest had a greater number of trees initially? By how many?

11) Assuming the population growth models continue to represent the growth of the forests, which forest will have a greater number of trees after 20 years? By how many?

Answer:

After $t = 20$ years, forest A will have 43 more trees than forest B.

12) Assuming the population growth models continue to represent the growth of the forests, which forest will have a greater number of trees after 100 years? By how many?

13) Discuss the above results from the previous four exercises. Assuming the population growth models continue to represent the growth of the forests, which forest will have the greater number of trees in the long run? Why? What are some factors that might influence the long-term validity of the exponential growth model?

Answer:

Answers will vary. Sample response: For a number of years, the population of forest A will increasingly exceed forest B, but because forest B actually grows at a faster rate, the population will eventually become larger than forest A and will remain that way as long as the population growth models hold. Some factors that might influence the long-term validity of the exponential growth model are drought, an epidemic that culls the population, and other environmental and biological factors.

For the following exercises, determine whether the equation represents exponential growth, exponential decay, or neither. Explain.

14) $y = 300(1 - t)^5$

15) $y = 220(1.06)^x$

Answer:

exponential growth; The growth factor, 1.06 is greater than 1.

16) $y = 16.5(1.025)^{\frac{1}{2}}$

17) $y = 11,701(0.97)^t$

Answer:

exponential decay; The decay factor, 0.97 is between 0 and 1

For the following exercises, find the formula for an exponential function that passes through the two points given.

18) $(0, 6)$ and $(3, 750)$

19) $(0, 2000)$ and $(2, 20)$

Answer:

$f(x) = 2000(0.1)^x$

20) $(-1, \frac{3}{2})$ and $(3, 24)$

21) \((−2,6)\) and $(3, 1)$

Answer:

$f(x) = (\frac{1}{6})^{-\frac{5}{6}} (\frac{1}{6})^{\frac{x}{6}} \approx 2.93(0.699)^x$

22) $(3, 1)$ and $(5, 4)$

For the following exercises, determine whether the table could represent a function that is linear, exponential, or neither. If it appears to be exponential, find a function that passes through the points.

23)

x	1	2	3	4
$f(x)$	70	40	10	-20

Answer:

Linear

24)

x	1	2	3	4
$h(x)$	70	49	34.3	24.01

25)

x	1	2	3	4
$m(x)$	80	61	42.9	25.61

Answer:

Neither

26)

x	1	2	3	4
$f(x)$	10	20	40	80

27)

x	1	2	3	4
$g(x)$	-3.25	2	7.25	12.5

Answer:

Linear

For the following exercises, use the compound interest formula, $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$

28) After a certain number of years, the value of an investment account is represented by the equation $10,250\left(1 + \frac{0.04}{12}\right)^{120}$. What is the value of the account?

29) What was the initial deposit made to the account in the previous exercise?

Answer:

$10,250

30) How many years had the account from the previous exercise been accumulating interest?

31) An account is opened with an initial deposit of $6,500 and earns 3.6% interest compounded semi-annually. What will the account be worth in 20 years?

Answer:

$13,268.58

32) How much more would the account in the previous exercise have been worth if the interest were compounding weekly?

33) Solve the compound interest formula for the principal, P.

Answer:

$P = A(t) \cdot (1 + \frac{r}{n})^{-nt}$

34) Use the formula found in the previous exercise to calculate the initial deposit of an account that is worth $14,472.74 after earning 5.5% interest compounded monthly for 5 years. (Round to the nearest dollar.)

35) How much more would the account in the previous two exercises be worth if it were earning interest for 5 more years?

Answer:

$4,572.56

36) Use properties of rational exponents to solve the compound interest formula for the interest rate, r.

37) Use the formula found in the previous exercise to calculate the interest rate for an account that was compounded semi-annually, had an initial deposit of $9,000 and was worth $13,373.53 after 10 years.

Answer:

4%

38) Use the formula found in the previous exercise to calculate the interest rate for an account that was compounded monthly, had an initial deposit of $\$5,500$, and was worth $\$38,455$ after 30 years.

For the following exercises, determine whether the equation represents continuous growth, continuous decay, or neither. Explain.

39) $y = 3742(e)^{0.75t}$

Answer:

continuous growth; the growth rate is greater than 0.

40) $y = 150(e)^{\frac{3.25}{t}}$

41) $y = 2.25(e)^{-2t}$

Answer:

continuous decay; the growth rate is less than 0.

42) Suppose an investment account is opened with an initial deposit of $\$12,000$ earning 7.2% interest compounded continuously. How much will the account be worth after 30 years?

43) How much less would the account from Exercise 42 be worth after 30 years if it were compounded monthly instead?

Answer:

$\$669.42$

Numeric

For the following exercises, evaluate each function. Round answers to four decimal places, if necessary.

44) $f(x) = 2(5)^x$ for $f(-3)$

45) $f(x) = -4^{2x+3}$ for $f(-1)$

Answer:

$f(-1) = -4$

46) $f(x) = e^x$, for $f(3)$

47) $f(x) = -2e^{x-1}$, for $f(-1)$

Answer:

$f(-1) \approx -0.2707$

48) $f(x) = 2.7(4)^{-x+1} + 1.5$, for $f(-2)$

49) $f(x) = 1.2e^{2x} - 0.3$, for $f(3)$

Answer:

$f(3) \approx 483.8146$

50) $f(x) = -\frac{3}{2}(3)^{-x} + \frac{3}{2}$, for $f(2)$

Technology

For the following exercises, use a graphing calculator to find the equation of an exponential function given the points on the curve.

51) $(0, 3)$ and $(3, 375)$

Answer:

$y = 3 \cdot 5^x$

52) $(3, 222.62)$ and $(10, 77.456)$

53) $(20, 29.495)$ and $(150, 730.89)$

Answer:

$y \approx 18 \cdot 1.025^x$

54) $(5, 2.909)$ and $(13, 0.005)$

55) ((11,310.035)\) and $(25, 356.3652)$

Answer:

$y \approx 0.2 \cdot 1.95^x$

Extensions

56) The annual percentage yield (APY) of an investment account is a representation of the actual interest rate earned on a compounding account. It is based on a compounding period of one year. Show that the APY of an account that compounds monthly can be found with the formula $APY = \left(1 + \frac{r}{12}\right)^{12} - 1$

57) Repeat the previous exercise to find the formula for the APY of an account that compounds daily. Use the results from this and the previous exercise to develop a function $I(n)$ for the APY of any account that compounds n times per year.

Answer:

$$APY = \frac{A(t) - a}{a}$$
$$= \frac{a\left(1 + \frac{r}{365}\right)^{365(1)} - a}{a} \; ; I(n) = \left(1 + \frac{r}{n}\right)^{n} - 1$$
$$= \frac{a\left[\left(1 + \frac{r}{365}\right)^{365} - 1\right]}{a}$$
$$= \left(1 + \frac{r}{365}\right)^{365} - 1$$

58) Recall that an exponential function is any equation written in the form $f(x) = a \cdot b^x$ such that a and b are positive numbers and $b \neq 1$. Any positive number b can be written as $b = e^n$ for some value of n. Use this fact to rewrite the formula for an exponential function that uses the number e as a base.

59) In an exponential decay function, the base of the exponent is a value between 0 and 1. Thus, for some number $b > 1$, the exponential decay function can be written as $f(x) = a \cdot \left(\frac{1}{b}\right)^x$. Use this formula, along with the fact that $b = e^n$, to show that an exponential decay function takes the form $f(x) = a \cdot (e)^{-nx}$ for some positive number n.

Answer:

Let f be the exponential decay function $f(x) = a \cdot \left(\frac{1}{b}\right)^x$ such that $b > 1$. Then for some number $n > 0$,

$$f(x) = a \cdot \left(\frac{1}{b}\right)^x$$
$$= a\left(b^{-1}\right)^x$$
$$= a\left(\left(e^n\right)^{-1}\right)^x$$
$$= a\left(e^{-n}\right)^x$$
$$= a(e)^{-nx}$$

60) The formula for the amount A in an investment account with a nominal interest rate r at any time t is given by $A(t) = a(e)^{rt}$, where a is the amount of principal initially deposited into an account that compounds continuously. Prove that the percentage of interest earned to principal at any time t can be calculated with th $= e^{rt} - 1$

Real-World Applications

61) The fox population in a certain region has an annual growth rate of 9% per year. In the year 2012, there were $23,900$ fox counted in the area. What is the fox population predicted to be in the year 2020?

Answer:

$47,622$ fox

62) A scientist begins with 100 milligrams of a radioactive substance that decays exponentially. After 35 hours, 50mg of the substance remains. How many milligrams will remain after 54 hours?

63) In the year 1985, a house was valued at $\$110,000$. By the year 2005, the value had appreciated to $\$145,000$. What was the annual growth rate between 1985 and 2005? Assume that the value continued to grow by the same percentage. What was the value of the house in the year 2010?

Answer:

1.39%; $\$155,368.09$

64) A car was valued at $\$38,000$ in the year 2007. By 2013, the value had depreciated to $\$11,000$ If the car's value continues to drop by the same percentage, what will it be worth by 2017?

65) Jamal wants to save $\$54,000$ for a down payment on a home. How much will he need to invest in an account with 8.2% APR, compounding daily, in order to reach his goal in 5 years?

Answer:

$\$35,838.76$

66) Kyoko has $\$10,000$ that she wants to invest. Her bank has several investment accounts to choose from, all compounding daily. Her goal is to have $\$15,000$ by the time she finishes graduate school in 6 years. To the nearest hundredth of a percent, what should her minimum annual interest rate be in order to reach her goal? (Hint: solve the compound interest formula for the interest rate.)

67) Alyssa opened a retirement account with 7.25% APR in the year 2000. Her initial deposit was $\$13,500$. How much will the account be worth in 2025 if interest compounds monthly? How much more would she make if interest compounded continuously?

Answer:

$\$82,247.78$; $\$449.75$

68) An investment account with an annual interest rate of 7% was opened with an initial deposit of $\$4,000$ Compare the values of the account after 9 years when the interest is compounded annually, quarterly, monthly, and continuously.

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4.2: Graphs of Exponential Functions

Skills to Develop

- Graph exponential functions.
- Graph exponential functions using transformations.

As we discussed in the previous section, exponential functions are used for many real-world applications such as finance, forensics, computer science, and most of the life sciences. Working with an equation that describes a real-world situation gives us a method for making predictions. Most of the time, however, the equation itself is not enough. We learn a lot about things by seeing their pictorial representations, and that is exactly why graphing exponential equations is a powerful tool. It gives us another layer of insight for predicting future events.

Graphing Exponential Functions

Before we begin graphing, it is helpful to review the behavior of exponential growth. Recall the table of values for a function of the form $f(x) = b^x$ whose base is greater than one. We'll use the function $f(x) = 2^x$. Observe how the output values in Table 4.2.1 change as the input increases by 1.

***Table* 4.2.1**

x	-3	-2	-1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Each output value is the product of the previous output and the base, 2. We call the base 2 the *constant ratio*. In fact, for any exponential function with the form $f(x) = ab^x$, b is the constant ratio of the function. This means that as the input increases by 1, the output value will be the product of the base and the previous output, regardless of the value of a.

Notice from the table that

- the output values are positive for all values of x;
- as x increases, the output values increase without bound; and
- as x decreases, the output values grow smaller, approaching zero.

Figure 4.2.1 shows the exponential growth function $f(x) = 2^x$.

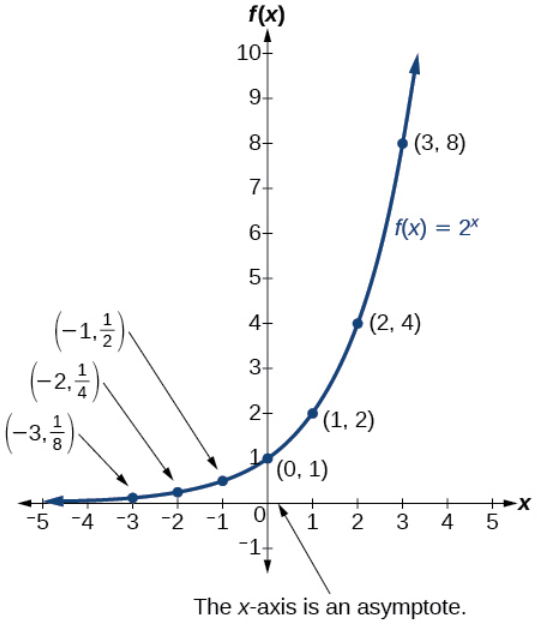


***Figure* 4.2.1:** *Notice that the graph gets close to the x-axis, but never touches it.*

The domain of $f(x) = 2^x$ is all real numbers, the range is $(0, \infty)$, and the horizontal asymptote is $y = 0$. Note that, unlike a horizontal asymptote for a rational function (see Section 3.7), the graph approaches $y = 0$ only at one end.

To get a sense of the behavior of exponential decay, we can create a table of values for a function of the form $f(x) = b^x$ whose base is between zero and one. We'll use the function $g(x) = \left(\frac{1}{2}\right)^x$. Observe how the output values in Table 4.2.2 change as the input increases by 1.

Table 4.2.2

x	-3	-2	-1	0	1	2	3
$g(x) = \left(\frac{1}{2}\right)^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

Again, because the input is increasing by 1, each output value is the product of the previous output and the base, or constant ratio $\frac{1}{2}$.

Notice from the table that

- the output values are positive for all values of x;
- as x increases, the output values grow smaller, approaching zero; and
- as x decreases, the output values grow without bound.

Figure 4.2.2 shows the exponential decay function, $g(x) = \left(\frac{1}{2}\right)^x$.

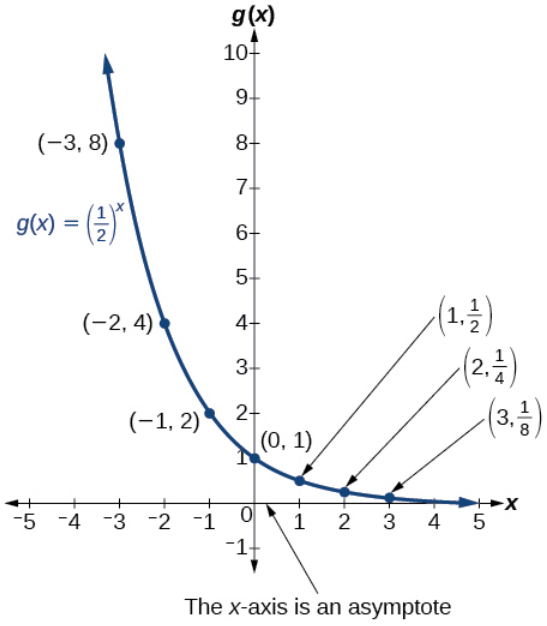


Figure 4.2.2

The domain of $g(x) = \left(\frac{1}{2}\right)^x$ is all real numbers, the range is $(0, \infty)$, and the horizontal asymptote is $y = 0$.

CHARACTERISTICS OF THE GRAPH OF THE toolkit FUNCTION $f(x) = b^x$

An exponential function with the form $f(x) = b^x$, $b > 0$, $b \neq 1$, has these characteristics:

- one-to-one function
- horizontal asymptote: $y = 0$
- domain: $(-\infty, \infty)$
- range: $(0, \infty)$
- x-intercept: none
- y-intercept: $(0, 1)$

- increasing if $b > 1$
- decreasing if $b < 1$

Figure 4.2.3 compares the graphs of exponential growth and decay functions.

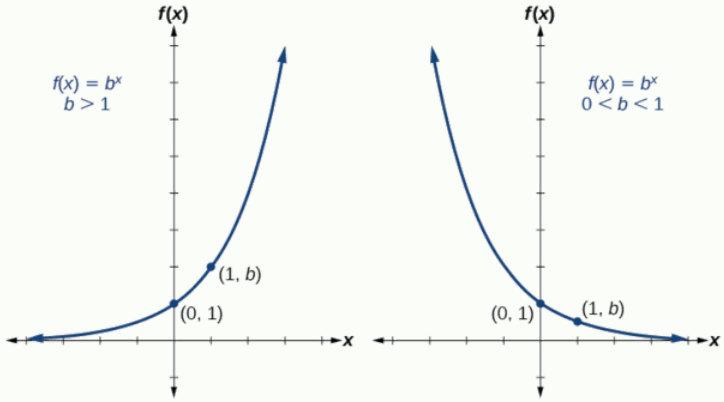


***Figure* 4.2.3**

How To: Given an exponential function of the form $f(x) = b^x$, graph the function

1. Plot at least 3 points of the graph by finding 3 input-output pairs, including the \(y\)-intercept $(0, 1)$.
2. Draw a smooth curve through the points.
3. State the domain, $(-\infty, \infty)$, the range, $(0, \infty)$, and the horizontal asymptote, $y = 0$.

Example 4.2.1: Sketching the Graph of an Exponential Function of the Form $f(x) = b^x$

Sketch a graph of $f(x) = 0.25^x$. State the domain, range, and asymptote.

Solution

Since $b = 0.25$ is between zero and one, we know the function is decreasing. The end behavior of the graph is as follows: as $x \to -\infty$, $y \to \infty$, and as $x \to \infty$, $y \to 0$, so the graph has an asymptote $y = 0$.

- Find two other points.
- Plot the y-intercept, $(0, 1)$, along with the two other points. We will use $(-1, 4)$ and $(1, 0.25)$.

Draw a smooth curve connecting the points as in Figure 4.2.4.

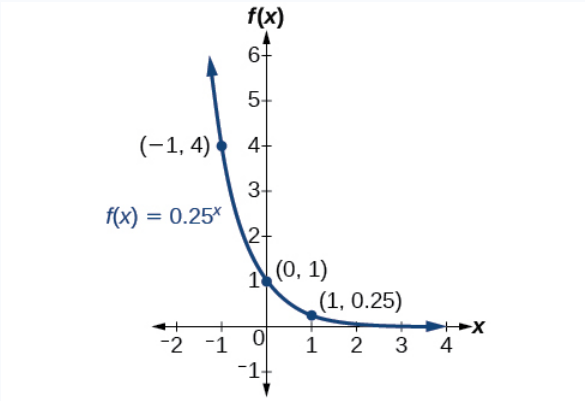


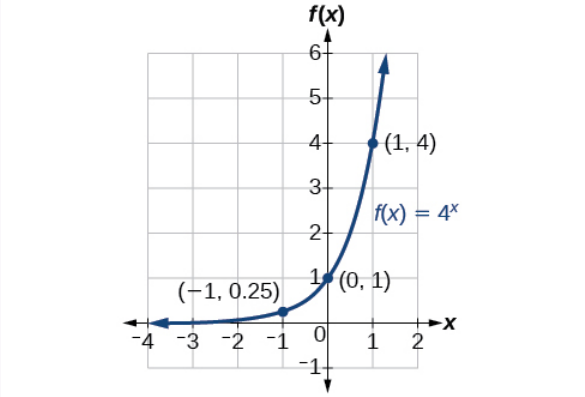
***Figure* 4.2.4**

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

Try It 4.2.1 Sketch the graph of $f(x) = 4^x$. State the domain, range, and asymptote.

Answer

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.



Graphing Transformations of Exponential Functions

Transformations of exponential graphs behave similarly to those of other functions. Just as with other toolkit functions, we can apply the four types of transformations—shifts, reflections, stretches, and compressions—to the toolkit function $f(x) = b^x$ without loss of shape. For instance, just as the quadratic function maintains its parabolic shape when shifted, reflected, stretched, or compressed, the exponential function also maintains its general shape regardless of the transformations applied.

Graphing a Vertical Shift

The first transformation occurs when we add a constant d to the toolkit function $f(x) = b^x$, giving us a vertical shift d units in the same direction as the sign. For example, if we begin by graphing a toolkit function, $f(x) = 2^x$, we can then graph two vertical shifts alongside it, using $d = 3$: the upward shift, $g(x) = 2^x + 3$ and the downward shift, $h(x) = 2^x - 3$. Both vertical shifts are shown in Figure 4.2.5.

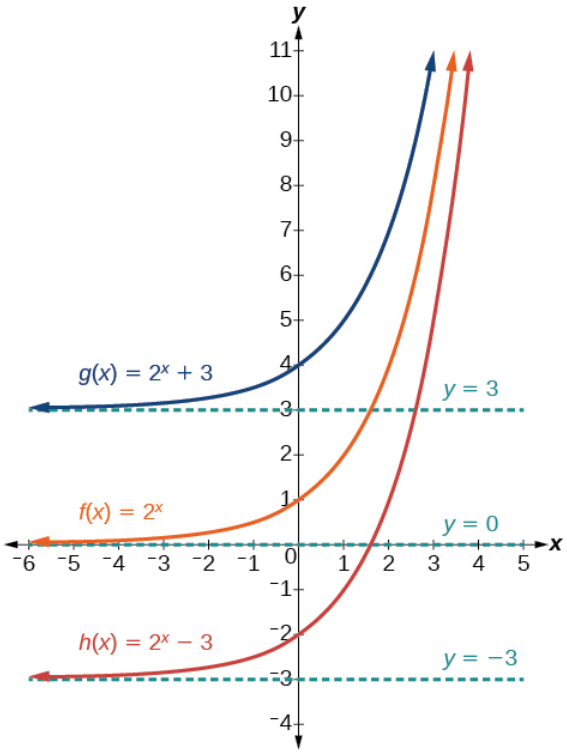


***Figure* 4.2.5**

Observe the results of shifting $f(x) = 2^x$ vertically:

- The domain, $(-\infty, \infty)$, remains unchanged.
- When the function is shifted up 3 units to $g(x) = 2^x + 3$:
  - The y-intercept shifts up 3 units to $(0, 4)$.
  - The asymptote shifts up 3 units to $y = 3$.
  - The range becomes $(3, \infty)$.
- When the function is shifted down 3 units to $h(x) = 2^x - 3$:
  - The y-intercept shifts down 3 units to $(0, -2)$.
  - The asymptote also shifts down 3 units to $y = -3$.
  - The range becomes $(-3, \infty)$.

Graphing a Horizontal Shift

The next transformation occurs when we add a constant c to the input of the toolkit function $f(x) = b^x$, giving us a horizontal shift c units in the *opposite* direction of the sign. For example, if we begin by graphing the toolkit function $f(x) = 2^x$, we can then graph two horizontal shifts alongside it, using $c = 3$: the shift left, $g(x) = 2^{x+3}$, and the shift right, $h(x) = 2^{x-3}$. $h(x) = 2^{x-3}$. Both horizontal shifts are shown in Figure $4.2.6$.

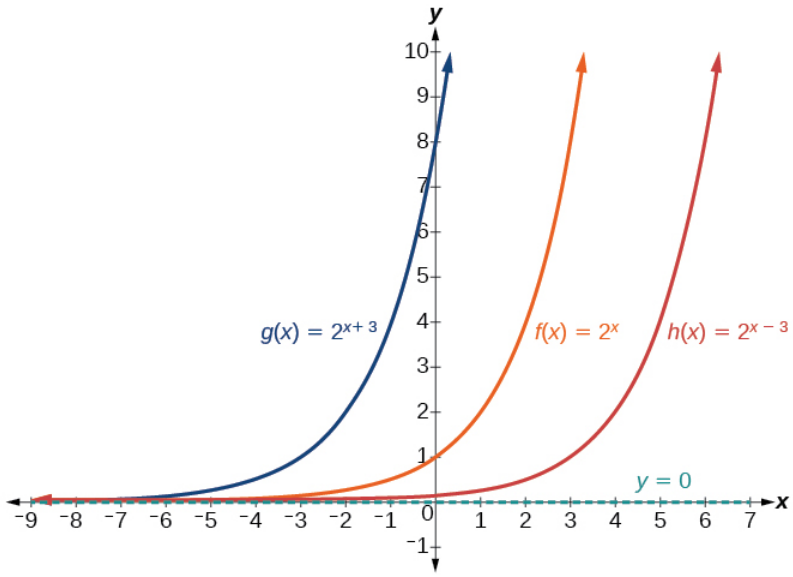


***Figure* 4.2.6**

Observe the results of shifting $f(x) = 2^x$ horizontally:

- The domain, $(-\infty, \infty)$, remains unchanged.
- The asymptote, $y = 0$, remains unchanged.
- The y-intercept shifts such that:
  - When the function is shifted left 3 units to $g(x) = 2^{x+3}$,the y-intercept becomes $(0, 8)$. This is because $2^{x+3} = 2^3 2^x = (8)2^x$, so the initial value of the function is 8.
  - When the function is shifted right 3 units to $h(x) = 2^{x-3}$,the y-intercept becomes $\left(0, \frac{1}{8}\right)$. Again, see that $2^{x-3} = \left(\frac{1}{8}\right)2^x$, so the initial value of the function is $\frac{1}{8}$.

SHIFTS OF THE toolkit FUNCTION $f(x) = b^x$

For any constants c and d,the function $f(x) = b^{x+c} + d$ shifts the toolkit function $f(x) = b^x$

- vertically d units, in the *same* direction of the sign of d.
- horizontally c units, in the *opposite* direction of the sign of c.
- The y-intercept becomes $(0, b^c + d)$.
- The horizontal asymptote becomes $y = d$.

- The range becomes (d, ∞).
- The domain, $(-\infty, \infty)$,remains unchanged.

How to: Given an exponential function with the form $f(x) = b^{x+c} + d$, graph the translation

1. Draw the horizontal asymptote $y = d$.
2. Shift the graph of $f(x) = b^x$ left c units if c is positive, and right c units if c is negative.
3. Shift the graph of $f(x) = b^x$ up d units if d is positive, and down d units if d is negative.
4. State the domain, $(-\infty, \infty)$, the range, (d, ∞), and the horizontal asymptote $y = d$.

Example 4.2.2: Graphing a Shift of an Exponential Function

Graph $f(x) = 2^{x+1} - 3$. State the domain, range, and asymptote.

Solution

We have an exponential equation of the form $f(x) = b^{x+c} + d$, with $b = 2$, $c = 1$, and $d = -3$.

Draw the horizontal asymptote $y = d$, so draw $y = -3$.

Shift the graph of $f(x) = b^x$ left 1 unit and down 3 units.

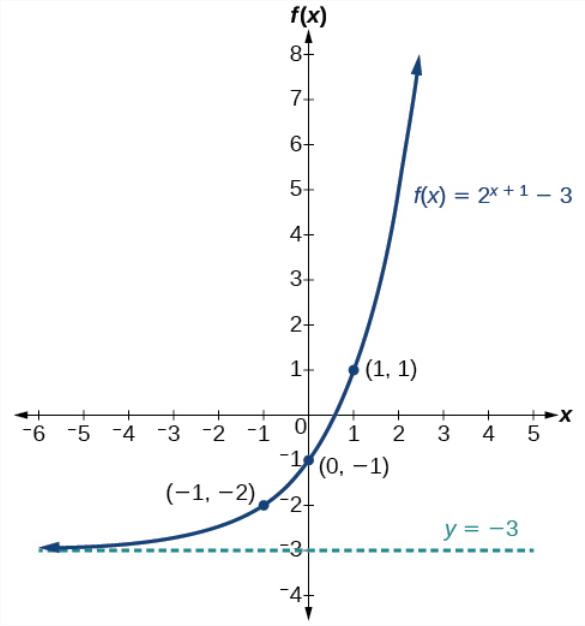


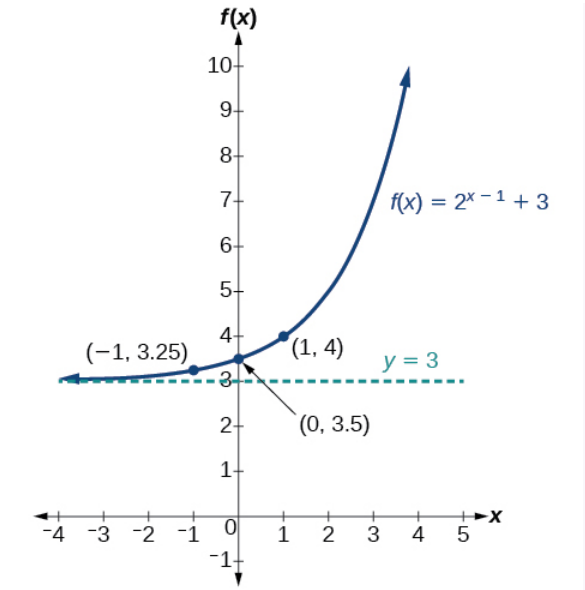
***Figure* 4.2.7**

The domain is $(-\infty, \infty)$; the range is $(-3, \infty)$; the horizontal asymptote is $y = -3$.

4.2.2 Graph $f(x) = 2^{x-1} + 3$. State domain, range, and asymptote.

Answer

The domain is $(-\infty, \infty)$; the range is $(3, \infty)$; the horizontal asymptote is $y = 3$.



How To: Given an equation of the form $f(x) = b^{x+c} + d$, use a graphing calculator to approximate the solution for x

- Press **[Y=]**. Enter the given exponential equation in the line headed "$\mathbf{Y_1}$=".
- Enter the given value for $f(x)$ in the line headed "$\mathbf{Y_2}$=".
- Press **[WINDOW]**. Adjust the y-axis so that it includes the value entered for "$\mathbf{Y_2}$=".
- Press **[GRAPH]** to observe the graph of the exponential function along with the line for the specified value of $f(x)$.
- To find the value of x,we compute the point of intersection. Press **[2ND]** then **[CALC]**. Select "intersect" and press **[ENTER]** three times. The point of intersection gives the value of $x$for the indicated value of the function.

Example 4.2.3: Approximating the Solution of an Exponential Equation

Solve $42 = 1.2(5)^x + 2.8$ graphically. Round to the nearest thousandth.

Solution

Press **[Y=]** and enter $1.2(5)^x + 2.8$ next to $\mathbf{Y_1}$=. Then enter 42 next to **Y2**=. For a window, use the values -3 to 3 for x and -5 to 55 for y. Press **[GRAPH]**. The graphs should intersect somewhere near $x = 2$.

For a better approximation, press **[2ND]** then **[CALC]**. Select **[5: intersect]** and press **[ENTER]** three times. The x-coordinate of the point of intersection is displayed as 2.1661943. (Your answer may be different if you use a different window or use a different value for **Guess?**) To the nearest thousandth, $x \approx 2.166$.

Try It 4.2.3 Solve $4 = 7.85(1.15)^x - 2.27$ graphically. Round to the nearest thousandth.

Answer

$x \approx -1.608$

Graphing a Stretch or Compression

While horizontal and vertical shifts involve adding constants to the input or to the function itself, a **stretch** or **compression** occurs when we multiply the toolkit function $f(x) = b^x$ by a constant $|a| > 0$. For example, if we begin by graphing the toolkit function $f(x) = 2^x$, we can then graph the stretch, using $a = 3$, to get $g(x) = 3(2)^x$ as shown on the left in Figure 4.2.8, and the compression, using $a = \frac{1}{3}$, to get $h(x) = \frac{1}{3}(2)^x$ as shown on the right in Figure 4.2.8.

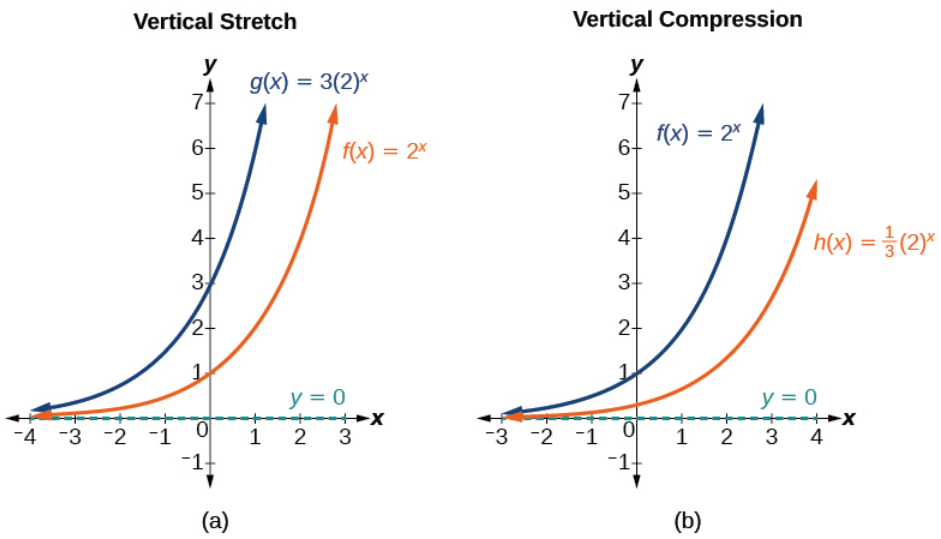


Figure 4.2.8: *(a) $g(x) = 3(2)^x$ stretches the graph of $f(x) = 2^x$ vertically by a factor of 3. (b) $h(x) = \frac{1}{3}(2)^x$ compresses the graph of $f(x) = 2^x$ vertically by a factor of $\frac{1}{3}$.*

STRETCHES AND COMPRESSIONS OF THE toolkit FUNCTION $f(x) = b^x$

For any factor $a > 0$,the function $f(x) = a(b)^x$

- is stretched vertically by a factor of a if $|a| > 1$.
- is compressed vertically by a factor of a if $|a| < 1$.
- has a y-intercept of $(0, a)$.
- has a horizontal asymptote at $y = 0$, a range of $(0, \infty)$, and a domain of $(-\infty, \infty)$, which are unchanged from the parent function.

Example 4.2.4: Graphing the Stretch of an Exponential Function

Sketch a graph of $f(x) = 4\left(\frac{1}{2}\right)^x$. State the domain, range, and asymptote.

Solution

Before graphing, identify the behavior and key points on the graph.

- Since $b = \frac{1}{2}$ is between zero and one, the graph will increase without bound as $x \to -\infty$, and the graph will approach the x-axis as $x \to \infty$.
- Since $a = 4$,the graph of $f(x) = \left(\frac{1}{2}\right)^x$ will be stretched by a factor of 4.
- Plot the y-intercept, $(0, 4)$, along with two other points. We will use $(-1, 8)$ and $(1, 2)$.

Draw a smooth curve connecting the points, as shown in Figure 4.2.9.

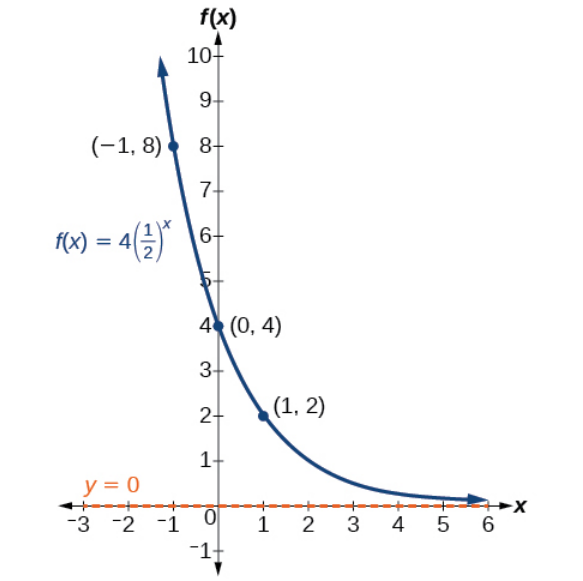


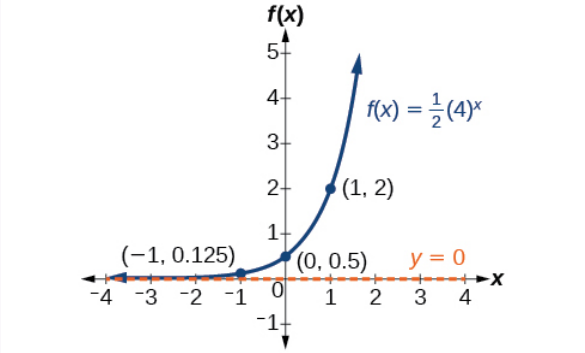
***Figure* 4.2.9**

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

4.2.4 Sketch the graph of $f(x) = \frac{1}{2}(4)^x$. State the domain, range, and asymptote.

Answer

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.



Graphing Reflections

In addition to shifting, compressing, and stretching a graph, we can also reflect it across the *x*-axis or the *y*-axis. When we multiply the toolkit function $f(x) = b^x$ by -1,we get a reflection across the *x*-axis. When we multiply the input by -1,we get a reflection across the *y*-axis. For example, if we begin by graphing the toolkit function $f(x) = 2^x$, we can then graph the two reflections alongside it. The reflection across the x-axis, $g(x) = -2^x$, is shown on the left side of Figure 4.2.10, and the reflection across the y-axis $h(x) = 2^{-x}$, is shown on the right side of Figure 4.2.10.

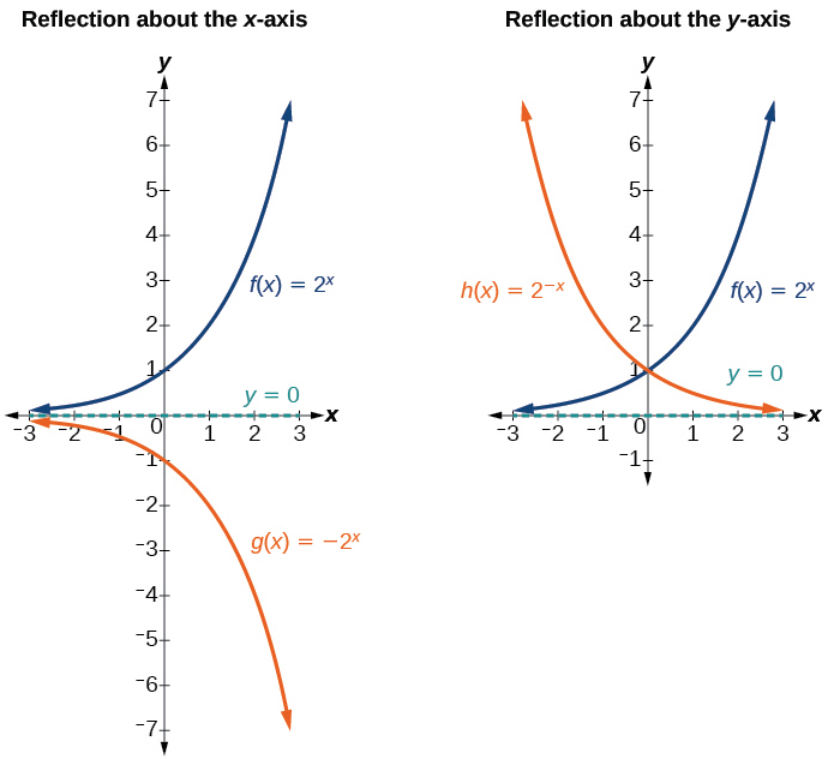


***Figure* 4.2.10:** *(a)* $g(x) = -2^x$ *reflects the graph of* $f(x) = 2^x$ *across the x-axis. (b)* $g(x) = 2^{-x}$ *reflects the graph of* $f(x) = 2^x$ *across the \(y\)-axis.*

REFLECTIONS OF THE toolkit FUNCTION $f(x) = b^x$

The function $f(x) = -b^x$

- reflects the toolkit function $f(x) = b^x$ across the *x*-axis.
- has a *y*-intercept of $(0, -1)$.
- has a range of $(-\infty, 0)$
- has a horizontal asymptote at $y = 0$ and domain of $(-\infty, \infty)$,which are unchanged from the parent function.

The function $f(x) = b^{-x}$

- reflects the toolkit function $f(x) = b^x$ across the y-axis.
- has a y-intercept of $(0, 1)$, a horizontal asymptote at $y = 0$, a range of $(0, \infty)$, and a domain of $(-\infty, \infty)$, which are unchanged from the toolkit function.

Example 4.2.5: Writing and Graphing the Reflection of an Exponential Function

Find and graph the equation for a function $g(x)$ that reflects $f(x) = (\frac{1}{4})^x$ across the *x*-axis. State its domain, range, and asymptote.

Solution

Since we want to reflect the parent function $f(x) = (\frac{1}{4})^x$ about the *x*-axis, we multiply $f(x)$ by -1 to get, $g(x) = -(\frac{1}{4})^x$. Plot the *y*-intercept, $(0, -1)$,along with two other points. We will use $(-1, -4)$ and $(1, -0.25)$.

Draw a smooth curve connecting the points:

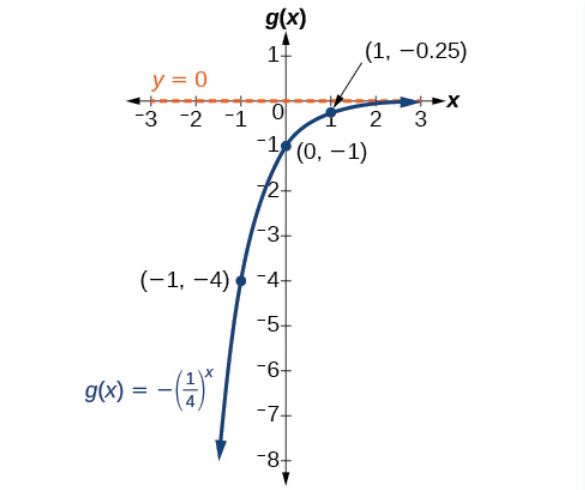


Figure 4.2.11

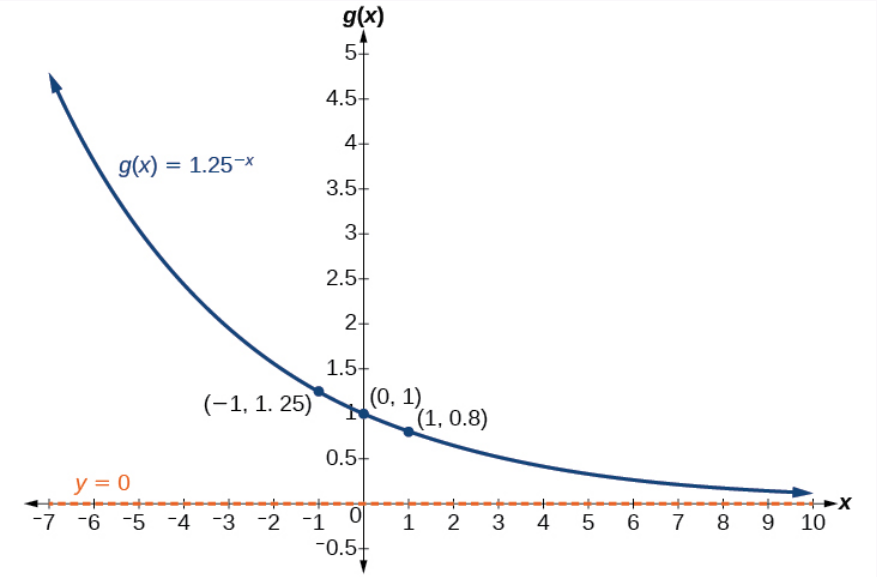
The domain is $(-\infty, \infty)$; the range is $(-\infty, 0)$; the horizontal asymptote is $y = 0$.

4.2.5

Find and graph the equation for a function, $g(x)$, that reflects $f(x) = 1.25^x$ across the y-axis. State its domain, range, and asymptote.

Answer

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.



Summarizing Translations of the Exponential Function

Now that we have worked with each type of translation for the exponential function, we can summarize them in **Table 4.2.3** to arrive at the general equation for translating exponential functions.

Table 4.2.3: Translations of the Parent Function $f(x) = b^x$

Translation	Form
Shift • Horizontally c units to the left • Vertically d units up	$f(x)=b^{x+c}+d$
Stretch and Compress • Stretch if $\|a\|>1$ • Compression if $0<\|a\|<1$	$f(x)=ab^x$
Reflect across the x-axis	$f(x)=-b^x$
Reflect across the y-axis	$f(x)=b^{-x}=\left(\frac{1}{b}\right)^x$
General equation for all translations	$f(x)=ab^{x+c}+d$

TRANSLATIONS OF EXPONENTIAL FUNCTIONS

A translation of an exponential function has the form

$$f(x)=ab^{x+c}+d$$

Where the toolkit function, $y=b^x$, $b>1$,is

- shifted horizontally c units to the left.
- stretched vertically by a factor of $|a|$ if $|a|>0$.
- compressed vertically by a factor of $|a|$ if $0<|a|<1$.
- shifted vertically d units.
- reflected across the x-axis when $a<0$.

Note that the order of the shifts, transformations, and reflections follows the order of operations.

Example 4.2.6: Writing a Function from a Description

Write the equation for the function described below. Give the horizontal asymptote, the domain, and the range.

$f(x)=e^x$ is vertically stretched by a factor of 2, reflected across the y-axis, and then shifted up 4 units.

Solution

We want to find an equation of the general form $f(x)=ab^{x+c}+d$. We use the description provided to find a, b, c, and d.

- We are given the toolkit function $f(x)=e^x$, so $b=e$. Note: e is a number, not a variable. It was defined in Section 4.1. Write down its definition, hand it it to your professor, and make her give you extra credit!
- The function is stretched by a factor of 2, so $a=2$.
- The function is reflected about the y-axis. We replace x with $-x$ to get: e^{-x}.
- There is no horizontal shift, so $c=0$.
- The graph is shifted vertically 4 units, so $d=4$.

Substituting in the general form we get,

$$\begin{aligned} f(x) &= ab^{x+c}+d \\ &= 2e^{-x+0}+4 \\ &= 2e^{-x}+4 \end{aligned}$$

The domain is $(-\infty,\infty)$; the range is $(4,\infty)$; the horizontal asymptote is $y=4$.

4.2.6 Write the equation for function described below. Give the horizontal asymptote, the domain, and the range.

$f(x)=e^x$ is compressed vertically by a factor of $\frac{1}{3}$, reflected across the x-axis and then shifted down 2 units.

Answer

$f(x) = -\frac{1}{3}e^x - 2$; the domain is $(-\infty, \infty)$; the range is $(-\infty, 2)$; the horizontal asymptote is $y = -2$.

Media

Access this online resource for additional instruction and practice with graphing exponential functions.

- Graph Exponential Functions

Key Equations

General form for the translation of the toolkit function $f(x) = b^x$:	$f(x) = ab^{x+c} + d$

Key Concepts

- The graph of the function $f(x) = b^x$ has a *y*-intercept at $(0, 1)$, domain $(-\infty, \infty)$, range $(0, \infty)$, and horizontal asymptote $y = 0$.
- If $b > 1$, the function is increasing. The end behavior of the graph to the left: y will approach the asymptote $y = 0$, and to the right: y will increase without bound.
- If $0 < b < 1$, the function is decreasing. The end behavior of the graph to the left: $y$will increase without bound, and to the right: y will approach the asymptote $y = 0$.
- The equation $f(x) = b^x + d$ represents a vertical shift of the toolkit function $f(x) = b^x$.
- The equation $f(x) = b^{x+c}$ represents a horizontal shift of the toolkit function $f(x) = b^x$.
- Approximate solutions of the equation $f(x) = b^{x+c} + d$ can be found using a graphing calculator.
- The equation $f(x) = ab^x$, where $a > 0$, represents a vertical stretch if $|a| > 1$ or compression if $0 < |a| < 1$ of the toolkit function $f(x) = b^x$.
- When the toolkit function $f(x) = b^x$ is multiplied by -1, the result, $f(x) = -b^x$, is a reflection across the *x*-axis. When the input is multiplied by -1,the result, $f(x) = b^{-x}$, is a reflection across the *y*-axis.
- All translations of the exponential function can be summarized by the general equation $f(x) = ab^{x+c} + d$.
- Using the general equation $f(x) = ab^{x+c} + d$, we can write the equation of a function given its description.

Contributors

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4.2E: Exercises

Verbal

1) What role does the horizontal asymptote of an exponential function play in telling us about the end behavior of the graph?

Answer:

An asymptote is a line that the graph of a function approaches, as x either increases or decreases without bound. The horizontal asymptote of an exponential function tells us the limit of the function's values as the independent variable gets either extremely large or extremely small.

2) What is the advantage of knowing how to recognize transformations of the graph of a parent function algebraically?

Algebraic

3) The graph of $f(x) = 3^x$ is reflected about the y-axis and stretched vertically by a factor of 4. What is the equation of the new function, $g(x)$? State its y-intercept, domain, and range.

Answer:

$g(x) = 4(3)^{-x}$ $(0, 4)$ Domain: all real numbers; Range: all real numbers greater than 0.

4) The graph of $f(x) = \left(\frac{1}{2}\right)^{-x}$ is reflected about the y-axis and compressed vertically by a factor of $\left(\frac{1}{6}\right)$. What is the equation of the new function, $g(x)$? State its y-intercept, domain, and range.

5) The graph of $f(x) = 10^x$ is reflected about the x-axis and shifted upward 7 units. What is the equation of the new function, $g(x)$? State its y-intercept, domain, and range.

Answer:

$g(x) = -10^x + 7$; y-intercept: $(0, 6)$; Domain: all real numbers; Range: all real numbers less than 7.

6) The graph of $f(x) = (1.68)^x$ is shifted right 3 units, stretched vertically by a factor of 2, reflected about the x-axis, and then shifted downward 3 units. What is the equation of the new function, $g(x)$? State its y-intercept (to the nearest thousandth), domain, and range.

Exercise 4.2E.7

The graph of $f(x) = 2\left(\frac{1}{4}\right)^{x-2}$ is shifted left 2 units, stretched vertically by a factor of 4, reflected across the x-axis, and then shifted downward 4 units. What is the equation of the new function, $g(x)$? State its y-intercept, domain, and range.

Answer

$g(x) = -8\left(\frac{1}{4}\right)^x - 4$; y-intercept: $(0, -12)$; Domain: all real numbers; Range: all real numbers less than 0.

Graphical

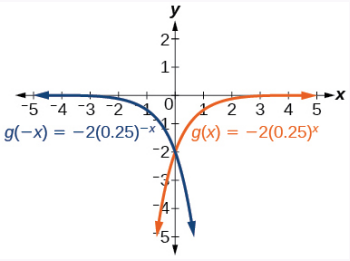
For the following exercises, graph the function and its reflection across the y-axis on the same axes, and give the y-intercept.

8) $f(x) = 3\left(\frac{1}{2}\right)^x$

Exercise 4.2E.9

$g(x) = -2(0.25)^x$

Answer



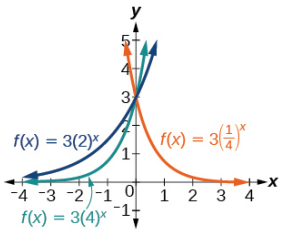
y-intercept: $(0, -2)$

10) $h(x) = 6(1.75)^{-x}$

For the following exercises, graph each set of functions on the same axes.

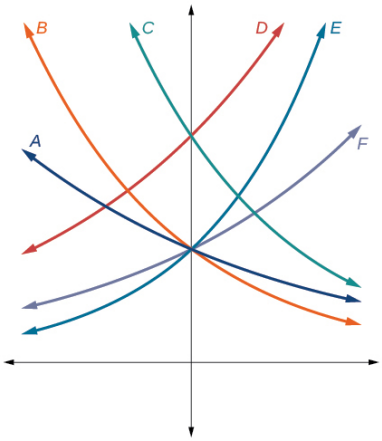
11) $f(x) = 3\left(\frac{1}{4}\right)^x, g(x) = 3(2)^x, h(x) = 3(4)^x$

Answer:



12) $f(x) = \frac{1}{4}(3)^x, g(x) = 2(3)^x, h(x) = 4(3)^x$

For the following exercises, match each function with one of the graphs in Figure below.



13) $f(x) = 2(0.69)^x$

Answer:

B

14) $f(x) = 2(1.28)^x$

15) $f(x) = 2(0.81)^x$

Answer:

A

16) $f(x) = 4(1.28)^x$

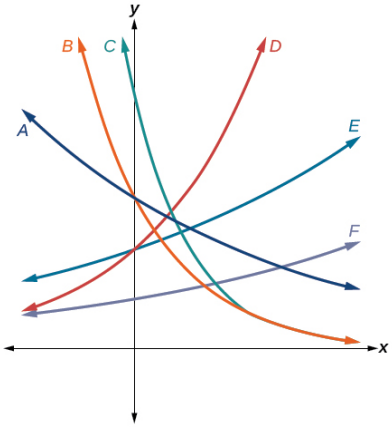
17) $f(x) = 2(1.59)^x$

Answer:

E

18) $f(x) = 4(0.69)^x$

For the following exercises, use the graphs shown in Figure below. All have the form $f(x) = ab^x$.



https://math.libretexts.org/@go/page/31099

Exercise 4.2E.19

Which graph has the largest value for b ?

Answer

D

Exercise 4.2E.20

Which graph has the smallest value for b?

Exercise 4.2E.21

Which graph has the largest value for a?

Answer

C

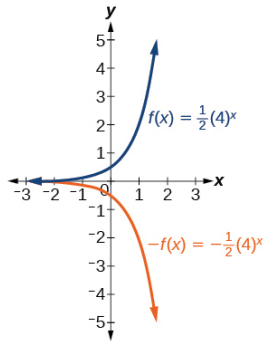
Exercise 4.2E.22

Which graph has the smallest value for a?

For the following exercises, graph the function and its reflection about the x-axis on the same axes.

23) $f(x) = \frac{1}{2}(4)^x$

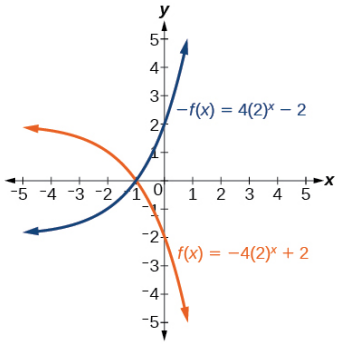
Answer:



24) $f(x) = 3(0.75)^x - 1$

25) $f(x) = -4(2)^x + 2$

Answer:



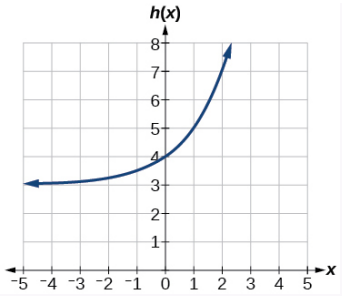
For the following exercises, graph the transformation of $f(x) = 2^x$. Give the horizontal asymptote, the domain, and the range.

26) $f(x) = 2^{-x}$

Exercise 4.2E.27

$h(x) = 2^x + 3$

Answer



Horizontal asymptote: $y = 3$ Domain: all real numbers; Range: all real numbers strictly greater than 3.

28) $f(x) = 2^{x-2}$

For the following exercises, describe the end behavior of the graphs of the functions.

29) $f(x) = -5(4)^x - 1$

Answer:

As $x \to \infty, f(x) \to -\infty$

As $x \to -\infty, f(x) \to -1$

30) $f(x) = 3\left(\frac{1}{2}\right)^x - 2$

31) $f(x) = 3(4)^{-x} + 2$

Answer:

As $x \to \infty, f(x) \to 2$

As $x \to -\infty, f(x) \to \infty$

For the following exercises, start with the graph of $f(x) = 4^x$. Then write a function that results from the given transformation.

32) Shift $f(x)$ 4 units upward

33) Shift $f(x)$ 3 units downward

Answer:

$f(x) = 4^x - 3$

34) Shift $f(x)$ 2 units left

35) Shift $f(x)$ 5 units right

Answer:

$f(x) = 4^{x-5}$

36) Reflect $f(x)$ about the x-axis

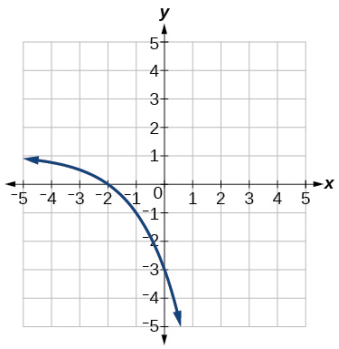
37) Reflect $f(x)$ about the y-axis

Answer:

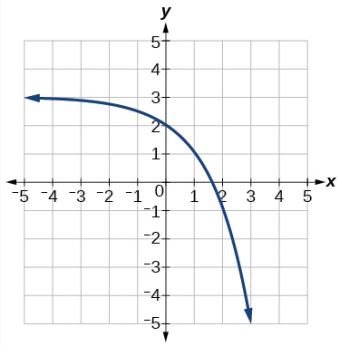
$f(x) = 4^{-x}$

For the following exercises, each graph is a transformation of $f(x) = 2^x$. Write an equation describing the transformation.

38)



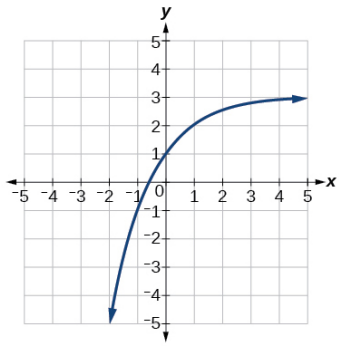
Exercise 4.2E.39



Answer

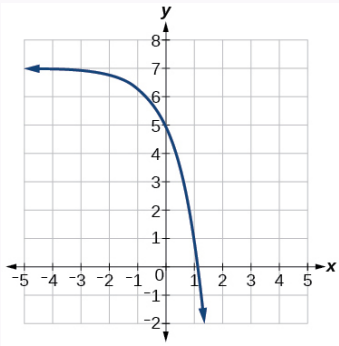
$y = -2^x + 3$

40)



For the following exercises, find an exponential equation for the graph.

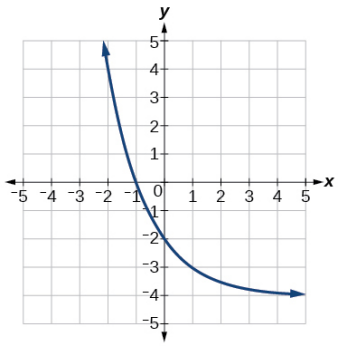
Exercise 4.2E.41



Answer

$y = -2(3)^x + 7$

42)



Numeric

For the following exercises, evaluate the exponential functions for the indicated value of x.

Exercise 4.2E.43

$g(x) = \frac{1}{3}(7)^{x-2}$, for $x = 6$.

Answer

$g(6) = 800 + \frac{1}{3} \approx 800.3333$

44) $f(x) = 4(2)^{x-1} - 2$ for $f(5)$.

45) $h(x) = -\frac{1}{2}\left(\frac{1}{2}\right)^{x} + 6$ for $h(-7)$.

Answer:

$h(-7) = -58$

Technology

For the following exercises, use a graphing calculator to approximate the solutions of the equation. Round to the nearest thousandth.

46) $-50 = \left(\frac{1}{2}\right)^{-x}$

47) $116 = \left(\frac{1}{4}\right)\left(\frac{1}{8}\right)^{x}$

Answer:

$x \approx -2.953$

48) $12 = 2(3)^{x} + 1$

49) $5 = 3\left(\frac{1}{2}\right)^{x-1} - 2$

Answer:

$x \approx -0.222$

50) $-30 = -4(2)^{x+2} + 2$

Extensions

51) Explore and discuss the graphs of $F(x) = (b)^{x}$ and $G(x) = \left(\frac{1}{b}\right)^{x}$. Then make a conjecture about the relationship between the graphs of the functions b^{x} and $\left(\frac{1}{b}\right)^{x}$ for any real number $b > 0$.

Answer:

The graph of $G(x) = \left(\frac{1}{b}\right)^{x}$ is the refelction about the y-axis of the graph of $F(x) = (b)^{x}$; For any real number $b > 0$ and function $f(x) = (b)^{x}$ the graph of $\left(\frac{1}{b}\right)^{x}$ is the the reflection about the y-axis, $F(-x)$.

52) Prove the conjecture made in the previous exercise.

53) Explore and discuss the graphs of $f(x) = 4^{x}$, $g(x) = 4^{x-2}$, and $h(x) = \left(\frac{1}{16}\right)4^{x}$. Then make a conjecture about the relationship between the graphs of the functions b^{x} and $\left(\frac{1}{b^{n}}\right)b^{x}$ for any real number n and real number $b > 0$.

Answer:

The graphs of $g(x)$ and $h(x)$ are the same and are a horizontal shift to the right of the graph of $f(x)$; For any real number n, real number $b > 0$, and function $f(x) = b^{x}$ the graph of $\left(\frac{1}{b^{n}}\right)b^{x}$ is the horizontal shift $f(x - n)$.

54) Prove the conjecture made in the previous exercise.

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4.3: Logarithmic Functions

Skills to Develop

- Convert from logarithmic to exponential form.
- Convert from exponential to logarithmic form.
- Evaluate logarithms.
- Use common logarithms.
- Use natural logarithms.

In 2010, a major earthquake struck Haiti, destroying or damaging over 285,000 homes. One year later, another, stronger earthquake devastated Honshu, Japan, destroying or damaging over 332,000 buildings, like those shown in Figure **4.3.1**. Even though both caused substantial damage, the earthquake in 2011 was 100 times stronger than the earthquake in Haiti. How do we know? The magnitudes of earthquakes are measured on a scale known as the Richter Scale. The Haitian earthquake registered a 7.0 on the Richter Scale whereas the Japanese earthquake registered a 9.0.



Figure* 4.3.1*: *Devastation of March 11, 2011 earthquake in Honshu, Japan. (credit: Daniel Pierce).*

The Richter Scale is a base-ten logarithmic scale. In other words, an earthquake of magnitude 8 is not twice as great as an earthquake of magnitude 4. It is

$$10^{8-4} = 10^4 = 10,000$$

times as great! In this lesson, we will investigate the nature of the Richter Scale and the base-ten function upon which it depends. Physically, how is the magnitude of an earthquake determined? If you want to do a little research on this and write up your results, your professor will give you extra credit when you hand it in.

Converting from Logarithmic to Exponential Form

In order to analyze the magnitude of earthquakes or compare the magnitudes of two different earthquakes, we need to be able to convert between logarithmic and exponential form. For example, suppose the amount of energy released from one earthquake were 500 times greater than the amount of energy released from another. We want to calculate the difference in magnitude. The equation that represents this problem is $10^x = 500$, where x represents the difference in magnitudes on the Richter Scale. How would we solve for x?

We have not yet learned a method for solving exponential equations. None of the algebraic tools discussed so far is sufficient to solve $10^x = 500$. We know that $10^2 = 100$ and $10^3 = 1000$, so it is clear that x must be some value between 2 and 3, since $y = 10^x$ is increasing. We can examine a graph, as in Figure **4.3.1**, to better estimate the solution.

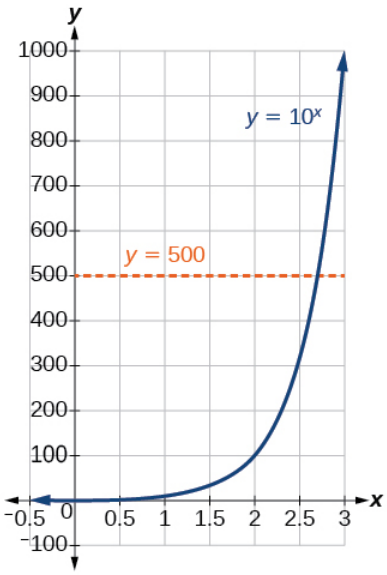


***Figure* 4.3.2**

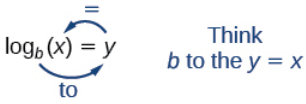
Estimating from a graph, however, is imprecise. To find an algebraic solution, we must introduce a new function. Observe that the graph in Figure **4.3.2** passes the horizontal line test. The exponential function $y = b^x$ is one-to-one, so its inverse, $x = b^y$ is also a function. As is the case with all inverse functions, we simply interchange x and y and solve for y to find the inverse function. To represent y as a function of x, we use a logarithmic function of the form $y = \log_b(x)$. The base b **logarithm** of a number is the exponent by which we must raise b to get that number.

We read a logarithmic expression as, "The logarithm with base b of x is equal to y," or, simplified, "log base b of x is y." We can also say, "b raised to the power of y is x," because logs are exponents. For example, the base 2 logarithm of 32 is 5, because 5 is the exponent we must apply to 2 to get 32. Since $2^5 = 32$, we can write $\log_2 32 = 5$. We read this as "log base 2 of 32 is 5."

We can express the relationship between logarithmic form and its corresponding exponential form as follows:

$$\log_b(x) = y \Leftrightarrow b^y = x, b > 0, b \neq 1 \tag{4.3.1}$$

Note that the base b is always positive.



Because logarithm is a function, it is most correctly written as $\log_b(x)$, using parentheses to denote function evaluation, just as we would with $f(x)$. However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written without parentheses, as $\log_b x$. Note that many calculators require parentheses around the x.

We can illustrate the notation of logarithms as follows:

Notice that, comparing the logarithm function and the exponential function, the input and the output are switched. This means $y = \log^b(x)$ and $y = b^x$ are inverse functions.

DEFINITION OF THE LOGARITHMIC FUNCTION

A logarithm base b of a positive number x satisfies the following definition.

For $x > 0$, $b > 0$, $b \neq 1$,

$$y = \log_b(x) \text{ is equivalent to } b^y = x \tag{4.3.2}$$

where,

- we read $\log_b(x)$ as, “the logarithm with base b of x” or the “log base b of x."
- the logarithm y is the exponent to which b must be raised to get x.

Also, since the logarithmic and exponential functions switch the x and y values, the domain and range of the exponential function are interchanged for the logarithmic function. Therefore,

- the domain of the logarithm function with base b is $(0, \infty)$.
- the range of the logarithm function with base b is $(-\infty, \infty)$.

Q&A: Can we take the logarithm of a negative number?

No. Because the base of an exponential function is always positive, no power of that base can ever be negative. We can never take the logarithm of a negative number. Also, we cannot take the logarithm of zero. Calculators may output a log of a negative number when in complex mode, but the log of a negative number is not a real number.

How To: Given an equation in logarithmic form $\log_b(x) = y$, convert it to exponential form

1. Examine the equation $y = \log_b x$ and identify b, y,and x.
2. Rewrite $\log_b x = y$ as $b^y = x$.

Example 4.3.1: Converting from Logarithmic Form to Exponential Form

Write the following logarithmic equations in exponential form.

a. $\log_6(\sqrt{6}) = \frac{1}{2}$
b. $\log_3(9) = 2$

Solution

First, identify the values of b, y,and x. Then, write the equation in the form $b^y = x$.

a. $\log_6(\sqrt{6}) = \frac{1}{2}$

Here, $b = 6$, $y = \frac{1}{2}$, and $x = \sqrt{6}$. Therefore, the equation $\log_6(\sqrt{6}) = \frac{1}{2}$ is equivalent to

$$6^{1/2} = \sqrt{6}$$

b. $\log_3(9) = 2$

Here, $b = 3$, $y = 2$,and $x = 9$. Therefore, the equation $\log_3(9) = 2$ is equivalent to

$$3^2 = 9$$

Try It 4.3.1

Write the following logarithmic equations in exponential form.

a. $\log_{10}(1,000,000) = 6$
b. $\log_5(25) = 2$

Answer

$\log_{10}(1,000,000) = 6$ is equivalent to $10^6 = 1,000,000$
$\log_5(25) = 2$ is equivalent to $5^2 = 25$

Converting from Exponential to Logarithmic Form

To convert from exponents to logarithms, we follow the same steps in reverse. We identify the base b, exponent x, and output y. Then we write $x = \log_b(y)$.

Example 4.3.2: Converting from Exponential Form to Logarithmic Form

Write the following exponential equations in logarithmic form.

a. $2^3 = 8$
b. $5^2 = 25$
c. $10^{-4} = \frac{1}{10,000}$

Solution

First, identify the values of b, y,and x. Then, write the equation in the form $x = \log_b(y)$.

a. $2^3 = 8$

Here, $b = 2$, $x = 3$,and $y = 8$. Therefore, the equation $2^3 = 8$ is equivalent to $\log_2(8) = 3$.

b. $5^2 = 25$

Here, $b = 5$, $x = 2$,and $y = 25$. Therefore, the equation $5^2 = 25$ is equivalent to $\log_5(25) = 2$.

c. $10^{-4} = \frac{1}{10,000}$

Here, $b = 10$, $x = -4$,and $y = \frac{1}{10,000}$. Therefore, the equation $10^{-4} = \frac{1}{10,000}$ is equivalent to $\log_{10}\left(\frac{1}{10,000}\right) = -4$.

4.3.2

Write the following exponential equations in logarithmic form.

a. $3^2 = 9$
b. $5^3 = 125$
c. $2^{-1} = \frac{1}{2}$

Answer

a. $3^2 = 9$ is equivalent to $\log_3(9) = 2$

b. $5^3 = 125$ is equivalent to $\log_5(125) = 3$

c. $2^{-1} = \frac{1}{2}$ is equivalent to $\log_2\left(\frac{1}{2}\right) = -1$

Evaluating Logarithms

Knowing the squares, cubes, and roots of numbers allows us to evaluate many logarithms mentally. For example, consider $\log_2 8$. We ask, "To what exponent must 2 be raised in order to get 8?" Because we already know $2^3 = 8$, it follows that $\log_2 8 = 3$.

Now consider solving $\log_7 49$ and $\log_3 27$ mentally.

- We ask, "To what exponent must 7 be raised in order to get 49?" We know $7^2 = 49$. Therefore, $\log_7 49 = 2$.
- We ask, "To what exponent must 3 be raised in order to get 27?" We know $3^3 = 27$. Therefore, $\log_3 27 = 3$.

Even some seemingly more complicated logarithms can be evaluated without a calculator. For example, let's evaluate $\log_{\frac{2}{3}} \frac{4}{9}$ mentally.

- We ask, "To what exponent must $\frac{2}{3}$ be raised in order to get $\frac{4}{9}$? " We know $2^2 = 4$ and $3^2 = 9$, so

$$\left(\frac{2}{3}\right)^2 = \frac{4}{9}.$$

Therefore,

$$\log_{\frac{2}{3}}\left(\frac{4}{9}\right) = 2.$$

How To: Given a logarithm of the form $y = \log_b(x)$, evaluate it mentally

1. Rewrite the argument x as a power of b: $b^y = x$.

2. Use previous knowledge of powers of b identify y by asking, "To what exponent should b be raised in order to get x?"

Example 4.3.3: Solving Logarithms Mentally

Solve $y = \log_4(64)$ without using a calculator.

Solution

First we rewrite the logarithm in exponential form: $4^y = 64$. Next, we ask, "To what exponent must 4 be raised in order to get 64?"

We know

$$4^3 = 64$$

Therefore,

$$\log_4(64) = 3$$

4.3.3

Solve $y = \log_{121}(11)$ without using a calculator.

Answer

$\log_{121}(11) = \frac{1}{2}$ (recalling that $\sqrt{121} = (121)^{1/2} = 11$)

Example 4.3.4: Evaluating the Logarithm of a Reciprocal

Evaluate $y = \log_3\left(\frac{1}{27}\right)$ without using a calculator.

Solution

First we rewrite the logarithm in exponential form: $3^y = \frac{1}{27}$. Next, we ask, "To what exponent must 3 be raised in order to get $\frac{1}{27}$?"

We know $3^3 = 27$,but what must we do to get the reciprocal, $\frac{1}{27}$? Recall from working with exponents that $b^{-a} = \frac{1}{b^a}$. We use this information to write

$$\begin{aligned} 3^{-3} &= \frac{1}{3^3} \\ &= \frac{1}{27} \end{aligned}$$

Therefore, $\log_3\left(\frac{1}{27}\right) = -3$.

4.3.4

Evaluate $y = \log_2\left(\frac{1}{32}\right)$ without using a calculator.

Answer

$\log_2\left(\frac{1}{32}\right) = -5$

Using Common Logarithms

Sometimes we may see a logarithm written without a base. In this case, we assume that the base is 10. In other words, the expression $\log(x)$ means $\log_{10}(x)$. We call a base-10 logarithm a *common logarithm*. Common logarithms are used to measure the Richter Scale mentioned at the beginning of the section. Scales for measuring the brightness of stars and the pH of acids and bases also use common logarithms.

DEFINITION OF THE COMMON LOGARITHM

A common logarithm is a logarithm with base 10. We write $\log_{10}(x)$ simply as $\log(x)$. The common logarithm of a positive number x satisfies the following definition.

For $x > 0$,

$$y = \log(x) \text{ is equivalent to } 10^y = x. \tag{4.3.3}$$

We read $\log(x)$ as, "the logarithm with base 10 of x " or "log base 10 of x."

The logarithm y is the exponent to which 10 must be raised to get x.

How To: Given a common logarithm of the form $y = \log(x)$, evaluate it mentally

1. Rewrite the argument x as a power of 10: $10^y = x$.
2. Use previous knowledge of powers of 10 to identify y by asking, "To what exponent must 10 be raised in order to get x?"

Example 4.3.5: Finding the Value of a Common Logarithm Mentally

Evaluate $y = \log(1000)$ without using a calculator.

Solution

First we rewrite the logarithm in exponential form: $10^y = 1000$. Next, we ask, "To what exponent must 10 be raised in order to get 1000?" We know

$$10^3 = 1000$$

Therefore, $\log(1000) = 3$.

Try It 4.3.5

Evaluate $y = \log(1{,}000{,}000)$.

Answer

$\log(1{,}000{,}000)=6$

How To: Given a common logarithm with the form $y = \log(x)$,evaluate it using a calculator

1. Press **[LOG]**.
2. Enter the value given for x,followed by **[ )]** (for some calculators).
3. Press **[ENTER]**.

Example 4.3.6: Finding the Value of a Common Logarithm Using a Calculator

Evaluate $y = \log(321)$ to four decimal places using a calculator.

Solution

- Press **[LOG]**.
- Enter 321, followed by **[ )]**.
- Press **[ENTER]**.

Rounding to four decimal places, $\log(321) \approx 2.5065$.

Analysis

Note that $10^2 = 100$ and that $10^3 = 1000$. Since 321 is between 100 and 1000, we know that $\log(321)$ must be between $\log(100)$ and $\log(1000)$. This gives us the following:

$$100 < 321 < 1000$$

$$2 < 2.5065 < 3$$

4.3.6

Evaluate $y = \log(123)$ to four decimal places using a calculator.

Answer

$\log(123) \approx 2.0899$

Example 4.3.7: Rewriting and Solving a Real-World Exponential Model

The amount of energy released from one earthquake was 500 times greater than the amount of energy released from another. The equation $10^x = 500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

Solution

We begin by rewriting the exponential equation in logarithmic form.

$$10^x = 500$$

$\log(500) = x$ Use the definition of the common log.

Next we evaluate the logarithm using a calculator:

- Press **[LOG]**.
- Enter 500,followed by **[ )]**.
- Press **[ENTER]**.
- To the nearest thousandth, $\log(500) \approx 2.699$.

The difference in magnitudes on the Richter Scale was about 2.699.

4.3.7

The amount of energy released from one earthquake was 8,500 times greater than the amount of energy released from another. The equation $10^x = 8500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

Answer

The difference in magnitudes on the Richter Scale was about 3.929.

Using Natural Logarithms

The most frequently used base for logarithms is the irrational number e, which was introduced in Section 4.1. If you can write the value of e approximated to the nearest thousandth, give it to your professor and make her give you extra credit. Base e logarithms are important in calculus and some scientific applications; they are called **natural logarithms**. The base e logarithm, $\log_e(x)$, has its own notation, $\ln(x)$. Most values of $\ln(x)$ can be found only using a calculator. The major exception is that, because the logarithm of 1 is always 0 in any base, $\ln 1 = 0$. For other natural logarithms, we can use the $\ln$ key that can be found on most scientific calculators. We can also find the natural logarithm of any power of e using the inverse property of logarithms.

DEFINITION OF THE NATURAL LOGARITHM

A natural logarithm is a logarithm with base e. We write $\log_e(x)$ simply as $\ln(x)$. The natural logarithm of a positive number x satisfies the following definition.

For $x > 0$,

$$y = \ln(x) \text{ is equivalent to } e^y = x.$$

We read $\ln(x)$ as, “the logarithm with base e of x” or “the natural logarithm of x.”

The logarithm y is the exponent to which e must be raised to get x.

Since the functions $y = e^x$ and $y = \ln(x)$ are inverse functions, $\ln(e^x) = x$ for all x and $e^{\ln(x)} = x$ for $x > 0$.

How To: Given a natural logarithm with the form $y = \ln(x)$, evaluate it using a calculator

1. Press **[LN]**.
2. Enter the value given for x, followed by **[ )]**.
3. Press **[ENTER]**.

Example 4.3.8: Evaluating a Natural Logarithm Using a Calculator

Evaluate $y = \ln(500)$ to four decimal places using a calculator.

Solution

- Press **[LN]**.
- Enter 500,followed by **[ )]**.
- Press **[ENTER]**.

Rounding to four decimal places, $\ln(500) \approx 6.2146$

Try It 4.3.8

Evaluate $\ln(-500)$.

Answer

It is not possible to take the logarithm of a negative number in the set of real numbers.

Media

Access this online resource for additional instruction and practice with logarithms.

- Introduction to Logarithms

Key Equations

Definition of the logarithmic function	For $x > 0$, $b > 0$, $b \neq 1$, $y = \log_b(x)$ is equivalent to $b^y = x$.
Definition of the common logarithm	For $x > 0$, $y = \log(x)$ is equivalent to $10^y = x$.
Definition of the natural logarithm	For $x > 0$, $y = \ln(x)$ is equivalent to $e^y = x$.

Key Concepts

- The inverse of an exponential function is a logarithmic function, and the inverse of a logarithmic function is an exponential function.
- Logarithmic equations can be written in an equivalent exponential form, using the definition of a logarithm. See Example 4.3.1.
- Exponential equations can be written in their equivalent logarithmic form using the definition of a logarithm See Example 4.3.2.
- Logarithmic functions with base b can be evaluated mentally using previous knowledge of powers of b. See Example 4.3.3 and Example 4.3.4.
- Common logarithms can be evaluated mentally using previous knowledge of powers of 10. See Example 4.3.5.
- When common logarithms cannot be evaluated mentally, a calculator can be used. See Example 4.3.6.
- Real-world exponential problems with base 10 can be rewritten as a common logarithm and then evaluated using a calculator. See Example 4.3.7 .
- Natural logarithms can be evaluated using a calculator. See Example 4.3.8.

Contributors

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4.3E: Exercises

4.3: Logarithmic Functions

The inverse of an exponential function is a logarithmic function, and the inverse of a logarithmic function is an exponential function.

Verbal

1) What is a base b logarithm? Discuss the meaning by interpreting each part of the equivalent equations $b^y = x$ and $\log_b x = y$ for $b > 0, b \neq 1$.

Answer

A logarithm is an exponent. Specifically, it is the exponent to which a base b is raised to produce a given value. In the expressions given, the base b has the same value. The exponent, $y$in the expression b^y can also be written as the logarithm, $\log_b x = y$and the value of x is the result of raising b to the power of y.

2) How is the logarithmic function $f(x) = \log_b x$ related to the exponential function $g(x) = b^x$? What is the result of composing these two functions?

3) How can the logarithmic equation $\log_b x = y$ be solved for x using the properties of exponents?

Answer

Since the equation of a logarithm is equivalent to an exponential equation, the logarithm can be converted to the exponential equation $b^y = x$ and then properties of exponents can be applied to solve for x

4) Discuss the meaning of the common logarithm. What is its relationship to a logarithm with base b, and how does the notation differ?

5) Discuss the meaning of the natural logarithm. What is its relationship to a logarithm with base b, and how does the notation differ?

Answer

The natural logarithm is a special case of the logarithm with base b in that the natural log always has base e. Rather than notating the natural logarithm as $\log_e(x)$the notation used is $\ln(x)$

Algebraic

For the following exercises, rewrite each equation in exponential form.

6) $\log_4(q) = m$

7) $\log_a(b) = c$

Answer

$a^c = b$

8) $\log_{16}(y) = x$

9) $\log_x(64) = y$

Answer

$x^y = 64$

10) $\log_y(x) = -11$

Exercise 4.3E.11

$\log_{15}(a) = b$

Answer

$15^b = a$

12) $\log_y(137) = x$

13) $\log_{13}(142) = a$

Answer

$13^a = 142$

14) $\log(v) = t$

15) $\ln(w) = n$

Answer

$e^n = w$

For the following exercises, rewrite each equation in logarithmic form.

16) $4^x = y$

17) $c^d = k$

Answer

$\log_c(k) = d$

18) $m^{-7} = n$

Exercise 4.3E.19

$19^x = y$

Answer

$\log_{19}(y) = x$

20) $x^{-\frac{10}{13}} = y$

21) $n^4 = 103$

Answer

$\log_n(103) = 4$

22) $\left(\frac{7}{5}\right)^m = n$

23) $y^x = \frac{39}{100}$

Answer

$\log_y\left(\frac{39}{100}\right) = x$

24) $10^a = b$

25) $e^k = h$

Answer

$\ln(w) = n$

For the following exercises, solve for x by converting the logarithmic equation to exponential form.

26) $\log_3(x) = 2$

27) $\log_2(x) = -3$

Answer

$x = 2^{-3} = \frac{1}{8}$

28) $\log_5(x) = 2$

29) $\log_3(x) = 3$

Answer

$x = 3^3 = 27$

30) $\log_2(x) = 6$

31) $\log_9(x) = \frac{1}{2}$

Answer

$x = 9^{\frac{1}{2}} = 3$

32) $\log_{18}(x) = 2$

Exercise $4.3E.33$

$\log_6(x) = -3$

Answer

$x = 6^{-3} = \frac{1}{216}$

34) $\log(x) = 3$

35) $\ln(x) = 2$

Answer

$x = e^2$

For the following exercises, use the definition of common and natural logarithms to simplify.

36) $\log(100^8)$

37) $10^{\log(32)}$

Answer

32

38) $2\log(.0001)$

39) $e^{\ln(1.06)}$

Answer

1.06

40) $\ln(e^{-5.03})$

41) $e^{\ln(10.125)}+4$

Answer

14.125

Numeric

For the following exercises, evaluate the base b logarithmic expression without using a calculator.

42) $\log_3\left(\frac{1}{27}\right)$

Exercise 4.3E.43

$\log_6(\sqrt{6})$

Answer

$\frac{1}{2}$

44) $\log_2\left(\frac{1}{8}\right)+4$

45) $6\log_8(4)$

Answer

4

For the following exercises, evaluate the common logarithmic expression without using a calculator.

46) $\log(10,000)$

47) $\log(0.001)$

Answer

-3

48) $\log(1)+7$

49) $2\log(100^{-3})$

Answer

-12

For the following exercises, evaluate the natural logarithmic expression without using a calculator.

50) $\ln\left(e^{\frac{1}{3}}\right)$

51) $\ln(1)$

Answer

0

52) $\ln\left(e^{-0.225}\right)-3$

53) $25\ln\left(e^{\frac{2}{5}}\right)$

Answer

10

Technology

For the following exercises, evaluate each expression using a calculator. Round to the nearest thousandth.

54) $\log(0.04)$

55) $\ln(15)$

Answer

2.708

56) $\ln\left(\frac{4}{5}\right)$

Exercise 4.3E.57

$\log(\sqrt{2})$

Answer

0.151

58) $\ln(\sqrt{2})$

Extensions

59) Is $x = 0$ in the domain of the function $f(x) = \log x$? If so, what is the value of the function when $x = 0$? Verify the result.

Answer

No, the function has no defined value for $x = 0$ To verify, suppose $x = 0$ is in the domain of the function $f(x) = \log(x)$ Then there is some number n such that $n = \log(0)$ Rewriting as an exponential equation gives: $10^n = 0$ which is impossible since no such real number n exists. Therefore, $x = 0$ is not the domain of the function $f(x) = \log(x)$.

60) Is $f(x) = 0$ in the range of the function $f(x) = \log(x)$? If so, for what value of x? Verify the result.

61) Is there a number x such that $\ln x = 2$? If so, what is that number? Verify the result.

Answer

Yes. Suppose there exists a real number x such that $\ln x = 2$Rewriting as an exponential equation gives $x = e^2$ which is a real number. To verify, let $x = e^2$Then, by definition, $\ln(x) = \ln(e^2) = 2$

62) Is the following true: $\frac{\log_3(27)}{\log_4\left(\frac{1}{64}\right)} = -1$ Verify the result.

63) Is the following true: $\frac{\ln(e^{1.725})}{\ln(1)} = 1.725$ Verify the result.

Answer

No; $\ln(1) = 0$, so $\frac{\ln(e^{1.725})}{\ln(1)} = 1.725$ is undefined.

Real-World Applications

64) The exposure index EI for a 35 millimeter camera is a measurement of the amount of light that hits the film. It is determined by the equation $EI = \log_2\left(\frac{f^2}{t}\right)$, where f is the "f-stop" setting on the camera, and t is the exposure time in seconds. Suppose the f-stop setting is 8 and the desired exposure time is 2 seconds. What will the resulting exposure index be?

65) Refer to the previous exercise. Suppose the light meter on a camera indicates an EI of -2, and the desired exposure time is 16 seconds. What should the f-stop setting be?

Answer

2

66) The intensity levels I of two earthquakes measured on a seismograph can be compared by the formula $\log\left(\frac{I_1}{I_2}\right) = M_1 - M_2$ where M is the magnitude given by the Richter Scale. In August 2009, an earthquake of magnitude 6.1 hit Honshu, Japan. In March 2011, that same region experienced yet another, more devastating earthquake, this time with a magnitude of 9.0. How many times greater was the intensity of the 2011 earthquake? Round to the nearest whole number.

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4.4: Graphs of Logarithmic Functions

Skills to Develop

- Identify the domain of a logarithmic function.
- Graph logarithmic functions.

In Graphs of Exponential Functions, we saw how creating a graphical representation of an exponential model gives us another layer of insight for predicting future events. How do logarithmic graphs give us insight into situations? Because every logarithmic function is the inverse function of an exponential function, we can think of every output on a logarithmic graph as the input for the corresponding inverse exponential equation. In other words, logarithms give the *cause* for an *effect*.

To illustrate, suppose we invest $\$2500$ in an account that offers an annual interest rate of 5,compounded continuously. We already know that the balance in our account for any year t can be found with the equation $A = 2500e^{0.05t}$.

But what if we wanted to know the year for any balance? We would need to create a corresponding new function by interchanging the input and the output; thus we would need to create a logarithmic model for this situation. By graphing the model, we can see the output (year) for any input (account balance). For instance, what if we wanted to know how many years it would take for our initial investment to double? Figure 4.4.1 shows this point on the logarithmic graph.

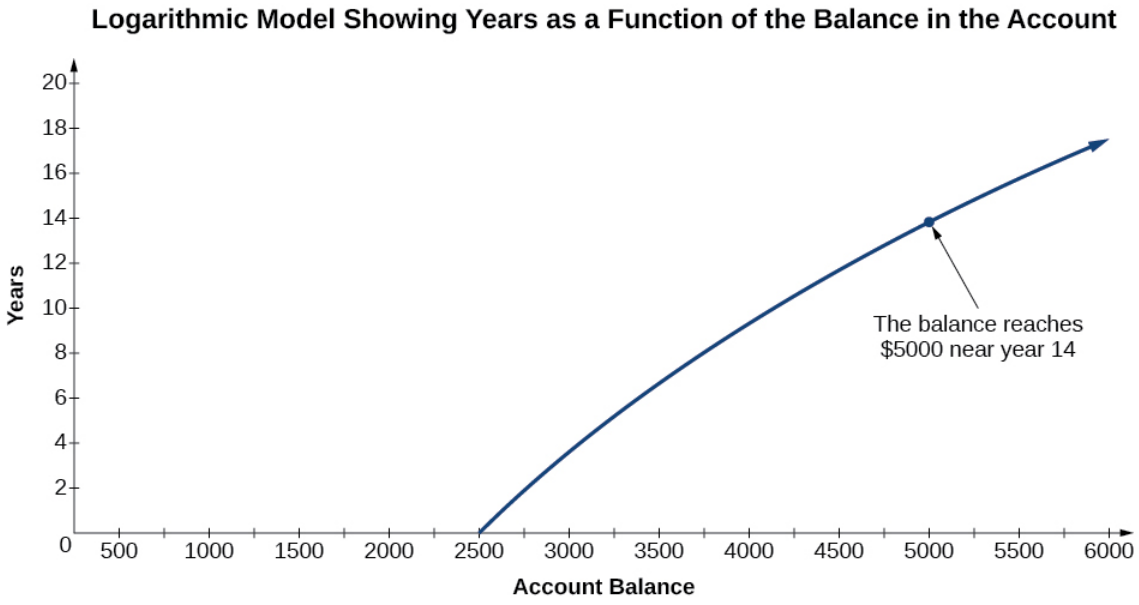


***Figure* 4.4.1**

In this section we will discuss the values for which a logarithmic function is defined, and then turn our attention to graphing the family of logarithmic functions.

Finding the Domain of a Logarithmic Function

Before working with graphs, we will take a look at the domain (the set of input values) for which the logarithmic function is defined.

Recall that the exponential function is defined as $y = b^x$ for any real number x and constant $b > 0$, $b \neq 1$, where

- The domain of y is $(-\infty, \infty)$.
- The range of y is $(0, \infty)$.

In the last section we learned that the logarithmic function $y = \log_b(x)$ is the inverse of the exponential function $y = b^x$. So, as inverse functions:

- The domain of $y = \log_b(x)$ is the range of $y = b^x$: $(0, \infty)$.
- The range of $y = \log_b(x)$ is the domain of $y = b^x$: $(-\infty, \infty)$.

Transformations of the toolkit function $y = \log_b(x)$ behave similarly to those of other functions. Just as with other toolkit functions, we can apply the four types of transformations—shifts, stretches, compressions, and reflections—to the toolkit function without loss of shape.

In Graphs of Exponential Functions we saw that certain transformations can change the *range* of $y = b^x$. Similarly, applying transformations to the toolkit function $y = \log_b(x)$ can change the *domain*. When finding the domain of a logarithmic function, therefore, it is important to remember that the domain consists *only of positive real numbers*. That is, the input value, or argument, of the logarithmic function must be greater than zero.

For example, consider $f(x) = \log_4(2x - 3)$. This function is defined for any values of x such that the argument, in this case $2x - 3$,is greater than zero. To find the domain, we set up an inequality and solve for x:

$$\begin{aligned} 2x - 3 &> 0 && \text{Show the argument greater than zero} \\ 2x &> 3 && \text{Add 3} \\ x &> 1.5 && \text{Divide by 2} \end{aligned}$$

In interval notation, the domain of $f(x) = \log_4(2x - 3)$ is $(1.5, \infty)$.

How To: Given a logarithmic function, identify the domain

1. Set up an inequality showing the argument greater than zero.
2. Solve for x.
3. Write the domain in interval notation.

Example 4.4.1: Identifying the Domain of a Logarithmic Shift

What is the domain of $f(x) = \log_2(x + 3)$?

Solution

The logarithmic function is defined only when the input is positive, so this function is defined when $x + 3 > 0$. Solving this inequality,

$$\begin{aligned} x + 3 &> 0 && \text{The input must be positive} \\ x &> -3 && \text{Subtract 3} \end{aligned}$$

The domain of $f(x) = \log_2(x + 3)$ is $(-3, \infty)$.

Try It 4.4.1

What is the domain of $f(x) = \log_5(x - 2) + 1$?

Answer

$(2, \infty)$

Graphing Logarithmic Functions

Now that we have a feel for the set of values for which a logarithmic function is defined, we move on to graphing logarithmic functions. The family of logarithmic functions includes the toolkit function $y = \log_b(x)$ along with all its transformations: shifts, stretches, compressions, and reflections.

We begin with the toolkit function $y = \log_b(x)$. Because every logarithmic function of this form is the inverse of an exponential function with the form $y = b^x$, their graphs will be reflections of each other across the line $y = x$. To illustrate this, we can observe the relationship between the input and output values of $y = 2^x$ and its equivalent $x = \log_2(y)$ in Table 4.4.1.

Table 4.4.1

x	-3	-2	-1	0	1	2	3
$2^x = y$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$\log_2(y) = x$	-3	-2	-1	0	1	2	3

Using the inputs and outputs from Table 4.4.1, we can build another table to observe the relationship between points on the graphs of the inverse functions $f(x) = 2^x$ and $g(x) = \log_2(x)$. See Table 4.4.2.

Table 4.4.2

$f(x) = 2^x$	$\left(-3, \frac{1}{8}\right)$	$\left(-2, \frac{1}{4}\right)$	$\left(-1, \frac{1}{2}\right)$	$(0, 1)$	$(1, 2)$	$(2, 4)$	$(3, 8)$
$g(x) = \log_2(x)$	$\left(\frac{1}{8}, -3\right)$	$\left(\frac{1}{4}, -2\right)$	$\left(\frac{1}{2}, -1\right)$	$(1, 0)$	$(2, 1)$	$(4, 2)$	$(8, 3)$

As expected, the x- and y-coordinates are reversed for the inverse functions. Figure 4.4.2 shows the graph of f and g.

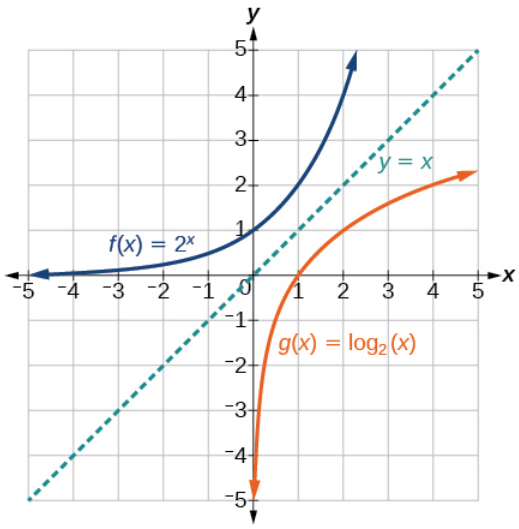


***Figure* 4.4.2:** *Notice that the graphs of $f(x) = 2^x$ and $g(x) = \log_2(x)$ are reflections about the line $y = x$.*

Observe the following from the graph:

- $f(x) = 2^x$ has a y-intercept at $(0, 1)$ and $g(x) = \log_2(x)$ has an x-intercept at $(1, 0)$.
- The domain of $f(x) = 2^x$, $(-\infty, \infty)$, is the same as the range of $g(x) = \log_2(x)$.
- The range of $f(x) = 2^x$, $(0, \infty)$, is the same as the domain of $g(x) = \log_2(x)$.

CHARACTERISTICS OF THE GRAPH OF THE toolkit FUNCTION, $f(x) = \log_b(x)$

For any real number x and constant $b > 0$, $b \neq 1$, we can see the following characteristics in the graph of $f(x) = \log_b(x)$:

- one-to-one function
- vertical asymptote: $x = 0$
- domain: $(0, \infty)$
- range: $(-\infty, \infty)$
- x-intercept: $(1, 0)$ and key point $(b, 1)$
- y-intercept: none
- increasing if $b > 1$
- decreasing if $0 < b < 1$

See Figure 4.4.3.

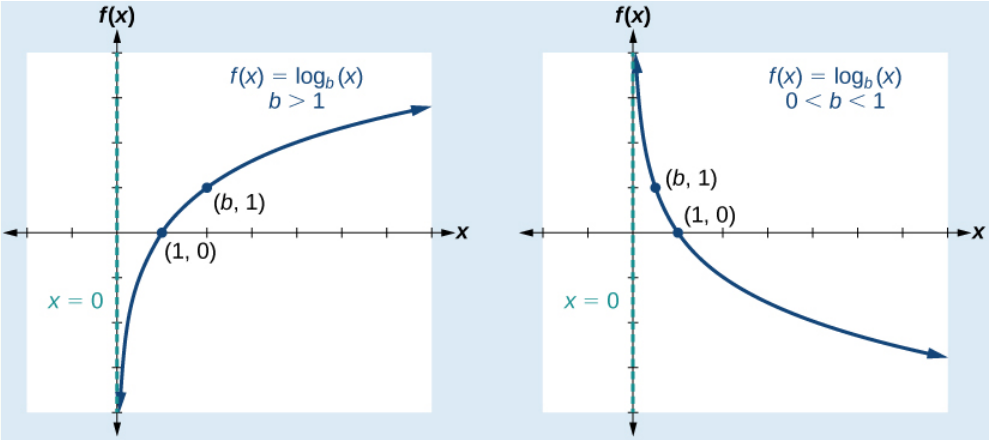


***Figure* 4.4.3**

Figure **4.4.4** shows how changing the base b in $f(x) = \log_b(x)$ can affect the graphs. Observe that the graphs compress vertically as the value of the base increases. (Note: recall that the function $\ln(x)$ has base $e \approx 2.718$.)

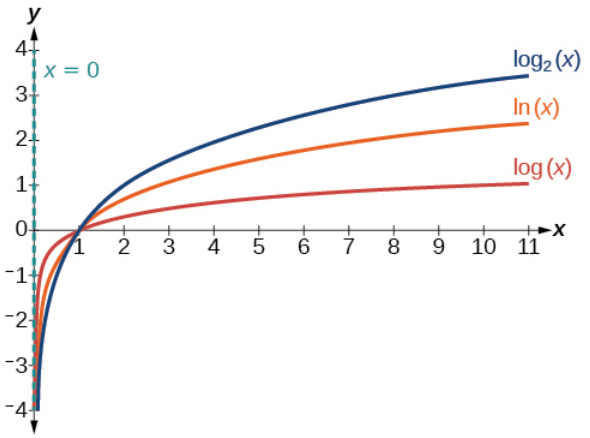


***Figure* 4.4.4:** *The graphs of three logarithmic functions with different bases, all greater than 1.*

How To: Given a logarithmic function with the form $f(x) = \log_b(x)$, graph the function.

1. Draw and label the vertical asymptote, $x = 0$.
2. Plot the *x*-intercept, $(1, 0)$.
3. Plot the key point $(b, 1)$.
4. Draw a smooth curve through the points.
5. State the domain, $(0, \infty)$,the range, $(-\infty, \infty)$, and the vertical asymptote, $x = 0$.

Example 4.4.2: Graphing a Logarithmic Function with the Form $f(x) = log_b(x)$

Graph $f(x) = \log_5(x)$. State the domain, range, and asymptote.

Solution

Before graphing, identify the behavior and key points for the graph.

- Since $b = 5$ is greater than one, we know the function is increasing. The left tail of the graph will approach the vertical asymptote $x = 0$, and the right tail will increase slowly without bound.
- The x-intercept is $(1, 0)$.
- The key point $(5, 1)$ is on the graph.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points (see Figure **4.4.5**).

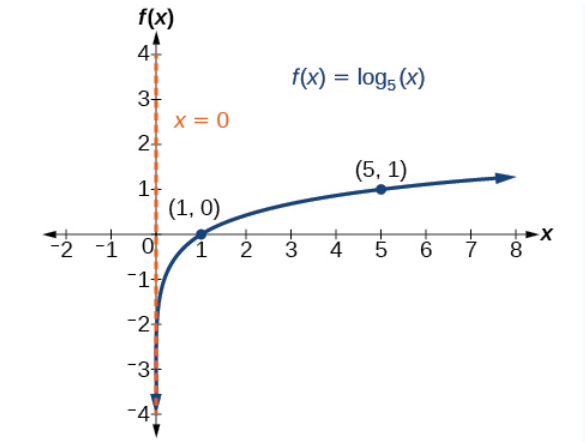


***Figure* 4.4.5**

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

4.4.2

Graph $f(x) = \log_{\frac{1}{5}}(x)$. State the domain, range, and asymptote.

Answer

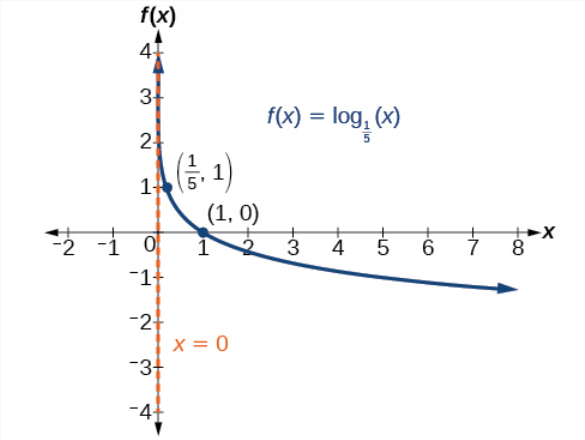


***Figure* 4.4.6**

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Graphing Transformations of Logarithmic Functions

As we mentioned in the beginning of the section, transformations of logarithmic graphs behave similarly to those of other toolkit functions. We can shift, stretch, compress, and reflect the toolkit function $y = \log_b(x)$ without loss of shape.

Graphing a Horizontal Shift of $f(x) = log_b(x)$

When a positive constant c is added or subtracted to the input of the toolkit function $f(x) = \log_b(x)$, the result is a horizontal shift c units *left* if c is positive and c units *right* if c is negative. To visualize horizontal shifts, we can observe the general graph of the toolkit function $f(x) = \log_b(x)$ and for $c > 0$ alongside the shift left, $g(x) = \log_b(x + c)$, and the shift right, $h(x) = \log_b(x - c)$. See Figure 4.4.7.

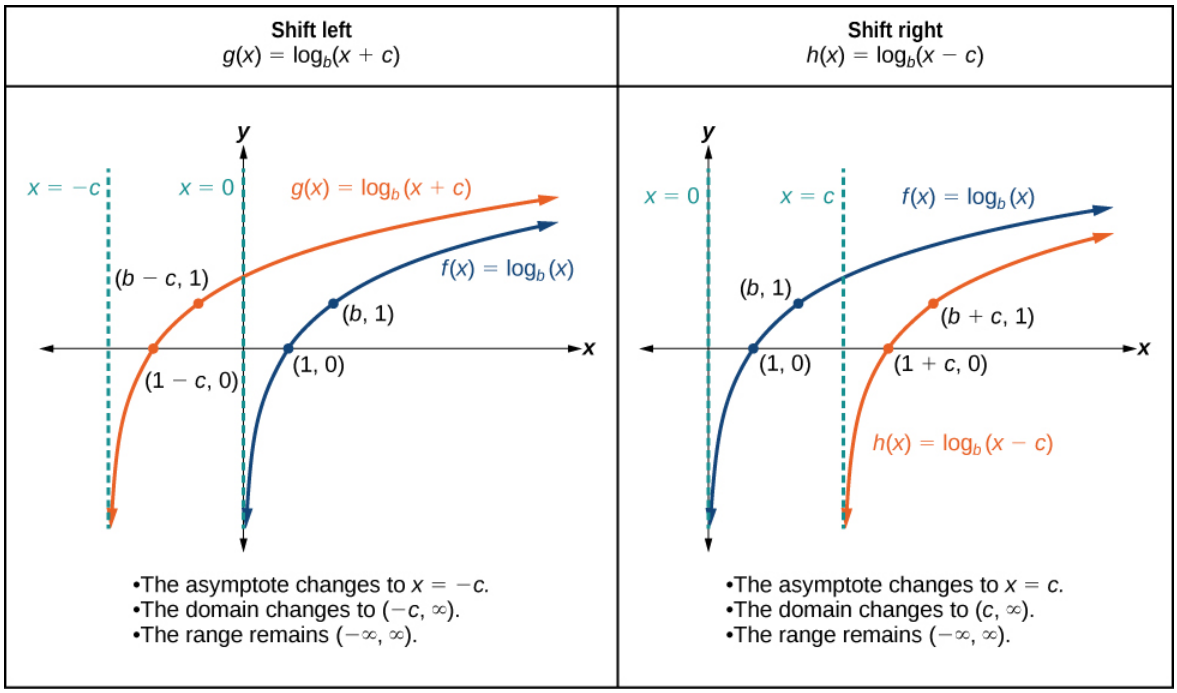


***Figure* 4.4.7**

HORIZONTAL SHIFTS OF THE toolkit FUNCTION $y = \log_b(x)$

For any constant c, the function $f(x) = \log_b(x + c)$

- shifts the toolkit function $y = \log_b(x)$ left c units if $c > 0$.
- shifts the toolkit function $y = \log_b(x)$ right $|c|$ units if $c < 0$.
- has the vertical asymptote $x = -c$.
- has domain $(-c, \infty)$.
- has range $(-\infty, \infty)$.

How To: Given a logarithmic function with the form $f(x) = \log_b(x + c)$, graph the translation.

1. Identify the horizontal shift:
   - If $c > 0$,shift the graph of $f(x) = \log_b(x)$ left c units.
   - If $c < 0$,shift the graph of $f(x) = \log_b(x)$ right $|c|$ units.
2. Draw the vertical asymptote $x = -c$.
3. Identify three key points from the toolkit function. Find new coordinates for the shifted functions by subtracting c from the x coordinate.
4. Label the three points.
5. The domain is $(-c, \infty)$,the range is $(-\infty, \infty)$, and the vertical asymptote is $x = -c$.

Example 4.4.3: Graphing a Horizontal Shift of the toolkit Function $y = \log_b(x)$

Sketch the horizontal shift $f(x) = \log_3(x - 2)$ alongside its toolkit function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

Solution

Since the function is $f(x) = \log_3(x - 2)$, we notice $x + (-2) = x$–2.

Thus $c = -2$, so $c < 0$. This means we will shift the function $f(x) = \log_3(x)$ right 2 units.

The vertical asymptote is $x = -(-2)$ or $x = 2$.

Consider the three key points from the toolkit function, $\left(\frac{1}{3}, -1\right)$, $(1, 0)$,and $(3, 1)$.

The new coordinates are found by adding 2 to the x coordinates.

Label the points $\left(\frac{7}{3}, -1\right)$, $(3, 0)$,and $(5, 1)$.

The domain is $(2, \infty)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 2$.

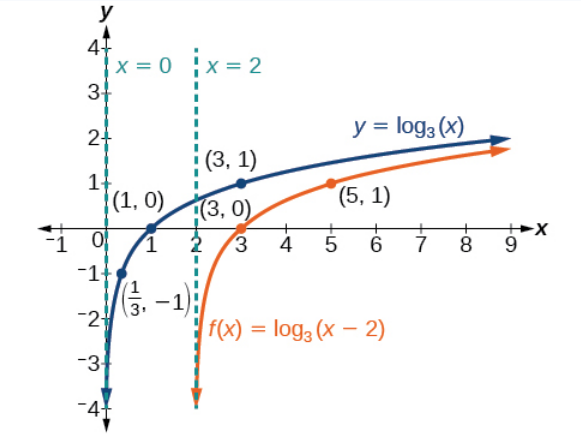


***Figure* 4.4.8**

4.4.3

Sketch a graph of $f(x) = \log_3(x + 4)$ alongside its toolkit function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

Answer

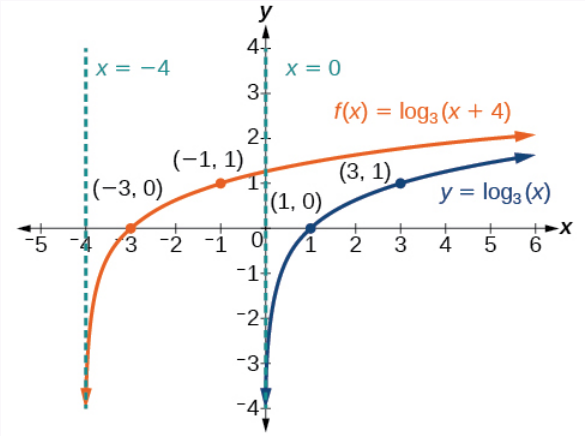


***Figure* 4.4.9**

The domain is $(-4, \infty)$,the range $(-\infty, \infty)$,and the asymptote $x = -4$.

Graphing a Vertical Shift of $y = \log_b(x)$

When a positive constant d is added or subtracted to the toolkit function $f(x) = \log_b(x)$,the result is a vertical shift d units up if d is added and d units down if d is subtracted.

VERTICAL SHIFTS OF THE toolkit FUNCTION $y = \log_b(x)$

For any constant d, the function $f(x) = \log_b(x) + d$

- shifts the toolkit function $y = \log_b(x)$ up d units if $d > 0$.

- shifts the toolkit function $y = \log_b(x)$ down $|d|$ units if $d < 0$.
- has the vertical asymptote $x = 0$.
- has domain $(0, \infty)$.
- has range $(-\infty, \infty)$.

How To: Given a logarithmic function with the form $f(x) = \log_b(x) + d$, graph the translation.

1. Identify the vertical shift:
   - If $d > 0$, shift the graph of $f(x) = \log_b(x)$ up d units.
   - If $d < 0$, shift the graph of $f(x) = \log_b(x)$ down $|d|$ units.
2. Draw the vertical asymptote $x = 0$.
3. Identify three key points from the toolkit function. Find new coordinates for the shifted functions by adding d to the y-coordinate.
4. Label the three points.
5. The domain is $(0, \infty)$, the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

Example 4.4.4: Graphing a Vertical Shift of the toolkit Function $y = log_b(x)$

Sketch a graph of $f(x) = \log_3(x) - 2$ alongside its toolkit function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution

Since the function is $f(x) = \log_3(x) - 2$,we notice $d = -2$. Thus $d < 0$.

This means we will shift the function $f(x) = \log_3(x)$ down 2 units.

The vertical asymptote is $x = 0$.

Consider these three key points from the toolkit function, $\left(\frac{1}{3}, -1\right)$, $(1, 0)$,and $(3, 1)$.

The new coordinates are found by subtracting 2 from the y-coordinates.

Label the points $\left(\frac{1}{3}, -3\right)$, $(1, -2)$, and $(3, -1)$.

The domain is $(0, \infty)$,the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

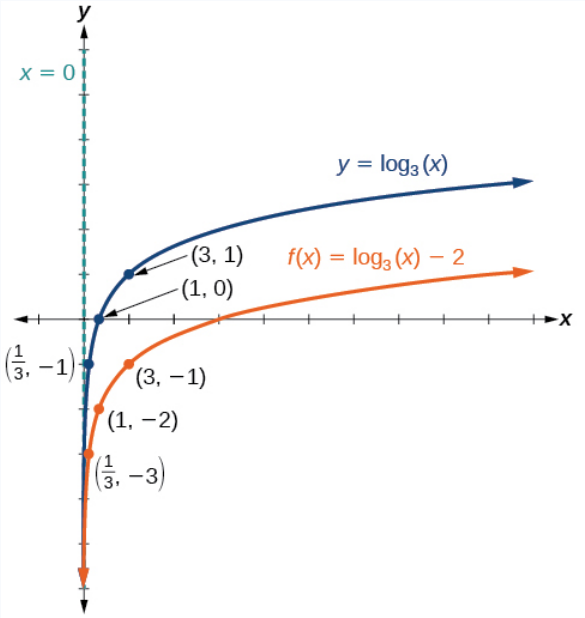


***Figure* 4.4.10**

The domain is $(0, \infty)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

4.4.4

Sketch a graph of $f(x) = \log_2(x) + 2$ alongside its toolkit function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Answer

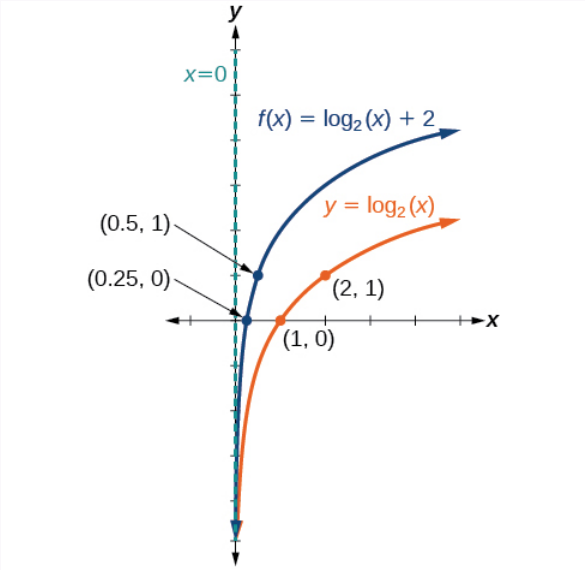


***Figure* 4.4.11**

The domain is $(0, \infty)$,the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Good job, trying this problem, and checking your answer! If you look carefully at the graph of the shifted function f, you will see that the labeled points are *not* vertical shifts of the two labeled points on the toolkit function. What are the correct points of the vertical shifts? Show them to your professor and make her give you extra credit!

Graphing Stretches and Compressions of $y = \log_b(x)$

When the toolkit function $f(x) = \log_b(x)$ is multiplied by a constant $a > 0$, the result is a vertical stretch or compression of the original graph.

VERTICAL STRETCHES AND COMPRESSIONS OF THE toolkit FUNCTION $y = \log_b(x)$

For any constant $a > 0$, the function $f(x) = a \log_b(x)$

- stretches the toolkit function $y = \log_b(x)$ vertically by a factor of a if $a > 1$.
- compresses the toolkit function $y = \log_b(x)$ vertically by a factor of a if $0 < a < 1$.
- has the vertical asymptote $x = 0$.
- has the x-intercept $(1, 0)$.
- has domain $(0, \infty)$.
- has range $(-\infty, \infty)$.

How To: Given a logarithmic function with the form $f(x) = a \log_b(x)$, $a > 0$, graph the translation.

1. Identify the vertical stretch or compressions:
   - If $a > 1$, the graph of $f(x) = \log_b(x)$ is stretched by a factor of a units.
   - If $0 < a < 1$, the graph of $f(x) = \log_b(x)$ is compressed by a factor of a units.
2. Draw the vertical asymptote $x = 0$.

3. Identify three key points from the toolkit function. Find new coordinates for the shifted functions by multiplying the y-coordinates by a.
4. Label the three points.
5. The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Example 4.4.5: Graphing a Stretch or Compression of the toolkit Function $y = log_b(x)$

Sketch a graph of $f(x) = 2\log_4(x)$ alongside its toolkit function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution

Since the function is $f(x) = 2\log_4(x)$,we will notice $a = 2$.

This means we will stretch the function $f(x) = \log_4(x)$ by a factor of 2.

The vertical asymptote is $x = 0$.

Consider three key points from the toolkit function, $(2, 1/2)$, $(1, 0)$, and $(4, 1)$.

The new coordinates are found by multiplying the y-coordinates by 2.

Label the points $(2, 1)$, $(1, 0)$, and $(4, 2)$. (If you are looking carefully, you will see that one of the key points from the toolkit function has not been labeled, and one of the new points on the stretched function have not been labeled. Give your professor the coordinates of the two missing points, and get extra credit!)

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$. See Figure 4.4.12.

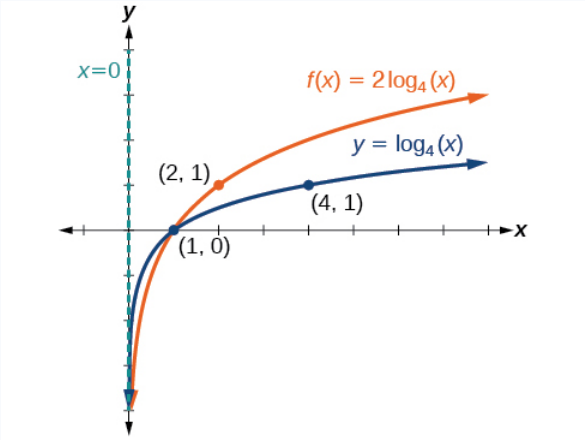


***Figure* 4.4.12**

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

4.4.5

Sketch a graph of $f(x) = \frac{1}{2}\log_4(x)$ alongside its toolkit function. Include some key points and asymptote on the graph. State the domain, range, and asymptote.

Answer

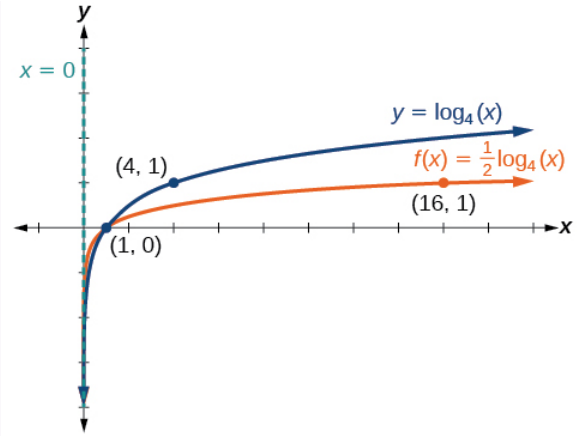


***Figure* 4.4.13**

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.
Again with the missing labels for key points! Can you name them?

Example 4.4.6: Combining a Shift and a Stretch

Sketch a graph of $f(x) = 5\log(x + 2)$. State the domain, range, and asymptote.

Solution

Remember: what happens inside parentheses happens first. First, we move the graph left 2 units, then stretch the function vertically by a factor of 5, as in Figure $4.4.16$. The vertical asymptote will be shifted to $x = -2$. The x-intercept will be $(-1, 0)$. The domain will be $(-2, \infty)$. Two points will help give the shape of the graph: $(-1, 0)$ and $(8, 5)$. We chose $x = 8$ as the x-coordinate of one point to graph because when $x = 8$, $x + 2 = 10$, the base of the common logarithm.

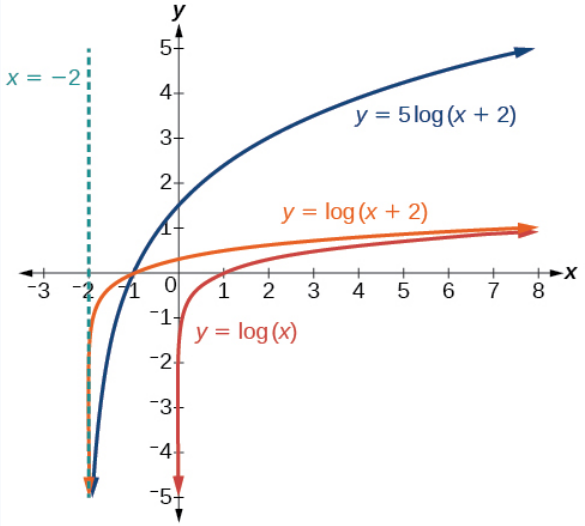


***Figure* 4.4.14**

The domain is $(-2, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = -2$.

4.4.6

Sketch a graph of the function $f(x) = 3\log(x - 2) + 1$. State the domain, range, and asymptote.

Answer

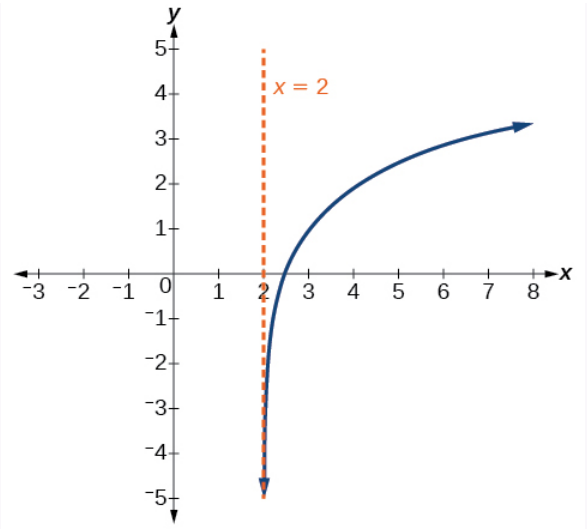


***Figure* 4.4.15**

The domain is $(2, \infty)$,the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 2$.

Graphing Reflections of $f(x) = \log_b(x)$

When the toolkit function $f(x) = \log_b(x)$ is multiplied by -1,the result is a reflection across the x-axis. When the *input* is multiplied by -1,the result is a reflection across the y-axis.

REFLECTIONS OF THE toolkit FUNCTION $y = \log_b(x)$

The function $f(x) = -\log_b(x)$

- reflects the toolkit function $y = \log_b(x)$ across the x-axis.
- has domain, $(0, \infty)$, range, $(-\infty, \infty)$, and vertical asymptote, $x = 0$, which are unchanged from the toolkit function.

The function $f(x) = \log_b(-x)$

- reflects the toolkit function $y = \log_b(x)$ across the y-axis.
- has domain $(-\infty, 0)$.
- has range, $(-\infty, \infty)$, and vertical asymptote, $x = 0$, which are unchanged from the toolkit function.

How To: Given a logarithmic function with the toolkit function $f(x) = \log_b(x)$, graph a translation.

***Table* 4.4.3**

If $f(x) = -\log_b(x)$	If $f(x) = \log_b(-x)$
1. Draw the vertical asymptote, $x = 0$.	1. Draw the vertical asymptote, $x = 0$.
2. Plot the x-intercept, $(1, 0)$.	2. Plot the x-intercept, $(-1, 0)$.
3. Reflect the graph of the toolkit function $f(x) = \log_b(x)$ across the x-axis.	3. Reflect the graph of the toolkit function $f(x) = \log_b(x)$ across the y-axis.
4. Draw a smooth curve through the points.	4. Draw a smooth curve through the points.
5. State the domain, $(0, \infty)$, the range, $(-\infty, \infty)$, and the vertical asymptote $x = 0$.	5. State the domain, $(-\infty, 0)$, the range, $(-\infty, \infty)$, and the vertical asymptote $x = 0$.

Example 4.4.7: Graphing a Reflection of a Logarithmic Function

Sketch a graph of $f(x) = \log(-x)$ alongside its toolkit function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution

Before graphing $f(x) = \log(-x)$, identify the behavior and key points for the graph.

- Since $b = 10$ is greater than one, we know that the toolkit function is increasing. Since the *input* value is multiplied by -1, $f(x)$ is a reflection of the toolkit graph across the y-axis. Thus, $f(x) = \log(-x)$ will be decreasing as x moves from negative infinity to zero, and the right tail of the graph will approach the vertical asymptote $x = 0$.
- The x-intercept is $(-1, 0)$.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points.

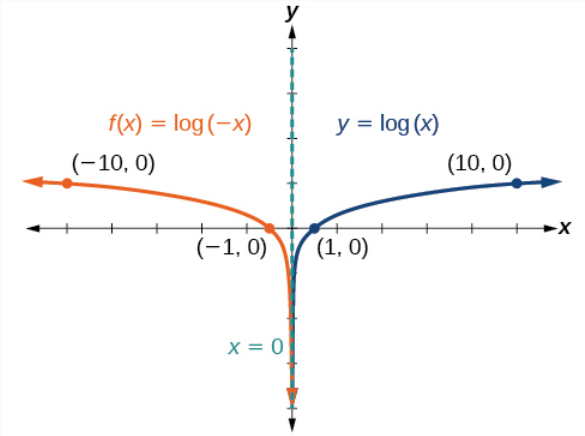


***Figure* 4.4.16**

The domain is $(-\infty, 0)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

4.4.7

Graph $f(x) = -\log(-x)$. State the domain, range, and asymptote.

Answer

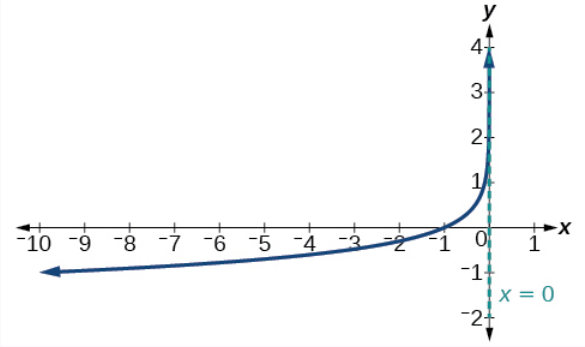


***Figure* 4.4.17**

The domain is $(-\infty, 0)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

Given a logarithmic equation, use a graphing calculator to approximate solutions.

1. Press **[Y=]**. Enter the given logarithm equation or equations as $\mathbf{Y_1}$= and, if needed, $\mathbf{Y_2}$=.
2. Press **[GRAPH]** to observe the graphs of the curves and use **[WINDOW]** to find an appropriate view of the graphs, including their point(s) of intersection.

3. To find the value of x, we compute the point of intersection. Press **[2ND]** then **[CALC]**. Select "intersect" and press **[ENTER]** three times. The point of intersection gives the value of x,for the point(s) of intersection.

Example 4.4.8: Approximating the Solution of a Logarithmic Equation

Solve $4\ln(x)+1=-2\ln(x-1)$ graphically. Round to the nearest thousandth.

Solution

Press **[Y=]** and enter $4\ln(x)+1$ next to Y_1=. Then enter $-2\ln(x-1)$ next to Y_2=. For a window, use the values 0 to 5 for $x\backslash 0 \; and \backslash (-10$ to 10 for y. Press **[GRAPH]**. The graphs should intersect somewhere a little to right of $x=1$.

For a better approximation, press **[2ND]** then **[CALC]**. Select **[5: intersect]** and press **[ENTER]** three times. The x-coordinate of the point of intersection is displayed as 1.3385297. (Your answer may be different if you use a different window or use a different value for **Guess?**) So, to the nearest thousandth, $x\approx 1.339$.

4.4.8

Solve $5\log(x+2)=4-\log(x)$ graphically. Round to the nearest thousandth.

Answer

$x\approx 3.049$

Summarizing Translations of the Logarithmic Function

Now that we have worked with each type of translation for the logarithmic function, we can summarize each in Table 4.4.4 to arrive at the general equation for translating exponential functions.

***Table* 4.4.4**

Translations of the toolkit Function $y=\log_b(x)$	
Translation	Form
Shift Horizontally c units to the left or right Vertically d units up or down	$y=\log_b(x+c)+d$
Stretch and Compress Stretch if $\lvert a\rvert>1$ Compression if $\lvert a\rvert<1$	$y=a\log_b(x)$
Reflect about the x-axis	$y=-\log_b(x)$
Reflect about the y-axis	$y=\log_b(-x)$
General equation for all translations	$y=a\log_b(x+c)+d$

TRANSLATIONS OF LOGARITHMIC FUNCTIONS

All translations of the toolkit logarithmic function, $y=\log_b(x)$, have the form

$$f(x)=a\log_b(x+c)+d$$

where the toolkit function, $y=\log_b(x)$, $b>1$,is

- shifted vertically d units up or down.
- shifted horizontally to the left or right c units.
- stretched vertically by a factor of $\lvert a\rvert$ if $\lvert a\rvert>0$.
- compressed vertically by a factor of $\lvert a\rvert$ if $0<\lvert a\rvert<1$.
- reflected across the x-axis when $a<0$.

For $f(x) = \log(-x)$, the graph of the toolkit function is reflected across the y-axis.

Example 4.4.9: Finding the Vertical Asymptote of a Logarithm Graph

What is the vertical asymptote of $f(x) = -2\log_3(x+4)+5$?

Solution

The vertical asymptote is at $x = -4$.

Analysis

The coefficient, the base, and the upward translation do not affect the asymptote. The shift of the curve 4 units to the left shifts the vertical asymptote to $x = -4$.

4.4.9

What is the vertical asymptote of $f(x) = 3 + \ln(x-1)$?

Answer

$x = 1$

Example 4.4.10: Finding the Equation from a Graph

Find a possible equation for the common logarithmic function graphed in Figure 4.4.18.

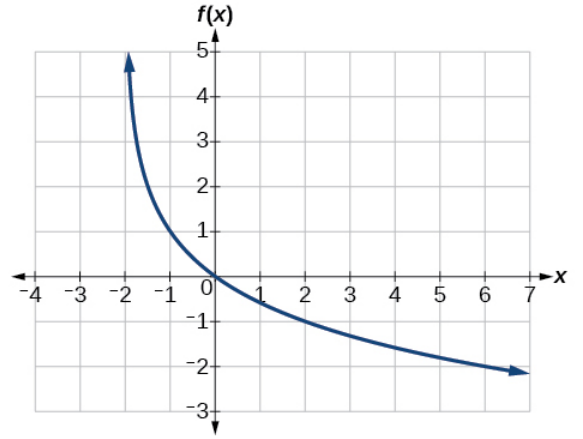


Figure 4.4.18

Solution

This graph has a vertical asymptote at $x = -2$, hence has been horizontally shifted 2 units to the left, and has been vertically reflected. We do not know yet the vertical shift or the vertical stretch. We know so far that the equation will have form:

$$f(x) = -a\log(x+2)+k$$

It appears the graph passes through the points $(-1,1)$ and $(2,-1)$. Substituting $(-1,1)$,

$$\begin{aligned} 1 &= -a\log(-1+2)+k \quad \text{Substitute } (-1,1) \\ 1 &= -a\log(1)+k \quad \text{Arithmetic} \\ 1 &= 0+k \quad \text{Evaluate } \log(1) \\ 1 &= k \end{aligned}$$

Next, substituting in $(2,-1)$,

$$\begin{aligned} -1 &= -a\log(2+2)+1 \quad \text{Substitute } (2,-1) \\ -2 &= -a\log(4) \quad \text{Arithmetic} \\ a &= \frac{2}{\log(4)} \quad \text{Solve for } a \end{aligned}$$

This gives us the equation $f(x) = -\frac{2}{\log(4)}\log(x+2)+1$.

Analysis

We can verify this answer by comparing the function values in Table 4.4.5 with the points on the graph in Figure 4.4.21.

***Table* 4.4.5**

x	−1	0	1	2	3
$f(x)$	1	0	−0.58496	−1	−1.3219
x	4	5	6	7	8
$f(x)$	−1.5850	−1.8074	−2	−2.1699	−2.3219

4.4.10

Give the equation of the natural logarithm graphed in Figure 4.4.19.

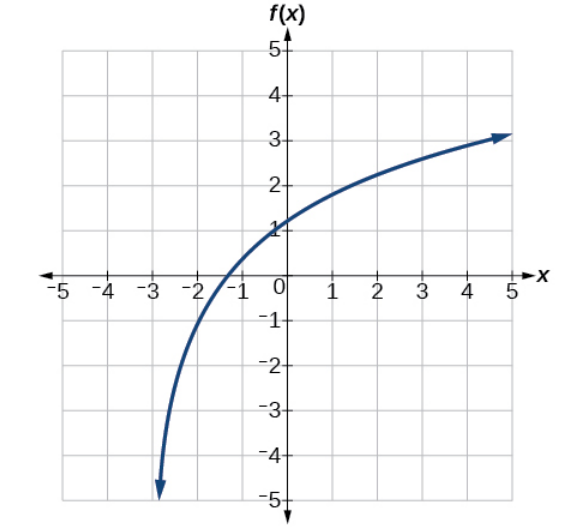


***Figure* 4.4.19**

Answer

$f(x) = 2\ln(x+3)-1$

Partial analysis: There is a vertical asymptote at $x = -3$, so the graph of $y = \ln(x)$ has been shifted 3 units left. Since $b = e \approx 2.7$, the graph has not been reflected across the x-axis.

: Is it possible to tell the domain and range and describe the end behavior of a function just by looking at the graph?

Yes, if we know the function is a general logarithmic function. For example, look at the graph in Figure 4.4.19. The graph approaches $x = -3$ (or thereabouts) more and more closely, so $x = -3$ is, or is very close to, the vertical asymptote. It approaches from the right, so the domain is all points to the right, $x|x > -3$. The range, as with all general logarithmic functions, is all real numbers. And we can see the end behavior because the graph goes down as it goes left and up as it goes right. The end behavior is: as $x \to -3^+$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$.

Media

Access these online resources for additional instruction and practice with graphing logarithms.

- Graph an Exponential Function and Logarithmic Function

- Match Graphs with Exponential and Logarithmic Functions
- Find the Domain of Logarithmic Functions

Key Equations

General Form for the Translation of the toolkit Logarithmic Function $f(x)=\log_b(x)$	$f(x)=a\log_b(x+c)+d$

Key Concepts

- To find the domain of a logarithmic function, set up an inequality showing the argument greater than zero, and solve for x. See Example 4.4.1 and Example 4.4.2.
- The graph of the toolkit function $f(x)=\log_b(x)$ has an x-intercept at $(1,0)$, domain $(0,\infty)$, range $(-\infty,\infty)$,vertical asymptote $x=0$, and
  - if $b>1$, the function is increasing.
  - if $0<b<1$, the function is decreasing.

  See Figure 4.4.3.
- The equation $f(x)=\log_b(x+c)$ shifts the toolkit function $y=\log_b(x)$ horizontally.
  - left c units if $c>0$.
  - right c units if $c<0$.

  See Example 4.4.3.
- The equation $f(x)=\log_b(x)+d$ shifts the toolkit function $y=\log_b(x)$ vertically.
  - up d units if $d>0$.
  - down d units if $d<0$.

  See Example 4.4.4.
- For any constant $a>0$, the equation $f(x)=a\log_b(x)$.
  - stretches the toolkit function $y=\log_b(x)$ vertically by a factor of a if $|a|>1$.
  - compresses the toolkit function $y=\log_b(x)$ vertically by a factor of a if $|a|<1$.

  See Example 4.4.5 and Example 4.4.6.
- When the toolkit function $y=\log_b(x)$ is multiplied by -1, the result is a reflection across the x-axis. When the input is multiplied by -1, the result is a reflection across the y-axis.
  - The equation $f(x)=-\log_b(x)$ represents a reflection of the toolkit function across the x-axis.
  - The equation $f(x)=\log_b(-x)$ represents a reflection of the toolkit function across the y-axis.

  See Example 4.4.7.
- A graphing calculator may be used to approximate solutions to some logarithmic equations. See Example 4.4.8.
- All translations of the logarithmic function can be summarized by the general equation $f(x)=a\log_b(x+c)+d$. See Table 4.4.4.
- Given an equation with the general form $f(x)=a\log_b(x+c)+d$,we can identify the vertical asymptote $x=-c$ for the transformation. See Example 4.4.9.
- Using the general equation $f(x)=a\log_b(x+c)+d$, we can write the equation of a logarithmic function given its graph. See Example 4.4.10.

Contributors

- Lynn Marecek (Santa Ana College) and MaryAnne Anthony-Smith (formerly of Santa Ana College). This content produced by OpenStax and is licensed under a Creative Commons Attribution License 4.0 license.

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4.4E: Exercises

4.4: Graphs of Logarithmic Functions

Verbal

1) The inverse of every logarithmic function is an exponential function and vice-versa. What does this tell us about the relationship between the coordinates of the points on the graphs of each?

Answer

Since the functions are inverses, their graphs are mirror images about the line $y-x$So for every point (a,b) on the graph of a logarithmic function, there is a corresponding point (b,a) on the graph of its inverse exponential function.

2) What type(s) of translation(s), if any, affect the range of a logarithmic function?

Exercise 4.4E.3

What type(s) of translation(s), if any, affect the domain of a logarithmic function?

Answer

Shifting the function right or left and reflecting the function about the y-axis will affect its domain.

4) Consider the general logarithmic function $f(x)=\log_b(x)$. Why can't x be zero?

5) Does the graph of a general logarithmic function have a horizontal asymptote? Explain.

Answer

No. A horizontal asymptote would suggest a limit on the range, and the range of any logarithmic function in general form is all real numbers.

Algebraic

For the following exercises, state the domain and range of the function.

6) $f(x)=\log_3(x+4)$

Exercise 4.4E.7

$h(x)=\ln\left(\frac{1}{2}-x\right)$

Answer

Domain: $\left(-\infty,\frac{1}{2}\right)$; Range: $(-\infty,\infty)$

8) $g(x)=\log_5(2x+9)-2$

9) $h(x)=\ln(4x+17)-5$

Answer

Domain: $\left(-\frac{17}{4},\infty\right)$; Range: $(-\infty,\infty)$

10) $f(x)=\log_2(12-3x)-3$

For the following exercises, state the domain and the vertical asymptote of the function.

11) $f(x)=\log_b(x-5)$

Answer

Domain: $(5, \infty)$; Vertical asymptote: $x = 5$

12) $g(x) = \ln(3 - x)$

13) $f(x) = \log(3x + 1)$

Answer

Domain: $\left(-\frac{1}{3}, \infty\right)$; Vertical asymptote: $x = -\frac{1}{3}$

14) $f(x) = 3\log(-x) + 2$

15) $g(x) = -\ln(3x + 9) - 7$

Answer

Domain: $(-3, \infty)$; Vertical asymptote: $x = -3$

For the following exercises, state the domain, vertical asymptote, and end behavior of the function.

16) $f(x) = \ln(2 - x)$

17) $f(x) = \log\left(x - \frac{3}{7}\right)$

Answer

Domain: $\left(\frac{3}{7}, \infty\right)$

Vertical asymptote: $x = \frac{3}{7}$

End behavior: as $x \to \left(\frac{3}{7}\right)^+$, $f(x) \to -\infty$ and as $x \to \infty, f(x) \to \infty$

18) $h(x) = -\log(3x - 4) + 3$

Exercise 4.4E.19

$g(x) = \ln(2x + 6) - 5$

Answer

Domain: $(-3, \infty)$

Vertical asymptote: $x = -3$

End behavior: as $x \to -3^+$, $f(x) \to -\infty$ and as $x \to \infty, f(x) \to \infty$

20) $f(x) = \log_3(15 - 5x) + 6$

For the following exercises, state the domain, range, and x- and y-intercepts, if they exist. If they do not exist, write DNE.

Exercise 4.4E.21

$h(x) = \log_4(x - 1) + 1$

Answer

Domain: $(1, \infty)$

Range: $-\infty, \infty$

Vertical asymptote: $x = 1$

x-intercept: $\left(\frac{5}{4}, 0\right)$

y-intercept: DNE

22) $f(x) = \log(5x + 10) + 3$

23) $g(x) = \ln(-x) - 2$

Answer

Domain: $(-\infty, 0)$

Range: $-\infty, \infty$

Vertical asymptote: $x = 0$

x-intercept: $(-e^2, 0)$

y-intercept: DNE

24) $f(x) = \log_2(x + 2) - 5$

25) $h(x) = 3\ln(x) - 9$

Answer

Domain: $(0, \infty)$

Range: $-\infty, \infty$

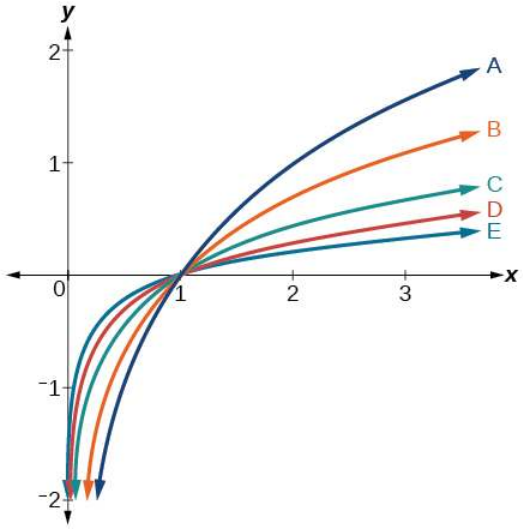
Vertical asymptote: $x = 0$

x-intercept: $(e^3, 0)$

y-intercept: DNE

Graphical

For the following exercises, match each function in Figure below with the letter corresponding to its graph.



26) $d(x) = \log(x)$

27) $f(x) = \ln(x)$

Answer

B

28) $g(x) = \log_2(x)$

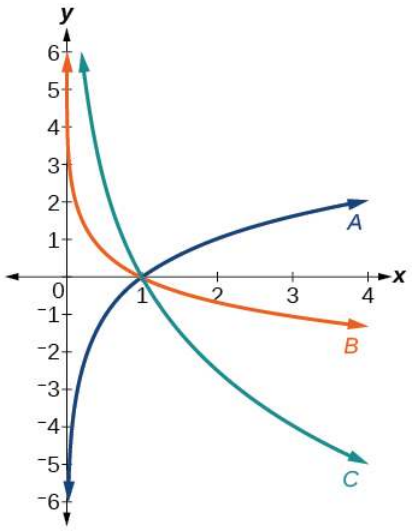
29) $h(x) = \log_5(x)$

Answer

C

30) $j(x) = \log_{25}(x)$

For the following exercises, match each function in Figure with the letter corresponding to its graph.



31) $f(x) = \log_{\frac{1}{3}}(x)$

Answer

B

32) $g(x) = \log_2(x)$

33) $h(x) = \log_{\frac{3}{4}}(x)$

Answer

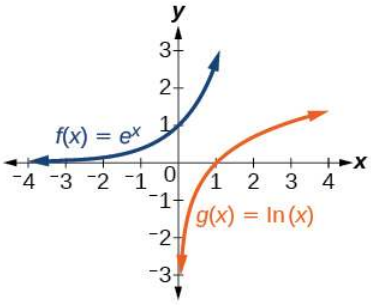
C

For the following exercises, sketch the graphs of each pair of functions on the same axis.

34) $f(x) = \log(x)$ and $g(x) = 10^x$

35) $f(x) = e^x$ and $g(x) = \ln(x)$

Answer



For the following exercises, match each function in Figure with the letter corresponding to its graph.

36) $f(x) = \log_4(-x + 2)$

37) $g(x) = -\log_4(x + 2)$

Answer

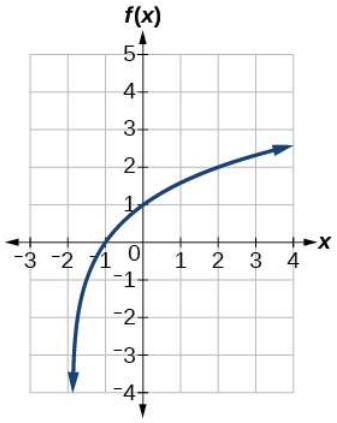
C

38) $h(x) = \log_4(x + 2)$

For the following exercises, sketch the graph of the indicated function.

39) $f(x) = \log_2(x + 2)$

Answer

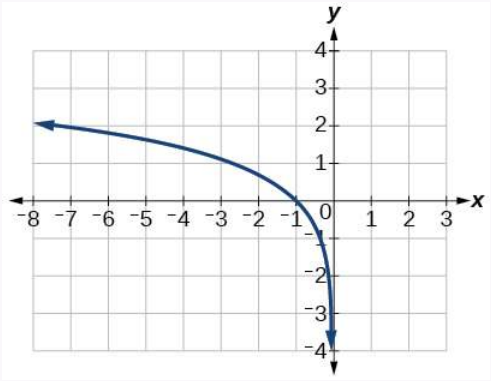


40) $f(x) = 2\log(x)$

Exercise 4.4E.41

$f(x) = \ln(-x)$

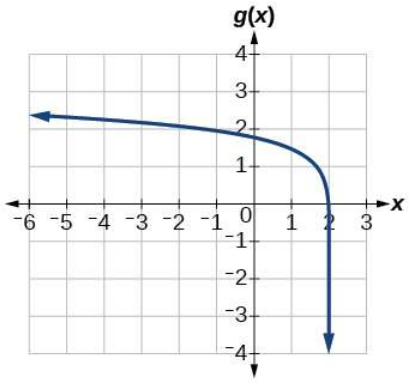
Answer



42) $g(x) = \log(4x + 16) + 4$

43) $g(x) = \log(6 - 3x) + 1$

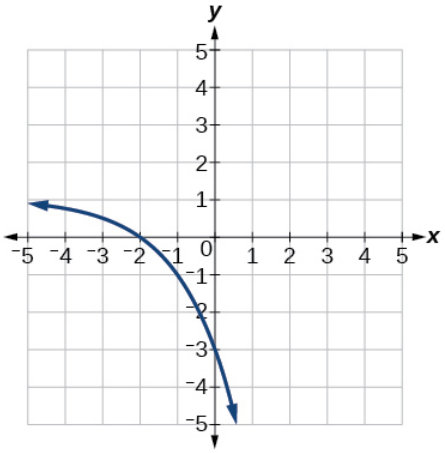
Answer



44) $h(x) = -\frac{1}{2}\log(x+1) - 3$

For the following exercises, write a logarithmic equation corresponding to the graph shown.

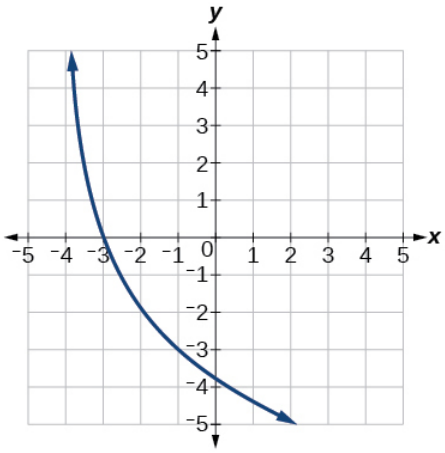
45) Use $y = \log_2(x)$ as the parent function.



Answer

$f(x) = \log_2(-(x-1))$

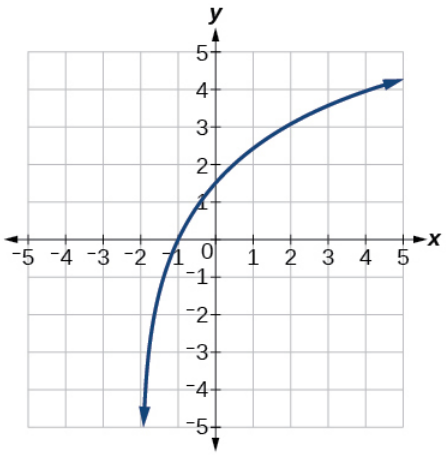
46) Use $f(x) = \log_3(x)$ as the parent function.



Exercise 4.4E.47

Use $f(x) = \log_4(x)$ as the parent function.

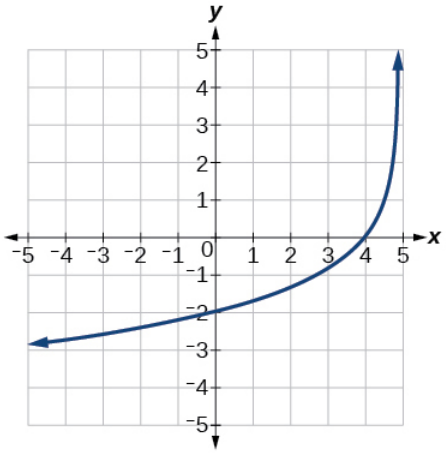
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Answer

$f(x) = 3\log_4(x+2)$

48) Use $f(x) = \log_5(x)$ as the parent function.



Technology

For the following exercises, use a graphing calculator to find approximate solutions to each equation.

49) $\log(x-1)+2 = \ln(x-1)+2$

Answer

$x = 2$

50) $\log(2x-3)+2 = -\log(2x-3)+5$

51) $\ln(x-2)+2 = -\ln(x+1)$

Answer

$x \approx 2.303$

52) $2\ln(5x+1) = \frac{1}{2}\ln(-5x)+1$

53) $\frac{1}{3}\log(1-x) = \log(x+1)+\frac{1}{3}$

Answer

$x \approx -0.472$

Extensions

54) Let b be any positive real number such that $b \neq 1$. What must $\log_b 1$ be equal to? Verify the result.

55) Explore and discuss the graphs of $f(x) = \log_{\frac{1}{2}}(x)$ and $g(x) = -\log_2(x)$. Make a conjecture based on the result.

Answer

The graphs of $f(x) = \log_{\frac{1}{2}}(x)$ and $g(x) = -\log_2(x)$ appear to be the same;

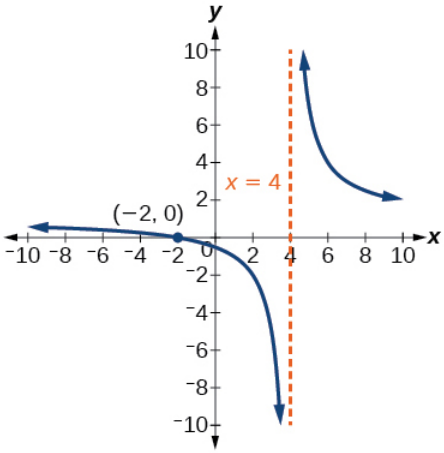
Conjecture: for any positive base $b \neq 1$, $\log_b(x) = \log_{\frac{1}{b}}(x)$

56) Prove the conjecture made in the previous exercise.

57) What is the domain of the function $f(x) = \ln\left(\frac{x+2}{x-4}\right)$? Discuss the result.

Answer

Recall that the argument of a logarithmic function must be positive, so we determine where $\frac{x+2}{x-4} > 0$. From the graph of the function $f(x) = \frac{x+2}{x-4}$, note that the graph lies above the x-axis on the interval $(-\infty, -2)$ and again to the right of the vertical asymptote, that is $(4, \infty)$ Therefore, the domain is $(-\infty, -2) \cup (4, \infty)$



58) Use properties of exponents to find the x-intercepts of the function $f(x) = \log(x^2 + 4x + 4)$ algebraically. Show the steps for solving, and then verify the result by graphing the function.

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4.5: Logarithmic Properties

Skills to Develop

- Use the product rule for logarithms.
- Use the quotient rule for logarithms.
- Use the power rule for logarithms.
- Expand logarithmic expressions.
- Condense logarithmic expressions.
- Use the change-of-base formula for logarithms.

In chemistry, the pH scale is used as a measure of the acidity or alkalinity of a substance. Substances with a pH less than 7 are considered acidic, and substances with a pH greater than 7 are said to be alkaline. Our bodies, for instance, must maintain a pH close to 7.35 in order for enzymes to work properly. To get a feel for what is acidic and what is alkaline, consider the following pH levels of some common substances:

- Battery acid: 0.8
- Stomach acid: 2.7
- Orange juice: 3.3
- Pure water: 7 at $25°C$
- Human blood: 7.35
- Fresh coconut: 7.8
- Sodium hydroxide (lye): 14

To determine whether a solution is acidic or alkaline, we find its pH, which is a measure of the number of active positive hydrogen ions in the solution. The pH is defined by the following formula, where $[H^+]$ is the concentration of hydrogen ion in the solution

$$pH = -\log([H^+]) \tag{4.5.1}$$

$$= \log\left(\frac{1}{[H^+]}\right) \tag{4.5.2}$$

The equivalence of Equations 4.5.1 and 4.5.2 is one of the logarithm properties we will examine in this section.

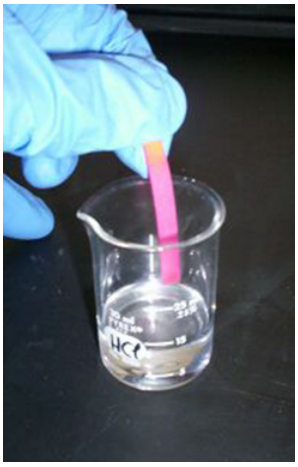


***Figure* 4.5.1:** *The pH of hydrochloric acid is tested with litmus paper. (credit: David Berardan).*

Using the Product Rule for Logarithms

Recall that the logarithmic and exponential functions "undo" each other. This means that logarithms have similar properties to exponents. Some important properties of logarithms are given here. First, the following properties are easy to prove.

$$\log_b 1 = 0 \tag{4.5.3}$$

$$\log_b b = 1 \tag{4.5.4}$$

For example, $\log_5 1 = 0$ since $5^0 = 1$. And $\log_5 5 = 1$ since $5^1 = 5$.

Next, we have the inverse property.

$$\log_b(b^x) = x \tag{4.5.5}$$

$$b^{\log_b x} = x,\ x > 0 \tag{4.5.6}$$

For example, to evaluate $\log(100)$,we can rewrite the logarithm as $\log_{10}(10^2)$, and then apply the inverse property $\log_b(b^x) = x$ to get $\log_{10}(10^2) = 2$.

To evaluate $e^{\ln(7)}$,we can rewrite the logarithm as $e^{\log_e 7}$, to see that we can apply the inverse property $b^{\log_b x} = x$ to get $e^{\log_e 7} = 7$.

Finally, we have the one-to-one property.

$$\text{if } \log_b M = \log_b N, \text{ then} M = N \tag{4.5.7}$$

We can use the one-to-one property to solve the equation $\log_3(3x) = \log_3(2x + 5)$ for x. Since the bases are the same, we can apply the one-to-one property by setting the arguments equal and solving for x:

$3x = 2x + 5$ Set the arguments equal.

$x = 5$ Subtract $2x$.

But what about the equation $\log_3(3x) + \log_3(2x + 5) = 2$? The one-to-one property does not help us in this instance. Before we can solve an equation like this, we need a method for combining terms on the left side of the equation.

Recall that we use the *product rule of exponents* to combine the product of exponents by adding: $x^a x^b = x^{a+b}$. We have a similar property for logarithms, called the **product rule for logarithms**, which says that the logarithm of a product is equal to a sum of logarithms. The product rule for logarithms can be used to rewrite a logarithm of a product as the sum of individual logarithms.

Because logs are exponents, and we multiply like bases, we can add the exponents. We will use the inverse property to derive the product rule below.

Theorem 4.5.1 THE PRODUCT RULE FOR LOGARITHMS

Given any real number x and positive real numbers M, N, and b, where $b \neq 1$,

$$\log_b(MN) = \log_b(M) + \log_b(N). \tag{4.5.8}$$

Proof

Let $m = \log_b M$ and $n = \log_b N$. In exponential form, these equations are $b^m = M$ and $b^n = N$. It follows that

$$\begin{aligned} \log_b(MN) &= \log_b(b^m b^n) && \text{Substitute for M and N} \\ &= \log_b(b^{m+n}) && \text{Apply the product rule for exponents} \\ &= m + n && \text{Apply the inverse property of logs} \\ &= \log_b(M) + \log_b(N) && \text{Substitute for m and n} \end{aligned}$$

Extra credit: Does this rule work if M and N are both negative? Why, or why not? Bring your explanation to your professor, and you will get extra credit.

Note that repeated applications of the product rule for logarithms allow us to simplify the logarithm of the product of any number of factors. For example, consider $\log_b(wxyz)$. Using the product rule for logarithms, we can rewrite this logarithm of a product as the sum of logarithms of its factors:

$$\log_b(wxyz) = \log_b w + \log_b x + \log_b y + \log_b z$$

How To: Given the logarithm of a product, use the product rule of logarithms to write an equivalent sum of logarithms

1. Factor the argument completely, expressing each whole number factor as a product of primes.
2. Write the equivalent expression by summing the logarithms of each factor.

Example 4.5.1: Using the Product Rule for Logarithms

Expand $\log_3(30x(3x+4))$.

Solution

We begin by factoring the argument completely, expressing 30 as a product of primes.

$$\log_3(30x(3x+4)) = \log_3(2\cdot 3\cdot 5\cdot x\cdot(3x+4))$$

Next we write the equivalent equation by summing the logarithms of each factor.

$$\log_3(30x(3x+4)) = \log_3(2)+\log_3(3)+\log_3(5)+\log_3(x)+\log_3(3x+4)$$

4.5.1

Expand $\log_b(8k)$.

Answer

$$\log_b 2+\log_b 2+\log_b 2+\log_b k = 3\log_b 2+\log_b k$$

Using the Quotient Rule for Logarithms

For quotients, we have a similar rule for logarithms. Recall that we use the *quotient rule of exponents* to combine the quotient of exponents by subtracting: $x^{\frac{a}{b}} = x^{a-b}$. The **quotient rule for logarithms** says that the logarithm of a quotient is equal to a difference of logarithms. The quotient rule for logarithms can be used to rewrite a logarithm of a quotient as the difference of individual logarithms.

Just as with the product rule, we can use the inverse property to derive the quotient rule.

Theorem 4.5.2 THE QUOTIENT RULE FOR LOGARITHMS

Given any real number x and positive real numbers M, N, and b, b, where $b\neq 1$,

$$\log_b\left(\frac{M}{N}\right) = \log_b(M)-\log_b(N). \tag{4.5.9}$$

Proof

Let $m=\log_b M$ and $n=\log_b N$. In exponential form, these equations are $b^m=M$ and $b^n=N$. It follows that

$$\begin{aligned}\log_b\left(\frac{M}{N}\right) &= \log_b\left(\frac{b^m}{b^n}\right) && \text{Substitute for M and N}\\ &= \log_b(b^{m-n}) && \text{Apply the quotient rule for exponents}\\ &= m-n && \text{Apply the inverse property of logs}\\ &= \log_b(M)-\log_b(N) && \text{Substitute for m and n}\end{aligned}$$

For example, to expand $\log\left(\frac{2x^2+6x}{3x+9}\right)$, we must first express the quotient in lowest terms. Factoring and canceling we get,

$$\begin{aligned}\log\left(\frac{2x^2+6x}{3x+9}\right) &= \log\left(\frac{2x(x+3)}{3(x+3)}\right) && \text{Factor the numerator and denominator}\\ &= \log\left(\frac{2x}{3}\right) && \text{Cancel the common factors}\end{aligned}$$

Next we apply the quotient rule by subtracting the logarithm of the denominator from the logarithm of the numerator. Then we apply the product rule.

$$\begin{aligned}\log\left(\frac{2x}{3}\right) &= \log(2x)-\log(3)\\ &= \log(2)+\log(x)-\log(3)\end{aligned}$$

How To: Given the logarithm of a quotient, use the quotient rule of logarithms to write an equivalent difference of logarithms

1. Express the argument in lowest terms by factoring the numerator and denominator and canceling common terms.
2. Write the equivalent expression by subtracting the logarithm of the denominator from the logarithm of the numerator.
3. Check to see that each term is fully expanded. If not, apply the product rule for logarithms to expand completely.

Example 4.5.2: Using the Quotient Rule for Logarithms

Expand $\log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right)$.

Solution

First we note that the quotient is factored and in lowest terms, so we apply the quotient rule.

$$\log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right) = \log_2(15x(x-1)) - \log_2((3x+4)(2-x))$$

Notice that the resulting terms are logarithms of products. To expand completely, we apply the product rule, noting that the prime factors of 15 are 3 and 5.

$$\begin{aligned}\log_2(15x(x-1)) - \log_2((3x+4)(2-x)) &= [\log_2(3)+\log_2(5)+\log_2(x)+\log_2(x-1)] \\ &\quad -[\log_2(3x+4)+\log_2(2-x)] \\ &= \log_2(3)+\log_2(5)+\log_2(x)+\log_2(x-1) \\ &\quad -\log_2(3x+4)-\log_2(2-x)\end{aligned}$$

Analysis

There are exceptions to consider in this and later examples. First, because denominators must never be zero, this expression is not defined for $x=-\frac{4}{3}$ and $x=2$. Also, since the argument of a logarithm must be positive, we note as we observe the expanded logarithm, that $x>0$, $x>1$, $x>-\frac{4}{3}$, and $x<2$. Combining these conditions yields the condition $1<x<2$.

Try It 4.5.2

Expand $\log_3\left(\frac{7x^2+21x}{7x(x-1)(x-2)}\right)$.

Answer

$\log_3(x+3)-\log_3(x-1)-\log_3(x-2)$

Q&A: Can we use the product or quotient rules for logarithms to rewrite $(\log_b M)(\log_b N)$ or $\frac{\log_b M}{\log_b N}$?

*No. These rules apply if we are applying a logarithm to a product or to a quotient, **not** when we are multiplying or dividing logarithms.*

Using the Power Rule for Logarithms

We've explored the product rule and the quotient rule, but how can we take the logarithm of a power, such as x^2? One method is as follows:

$$\begin{aligned}\log_b(x^2) &= \log_b(x\cdot x) \\ &= \log_b x+\log_b x \\ &= 2\log_b x\end{aligned}$$

Notice that we used the **product rule for logarithms** to find a solution for the example above. By doing so, we have derived the **power rule for logarithms**. Keep in mind that, although the input to a logarithm may not be written as a power, we may be able to change it to a power. For example,

$$100=10^2 \quad \sqrt{3}=3^{\frac{1}{2}} \quad \frac{1}{e}=e^{-1}$$

THE POWER RULE FOR LOGARITHMS

The power rule for logarithms can be used to rewrite the logarithm of a positive number M raised to an exponent as the product of the exponent times the logarithm of M.

$$\log_b(M^n) = n\log_b M \tag{4.5.10}$$

How To: Given the logarithm of a power, use the power rule of logarithms to write an equivalent product

1. Express the argument as a power, if needed.
2. Write the equivalent expression by multiplying the exponent times the logarithm of the base.

Example 4.5.3: Rewriting a logarithmic Expression Using the Power Rule

Rewrite $\log_3(25)$ using the power rule for logs.

Solution

Expressing the argument as a power, we get $\log_3(25) = \log_3(5^2)$.

Next we identify the exponent, 2, and the base, 5, and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.

$$\log_3(5^2) = 2\log_3(5)$$

Try It 4.5.3

Rewrite $\ln\left(\frac{1}{x^2}\right)$ using the power rule for logs.

Answer

$-2\ln(x)$

Example 4.5.4: Using the Power Rule in Reverse

Rewrite $4\ln(x)$ using the power rule for logs to a single logarithm with a leading coefficient of 1.

Solution

Because the logarithm of a power is the product of the exponent times the logarithm of the base, it follows that the product of a number and a logarithm can be written as a power. For the expression $4\ln(x)$, we identify the factor, 4, as the exponent and the argument, x, as the base, and rewrite the product as a logarithm of a power: $4\ln(x) = \ln(x^4)$.

Try It 4.5.4

Rewrite $2\log_3 4$ using the power rule for logs to a single logarithm with a coefficient of 1.

Answer

$\log_3 16$

Expanding Logarithmic Expressions

Taken together, the product rule, quotient rule, and power rule are often called “laws of logs.” Sometimes we apply more than one rule in order to simplify an expression. For example:

$$\begin{aligned}\log_b\left(\frac{6x}{y}\right) &= \log_b(6x) - \log_b y \\ &= \log_b 6 + \log_b x - \log_b y\end{aligned}$$

We can use the power rule to expand logarithmic expressions involving negative and fractional exponents. Here is an alternate proof of the quotient rule for logarithms using the fact that a reciprocal is a negative power:

$$\begin{aligned}\log_b\left(\frac{A}{C}\right) &= \log_b(AC^{-1}) \\ &= \log_b(A)+\log_b(C^{-1}) \\ &= \log_b A+(-1)\log_b C \\ &= \log_b A-\log_b C\end{aligned}$$

We can also apply the product rule to express a sum or difference of logarithms as the logarithm of a product.

With practice, we can look at a logarithmic expression and expand it mentally, writing the final answer. Remember, however, that we can **only** do this with products, quotients, powers, and roots—**never** with addition or subtraction inside the argument of the logarithm.

Example 4.5.5: Expanding Logarithms Using Product, Quotient, and Power Rules

Rewrite $ln\left(\frac{x^4y}{7}\right)$ as a sum or difference of logs.

Solution

First, because we have a quotient of two expressions, we can use the quotient rule:

$$\ln\left(\frac{x^4y}{7}\right) = \ln(x^4y)-\ln(7)$$

Then seeing the product in the first term, we use the product rule:

$$\ln(x^4y)-\ln(7) = \ln(x^4)+\ln(y)-\ln(7)$$

Finally, we use the power rule on the first term:

$$\ln(x^4)+\ln(y)-\ln(7) = 4\ln(x)+\ln(y)-\ln(7)$$

4.5.5

Expand $\log\left(\frac{x^2y^3}{z^4}\right)$.

Answer

$2\log x+3\log y-4\log z$

Example 4.5.6: Using the Power Rule for Logarithms to Simplify the Logarithm of a Radical Expression

Expand $\log(\sqrt{x})$.

Solution

$$\begin{aligned}\log(\sqrt{x}) &= \log\left(x^{1/2}\right) \\ &= \frac{1}{2}\log x\end{aligned}$$

4.5.6

Expand $\ln(\sqrt[3]{x^2})$.

Answer

$\frac{2}{3}\ln x$

: Can we expand $\ln(x^2+y^2)$?

No. There is no way to expand the logarithm of a sum or difference inside the argument of the logarithm.

Example 4.5.7: Expanding Complicated Logarithmic Expressions

Expand $\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right)$.

Solution

We can expand by applying the Product and Quotient Rules.

$$\begin{aligned}\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right) &= \log_6 64+\log_6 x^3+\log_6(4x+1)-\log_6(2x-1)\\ &\qquad \text{Apply the Quotient Rule}\\ &= \log_6 2^6+\log_6 x^3+\log_6(4x+1)-\log_6(2x-1)\\ &\qquad \text{Simplify by writing 64 as } 2^6\\ &= 6\log_6 2+3\log_6 x+\log_6(4x+1)-\log_6(2x-1)\\ &\qquad \text{Apply the Power Rule}\end{aligned}$$

4.5.7

Expand $\ln\left(\frac{\sqrt{(x-1)(2x+1)^2}}{(x^2-9)}\right)$.

Answer

$\frac{1}{2}\ln(x-1)+\ln(2x+1)-\ln(x+3)-\ln(x-3)$

Good job! You completed an exercise that has relatively little meaning as a useful exercise. Tell your professor to give you extra credit!

Condensing Logarithmic Expressions

We can use the rules of logarithms we just learned to condense sums, differences, and products with the same base as a single logarithm. It is important to remember that the logarithms must have the same base to be combined. We will learn later how to change the base of any logarithm before condensing.

How To: Given a sum, difference, or product of logarithms with the same base, write an equivalent expression as a single logarithm

1. Apply the power property first. Identify terms that are products of factors and a logarithm, and rewrite each as the logarithm of a power.
2. Next apply the product property. Rewrite sums of logarithms as the logarithm of a product.
3. Apply the quotient property last. Rewrite differences of logarithms as the logarithm of a quotient.

Example 4.5.8: Using the Product and Quotient Rules to Combine Logarithms

Write $\log_3(5)+\log_3(8)-\log_3(2)$ as a single logarithm.

Solution

Using the product and quotient rules

$$\log_3(5)+\log_3(8)=\log_3(5\cdot 8)=\log_3(40)$$

This reduces our original expression to

$$\log_3(40)-\log_3(2)$$

Then, using the quotient rule

$$\log_3(40)-\log_3(2)=\log_3\left(\frac{40}{2}\right)=\log_3(20)$$

4.5.8

Condense $\log 3-\log 4+\log 5-\log 6$.

Answer

$\log\left(\frac{3\cdot 5}{4\cdot 6}\right)$; can also be written $\log\left(\frac{5}{8}\right)$ by reducing the fraction to lowest terms.

Example 4.5.9: Condensing Complicated Logarithmic Expressions

Condense $\log_2(x^2)+\frac{1}{2}\log_2(x-1)-3\log_2((x+3)^2)$.

Solution

We apply the power rule first:

$$\log_2(x^2)+\frac{1}{2}\log_2(x-1)-3\log_2((x+3)^2)=\log_2(x^2)+\log_2(\sqrt{x-1})-\log_2((x+3)^6)$$

Next we apply the product rule to the sum:

$$\log_2(x^2)+\log_2(\sqrt{x-1})-\log_2((x+3)^6)=\log_2(x^2\sqrt{x-1})-\log_2((x+3)^6)$$

Finally, we apply the quotient rule to the difference:

$$\log_2(x^2\sqrt{x-1})-\log_2((x+3)^6)=\log_2\left(\frac{x^2\sqrt{x-1}}{(x+3)^6}\right)$$

4.5.9

Condense $4(3\log(x)+\log(x+5)-\log(2x+3))$.

Answer

$\log\left(\frac{x^{12}(x+5)^4}{(2x+3)^4}\right)$; this answer could also be written $\log\left(\frac{x^3(x+5)}{(2x+3)}\right)^4$

Example 4.5.10: Applying of the Laws of Logs

Recall that, in chemistry, $pH=-\log[H+]$. If the concentration of hydrogen ions in a liquid is doubled, what is the effect on pH?

Solution

Suppose C is the original concentration of hydrogen ions, and P is the original pH of the liquid. Then $P=-\log(C)$. If the concentration is doubled, the new concentration is $2C$. Then the pH of the new liquid is

$$pH=-\log(2C)$$

Using the product rule of logs

$$pH=-\log(2C)=-(\log(2)+\log(C))=-\log(2)-\log(C)$$

Since $P=-\log(C)$,the new pH is

$$pH=P-\log(2)\approx P-0.301$$

4.5.10

When the concentration of hydrogen ions is doubled, the pH decreases by about 0.301.

How does the pH change when the concentration of positive hydrogen ions is decreased by half?

Answer

The pH increases by about 0.301.

Using the Change-of-Base Formula for Logarithms

Most calculators can evaluate only common and natural logs. In order to evaluate logarithms with a base other than 10 or e, we use the **change-of-base formula**, which allows us to rewrite a logarithm base b as the quotient of logarithms of any other base. When

using a calculator, we can change any logarithm to common or natural logs.

To derive the change-of-base formula, we use the **one-to-one** property and **power rule for logarithms**.

Theorem 4.5.3 THE CHANGE-OF-BASE FORMULA

Given any positive real numbers M, b, and n, where $n \neq 1$ and $b \neq 1$,

$$\log_b M = \frac{\log_b M}{\log_n b} \tag{4.5.11}$$

Proof

Let $y = \log_b M$. By taking the log base n of both sides of the equation, we arrive at an exponential form, namely $b^y = M$. It follows that

$$\begin{aligned} \log_n(b^y) &= \log_n M && \text{Apply the one-to-one property} \\ y\log_n b &= \log_n M && \text{Apply the power rule for logarithms} \\ y &= \frac{\log_n M}{\log_n b} && \text{Isolate y} \\ \log_b M &= \frac{\log_n M}{\log_n b} && \text{Substitute for y} \end{aligned}$$

The change-of-base formula can be used to calculate a logarithm with any base. For example, to evaluate $\log_5 36$ using a calculator, we must first rewrite the expression as a quotient of common or natural logs. We will use the common log.

$$\begin{aligned} \log_5 36 &= \frac{\log(36)}{\log(5)} && \text{Apply the change of base formula using base 10} \\ &\approx 2.2266 && \text{Use a calculator to evaluate to 4 decimal places} \end{aligned}$$

How To: Given a logarithm with the form $\log_b M$, use the change-of-base formula to rewrite it as a quotient of logs with any positive base n, where $n \neq 1$

1. Choose the new base n, remembering that the common log, $\log(x)$, has base 10, and the natural log, $\ln(x)$,has base e.
2. Rewrite the log as a quotient using the change-of-base formula:
   - The numerator of the quotient will be a logarithm with base n and argument M.
   - The denominator of the quotient will be a logarithm with base n and argument b.

Example 4.5.11: Changing Logarithmic Expressions to Expressions Involving Only Natural Logs

Change $\log_5 3$ to a quotient of natural logarithms and evaluate to 4 decimal places.

Solution

Because we will be expressing $\log_5 3$ as a quotient of natural logarithms, the new base, $n = e$.

We rewrite the log as a quotient using the change-of-base formula. The numerator of the quotient will be the natural log with argument 3. The denominator of the quotient will be the natural log with argument 5.

$$\log_b M = \frac{\ln M}{\ln b}$$

$$\log_5 3 = \frac{\ln 3}{\ln 5} \approx 0.6826$$

Try It 4.5.11

Change $\log_{0.5} 8$ to a quotient of natural logarithms and evaluate.

Answer

$$\frac{\ln 8}{\ln 0.5} = -3.0000$$

: Can we change common logarithms to natural logarithms?

Yes. Remember that $\log 9$ *means* $\log_{10} 9$. *So,* $\log 9 = \dfrac{\ln 9}{\ln 10}$.

4.5.12

Evaluate $\log_5(100)$ using the change-of-base formula.

Answer

$\dfrac{\ln 100}{\ln 5} \approx 2.861$

Media

Access these online resources for additional instruction and practice with laws of logarithms.

- The Properties of Logarithms
- Expand Logarithmic Expressions
- Evaluate a Natural Logarithmic Expression

Key Equations

The Product Rule for Logarithms	$\log_b(MN) = \log_b(M) + \log_b(N)$
The Quotient Rule for Logarithms	$\log_b(\frac{M}{N}) = \log_b M - \log_b N$
The Power Rule for Logarithms	$\log_b(M^n) = n\log_b M$
The Change-of-Base Formula	$\log_b M = \dfrac{\log_n M}{\log_n b}$ $n > 0, n \neq 1, b > 0, b \neq 1$

Key Concepts

- We can use the product rule of logarithms to rewrite the log of a product as a sum of logarithms. See Example **4.5.1**.
- We can use the quotient rule of logarithms to rewrite the log of a quotient as a difference of logarithms. See Example **4.5.2**.
- We can use the power rule for logarithms to rewrite the log of a power as the product of the exponent and the log of its base. See Example **4.5.3**.
- We can use the product rule, the quotient rule, and the power rule together to expand a logarithm with a complex input. See Example **4.5.5,** Example **4.5.6,** and Example **4.5.7**.
- The rules of logarithms can also be used to condense sums, differences, and products with the same base as a single logarithm. See Example **4.5.8** and Example **4.5.9**.
- We can convert a logarithm with any base to a quotient of logarithms with any other base using the change-of-base formula. See Example **4.5.11**.

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4.5E: Exercises

4.5: Logarithmic Properties

Verbal

1) How does the power rule for logarithms help when solving logarithms with the form $\log_b(\sqrt[n]{x})$?

Answer

Any root expression can be rewritten as an expression with a rational exponent so that the power rule can be applied, making the logarithm easier to calculate. Thus, $\log_b\left(x^{\frac{1}{n}}\right)=\frac{1}{n}\log_b(x)$.

2) What does the change-of-base formula do? Why is it useful when using a calculator?

Algebraic

For the following exercises, expand each logarithm as much as possible. Rewrite each expression as a sum, difference, or product of logs.

3) $\log_b(7x\cdot 2y)$

Answer

$\log_b(2)+\log_b(7)+\log_b(x)+\log_b(y)$

4) $\ln(3ab\cdot 5c)$

5) $\log_b\left(\frac{13}{17}\right)$

Answer

$\log_b(13)-\log_b(17)$

6) $\log_4\left(\frac{\frac{x}{z}}{w}\right)$

7) $\ln\left(\frac{1}{4^k}\right)$

Answer

$-k\ln(4)$

8) $\log_2(y^x)$

For the following exercises, condense to a single logarithm if possible.

9) $\ln(7)+\ln(x)+\ln(y)$

Answer

$\ln(7xy)$

10) $\log_3(2)+\log_3(a)+\log_3(11)+\log_3(b)$

11) $\log_b(28)-\log_b(7)$

Answer

$\log_b(4)$

12) $\ln(a)-\ln(d)-\ln(c)$

13) $-\log_b\left(\frac{1}{7}\right)$

Answer

$\log_b(7)$

14) $\frac{1}{3}\ln(8)$

For the following exercises, use the properties of logarithms to expand each logarithm as much as possible. Rewrite each expression as a sum, difference, or product of logs.

15) $\log\left(\frac{x^{15}y^{13}}{z^{19}}\right)$

Answer

$15\log(x)+13\log(y)-19\log(z)$

16) $\ln\left(\frac{a^{-2}}{b^{-4}c^5}\right)$

17) $\log\left(\sqrt{x^3y^{-4}}\right)$

Answer

$\frac{3}{2}\log(x)-2\log(y)$

18) $\ln\left(y\sqrt{\frac{y}{1-y}}\right)$

19) $\log\left(x^2y^3\sqrt[3]{x^2y^5}\right)$

Answer

$\frac{8}{3}\log(x)+\frac{14}{3}\log(y)$

For the following exercises, condense each expression to a single logarithm using the properties of logarithms.

20) $\log\left(2x^4\right)+\log\left(3x^5\right)$

21) $\ln\left(6x^9\right)-\ln\left(3x^2\right)$

Answer

$\ln\left(2x^7\right)$

22) $2\log(x)+3\log(x+1)$

23) $\log(x)-\frac{1}{2}\log(y)+3\log(z)$

Answer

$\log\left(\frac{xz^3}{\sqrt{y}}\right)$

24) $4\log_7(c)+\frac{\log_7(a)}{3}+\frac{\log_7(b)}{3}$

For the following exercises, rewrite each expression as an equivalent ratio of logs using the indicated base.

25) $\log_7(15)$ to base e

Answer

$\log_7(15)=\frac{\ln(15)}{\ln(7)}$

26) $\log_{14}(55.875)$ to base 10

For the following exercises, suppose $\log_5(6) = a$ and $\log_5(11) = b$. Use the change-of-base formula along with properties of logarithms to rewrite each expression in terms of a and b. Show the steps for solving.

27) $\log_{11}(5)$

Answer

$$\log_{11}(5) = \frac{\log_5(5)}{\log_5(11)} = \frac{1}{b}$$

28) $\log_6(55)$

29) $\log_{11}\left(\frac{6}{11}\right)$

Answer

$$\log_{11}\left(\frac{6}{11}\right) = \frac{\log_{11}\left(\frac{6}{11}\right)}{\log_5(11)} = \frac{\log_5(6) - \log_5(11)}{\log_5(11)} = \frac{a-b}{b} = \frac{a}{b} - 1$$

Numeric

For the following exercises, use properties of logarithms to evaluate without using a calculator.

30) $\log_3\left(\frac{1}{9}\right) - 3\log_3(3)$

31) $6\log_8(2) + \frac{\log_8(64)}{3\log_8(4)}$

Answer

3

32) $2\log_9(3) - 4\log_9(3) + \log_9\left(\frac{1}{729}\right)$

For the following exercises, use the change-of-base formula to evaluate each expression as a quotient of natural logs. Use a calculator to approximate each to five decimal places.

33) $\log_3(22)$

Answer

2.81359

34) $\log_8(65)$

35) $\log_6(5.38)$

Answer

0.93913

36) $\log_4\left(\frac{15}{2}\right)$

37) $\log_{\frac{1}{2}}(4.7)$

Answer

−2.23266

Extensions

38) Use the product rule for logarithms to find all x values such that $\log_{12}(2x+6) + \log_{12}(x+2) = 2$. Show the steps for solving.

39) Use the quotient rule for logarithms to find all x values such that $\log_6(x+2)-\log_6(x-3)=1$. Show the steps for solving.

Answer

Rewriting as an exponential equation and solving for x:

$$
\begin{aligned}
6^1 &= \frac{x+2}{x-3} \\
0 &= \frac{x+2}{x-3} - 6 \\
0 &= \frac{x+2}{x-3} - \frac{6(x-3)}{(x-3)} \\
0 &= \frac{x+2-6x+18}{x-3} \\
0 &= \frac{x-4}{x-3} \\
x &= 4
\end{aligned}
$$

Checking, we find that $\log_6(4+2)-\log_6(4-3)=\log_6(6)-\log_6(1)$ is defined, so $x=4$

40) Can the power property of logarithms be derived from the power property of exponents using the equation $b^x=m$? If not, explain why. If so, show the derivation.

41) Prove that $\log_b(n)=\frac{1}{\log_n(n)}$ for any positive integers $b>1$ and $n>1$.

Answer

Let b and n be positive integers greater than 1. Then, by the change-of-base formula, $\log_b(n)=\frac{\log_n(n)}{\log_n(b)}=\frac{1}{\log_n(b)}$

42) Does $\log_{81}(2401)=\log_3(7)$? Verify the claim algebraically.

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4.6: Exponential and Logarithmic Equations

Skills to Develop

- Use like bases to solve exponential equations.
- Use logarithms to solve exponential equations.
- Use the definition of a logarithm to solve logarithmic equations.
- Use the one-to-one property of logarithms to solve logarithmic equations.
- Solve applied problems involving exponential and logarithmic equations.

In 1859, an Australian landowner named Thomas Austin released 24 rabbits into the wild for hunting. Because Australia had few predators and ample food, the rabbit population exploded. In fewer than ten years, the rabbit population numbered in the millions.



Figure* 4.6.1*: *Wild rabbits in Australia. The rabbit population grew so quickly in Australia that the event became known as the "rabbit plague." (credit: Richard Taylor, Flickr)*

Uncontrolled population growth, as in the wild rabbits in Australia, can be modeled with exponential functions. Equations resulting from those exponential functions can be solved to analyze and make predictions about exponential growth. In this section, we will learn techniques for solving exponential functions.

Using Like Bases to Solve Exponential Equations

The first technique involves two functions with like bases. Recall that the one-to-one property of exponential functions tells us that, for any real numbers b, S, and T, where $b > 0$, $b \neq 1$, if $b^S = b^T$ then $S = T$.

In other words, when an **exponential equation** has the same base on each side, the exponents must be equal. This also applies when the exponents are algebraic expressions. Therefore, we can solve many exponential equations by using the rules of exponents to rewrite each side as a power with the same base. Then, we use the fact that exponential functions are one-to-one to set the exponents equal to one another, and solve for the unknown.

For example, consider the equation $3^{4x-7} = \dfrac{3^{2x}}{3}$. To solve for x, we use the division property of exponents to rewrite the right side so that both sides have the common base, 3. Then we apply the one-to-one property of exponents by setting the exponents equal to one another and solving for x:

$$
\begin{aligned}
3^{4x-7} &= \frac{3^{2x}}{3} \\
3^{4x-7} &= \frac{3^{2x}}{3^1} && \text{Rewrite 3 as } 3^1 \\
3^{4x-7} &= 3^{2x-1} && \text{Use the division property of exponents} \\
4x-7 &= 2x-1 && \text{Apply the one-to-one property of exponents} \\
2x &= 6 && \text{Subtract 2x and add 7 to both sides} \\
x &= 3 && \text{Divide by 2}
\end{aligned}
$$

USING THE ONE-TO-ONE PROPERTY OF EXPONENTIAL FUNCTIONS TO SOLVE EXPONENTIAL EQUATIONS

For any algebraic expressions S and T, and any positive real number $b \neq 1$,

$$\text{if } b^S = b^T \text{ then } S = T \tag{4.6.1}$$

How To: Given an exponential equation with the form $b^S = b^T$, where S and T are algebraic expressions with an unknown, solve for the unknown.

1. Use the rules of exponents to simplify, if necessary, so that the resulting equation has the form $b^S = b^T$.
2. Use the one-to-one property to set the exponents equal.
3. Solve the resulting equation, $S = T$, for the unknown.

Example 4.6.1: Solving an Exponential Equation with a Common Base

Solve $2^{x-1} = 2^{2x-4}$.

Solution

$$\begin{aligned} 2^{x-1} &= 2^{2x-4} && \text{The common base is 2} \\ x-1 &= 2x-4 && \text{By the one-to-one property the exponents must be equal} \\ x &= 3 && \text{Solve for x} \end{aligned}$$

Try It 4.6.1

Solve $5^{2x} = 5^{3x+2}$.

Answer

$x = -2$

Rewriting Equations So All Powers Have the Same Base

Sometimes the *common base* for an exponential equation is not explicitly shown. In these cases, we simply rewrite the terms in the equation as powers with a common base, and solve using the one-to-one property.

For example, consider the equation $256 = 4^{x-5}$. We can rewrite both sides of this equation as a power of 2. Then we apply the rules of exponents, along with the one-to-one property, to solve for x:

$$\begin{aligned} 256 &= 4^{x-5} \\ 2^8 &= (2^2)^{x-5} && \text{The common base is 2} \\ 2^8 &= 2^{2x-10} && \text{Rewrite each side as a power with base 2} \\ 8 &= 2x-10 && \text{Use the one-to-one property of exponents} \\ 18 &= 2x && \text{Add 10 to both sides} \\ x &= 9 && \text{Divide by 2} \end{aligned}$$

Example 4.6.2: Solving Equations by Rewriting Them to Have a Common Base

Solve $8^{x+2} = 16^{x+1}$.

Solution

$$\begin{aligned} 8^{x+2} &= 16^{x+1} \\ (2^3)^{x+2} &= (2^4)^{x+1} && \text{Write 8 and 16 as powers of 2} \\ 2^{3x+6} &= 2^{4x+4} && \text{To take a power of a power, multiply exponents} \\ 3x+6 &= 4x+4 && \text{Use the one-to-one property to set the exponents equal} \\ x &= 2 && \text{Solve for x} \end{aligned}$$

Try It 4.6.2

Solve $5^{2x} = 25^{3x+2}$.

Answer

$x = -1$

Example 4.6.3: Solving Equations by Rewriting Roots with Fractional Exponents to Have a Common Base

Solve $2^{5x} = \sqrt{2}$.

Solution

$$\begin{aligned} 2^{5x} &= 2^{1/2} && \text{Write the square root of 2 as a power of 2} \\ 5x &= \frac{1}{2} && \text{Use the one-to-one property} \\ x &= \frac{1}{10} && \text{Solve for x} \end{aligned}$$

Try It 4.6.3

Solve $5^x = \sqrt{5}$.

Answer

$x = \frac{1}{2}$

Q&A: Do all exponential equations have a solution? If not, how can we tell if there is a solution during the problem-solving process?

No. Recall that the range of an exponential function is always positive. While solving the equation, we may obtain an expression that is undefined.

Example 4.6.4: Solving an Equation with Positive and Negative Powers

Solve $3^{x+1} = -2$.

Solution

This equation has no solution. There is no real value of x that will make the equation a true statement because any power of a positive number is positive.

Analysis

Figure 4.6.2 shows that the two graphs do not cross so the left side is never equal to the right side. Thus the equation has no solution.

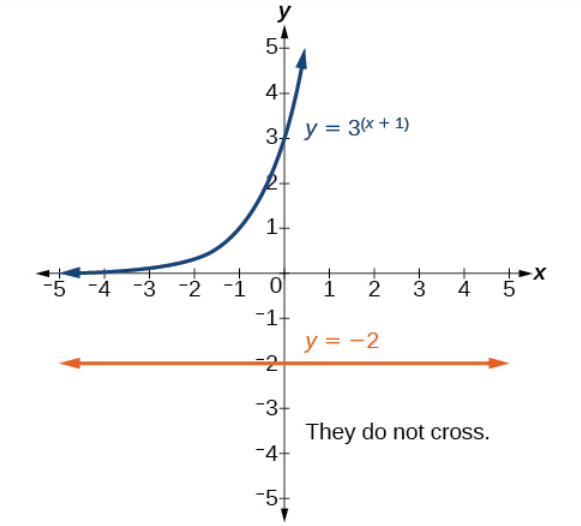


Figure 4.6.2

Solving Exponential Equations Using Logarithms

Sometimes the terms of an exponential equation cannot be rewritten with a common base. In these cases, we solve by taking the logarithm of each side.

How To: Given an exponential equation in which a common base cannot be found, solve for the unknown

1. Apply a logarithm function to both sides of the equation.
   - If one of the terms in the equation has base 10, use the common logarithm.
   - If none of the terms in the equation has base 10, use the natural logarithm.
2. Use the rules of logarithms to solve for the unknown.

Example 4.6.5: Solving an Equation Containing Powers of Different Bases

Solve $5^{x+2} = 4^x$.

Solution

$$\begin{aligned}
5^{x+2} &= 4^x && \text{There is no easy way to get the powers to have the same base}\\
\ln 5^{x+2} &= \ln 4^x && \text{Take ln of both sides}\\
(x+2)\ln 5 &= x\ln 4 && \text{Use laws of logs}\\
x\ln 5 + 2\ln 5 &= x\ln 4 && \text{Use the distributive law}\\
x\ln 5 - x\ln 4 &= -2\ln 5 && \text{Get terms containing x on one side, terms without x on the other}\\
x(\ln 5 - \ln 4) &= -2\ln 5 && \text{On the left hand side, factor out an x}\\
x\ln\left(\frac{5}{4}\right) &= \ln\left(\frac{1}{25}\right) && \text{Use the laws of logs}\\
x &= \frac{\ln\left(\frac{1}{25}\right)}{\ln\left(\frac{5}{4}\right)} && \text{Divide by the coefficient of x}
\end{aligned}$$

Analysis

If we want a decimal approximation of this value, we can back up a little, and calculate $\frac{-2\ln 5}{\ln\left(\frac{5}{4}\right)} \approx -14.425$.

Try It 4.6.4

Solve $2^x = 3^{x+1}$.

Answer

$$x = \frac{\ln 3}{\ln\left(\frac{2}{3}\right)}$$

Q&A: Is there any way to solve $2^x = 3^x$?

Yes. The solution is $x = 0$. Look at the graphs of $y = 2^x$ and $y = 3^x$ to see that they intersect only at $x = 0$.

Equations Containing e

One common type of exponential equation uses the base e. This constant occurs again and again in nature, in mathematics, in science, in engineering, and in finance. When we have an equation with a base e on either side, we can use the natural logarithm to solve it.

How To: Given an equation of the form $y = Ae^{kt}$, solve for t.

1. Divide both sides of the equation by A.

2. Apply the natural logarithm of both sides of the equation.
3. Divide both sides of the equation by k.

Example 4.6.6: Solve an Equation of the Form $y = Ae^{kt}$

Solve $100 = 20e^{2t}$.

Solution

$$\begin{aligned} 100 &= 20e^{2t} \\ 5 &= e^{2t} && \text{Divide by the coefficient of } e^{2t} \\ \ln 5 &= 2t && \text{Take ln of both sides. Use the fact that } ln(x) \text{ and } e^x \text{ are inverse functions} \\ t &= \frac{\ln 5}{2} && \text{Divide by the coefficient of t} \end{aligned}$$

Analysis

Using laws of logs, we can also write this answer in the form $t = \ln \sqrt{5}$. If we want a decimal approximation of the answer, we can use a calculator to find $\frac{\ln 5}{2} \approx 0.805$.

4.6.5

Solve $3e^{0.5t} = 11$.

Answer

$t = 2\ln\left(\frac{11}{3}\right)$ or $\ln\left(\frac{11}{3}\right)^2$

: Does every equation of the form $y = Ae^{kt}$ have a solution?

No. There will be a solution when $k \neq 0$, or when y and A are either both 0 or neither 0, and they have the same sign. An example of an equation that has no solution is $2 = -3e^t$.

Example 4.6.7: Solving an Equation That Can Be Simplified to the Form $y = Ae^{kt}$

Solve $4e^{2x} + 5 = 12$.

Solution

$$\begin{aligned} 4e^{2x} + 5 &= 12 \\ 4e^{2x} &= 7 && \text{Combine like terms} \\ e^{2x} &= \frac{7}{4} && \text{Divide by the coefficient of } e^{2x} \\ 2x &= \ln\left(\frac{7}{4}\right) && \text{Take ln of both sides} \\ x &= \frac{1}{2}\ln\left(\frac{7}{4}\right) \approx 0.280 && \text{Solve for x} \end{aligned}$$

4.6.6

Solve $3 + e^{2t} = 7e^{2t}$.

Answer

$t = \ln\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}\ln(2) \approx -0.347$

Extraneous Solutions

Sometimes the methods used to solve an equation introduce an **extraneous solution**, which is a solution that is correct algebraically but does not satisfy the conditions of the original equation. One such situation may arise when a logarithm is taken on

both sides of the equation. In such cases, remember that the argument of any logarithm must be positive. If the number we are evaluating in a logarithm function is negative, there is no output.

Example 4.6.8: Solving Exponential Functions in Quadratic Form

Solve $e^{2x} - e^x = 56$.

Solution:

$$\begin{aligned} e^{2x} - e^x &= 56 \\ e^{2x} - e^x - 56 &= 0 && \text{Get one side of the equation equal to zero} \\ (e^x + 7)(e^x - 8) &= 0 && \text{Factor by the reverse FOIL method} \\ e^x + 7 = 0 \text{ or } e^x - 8 &= 0 && \text{If a product is zero, then one factor must be zero} \\ e^x = -7 \text{ or } e^x &= 8 && \text{Isolate the exponentials} \\ e^x &= 8 && \text{Reject the equation in which the power equals a negative number} \\ x &= \ln 8 && \text{Solve the equation in which the power equals a positive number} \end{aligned}$$

Analysis

When we plan to use factoring to solve a problem, we always get zero on one side of the equation, because zero has the unique property that when a product is zero, one or both of the factors must be zero. We reject the equation $e^x = -7$ because a positive number never equals a negative number. The solution $\ln(-7)$ is not a real number, and in the real number system this solution is rejected as an extraneous solution.

4.6.7

Solve $e^{2x} = e^x + 2$.

Answer

$x = \ln 2 \approx 0.693$

: Does every logarithmic equation have a solution?

No. Keep in mind that we can only apply the logarithm to a positive number. Always check for extraneous solutions.

Using the Definition of a Logarithm to Solve Logarithmic Equations

We have already seen that every logarithmic equation $\log_b(x) = y$ is equivalent to the exponential equation $b^y = x$. We can use this fact, along with the rules of logarithms, to solve logarithmic equations where the argument is an algebraic expression.

For example, consider the equation $\log_2(2) + \log_2(3x - 5) = 3$. To solve this equation, we can use rules of logarithms to rewrite the left side in compact form and then apply the definition of logs to solve for x:

$$\begin{aligned} \log_2(2) + \log_2(3x - 5) &= 3 \\ \log_2(2(3x - 5)) &= 3 && \text{Apply the product rule of logarithms} \\ \log_2(6x - 10) &= 3 && \text{Distribute} \\ 2^3 &= 6x - 10 && \text{Apply the definition of a logarithm} \\ 8 &= 6x - 10 && \text{Calculate } 2^3 \\ 18 &= 6x && \text{Add 10 to both sides} \\ x &= 3 && \text{Divide by 6} \end{aligned}$$

USING THE DEFINITION OF A LOGARITHM TO SOLVE LOGARITHMIC EQUATIONS

For any algebraic expression S and real numbers b and c,where $b > 0$, $b \neq 1$,

$$\log_b(S) = c \text{ if and only if } b^c = S \tag{4.6.2}$$

Example 4.6.9: Using Algebra to Solve a Logarithmic Equation

Solve $2\ln x + 3 = 7$.

Solution

$$
\begin{aligned}
2\ln x + 3 &= 7 \\
2\ln x &= 4 && \text{Subtract 3} \\
\ln x &= 2 && \text{Divide by 2} \\
x &= e^2 \approx 7.389 && \text{Rewrite in exponential form}
\end{aligned}
$$

4.6.8

Solve $6 + \ln x = 10$.

Answer

$x = e^4 \approx 54.598$

Example 4.6.10: Using Algebra Before and After Using the Definition of the Natural Logarithm

Solve $2\ln(6x) = 7$.

Solution:

$$
\begin{aligned}
2\ln(6x) &= 7 \\
\ln(6x) &= \frac{7}{2} && \text{Divide by 2} \\
6x &= e^{7/2} && \text{Use the definition of ln} \\
x &= \frac{1}{6}e^{7/2} \approx 5.519 && \text{Divide by 6}
\end{aligned}
$$

4.6.9

Solve $2\ln(x+1) = 10$.

Answer

$x = e^5 - 1$

Example 4.6.11: Using a Graph to Understand the Solution to a Logarithmic Equation

Solve $\ln x = 3$.

Solution

$$
\begin{aligned}
\ln x &= 3 \\
x &= e^3 && \text{Use the definition of the natural logarithm}
\end{aligned}
$$

Figure 4.6.3 represents the graph of the equation. On the graph, the x-coordinate of the point at which the two graphs intersect is close to 20. In other words $e^3 \approx 20$. A calculator gives a better approximation: $e^3 \approx 20.0855$.

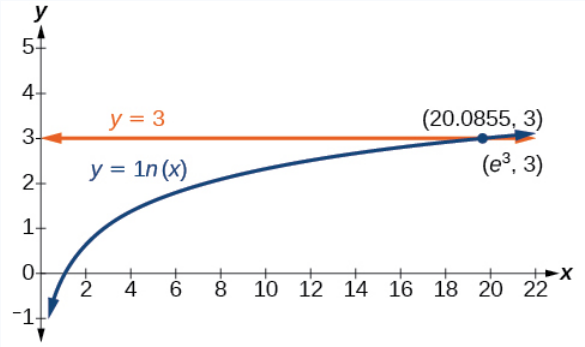


Figure 4.6.3: *The graphs of $y = \ln x$ and $y = 3$ cross at the point $(e^3, 3)$, which is approximately $(20.0855, 3)$.*

Using the One-to-One Property of Logarithms to Solve Logarithmic Equations

As with exponential equations, we can use the one-to-one property to solve logarithmic equations. The one-to-one property of logarithmic functions tells us that, for any real numbers $x > 0$, $S > 0$, $T > 0$ and any positive real number b, where $b \neq 1$,

$$\text{If } \log_b S = \log_b T \text{ then } S = T.$$

For example,

$$\text{If } \log_2(x-1) = \log_2(8), \text{ then } x - 1 = 8.$$

So, if $x - 1 = 8$, then we can solve for x, and we get $x = 9$. To check, we can substitute $x = 9$ into the original equation: $\log_2(9-1) = \log_2(8) = 3$. In other words, when a logarithmic equation has the same base on each side, the arguments must be equal. This also applies when the arguments are algebraic expressions. Therefore, when given an equation with logs of the same base on each side, we can use rules of logarithms to rewrite each side as a single logarithm. Then we use the fact that logarithmic functions are one-to-one to set the arguments equal to one another and solve for the unknown.

For example, consider the equation $\log(3x-2) - \log(2) = \log(x+4)$. To solve this equation, we can use the rules of logarithms to rewrite the left side as a single logarithm, and then apply the one-to-one property to solve for x:

$$\begin{aligned} \log(3x-2) - \log(2) &= \log(x+4) \\ \log\left(\frac{3x-2}{2}\right) &= \log(x+4) && \text{Apply the quotient rule of logarithms} \\ \frac{3x-2}{2} &= x+4 && \text{Apply the one to one property of a logarithm} \\ 3x-2 &= 2x+8 && \text{Multiply both sides of the equation by 2} \\ x &= 10 && \text{Subtract 2x and add 2} \end{aligned}$$

To check the result, substitute $x = 10$ into $\log(3x-2) - \log(2) = \log(x+4)$.

$$\begin{aligned} \log(3(10)-2) - \log(2) &= \log((10)+4) \\ \log(28) - \log(2) &= \log(14) \\ \log\left(\frac{28}{2}\right) &= \log(14) && \text{The solution checks} \end{aligned}$$

USING THE ONE-TO-ONE PROPERTY OF LOGARITHMS TO SOLVE LOGARITHMIC EQUATIONS

For any algebraic expressions S and T and any positive real number b, where $b \neq 1$,

$$\text{If } \log_b S = \log_b T \text{ then } S = T \tag{4.6.3}$$

Note: when solving an equation involving logarithms, always check to see if the answer is correct or if it is an extraneous solution.

How To: Given an equation containing logarithms, solve it using the one-to-one property

1. Use the rules of logarithms to combine like terms, if necessary, so that the resulting equation has the form $\log_b S = \log_b T$.
2. Use the one-to-one property to set the arguments equal.
3. Solve the resulting equation, $S = T$, for the unknown.

Example 4.6.12: Solving an Equation Using the One-to-One Property of Logarithms

Solve $\ln(x^2) = \ln(2x+3)$.

Solution

$$\ln(x^2) = \ln(2x+3)$$
$$x^2 = 2x+3 \quad \text{Use the one-to-one property of the logarithm}$$
$$x^2 - 2x - 3 = 0 \quad \text{Get zero on one side before factoring}$$
$$(x-3)(x+1) = 0 \quad \text{Factor using FOIL}$$
$$x-3=0 \quad \text{or } x+1=0 \quad \text{If a product is zero, one of the factors must be zero}$$
$$x = 3 \quad \text{or}$$
$$x = -1 \quad \text{Solve for x}$$

Analysis

There are two solutions: 3 or -1. The solution -1 is negative, but it checks when substituted into the original equation because the argument of the logarithm functions is still positive.

4.6.10

Solve $\ln(x^2) = \ln 1$.

Answer

$x = 1$ or $x = -1$

Solving Applied Problems Using Exponential and Logarithmic Equations

In previous sections, we learned the properties and rules for both exponential and logarithmic functions. We have seen that any exponential function can be written as a logarithmic function and vice versa. We have used exponents to solve logarithmic equations and logarithms to solve exponential equations. We are now ready to combine our skills to solve equations that model real-world situations, whether the unknown is in an exponent or in the argument of a logarithm.

One such application is in science, calculating the time it takes for half of the unstable material in a sample of a radioactive substance to decay, called its **half-life**. Table 4.6.1 lists the half-life for several of the more common radioactive substances.

***Table* 4.6.1**

Substance	*Use*	*Half-life*
gallium-67	*nuclear medicine*	*80 hours*
cobalt-60	*manufacturing*	*5.3 years*
technetium-99m	*nuclear medicine*	*6 hours*
americium-241	*construction*	*432 years*
carbon-14	*archeological dating*	*5,715 years*
uranium-235	*atomic power*	*703,800,000 or* 7.038×10^8 *years*

We can see how widely the half-lives for these substances vary. Knowing the half-life of a substance allows us to calculate the amount remaining after a specified time. We can use the formula for radioactive decay:

$$A(t) = A_0 e^{\frac{\ln(0.5)t}{T}} \tag{4.6.4}$$
$$A(t) = A_0 (e^{\ln(0.5)})^{\frac{t}{T}} \tag{4.6.5}$$
$$A(t) = A_0 \left(\frac{1}{2}\right)^{\frac{t}{T}} \tag{4.6.6}$$

where

- A_0 is the amount initially present
- T is the half-life of the substance
- t is the time period over which the substance is studied

- y is the amount of the substance present after time t

Example 4.6.13: Using the Formula for Radioactive Decay to Find the Quantity of a Substance

How long will it take for ten percent of a 1000-gram sample of uranium-235 to decay?

Solution

$$
\begin{aligned}
y &= 1000e^{\frac{\ln(0.5)}{703,800,000}t} \\
900 &= 1000e^{\frac{\ln(0.5)}{703,800,000}t} && \text{After 10\% decays, 900 grams are left} \\
0.9 &= e^{\frac{\ln(0.5)}{703,800,000}t} && \text{Divide by 1000} \\
\ln(0.9) &= \ln\left(e^{\frac{\ln(0.5)}{703,800,000}t}\right) && \text{Take ln of both sides} \\
\ln(0.9) &= \frac{\ln(0.5)}{703,800,000}t && \ln(e^M) = M \\
t &= 703,800,000 \times \frac{\ln(0.9)}{\ln(0.5)} && \text{Solve for t} \\
t &\approx 106,979,777 && \text{years}
\end{aligned}
$$

Analysis

Ten percent of 1000 grams is 100 grams. If 100 grams decay, the amount of uranium-235 remaining is 900 grams.

4.6.11

How long will it take before twenty percent of our 1000-gram sample of uranium-235 has decayed?

Answer

$t = 703,800,000 \times \frac{\ln(0.8)}{\ln(0.5)}$ years ≈ 226,572,993 years.

Media

Access these online resources for additional instruction and practice with exponential and logarithmic equations.

- Solving Logarithmic Equations
- Solving Exponential Equations with Logarithms

Key Equations

One-to-one property for exponential functions	For any algebraic expressions S and T and any positive real number b, where if $b^S = b^T$ then $S = T$.
Definition of a logarithm	For any algebraic expression S and positive real numbers b and c, where $b \neq 1$, $\log_b(S) = c$ if and only if $b^c = S$.
One-to-one property for logarithmic functions	For any algebraic expressions S and T and any positive real number b, where $b \neq 1$, if $\log_b S = \log_b T$ then $S = T$.

Key Concepts

- We can solve many exponential equations by using the rules of exponents to rewrite each side as a power with the same base. Then we use the fact that exponential functions are one-to-one to set the exponents equal to one another and solve for the unknown.
- When we are given an exponential equation where the bases are explicitly shown as being equal, set the exponents equal to one another and solve for the unknown. See Example 4.6.1.

- When we are given an exponential equation where the bases are *not* explicitly shown as being equal, rewrite each side of the equation as powers of the same base, then set the exponents equal to one another and solve for the unknown. See Example **4.6.2**, Example **4.6.3**, and Example **4.6.4**.
- When an exponential equation cannot be rewritten with a common base, solve by taking the logarithm of each side. See Example **4.6.5**.
- We can solve exponential equations with base e,by applying the natural logarithm of both sides because exponential and logarithmic functions are inverses of each other. See Example **4.6.6** and Example **4.6.7**.
- After solving an exponential equation, check each solution in the original equation to find and eliminate any extraneous solutions. See Example **4.6.8**.
- When given an equation of the form $\log_b(S) = c$, where S is an algebraic expression, we can use the definition of a logarithm to rewrite the equation as the equivalent exponential equation $b^c = S$, and solve for the unknown. See Example **4.6.9** and Example **4.6.10**.
- We can also use graphing to solve equations with the form $\log_b(S) = c$. We graph both equations $y = \log_b(S)$ and $y = c$ on the same coordinate plane and identify the solution as the *x*-value of the intersecting point. See Example **4.6.11**.
- When given an equation of the form $\log_b S = \log_b T$, where S and T are algebraic expressions, we can use the one-to-one property of logarithms to solve the equation $S = T$ for the unknown. See Example **4.6.12**.
- Combining the skills learned in this and previous sections, we can solve equations that model real world situations, whether the unknown is in an exponent or in the argument of a logarithm. See Example **4.6.13**.

Contributors

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4.6E: Exercises

4.6: Exponential and Logarithmic Equations

Uncontrolled population growth can be modeled with exponential functions. Equations resulting from those exponential functions can be solved to analyze and make predictions about exponential growth. In this section, we will learn techniques for solving exponential functions.

Verbal

1) How can an exponential equation be solved?

Answer

Determine first if the equation can be rewritten so that each side uses the same base. If so, the exponents can be set equal to each other. If the equation cannot be rewritten so that each side uses the same base, then apply the logarithm to each side and use properties of logarithms to solve.

2) When does an extraneous solution occur? How can an extraneous solution be recognized?

3) When can the one-to-one property of logarithms be used to solve an equation? When can it not be used?

Answer

The one-to-one property can be used if both sides of the equation can be rewritten as a single logarithm with the same base. If so, the arguments can be set equal to each other, and the resulting equation can be solved algebraically. The one-to-one property cannot be used when each side of the equation cannot be rewritten as a single logarithm with the same base.

Algebraic

For the following exercises, use like bases to solve the exponential equation.

4) $4^{-3v-2} = 4^{-v}$

5) $64 \cdot 4^{3x} = 16$

Answer

$x = -\frac{1}{3}$

6) $3^{2x+1} \cdot 3^{x} = 243$

7) $2^{-3n} \cdot \frac{1}{4} = 2^{n+2}$

Answer

$n = -1$

8) $625 \cdot 5^{3x+3} = 125$

9) $\frac{36^{3b}}{36^{2b}} = 216^{2-b}$

Answer

$b = \frac{6}{5}$

10) $\left(\frac{1}{64}\right)^{3n} \cdot 8 = 2^{6}$

For the following exercises, use logarithms to solve.

11) $9^{x-10} = 1$

Answer

$x = 10$

12) $2e^{6x} = 13$

13) $e^{r+10} - 10 = -42$

Answer

No solution

14) $2 \cdot 10^{9a} = 29$

15) $-8 \cdot 10^{p+7} - 7 = -24$

Answer

$p = \log\left(\frac{17}{8}\right) - 7$

16) $7e^{3n-5} + 5 = -89$

17) $e^{-3k} + 6 = 44$

Answer

$k = -\frac{\ln(38)}{3}$

18) $-5e^{9x-8} - 8 = -62$

19) $-6e^{9x+8} + 2 = -74$

Answer

$x = \frac{\frac{\ln(38)}{3} - 8}{9}$

20) $2^{x+1} = 5^{2x-1}$

21) $e^{2x} - e^{x} - 132 = 0$

Answer

$x = \ln 12$

22) $7e^{8x+8} - 5 = -95$

23) $10e^{8x+3} + 2 = 8$

Answer

$x = \frac{\frac{\ln(3)}{5} - 3}{8}$

24) $4e^{3x+3} - 7 = 53$

25) $8e^{-5x-2} - 4 = -90$

Answer

No solution

26) $3^{2x+1} = 7^{x-2}$

27) $e^{2x} - e^{x} - 6 = 0$

Answer

$x = \ln 3$

28) $3e^{3-3x}+6=-31$

For the following exercises, use the definition of a logarithm to rewrite the equation as an exponential equation.

29) $\log\left(\frac{1}{100}\right)=-2$

Answer

$10^{-2}=\frac{1}{100}$

30) $\log_{324}(18)=\frac{1}{2}$

For the following exercises, use the definition of a logarithm to solve the equation.

31) $5\log_7 n=10$

Answer

$n=49$

32) $-8\log_9 x=16$

33) $4+\log_2(9k)=2$

Answer

$k=\frac{1}{36}$

34) $2\log(8n+4)+6=10$

35) $10-4\ln(9-8x)=6$

Answer

$x=\frac{9-e}{8}$

For the following exercises, use the one-to-one property of logarithms to solve.

36) $\ln(10-3x)=\ln(-4x)$

37) $\log_{13}(5n-2)=\log_{13}(8-5n)$

Answer

$n=1$

38) $\log(x+3)-\log(x)=\log(74)$

39) $\ln(-3x)=\ln(x^2-6x)$

Answer

No solution

40) $\log_4(6-m)=\log_4(3m)$

41) $\ln(x-2)-\ln(x)=\ln(54)$

Answer

No solution

42) $\log_9(2n^2-14n)=\log_9(-45+n^2)$

43) $\ln(x^2-10)+\ln(9)=\ln(10)$

Answer

$x = \pm\frac{10}{3}$

For the following exercises, solve each equation for x.

44) $\log(x+12) = \log(x) + \log(12)$

45) $\ln(x) + \ln(x-3) = \ln(7x)$

Answer

$x = 10$

46) $\log_2(7x+6) = 3$

47) $\ln(7) + \ln(2-4x^2) = \ln(14)$

Answer

$x = 0$

48) $\log_8(x+6) - \log_8(x) = \log_8(58)$

49) $\ln(3) - \ln(3-3x) = \ln(4)$

Answer

$x = \frac{3}{4}$

50) $\log_3(3x) - \log_3(6) = \log_3(77)$

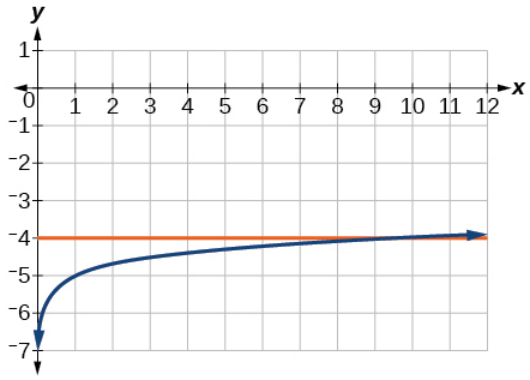
Graphical

For the following exercises, solve the equation for x, if there is a solution. Then graph both sides of the equation, and observe the point of intersection (if it exists) to verify the solution.

51) $\log_9(x) - 5 = -4$

Answer

$x = 9$

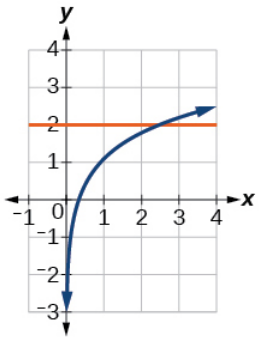


52) $\log_3(x) + 3 = 2$

53) $\ln(3x) = 2$

Answer

$x = \frac{e^2}{3} \approx 2.5$

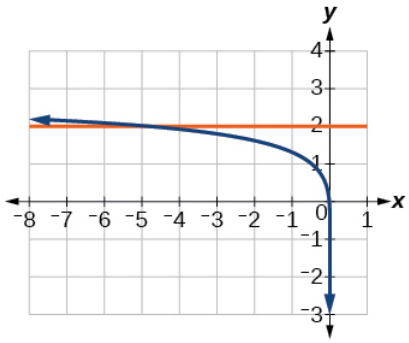


54) $\ln(x-5)=1$

55) $\log(4)+\log(-5x)=2$

Answer

$x=-5$

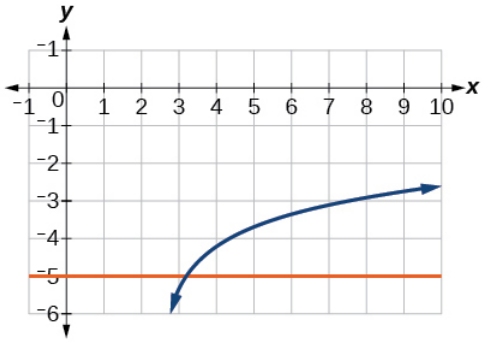


56) $-7+\log_3(4-x)=-6$

57) $\ln(4x-10)-6=-5$

Answer

$x=\frac{e+10}{4}\approx 3.2$



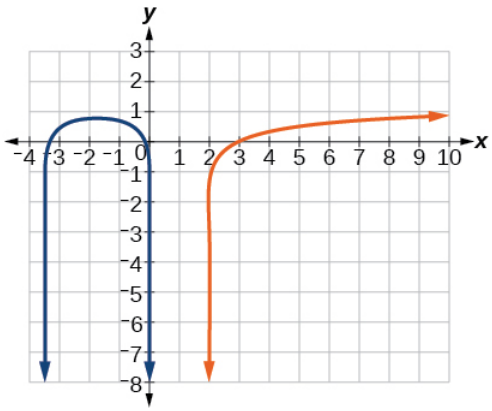
58) $\log(4-2x)=\log(-4x)$

59) $\log_{11}(-2x^2-7x)=\log_{11}(x-2)$

Answer

No solution

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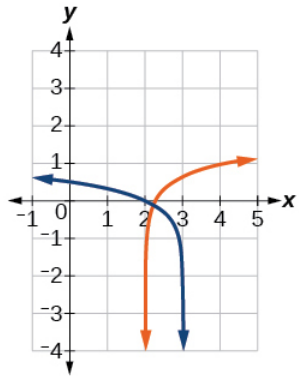


60) $\ln(2x+9) = \ln(-5x)$

61) $\log_9(3-x) = \log_9(4x-8)$

Answer

$x = \frac{11}{5} \approx 2.2$

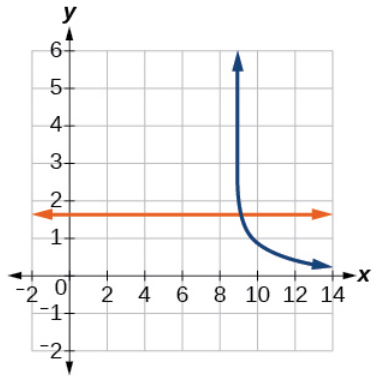


62) $\log(x^2+13) = \log(7x+3)$

63) $\frac{3}{\log_2(10)} - \log(x-9) = \log(44)$

Answer

$x = \frac{101}{11} \approx 9.2$



64) $\ln(x) - \ln(x+3) = \ln(6)$

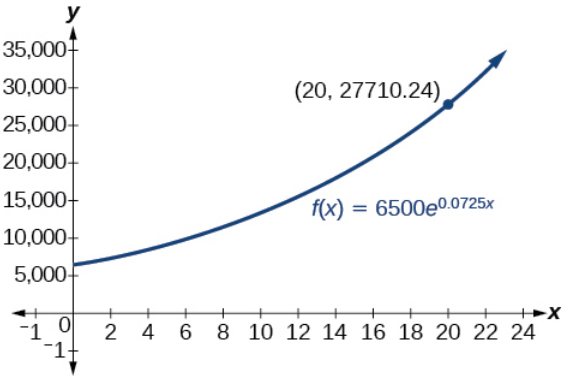
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For the following exercises, solve for the indicated value, and graph the situation showing the solution point.

65) An account with an initial deposit of $\$6,500$ earns 7.25% annual interest, compounded continuously. How much will the account be worth after 20 years?

Answer

about $\$27,710.24$

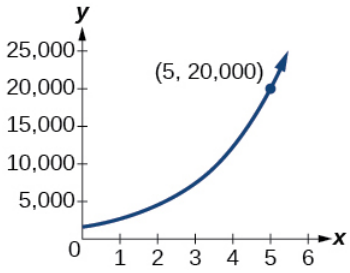


66) The formula for measuring sound intensity in decibels D is defined by the equation $D = 10\log\left(\frac{I}{I_0}\right)$, where I is the intensity of the sound in watts per square meter and $I_0 = 10^{-12}$ is the lowest level of sound that the average person can hear. How many decibels are emitted from a jet plane with a sound intensity of $8 \cdot 3 \cdot 10^2$ watts per square meter?

67) The population of a small town is modeled by the equation $P = 1650e^{0.5t}$ where t is measured in years. In approximately how many years will the town's population reach $20,000$?

Answer

about 5 years



Technology

For the following exercises, solve each equation by rewriting the exponential expression using the indicated logarithm. Then use a calculator to approximate the variable to 3 decimal places.

68) $1000(1.03)^t = 5000$ using the common log.

69) $e^{5x} = 17$ using the natural log.

Answer

$\frac{\ln(17)}{5} \approx 0.567$

70) $3(1.04)^{3t} = 8$ using the common log

71) $3^{4x-5} = 38$ using the common log

Answer

$$x = \frac{\log(38 + 5\log(3))}{4\log(3)} \approx 2.078$$

72) $50e^{-0.12t} = 10$ using the natural log

For the following exercises, use a calculator to solve the equation. Unless indicated otherwise, round all answers to the nearest ten-thousandth.

73) $7e^{3x-5} + 7.9 = 47$

Answer

$x \approx 2.2401$

74) $\ln(3) + \ln(4.4x + 6.8) = 2$

75) $\log(-0.7x - 9) = 1 + 5\log(5)$

Answer

$x \approx -44655.7143$

76) Atmospheric pressure P in pounds per square inch is represented by the formula $P = 14.7e^{-0.21x}$, where x is the number of miles above sea level. To the nearest foot, how high is the peak of a mountain with an atmospheric pressure of 8.369 pounds per square inch? (Hint: there are 5280 feet in a mile)

77) The magnitude M of an earthquake is represented by the equation $M = \frac{2}{3}\log\left(\frac{E}{E_0}\right)$ where E is the amount of energy released by the earthquake in joules $E_0 = 10^{4.4}$ and is the assigned minimal measure released by an earthquake. To the nearest hundredth, what would the magnitude be of an earthquake releasing $1.4 \cdot 10^{13}$ joules of energy?

Answer

about 5.83

Extensions

78) Use the definition of a logarithm along with the one-to-one property of logarithms to prove that $b^{\log_b x} = x$.

79) Recall the formula for continually compounding interest, $y = Ae^{kt}$. Use the definition of a logarithm along with properties of logarithms to solve the formula for time t such that t is equal to a single logarithm.

Answer

$$t = \ln\left(\left(\frac{y}{A}\right)^{\frac{1}{k}}\right)$$

80) Recall the compound interest formula $A = a\left(1 + \frac{r}{k}\right)^{kt}$. Use the definition of a logarithm along with properties of logarithms to solve the formula for time t.

81) Newton's Law of Cooling states that the temperature T of an object at any time t can be described by the equation $T = T_s + (T_0 - T_s)e^{-kt}$, where T_s is the temperature of the surrounding environment, T_0 is the initial temperature of the object, and k is the cooling rate. Use the definition of a logarithm along with properties of logarithms to solve the formula for time t such that t is equal to a single logarithm.

Answer

$$t = \ln\left(\left(\frac{T - T_s}{T_0 - T_s}\right)^{-\frac{1}{k}}\right)$$

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CHAPTER OVERVIEW

5: Trigonometric Functions

The trigonometric functions are functions of an angle. and relate the angles of a triangle to the lengths of its sides. They are important in the study of triangles and modeling periodic phenomena, among many other applications.

5.0: Prelude to Trigonometric Functions
5.0E: Exercises
5.1: Angles
5.1E: Exercises
5.2: Unit Circle - Sine and Cosine Functions
5.2E: Exercises
5.3: The Other Trigonometric Functions
5.3E: Exercises
5.4: Right Triangle Trigonometry
5.4E: Exercises

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5.0: Prelude to Trigonometric Functions

Life is dense with phenomena that repeat in regular intervals. Each day, for example, the tides rise and fall in response to the gravitational pull of the moon. Similarly, the progression from day to night occurs as a result of Earth's rotation, and the pattern of the seasons repeats in response to Earth's revolution around the sun. Outside of nature, many stocks that mirror a company's profits are influenced by changes in the economic business cycle.



The tide rises and falls at regular, predictable intervals. (credit: Andrea Schaffer, Flickr)

In mathematics, a function that repeats its values in regular intervals is known as a periodic function. The graphs of such functions show a general shape reflective of a pattern that keeps repeating. This means the graph of the function has the same output at exactly the same place in every cycle. And this translates to all the cycles of the function having exactly the same length. So, if we know all the details of one full cycle of a true periodic function, then we know the state of the function's outputs at all times, future and past. In this chapter, we will investigate various examples of periodic functions.

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5.0E: Exercises

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5.1: Angles

Skills to Develop

- Draw angles in standard position.
- Convert between degrees and radians.
- Find coterminal angles.
- Find the length of a circular arc.
- Use linear and angular speed to describe motion on a circular path.

A golfer swings to hit a ball over a sand trap and onto the green. An airline pilot maneuvers a plane toward a narrow runway. A dress designer creates the latest fashion. What do they all have in common? They all work with angles, and so do all of us at one time or another. Sometimes we need to measure angles exactly with instruments. Other times we estimate them or judge them by eye. Either way, the proper angle can make the difference between success and failure in many undertakings. In this section, we will examine properties of angles.

Drawing Angles in Standard Position

Properly defining an angle first requires that we define a *ray*. A ray consists of one point on a line and all points extending in one direction from that point. The first point is called the endpoint of the ray. We can refer to a specific ray by stating its *endpoint* and any other point on it. The ray in Figure 5.1.1 can be named as ray EF, or in symbol form $\overrightarrow{EF}$.

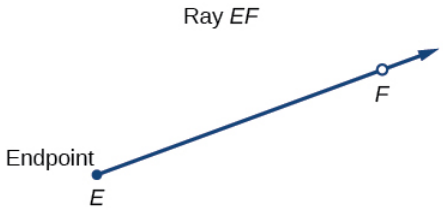


***Figure* 5.1.1**

An *angle* is the union of two rays having a common endpoint. The endpoint is called the **vertex** of the angle, and the two rays are the sides of the angle. The angle in Figure 5.1.2 is formed from $\overrightarrow{ED}$ and $\overrightarrow{EF}$. Angles can be named using a point on each ray and the vertex, such as angle *DEF*, or in symbol form $\angle DEF$.

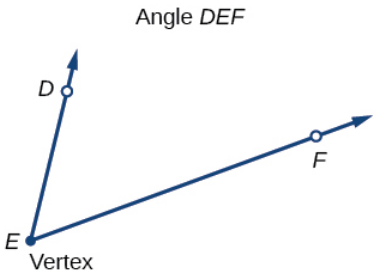


***Figure* 5.1.2**

Greek letters are often used as variables for the measure of an angle. Table 5.1.1 is a list of Greek letters commonly used to represent angles, and a sample angle is shown in Figure 5.1.3.

***Table* 5.1.1**

θ	φ or ϕ	α	β	γ
theta	phi	alpha	beta	gamma

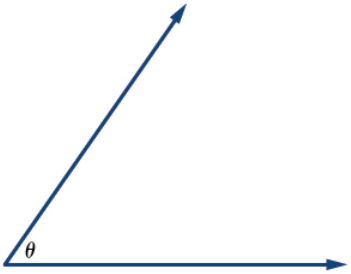


***Figure* 5.1.3**: *Angle theta, shown as $\angle\theta$*

Angle creation is a dynamic process. We start with two rays lying on top of one another. We leave one fixed in place, and rotate the other. The fixed ray is the **initial side,** and the rotated ray is the **terminal side**. In order to identify the different sides, we indicate the rotation with a small arc and arrow close to the vertex as in Figure 5.1.4.

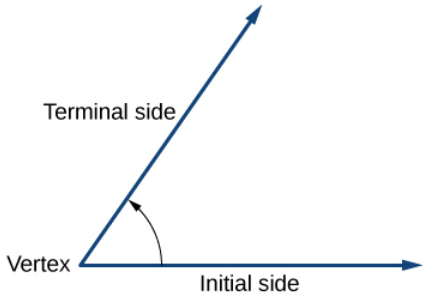


***Figure* 5.1.4**

As we discussed at the beginning of the section, there are many applications for angles, but in order to use them correctly, we must be able to measure them. The **measure of an angle** is the amount of rotation from the initial side to the terminal side. Probably the most familiar unit of angle measurement is the degree. One **degree** is $\frac{1}{360}$ of a circular rotation, so a complete circular rotation contains 360 degrees. An angle measured in degrees should always include the unit “degrees” after the number, or include the degree symbol °. For example, 90 degrees = 90°.

To formalize our work, we will begin by drawing angles on an x-y- coordinate plane. Angles can occur in any position on the coordinate plane, but for the purpose of comparison, the convention is to illustrate them in the same position whenever possible. An angle is in **standard position** if its vertex is located at the origin, and its initial side extends along the positive x-axis. See Figure 5.1.5.

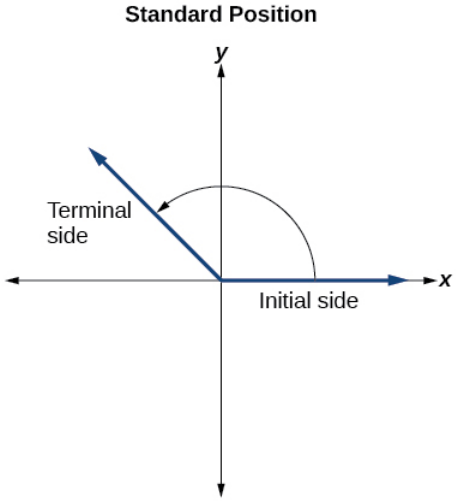


***Figure* 5.1.5**

If the angle is measured in a counterclockwise direction from the initial side to the terminal side, the angle is said to be a **positive angle**. If the angle is measured in a clockwise direction, the angle is said to be a **negative angle**.

Drawing an angle in standard position always starts the same way - draw the initial side along the positive x-axis. To place the terminal side of the angle, we must calculate the fraction of a full rotation the angle represents. We do that by dividing the angle measure in degrees by 360°. For example, to draw a 90° angle, we calculate that $\frac{90°}{360°} = \frac{1}{4}$. So, the terminal side will be one-fourth of the way around the circle, moving counterclockwise from the positive x-axis. To draw a 360° angle, we calculate that $\frac{360°}{360°} = 1$. So the terminal side will be 1 complete rotation around the circle, moving counterclockwise from the positive x-axis. In this case, the initial side and the terminal side overlap. See Figure 5.1.6 .

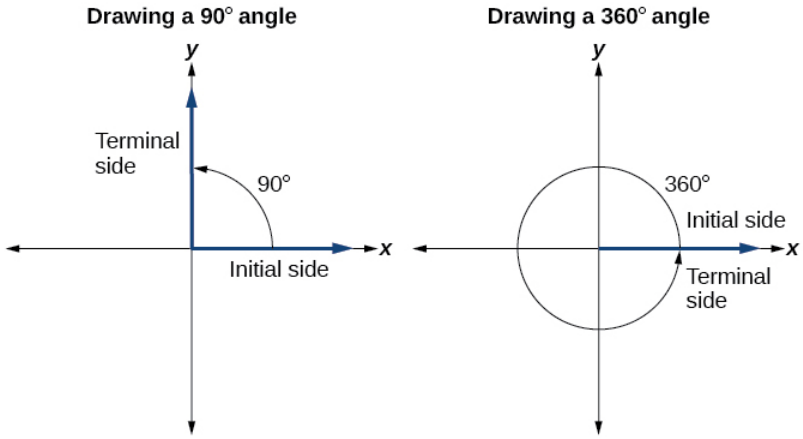


***Figure* 5.1.6**

Recall that the x- and y-axes divide the coordinate plane into four quarters called quadrants. We label these quadrants to mimic the direction a positive angle would sweep. The four quadrants are labeled I, II, III, and IV. We have a special type of angle in standard position whose terminal side lies on an axis, a **quadrantal angle**. This type of angle can have a measure of 0°, 90°, 180°, 270° or 360°. See Figure 5.1.7.

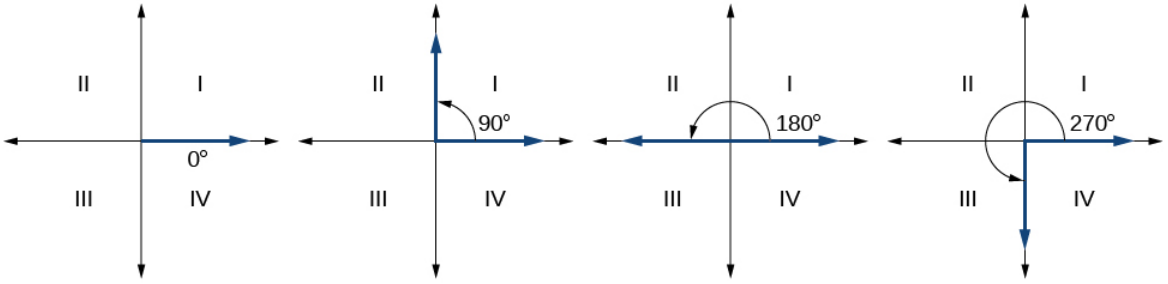


***Figure* 5.1.7**: *Quadrantal angles are angles in standard position whose terminal side lies along an axis.*

QUADRANTAL ANGLES

Quadrantal angles are angles in standard position whose terminal side lies on an axis, including 0°, 90°, 180°, 270°, or 360°.

How To: Given an angle measure in degrees, draw the angle in standard position

1. Express the angle measure as a fraction of 360°.
2. Reduce the fraction to simplest form.
3. Draw an angle that contains that same fraction of the circle, beginning on the positive x-axis and moving counterclockwise for positive angles and clockwise for negative angles.

Example 5.1.1: Drawing an Angle in Standard Position Measured in Degrees

1. Sketch an angle of 30° in standard position.
2. Sketch an angle of −135° in standard position.

Solution

1. Divide the angle measure by 360°.

$$\frac{30°}{360°} = \frac{1}{12} \tag{5.1.1}$$

To rewrite the fraction in a more familiar fraction, we can recognize that

$$\frac{1}{12} = \frac{1}{3}\left(\frac{1}{4}\right) \tag{5.1.2}$$

One-twelfth equals one-third of a quarter, so by dividing a quarter rotation into thirds, we can sketch a line at 30° as in Figure 5.1.8.

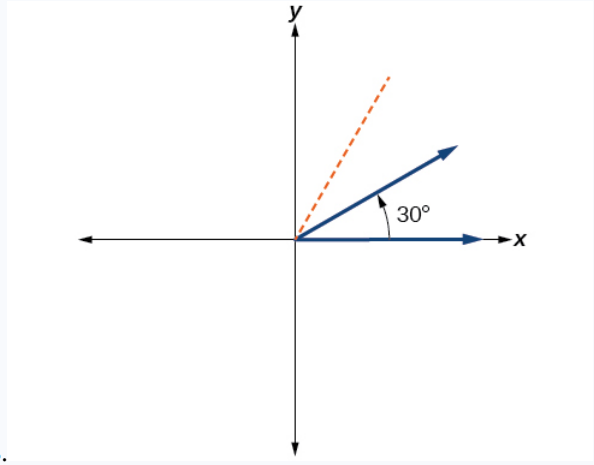


***Figure* 5.1.8**

2. Divide the angle measure by 360°.

$$\frac{-135°}{360°} = -\frac{3}{8} \tag{5.1.3}$$

In this case, we can recognize that

$$-\frac{3}{8} = -\frac{3}{2}\left(\frac{1}{4}\right) \tag{5.1.4}$$

Negative three-eighths is one and one-half times a quarter, so we place a line by moving clockwise one full quarter and one-half of another quarter, as in Figure 5.1.9.

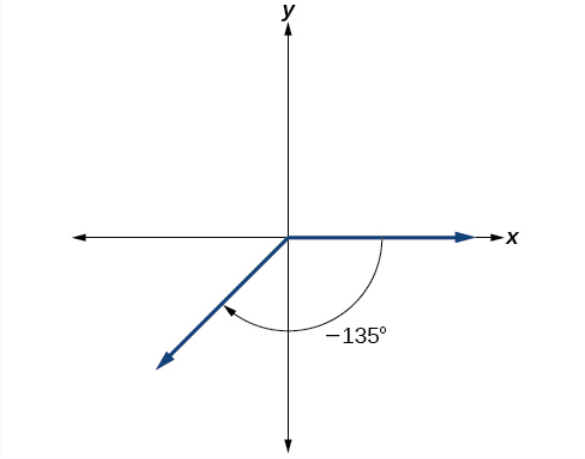


***Figure* 5.1.9**

5.1.1

Show an angle of 240° on a circle in standard position.

Answer

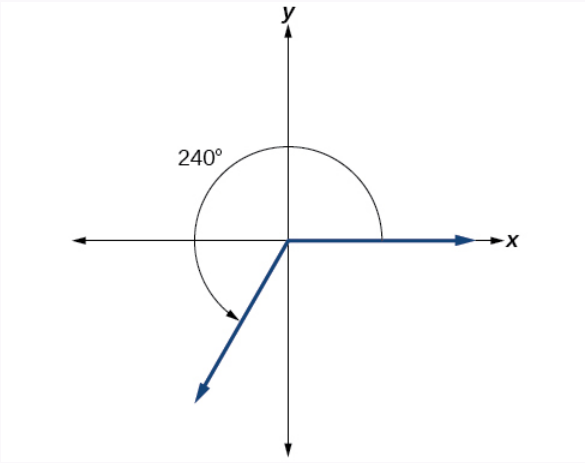


***Figure* 5.1.10**

Converting Between Degrees and Radians

Dividing a circle into 360 parts is an arbitrary choice, although it creates the familiar degree measurement. We may choose other ways to divide a circle. To find another unit, think of the process of drawing a circle. Imagine that you stop before the circle is completed. The portion that you drew is referred to as an arc. An **arc** may be a portion of a full circle, a full circle, or more than a full circle, represented by more than one full rotation. The length of the arc around an entire circle is called the **circumference** of that circle.

The circumference of a circle is $C = 2\pi r$. If we divide both sides of this equation by r, we create the ratio of the circumference to the radius, which is always 2π regardless of the length of the radius. So the circumference of any circle is $2\pi \approx 6.28$ times the length of the radius. That means that if we took a string as long as the radius and used it to measure consecutive lengths around the circumference, there would be room for six full string-lengths and a little more than a quarter of a seventh string-length, as shown in Figure 5.1.11.

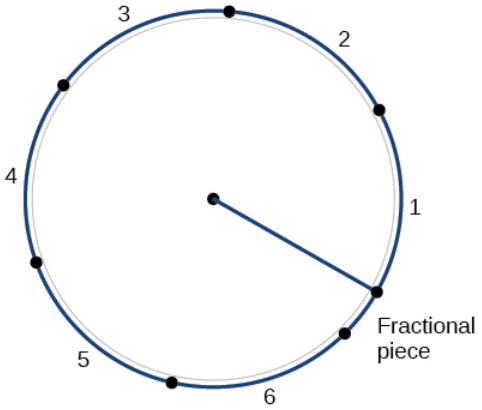


***Figure* 5.1.11**

This brings us to our new angle measure. One **radian** is the measure of a central angle of a circle that intercepts an arc equal in length to the radius of that circle. A central angle is an angle formed at the center of a circle by two radii. Because the total circumference equals 2π times the radius, a full circular rotation is 2π radians. So

$$2\pi \text{ radians } = 360^\circ \tag{5.1.5}$$

$$\pi \text{ radians } = \frac{360^\circ}{2} = 180^\circ \tag{5.1.6}$$

$$1 \text{ radian } = \frac{180^\circ}{\pi} \approx 57.3^\circ \tag{5.1.7}$$

See Figure 5.1.12. Note that when an angle is described without a specific unit, it refers to radian measure. For example, an angle measure of 3 indicates 3 radians. In fact, radian measure can be considered dimensionless, since it is the quotient of a length (circumference) divided by a length (radius) and the length units cancel out.

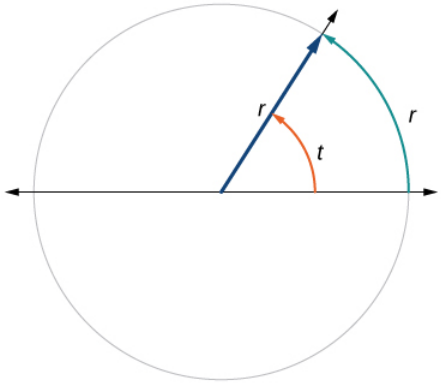


***Figure* 5.1.12**: *The angle t sweeps out a measure of one radian. Note that the length of the intercepted arc is the same as the length of the radius of the circle.*

Relating Arc Lengths to Radius

An **arc length** s is the length of the curve along the arc. Just as the full circumference of a circle always has a constant ratio to the radius, the arc length produced by any given angle also has a constant relation to the radius, regardless of the length of the radius.

This ratio, called the **radian measure**, is the same regardless of the radius of the circle—it depends only on the angle. This property allows us to define a measure of any angle as the ratio of the arc length s s to the radius r. See Figure 5.1.13.

$$s = r\theta \tag{5.1.8}$$

$$\theta = \frac{s}{r} \tag{5.1.9}$$

If $s = r$, then $\theta = \frac{r}{r} = 1$ radian.

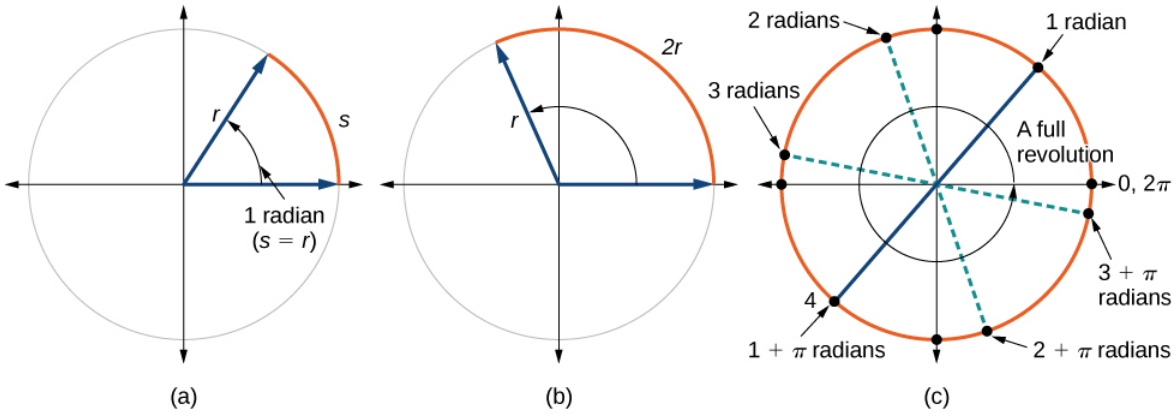


***Figure* 5.1.13**: *(a) In an angle of 1 radian, the arc length s equals the radius r. (b) An angle of 2 radians has an arc length $s = 2r$. (c) A full revolution is 2π or about 6.28 radians.*

To elaborate on this idea, consider two circles, one with radius 2 and the other with radius 3. Recall the circumference of a circle is $C = 2\pi r$,where r is the radius. The smaller circle then has circumference $2\pi(2) = 4\pi$ and the larger has circumference $2\pi(3) = 6\pi$. Now we draw a 45° angle on the two circles, as in Figure 5.1.14.

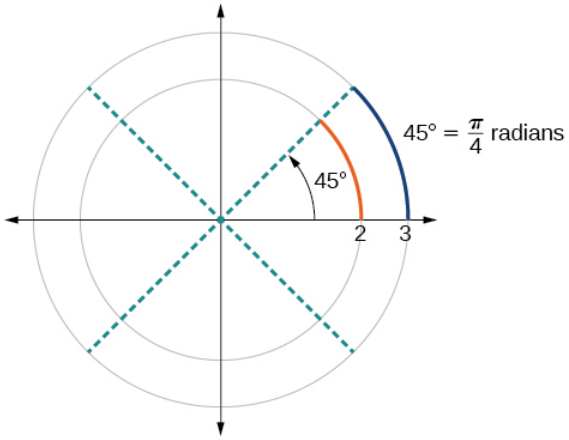


***Figure* 5.1.14**: *A 45° angle contains one-eighth of the circumference of a circle, regardless of the radius.*

Notice what happens if we find the ratio of the arc length divided by the radius of the circle.

$$\text{Smaller circle: } \frac{\frac{1}{2}\pi}{2} = \frac{1}{4}\pi \tag{5.1.10}$$

$$\text{Larger circle: } \frac{\frac{3}{4}\pi}{3} = \frac{1}{4}\pi \tag{5.1.11}$$

Since both ratios are $\frac{1}{4}\pi$, the angle measures of both circles are the same, even though the arc length and radius differ.

definition: RADIANS

One **radian** is the measure of the central angle of a circle such that the length of the arc between the initial side and the terminal side is equal to the radius of the circle. A full revolution (360°) equals 2π radians. A half revolution (180°) is equivalent to π radians.

The **radian measure** of an angle is the ratio of the length of the arc subtended by the angle to the radius of the circle. In other words, if s is the length of an arc of a circle, and r is the radius of the circle, then the central angle containing that arc measures $\frac{s}{r}$ radians. In a circle of radius 1, the radian measure corresponds to the length of the arc.

: A measure of 1 radian looks to be about 60°. Is that correct?

Yes. It is approximately 57.3°. Because 2π *radians equals 360°,* 1 *radian equals* $\frac{360°}{2\pi} \approx 57.3°$.

Using Radians

Because **radian** measure is the ratio of two lengths, it is a unitless measure. For example, in Figure 5.1.14, suppose the radius were 2 inches and we measured a distance along the arc of 2 inches as well. When we calculate the radian measure of the angle, the "inches" cancel, and we have a result without units. Therefore, it is not necessary to write the label "radians" after a radian measure, and if we see an angle that is not labeled with "degrees" or the degree symbol, we can assume that it is a radian measure.

Considering the most basic case, the **unit circle** (a circle with radius 1), we know that 1 rotation equals 360 degrees, 360°. We can also track one rotation around a circle by finding the circumference, $C = 2\pi r$, and for the unit circle $C = 2\pi$. These two different ways to rotate around a circle give us a way to convert from degrees to radians.

$$\begin{aligned} 1 \text{ rotation} &= 360° = 2\pi \text{ radians} \\ \frac{1}{2} \text{ rotation} &= 180° = \pi \text{ radians} \\ \frac{1}{4} \text{ rotation} &= 90° = \frac{\pi}{2} \text{ radians} \end{aligned} \tag{5.1.12}$$

Identifying Special Angles Measured in Radians

In addition to knowing the measurements in degrees and radians of a quarter revolution, a half revolution, and a full revolution, there are other frequently encountered angles in one revolution of a circle with which we should be familiar. It is common to encounter multiples of 30, 45, 60, and 90 degrees. These values are shown in Figure 5.1.15. Becoming familiar with these angles will be very useful as we study the properties associated with angles.

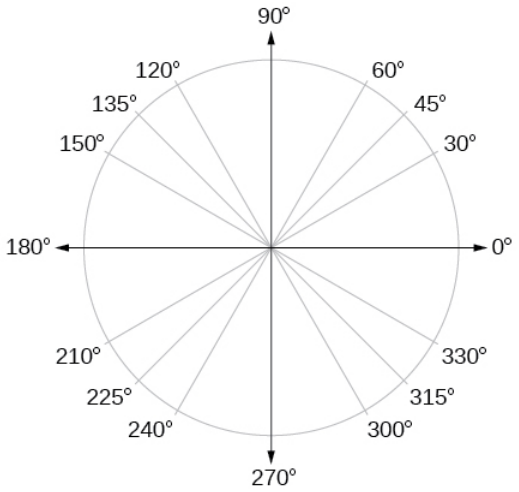


***Figure* 5.1.15**: *Commonly encountered angles measured in degrees*

Now, we can list the corresponding radian values for the common measures of a circle corresponding to those listed in Figure 5.1.15, which are shown in Figure 5.1.16. Be sure you can verify each of these measures.

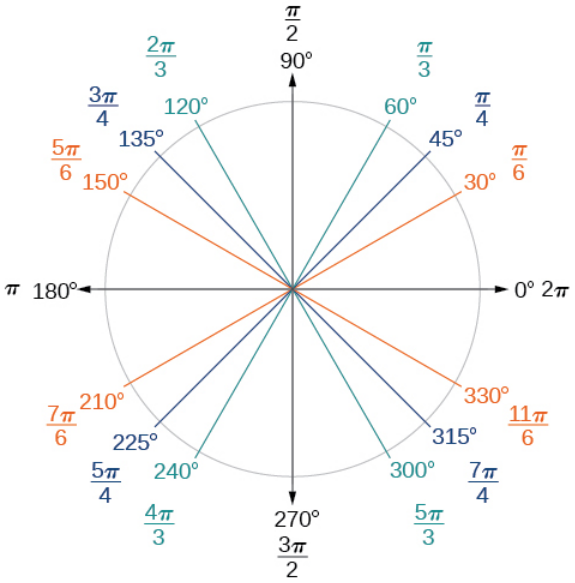


***Figure* 5.1.16**: *Commonly encountered angles measured in radians*

Example 5.1.2: Finding a Radian Measure

Find the radian measure of one-third of a full rotation.

Solution

For any circle, the arc length along such a rotation would be one-third of the circumference. We know that

$$\text{1 rotation} = 2\pi r \tag{5.1.13}$$

So,

$$s = \frac{1}{3}(2\pi r) \tag{5.1.14}$$

$$= \frac{2\pi r}{3} \tag{5.1.15}$$

The radian measure would be the arc length divided by the radius.

$$\text{radian measure} = \frac{\frac{2\pi r}{3}}{r} \tag{5.1.16}$$

$$= \frac{2\pi r}{3r} \tag{5.1.17}$$

$$= \frac{2\pi}{3} \tag{5.1.18}$$

5.1.2

Find the radian measure of three-fourths of a full rotation.

Answer

$\frac{3\pi}{2}$

Converting between Radians and Degrees

Because degrees and radians both measure angles, we need to be able to convert between them. We can easily do so using a proportion.

$$\frac{\theta}{180} = \frac{\theta^R}{\pi} \tag{5.1.19}$$

This proportion shows that the measure of angle θ in degrees divided by 180 equals the measure of angle θ in radians divided by π. Or, phrased another way, degrees is to 180 as radians is to π.

$$\frac{\text{Degrees}}{180} = \frac{Radians}{\pi} \tag{5.1.20}$$

CONVERTING BETWEEN RADIANS AND DEGREES

To convert between degrees and radians, use the proportion

$$\frac{\theta}{180} = \frac{\theta^R}{\pi} \tag{5.1.21}$$

Example 5.1.3: Converting Radians to Degrees

Convert each radian measure to degrees.

1. $\frac{\pi}{6}$
2. 3

Solution

Because we are given radians and we want degrees, we should set up a proportion and solve it.

1. We use the proportion, substituting the given information.

$$\frac{\theta}{180} = \frac{\theta^R}{\pi} \tag{5.1.22}$$

$$\frac{\theta}{180} = \frac{\frac{\pi}{6}}{\pi} \tag{5.1.23}$$

$$\theta = \frac{180}{6} \tag{5.1.24}$$

$$\theta = 30^\circ \tag{5.1.25}$$

2. We use the proportion, substituting the given information.

$$\frac{\theta}{180} = \frac{\theta^R}{\pi} \tag{5.1.26}$$

$$\frac{\theta}{180} = \frac{3}{\pi} \tag{5.1.27}$$

$$\theta = \frac{3(180)}{\pi} \tag{5.1.28}$$

$$\theta \approx 172^\circ \tag{5.1.29}$$

5.1.3

Convert $-\frac{3\pi}{4}$ radians to degrees.

Answer

-135°

Example 5.1.4: Converting Degrees to Radians

Convert 15 degrees to radians.

Solution

In this example, we start with degrees and want radians, so we again set up a proportion and solve it, but we substitute the given information into a different part of the proportion.

$$\frac{\theta}{180} = \frac{\theta^R}{\pi} \tag{5.1.30}$$

$$\frac{15}{180} = \frac{\theta^R}{\pi} \tag{5.1.31}$$

$$\frac{15\pi}{180} = \theta^R \tag{5.1.32}$$

$$\frac{\pi}{12} = \theta^R \tag{5.1.33}$$

Analysis

Another way to think about this problem is by remembering that $30^\circ = \frac{\pi}{6}$. Because $15^\circ = \frac{1}{2}(30^\circ)$, ,we can find that $\frac{1}{2}(\frac{\pi}{6})$ is $\frac{\pi}{12}$.

5.1.4

Convert 126° to radians.

Answer

$\frac{7\pi}{10}$

Finding Coterminal Angles

Converting between degrees and radians can make working with angles easier in some applications. For other applications, we may need another type of conversion. Negative angles and angles greater than a full revolution are more awkward to work with than those in the range of 0° to 360°, or 0 to 2π. It would be convenient to replace those out-of-range angles with a corresponding angle within the range of a single revolution.

It is possible for more than one angle to have the same terminal side. Look at Figure 5.1.17. The angle of 140° is a **positive angle**, measured counterclockwise. The angle of –220° is a **negative angle**, measured clockwise. But both angles have the same terminal side. If two angles in standard position have the same terminal side, they are **coterminal** angles. Every angle greater than 360° or less than 0° is coterminal with an angle between 0° and 360°, and it is often more convenient to find the coterminal angle within the range of 0° to 360° than to work with an angle that is outside that range.

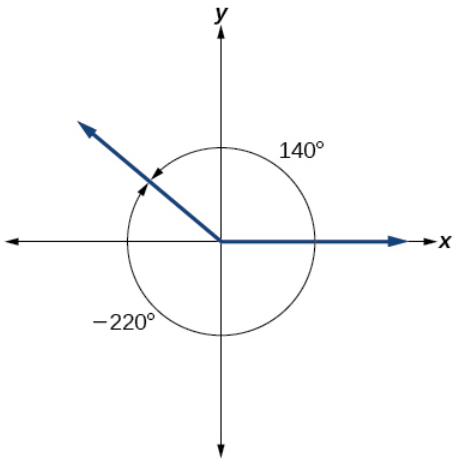


***Figure* 5.1.17**: *An angle of 140° and an angle of –220° are coterminal angles.*

Any angle has infinitely many **coterminal angles** because each time we add 360° to that angle—or subtract 360° from it—the resulting value has a terminal side in the same location. For example, 100° and 460° are coterminal for this reason, as is −260°. Recognizing that any angle has infinitely many coterminal angles explains the repetitive shape in the graphs of trigonometric functions.

An angle's reference angle is the measure of the smallest acute angle t formed by the terminal side of the angle t and the horizontal axis. (*Acute angles* are angles between $0°$ and $90°$.) Acute angles in standard position have terminal sides that lie in the first quadrant and are their own reference angles. They can be used as models for angles in other quadrants. See Figure 5.1.18 for examples of reference angles for angles in different quadrants. Note that reference angles are **not** generally in standard position. We will make use of reference angles in Section 5.2.

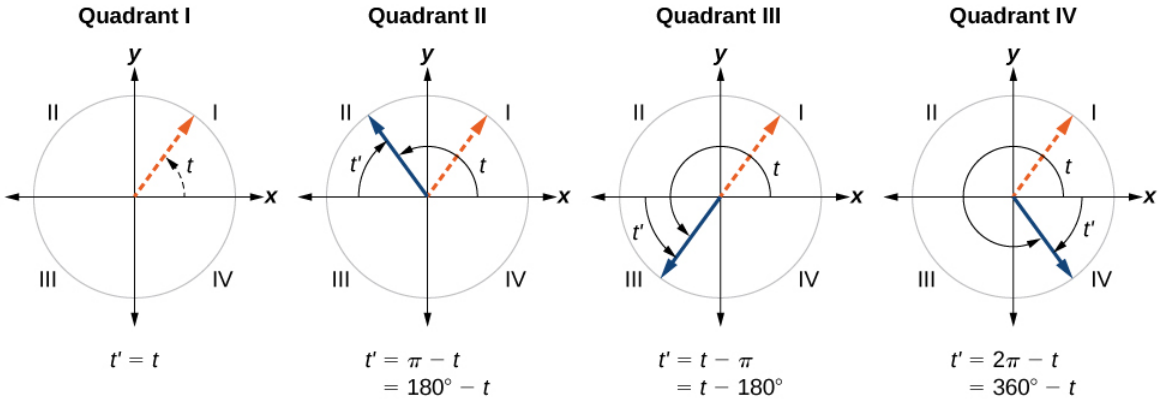


***Figure* 5.1.18**

definition: COTERMINAL AND REFERENCE ANGLES

- *Coterminal angles* are two angles in standard position that have the same terminal side.
- An angle's *reference angle* is the size of the smallest acute angle, t', formed by the terminal side of the angle t and the horizontal axis.

How To: Given an angle greater than 360°, find a coterminal angle between 0° and 360°

1. Subtract 360° from the given angle.
2. If the result is still greater than 360°, subtract 360° again till the result is between 0° and 360°.
3. The resulting angle is coterminal with the original angle.

Example 5.1.5: Finding an Angle Coterminal with an Angle of Measure Greater Than 360°

Find the least positive angle θ that is coterminal with an angle measuring 800°, where $0° \le \theta < 360°$.

Solution

An angle with measure 800° is coterminal with an angle with measure 800 − 360 = 440°, but 440° is still greater than 360°, so we subtract 360° again to find another coterminal angle: 440 − 360 = 80°.

The angle $\theta = 80°$ is coterminal with 800°. To put it another way, 800° equals 80° plus two full rotations, as shown in Figure 5.1.19.

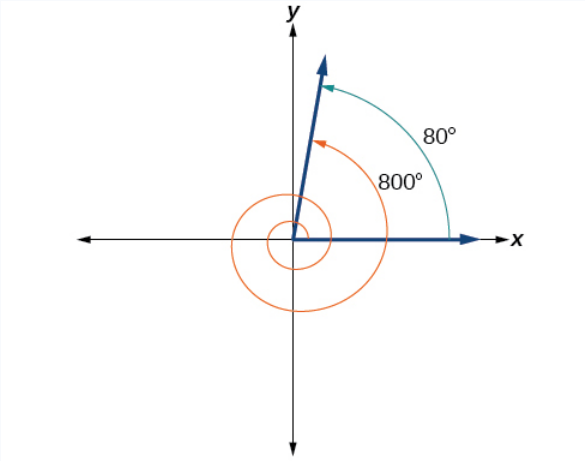


***Figure* 5.1.19**

5.1.5

Find an angle α that is coterminal with an angle measuring 870°, where $0° \leq \alpha < 360°$.

Answer

$\alpha = 150°$

How To: Given an angle with measure less than 0°, find a coterminal angle having a measure between 0° and 360°.

1. Add 360° to the given angle.
2. If the result is still less than 0°, add 360° again until the result is between 0° and 360°.
3. The resulting angle is coterminal with the original angle.

Example 5.1.6: Finding an Angle Coterminal with an Angle Measuring Less Than 0°

Show the angle with measure −45° on a circle and find a positive coterminal angle α such that $0° \leq \alpha < 360°$.

Solution

Since 45° is half of 90°, we can start at the positive horizontal axis and measure clockwise half of a 90° angle.

Because we can find coterminal angles by adding or subtracting a full rotation of 360°, we can find a positive coterminal angle here by adding 360°:

$$-45° + 360° = 315° \tag{5.1.34}$$

We can then show the angle on a circle, as in Figure 5.1.20.

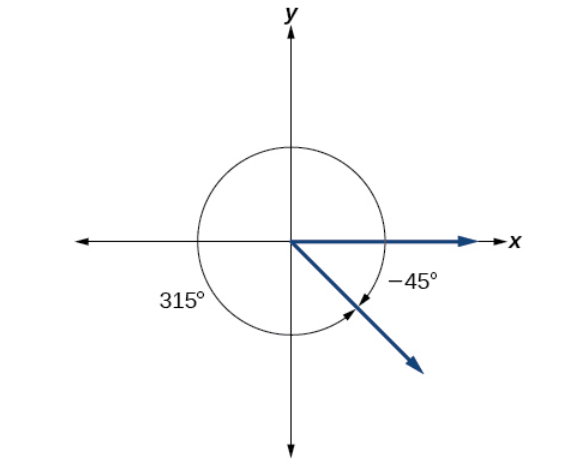


***Figure* 5.1.20**

5.1.6

Find an angle β that is coterminal with an angle measuring −300° such that $0° \leq \beta < 360°$.

Answer

$\beta = 60°$

Finding Coterminal Angles Measured in Radians

We can find **coterminal** angles measured in radians in much the same way as we have found them using degrees. In both cases, we find coterminal angles by adding or subtracting one or more full rotations.

How To: Given an angle greater than 2π, find a coterminal angle between 0 and 2π.

1. Subtract 2π from the given angle.
2. If the result is still greater than 2π, subtract 2π again until the result is between 0 and 2π.
3. The resulting angle is coterminal with the original angle.

Example 5.1.7: Finding Coterminal Angles Using Radians

Find an angle β that is coterminal with $\frac{19\pi}{4}$, where $0 \leq \beta < 2\pi$.

Solution

When working in degrees, we found coterminal angles by adding or subtracting 360 degrees, a full rotation. Likewise, in radians, we can find coterminal angles by adding or subtracting full rotations of 2π radians:

$$\frac{19\pi}{4} - 2\pi = \frac{19\pi}{4} - \frac{8\pi}{4} \tag{5.1.35}$$

$$= \frac{11\pi}{4} \tag{5.1.36}$$

The angle $\frac{11\pi}{4}$ is coterminal, but not less than 2π, so we subtract another rotation:

$$\frac{11\pi}{4} - 2\pi = \frac{11\pi}{4} - \frac{8\pi}{4} \tag{5.1.37}$$

$$= \frac{3\pi}{4} \tag{5.1.38}$$

The angle $\frac{3\pi}{4}$ is coterminal with $\frac{19\pi}{4}$, as shown in Figure 5.1.21.

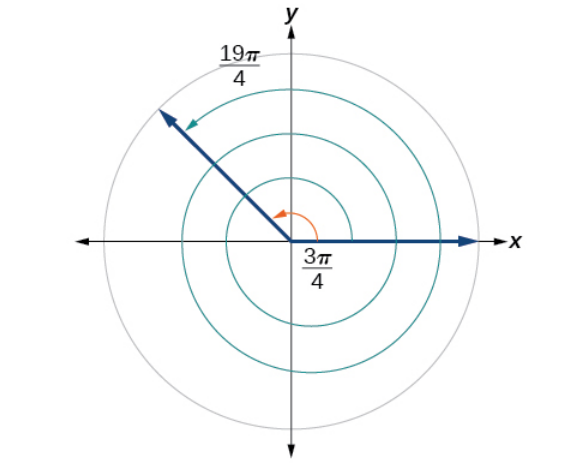


***Figure* 5.1.21**

5.1.7

Find an angle of measure θ that is coterminal with an angle of measure $-\frac{17\pi}{6}$ where $0 \leq \theta < 2\pi$.

Answer

$\frac{7\pi}{6}$

Determining the Length of an Arc

Recall that the **radian measure** θ of an angle was defined as the ratio of the **arc length** s of a circular arc to the radius r of the circle, $\theta = \frac{s}{r}$. From this relationship, we can find arc length along a circle, given an angle.

ARC LENGTH ON A CIRCLE

In a circle of radius r, the length of an arc s subtended by an angle with measure θ in radians, shown in Figure 5.1.22, is

$$s = r\theta \tag{5.1.39}$$

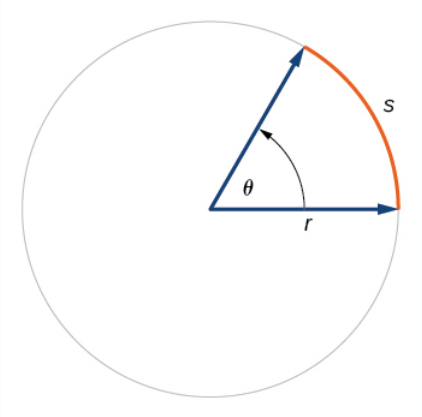


***Figure* 5.1.22**

Given a circle of radius r, calculate the length s of the arc subtended by a given angle of measure θ.

1. If necessary, convert θ to radians.
2. Multiply the radius r by the radian measure of θ : $s = r\theta$.

Example 5.1.8: Finding the Length of an Arc

Assume the orbit of Mercury around the sun is a perfect circle. Mercury is approximately 36 million miles from the sun.

1. In one Earth day, Mercury completes 0.0114 of its total revolution. How many miles does it travel in one day?
2. Use your answer from part (a) to determine the radian measure for Mercury's movement in one Earth day.
3. In Earth days, how long is Mercury's year?

Solution

1. Let's begin by finding the circumference of Mercury's orbit.

$$\begin{aligned} C &= 2\pi r && (5.1.40)\\ &= 2\pi(36 \text{ million miles}) && (5.1.41)\\ &\approx 226 \text{ million miles} && (5.1.42) \end{aligned}$$

Since Mercury completes 0.0114 of its total revolution in one Earth day, we can now find the distance traveled:

$$(0.0114)226 \text{ million miles} = 2.58 \text{ million miles} \quad (5.1.43)$$

2. Now, we convert to radians:

$$\begin{aligned} \text{radian} &= \frac{\text{arc length}}{\text{radius}} && (5.1.44)\\ &= \frac{2.58 \text{ million miles}}{36 \text{ million miles}} && (5.1.45)\\ &= 0.0717 && (5.1.46) \end{aligned}$$

3. A year is the length of time it takes for the planet to make one full revolution around the sun; in other words, to travel through a radian measure of 2π. We can divide 2π by 0.0717 to get $\frac{2\pi}{0.0717} \approx 87.63$ Earth days.

5.1.8

Find the arc length along a circle of radius 10 units subtended by an angle of 215°.

Answer

$\frac{215\pi}{18} = 37.525$ units

Finding the Area of a Sector of a Circle

In addition to arc length, we can also use angles to find the area of a **sector of a circle**. A sector is a region of a circle bounded by two radii and the intercepted arc, like a slice of pizza or pie. Recall that the area of a circle with radius r can be found using the formula $A = \pi r^2$. If the two radii form an angle of θ, measured in radians, then $\frac{\theta}{2\pi}$ is the ratio of the angle measure to the measure of a full rotation and is also, therefore, the ratio of the area of the sector to the area of the circle. Thus, the **area of a sector** is the fraction $\frac{\theta}{2\pi}$ multiplied by the entire area. (Always remember that this formula only applies if θ is in radians.)

$$\begin{aligned} \text{Area of sector} &= \left(\frac{\theta}{2\pi}\right)\pi r^2 && (5.1.47)\\ &= \frac{\theta \pi r^2}{2\pi} && (5.1.48)\\ &= \frac{1}{2}\theta r^2 && (5.1.49) \end{aligned}$$

AREA OF A SECTOR

The area of a **sector of a circle** with radius r subtended by an angle θ, measured in radians, is

$$A = \frac{1}{2}\theta r^2 \quad (5.1.50)$$

See Figure 5.1.23.

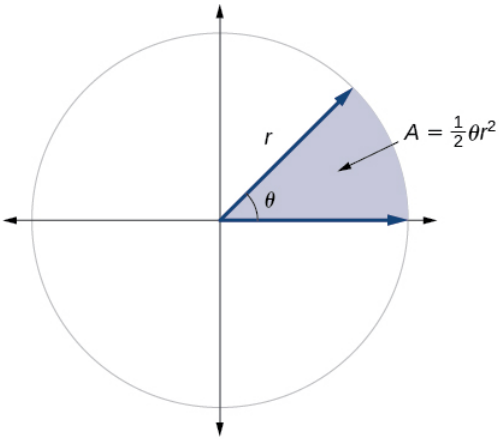


***Figure* 5.1.23**: *The area of the sector equals half the square of the radius times the central angle measured in radians.*

How To: Given a circle of radius r, find the area of a sector defined by a given angle θ.

1. If necessary, convert θ to radians.
2. Multiply half the radian measure of θ by the square of the radius r : $A = \frac{1}{2}\theta r^2$.

Example 5.1.9: Finding the Area of a Sector

An automatic lawn sprinkler sprays a distance of 20 feet while rotating 30 degrees, as shown in Figure 5.1.24. What is the area of the sector of grass the sprinkler waters?

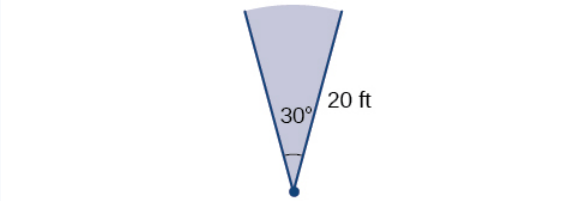


***Figure* 5.1.24**: *The sprinkler sprays 20 ft within an arc of 30°.*

Solution

First, we need to convert the angle measure into radians. Because 30 degrees is one of our special angles, we already know the equivalent radian measure, but we can also convert:

$$30 \text{ degrees} = 30 \cdot \frac{\pi}{180} \tag{5.1.51}$$

$$= \frac{\pi}{6} \text{ radians} \tag{5.1.52}$$

The area of the sector is then

$$\text{Area} = \frac{1}{2}(\frac{\pi}{6})(20)^2 \tag{5.1.53}$$

$$\approx 104.72 \tag{5.1.54}$$

So the area is about 104.72 ft^2.

Try It 5.1.9

In central pivot irrigation, a large irrigation pipe on wheels rotates around a center point. A farmer has a central pivot system with a radius of 400 meters. If water restrictions only allow her to water 150 thousand square meters a day, what angle should she set the system to cover? Write the answer in radian measure to two decimal places.

Answer

1.88

Use Linear and Angular Speed to Describe Motion on a Circular Path

In addition to finding the area of a sector, we can use angles to describe the speed of a moving object. An object traveling in a circular path has two types of speed. **Linear speed** is speed along a straight path and can be determined by the distance it travels (its **displacement**) in a given time interval. For instance, if a wheel with radius 5 inches rotates once a second, a point on the edge of the wheel moves a distance equal to the circumference, or 10π inches, every second. So the linear speed of the point is 10π in/s. The equation for linear speed is as follows, where v is linear speed, s is displacement, and t is time.

$$v = \frac{s}{t} \tag{5.1.55}$$

Angular speed results from circular motion and can be determined by the angle through which a point rotates in a given time interval. In other words, angular speed is angular rotation per unit time. So, for instance, if a gear makes a full rotation every 4 seconds, we can calculate its angular speed as $\frac{360 \text{ degrees}}{4 \text{ seconds}} = 90$ degrees per second. Angular speed can be given in radians per second, rotations per minute, or degrees per hour for example. The equation for angular speed is as follows, where ω (read as omega) is angular speed, θ is the angle traversed, and t is time.

$$\omega = \frac{\theta}{t} \tag{5.1.56}$$

Combining the definition of angular speed with the arc length equation, $s = r\theta$, we can find a relationship between angular and linear speeds. The angular speed equation can be solved for θ, giving $\theta = \omega t$. Substituting this into the arc length equation gives:

$$s = r\theta \tag{5.1.57}$$
$$= r\omega t \tag{5.1.58}$$

Substituting this into the linear speed equation gives:

$$v = \frac{s}{t} \quad = \frac{r\omega t}{t} = r\omega \tag{5.1.59}$$

definition: ANGULAR AND LINEAR SPEED

As a point moves along a circle of radius r, its **angular speed**, ω, is the angular rotation θ per unit time, t.

$$\omega = \frac{\theta}{t} \tag{5.1.60}$$

The **linear speed.** v, of the point can be found as the distance traveled, arc length s, per unit time, t.

$$v = \frac{s}{t} \tag{5.1.61}$$

When the angular speed is measured in radians per unit time, linear speed and angular speed are related by the equation

$$v = r\omega \tag{5.1.62}$$

This equation states that the angular speed in radians, ω, representing the amount of rotation occurring in a unit of time, can be multiplied by the radius r to calculate the total arc length traveled in a unit of time, which is the definition of linear speed.

How To: Given the amount of angle rotation and the time elapsed, calculate the angular speed

1. If necessary, convert the angle measure to radians.
2. Divide the angle in radians by the number of time units elapsed: $\omega = \frac{\theta}{t}$.
3. The resulting speed will be in radians per time unit.

Example 5.1.10: Finding Angular Speed

A water wheel, shown in Figure 5.1.25, completes 1 rotation every 5 seconds. Find the angular speed in radians per second.

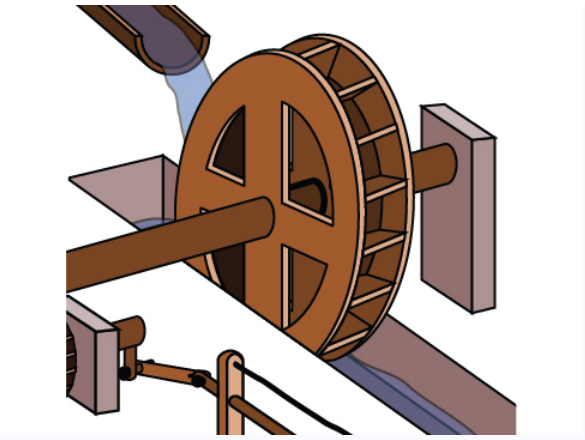


Figure 5.1.25

Solution

The wheel completes 1 rotation, or passes through an angle of 2π radians in 5 seconds, so the angular speed would be $\omega = \frac{2\pi}{5} \approx 1.257$ radians per second.

5.1.10

An old vinyl record is played on a turntable rotating clockwise at a rate of 45 rotations per minute. Find the angular speed in radians per second.

Answer

$-\frac{3\pi}{2}$ rad/s

Given the radius of a circle, an angle of rotation, and a length of elapsed time, determine the linear speed

1. Convert the total rotation to radians if necessary.
2. Divide the total rotation in radians by the elapsed time to find the angular speed: apply $\omega = \frac{\theta}{t}$.
3. Multiply the angular speed by the length of the radius to find the linear speed, expressed in terms of the length unit used for the radius and the time unit used for the elapsed time: apply $v = r\omega$.

Example 5.1.11: Finding a Linear Speed

A bicycle has wheels 28 inches in diameter. A tachometer determines the wheels are rotating at 180 RPM (revolutions per minute). Find the speed the bicycle is traveling down the road.

Solution

The linear speed of the outside of the tires is the speed at which the bicycle travels down the road. Here, we have an angular speed and need to find the corresponding linear speed.

We begin by converting from rotations per minute to radians per minute. It can be helpful to utilize the units (*dimensional analysis*) to make this conversion:

$$180\frac{\cancel{\text{rotations}}}{\text{minute}} \cdot \frac{2\pi \text{ radians}}{\cancel{\text{rotation}}} = 360\pi\frac{\text{radians}}{\text{minute}} \tag{5.1.63}$$

Using the formula from above along with the radius of the wheels, we can find the linear speed:

$$v = (14 \text{ inches})(360\pi\frac{\text{radians}}{\text{minute}}) \tag{5.1.64}$$

$$= 5040\pi\frac{\text{inches}}{\text{minute}} \tag{5.1.65}$$

Remember that radians are a unitless measure, so it is not necessary to include them.

Finally, we may wish to convert this linear speed into a more familiar measurement, like miles per hour. Again, we use dimensional analysis to help with the conversion.

$$5040\pi \frac{\cancel{\text{inches}}}{\cancel{\text{minute}}} \cdot \frac{1\ \cancel{\text{feet}}}{12\ \cancel{\text{inches}}} \cdot \frac{1\ \text{mile}}{5280\ \cancel{\text{feet}}} \cdot \frac{60\ \cancel{\text{minutes}}}{1\ \text{hour}} \approx 14.99\ \text{miles per hour (mph)} \tag{5.1.66}$$

5.1.11

A satellite is rotating around Earth at 0.25 radians per hour at an altitude of 242 km above Earth. If the radius of Earth is 6378 kilometers, find the linear speed of the satellite in kilometers per hour.

Answer

1655 kilometers per hour

Media

Access these online resources for additional instruction and practice with angles, arc length, and areas of sectors.

- Angles in Standard Position
- Angle of Rotation
- Coterminal Angles
- Determining Coterminal Angles
- Positive and Negative Coterminal Angles
- Radian Measure
- Coterminal Angles in Radians
- Arc Length and Area of a Sector

Key Equations

arc length	$s = r\theta$
area of a sector	$A = \frac{1}{2}\theta r^2$
angular speed	$\omega = \frac{\theta}{t}$
linear speed	$v = \frac{s}{t}$
linear speed related to angular speed	$v = r\omega$

Key Concepts

- An angle is formed from the union of two rays, by keeping the initial side fixed and rotating the terminal side. The amount of rotation determines the measure of the angle.
- An angle is in standard position if its vertex is at the origin and its initial side lies along the positive *x*-axis. A positive angle is measured counterclockwise from the initial side and a negative angle is measured clockwise.
- To draw an angle in standard position, draw the initial side along the positive *x*-axis and then place the terminal side according to the fraction of a full rotation the angle represents. See **Example**.
- In addition to degrees, the measure of an angle can be described in radians. See **Example**.
- To convert between degrees and radians, use the proportion $\frac{\theta}{180} = \frac{\theta^R}{\pi}$. See Example and Example.
- Two angles that have the same terminal side are called coterminal angles.
- We can find coterminal angles by adding or subtracting 360° or 2π. See Example and Example.
- Coterminal angles can be found using radians just as they are for degrees. See **Example**.
- The length of a circular arc is a fraction of the circumference of the entire circle. See **Example**.

- The area of sector is a fraction of the area of the entire circle. See **Example**.
- An object moving in a circular path has both linear and angular speed.
- The angular speed of an object traveling in a circular path is the measure of the angle through which it turns in a unit of time. See **Example**.
- The linear speed of an object traveling along a circular path is the distance it travels in a unit of time. See **Example**.

Glossary

angle
the union of two rays having a common endpoint

angular speed
the angle through which a rotating object travels in a unit of time

arc length
the length of the curve formed by an arc

area of a sector
area of a portion of a circle bordered by two radii and the intercepted arc; the fraction $\frac{\theta}{2\pi}$ multiplied by the area of the entire circle

coterminal angles
description of positive and negative angles in standard position sharing the same terminal side

degree
a unit of measure describing the size of an angle as one-360th of a full revolution of a circle

initial side
the side of an angle from which rotation begins

linear speed
the distance along a straight path a rotating object travels in a unit of time; determined by the arc length

measure of an angle
the amount of rotation from the initial side to the terminal side

negative angle
description of an angle measured clockwise from the positive x-axis

positive angle
description of an angle measured counterclockwise from the positive x-axis

quadrantal angle
an angle whose terminal side lies on an axis

radian measure
the ratio of the arc length formed by an angle divided by the radius of the circle

radian
the measure of a central angle of a circle that intercepts an arc equal in length to the radius of that circle

ray
one point on a line and all points extending in one direction from that point; one side of an angle

reference angle

the measure of the acute angle formed by the terminal side of the angle and the horizontal axis

standard position

the position of an angle having the vertex at the origin and the initial side along the positive *x*-axis

terminal side

the side of an angle at which rotation ends

vertex

the common endpoint of two rays that form an angle

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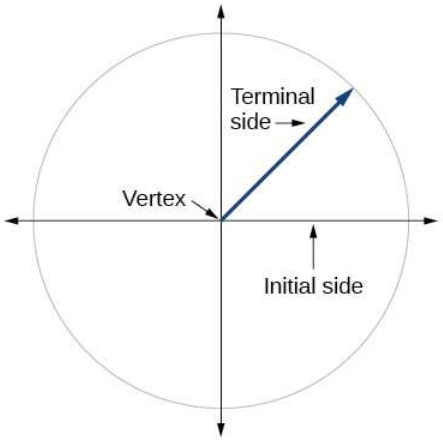
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5.1E: Exercises

Verbal

1) Draw an angle in standard position. Label the vertex, initial side, and terminal side.

Answer



2) Explain why there are an infinite number of angles that are coterminal to a certain angle.

3) State what a positive or negative angle signifies, and explain how to draw each.

Answer

Whether the angle is positive or negative determines the direction. A positive angle is drawn in the counterclockwise direction, and a negative angle is drawn in the clockwise direction.

4) How does radian measure of an angle compare to the degree measure? Include an explanation of 1 radian in your paragraph.

5) Explain the differences between linear speed and angular speed when describing motion along a circular path.

Answer

Linear speed is a measurement found by calculating distance of an arc compared to time. Angular speed is a measurement found by calculating the angle of an arc compared to time.

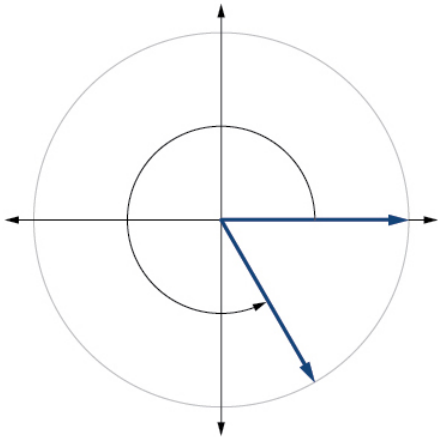
Graphical

For exercises 6-21, draw an angle in standard position with the given measure.

6) $30°$

7) $300°$

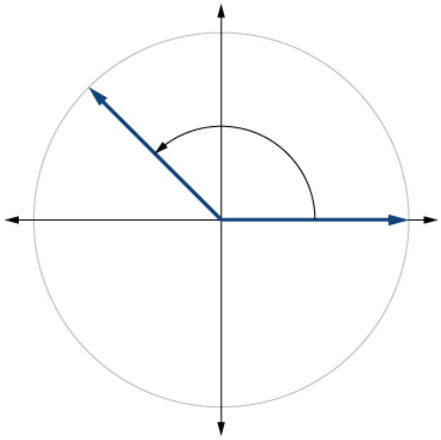
Answer



8) $-80°$

9) $135°$

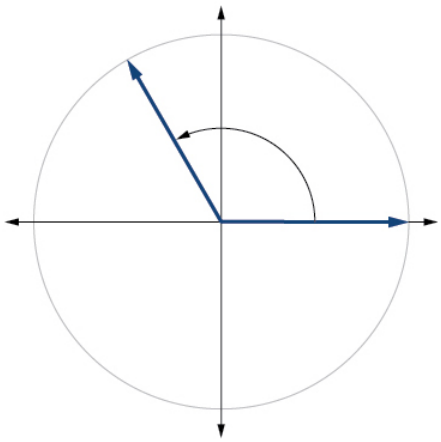
Answer



10) $-150°$

11) $\frac{2\pi}{3}$

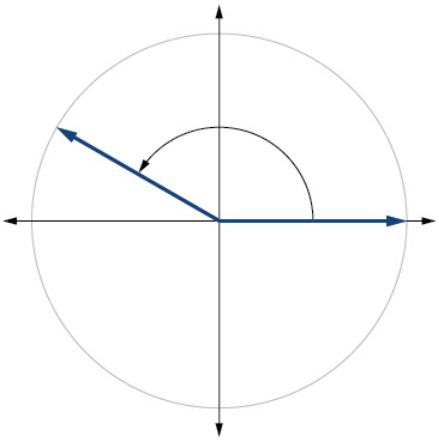
Answer



12) $\frac{7\pi}{4}$

13) $\frac{5\pi}{6}$

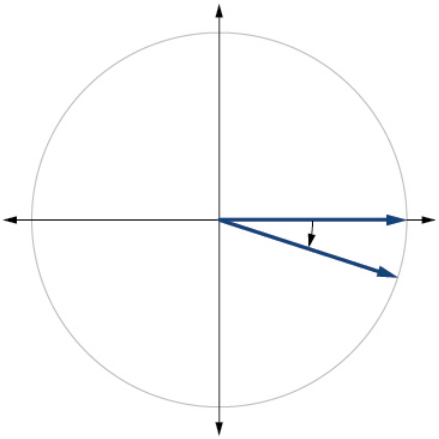
Answer



14) $\frac{\pi}{2}$

15) $-\frac{\pi}{10}$

Answer



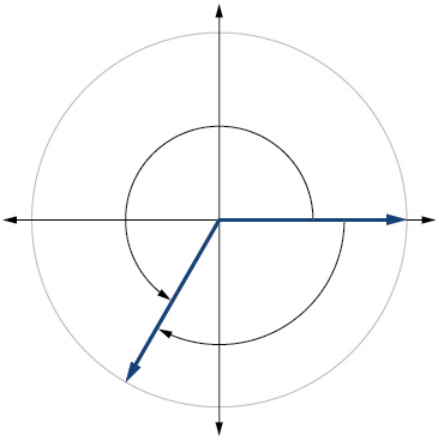
16) 415°

17) -120°

Answer

240°

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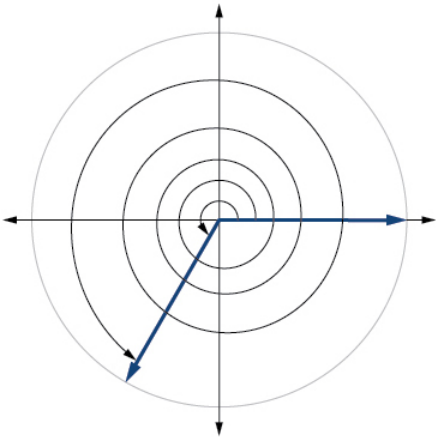


18) -315°

19) $\frac{22\pi}{3}$

Answer

$\frac{4\pi}{3}$

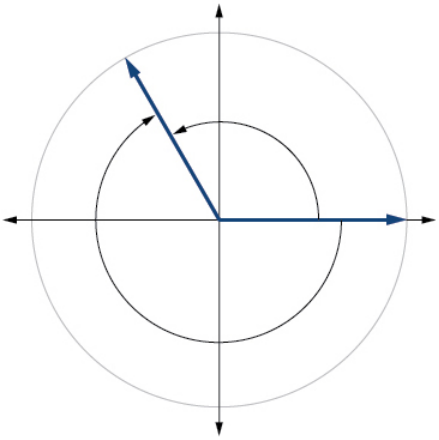


20) $-\frac{\pi}{6}$

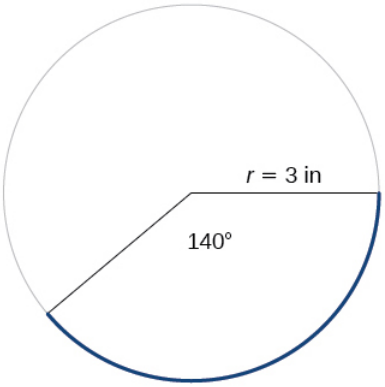
21) $-\frac{4\pi}{3}$

Answer

$\frac{2\pi}{3}$



For the exercises 22-23, refer to Figure below. Round to two decimal places.



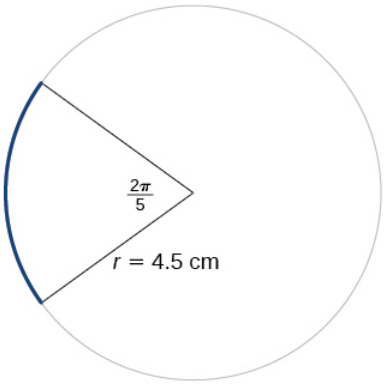
22) Find the arc length.

23) Find the area of the sector.

Answer

$\frac{27\pi}{2} \approx 11.00 \text{ in}^2$

For exercises 24-25, refer to Figure below. Round to two decimal places.



24) Find the arc length.

25) Find the area of the sector.

Answer

$\frac{81\pi}{20} \approx 12.72 \text{ cm}^2$

Algebraic

For exercises 26-32, convert angles in radians to degrees.

26) $\frac{3\pi}{4}$ radians

27) $\frac{\pi}{9}$ radians

Answer

20°

28) $-\frac{5\pi}{4}$ radians

29) $\frac{\pi}{3}$ radians

Answer

60°

30) $-\frac{7\pi}{3}$ radians

31) $-\frac{5\pi}{12}$ radians

Answer

-75°

32) $\frac{11\pi}{6}$ radians

For exercises 33-39, convert angles in degrees to radians.

33) 90°

Answer

$\frac{\pi}{2}$ radians

34) 100°

35) -540°

Answer

-3π radians

36) -120°

37) 180°

Answer

π radians

38) -315°

39) 150°

Answer

$\frac{5\pi}{6}$ radians

For exercises 40-45, use to given information to find the length of a circular arc. Round to two decimal places.

40) Find the length of the arc of a circle of radius 12 inches subtended by a central angle of $\frac{\pi}{4}$ radians.

41) Find the length of the arc of a circle of radius 5.02 miles subtended by the central angle of $\frac{\pi}{3}$.

Answer

$\frac{5.02\pi}{3} \approx 5.26$ miles

42) Find the length of the arc of a circle of diameter 14 meters subtended by the central angle of $\frac{5\pi}{6}$.

43) Find the length of the arc of a circle of radius 10 centimeters subtended by the central angle of $50°$.

Answer

$\frac{25\pi}{9} \approx 8.73$ centimeters

44) Find the length of the arc of a circle of radius 5 inches subtended by the central angle of 220^{circ}.

45) Find the length of the arc of a circle of diameter 12 meters subtended by the central angle is 63^{circ}.

Answer

$\frac{21\pi}{10} \approx 6.60$ meters

For exercises 46-49, use the given information to find the area of the sector. Round to four decimal places.

46) A sector of a circle has a central angle of $45°$ and a radius 6 cm.

47) A sector of a circle has a central angle of $30°$ and a radius of 20 cm.

Answer

$104.7198\ cm^2$

48) A sector of a circle with diameter 10 feet and an angle of $\frac{\pi}{2}$ radians.

49) A sector of a circle with radius of 0.7 inches and an angle of π radians.

Answer

$0.7697\ in^2$

For exercises 50-53, find the angle between $0°$ and $360°$ that is coterminal to the given angle.

50) $-40°$

51) $-110°$

Answer

$250°$

52) $700°$

53) $1400°$

Answer

$320°$

For exercises 54-57, find the angle between 0 and 2π in radians that is coterminal to the given angle.

54) $-\frac{\pi}{9}$

55) $\frac{10\pi}{3}$

Answer

$\frac{4\pi}{3}$

56) $\frac{13\pi}{6}$

57) $\frac{44\pi}{9}$

Answer

$\frac{8\pi}{9}$

Real-World Applications

58) A truck with 32-inch diameter wheels is traveling at 60 mi/h. Find the angular speed of the wheels in rad/min. How many revolutions per minute do the wheels make?

59) A bicycle with 24-inch diameter wheels is traveling at 15 mi/h. Find the angular speed of the wheels in rad/min. How many revolutions per minute do the wheels make?

Answer

1320 rad 210.085 RPM

60) A wheel of radius 8 inches is rotating $15°/s$. What is the linear speed v, the angular speed in RPM, and the angular speed in rad/s?

61) A wheel of radius 14 inches is rotating 0.5rad/s. What is the linear speed v, the angular speed in RPM, and the angular speed in deg/s?

Answer

7 in./s, 4.77 RPM, 28.65 deg/s

62) A CD has diameter of 120 millimeters. When playing audio, the angular speed varies to keep the linear speed constant where the disc is being read. When reading along the outer edge of the disc, the angular speed is about 200 RPM (revolutions per minute). Find the linear speed.

63) When being burned in a writable CD-R drive, the angular speed of a CD is often much faster than when playing audio, but the angular speed still varies to keep the linear speed constant where the disc is being written. When writing along the outer edge of the disc, the angular speed of one drive is about 4800 RPM (revolutions per minute). Find the linear speed if the CD has diameter of 120 millimeters.

Answer

$1,809,557.37\ \text{mm/min} = 30.16\ \text{m/s}$

64) A person is standing on the equator of Earth (radius 3960 miles). What are his linear and angular speeds?

65) Find the distance along an arc on the surface of Earth that subtends a central angle of 5 minutes ($1\ \text{minute} = \frac{1}{60}\ \text{degree}$). The radius of Earth is 3960 miles.

Answer

5.76 miles

66) Find the distance along an arc on the surface of Earth that subtends a central angle of 7 minutes ($1 \text{ minute} = \frac{1}{60} \text{ degree}$). The radius of Earth is 3960 miles.

67) Consider a clock with an hour hand and minute hand. What is the measure of the angle the minute hand traces in 20 minutes?

Answer

$120°$

Extensions

68) Two cities have the same longitude. The latitude of city A is 9.00 degrees north and the latitude of city B is 30.00 degree north. Assume the radius of the earth is 3960 miles. Find the distance between the two cities.

69) A city is located at 40 degrees north latitude. Assume the radius of the earth is 3960 miles and the earth rotates once every 24 hours. Find the linear speed of a person who resides in this city.

Answer

794 miles per hour

70) A city is located at 75 degrees north latitude. Assume the radius of the earth is 3960 miles and the earth rotates once every 24 hours. Find the linear speed of a person who resides in this city.

71) Find the linear speed of the moon if the average distance between the earth and moon is $239,000$ miles, assuming the orbit of the moon is circular and requires about 28 days. Express answer in miles per hour.

Answer

$2,234$ miles per hour

72) A bicycle has wheels 28 inches in diameter. A tachometer determines that the wheels are rotating at 180 RPM (revolutions per minute). Find the speed the bicycle is traveling down the road.

73) A car travels 3 miles. Its tires make 2640 revolutions. What is the radius of a tire in inches?

Answer

11.5 inches

74) A wheel on a tractor has a 24-inch diameter. How many revolutions does the wheel make if the tractor travels 4 miles?

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5.2: Unit Circle - Sine and Cosine Functions

Skills to Develop

- Find function values for the sine and cosine of $30°$ or $\frac{\pi}{6}$, $45°$ or $\frac{\pi}{4}$, and $60°$ or $\frac{\pi}{3}$.
- Identify the domain and range of sine and cosine functions.
- Find reference angles.
- Use reference angles to evaluate trigonometric functions.

Looking for a thrill? Then consider a ride on the Singapore Flyer, the world's tallest Ferris wheel. Located in Singapore, the Ferris wheel soars to a height of 541 feet—a little more than a tenth of a mile! Described as an observation wheel, riders enjoy spectacular views as they travel from the ground to the peak and down again in a repeating pattern. In this section, we will examine this type of revolving motion around a circle. To do so, we need to define the type of circle first, and then place that circle on a coordinate system. Then we can discuss circular motion in terms of the coordinate pairs.



***Figure* 5.2.1:** *The Singapore Flyer is the world's tallest Ferris wheel. (credit: "Vibin JK"/Flickr)*

Finding Function Values for the Sine and Cosine

To define our trigonometric functions, we begin by drawing a unit circle, a circle centered at the origin with radius 1, as shown in Figure 5.2.2. The angle (in radians) that t intercepts forms an arc of length s. Using the formula $s = rt$, and knowing that $r = 1$, we see that for a **unit circle**, $s = t$.

For any angle t, we can label the intersection of the terminal side and the unit circle by its coordinates, (x, y). The coordinates x and y will be the outputs of the trigonometric functions $f(t) = \cos t$ and $f(t) = \sin t$, respectively. This means $x = \cos t$ and $y = \sin t$. The actual function names are "cosine" and "sine", but we use the notation $\cos(t)$ and $\sin(t)$.

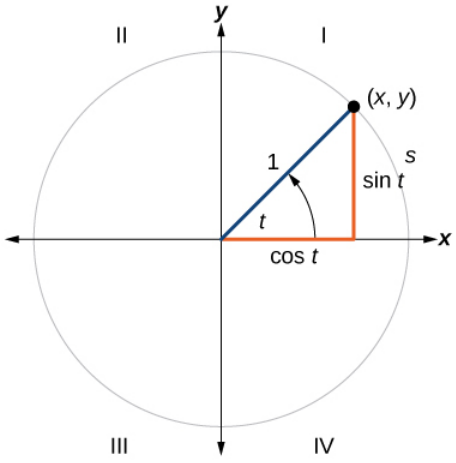


***Figure* 5.2.2:** Unit circle where the central angle is t radians

UNIT CIRCLE

A unit circle has a center at $(0,0)$ and radius 1. In a unit circle, the length of the intercepted arc is equal to the radian measure of the central angle 1.

Let (x,y) be the endpoint on the unit circle of an arc of arc length s. The (x,y) coordinates of this point depend on the angle. Hence, each coordinate can be described as a function of the angle.

Defining Sine and Cosine Functions

Now that we have our unit circle labeled, we can learn how the (x,y) coordinates relate to the **arc length** and **angle**. The **sine function** relates a real number t to the y-coordinate of the point where the corresponding angle intercepts the unit circle. More precisely, the sine of an angle t equals the y-value of the endpoint on the unit circle of an arc of length t. Like all functions, the sine function has an input and an output. Its input is the measure of the angle; its output is the y-coordinate of the corresponding point on the unit circle. In Figure 5.2.3, the sine is equal to y.

Similarly, the output of the **cosine function** of an angle t equals the x-value of the endpoint on the unit circle of an arc of length t. In Figure 5.2.3, the cosine is equal to x.

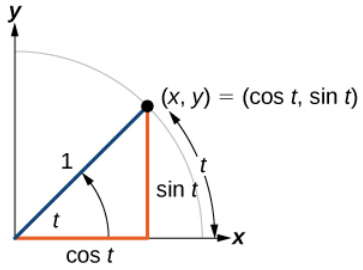


***Figure* 5.2.3**

Because it is understood that sine and cosine are functions, we do not always write them with parentheses: $\sin t$ is the same as $\sin(t)$ and $\cos t$ is the same as $\cos(t)$. Likewise, $\cos^2 t$ is a commonly used shorthand notation for $(\cos(t))^2$. Be aware that many calculators and computers do not recognize the shorthand notation. When in doubt, use the extra parentheses when entering calculations into a calculator or computer.

definition: SINE AND COSINE FUNCTIONS

If t is a real number and a point (x,y) on the unit circle corresponds to an angle in standard position of t radians, then

$$\cos t = x \tag{5.2.1}$$
$$\sin t = y. \tag{5.2.2}$$

How To: Given a point $P(x,y)$ on the unit circle corresponding to an angle of t, find the sine and cosine

1. The sine of t is equal to the y-coordinate of point P: $\sin\ t = y$.
2. The cosine of t is equal to the x-coordinate of point P: $\cos\ t = x$.

Example 5.2.1: Finding Function Values for Sine and Cosine

Point P is a point on the unit circle corresponding to an angle of t, as shown in Figure 5.2.4. Find $\cos(t)$ and $\sin(t)$.

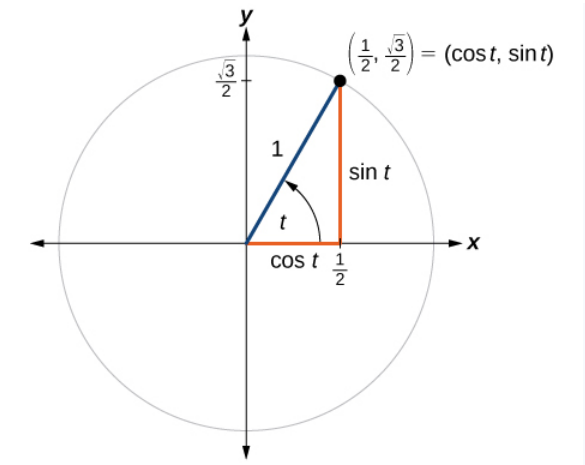


***Figure* 5.2.4**

Solution

We know that $\cos t$ is the x-coordinate of the corresponding point on the unit circle and $\sin t$ is the y-coordinate of the corresponding point on the unit circle. So:

$$x = \cos t = \frac{1}{2} \tag{5.2.3}$$

$$y = \sin t = \frac{\sqrt{3}}{2} \tag{5.2.4}$$

Analysis: Because the unit circle has radius 1, we can think of both the x- and y-coordinates as ratios of a right triangle, if angle t is acute. The length of the hypotenuse is 1, and the legs have length x and y. Thus,

\[\begin{align \cos t = x &= \frac{x}{1} \ \sin t = y &= \frac{y}{1}.\]

For a given angle t, would the x, y-coordinates change if we used a circle of radius 2, or 3? Yes. Would the ratios change? No.

5.2.1:

A certain angle t corresponds to a point on the unit circle at $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ as shown in Figure 5.2.5. Find $\cos t$ and $\sin t$.

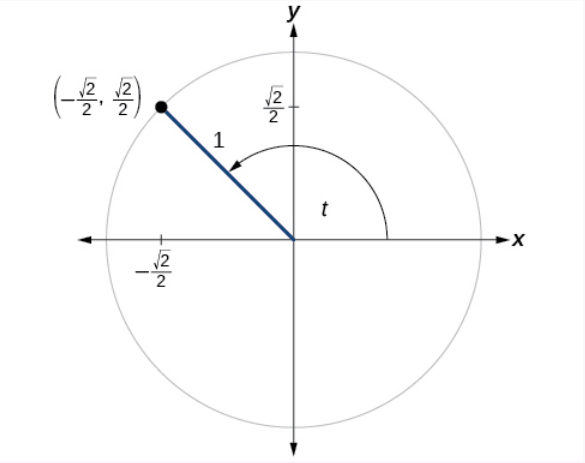


***Figure* 5.2.5**

Answer

$\cos(t) = -\frac{\sqrt{2}}{2}, \sin(t) = \frac{\sqrt{2}}{2}$

Finding Sines and Cosines of Angles on an Axis

For quadrantral angles, the corresponding point on the unit circle falls on the *x*- or *y*-axis. In that case, we can easily calculate cosine and sine from the values of x and y.

Example 5.2.2: Calculating Sines and Cosines along an Axis

Find $\cos(90°)$ and $\sin(90°)$.

Solution

Moving $90°$ counterclockwise around the unit circle from the positive *x*-axis brings us to the top of the circle, where the (x, y) coordinates are (0, 1), as shown in Figure 5.2.6.

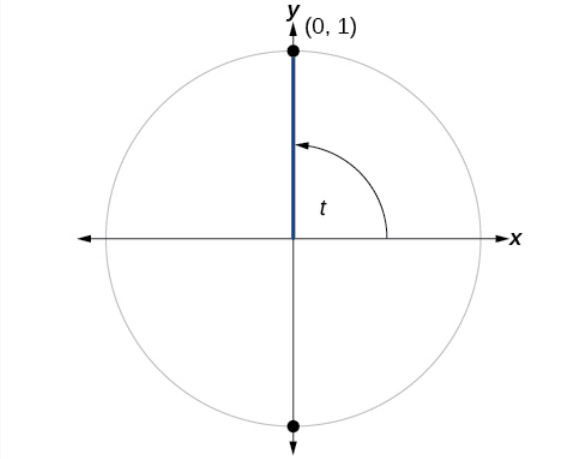


***Figure* 5.2.6**

Using our definitions of cosine and sine,

$$x = \cos t = \cos(90°) = 0 \quad (5.2.5)$$
$$y = \sin t = \sin(90°) = 1 \quad (5.2.6)$$

The cosine of 90° is 0; the sine of 90° is 1.

5.2.2

Find cosine and sine of the angle π.

Answer

$\cos(\pi) = -1,\ \sin(\pi) = 0$

The Pythagorean Identity

Now that we have defined sine and cosine, we will learn how they relate to each other and the unit circle. The equation for the unit circle is $x^2 + y^2 = 1$. Because $x = \cos t$ and $y = \sin t$, we can substitute for x and y to get $\cos^2 t + \sin^2 t = 1$. This equation, $\cos^2 t + \sin^2 t = 1$, is known as the **Pythagorean Identity**. See Figure 5.2.7.

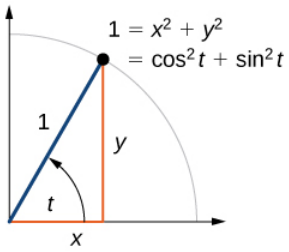


***Figure* 5.2.7**

We can use the Pythagorean Identity to find the cosine of an angle if we know the sine, and vice versa. However, because the equation yields two solutions, we need additional knowledge of the angle to choose the solution with the correct sign. If we know the quadrant where the angle is, we can easily choose the correct solution.

PYTHAGOREAN IDENTITY

The **Pythagorean Identity** states that, for any real number t,

$$\cos^2 t + \sin^2 t = 1 \tag{5.2.7}$$

How To: Given the sine of some angle t and its quadrant location, find the cosine of t.

1. Substitute the known value of $\sin(t)$ into the Pythagorean Identity.
2. Solve for $\cos(t)$.
3. Choose the solution with the appropriate sign for the x-values in the quadrant where t is located.

Example 5.2.3: Finding a Cosine from a Sine or a Sine from a Cosine

If $\sin(t) = \frac{3}{7}$ and t terminates in the second quadrant, find $\cos(t)$.

Solution

If we drop a vertical line from the point on the unit circle corresponding to the terminal side of t, we create a right triangle, from which we can see that the Pythagorean Identity is simply one case of the Pythagorean Theorem. See Figure 5.2.8.

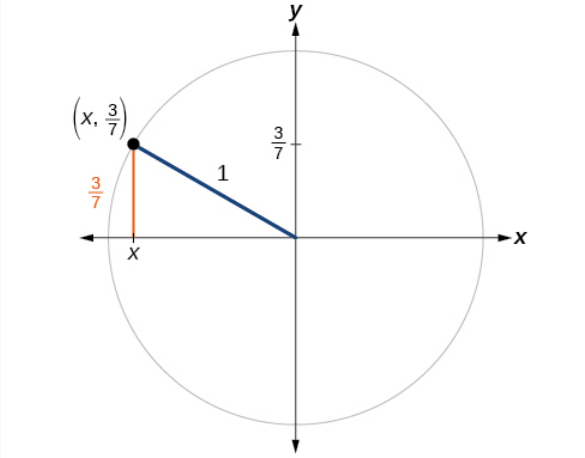


***Figure* 5.2.8**

Substituting the known value for sine into the Pythagorean Identity,

$$\cos^2(t) + \sin^2(t) = 1 \tag{5.2.8}$$

$$\cos^2(t) + \frac{9}{49} = 1 \tag{5.2.9}$$

$$\cos^2(t) = \frac{40}{49} \tag{5.2.10}$$

$$\cos(t) = \pm\sqrt{\frac{40}{49}} = \pm\frac{\sqrt{40}}{7} = \pm\frac{2\sqrt{10}}{7} \tag{5.2.11}$$

Because the angle terminates in the second quadrant, we know the x-value is a negative real number, so the cosine is also negative. So

$$\cos(t) = -\frac{2\sqrt{10}}{7}. \tag{5.2.12}$$

5.2.3

If $\cos(t) = \frac{24}{25}$ and tt is in the fourth quadrant, find $\sin(t)$.

Answer

$\sin(t) = -\frac{7}{25}$

Finding Sines and Cosines of Special Angles

We have already learned some properties of the special angles, such as the conversion from radians to degrees. We can also calculate sines and cosines of the special angles using the **Pythagorean Identity** and our knowledge of triangles.

Finding Sines and Cosines of 45° Angles

First, we will look at angles of $45°$ or $\frac{\pi}{4}$, as shown in Figure 5.2.9 . A $45°$–$45°$–$90°$ triangle is an isosceles triangle, so the x- and y-coordinates of the corresponding point on the circle are the same. Because the x- and y-values are the same, the sine and cosine values will also be equal.

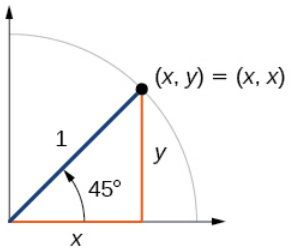


***Figure* 5.2.9**

At $t = \frac{\pi}{4}$, which is 45 degrees, the radius of the unit circle bisects the first quadrant. This means the radius lies along the line $y = x$. A unit circle has a radius equal to 1. So, the right triangle formed below the line $y = x$ has sides x and y $(y = x)$, and a radius = 1. See Figure 5.2.10.

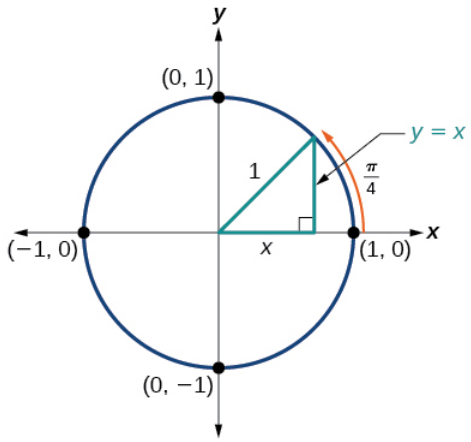


***Figure* 5.2.10**

From the Pythagorean Theorem we get

$$x^2 + y^2 = 1 \tag{5.2.13}$$

Substituting $y = x$, we get

$$x^2 + x^2 = 1 \tag{5.2.14}$$

Combining like terms we get

$$2x^2 = 1 \tag{5.2.15}$$

And solving for x, we get

$$x^2 = \frac{1}{2} \tag{5.2.16}$$

$$x = \pm\frac{1}{\sqrt{2}} \tag{5.2.17}$$

In Quadrant I, $x = \frac{1}{\sqrt{2}}$.

At $t = \frac{\pi}{4}$ or $45°$,

$$(x, y) = (x, x) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \tag{5.2.18}$$

$$x = \frac{1}{\sqrt{2}},\ y = \frac{1}{\sqrt{2}} \tag{5.2.19}$$

$$\cos t = \frac{1}{\sqrt{2}},\ \sin t = \frac{1}{\sqrt{2}} \tag{5.2.20}$$

If we then rationalize the denominators, we get

$$\cos t = \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} \tag{5.2.21}$$

$$= \frac{\sqrt{2}}{2} \approx 0.707 \tag{5.2.22}$$

$$\sin t = \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} \tag{5.2.23}$$

$$= \frac{\sqrt{2}}{2} \approx 0.707 \tag{5.2.24}$$

Therefore, the (x, y) coordinates of a point on a circle of radius 1 at an angle of $45°$ are $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Finding Sines and Cosines of 30° and 60° Angles

Next, we will find the cosine and sine at an angle of $30°$, or $\frac{\pi}{6}$. First, we will draw a triangle inside a circle with one side at an angle of $30°$,and another at an angle of $-30°$, as shown in Figure 5.2.11. If the resulting two right triangles are combined into one large triangle, notice that all three angles of this larger triangle will be $60°$, as shown in Figure 5.2.12.

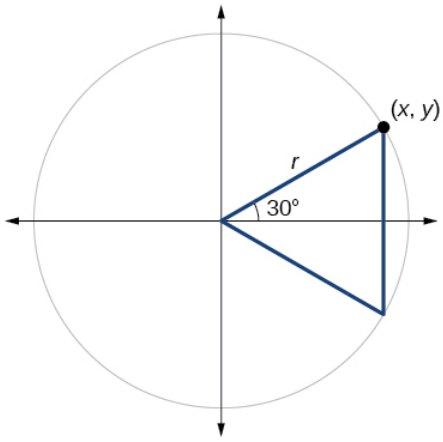


***Figure* 5.2.11**

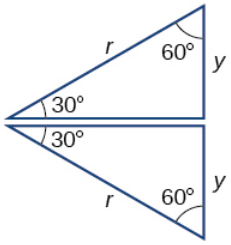


***Figure* 5.2.12**

Because all the angles are equal, the sides are also equal. The vertical line has length $2y$, and since the sides are all equal, we can also conclude that $r = 2y$ or $y = \frac{1}{2}r$. Since $\sin\, t = y$,

$$\sin(\frac{\pi}{6}) = \frac{1}{2}r \tag{5.2.25}$$

And since $r = 1$ in our **unit circle**,

$$\sin(\frac{\pi}{6}) = \frac{1}{2}(1) \tag{5.2.26}$$

$$= \frac{1}{2} \tag{5.2.27}$$

Using the Pythagorean Identity, we can find the cosine value.

$$\cos^2\tfrac{\pi}{6} + \sin^2\left(\tfrac{\pi}{6}\right) = 1$$

$$\cos^2\left(\tfrac{\pi}{6}\right) + \left(\tfrac{1}{2}\right)^2 = 1$$

$$\cos^2\left(\tfrac{\pi}{6}\right) = \tfrac{3}{4} \qquad \text{Use the square root property.}$$

$$\cos\left(\tfrac{\pi}{6}\right) = \tfrac{\pm\sqrt{3}}{\pm\sqrt{4}} = \tfrac{\sqrt{3}}{2} \qquad \text{Since\(y\)is positive, choose the positive root.}$$

The (x, y) coordinates for the point on a circle of radius 1 at an angle of $30°$ are $(\frac{\sqrt{3}}{2}, \frac{1}{2})$. At $t = \frac{\pi}{3}$ (or 60°), the radius of the unit circle, 1, serves as the hypotenuse of a 30-60-90 degree right triangle, BAD, as shown in Figure 5.2.13. Angle A has measure 60°.

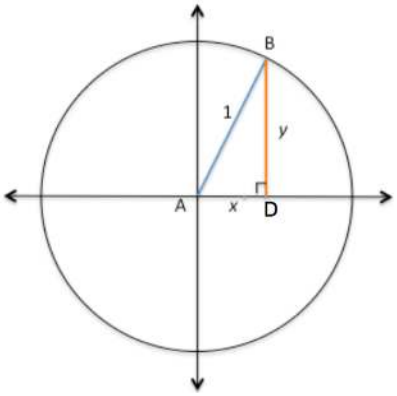


***Figure* 5.2.13**

The measure of angle ABD is 30°. Notice that AD is the x-coordinate of point B, which is at the intersection of the terminal side of the 60° angle and the unit circle. This gives us a triangle BAD with hypotenuse of 1 and side x of length $\frac{1}{2}$.

From the Pythagorean Theorem, we get

$$x^2 + y^2 = 1 \tag{5.2.28}$$

Substituting $x = \frac{1}{2}$, we get

$$\left(\frac{1}{2}\right)^2 + y^2 = 1 \tag{5.2.29}$$

Solving for y, we get

$$\frac{1}{4} + y^2 = 1 \tag{5.2.30}$$

$$y^2 = 1 - \frac{1}{4} \tag{5.2.31}$$

$$y^2 = \frac{3}{4} \tag{5.2.32}$$

$$y = \pm\frac{\sqrt{3}}{2} \tag{5.2.33}$$

Since $t = \frac{\pi}{3}$ has the terminal side in Quadrant I where the y-coordinate is positive, we choose $y = \frac{\sqrt{3}}{2}$, the positive value.

At $t = \frac{\pi}{3}$ (60°), the (x, y) coordinates for the point on a circle of radius 1 at an angle of 60° are $(\frac{1}{2}, \frac{\sqrt{3}}{2})$, so we can find the sine and cosine.

$$(x, y) = \left(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\right)$$
$$x = \tfrac{1}{2}, y = \tfrac{\sqrt{3}}{2}$$
$$\cos t = \tfrac{1}{2}, \sin t = \tfrac{\sqrt{3}}{2}$$

We have now found the cosine and sine values for all of the most commonly encountered angles in the first quadrant of the unit circle. Table 5.2.1 summarizes these values.

***Table* 5.2.1**

Angle	0	$\frac{\pi}{6}$, or 30°	$\frac{\pi}{4}$, or 45°	$\frac{\pi}{3}$, or 60°	$\frac{\pi}{2}$, or 90°
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Figure shows the common angles in the first quadrant of the unit circle.

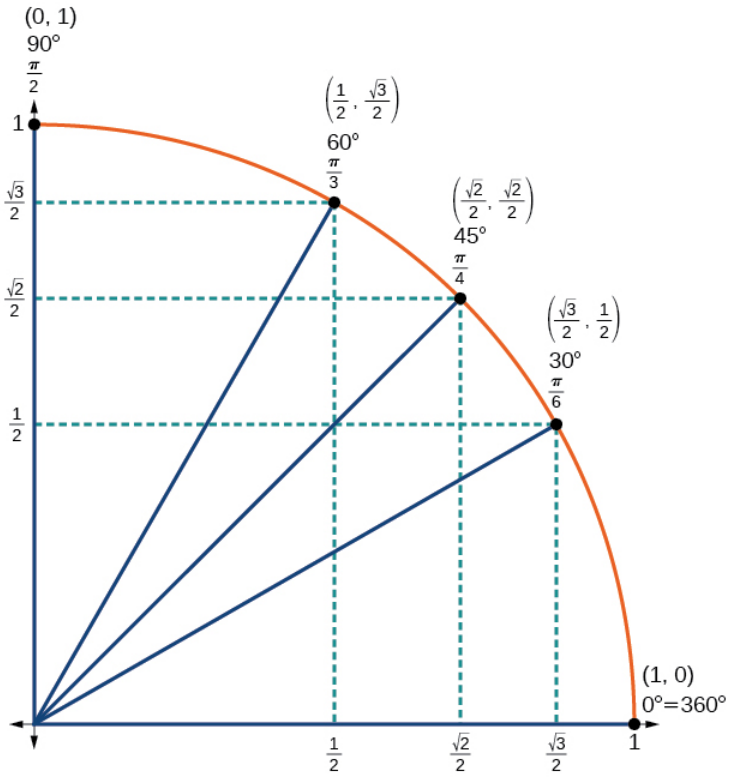


***Figure* 5.2.14**

Using a Calculator to Find Sine and Cosine

To find the cosine and sine of angles other than the **special angles**, we turn to a computer or calculator. **Be aware**: Most calculators can be set into "degree" or "radian" mode, which tells the calculator the units for the input value. When we evaluate $\cos(30)$ on our calculator, it will evaluate it as the cosine of 30 degrees if the calculator is in degree mode, or the cosine of 30 radians if the calculator is in radian mode.

How To: Given an angle in radians, use a graphing calculator to find the cosine

1. If the calculator has degree mode and radian mode, set it to radian mode.
2. Press the COS key.
3. Enter the radian value of the angle and press the close-parentheses key ")".
4. Press ENTER.

Example 5.2.4: Using a Graphing Calculator to Find Sine and Cosine

Evaluate $\cos(\frac{5\pi}{3})$ using a graphing calculator or computer.

Solution

Enter the following keystrokes:

$\text{COS}(5 \times \pi \div 3)\text{ENTER}$

$$\cos(\frac{5\pi}{3}) = 0.5 \tag{5.2.34}$$

Analysis

We can find the cosine or sine of an angle in degrees directly on a calculator with degree mode. For calculators or software that use only radian mode, we can find the sign of $20°$, for example, by including the conversion factor to radians as part of the input:

$$\text{SIN}(20 \times \pi \div 180)\text{ENTER} \tag{5.2.35}$$

Exercise 5.2.4

Evaluate $\sin\left(\frac{\pi}{3}\right)$.

Solution

approximately 0.866025403

Identifying the Domain and Range of Sine and Cosine Functions

Now that we can find the sine and cosine of an angle, we need to discuss their domains and ranges. What are the domains of the sine and cosine functions? That is, what are the smallest and largest numbers that can be inputs of the functions? Because angles smaller than 0 and angles larger than 2π 2π can still be graphed on the unit circle and have real values of x, y, and r, there is no lower or upper limit to the angles that can be inputs to the sine and cosine functions. The input to the sine and cosine functions is the rotation from the positive *x*-axis, and that may be any real number.

What are the ranges of the sine and cosine functions? What are the least and greatest possible values for their output? We can see the answers by examining the **unit circle**, as shown in Figure. The bounds of the *x*-coordinate are $[-1, 1]$. The bounds of the *y*-coordinate are also $[-1, 1]$. Therefore, the range of both the sine and cosine functions is $[-1, 1]$.

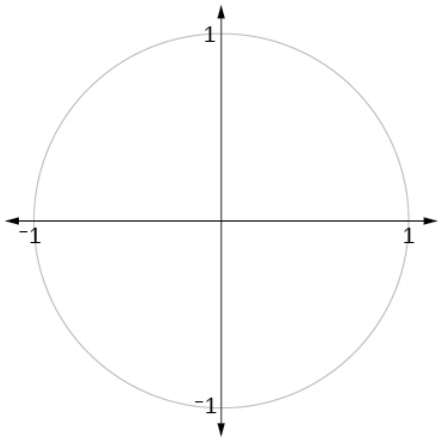


***Figure* 5.2.15**

Finding Reference Angles

We have discussed finding the sine and cosine for angles in the first quadrant, but what if our angle is in another quadrant? For any given angle in the first quadrant, there is an angle in the second quadrant with the same sine value. Because the sine value is the *y*-coordinate on the unit circle, the other angle with the same sine will share the same *y*-value, but have the opposite *x*-value. Therefore, its cosine value will be the opposite of the first angle's cosine value.

Likewise, there will be an angle in the fourth quadrant with the same cosine as the original angle. The angle with the same cosine will share the same *x*-value but will have the opposite *y*-value. Therefore, its sine value will be the opposite of the original angle's sine value.

As shown in Figure, angle α has the same sine value as angle t; the cosine values are opposites. Angle β has the same cosine value as angle t; the sine values are opposites.

$$\begin{aligned} \sin(t) &= \sin(\alpha) \quad \text{and} \quad \cos(t) = -\cos(\alpha) \\ \sin(t) &= -\sin(\beta) \quad \text{and} \quad \cos(t) = \cos(\beta) \end{aligned}$$

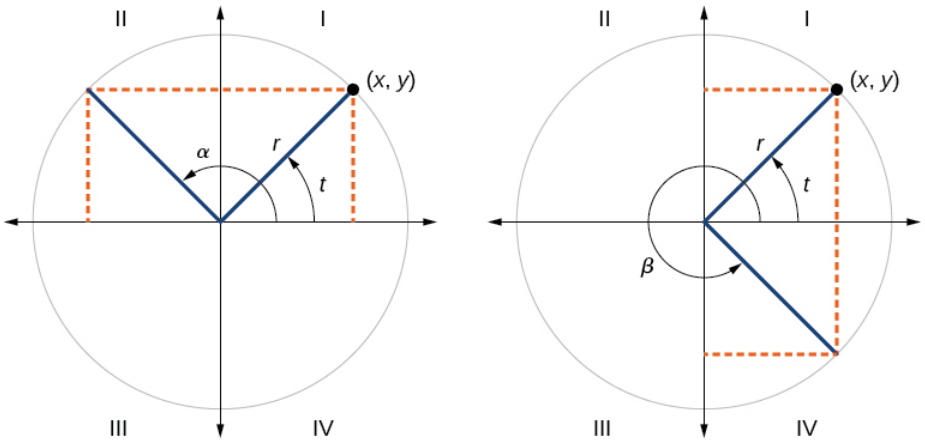


***Figure* 5.2.16**

Recall that an angle's reference angle is the acute angle, t, formed by the terminal side of the angle t and the horizontal axis. A reference angle is always an angle between 0 and $90°$, or 0 and $\frac{\pi}{2}$ radians. As we can see from Figure, for any angle in quadrants II, III, or IV, there is a reference angle in quadrant I.

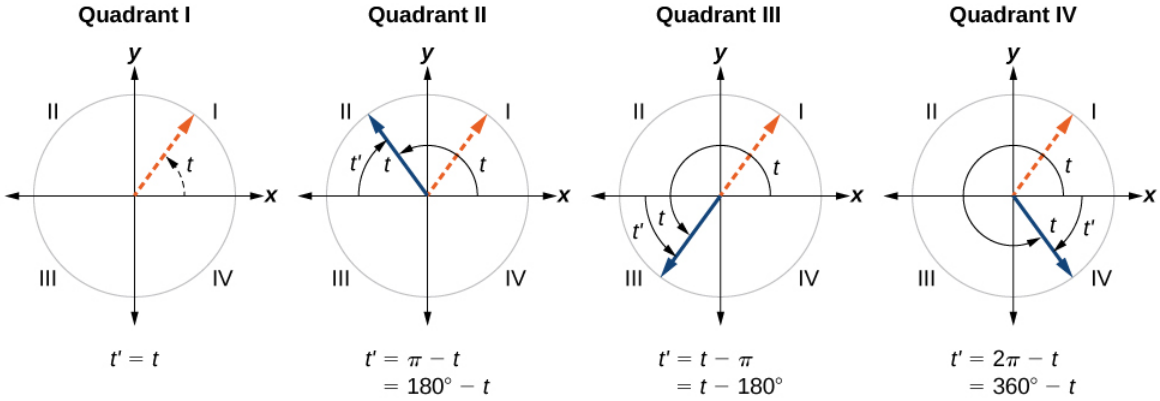


***Figure* 5.2.17**

How To: Given an angle between 0 and 2π, find its reference angle

1. An angle in the first quadrant is its own reference angle.
2. For an angle in the second or third quadrant, the reference angle is $|\pi - t|$ or $|180° - t|$.
3. For an angle in the fourth quadrant, the reference angle is $2\pi - t$ or $360° - t$.
4. If an angle is less than 0 or greater than 2π, add or subtract 2π as many times as needed to find an equivalent angle between 0 and 2π.

Example 5.2.5: Finding a Reference Angle

Find the reference angle of $225°$ as shown in Figure.

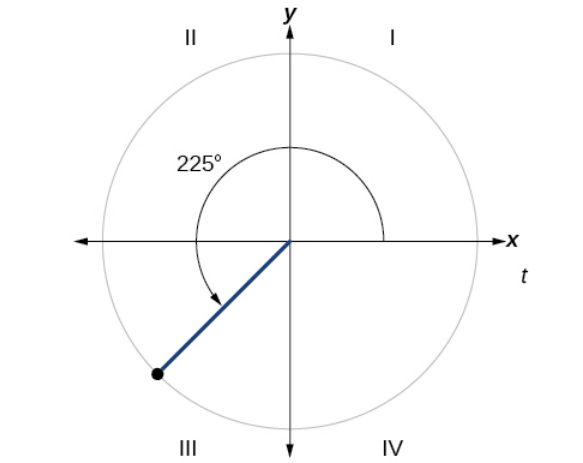


***Figure* 5.2.18**

Solution

Because $225°$ is in the third quadrant, the reference angle is

$$|(180°-225°)| = |-45°| = 45° \tag{5.2.36}$$

Exercise 5.2.5:

Find the reference angle of $\frac{5\pi}{3}$.

Solution

$\frac{\pi}{3}$

Using Reference Angles

Now let's take a moment to reconsider the Ferris wheel introduced at the beginning of this section. Suppose a rider snaps a photograph while stopped twenty feet above ground level. The rider then rotates three-quarters of the way around the circle. What is the rider's new elevation? To answer questions such as this one, we need to evaluate the sine or cosine functions at angles that are greater than 90 degrees or at a **negative angle**. Reference angles make it possible to evaluate trigonometric functions for angles outside the first quadrant. They can also be used to find (x, y) coordinates for those angles. We will use the **reference angle** of the angle of rotation combined with the quadrant in which the terminal side of the angle lies.

Using Reference Angles to Evaluate Trigonometric Functions

We can find the cosine and sine of any angle in any quadrant if we know the cosine or sine of its reference angle. The absolute values of the cosine and sine of an angle are the same as those of the reference angle. The sign depends on the quadrant of the original angle. The cosine will be positive or negative depending on the sign of the *x*-values in that quadrant. The sine will be positive or negative depending on the sign of the *y*-values in that quadrant.

USING REFERENCE ANGLES TO FIND COSINE AND SINE

Angles have cosines and sines with the same absolute value as cosines and sines of their reference angles. The sign (positive or negative) can be determined from the quadrant of the angle.

How To: Given an angle in standard position, find the reference angle, and the cosine and sine of the original angle

1. Measure the angle between the terminal side of the given angle and the horizontal axis. That is the reference angle.
2. Determine the values of the cosine and sine of the reference angle.
3. Give the cosine the same sign as the *x*-values in the quadrant of the original angle.
4. Give the sine the same sign as the *y*-values in the quadrant of the original angle.

Example 5.2.6: Using Reference Angles to Find Sine and Cosine

1. Using a reference angle, find the exact value of $\cos(150°)$ and $\sin(150°)$.
2. Using the reference angle, find $\cos\frac{5\pi}{4}$ and $\sin\frac{5\pi}{4}$.

Solution

This tells us that 150° has the same sine and cosine values as 30°, except for the sign. We know that

1. 150° is located in the second quadrant. The angle it makes with the *x*-axis is 180° − 150° = 30°, so the reference angle is 30°.

$$\cos(30°) = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin(30°) = \frac{1}{2}.$$

Since 150° is in the second quadrant, the *x*-coordinate of the point on the circle is negative, so the cosine value is negative. The *y*-coordinate is positive, so the sine value is positive.

$$\cos(150°) = -\frac{\sqrt{3}}{2} \quad \text{and} \quad \sin(150°) = \frac{1}{2}$$

2. $\frac{5\pi}{4}$ is in the third quadrant. Its reference angle is $\frac{5\pi}{4} - \pi = \frac{\pi}{4}$. The cosine and sine of $\frac{\pi}{4}$ are both $\frac{sqrt2}{2}$. In the third quadrant, both x x and y y are negative, so:

$$\cos\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} \quad \text{and} \quad \sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} \tag{5.2.37}$$

Exercise 5.2.6

1. Use the reference angle of $315°$ to find $\cos(315°)$ and $\sin(315°)$.
2. Use the reference angle of $-\frac{\pi}{6}$ to find $\cos(-\frac{\pi}{6})$ and $\sin(-\frac{\pi}{6})$.

Solution

$\cos(315°) = \frac{\sqrt{2}}{2}, \sin(315°) = \frac{-\sqrt{2}}{2}$

$\cos(-\frac{\pi}{6}) = \frac{\sqrt{3}}{2}, \sin(-\frac{\pi}{6}) = -\frac{1}{2}$

Using Reference Angles to Find Coordinates

Now that we have learned how to find the cosine and sine values for special angles in the first quadrant, we can use symmetry and reference angles to fill in cosine and sine values for the rest of the special angles on the unit circle. They are shown in Figure. Take time to learn the (x, y) coordinates of all of the major angles in the first quadrant.

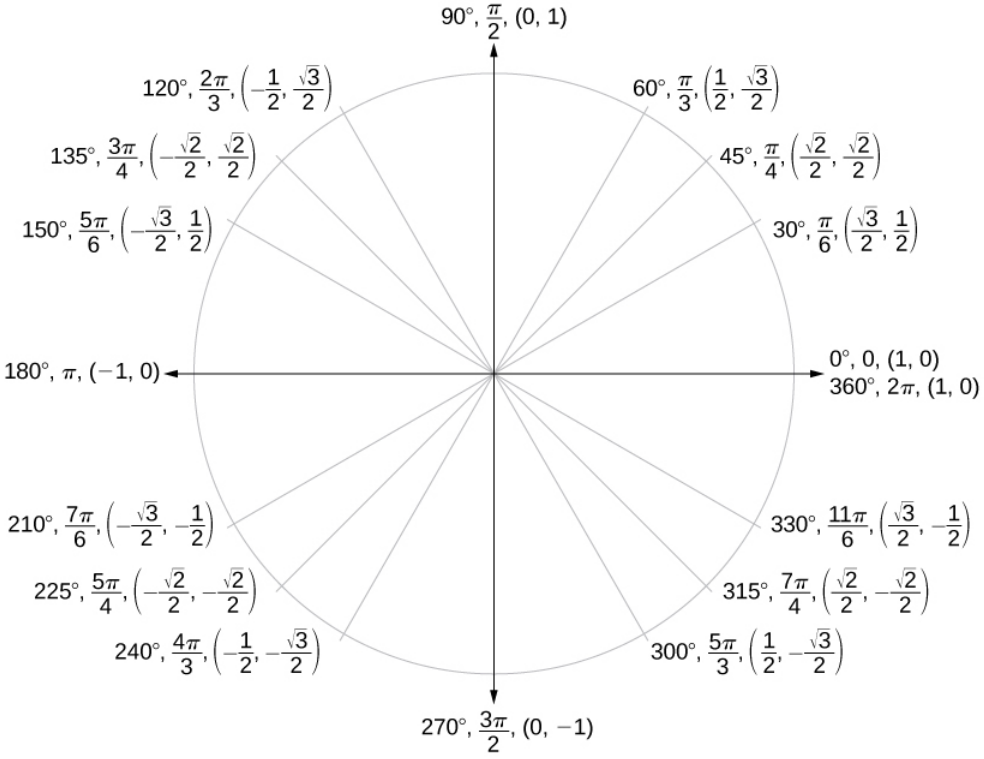


***Figure* 5.2.19:** *Special angles and coordinates of corresponding points on the unit circle*

In addition to learning the values for special angles, we can use reference angles to find (x, y) coordinates of any point on the unit circle, using what we know of reference angles along with the **identities**

$$x = \cos\, t \tag{5.2.38}$$
$$y = \sin\, t \tag{5.2.39}$$

First we find the reference angle corresponding to the given angle. Then we take the sine and cosine values of the **reference angle**, and give them the signs corresponding to the *y*- and *x*-values of the quadrant.

How To: Given the angle of a point on a circle and the radius of the circle, find the (x, y) coordinates of the point.

Find the reference angle by measuring the smallest angle to the *x*-axis.

1. Find the cosine and sine of the reference angle.
2. Determine the appropriate signs for x and y in the given quadrant.

Example 5.2.7: Using the Unit Circle to Find Coordinates

Find the coordinates of the point on the unit circle at an angle of $\frac{7\pi}{6}$.

Solution

We know that the angle $\frac{7\pi}{6}$ is in the third quadrant.

First, let's find the reference angle by measuring the angle to the *x*-axis. To find the reference angle of an angle whose terminal side is in quadrant III, we find the difference of the angle and π. π.

$$\frac{7\pi}{6} - \pi = \frac{\pi}{6} \tag{5.2.40}$$

Next, we will find the cosine and sine of the reference angle:

$$\cos\left(\frac{\pi}{6}\right) = \frac{3}{2} \qquad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \tag{5.2.41}$$

We must determine the appropriate signs for x and y in the given quadrant. Because our original angle is in the third quadrant, where both x x and y y are negative, both cosine and sine are negative.

$$\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2} \tag{5.2.42}$$

$$\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2} \tag{5.2.43}$$

Now we can calculate the (x, y) coordinates using the identities $x = \cos\,\theta$ and $y = \sin\,\theta$.

The coordinates of the point are $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ on the unit circle.

Exercise 5.2.7:

Find the coordinates of the point on the unit circle at an angle of $\frac{5\pi}{3}$.

Solution

$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

media

Access these online resources for additional instruction and practice with sine and cosine functions.

- Trigonometric Functions Using the Unit Circle
- Sine and Cosine from the Unit Circle
- Sine and Cosine from the Unit Circle and Multiples of Pi Divided by Six
- Sine and Cosine from the Unit Circle and Multiples of Pi Divided by Four
- Trigonometric Functions Using Reference Angles

Key Equations

Cosine	$\cos\,t = x$
Sine	$\sin\,t = y$
Pythagorean Identity	$\cos^2 t + \sin^2 t = 1$

Key Concepts

- Finding the function values for the sine and cosine begins with drawing a unit circle, which is centered at the origin and has a radius of 1 unit.
- Using the unit circle, the sine of an angle t t equals the y-value of the endpoint on the unit circle of an arc of length t t whereas the cosine of an angle t t equals the x-value of the endpoint. See Example.
- The sine and cosine values are most directly determined when the corresponding point on the unit circle falls on an axis. See Example.
- When the sine or cosine is known, we can use the Pythagorean Identity to find the other. The Pythagorean Identity is also useful for determining the sines and cosines of special angles. See Example.
- Calculators and graphing software are helpful for finding sines and cosines if the proper procedure for entering information is known. See Example.
- The domain of the sine and cosine functions is all real numbers.
- The range of both the sine and cosine functions is $[-1, 1]$.
- The sine and cosine of an angle have the same absolute value as the sine and cosine of its reference angle.
- The signs of the sine and cosine are determined from the x- and y-values in the quadrant of the original angle.
- An angle's reference angle is the size angle, t, formed by the terminal side of the angle t and the horizontal axis. See Example.
- Reference angles can be used to find the sine and cosine of the original angle. See Example.
- Reference angles can also be used to find the coordinates of a point on a circle. See Example.

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Glossary

cosine function
the x-value of the point on a unit circle corresponding to a given angle

Pythagorean Identity
a corollary of the Pythagorean Theorem stating that the square of the cosine of a given angle plus the square of the sine of that angle equals 1

sine function
the y-value of the point on a unit circle corresponding to a given angle

unit circle
a circle with a center at $(0, 0)$ and radius 1.

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5.2E: Exercises

Verbal

1) Describe the unit circle.

Answer

The unit circle is a circle of radius 1 centered at the origin.

2) What do the x- and y-coordinates of the points on the unit circle represent?

3) Discuss the difference between a coterminal angle and a reference angle.

Answer

Coterminal angles are angles that share the same terminal side. A reference angle is the size of the smallest acute angle, t, formed by the terminal side of the angle t and the horizontal axis.

4) Explain how the cosine of an angle in the second quadrant differs from the cosine of its reference angle in the unit circle.

5) Explain how the sine of an angle in the second quadrant differs from the sine of its reference angle in the unit circle.

Answer

The sine values are equal.

Algebraic

For the exercises 6-9, use the given sign of the sine and cosine functions to find the quadrant in which the terminal point determined by t lies.

6) $\sin(t) < 0$ and $\cos(t) < 0$

7) $\sin(t) > 0$ and $\cos(t) > 0$

Answer

I

8) $\sin(t) > 0$ and $\cos(t) < 0$

9) $\sin(t) < 0$ and $\cos(t) > 0$

Answer

IV

For the exercises 10-22, find the exact value of each trigonometric function.

10) $\sin \frac{\pi}{2}$

11) $\sin \frac{\pi}{3}$

Answer

$\frac{\sqrt{3}}{2}$

12) $\cos \frac{\pi}{2}$

13) $\cos \frac{\pi}{3}$

Answer

$\frac{1}{2}$

14) $\sin \frac{\pi}{4}$

15) $\cos \frac{\pi}{4}$

Answer

$\frac{\sqrt{2}}{2}$

16) $\sin \frac{\pi}{6}$

17) $\sin \pi$

Answer

0

18) $\sin \frac{3\pi}{2}$

19) $\cos \pi$

Answer

-1

20) $\cos 0$

21) $cos \frac{\pi}{6}$

Answer

$\frac{\sqrt{3}}{2}$

22) $\sin 0$

Numeric

For the exercises 23-33, state the reference angle for the given angle.

23) $240°$

Answer

$60°$

24) $-170°$

25) $100°$

Answer

$80°$

26) $-315°$

27) $135°$

Answer

$45°$

28) $\frac{5\pi}{4}$

29) $\frac{2\pi}{3}$

Answer

$\frac{\pi}{3}$

30) $\frac{5\pi}{6}$

31) $-\frac{11\pi}{3}$

Answer

$\frac{\pi}{3}$

32) $\frac{-7\pi}{4}$

33) $\frac{-\pi}{8}$

Answer

$\frac{\pi}{8}$

For exercises 34-49, find the reference angle, the quadrant of the terminal side, and the sine and cosine of each angle. If the angle is not one of the angles on the unit circle, use a calculator and round to three decimal places.

34) $225°$

Exercise $5.2E.35$

$300°$

Answer

$60°$, Quadrant IV, $\sin(300°) = -\frac{\sqrt{3}}{2}, \cos(300°) = \frac{1}{2}$

36) $320°$

37) $135°$

Answer

$45°$, Quadrant II, $\sin(135°) = \frac{\sqrt{2}}{2}, \cos(135°) = -\frac{\sqrt{2}}{2}$

38) $210°$

39) $120°$

Answer

$60°$, Quadrant II, $\sin(120°) = \frac{\sqrt{3}}{2}$, $\cos(120°) = -\frac{1}{2}$

40) $250°$

41) $150°$

Answer

$30°$, Quadrant II, $\sin(150°) = \frac{1}{2}$, $\cos(150°) = -\frac{\sqrt{3}}{2}$

42) $\frac{5\pi}{4}$

Exercise 5.2E.43

$\frac{7\pi}{6}$

Answer

$\frac{\pi}{6}$, Quadrant III, $\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$, $\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

44) $\frac{5\pi}{3}$

45) $\frac{3\pi}{4}$

Answer

$\frac{\pi}{4}$, Quadrant II, $\sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$, $\cos\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{2}}{2}$

46) $\frac{4\pi}{3}$

47) $\frac{2\pi}{3}$

Answer

$\frac{\pi}{3}$, Quadrant II, $\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$, $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$

48) $\frac{5\pi}{6}$

49) $\frac{7\pi}{4}$

Answer

$\frac{\pi}{4}$, Quadrant IV, $\sin\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2}$, $\cos\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2}$

For the exercises 50-59, find the requested value.

50) If $\cos(t) = \frac{1}{7}$ and t is in the 4th quadrant, find $\sin(t)$.

51) If $\cos(t) = \frac{2}{9}$ and t is in the 1st quadrant, find $\sin(t)$.

Answer

$\frac{\sqrt{77}}{9}$

52) If $\sin(t) = \frac{3}{8}$ and t is in the 2nd quadrant, find $\cos(t)$.

Exercise 5.2E.53

If $\sin(t) = -\frac{1}{4}$ and t is in Quadrant III, find $\cos(t)$.

Answer

$-\frac{\sqrt{15}}{4}$

54) Find the coordinates of the point on a circle with radius 15 corresponding to an angle of $220°$.

55) Find the coordinates of the point on a circle with radius 20 corresponding to an angle of $120°$.

Answer

$(-10, 10\sqrt{3})$

56) Find the coordinates of the point on a circle with radius 8 corresponding to an angle of $\frac{7\pi}{4}$.

57) Find the coordinates of the point on a circle with radius 16 corresponding to an angle of $\frac{5\pi}{9}$.

Answer

$(-2.778, 15.757)$

58) State the domain of the sine and cosine functions.

59) State the range of the sine and cosine functions.

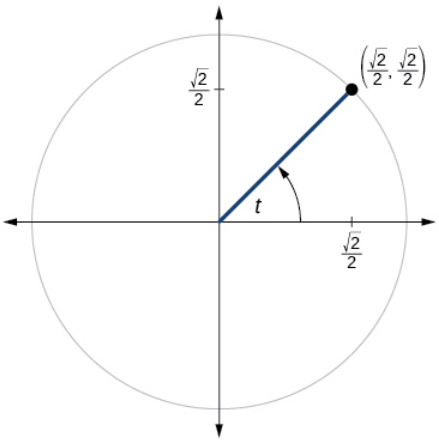
Answer

$[-1, 1]$

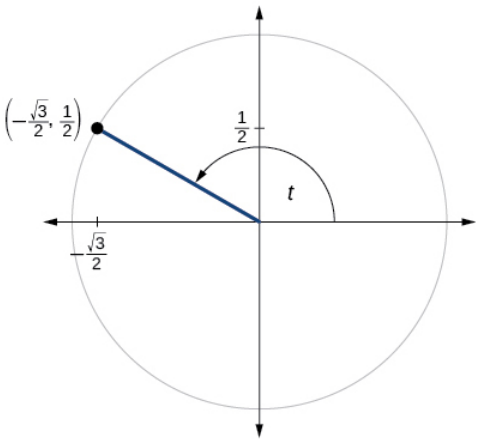
Graphical

For the exercises 60-79, use the given point on the unit circle to find the value of the sine and cosine of t.

60)



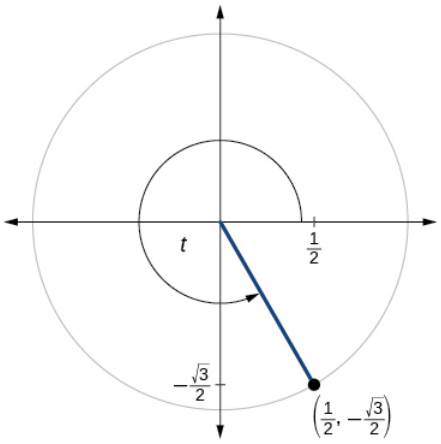
61)



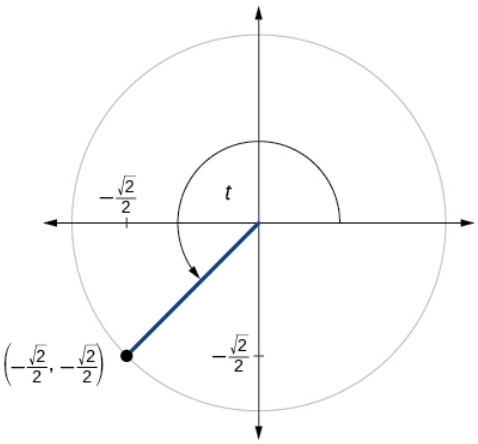
Answer

$\sin t = \frac{1}{2}, \cos t = -\frac{\sqrt{3}}{2}$

62)



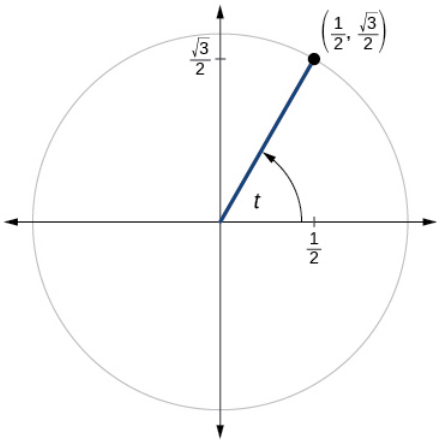
63)



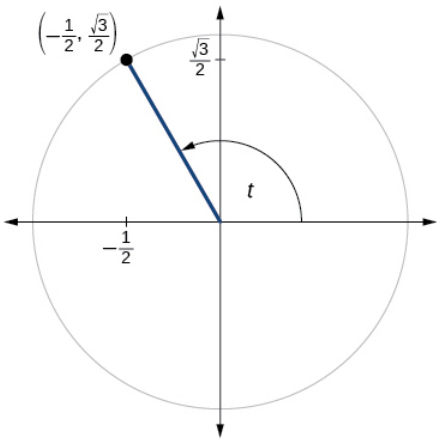
Answer

$\sin t = -\frac{\sqrt{2}}{2}, \cos t = -\frac{\sqrt{2}}{2}$

64)



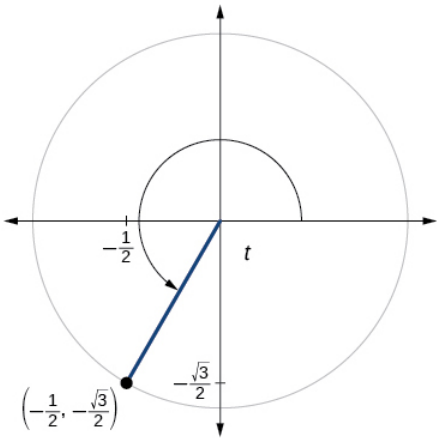
65)



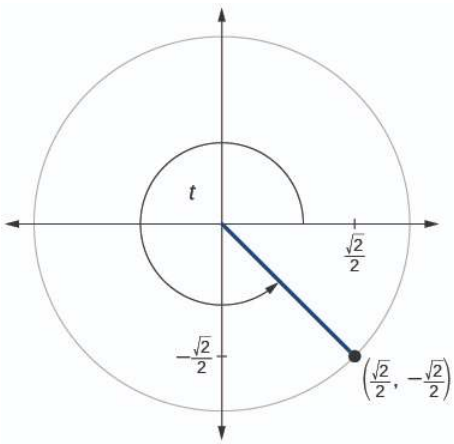
Answer

$$\sin t = \frac{\sqrt{3}}{2}, \cos t = -\frac{1}{2}$$

66)



67)

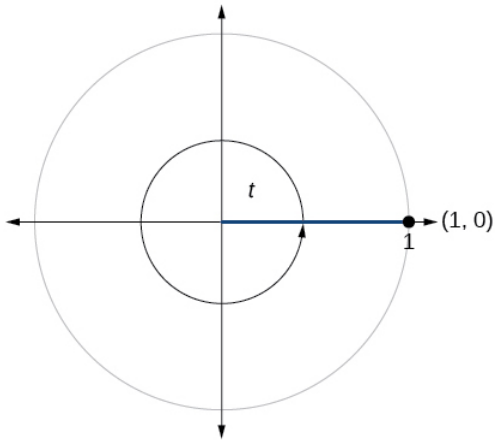


Answer

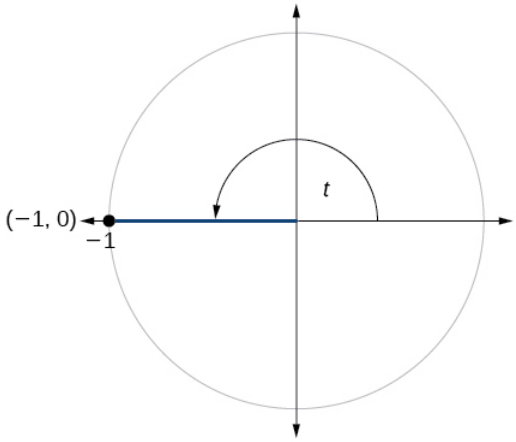
$$\sin t = -\frac{\sqrt{2}}{2}, \cos t = \frac{\sqrt{2}}{2}$$

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68)



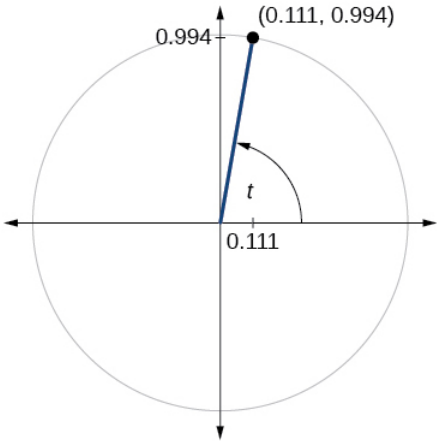
69)



Answer

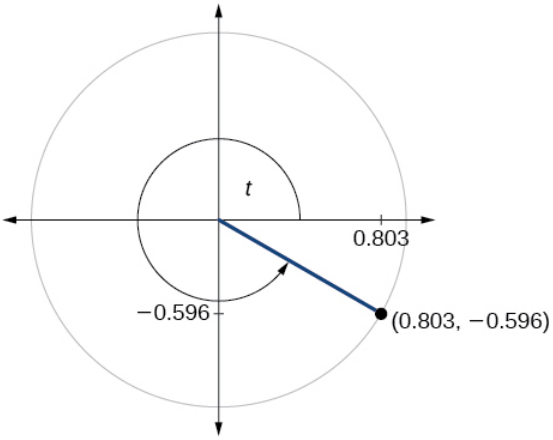
$\sin t = 0, \cos t = -1$

70)



71)

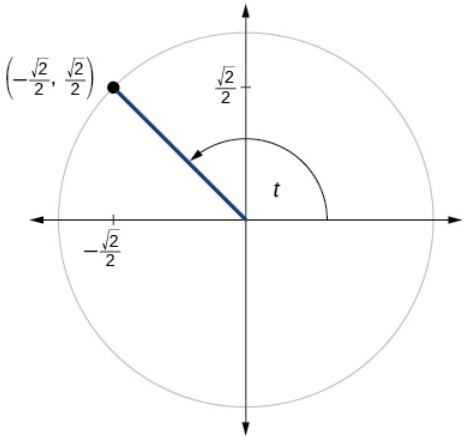
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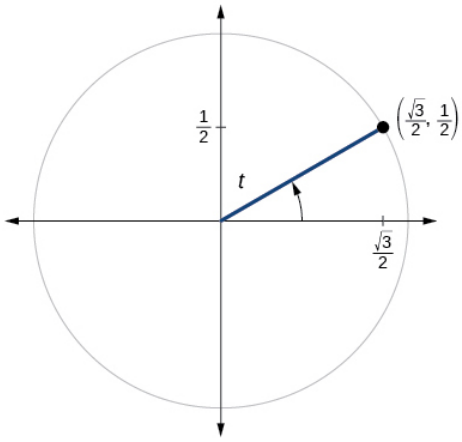
Answer

$\sin t = -0.596, \cos t = 0.803$

72)



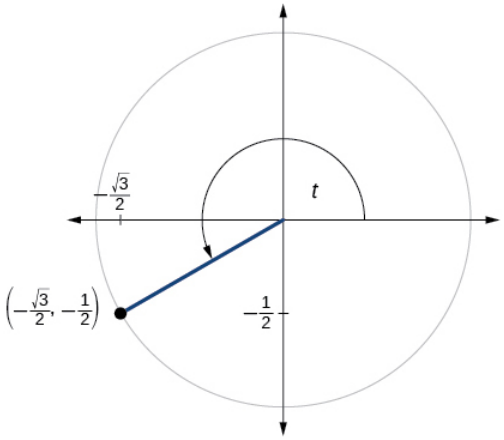
73)



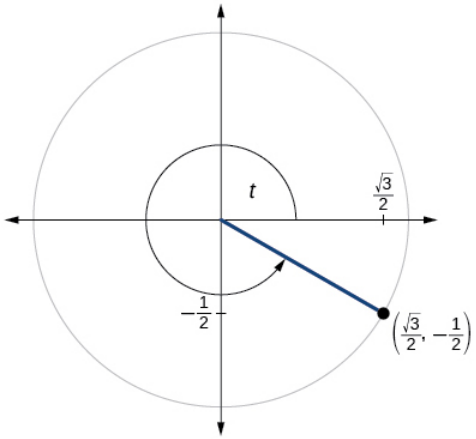
Answer

$\sin t = \frac{1}{2}, \cos t = \frac{\sqrt{3}}{2}$

74)



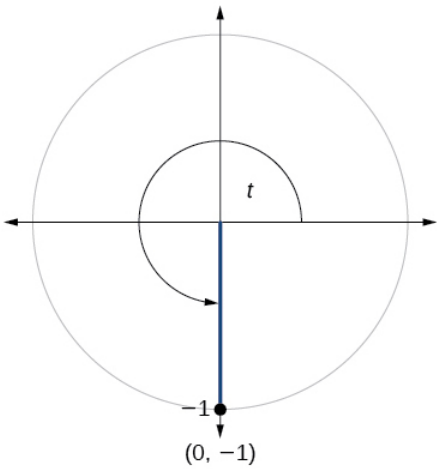
75)



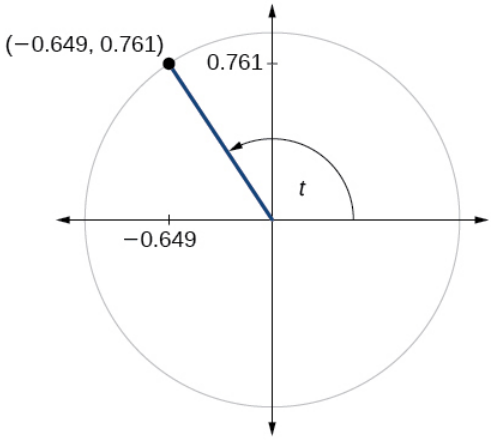
Answer

$\sin t = -\frac{1}{2}, \cos t = \frac{\sqrt{3}}{2}$

76)



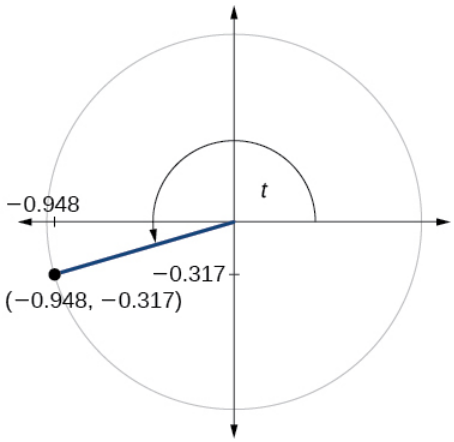
77)



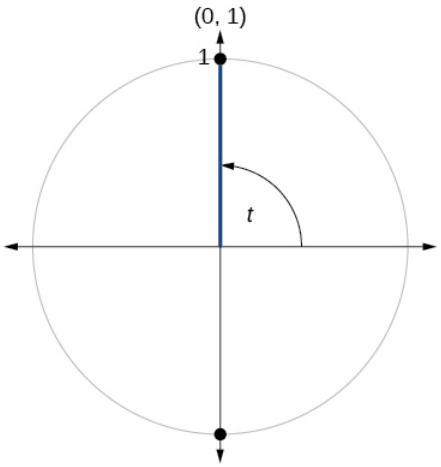
Answer

$\sin t = 0.761, \cos t = -0.649$

78)



79)



Answer

$\sin t = 1, \cos t = 0$

Technology

For the exercises 80-89, use a graphing calculator to evaluate.

80) $\sin\frac{5\pi}{9}$

81) $cos\frac{5\pi}{9}$

Answer

-0.1736

82) $\sin\frac{\pi}{10}$

83) $\cos\frac{\pi}{10}$

Answer

0.9511

84) $\sin\frac{3\pi}{4}$

85) $\cos\frac{3\pi}{4}$

Answer

-0.7071

86) $\sin 98°$

87) $\cos 98°$

Answer

-0.1392

88) $\cos 310°$

89) $\sin 310°$

Answer

-0.7660

Extensions

For the exercises 90-99, evaluate.

90) $\sin\left(\frac{11\pi}{3}\right)\cos\left(\frac{-5\pi}{6}\right)$

91) $\sin\left(\frac{3\pi}{4}\right)\cos\left(\frac{5\pi}{3}\right)$

Answer

$\frac{\sqrt{2}}{4}$

92) $\sin\left(-\frac{4\pi}{3}\right)\cos\left(\frac{\pi}{2}\right)$

93) $\sin\left(\frac{-9\pi}{4}\right)\cos\left(\frac{-\pi}{6}\right)$

Answer

$-\frac{\sqrt{6}}{4}$

94) $\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{-\pi}{3}\right)$

95) $\sin\left(\frac{7\pi}{4}\right)\cos\left(\frac{-2\pi}{3}\right)$

Answer

$\frac{\sqrt{2}}{4}$

96) $\cos\left(\frac{5\pi}{6}\right)\cos\left(\frac{2\pi}{3}\right)$

97) $\cos\left(\frac{-\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)$

Answer

$\frac{\sqrt{2}}{4}$

98) $\sin\left(\frac{-5\pi}{4}\right)\sin\left(\frac{11\pi}{6}\right)$

99) $\sin(\pi)\sin\left(\frac{\pi}{6}\right)$

Answer

0

Real-World Applications

For the exercises 100-104, use this scenario: A child enters a carousel that takes one minute to revolve once around. The child enters at the point $(0,1)$, that is, on the due north position. Assume the carousel revolves counter clockwise.

100) What are the coordinates of the child after 45 seconds?

101) What are the coordinates of the child after 90 seconds?

Answer

$(0,-1)$

102) What is the coordinates of the child after 125 seconds?

103) When will the child have coordinates $(0.707,-0.707)$ if the ride lasts 6 minutes? (There are multiple answers.)

Answer

37.5 seconds, 97.5 seconds, 157.5 seconds, 217.5 seconds, 277.5 seconds, 337.5 seconds

104) When will the child have coordinates $(-0.866,-0.5)$ if the ride last 6 minutes?

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5.3: The Other Trigonometric Functions

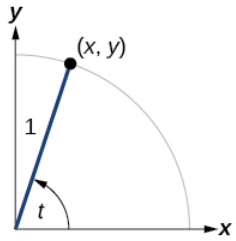
Skills to Develop

- Find exact values of the trigonometric functions secant, cosecant, tangent, and cotangent of $\frac{\pi}{3}$, $\frac{\pi}{4}$, and $\frac{\pi}{6}$.
- Use reference angles to evaluate the trigonometric functions secant, tangent, and cotangent.
- Use properties of even and odd trigonometric functions.
- Recognize and use fundamental identities.
- Evaluate trigonometric functions with a calculator.

A wheelchair ramp that meets the standards of the Americans with Disabilities Act must make an angle with the ground whose tangent is $\frac{1}{12}$ or less, regardless of its length. A tangent represents a ratio, so this means that for every 1 inch of rise, the ramp must have 12 inches of run. Trigonometric functions allow us to specify the shapes and proportions of objects independent of exact dimensions. We have already defined the sine and cosine functions of an angle. Though sine and cosine are the trigonometric functions most often used, there are four others. Together they make up the set of six trigonometric functions. In this section, we will investigate the remaining functions.

Finding Exact Values of the Trigonometric Functions Secant, Cosecant, Tangent, and Cotangent

To define the remaining functions, we will once again draw a unit circle with a point (x, y) corresponding to an angle of t, t,as shown in Figure. As with the sine and cosine, we can use the (x, y) coordinates to find the other functions.



The first function we will define is the tangent. The **tangent** of an angle is the ratio of the *y*-value to the *x*-value of the corresponding point on the unit circle. In Figure, the tangent of angle t is equal to $\frac{y}{x}, x \neq 0$. Because the *y*-value is equal to the sine of t, and the *x*-value is equal to the cosine of t, the tangent of angle t can also be defined as $\frac{\sin t}{\cos t}, \cos t \neq 0$. The tangent function is abbreviated as $\tan$. The remaining three functions can all be expressed as reciprocals of functions we have already defined.

- The **secant** function is the reciprocal of the cosine function. In Figure, the secant of angle t is equal to $\frac{1}{\cos t} = \frac{1}{x}, x \neq 0$. The secant function is abbreviated as $\sec$.
- The **cotangent** function is the reciprocal of the tangent function. In Figure, the cotangent of angle t is equal to $\frac{\cos t}{\sin t} = \frac{x}{y}, y \neq 0$. The cotangent function is abbreviated as $\cot$.
- The **cosecant** function is the reciprocal of the sine function. In Figure, the cosecant of angle t is equal to $\frac{1}{\sin t} = \frac{1}{y}, y \neq 0$. The cosecant function is abbreviated as $\csc$.

TANGENT, SECANT, COSECANT, AND COTANGENT FUNCTIONS

If t is a real number and (x, y) is a point where the terminal side of an angle of t radians intercepts the unit circle, then

$$\tan t = \frac{y}{x}, x \neq 0 \tag{5.3.1}$$

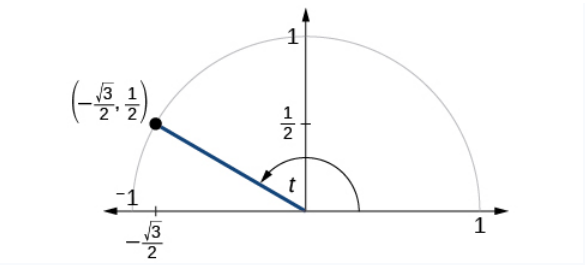
$$\sec t = \frac{1}{x}, x \neq 0 \tag{5.3.2}$$

$$\csc t = \frac{1}{y}, y \neq 0 \tag{5.3.3}$$

$$\cot t = \frac{x}{y}, y \neq 0 \tag{5.3.4}$$

Example 5.3.1: Finding Trigonometric Functions from a Point on the Unit Circle

The point $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$ is on the unit circle, as shown in Figure. Find $\sin t, \cos t, \tan t, \sec t, \csc t,$ and $\cot t$.



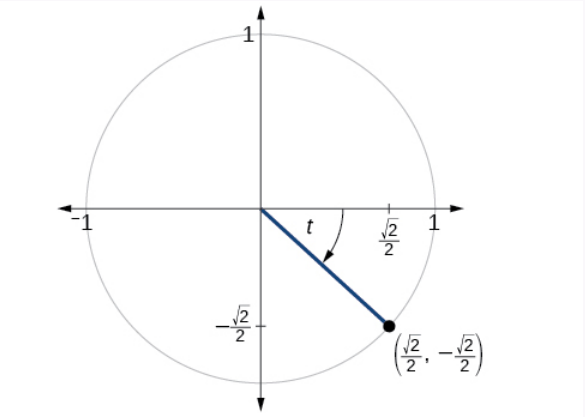
Solution

Because we know the (x, y) coordinates of the point on the unit circle indicated by angle t, we can use those coordinates to find the six functions:

$$\sin t = y = \frac{1}{2}$$
$$\cos t = x = -\frac{\sqrt{3}}{2}$$
$$\tan t = \frac{y}{x} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{2}\left(-\frac{2}{\sqrt{3}}\right) = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$
$$\sec t = \frac{1}{x} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$
$$\csc t = \frac{1}{y} = \frac{1}{\frac{1}{2}} = 2$$
$$\cot t = \frac{x}{y} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\frac{\sqrt{3}}{2}\left(\frac{2}{1}\right) = -\sqrt{3}$$

Exercise 5.3.1:

The point $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ is on the unit circle, as shown in Figure. Find $\sin t, \cos t, \tan t, \sec t, \csc t$, and $\cot t$.



Solution

$\sin t = -\frac{\sqrt{2}}{2}, \cos t = \frac{\sqrt{2}}{2}, \tan t = -1, \sec t = \sqrt{2}, \csc t = -\sqrt{2}, \cot t = -1$

Example 5.3.2: Finding the Trigonometric Functions of an Angle

Find $\sin t, \cos t, \tan t, \sec t, \csc t$, and $\cot t$ when $t = \frac{\pi}{6}$.

Solution

We have previously used the properties of equilateral triangles to demonstrate that $\sin \frac{\pi}{6} = \frac{1}{2}$ and $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$. We can use these values and the definitions of tangent, secant, cosecant, and cotangent as functions of sine and cosine to find the remaining function values.

$$\tan\frac{\pi}{6} = \frac{\sin\frac{\pi}{6}}{\cos\frac{\pi}{6}} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\sec\frac{\pi}{6} = \frac{1}{\cos\frac{\pi}{6}} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\csc\frac{\pi}{6} = \frac{1}{\sin\frac{\pi}{6}} = \frac{1}{\frac{1}{2}} = 2$$

$$\cot\frac{\pi}{6} = \frac{\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

Exercise 5.3.2:

Find $\sin t, \cos t, \tan t, \sec t, \csc t,$ and $\cot t$ when $t = \frac{\pi}{3}$.

Solution

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \tag{5.3.5}$$

$$\cos\frac{\pi}{3} = \frac{1}{2} \tag{5.3.6}$$

$$\tan\frac{\pi}{3} = \sqrt{3} \tag{5.3.7}$$

$$\sec\frac{\pi}{3} = 2 \tag{5.3.8}$$

$$\csc\frac{\pi}{3} = \frac{2\sqrt{3}}{3} \tag{5.3.9}$$

$$\cot\frac{\pi}{3} = \frac{\sqrt{3}}{3} \tag{5.3.10}$$

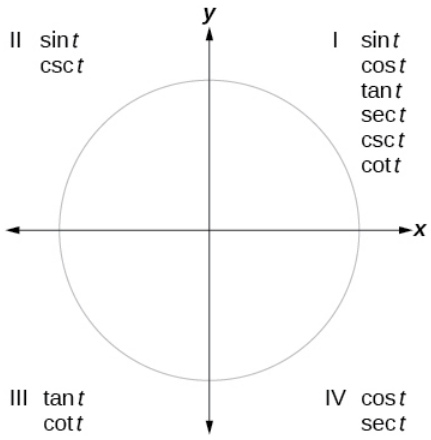
Because we know the sine and cosine values for the common first-quadrant angles, we can find the other function values for those angles as well by setting x x equal to the cosine and y y equal to the sine and then using the definitions of tangent, secant, cosecant, and cotangent. The results are shown in Table.

Angle	0	$\frac{\pi}{6}$, or 30°	$\frac{\pi}{4}$, or 45°	$\frac{\pi}{3}$, or 60°	$\frac{\pi}{2}$, or 90°
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
Tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined
Secant	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	Undefined
Cosecant	Undefined	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
Cotangent	Undefined	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

Using Reference Angles to Evaluate Tangent, Secant, Cosecant, and Cotangent

We can evaluate trigonometric functions of angles outside the first quadrant using reference angles as we have already done with the sine and cosine functions. The procedure is the same: Find the **reference angle** formed by the terminal side of the given angle with the horizontal axis. The trigonometric function values for the original angle will be the same as those for the reference angle, except for the positive or negative sign, which is determined by *x*- and *y*-values in the original quadrant. Figure shows which functions are positive in which quadrant.

To help us remember which of the six trigonometric functions are positive in each quadrant, we can use the mnemonic phrase "A Smart Trig Class." Each of the four words in the phrase corresponds to one of the four quadrants, starting with quadrant I and rotating counterclockwise. In quadrant I, which is "**A**," **a**ll of the six trigonometric functions are positive. In quadrant II, "**S**mart," only **s**ine and its reciprocal function, cosecant, are positive. In quadrant III, "**T**rig," only **t**angent and its reciprocal function, cotangent, are positive. Finally, in quadrant IV, "**C**lass," only **c**osine and its reciprocal function, secant, are positive.



HOWTO: Given an angle not in the first quadrant, use reference angles to find all six trigonometric functions

1. Measure the angle formed by the terminal side of the given angle and the horizontal axis. This is the reference angle.
2. Evaluate the function at the reference angle.
3. Observe the quadrant where the terminal side of the original angle is located. Based on the quadrant, determine whether the output is positive or negative.

Example 5.3.3: Using Reference Angles to Find Trigonometric Functions

Use reference angles to find all six trigonometric functions of $-\frac{5\pi}{6}$.

Solution

The angle between this angle's terminal side and the x-axis is $\frac{\pi}{6}$, so that is the reference angle. Since $-\frac{5\pi}{6}$ is in the third quadrant, where both x and y are negative, cosine, sine, secant, and cosecant will be negative, while tangent and cotangent will be positive.

$$\cos(-\frac{5\pi}{6}) = -\frac{\sqrt{3}}{2}, \sin(-\frac{5\pi}{6}) = -\frac{1}{2}, \tan(-\frac{5\pi}{6}) = \frac{\sqrt{3}}{3} \tag{5.3.11}$$

$$\sec(-\frac{5\pi}{6}) = -\frac{2\sqrt{3}}{3}, \csc(-\frac{5\pi}{6}) = -2, \cot(-\frac{5\pi}{6}) = \sqrt{3} \tag{5.3.12}$$

Exercise 5.3.3

Use reference angles to find all six trigonometric functions of $-\frac{7\pi}{4}$.

Solution

$\sin(-\frac{7\pi}{4}) = \frac{\sqrt{2}}{2}, \cos(\frac{-7\pi}{4}) = \frac{\sqrt{2}}{2}, \tan(\frac{-7\pi}{4}) = 1,$

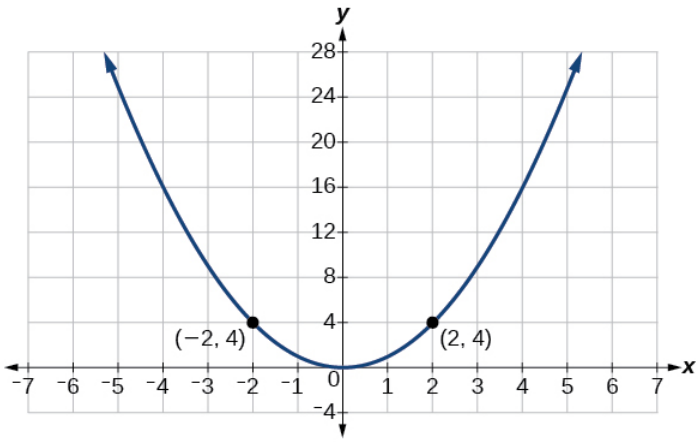
$\sec(\frac{-7\pi}{4}) = \sqrt{2}, \csc(\frac{-7\pi}{4}) = \sqrt{2}, \cot(\frac{-7\pi}{4}) = 1$

Using Even and Odd Trigonometric Functions

To be able to use our six trigonometric functions freely with both positive and negative angle inputs, we should examine how each function treats a negative input. As it turns out, there is an important difference among the functions in this regard.

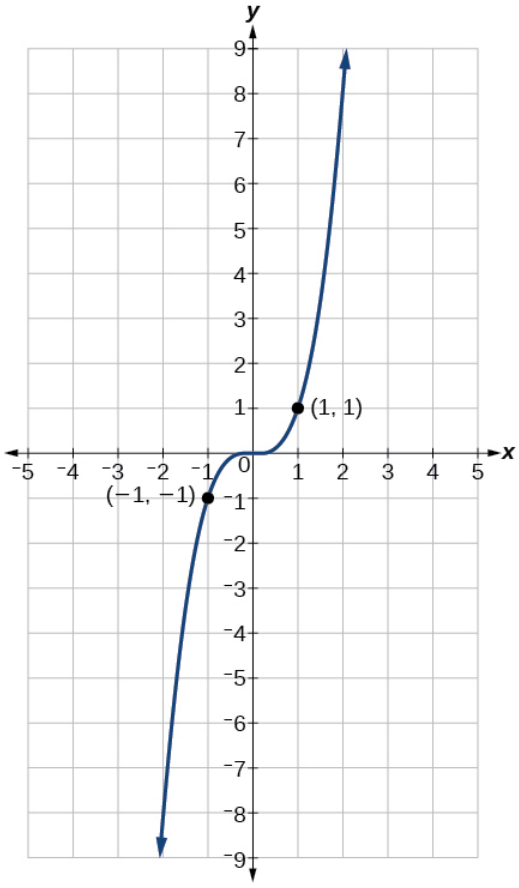
Consider the function $f(x) = x^2$, shown in Figure. The graph of the function is symmetrical about the y-axis. All along the curve, any two points with opposite x-values have the same function value. This matches the result of calculation: $(4)^2 = (-4)^2, (-5)^2$

$= (5)^2$, and so on. So $f(x) = x^2$ is an **even function**, a function such that two inputs that are opposites have the same output. That means $f(-x) = f(x)$.



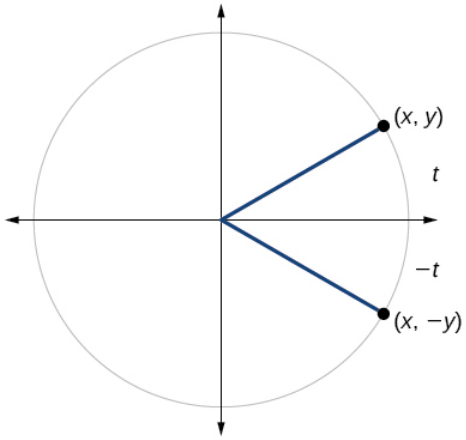
The function $f(x) = x^2$ is an even function.

Now consider the function $f(x) = x^3$, shown in Figure. The graph is not symmetrical about the *y*-axis. All along the graph, any two points with opposite *x*-values also have opposite *y*-values. So $f(x) = x^3$ is an **odd function**, one such that two inputs that are opposites have outputs that are also opposites. That means $f(-x) = -f(x)$.



The function $f(x) = x^3$ is an odd function.

We can test whether a trigonometric function is even or odd by drawing a **unit circle** with a positive and a negative angle, as in Figure. The sine of the positive angle is y. The sine of the negative angle is $-y$. The **sine function**, then, is an odd function. We can test each of the six trigonometric functions in this fashion. The results are shown in Table.



$\sin t = y$ (5.3.13)	$\cos t = x$ (5.3.16)	$\tan(t) = \frac{y}{x}$ (5.3.19)
$\sin(-t) = -y$ (5.3.14)	$\cos(-t) = x$ (5.3.17)	$\tan(-t) = -\frac{y}{x}$ (5.3.20)
$\sin t \neq sin(-t)$ (5.3.15)	$\cos t = \cos(-t)$ (5.3.18)	$\tan t \neq \tan(-t)$ (5.3.21)
$\sec t = \frac{1}{x}$ (5.3.22)	$\csc t = \frac{1}{y}$ (5.3.25)	$\cot t = \frac{x}{y}$ (5.3.28)
$\sec(-t) = \frac{1}{x}$ (5.3.23)	$\csc(-t) = \frac{1}{-y}$ (5.3.26)	$\cot(-t) = \frac{x}{-y}$ (5.3.29)
$\sec t = \sec(-t)$ (5.3.24)	$\csc t \neq \csc(-t)$ (5.3.27)	$\cot t \neq \cot(-t)$ (5.3.30)

EVEN AND ODD TRIGONOMETRIC FUNCTIONS

An **even function** is one in which $f(-x) = f(x)$.

An **odd function** is one in which $f(-x) = -f(x)$.

Cosine and secant are even:

$$\cos(-t) = \cos t \quad (5.3.31)$$
$$\sec(-t) = \sec t \quad (5.3.32)$$

Sine, tangent, cosecant, and cotangent are odd:

$$\sin(-t) = -\sin t \quad (5.3.33)$$
$$\tan(-t) = -\tan t \quad (5.3.34)$$
$$\csc(-t) = -\csc t \quad (5.3.35)$$
$$\cot(-t) = -\cot t \quad (5.3.36)$$

Example 5.3.4: Using Even and Odd Properties of Trigonometric Functions

If the secant of angle t t is 2, what is the secant of $-t$?

Solution

Secant is an even function. The secant of an angle is the same as the secant of its opposite. So if the secant of angle t is 2, the secant of $-t$ is also 2.

Exercise 5.3.4:

If the cotangent of angle t is $\sqrt{3}$, what is the cotangent of $-t$?

Solution

$-\sqrt{3}$

Recognizing and Using Fundamental Identities

We have explored a number of properties of trigonometric functions. Now, we can take the relationships a step further, and derive some fundamental identities. Identities are statements that are true for all values of the input on which they are defined. Usually, identities can be derived from definitions and relationships we already know. For example, the Pythagorean Identity we learned earlier was derived from the Pythagorean Theorem and the definitions of sine and cosine.

FUNDAMENTAL IDENTITIES

We can derive some useful **identities** from the six trigonometric functions. The other four trigonometric functions can be related back to the sine and cosine functions using these basic relationships:

$$\tan t = \frac{\sin t}{\cos t} \tag{5.3.37}$$

$$\sec t = \frac{1}{\cos t} \tag{5.3.38}$$

$$\csc t = \frac{1}{\sin t} \tag{5.3.39}$$

$$\cot t = \frac{1}{\tan t} = \frac{\cos t}{\sin t} \tag{5.3.40}$$

Example 5.3.5: Using Identities to Evaluate Trigonometric Functions

1. Given $\sin(45°) = \frac{\sqrt{2}}{2}, \cos(45°) = \frac{\sqrt{2}}{2}$, evaluate $\tan(45°)$.
2. Given $\sin(\frac{5\pi}{6}) = \frac{1}{2}, \cos(\frac{5\pi}{6}) = -\frac{\sqrt{3}}{2}$, evaluate $\sec(\frac{5\pi}{6})$.

Solution

Because we know the sine and cosine values for these angles, we can use identities to evaluate the other functions.

$$\tan(45°) = \frac{\sin(45°)}{\cos(45°)} \tag{5.3.41}$$

1.
$$= \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} \tag{5.3.42}$$

$$= 1 \tag{5.3.43}$$

$$\sec(\frac{5\pi}{6}) = \frac{1}{\cos(\frac{5\pi}{6})} \tag{5.3.44}$$

$$= \frac{1}{-\frac{\sqrt{3}}{2}} \tag{5.3.45}$$

2.
$$= \frac{-2\sqrt{3}}{1} \tag{5.3.46}$$

$$= \frac{-2}{\sqrt{3}} \tag{5.3.47}$$

$$= -\frac{2\sqrt{3}}{3} \tag{5.3.48}$$

Exercise 5.3.5:

Evaluate $\csc(\frac{7\pi}{6})$.

Solution

-2

Example 5.3.6: Using Identities to Simplify Trigonometric Expressions

Simplify $\frac{\sec t}{\tan t}$.

Solution

We can simplify this by rewriting both functions in terms of sine and cosine.

$$\begin{aligned}\frac{\sec t}{\tan t} &= \frac{1/\cos t}{\sin t/\cos t} && \text{To divide the functions, we multiply by the reciprocal.}\\ &= \frac{1}{\cos t}\frac{\cos t}{\sin t} && \text{Divide out the cosines.}\\ &= \frac{1}{\sin t} && \text{Simplify and use the identity.}\\ &= \csc t\end{aligned} \tag{5.3.49}$$

By showing that $\frac{\sec t}{\tan t}$ can be simplified to $\csc t$,we have, in fact, established a new identity.

$$\frac{\sec t}{\tan t} = \csc t \tag{5.3.50}$$

Exercise 5.3.6:

Simplify $(\tan t)(\cos t)$.

Solution

$\sin t$

Alternate Forms of the Pythagorean Identity

We can use these fundamental identities to derive alternative forms of the **Pythagorean Identity**, $\cos^2 t + \sin^2 t = 1$. One form is obtained by dividing both sides by $\cos^2 t$:

$$\frac{\cos^2 t}{\cos^2 t} + \frac{\sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t} \tag{5.3.51}$$

$$1 + \tan^2 t = \sec^2 t \tag{5.3.52}$$

The other form is obtained by dividing both sides by $\sin^2 t$:

$$\frac{\cos^2 t}{\sin^2 t} + \frac{\sin^2 t}{\sin^2 t} = \frac{1}{\sin^2 t} \tag{5.3.53}$$

$$\cot^2 t + 1 = \csc^2 t \tag{5.3.54}$$

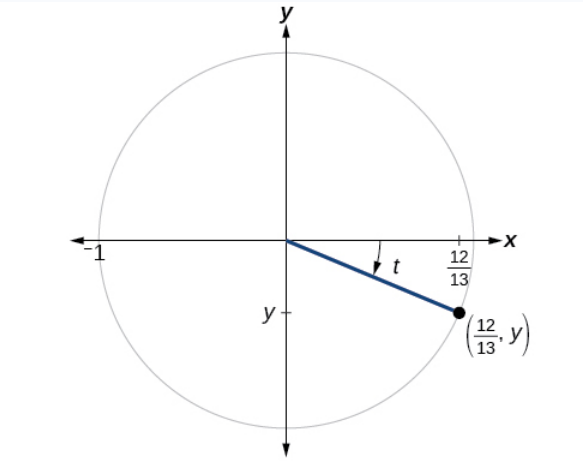
ALTERNATE FORMS OF THE PYTHAGOREAN IDENTITY

$$1 + \tan^2 t = \sec^2 t \tag{5.3.55}$$

$$\cot^2 t + 1 = \csc^2 t \tag{5.3.56}$$

Example 5.3.7: Using Identities to Relate Trigonometric Functions

If cos(t)=1213 cos(t)=1213 and t t is in quadrant IV, as shown in Figure, find the values of the other five trigonometric functions.



Solution

We can find the sine using the Pythagorean Identity, $\cos^2 t + \sin^2 t = 1$, and the remaining functions by relating them to sine and cosine.

$$(\frac{12}{13})^2 + \sin^2 t = 1 \tag{5.3.57}$$

$$\sin^2 t = 1 - (\frac{12}{13})^2 \tag{5.3.58}$$

$$\sin^2 t = 1 - \frac{144}{169} \tag{5.3.59}$$

$$\sin^2 t = \frac{25}{169} \tag{5.3.60}$$

$$\sin t = \pm\sqrt{\frac{25}{169}} \tag{5.3.61}$$

$$\sin t = \pm\frac{\sqrt{25}}{\sqrt{169}} \tag{5.3.62}$$

$$\sin t = \pm\frac{5}{13} \tag{5.3.63}$$

The sign of the sine depends on the *y*-values in the quadrant where the angle is located. Since the angle is in quadrant IV, where the *y*-values are negative, its sine is negative, $-\frac{5}{13}$.

The remaining functions can be calculated using identities relating them to sine and cosine.

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{5}{13}}{\frac{12}{13}} = -\frac{5}{12} \tag{5.3.64}$$

$$\sec t = \frac{1}{\cos t} = \frac{1}{\frac{12}{13}} = \frac{13}{12} \tag{5.3.65}$$

$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{5}{13}} = -\frac{13}{5} \tag{5.3.66}$$

$$\cot t = \frac{1}{\tan t} = \frac{1}{-\frac{5}{12}} = -\frac{12}{5} \tag{5.3.67}$$

Exercise 5.3.7:

If $\sec(t) = -\frac{17}{8}$ and $0 < t < \pi$, find the values of the other five functions.

Solution

$\cos t = -\frac{8}{17}, \sin t = \frac{15}{17}, \tan t = -\frac{15}{8}$

$\csc t = \frac{17}{15}, \cot t = -\frac{8}{15}$

As we discussed in the chapter opening, a function that repeats its values in regular intervals is known as a **periodic function**. The trigonometric functions are periodic. For the four trigonometric functions, sine, cosine, cosecant and secant, a revolution of one circle, or 2π,will result in the same outputs for these functions. And for tangent and cotangent, only a half a revolution will result in the same outputs.

Other functions can also be periodic. For example, the lengths of months repeat every four years. If x x represents the length time, measured in years, and $f(x)$ represents the number of days in February, then $f(x + 4) = f(x)$. This pattern repeats over and over through time. In other words, every four years, February is guaranteed to have the same number of days as it did 4 years earlier. The positive number 4 is the smallest positive number that satisfies this condition and is called the period. A **period** is the shortest interval over which a function completes one full cycle—in this example, the period is 4 and represents the time it takes for us to be certain February has the same number of days.

PERIOD OF A FUNCTION

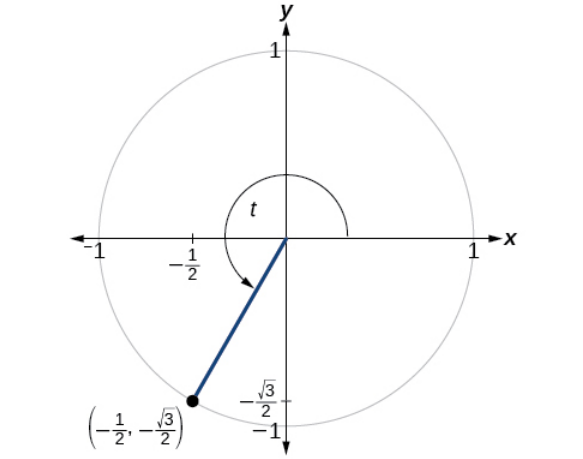
The **period** P of a repeating function f f is the number representing the interval such that $f(x + P) = f(x)$ for any value of x.

The period of the cosine, sine, secant, and cosecant functions is 2π.

The period of the tangent and cotangent functions is π.

Example 5.3.8: Finding the Values of Trigonometric Functions

Find the values of the six trigonometric functions of angle t t based on Figure.



Solution

$$\sin t = y = -\frac{\sqrt{3}}{2} \tag{5.3.68}$$

$$\cos t = x = -\frac{1}{2} \tag{5.3.69}$$

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \sqrt{3} \tag{5.3.70}$$

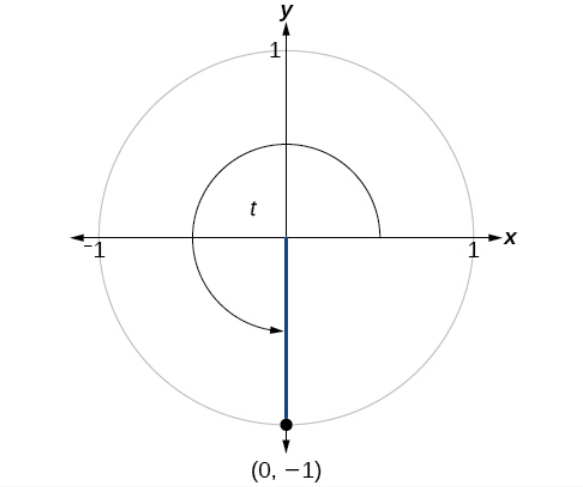
$$\sec t = \frac{1}{\cos t} = \frac{1}{-\frac{1}{2}} = -2 \tag{5.3.71}$$

$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2\sqrt{3}}{3} \tag{5.3.72}$$

$$\cot t = \frac{1}{\tan t} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \tag{5.3.73}$$

Exercise 5.3.8

Find the values of the six trigonometric functions of angle t t based on Figure.



Solution

$$\sin t = -1, \cos t = 0, \tan t = \text{Undefined} \tag{5.3.74}$$

$$\tag{5.3.75}$$

$$sect = \text{Undefined}, \csc t = -1, \cot t = 0 \tag{5.3.76}$$

Example 5.3.9: Finding the Value of Trigonometric Functions

If $\sin(t) = -\frac{\sqrt{3}}{2}$ and $\cos(t) = \frac{1}{2}$,find $\sec(t), \csc(t), \tan(t), \cot(t)$.

Solution

$$\sec t = \frac{1}{\cos t} = \frac{1}{\frac{1}{2}} = 2 \tag{5.3.77}$$

$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{\sqrt{3}}{2}} - \frac{2\sqrt{3}}{3} \tag{5.3.78}$$

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3} \tag{5.3.79}$$

$$\cot t = \frac{1}{\tan t} = \frac{1}{-\sqrt{3}} = -\frac{\sqrt{3}}{3} \tag{5.3.80}$$

Exercise 5.3.9:

If $\sin(t) = \frac{\sqrt{2}}{2}$ and $\cos(t) = \frac{\sqrt{2}}{2}$, find $\sec(t), \csc(t), \tan(t)$, and $\cot(t)$.

Solution

$\sec t = \sqrt{2}, \csc t = \sqrt{2}, \tan t = 1, \cot t = 1$

Evaluating Trigonometric Functions with a Calculator

We have learned how to evaluate the six trigonometric functions for the common first-quadrant angles and to use them as reference angles for angles in other quadrants. To evaluate trigonometric functions of other angles, we use a scientific or graphing calculator or computer software. If the calculator has a degree mode and a radian mode, confirm the correct mode is chosen before making a calculation.

Evaluating a tangent function with a scientific calculator as opposed to a graphing calculator or computer algebra system is like evaluating a sine or cosine: Enter the value and press the TAN key. For the reciprocal functions, there may not be any dedicated keys that say CSC, SEC, or COT. In that case, the function must be evaluated as the reciprocal of a sine, cosine, or tangent.

If we need to work with degrees and our calculator or software does not have a degree mode, we can enter the degrees multiplied by the conversion factor $\frac{\pi}{180}$ to convert the degrees to radians. To find the secant of $30°$, we could press

$$(\text{for a scientific calculator}) : \frac{1}{30 \times \frac{\pi}{180}} \text{COS} \tag{5.3.81}$$

or

$$(\text{for a graphing calculator}) : \frac{1}{\cos(\frac{30\pi}{180})} \tag{5.3.82}$$

how to: Given an angle measure in radians, use a scientific calculator to find the cosecant

1. If the calculator has degree mode and radian mode, set it to radian mode.
2. Enter: 1 /
3. Enter the value of the angle inside parentheses.
4. Press the SIN key.
5. Press the = key.

how to: Given an angle measure in radians, use a graphing utility/calculator to find the cosecant

1. If the graphing utility has degree mode and radian mode, set it to radian mode.
2. Enter: 1 /
3. Press the SIN key.

4. Enter the value of the angle inside parentheses.
5. Press the ENTER key.

Example 5.3.10: Evaluating the Cosecant Using Technology

Evaluate the cosecant of $\frac{5\pi}{7}$.

Solution

For a scientific calculator, enter information as follows:

$$1/(5 \times \pi/7)\text{SIN} = \tag{5.3.83}$$

$$\csc(\frac{5\pi}{7}) \approx 1.279 \tag{5.3.84}$$

Exercise 5.3.10:

Evaluate the cotangent of $-\frac{\pi}{8}$.

—

≈ -2.414

media

Access these online resources for additional instruction and practice with other trigonometric functions.

- Determining Trig Function Values
- More Examples of Determining Trig Functions
- Pythagorean Identities
- Trig Functions on a Calculator

Key Equations

Tangent function	$\tan t = \frac{\sin t}{\cos t}$
Secant function	$\sec t = \frac{1}{\cos t}$
Cosecant function	$\csc t = \frac{1}{\sin t}$
Cotangent function	$\cot t = \frac{1}{\tan t} = \frac{\cos t}{\sin t}$

Key Concepts

- The tangent of an angle is the ratio of the *y*-value to the *x*-value of the corresponding point on the unit circle.
- The secant, cotangent, and cosecant are all reciprocals of other functions. The secant is the reciprocal of the cosine function, the cotangent is the reciprocal of the tangent function, and the cosecant is the reciprocal of the sine function.
- The six trigonometric functions can be found from a point on the unit circle. See Example.
- Trigonometric functions can also be found from an angle. See Example.
- Trigonometric functions of angles outside the first quadrant can be determined using reference angles. See Example.
- A function is said to be even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$.
- Cosine and secant are even; sine, tangent, cosecant, and cotangent are odd.
- Even and odd properties can be used to evaluate trigonometric functions. See Example.
- The Pythagorean Identity makes it possible to find a cosine from a sine or a sine from a cosine.
- Identities can be used to evaluate trigonometric functions. See Example and Example.
- Fundamental identities such as the Pythagorean Identity can be manipulated algebraically to produce new identities. See Example.
- The trigonometric functions repeat at regular intervals.
- The period P of a repeating function f f is the smallest interval such that $f(x + P) = f(x)$ for any value of x.

- The values of trigonometric functions of special angles can be found by mathematical analysis.
- To evaluate trigonometric functions of other angles, we can use a calculator or computer software. See Example.

Contributors

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Glossary

cosecant
the reciprocal of the sine function: on the unit circle, $\csc t = \frac{1}{y}, y \neq 0$

cotangent
the reciprocal of the tangent function: on the unit circle, $\cot t = \frac{x}{y}, y \neq 0$

identities
statements that are true for all values of the input on which they are defined

period
the smallest interval P of a repeating function f such that $f(x + P) = f(x)$

secant
the reciprocal of the cosine function: on the unit circle, $\sec t = \frac{1}{x}, x \neq 0$

tangent
the quotient of the sine and cosine: on the unit circle, $\tan t = \frac{y}{x}, x \neq 0$

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5.3E: Exercises

5.3: The Other Trigonometric Functions

Verbal

1) On an interval of $[0, 2\pi)$, can the sine and cosine values of a radian measure ever be equal? If so, where?

Answer

Yes, when the reference angle is $\frac{\pi}{4}$ and the terminal side of the angle is in quadrants I and III. Thus, at $x = \frac{\pi}{4}, \frac{5\pi}{4}$, the sine and cosine values are equal.

2) What would you estimate the cosine of π degrees to be? Explain your reasoning.

3) For any angle in quadrant II, if you knew the sine of the angle, how could you determine the cosine of the angle?

Answer

Substitute the sine of the angle in for y in the Pythagorean Theorem $x^2 + y^2 = 1$. Solve for x and take the negative solution.

4) Describe the secant function.

5) Tangent and cotangent have a period of π. What does this tell us about the output of these functions?

Answer

The outputs of tangent and cotangent will repeat every π units.

Algebraic

For the exercises 6-17, find the exact value of each expression.

6) $\tan \frac{\pi}{6}$

7) $\sec \frac{\pi}{6}$

Answer

$\frac{2\sqrt{3}}{3}$

8) $\csc \frac{\pi}{6}$

9) $\cot \frac{\pi}{6}$

Answer

$\sqrt{3}$

10) $\tan \frac{\pi}{4}$

11) $\sec \frac{\pi}{4}$

Answer

$\sqrt{2}$

12) $\csc \frac{\pi}{4}$

13) $\cot \frac{\pi}{4}$

Answer

1

14) $\tan \frac{\pi}{3}$

15) $\sec \frac{\pi}{3}$

Answer

2

16) $\csc \frac{\pi}{3}$

17) $\cot \frac{\pi}{3}$

Answer

$\frac{\sqrt{3}}{3}$

For the exercises 18-48, use reference angles to evaluate the expression.

18) $\tan \frac{5\pi}{6}$

19) $\sec \frac{7\pi}{6}$

Answer

$-\frac{2\sqrt{3}}{3}$

20) $\csc \frac{11\pi}{6}$

21) $\cot \frac{13\pi}{6}$

Answer

$\sqrt{3}$

22) $\tan \frac{7\pi}{4}$

23) $\sec \frac{3\pi}{4}$

Answer

$-\sqrt{2}$

24) $\csc \frac{5\pi}{4}$

25) $\cot \frac{11\pi}{4}$

Answer

-1

26) $\tan \frac{8\pi}{3}$

27) $\sec \frac{4\pi}{3}$

Answer

-2

28) $\csc \frac{2\pi}{3}$

29) $\cot \frac{5\pi}{3}$

Answer

$-\frac{\sqrt{3}}{3}$

30) $\tan 225°$

31) $\sec 300°$

Answer

2

32) $\csc 150°$

33) $\cot 240°$

Answer

$\frac{\sqrt{3}}{3}$

34) $\tan 330°$

35) $\sec 120°$

Answer

-2

36) $\csc 210°$

37) $\cot 315°$

Answer

-1

38) If $\sin t = \frac{3}{4}$, and t is in quadrant II, find $\cos t, \sec t, \csc t, \tan t, \cot t$.

39) If $\cos t = -\frac{1}{3}$, and t is in quadrant III, find $\sin t, \sec t, \csc t, \tan t, \cot t$.

Answer

If $\sin t = -\frac{2\sqrt{2}}{3}, \sec t = -3, \csc t = -\csc t = -\frac{3\sqrt{2}}{4}, \tan t = 2\sqrt{2}, \cot t = \frac{\sqrt{2}}{4}$

40) If $\tan t = \frac{12}{5}$, and $0 \le t < \frac{\pi}{2}$, find $\sin t, \cos t, \sec t, \csc t$, and $\cot t$.

41) If $\sin t = \frac{\sqrt{3}}{2}$ and $\cos t = \frac{1}{2}$, find $\sec t, \csc t, \tan t$, and $\cot t$.

Answer

$\sec t = 2, \csc t = \csc t = \frac{2\sqrt{3}}{3}, \tan t = \sqrt{3}, \cot t = \frac{\sqrt{3}}{3}$

42) If $\sin 40° \approx 0.643 \ \cos 40° \approx 0.766 \ \sec 40°, \csc 40°, \tan 40°, \text{ and } \cot 40°$.

43) If $\sin t = \frac{\sqrt{2}}{2}$, what is the $\sin(-t)$?

Answer

$-\frac{\sqrt{2}}{2}$

44) If $\cos t = \frac{1}{2}$, what is the $\cos(-t)$?

45) If $\sec t = 3.1$, what is the $\sec(-t)$?

Answer

3.1

46) If $\csc t = 0.34$, what is the $\csc(-t)$?

47) If $\tan t = -1.4$, what is the $\tan(-t)$?

Answer

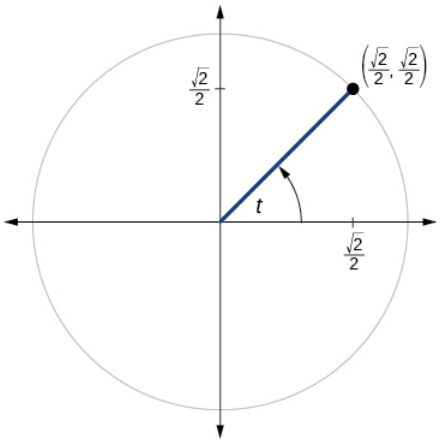
1.4

48) If $\cot t = 9.23$, what is the $\cot(-t)$?

Graphical

For the exercises 49-51, use the angle in the unit circle to find the value of the each of the six trigonometric functions.

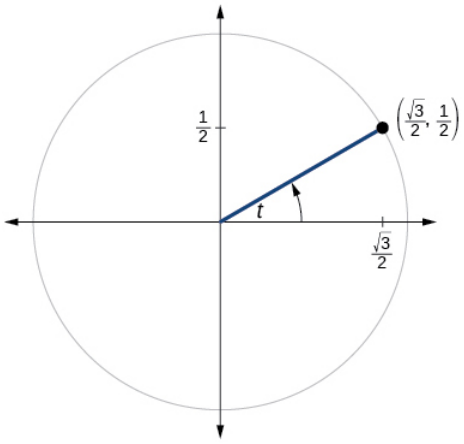
49)



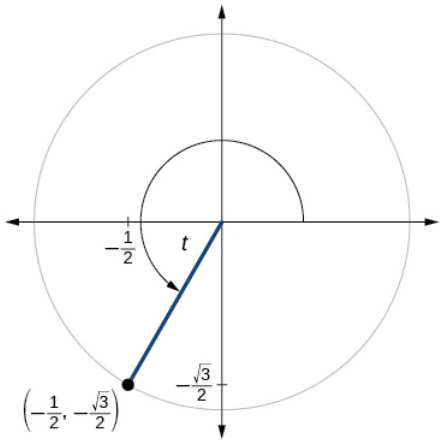
Answer

$\sin t = \frac{\sqrt{2}}{2}, \cos t = \frac{\sqrt{2}}{2}, \tan t = 1, \cot t = 1, \sec t = \sqrt{2}, \csc t = \csc t = \sqrt{2}$

50)



51)



Answer

$$\sin t = -\frac{\sqrt{3}}{2}, \cos t = -\frac{1}{2}, \tan t = \sqrt{3}, \cot t = \frac{\sqrt{3}}{3}, \sec t = -2, \csc t = -\csc t = -\frac{2\sqrt{3}}{3}$$

Technology

For the exercises 52-61, use a graphing calculator to evaluate.

52) $\csc \frac{5\pi}{9}$

53) $\cot \frac{4\pi}{7}$

Answer

-0.228

54) $\sec \frac{\pi}{10}$

55) $\tan \frac{5\pi}{8}$

Answer

-2.414

56) $\sec \frac{3\pi}{4}$

57) $\csc \frac{\pi}{4}$

Answer

1.414

58) $\tan 98°$

59) $\cot 33°$

Answer

1.540

60) $\cot 140°$

61) $\sec 310°$

Answer

1.556

Extensions

For the exercises 62-69, use identities to evaluate the expression.

62) If $\tan(t) \approx 2.7$, and $\sin(t) \approx 0.94$, find $\cos(t)$.

63) If $\tan(t) \approx 1.3$, and $\cos(t) \approx 0.61$, find $\sin(t)$.

Answer

$\sin(t) \approx 0.79$

64) If $\csc(t) \approx 3.2$, and $\csc(t) \approx 3.2$, and $\cos(t) \approx 0.95$, find $\tan(t)$.

65) If $\cot(t) \approx 0.58$, and $\cos(t) \approx 0.5$, find $\csc(t)$.

Answer

$\csc(t) \approx 1.16$

66) Determine whether the function $f(x) = 2\sin x \cos x$ is even, odd, or neither.

67) Determine whether the function $f(x) = 3\sin^2 x \cos x + \sec x$ is even, odd, or neither.

Answer

even

68) Determine whether the function $f(x) = \sin x - 2\cos^2 x$ is even, odd, or neither.

69) Determine whether the function $f(x) = \csc^2 x + \sec x$ is even, odd, or neither.

Answer

even

For the exercises 70-71, use identities to simplify the expression.

70) $\csc t \tan t$

71) $\dfrac{\sec t}{\csc t}$

Answer

$\dfrac{\sin t}{\cos t} = \tan t$

Real-World Applications

72) The amount of sunlight in a certain city can be modeled by the function $h = 15\cos\left(\dfrac{1}{600}d\right)$, where h represents the hours of sunlight, and d is the day of the year. Use the equation to find how many hours of sunlight there are on February 10, the 42nd day of the year. State the period of the function.

73) The amount of sunlight in a certain city can be modeled by the function $h = 16\cos\left(\dfrac{1}{500}d\right)$, where h represents the hours of sunlight, and d is the day of the year. Use the equation to find how many hours of sunlight there are on September 24, the 267th day of the year. State the period of the function.

Answer

13.77 hours, period: 1000π

74) The equation $P = 20\sin(2\pi t) + 100$ models the blood pressure, P, where t represents time in seconds.

a. Find the blood pressure after 15 seconds.
b. What are the maximum and minimum blood pressures?

75) The height of a piston, h, in inches, can be modeled by the equation $y = 2\cos x + 6$, where x represents the crank angle. Find the height of the piston when the crank angle is $55°$.

Answer

7.73 inches

76) The height of a piston, h,in inches, can be modeled by the equation $y = 2\cos x + 5$, where x represents the crank angle. Find the height of the piston when the crank angle is $55°$.

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5.4: Right Triangle Trigonometry

Skills to Develop

- Use right triangles to evaluate trigonometric functions.
- Find function values for 30° $\left(\frac{\pi}{6}\right)$, 45° $\left(\frac{\pi}{4}\right)$, and 60° $\left(\frac{\pi}{3}\right)$.
- Use equal cofunctions of complementary angles.
- Use the definitions of trigonometric functions of any angle.
- Use right-triangle trigonometry to solve applied problems.

Mt. Everest, which straddles the border between China and Nepal, is the tallest mountain in the world. Measuring its height is no easy task and, in fact, the actual measurement has been a source of controversy for hundreds of years. The measurement process involves the use of triangles and a branch of mathematics known as trigonometry. In this section, we will define a new group of functions known as trigonometric functions, and find out how they can be used to measure heights, such as those of the tallest mountains.

We have previously defined the sine and cosine of an angle in terms of the coordinates of a point on the unit circle intersected by the terminal side of the angle:

$$\cos(t) = x \tag{5.4.1}$$
$$\sin(t) = y \tag{5.4.2}$$

In this section, we will see another way to define trigonometric functions using properties of *right triangles*.

Using Right Triangles to Evaluate Trigonometric Functions

In earlier sections, we used a unit circle to define the *trigonometric functions*. In this section, we will extend those definitions so that we can apply them to right triangles. The value of the sine or cosine function of t is its value at t radians. First, we need to create our right triangle. Figure 5.4.1 shows a point on a unit circle of radius 1. If we drop a vertical line segment from the point (x, y) to the x-axis, we have a right triangle whose vertical side has length y and whose horizontal side has length x. We can use this right triangle to redefine sine, cosine, and the other trigonometric functions as ratios of the sides of a right triangle.

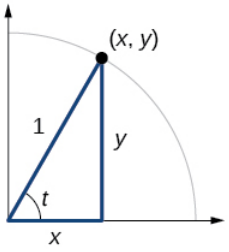


Figure 5.4.1

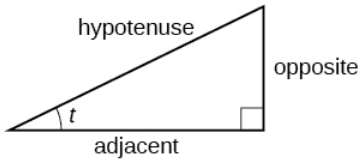
We know

$$\cos(t) = \frac{x}{1} = x \tag{5.4.3}$$

Likewise, we know

$$\sin(t) = \frac{y}{1} = y \tag{5.4.4}$$

These ratios still apply to the sides of a right triangle when no unit circle is involved and when the triangle is not in standard position and is not being graphed using (x, y) coordinates. To be able to use these ratios freely, we will give the sides more general names: Instead of x,we will call the side between the given angle and the right angle the **adjacent side** to angle t. (Adjacent means "next to.") Instead of y,we will call the side most distant from the given angle the **opposite side** from angle t. And instead of 1,we will call the side of a right triangle opposite the right angle the **hypotenuse**. These sides are labeled in Figure 5.4.2.



5.4.2: *The sides of a right triangle in relation to angle* t.

Understanding Right Triangle Relationships

Given a right triangle with an acute angle of t,

$$\sin(t) = \frac{\text{opposite}}{\text{hypotenuse}} \tag{5.4.5}$$

$$\cos(t) = \frac{\text{adjacent}}{\text{hypotenuse}} \tag{5.4.6}$$

$$\tan(t) = \frac{\text{opposite}}{\text{adjacent}} \tag{5.4.7}$$

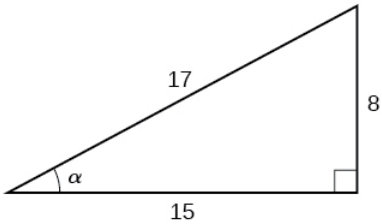
A common mnemonic for remembering these relationships is SohCahToa, formed from the first letters of "Sine is opposite over hypotenuse, Cosine is adjacent over hypotenuse, Tangent is opposite over adjacent."

How To: Given the side lengths of a right triangle and one of the acute angles, find the sine, cosine, and tangent of that angle

1. Find the sine as the ratio of the opposite side to the hypotenuse.
2. Find the cosine as the ratio of the adjacent side to the hypotenuse.
3. Find the tangent is the ratio of the opposite side to the adjacent side.

Example 5.4.1: Evaluating a Trigonometric Function of a Right Triangle

Given the triangle shown in Figure 5.4.3, find the value of $\cos(\alpha)$.



5.4.3

Solution

The side adjacent to the angle is 15, and the hypotenuse of the triangle is 17, so:

$$\cos(\alpha) = \frac{\text{adjacent}}{\text{hypotenuse}} \tag{5.4.8}$$

$$= \frac{15}{17} \tag{5.4.9}$$

Try It 5.4.1 Given the triangle shown in Figure 5.4.4, find the value of $\sin t$.

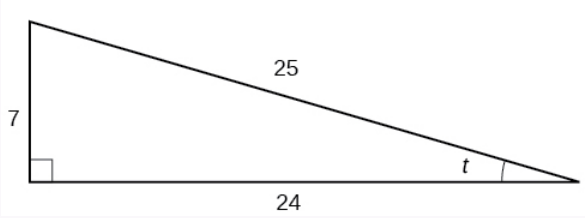


Figure 5.4.4

Answer

$\frac{7}{25}$

Relating Angles and Their Functions

When working with right triangles, the same rules apply regardless of the orientation of the triangle. In fact, we can evaluate the six trigonometric functions of either of the two acute angles in the triangle in Figure 5.4.5. The side opposite one acute angle is the side adjacent to the other acute angle, and vice versa.

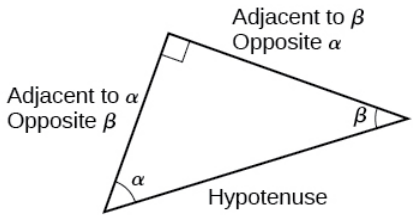


Figure 5.4.5*: The side adjacent to one angle is opposite the other.*

We will be asked to find all six trigonometric functions for a given angle in a triangle. Our strategy is to find the sine, cosine, and tangent of the angles first. Then, we can find the other trigonometric functions easily because we know that the reciprocal of sine is cosecant, the reciprocal of cosine is secant, and the reciprocal of tangent is cotangent.

How To: Given the side lengths of a right triangle, evaluate the six trigonometric functions of one of the acute angles.

1. If needed, draw the right triangle and label the angle provided.
2. Identify the angle, the adjacent side, the side opposite the angle, and the hypotenuse of the right triangle.
3. Find the required function:
   - sine as the ratio of the opposite side to the hypotenuse
   - cosine as the ratio of the adjacent side to the hypotenuse
   - tangent as the ratio of the opposite side to the adjacent side
   - secant as the ratio of the hypotenuse to the adjacent side
   - cosecant as the ratio of the hypotenuse to the opposite side
   - cotangent as the ratio of the adjacent side to the opposite side

Example 5.4.2: Evaluating Trigonometric Functions of Angles Not in Standard Position

Using the triangle shown in Figure 5.4.6, evaluate $\sin(\alpha), \cos(\alpha), \tan(\alpha), \sec(\alpha), \csc(\alpha)$, and $\cot(\alpha)$.

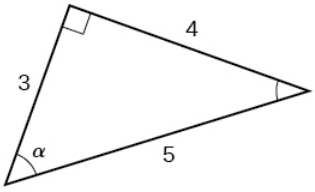


Figure 5.4.6

Solution

$$\sin(\alpha) = \frac{\text{opposite } \alpha}{\text{hypotenuse}} = \frac{4}{5} \tag{5.4.10}$$

$$\cos(\alpha) = \frac{\text{adjacent to } \alpha}{\text{hypotenuse}} = \frac{3}{5} \tag{5.4.11}$$

$$\tan(\alpha) = \frac{\text{opposite } \alpha}{\text{adjacent to } \alpha} = \frac{4}{3} \tag{5.4.12}$$

$$\sec(\alpha) = \frac{\text{hypotenuse}}{\text{adjacent to } \alpha} = \frac{5}{3} \tag{5.4.13}$$

$$\csc(\alpha) = \frac{\text{hypotenuse}}{\text{opposite } \alpha} = \frac{5}{4} \tag{5.4.14}$$

$$\cot(\alpha) = \frac{\text{adjacent to } \alpha}{\text{opposite } \alpha} = \frac{3}{4} \tag{5.4.15}$$

5.4.2 Using the triangle shown in Figure 5.4.7, evaluate $\sin(t), \cos(t), \tan(t), \sec(t), \csc(t)$, and $\cot(t)$.

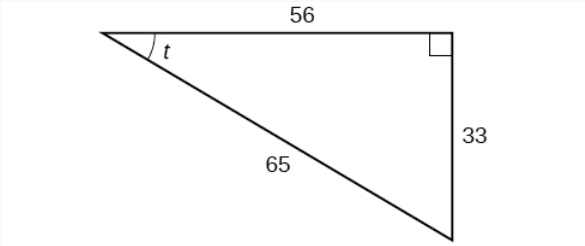


Figure 5.4.7

Answer

$$\sin(t) = \frac{33}{65}, \cos(t) = \frac{56}{65}, \tan(t) = \frac{33}{56}, \tag{5.4.16}$$

(5.4.17)

$$\sec(t) = \frac{65}{56}, \csc(t) = \frac{65}{33}, \cot(t) = \frac{56}{33} \tag{5.4.18}$$

Finding Trigonometric Functions of Special Angles Using Side Lengths

We have already discussed the trigonometric functions as they relate to the **special angles** on the unit circle. Now, we can use those relationships to evaluate triangles that contain those special angles. We do this because when we evaluate the special angles in trigonometric functions, they have relatively friendly values, values that contain either no or just one square root in the ratio. Therefore, these are the angles often used in math and science problems. We will use multiples of $30°, 60°$, and $45°$, however, remember that when dealing with right triangles, we are limited to angles between $0°$ and $90°$.

Suppose we have a $30°, 60°, 90°$ triangle, which can also be described as a $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$ triangle. The sides have lengths in the relation $s, \sqrt{3}s, 2s$. The sides of a $45°, 45°, 90°$ triangle, which can also be described as a $\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}$ triangle, have lengths in the relation $s, s, \sqrt{2}s$. These relations are shown in Figure 5.4.8. Note that we have placed our triangles inside a circle centered at the origin, with one of the special angles in standard position. Each circle has an unknown radius, *not necessarily 1*, which becomes the hypotenuse of each triangle.

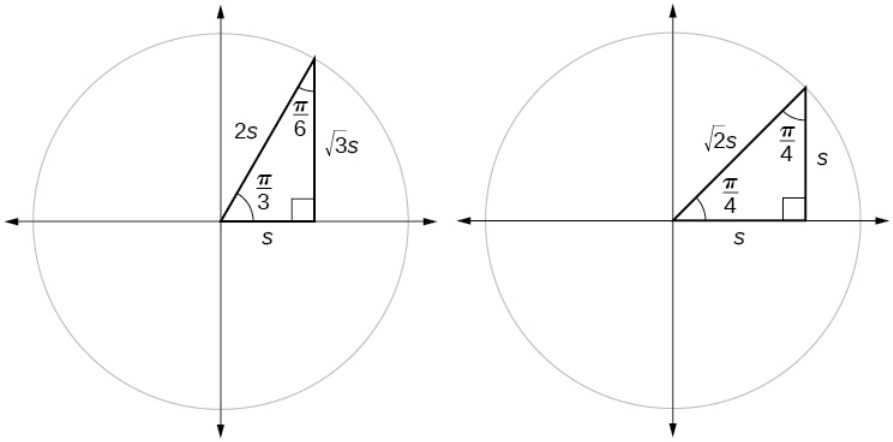


Figure 5.4.8*: Side lengths of special triangles*

We can then use the ratios of the side lengths to evaluate trigonometric functions of special angles.

How To: Given trigonometric functions of a special angle, evaluate using side lengths.

1. Use the side lengths shown in Figure 5.4.8 for the special angle you wish to evaluate.
2. Use the ratio of side lengths appropriate to the function you wish to evaluate.

Example 5.4.3: Evaluating Trigonometric Functions of Special Angles Using Side Lengths

Find the exact value of the trigonometric functions of $\frac{\pi}{3}$, using side lengths.

Solution

$$\sin\left(\frac{\pi}{3}\right) = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}s}{2s} = \frac{\sqrt{3}}{2} \tag{5.4.19}$$

$$\cos\left(\frac{\pi}{3}\right) = \frac{\text{adj}}{\text{hyp}} = \frac{s}{2s} = \frac{1}{2} \tag{5.4.20}$$

$$\tan\left(\frac{\pi}{3}\right) = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{3}s}{s} = \sqrt{3} \tag{5.4.21}$$

$$\sec\left(\frac{\pi}{3}\right) = \frac{\text{hyp}}{\text{adj}} = \frac{2s}{s} = 2 \tag{5.4.22}$$

$$\csc\left(\frac{\pi}{3}\right) = \frac{\text{hyp}}{\text{opp}} = \frac{2s}{\sqrt{3}s} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \tag{5.4.23}$$

$$\cot\left(\frac{\pi}{3}\right) = \frac{\text{adj}}{\text{opp}} = \frac{s}{\sqrt{3}s} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \tag{5.4.24}$$

Analysis

Since the circles in Figure 5.4.8 are no longer unit circles, $\sin\left(\frac{\pi}{3}\right)$ and $\cos\left(\frac{\pi}{3}\right)$ cannot be determined from the (x, y)-coordinates of the point on the circumference of the circle. On the other hand, we can use the sine and cosine ratios to compute those (x, y)-coordinates: $y = 2s\sin\left(\frac{\pi}{3}\right)$, and $x = 2s\cos\left(\frac{\pi}{3}\right)$. In general, given a circle of radius r centered at the origin and a point P on the circumference, the (x, y)-coordinates of P are $(r\cos\theta, r\sin\theta)$, where θ is the angle in standard position whose terminal side passes through P.

Try It 5.4.3 Find the exact value of the trigonometric functions of $\frac{\pi}{4}$, using side lengths.

Find the exact value of the trigonometric functions of $\frac{\pi}{4}$, using side lengths.

Answer

$$\sin\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2},\cos\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2},\tan\left(\frac{\pi}{4}\right)=1,$$
$$\sec\left(\frac{\pi}{4}\right)=\sqrt{2},\csc\left(\frac{\pi}{4}\right)=\sqrt{2},\cot\left(\frac{\pi}{4}\right)=1$$

Using Equal Cofunction of Complements

If we look more closely at the relationship between the sine and cosine of the special angles relative to the unit circle, we will notice a pattern. In a right triangle with angles of $\frac{\pi}{6}$ and $\frac{\pi}{3}$, we see that the sine of $\frac{\pi}{3}$, namely $\frac{\sqrt{3}}{2}$, is also the cosine of $\frac{\pi}{6}$, while the sine of $\frac{\pi}{6}$, namely $\frac{1}{2}$, is also the cosine of $\frac{\pi}{3}$. See Figure 5.4.9.

$$\sin\frac{\pi}{3}=\cos\frac{\pi}{6}=\frac{\sqrt{3}s}{2s}=\frac{\sqrt{3}}{2} \tag{5.4.25}$$
$$\sin\frac{\pi}{6}=\cos\frac{\pi}{3}=\frac{s}{2s}=\frac{1}{2} \tag{5.4.26}$$

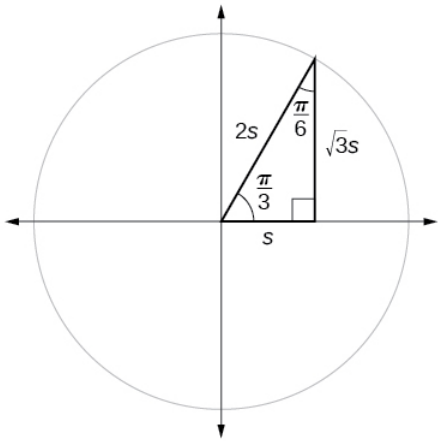
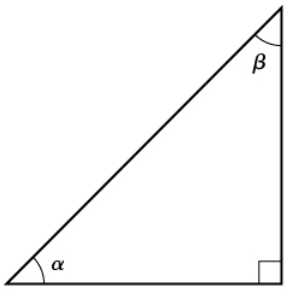


Figure **5.4.9***: The sine of* $\frac{\pi}{3}$ *equals the cosine of* $\frac{\pi}{6}$ *and vice versa.*

This result should not be surprising because, as we see from Figure 5.4.9, the side opposite the angle of $\frac{\pi}{3}$ is also the side adjacent to $\frac{\pi}{6}$, so $\sin\left(\frac{\pi}{3}\right)$ and $\cos\left(\frac{\pi}{6}\right)$ are exactly the same ratio of the same two sides, $\sqrt{3}s$ and $2s$. Similarly, $\cos\left(\frac{\pi}{3}\right)$ and $\sin\left(\frac{\pi}{6}\right)$ are also the same ratio using the same two sides, s and $2s$.

The interrelationship between the sines and cosines of $\frac{\pi}{6}$ and $\frac{\pi}{3}$ also holds for the two acute angles in any right triangle, since in every case, the ratio of the same two sides would constitute the sine of one angle and the cosine of the other. Since the three angles of a triangle add to π, and the right angle is $\frac{\pi}{2}$, the remaining two angles must also add up to $\frac{\pi}{2}$. That means that a right triangle can be formed with any two angles that add to $\frac{\pi}{2}$ — in other words, any two complementary angles. So we may state a *cofunction identity*: If any two angles are complementary, the sine of one is the cosine of the other, and vice versa. This identity is illustrated in Figure 5.4.10. In fact this identity, as well as the others given in Table 5.4.1, hold true for any angle, not just acute angles.



$\sin\alpha = \cos\beta$
$\sin\beta = \cos\alpha$

Figure **5.4.10***: Cofunction identity of sine and cosine of complementary angles*

Using this identity, we can state without calculating, for instance, that the sine of $\frac{\pi}{12}$ equals the cosine of $\frac{5\pi}{12}$, and that the sine of $\frac{5\pi}{12}$ equals the cosine of $\frac{\pi}{12}$. We can also state that if, for a certain angle t, $\cos t = \frac{5}{13}$, then $\sin(\frac{\pi}{2} - t) = \frac{5}{13}$ as well. Cofunction identities also hold true for secant and cosecant, and for tangent and cotangent.

COFUNCTION IDENTITIES

The **cofunction identities** for any angle t given in radians are listed in Table 5.4.1.

Table 5.4.1

$\cos t = \sin(\frac{\pi}{2} - t)$	(5.4.27)	$\sin t = \cos(\frac{\pi}{2} - t)$	(5.4.28)
$\tan t = \cot(\frac{\pi}{2} - t)$	(5.4.29)	$\cot t = \tan(\frac{\pi}{2} - t)$	(5.4.30)
$\sec t = \csc(\frac{\pi}{2} - t)$	(5.4.31)	$\csc t = \sec(\frac{\pi}{2} - t)$	(5.4.32)

How To: Given the sine and cosine of an angle, find the sine or cosine of its complement.

1. To find the sine of the complementary angle, find the cosine of the original angle.
2. To find the cosine of the complementary angle, find the sine of the original angle.

Example 5.4.4:

Using Cofunction Identities

If $\sin t = \frac{5}{12}$, find $\cos\left(\frac{\pi}{2} - t\right)$.

Solution

According to the cofunction identities for sine and cosine,

$$\sin(t) = \cos(\frac{\pi}{2} - t). \tag{5.4.33}$$

So

$$\cos(\frac{\pi}{2} - t) = \frac{5}{12}. \tag{5.4.34}$$

Try It: 5.4.4:

If $\csc(\frac{\pi}{6}) = 2$, find $\sec(\frac{\pi}{3})$.

Answer

2

Using Trigonometric Functions

In previous examples, we evaluated the sine and cosine in triangles where we knew all three sides. But the real power of right-triangle trigonometry emerges when we look at triangles in which we know an angle but do not know all the sides.

How To: Given a right triangle, the length of one side, and the measure of one acute angle, find the remaining sides.

1. For each side, select the trigonometric function that has the unknown side as either the numerator or the denominator. The known side will in turn be the denominator or the numerator.
2. Write an equation setting the function value of the known angle equal to the ratio of the corresponding sides.

3. Using the value of the trigonometric function and the known side length, solve for the missing side length.

Example 5.4.5: Finding Missing Side Lengths Using Trigonometric Ratios

Find the unknown sides of the triangle in Figure 5.4.11.

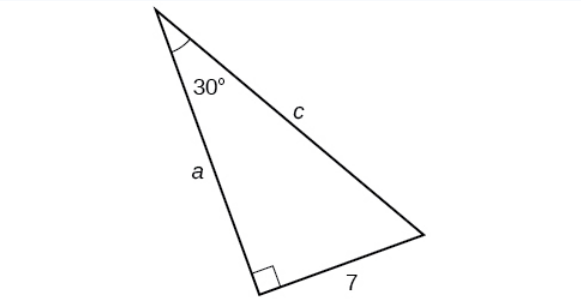


Figure **5.4.11**

Solution

We know the angle and the opposite side, so we can use the tangent to find the adjacent side.

$$\tan(30°) = \frac{7}{a} \tag{5.4.35}$$

We rearrange to solve for a.

$$a = \frac{7}{\tan(30°)} \tag{5.4.36}$$

$$= 12.1 \tag{5.4.37}$$

We can use the sine to find the hypotenuse.

$$\sin(30°) = \frac{7}{c} \tag{5.4.38}$$

Again, we rearrange to solve for c.

$$c = \frac{7}{\sin(30°)} = 14 \tag{5.4.39}$$

5.4.5:

A right triangle has one angle of $\frac{\pi}{3}$ and a hypotenuse of 20. Find the unknown sides and angle of the triangle.

Answer

$\text{adjacent} = 10; \text{opposite} = 10\sqrt{3};$ missing angle is $\frac{\pi}{6}$

Using Right Triangle Trigonometry to Solve Applied Problems

Right-triangle trigonometry has many practical applications. For example, the ability to compute the lengths of sides of a triangle makes it possible to find the height of a tall object without climbing to the top or having to extend a tape measure along its height. We do so by measuring a distance from the base of the object to a point on the ground some distance away, where we can look up to the top of the tall object at an angle. The **angle of elevation** of an object above an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. The right triangle this position creates has sides that represent the unknown height, the measured distance from the base, and the angled line of sight from the ground to the top of the object. Knowing the measured distance to the base of the object and the angle of the line of sight, we can use trigonometric functions to calculate the unknown height. Similarly, we can form a triangle from the top of a tall object by looking downward. The **angle of depression** of an object below an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. See Figure 5.4.12.

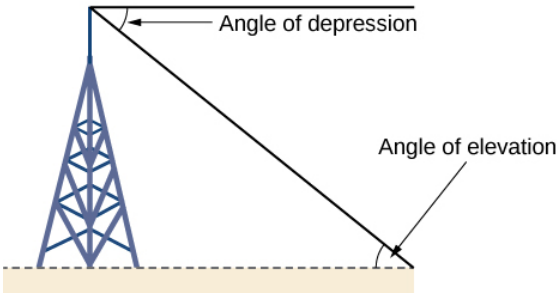


Figure **5.4.12**

How To: Given a tall object, measure its height indirectly.

1. Make a sketch of the problem situation to keep track of known and unknown information.
2. Lay out a measured distance from the base of the object to a point where the top of the object is clearly visible.
3. At the other end of the measured distance, look up to the top of the object. Measure the angle the line of sight makes with the horizontal.
4. Write an equation relating the unknown height, the measured distance, and the tangent of the angle of the line of sight.
5. Solve the equation for the unknown height.

Example 5.4.6: Measuring a Distance Indirectly

To find the height of a tree, a person walks to a point 30 feet from the base of the tree. She measures an angle of 57° between a line of sight to the top of the tree and the ground, as shown in Figure 5.4.13. Find the height of the tree.

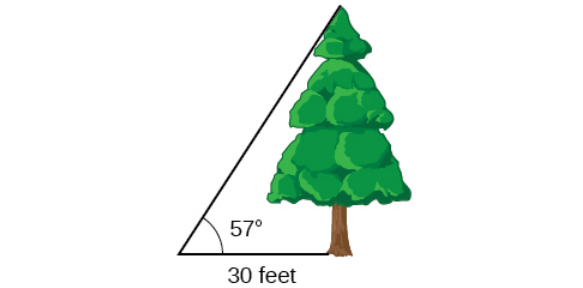


Figure **5.4.13**

Solution

We know that the angle of elevation is $57°$ and the adjacent side is 30 ft long. The opposite side is the unknown height.

The trigonometric function relating the side opposite to an angle and the side adjacent to the angle is the tangent. So we will state our information in terms of the tangent of $57°$, letting h be the unknown height.

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan(57°) = \frac{h}{30} \quad \text{Solve for } h.$$

$$h = 30\tan(57°) \quad \text{Multiply.}$$

$$h \approx 46.2 \quad \text{Use a calculator.}$$

The tree is approximately 46 feet tall.

Try It 5.4.6:

How long a ladder is needed to reach a windowsill 50 feet above the ground if the ladder rests against the building making an angle of $\frac{5\pi}{12}$ with the ground? Round to the nearest foot.

Answer

About 52 ft

media:

Access these online resources for additional instruction and practice with right triangle trigonometry.

- Finding Trig Functions on Calculator
- Finding Trig Functions Using a Right Triangle
- Relate Trig Functions to Sides of a Right Triangle
- Determine Six Trig Functions from a Triangle
- Determine Length of Right Triangle Side

Visit this website for additional practice questions from Learningpod.

Key Equations

Cofunction Identities	$\cos t = \sin(\frac{\pi}{2} - t)$ (5.4.40) $\sin t = \cos(\frac{\pi}{2} - t)$ (5.4.41) $\tan t = \cot(\frac{\pi}{2} - t)$ (5.4.42) $\cot t = \tan(\frac{\pi}{2} - t)$ (5.4.43) $\sec t = \csc(\frac{\pi}{2} - t)$ (5.4.44) $\csc t = \sec(\frac{\pi}{2} - t)$ (5.4.45)

Key Concepts

- We can define trigonometric functions as ratios of the side lengths of a right triangle. See Example.
- The same side lengths can be used to evaluate the trigonometric functions of either acute angle in a right triangle. See Example.
- We can evaluate the trigonometric functions of special angles, knowing the side lengths of the triangles in which they occur. See Example.
- Any two complementary angles could be the two acute angles of a right triangle.
- If two angles are complementary, the cofunction identities state that the sine of one equals the cosine of the other and vice versa. See Example.
- We can use trigonometric functions of an angle to find unknown side lengths.
- Select the trigonometric function representing the ratio of the unknown side to the known side. See Example.
- Right-triangle trigonometry permits the measurement of inaccessible heights and distances.
- The unknown height or distance can be found by creating a right triangle in which the unknown height or distance is one of the sides, and another side and angle are known. See Example.

Glossary

adjacent side

in a right triangle, the side between a given angle and the right angle

angle of depression

the angle between the horizontal and the line from the object to the observer's eye, assuming the object is positioned lower than the observer

angle of elevation

the angle between the horizontal and the line from the object to the observer's eye, assuming the object is positioned higher than the observer

opposite side

in a right triangle, the side most distant from a given angle

hypotenuse

the side of a right triangle opposite the right angle

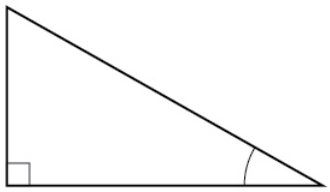
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5.4E: Exercises

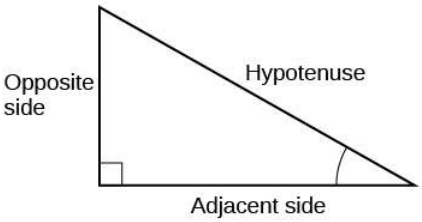
5.4: Right Triangle Trigonometry

Verbal

1) For the given right triangle, label the adjacent side, opposite side, and hypotenuse for the indicated angle.



Answer



2) When a right triangle with a hypotenuse of 1 is placed in the unit circle, which sides of the triangle correspond to the x- and y-coordinates?

3) The tangent of an angle compares which sides of the right triangle?

Answer

The tangent of an angle is the ratio of the opposite side to the adjacent side.

4) What is the relationship between the two acute angles in a right triangle?

5) Explain the cofunction identity.

Answer

For example, the sine of an angle is equal to the cosine of its complement; the cosine of an angle is equal to the sine of its complement.

Algebraic

For the exercises 6-9, use cofunctions of complementary angles.

6) $\cos(34°) = \sin(\_°)$

7) $\cos(\frac{\pi}{3}) = \sin(\_\_)$

Answer

$\frac{\pi}{6}$

8) $\csc(21°) = \sec(\_\_°)$

9) $\tan(\frac{\pi}{4}) = \cot(\_)$

Answer

$\frac{\pi}{4}$

For the exercises 10-16, find the lengths of the missing sides if side a is opposite angle A, side b is opposite angle B, and side c is the hypotenuse.

10) $\cos B = \frac{4}{5}, a = 10$

11) $\sin B = \frac{1}{2}, a = 20$

Answer

$b = \frac{20\sqrt{3}}{3}, c = \frac{40\sqrt{3}}{3}$

12) $\tan A = \frac{5}{12}, b = 6$

13) $\tan A = 100, b = 100$

Answer

$a = 10,000, c = 10,000.5$

14) $\sin B = \frac{1}{\sqrt{3}}, a = 2$

15) $a = 5, \measuredangle A = 60°$

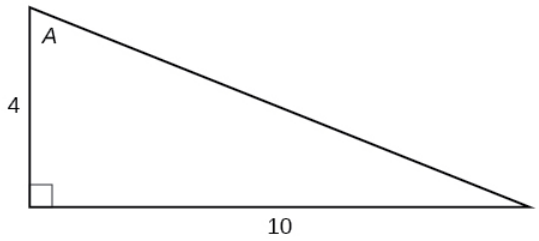
Answer

$b = \frac{5\sqrt{3}}{3}, c = \frac{10\sqrt{3}}{3}$

16) $c = 12, \measuredangle A = 45°$

Graphical

For the exercises 17-22, use Figure below to evaluate each trigonometric function of angle A.



17) $\sin A$

Answer

$\frac{5\sqrt{29}}{29}$

18) $\cos A$

19) $\tan A$

Answer

$\frac{5}{2}$

20) $\csc A$

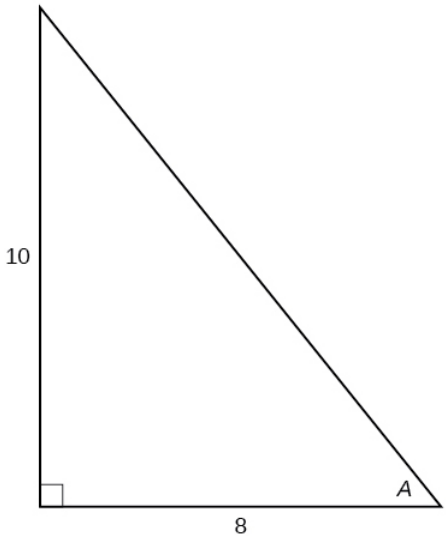
21) $\sec A$

Answer

$\frac{\sqrt{29}}{2}$

22) $\cot A$

For the exercises 23-,28 use Figure below to evaluate each trigonometric function of angle A.



23) $\sin A$

Answer

$\frac{5\sqrt{41}}{41}$

24) $\cos A$

25) $\tan A$

Answer

$\frac{5}{4}$

26) $\csc A$

27) $\sec A$

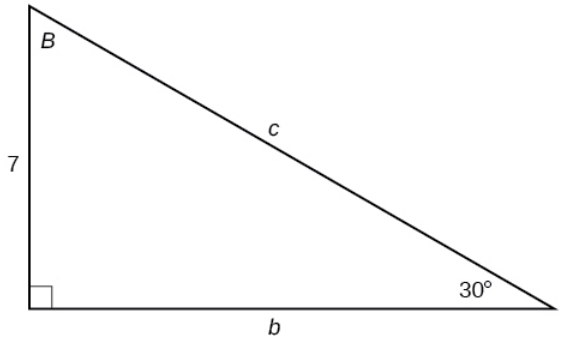
Answer

$\frac{\sqrt{41}}{4}$

28) $\cot A$

For the exercises 29-31, solve for the unknown sides of the given triangle.

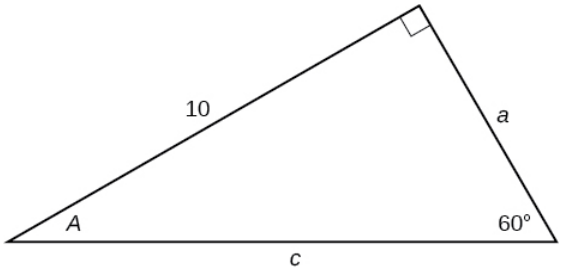
29)



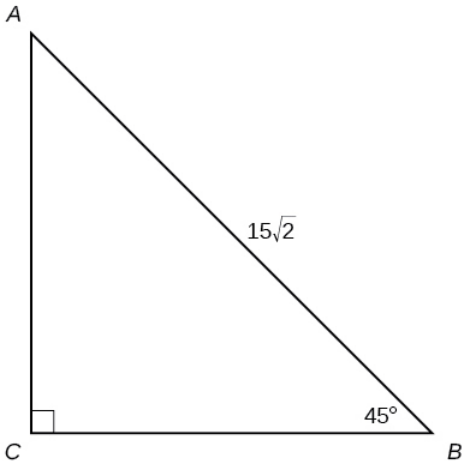
Answer

$c = 14, b = 7\sqrt{3}$

30)



31)



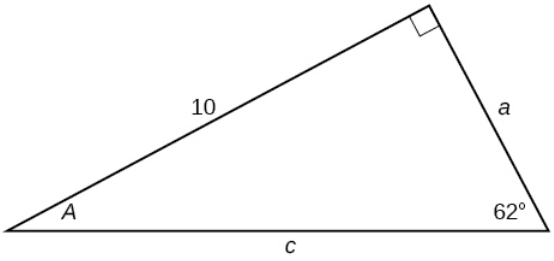
Answer

$a = 15, b = 15$

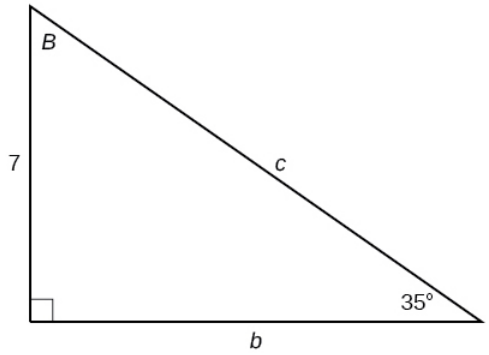
Technology

For the exercises 32-41, use a calculator to find the length of each side to four decimal places.

32)



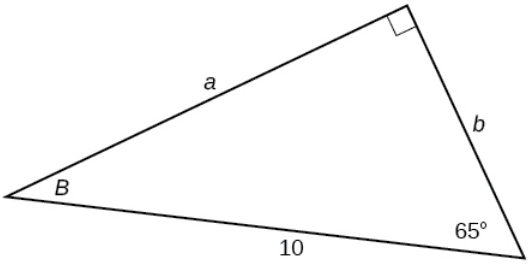
33)



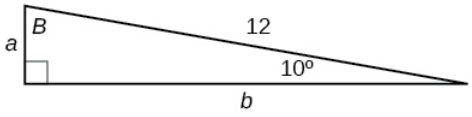
Answer

$b = 9.9970, c = 12.2041$

34)



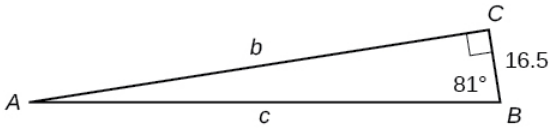
35)



Answer

$a = 2.0838, b = 11.8177$

36)



37) $b = 15, \measuredangle B = 15°$

Answer

$a = 55.9808, c = 57.9555$

38) $c = 200, \measuredangle B = 5°$

39) $c = 50, \measuredangle B = 21°$

Answer

$a = 46.6790, b = 17.9184$

40) $a = 30, \measuredangle A = 27°$

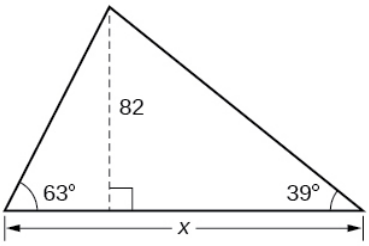
41) $b = 3.5, \measuredangle A = 78°$

Answer

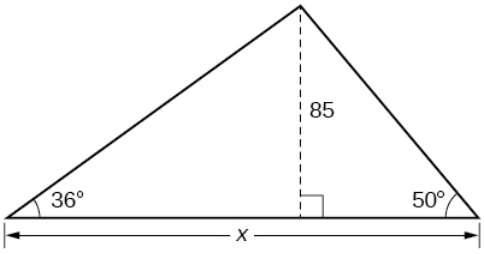
$a = 16.4662, c = 16.8341$

Extensions

42) Find x.



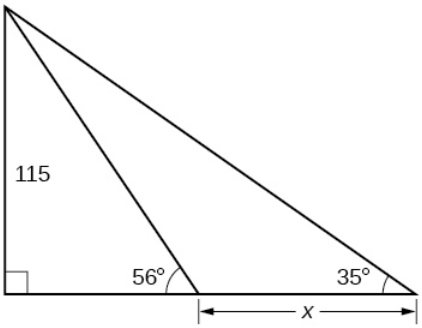
43) Find x.



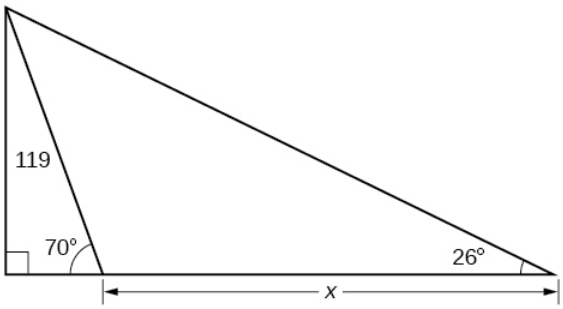
Answer

188.3159

44) Find x.



45) Find x.



Answer

200.6737

46) A radio tower is located 400 feet from a building. From a window in the building, a person determines that the angle of elevation to the top of the tower is 36°, and that the angle of depression to the bottom of the tower is 23°. How tall is the tower?

47) A radio tower is located 325 feet from a building. From a window in the building, a person determines that the angle of elevation to the top of the tower is 43°, and that the angle of depression to the bottom of the tower is 31°. How tall is the tower?

Answer

498.3471 ft

48) A 200-foot tall monument is located in the distance. From a window in a building, a person determines that the angle of elevation to the top of the monument is 15°, and that the angle of depression to the bottom of the tower is 2°. How far is the person from the monument?

49) A 400-foot tall monument is located in the distance. From a window in a building, a person determines that the angle of elevation to the top of the monument is 18°, and that the angle of depression to the bottom of the monument is 3°. How far is the person from the monument?

Answer

1060.09 ft

50) There is an antenna on the top of a building. From a location 300 feet from the base of the building, the angle of elevation to the top of the building is measured to be 40°. From the same location, the angle of elevation to the top of the antenna is measured to be 43°. Find the height of the antenna.

51) There is lightning rod on the top of a building. From a location 500 feet from the base of the building, the angle of elevation to the top of the building is measured to be 36°. From the same location, the angle of elevation to the top of the lightning rod is measured to be 38°. Find the height of the lightning rod.

Answer

27.372 ft

Real-World Applications

52) A 33-ft ladder leans against a building so that the angle between the ground and the ladder is 80°. How high does the ladder reach up the side of the building?

53) A 23-ft ladder leans against a building so that the angle between the ground and the ladder is 80°. How high does the ladder reach up the side of the building?

Answer

22.6506 ft

54) The angle of elevation to the top of a building in New York is found to be 9 degrees from the ground at a distance of 1 mile from the base of the building. Using this information, find the height of the building.

55) The angle of elevation to the top of a building in Seattle is found to be 2 degrees from the ground at a distance of 2 miles from the base of the building. Using this information, find the height of the building.

Answer

368.7633 ft

56) Assuming that a 370-foot tall giant redwood grows vertically, if I walk a certain distance from the tree and measure the angle of elevation to the top of the tree to be $60°$, how far from the base of the tree am I?

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CHAPTER OVERVIEW

6: Periodic Functions

In this chapter, we will investigate graphs of sine, cosine, and other trigonometric functions.

6.0: Prelude to Periodic Functions

6.0E: Exercises

6.1: Graphs of the Sine and Cosine Functions

6.1E: Exercises

6.2: Graphs of the Other Trigonometric Functions

6.2E: Exercises

6.3: Inverse Trigonometric Functions

6.3E: Exercises

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6.0: Prelude to Periodic Functions

Each day, the sun rises in an easterly direction, approaches some maximum height relative to the celestial equator, and sets in a westerly direction. The celestial equator is an imaginary line that divides the visible universe into two halves in much the same way Earth's equator is an imaginary line that divides the planet into two halves. The exact path the sun appears to follow depends on the exact location on Earth, but each location observes a predictable pattern over time.



Figure **6.0.1**: (credit: "Maxxer_", Flickr)

The pattern of the sun's motion throughout the course of a year is a periodic function. Creating a visual representation of a periodic function in the form of a graph can help us analyze the properties of the function. In this chapter, we will investigate graphs of sine, cosine, and other trigonometric functions.

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6.0E: Exercises

No Exercises for Section 6.0E!

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6.1: Graphs of the Sine and Cosine Functions

Skills to Develop

- Graph variations of $y = \sin(x)$ and $y = \cos(x)$.
- Use phase shifts of sine and cosine curves.

White light, such as the light from the sun, is not actually white at all. Instead, it is a composition of all the colors of the rainbow in the form of waves. The individual colors can be seen only when white light passes through an optical prism that separates the waves according to their wavelengths to form a rainbow.



Figure* 6.1.1*: *Light can be separated into colors because of its wavelike properties. (credit: "wonderferret"/ Flickr)*

Light waves can be represented graphically by the sine function. In the chapter on Trigonometric Functions, we examined trigonometric functions such as the sine function. In this section, we will interpret and create graphs of sine and cosine functions.

Graphing Sine and Cosine Functions

Recall that the sine and cosine functions relate real number values to the x- and y-coordinates of a point on the unit circle. So what do they look like on a graph on a coordinate plane? Let's start with the sine function. To graph $y = \sin(x)$, note that we are making a change in the meaning of the variables x and y from the meaning in Chapter 5. The variable x becomes the symbol representing the input value of the sine function, so x now represents an angle in standard position; while y represents the output, which is the vertical coordinate of a point on the unit circle. We can create a table of values and use them to sketch a graph. Table **6.1.1** lists some of the values for the sine function on a unit circle.

***Table* 6.1.1**

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Plotting the points from the table and continuing along the *x*-axis gives the shape of the sine function. See Figure **6.1.2**.

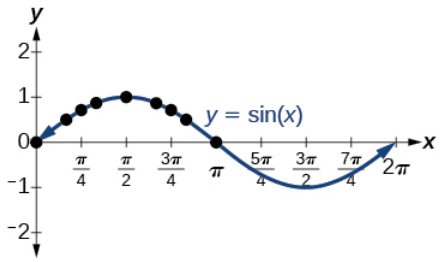


***Figure* 6.1.2:** *The sine function*

Notice how the sine values are positive between 0 and π, which correspond to the values of the sine function in quadrants I and II on the unit circle, and the sine values are negative between π and 2π, which correspond to the values of the sine function in quadrants III and IV on the unit circle. See Figure **6.1.3**.

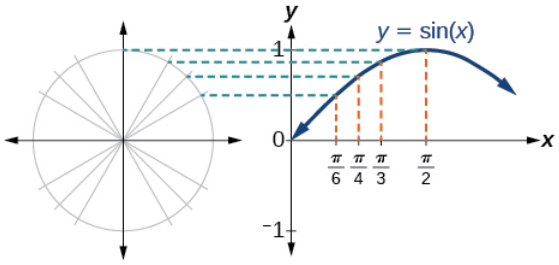


***Figure* 6.1.3:** *Plotting values of the sine function*

Now let's take a similar look at the **cosine function**. Again, we can create a table of values and use them to sketch a graph. This time, x again represents an angle, and y represents the horizontal coordinate of a point on the unit circle. Table **6.1.2** lists some of the values for the cosine function on a unit circle.

***Table* 6.1.2**

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

As with the sine function, we can plot points to create a graph of the cosine function as in Figure **6.1.4**.

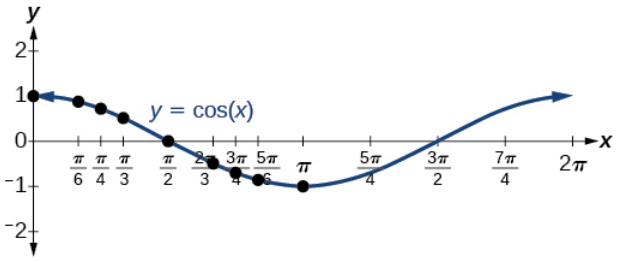


***Figure* 6.1.4:** *The cosine function*

Because we can evaluate the sine and cosine of any real number, both of these functions are defined for all real numbers. By thinking of the sine and cosine values as coordinates of points on a unit circle, it becomes clear that the range of both functions must be the interval $[-1, 1]$.

In both graphs, the shape of the graph repeats after 2π, which means the functions are **periodic** with a period of 2π. A **periodic function** is a function for which a specific **horizontal shift**, P, results in a function equal to the original function: $f(x + P) = f(x)$ for all values of x in the domain of f. When this occurs, we call the smallest such horizontal shift P (with $P > 0$) the **period** of the function. Figure **6.1.5** shows several periods of the sine and cosine functions. Any portion of the graph shown on one period $[x, x + P]$ is called a **cycle**. The graph of a periodic function should always include at least one full cycle.

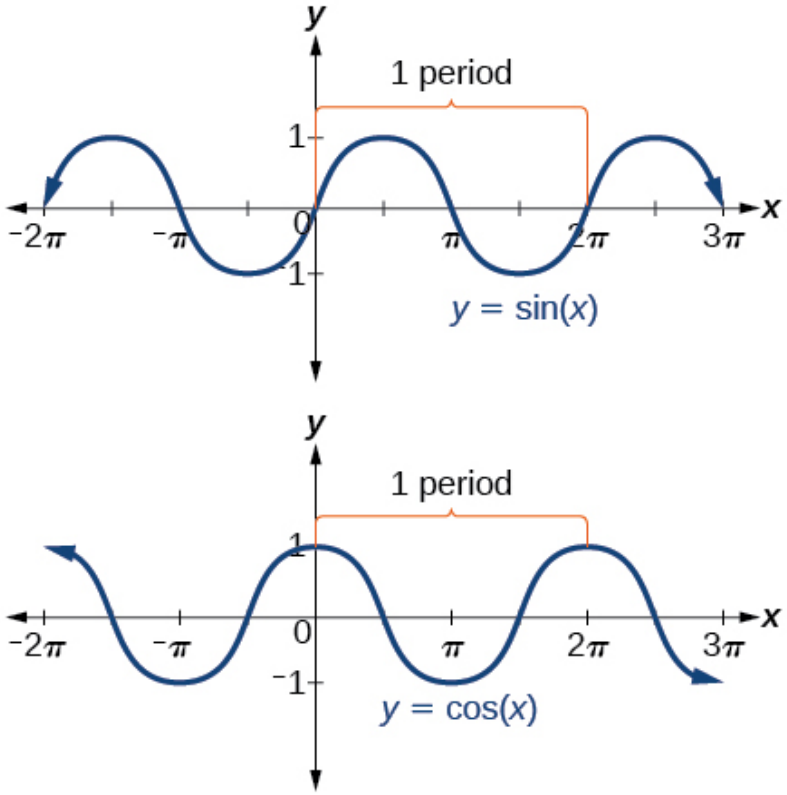


***Figure* 6.1.5**

Looking again at the sine and cosine functions on a domain centered at the y-axis helps reveal symmetries. As we can see in Figure **6.1.6**, the sine function is symmetric about the origin. Recall from The Other Trigonometric Functions that we determined from the unit circle that the sine function is an odd function because $\sin(-x) = -\sin\ x$. Now we can clearly see this property from the graph.

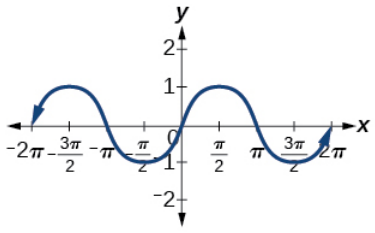


***Figure* 6.1.6:** *Odd symmetry of the sine function*

Figure **6.1.7** shows that the cosine function is symmetric across the y-axis. Again, we determined that the cosine function is an even function. Now we can see from the graph that $\cos(-x) = \cos\ x$.

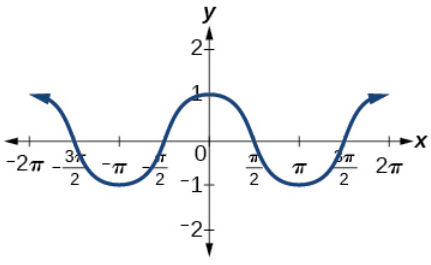


***Figure* 6.1.7:** *Even symmetry of the cosine function*

CHARACTERISTICS OF SINE AND COSINE FUNCTIONS

The sine and cosine functions have several distinct characteristics:

- They are periodic functions with a period of 2π.
- The domain of each function is $(-\infty, \infty)$ and the range is $[-1, 1]$.
- The graph of $y = \sin\ x$ is symmetric about the origin, because it is an odd function.
- The graph of $y = \cos\ x$ is symmetric across the y-axis, because it is an even function.

Investigating Sinusoidal Functions

As we can see, sine and cosine functions have a regular period and range. If we watch ocean waves or ripples on a pond, we will see that they resemble the sine or cosine functions. However, they are not necessarily identical. Some are taller or longer than others. A function that has the same general shape as a sine or cosine function is known as a **sinusoidal function**. The general forms of sinusoidal functions are

$$y = A\sin(Bx - C) + D$$

and

$$y = A\cos(Bx - C) + D$$

Using the identities $\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$, we can always rewrite a sinusoidal function in an equivalent form where the coefficient of x is a positive number (for example, $\sin(-3x) = -\sin(3x)$); thus, we will only consider the case where $B > 0$.

Determining the Period of Sinusoidal Functions

Looking at the forms of sinusoidal functions, we can see that they are transformations of the sine and cosine functions. We can use what we know about transformations to determine the period.

In the general formula, B is related to the period by $P = \frac{2\pi}{|B|}$. If $B > 1$, then the period is less than 2π and the function undergoes a horizontal compression, whereas if $B < 1$, then the period is greater than 2π and the function undergoes a horizontal stretch. For example, $f(x) = \sin(x)$, $B = 1$, so the period is 2π,which we knew. If $f(x) = \sin(2x)$, then $B = 2$, so the period is π and the graph is compressed. If $f(x) = \sin\left(\frac{x}{2}\right)$, then $B = \frac{1}{2}$, so the period is 4π and the graph is stretched. Notice in Figure 6.1.8 how the period is indirectly related to B.

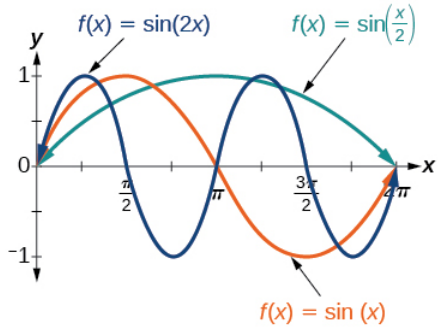


***Figure* 6.1.8**

PERIOD OF SINUSOIDAL FUNCTIONS

Letting $C = 0$ and $D = 0$ in the general form equations of the sine and cosine functions, we obtain the forms

- $y = A\sin(Bx)$
- $y = A\cos(Bx)$

The period is $\frac{2\pi}{B}$.

Example 6.1.1: Identifying the Period of a Sine or Cosine Function

Determine the period of the function $f(x) = \sin\left(\frac{\pi}{6}x\right)$.

Solution

Let's begin by comparing the equation to the general form $y = A\sin(Bx)$.

In the given equation, $B = \frac{\pi}{6}$, so the period will be

$$\begin{aligned} P &= \frac{2\pi}{B} \\ &= \frac{2\pi}{\frac{\pi}{6}} \\ &= 2\pi \cdot \frac{6}{\pi} \\ &= 12 \end{aligned}$$

6.1.1

Determine the period of the function $g(x) = \cos(\frac{x}{3})$.

Answer

6π

Determining Amplitude

Returning to the general formula for a sinusoidal function, we have analyzed how the variable B relates to the period. Now let's turn to the variable A so we can analyze how it is related to the **amplitude**, or greatest distance from rest. $|A|$ represents the vertical stretch factor, and its absolute value $|A|$ is the amplitude. The local maxima will be a distance $|A|$ above the vertical **midline** of the graph, which is the line $x = D$; because $D = 0$ in this case, the midline is the x-axis. The local minima will be the same distance below the midline. If $|A| > 1$, the function is stretched. For example, the amplitude of $f(x) = 4sinx$ is twice the amplitude of

$$f(x) = 2\sin x$$

If $|A| < 1$, the function is compressed. Figure 6.1.9 compares several sine functions with different amplitudes.

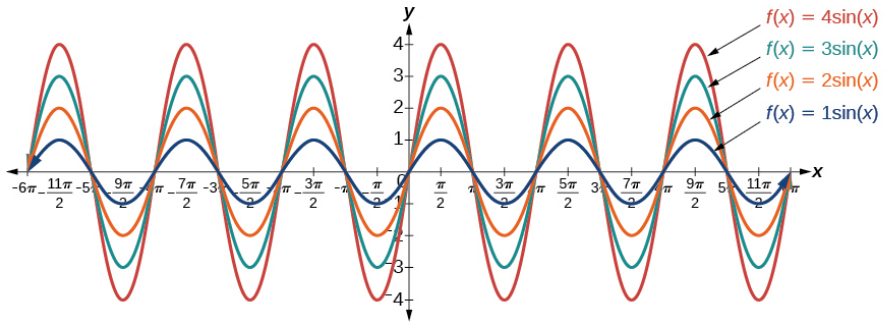


***Figure* 6.1.9**

AMPLITUDE OF SINUSOIDAL FUNCTIONS

Letting $C = 0$ and $D = 0$ in the general form equations of the sine and cosine functions, we obtain the forms

- $y = A\sin(Bx)$
- $y = A\cos(Bx)$

The amplitude is $|A|$, and the vertical height from the midline is $|A|$. In addition, notice in the example that

$$|A| = amplitude = \frac{1}{2}|\text{maximum} - \text{minimum}|$$

Example 6.1.2: Identifying the Amplitude of a Sine or Cosine Function

What is the amplitude of the sinusoidal function $f(x) = -4\sin(x)$? Is the function stretched or compressed vertically?

Solution

Let's begin by comparing the function to the simplified form $y = A\sin(Bx)$.

In the given function, $A = -4$, so the amplitude is $|A| = |-4| = 4$. The function is stretched.

Analysis

The negative value of A results in a reflection across the x-axis of the sine function, as shown in Figure **6.1.10**.

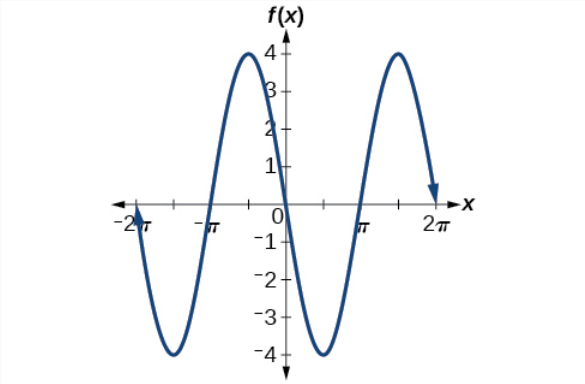


***Figure* 6.1.10**

6.1.2

What is the amplitude of the sinusoidal function $f(x) = \frac{1}{2}\sin(x)$? Is the function stretched or compressed vertically?

Answer

$\frac{1}{2}$; compressed

Analyzing Graphs of Variations of $y = \sin\ x$ and $y = \cos\ x$

Now that we understand how A and B relate to the general form equation for the sine and cosine functions, we will explore the variables C and D. Recall the general form:

$$y = A\sin(Bx - C) + D \qquad \text{and} \qquad y = A\cos(Bx - C) + D$$

which can be written equivalently as

$$y = A\sin\left(B\left(x - \frac{C}{B}\right)\right) + D \qquad \text{and} \qquad y = A\cos\left(B\left(x - \frac{C}{B}\right)\right) + D$$

The value $\frac{C}{B}$ for a sinusoidal function is the horizontal displacement of the basic sine or cosine function, which is called the **phase shift**. If $C > 0$, the graph shifts to the right. If $C < 0$, the graph shifts to the left. The greater the value of $\frac{|C|}{B}$, the more the graph is shifted. Figure **6.1.11** shows that the graph of $f(x) = \sin(x - \pi)$ shifts to the right by π units, which is a larger shift than we see in the graph of $f(x) = \sin\left(x - \frac{\pi}{4}\right)$, which shifts to the right by $\frac{\pi}{4}$ units. ***Extra credit***: A phase shift of $\frac{C(1 + 2\pi n)}{B}$, where $n \in \mathbb{Z}$, results in the same graph as a phase shift of $\frac{C}{B}$. If you can show that this is true, take it to your professor; she will give you *extra credit*.

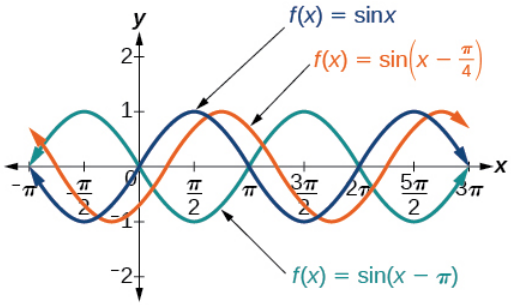


Figure 6.1.11

While C relates to the horizontal shift, D indicates the vertical shift from the graph of the Toolkit Functions $y = \sin x$ and $y = \cos x$. See Figure 6.1.12. The function $y = \sin(x) + D$ has its midline at $y = D$.

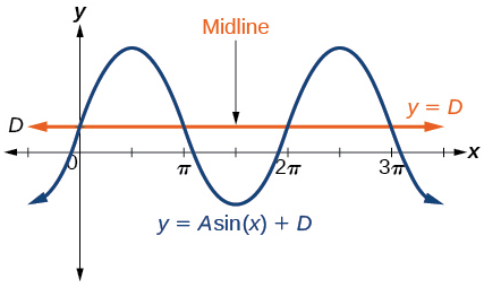


Figure 6.1.12

Any value of D other than zero shifts the graph up or down. Figure 6.1.13 compares $f(x) = \sin x$ with $f(x) = \sin x + 2$, which is shifted 2 units up on a graph.

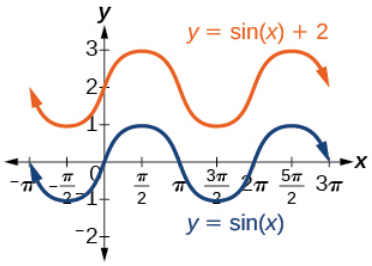


Figure 6.1.13

VARIATIONS OF SINE AND COSINE FUNCTIONS

Given an equation in the form $f(x) = A\sin(Bx - C) + D$ or $f(x) = A\cos(Bx - C) + D$, $\frac{C}{B}$ is the **phase shift** and D is the **vertical shift.**

Example 6.1.3: Identifying the Phase Shift of a Function

Determine the direction and magnitude of the phase shift for $f(x) = \sin\left(x + \frac{\pi}{6}\right) - 2$.

Solution

Let's begin by comparing the equation to the general form $y = A\sin(Bx - C) + D$.

In the given equation, notice that $B = 1$ and $C = -\frac{\pi}{6}$. So the phase shift is

$$\begin{aligned} \frac{C}{B} &= -\frac{\frac{\pi}{6}}{1} \\ &= -\frac{\pi}{6} \end{aligned}$$

or $\frac{\pi}{6}$ units to the left.

Analysis

We must pay attention to the sign in the equation for the general form of a sinusoidal function. The equation shows a minus sign before C. Therefore $f(x) = \sin(x + \frac{\pi}{6}) - 2$ can be rewritten as $f(x) = \sin\left(x - \left(-\frac{\pi}{6}\right)\right) - 2$. If the value of C is negative, the shift is to the left.

6.1.3

Determine the direction and magnitude of the phase shift for $f(x) = 3\cos\left(x - \frac{\pi}{2}\right)$.

Answer

Phase shift: $\frac{\pi}{2}$; direction: right

Example 6.1.4: Identifying the Vertical Shift of a Function

Determine the direction and magnitude of the vertical shift for $f(x) = \cos(x) - 3$.

Solution

Let's begin by comparing the equation to the general form $y = A\cos(Bx - C) + D$.

In the given equation, $D = -3$ so the shift is 3 units downward.

Try It 6.1.4

Determine the direction and magnitude of the vertical shift for $f(x) = 3\sin(x) + 2$.

Answer

2 units up

How To: Given a sinusoidal function in the form $f(x) = A\sin(Bx - C) + D$, identify the midline, amplitude, period, and phase shift

1. Determine the amplitude as $|A|$.
2. Determine the period as $P = \frac{2\pi}{B}$.
3. Determine the phase shift as $\frac{C}{B}$.
4. Determine the midline as $y = D$.

Example 6.1.5: Identifying the Variations of a Sinusoidal Function from its Equation

Determine the midline, amplitude, period, and phase shift of the function $y = 3\sin(2x) + 1$.

Solution

Let's begin by comparing the equation to the general form $y = A\sin(Bx - C) + D$.

$A = 3$, so the amplitude is $|A| = 3$.

Next, $B = 2$, so the period is $P = \frac{2\pi}{B} = \frac{2\pi}{2} = \pi$.

There is no added constant inside the parentheses, so $C = 0$ and the phase shift is $\frac{C}{B} = \frac{0}{2} = 0$.

Finally, $D = 1$, so the midline is $y = 1$.

Analysis

Inspecting the graph, we can determine that the period is π, the midline is $y = 1$, and the amplitude is 3. See Figure 6.1.14.

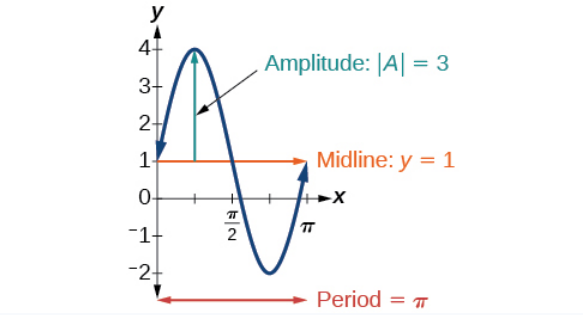


***Figure* 6.1.14**

6.1.5

Determine the midline, amplitude, period, and phase shift of the function $y = \frac{1}{2}\cos\left(\frac{x}{3} - \frac{\pi}{3}\right)$.

Answer

midline: $y = 0$; amplitude: $|A| = \frac{1}{2}$; period: $P = \frac{2\pi}{B} = 6\pi$; phase shift: $\frac{C}{B} = \pi$ - see Figure 6.1.15

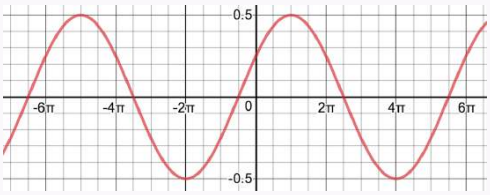


Figure 6.1.15

Example 6.1.6: Identifying the Equation for a Sinusoidal Function from a Graph

Determine a formula for the cosine function in Figure 6.1.16.

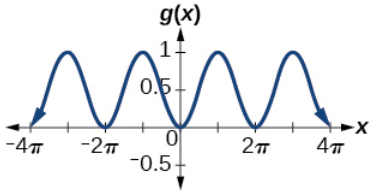


Figure 6.1.16

Solution

To determine the equation, we need to identify each value in the general form of a sinusoidal function.

$$y = A\sin(Bx - C) + D$$

$$y = A\cos(Bx - C) + D$$

The graph could represent either a sine or a cosine function that is shifted and/or reflected. When $x = 0$, the graph has an extreme point, $(0, 0)$. Since the cosine function has an extreme point for $x = 0$, let us write our equation in terms of a cosine function, so that we can take $C = 0$. Since the extreme point is a minimum, unlike the Toolkit function $\cos x$ which has a maximum value at $x = 0$, we can deduce that $A < 0$, so that the graph has been vertically reflected.

Let's start with the midline. We can see that the graph rises and falls an equal distance above and below $y = 0.5$. This value, which is the midline, is D in the equation, so $D = 0.5$.

The greatest distance above and below the midline is the amplitude. The maxima are 0.5 units above the midline and the minima are 0.5 units below the midline. So $|A| = 0.5$. Another way we could have determined the amplitude is by recognizing that the difference between the height of local maxima and minima is 1, so $|A| = \frac{1}{2}$ or 0.5. Also, before the vertical shift, the graph is reflected across the x-axis so that $A = -0.5$.

The graph is not horizontally stretched or compressed, so $B = 1$; and the graph is not shifted horizontally, so $C = 0$.

Putting this all together,

$$g(x) = -0.5\cos(x) + 0.5$$

6.1.6

Determine a formula for the sine function in Figure 6.1.17.

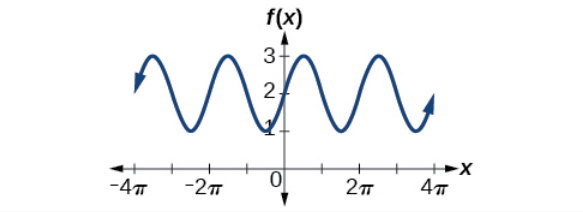


***Figure* 6.1.17**

Answer

$f(x) = \sin(x) + 2$

Example 6.1.7: Identifying the Equation for a Sinusoidal Function from a Graph

Determine an equation for the sinusoidal function in Figure **6.1.18**.

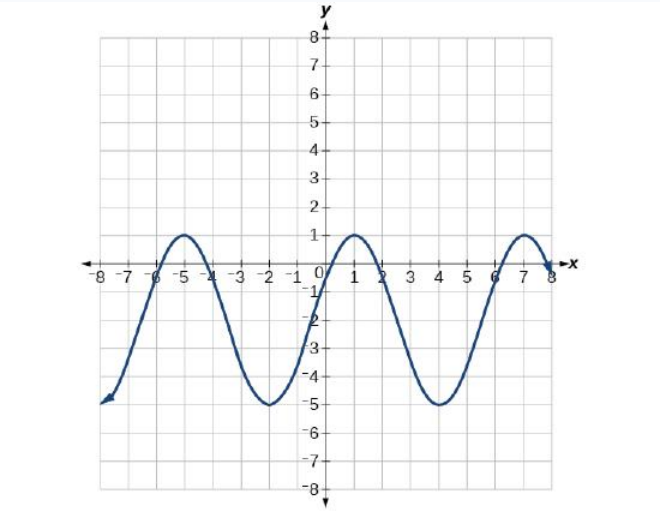


***Figure* 6.1.18**

Solution

With the highest value at 1 and the lowest value at -5, the midline will be halfway between at -2. So $D = -2$.

The distance from the midline to the highest or lowest value gives an amplitude of $|A| = 3$.

The period of the graph is 6, which can be measured from the peak at $x = 1$ to the next peak at $x = 7$, or from the distance between the lowest points. Therefore, $P = \frac{2\pi}{B} = 6$. Using the positive value for B, we find that

$$\begin{aligned} B &= \frac{2\pi}{P} \\ &= \frac{2\pi}{6} \\ &= \frac{\pi}{3} \end{aligned}$$

So far, our equation is either $y = \pm 3 \sin\left(\frac{\pi}{3}x - C\right) - 2$ or $y = \pm 3 \cos\left(\frac{\pi}{3}x - C\right) - 2$. For the shape and shift, we have more than one option. We could write this as any one of the following:

- a positive or negative cosine shifted to the left or right
- a positive or negative sine shifted to the left or right

While any of these would be correct, we will choose to use a positive cosine function. The shift is an integer value of 1 unit to the right. So $\frac{C}{B} = 1$, or $\frac{C}{\frac{\pi}{3}} = 1$, or $C = \frac{\pi}{3}$. Our function becomes

$$y = 3\cos\left(\frac{\pi}{3}x - \frac{\pi}{3}\right) - 2.$$

6.1.7

Write a formula for the function graphed in Figure **6.1.19**.

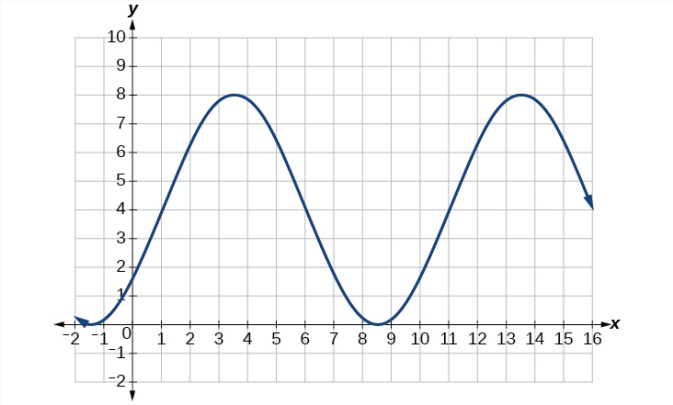


***Figure* 6.1.19**

Answer

two possibilities: $y = 4\sin\left(\frac{\pi}{5}x + \frac{3\pi}{10}\right) + 4$ or $y = 4\cos\left(\frac{\pi}{5}x - \frac{7\pi}{10}\right) + 4$

Graphing Variations of $y = \sin\ x$ and $y = \cos\ x$

We have learned about types of variations of sine and cosine functions and used that information to write equations from graphs. Now we will use the same information to create graphs from equations.

To start, instead of focusing on the general form equations

$$y = A\sin(Bx - C) + D \text{ and } y = A\cos(Bx - C) + D$$

we will let $C = 0$ and $D = 0$ and work with a simplified form of the equations. Since we know that $\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$, we can always write B as a positive number, with A either positive or negative, as required.

How To: Given the function $y = A\sin(Bx)$, sketch one cycle of its graph.

1. Identify the amplitude, $|A|$.
2. Identify the period, $P = \frac{2\pi}{B}$.
3. Start at the origin, with the function increasing to the right if A is positive, or decreasing to the right if A is negative.
4. At $x = \frac{\pi}{2B}$ there is a local maximum for $A > 0$ with $y = A$ or a minimum for $A < 0$, with $y = -A$.
5. The curve returns to the x-axis at $x = \frac{\pi}{B}$.
6. There is a local minimum for $A > 0$ (or a maximum for $A < 0$) at $x = \frac{3\pi}{2B}$ with $y = -A$ (or $y = A$).
7. The curve returns again to the x-axis at $x = \frac{2\pi}{B}$.

How To: Given the function $y = A\cos(Bx)$, sketch one cycle of its graph.

1. Identify the amplitude, $|A|$.
2. Identify the period, $P = \frac{2\pi}{B}$.
3. Start at $x = 0$, with $y = A$ and the function decreasing to the right if A is positive, or $y = -A$ and the function increasing to the right if A is negative. There is a local maximum for $A > 0$ or a minimum for $A < 0$.
4. The curve returns to the x-axis at $x = \frac{\pi}{2B}$.

5. There is a local minimum for $A > 0$ (or a maximum for $A < 0$) at $x = \frac{\pi}{B}$ with $y = -A$ (or $y = A$).
6. The curve returns again to the x-axis at $x = \frac{3\pi}{2B}$.
7. There is a local maximum for $A > 0$ or a minimum for $A < 0$ at $x = \frac{2\pi}{B}$.

Example 6.1.8: Graphing a Function and Identifying the Amplitude and Period

Sketch a graph of $f(x) = -2\sin\left(\frac{\pi x}{2}\right)$.

Solution

Let's begin by comparing the equation to the form $y = A\sin(Bx)$.

- *Step 1.* We can see from the equation that $A = -2$,so the amplitude is 2.

  $|A| = 2$

- *Step 2.* The equation shows that $B = \frac{\pi}{2}$, so the period is

$$\begin{aligned} P &= \frac{2\pi}{\frac{\pi}{2}} \\ &= 2\pi \cdot \frac{2}{\pi} \\ &= 4 \end{aligned}$$

- *Step 3.* Because A is negative, the graph descends as we move to the right of the origin.
- *Step 4.* The x-intercepts are at the beginning of one period, $x = 0$, at $x = 2$ (halfway through the cycle) and at the end of one cycle at $x = 4$.

The quarter points include the minimum at $x = 1$ and the maximum at $x = 3$. A local minimum will occur 2 units below the midline at $x = 1$, and a local maximum will occur at 2 units above the midline at $x = 3$. Figure 6.1.20 shows the graph of the function. Note that by letting B equal a multiple of π, the scale of the x-axis has changed to integer units.

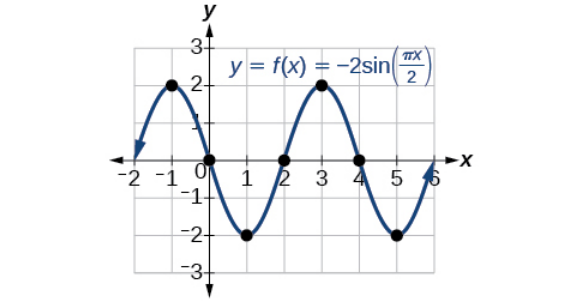


***Figure* 6.1.20**

6.1.8

Sketch a graph of $g(x) = -0.8\cos(2x)$. Determine the midline, amplitude, period, and phase shift.

Answer

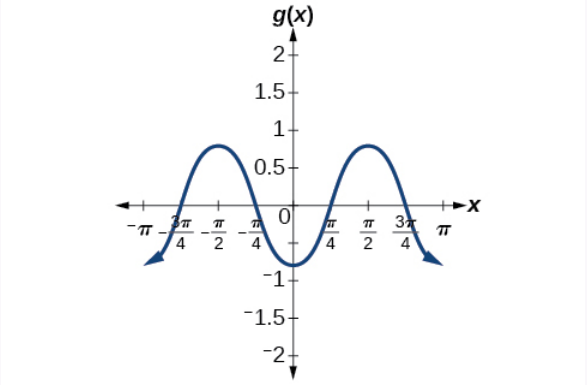


***Figure* 6.1.20**

midline: $y = 0$; amplitude: $|A| = 0.8$; period: $P = \frac{2\pi}{B} = \pi$; phase shift: $\frac{C}{B} = 0$, or none

How To: Given a sinusoidal function with a phase shift and a vertical shift, sketch its graph

1. Express the function in the general form $y = A\sin(Bx - C) + D$ or $y = A\cos(Bx - C) + D$.
2. Identify the amplitude, $|A|$.
3. Identify the period, $P = \frac{2\pi}{B}$.
4. Identify the phase shift, $\frac{C}{B}$.
5. Draw the graph of $f(x) = A\sin(Bx)$ shifted to the right or left by $\frac{C}{B}$ and up or down by D.

Example 6.1.9: Graphing a Transformed Sinusoid

Sketch a graph of $f(x) = 3\sin\left(\frac{\pi x}{4} - \frac{\pi}{4}\right)$.

Solution

- *Step 1.* The function is already written in general form: $f(x) = 3\sin\left(\frac{\pi x}{4} - \frac{\pi}{4}\right)$. $C = \frac{\pi}{4}$ and $D = 0$. This graph will have the shape of a sine function, starting somewhere along the midline ($y = 0$) and increasing to the right, since $A > 0$.
- *Step 2.* $|A| = |3| = 3$. The amplitude is 3.
- *Step 3.* Since $B = \frac{\pi}{4}$, we determine the period as follows.

$$\begin{aligned} P &= \frac{2\pi}{B} \\ &= \frac{2\pi}{\frac{\pi}{4}} \\ &= 2\pi \cdot \frac{4}{\pi} \\ &= 8 \end{aligned}$$

  The period is 8.

- *Step 4.* Since $C = \frac{\pi}{4}$, the phase shift is

$$\frac{C}{B} = \frac{\frac{\pi}{4}}{\frac{\pi}{4}} = 1$$

  .

  The phase shift is 1 unit to the right.

- *Step 5.* Figure $6.1.22$ shows the graph of the function.

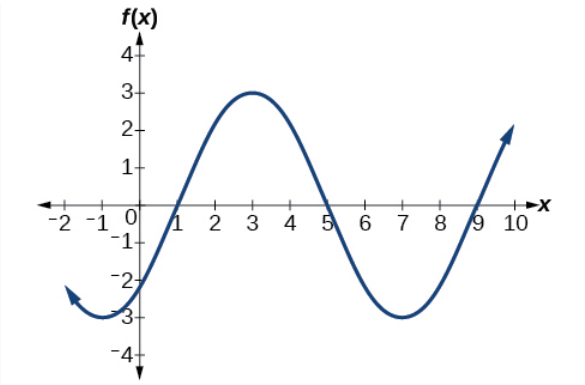


***Figure* 6.1.22:** *A horizontally compressed, vertically stretched, and horizontally shifted sinusoid*

6.1.9

Draw a graph of $g(x) = -2\cos\left(\frac{\pi}{3}x + \frac{\pi}{6}\right)$. Determine the midline, amplitude, period, and phase shift.

Answer

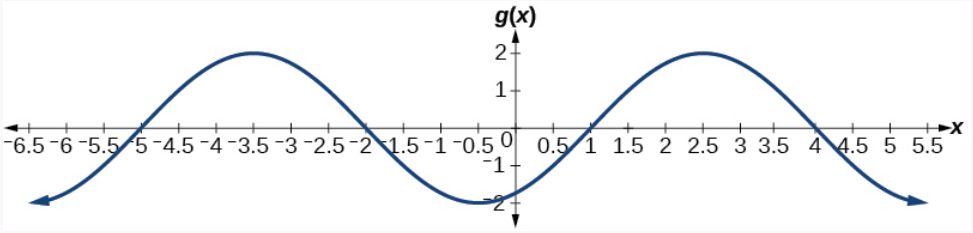


***Figure* 6.1.23**

midline: $y = 0$; amplitude: $|A| = 2$; period: $P = \frac{2\pi}{B} = 6$; phase shift: $\frac{C}{B} = -\frac{1}{2}$

Example 6.1.10: Identifying the Properties of a Sinusoidal Function

Given $y = -2\cos\left(\frac{\pi}{2}x + \pi\right) + 3$, determine the amplitude, period, phase shift, and horizontal shift. Then graph the function.

Solution

Begin by comparing the equation to the general form.

$$y = A\cos(Bx - C) + D$$

- *Step 1.* The function is already written in general form.
- *Step 2.* Since $A = -2$, the amplitude is $|A| = 2$. The graph of the cosine function has been reflected about the x-axis.
- *Step 3.* $B = \frac{\pi}{2}$, so the period is $P = \frac{2\pi}{B} = \frac{2\pi}{\frac{\pi}{2}} = 2\pi \cdot \frac{2}{\pi} = 4$. The period is 4.
- *Step 4.* $C = -\pi$, so we calculate the phase shift as $\frac{C}{B} = -\frac{\pi}{\frac{\pi}{2}} = -\pi \cdot \frac{2}{\pi} = -2$. The phase shift is -2.
- *Step 5.* $D = 3$, so the midline is $y = 3$, and the vertical shift is up 3.

Figure 6.1.24 shows two cycles of the graph of the function.

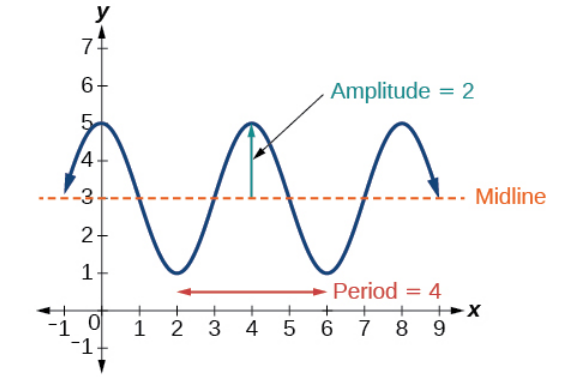


***Figure* 6.1.24**

Using Transformations of Sine and Cosine Functions

We can use the transformations of sine and cosine functions in numerous applications. As mentioned in Section 5.2, **circular motion** can be modeled using either the sine or cosine function.

Example 6.1.11: Finding the Vertical Component of Circular Motion

A point rotates around a circle of radius 3 centered at the origin. Sketch a graph of the y-coordinate of the point as a function of the angle of rotation.

Solution

Recall from Section 5.4 that, for a point on a circle of radius r, the y-coordinate of the point is $y = r\sin(x)$, so in this case, we get the equation $y(x) = 3\sin(x)$. The constant 3 causes a vertical stretch of the y-values of the function by a factor of 3, which we can see in the graph in Figure 6.1.25.

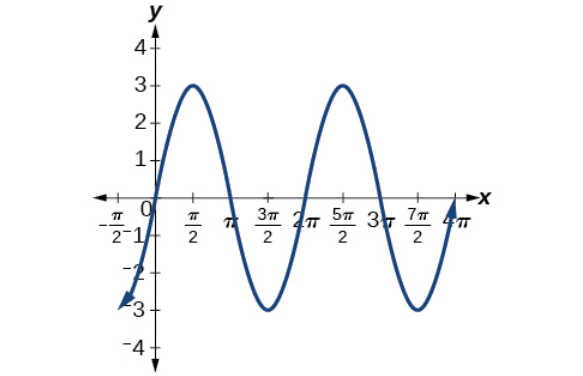


***Figure* 6.1.25**

Analysis

Notice that the period of the function is still 2π; as we travel around the circle, we return to the point $(3, 0)$ for $x = 2\pi, 4\pi, 6\pi,$....Because the outputs of the graph will now oscillate between -3 and 3, the amplitude of the sine wave is 3.

Example 6.1.12: Finding the Vertical Component of Circular Motion

A circle with radius 3 ft is mounted with its center 4 ft off the ground. The point closest to the ground is labeled P, as shown in Figure 6.1.26. Sketch a graph of the height above the ground of the point P as the circle is rotated; then find a function that gives the height in terms of the angle of rotation.

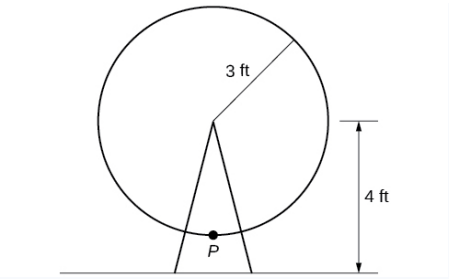


***Figure* 6.1.26**

Solution

Sketching the height, we note that it will start 1 ft above the ground, then increase up to 7 ft above the ground, and continue to oscillate 3 ft above and below the center value of 4 ft, as shown in Figure 6.1.27.

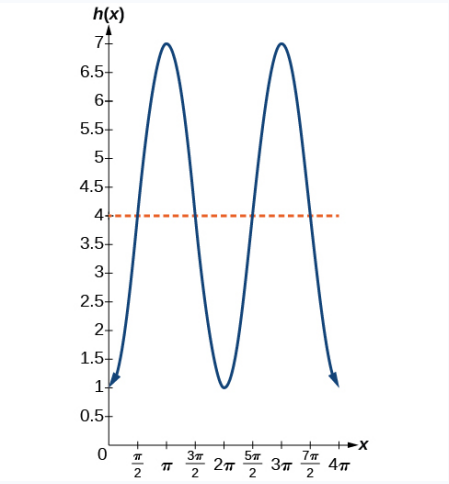


***Figure* 6.1.27**

Although we could use a transformation of either the sine or cosine function, we start by looking for characteristics that would make one function easier to use than the other. Let's use a cosine function because it starts at the highest or lowest value, while a **sine function** starts at the middle value. A standard cosine starts at the highest value, and this graph starts at the lowest value, so we need to incorporate a vertical reflection.

Second, we see that the graph oscillates 3 units above and below the center, while a basic cosine has an amplitude of 1, so this graph has been vertically stretched by 3, as in the last example.

Finally, to move the center of the circle up to a height of 4, the graph has been vertically shifted up by 4. Putting these transformations together, we find that

$$y = -3\cos(x) + 4$$

6.1.10

A weight is attached to a spring that is then hung from a board, as shown in Figure 6.1.28. As the spring oscillates up and down, the position y of the weight relative to the board ranges from –1 in. (at time $x = 0$) to –7 in. (at time $x = \pi$) below the board. Assume the position of y is given as a sinusoidal function of x. Sketch a graph of the function, and then find a cosine function that gives the position y in terms of x.

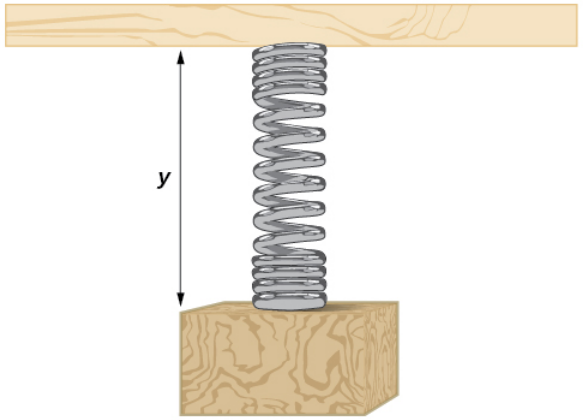


Figure 6.1.28

Answer

$y = 3\cos(x) - 4$

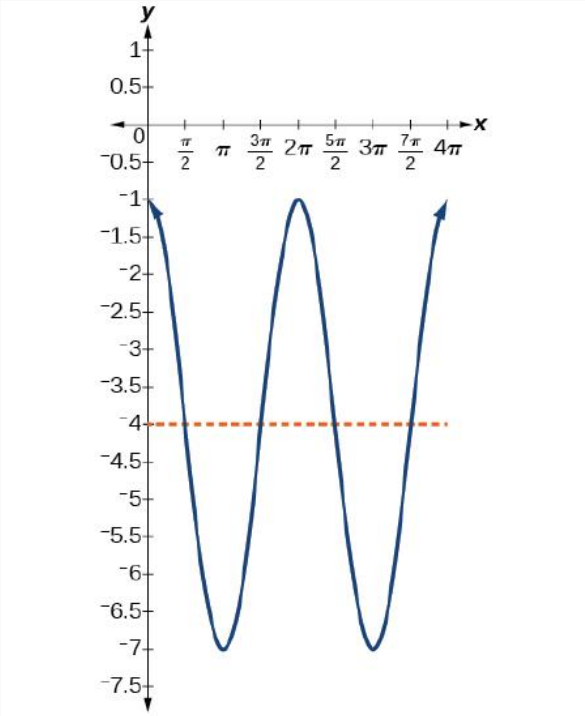


Figure 6.1.29

Example 6.1.13: Determining a Rider's Height on a Ferris Wheel

The London Eye is a huge Ferris wheel with a diameter of 135 meters (443 feet). It completes one rotation every 30 minutes. Riders board at the lowest point from a platform 2 meters above the ground. Express a rider's height above ground as a function of time in minutes.

Solution

With a diameter of 135 m, the wheel has a radius of 67.5 m. The height will oscillate with amplitude 67.5 m above and below the center.

Passengers board 2 m above ground level, so the center of the wheel must be located $67.5 + 2 = 69.5$ m above ground level. The midline of the oscillation will be at 69.5 m.

The wheel takes 30 minutes to complete 1 revolution, so the height will oscillate with a period of 30 minutes.

Lastly, because the rider boards at the lowest point, the height will start at the smallest value and increase, following the shape of a vertically reflected cosine curve.

- Amplitude: 67.5, so $A = 67.5$
- Midline: 69.5, so $D = 69.5$
- Period: 30, so $B = \frac{2\pi}{30} = \frac{\pi}{15}$
- Shape: $-\cos(t)$

An equation for the rider's height would be

$$y = -67.5\cos\left(\frac{\pi}{15}t\right) + 69.5$$

where t is in minutes and y is measured in meters.

Media

Access these online resources for additional instruction and practice with graphs of sine and cosine functions.

- Amplitude and Period of Sine and Cosine
- Translations of Sine and Cosine
- Graphing Sine and Cosine Transformations
- Graphing the Sine Function

Key Equations

Sinusoidal functions	$f(x) = A\sin(Bx - C) + D$ $f(x) = A\cos(Bx - C) + D$

Key Concepts

- Periodic functions repeat after a given value. The smallest such value is the period. The basic sine and cosine functions have a period of 2π.
- The function $\sin x$ is odd, so its graph is symmetric about the origin. The function $\cos x$ is even, so its graph is symmetric about the *y*-axis.
- The graph of a sinusoidal function has the same general shape as a sine or cosine function.
- In the general formula for a sinusoidal function, the period is $P = \frac{2\pi}{|B|}$. See Example 6.1.1.
- In the general formula for a sinusoidal function, $|A|$ represents amplitude. If $|A| > 1$, the function is stretched, whereas if $|A| < 1$, the function is compressed. See Example 6.1.2.
- The value $\frac{C}{B}$ in the general formula for a sinusoidal function indicates the phase shift. See Example 6.1.3.
- The value D in the general formula for a sinusoidal function indicates the vertical shift from the midline. See Example 6.1.4.
- Combinations of variations of sinusoidal functions can be detected from an equation. See Example 6.1.5.
- The equation for a sinusoidal function can be determined from a graph. See Example 6.1.6 and Example 6.1.7.
- A sinusoidal function can be graphed by identifying its amplitude, period, phase shift, and horizontal shift. See Example 6.1.8, Example 6.1.9 and Example 6.1.10.
- Sinusoidal functions can be used to solve real-world problems. See Example 6.1.11, Example 6.1.12, and Example 6.1.13.

Contributors

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6.1E: Exercises

6.1: Graphs of the Sine and Cosine Functions

In the chapter on Trigonometric Functions, we examined trigonometric functions such as the sine function. In this section, we will interpret and create graphs of sine and cosine functions

Verbal

1) Why are the sine and cosine functions called periodic functions?

Answer

The sine and cosine functions have the property that $f(x+P)=f(x)$ for a certain P. This means that the function values repeat for every P units on the x-axis.

2) How does the graph of $y=\sin x$ compare with the graph of $y=\cos x$? Explain how you could horizontally translate the graph of $y=\sin x$ to obtain $y=\cos x$.

3) For the equation $A\cos(Bx+C)+D$,what constants affect the range of the function and how do they affect the range?

Answer

The absolute value of the constant A (amplitude) increases the total range and the constant D (vertical shift) shifts the graph vertically.

4) How does the range of a translated sine function relate to the equation $y=A\sin(Bx+C)+D$?

5) How can the unit circle be used to construct the graph of $f(t)=\sin t$?

Answer

At the point where the terminal side of t intersects the unit circle, you can determine that the $\sin t$ equals the y-coordinate of the point.

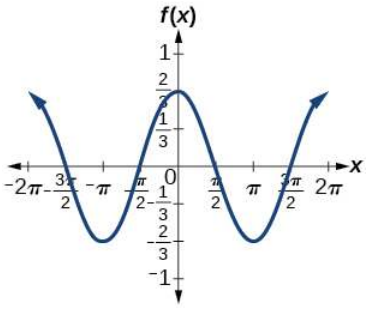
Graphical

For the following exercises, graph two full periods of each function and state the amplitude, period, and midline. State the maximum and minimum y-values and their corresponding x-values on one period for $x>0$. Round answers to two decimal places if necessary.

6) $f(x)=2\sin x$

7) $f(x)=\frac{2}{3}\cos x$

Answer



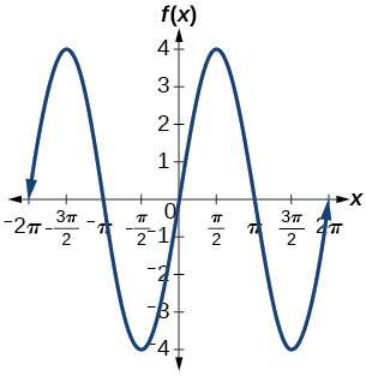
amplitude: $\frac{2}{3}$period: 2πmidline: $y=0$maximum: $y=\frac{2}{3}$ occurs at $x=0$minimum: $y=-\frac{2}{3}$ occurs at $x=\pi$for one period, the graph starts at 0 and ends at 2π.

8) $f(x) = -3\sin x$

Exercise 6.1E.9

$f(x) = 4\sin x$

Answer



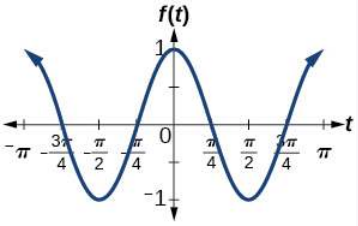
amplitude: 4; period: 2π midline: $y = 0$ maximum $y = 4$ occurs at $x = \frac{\pi}{2}$ minimum: $y = -4$ occurs at $x = \frac{3\pi}{2}$ one full period occurs from $x = 0$ to $x = 2\pi$

10) $f(x) = 2\cos x$

Exercise 6.1E.11

$f(x) = \cos(2x)$

Answer

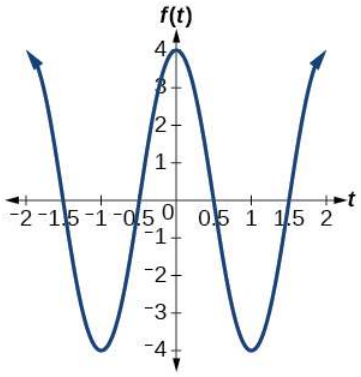


amplitude: 1; period: π midline: $y = 0$ maximum: $y = 1$ occurs at $x = \pi$ minimum: $y = -1$ occurs at $x = \frac{\pi}{2}$ one full period is graphed from $x = 0$ to $x = \pi$

12) $f(x) = 2\sin\left(\frac{1}{2}x\right)$

13) $f(x) = 4\cos(\pi x)$

Answer

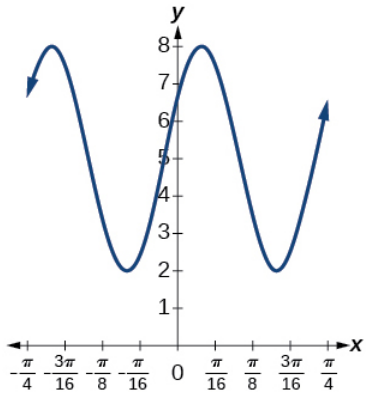


amplitude: 4; period: 2; midline: $y = 0$maximum: $y = 4$ occurs at $x = 0$minimum: $y = -4$ occurs at $x = 1$

14) $f(x) = 3\cos\left(\frac{6}{5}x\right)$

15) $y = 3\sin(8(x + 4)) + 5$

Answer

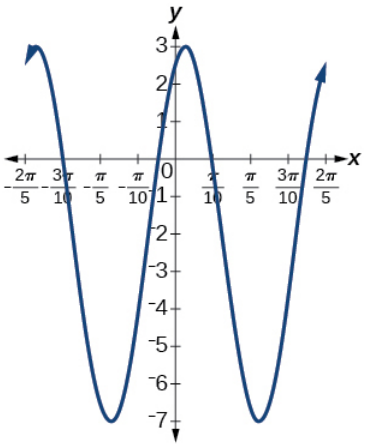


amplitude: 3; period: $\frac{\pi}{2}$ midline: $y = 5$ maximum: $y = 8$ occurs at $x = 0.12$ minimum: $y = 2$ occurs at $x = 0.516$ horizontal shift: -4 vertical translation 5; one period occurs from $x = 0$ to $x = \frac{\pi}{4}$

16) $y = 2\sin(3x - 21) + 4$

17) $y = 5\sin(5x + 20) - 2$

Answer



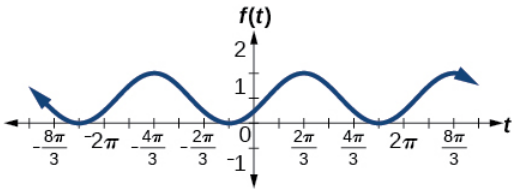
amplitude: 5; period: $\frac{2\pi}{5}$ midline: $y=-2$ maximum: $y=3$ occurs at $x=0.08$ minimum: $y=-7$ occurs at $x=0.71$ phase shift: -4 vertical translation: -2 one full period can be graphed on $x=0$ to $x=\frac{2\pi}{5}$

For the following exercises, graph one full period of each function, starting at $x=0$. For each function, state the amplitude, period, and midline. State the maximum and minimum y-values and their corresponding x-values on one period for $x>0$. State the phase shift and vertical translation, if applicable. Round answers to two decimal places if necessary.

18) $f(t)=2\sin\left(t-\frac{5\pi}{6}\right)$

19) $f(t)=-\cos\left(t+\frac{\pi}{3}\right)+1$

Answer

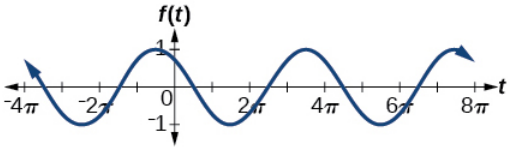


amplitude: 1; period: 2π midline: $y=1$ maximum: $y=2$ occurs at $x=2.09$ maximum: $y=2$ occurs at $t=2.09$ minimum: $y=0$ occurs at $t=5.24$ phase shift: $-\frac{\pi}{3}$ vertical translation: 1; one full period is from $t=0$ to $t=2\pi$

20) $f(t)=4\cos\left(2\left(t+\frac{\pi}{4}\right)\right)-3$

21) $f(t)=-\sin\left(\frac{1}{2}t+\frac{5\pi}{3}\right)$

Answer

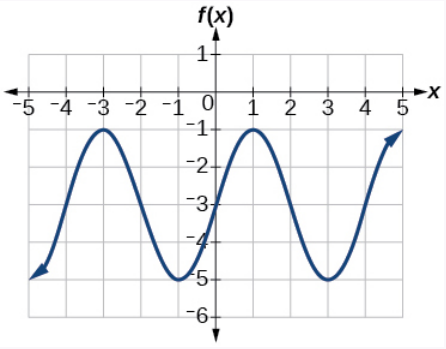


amplitude: 1; period: 4π midline: $y=0$ maximum: $y=1$ occurs at $t=11.52$ minimum: $y=-1$ occurs at $t=5.24$ phase shift: $-\frac{10\pi}{3}$ vertical shift: 0

22) $f(x)=4\sin\left(\frac{\pi}{2}(x-3)\right)+7$

Exercise 6.1E.23

Determine the amplitude, midline, period, and an equation involving the sine function for the graph shown in Figure below.

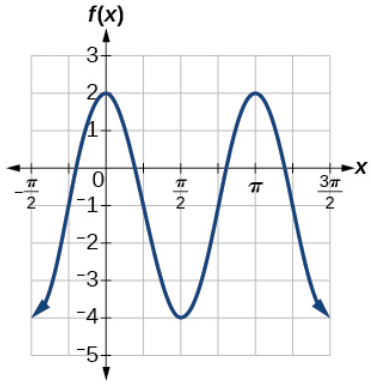


Answer

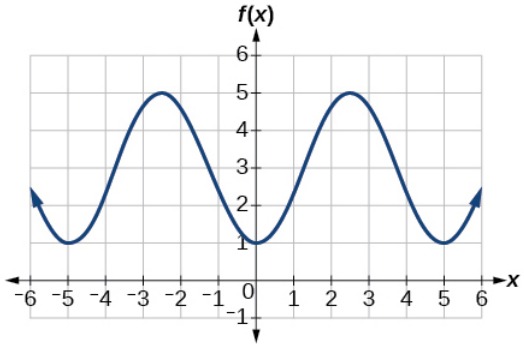
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amplitude: 2; midline: $y=-3$; period: 4; equation: $f(x)=2\sin\left(\frac{\pi}{2}x\right)-3$

24) Determine the amplitude, period, midline, and an equation involving cosine for the graph shown in the Figure below.



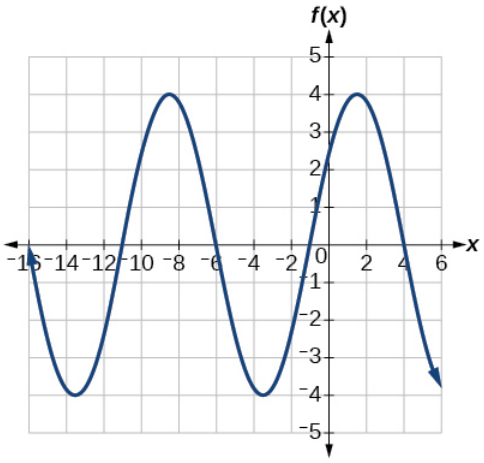
25) Determine the amplitude, period, midline, and an equation involving cosine for the graph shown in the Figure below.



Answer

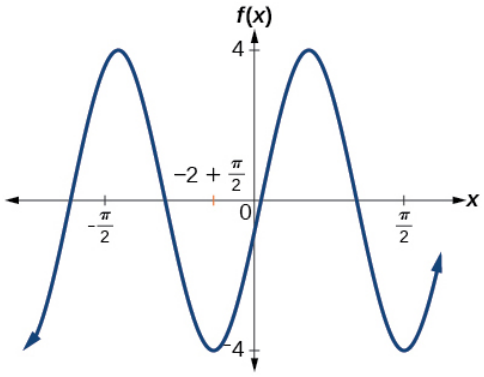
amplitude: 2; period: 5; midline: $y=3$ equation: $f(x)=-2\cos\left(\frac{2\pi}{5}x\right)+3$

26) Determine the amplitude, period, midline, and an equation involving sine for the graph shown in the Figure below.



27) Determine the amplitude, period, midline, and an equation involving cosine for the graph shown in the Figure below.

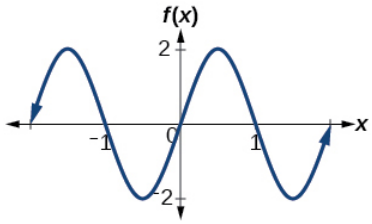
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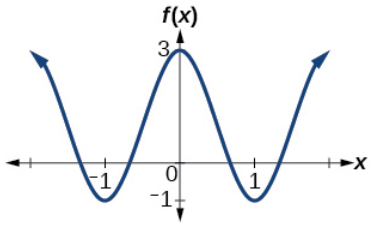
Answer

amplitude: 4; period: 2; midline: $y = 0$ equation: $f(x) = -4\cos\left(\pi\left(x - \frac{\pi}{2}\right)\right)$

28) Determine the amplitude, period, midline, and an equation involving sine for the graph shown in the Figure below.



29) Determine the amplitude, period, midline, and an equation involving cosine for the graph shown in the Figure below.



Answer

amplitude: 2; period: 2; midline $y = 1$ equation: $f(x) = 2\cos(\pi x) + 1$

30) Determine the amplitude, period, midline, and an equation involving sine for the graph shown in the Figure below.

Algebraic

For the following exercises, let $f(x) = \sin x$.

31) On $[0, 2\pi)$, solve $f(x) = 0$.

32) On $[0, 2\pi)$, solve $f(x) = \frac{1}{2}$.

Answer

$\frac{\pi}{6}, \frac{5\pi}{6}$

33) Evaluate $f\left(\frac{\pi}{2}\right)$.

34) On $[0, 2\pi)$, $f(x) = \frac{\sqrt{2}}{2}$. Find all values of x.

Answer

$\frac{\pi}{4}, \frac{3\pi}{4}$

35) On $[0, 2\pi)$, the maximum value(s) of the function occur(s) at what x-value(s)?

36) On $[0, 2\pi)$, the minimum value(s) of the function occur(s) at what x-value(s)?

Answer

$\frac{3\pi}{2}$

37) Show that $f(-x) = -f(x)$. This means that $f(x) = \sin x$ is an odd function and possesses symmetry with respect to _____.

For the following exercises, let $f(x) = \cos x$

38) On $[0, 2\pi)$, solve the equation $f(x) = \cos x = 0$

Answer

$\frac{\pi}{2}, \frac{3\pi}{2}$

39) On $[0, 2\pi)$, solve $f(x) = \frac{1}{2}$.

40) On $[0, 2\pi)$, find the x-intercepts of $f(x) = \cos x$.

Answer

$\frac{\pi}{2}, \frac{3\pi}{2}$

41) On $[0, 2\pi)$, find the x-values at which the function has a maximum or minimum value.

42) On $[0, 2\pi)$, solve the equation $f(x) = \frac{\sqrt{3}}{2}$.

Answer

$\frac{\pi}{6}, \frac{11\pi}{6}$

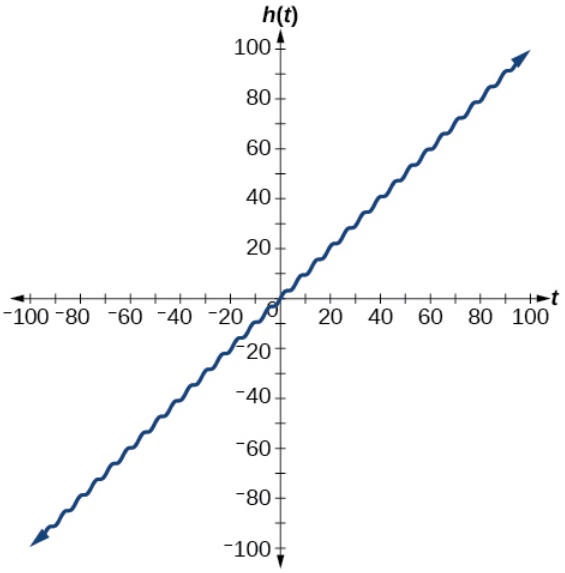
Technology

43) Graph $h(x) = x + \sin x$ on $[0, 2\pi]$. Explain why the graph appears as it does.

44) Graph $h(x) = x + \sin x$ on $[-100, 100]$. Did the graph appear as predicted in the previous exercise?

Answer

The graph appears linear. The linear functions dominate the shape of the graph for large values of x.

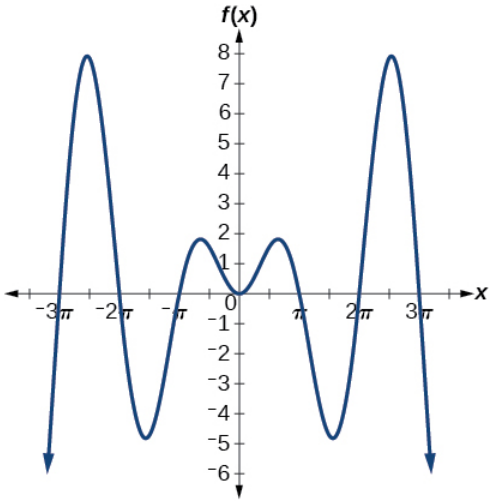


45) Graph $f(x) = x \sin x$ on $[0, 2\pi]$ and verbalize how the graph varies from the graph of $f(x) = x \sin x$.

46) Graph $f(x) = x \sin x$ on the window $[-10, 10]$ and explain what the graph shows.

Answer

The graph is symmetric with respect to the y-axis and there is no amplitude because the function is not periodic.



47) Graph $f(x) = \dfrac{\sin x}{x}$ on the window $[-5\pi, 5\pi]$ and explain what the graph shows.

Real-World Applications

48) A Ferris wheel is 25 meters in diameter and boarded from a platform that is 1 meter above the ground. The six o'clock position on the Ferris wheel is level with the loading platform. The wheel completes 1 full revolution in 10 minutes. The function $h(t)$ gives a person's height in meters above the ground t minutes after the wheel begins to turn.

a. Find the amplitude, midline, and period of $h(t)$.
b. Find a formula for the height function $h(t)$.
c. How high off the ground is a person after 5 minutes?

Answer

a. Amplitude: 12.5; period: 10; midline: $y = 13.5$

b. $h(t) = 12.5 \sin\left(\frac{\pi}{5}(t - 2.5)\right) + 13.5$

c. 26 ft

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6.2: Graphs of the Other Trigonometric Functions

Skills to Develop

- Analyze the graph of $y = \tan x$.
- Graph variations of $y = \tan x$.
- Analyze the graphs of $y = \sec x$ and $y = \csc x$.
- Graph variations of $y = \sec x$ and $y = \csc x$.
- Analyze the graph of $y = \cot x$.
- Graph variations of $y = \cot x$.

We know the tangent function can be used to find distances, such as the height of a building, mountain, or flagpole. But what if we want to measure repeated occurrences of distance? Imagine, for example, a police car parked next to a warehouse. The rotating light from the police car would travel across the wall of the warehouse in regular intervals. If the input is time, the output would be the distance the beam of light travels. The beam of light would repeat the distance at regular intervals. The tangent function can be used to approximate this distance. Asymptotes would be needed to illustrate the repeated cycles when the beam runs parallel to the wall because, seemingly, the beam of light could appear to extend forever. The graph of the tangent function would clearly illustrate the repeated intervals. In this section, we will explore the graphs of the tangent and other trigonometric functions.

Analyzing the Graph of $y = \tan x$

We will begin with the graph of the tangent function, plotting points as we did for the sine and cosine functions. Recall that

$$\tan x = \frac{\sin x}{\cos x}.$$

The period of the tangent function is π because the graph repeats itself on intervals of $k\pi$ where k is a constant. If we graph the tangent function on $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, we can see the behavior of the graph on one complete cycle. If we look at any larger interval, we will see that the characteristics of the graph repeat.

We can determine whether tangent is an odd or even function by using the definition of tangent.

$$\begin{aligned} \tan(-x) &= \frac{\sin(-x)}{\cos(-x)} && \text{Definition of tangent} \\ &= \frac{-\sin x}{\cos x} && \text{Sine is an odd function, cosine is even} \\ &= -\frac{\sin x}{\cos x} && \text{The quotient of an odd and an even function is odd} \\ &= -\tan x && \text{Definition of tangent} \end{aligned}$$

Therefore, tangent is an odd function. We can further analyze the graphical behavior of the tangent function by looking at values for some of the special angles, as listed in Table 6.2.1.

Table 6.2.1

x	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\tan x$	undefined	$-\sqrt{3}$	–1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

These points will help us draw our graph, but we need to determine how the graph behaves where it is undefined. If we look more closely at values when $\frac{\pi}{3} < x < \frac{\pi}{2}$, we can use a table to look for a trend. Because $\frac{\pi}{3} \approx 1.05$ and $\frac{\pi}{2} \approx 1.57$, we will evaluate x at radian measures $1.05 < x < 1.57$ as shown in Table 6.2.2.

Table 6.2.2

x	1.3	1.5	1.55	1.56
$\tan x$	3.6	14.1	48.1	92.6

As x approaches $\frac{\pi}{2}$, the outputs of the function get larger and larger. Because $y = \tan\ x$ is an odd function, we see the corresponding table of negative values in Table 6.2.3.

Table 6.2.3

x	−1.3	−1.5	−1.55	−1.56
$\tan x$	−3.6	−14.1	−48.1	−92.6

We can see that, as x approaches $-\frac{\pi}{2}$, the outputs get smaller and smaller. Remember that there are some values of x for which $\cos\ x = 0$. For example, $\cos\left(\frac{\pi}{2}\right) = 0$ and $\cos\left(\frac{3\pi}{2}\right) = 0$. At these values, the tangent function is undefined, so the graph of $y = \tan\ x$ has discontinuities at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$. At these values, the graph of the tangent has vertical asymptotes. Figure 6.2.1 represents the graph of $y = \tan\ x$. The tangent is positive from 0 to $\frac{\pi}{2}$ and from π to $\frac{3\pi}{2}$, corresponding to quadrants I and III of the unit circle.

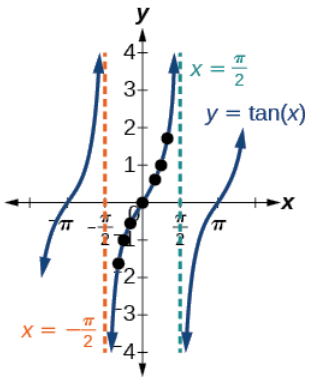


***Figure* 6.2.1:** *Graph of the tangent function*

Graphing Variations of $y = \tan\ x$

As with the sine and cosine functions, the *tangent function* can be described by a general equation.

$$y = A\tan(Bx)$$

We can identify horizontal and vertical stretches and compressions using values of A and B. The horizontal stretch can typically be determined from the period of the graph. With tangent graphs, it is often necessary to determine a vertical stretch using a point on the graph.

Because there are no maximum or minimum values of a tangent function, the term *amplitude* cannot be interpreted as it is for the sine and cosine functions. Instead, we will use the phrase *stretching/compressing factor* when referring to the constant A.

FEATURES OF THE GRAPH OF $y = A\tan(Bx)$

- The stretching factor is $|A|$.
- The period is $P = \frac{\pi}{|B|}$.
- The domain is all real numbers x,where $x \neq \frac{\pi}{2|B|} + \frac{\pi}{|B|}k$ such that k is an integer.
- The range is $(-\infty, \infty)$.
- The asymptotes occur at $x = \frac{\pi}{2|B|} + \frac{\pi}{|B|}k$ where k is an integer.
- $y = A\tan(Bx)$ is an odd function.

Graphing One Period of a Stretched or Compressed Tangent Function

We can use what we know about the properties of the **tangent function** to quickly sketch a graph of any stretched and/or compressed tangent function of the form $f(x) = A\tan(Bx)$. We focus on a single **period** of the function including the origin, because the periodic property enables us to extend the graph to the rest of the function's domain if we wish. Our limited domain is then the interval $\left(-\frac{P}{2}, \frac{P}{2}\right)$ and the graph has vertical asymptotes at $\pm\frac{P}{2}$ where $P = \frac{\pi}{B}$. On $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the graph will come up

from the left asymptote at $x = -\frac{\pi}{2}$, cross through the origin, and continue to increase as it approaches the right asymptote at $x = \frac{\pi}{2}$. To make the function approach the asymptotes at the correct rate, we also need to set the vertical scale by actually evaluating the function for at least one point that the graph will pass through. For example, we can use

$$f\left(\frac{P}{4}\right) = A\tan\left(B\frac{P}{4}\right) = A\tan\left(B\frac{\pi}{4B}\right) = A$$

because $\tan\left(\frac{\pi}{4}\right) = 1$.

Howto: Given the function $f(x) = A\tan(Bx)$, graph one period.

1. Identify the stretching factor, $|A|$.
2. Identify B and determine the period, $P = \frac{\pi}{|B|}$.
3. Draw vertical asymptotes at $x = -\frac{P}{2}$ and $x = \frac{P}{2}$.
4. For $A > 0$, the graph approaches the left asymptote at negative output values and the right asymptote at positive output values (reverse for $A < 0$).
5. Plot reference points at $\left(\frac{P}{4}, A\right)$, $(0, 0)$, and $\left(-\frac{P}{4}, -A\right)$, and draw the graph through these points.

Example 6.2.1: Sketching a Compressed Tangent

Sketch a graph of one period of the function $y = 0.5\tan\left(\frac{\pi}{2}x\right)$.

Solution

First, we identify A and B.

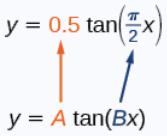


***Figure* 6.2.2**

Because $A = 0.5$ and $B = \frac{\pi}{2}$, we can find the stretching/compressing factor and period. The period is $\frac{\pi}{\frac{\pi}{2}} = 2$, so the asymptotes are at $x = \pm 1$. At a quarter period from the origin, we have

$$\begin{aligned} f(0.5) &= 0.5\tan\left(\frac{0.5\pi}{2}\right) \\ &= 0.5\tan\left(\frac{\pi}{4}\right) \\ &= 0.5 \end{aligned}$$

This means the curve must pass through the points $(0.5, 0.5)$, $(0, 0)$,and $(-0.5, -0.5)$. The only inflection point is at the origin. Figure 6.2.3 shows the graph of one period of the function.

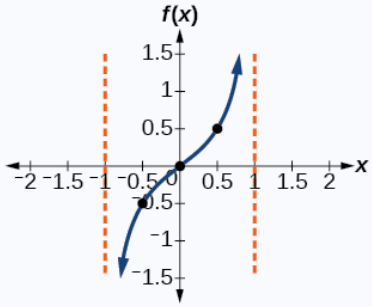


***Figure* 6.2.3**

6.2.1

Sketch a graph of $f(x) = 3\tan\left(\frac{\pi}{6}x\right)$.

Answer

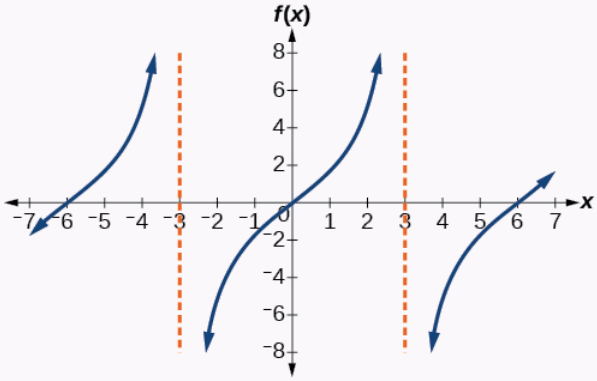


***Figure* 6.2.4**

Graphing One Period of a Shifted Tangent Function

Now that we can graph a **tangent function** that is stretched or compressed, we will add a vertical and/or horizontal (or phase) shift. In this case, we add C and D to the general form of the tangent function.

$$f(x) = A\tan(Bx - C) + D$$

The graph of a transformed tangent function is different from the basic tangent function $\tan x$ in several ways:

FEATURES OF THE GRAPH OF $y = A\tan(Bx - C) + D$

- The stretching factor is $|A|$.
- The period is $\frac{\pi}{|B|}$.
- The domain is $x \neq \frac{C}{B} + \frac{\pi}{|B|}k$,where k is an integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \frac{C}{B} + \frac{\pi}{|B|}k$,where k is an odd integer.
- There is no amplitude.
- $y = A\tan(Bx)$ is and odd function because it is the qoutient of odd and even functions(sin and cosine perspectively).

Howto: Given the function $y = A\tan(Bx - C) + D$, sketch the graph of one period.

1. Express the function given in the form $y = A\tan(Bx - C) + D$.
2. Identify the **stretching/compressing** factor, $|A|$.
3. Identify B and determine the period, $P = \frac{\pi}{|B|}$.
4. Identify C and determine the phase shift, $\frac{C}{B}$.
5. Draw the graph of $y = A\tan(Bx)$ shifted to the right by $\frac{C}{B}$ and up by D.
6. Sketch the vertical asymptotes, which occur at $x = \frac{C}{B} + \frac{\pi}{2|B|}k$,where k is an odd integer.
7. Plot any three reference points and draw the graph through these points.

Example 6.2.2: Graphing One Period of a Shifted Tangent Function

Graph one period of the function $y = -2\tan(\pi x + \pi) - 1$.

Solution

- *Step 1.* The function is already written in the form $y = A\tan(Bx - C) + D$.
- *Step 2.*\(A=−2\), so the stretching factor is $|A| = 2$.
- *Step 3.* \(B=\pi\), so the period is $P = \frac{\pi}{|B|} = \frac{\pi}{pi} = 1$.
- *Step 4.* \(C=−\pi\), so the phase shift is $CB = \frac{-\pi}{\pi} = -1$.
- *Step 5-7.* The asymptotes are at $x = -\frac{3}{2}$ and $x = -\frac{1}{2}$ and the three recommended reference points are $(-1.25, 1)$, $(-1, -1)$, and $(-0.75, -3)$. The graph is shown in Figure 6.2.5.

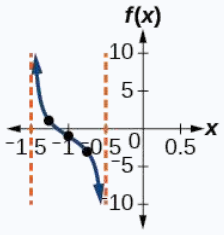


***Figure* 6.2.5**

Analysis

Note that this is a decreasing function because $A < 0$.

Exercise 6.2.2

How would the graph in Example 6.2.2 look different if we made $A = 2$ instead of -2?

Answer

It would be reflected across the line $y = -1$, becoming an increasing function.

Howto: Given the graph of a tangent function, identify horizontal and vertical stretches.

1. Find the period P from the spacing between successive vertical asymptotes or *x*-intercepts.
2. Write $f(x) = A\tan\left(\frac{\pi}{P}x\right)$.
3. Determine a convenient point $(x, f(x))$ on the given graph and use it to determine A.

Example 6.2.3: Identifying the Graph of a Stretched Tangent

Find a formula for the function graphed in Figure 6.2.6.

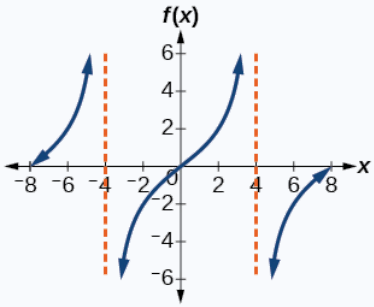


***Figure* 6.2.6:** *A stretched tangent function*

Solution

The graph has the shape of a tangent function.

- *Step 1.* One cycle extends from –4 to 4, so the period is $P = 8$. Since $P = \frac{\pi}{|B|}$, we have $B = \frac{\pi}{P} = \frac{\pi}{8}$.
- *Step 2.* The equation must have the form $f(x) = A\tan\left(\frac{\pi}{8}x\right)$.

- *Step 3.* To find the vertical stretch A,we can use the point $(2, 2)$.

$$\begin{aligned} 2 &= A\tan\left(\frac{\pi}{8}\cdot 2\right) \\ &= A\tan\left(\frac{\pi}{4}\right) \end{aligned}$$

Because $\tan\left(\frac{\pi}{4}\right) = 1$, $A = 2$.

This function would have a formula $f(x) = 2\tan\left(\frac{\pi}{8}x\right)$.

Exercise 6.2.3

Find a formula for the function in Figure 6.2.7.

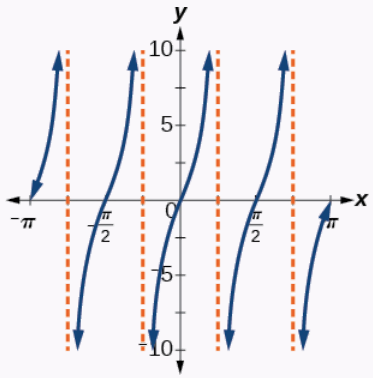


***Figure* 6.2.7**

Answer

$g(x) = 4\tan(2x)$

Analyzing the Graphs of $y = \sec x$ and $y = \csc x$

The secant was defined by the reciprocal identity $\sec x = \frac{1}{\cos x}$. Notice that the function is undefined when the cosine is 0, leading to vertical asymptotes at $\frac{\pi}{2}$, $\frac{3\pi}{2}$ etc. Because the cosine is never more than 1 in absolute value, the secant, being the reciprocal, will never be less than 1 in absolute value.

We can graph $y = \sec x$ by observing the graph of the cosine function because these two functions are reciprocals of one another. See Figure 6.2.8. The graph of the cosine is shown as a dashed orange wave so we can see the relationship. Where the graph of the cosine function decreases, the graph of the secant function increases. Where the graph of the cosine function increases, the graph of the secant function decreases. When the cosine function is zero, the secant is undefined.

The secant graph has vertical asymptotes at each value of x where the cosine graph crosses the x-axis; we show these in the graph below with dashed vertical lines, but will not show all the asymptotes explicitly on all later graphs involving the secant and cosecant.

Note that, because cosine is an even function, secant is also an even function. That is, $\sec(-x) = \sec x$.

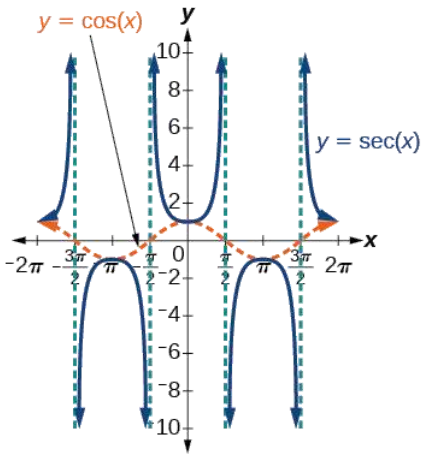


***Figure* 6.2.8:** *Graph of the secant function,* $f(x) = \sec x = \dfrac{1}{\cos x}$

As we did for the tangent function, we will again refer to the constant $|A|$ as the stretching factor, not the amplitude.

FEATURES OF THE GRAPH OF $y = A\sec(Bx)$

- The stretching factor is $|A|$.
- The period is $\dfrac{2\pi}{|B|}$.
- The domain is $x \neq \dfrac{\pi}{2|B|}k$, where k is an odd integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \dfrac{\pi}{2|B|}k$, where k is an odd integer.
- There is no amplitude.
- $y = A\sec(Bx)$ is an even function because cosine is an even function.

Similar to the secant, the **cosecant** is defined by the reciprocal identity $\csc x = \dfrac{1}{\sin x}$. Notice that the function is undefined when the sine is 0, leading to a vertical asymptote in the graph at 0, π, etc. Since the sine is never more than 1 in absolute value, the cosecant, being the reciprocal, will never be less than 1 in absolute value.

We can graph $y = \csc x$ by observing the graph of the sine function because these two functions are reciprocals of one another. See Figure $6.2.7$. The graph of sine is shown as a dashed orange wave so we can see the relationship. Where the graph of the sine function decreases, the graph of the cosecant function increases. Where the graph of the sine function increases, the graph of the **cosecant function** decreases.

The cosecant graph has vertical asymptotes at each value of x where the sine graph crosses the x-axis; we show these in the graph below with dashed vertical lines.

Note that, since sine is an odd function, the cosecant function is also an odd function. That is, $\csc(-x) = -\csc x$.

The graph of cosecant, which is shown in Figure $6.2.9$, is similar to the graph of secant.

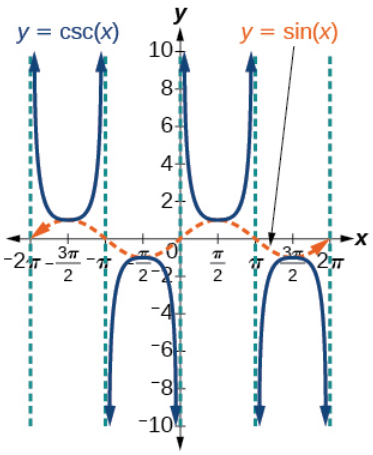


***Figure* 6.2.9:** *The graph of the cosecant function,* $f(x) = \csc x = \frac{1}{\sin x}$

FEATURES OF THE GRAPH OF $y = A\csc(Bx)$

- The stretching factor is $|A|$.
- The period is $\frac{2\pi}{|B|}$.
- The domain is $x \neq \frac{\pi}{|B|}k$, where k is an integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The asymptotes occur at $x = \frac{\pi}{|B|}k$, where k is an integer.
- $y = A\csc(Bx)$ is an odd function because sine is an odd function.

Graphing Variations of $y = \sec x$ and $y = \csc x$

For shifted, compressed, and/or stretched versions of the secant and cosecant functions, we can follow similar methods to those we used for tangent and cotangent. That is, we locate the vertical asymptotes and also evaluate the functions for a few points (specifically the local extrema). If we want to graph only a single period, we can choose the interval for the period in more than one way. The procedure for secant is very similar, because the cofunction identity means that the secant graph is the same as the cosecant graph shifted half a period to the left. Vertical and phase shifts may be applied to the cosecant function in the same way as for the secant and other functions.The equations become the following.

$$y = A\sec(Bx - C) + D \tag{6.2.1}$$

$$y = A\csc(Bx - C) + D \tag{6.2.2}$$

FEATURES OF THE GRAPH OF $y = A\sec(Bx - C) + D$

- The stretching factor is $|A|$.
- The period is $\frac{2\pi}{|B|}$.
- The domain is $x \neq \frac{C}{B} + \frac{\pi}{2|B|}k$,where k is an odd integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \frac{C}{B} + \frac{\pi}{2|B|}k$,where k is an odd integer.
- There is no amplitude.
- $y = A\sec(Bx)$ is an even function because cosine is an even function.

FEATURES OF THE GRAPH OF $y = A\csc(Bx - C) + D$

1. The stretching factor is $|A|$.
2. The period is $\frac{2\pi}{|B|}$.
3. The domain is $x \neq \frac{C}{B} + \frac{\pi}{2|B|}k$,where k is an integer.

4. The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
5. The vertical asymptotes occur at $x = \frac{C}{B} + \frac{\pi}{|B|}k$,where k is an integer.
6. There is no amplitude.
7. $y = A\csc(Bx)$ is an odd function because sine is an odd function.

HOWTO: Given a function of the form $y = A\sec(Bx)$, graph one period

1. Express the function given in the form $y = A\sec(Bx)$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $P = \frac{2\pi}{|B|}$.
4. Sketch the graph of $y = A\cos(Bx)$.
5. Use the reciprocal relationship between $y = \cos x$ and $y = \sec x$ to draw the graph of $y = A\sec(Bx)$.
6. Sketch the asymptotes.
7. Plot any two reference points and draw the graph through these points.

Example 6.2.4: Graphing a Variation of the Secant Function

Graph one period of $f(x) = 2.5\sec(0.4x)$.

Solution

- *Step 1.* The given function is already written in the general form, $y = A\sec(Bx)$.
- *Step 2.* $A = 2.5$ so the stretching factor is 2.5.
- *Step 3.* $B = 0.4$ so $P = \frac{2\pi}{0.4} = 5\pi$. The period is 5π units.
- *Step 4.* Sketch the graph of the function $g(x) = 2.5\cos(0.4x)$.
- *Step 5.* Use the reciprocal relationship of the cosine and secant functions to draw the cosecant function.
- *Steps 6–7.* Sketch two asymptotes at $x = 1.25\pi$ and $x = 3.75\pi$. We can use two reference points, the local minimum at $(0, 2.5)$ and the local maximum at $(2.5\pi, -2.5)$. Figure 6.2.10 shows the graph.

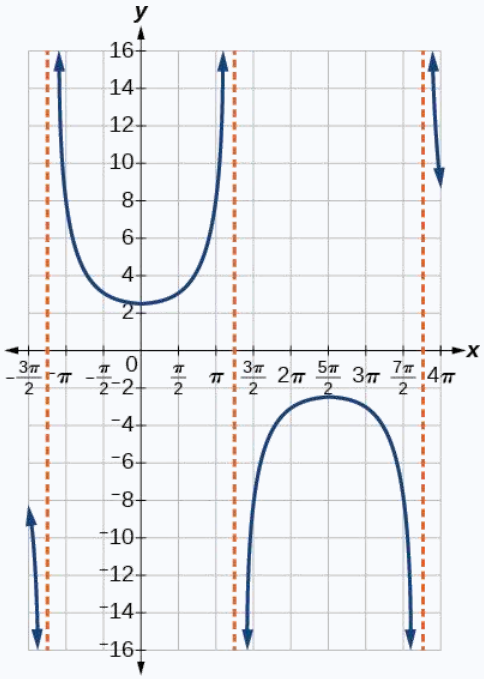


***Figure* 6.2.10**

Exercise 6.2.4

Graph one period of $f(x) = -2.5\sec(0.4x)$.

Answer

This is a vertical reflection of the preceding graph because A is negative.

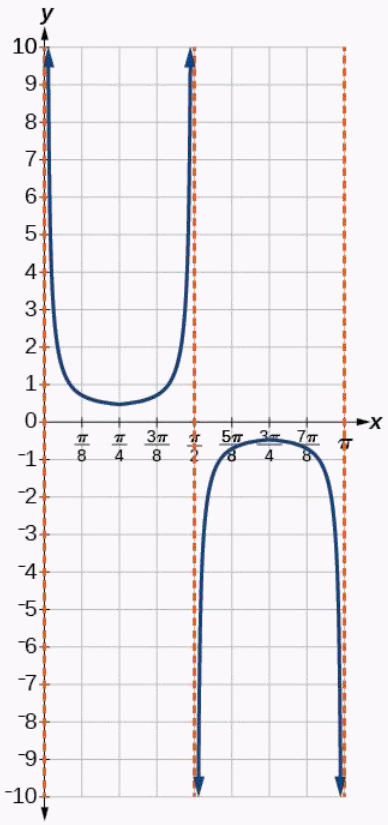


***Figure* 6.2.11**

Q&A: Do the vertical shift and stretch/compression affect the secant's range?

Yes. The range of $f(x) = A\sec(Bx - C) + D$ is $(-\infty, -|A| + D] \cup [|A| + D, \infty)$.

Howto: Given a function of the form $f(x) = A\sec(Bx - C) + D$, graph one period.

1. Express the function given in the form $y = A\sec(Bx - C) + D$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $\frac{2\pi}{|B|}$.
4. Identify C and determine the phase shift, $\frac{C}{B}$.
5. Draw the graph of $y = A\sec(Bx)$. but shift it to the right by $\frac{C}{B}$ and up by D.
6. Sketch the vertical asymptotes, which occur at $x = \frac{C}{B} + \frac{\pi}{2|B|}k$, where k is an odd integer.

Example 6.2.5: Graphing a Variation of the Secant Function

Graph one period of $y = 4\sec\left(\frac{\pi}{3x} - \frac{\pi}{2}\right) + 1$.

Solution

- *Step 1.* Express the function given in the form $y = 4\sec\left(\frac{\pi}{3x} - \frac{\pi}{2}\right) + 1$.
- *Step 2.* The stretching/compressing factor is $|A| = 4$.
- *Step 3.* The period is

$$\begin{aligned}\frac{2\pi}{|B|} &= \frac{2\pi}{\frac{\pi}{3}} \\ &= 2\pi \cdot \frac{3}{\pi} \\ &= 6\end{aligned}$$

- *Step 4.* The phase shift is

$$\begin{aligned}\frac{C}{B} &= \frac{\frac{\pi}{2}}{\frac{\pi}{3}} \\ &= \frac{\pi}{2} \cdot \frac{3}{\pi} \\ &= 1.5\end{aligned}$$

- *Step 5.* Draw the graph of $y = A\sec(Bx)$, but shift it to the right by $\frac{C}{B} = 1.5$ and up by $D = 6$.
- *Step 6.* Sketch the vertical asymptotes, which occur at $x = 0$, $x = 3$, and $x = 6$. There is a local minimum at $(1.5, 5)$ and a local maximum at $(4.5, -3)$. Figure 6.2.12 shows the graph.

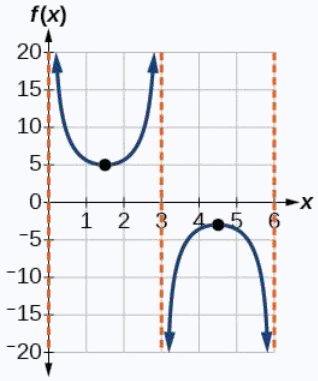


***Figure* 6.2.12**

Exercise 6.2.5

Graph one period of $f(x) = -6\sec(4x + 2) - 8$.

Answer

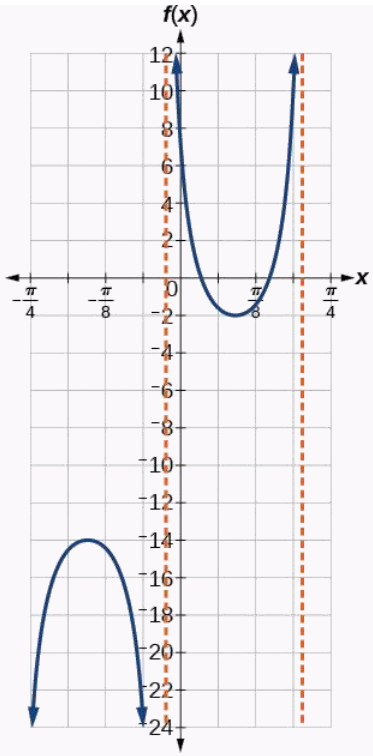


***Figure* 6.2.13**

Q&A: The domain of $\csc x$ was given to be all x such that $x \neq k\pi$ for any integer k. Would the domain of $y = A\csc(Bx - C) + D$ be $x \neq \frac{C + k\pi}{B}$?

Yes. The excluded points of the domain follow the vertical asymptotes. Their locations show the horizontal shift and compression or expansion implied by the transformation to the original function's input.

Howto: Given a function of the form $y = A\csc(Bx)$, graph one period.

1. Express the function given in the form $y = A\csc(Bx)$.
2. $|A|$.
3. Identify B and determine the period, $P = \frac{2\pi}{|B|}$.
4. Draw the graph of $y = A\sin(Bx)$.
5. Use the reciprocal relationship between $y = \sin x$ and $y = \csc x$ to draw the graph of $y = A\csc(Bx)$.
6. Sketch the asymptotes.
7. Plot any two reference points and draw the graph through these points.

Example 6.2.6: Graphing a Variation of the Cosecant Function

Graph one period of $f(x) = -3\csc(4x)$.

Solution

- *Step 1.* The given function is already written in the general form, $y = A\csc(Bx)$.
- *Step 2.* $|A| = |-3| = 3$,so the stretching factor is 3.
- *Step 3.* $B = 4$,so $P = \frac{2\pi}{4} = \frac{\pi}{2}$. The period is $\frac{\pi}{2}$ units.
- *Step 4.* Sketch the graph of the function $g(x) = -3\sin(4x)$.
- *Step 5.* Use the reciprocal relationship of the sine and cosecant functions to draw the cosecant function.

- *Steps 6–7.* Sketch three asymptotes at $x = 0$, $x = \frac{\pi}{4}$, and $x = \frac{\pi}{2}$. We can use two reference points, the local maximum at $\left(\frac{\pi}{8}, -3\right)$ and the local minimum at $\left(\frac{3\pi}{8}, 3\right)$. Figure 6.2.14 shows the graph.

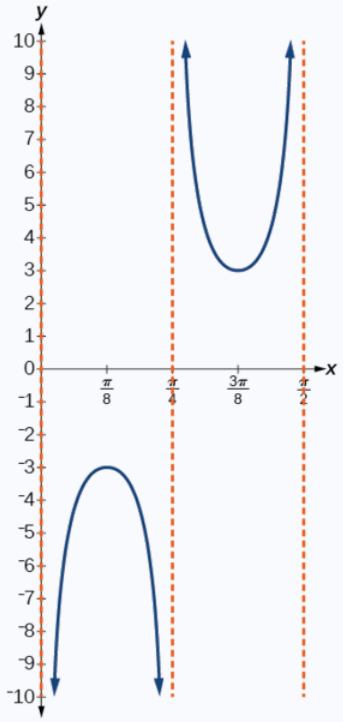


***Figure* 6.2.14**

Exercise 6.2.6

Graph one period of $f(x) = 0.5\csc(2x)$.

Answer

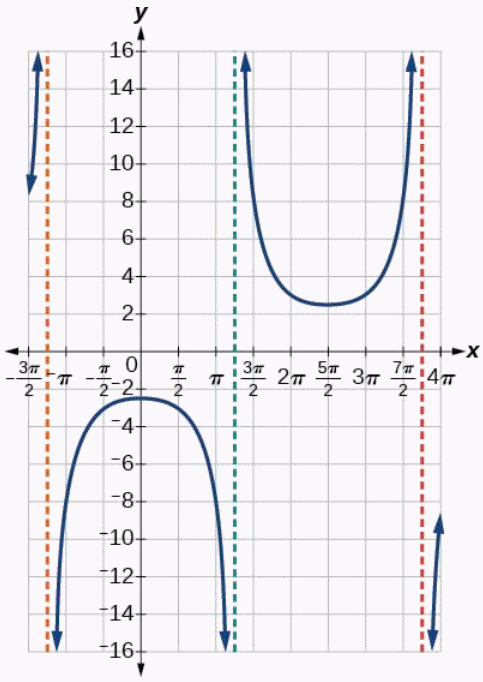


Figure 6.2.15

Howto: Given a function of the form $f(x) = A\csc(Bx - C) + D$, graph one period

1. Express the function given in the form $y = A\csc(Bx - C) + D$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $\frac{2\pi}{|B|}$.
4. Identify C and determine the phase shift, $\frac{C}{B}$.
5. Draw the graph of $y = A\csc(Bx)$ but shift it to the right by and up by D.
6. Sketch the vertical asymptotes, which occur at $x = \frac{C}{B} + \frac{\pi}{|B|}k$,where k is an integer.

Example 6.2.7: Graphing a Vertically Stretched, Horizontally Compressed, and Vertically Shifted Cosecant

Sketch a graph of $y = 2\csc\left(\frac{\pi}{2}x\right) + 1$. What are the domain and range of this function?

Solution

- *Step 1.* Express the function given in the form $y = 2\csc\left(\frac{\pi}{2}x\right) + 1$.
- *Step 2.* Identify the stretching/compressing factor, $|A| = 2$.
- *Step 3.* The period is $\frac{2\pi}{|B|} = \frac{2\pi}{\frac{\pi}{2}} = 2\pi \cdot \frac{2}{\pi} = 4$.
- *Step 4.* The phase shift is $\frac{0}{\frac{\pi}{2}} = 0$.
- *Step 5.* Draw the graph of $y = A\csc(Bx)$ but shift it up $D = 1$.
- *Step 6.* Sketch the vertical asymptotes, which occur at $x = 0$, $x = 2$, $x = 4$.

The graph for this function is shown in Figure 6.2.16.

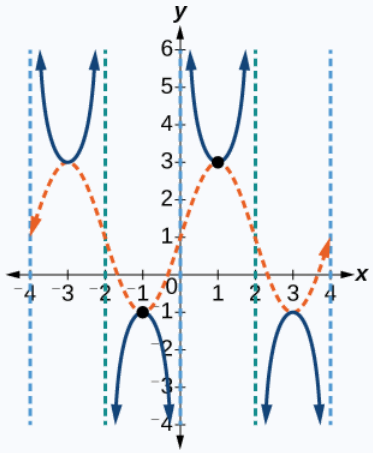


***Figure* 6.2.16:** *A transformed cosecant function*

Analysis

The vertical asymptotes shown on the graph mark off one period of the function, and the local extrema in this interval are shown by dots. Notice how the graph of the transformed cosecant relates to the graph of $f(x) = 2\sin\left(\frac{\pi}{2}x\right) + 1$,shown as the orange dashed wave.

Exercise 6.2.7

Given the graph of $f(x) = 2\cos\left(\frac{\pi}{2}x\right) + 1$ shown in Figure 6.2.17, sketch the graph of $g(x) = 2\sec\left(\frac{\pi}{2}x\right) + 1$ on the same axes.

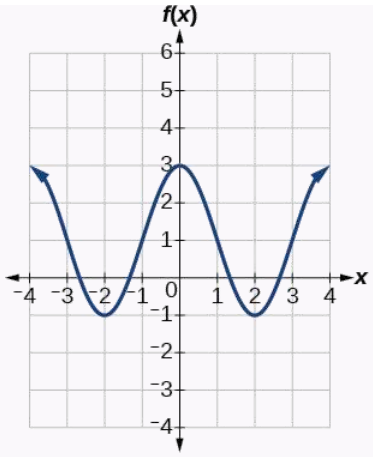


Figure 6.2.17

Answer

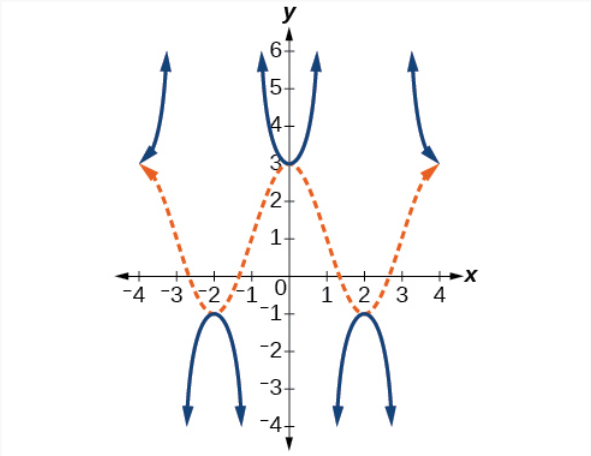


Figure 6.2.18

Analyzing the Graph of $y = \cot x$

The last trigonometric function we need to explore is cotangent. The cotangent is defined by the reciprocal identity $\cot x = \frac{1}{\tan x}$. Notice that the function is undefined when the tangent function is 0, leading to a vertical asymptote in the graph at 0, π, etc. Since the output of the tangent function is all real numbers, the output of the cotangent function is also all real numbers.

We can graph $y = \cot x$ by observing the graph of the tangent function because these two functions are reciprocals of one another. See Figure 6.2.19. Where the graph of the tangent function decreases, the graph of the cotangent function increases. Where the graph of the tangent function increases, the graph of the cotangent function decreases.

The cotangent graph has vertical asymptotes at each value of x where $\tan x = 0$; we show these in the graph below with dashed lines. Since the cotangent is the reciprocal of the tangent, $\cot x$ has vertical asymptotes at all values of x where $\tan x = 0$, and $\cot x = 0$ at all values of x where $\tan x$ has its vertical asymptotes.

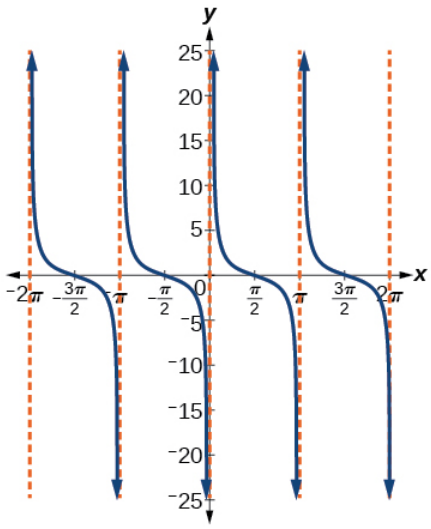


Figure* 6.2.19*: *The cotangent function*

FEATURES OF THE GRAPH OF $y = A\cot(BX)$

- The stretching factor is $|A|$.
- The period is $P = \frac{\pi}{|B|}$.
- The domain is $x \neq \frac{\pi}{|B|}k$, where k is an integer.
- The range is $(-\infty, \infty)$.
- The asymptotes occur at $x = \frac{\pi}{|B|}k$, where k is an integer.
- $y = A\cot(Bx)$ is an odd function.

Graphing Variations of $y = \cot x$

We can transform the graph of the cotangent in much the same way as we did for the tangent. The equation becomes the following.

$$y = A\cot(Bx - C) + D \tag{6.2.3}$$

PROPERTIES OF THE GRAPH OF $y = A\cot(Bx - c) + D$

- The stretching factor is $|A|$.
- The period is $\frac{\pi}{|B|}$
- The domain is $x \neq \frac{C}{B} + \frac{\pi}{|B|}k$,where k is an integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \frac{C}{B} + \frac{\pi}{|B|}k$,where k is an integer.
- There is no amplitude.
- $y = A\cot(Bx)$ is an odd function because it is the quotient of even and odd functions (cosine and sine, respectively)

Howto: Given a modified cotangent function of the form $f(x) = A\cot(Bx)$,graph one period.

1. Express the function in the form $f(x) = A\cot(Bx)$.
2. Identify the stretching factor, $|A|$.
3. Identify the period, $P = \frac{\pi}{|B|}$.
4. Draw the graph of $y = A\tan(Bx)$.
5. Plot any two reference points.
6. Use the reciprocal relationship between tangent and cotangent to draw the graph of $y = Acot(Bx)$.
7. Sketch the asymptotes.

Example 6.2.8: Graphing Variations of the Cotangent Function

Determine the stretching factor, period, and phase shift of $y = 3\cot(4x)$, and then sketch a graph.

Solution

- *Step 1.* Expressing the function in the form $f(x) = A\cot(Bx)$ gives $f(x) = 3\cot(4x)$.
- *Step 2.* The stretching factor is $|A| = 3$.
- *Step 3.* The period is $P = \frac{\pi}{4}$.
- *Step 4.* Sketch the graph of $y = 3\tan(4x)$.
- *Step 5.* Plot two reference points. Two such points are $\left(\frac{\pi}{16}, 3\right)$ and $\left(\frac{3\pi}{16}, -3\right)$.
- *Step 6.* Use the reciprocal relationship to draw $y = 3\cot(4x)$.
- *Step 7.* Sketch the asymptotes, $x = 0$, $x = \frac{\pi}{4}$.

The orange graph in Figure 6.2.20 shows $y = 3\tan(4x)$ and the blue graph shows $y = 3\cot(4x)$.

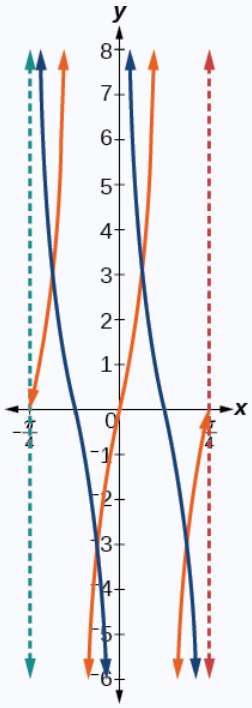


***Figure* 6.2.20**

Howto: Given a modified cotangent function of the form $f(x) = A\cot(Bx - C) + D$, graph one period.

1. Express the function in the form $f(x) = A\cot(Bx - C) + D$.
2. Identify the stretching factor, $|A|$.
3. Identify the period, $P = \frac{\pi}{|B|}$.
4. Identify the phase shift, $\frac{C}{B}$.
5. Draw the graph of $y = A\tan(Bx)$ shifted to the right by $\frac{C}{B}$ and up by D.
6. Sketch the asymptotes $x = \frac{C}{B} + \frac{\pi}{|B|}k$, where k is an integer.
7. Plot any three reference points and draw the graph through these points.

Example 6.2.9: Graphing a Modified Cotangent

Sketch a graph of one period of the function $f(x) = 4\cot\left(\frac{\pi}{8}x - \frac{\pi}{2}\right) - 2$.

Solution

- *Step 1.* The function is already written in the general form $f(x) = A\cot(Bx - C) + D$.
- *Step 2.* $A = 4$,so the stretching factor is 4.
- *Step 3.* $B = \frac{\pi}{8}$, so the period is $P = \frac{\pi}{|B|} = \frac{\pi}{\frac{\pi}{8}} = 8$.
- *Step 4.* $C = \frac{\pi}{2}$,so the phase shift is $CB = \frac{\frac{\pi}{2}}{\frac{\pi}{8}} = 4$.
- *Step 5.* We draw $f(x) = 4\tan\left(\frac{\pi}{8}x - \frac{\pi}{2}\right) - 2$.
- *Step 6-7.* Three points we can use to guide the graph are $(6, 2)$, $(8, -2)$, and $(10, -6)$. We use the reciprocal relationship of tangent and cotangent to draw $f(x) = 4\cot\left(\frac{\pi}{8}x - \frac{\pi}{2}\right) - 2$.
- *Step 8.* The vertical asymptotes are $x = 4$ and $x = 12$.

The graph is shown in Figure 6.2.21.

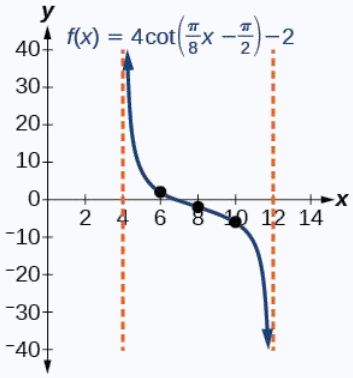


***Figure* 6.2.21:** *One period of a modified cotangent function*

Using the Graphs of Trigonometric Functions to Solve Real-World Problems

Many real-world scenarios represent periodic functions and may be modeled by trigonometric functions. As an example, let's return to the scenario from the section opener. Have you ever observed the beam formed by the rotating light on a police car and wondered about the movement of the light beam itself across the wall? The periodic behavior of the distance the light shines as a function of time is obvious, but how do we determine the distance? We can use the **tangent function**.

Example 6.2.10: Using Trigonometric Functions to Solve Real-World Scenarios

Suppose the function $y = 5\tan(\frac{\pi}{4}t)$ marks the distance in the movement of a light beam from the top of a police car across a wall where t is the time in seconds and y is the distance in feet from a point on the wall directly across from the police car.

a. Find and interpret the stretching factor and period.
b. Graph on the interval $[0, 5]$.
c. Evaluate $f(1)$ and discuss the function's value at that input.

Solution

a. We know from the general form of $y = A\tan(Bt)$ that $|A|$ is the stretching factor and $\frac{\pi}{B}$ is the period.

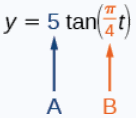


***Figure* 6.2.22**

We see that the stretching factor is 5. This means that the beam of light will have moved 5 ft after half the period.

The period is $\frac{\pi}{\frac{\pi}{4}} = \frac{\pi}{1} \cdot \frac{4}{\pi} = 4$. This means that every 4 seconds, the beam of light sweeps the wall. The distance from the spot across from the police car grows larger as the police car approaches.

b. To graph the function, we draw an asymptote at $t = 2$ and use the stretching factor and period. See Figure 6.2.23

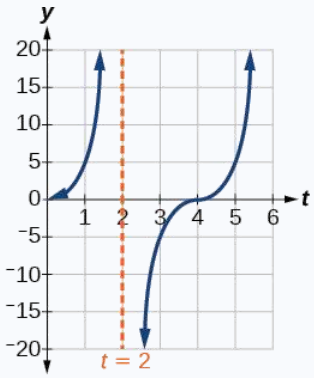


***Figure* 6.2.23**

c. period: $f(1) = 5\tan(\frac{\pi}{4}(1)) = 5(1) = 5$; after 1 second, the beam of has moved 5 ft from the spot across from the police car.

Media

Access these online resources for additional instruction and practice with graphs of other trigonometric functions.

- Graphing the Tangent
- Graphing Cosecant and Secant
- Graphing the Cotangent

Key Equations

Shifted, compressed, and/or stretched tangent function	$y = A\tan(Bx - C) + D$
Shifted, compressed, and/or stretched secant function	$y = A\sec(Bx - C) + D$
Shifted, compressed, and/or stretched cosecant function	$y = A\csc(Bx - C) + D$
Shifted, compressed, and/or stretched cotangent function	$y = A\cot(Bx - C) + D$

Key Concepts

- The tangent function has period π.
- $f(x) = A\tan(Bx - C) + D$ is a tangent with vertical and/or horizontal stretch/compression and shift. See Example 6.2.1, Example 6.2.2, and Example 6.2.3.
- The secant and cosecant are both periodic functions with a period of 2π. $f(x) = A\sec(Bx - C) + D$ gives a shifted, compressed, and/or stretched secant function graph. See Example 6.2.4 and Example 6.2.5.
- $f(x) = A\csc(Bx - C) + D$ gives a shifted, compressed, and/or stretched cosecant function graph. See Example 6.2.6 and Example 6.2.7.
- The cotangent function has period π and vertical asymptotes at $0, \pm\pi, \pm 2\pi, \ldots$.
- The range of cotangent is $(-\infty, \infty)$, and the function is decreasing at each point in its range.
- The cotangent is zero at $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$.
- $f(x) = A\cot(Bx - C) + D$ is a cotangent with vertical and/or horizontal stretch/compression and shift. See Example 6.2.8 and Example 6.2.9.
- Real-world scenarios can be solved using graphs of trigonometric functions. See Example 6.2.10.

Contributors

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6.2E: Exercises

6.2: Graphs of the Other Trigonometric Functions

This section addresses the graphing of the Tangent, Cosecant, Secant, and Cotangent curves.

Verbal

1) Explain how the graph of the sine function can be used to graph $y = \csc x$.

Answer

Since $y = \csc x$ is the reciprocal function of $y = \sin x$ you can plot the reciprocal of the coordinates on the graph of $y = \sin x$ obtain the y-coordinates of $y = \csc x$ The x-intercepts of the graph $y = \sin x$ are the vertical asymptotes for the graph of $y = \csc x$.

2) How can the graph of $y = \cos x$ be used to construct the graph of $y = \sec x$?

3) Explain why the period of $y = \tan x$ is equal to π.

Answer

Answers will vary. Using the unit circle, one can show that $y = \tan(x + \pi) = \tan x$.

4) Why are there no intercepts on the graph of $y = \csc x$?

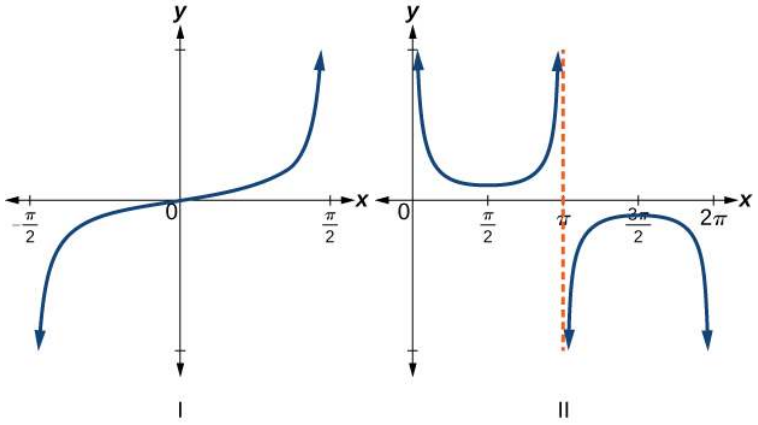
5) How does the period of $y = \csc x$ compare with the period of $y = \sin x$?

Answer

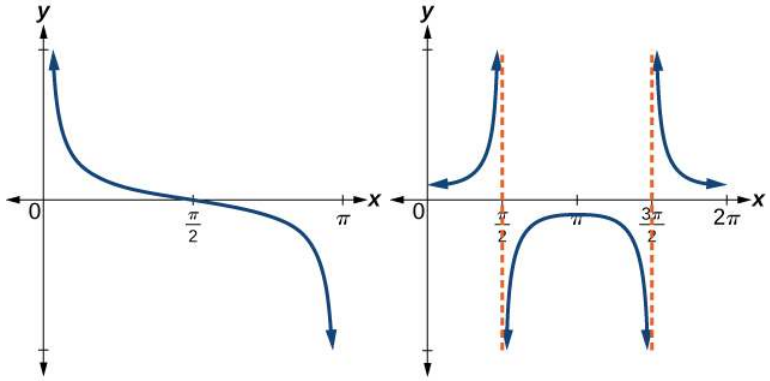
The period is the same: 2π

Algebraic

For exercises 6-9, match each trigonometric function with one of the following graphs.



I II



III IV

Exercise 6.2E.6

$f(x) = \tan x$

Answer

I

Exercise 6.2E.7

$f(x) = \sec x$

Answer

IV

Exercise 6.2E.8

$f(x) = \csc x$

Answer

II

Exercise 6.2E.9

$f(x) = \cot x$

Answer

III

For exercises 10-16, find the period and horizontal shift of each of the functions.

10) $f(x) = 2\tan(4x - 32)$

11) $h(x) = 2\sec\left(\frac{\pi}{4}(x+1)\right)$

Answer

period: 8; horizontal shift: 1 unit to left

12) $m(x) = 6\csc\left(\frac{\pi}{3}x + \pi\right)$

13) If $\tan x = -1.5$, find $\tan(-x)$.

Answer

1.5

14) If $\sec x = 2$, find $\sec(-x)$.

15) If $\csc x = -5$, find $\csc(-x)$.

Answer

5

16) If $x \sin x = 2$, find $(-x)\sin(-x)$.

For exercises 17-18, rewrite each expression such that the argument x is positive.

Exercise $6.2E.17$

$\cot(-x)\cos(-x) + \sin(-x)$

Answer

$-\cot x \cos x - \sin x$

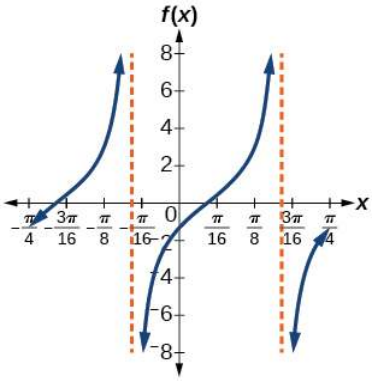
18) $\cos(-x) + \tan(-x)\sin(-x)$

Graphical

For the exercises 19-36, sketch two periods of the graph for each of the following functions. Identify the stretching factor, period, and asymptotes.

19) $f(x) = 2\tan(4x - 32)$

Answer



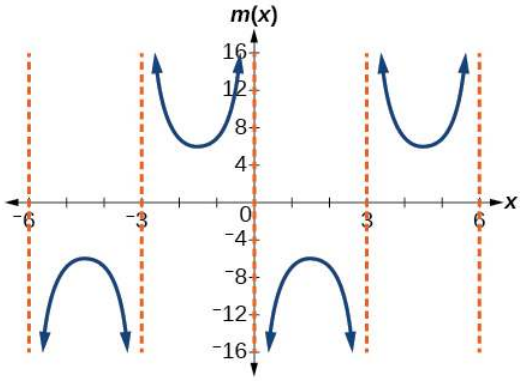
stretching factor: 2; period: $\frac{\pi}{3}$ asymptotes: $x = \frac{1}{4}\left(\frac{\pi}{2} + \pi k\right) + 8$, where k is an integer

20) $h(x) = 2\sec\left(\frac{\pi}{4}(x+1)\right)$

21) $m(x) = 6\csc\left(\frac{\pi}{3}x + \pi\right)$

Answer

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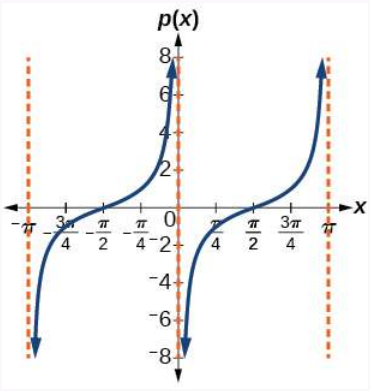
stretching factor: 6; period: 6; asymptotes: $x = k$, where k is an integer

22) $j(x) = \tan\left(\frac{\pi}{2}x\right)$

Exercise 6.2E.23

$p(x) = \tan\left(x - \frac{\pi}{2}\right)$

Answer

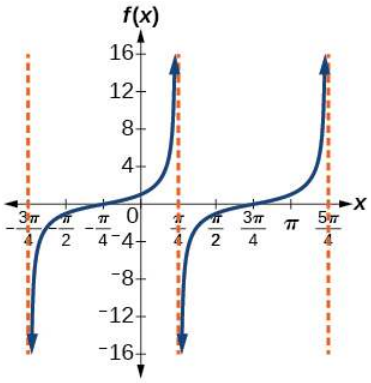


stretching factor: 1; period: π asymptotes: $x = \pi k$, where k is an integer

24) $f(x) = 4\tan(x)$

25) $f(x) = \tan\left(x + \frac{\pi}{4}\right)$

Answer



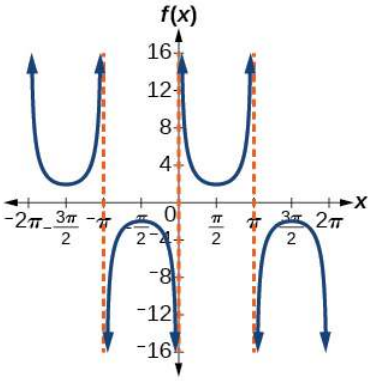
Stretching factor: 1; period: π asymptotes: $x = \frac{\pi}{4} + \pi k$, where k is an integer

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26) $f(x) = \pi \tan(\pi x - \pi) - \pi$

27) $f(x) = 2 \csc(x)$

Answer

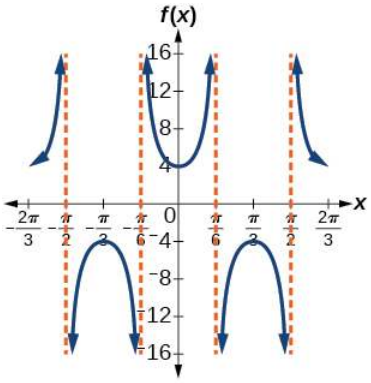


stretching factor: 2; period: 2π asymptotes: $x = \pi k$, where k is an integer

28) $f(x) = -\frac{1}{4}\csc(x)$

29) $f(x) = 4 \sec(3x)$

Answer

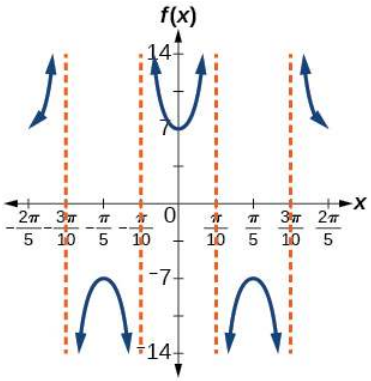


stretching factor: 4; period: $\frac{2\pi}{3}$ asymptotes: $x = \frac{\pi}{6}k$, where k is an odd integer

30) $f(x) = -3 \cot(2x)$

31) $f(x) = 7 \sec(5x)$

Answer

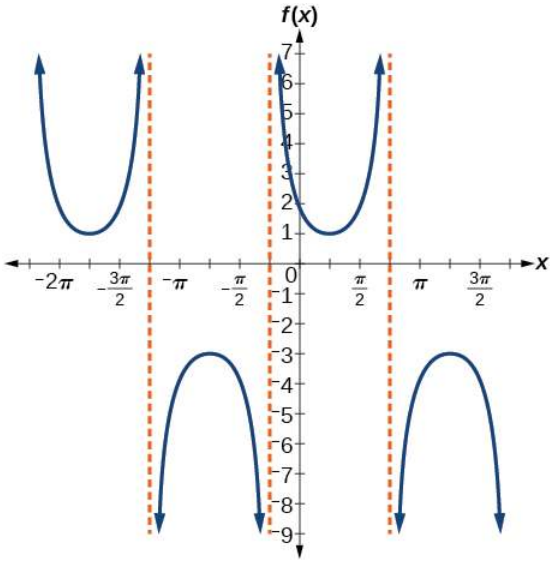


stretching factor: 7; period: $\frac{2\pi}{5}$ asymptotes: $x = \frac{\pi}{10}k$, where k is an odd integer

32) $f(x) = \frac{9}{10}\csc(\pi x)$

33) $f(x) = 2\csc\left(x + \frac{\pi}{4}\right) - 1$

Answer

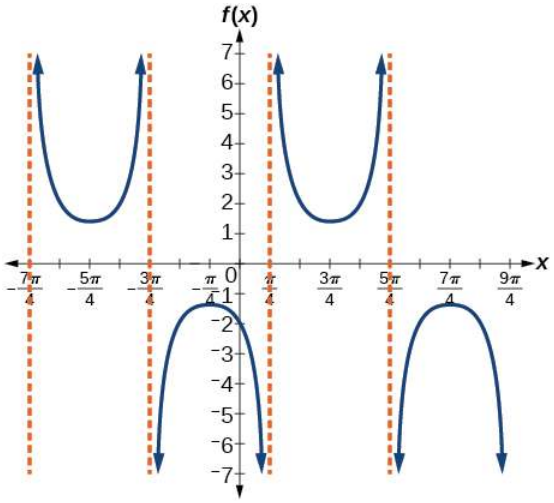


Stretching factor: 2; period: 2π ; asymptotes: $x = -\frac{\pi}{4} + \pi k$, where k is an integer

34) $f(x) = -\sec\left(x - \frac{\pi}{3}\right) - 2$

35) $f(x) = \frac{7}{5}\csc\left(x - \frac{\pi}{4}\right)$

Answer



Stretching factor: $\frac{7}{5}$; period: 2π; asymptotes: $x = \frac{\pi}{4} + \pi k$, where k is an integer

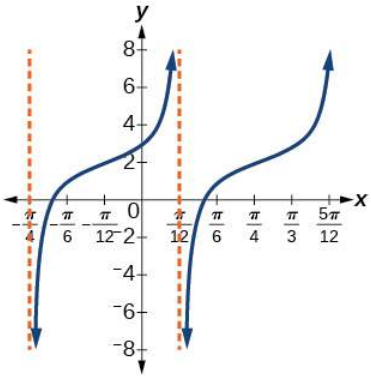
36) $f(x) = 5\left(\cot\left(x + \frac{\pi}{2}\right) - 3\right)$

For the exercises 37-38, find and graph two periods of the periodic function with the given stretching factor, $|A|$, period, and phase shift.

37) A tangent curve, $A = 1$, period of $\frac{\pi}{3}$; and phase shift $(h, k) = \left(\frac{\pi}{4}, 2\right)$

Answer

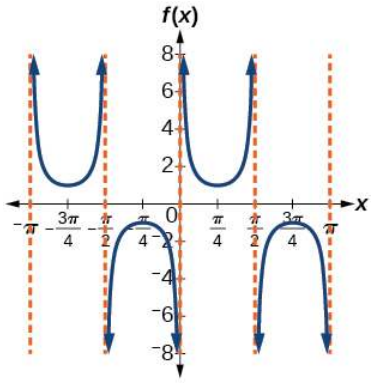
$y = \tan\left(3\left(x - \frac{\pi}{4}\right)\right) + 2$



38) A tangent curve, $A = -2$, period of $\frac{\pi}{4}$; and phase shift $(h, k) = \left(-\frac{\pi}{4}, -2\right)$

For the exercises 39-45, find an equation for the graph of each function.

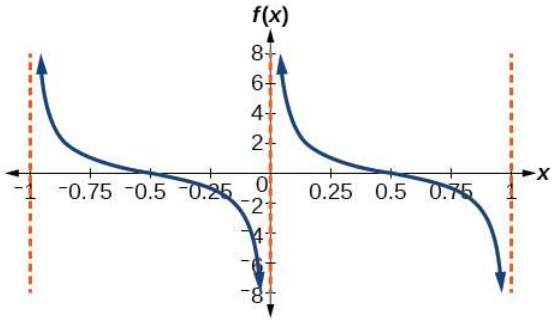
39)



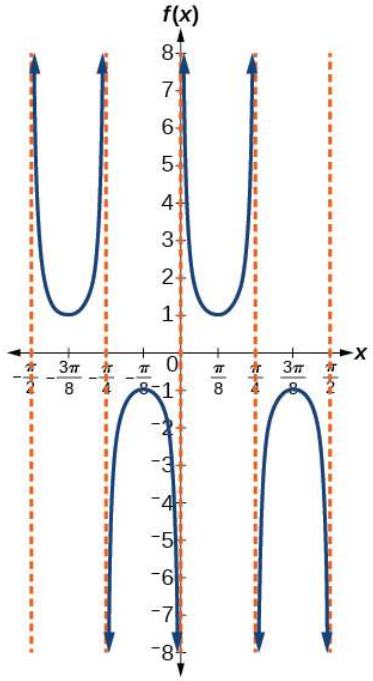
Answer

$f(x) = \csc(2x)$

40)



41)

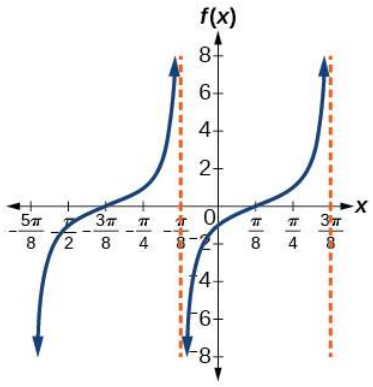


Answer

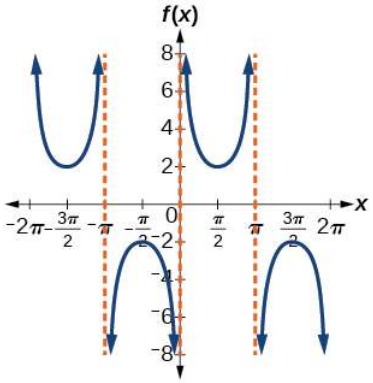
$f(x) = \csc(4x)$

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42)



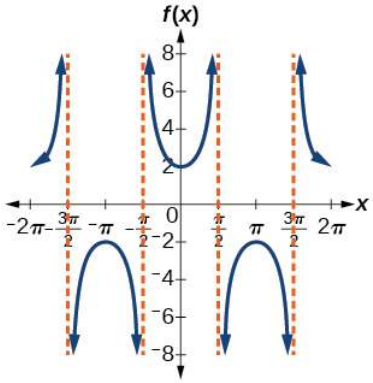
43)



Answer

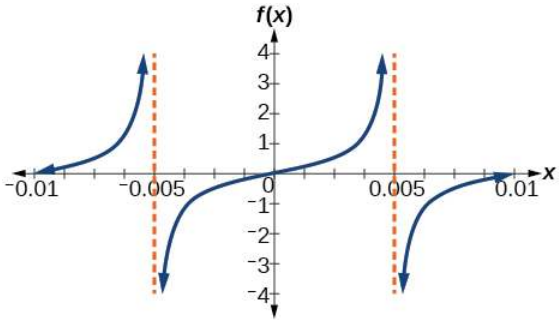
$f(x) = 2\csc x$

44)



45)

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Answer

$f(x) = \frac{1}{2}\tan(100\pi x)$

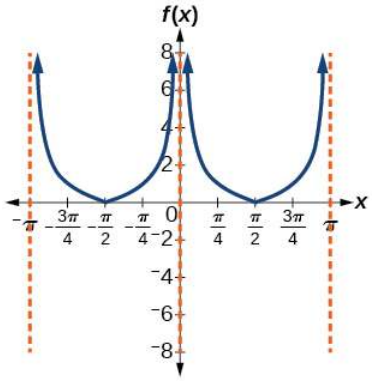
Technology

For the exercises 46-53, use a graphing calculator to graph two periods of the given function. Note: most graphing calculators do not have a cosecant button; therefore, you will need to input $\csc x$ as $\frac{1}{\sin x}$

46) $f(x) = |\csc(x)|$

47) $f(x) = |\cot(x)|$

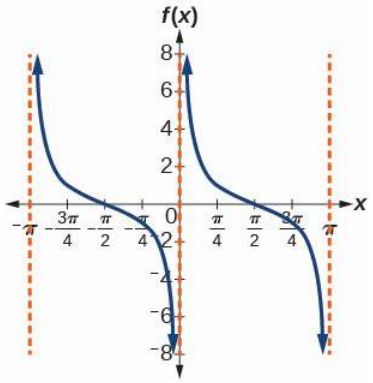
Answer



48) $f(x) = 2^{\csc(x)}$

49) $f(x) = \frac{\csc(x)}{\sec(x)}$

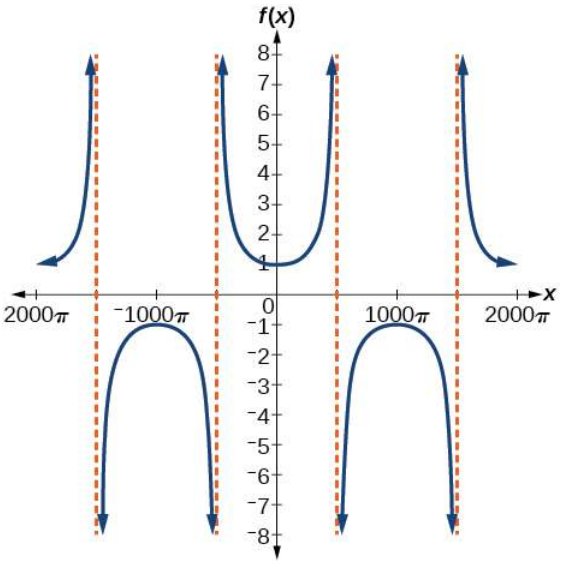
Answer



50) Graph $f(x) = 1 + \sec^2(x) - \tan^2(x)$. What is the function shown in the graph?

51) $f(x) = \sec(0.001x)$

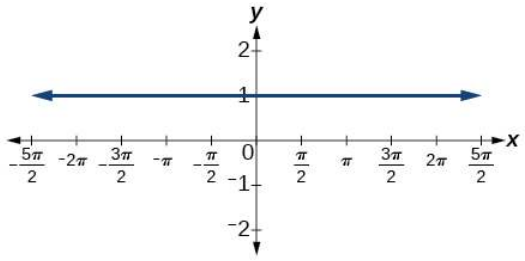
Answer



52) $f(x) = \cot(100\pi x)$

53) $f(x) = \sin^2 x + \cos^2 x$

Answer



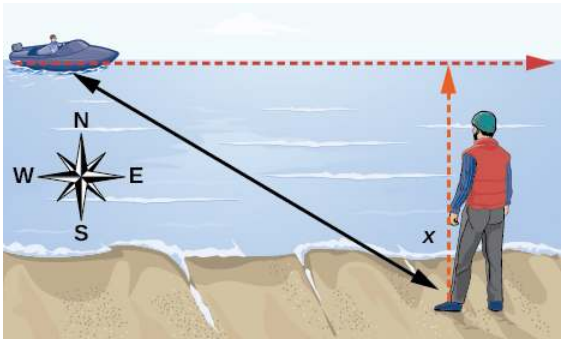
Real-World Applications

54) The function $f(x) = 20\tan\left(\frac{\pi}{10}x\right)$ marks the distance in the movement of a light beam from a police car across a wall for time x, in seconds, and distance $f(x)$, in feet.

a. Graph on the interval $[0, 5]$
b. Find and interpret the stretching factor, period, and asymptote.
c. Evaluate $f(10)$ and $f(2.5)$ and discuss the function's values at those inputs.

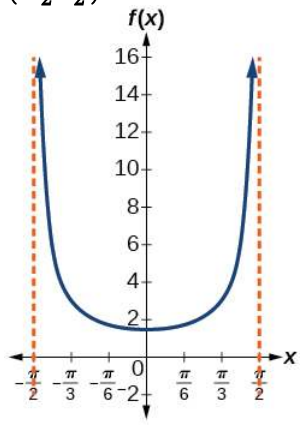
55) Standing on the shore of a lake, a fisherman sights a boat far in the distance to his left. Let x, measured in radians, be the angle formed by the line of sight to the ship and a line due north from his position. Assume due north is 0 and x is measured negative to the left and positive to the right. (See Figure below.) The boat travels from due west to due east and, ignoring the curvature of the Earth, the distance $d(x)$, in kilometers, from the fisherman to the boat is given by the function $d(x) = 1.5\sec(x)$.

a. What is a reasonable domain for $d(x)$?
b. Graph $d(x)$ on this domain.
c. Find and discuss the meaning of any vertical asymptotes on the graph of $d(x)$.
d. Calculate and interpret $d\left(-\frac{\pi}{3}\right)$. Round to the second decimal place.
e. Calculate and interpret $d\left(\frac{\pi}{6}\right)$. Round to the second decimal place.
f. What is the minimum distance between the fisherman and the boat? When does this occur?



Answer

a. $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

b. 

c. $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$ the distance grows without bound as $|x|$ approaches $\frac{\pi}{2}$—i.e., at right angles to the line representing due north, the boat would be so far away, the fisherman could not see it;

d. 3; when $x = -\frac{\pi}{3}$ the boat is 3 km away;

e. 1.73; when $x = \frac{\pi}{6}$ the boat is about 1.73 km away;

f. 1.5 km; when $x = 0$

56) A laser rangefinder is locked on a comet approaching Earth. The distance $g(x)$, in kilometers, of the comet after x days, for x in the interval 0 to 30 days, is given by $g(x) = 250,000 \csc\left(\frac{\pi}{30}x\right)$.

a. Graph $g(x)$ on the interval $[0, 35]$.
b. Evaluate $g(5)$ and interpret the information.
c. What is the minimum distance between the comet and Earth? When does this occur? To which constant in the equation does this correspond?
d. Find and discuss the meaning of any vertical asymptotes.

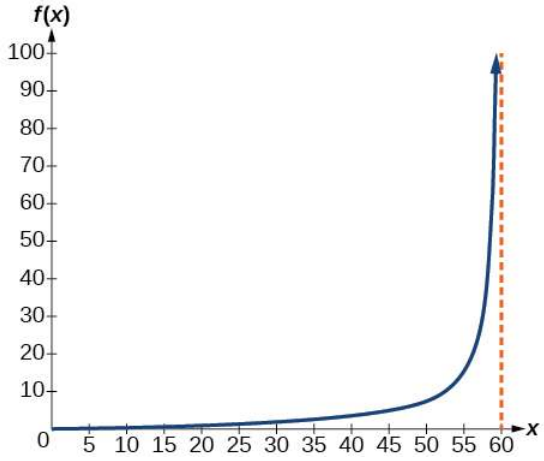
57) A video camera is focused on a rocket on a launching pad 2 miles from the camera. The angle of elevation from the ground to the rocket after x seconds is $\frac{\pi}{120}x$.

a. Write a function expressing the altitude $h(x)$, in miles, of the rocket above the ground after x seconds. Ignore the curvature of the Earth.
b. Graph $h(x)$ on the interval $(0, 60)$.
c. Evaluate and interpret the values $h(0)$ and $h(30)$.
d. What happens to the values of $h(x)$ as x approaches 60 seconds? Interpret the meaning of this in terms of the problem.

Answer

a. $h(x) = 2\tan\left(\frac{\pi}{120}x\right)$

b.



c. $h(0) = 0$ after 0 seconds, the rocket is 0 mi above the ground; $h(30) = 2$ after 30 seconds, the rockets is 2 mi high;

d. As x approaches 60 seconds, the values of $h(x)$ grow increasingly large. The distance to the rocket is growing so large that the camera can no longer track it.

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6.3: Inverse Trigonometric Functions

Skills to Develop

- Understand and use the inverse sine, cosine, and tangent functions.
- Find the exact value of expressions involving the inverse sine, cosine, and tangent functions.
- Use a calculator to evaluate inverse trigonometric functions.
- Find exact values of composite functions with inverse trigonometric functions.

For any right triangle, given one other angle and the length of one side, we can figure out what the other angles and sides are. But what if we are given only two sides of a right triangle? We need a procedure that leads us from a ratio of sides to an angle. This is where the notion of an inverse to a trigonometric function comes into play. In this section, we will explore the inverse trigonometric functions.

Understanding and Using the Inverse Sine, Cosine, and Tangent Functions

In order to use inverse trigonometric functions, we need to understand that an inverse trigonometric function "undoes" what the original trigonometric function "does," as is the case with any other function and its inverse. Bear in mind that the sine, cosine, and tangent functions are not one-to-one functions. The graph of each function would fail the horizontal line test. In fact, no periodic function can be one-to-one because each output in its range corresponds to at least one input in every period, and there are an infinite number of periods. As with other functions that are not one-to-one, we will need to restrict the **domain** of each function to yield a new function that is one-to-one. Once this is done, the domain of the inverse function is the range of the original function, and the range of the inverse function is the restricted domain of the original function.

In previous sections, we evaluated the trigonometric functions at various angles, but at times we need to know what angle would yield a specific sine, cosine, or tangent value. For this, we need inverse functions. Recall that, for a one-to-one function, if $f(a) = b$, then an inverse function would satisfy $f^{-1}(b) = a$.

For example, if $f(x) = \sin\ x$, then we would write $f^{-1}(x) = \sin^{-1}x$. Be aware that $\sin^{-1}x$ does **not** mean $\frac{1}{\sin\ x}$. The following examples illustrate the inverse trigonometric functions:

- Since $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, then $\frac{\pi}{6} = \sin^{-1}\left(\frac{1}{2}\right)$.
- Since $\cos(\pi) = -1$, then $\pi = \cos^{-1}(-1)$.
- Since $\tan\left(\frac{\pi}{4}\right) = 1$, then $\frac{\pi}{4} = \tan^{-1}(1)$.

To create the inverse functions, we choose a restricted domain for each function that includes the number 0. Figure **6.3.1** shows the graph of the sine function limited to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and the graph of the cosine function limited to $[0, \pi]$.

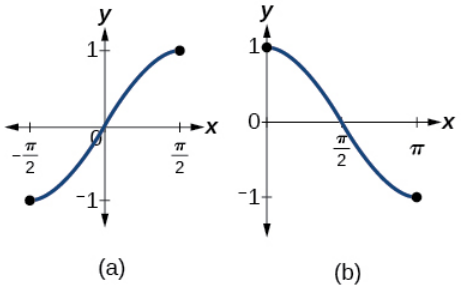


Figure* 6.3.1*: *(a) Sine function on a restricted domain of* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$*; (b) Cosine function on a restricted domain of* $[0, \pi]$

Figure **6.3.2** shows the graph of the tangent function limited to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

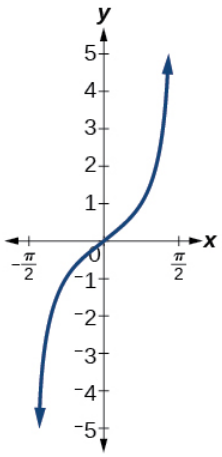


***Figure* 6.3.2:** *Tangent function on a restricted domain of* $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

These conventional choices for the restricted domain are somewhat arbitrary, but they have important, helpful characteristics. Each domain includes the origin and some positive values; the range for each function remains the same; and most importantly, each results in a one-to-one function that is invertible. The conventional choice for the restricted domain of the tangent function also has the useful property that it extends from one **vertical asymptote** to the next instead of being divided into two parts by an asymptote, so it is continuous.

On these restricted domains, we can define the inverse trigonometric functions.

- The inverse sine function $y = \sin^{-1} x$ means $x = \sin\ y$. The inverse sine function is sometimes called the **arcsine** function, and notated $\arcsin\ x$.

$$y = \sin^{-1} x \text{ has domain } [-1, 1] \text{ and range } \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right]$$

- The inverse cosine function $y = \cos^{-1} x$ means $x = \cos\ y$. The inverse cosine function is sometimes called the **arccosine** function, and notated $\arccos\ x$.

$$y = \cos^{-1} x \text{ has domain } [-1, 1] \text{ and range } [0, \pi].$$

- The inverse tangent function $y = \tan^{-1} x$ means $x = \tan\ y$. The inverse tangent function is sometimes called the **arctangent** function, and notated $\arctan\ x$.

$$y = \tan^{-1} x \text{ has domain } (-\infty, \infty) \text{ and range } \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right).$$

The graphs of the inverse functions are shown in Figures 6.3.3 - 6.3.5. Each graph of the inverse trigonometric function is a reflection of the graph of the original function about the line $y = x$. ***Extra credit***: the graph of $y = \tan\ x$ has two vertical asymptotes. Extend the graphs in Figure 6.3.5 to show these asymptotes, as well as the corresponding horizontal asymptotes for the graph of $y = \arctan\ x$. Show your more accurate graph to your professor for extra credit.

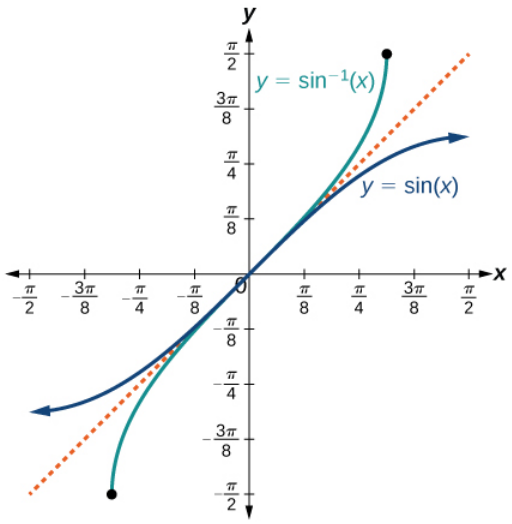


***Figure* 6.3.3:** *The sine function and inverse sine (or arcsine) function*

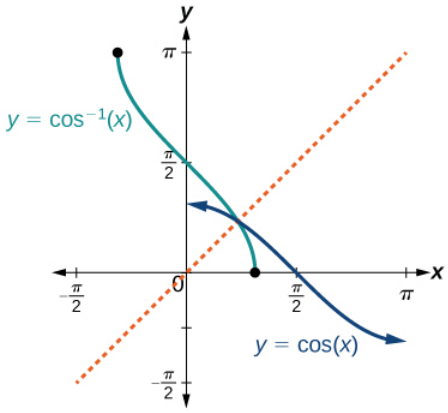


***Figure* 6.3.4:** *The cosine function and inverse cosine (or arccosine) function*

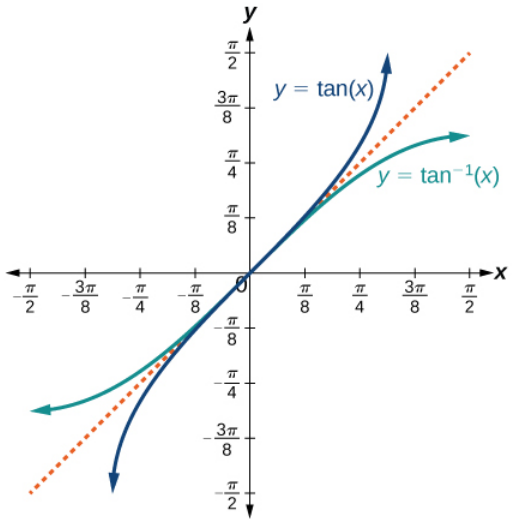


***Figure* 6.3.5:** *The tangent function and inverse tangent (or arctangent) function*

equivalent forms FOR INVERSE SINE, COSINE, AND TANGENT FUNCTIONS

For angles in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\sin y = x$ can be written equivalently as $\sin^{-1} x = y$.

For angles in the interval $[0, \pi]$, $\cos y = x$ can be written equivalently as $\cos^{-1} x = y$.

For angles in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\tan y = x$ can be written equivalently as $\tan^{-1} x = y$.

Example 6.3.1: reWrite a trigonometric equation using an Inverse trigonometric Function

Given $\sin\left(\frac{5\pi}{12}\right) \approx 0.96593$, rewrite in an equivalent form using the inverse sine.

Solution

Use the equivalent form for the inverse sine. If $\sin y = x$, then $\sin^{-1} x = y$.

In this problem, $x = 0.96593$, and the angle $y = \frac{5\pi}{12}$.

$\sin^{-1}(0.96593) \approx \frac{5\pi}{12}$

6.3.1

Given $\cos(0.5) \approx 0.8776$, rewrite in an equivalent form using the inverse cosine.

Answer

$\arccos(0.8776) \approx 0.5$

Finding the Exact Value of Expressions Involving the Inverse Sine, Cosine, and Tangent Functions

Now that we have defined inverse trigonometric functions, we will learn to evaluate them. For most values in their domains, we must evaluate inverse trigonometric functions by using a calculator, interpolating from a table, or using some other numerical technique. However, just as we did with the original trigonometric functions, we can give exact values for the inverse functions when we are using the special angles, specifically $\frac{\pi}{6}$ (or 30°), $\frac{\pi}{4}$ (or 45°), and $\frac{\pi}{3}$ (or 60°), and their reflections into other quadrants.

How To: Given a "special" input value, evaluate an inverse trigonometric function.

1. Find angle x for which the original trigonometric function has an output equal to the given input for the inverse trigonometric function.
2. If x is not in the defined range of the inverse, find another angle x' that is in the defined range and has the same sine, cosine, or tangent as x, depending on which corresponds to the given inverse function.

Example 6.3.2: Evaluating Inverse Trigonometric Functions for Special Input Values

Evaluate each of the following.

a. $\sin^{-1}\left(\frac{1}{2}\right)$

b. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

c. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

d. $\tan^{-1}(1)$

Solution

a. Evaluating $\sin^{-1}\left(\frac{1}{2}\right)$ is the same as determining the angle that would have a sine value of $\frac{1}{2}$. In other words, what angle x would satisfy $\sin(x) = \frac{1}{2}$? There are multiple values that would satisfy this relationship, such as $\frac{\pi}{6}$ and $\frac{5\pi}{6}$, but we know we need the angle in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so the answer will be $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$. Remember that the inverse is a function, so for each input, we will get exactly one output.

b. To evaluate $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$, we know that $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$ both have a sine value of $-\frac{\sqrt{2}}{2}$, but neither is in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. For that, we need the negative angle coterminal with $\frac{7\pi}{4}$: $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)=-\frac{\pi}{4}$.

c. To evaluate $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$, we are looking for an angle in the interval $[0,\pi]$ with a cosine value of $-\frac{\sqrt{3}}{2}$. The angle that satisfies this is $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\frac{5\pi}{6}$.

d. Evaluating $\tan^{-1}(1)$, we are looking for an angle in the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ with a tangent value of 1. The correct angle is $\tan^{-1}(1)=\frac{\pi}{4}$.

6.3.2

Evaluate each of the following.

a. $\sin^{-1}(-1)$
b. $\tan^{-1}(-1)$
c. $\cos^{-1}(-1)$
d. $\cos^{-1}\left(\frac{1}{2}\right)$

Answer

a. $-\frac{\pi}{2}$
b. $-\frac{\pi}{4}$
c. π
d. $\frac{\pi}{3}$

Using a Calculator to Evaluate Inverse Trigonometric Functions

To evaluate **inverse trigonometric functions** that do not involve the special angles discussed previously, we will need to use a calculator or other type of technology. Most scientific calculators and calculator-emulating applications have specific keys or buttons for the inverse sine, cosine, and tangent functions. These may be labeled, for example, SIN^{-1}, ARCSIN, or ASIN.

In Chapter 5, we worked with trigonometry on a right triangle to solve for the sides of a triangle given one side and an additional angle. Using the inverse trigonometric functions, we can solve for the angles of a right triangle given two sides, and we can use a calculator to find the values to several decimal places.

In the following examples, the answers will be interpreted as angles and we will use θ as the dependent variable. The value displayed on the calculator may be in degrees or radians, so be sure to set the mode appropriate to the application.

Example 6.3.3: Evaluating the Inverse Sine on a Calculator

Evaluate $\sin^{-1}(0.97)$ using a calculator.

Solution

Because the output of the inverse function is an angle, the calculator will give us a degree value if in degree mode and a radian value if in radian mode. Calculators also use the same domain restrictions on the angles as we are using.

In radian mode, $\sin^{-1}(0.97)\approx 1.3252$. In degree mode, $\sin^{-1}(0.97)\approx 75.93°$. Note that in calculus and beyond we will use radians in almost all cases.

6.3.3

Evaluate $\cos^{-1}(-0.4)$ using a calculator.

Answer

1.9823 or $113.58°$

How To: Given two sides of a right triangle like the one shown in Figure 6.3.6, find an angle.

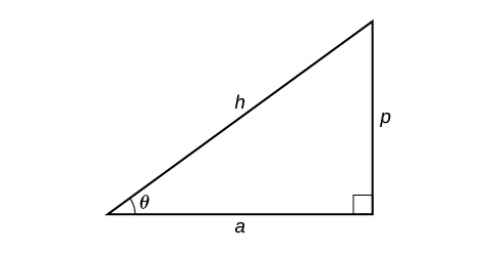


***Figure* 6.3.6**

1. If one given side is the hypotenuse of length h and the side of length a adjacent to the desired angle is given, use the equation $\theta = \cos^{-1}\left(\frac{a}{h}\right)$.
2. If one given side is the hypotenuse of length h and the side of length p opposite to the desired angle is given, use the equation $\theta = \sin^{-1}\left(\frac{p}{h}\right)$.
3. If the two legs (the sides adjacent to the right angle) are given, then use the equation $\theta = \tan^{-1}\left(\frac{p}{a}\right)$.

Example 6.3.4: Applying the Inverse Cosine to a Right Triangle

Solve the triangle in Figure **6.3.7** for the angle θ.

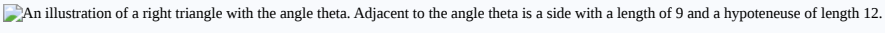


***Figure* 6.3.7**

Solution

Because we know the hypotenuse and the side adjacent to the angle, it makes sense for us to use the cosine function.

$$\begin{aligned} \cos\theta &= \frac{9}{12} \\ \theta &= \cos^{-1}\left(\frac{9}{12}\right) \quad \text{Apply definition of the inverse} \\ \theta &\approx 0.7227 \quad \text{or about } 41.41° \end{aligned}$$

Try It 6.3.4

Solve the triangle in Figure **6.3.8** for the angle θ.

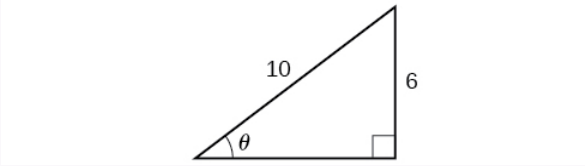


***Figure* 6.3.8**

Answer

$\sin^{-1}(0.6) \approx 0.6435 \text{ rad} \approx 36.87°$

Finding Exact Values of Composite Functions with Inverse Trigonometric Functions

There are times when we need to compose a trigonometric function with an inverse trigonometric function. In these cases, we can usually find exact values for the resulting expressions without resorting to a calculator. Even when the input to the composite

function is a variable or an expression, we can often find an expression for the output. To help sort out different cases, let $f(x)$ and $g(x)$ be two different trigonometric functions belonging to the set{ $\sin(x), \cos(x), \tan(x)$ } and let $f^{-1}(y)$ and $g^{-1}(y)$ be their inverses.

Evaluating Compositions of the Form $f(f^{-1}(y))$ and $f^{-1}(f(x))$

For any trigonometric function, $f(f^{-1}(y)) = y$ for all y in the proper domain for the given function. This follows from the definition of the inverse and from the fact that the range of f was defined to be identical to the domain of f^{-1}. However, we have to be a little more careful with expressions of the form $f^{-1}(f(x))$.

COMPOSITIONS OF A TRIGONOMETRIC FUNCTION AND ITS INVERSE

$$\begin{aligned}
\sin(\sin^{-1}x) &= x \quad \text{for } -1 \le x \le 1 \\
\cos(\cos^{-1}x) &= x \quad \text{for } -1 \le x \le 1 \\
\tan(\tan^{-1}x) &= x \quad \text{for } -\infty < x < \infty \\
\sin^{-1}(\sin x) &= x \quad \text{only for } -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\
\cos^{-1}(\cos x) &= x \quad \text{only for } 0 \le x \le \pi \\
\tan^{-1}(\tan x) &= x \quad \text{only for } -\frac{\pi}{2} < x < \frac{\pi}{2}
\end{aligned}$$

Q&A: Is it correct that $\sin^{-1}(\sin x) = x$?

No. This equation is correct if x belongs to the restricted domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, but sine is defined for all real input values, and for x outside the restricted interval, the equation is not correct because its inverse always returns a value in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The situation is similar for cosine and tangent and their inverses. For example, $\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right) = \frac{\pi}{4}$.

How To: Given an expression of the form $f^{-1}(f(\theta))$ where $f(\theta) = \sin\theta$, $\cos\theta$, or $\tan\theta$, evaluate.

1. If θ is in the restricted domain of f, then $f^{-1}(f(\theta)) = \theta$.
2. If not, then find an angle ϕ within the restricted domain of f such that $f(\phi) = f(\theta)$. Then $f^{-1}(f(\theta)) = \phi$.

Example 6.3.5: Using Inverse Trigonometric Functions

Evaluate the following:

a. $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$
b. $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$
c. $\cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right)$
d. $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right)$

Solution

a. $\frac{\pi}{3}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$.
b. $\frac{2\pi}{3}$ is not in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, but $\sin\left(\frac{2\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right)$, so $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = \frac{\pi}{3}$.
c. $\frac{2\pi}{3}$ is in $[0, \pi]$, so $\cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right) = \frac{2\pi}{3}$.
d. $-\frac{\pi}{3}$ is not in $[0, \pi]$, but $\cos\left(-\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right)$ because cosine is an even function. $\frac{\pi}{3}$ is in $[0, \pi]$, so $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$.

Try It 6.3.5

Evaluate a) $\tan^{-1}\left(\tan\left(\frac{\pi}{8}\right)\right)$ and b) $\tan^{-1}\left(\tan\left(\frac{11\pi}{9}\right)\right)$.

Answer

a) $\frac{\pi}{8}$; b) $\frac{2\pi}{9}$

Evaluating Compositions of the Form $f^{-1}(g(x))$

We will now explore how to evaluate a composition of a trigonometric function and the inverse of another trigonometric function. We will begin with compositions of the form $f^{-1}(g(x))$. For special values of x,we can exactly evaluate the inner function and then the outer, inverse function. However, we can find a more general approach by using the cofunction identities. Although these hold true for any angle θ, for a nice visual illustration, we will consider the two acute angles of a right triangle where one is θ, making the other $\frac{\pi}{2} - \theta$. Consider the sine and cosine of each angle of the right triangle in Figure **6.3.9**.

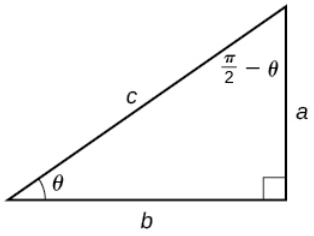


Figure* 6.3.9*: *Right triangle illustrating the cofunction relationships*

Because $\cos\theta = \frac{b}{c} = sin\left(\frac{\pi}{2} - \theta\right)$, we have $\sin^{-1}(\cos\theta) = \frac{\pi}{2} - \theta$ if $0 \le \theta \le \pi$. If θ is not in this domain, then we need to find another angle that has the same cosine as θ and does belong to the restricted domain; we then subtract this angle from $\frac{\pi}{2}$. Similarly, $\sin\theta = \frac{a}{c} = \cos\left(\frac{\pi}{2} - \theta\right)$, so $\cos^{-1}(\sin\theta) = \frac{\pi}{2} - \theta$ if $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. These are just the function-cofunction relationships presented in another way.

How To: Given functions of the form $\sin^{-1}(\cos x)$ and $\cos^{-1}(\sin x)$, evaluate them.

1. If x is in $[0, \pi]$, then $\sin^{-1}(\cos x) = \frac{\pi}{2} - x$.
2. If x is not in $[0, \pi]$, then find another angle x' in $[0, \pi]$ such that $\cos x' = \cos x$.

$$\sin^{-1}(\cos x) = \frac{\pi}{2} - x' \tag{6.3.1}$$

3. If x is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then $\cos^{-1}(\sin x) = \frac{\pi}{2} - x$.
4. If x is not in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then find another angle x' in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin x' = \sin x$.

$$\cos^{-1}(\sin x) = \frac{\pi}{2} - x' \tag{6.3.2}$$

Example 6.3.6: Evaluating the Composition of an Inverse Sine with a Cosine

Evaluate $\sin^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right)$

a. by direct evaluation.
b. by the method described previously.

Solution

a. Here, we can directly evaluate the inside of the composition.

$$\begin{aligned}\cos\left(\frac{13\pi}{6}\right) &= \cos\left(\frac{\pi}{6} + 2\pi\right) \\ &= \cos\left(\frac{\pi}{6}\right) \\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

Now, we can evaluate the inverse function as we did earlier.

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3} \tag{6.3.3}$$

b. We have $x = \frac{13\pi}{6}$, $x' = \frac{\pi}{6}$, and

$$\sin^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right) = \frac{\pi}{2} - \frac{\pi}{6}$$
$$= \frac{\pi}{3}$$

6.3.6

Evaluate $\cos^{-1}\left(\sin\left(-\frac{11\pi}{4}\right)\right)$.

Answer

$\frac{3\pi}{4}$

Evaluating Compositions of the Form $f(g^{-1}(x))$

To evaluate compositions of the form $f(g^{-1}(x))$, where f and g are any two of the functions sine, cosine, or tangent and x is any input in the domain of g^{-1}, we have exact formulas, such as $\sin(\cos^{-1}x) = \sqrt{1-x^2}$. When we need to use them, we can derive these formulas by using the trigonometric relations between the angles and sides of a right triangle, together with the use of Pythagoras's relation between the lengths of the sides. We can use the Pythagorean identity, $\sin^2 x + \cos^2 x = 1$, to solve for one when given the other. We can also use the inverse trigonometric functions to find compositions involving algebraic expressions.

Example 6.3.7: Evaluating the Composition of a Sine with an Inverse Cosine

Find an exact value for $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right)$.

Solution

Beginning with the inside, we can say there is some angle such that $\theta = \cos^{-1}\left(\frac{4}{5}\right)$, which means $\cos\theta = \frac{4}{5}$, and we are looking for $\sin\theta$. We can use the Pythagorean identity to do this.

$$\sin^2\theta + \cos^2\theta = 1 \quad \text{Use our known value for cosine}$$
$$\sin^2\theta + \left(\frac{4}{5}\right)^2 = 1 \quad \text{Solve for sine}$$
$$\sin^2\theta = 1 - \frac{16}{25}$$
$$\sin\theta = \pm\frac{9}{25}$$
$$= \pm\frac{3}{5}$$

Since $\theta = \cos^{-1}\left(\frac{4}{5}\right)$ is in Quadrant I, $\sin\theta$ must be positive, so the solution is $\frac{3}{5}$. See Figure 6.3.10.

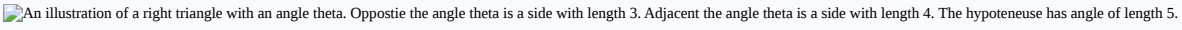


***Figure* 6.3.10:** *Right triangle illustrating that if* $\cos\theta = \frac{4}{5}$, *then* $\sin\theta = \frac{3}{5}$

More generally, we know that the inverse cosine always gives an angle on the interval $[0, \pi]$, so we know that the sine of that angle must be positive. Thus, given x in $[-1, 1]$, $\sin(cos^{-1}(x)) = \sqrt{1-x^2}$.

6.3.7

Evaluate $\cos\left(\tan^{-1}\left(\frac{5}{12}\right)\right)$.

Answer

$\frac{12}{13}$

Example 6.3.8: Evaluating the Composition of a Sine with an Inverse Tangent

Find an exact value for $\sin\left(\tan^{-1}\left(\frac{7}{4}\right)\right)$.

Solution

Here we will use a different technique. From the inside, we know there is an angle such that $\tan\theta = \frac{7}{4}$. We can envision this as the opposite and adjacent sides on a right triangle, as shown in Figure 6.3.11.

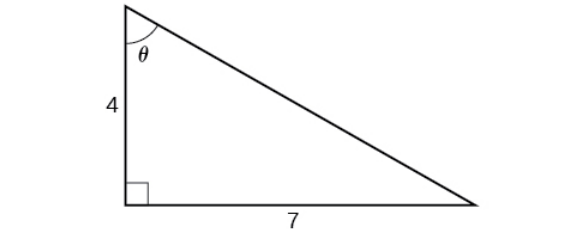


***Figure* 6.3.11:** *A right triangle with two sides known*

Using the Pythagorean Theorem, we can find the hypotenuse of this triangle.

$$4^2 + 7^2 = (hypotenuse)^2$$
$$hypotenuse = \sqrt{65}$$

Now, we can evaluate the sine of the angle as the opposite side divided by the hypotenuse.

$$\sin\theta = \frac{7}{\sqrt{65}}$$

This gives us our desired composition.

$$\begin{aligned}\sin\left(\tan^{-1}\left(\frac{7}{4}\right)\right) &= \sin\theta \\ &= \frac{7}{\sqrt{65}} \\ &= \frac{7\sqrt{65}}{65}\end{aligned}$$

6.3.8

Evaluate $\cos\left(\sin^{-1}\left(\frac{7}{9}\right)\right)$.

Answer

$\frac{4\sqrt{2}}{9}$

Example 6.3.9: Finding the Cosine of the Inverse Sine of an Algebraic Expression

Find a simplified expression for $\cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right)$ for $-3 \le x \le 3$.

Solution

We know there is an angle θ such that $\sin\theta = \frac{x}{3}$.

$$\sin^2\theta + \cos^2\theta = 1 \quad \text{Use the Pythagorean Theorem}$$
$$\left(\frac{x}{3}\right)^2 + \cos^2\theta = 1 \quad \text{Solve for cosine}$$
$$\cos^2\theta = 1 - \frac{x^2}{9}$$
$$\cos\theta = \pm\sqrt{\frac{9-x^2}{9}}$$
$$= \pm\sqrt{\frac{9-x^2}{3}}$$

Because we know that the inverse sine must give an angle on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we can deduce that the cosine of that angle must be positive.

$$\cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right)=\sqrt{\frac{9-x^2}{3}}$$

6.3.9

Find a simplified expression for $\sin(\tan^{-1}(4x))$ for $-\frac{1}{4}\le x\le\frac{1}{4}$.

Answer

$$\frac{4x}{\sqrt{16x^2+1}}$$

Media

Access this online resource for additional instruction and practice with inverse trigonometric functions.

- Evaluate Expressions Involving Inverse Trigonometric Functions

Visit this website for additional practice questions from Learningpod.

Key Concepts

- An inverse function is one that "undoes" another function. The domain of an inverse function is the range of the original function and the range of an inverse function is the domain of the original function.
- Because the trigonometric functions are not one-to-one on their natural domains, inverse trigonometric functions are defined for restricted domains.
- For any trigonometric function $f(x)$, if $x=f^{-1}(y)$, then $f(x)=y$. However, $f(x)=y$ only implies $x=f^{-1}(y)$ if x is in the restricted domain of f. See Example **6.3.1**.
- Special angles are the outputs of inverse trigonometric functions for special input values; for example, $\frac{\pi}{4}=\tan^{-1}(1)$ and $\frac{\pi}{6}=\sin^{-1}(\frac{1}{2})$.See Example **6.3.2**.
- A calculator will return an angle within the restricted domain of the original trigonometric function. See Example **6.3.3**.
- Inverse functions allow us to find an angle when given two sides of a right triangle. See Example **6.3.4**.
- If the inside function is a trigonometric function, and $0\le x\le\pi$, then $\sin^{-1}(\cos x)=\frac{\pi}{2}-x$. If $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$, then $\cos^{-1}(\sin x)=\frac{\pi}{2}-x$. If x is not in the restricted domain, then an appropriate x' must be chosen. See Example **6.3.6**.
- In function composition, if the inside function is an inverse trigonometric function, then there are exact expressions; for example,$\sin(\cos^{-1}(x))=\sqrt{1-x^2}$. See Example **6.3.7**.
- When evaluating the composition of a trigonometric function with an inverse trigonometric function, draw a reference triangle to assist in determining the ratio of sides that represents the output of the trigonometric function. See Example **6.3.8**.
- When evaluating the composition of a trigonometric function with an inverse trigonometric function, you may use trig identities to assist in determining the ratio of sides. See Example **6.3.9**.

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6.3E: Exercises

6.3: Inverse Trigonometric Functions

In this section, we will explore the inverse trigonometric functions. Inverse trigonometric functions "undoes" what the original trigonometric function "does," as is the case with any other function and its inverse. In other words, the domain of the inverse function is the range of the original function, and vice versa.

Verbal

1) Why do the functions $f(x) = \sin^{-1} x$ and $g(x) = \cos^{-1} x$ have different ranges?

Answer

The function $y = \sin x$ is one-to-one on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ thus, this interval is the range of the inverse function of $y = \sin x$, $f(x) = \sin^{-1} x$ The function $y = \cos x$ is one-to-one on $[0, \pi]$ thus, this interval is the range of the inverse function of $y = \cos x$, $f(x) = \cos^{-1} x$

2) Since the functions $y = \cos x$ and $y = \cos^{-1} x$ are inverse functions, why is $\cos^{-1}\left(\cos\left(-\frac{\pi}{6}\right)\right)$ not equal to $-\frac{\pi}{6}$?

3) Explain the meaning of $\frac{\pi}{6} = \arcsin(0.5)$.

Answer

$\frac{\pi}{6}$ is the radian measure of an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is 0.5.

4) Most calculators do not have a key to evaluate $\sec^{-1}(2)$. Explain how this can be done using the cosine function or the inverse cosine function.

5) Why must the domain of the sine function, $\sin x$, be restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for the inverse sine function to exist?

Answer

In order for any function to have an inverse, the function must be one-to-one and must pass the horizontal line test. The regular sine function is not one-to-one unless its domain is restricted in some way. Mathematicians have agreed to restrict the sine function to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so that it is one-to-one and possesses an inverse.

6) Discuss why this statement is incorrect: $\arccos(\cos x) = x$ for all x.

7) Determine whether the following statement is true or false and explain your answer: $\arccos(-x) = \pi - \arccos x$

Answer

True . The angle, θ_1 that equals $\arccos(-x)$, $x > 0$, will be a second quadrant angle with reference angle, θ_2, where θ_2 equals $\arccos x$, $x > 0$. Since θ_2 is the reference angle for θ_1, $\theta_2 = \pi - \theta_1$ and $\arccos(-x) = \pi - \arccos x-$

Algebraic

For the exercises 8-16, evaluate the expressions.

8) $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$

9) $\sin^{-1}\left(-\frac{1}{2}\right)$

Answer

$-\frac{\pi}{6}$

10) $\cos^{-1}\left(-\frac{1}{2}\right)$

11) $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

Answer

$\frac{3\pi}{4}$

12) $\tan^{-1}(1)$

13) $\tan^{-1}(-\sqrt{3})$

Answer

$-\frac{\pi}{3}$

14) $\tan^{-1}(-1)$

15) $\tan^{-1}(\sqrt{3})$

Answer

$\frac{\pi}{3}$

16) $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

For the exercises 17-21, use a calculator to evaluate each expression. Express answers to the nearest hundredth.

17) $\cos^{-1}(-0.4)$

Answer

1.98

18) $\arcsin(0.23)$

19) $\arccos\left(\frac{3}{5}\right)$

Answer

0.93

20) $\cos^{-1}(-0.8)$

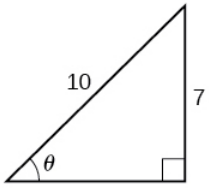
21) $\tan^{-1}(6)$

Answer

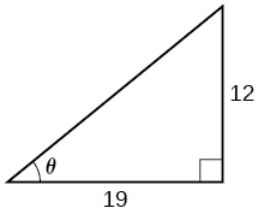
1.41

For the exercises 22-23, find the angle θ in the given right triangle. Round answers to the nearest hundredth.

22)



23)



Answer

0.56 radians

For the exercises 24-36, find the exact value, if possible, without a calculator. If it is not possible, explain why.

24) $\sin^{-1}(\cos(\pi))$

25) $\tan^{-1}(\sin(\pi))$

Answer

0

26) $\cos^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$

27) $\tan^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$

Answer

0.71

28) $\sin^{-1}\left(\cos\left(\frac{-\pi}{2}\right)\right)$

29) $\tan^{-1}\left(\sin\left(\frac{4\pi}{3}\right)\right)$

Answer

−0.71

30) $\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right)$

31) $\tan^{-1}\left(\sin\left(\frac{-5\pi}{2}\right)\right)$

Answer

$-\frac{\pi}{4}$

32) $\cos\left(\sin^{-1}\left(\frac{4}{5}\right)\right)$

33) $\sin\left(\cos^{-1}\left(\frac{3}{5}\right)\right)$

Answer

0.8

34) $\sin\left(\tan^{-1}\left(\frac{4}{3}\right)\right)$

35) $\cos\left(\tan^{-1}\left(\frac{12}{5}\right)\right)$

Answer

$\frac{5}{13}$

36) $\cos\left(\sin^{-1}\left(\frac{1}{2}\right)\right)$

For the exercises 37-41, find the exact value of the expression in terms of x with the help of a reference triangle.

37) $\tan\left(\sin^{-1}(x-1)\right)$

Answer

$\frac{x-1}{\sqrt{-x^2+2x}}$

38) $\sin\left(\sin^{-1}(1-x)\right)$

39) $\cos\left(\sin^{-1}\left(\frac{1}{x}\right)\right)$

Answer

$\frac{\sqrt{x^2-1}}{x}$

40) $\cos\left(\tan^{-1}(3x-1)\right)$

41) $\tan\left(\sin^{-1}\left(x+\frac{1}{2}\right)\right)$

Answer

$\frac{x+0.5}{\sqrt{-x^2-x+\frac{3}{4}}}$

Extensions

For the exercise 42, evaluate the expression without using a calculator. Give the exact value.

42) $\frac{\sin^{-1}\left(\frac{1}{2}\right)-\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)+\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)-\cos^{-1}(1)}{\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)-\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)+\cos^{-1}\left(\frac{1}{2}\right)-\sin^{-1}(0)}$

For the exercises 43-47, find the function if $\sin t = \frac{x}{x+1}$

43) $\cos t$

Answer

$\frac{\sqrt{2x+1}}{x+1}$

44) $\sec t$

45) $\cot t$

Answer

$\frac{\sqrt{2x+1}}{x}$

46) $\cos\left(\sin^{-1}\left(\frac{x}{x+1}\right)\right)$

47) $\tan^{-1}\left(\frac{x}{\sqrt{2x+1}}\right)$

Answer

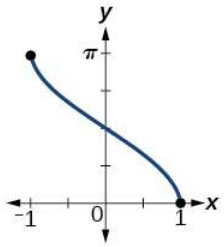
t

Graphical

48) Graph $y = \sin^{-1} x$ and state the domain and range of the function.

49) Graph $y = \arccos x$ and state the domain and range of the function.

Answer



domain $[-1, 1]$ range $[0, \pi]$

50) Graph one cycle of $y = \tan^{-1} x$ and state the domain and range of the function.

51) For what value of x does $\sin x = \sin^{-1} x$? Use a graphing calculator to approximate the answer.

Answer

approximately $x = 0.00$

52) For what value of x does $\cos x = \cos^{-1} x$? Use a graphing calculator to approximate the answer.

Real-World Applications

53) Suppose a 13-foot ladder is leaning against a building, reaching to the bottom of a second-floor window 12 feet above the ground. What angle, in radians, does the ladder make with the building?

Answer

0.395 radians

54) Suppose you drive 0.6 miles on a road so that the vertical distance changes from 0 to 150 feet. What is the angle of elevation of the road?

55) An isosceles triangle has two congruent sides of length 9 inches. The remaining side has a length of 8 inches. Find the angle that a side of 9 inches makes with the 8-inch side.

Answer

1.11 radians

56) Without using a calculator, approximate the value of $\arctan(10,000)$. Explain why your answer is reasonable.

57) A truss for the roof of a house is constructed from two identical right triangles. Each has a base of 12 feet and height of 4 feet. Find the measure of the acute angle adjacent to the 4-foot side.

Answer

1.25 radians

58) The line $y = \frac{3}{5}x$ passes through the origin in the x, y-plane. What is the measure of the angle that the line makes with the positive x-axis?

59) The line $y = \frac{-3}{7}x$ passes through the origin in the x, y-plane. What is the measure of the angle that the line makes with the negative x-axis?

Answer

0.405 radians

60) What percentage grade should a road have if the angle of elevation of the road is 4 degrees? (The percentage grade is defined as the change in the altitude of the road over a 100-foot horizontal distance. For example a 5% grade means that the road rises 5 feet for every 100 feet of horizontal distance.)

61) A 20-foot ladder leans up against the side of a building so that the foot of the ladder is 10 feet from the base of the building. If specifications call for the ladder's angle of elevation to be between 35 and 45 degrees, does the placement of this ladder satisfy safety specifications?

Answer

No. The angle the ladder makes with the horizontal is 60 degrees.

62) Suppose a 15-foot ladder leans against the side of a house so that the angle of elevation of the ladder is 42 degrees. How far is the foot of the ladder from the side of the house?

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CHAPTER OVERVIEW

7: Trigonometric Identities and Equations

In this chapter, we discuss how to manipulate trigonometric equations algebraically by applying various formulas and trigonometric identities. We will also investigate some of the ways that trigonometric equations are used to model real-life phenomena.

7.0: Prelude to Trigonometric Identities and Equations

7.0E: Exercises

7.1: Simplifying Trigonometric Expressions with Identities

7.1E: Exercises

7.2: Sum and Difference Identities

7.2E: Exercises

7.3: Double-Angle, Half-Angle, and Reduction Formulas

7.3E: Exercises

7.5: Solving Trigonometric Equations

7.5E: Exercises

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7.0: Prelude to Trigonometric Identities and Equations

Math is everywhere, even in places we might not immediately recognize. For example, mathematical relationships describe the transmission of images, light, and sound. The sinusoidal graph in Figure **7.0.1** models music playing on a phone, radio, or computer. Such graphs are described using trigonometric equations and functions. In this chapter, we discuss how to manipulate trigonometric equations algebraically by applying various formulas and trigonometric identities. We will also investigate some of the ways that trigonometric equations are used to model real-life phenomena.

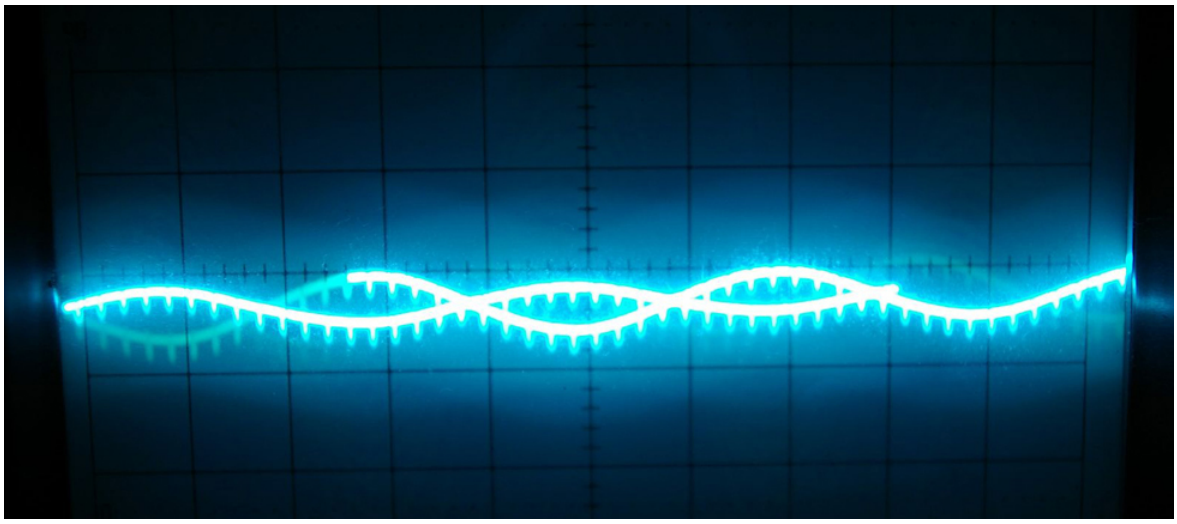


Figure* 7.0.1*: *A sine wave models disturbance. (credit: modification of work by Mikael Altemark, Flickr).*

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7.0E: Exercises

No Exercises for Section 7.0!

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7.1: Simplifying Trigonometric Expressions with Identities

Skills to Develop

- Verify the fundamental trigonometric identities.
- Simplify trigonometric expressions using algebra and the identities.

In espionage movies, we see international spies with multiple passports, each claiming a different identity. However, we know that each of those passports represents the same person. The trigonometric identities act in a similar manner to multiple passports—there are many ways to represent the same trigonometric expression. Just as a spy will choose an Italian passport when traveling to Italy, we choose the identity that applies to the given scenario when solving a trigonometric equation.



Figure* 7.1.1*: *International passports and travel documents*

In this section, we will begin an examination of the fundamental trigonometric identities, including how we can verify them and how we can use them to simplify trigonometric expressions.

Verifying the Fundamental Trigonometric Identities

Identities enable us to simplify complicated expressions. They are the basic tools of trigonometry used in solving trigonometric equations, just as factoring, finding common denominators, and using special formulas are the basic tools of solving algebraic equations. In fact, we use algebraic techniques constantly to simplify trigonometric expressions. Basic properties and formulas of algebra, such as the difference of squares formula and the perfect squares formula, will simplify the work involved with trigonometric expressions and equations. We already know that all of the trigonometric functions are related because they all are defined in terms of the unit circle. Consequently, any trigonometric identity can be written in many ways.

To verify the trigonometric identities, we usually start with the more complicated side of the equation and essentially rewrite the expression until it has been transformed into the same expression as the other side of the equation. Sometimes we have to factor expressions, expand expressions, find common denominators, or use other algebraic strategies to obtain the desired result. In this first section, we will work with the fundamental identities: the **Pythagorean identities**, the even-odd identities, the reciprocal identities, and the quotient identities.

We will begin with the **Pythagorean identities** (Table 7.1.1), which are equations involving trigonometric functions based on the properties of a right triangle. We have already seen and used the first of these identifies, but now we will also use additional identities.

Table* 7.1.1*: *Pythagorean Identities*

$\sin^2\theta+\cos^2\theta=1$	$1+\cot^2\theta=\csc^2\theta$	$1+\tan^2\theta=\sec^2\theta$

The second and third identities can be obtained by manipulating the first. The identity $1+\cot^2\theta=\csc^2\theta$ is found by rewriting the left side of the equation in terms of sine and cosine.

Prove: $1+\cot^2\theta=\csc^2\theta$.

$$
\begin{aligned}
1+\cot^2\theta &= \left(1+\frac{\cos^2\theta}{\sin^2\theta}\right) && \text{Rewrite the left side}\\
&= \left(\frac{\sin^2\theta}{\sin^2\theta}\right)+\left(\frac{\cos^2\theta}{\sin^2\theta}\right) && \text{Write both terms with a common denominator}\\
&= \frac{\sin^2\theta+\cos^2\theta}{\sin^2\theta}\\
&= \frac{1}{\sin^2\theta}\\
&= \csc^2\theta
\end{aligned}
$$

Similarly, $1+\tan^2\theta=\sec^2\theta$ can be obtained by rewriting the left side of this identity in terms of sine and cosine:

$$
\begin{aligned}
1+\tan^2\theta &= 1+\left(\frac{\sin^2\theta}{\cos^2\theta}\right) && \text{Rewrite left side}\\
&= \left(\frac{\cos^2\theta}{\cos^2\theta}\right)+\left(\frac{\sin^2\theta}{\cos^2\theta}\right) && \text{Write both terms with a common denominator}\\
&= \frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}\\
&= \frac{1}{\cos^2\theta}\\
&= \sec^2\theta.
\end{aligned}
$$

In Section 5.3, we determined which trigonometric functions are odd and which are even. Table **7.1.2** shows these relationships as **even-odd identities**.

***Table* 7.1.2:** *Even-Odd Identities*

$\tan(-\theta)=-\tan\theta$	$\sin(-\theta)=-\sin\theta$	$\cos(-\theta)=\cos\theta$
$\cot(-\theta)=-\cot\theta$	$\csc(-\theta)=-\csc\theta$	$\sec(-\theta)=\sec\theta$

Recall that an **odd function** is one in which $f(-x)=-f(x)$ for all x in the domain of f. The sine function is an odd function because $\sin(-\theta)=-\sin\theta$. The graph of an odd function is symmetric about the origin. For example, consider corresponding inputs of $\frac{\pi}{2}$ and $-\frac{\pi}{2}$. The output of $\sin\left(\frac{\pi}{2}\right)$ is the opposite of the output of $\sin\left(-\frac{\pi}{2}\right)$. Thus,

$$
\begin{aligned}
\sin\left(\frac{\pi}{2}\right) &= 1\\
\sin\left(-\frac{\pi}{2}\right) &= -\sin\left(\frac{\pi}{2}\right)\\
&= -1
\end{aligned}
$$

This is shown in Figure **7.1.2**.

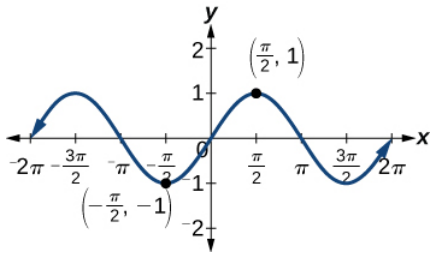


***Figure* 7.1.2:** *Graph of* $y=\sin\theta$

Recall that an **even function** is one in which

$$f(-x) = f(x) \text{ for all } x \text{ in the domain of } f$$

The graph of an even function is symmetric across the y-axis. The cosine function is an even function because $\cos(-\theta) = \cos\theta$. For example, consider corresponding inputs $\frac{\pi}{4}$ and $-\frac{\pi}{4}$. The output of $\cos\left(\frac{\pi}{4}\right)$ is the same as the output of $\cos\left(-\frac{\pi}{4}\right)$. Thus,

$$\cos\left(-\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right)$$
$$\approx 0.707$$

See Figure **7.1.3**.

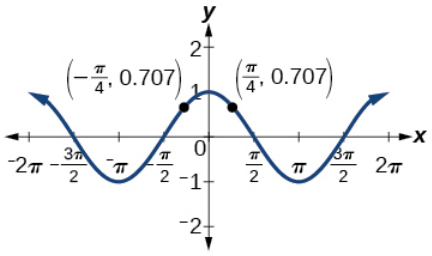


***Figure* 7.1.3:** *Graph of $y = \cos\theta$*

The other even-odd identities follow from the even and odd nature of the sine and cosine functions. For example, consider the tangent identity, $\tan(-\theta) = -\tan\theta$. We can interpret the tangent of a negative angle as

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta.$$

Tangent is therefore an odd function.

The cotangent identity, $\cot(-\theta) = -\cot\theta$, also follows from the sine and cosine identities. We can interpret the cotangent of a negative angle as

$$\cot(-\theta) = \frac{\cos(-\theta)}{\sin(-\theta)} = \frac{\cos\theta}{-\sin\theta} = -\cot\theta.$$

Cotangent is therefore an odd function.

The cosecant function is the reciprocal of the sine function, which means that the cosecant of a negative angle will be interpreted as

$$\csc(-\theta) = \frac{1}{\sin(-\theta)} = \frac{1}{-\sin\theta} = -\csc\theta.$$

The cosecant function is therefore odd.

Finally, the secant function is the reciprocal of the cosine function, and the secant of a negative angle is interpreted as

$$\sec(-\theta) = \frac{1}{\cos(-\theta)} = \frac{1}{\cos\theta} = \sec\theta.$$

The secant function is therefore even.

To sum up, only two of the trigonometric functions, cosine and secant, are even. The other four functions are odd, verifying the even-odd identities.

The next set of fundamental identities is the set of **reciprocal identities**, which, as their name implies, relate trigonometric functions that are reciprocals of each other (Table **7.1.3**). We first encountered these identities in Section 5.3.

***Table* 7.1.3:** *Reciprocal Identities*

$\sin\theta = \frac{1}{\csc\theta}$	$\csc\theta = \frac{1}{\sin\theta}$
$\cos\theta = \frac{1}{\sec\theta}$	$\sec\theta = \frac{1}{\cos\theta}$
$\tan\theta = \frac{1}{\cot\theta}$	$\cot\theta = \frac{1}{\tan\theta}$

The final set of identities is the set of **quotient identities,** which define relationships among certain trigonometric functions and can be very helpful in verifying other identities (Table 7.1.4).

***Table** 7.1.4: Quotient Identities*

$\tan\theta = \frac{\sin\theta}{\cos\theta}$	$\cot\theta = \frac{\cos\theta}{\sin\theta}$

The reciprocal and quotient identities are derived from the definitions of the basic trigonometric functions.

SUMMARIZING TRIGONOMETRIC IDENTITIES

The Pythagorean identities are based on the properties of a right triangle.

$$\cos^2\theta + \sin^2\theta = 1$$

$$1 + \cot^2\theta = \csc^2\theta$$

$$1 + \tan^2\theta = \sec^2\theta$$

The even-odd identities relate the value of a trigonometric function at a given angle to the value of the function at the opposite angle.

$$\tan(-\theta) = -\tan\theta$$

$$\cot(-\theta) = -\cot\theta$$

$$\sin(-\theta) = -\sin\theta$$

$$\csc(-\theta) = -\csc\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\sec(-\theta) = \sec\theta$$

The reciprocal identities define reciprocals of the trigonometric functions.

$$\sin\theta = \frac{1}{\csc\theta}$$

$$\cos\theta = \frac{1}{\sec\theta}$$

$$\tan\theta = \frac{1}{\cot\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}$$

The quotient identities define the relationship among the trigonometric functions.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

Verifying other Trigonometric Identities

How To: Given a trigonometric identity, verify that it is true

1. Work on one side of the equation. It is usually better to start with the more complex side, as it is easier to simplify than to build.
2. Look for opportunities to factor expressions, square a binomial, or add fractions.
3. Noting which functions are in the final expression, look for opportunities to use the identities and make the proper substitutions.
4. If these steps do not yield the desired result, try converting all terms to sines and cosines.

Example 7.1.1: Verifying a Trigonometric Identity

Verify $\tan\theta\cos\theta = \sin\theta$.

Solution

We will start on the left side, as it is the more complicated side:

$$\begin{aligned}\tan\theta\cos\theta &= \left(\frac{\sin\theta}{\cos\theta}\right)\cos\theta \\ &= \sin\theta.\end{aligned}$$

Analysis

This identity was fairly simple to verify, as it only required writing $\tan\theta$ in terms of $\sin\theta$ and $\cos\theta$.

Try It 7.1.1

Verify the identity $\csc\theta\cos\theta\tan\theta = 1$.

Answer

$$\begin{aligned}\csc\theta\cos\theta\tan\theta &= \left(\frac{1}{\sin\theta}\right)\cos\theta\left(\frac{\sin\theta}{\cos\theta}\right) \\ &= \frac{\cos\theta}{\sin\theta}\left(\frac{\sin\theta}{\cos\theta}\right) \\ &= \frac{\sin\theta\cos\theta}{\sin\theta\cos\theta} \\ &= 1\end{aligned}$$

Example 7.1.2: Verifying an Equivalency Using the Even-Odd Identities

Verify the following equivalency using the even-odd identities:

$$(1+\sin x)[1+\sin(-x)] = \cos^2 x$$

Solution

Working on the left side of the equation, we have

$$\begin{aligned}(1+\sin x)[1+\sin(-x)] &= (1+\sin x)(1-\sin x), \quad \text{since } \sin(-x) = -\sin x \\ &= 1-\sin^2 x \quad \text{(difference of squares)} \\ &= \cos^2 x, \quad \text{since } \sin^2\theta + \cos^2\theta = 1.\end{aligned}$$

Example 7.1.3: Verifying a Trigonometric Identity Involving $\sec^2\theta$

Verify the identity $\dfrac{\sec^2\theta-1}{\sec^2\theta}=\sin^2\theta$

Solution

As the left side is more complicated, let's begin there.

$$\begin{aligned}\frac{\sec^2\theta-1}{\sec^2\theta}&=\frac{\sec^2\theta}{\sec^2\theta}-\frac{1}{\sec^2\theta}\\&=1-\cos^2\theta\\&=\sin^2\theta\end{aligned}$$

There is more than one way to verify an identity. Here is another possibility. Again, we can start with the left side.

$$\begin{aligned}\frac{\sec^2\theta-1}{\sec^2\theta}&=\frac{(\tan^2\theta+1)-1}{\sec^2\theta}, && \text{since } \sec^2\theta=\tan^2\theta+1\\&=\frac{\tan^2\theta}{\sec^2\theta}\\&=\tan^2\theta\left(\frac{1}{\sec^2\theta}\right)\\&=\tan^2\theta\left(\cos^2\theta\right), && \text{since } \cos^2\theta=\frac{1}{\sec^2\theta}\\&=\left(\frac{\sin^2\theta}{\cos^2\theta}\right)\cos^2\theta, && \text{since } \tan^2\theta=\frac{\sin^2\theta}{\cos^2\theta}\\&=\sin^2\theta\end{aligned}$$

Analysis

In the first method, we split the fraction, putting both terms in the numerator over a common denominator. In the second method, we used the identity $\sec^2\theta=\tan^2\theta+1$ and continued to simplify. This problem illustrates that there are multiple ways we can verify an identity. Employing some creativity can sometimes simplify a procedure. As long as the substitutions are correct, the answer will be the same.

Try It 7.1.2

Show that $\dfrac{\cot\theta}{\csc\theta}=\cos\theta$.

Answer

$$\begin{aligned}\frac{\cot\theta}{\csc\theta}&=\frac{\frac{\cos\theta}{\sin\theta}}{\frac{1}{\sin\theta}}\\&=\frac{\cos\theta}{\sin\theta}\cdot\frac{\sin\theta}{1}\\&=\cos\theta\end{aligned}$$

Example 7.1.4: Creating an Identity

Create an identity for the expression $2\tan\theta\sec\theta$ by rewriting strictly in terms of sine.

Solution

There are a number of ways to begin, but here we will use the quotient and reciprocal identities to rewrite the expression:

$$\begin{aligned} 2\tan\theta\sec\theta &= 2\left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{1}{\cos\theta}\right) \\ &= \frac{2\sin\theta}{\cos^2\theta} \\ &= \frac{2\sin\theta}{1-\sin^2\theta} \qquad \text{Substitute } 1-\sin^2\theta \text{ for } \cos^2\theta \end{aligned}$$

Thus,

$$2\tan\theta\sec\theta = \frac{2\sin\theta}{1-\sin^2\theta}$$

Example 7.1.5: Verifying an Identity Using Algebra and Even/Odd Identities

Verify the identity:

$$\frac{\sin^2(-\theta)-\cos^2(-\theta)}{\sin(-\theta)-\cos(-\theta)} = \cos\theta - \sin\theta$$

Solution

Let's start with the left side and simplify:

$$\begin{aligned} \frac{\sin^2(-\theta)-\cos^2(-\theta)}{\sin(-\theta)-\cos(-\theta)} &= \frac{[\sin(-\theta]^2-[\cos(-\theta)]^2}{\sin(-\theta)-\cos(-\theta)} \\ &= \frac{(-\sin\theta)^2-(\cos\theta)^2}{-\sin\theta-\cos\theta}, \quad \text{since } \sin(-x)=-\sin x \text{ and } \cos(-x)=\cos x \\ &= \frac{(\sin\theta)^2-(\cos\theta)^2}{-\sin\theta-\cos\theta} \\ &= \frac{(\sin\theta-\cos\theta)(\sin\theta+\cos\theta)}{-(\sin\theta+\cos\theta)} \\ &\qquad \text{(difference of squares in the numerator)} \\ &= \cos\theta-\sin\theta \end{aligned}$$

7.1.3

Verify the identity $\frac{\sin^2\theta-1}{\tan\theta\sin\theta-\tan\theta} = \frac{\sin\theta+1}{\tan\theta}$.

Answer

$$\begin{aligned} \frac{\sin^2\theta-1}{\tan\theta\sin\theta-\tan\theta} &= \frac{(\sin\theta+1)(\sin\theta-1)}{\tan\theta(\sin\theta-1)} \\ &= \frac{\sin\theta+1}{\tan\theta} \end{aligned}$$

Example 7.1.6: Verifying an Identity Involving Cosines and Cotangents

Verify the identity: $(1-\cos^2 x)(1+\cot^2 x)=1$.

Solution:

$$
\begin{aligned}
(1-\cos^2 x)(1+\cot^2 x) &= (1-\cos^2 x)\left(1+\frac{\cos^2 x}{\sin^2 x}\right) \\
&= (1-\cos^2 x)\left(\frac{\sin^2 x}{\sin^2 x}+\frac{\cos^2 x}{\sin^2 x}\right) \\
&\qquad \text{Rewrite using a common denominator} \\
&= (1-\cos^2 x)\left(\frac{\sin^2 x+\cos^2 x}{\sin^2 x}\right) \\
&= (\sin^2 x)\left(\frac{1}{\sin^2 x}\right) \\
&= 1
\end{aligned}
$$

Example 7.1.7: Simplify an expression by Rewriting

Simplify the expression by rewriting:

$$\csc^2\theta - \cot^2\theta.$$

Solution:

We will use Ratio Identities and a Pythagorean Identity:

$$
\begin{aligned}
\csc^2\theta - \cot^2\theta &= \frac{1}{\sin^2\theta} - \frac{\cos^2\theta}{\sin^2\theta} \\
&= \frac{1-\cos^2\theta}{\sin^2\theta} \\
&= \frac{\sin^2\theta}{\sin^2\theta} \\
&= 1
\end{aligned}
$$

7.1.4

Use algebraic techniques to verify the identity: $\dfrac{\cos\theta}{1+\sin\theta} = \dfrac{1-\sin\theta}{\cos\theta}$.

(Hint: Work on the left side. Multiply the numerator and denominator by $1-\sin\theta$.)

Answer

$$
\begin{aligned}
\frac{\cos\theta}{1+\sin\theta}\left(\frac{1-\sin\theta}{1-\sin\theta}\right) &= \frac{\cos\theta(1-\sin\theta)}{1-\sin^2\theta} \\
&= \frac{\cos\theta(1-\sin\theta)}{\cos^2\theta} \\
&= \frac{1-\sin\theta}{\cos\theta}
\end{aligned}
$$

Media

Access these online resources for additional instruction and practice with the fundamental trigonometric identities.

- Fundamental Trigonometric Identities
- Verifying Trigonometric Identities

Key Equations

Pythagorean identities	$\cos^2\theta + \sin^2\theta = 1$ $1 + \cot^2\theta = \csc^2\theta$ $1 + \tan^2\theta = \sec^2\theta$
Even-odd identities	$\tan(-\theta) = -\tan\theta$ $\cot(-\theta) = -\cot\theta$ $\sin(-\theta) = -\sin\theta$ $\csc(-\theta) = -\csc\theta$ $\cos(-\theta) = \cos\theta$ $\sec(-\theta) = \sec\theta$
Reciprocal identities	$\sin\theta = \frac{1}{\csc\theta}$ $\cos\theta = \frac{1}{\sec\theta}$ $\tan\theta = \frac{1}{\cot\theta}$ $\csc\theta = \frac{1}{\sin\theta}$ $\sec\theta = \frac{1}{\cos\theta}$ $\cot\theta = \frac{1}{\tan\theta}$
Quotient identities	$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Key Concepts

- There are multiple ways to represent a trigonometric expression. Verifying the identities illustrates how expressions can be rewritten to simplify a problem.
- Simplifying one side of the equation to equal the other side is a method for verifying an identity. See Example **7.1.1** and Example **7.1.2**.
- The approach to verifying an identity depends on the nature of the identity. It is often useful to begin on the more complex side of the equation. See Example **7.1.3**.
- We can create an identity from a given expression. See Example **7.1.4**.
- Verifying an identity may involve algebra with the fundamental identities. See Example **7.1.5** and Example **7.1.6**.

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7.1E: Exercises

Verbal

1) We know $g(x) = \cos x$ is an even function, and $f(x) = \sin x$ and $h(x) = \tan x$are odd functions. What about $G(x) = \cos^2 x$, $F(x) = \sin^2 x$ and $H(x) = \tan^2 x$? Are they even, odd, or neither? Why?

Answer

All three functions, F, G, and H are even.

This is because

$F(-x) = \sin(-x)\sin(-x) = (-\sin x)(-\sin x) = \sin^2 x = F(x), G(-x) = \cos(-x)\cos(-x) = \cos x \cos x = cos^2 x = H(-x)$
$= \tan(-x)\tan(-x) = (-\tan x)(-\tan x) = \tan 2x = H(x)$

2) Examine the graph of $f(x) = \sec x$ on the interval $[-\pi, \pi]$. How can we tell whether the function is even or odd by only observing the graph of $f(x) = \sec x$?

3) After examining the reciprocal identity for $\sec t$, explain why the function is undefined at certain points.

Answer

When $\cos t = 0$ then $\sec t = 10$ which is undefined.

4) All of the Pythagorean identities are related. Describe how to manipulate the equations to get from $\sin^2 t + \cos^2 t = 1$ to the other forms.

Algebraic

For the exercises 5-15, use the fundamental identities to fully simplify the expression.

5) $\sin x \cos x \sec x$

Answer

$\sin x$

6) $\sin(-x)\cos(-x)\csc(-x)$

7) $\tan x \sin x + \sec x \cos^2 x$

Answer

$\sec x$

8) $\csc x + \cos x \cot(-x)$

9) $\dfrac{\cot t + \tan t}{\sec(-t)}$

Answer

$\csc x$

10) $3\sin^3 t \csc t + \cos^2 t + 2\cos(-t)\cos t$

11) $-\tan(-x)\cot(-x)$

Answer

-1

12) $\dfrac{-\sin(-x)\cos x \sec x \csc x \tan x}{\cot x}$

13) $\frac{1+\tan^2\theta}{\csc^2\theta}+\sin^2\theta+\frac{1}{\sec^\theta}$

Answer

$\sec^2 x$

14) $\left(\frac{\tan x}{\csc^2 x}+\frac{\tan x}{\sec^2 x}\right)\left(\frac{1+\tan x}{1+\cot x}\right)-\frac{1}{\cos^2 x}$

15) $\frac{1-\cos^2 x}{\tan^2 x}+2\sin^2 x$

Answer

$\sin^2 x+1$

For the exercises 16-28, simplify the first trigonometric expression by writing the simplified form in terms of the second expression.

16) $\frac{\tan x+\cot x}{\csc x};\cos x$

17) $\frac{\sec x+\csc x}{1+\tan x};\sin x$

Answer

$\frac{1}{\sin x}$

18) $\frac{\cos x}{1+\sin x}+\tan x;\cos x$

19) $\frac{1}{\sin x\cos x}-\cot x;\cot x$

Answer

$\frac{1}{\cot x}$

20) $\frac{1}{1-\cos x}-\frac{\cos x}{1+\cos x};\csc x$

21) $(\sec x+\csc x)(\sin x+\cos x)-2-\cot x;\tan x$

Answer

$\tan x$

22) $\frac{1}{\csc x-\sin x};\sec x$ and $\tan x$

23) $\frac{1-\sin x}{1+\sin x}-\frac{1+\sin x}{1-\sin x};\sec x$ and $\tan x$

Answer

$-4\sec x\tan x$

24) $\tan x;\sec x$

25) $\sec x;\cot x$

Answer

$\pm\sqrt{\frac{1}{\cot^2 x}+1}$

26) $\sec x;\sin x$

27) $\cot x;\sin x$

Answer

$$\frac{\pm\sqrt{1-\sin^2 x}}{\sin x}$$

28) $\cot x; \csc x$

For the exercises 29-33, verify the identity.

29) $\cos x - \cos^3 x = \cos x \sin^2 x$

Answer

Answers will vary. Sample proof:

$$\begin{aligned}\cos x - \cos^3 x &= \cos x(1-\cos^2 x)\\ &= \cos x \sin^x\end{aligned}$$

30) $\cos x(\tan x - \sec(-x)) = \sin x - 1$

31) $\frac{1+\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = 1 + 2\tan^2 x$

Answer

Answers will vary. Sample proof:

$$\begin{aligned}\frac{1+\sin^2 x}{\cos^2 x} &= \frac{1}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}\\ &= \sec^2 x + \tan^2 x\\ &= \tan^2 x + 1 + \tan^2 x\\ &= 1 + 2\tan^2 x\end{aligned}$$

32) $(\sin x + \cos x)^2 = 1 + 2\sin x \cos x$

33) $\cos^2 x - \tan^2 x = 2 - \sin^2 x - \sec^2 x$

Answer

Answers will vary. Sample proof:

$$\begin{aligned}\cos^2 x - \tan^2 x &= 1 - \sin^2 x - (\sec^2 x - 1)\\ &= 1 - \sin^2 x - \sec^2 x + 1\\ &= 2 - \sin^2 x - \sec^2 x\end{aligned}$$

Extensions

For the exercises 34-39, prove or disprove the identity.

34) $\frac{1}{1+\cos x} - \frac{1}{1-\cos(-x)} = -2\cot x \csc x$

35) $\csc^2 x(1+\sin^2 x) = \cot^2 x$

Answer

False

36) $\left(\frac{\sec^2(-x) - \tan^2 x}{\tan x}\right)\left(\frac{2+2\tan x}{2+2\cot x}\right) - 2\sin^2 x = \cos 2x$

37) $\frac{\tan x}{\sec x}\sin(-x) = \cos^2 x$

Answer

False

38) $\frac{\sec(-x)}{\tan x + \cot x} = -\sin(-x)$

39) $\dfrac{1+\sin x}{\cos x} = \dfrac{\cos x}{1+\sin(-x)}$

Answer

Proved with negative and Pythagorean identities

For the exercises 40-, determine whether the identity is true or false. If false, find an appropriate equivalent expression.

40) $\dfrac{\cos^2\theta - \sin^2\theta}{1-\tan^\theta} = \sin^2\theta$

41) $3\sin^2\theta + 4\cos^2\theta = 3 + \cos^2\theta$

Answer

True

$$\begin{aligned} 3\sin^2\theta + 4\cos^2\theta &= 3\sin^2\theta + 3\cos^2\theta + \cos^2\theta \\ &= 3\left(\sin^2\theta + \cos^2\theta\right) + \cos^2\theta \\ &= 3 + \cos^2\theta \end{aligned}$$

42) $\dfrac{\sec\theta + \tan\theta}{\cot\theta + \cos^\theta} = \sec^2\theta$

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7.2: Sum and Difference Identities

Skills to Develop

- Use sum and difference formulas for cosine.
- Use sum and difference formulas for sine.
- Use sum and difference formulas for tangent.
- Use sum and difference formulas for cofunctions.
- Use sum and difference formulas to verify identities.

How can the height of a mountain be measured? What about the distance from Earth to the sun? Like many seemingly impossible problems, we rely on mathematical formulas to find the answers. The trigonometric identities, commonly used in mathematical proofs, have had real-world applications for centuries, including their use in calculating long distances.

The trigonometric identities we will examine in this section can be traced to a Persian astronomer who lived around 950 AD, but the ancient Greeks discovered these same formulas much earlier and stated them in terms of chords. These are special equations or postulates, true for all values input to the equations, and with innumerable applications.



Figure* 7.2.1*: *Mount McKinley, in Denali National Park, Alaska, rises 20,237 feet (6,168 m) above sea level. It is the highest peak in North America. (credit: Daniel A. Leifheit, Flickr)*

In this section, we will learn techniques that will enable us to solve problems such as the ones presented above. The formulas that follow will simplify many trigonometric expressions and equations. Keep in mind that, throughout this section, the term *formula* is used synonymously with the word *identity*.

Using the Sum and Difference Formulas for Cosine

Finding the exact value of the sine, cosine, or tangent of an angle is often easier if we can rewrite the given angle in terms of two angles that have known trigonometric values. We can use the special angles, which we review in the unit circle shown in Figure 7.2.2.

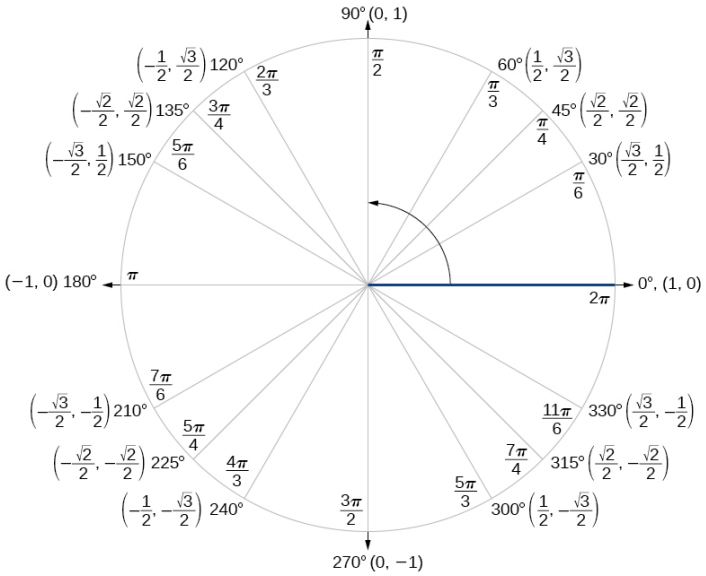


Figure 7.2.2: The Unit Circle

We will begin with the sum and difference formulas for cosine, so that we can find the cosine of a given angle if we can break it up into the sum or difference of two of the special angles (Table **7.2.1**).

***Table* 7.2.1**

Sum formula for cosine	$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$
Difference formula for cosine	$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

First, we will (almost) prove the difference formula for cosines. Let's consider two points on the unit circle (Figure **7.2.3**). Point P is at an angle α from the positive x-axis with coordinates $(\cos\alpha,\sin\alpha)$ and point Q is at an angle of β from the positive x-axis with coordinates $(\cos\beta,\sin\beta)$. Note the measure of angle POQ is $\alpha-\beta$ (Note: angle $P0Q$ is **not** in standard position).

Label two more points: A at an angle of $(\alpha-\beta)$ in standard position from the positive x-axis with coordinates $(\cos(\alpha-\beta),\sin(\alpha-\beta))$; and point B with coordinates $(1,0)$. Triangle $P0Q$ is a rotation of triangle $A0B$ and thus the distance from P to Q is the same as the distance from A to B.

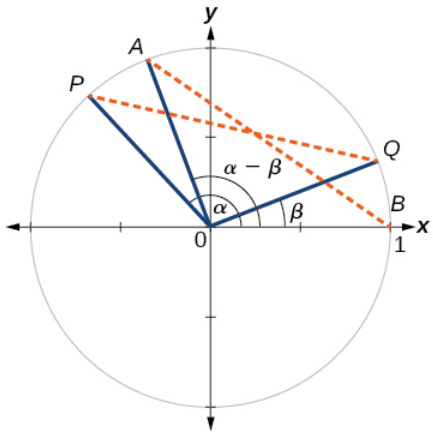


***Figure* 7.2.3**

We can find the distance from P to Q using the distance formula.

$$d_{PQ} = \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2}$$

$$= \sqrt{\cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta}$$

Apply Pythagorean identity and simplify.

$$= \sqrt{(\cos^2\alpha + \sin^2\alpha) + (\cos^2\beta + \sin^2\beta) - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta}$$

$$= \sqrt{1 + 1 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta}$$

$$= \sqrt{2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta}$$

Similarly, using the distance formula we can find the distance from A to B.

$$d_{AB} = \sqrt{(\cos(\alpha - \beta) - 1)^2 + (\sin(\alpha - \beta) - 0)^2}$$

$$= \sqrt{\cos^2(\alpha - \beta) - 2\cos(\alpha - \beta) + 1 + \sin^2(\alpha - \beta)}$$

Apply Pythagorean identity and simplify

$$= \sqrt{(\cos^2(\alpha - \beta) + \sin^2(\alpha - \beta)) - 2\cos(\alpha - \beta) + 1}$$

$$= \sqrt{1 - 2\cos(\alpha - \beta) + 1}$$

$$= \sqrt{2 - 2\cos(\alpha - \beta)}$$

Since $d_{PQ} = d_{AB}$, we have

$$\sqrt{2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta} = \sqrt{2 - 2\cos(\alpha - \beta)}$$

Square both sides of this equation to get

$$2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta = 2 - 2\cos(\alpha - \beta)$$

Subtract 2 from both sides and divide both sides by -2.

$$\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta)$$

Thus, we have the difference formula for cosine. We could use similar methods to derive the cosine of the sum of two angles.

Analysis: To completely prove the formula, we would have to consider other possibilities for α and β, such as $\alpha > \pi$ and $\beta < 0$, and we would have to show that the law always holds. For this course, we will accept the idea of this (almost) proof.

SUM AND DIFFERENCE FORMULAS FOR COSINE

These formulas can be used to calculate the cosine of sums and differences of angles.

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

How To: Given two angles, find the cosine of the sum or difference of the angles

1. Write the sum or difference formula for cosine.
2. Substitute the values of the given angles into the formula.
3. Simplify.

Example 7.2.1: Finding the Exact Value Using the Formula for the Cosine of the Difference of Two Angles

Using the formula for the cosine of the difference of two angles, find the exact value of $\cos\left(\frac{5\pi}{4} - \frac{\pi}{6}\right)$.

Solution

Begin by writing the formula for the cosine of the difference of two angles. Then substitute the given values.

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

$$\cos\left(\frac{5\pi}{4}-\frac{\pi}{6}\right)=\cos\left(\frac{5\pi}{4}\right)\cos\left(\frac{\pi}{6}\right)+\sin\left(\frac{5\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)$$

$$=\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)-\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$

$$=-\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}$$

$$=\frac{-\sqrt{6}-\sqrt{2}}{4}$$

If you have a scientific calculator you can check the answer, making sure you use radian mode.

7.2.1:

Find the exact value of $\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$.

Answer

$\frac{\sqrt{2}+\sqrt{6}}{4}$

Example 7.2.2: Finding the Exact Value Using the Formula for the Sum of Two Angles for Cosine

Find the exact value of $\cos(75°)$.

Solution

Since $75°=45°+30°$, we can evaluate $\cos(75°)$ as $\cos(45°+30°)$.

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$\cos(45°+30°)=\cos(45°)\cos(30°)-\sin(45°)\sin(30°)$$

$$=\frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right)-\frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)$$

$$=\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}$$

$$=\frac{\sqrt{6}-\sqrt{2}}{4}$$

If you have a scientific calculator you can check the answer, making sure you use degree mode.

Analysis

We could have also solved this problem using the fact that $75°=135°-60°$.

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

$$\cos(135°-60°)=\cos(135°)\cos(60°)+\sin(135°)\sin(60°)$$

$$=\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)+\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$=\left(-\frac{\sqrt{2}}{4}\right)+\left(\frac{\sqrt{6}}{4}\right)$$

$$=\frac{\sqrt{6}-\sqrt{2}}{4}$$

7.2.2

Find the exact value of $\cos(105°)$.

Answer

$$\cos(105°) = \cos(60°+45°) = \cos 60°\cos 45° - \sin 60°\sin 45° = \frac{\sqrt{2}-\sqrt{6}}{4}$$

Using the Sum and Difference Formulas for Sine

The sum and difference formulas for sine can be derived in the same manner as those for cosine, and they are similar to the cosine formulas. (In what ways are they different?)

SUM AND DIFFERENCE FORMULAS FOR SINE

These formulas can be used to calculate the sines of sums and differences of angles.

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

How To: Given two angles, find the sine of the sum or difference of the angles.

1. Write the sum or difference formula for sine.
2. Substitute the given angles into the formula.
3. Simplify.

Example 7.2.3: Using Sum and Difference Identities to Evaluate the sine of the Difference of Angles

Use the sum and difference identities to evaluate the difference of the angles and show that part *a* equals part *b*.

a. $\sin(45°+30°)$
b. $\sin(120°-45°)$

Solution

a. Let's begin by writing the sum formula and substitute the given angles.

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(45°+30°) = \sin(45°)\cos(30°) + \cos(45°)\sin(30°)$$

Next, we need to find the values of the trigonometric expressions.

$$\sin(45°) = \frac{\sqrt{2}}{2},\ \cos(30°) = \frac{\sqrt{3}}{2},\ \cos(45°) = \frac{\sqrt{2}}{2},\ \sin(30°) = \frac{1}{2}$$

Now we can substitute these values into the equation and simplify.

$$\sin(45°+30°) = \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)$$
$$= \frac{\sqrt{6}+\sqrt{2}}{4}$$

b. Now we write the difference formula and substitute the given angles.

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\sin(120°-45°) = \sin(120°)\cos(45°) - \cos(120°)\sin(45°)$$

Next, we find the values of the trigonometric expressions.

$$\sin(120°) = \frac{\sqrt{3}}{2},\ \cos(45°) = \frac{\sqrt{2}}{2},\ \cos(120°) = -\frac{1}{2},\ \sin(45°) = \frac{\sqrt{2}}{2}$$

Now we can substitute these values into the equation and simplify.

$$\sin(120^\circ - 45^\circ) = \frac{\sqrt{3}}{2}\left(\frac{\sqrt{2}}{2}\right) - \left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$
$$= \frac{\sqrt{6}+\sqrt{2}}{4}$$

Analysis

In both cases, we are calculating $\sin 75^\circ$, so it is not surprising that we get the same answer.

Example 7.2.4: Finding the Exact Value of an Expression Involving an Inverse Trigonometric Function

Find the exact value of $\sin\left(\cos^{-1}\frac{1}{2} + \sin^{-1}\frac{3}{5}\right)$. Then check the answer with a graphing calculator.

Solution

The pattern displayed in this problem is $\sin(\alpha+\beta)$. Let $\alpha = \cos^{-1}\left(\frac{1}{2}\right)$ and $\beta = \sin^{-1}\left(\frac{3}{5}\right)$. Then we can write

$$\cos\alpha = \frac{1}{2},\ 0 \le \alpha \le \pi$$
$$\sin\beta = \frac{3}{5},\ -\frac{\pi}{2} \le \beta \le \frac{\pi}{2}$$

We will use the Pythagorean identities to find $\sin\alpha$ and $\cos\beta$

$$\sin\alpha = \sqrt{1-\cos^2\alpha}$$
$$= \sqrt{1-\frac{1}{4}}$$
$$= \sqrt{\frac{3}{4}}$$
$$= \frac{\sqrt{3}}{2}$$
$$\cos\beta = \sqrt{1-\sin^2\beta}$$
$$= \sqrt{1-\frac{9}{25}}$$
$$= \sqrt{\frac{16}{25}}$$
$$= \frac{4}{5}$$

Using the sum formula for sine,

$$\sin\left(\cos^{-1}\frac{1}{2} + \sin^{-1}\frac{3}{5}\right) = \sin(\alpha+\beta)$$
$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$= \frac{\sqrt{3}}{2}\cdot\frac{4}{5} + \frac{1}{2}\cdot\frac{3}{5}$$
$$= \frac{4\sqrt{3}+3}{10}$$

Using the Sum and Difference Formulas for Tangent

Finding exact values for the tangent of the sum or difference of two angles is a little more complicated, but again, it is a matter of recognizing the pattern.

Finding the sum of two angles formula for tangent involves taking the quotient of the sum formulas for sine and cosine and simplifying. Recall, $\tan x = \frac{\sin x}{\cos x}$, $\cos x \neq 0$.

Let's derive the sum formula for tangent.

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} \\ &= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta} \\ &= \frac{\dfrac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \\ &= \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \\ &= \frac{\dfrac{\sin\alpha}{\cos\alpha}+\dfrac{\sin\beta}{\cos\beta}}{1-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \\ &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\end{aligned}$$

We can derive the difference formula for tangent in a similar way.

SUM AND DIFFERENCE FORMULAS FOR TANGENT

These formulas can be used to find the sum and difference for tangent:

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

How To: Given two angles, find the tangent of the sum or difference of the angles

1. Write the sum or difference formula for tangent.
2. Substitute the given angles into the formula.
3. Simplify.

Example 7.2.5: Finding the Exact Value of an Expression Involving Tangent

Find the exact value of $\tan\left(\frac{\pi}{6}+\frac{\pi}{4}\right)$.

Solution

Let's first write the sum formula for tangent and then substitute the given angles into the formula.

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$\tan\left(\frac{\pi}{6}+\frac{\pi}{4}\right)=\frac{\tan\left(\frac{\pi}{6}\right)+\tan\left(\frac{\pi}{4}\right)}{1-\left(\tan\left(\frac{\pi}{6}\right)\right)\left(\tan\left(\frac{\pi}{4}\right)\right)}$$

Next, we determine the individual function values within the formula:

$$\tan\left(\frac{\pi}{6}\right)=\frac{1}{\sqrt{3}},\quad \tan\left(\frac{\pi}{4}\right)=1$$

So we have

$$\begin{aligned}\tan\left(\frac{\pi}{6}+\frac{\pi}{4}\right)&=\frac{\frac{1}{\sqrt{3}}+1}{1-\left(\frac{1}{\sqrt{3}}\right)(1)}\\&=\frac{\frac{1+\sqrt{3}}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}}\\&=\frac{1+\sqrt{3}}{\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}-1}\right)\\&=\frac{\sqrt{3}+1}{\sqrt{3}-1}\end{aligned}$$

7.2.3:

Find the exact value of $\tan\left(\frac{2\pi}{3}+\frac{\pi}{4}\right)$.

Answer

$$\frac{1-\sqrt{3}}{1+\sqrt{3}}$$

Example 7.2.6: evaluating Multiple trigonometric functions using Sums and Differences of Angles

Given $\sin\alpha=\frac{3}{5}$, $0<\alpha<\frac{\pi}{2}$, $\cos\beta=-\frac{5}{13}$, $\pi<\beta<\frac{3\pi}{2}$, find

a. $\sin(\alpha+\beta)$
b. $\cos(\alpha+\beta)$
c. $\tan(\alpha+\beta)$
d. $\tan(\alpha-\beta)$

Solution

We can use the sum and difference formulas to identify the sum or difference of angles when the ratio of sine, cosine, or tangent is provided for each of the individual angles. To do so, we construct what is called a reference triangle to help find each component of the sum and difference formulas.

a. To find $\sin(\alpha+\beta)$,we begin with $\sin\alpha=\frac{3}{5}$ and $0<\alpha<\frac{\pi}{2}$. The side opposite α has length 3, the hypotenuse has length 5, and α is in the first quadrant. See Figure 7.2.4. Using the Pythagorean Theorem, we can find the length of side a:

$$\begin{aligned}a^2+3^2&=5^2\\a^2&=16\\a&=4\end{aligned}$$

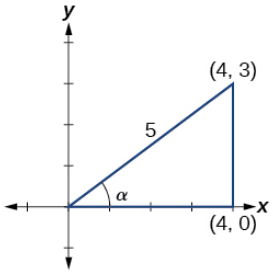


Figure 7.2.4

Since $\cos\beta = -\frac{5}{13}$ and $\pi < \beta < \frac{3\pi}{2}$, the side adjacent to β is -5,the hypotenuse is 13, and β is in the third quadrant. See Figure 7.2.5. Again, using the Pythagorean Theorem, we have

$$\begin{aligned}(-5)^2 + a^2 &= 13^2\\ 25 + a^2 &= 169\\ a^2 &= 144\\ a &= \pm 12\end{aligned}$$

Since β is in the third quadrant, $a = -12$.

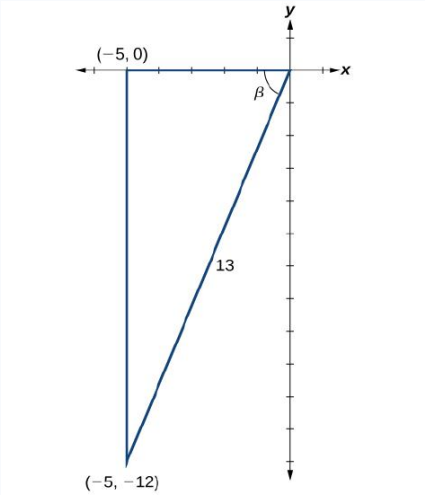


Figure 7.2.5

The next step is finding the cosine of α and the sine of β. The cosine of α is the adjacent side over the hypotenuse. We can find it from the triangle in Figure 7.2.5: $\cos\alpha = \frac{4}{5}$. We can also find the sine of β from the triangle in Figure 7.2.5, as opposite side over the hypotenuse: $\sin\beta = -\frac{12}{13}$. Now we are ready to evaluate $\sin(\alpha + \beta)$.

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\\ &= \left(\frac{3}{5}\right)\left(-\frac{5}{13}\right) + \left(\frac{4}{5}\right)\left(-\frac{12}{13}\right)\\ &= -\frac{15}{65} - \frac{48}{65}\\ &= -\frac{63}{65}\end{aligned}$$

b. We can find $\cos(\alpha+\beta)$ in a similar manner. We substitute the values according to the formula.

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= \left(\frac{4}{5}\right)\left(-\frac{5}{13}\right) - \left(\frac{3}{5}\right)\left(-\frac{12}{13}\right) \\ &= -\frac{20}{65} + \frac{36}{65} \\ &= \frac{16}{65}\end{aligned}$$

c. For $\tan(\alpha+\beta)$,if $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$, then

$$\tan\alpha = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$$

If $\sin\beta = -\frac{12}{13}$ and $\cos\beta = -\frac{5}{13}$, then

$$\tan\beta = \frac{-\frac{12}{13}}{-\frac{5}{13}} = \frac{12}{5}$$

Then,

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \\ &= \frac{\frac{3}{4}+\frac{12}{5}}{1-\frac{3}{4}(\frac{12}{5})} \\ &= \frac{\frac{63}{20}}{-\frac{16}{20}} \\ &= -\frac{63}{16}\end{aligned}$$

d. To find $\tan(\alpha-\beta)$, we have the values we need. We can substitute them in and evaluate.

$$\begin{aligned}\tan(\alpha-\beta) &= \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta} \\ &= \frac{\frac{3}{4}-\frac{12}{5}}{1+\frac{3}{4}(\frac{12}{5})} \\ &= \frac{-\frac{33}{20}}{\frac{56}{20}} \\ &= -\frac{33}{56}\end{aligned}$$

Analysis

A common mistake when addressing problems such as this one is that we may be tempted to think that α and β are angles in the same triangle, which of course, they are not. Also note that

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$$

Using Sum and Difference Formulas for Cofunctions

We can use the sine, cosine, and tangent functions for the sums and differences of angles, to evaluate their cofunctions. You may recall from Right Triangle Trigonometry that, if the sum of two positive angles is $\frac{\pi}{2}$, those two angles are complements, and the

sum of the two acute angles in a right triangle is $\frac{\pi}{2}$, so they are also complements. In Figure **7.2.6**, notice that if one of the acute angles is labeled as θ,then the other acute angle must be labeled $\left(\frac{\pi}{2}-\theta\right)$.

Notice also that $\sin\theta=\cos\left(\frac{\pi}{2}-\theta\right)$,which is opposite over hypotenuse. Thus, when two angles are complementary, we can say that the sine of θ equals the cofunction of the complement of θ. Similarly, tangent and cotangent are cofunctions, and secant and cosecant are cofunctions.

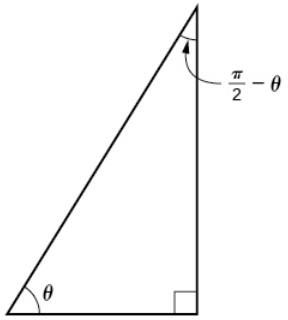


***Figure* 7.2.6**

From these relationships, the cofunction identities are formed. Recall that you first encountered these identities in The Unit Circle: Sine and Cosine Functions.

COFUNCTION IDENTITIES

The cofunction identities are summarized in Table **7.2.2**.

Table 7.2.2

$\sin\theta=\cos\left(\frac{\pi}{2}-\theta\right)$	$\cos\theta=\sin\left(\frac{\pi}{2}-\theta\right)$
$\tan\theta=\cot\left(\frac{\pi}{2}-\theta\right)$	$\cot\theta=\tan\left(\frac{\pi}{2}-\theta\right)$
$\sec\theta=\csc\left(\frac{\pi}{2}-\theta\right)$	$\csc\theta=\sec\left(\frac{\pi}{2}-\theta\right)$

Notice that the formulas in the table may also be derived algebraically using the sum and difference formulas. For example, using

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

We can write

$$\begin{aligned}\cos\left(\frac{\pi}{2}-\theta\right)&=\cos\frac{\pi}{2}\cos\theta+\sin\frac{\pi}{2}\sin\theta\\&=(0)\cos\theta+(1)\sin\theta\\&=\sin\theta\end{aligned}$$

Using the Sum and Difference Formulas to Verify Identities

Verifying an identity means demonstrating that the equation holds for all values of the variable. It helps to be very familiar with the identities or to have a list of them accessible while working the problems. Reviewing the general rules presented earlier may help simplify the process of verifying an identity.

How To: Given an identity, verify using sum and difference formulas

1. Begin with the expression on the side of the equal sign that appears most complex. Rewrite that expression until it matches the other side of the equal sign.
2. Look for opportunities to use the sum and difference formulas.
3. Rewrite sums or differences of quotients as single quotients.
4. If the process becomes cumbersome, rewrite the expression in terms of sines and cosines.

Example 7.2.7A: Verifying an Identity Involving Sine

Verify the identity $\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta$.

Solution

We will work on the left side of the equation, which includes the sines of the sum and the difference of angles. We can rewrite each using the sum and difference formulas.

$$\begin{aligned}\sin(\alpha+\beta)+\sin(\alpha-\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta+\sin\alpha\cos\beta-\cos\alpha\sin\beta\\ &= 2\sin\alpha\cos\beta\end{aligned}$$

The identity is verified.

Example 7.2.7B: Verifying an Identity Involving Tangent

Verify the following identity.

$$\frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}=\tan\alpha-\tan\beta$$

Solution

We begin by rewriting the numerator on the left side of the equation.

$$\begin{aligned}\frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta} &= \frac{\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\cos\alpha\cos\beta}\\ &= \frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}-\frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta} \qquad \text{Rewrite as two fractions with the same denominator}\\ &= \frac{\sin\alpha}{\cos\alpha}-\frac{\sin\beta}{\cos\beta} \qquad \text{Cancel}\\ &= \tan\alpha-\tan\beta \qquad \text{Rewrite in terms of tangent}\end{aligned}$$

7.2.4:

Verify the identity: $\tan(\pi-\theta)=-\tan\theta$.

Answer

$$\begin{aligned}\tan(\pi-\theta) &= \frac{\tan(\pi)-\tan\theta}{1+\tan(\pi)\tan\theta}\\ &= \frac{0-\tan\theta}{1+0\cdot\tan\theta}\\ &= -\tan\theta\end{aligned}$$

Example 7.2.8: Investigating a Guy-wire Application

For a climbing wall, a guy-wire R is attached 47 feet high on a vertical pole. Added support is provided by another guy-wire S attached 40 feet above ground on the same pole. If the wires are attached to the ground 50 feet from the pole, find the angle α between the wires. See Figure 7.2.8.

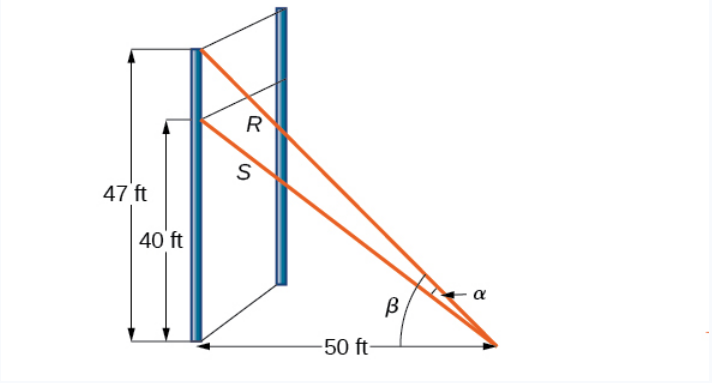


***Figure* 7.2.7**

Solution

Let's first summarize the information we can gather from the diagram. As only the sides adjacent to the right angle are known, we can use the tangent function. Notice that $\tan\beta = \frac{47}{50}$, and $\tan(\beta - \alpha) = \frac{40}{50} = \frac{4}{5}$. We can then use difference formula for tangent.

$$\tan(\beta - \alpha) = \frac{\tan\beta - \tan\alpha}{1 + \tan\beta\tan\alpha}$$

Now, substituting the values we know into the formula, we have

$$\frac{4}{5} = \frac{\frac{47}{50} - \tan\alpha}{1 + \frac{47}{50}\tan\alpha}$$

$$4(1 + \frac{47}{50}\tan\alpha) = 5(\frac{47}{50} - \tan\alpha)$$

Use the distributive property, and then simplify the functions.

$$4(1) + 4(\frac{47}{50})\tan\alpha = 5(\frac{47}{50}) - 5\tan\alpha$$

$$4 + 3.76\tan\alpha = 4.7 - 5\tan\alpha$$

$$5\tan\alpha + 3.76\tan\alpha = 0.7$$

$$8.76\tan\alpha = 0.7$$

$$\tan\alpha \approx 0.07991$$

$$tan^{-1}(0.07991) \approx .079741$$

Now we can calculate the angle in degrees.

$$\alpha \approx 0.079741(\frac{180}{\pi})$$

$$\approx 4.57°$$

Media

Access these online resources for additional instruction and practice with sum and difference identities.

- Sum and Difference Identities for Cosine
- Sum and Difference Identities for Sine
- Sum and Difference Identities for Tangent

Key Equations

Sum Formula for Cosine	$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$
Difference Formula for Cosine	$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$
Sum Formula for Sine	$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$
Difference Formula for Sine	$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$
Sum Formula for Tangent	$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$
Difference Formula for Tangent	$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$
Cofunction identities	$\sin\theta=\cos(\frac{\pi}{2}-\theta)$ $\cos\theta=\sin(\frac{\pi}{2}-\theta)$ $\tan\theta=\cot(\frac{\pi}{2}-\theta)$ $\cot\theta=\tan(\frac{\pi}{2}-\theta)$ $\sec\theta=\csc(\frac{\pi}{2}-\theta)$ $\csc\theta=\sec(\frac{\pi}{2}-\theta)$

Key Concepts

- The sum formula for cosines states that the cosine of the sum of two angles equals the product of the cosines of the angles minus the product of the sines of the angles. The difference formula for cosines states that the cosine of the difference of two angles equals the product of the cosines of the angles plus the product of the sines of the angles.
- The sum formula for sines states that the sine of the sum of two angles equals the product of the sine of the first angle and cosine of the second angle plus the product of the cosine of the first angle and the sine of the second angle. The difference formula for sines states that the sine of the difference of two angles equals the product of the sine of the first angle and cosine of the second angle minus the product of the cosine of the first angle and the sine of the second angle.
- The sum formula for tangent states that the tangent of the sum of two angles equals the sum of the tangents of the angles divided by 1 minus the product of the tangents of the angles. The difference formula for tangent states that the tangent of the difference of two angles equals the difference of the tangents of the angles divided by 1 plus the product of the tangents of the angles. See Example .
- The sum and difference formulas can be used to find the exact values of the sine, cosine, or tangent of an angle. See Examples **7.2.1**, **7.2.2**, **7.2.3**, and **7.2.5**.
- The sum and difference formulas for sine and cosine can also be used for inverse trigonometric functions. See Example **7.2.4**.
- The Pythagorean Theorem along with the sum and difference formulas can be used to find multiple sums and differences of angles. See Example **7.2.6**.
- Sum and difference formulas are useful in verifying identities. See Example **7.2.7*A*** and Example **7.2.7*B***.
- Application problems are often easier to solve by using sum and difference formulas. See Example **7.2.8**.

Contributors

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7.2E: Exercises

Verbal

1) Explain the basis for the cofunction identities and when they apply.

Answer

The cofunction identities apply to complementary angles. Viewing the two acute angles of a right triangle, if one of those angles measures x the second angle measures $\frac{\pi}{2}-x$ Then $\sin x=\cos\left(\frac{\pi}{2}-x\right)$ The same holds for the other cofunction identities. The key is that the angles are complementary.

2) Is there only one way to evaluate $\cos\left(\frac{5\pi}{4}\right)$? Explain how to set up the solution in two different ways, and then compute to make sure they give the same answer.

3) Explain to someone who has forgotten the even-odd properties of sinusoidal functions how the addition and subtraction formulas can determine this characteristic for $f(x)=\sin(x)$ and $g(x)=\cos(x)$. (Hint: $0-x=-x$)

Answer

$\sin(-x)=-\sin x$, so $\sin x$ is odd. $\cos(-x)=\cos(0-x)=\cos x$, so $\cos x$ is even.

Algebraic

For exercises 4-9, find the exact value.

4) $\cos\left(\frac{7\pi}{12}\right)$

5) $\cos\left(\frac{\pi}{12}\right)$

Answer

$\frac{\sqrt{2}+\sqrt{6}}{4}$

6) $\sin\left(\frac{5\pi}{12}\right)$

7) $\sin\left(\frac{11\pi}{12}\right)$

Answer

$\frac{\sqrt{6}-\sqrt{2}}{4}$

8) $\tan\left(-\frac{\pi}{12}\right)$

9) $\tan\left(\frac{19\pi}{12}\right)$

Answer

$-2-\sqrt{3}$

For exercises 10-13, rewrite in terms of $\sin x$ and $\cos x$

10) $\sin\left(x+\frac{11\pi}{6}\right)$

11) $\sin\left(x-\frac{3\pi}{4}\right)$

Answer

$-\frac{\sqrt{2}}{2}\sin x-\frac{\sqrt{2}}{2}\cos x$

12) $\cos\left(x-\frac{5\pi}{6}\right)$

13) $\cos\left(x+\frac{2\pi}{3}\right)$

Answer

$-\frac{1}{2}\cos x-\frac{\sqrt{3}}{2}\sin x$

For exercises 14-19, simplify the given expression.

14) $\csc\left(\frac{\pi}{2}-t\right)$

15) $\sec\left(\frac{\pi}{2}-\theta\right)$

Answer

$\csc\theta$

16) $\cot\left(\frac{\pi}{2}-x\right)$

17) $\tan\left(\frac{\pi}{2}-x\right)$

Answer

$\cot x$

18) $\sin(2x)\cos(5x)-\sin(5x)\cos(2x)$

19) $\frac{\tan\left(\frac{3}{2}x\right)-\tan\left(\frac{7}{5}x\right)}{1+\tan\left(\frac{3}{2}x\right)\tan\left(\frac{7}{5}x\right)}$

Answer

$\tan\left(\frac{x}{10}\right)$

For exercises 20-21, find the requested information.

20) Given that $\sin a=\frac{2}{3}$ and $\cos b=-\frac{1}{4}$, with a and b both in the interval $\left[\frac{\pi}{2},\pi\right)$, find $\sin(a+b)$ and $\cos(a-b)$.

21) Given that $\sin a=\frac{4}{5}$ and $\cos b=\frac{1}{3}$, with a and b both in the interval $\left[0,\frac{\pi}{2}\right)$, find $\sin(a-b)$ and $\cos(a+b)$.

Answer

$\sin(a-b)=\left(\frac{4}{5}\right)\left(\frac{1}{3}\right)-\left(\frac{3}{5}\right)\left(\frac{2\sqrt{2}}{3}\right)=\frac{4-6\sqrt{2}}{15}$

$\cos(a+b)=\left(\frac{3}{5}\right)\left(\frac{1}{3}\right)-\left(\frac{4}{5}\right)\left(\frac{2\sqrt{2}}{3}\right)=\frac{3-8\sqrt{2}}{15}$

For exercises 22-24, find the exact value of each expression.

22) $\sin\left(\cos^{-1}(0)-\cos^{-1}\left(\frac{1}{2}\right)\right)$

23) $\cos\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)+\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$

Answer

$\frac{\sqrt{2}-\sqrt{6}}{4}$

24) $\tan\left(\sin^{-1}\left(\frac{1}{2}\right)-\cos^{-1}\left(\frac{1}{2}\right)\right)$

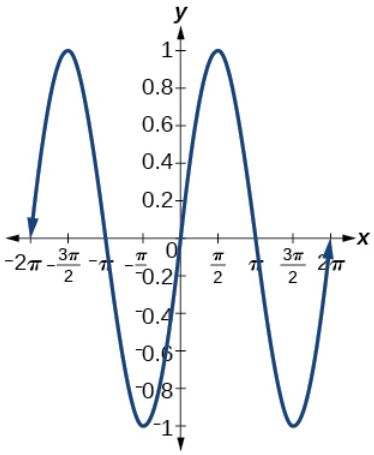
Graphical

For exercises 25-32, simplify the expression, and then graph both expressions as functions to verify the graphs are identical.

25) $\cos\left(\frac{\pi}{2}-x\right)$

Answer

$\sin x$

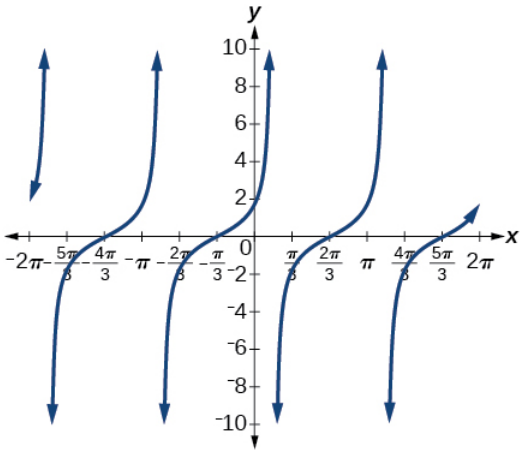


26) $\sin(\pi-x)$

27) $\tan\left(\frac{\pi}{3}+x\right)$

Answer

$\cot\left(\frac{\pi}{6}-x\right)$

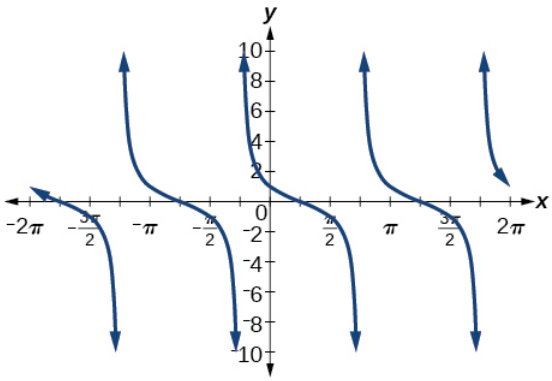


28) $\sin\left(\frac{\pi}{3}+x\right)$

29) $\tan\left(\frac{\pi}{4}-x\right)$

Answer

$\cot\left(\frac{\pi}{4}+x\right)$

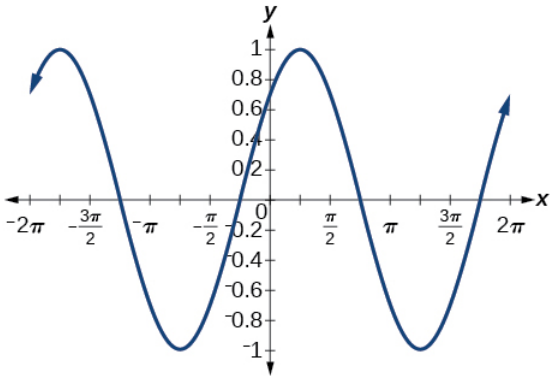


30) $\cos\left(\frac{7\pi}{6}+x\right)$

31) $\sin\left(\frac{\pi}{4}+x\right)$

Answer

$\frac{\sin x}{\sqrt{2}}+\frac{\cos x}{\sqrt{2}}$



32) $\cos\left(\frac{5\pi}{4}+x\right)$

For exercises 33-41, use a graph to determine whether the functions are the same or different. If they are the same, show why. If they are different, replace the second function with one that is identical to the first. (Hint: think $2x=x+x$)

33) $f(x)=\sin(4x)-\sin(3x)\cos x, g(x)=\sin x\cos(3x)$

Answer

They are the same.

34) $f(x)=\cos(4x)+\sin x\sin(3x), g(x)=-\cos x\cos(3x)$

35) $f(x)=\sin(3x)\cos(6x), g(x)=-\sin(3x)\cos(6x)$

Answer

They are different, try $g(x)=\sin(9x)-\cos(3x)\sin(6x)$

36) $f(x)=\sin(4x), g(x)=\sin(5x)\cos x-\cos(5x)\sin x$

37) $f(x)=\sin(2x), g(x)=2\sin x\cos x$

Answer

They are the same.

38) $f(\theta) = \cos(2\theta), g(\theta) = \cos^2\theta - \sin^2\theta$

39) $f(\theta) = \tan(2\theta), g(\theta) = \dfrac{\tan\theta}{1+\tan^2\theta}$

Answer

They are different, try $g(\theta) = \dfrac{2\tan\theta}{1-\tan^2\theta}$

40) $f(x) = \sin(3x)\sin x, g(x) = \sin^2(2x)\cos^2 x - \cos^2(2x)\sin 2x$

41) $f(x) = \tan(-x), g(x) = \dfrac{\tan x - \tan(2x)}{1-\tan x\tan(2x)}$

Answer

They are different, try $g(x) = \dfrac{\tan x - \tan(2x)}{1+\tan x\tan(2x)}$

Technology

For the exercises 42-46, find the exact value algebraically, and then confirm the answer with a calculator to the fourth decimal point.

42) $\sin(75°)$

43) $\sin(195°)$

Answer

$-\dfrac{\sqrt{3}-1}{2\sqrt{2}}$, or -0.2588

44) $\cos(165°)$

45) $\cos(345°)$

Answer

$\dfrac{1+\sqrt{3}}{2\sqrt{2}}$, or -0.9659

46) $\tan(-15°)$

Extensions

For the exercises 47-51, prove the identities provided.

47) $\tan\left(x+\dfrac{\pi}{4}\right) = \dfrac{\tan x+1}{1-\tan x}$

Answer

$$\tan\left(x+\frac{\pi}{4}\right) =$$
$$\frac{\tan x+\tan\left(\frac{\pi}{4}\right)}{1-\tan x\tan\left(\frac{\pi}{4}\right)} =$$
$$\frac{\tan x+1}{1-\tan x(1)} = \frac{\tan x+1}{1-\tan x}$$

48) $\dfrac{\tan(a+b)}{\tan(a-b)} = \dfrac{\sin a\cos a+\sin b\cos b}{\sin a\cos a-\sin b\cos b}$

49) $\dfrac{\cos(a+b)}{\cos a\cos b} = 1-\tan a\tan b$

Answer

$$\frac{\cos(a+b)}{\cos a \cos b} =$$
$$\frac{\cos a \cos b}{\cos a \cos b} - \frac{\sin a \sin b}{\cos a \cos b} = 1 - \tan a \tan b$$

50) $\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y$

51) $\frac{\cos(x+h)-\cos(x)}{h} = \cos x \frac{\cos h - 1}{h} - \sin x \frac{\sin h}{h}$

Answer

$$\frac{\cos(x+h)-\cos(x)}{h} =$$
$$\frac{\cos x \cosh - \sin x \sinh - \cos x}{h} =$$
$$\frac{\cos x(\cosh - 1) - \sin x(\sinh - 1)}{h} = \cos x \frac{\cos h - 1}{h} - \sin x \frac{\sin h}{h}$$

For the exercises 52-, prove or disprove the statements.

52) $\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$

53) $\tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$

Answer

True

54) $\frac{\tan(x+y)}{1+\tan x \tan x} = \frac{\tan x + \tan y}{1 - \tan^2 x \tan^2 y}$

55) If α, β, and γ are angles in the same triangle, then prove or disprove $\sin(\alpha+\beta) = \sin \gamma$.

Answer

True. Note that $\sin(\alpha+\beta) = \sin(\pi - \gamma)$ and expand the right hand side.

56) If α, β , and γ are angles in the same triangle, then prove or disprove $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha \tan\beta \tan\gamma$.

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7.3: Double-Angle, Half-Angle, and Reduction Formulas

Skills to Develop

- Use double-angle formulas to find exact values.
- Use double-angle formulas to verify identities.
- Use reduction formulas to simplify an expression.
- Use half-angle formulas to find exact values.

Bicycle ramps made for competition (see Figure 7.3.1) must vary in height depending on the skill level of the competitors. For advanced competitors, the angle formed by the ramp and the ground should be θ such that $\tan\theta = \frac{5}{3}$. The angle is divided in half for novices. What is the steepness of the ramp for novices? In this section, we will investigate three additional categories of identities that we can use to answer questions such as this one.

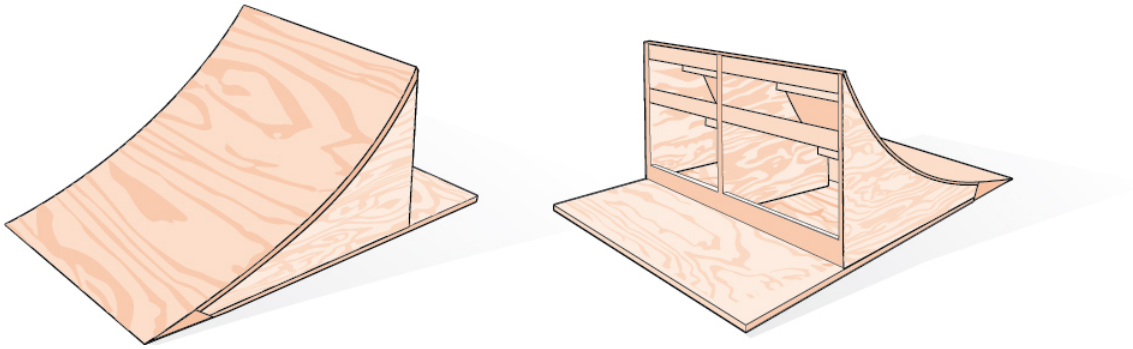


***Figure* 7.3.1:** *Bicycle ramps for advanced riders have a steeper incline than those designed for novices.*

Using Double-Angle Formulas to Find Exact Values

In the previous section, we used addition and subtraction formulas for trigonometric functions. Now, we take another look at those same formulas. The **double-angle formulas** are a special case of the sum formulas, where $\alpha = \beta$. Deriving the double-angle formula for sine begins with the sum formula,

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

If we let $\alpha = \beta = \theta$, then we have

$$\begin{aligned}\sin(\theta+\theta) &= \sin\theta\cos\theta + \cos\theta\sin\theta\\ \sin(2\theta) &= 2\sin\theta\cos\theta.\end{aligned}$$

There are three options for the double-angle formula for cosine. First, starting from the sum formula, $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$,and letting $\alpha = \beta = \theta$, we have

$$\begin{aligned}\cos(\theta+\theta) &= \cos\theta\cos\theta - \sin\theta\sin\theta\\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta.\end{aligned}$$

Using one of the Pythagorean Identities, we can expand this double-angle formula for cosine and get two more variations. The first variation is:

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta\\ &= (1-\sin^2\theta) - \sin^2\theta\\ &= 1 - 2\sin^2\theta.\end{aligned}$$

The second variation is:

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta\\ &= \cos^2\theta - (1-\cos^2\theta)\\ &= 2\cos^2\theta - 1\end{aligned}$$

To derive the double-angle formula for tangent, replacing $\alpha = \beta = \theta$ in the sum formula gives

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$
$$\tan(\theta + \theta) = \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta}$$
$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}.$$

DOUBLE-ANGLE FORMULAS

The *double-angle formulas* are summarized as follows:

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$
$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$

How to: Given the tangent of an angle and the quadrant in which it is located, use the double-angle formulas to find the exact value

1. Draw a triangle to reflect the given information.
2. Determine the correct double-angle formula.
3. Substitute values into the formula based on the triangle.
4. Simplify.

Example 7.3.1: Using a Double-Angle Formula to Find the Exact Value Involving Tangent

Given that $\tan\theta = -\frac{3}{4}$ and θ is in quadrant II, find the following:

a. $\sin(2\theta)$
b. $\cos(2\theta)$
c. $\tan(2\theta)$

Solution

If we draw a triangle to reflect the information given, we can find the values needed to solve the problems on the image. We are given $\tan\theta = -\frac{3}{4}$,such that θ is in quadrant II. The tangent of an angle is equal to the opposite side over the adjacent side, and because θ is in the second quadrant, the adjacent side is on the *x*-axis and is negative. Use the **Pythagorean Theorem** to find the length of the hypotenuse:

$$(-4)^2 + (3)^2 = c^2$$
$$16 + 9 = c^2$$
$$25 = c^2$$
$$c = 5$$

Now we can draw a triangle similar to the one shown in Figure **7.3.2**.

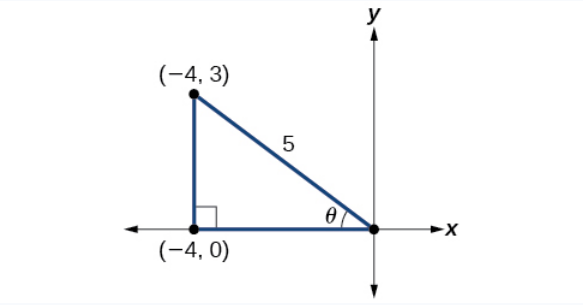


***Figure* 7.3.2**

a. Let's begin by writing the double-angle formula for sine.

$$\sin(2\theta) = 2 \sin \theta \cos \theta$$

We see that we to need to find $\sin \theta$ and $\cos \theta$. Based on Figure 7.3.2, we see that the hypotenuse equals 5, so $\sin\theta = 35$, $\sin \theta = 35$, and $\cos\theta = -45$. Substitute these values into the equation, and simplify.

Thus,

$$\begin{aligned} \sin(2\theta) &= 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right) \\ &= -\frac{24}{25} \end{aligned}$$

b. Write the double-angle formula for cosine.

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

Again, substitute the values of the sine and cosine into the equation, and simplify.

$$\begin{aligned} \cos(2\theta) &= \left(-\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 \\ &= \frac{16}{25} - \frac{9}{25} \\ &= \frac{7}{25} \end{aligned}$$

c. Write the double-angle formula for tangent.

$$\tan(2\theta) = \frac{2 \tan \theta}{1 - \tan^2\theta}$$

In this formula, we need the tangent, which we were given as $\tan \theta = -\frac{3}{4}$. Substitute this value into the equation, and simplify.

$$\begin{aligned} \tan(2\theta) &= \frac{2\left(-\frac{3}{4}\right)}{1 - \left(-\frac{3}{4}\right)^2} \\ &= \frac{-\frac{3}{2}}{1 - \frac{9}{16}} \\ &= -\frac{3}{2}\left(\frac{16}{7}\right) \\ &= -\frac{24}{7} \end{aligned}$$

Exercise 7.3.1

Given $\sin \alpha = \frac{5}{8}$,with θ in quadrant I, find $\cos(2\alpha)$.

Answer

$$\cos(2\alpha) = \frac{7}{32}$$

Example 7.3.2: Using the Double-Angle Formula for Cosine without Exact Values

Use the double-angle formula for cosine to write $\cos(6x)$ in terms of $cos(3x)$.

Solution

$$\begin{aligned} \cos(6x) &= \cos(3x + 3x) \\ &= \cos 3x \cos 3x - \sin 3x \sin 3x \\ &= \cos^2 3x - \sin^2 3x \end{aligned}$$

Analysis

This example illustrates that we can use the double-angle formula without having exact values. It emphasizes that the pattern is what we need to remember and that identities are true for all values in the domain of the trigonometric function.

Using Double-Angle Formulas to Verify Identities

Establishing identities using the double-angle formulas is performed using the same steps we used to derive the sum and difference formulas. Choose the more complicated side of the equation and rewrite it until it matches the other side.

Example 7.3.3: Using the Double-Angle Formulas to Verify an Identity

Verify the following identity using double-angle formulas: $1+\sin(2\theta)=(\sin\theta+\cos\theta)^2$

Solution

We will work on the right side of the equal sign and rewrite the expression until it matches the left side.

$$\begin{aligned}(\sin\theta+\cos\theta)^2 &= \sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta\\ &= (\sin^2\theta+\cos^2\theta)+2\sin\theta\cos\theta\\ &= 1+2\sin\theta\cos\theta\\ &= 1+\sin(2\theta)\end{aligned}$$

Analysis

This process is not complicated, as long as we recall the perfect square formula from algebra:

$$(a\pm b)^2=a^2\pm 2ab+b^2$$

where $a=\sin\theta$ and $b=\cos\theta$. Part of being successful in mathematics is the ability to recognize patterns. While the terms or symbols may change, the algebra remains consistent.

Exercise 7.3.2

Verify the identity: $\cos^4\theta-\sin^4\theta=\cos(2\theta)$.

Answer

$\cos^4\theta-\sin^4\theta=(\cos^2\theta+\sin^2\theta)(\cos^2\theta-\sin^2\theta)=\cos(2\theta)$

Example 7.3.4: Verifying a Double-Angle Identity for Tangent

Verify the identity: $\tan(2\theta)=2\cot\theta-\tan\theta$

Solution

In this case, we will work with the left side of the equation and simplify or rewrite until it equals the right side of the equation.

$$\begin{aligned}\tan(2\theta) &= \frac{2\tan\theta}{1-\tan^2\theta} && \text{Double-angle formula}\\ &= \frac{2\tan\theta\left(\frac{1}{\tan\theta}\right)}{(1-\tan^2\theta)\left(\frac{1}{\tan\theta}\right)} && \text{Multiply by a term that results in desired numerator}\\ &= \frac{2}{\frac{1}{\tan\theta}-\frac{\tan^2\theta}{\tan\theta}}\\ &= \frac{2}{\cot\theta-\tan\theta} && \text{Use reciprocal identity for } \frac{1}{\tan\theta}\end{aligned}$$

Analysis

Here is a case where the more complicated side of the initial equation appeared on the right, but we chose to work the left side. However, if we had chosen the left side to rewrite, we would have been working backwards to arrive at the equivalency. For example, suppose that we wanted to show

$$\frac{2\tan\theta}{1-\tan^2\theta} = \frac{2}{\cot\theta - \tan\theta}$$

Lets work on the right side

$$\begin{aligned}\frac{2}{\cot\theta - \tan\theta} &= \frac{2}{\frac{1}{\tan\theta} - \tan\theta}\left(\frac{\tan\theta}{\tan\theta}\right)\\ &= \frac{2\tan\theta}{\frac{1}{\tan\theta}(\tan\theta) - \tan\theta(\tan\theta)}\\ &= \frac{2\tan\theta}{1-\tan^2\theta}\end{aligned}$$

When using the identities to simplify a trigonometric expression or solve a trigonometric equation, there are usually several paths to a desired result. There is no set rule as to what side should be manipulated. However, we should begin with the guidelines set forth earlier.

Exercise 7.3.3

Verify the identity: $\cos(2\theta)\cos\theta = \cos^3\theta - \cos\theta\sin^2\theta$.

Answer

$\cos(2\theta)\cos\ \theta = (\cos^2\theta - \sin^2\theta)\cos\ \theta = \cos^3\theta - \cos\ \theta\sin^2\theta$

Use Reduction Formulas to Simplify an Expression

The double-angle formulas can be used to derive the **reduction formulas,** which are formulas we can use to reduce the power of a given expression involving even powers of sine or cosine. They allow us to rewrite the even powers of sine or cosine in terms of the first power of cosine. These formulas are especially important in higher-level math courses, calculus in particular. Also called the power-reducing formulas, three identities are included and are easily derived from the double-angle formulas.

We can use two of the three double-angle formulas for cosine to derive the reduction formulas for sine and cosine. Let's begin with $\cos(2\theta) = 1 - 2\sin^2\theta$. Solve for $\sin^2\theta$:

$$\begin{aligned}\cos(2\theta) &= 1 - 2\sin^2\theta\\ 2\sin^2\theta &= 1 - \cos(2\theta)\\ \sin^2\theta &= \frac{1-\cos(2\theta)}{2}\end{aligned}$$

Next, we use the formula $\cos(2\theta) = 2\cos^2\theta - 1$. Solve for $\cos^2\theta$:

$$\begin{aligned}\cos(2\theta) &= 2\cos^2\theta - 1\\ 1 + \cos(2\theta) &= 2\cos^2\theta\\ \frac{1+\cos(2\theta)}{2} &= \cos^2\theta\end{aligned}$$

The last reduction formula is derived by writing tangent in terms of sine and cosine:

$$\begin{aligned}\tan^2\theta &= \frac{\sin^2\theta}{\cos^2\theta}\\ &= \frac{\frac{1-\cos(2\theta)}{2}}{\frac{1+\cos(2\theta)}{2}} && \text{(Substitute the reduction formulas)}\\ &= \left(\frac{1-\cos(2\theta)}{2}\right)\left(\frac{2}{1+\cos(2\theta)}\right)\\ &= \frac{1-\cos(2\theta)}{1+\cos(2\theta)}\end{aligned}$$

REDUCTION FORMULAS

The reduction formulas are summarized as follows:

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2} \tag{7.3.1}$$

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2} \tag{7.3.2}$$

$$\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)} \tag{7.3.3}$$

Example 7.3.5: Writing an Equivalent Expression Not Containing Powers Greater Than 1

Write an equivalent expression for $\cos^4 x$ that does not involve any powers of sine or cosine greater than 1.

Solution

We will apply the reduction formula for cosine twice.

$$\begin{aligned}
\cos^4 x &= (\cos^2 x)^2 \\
&= \left(\frac{1+\cos(2x)}{2}\right)^2 && \text{Substitute reduction formula} \\
&= \frac{1}{4}(1+2\cos(2x)+\cos^2(2x)) \\
&= \frac{1}{4}+\frac{1}{2}\cos(2x)+\frac{1}{4}\left(\frac{1+\cos^2(2x)}{2}\right) && \text{Substitute reduction formula for } \cos^2 x \\
&= \frac{1}{4}+\frac{1}{2}\cos(2x)+\frac{1}{8}+\frac{1}{8}\cos(4x) \\
&= \frac{3}{8}+\frac{1}{2}\cos(2x)+\frac{1}{8}\cos(4x)
\end{aligned}$$

Analysis

The solution is found by using the reduction formula twice, as noted, and the perfect square formula from algebra.

Example 7.3.6: Using the Power-Reducing Formulas to Prove an Identity

Use the power-reducing formulas to prove $\sin^3(2x) = \left[\frac{1}{2}\sin(2x)\right][1-\cos(4x)]$

Solution

We will work on simplifying the left side of the equation:

$$\begin{aligned}
\sin^3(2x) &= [\sin(2x)][\sin^2(2x)] \\
&= \sin(2x)\left[\frac{1-\cos(4x)}{2}\right] && \text{Substitute the power-reduction formula.} \\
&= \sin(2x)\left(\frac{1}{2}\right)[1-\cos(4x)] \\
&= \frac{1}{2}[\sin(2x)][1-\cos(4x)]
\end{aligned}$$

Analysis

Note that in this example, we substituted $\frac{1-\cos(4x)}{2}$ for $\sin^2(2x)$. The formula states $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$

We let $\theta = 2x$, so $2\theta = 4x$.

Exercise 7.3.4

Use the power-reducing formulas to prove that $10\cos^4 x = \frac{15}{4} + 5\cos(2x) + \frac{5}{4}\cos(4x)$.

Answer

$$\begin{aligned}
10\cos^4 x &= 10(\cos^2 x)^2 \\
&= 10\left[\frac{1+\cos(2x)}{2}\right]^2 \qquad \text{Substitute reduction formula for } \cos^2 x \\
&= \frac{10}{4}[1+2\cos(2x)+\cos^2(2x)] \\
&= \frac{10}{4}+\frac{10}{2}\cos(2x)+\frac{10}{4}\left(\frac{1+\cos^2(2x)}{2}\right) \qquad \text{Substitute reduction formula for } \cos^2 x \\
&= \frac{10}{4}+\frac{10}{2}\cos(2x)+\frac{10}{8}+\frac{10}{8}\cos(4x) \\
&= \frac{30}{8}+5\cos(2x)+\frac{10}{8}\cos(4x) \\
&= \frac{15}{4}+5\cos(2x)+\frac{5}{4}\cos(4x)
\end{aligned}$$

Using Half-Angle Formulas to Find Exact Values

The next set of identities is the set of **half-angle formulas**, which can be derived from the reduction formulas and we can use when we have an angle that is half the size of a special angle. If we replace θ with $\frac{\alpha}{2}$,the half-angle formula for sine is found by simplifying the equation and solving for $\sin\left(\frac{\alpha}{2}\right)$. Note that the half-angle formulas are preceded by a $\pm$ sign. This does not mean that both the positive and negative expressions are valid. Rather, it depends on the quadrant in which $\frac{\alpha}{2}$ terminates.

The half-angle formula for sine is derived as follows:

$$\begin{aligned}
\sin^2\theta &= \frac{1-\cos(2\theta)}{2} \\
\sin^2\left(\frac{\alpha}{2}\right) &= \frac{1-\left(\cos 2\cdot\frac{\alpha}{2}\right)}{2} \\
&= \frac{1-\cos\alpha}{2} \\
\sin\left(\frac{\alpha}{2}\right) &= \pm\sqrt{\frac{1-\cos\alpha}{2}}
\end{aligned}$$

To derive the half-angle formula for cosine, we have

$$\begin{aligned}
\cos^2\theta &= \frac{1+\cos(2\theta)}{2} \\
\cos^2\left(\frac{\alpha}{2}\right) &= \frac{1+\cos\left(2\cdot\frac{\alpha}{2}\right)}{2} \\
&= \frac{1+\cos\alpha}{2} \\
\cos\left(\frac{\pi}{2}\right) &= \pm\sqrt{\frac{1+\cos\alpha}{2}}
\end{aligned}$$

For the tangent identity, we have

$$\begin{aligned}
\tan^2\theta &= \frac{1-\cos(2\theta)}{1+\cos(2\theta)} \\
\tan^2\left(\frac{\alpha}{2}\right) &= \frac{1-\cos\left(2\cdot\frac{\alpha}{2}\right)}{1+\cos\left(2\cdot\frac{\alpha}{2}\right)} \\
\tan\left(\frac{\alpha}{2}\right) &= \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}
\end{aligned}$$

HALF-ANGLE FORMULAS

The **half-angle formulas** are as follows:

$$\sin\left(\frac{\alpha}{2}\right)=\pm\sqrt{\frac{1-\cos\alpha}{2}} \tag{7.3.4}$$

$$\cos\left(\frac{\alpha}{2}\right)=\pm\sqrt{\frac{1+\cos\alpha}{2}} \tag{7.3.5}$$

$$\tan\left(\frac{\alpha}{2}\right)=\pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}=\frac{\sin\alpha}{1+\cos\alpha}=\frac{1-\cos\alpha}{\sin\alpha} \tag{7.3.6}$$

Example 7.3.7

Using a Half-Angle Formula to Find the Exact Value of a Sine Function. Find $\sin(15°)$ using a half-angle formula.

Solution

Since $15°=\frac{30°}{2}$,we use the half-angle formula for sine (Equation 7.3.4):

$$\begin{aligned}\sin\frac{30°}{2}&=\sqrt{\frac{1-\cos 30°}{2}}\\&=\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}\\&=\sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}}\\&=\sqrt{\frac{2-\sqrt{3}}{4}}\\&=\frac{\sqrt{2-\sqrt{3}}}{2}\end{aligned}$$

Remember that we can check the answer with a graphing calculator.

Analysis

Notice that we used only the positive root because $\sin(15°)$ is positive.

Howto: Given the tangent of an angle and the quadrant in which the angle lies, find the exact values of trigonometric functions of half of the angle.

1. Draw a triangle to represent the given information.
2. Determine the correct half-angle formula.
3. Substitute values into the formula based on the triangle.
4. Simplify.

Example 7.3.8: Finding Exact Values Using Half-Angle Identities

Given that $\tan\alpha=\frac{8}{15}$ and α lies in quadrant III, find the exact value of the following:

a. $\sin\left(\frac{\alpha}{2}\right)$
b. $\cos\left(\frac{\alpha}{2}\right)$
c. $\tan\left(\frac{\alpha}{2}\right)$

Solution

Using the given information, we can draw the triangle shown in Figure 7.3.3. Using the Pythagorean Theorem, we find the hypotenuse to be 17. Therefore, we can calculate $\sin\alpha=-\frac{8}{17}$ and $\cos\alpha=-\frac{15}{17}$.

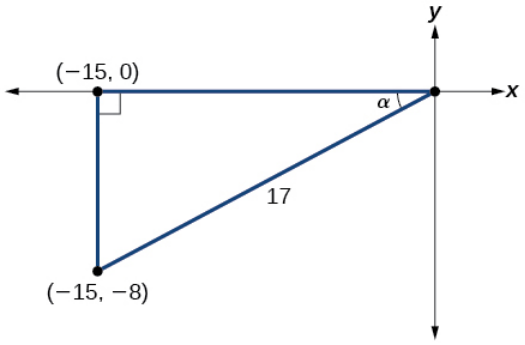


***Figure* 7.3.3**

a. Before we start, we must remember that if α is in quadrant III, then $180° < \alpha < 270°$, so $\frac{180°}{2} < \frac{\alpha}{2} < \frac{270°}{2}$. This means that the terminal side of $\frac{\alpha}{2}$ is in quadrant II, since $90° < \frac{\alpha}{2} < 135°$. To find $\sin\frac{\alpha}{2}$, we begin by writing the half-angle formula for sine. Then we substitute the value of the cosine we found from the triangle in Figure **7.3.3** and simplify.

$$\begin{aligned}\sin\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{2}}\\ &= \pm\sqrt{\frac{1-\left(-\frac{15}{17}\right)}{2}}\\ &= \pm\sqrt{\frac{\frac{32}{17}}{2}}\\ &= \pm\sqrt{\frac{32}{17}\cdot\frac{1}{2}}\\ &= \pm\sqrt{\frac{16}{17}}\\ &= \pm\frac{4}{\sqrt{17}}\\ &= \frac{4\sqrt{17}}{17}\end{aligned}$$

We choose the positive value of $\sin\frac{\alpha}{2}$ because the angle terminates in quadrant II and sine is positive in quadrant II.

b. To find $\cos\frac{\alpha}{2}$, we will write the half-angle formula for cosine, substitute the value of the cosine we found from the triangle in Figure **7.3.3**, and simplify.

$$\begin{aligned}\cos\frac{\alpha}{2} &= \pm\sqrt{\frac{1+\cos\alpha}{2}}\\ &= \pm\sqrt{\frac{1+\left(-\frac{15}{17}\right)}{2}}\\ &= \pm\sqrt{\frac{\frac{2}{17}}{2}}\\ &= \pm\sqrt{\frac{2}{17}\cdot\frac{1}{2}}\\ &= \pm\sqrt{\frac{1}{17}}\\ &= -\frac{\sqrt{17}}{17}\end{aligned}$$

We choose the negative value of $\cos\frac{\alpha}{2}$ because the angle is in quadrant II because cosine is negative in quadrant II.

c. To find $\tan\frac{\alpha}{2}$, we write the half-angle formula for tangent. Again, we substitute the value of the cosine we found from the triangle in Figure **7.3.3** and simplify.

$$\begin{aligned}\tan\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}\\ &= \pm\sqrt{\frac{1-\left(-\frac{15}{17}\right)}{1+\left(-\frac{15}{17}\right)}}\\ &= \pm\sqrt{\frac{\frac{32}{17}}{\frac{2}{17}}}\\ &= \pm\sqrt{\frac{32}{2}}\\ &= -\sqrt{16}\\ &= -4\end{aligned}$$

We choose the negative value of $\tan\frac{\alpha}{2}$ because $\frac{\alpha}{2}$ lies in quadrant II, and tangent is negative in quadrant II.

Exercise 7.3.5

Given that $\sin\alpha = -\frac{4}{5}$ and α lies in quadrant IV, find the exact value of $\cos\left(\frac{\alpha}{2}\right)$.

Answer

$-\frac{2}{\sqrt{5}}$

Example 7.3.9: Finding the Measurement of a Half Angle

Now, we will return to the problem posed at the beginning of the section. A bicycle ramp is constructed for high-level competition with an angle of θ formed by the ramp and the ground. Another ramp is to be constructed half as steep for novice competition. If $\tan\theta = 53$ for higher-level competition, what is the measurement of the angle for novice competition?

Solution

Since the angle for novice competition measures half the steepness of the angle for the high level competition, and $\tan\theta = \frac{5}{3}$ for high competition, we can find $\cos\theta$ from the right triangle and the Pythagorean theorem so that we can use the half-angle identities. See Figure 7.3.4.

$$\begin{aligned}3^2 + 5^2 &= 34\\ c &= \sqrt{34}\end{aligned}$$

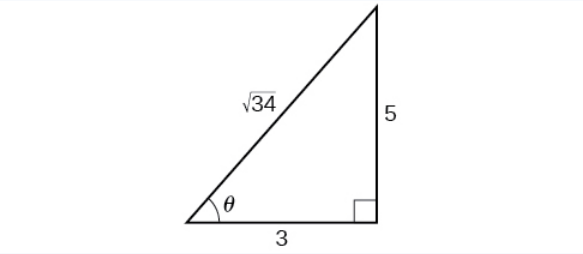


***Figure* 7.3.4**

We see that $\cos\theta = \frac{3}{\sqrt{34}} = \frac{3\sqrt{34}}{34}$. We can use the half-angle formula for tangent: $\tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$. Since $\tan\theta$ is in the first quadrant, so is $\tan\frac{\theta}{2}$.

$$\tan\frac{\theta}{2} = \sqrt{\frac{1-\frac{3\sqrt{34}}{34}}{1+\frac{3\sqrt{34}}{34}}}$$
$$= \sqrt{\frac{\frac{34-3\sqrt{34}}{34}}{\frac{34+3\sqrt{34}}{34}}}$$
$$= \sqrt{\frac{34-3\sqrt{34}}{34+3\sqrt{34}}}$$
$$\approx 0.57$$

We can take the inverse tangent to find the angle: $\tan^{-1}(0.57) \approx 29.7°$. So the angle of the ramp for novice competition is $\approx 29.7°$.

Media

Access these online resources for additional instruction and practice with double-angle, half-angle, and reduction formulas.

- Double-Angle Identities
- Half-Angle Identities

Key Equations

Double-angle formulas	$\sin(2\theta) = 2\sin\theta\cos\theta$ $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ $= 1 - 2\sin^2\theta$ $= 2\cos^2\theta - 1$ $\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$
Reduction formulas	$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$ $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$ $\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$
Half-angle formulas	$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$ $\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$ $\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$ $= \frac{\sin\alpha}{1+\cos\alpha}$ $= \frac{1-\cos\alpha}{\sin\alpha}$

Key Concepts

- Double-angle identities are derived from the sum formulas of the fundamental trigonometric functions: sine, cosine, and tangent. See Example **7.3.1**, Example **7.3.2**, Example **7.3.3**, and Example **7.3.4**.
- Reduction formulas are especially useful in calculus, as they allow us to reduce the power of the trigonometric term. See Example **7.3.5** and Example **7.3.6**.
- Half-angle formulas allow us to find the value of trigonometric functions involving half-angles, whether the original angle is known or not. See Example **7.3.7**, Example **7.3.8**, and Example **7.3.9**.

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7.3E: Exercises

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7.5: Solving Trigonometric Equations

Skills to Develop

- Solve linear trigonometric equations in sine and cosine.
- Solve equations involving a single trigonometric function.
- Solve trigonometric equations using a calculator.
- Solve trigonometric equations that are quadratic in form.
- Solve trigonometric equations using fundamental identities.
- Solve trigonometric equations with multiple angles.
- Solve right triangle problems.

Thales of Miletus (circa 625–547 BC) is known as the founder of geometry. The legend is that he calculated the height of the Great Pyramid of Giza in Egypt using the theory of *similar triangles*, which he developed by measuring the shadow of his staff. Based on proportions, this theory has applications in a number of areas, including fractal geometry, engineering, and architecture. Often, the angle of elevation and the angle of depression are found using similar triangles.



***Figure* 7.5.1:** *Egyptian pyramids standing near a modern city. (credit: Oisin Mulvihill)*

In earlier sections of this chapter, we looked at trigonometric identities. Identities are true for all values in the domain of the variable. In this section, we begin our study of trigonometric equations to study real-world scenarios such as the finding the dimensions of the pyramids.

Solving Linear Trigonometric Equations in Sine and Cosine

Trigonometric equations are, as the name implies, equations that involve trigonometric functions. Similar in many ways to solving polynomial equations or rational equations, only specific values of the variable will be solutions, if there are solutions at all. Often we will solve a trigonometric equation over a specified interval. However, just as often, we will be asked to find all possible solutions, and as trigonometric functions are periodic, solutions are repeated within each period. In other words, trigonometric equations may have an infinite number of solutions. Additionally, like rational equations, the domain of the function must be considered before we assume that any solution is valid. The **period** of both the sine function and the cosine function is 2π. In other words, every 2π units, the y-values repeat. If we need to find all possible solutions, then we must add $2\pi k$,where k is an integer, to the initial solution. Recall the rule that gives the format for stating all possible solutions for a function where the period is 2π:

$$\sin\theta = \sin(\theta \pm 2k\pi) \tag{7.5.1}$$

There are similar rules for indicating all possible solutions for the other trigonometric functions. Solving trigonometric equations requires the same techniques as solving algebraic equations. We read the equation from left to right, horizontally, like a sentence. We look for known patterns, factor, find common denominators, and substitute certain expressions with a variable to make solving a more straightforward process. However, with trigonometric equations, we also have the advantage of using the identities we developed in the previous sections.

Example 7.5.1A: Solving a Linear Trigonometric Equation Involving the Cosine Function

Find all possible exact solutions for the equation $\cos\theta = \frac{1}{2}$.

Solution

From the unit circle, we know that

$$\cos\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

These are the solutions in the interval $[0, 2\pi]$. All possible solutions are given by

$$\theta = \frac{\pi}{3} \pm 2k\pi \quad \text{and} \quad \theta = \frac{5\pi}{3} \pm 2k\pi$$

where k is an integer.

Example 7.5.1B: Solving a Linear Equation Involving the Sine Function

Find all possible exact solutions for the equation $\sin t = \frac{1}{2}$.

Solution

Solving for all possible values of t means that solutions include angles beyond the period of 2π. From the section on Sum and Difference Identities, we can see that the solutions are $t = \frac{\pi}{6}$ and $t = \frac{5\pi}{6}$. But the problem is asking for all possible values that solve the equation. Therefore, the answer is

$$t = \frac{\pi}{6} \pm 2\pi k \quad \text{and} \quad t = \frac{5\pi}{6} \pm 2\pi k$$

where k is an integer.

Howto: Given a trigonometric equation, solve using algebra

1. Look for a pattern that suggests an algebraic property, such as the difference of squares or a factoring opportunity.
2. Substitute the trigonometric expression with a single variable, such as x or u.
3. Solve the equation the same way an algebraic equation would be solved.
4. Substitute the trigonometric expression back in for the variable in the resulting expressions.
5. Solve for the angle.

Example 7.5.2: Solve the Linear Trigonometric Equation

Solve the equation exactly: $2\cos\theta - 3 = -5$, $0 \le \theta < 2\pi$.

Solution

Use algebraic techniques to solve the equation.

$$\begin{aligned} 2\cos\theta - 3 &= -5 \\ 2\cos\theta &= -2 \\ \cos\theta &= -1 \\ \theta &= \pi \end{aligned}$$

Exercise 7.5.2

Solve exactly the following linear equation on the interval $[0, 2\pi)$: $2\sin x + 1 = 0$.

Answer

$x = \frac{7\pi}{6}, \frac{11\pi}{6}$

Solving Equations Involving a Single Trigonometric Function

When we are given equations that involve only one of the six trigonometric functions, their solutions involve using algebraic techniques and the unit circle (see [link]). We need to make several considerations when the equation involves trigonometric functions other than sine and cosine. Problems involving the reciprocals of the primary trigonometric functions need to be viewed from an algebraic perspective. In other words, we will write the reciprocal function, and solve for the angles using the function. Also, an equation involving the tangent function is slightly different from one containing a sine or cosine function. First, as we know, the period of tangent is π,not 2π. Further, the domain of tangent is all real numbers with the exception of odd integer multiples of $\frac{\pi}{2}$,unless, of course, a problem places its own restrictions on the domain.

Example $7.5.3A$: Solving a Problem Involving a Single Trigonometric Function

Solve the problem exactly: $2\sin^2\theta - 1 = 0, 0 \le \theta < 2\pi$.

Solution

As this problem is not easily factored, we will solve using the square root property. First, we use algebra to isolate $\sin\theta$. Then we will find the angles.

$$\begin{aligned} 2\sin^2\theta - 1 &= 0 \\ 2\sin^2\theta &= 1 \\ \sin^2\theta &= \frac{1}{2} \\ \sqrt{\sin^2\theta} &= \pm\sqrt{\frac{1}{2}} \\ \sin\theta &= \pm\frac{1}{\sqrt{2}} \\ &= \pm\frac{\sqrt{2}}{2} \\ \theta &= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \end{aligned}$$

Example $7.5.3B$: Solving a Trigonometric Equation Involving Cosecant

Solve the following equation exactly: $\csc\theta = -2, 0 \le \theta < 4\pi$.

Solution

We want all values of θ for which $\csc\theta = -2$ over the interval $0 \le \theta < 4\pi$.

$$\begin{aligned} \csc\theta &= -2 \\ \frac{1}{\sin\theta} &= -2 \\ \sin\theta &= -\frac{1}{2} \\ \theta &= \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6} \end{aligned}$$

Analysis

As $\sin\theta = -\frac{1}{2}$, notice that all four solutions are in the third and fourth quadrants.

Example $7.5.3C$: Solving an Equation Involving Tangent

Solve the equation exactly: $\tan\left(\theta - \frac{\pi}{2}\right) = 1, 0 \le \theta < 2\pi$.

Solution

Recall that the tangent function has a period of π. On the interval $[0, \pi)$,and at the angle of $\frac{\pi}{4}$,the tangent has a value of 1. However, the angle we want is $\left(\theta - \frac{\pi}{2}\right)$. Thus, if $\tan\left(\frac{\pi}{4}\right) = 1$,then

$$\theta - \frac{\pi}{2} = \frac{\pi}{4}$$
$$\theta = \frac{3\pi}{4} \pm k\pi$$

Over the interval $[0, 2\pi)$,we have two solutions:

$\theta = \frac{3\pi}{4}$ and $\theta = \frac{3\pi}{4} + \pi = \frac{7\pi}{4}$

Exercise 7.5.3

Find all solutions for $\tan x = \sqrt{3}$.

Answer

$\frac{\pi}{3} \pm \pi k$

Example 7.5.4: Identify all Solutions to the Equation Involving Tangent

Identify all exact solutions to the equation $2(\tan x + 3) = 5 + \tan x$, $0 \le x < 2\pi$.

Solution

We can solve this equation using only algebra. Isolate the expression $\tan x$ on the left side of the equals sign.

$$\begin{aligned} 2(\tan x) + 2(3) &= 5 + \tan x \\ 2\tan x + 6 &= 5 + \tan x \\ 2\tan x - \tan x &= 5 - 6 \\ \tan x &= -1 \end{aligned}$$

There are two angles on the unit circle that have a tangent value of -1: $\theta = \frac{3\pi}{4}$ and $\theta = \frac{7\pi}{4}$.

Solve Trigonometric Equations Using a Calculator

Not all functions can be solved exactly using only the unit circle. When we must solve an equation involving an angle other than one of the special angles, we will need to use a calculator. Make sure it is set to the proper mode, either degrees or radians, depending on the criteria of the given problem.

Example 7.5.5A: Using a Calculator to Solve a Trigonometric Equation Involving Sine

Use a calculator to solve the equation $\sin\theta = 0.8$,where θ is in radians.

Solution

Make sure mode is set to radians. To find θ, use the inverse sine function. On most calculators, you will need to push the 2ND button and then the SIN button to bring up the $\sin^{-1}$ function. What is shown on the screen is $\sin^{-1}$ .The calculator is ready for the input within the parentheses. For this problem, we enter $\sin^{-1}(0.8)$, and press ENTER. Thus, to four decimals places,

$\sin^{-1}(0.8) \approx 0.9273$

The solution is

$\theta \approx 0.9273 \pm 2\pi k$

The angle measurement in degrees is

$$\begin{aligned} \theta &\approx 53.1^\circ \\ \theta &\approx 180^\circ - 53.1^\circ \\ &\approx 126.9^\circ \end{aligned}$$

Analysis

Note that a calculator will only return an angle in quadrants I or IV for the sine function, since that is the range of the inverse sine. The other angle is obtained by using $\pi - \theta$.

Example 7.5.5B: Using a Calculator to Solve a Trigonometric Equation Involving Secant

Use a calculator to solve the equation $\sec\theta = -4$, giving your answer in radians.

Solution

We can begin with some algebra.

$$\sec\theta = -4$$
$$\frac{1}{\cos\theta} = -4$$
$$\cos\theta = -\frac{1}{4}$$

Check that the MODE is in radians. Now use the inverse cosine function

$$\cos^{-1}\left(-\frac{1}{4}\right) \approx 1.8235$$
$$\theta \approx 1.8235 + 2\pi k$$

Since $\frac{\pi}{2} \approx 1.57$ and $\pi \approx 3.14$,1.8235 is between these two numbers, thus $\theta \approx 1.8235$ is in quadrant II. Cosine is also negative in quadrant III. Note that a calculator will only return an angle in quadrants I or II for the cosine function, since that is the range of the inverse cosine. See Figure 7.5.2.



***Figure* 7.5.2**

So, we also need to find the measure of the angle in quadrant III. In quadrant III, the reference angle is $\theta' \approx \pi - 1.8235 \approx 1.3181$. The other solution in quadrant III is $\theta' \approx \pi + 1.3181 \approx 4.4597$.

The solutions are $\theta \approx 1.8235 \pm 2\pi k$ and $\theta \approx 4.4597 \pm 2\pi k$.

Exercise 7.5.5

Solve $\cos\theta = -0.2$.

Answer

$\theta \approx 1.7722 \pm 2\pi k$ and $\theta \approx 4.5110 \pm 2\pi k$

Solving Trigonometric Equations in Quadratic Form

Solving a *quadratic equation* may be more complicated, but once again, we can use algebra as we would for any quadratic equation. Look at the pattern of the equation. Is there more than one trigonometric function in the equation, or is there only one? Which trigonometric function is squared? If there is only one function represented and one of the terms is squared, think about the standard form of a quadratic. Replace the trigonometric function with a variable such as x or u. If substitution makes the equation look like a quadratic equation, then we can use the same methods for solving quadratics to solve the trigonometric equations.

For example, the equation $(\sin x + 1)(\sin x - 1) = 0$ resembles the equation $(x + 1)(x - 1) = 0$,which uses the factored form of the difference of squares. Using algebra makes finding a solution straightforward and familiar. We can set each factor equal to zero and solve. This is one example of recognizing algebraic patterns in trigonometric expressions or equations.

Example 7.5.7A: Writing the Trigonometric Expression as an Algebraic Expression

Write the following trigonometric expression as an algebraic expression: $2\cos^2\theta + \cos\theta - 1$.

Solution

Notice that the pattern displayed has the same form as a standard quadratic expression,$ax^2 + bx + c$. Letting $\cos\theta = x$,we can rewrite the expression as follows:

$$2x^2 + x - 1$$

This expression can be factored as $(2x+1)(x-1)$. If it were set equal to zero and we wanted to solve the equation, we would use the zero factor property and solve each factor for x. At this point, we would replace x with $\cos\theta$ and solve for θ.

Example 7.5.7B: Rewriting a Trigonometric Expression Using the Difference of Squares

Rewrite the trigonometric expression using the difference of squares: $4cos^2\theta - 1$.

Solution

Notice that both the coefficient and the trigonometric expression in the first term are squared, and the square of the number 1 is 1. This is the difference of squares.

$$\begin{aligned} 4\cos^2\theta - 1 &= (2\cos\theta)^2 - 1 \\ &= (2\cos\theta - 1)(2\cos\theta + 1) \end{aligned}$$

Analysis

If this expression were written in the form of an equation set equal to zero, we could solve each factor using the zero factor property. We could also use substitution like we did in the previous problem and let $\cos\theta = x$, rewrite the expression as $4x^2 - 1$, and factor $(2x-1)(2x+1)$. Then replace x with $\cos\theta$ and solve for the angle.

Exercise 7.5.4

Rewrite the trigonometric expression using the difference of squares: $25 - 9\sin^2\theta$.

Answer

This is a difference of squares formula: $25 - 9\sin^2\theta = (5 - 3\sin\theta)(5 + 3\sin\theta)$.

Example 7.5.6A: Solving a Trigonometric Equation in Quadratic Form

Solve the equation exactly: $\cos^2\theta + 3\cos\theta - 1 = 0,\ 0 \le \theta < 2\pi$.

Solution

We begin by using substitution and replacing $\cos\theta$ with x. It is not necessary to use substitution, but it may make the problem easier to solve visually. Let $\cos\theta = x$. We have

$$x^2 + 3x - 1 = 0$$

The equation cannot be factored, so we will use the **quadratic formula:** $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$$\begin{aligned} x &= \frac{-3 \pm \sqrt{(-3)^2 - 4(1)(-1)}}{2} \\ &= \frac{-3 \pm \sqrt{13}}{2} \end{aligned}$$

Replace x with $\cos\theta$ and solve.

$$\begin{aligned} \cos\theta &= \frac{-3 \pm \sqrt{13}}{2} \\ \theta &= \cos^{-1}\left(\frac{-3 + \sqrt{13}}{2}\right) \end{aligned}$$

Note that only the + sign is used. This is because we get an error when we solve $\theta = \cos^{-1}\left(\dfrac{-3 - \sqrt{13}}{2}\right)$ on a calculator, since the domain of the inverse cosine function is $[-1, 1]$. However, there is a second solution:

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{-3 + \sqrt{13}}{2}\right) \\ &\approx 1.26 \end{aligned}$$

This terminal side of the angle lies in quadrant I. Since cosine is also positive in quadrant IV, the second solution is

$$\theta = 2\pi - \cos^{-1}\left(\frac{-3+\sqrt{13}}{2}\right)$$
$$\approx 5.02$$

Example 7.5.6B: Solving a Trigonometric Equation in Quadratic Form by Factoring

Solve the equation exactly: $2\sin^2\theta - 5\sin\theta + 3 = 0,\ 0 \le \theta \le 2\pi$.

Solution

Using grouping, this quadratic can be factored. Either make the real substitution, $\sin\theta = u$,or imagine it, as we factor:

$$\begin{aligned} 2\sin^2\theta - 5\sin\theta + 3 &= 0 \\ (2\sin\theta - 3)(\sin\theta - 1) &= 0 \quad \text{Now set each factor equal to zero.} \\ 2\sin\theta - 3 &= 0 \\ 2\sin\theta &= 3 \\ \sin\theta &= \frac{3}{2} \\ \sin\theta - 1 &= 0 \\ \sin\theta &= 1 \end{aligned}$$

Next solve for θ: $\sin\theta \neq \frac{3}{2}$, as the range of the sine function is $[-1, 1]$. However, $\sin\theta = 1$, giving the solution $\theta = \frac{\pi}{2}$.

Analysis

Make sure to check all solutions on the given domain as some factors have no solution.

Exercise 7.5.6

Solve $\sin^2\theta = 2\cos\theta + 2,\ 0 \le \theta \le 2\pi$. [Hint: Make a substitution to express the equation only in terms of cosine.]

Answer

$\cos\theta = -1,\ \theta = \pi$

Example 7.5.7A: Solving a Trigonometric Equation Using Algebra

Solve exactly: $2\sin^2\theta + \sin\theta = 0;\ 0 \le \theta < 2\pi$

Solution

This problem should appear familiar as it is similar to a quadratic. Let $\sin\theta = x$. The equation becomes $2x^2 + x = 0$. We begin by factoring:

$$\begin{aligned} 2x^2 + x &= 0 \\ x(2x+1) &= 0 \quad \text{Set each factor equal to zero.} \\ x &= 0 \\ 2x + 1 &= 0 \\ x &= -\frac{1}{2} \end{aligned}$$

Then, substitute back into the equation the original expression $\sin\theta$ for x . Thus,

$$\begin{aligned} \sin\theta &= 0 \\ \theta &= 0, \pi \\ \sin\theta &= -\frac{1}{2} \\ \theta &= \frac{7\pi}{6}, \frac{11\pi}{6} \end{aligned}$$

The solutions within the domain $0 \le \theta < 2\pi$ are $\theta = 0, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}$.

If we prefer not to substitute, we can solve the equation by following the same pattern of factoring and setting each factor equal to zero.

$$\begin{aligned} \sin^2\theta + \sin\theta &= 0 \\ \sin\theta(2\sin\theta + 1) &= 0 \\ \sin\theta &= 0 \\ \theta &= 0, \pi \\ 2\sin\theta + 1 &= 0 \\ 2\sin\theta &= -1 \\ \sin\theta &= -\frac{1}{2} \\ \theta &= \frac{7\pi}{6}, \frac{11\pi}{6} \end{aligned}$$

Analysis

We can see the solutions on the graph in Figure 7.5.3. On the interval $0 \le \theta < 2\pi$,the graph crosses the x-axis four times, at the solutions noted. Notice that trigonometric equations that are in quadratic form can yield up to four solutions instead of the expected two that are found with quadratic equations. In this example, each solution (angle) corresponding to a positive sine value will yield two angles that would result in that value.

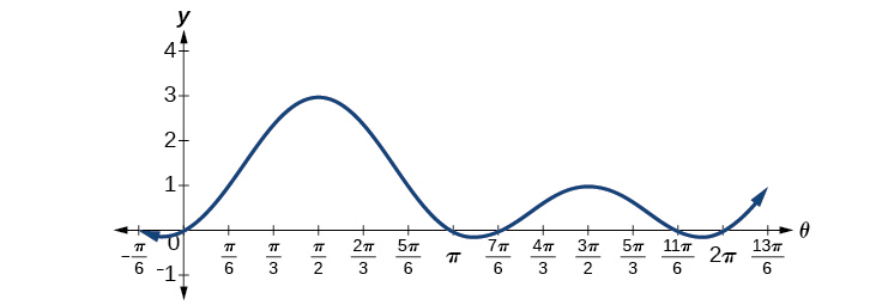


***Figure* 7.5.3**

We can verify the solutions on the unit circle in via the result in the section on Sum and Difference Identities as well.

Example 7.5.7*B*: Solving a Trigonometric Equation Quadratic in Form

Solve the equation quadratic in form exactly: $2\sin^2\theta - 3\sin\theta + 1 = 0,\ 0 \le \theta < 2\pi$.

Solution

We can factor using grouping. Solution values of θ can be found on the unit circle.

$$\begin{aligned} (2\sin\theta - 1)(\sin\theta - 1) &= 0 \\ 2\sin\theta - 1 &= 0 \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6}, \frac{5\pi}{6} \\ \sin\theta &= 1 \\ \theta &= \frac{\pi}{2} \end{aligned}$$

Exercise 7.5.7

Solve the quadratic equation $2\cos^2\theta + \cos\theta = 0$.

Answer

$\frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{3\pi}{2}$

Solving Trigonometric Equations Using Fundamental Identities

While algebra can be used to solve a number of trigonometric equations, we can also use the fundamental identities because they make solving equations simpler. Remember that the techniques we use for solving are not the same as those for verifying identities. The basic rules of algebra apply here, as opposed to rewriting one side of the identity to match the other side. In the next example, we use two identities to simplify the equation.

Example 7.5.8A: Use Identities to Solve an Equation

Use identities to solve exactly the trigonometric equation over the interval $0 \le x < 2\pi$.

$$\cos x \cos(2x) + \sin x \sin(2x) = \frac{\sqrt{3}}{2}$$

Solution

Notice that the left side of the equation is the difference formula for cosine.

$$\begin{aligned} \cos x \cos(2x) + \sin x \sin(2x) &= \frac{\sqrt{3}}{2} \\ \cos(x - 2x) &= \frac{\sqrt{3}}{2} && \text{Difference formula for cosine} \\ \cos(-x) &= \frac{\sqrt{3}}{2} && \text{Use the negative angle identity.} \\ \cos x &= \frac{\sqrt{3}}{2} \end{aligned}$$

From the unit circle in the section on Sum and Difference Identities, we see that $\cos x = \frac{\sqrt{3}}{2}$ when $x = \frac{\pi}{6}, \frac{11\pi}{6}$.

Example 7.5.8B: Solving the Equation Using a Double-Angle Formula

Solve the equation exactly using a double-angle formula: $\cos(2\theta) = \cos\theta$.

Solution

We have three choices of expressions to substitute for the double-angle of cosine. As it is simpler to solve for one trigonometric function at a time, we will choose the double-angle identity involving only cosine:

$$\begin{aligned} \cos(2\theta) &= \cos\theta \\ 2\cos^2\theta - 1 &= \cos\theta \\ 2\cos^2\theta - \cos\theta - 1 &= 0 \\ (2\cos\theta + 1)(\cos\theta - 1) &= 0 \\ 2\cos\theta + 1 &= 0 \\ \cos\theta &= -\frac{1}{2} \\ \cos\theta - 1 &= 0 \\ \cos\theta &= 1 \end{aligned}$$

So, if $\cos\theta = -\frac{1}{2}$,then $\theta = \frac{2\pi}{3} \pm 2\pi k$ and $\theta = \frac{4\pi}{3} \pm 2\pi k$; if $\cos\theta = 1$,then $\theta = 0 \pm 2\pi k$.

Example 7.5.8C: Solving an Equation Using an Identity

Solve the equation exactly using an identity: $3\cos\theta + 3 = 2\sin^2\theta$, $0 \le \theta < 2\pi$.

Solution

If we rewrite the right side, we can write the equation in terms of cosine:

$$
\begin{aligned}
3\cos\theta + 3 &= 2\sin^2\theta \\
3\cos\theta + 3 &= 2(1-\cos^2\theta) \\
3\cos\theta + 3 &= 2 - 2\cos^2\theta \\
2\cos^2\theta + 3\cos\theta + 1 &= 0 \\
(2\cos\theta + 1)(\cos\theta + 1) &= 0 \\
2\cos\theta + 1 &= 0 \\
\cos\theta &= -\frac{1}{2} \\
\theta &= \frac{2\pi}{3},\ \frac{4\pi}{3} \\
\cos\theta + 1 &= 0 \\
\cos\theta &= -1 \\
\theta &= \pi
\end{aligned}
$$

Our solutions are $\theta = \frac{2\pi}{3},\ \frac{4\pi}{3},\ \pi$.

Solving Trigonometric Equations with Multiple Angles

Sometimes it is not possible to solve a trigonometric equation with identities that have a multiple angle, such as $\sin(2x)$ or $\cos(3x)$. When confronted with these equations, recall that $y = \sin(2x)$ is a horizontal compression by a factor of 2 of the function $y = \sin x$. On an interval of 2π,we can graph two periods of $y = \sin(2x)$,as opposed to one cycle of $y = \sin x$. This compression of the graph leads us to believe there may be twice as many x-intercepts or solutions to $\sin(2x) = 0$ compared to $\sin x = 0$. This information will help us solve the equation.

Example 7.5.9: Solving a Multiple Angle Trigonometric Equation

Solve exactly: $\cos(2x) = \frac{1}{2}$ on $[0, 2\pi)$.

Solution

We can see that this equation is the standard equation with a multiple of an angle. If $\cos(\alpha) = \frac{1}{2}$,we know α is in quadrants I and IV. While $\theta = \cos^{-1}\frac{1}{2}$ will only yield solutions in quadrants I and II, we recognize that the solutions to the equation $\cos\theta = \frac{1}{2}$ will be in quadrants I and IV.

Therefore, the possible angles are $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$. So, $2x = \frac{\pi}{3}$ or $2x = \frac{5\pi}{3}$, which means that $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$. Does this make sense? Yes, because $\cos\left(2\left(\frac{\pi}{6}\right)\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$.

Are there any other possible answers? Let us return to our first step.

In quadrant I, $2x = \frac{\pi}{3}$, so $x = \frac{\pi}{6}$ as noted. Let us revolve around the circle again:

$$
\begin{aligned}
2x &= \frac{\pi}{3} + 2\pi \\
&= \frac{\pi}{3} + \frac{6\pi}{3} \\
&= \frac{7\pi}{3} \\
x &= \frac{7\pi}{6}
\end{aligned}
$$

One more rotation yields

$$
\begin{aligned}
2x &= \frac{\pi}{3} + 4\pi \\
&= \frac{\pi}{3} + \frac{12\pi}{3} \\
&= \frac{13\pi}{3}
\end{aligned}
$$

$x = \frac{13\pi}{6} > 2\pi$, so this value for x is larger than 2π, so it is not a solution on $[0, 2\pi)$.

In quadrant IV, $2x = \frac{5\pi}{3}$, so $x = \frac{5\pi}{6}$ as noted. Let us revolve around the circle again:

$$\begin{aligned} 2x &= \frac{5\pi}{3} + 2\pi \\ &= \frac{5\pi}{3} + \frac{6\pi}{3} \\ &= \frac{11\pi}{3} \end{aligned}$$

so $x = \frac{11\pi}{6}$.

One more rotation yields

$$\begin{aligned} 2x &= \frac{5\pi}{3} + 4\pi \\ &= \frac{5\pi}{3} + \frac{12\pi}{3} \\ &= \frac{17\pi}{3} \end{aligned}$$

$x = \frac{17\pi}{6} > 2\pi$,so this value for x is larger than 2π,so it is not a solution on $[0, 2\pi)$.

Our solutions are $x = \frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{7\pi}{6}$, and $\frac{11\pi}{6}$. Note that whenever we solve a problem in the form of $sin(nx) = c$, we must go around the unit circle n times.

Solving Right Triangle Problems

We can now use all of the methods we have learned to solve problems that involve applying the properties of right triangles and the Pythagorean Theorem. We begin with the familiar Pythagorean Theorem,

$$a^2 + b^2 = c^2 \tag{7.5.2}$$

and model an equation to fit a situation.

Example 7.5.10A: Using the Pythagorean Theorem to Model an Equation

One of the cables that anchors the center of the London Eye Ferris wheel to the ground must be replaced. The center of the Ferris wheel is 69.5 meters above the ground, and the second anchor on the ground is 23 meters from the base of the Ferris wheel. Approximately how long is the cable, and what is the angle of elevation (from ground up to the center of the Ferris wheel)? See Figure 7.5.4.

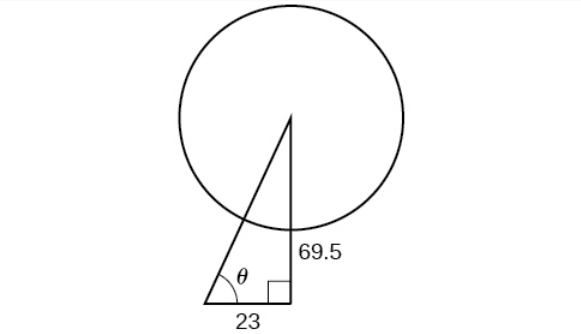


***Figure* 7.5.4**

Solution

Use the Pythagorean Theorem (Equation 7.5.2) and the properties of right triangles to model an equation that fits the problem. Using the information given, we can draw a right triangle. We can find the length of the cable with the Pythagorean Theorem.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ (23)^2 + (69.5)^2 &\approx 5359 \\ \sqrt{5359} &\approx 73.2\ m \end{aligned}$$

The angle of elevation is θ,formed by the second anchor on the ground and the cable reaching to the center of the wheel. We can use the tangent function to find its measure. Round to two decimal places.

$$\begin{aligned} \tan\theta &= 69.523 \\ \tan^{-1}(69.523) &\approx 1.2522 \\ &\approx 71.69^\circ \end{aligned}$$

The angle of elevation is approximately $71.7°$, and the length of the cable is 73.2 meters.

Example $7.5.10B$: Using the Pythagorean Theorem to Model an Abstract Problem

OSHA safety regulations require that the base of a ladder be placed 1 foot from the wall for every 4 feet of ladder length. Find the angle that a ladder of any length forms with the ground and the height at which the ladder touches the wall.

Solution

For any length of ladder, the base needs to be a distance from the wall equal to one fourth of the ladder's length. Equivalently, if the base of the ladder is "a" feet from the wall, the length of the ladder will be $4a$ feet. See Figure $7.5.5$.

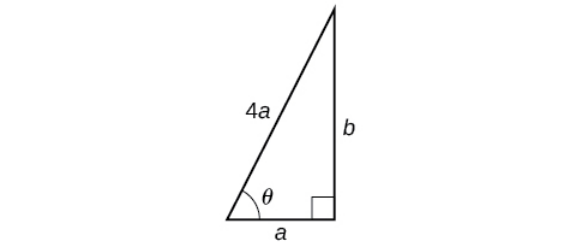


***Figure* 7.5.5**

The side adjacent to θ is a and the hypotenuse is $4a$. Thus,

$$\begin{aligned} \cos\theta &= \frac{a}{4a} \\ &= \frac{1}{4} \\ \cos^{-1}\left(\frac{1}{4}\right) &\approx 75.5^\circ \end{aligned}$$

The elevation of the ladder forms an angle of $75.5°$ with the ground. The height at which the ladder touches the wall can be found using the Pythagorean Theorem:

$$\begin{aligned} a^2 + b^2 &= (4a)^2 \\ b^2 &= (4a)^2 - a^2 \\ b^2 &= 16a^2 - a^2 \\ b^2 &= 15a^2 \\ b &= a\sqrt{15} \end{aligned}$$

Thus, the ladder touches the wall at $a\sqrt{15}$ feet from the ground.

Media

Access these online resources for additional instruction and practice with solving trigonometric equations.

- Solving Trigonometric Equations I
- Solving Trigonometric Equations II
- Solving Trigonometric Equations III
- Solving Trigonometric Equations IV
- Solving Trigonometric Equations V
- Solving Trigonometric Equations VI

Key Concepts

- When solving linear trigonometric equations, we can use algebraic techniques just as we do solving algebraic equations. Look for patterns, like the difference of squares, quadratic form, or an expression that lends itself well to substitution. See Example **7.5.1**, Example **7.5.2**, and Example **7.5.3**.
- Equations involving a single trigonometric function can be solved or verified using the unit circle. See Example **7.5.4**, Example **7.5.5**, and Example **7.5.6**, and Example **7.5.7**.
- We can also solve trigonometric equations using a graphing calculator. See Example **7.5.8** and Example **7.5.9**.
- Many equations appear quadratic in form. We can use substitution to make the equation appear simpler, and then use the same techniques we use solving an algebraic quadratic: factoring, the quadratic formula, etc. See Example **7.5.10**, Example **7.5.11**, Example **7.5.12**, and Example **7.5.13**.
- We can also use the identities to solve trigonometric equation. See Example **7.5.14**, Example **7.5.15**, and Example **7.5.16**.
- We can use substitution to solve a multiple-angle trigonometric equation, which is a compression of a standard trigonometric function. We will need to take the compression into account and verify that we have found all solutions on the given interval. See Example **7.5.17**.
- Real-world scenarios can be modeled and solved using the Pythagorean Theorem and trigonometric functions. See Example **7.5.18**.

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7.5E: Exercises

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CHAPTER OVERVIEW

9: Systems of Equations and Inequalities

In this chapter, we will investigate matrices and their inverses, and various ways to use matrices to solve systems of equations. First, however, we will study systems of equations on their own: linear and nonlinear, and then partial fractions.

9.4: Partial Fractions

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9.4: Partial Fractions

Skills to Develop

Decompose $\frac{P(x)}{Q(x)}$, where

- $Q(x)$ has only nonrepeated linear factors.
- $Q(x)$ has repeated linear factors.
- $Q(x)$ has a nonrepeated irreducible quadratic factor.
- $Q(x)$ has a repeated irreducible quadratic factor.

Earlier in this chapter, we studied systems of two equations in two variables, systems of three equations in three variables, and nonlinear systems. Here we introduce another way that systems of equations can be utilized—the decomposition of rational expressions. Fractions can be complicated; adding a variable in the denominator makes them even more so. The methods studied in this section will help simplify the concept of a rational expression.

Decomposing $\frac{P(x)}{Q(x)}$ where $Q(x)$ Has Only Nonrepeated Linear Factors

Recall the algebra regarding adding and subtracting rational expressions. These operations depend on finding a common denominator so that we can write the sum or difference as a single, simplified rational expression. In this section, we will look at **partial fraction decomposition**, which is the undoing of the procedure to add or subtract rational expressions. In other words, it is a return from the single simplified **rational expression** to the original expressions, called the **partial fractions**.

For example, suppose we add the following fractions:

$$\frac{2}{x-3}+\frac{-1}{x+2}$$

We would first need to find a common denominator: $(x+2)(x-3)$.

Next, we would write each expression with this common denominator and find the sum of the terms.

$$\begin{aligned}\frac{2}{x-3}\left(\frac{x+2}{x+2}\right)+\frac{-1}{x+2}\left(\frac{x-3}{x-3}\right)&=\frac{2x+4-x+3}{(x+2)(x-3)}\\&=\frac{x+7}{x^2-x-6}\end{aligned}$$

Partial fraction **decomposition** is the reverse of this procedure. We would start with the solution and rewrite (decompose) it as the sum of two fractions.

$$\underbrace{\frac{x+7}{x^2-x-6}}_{\text{Simplified sum}}=\underbrace{\frac{2}{x-3}+\frac{-1}{x+2}}_{\text{Partial fraction decomposition}}$$

We will investigate rational expressions with linear factors and quadratic factors in the denominator where the degree of the numerator is less than the degree of the denominator. Regardless of the type of expression we are decomposing, the first and most important thing to do is factor the denominator.

When the denominator of the simplified expression contains distinct linear factors, it is likely that each of the original rational expressions, which were added or subtracted, had one of the linear factors as the denominator. In other words, using the example above, the factors of x^2-x-6 are $(x-3)(x+2)$, the denominators of the decomposed rational expression. So we will rewrite the simplified form as the sum of individual fractions and use a variable for each numerator. Then, we will solve for each numerator using one of several methods available for partial fraction decomposition.

PARTIAL FRACTION DECOMPOSITION OF $\frac{P(x)}{Q(x)}$: $Q(x)$ HAS NONREPEATED LINEAR FACTORS

The *partial fraction decomposition* of $\frac{P(x)}{Q(x)}$ when $Q(x)$ has nonrepeated linear factors and the degree of $P(x)$ is less than the degree of $Q(x)$ is

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(a_1x+b_1)} + \frac{A_2}{(a_2x+b_2)} + \frac{A_3}{(a_3x+b_3)} + \cdots + \frac{A_n}{(a_nx+b_n)} \tag{9.4.1}$$

How to: Given a rational expression with distinct linear factors in the denominator, decompose it

1. Use a variable for the original numerators, usually A, B, or C, depending on the number of factors, placing each variable over a single factor. For the purpose of this definition, we use A_n for each numerator
$$\frac{P(x)}{Q(x)} = \frac{A_1}{(a_1x+b_1)} + \frac{A_2}{(a_2x+b_2)} + \frac{A_3}{(a_3x+b_3)} + \cdots + \frac{A_n}{(a_nx+b_n)}$$
2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

Example 9.4.1: Decomposing a Rational Function with Distinct Linear Factors

Decompose the given rational expression with distinct linear factors.

$$\frac{3x}{(x+2)(x-1)}$$

Solution

We will separate the denominator factors and give each numerator a symbolic label, like A, B ,or C.

$$\frac{3x}{(x+2)(x-1)} = \frac{A}{(x+2)} + \frac{B}{(x-1)}$$

Multiply both sides of the equation by the common denominator to eliminate the fractions:

$$(x+2)(x-1)\left[\frac{3x}{(x+2)(x-1)}\right] = (x+2)(x-1)\left[\frac{A}{(x+2)}\right] + (x+2)(x-1)\left[\frac{B}{(x-1)}\right]$$

The resulting equation is

$$3x = A(x-1) + B(x+2)$$

Expand the right side of the equation and collect like terms.

$$3x = Ax - A + Bx + 2B$$
$$3x = (A+B)x - A + 2B$$

Set up a system of equations associating corresponding coefficients.

$$3 = A + B$$
$$0 = -A + 2B$$

Add the two equations and solve for B.

$$3 = A + B$$
$$\underline{0 = -A + 2B}$$
$$3 = 0 + 3B$$
$$1 = B$$

Substitute $B = 1$ into one of the original equations in the system.

$$3 = A + 1$$
$$2 = A$$

Thus, the partial fraction decomposition is

$$\frac{3x}{(x+2)(x-1)} = \frac{2}{(x+2)} + \frac{1}{(x-1)}$$

Another method to use to solve for A or B is by considering the equation that resulted from eliminating the fractions and substituting a value for x that will make either the $A-$ or $B-$term equal 0. If we let $x = 1$, the

$A-$ term becomes 0 and we can simply solve for B.

$$3x = A(x-1) + B(x+2)$$
$$3(1) = A[(1)-1] + B[(1)+2]$$
$$3 = 0 + 3B$$
$$1 = B$$

Next, either substitute $B = 1$ into the equation and solve for A, or make the $B-$term 0 by substituting $x = -2$ into the equation.

$$3x = A(x-1) + B(x+2)$$
$$3(-2) = A[(-2)-1] + B[(-2)+2]$$
$$-6 = -3A + 0$$
$$\frac{-6}{-3} = A$$
$$2 = A$$

We obtain the same values for A and B using either method, so the decompositions are the same using either method.

$$\frac{3x}{(x+2)(x-1)} = \frac{2}{(x+2)} + \frac{1}{(x-1)}$$

Although this method is not seen very often in textbooks, we present it here as an alternative that may make some partial fraction decompositions easier. It is known as the **Heaviside method**, named after Charles Heaviside, a pioneer in the study of electronics.

Exercise 9.4.1

Find the partial fraction decomposition of the following expression.

$$\frac{x}{(x-3)(x-2)}$$

Answer

$$\frac{3}{x-3} - \frac{2}{x-2}$$

Decomposing $\frac{P(x)}{Q(x)}$ where $Q(x)$ Has Repeated Linear Factors

Some fractions we may come across are special cases that we can decompose into partial fractions with repeated linear factors. We must remember that we account for repeated factors by writing each factor in increasing powers.

PARTIAL FRACTION DECOMPOSITION OF $\frac{P(x)}{Q(x)}$: $Q(x)$ HAS REPEATED LINEAR FACTORS

The partial fraction decomposition of $\frac{P(x)}{Q(x)}$, when $Q(x)$ has a repeated linear factor occurringn n times and the degree of $P(x)$ is less than the degree of $Q(x)$, is

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(a_1x+b_1)} + \frac{A_2}{(a_2x+b_2)} + \frac{A_3}{(a_3x+b_3)} + \cdots + \frac{A_n}{(a_nx+b_n)} \tag{9.4.2}$$

Write the denominator powers in increasing order.

How to: decompose a rational expression with repeated linear factors

1. Use a variable like A, B, or C for the numerators and account for increasing powers of the denominators.

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(a_1x+b_1)} + \frac{A_2}{(a_2x+b_2)} + \frac{A_3}{(a_3x+b_3)} + \cdots + \frac{A_n}{(a_nx+b_n)} \tag{9.4.3}$$

2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

Example 9.4.2: Decomposing with Repeated Linear Factors

Decompose the given rational expression with repeated linear factors.

$$\frac{-x^2+2x+4}{x^3-4x^2+4x}$$

Solution

The denominator factors are $x(x-2)^2$. To allow for the repeated factor of $(x-2)$, the decomposition will include three denominators: x, $(x-2)$, and $(x-2)^2$. Thus,

$$\frac{-x^2+2x+4}{x^3-4x^2+4x} = \frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$$

Next, we multiply both sides by the common denominator.

$$x(x-2)^2\left[\frac{-x^2+2x+4x}{(x-2)^2}\right] = \left[\frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}\right]x(x-2)^2$$
$$-x^2+2x+4 = A(x-2)^2 + Bx(x-2) + Cx$$

On the right side of the equation, we expand and collect like terms.

$$\begin{aligned} -x^2+2x+4 &= A(x^2-4x+4) + B(x^2-2x) + Cx \\ &= Ax^2 - 4Ax + 4A + Bx^2 - 2Bx + Cx \\ &= (A+B)x^2 + (-4A-2B+C)x + 4A \end{aligned}$$

Next, we compare the coefficients of both sides. This will give the system of equations in three variables:

$$\begin{aligned} -x^2+2x+4 &= (A+B)x^2 + (-4A-2B+C)x + 4A \\ A+B &= -1 \\ -4A-2B+C &= 2 \\ 4A &= 4 \end{aligned}$$

Solving for A, we have

$$\begin{aligned} 4A &= 4 \\ A &= 1 \end{aligned}$$

Substitute $A=1$ into Equation ???.

$$\begin{aligned} A+B &= -1 \\ (1)+B &= -1 \\ B &= -2 \end{aligned}$$

Then, to solve for C, substitute the values for A and B into Equation ???.

$$-4A-2B+C=2$$
$$-4(1)-2(-2)+C=2$$
$$-4+4+C=2$$
$$C=2$$

Thus,

$$\frac{-x^2+2x+4}{x^3-4x^2+4x}=\frac{1}{x}-\frac{2}{(x-2)}+\frac{2}{(x-2)^2}$$

Exercise 9.4.2

Find the partial fraction decomposition of the expression with repeated linear factors.

$$\frac{6x-11}{(x-1)^2}$$

Answer

$$\frac{6}{x-1}-\frac{5}{(x-1)^2} \tag{9.4.4}$$

Decomposing $\frac{P(x)}{Q(x)}$, where $Q(x)$ Has a Nonrepeated Irreducible Quadratic Factor

So far, we have performed partial fraction decomposition with expressions that have had linear factors in the denominator, and we applied numerators A, B, or C representing constants. Now we will look at an example where one of the factors in the denominator is a quadratic expression that does not factor. This is referred to as an irreducible quadratic factor. In cases like this, we use a linear numerator such as $Ax+B$, $Bx+C$, etc.

DECOMPOSITION OF $\frac{P(x)}{Q(x)}$: $Q(x)$ HAS A NONREPEATED IRREDUCIBLE QUADRATIC FACTOR

The partial fraction decomposition of $\frac{P(x)}{Q(x)}$ such that $Q(x)$ has a nonrepeated irreducible quadratic factor and the degree of $P(x)$ is less than the degree of $Q(x)$ is written as

$$\frac{P(x)}{Q(x)}=\frac{A_1x+B_1}{(a_1x^2+b1_x+c_1)}+\frac{A_2x+B_2}{(a_2x^2+b_2x+c_2)}+\cdots+\frac{A_nx+B_n}{(a_nx^2+b_nx+c_n)} \tag{9.4.5}$$

The decomposition may contain more rational expressions if there are linear factors. Each linear factor will have a different constant numerator: A, B, C, and so on.

Howto: decompose a rational expression where the factors of the denominator are distinct, irreducible quadratic factors

1. Use variables such as A, B, or C for the constant numerators over linear factors, and linear expressions such as A_1x+B_1, A_2x+B_2, etc., for the numerators of each quadratic factor in the denominator.
$$\frac{P(x)}{Q(x)}=\frac{A}{ax+b}+\frac{A_1x+B_1}{(a_1x^2+b1_x+c_1)}+\frac{A_2x+B_2}{(a_2x^2+b_2x+c_2)}+\cdots+\frac{A_nx+B_n}{(a_nx^2+b_nx+c_n)}$$
2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

Example 9.4.3: Decomposing $\frac{P(x)}{Q(x)}$ When $Q(x)$ Contains a Nonrepeated Irreducible Quadratic Factor

Find a partial fraction decomposition of the given expression.

$$\frac{8x^2+12x-20}{(x+3)(x^2+x+2)}$$

Solution

We have one linear factor and one irreducible quadratic factor in the denominator, so one numerator will be a constant and the other numerator will be a linear expression. Thus,

$$\frac{8x^2+12x-20}{(x+3)(x^2+x+2)}=\frac{A}{(x+3)}+\frac{Bx+C}{(x^2+x+2)}$$

We follow the same steps as in previous problems. First, clear the fractions by multiplying both sides of the equation by the common denominator.

$$(x+3)(x^2+x+2)\left[\frac{8x^2+12x-20}{(x+3)(x^2+x+2)}\right]=\left[\frac{A}{(x+3)}+\frac{Bx+C}{(x^2+x+2)}\right](x+3)(x^2+x+2)$$

$$8x^2+12x-20=A(x^2+x+2)+(Bx+C)(x+3)$$

Notice we could easily solve for A by choosing a value forx x that will make the $Bx+C$ term equal 0. Let $x=-3$ and substitute it into the equation.

$$\begin{aligned}8x^2+12x-20&=A(x^2+x+2)+(Bx+C)(x+3)\\8(-3)^2+12(-3)-20&=A((-3)^2+(-3)+2)+(B(-3)+C)((-3)+3)\\16&=8A\\A&=2\end{aligned}$$

Now that we know the value of A, substitute it back into the equation. Then expand the right side and collect like terms.

$$\begin{aligned}8x^2+12x-20&=2(x^2+x+2)+(Bx+C)(x+3)\\8x^2+12x-20&=2x^2+2x+4+Bx^2+3B+Cx+3C\\8x^2+12x-20&=(2+B)x^2+(2+3B+C)x+(4+3C)\end{aligned}$$

Setting the coefficients of terms on the right side equal to the coefficients of terms on the left side gives the system of equations.

Multiple \label

Solve for B using Equation ??? and solve for C using Equation ???.

Multiple \label

Thus, the partial fraction decomposition of the expression is

$$\frac{8x^2+12x-20}{(x+3)(x^2+x+2)}=\frac{2}{(x+3)}+\frac{6x-8}{(x^2+x+2)}$$

Q&A: Could we have just set up a system of equations to solve the example above?

Yes, we could have solved it by setting up a system of equations without solving for A first. The expansion on the right would be:

$$\begin{aligned}8x^2+12x-20&=Ax^2+Ax+2A+Bx^2+3B+Cx+3C\\8x^2+12x-20&=(A+B)x^2+(A+3B+C)x+(2A+3C)\end{aligned}$$

So the system of equations would be:

$$\begin{aligned}A+B&=8\\A+3B+C&=12\\2A+3C&=-20\end{aligned}$$

Exercise 9.4.3

Find the partial fraction decomposition of the expression with a nonrepeating irreducible quadratic factor.

$$\frac{5x^2-6x+7}{(x-1)(x^2+1)}$$

Answer

$$\frac{3}{x-1}+\frac{2x-4}{x^2+1}$$

Decomposing $\frac{P(x)}{Q(x)}$ When $Q(x)$ Has a Repeated Irreducible Quadratic Factor

Now that we can decompose a simplified rational expression with an irreducible quadratic factor, we will learn how to do partial fraction decomposition when the simplified rational expression has repeated irreducible quadratic factors. The decomposition will consist of partial fractions with linear numerators over each irreducible quadratic factor represented in increasing powers.

DECOMPOSITION OF $\frac{P(x)}{Q(x)}$ WHEN $Q(X)$ HAS A REPEATED IRREDUCIBLE QUADRATIC FACTOR

The partial fraction decomposition of $\frac{P(x)}{Q(x)}$, when $Q(x)$ has a repeated irreducible quadratic factor and the degree of $P(x)$ is less than the degree of $Q(x)$, is

$$\frac{P(x)}{(ax^2+bx+c)^n} = \frac{A_1x+B_1}{(ax^2+bx+c)}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\frac{A_3x+B_3}{(ax^2+bx+c)^3}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n} \tag{9.4.6}$$

Write the denominators in increasing powers.

How to: decompose a rational expression that has a repeated irreducible factor

1. Use variables like A, B, or C for the constant numerators over linear factors, and linear expressions such as A_1x+B_1, A_2x+B_2, etc., for the numerators of each quadratic factor in the denominator written in increasing powers, such as
$$\frac{P(x)}{Q(x)}=\frac{A}{ax+b}+\frac{A_1x+B_1}{(ax^2+bx+c)}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$$
2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

Example 9.4.4: Decomposing a Rational Function with a Repeated Irreducible Quadratic Factor in the Denominator

Decompose the given expression that has a repeated irreducible factor in the denominator.

$$\frac{x^4+x^3+x^2-x+1}{x(x^2+1)^2}$$

Solution

The factors of the denominator are x, (x^2+1), and $(x^2+1)^2$. Recall that, when a factor in the denominator is a quadratic that includes at least two terms, the numerator must be of the linear form $Ax+B$. So, let's begin the decomposition.

$$\frac{x^4+x^3+x^2-x+1}{x(x^2+1)^2}=\frac{A}{x}+\frac{Bx+C}{(x^2+1)}+\frac{Dx+E}{(x^2+1)^2}$$

We eliminate the denominators by multiplying each term by $x(x^2+1)^2$. Thus,

$$x^4+x^3+x^2-x+1=A(x^2+1)^2+(Bx+C)(x)(x^2+1)+(Dx+E)(x)$$

$$x^4+x^3+x^2-x+1=A(x^4+2x^2+1)+Bx^4+Bx^2+Cx^3+Cx+Dx^2+Ex$$

Expand the right side.

$$=Ax^4+2Ax^2+A+Bx^4+Bx^2+Cx^3+Cx+Dx^2+Ex$$

Now we will collect like terms.

$$x^4+x^3+x^2-x+1=(A+B)x^4+(C)x^3+(2A+B+D)x^2+(C+E)x+A$$

Set up the system of equations matching corresponding coefficients on each side of the equal sign.

$$\begin{aligned} A+B&=1\\ C&=1\\ 2A+B+D&=1\\ C+E&=-1\\ A&=1 \end{aligned}$$

We can use substitution from this point. Substitute $A=1$ into the first equation.

$$\begin{aligned} 1+B&=1\\ B&=0 \end{aligned}$$

Substitute $A=1$ and $B=0$ into the third equation.

$$\begin{aligned} 2(1)+0+D&=1\\ D&=-1 \end{aligned}$$

Substitute $C=1$ into the fourth equation.

$$\begin{aligned} 1+E&=-1\\ E&=-2 \end{aligned}$$

Now we have solved for all of the unknowns on the right side of the equal sign. We have $A=1$, $B=0$, $C=1$, $D=-1$, and $E=-2$. We can write the decomposition as follows:

$$\frac{x^4+x^3+x^2-x+1}{x(x^2+1)^2}=\frac{1}{x}+\frac{1}{(x^2+1)}-\frac{x+2}{(x^2+1)^2}$$

Exercise 9.4.4

Find the partial fraction decomposition of the expression with a repeated irreducible quadratic factor.

$$\frac{x^3-4x^2+9x-5}{(x^2-2x+3)^2}$$

Answer

$$\frac{x-2}{x^2-2x+3}+\frac{2x+1}{(x^2-2x+3)^2}$$

Media

Access these online resources for additional instruction and practice with partial fractions.

- Partial Fraction Decomposition
- Partial Fraction Decomposition With Repeated Linear Factors
- Partial Fraction Decomposition With Linear and Quadratic Factors

Key Concepts

- Decompose $\frac{P(x)}{Q(x)}$ by writing the partial fractions as

$$\frac{A}{a_1x+b_1}+\frac{B}{a_2x+b_2}.$$

Solve by clearing the fractions, expanding the right side, collecting like terms, and setting corresponding coefficients equal to each other, then setting up and solving a system of equations (see Example **9.4.1**).

- The decomposition of $\frac{P(x)}{Q(x)}$ with repeated linear factors must account for the factors of the denominator in increasing powers (see Example **9.4.2**).
- The decomposition of $\frac{P(x)}{Q(x)}$ with a nonrepeated irreducible quadratic factor needs a linear numerator over the quadratic factor, as in $\frac{A}{x}+\frac{Bx+C}{(ax^2+bx+c)}$ (see Example **9.4.3**).
- In the decomposition of $\frac{P(x)}{Q(x)}$, where $Q(x)$ has a repeated irreducible quadratic factor, when the irreducible quadratic factors are repeated, powers of the denominator factors must be represented in increasing powers as

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

See Example **9.4.4**.

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CHAPTER OVERVIEW

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1.0: Prelude to Functions

Toward the end of the twentieth century, the values of stocks of internet and technology companies rose dramatically. As a result, the Standard and Poor's stock market average rose as well. The *Standard and Poor's Index* tracks the value of that initial investment of just under $100 over the 40 years. It shows that an investment that was worth less than $500 until about 1995 skyrocketed up to about $1100 by the beginning of 2000. That five-year period became known as the "dot-com bubble" because so many internet startups were formed. As bubbles tend to do, though, the dot-com bubble eventually burst. Many companies grew too fast and then suddenly went out of business. The result caused the sharp decline represented on the graph beginning at the end of 2000.

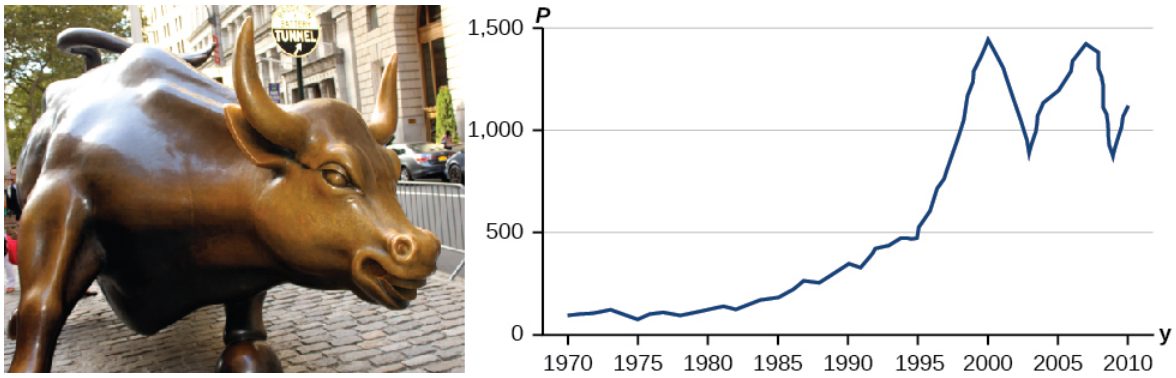


Figure 1.0.1*: Figure of a Bull and a graph of market prices. Standard and Poor's Index with dividends reinvested (credit "bull": modification of work by Prayitno Hadinata; credit "graph": modification of work by MeasuringWorth)*

Notice, as we consider this example, that there is a definite relationship between the year and stock market average. For any year we choose, we can determine the corresponding value of the stock market average. In this chapter, we will explore these kinds of relationships and their properties.

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1.0E: Exercises

This page is a draft and is under active development.

1.0E: Exercises

For this Section, there are no Exercises!

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1.1: Functions and Function Notation

Skills to Develop

- Determine whether a relation represents a function.
- Find the value of a function.
- Determine whether a function is one-to-one.
- Use the vertical line test to identify functions.
- Graph the functions listed in the library of functions.

A jetliner changes altitude as its distance from the starting point of a flight increases. The weight of a growing child increases with time. In each case, one quantity depends on another. There is a relationship between the two quantities that we can describe, analyze, and use to make predictions. In this section, we will analyze such relationships.

Determining Whether a Relation Represents a Function

A relation is a set of ordered pairs. The set of the first components of each ordered pair is called the **domain** and the set of the second components of each ordered pair is called the **range**. Consider the following set of ordered pairs. The first numbers in each pair are the first five **natural numbers**. The second number in each pair is twice that of the first.

$$\{(1,2),(2,4),(3,6),(4,8),(5,10)\}$$

The domain is $\{1,2,3,4,5\}$. The range is $\{2,4,6,8,10\}$.

Note that each value in the domain is also known as an **input** value, or **independent variable**, and is often labeled with the lowercase letter x. Each value in the range is also known as an output value, or **dependent variable**, and is often labeled lowercase letter y.

A function f is a relation that assigns a single value in the range to each value in the domain. In other words, no x-values are repeated. For our example that relates the first five natural numbers to numbers double their values, this relation is a function because each element in the domain, {1, 2, 3, 4, 5}, is paired with exactly one element in the range, $\{2,4,6,8,10\}$. The input value of a function is also known as the **argument**.

Now let's consider the set of ordered pairs that relates the terms "even" and "odd" to the first five natural numbers. It would appear as

$$\{(\text{odd},1),(\text{even},2),(\text{odd},3),(\text{even},4),(\text{odd},5)\}$$

Notice that each element in the domain, {even, odd} is not paired with exactly one element in the range, $\{1,2,3,4,5\}$. For example, the term "odd" corresponds to three values from the domain, $\{1,3,5\}$ and the term "even" corresponds to two values from the range, $\{2,4\}$. This violates the definition of a function, so this relation is not a function.

Figure 1.1.1 compares relations that are functions and not functions.

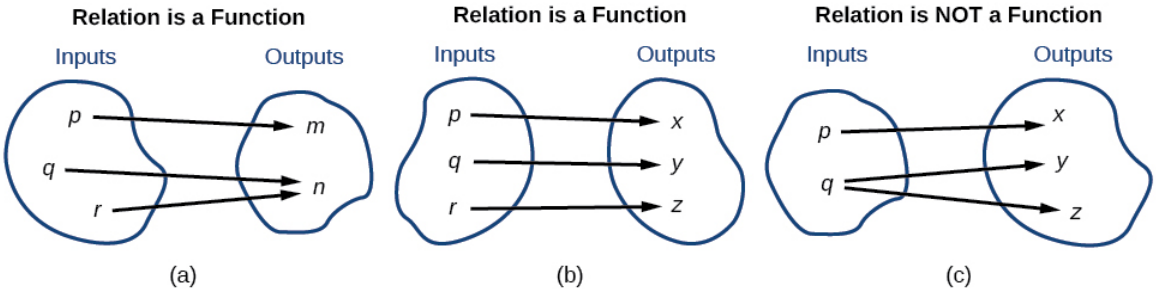


Figure* 1.1.1*: *(a) This relationship is a function. Each input is associated with a single output. Note that input q and r both give output n. (b) This relationship is also a function. Each input is associated with a single output. (c) This relationship is not a function because input q is associated with two different outputs.*

definition: Function

A **function** is a relation in which each possible input value leads to exactly one output value. We say "the output is a function of the input."

The **input** values make up the **domain**, and the **output** values make up the **range**.

How To: Given a relationship between two quantities, determine whether the relationship is a function

1. Identify the input values.
2. Identify the output values.
3. If each input value leads to only one output value, classify the relationship as a function. If any input value leads to two or more outputs, do not classify the relationship as a function.

Example 1.1.1: Determining If Menu Price Lists Are Functions

The coffee shop menu, shown in Figure **1.1.2** consists of items and their prices.

a. Is price a function of the item?
b. Is the item a function of the price?

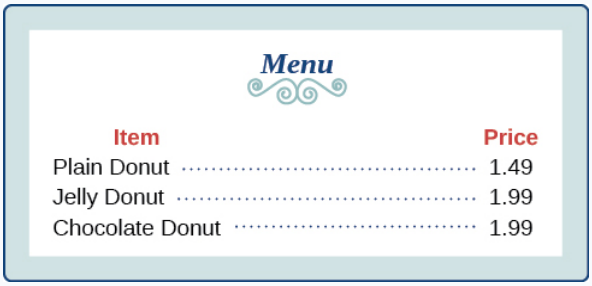


***Figure* 1.1.2:** *A menu of donut prices from a coffee shop where a plain donut is $1.49 and a jelly donut and chocolate donut are $1.99.*

Solution

a. Let's begin by considering the input as the items on the menu. The output values are then the prices. See Figure **1.1.3**.



***Figure* 1.1.3:** *A menu of donut prices from a coffee shop where a plain donut is $1.49 and a jelly donut and chocolate donut are $1.99.*

Each item on the menu has only one price, so the price is a function of the item.

a. Two items on the menu have the same price. If we consider the prices to be the input values and the items to be the output, then the same input value could have more than one output associated with it. See Figure **1.1.4**.

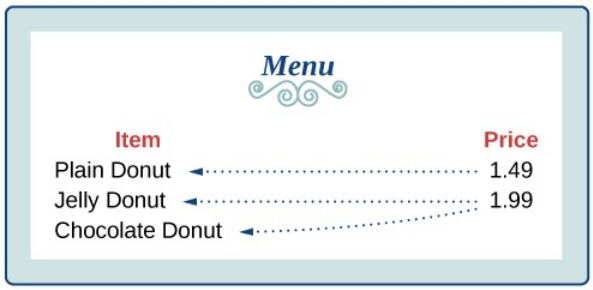


***Figure* 1.1.4:** *Association of the prices to the donuts.*

Therefore, the item is a not a function of price.

Example 1.1.2: Determining If Class Grade Rules Are Functions

In a particular math class, the overall percent grade corresponds to a grade point average. Is grade point average a function of the percent grade? Is the percent grade a function of the grade point average? Table **1.1.1** shows a possible rule for assigning grade points.

***Table* 1.1.1:** *Class grade points.*

Percent grade	0–56	57–61	62–66	67–71	72–77	78–86	87–91	92–100
Grade point average	0.0	1.0	1.5	2.0	2.5	3.0	3.5	4.0

Solution

For any percent grade earned, there is an associated grade point average, so the grade point average is a function of the percent grade. In other words, if we input the percent grade, the output is a specific grade point average.

In the grading system given, there is a range of percent grades that correspond to the same grade point average. For example, students who receive a grade point average of 3.0 could have a variety of percent grades ranging from 78 all the way to 86. Thus, percent grade is not a function of grade point average.

1.1.1

Table **1.1.2** lists the five greatest baseball players of all time in order of rank.

***Table* 1.1.2:** *Five greatest baseball players.*

Player	Rank
Babe Ruth	1
Willie Mays	2
Ty Cobb	3
Walter Johnson	4
Hank Aaron	5

a. Is the rank a function of the player name?

b. Is the player name a function of the rank?

Answer

a. yes

b. yes (Note: If two players had been tied for, say, 4th place, then the name would not have been a function of rank.)

Using Function Notation

Once we determine that a relationship is a function, we need to display and define the functional relationships so that we can understand and use them, and sometimes also so that we can program them into computers. There are various ways of representing functions. A standard function notation is one representation that facilitates working with functions.

To represent "height is a function of age," we start by identifying the descriptive variables h for height and a for age. The letters f, g,and h are often used to represent functions just as we use x, y,and z to represent numbers and A, B, and C to represent sets.

$$\begin{array}{ll} h \text{ is } f \text{ of } a & \text{We name the function } f\text{; height is a function of age.} \\ h = f(a) & \text{We use parentheses to indicate the function input.} \\ f(a) & \text{We name the function } f \text{ ; the expression is read as " } f \text{ of } a\text{."} \end{array} \tag{1.1.1}$$

Remember, we can use any letter to name the function; the notation $h(a)$ shows us that h depends on a. The value a must be put into the function h to get a result. The parentheses indicate that age is input into the function; they do not indicate multiplication.

We can also give an algebraic expression as the input to a function. For example $f(a+b)$ means "first add a and b, and the result is the input for the function f." The operations must be performed in this order to obtain the correct result.

Function Notation

The notation $y = f(x)$ defines a function named f. This is read as "y is a function of x." The letter x represents the input value, or independent variable. The letter y, or $f(x)$, represents the output value, or dependent variable.

Example 1.1.3: Using Function Notation for Days in a Month

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month.

Solution

The number of days in a month is a function of the name of the month, so if we name the function f, we write $\text{days} = \text{f(month)}$ or $\text{d} = \text{f(m)}$. The name of the month is the input to a "rule" that associates a specific number (the output) with each input.

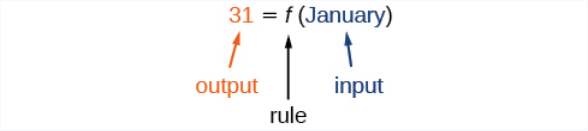


Figure 1.1.5: *The function* $31 = f(January)$ *where 31 is the output, f is the rule, and January is the input.*

For example, $\text{f(March)} = 31$, because March has 31 days. The notation $d = f(m)$ reminds us that the number of days, d (the output), is dependent on the name of the month, m (the input).

Analysis

Note that the inputs to a function do not have to be numbers; function inputs can be names of people, labels of geometric objects, or any other element that determines some kind of output. However, most of the functions we will work with in this book will have numbers as inputs and outputs.

Example 1.1.4: Interpreting Function Notation

A function $N = f(y)$ gives the number of police officers, N, in a town in year y. What does $f(2005) = 300$ represent?

Solution

When we read $f(2005) = 300$, we see that the input year is 2005. The value for the output, the number of police officers (N), is 300. Remember, $N = f(y)$. The statement $f(2005) = 300$ tells us that in the year 2005 there were 300 police officers in the town.

1.1.2

Use function notation to express the weight of a pig in pounds w as a function of its age in days d.

Answer

$w = f(d)$

Instead of a notation such as $y = f(x)$, could we use the same symbol for the output as for the function, such as $y = y(x)$, meaning "y is a function of x?"

Yes, this is often done, especially in applied subjects that use higher math, such as physics and engineering. However, in exploring math itself we like to maintain a distinction between a function such as f, which is a rule or procedure, and the output y we get by applying f to a particular input x. This is why we usually use notation such as $y = f(x), P = W(d)$, and so on.

Representing Functions Using Tables

A common method of representing functions is in the form of a table. The table rows or columns display the corresponding input and output values. In some cases, these values represent all we know about the relationship; other times, the table provides a few select examples from a more complete relationship.

Table 1.1.3 lists the input number of each month ($January = 1$, $February = 2$, and so on) and the output value of the number of days in that month. This information represents all we know about the months and days for a given year (that is not a leap year). Note that, in this table, we define a days-in-a-month function f where $D = f(m)$ identifies months by an integer rather than by name.

***Table* 1.1.3:** *Months and number of days per month.*

Month number, m (input)	1	2	3	4	5	6	7	8	9	10	11	12
Days in month, D (output)	31	28	31	30	31	30	31	31	30	31	30	31

Table 1.1.4 defines a function $Q = g(n)$ Remember, this notation tells us that g is the name of the function that takes the input n and gives the output Q.

Table 1.1.4: *Function $Q = g(n)$*

n	1	2	3	4	5
Q	8	6	7	6	8

Table 1.1.5 displays the age of children in years and their corresponding heights. This table displays just some of the data available for the heights and ages of children. We can see right away that this table does not represent a function because the same input value, 5 years, has two different output values, 40 in. and 42 in.

Table 1.1.5: *Age of children and their corresponding heights.*

Age in years, a (input	5	1 8 0
Height in inches, h (output)	40	3 4 4

How To: Given a table of input and output values, determine whether the table represents a function

1. Identify the input and output values.
2. Check to see if each input value is paired with only one output value. If so, the table represents a function.

Example 1.1.5: Identifying Tables that Represent Functions

Which table, Table 1.1.6, Table 1.1.7, or Table 1.1.8, represents a function (if any)?

Table 1.1.6

Input	Output
2	1
5	3
8	6

Table 1.1.7

Input	Output
-3	5
0	1
4	5

Table 1.1.8

Input	Output
1	0
5	2
5	4

Solution

Table 1.1.6 and Table 1.1.7 define functions. In both, each input value corresponds to exactly one output value. Table 1.1.8 does not define a function because the input value of 5 corresponds to two different output values.

When a table represents a function, corresponding input and output values can also be specified using function notation.

The function represented by Table 1.1.6 can be represented by writing

$$f(2) = 1,\ f(5) = 3, \text{ and } f(8) = 6 \tag{1.1.2}$$

Similarly, the statements

$$g(-3) = 5,\ g(0) = 1, \text{ and } g(4) = 5 \tag{1.1.3}$$

represent the function in Table 1.1.7.

Table 1.1.8 cannot be expressed in a similar way because it does not represent a function.

1.1.3

Does Table 1.1.9 represent a function?

Table 1.1.9

Input	Output
1	10
2	100
3	1000

Answer

yes

Representing Functions in Algebraic Form from an Equation

Some functions are defined by mathematical rules or procedures expressed in *equation* form. If it is possible to express the function output as a formula involving the input quantity, then we say the function can be written in **algebraic form**. For example, consider the equation $2n + 6p = 12$. If we can rewrite it so that p is the output of a formula involving n, then p is a function of n.

How To: Given a function in equation form, write its algebraic formula

1. Solve the equation to isolate the output variable on one side of the equal sign, with the other side as an expression that involves only the input variable.
2. Use all the usual algebraic methods for solving equations, such as adding or subtracting the same quantity to or from both sides, or multiplying or dividing both sides of the equation by the same quantity.

Example 1.1.6: Finding an Equation of a Function

Express the relationship $2n + 6p = 12$ as a function $p = f(n)$, if possible.

Solution:

To express the relationship in this form, we need to be able to write the relationship where p is a function of n, which means writing it as $p =$ [expression involving n].

$$2n + 6p = 12 \tag{1.1.4}$$

$$6p = 12 - 2n \quad \text{Subtract 2n from both sides.} \tag{1.1.5}$$

$$p = \frac{12 - 2n}{6} \quad \text{Divide both sides by 6 and simplify.} \tag{1.1.6}$$

$$p = \frac{12}{6} - \frac{2n}{6} \tag{1.1.7}$$

$$p = 2 - \frac{1}{3}n \tag{1.1.8}$$

Therefore, p as a function of n is written as

$$p = f(n) = 2 - \frac{1}{3}n$$

Example 1.1.7: Expressing the Equation of a Circle as a Function

Does the equation $x^2 + y^2 = 1$ represent a function with x as input and y as output? If so, express the relationship as a function $y = f(x)$.

Solution

First we subtract x^2 from both sides.

$$y^2 = 1 - x^2$$

We now try to solve for y in this equation.

$$y = \pm\sqrt{1 - x^2}$$

$$= +\sqrt{1 - x^2} \text{ and } -\sqrt{1 - x^2}$$

We get two outputs corresponding to the same input, so this relationship cannot be represented as a single function $y = f(x)$.

Given an input $x = 0.5$, can you find the two outputs? If you can, write it up (show your work) and give it to your instructor for ***extra credit***.

Analysis

It is important to note that not every relationship expressed by an equation can also be expressed as a function with a formula.

1.1.4

If $x - 8y^3 = 0$, express y as a function of x.

Answer

$$y = f(x) = \frac{\sqrt[3]{x}}{2}$$

Are there relationships expressed by an equation that do represent a function but which still cannot be represented by an algebraic formula?

Yes, this can happen. For example, given the equation $x = y + 2^y$, if we want to express y as a function of x, there is no simple algebraic formula involving only x that equals y. However, each x does determine a unique value for y, and there are mathematical procedures by which y can be found to any desired accuracy. In this case, we say that the equation gives an implicit (implied) rule for y as a function of x, even though the formula cannot be written explicitly.

Finding Input and Output Values of a Function

When we know an input value and want to determine the corresponding output value for a function, we evaluate the function. Evaluating will always produce one result because each input value of a function corresponds to exactly one output value.

When we know an output value and want to determine the input values that would produce that output value, we set the output equal to the function's formula and solve for the input. Solving can produce more than one solution because different input values can produce the same output value.

Evaluation of Functions in Algebraic Forms

When we have a function in formula form, it is usually a simple matter to evaluate the function. For example, the function $f(x) = 5 - 3x^2$ can be evaluated by squaring the input value, multiplying by 3, and then subtracting the product from 5.

Given the formula for a function, evaluate

1. Replace the input variable in the formula with the value provided.
2. Calculate the result.

Example 1.1.8: Evaluating Functions at Specific Values

1. Evaluate $f(x) = x^2 + 3x - 4$ at

   a. -2

   b. a

   c. $a + h$

2. Evaluate $\frac{f(a+h)-f(a)}{h}$

Solution

Replace the x in the function with each specified value.

a. Because the input value is a number, 2, we can use simple algebra to simplify.

$$\begin{aligned} f(-2) &= (-2)^2 + 3(-2) - 4 && (1.1.9)\\ &= 4 - 6 - 4 && (1.1.10)\\ &= -6 && (1.1.11) \end{aligned}$$

b. In this case, the input value is a letter so we cannot simplify the answer any further.

$$f(a) = a^2 + 3a - 4$$

c. With an input value of $a + h$, we must use the distributive property.

$$\begin{aligned} f(a+h) &= (a+h)^2 + 3(a+h) - 4 && (1.1.12)\\ &= a^2 + 2ah + h^2 + 3a + 3h - 4 && (1.1.13) \end{aligned}$$

d. In this case, we apply two different input values to the function, and then perform algebraic operations on the two results. We already found that

$$f(a+h) = a^2 + 2ah + h^2 + 3a + 3h - 4$$

and we know that

$$f(a) = a^2 + 3a - 4$$

Now we combine the results and simplify.

$$\begin{aligned} \frac{f(a+h)-f(a)}{h} &= \frac{(a^2 + 2ah + h^2 + 3a + 3h - 4) - (a^2 + 3a - 4)}{h} && (1.1.14)\\ &= \frac{(ah + h^2 + 3h)}{h} && (1.1.15)\\ &= \frac{h(a + h + 3)}{h} && \text{Factor out } h. \quad (1.1.16)\\ &= 2a + h + 3 && \text{Simplify.} \quad (1.1.17) \end{aligned}$$

Example 1.1.9: Evaluating Functions

Given the function $h(p) = p^2 + 2p$, evaluate $h(4)$.

Solution

To evaluate $h(4)$, we substitute the value 4 for the input variable p in the given function.

$$\begin{aligned} h(p) &= p^2 + 2p && (1.1.18)\\ h(4) &= (4)^2 + 2(4) && (1.1.19)\\ &= 16 + 8 && (1.1.20)\\ &= 24 && (1.1.21) \end{aligned}$$

Therefore, for an input of 4, we have an output of 24.

1.1.5

Given the function $g(m) = \sqrt{m-4}$, evaluate $g(5)$.

Answer

$g(5) = 1$

Example 1.1.10: Solving Functions

Given the function $h(p) = p^2 + 2p$, solve for $h(p) = 3$.

Solution

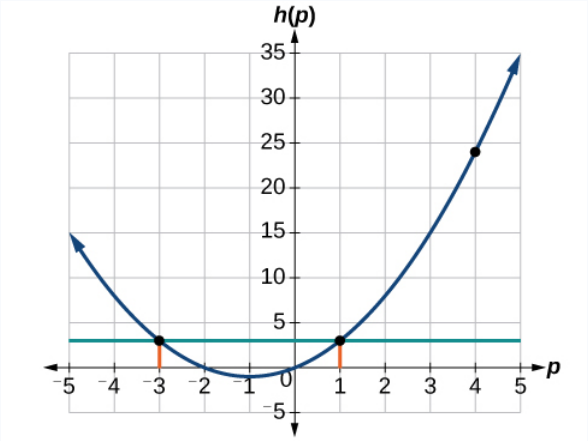
$$\begin{aligned} h(p) &= 3 \\ p^2 + 2p &= 3 \quad \text{Substitute the original function} \\ p^2 + 2p - 3 &= 0 \quad \text{Subtract 3 from each side.} \\ (p+3)(p-1) &= 0 \quad \text{Factor.} \end{aligned} \tag{1.1.22}$$

If $(p+3)(p-1) = 0$, either $(p+3) = 0$ or $(p-1) = 0$ (or both of them equal 0). We will set each factor equal to 0 and solve for p in each case.

$$(p+3) = 0,\ p = -3$$

$$(p-1) = 0,\ p = 1$$

This gives us two solutions. The output $h(p) = 3$ when the input is either $p = 1$ or $p = -3$. We can also verify by graphing as in Figure 1.1.6. The graph verifies that $h(1) = h(-3) = 3$ and $h(4) = 24$.



p	-3	-2	0	1	4
$h(p)$	3	0	0	3	24

***Figure* 1.1.6:** Graph of $h(p) = p^2 + 2p$

1.1.6

Given the function $g(m) = \sqrt{m-4}$, solve $g(m) = 0$. Note: the m-values that are solutions to this equation are called the **zeroes** of the function.

Answer

$m = 4$

Evaluating a Function Given in Tabular Form

As we saw above, we can represent functions in tables. Conversely, we can use information in tables to write functions, and we can evaluate functions using the tables. For example, how well do our pets recall the fond memories we share with them? There is an urban legend that a goldfish has a memory of 3 seconds, but this is just a myth. Goldfish can remember up to 3 months, while the beta fish has a memory of up to 5 months. And while a puppy's memory span is no longer than 30 seconds, the adult dog can remember for 5 minutes. This is meager compared to a cat, whose memory span lasts for 16 hours.

The function that relates the type of pet to the duration of its memory span is more easily visualized with the use of a table. See Table **1.1.10**.

***Table* 1.1.10**

Pet	Memory span, in hours
Puppy	0.008
Adult Dog	0.083
Cat	3
Goldfish	2160
Beta Fish	3600

At times, evaluating a function in table form may be more useful than using equations. Here let us call the function M. The domain of the function is the type of pet and the range is a real number representing the number of hours the pet's memory span lasts. We can evaluate the function M at the input value of "goldfish." We would write $M(goldfish) = 2160$. Notice that, to evaluate the function in table form, we identify the input value and the corresponding output value from the pertinent row of the table. The tabular form for function M seems ideally suited to this function, more so than writing it in paragraph or function form.

How To: Given a function represented by a table, identify specific output and input values

1. Find the given input in the row (or column) of input values.
2. Identify the corresponding output value paired with that input value.
3. Find the given output values in the row (or column) of output values, noting every time that output value appears.
4. Identify the input value(s) corresponding to the given output value.

Example 1.1.11: Evaluating and Solving a Tabular Function

Using Table **1.1.11**,

a. Evaluate $g(3)$.
b. Solve $g(n) = 6$.

Table 1.1.11

n	1	2	3	4	5
$g(n)$	8	6	7	6	8

Solution

a. Evaluating $g(3)$ means determining the output value of the function g for the input value of $n = 3$. The table output value corresponding to $n = 3$ is 7, so $g(3) = 7$.
b. Solving $g(n) = 6$ means identifying the input values, n, that produce an output value of 6. Table 1.1.11 shows two solutions: 2 and 4.

When we input 2 into the function g, our output is 6. When we input 4 into the function g, our output is also 6.

1.1.7

Using Table 1.1.11, evaluate $g(1)$.

Answer

$g(1) = 8$

Finding Function Values from a Graph

Evaluating a function using a graph also requires finding the corresponding output value for a given input value, only in this case, we find the output value by looking at the graph. Solving a function equation using a graph requires finding all instances of the given output value on the graph and observing the corresponding input value(s). However, note that without an actual formula or table, we can only *guess* at the coordinate values of most points. Coordinate values of a point provide an input value (the x-coordinate) and the corresponding output value (the y-coordinate).

Example 1.1.12: Reading Function Values from a Graph

Given the graph in Figure 1.1.7,

a. Evaluate $f(2)$.
b. Solve $f(x) = 4$.

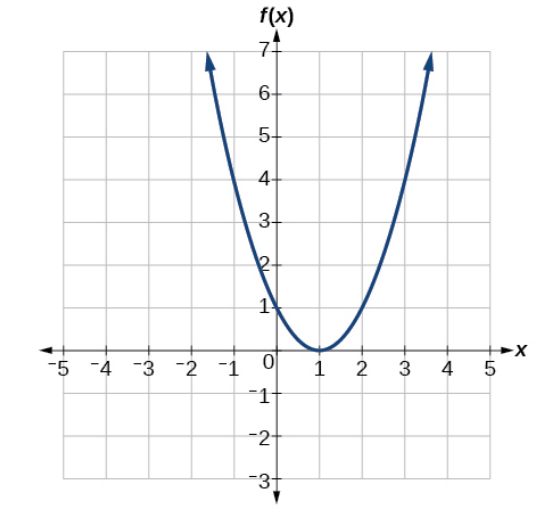


Figure* 1.1.7*: Graph of a parabola centered at $(1, 0)$.

Solution

To evaluate $f(2)$, locate the point on the curve where $x = 2$, then read the y-coordinate of that point. The point has coordinates $(2, 1)$, so $f(2) = 1$. See Figure 1.1.8.

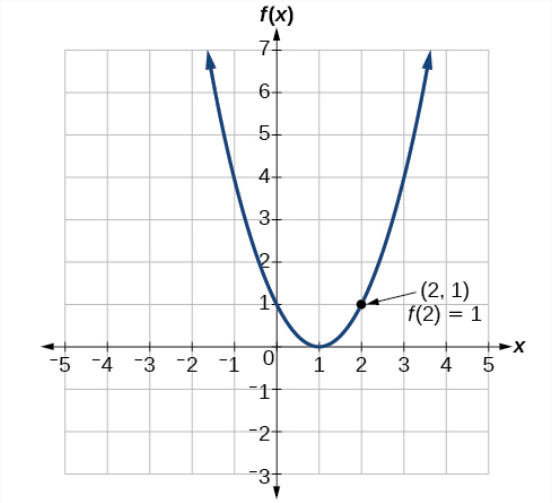


Figure* 1.1.8*: *Graph of a parabola centered at* $(1, 0)$ *with the labeled point* $(2, 1)$ *where* $f(2) = 1$.

To solve $f(x) = 4$, we find the output value 4 on the vertical axis. Moving horizontally along the line $y = 4$, we locate two points of the curve with output value 4: $(-1, 4)$ and $(3, 4)$. These points represent the two solutions to $f(x) = 4$: $x = -1$ or $x = 3$. This means $f(-1) = 4$ and $f(3) = 4$, or when the input is −1 or 3, the output is 4. See Figure 1.1.9.

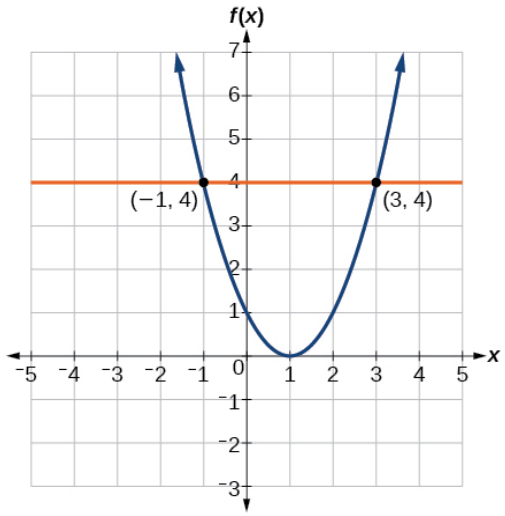


Figure* 1.1.9*: *Graph of a parabola with a vertex at* $(0,1)$ *and labeled points at* $(-1,4)$ *and* $(3,4)$*. A line at* $y=4$ *intersects the parabola at the labeled points.*

1.1.8

Given the graph in Figure **1.1.7**, solve $f(x)=1$.

Answer

$x=0$ or $x=2$

Determining Whether a Function is One-to-One

Some functions have a given output value that corresponds to two or more input values. For example, in the stock chart shown in the Figure **1.0.1** at the beginning of this chapter, the stock price was $1000 on five different dates, meaning that there were five different input values that all resulted in the same output value of $1000.

However, some functions have only one input value for each output value, as well as having only one output for each input. We call these functions one-to-one functions. As an example, consider a school that uses only letter grades and decimal equivalents, as listed in Table **1.1.12**.

Table* 1.1.12*: *Letter grades and decimal equivalents.*

Letter Grade	Grade Point Average
A	4.0
B	3.0
C	2.0
D	1.0

This grading system represents a one-to-one function, because each letter input yields one particular grade point average output and each grade point average corresponds to one input letter.

To visualize this concept, let's look again at the two simple functions sketched in Figure **1.1.1**(a) and Figure **1.1.1**(b). The function in part (a) shows a relationship that is not a one-to-one function because inputs q and r both give output n. The function in part (b) shows a relationship that is a one-to-one function because each output is associated with a single input.

definition: One-to-One Function

A **one-to-one function** is a function in which each output value corresponds to exactly one input value.

Example 1.1.13: Determining Whether a Relationship Is a One-to-One Function

Is the area of a circle a function of its radius? If yes, is the function one-to-one?

Solution

A circle of radius r has a unique area measure given by $A = \pi r^2$,so for any input, r, there is only one output, A. The area is a function of radius$r$.

If the function is one-to-one, the output value, the area, must correspond to a unique input value, the radius. Any area measure A is given by the formula $A = \pi r^2$. Because areas and radii are positive numbers, there is exactly one solution:$\sqrt{\frac{A}{\pi}}$. So the area of a circle is a one-to-one function of the circle's radius.

1.1.9

a. Is a balance a function of the bank account number?

b. Is a bank account number a function of the balance?

c. Is a balance a one-to-one function of the bank account number?

Answer

a. yes, because each bank account has a single balance at any given time;
b. no, because several bank account numbers may have the same balance;
c. no, because the same output may correspond to more than one input.

1.1.10

a. If each percent grade earned in a course translates to one letter grade, is the letter grade a function of the percent grade?
b. If so, is the function one-to-one?

Answer

a. Yes, letter grade is a function of percent grade;
b. No, it is not one-to-one. There are 100 different percent numbers we could get but only about five possible letter grades, so there cannot be only one percent number that corresponds to each letter grade.

Using the Vertical Line Test

As we have seen in some examples above, we can represent a function using a graph. Graphs display a great many input-output pairs in a small space. The visual information they provide often makes relationships easier to understand. By convention, graphs are typically constructed with the input values along the horizontal axis and the output values along the vertical axis.

The most common graphs name the input value x and the output y, and we say y is a function of x, or $y = f(x)$ when the function is named f. The graph of the function is the set of all points (x, y) in the plane that satisfies the equation $y = f(x)$. If the function is defined for only a few input values, then the graph of the function is only a few points, where the x-coordinate of each point is an input value and the y-coordinate of each point is the corresponding output value. For example, the black dots on the graph in Figure 1.1.10 tell us that $f(0) = 2$ and $f(6) = 1$. However, the set of all points (x, y) satisfying $y = f(x)$ is a curve. The curve shown includes $(0, 2)$ and $(6, 1)$ because the curve passes through those points

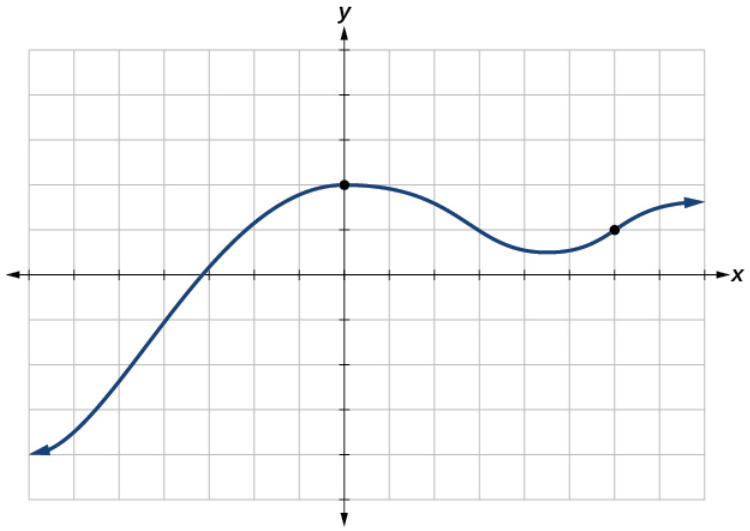


Figure* 1.1.10*: *Graph of a polynomial.*

The vertical line test can be used to determine whether a graph represents a function. If we can draw any vertical line that intersects a graph more than once, then the graph does not define a function because a function has only one output value for each input value. See Figure **1.1.11.**

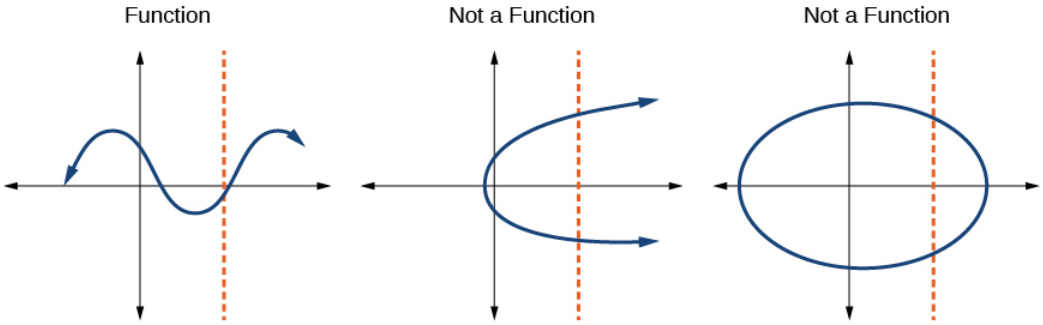


Figure* 1.1.11*: *Three graphs visually showing what is and is not a function.*

How To: Given a graph, use the vertical line test to determine if the graph represents a function

1. Inspect the graph to see if any vertical line drawn would intersect the curve more than once.
2. If there is any such line, determine that the graph does not represent a function.

Example 1.1.14: Applying the Vertical Line Test

Which of the graphs in Figure 1.1.12 represent(s) a function $y = f(x)$?

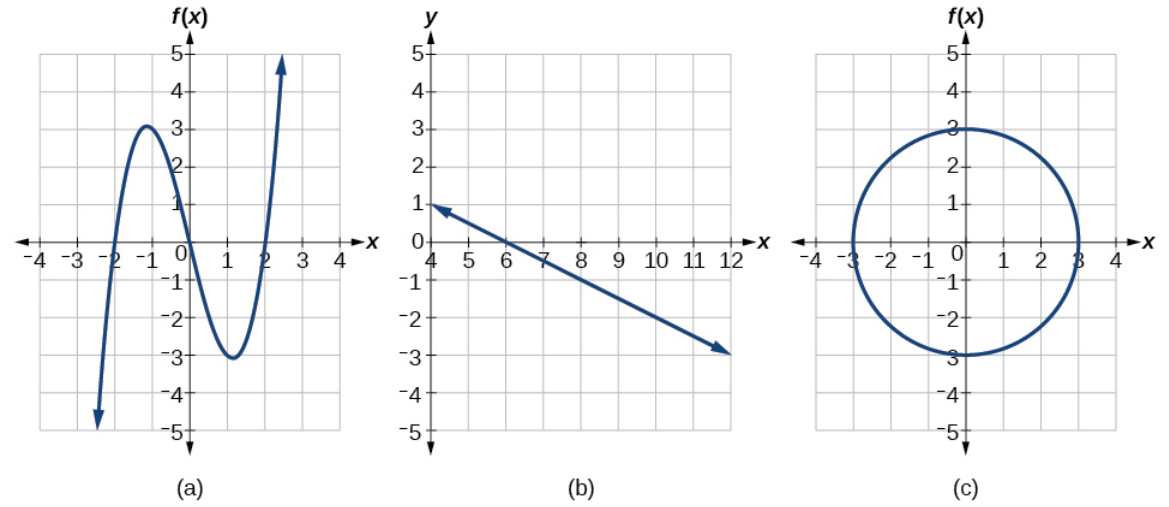


Figure* 1.1.12*: *Graph of a polynomial (a), a downward-sloping line (b), and a circle (c).*

Solution

If any vertical line intersects a graph more than once, the relation represented by the graph is not a function. Notice that any vertical line would pass through only one point of the two graphs shown in parts (a) and (b) of Figure **1.1.12**. From this we can conclude that these two graphs represent functions. The third graph does not represent a function. At almost all x-values, a vertical line intersects the graph at more than one point, as shown in Figure **1.1.13**.

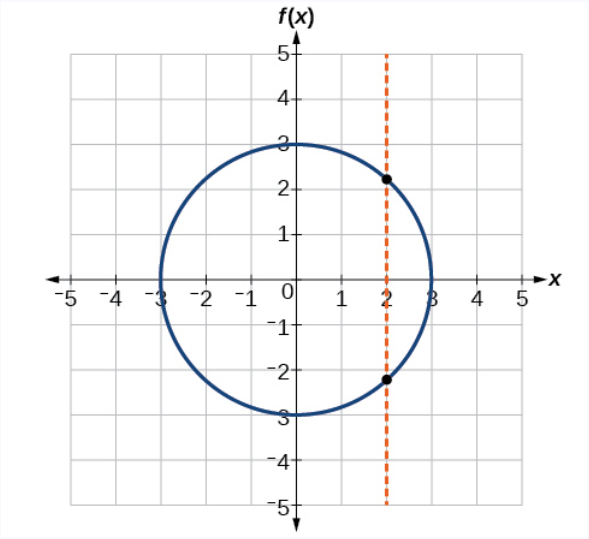


Figure* 1.1.13*: Graph of a circle.

1.1.11

Does the graph in Figure **1.1.14** represent a function?

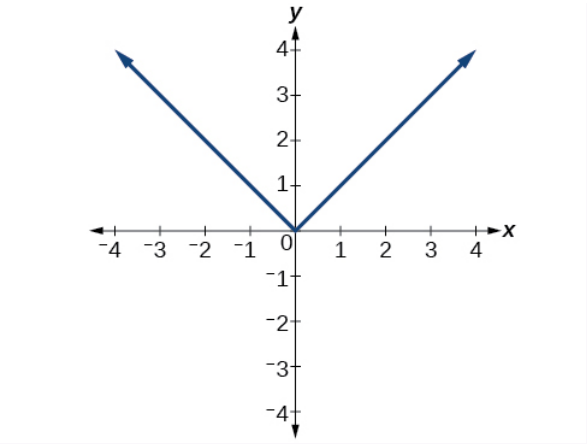


Figure* 1.1.14*: *Graph of absolute value function.*

Answer

yes

Using the Horizontal Line Test

Once we have determined that a graph defines a function, an easy way to determine if it is a one-to-one function is to use the **horizontal line test**. Draw horizontal lines through the graph. If any horizontal line intersects the graph more than once, then the graph does not represent a one-to-one function.

Why not? If a horizontal line intersects the graph of a function more than once, what must be true about the y-coordinates of the intersection points? If you can explain clearly, write it up and give it to your instructor for ***extra credit***.

How To: Given a graph of a function, use the horizontal line test to determine if the graph represents a one-to-one function

1. Inspect the graph to see if any horizontal line drawn would intersect the curve more than once.
2. If there is no such line, the function is one-to-one.

Example 1.1.15: Applying the Horizontal Line Test

Consider the functions shown in Figure 1.1.12(a) and Figure 1.1.12(b). Are either of the functions one-to-one?

Solution

The function in Figure 1.1.12 (a) is not one-to-one. The horizontal line shown in Figure 1.1.15 intersects the graph of the function at two points (and we can even find horizontal lines that intersect it at three points.)

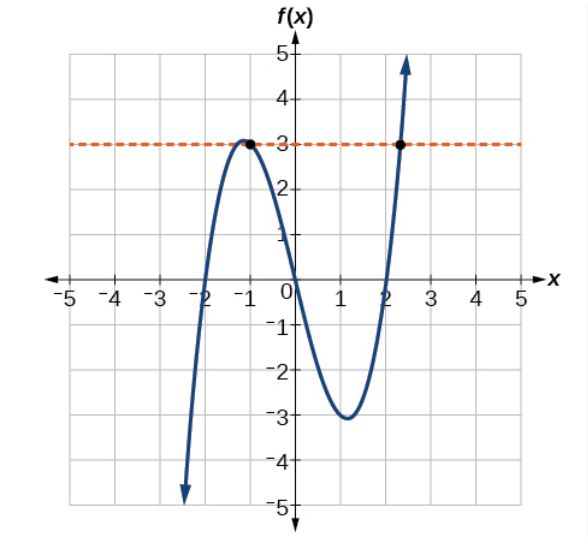


Figure* 1.1.15*: *Graph of a polynomial with a horizontal line crossing through 2 points*

The function in Figure(b) is one-to-one. Any horizontal line will intersect a diagonal line at most once.

1.1.12

Is the function shown in Figure **1.1.14** one-to-one?

Answer

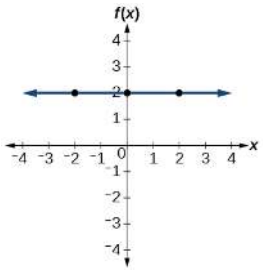
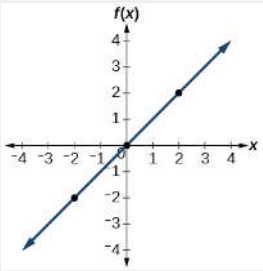
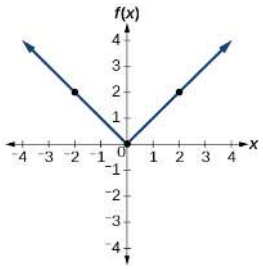
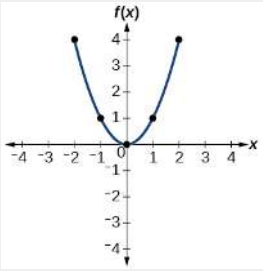
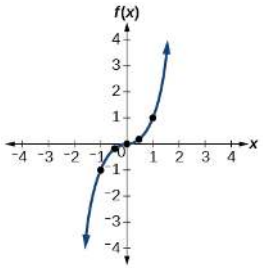
No, because it does not pass the horizontal line test.

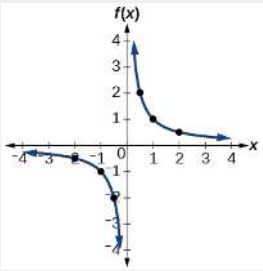
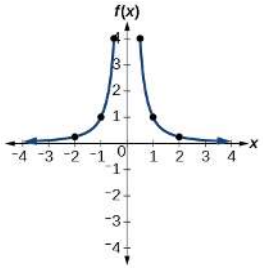
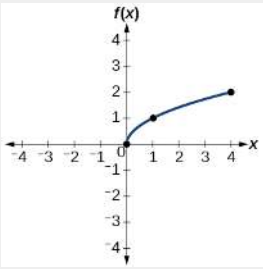
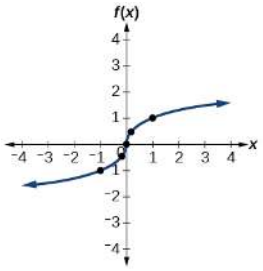
Identifying Basic Toolkit Functions

In this text, we will be exploring functions—the shapes of their graphs, their unique characteristics, their algebraic formulas, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of building-block elements. We call these our "toolkit functions," which form a set of basic named functions for which we know the graph, formula, and special properties. Some of these functions are programmed to individual buttons on many calculators. For these definitions we will use x as the input variable and $y = f(x)$ as the output variable.

We will see these toolkit functions, combinations of toolkit functions, their graphs, and their transformations frequently throughout this book. It will be very helpful if we can recognize these toolkit functions and their features quickly by name, formula, graph, and basic table properties. The graphs and sample table values are included with each function shown in Table **1.1.14**.

Table* 1.1.14*: *Toolkit Functions*

Name	Function	Graph
Constant	$f(x) = c$ where c is a constant	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>2</td></tr><tr><td>0</td><td>2</td></tr><tr><td>2</td><td>2</td></tr></table>
Identity	$f(x) = x$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>−2</td></tr><tr><td>0</td><td>0</td></tr><tr><td>2</td><td>2</td></tr></table>
Absolute Value	$f(x) = \lvert x \rvert$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>2</td></tr><tr><td>0</td><td>0</td></tr><tr><td>2</td><td>2</td></tr></table>
Quadratic	$f(x) = x^2$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−2</td><td>4</td></tr><tr><td>−1</td><td>1</td></tr><tr><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>4</td></tr></table>
Cubic	$f(x) = x^3$	 <table><tr><th>x</th><th>f(x)</th></tr><tr><td>−1</td><td>−1</td></tr><tr><td>−0.5</td><td>−0.125</td></tr><tr><td>0</td><td>0</td></tr><tr><td>0.5</td><td>0.125</td></tr><tr><td>1</td><td>1</td></tr></table>

Name	Function	Graph
reciprocal	$f(x) = \frac{1}{x}$	
		x | f(x) −2 | −0.5 −1 | −1 −0.5 | −2 0.5 | 2 1 | 1 2 | 0.5
Reciprocal squared	$f(x) = \frac{1}{x^2}$	
		x | f(x) −2 | 0.25 −1 | 1 −0.5 | 4 0.5 | 4 1 | 1 2 | 0.25
Square root	$f(x) = \sqrt{x}$	
		x | f(x) 0 | 0 1 | 1 4 | 2
Cube root	$f(x) = \sqrt[3]{x}$	
		x | f(x) −1 | −1 −0.125 | −0.5 0 | 0 0.125 | 0.5 1 | 1

Key Equations

- Constant function $f(x) = c$, where c is a constant
- Identity function $f(x) = x$
- Absolute value function $f(x) = |x|$
- Quadratic function $f(x) = x^2$
- Cubic function $f(x) = x^3$
- Reciprocal function $f(x) = \frac{1}{x}$
- Reciprocal squared function $f(x) = \frac{1}{x^2}$
- Square root function $f(x) = \sqrt{x}$
- Cube root function $f(x) = \sqrt[3]{x}$

Key Concepts

- A relation is a set of ordered pairs. A function is a specific type of relation in which each domain value, or input, leads to exactly one range value, or output. See Example 1.1.1 and Example 1.1.2.

- Function notation is a shorthand method for relating the input to the output in the form y=f(x). See Example 1.1.3 and Example 1.1.4.
- In tabular form, a function can be represented by rows or columns that relate to input and output values. See Example 1.1.5.
- An algebraic form of a function can be written from an equation. See Example 1.1.6 and Example 1.1.7.
- To evaluate a function, we determine an output value for a corresponding input value. Algebraic forms of a function can be evaluated by replacing the input variable with a given value. See Example 1.1.8 and Example 1.1.9.
- To solve for a specific function value, we determine the input values that yield the specific output value. See Example 1.1.10.
- Input and output values of a function can be identified from a table. See Example 1.1.11.
- Relating input values to output values on a graph is another way to evaluate a function. See Example 1.1.12.
- A function is one-to-one if each output value corresponds to only one input value. See Example 1.1.13.
- A graph represents a function if any vertical line drawn on the graph intersects the graph at no more than one point. See Example 1.1.14.
- The graph of a one-to-one function passes the horizontal line test. See Example 1.1.15.

Footnotes

1 http://www.baseball-almanac.com/lege.../lisn100.shtml. Accessed 3/24/2014.
2 http://www.kgbanswers.com/how-long-i...y-span/4221590. Accessed 3/24/2014.

Glossary

dependent variable
an output variable

domain
the set of all possible input values for a relation

function
a relation in which each input value yields a unique output value

horizontal line test
a method of testing whether a function is one-to-one by determining whether any horizontal line intersects the graph more than once

independent variable
an input variable

input
each object or value in a domain that relates to another object or value by a relationship known as a function

one-to-one function
a function for which each value of the output is associated with a unique input value

output
each object or value in the range that is produced when an input value is entered into a function

range
the set of output values that result from the input values in a relation

relation
a set of ordered pairs

vertical line test
a method of testing whether a graph represents a function by determining whether a vertical line intersects the graph no more than once

Contributors

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1.1E: Exercises

This page is a draft and is under active development.

1.1: Functions and Function Notation

Verbal

Exercise 1.1.1

What is the difference between a relation and a function?

Answer:

A relation is a set of ordered pairs. A function is a special kind of relation in which no two ordered pairs have the same first coordinate.

Exercise 1.1.2

What is the difference between the input and the output of a function?

Exercise 1.1.3

Why does the vertical line test tell us whether the graph of a relation represents a function?

Answer:

When a vertical line intersects the graph of a relation more than once, that indicates that for that input there is more than one output. At any particular input value, there can be only one output if the relation is to be a function.

Exercise 1.1.4

How can you determine if a relation is a one-to-one function?

Exercise 1.1.5

Why does the horizontal line test tell us whether the graph of a function is one-to-one?

Answer:

When a horizontal line intersects the graph of a function more than once, that indicates that for that output there is more than one input. A function is one-to-one if each output corresponds to only one input.

Algebraic

For the following exercises, determine whether the relation represents a function.

Exercise 1.1.6

$\{(a,b),(c,d),(a,c)\}$

Exercise 1.1.7

$\{(a,b),(b,c),(c,c)\}$

Answer:

function

For the following exercises, determine whether the relation represents y as a function of x.

Exercise 1.1.8

$5x + 2y = 10$

Exercise 1.1.9

$y = x^2$

Answer:

function

Exercise 1.1.10

$x = y^2$

Exercise 1.1.11

$3x^2 + y = 14$

Answer:

function

Exercise 1.1.12

$2x + y^2 = 6$

Exercise 1.1.13

$y = -2x^2 + 40x$

Answer:

function

Exercise 1.1.14

$y = \frac{1}{x}$

Exercise 1.1.15

$x = \frac{3y + 5}{7y - 1}$

Answer:

function

Exercise 1.1.16

$x = \sqrt{1 - y^2}$

Exercise 1.1.17

$y = \frac{3x + 5}{7x - 1}$

Answer:

function

Exercise 1.1.18

$x^2 + y^2 = 9$

Exercise 1.1.19

$2xy = 1$

Answer:

function

Exercise 1.1.20

$x = y^3$

Exercise 1.1.21

$y = x^3$

Answer:

function

Exercise 1.1.22

$y = \sqrt{1 - x^2}$

Exercise 1.1.23

$x = \pm\sqrt{1 - y}$

Answer:

function

Exercise 1.1.24

$y = \pm\sqrt{1 - x}$

Exercise 1.1.25

$y^2 = x^2$

Answer:

not a function

Exercise 1.1.26

$y^3 = x^2$

For the following exercises, evaluate the function f at the indicated values $f(-3), f(2), f(-a), -f(a), f(a+h)$.

Exercise 1.1.27

$f(x) = 2x - 5$

Answer:

$f(-3) = -11;\ f(2) = -1;\ f(-a) = -2a - 5;\ -f(a) = -2a + 5;\ f(a+h) = 2a + 2h - 5$

Exercise 1.1.28

$f(x) = -5x^2 + 2x - 1$

Exercise 1.1.29

$f(x) = \sqrt{2 - x} + 5$

Answer:

$f(-3) = \sqrt{5+5};\ f(2) = 5;\ f(-a) = \sqrt{2+a+5};\ -f(a) = -\sqrt{2-a} - 5;\ f(a+h) = \sqrt{2-a-h} + 5$

Exercise 1.1.30

$f(x) = \dfrac{6x-1}{5x+2}$

Exercise 1.1.31

$f(x) = |x-1| - |x+1|$

Answer:

$f(-3) = 2$; $f(2) = 1-3 = -2$; $f(-a) = |-a-1| - |-a+1|$; $-f(a) = -|a-1| + |a+1|$; $f(a+h) = |a+h-1| - |a+h+1|$

Exercise 1.1.32

Given the function $g(x) = 5 - x^2$, evaluate $g(x+h) - g(x))/h$, $h \neq 0$.

Exercise 1.1.33

Given the function $g(x) = x^2 + 2x$, evaluate $\dfrac{g(x) - g(a)}{x-a}$, $x \neq a$.

Answer:

$\dfrac{g(x) - g(a)}{x-a} = x + a + 2$, $x \neq a$

Exercise 1.1.34

Given the function $k(t) = 2t - 1$:

a. Evaluate $k(2)$.
b. Solve $k(t) = 7$.

Exercise 1.1.35

Given the function $f(x) = 8 - 3x$:

a. Evaluate $f(-2)$.
b. Solve $f(x) = -1$.

Answer:

a. $f(-2) = 14$
b. $x = 3$

Exercise 1.1.36

Given the function $p(c) = c^2 + c$:

a. Evaluate $p(-3)$.
b. Solve $p(c) = 2$.

Exercise 1.1.37

Given the function $f(x) = x^2 - 3x$:

a. Evaluate $f(5)$.
b. Solve $f(x) = 4$.

Answer:

a. $f(5) = 10$; b. $x = -1$ or $x = 4$

Exercise 1.1.38

Given the function $f(x) = \sqrt{x+2}$:

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a. Evaluate $f(7)$.
b. Solve $f(x) = 4$.

Exercise 1.1.39

Consider the relationship $3r + 2t = 18$.

a. Write the relationship as a function $r = f(t)$.
b. Evaluate $f(-3)$.
c. Solve $f(t) = 2$.

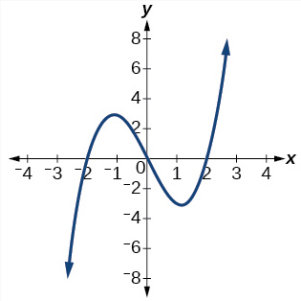
Answer:

a. $f(t) = 6 - \frac{2}{3}t$
b. $f(-3) = 8$
c. $t = 6$

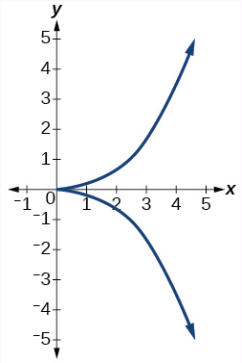
Graphical

For the following exercises, use the vertical line test to determine which graphs show relations that are functions.

Exercise 1.1.40



Exercise 1.1.41

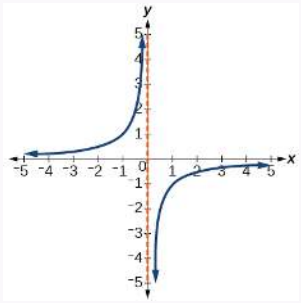


Answer:

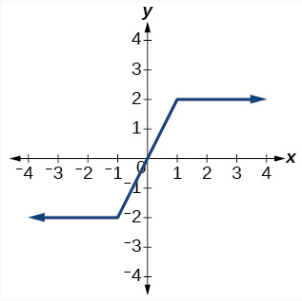
not a function

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Exercise 1.1.42



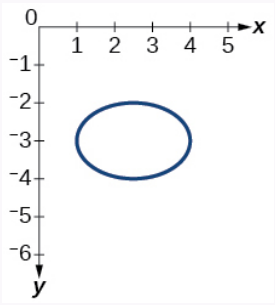
Exercise 1.1.43



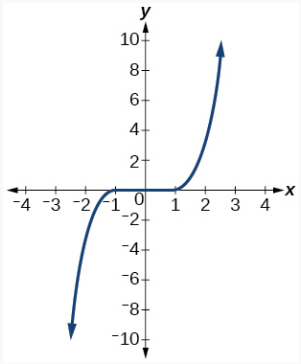
Answer:

function

Exercise 1.1.44



Exercise 1.1.45

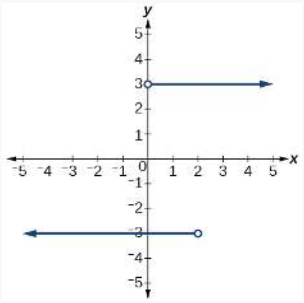


Answer:

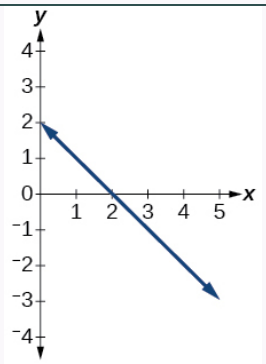
function

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Exercise 1.1.46



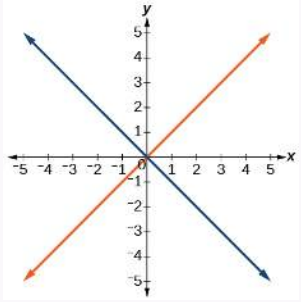
Exercise 1.1.47



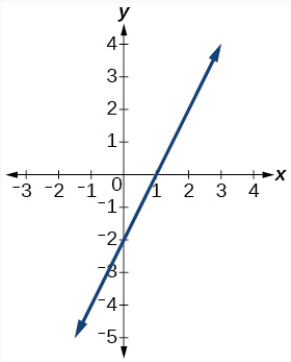
Answer:

function

Exercise 1.1.48



Exercise 1.1.49

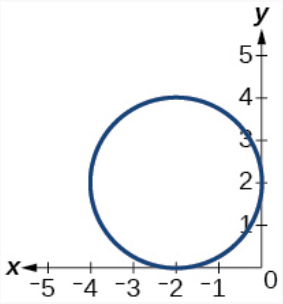


Answer:

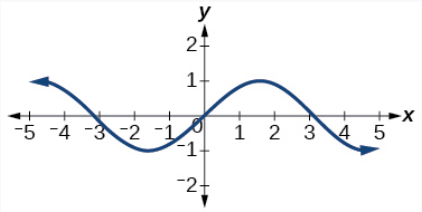
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function

Exercise 1.1.50



Exercise 1.1.51



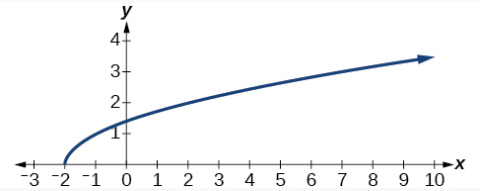
Answer:

function

Exercise 1.1.52

Given the following graph,

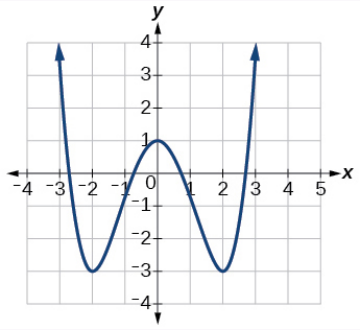
- Evaluate $f(-1)$.
- Solve for $f(x) = 3$.



Exercise 1.1.53

Given the following graph,

- Evaluate $f(0)$.
- Solve for $f(x) = -3$.



Answer:

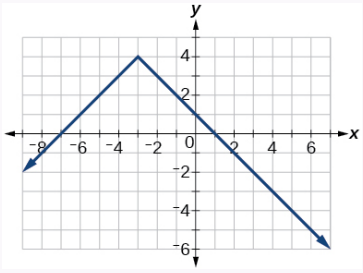
a. $f(0) = 1$

b. $f(x) = -3$, $x = -2$ or $x = 2$

Exercise 1.1.54

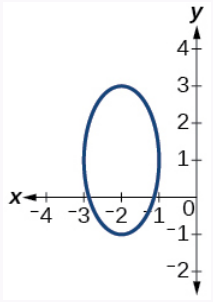
Given the following graph,

- Evaluate $f(4)$.
- Solve for $f(x) = 1$.



For the following exercises, determine if the given graph is a one-to-one function.

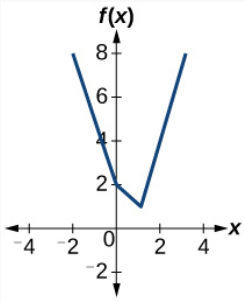
Exercise 1.1.55



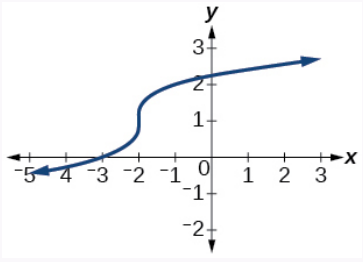
Answer:

not a function so it is also not a one-to-one function

Exercise 1.1.56



Exercise 1.1.57

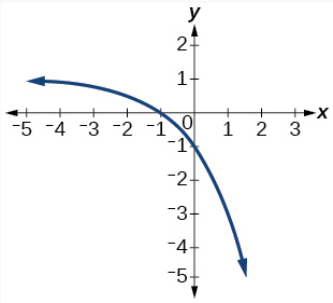


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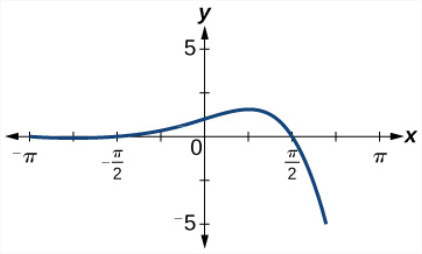
Answer:

one-to-one function

Exercise 1.1.58



Exercise 1.1.59



Answer:

function, but not one-to-one

Numeric

For the following exercises, determine whether the relation represents a function.

Exercise 1.1.60

$\{(-1,-1),(-2,-2),(-3,-3)\}$

Exercise 1.1.61

$\{(3,4),(4,5),(5,6)\}$

Answer:

function

Exercise 1.1.62

$(2,5),(7,11),(15,8),(7,9)\}$

For the following exercises, determine if the relation represented in table form represents y as a function of x.

Exercise 1.1.63

x	5	10	15
y	3	8	14

Answer:

function

Exercise 1.1.64

x	5	10	15
y	3	8	8

Exercise 1.1.65

x	5	10	10
y	3	8	14

Answer:

not a function

For the following exercises, use the function f represented in **Table 1.1.14**.

x	$f(x)$
0	74
1	28
2	1
3	53
4	56
5	3
6	36
7	45
8	14
9	47

Table 1.1.14

Exercise 1.1.66

Evaluate $f(3)$.

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Exercise 1.1.67

Solve $f(x) = 1$.

Answer:

$f(x) = 1, x = 2$

For the following exercises, evaluate the function f at the values $f(-2)$, $f(-1)$, $f(0)$, $f(1)$,and $f(2)$.

Exercise 1.1.68

$f(x) = 4 - 2x$

Exercise 1.1.69

$f(x) = 8 - 3x$

Answer:

$f(-2) = 14; f(-1) = 11; f(0) = 8; f(1) = 5; f(2) = 2$

Exercise 1.1.70

$f(x) = 8x^2 - 7x + 3$

Exercise 1.1.71

$f(x) = 3 + \sqrt{x+3}$

Answer:

$f(-2) = 4;\ f(-1) = 4.414;\ f(0) = 4.732;\ f(1) = 4.5;\ f(2) = 5.236$

Exercise 1.1.72

$f(x) = \dfrac{x-2}{x+3}$

Exercise 1.1.73

$f(x) = 3x$

Answer:

$f(-2) = 19;\ f(-1) = 13;\ f(0) = 1;\ f(1) = 3;\ f(2) = 9$

For the following exercises, evaluate the expressions, given functions f, g, and h:

- $f(x) = 3x - 2$
- $g(x) = 5 - x^2$
- $h(x) = -2x^2 + 3x - 1$

Exercise 1.1.74

$3f(1) - 4g(-2)$

Exercise 1.1.75

$f\left(\frac{7}{3}\right) - h(-2)$

Answer:

20

Technology

For the following exercises, graph $y = x^2$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.

Exercise 1.1.76

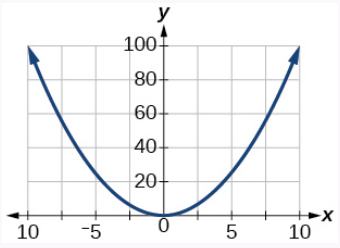
$[-0.1, 0.1]$

Exercise 1.1.77

$[-10, 10]$

Answer:

$[0, 100]$



Exercise 1.1.78

$[-100, 100]$

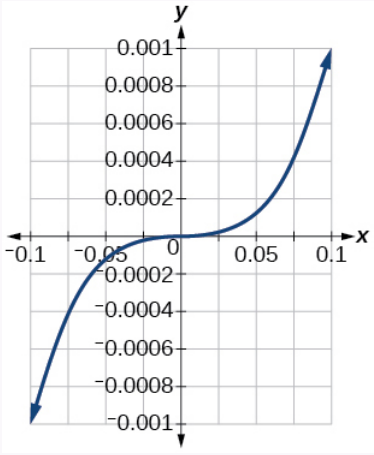
For the following exercises, graph $y = x^3$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.

Exercise 1.1.79

$[-0.1, 0.1]$

Answer:

$[-0.001, 0.001]$



Exercise 1.1.80

$[-10, 10]$

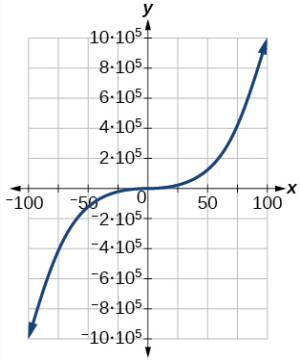
Exercise 1.1.81

$[-100, 100]$

Answer:

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$[-1,000,000, 1,000,000]$



For the following exercises, graph $y = \sqrt{x}$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.

Exercise 1.1.82

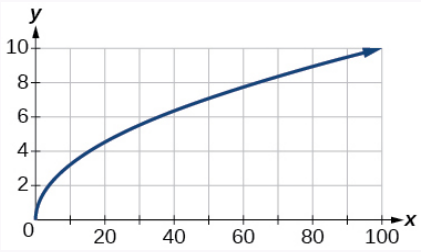
$[0, 0.01]$

Exercise 1.1.83

$[0, 100]$

Answer:

$[0, 10]$



Exercise 1.1.84

$[0, 10,000]$

For the following exercises, graph $y = \sqrt[3]{x}$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.

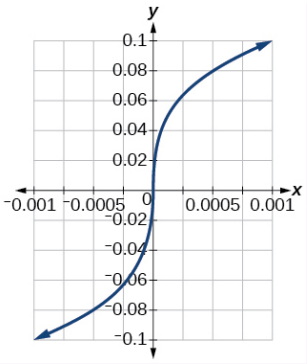
Exercise 1.1.85

$[-0.001, 0.001]$

Answer:

$[-0.1, 0.1]$

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Exercise 1.1.86

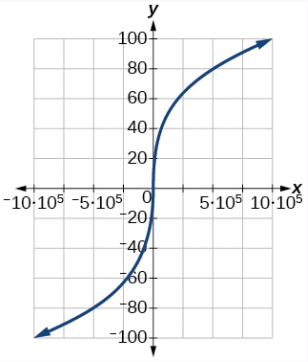
$[-1000, 1000]$

Exercise 1.1.87

$[-1,000,000, 1,000,000]$

Answer:

$[-100, 100]$



Real-World Applications

Exercise 1.1.88

The amount of garbage, G, produced by a city with population p is given by $G = f(p)$. G is measured in tons per week, and p is measured in thousands of people.

The town of Tola has a population of 40,000 and produces 13 tons of garbage each week. Express this information in terms of the function f.Tola has a population of 40,000 and produces 13 tons of garbage each week. Express this information in terms of the function f.
Explain the meaning of the statement $f(5) = 2$.

Exercise 1.1.89

The number of cubic yards of dirt, D, needed to cover a garden with area a square feet is given by $D = g(a)$.

a. A garden with area 5000 ft^2 requires 50 yd^3 of dirt. Express this information in terms of the function g.
b. Explain the meaning of the statement $g(100) = 1$.

Answer:

a. $g(5000) = 50$
b. The number of cubic yards of dirt required for a garden of 100 square feet is 1.

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Exercise 1.1.90

Let $f(t)$ be the number of ducks in a lake t years after 1990. Explain the meaning of each statement:

a. f(5)=30
b. f(10)=40

Exercise 1.1.91

Let $h(t)$ be the height above ground, in feet, of a rocket t seconds after launching. Explain the meaning of each statement:

a. $h(1) = 200$
b. $h(2) = 350$

Answer:

a. The height of a rocket above ground after 1 second is 200 ft.
b. the height of a rocket above ground after 2 seconds is 350 ft.

Exercise 1.1.92

Show that the function $f(x) = 3(x-5)^2 + 7$ is not one-to-one.

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1.2: Domain and Range

Skills to Develop

- Find the domain of a function defined by an equation.
- Find the domain and range of a function from a graph.
- Graph piecewise-defined functions.

If you're in the mood for a scary movie, you may want to check out one of the five most popular horror movies of all time—I am Legend, Hannibal, The Ring, The Grudge, and The Conjuring. Figure 1.2.1 shows the amount, in dollars, each of those movies grossed when they were released as well as the ticket sales for horror movies in general by year. Notice that we can use the data to create a function of the amount each movie earned or the total ticket sales for all horror movies by year. In creating various functions using the data, we can identify different independent and dependent variables, and we can analyze the data and the functions to determine the domain and range. For functions defined by an equation rather than by data, determining the domain and range requires a different kind of analysis. In this section, we will investigate methods for determining the domain and range of functions such as these.

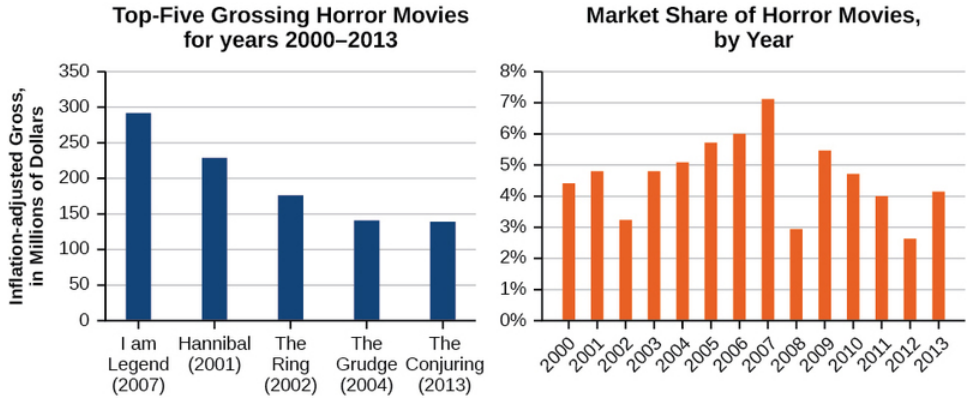


Figure* 1.2.1*: *Graph of the Top-Five Grossing Horror Movies for years 2000-2003, and a Graph of the Market Share of Horror Movies by Year. Based on data compiled by www.the-numbers.com.*

In Section 1.1, Functions and Function Notation, we were introduced to the concepts of **domain** and **range**. In this section, we will practice determining domains and ranges for specific functions. Keep in mind that in determining domains and ranges, we need to consider what is physically possible or meaningful in real-world examples, such as tickets sales and year in the horror movie example above. We also need to consider what is mathematically permitted. In this course, unless specifically stated otherwise, we are interested in functions whose inputs and outputs are always real numbers. For example, we cannot include any input value that leads us to take an even root of a negative number if the domain and range consist of real numbers. Or in a function expressed as a quotient, we cannot include any input value in the domain that would lead us to divide by 0.

We can visualize the domain as a "holding area" that contains "raw materials" for a "function machine" and the range as another "holding area" for the machine's outputs (Figure 1.2.2).

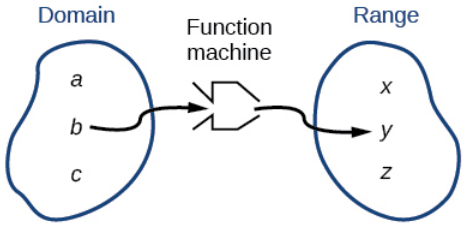


Figure* 1.2.2*: *Diagram of how a function relates two sets*

Example 1.2.1: Finding the Domain of a Function as a Set of Ordered Pairs

Find the domain and range of the following function: $\{(2,10),(3,10),(4,20),(5,30),(6,40)\}$.

Solution

First identify the input values. The input value is the first coordinate in an ordered pair. There are no restrictions, as the ordered pairs are simply listed. The domain is the set of the first coordinates of the ordered pairs.

$$D = \{2, 3, 4, 5, 6\}$$

Next identify the output values. The output value is the second coordinate in an ordered pair. The range is the set of the second coordinates of the ordered pairs. Do not list the same output twice.

$$R = \{10, 20, 30, 40\}$$

1.2.1

Find the domain and range of the function:

$$\{(-5, 4), (0, 0), (5, -4), (10, -8), (15, -12)\}$$

Answer

$D = \{-5, 0, 5, 10, 15\}$
$R = \{4, 0, -4, -8, -12\}$

Finding the Domain of a Function Defined by an Equation

Let's turn our attention to finding the domain of a function whose equation is provided. Finding the range is harder. We will see later on in this section how to find the range if we have a graph of the function. Oftentimes, finding the domain of such functions involves remembering three different forms. First, if the function has no denominator or an even root, consider whether the domain could be all real numbers. Second, if there is a denominator in the function's equation, exclude values from the domain that force the denominator to be zero. Third, if there is an even root, consider excluding values that would make the radicand negative.

We can often write the **domain** and **range** in **interval notation**, which uses values within brackets to describe a set of numbers. In interval notation, we use a square bracket [when the set includes the endpoint and a parenthesis (to indicate that the endpoint is either not included or the interval is unbounded. For example, if a person has $100 to spend, he or she might express the interval that is more than 0 and less than or equal to 100 and write $(0, 100]$. **Note:** This would not be quite correct, since this interval includes **all** real numbers between 0 and 100, and one cannot spend $\$\sqrt{5}$.

Recall the conventions for sets written using interval notation:

- The smallest term from the interval is written first.
- The largest term in the interval is written second, following a comma.
- All real numbers between the smallest and largest term are included in the set.
- Parentheses, "(" or ")," are used to signify that an endpoint is not included, called **exclusive**.
- Brackets, "[" or "]," are used to indicate that an endpoint is included, called **inclusive**.

See Figure **1.2.3** for a summary of interval notation.

Inequality	Interval Notation	Graph on Number Line	Description
$x > a$	(a, ∞)		x is greater than a
$x < a$	$(-\infty, a)$		x is less than a
$x \geq a$	$[a, \infty)$		x is greater than or equal to a
$x \leq a$	$(-\infty, a]$		x is less than or equal to a
$a < x < b$	(a, b)		x is strictly between a and b
$a \leq x < b$	$[a, b)$		x is between a and b, to include a
$a < x \leq b$	$(a, b]$		x is between a and b, to include b
$a \leq x \leq b$	$[a, b]$		x is between a and b, to include a and b

***Figure* 1.2.3:** *Summary of interval notation.*

How To: Given a function written in equation form, find the domain.

1. Identify where the input values appear in the equation.
2. Begin with the set of all real numbers as the possible domain.
3. Identify any restrictions on the input and exclude those values from the set of real numbers.
4. Write the domain in interval form, if possible.

Example 1.2.2: Finding the Domain of a Function written in equation form

Find the domain of the function $f(x) = x^2 - 1$.

Solution

The input value, shown by the variable x in the equation, is squared and then the result is decreased by one. Any real number may be squared and then be decreased by one, so there are no restrictions on the domain of this function. The domain is the set of real numbers.

In interval form the domain of f is $D = (-\infty, \infty)$, or alternatively $D = \mathbb{R}$.

1.2.2

Find the domain of the function:

$$f(x) = 5 - x + x^3$$

Answer

$D = (-\infty, \infty)$, or $\mathbb{R}$

How To: Given a function written in an equation form that includes a fraction, find the domain

1. Identify where the input values appear in the equation.
2. Begin with the set of all real numbers as the possible domain.
3. Identify any restrictions on the input. If the denominator contains an input value, set the denominator equal to zero and solve for x . The domain must be restricted to **exclude** all solutions to this equation.
4. Write the domain in interval form, making sure to exclude any restricted values from the domain.

Example 1.2.3: Finding the Domain of a Function Involving a Denominator

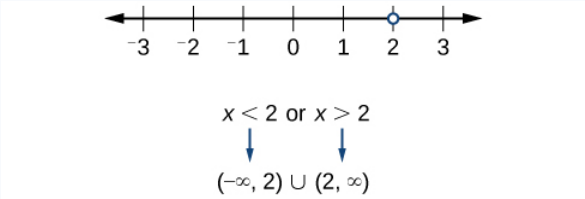
Find the domain of the function $f(x) = \dfrac{x+1}{2-x}$.

Solution

When there is a denominator, we want to exclude any x-values that force the denominator to be zero. To find these values, we set the denominator equal to 0 and solve for x.

$$\begin{aligned} 2 - x &= 0 \\ -x &= -2 \\ x &= 2 \end{aligned}$$

Now, we will exclude 2 from the domain. The domain consists of all real numbers that do not equal 2, which means $x < 2$ or $x > 2$. We can use a symbol known as the union, $\cup$, to combine the two sets.



In interval form, the domain of f is $D = (-\infty, 2) \cup (2, \infty)$.

1.2.3

Find the domain of the function:

$$f(x) = \frac{1 + 4x}{2x - 1}$$

Answer

$$D = (-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$$

Good for you, to try this problem! Get your professor to give you extra credit for **Try It 1.2.3**.

Given a function written as an equation that includes an even root, find the domain

1. Identify where the input values appear in the equation.
2. Identify any restrictions on the input. Since there is an even root, only include inputs that do not result in a negative number in the radicand. Set the radicand greater than or equal to zero and solve for x.
3. The solution(s) are the domain of the function. If possible, write the answer in interval form.

Example 1.2.4: Finding the Domain of a Function with an Even Root

Find the domain of the function:

$$f(x) = \sqrt{7 - x}.$$

Solution

When there is an even root in the formula, we exclude any real numbers that result in a negative number in the radicand.

Set the radicand greater than or equal to zero and solve for x.

$$\begin{aligned} 7 - x &\geq 0 \\ -x &\geq -7 \\ x &\leq 7 \end{aligned}$$

Now, we will include any real number less than or equal to 7 in the domain. The domain is $D = (-\infty, 7]$.

1.2.4

Find the domain of the function

$$f(x) = \sqrt{x + 52}.$$

Answer

$$[-52, \infty)$$

Can there be functions in which the domain and range do not intersect at all?

Yes. For example, the function $f(x) = -\sqrt{\frac{1}{x}}$ has the set of all positive real numbers as its domain but the set of all negative real numbers as its range. As a more extreme example, a function's inputs and outputs can be completely different categories (for example, names of weekdays as inputs and numbers as outputs, as on an attendance chart), in such cases the domain and range have no elements in common.

Using Different Notations to Specify Domain

In the previous examples, we used interval notation to describe the domain of functions. We can also use inequalities, or other statements that might define sets of values or data, to describe the domain in **set-builder notation**. For example, $\{x \mid 10 \leq x < 30\}$ describes the allowed values of x in set-builder notation. The braces "{}" are read as "the set of," and the vertical bar "|" is read as "such that," so we would read $\{x \mid 10 \leq x < 30\}$ as "the set of x-values such that 10 is less than or equal to x, and x is less than 30."

Figure 1.2.4 compares inequality notation, set-builder notation, and interval notation.

	Inequality Notation	Set-builder Notation	Interval Notation
	$5 < h \le 10$	$\{h \mid 5 < h \le 10\}$	$(5, 10]$
	$5 \le h < 10$	$\{h \mid 5 \le h < 10\}$	$[5, 10)$
	$5 < h < 10$	$\{h \mid 5 < h < 10\}$	$(5, 10)$
	$h < 10$	$\{h \mid h < 10\}$	$(-\infty, 10)$
	$h \ge 10$	$\{h \mid h \ge 10\}$	$[10, \infty)$
	All real numbers	$\mathbb{R}$	$(-\infty, \infty)$

***Figure* 1.2.4:** *Summary of notations for inequalities, set-builder, and interval.*

In set notation, there is a symbol $\cup$ to represent "or," and we say we are taking the union of the two sets. We saw this in earlier examples, to combine two unconnected intervals. As another example, we can take the union of the sets $\{2, 3, 5\}$ and $\{4, 5, 6\}$: $\{2, 3, 5\} \cup \{4, 6\} = \{2, 3, 4, 5, 6\}$. It is the set of all elements that belong to one or the other (or both) of the original two sets. For sets with a finite number of elements like these, the elements do not have to be listed in ascending order of numerical value. If the original two sets have some elements in common, those elements should be listed only once in the union set. For sets of real numbers on intervals, another example of a union is

$$\{x \mid |x| \ge 3\} = (-\infty, -3] \cup [3, \infty) \tag{1.2.1}$$

Set-Builder Notation compared with Interval Notation

Set-builder notation is a method of specifying a set of elements that satisfy a certain condition. It takes the form $\{x \mid \text{statement about } x\}$ which is read as "the set of all x such that the statement about x is true." For example, we can write

$$\{x \mid 4 < x \le 12\},$$

which is the set of all x such that x is greater than 4 and less than or equal to 12.

Interval notation is a way of describing sets that include all real numbers between a lower limit that may or may not be included and an upper limit that may or may not be included. The endpoint values are listed between brackets or parentheses. A square bracket indicates inclusion in the set, and a parenthesis indicates exclusion from the set. For example, the same set just given in set-builder notation can also be written in interval notation as

$$(4, 12].$$

How To: Given a line graph, describe the set of values using interval notation

1. Identify the intervals to be included in the set by determining where the graphed line overlays the real number line.
2. At the left end of each interval, use "[" for each end value to be included in the set (solid dot) or "(" for each excluded end value (open dot).

3. At the right end of each interval, use "]" for each end value to be included in the set (filled dot) or ")" for each excluded end value (open dot).
4. Use the union symbol ∪ to combine all intervals into one set.

Example 1.2.5: Describing Sets on the Real-Number Line

Describe the intervals of values shown in Figure **1.2.5** using inequality notation, set-builder notation, and interval notation.



***Figure* 1.2.5:** *Line graph of* $1 \le x \le 3$ *or* $5 < x$.

Solution

To describe the values x included in the intervals shown, we would say, "x is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5."

Set-builder Notation

$$\{x \mid 1 \le x \le 3 \text{ or } x > 5\}$$

Interval notation

$$[1, 3] \cup (5, \infty)$$

Remember that when writing or reading interval notation, using a square bracket means the boundary value is included in the set. Using a parenthesis means the boundary value is not included in the set.

1.2.5

Given Figure **1.2.6**, specify the graphed set in

a. words
b. set-builder notation
c. interval notation



***Figure* 1.2.6:** *Line graph of* $x \le -2$ *or* $-1 \le x < 3$.

Answer

a. Values that are less than or equal to –2, or values that are greater than or equal to –1 and less than 3
b. $\{x \mid x \le -2 \text{ or } -1 \le x < 3\}$
c. $(-\infty, -2] \cup [-1, 3)$

Finding Domain and Range from Graphs

Another way to identify the domain of a function is by using graphs. A graph is also an excellent way to identify the range of a function. Because the domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the x-axis. The range is the set of possible output values, which are shown on the y-axis. Keep in mind that if the graph continues beyond the portion of the graph we can see, the domain and range may be greater than the visible values. See Figure **1.2.7**.

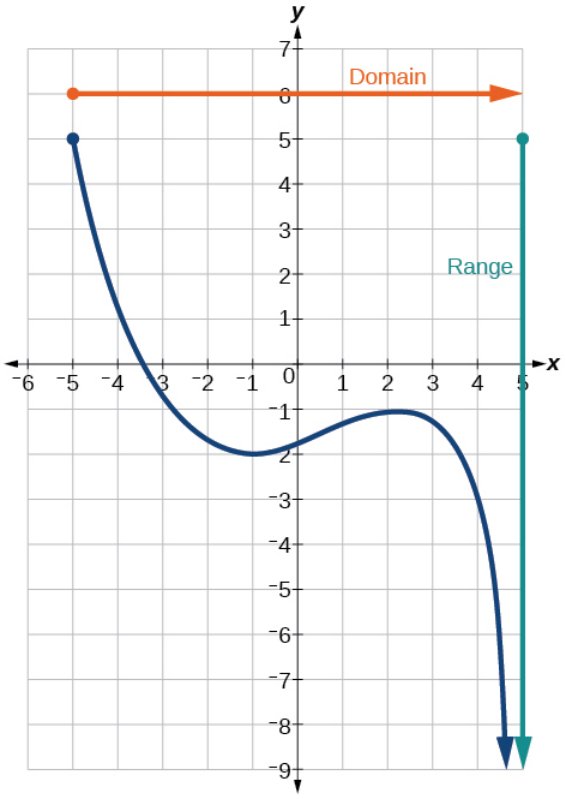


***Figure* 1.2.7:** *Graph of a polynomial that shows the x-axis contains the domain and the y-axis contains the range*

Observe that the graph extends horizontally from −5 to the right without bound, so the domain is $[-5, \infty)$. The vertical extent of the graph is all values from 5 and below, so the range is $(-\infty, 5]$. Note that the domain and range are always written from smaller to larger values, or from left to right for domain, and from the bottom of the graph to the top of the graph for range.

Example 1.2.6*A*: Finding Domain and Range from a Graph

Find the domain and range of the function f whose graph is shown in Figure 1.2.8.

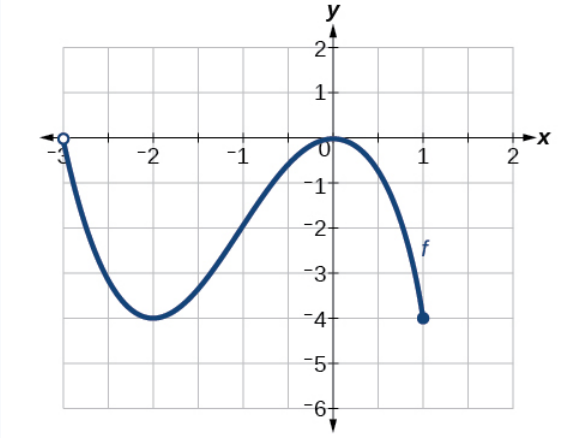


***Figure* 1.2.8:** *Graph of a function defined on (-3, 1].*

Solution

Observe that the horizontal extent of the graph is –3 to 1, not including $x = -3$ but including $x = 1$, so the domain of f is $(-3, 1]$.

The vertical extent of the graph is 0 to –4. The range is $[-4, 0]$. Although 0 is not the output for $x = -3$; in fact, there is no output for $x = -3$ since -3 is not part of the domain; we can see that the point $(0, 0)$ is on the graph, which means $f(0) = 0$, and

so 0 **is** in the range. See Figure **1.2.9**.

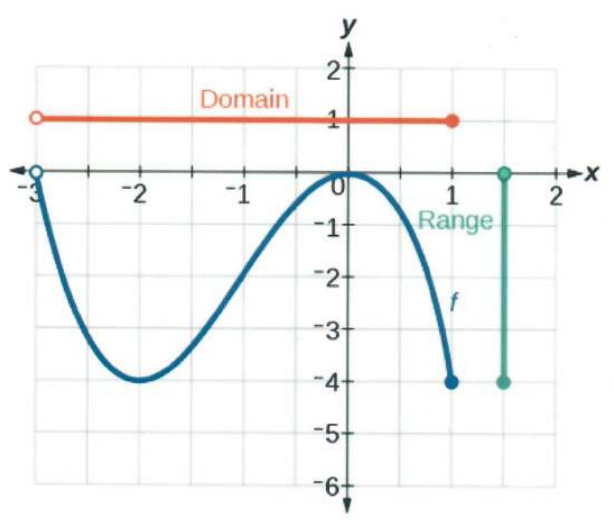


Figure* 1.2.9*: *Graph of the previous function showing the domain and range*

Example 1.2.6*B*: Finding Domain and Range from a Graph of Oil Production

Find the domain and range of the function f whose graph is shown in Figure **1.2.10**.

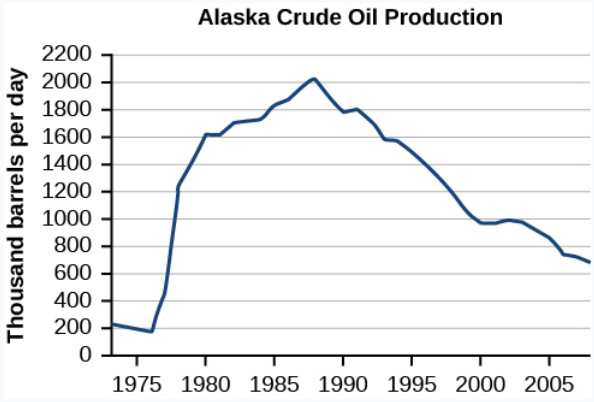


Figure* 1.2.10*: *Graph of the Alaska Crude Oil Production where the vertical axis is thousand barrels per day and the horizontal axis is years (credit: modification of work by the U.S. Energy Information Administration)*

Solution:

The input quantity along the horizontal axis is "years," which we represent with the variable t for time. The output quantity is "thousands of barrels of oil per day," which we represent with the variable b for barrels. The graph may continue to the left and right beyond what is viewed, but based on the portion of the graph that is visible, we can determine the domain as $1973 \le t \le 2008$ and the range as approximately $180 \le b \le 2010$. **Note:** This is not really correct, because not every real number is included in the domain, nor in the range. The graph represents a *mathematical model* of a real-life situation.

In interval notation, the domain is $[1973, 2008]$, and the range is about $[180, 2010]$. For the domain and the range, we approximate the smallest and largest values since they do not fall exactly on the grid lines.

1.2.6

Given Figure **1.2.11**, identify the approximate domain and range using interval notation.

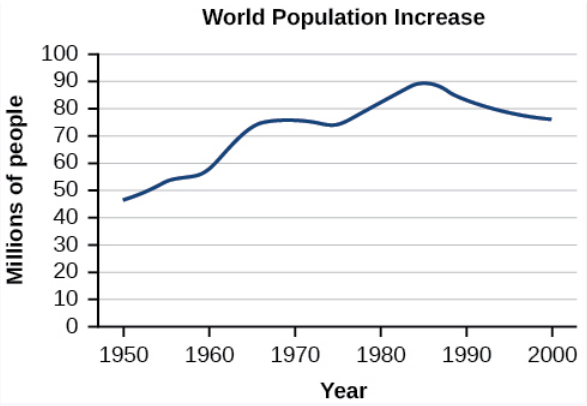


***Figure* 1.2.11:** *Graph of World Population Increase where the y-axis represents millions of people and the x-axis represents the year.*

Answer

domain =$[1950, 2000]$

range = $[4.7 \times 10^7, 9.0 \times 10^7]$

Can a function's domain and range be the same?

Yes. For example, the domain and range of the cube root function are both the set of all real numbers.

Finding Domains and Ranges of the Toolkit Functions

We will now return to our set of toolkit functions to determine the domain and range of each.

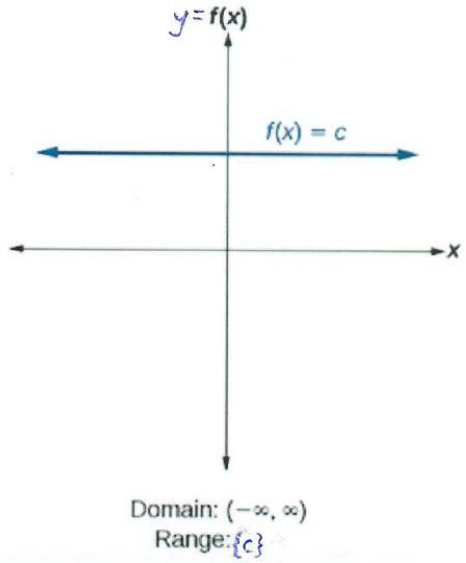


***Figure* 1.2.12:** *Constant function* $f(x) = c$.

For the **constant function** $f(x) = c$, the domain consists of all real numbers; there are no restrictions on the input. The only output value is the constant c, so the range is the set $\{c\}$ that contains this single element. In interval notation, this could be written as $[c, c]$, the interval that both begins and ends with c.

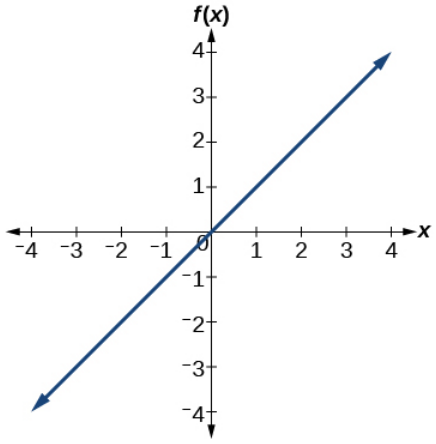


***Figure* 1.2.13:** *Identity function* $f(x) = x$.

For the **identity function** $f(x) = x$, there is no restriction on x. Both the domain and range are the set of all real numbers.

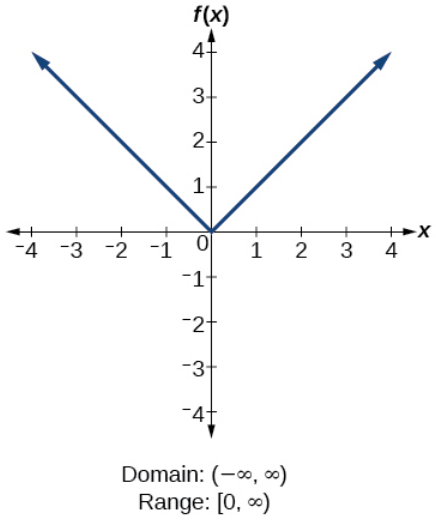


***Figure* 1.2.14:** *Absolute function* $f(x) = |x|$.

For the **absolute value function** $f(x) = |x|$, there is no restriction on x. However, because absolute value is defined as a distance from 0, the output can only be greater than or equal to 0.

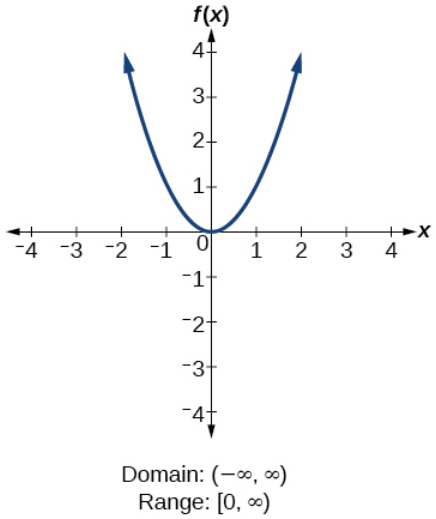


***Figure* 1.2.15:** *Quadratic function* $f(x) = x^2$.

For the **quadratic function** $f(x) = x^2$, the domain is all real numbers since the horizontal extent of the graph is the whole real number line. Because the graph does not include any negative values for the range, the range is only nonnegative real numbers.

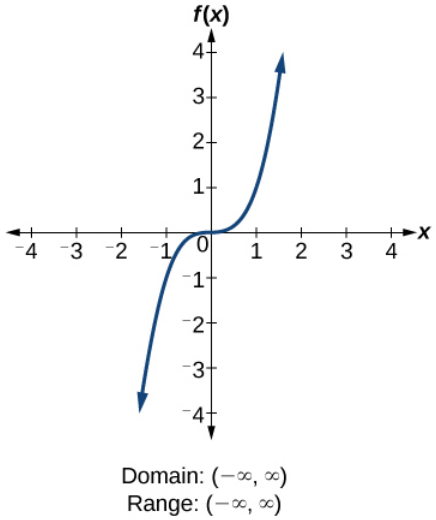
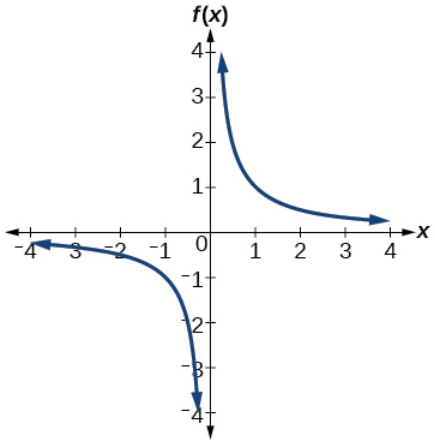


***Figure* 1.2.16:** *Cubic function* $f(x) = x^3$.

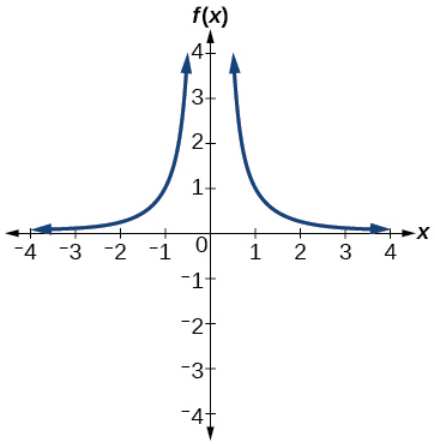
For the **cubic function** $f(x) = x^3$, the domain is all real numbers because the horizontal extent of the graph is the whole real number line. The same applies to the vertical extent of the graph, so the domain and range include all real numbers.



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$

***Figure* 1.2.17:** *Reciprocal function* $f(x) = \frac{1}{x}$.

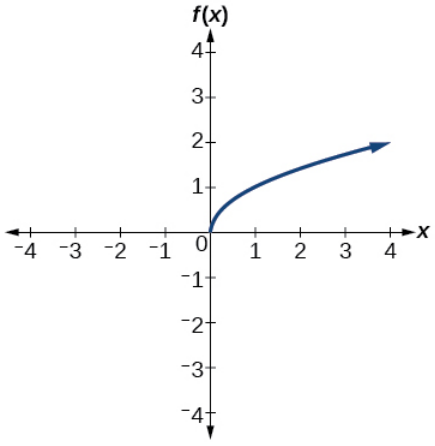
For the **reciprocal function** $f(x) = \frac{1}{x}$, we cannot divide by 0, so we must exclude 0 from the domain. Further, 1 divided by any value can never be 0, so the range also will not include 0. In set-builder notation, we could also write $\{x|x \neq 0\}$, the set of all real numbers that are not zero, for both the domain and the range.



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(0, \infty)$

***Figure* 1.2.18:** *Reciprocal squared function* $f(x) = \frac{1}{x^2}$

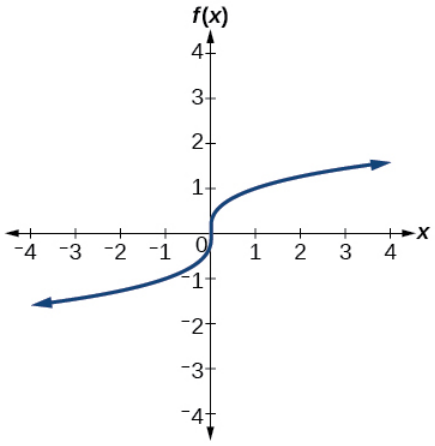
For the **reciprocal squared function** $f(x) = \frac{1}{x^2}$,we cannot divide by 0, so we must exclude 0 from the domain. There is also no x that can give an output of 0, so 0 is excluded from the range as well. Note that the output of this function is always positive due to the square in the denominator, so the range includes only positive numbers.



Domain: [0, ∞)
Range: [0, ∞)

***Figure* 1.2.19:** *Square root function* $f(x) = \sqrt{x}$.

For the **square root function** $f(x) = \sqrt{x}$, we cannot take the square root of a negative real number, so the domain must be 0 or greater. The range also excludes negative numbers because the square root symbol $\sqrt{x}$ is defined to be the positive square root of x, even though x also has a negative square root, denoted $-\sqrt{x}$.



Domain: (−∞, ∞)
Range: (−∞, ∞)

***Figure* 1.2.20:** *Cube root function* $f(x) = \sqrt[3]{x}$.

For the **cube root function** $f(x) = \sqrt[3]{x}$, the domain and range include all real numbers. Note that there is no problem taking a cube root, or any odd-integer root, of a negative number, and the resulting output is negative (it is an odd function).

How To: Given the formula for a function, determine the domain and range

1. Exclude from the domain any input values that result in division by zero.
2. Exclude from the domain any input values that have nonreal (or undefined) number outputs.
3. If possible, use the valid input values to determine the range of the output values.
4. Look at the function graph and/or table values to confirm the actual function behavior.

Example 1.2.7: Finding the Domain Using Toolkit Functions

Find the domain of $f(x) = 2x^3 - x$.

Solution

This function uses two Toolkit Functions as building blocks. There are no restrictions on the domain, as any real number may be cubed and then subtracted from the result.

The domain is $(-\infty, \infty)$.

Example 1.2.8: Finding the Domain

Find the domain of $f(x) = \frac{2}{x+1}$.

Solution

We cannot evaluate the function at $x = -1$ because division by zero is undefined. The domain is $(-\infty, -1) \cup (-1, \infty)$.

Example 1.2.9: Finding the Domain

Find the domain of $f(x) = 2\sqrt{x+4}$.

Solution

We cannot take the square root of a negative number, so the value inside the radical must be nonnegative.

$x + 4 \geq 0$ when $x \geq -4$

The domain of $f(x)$ is $[-4, \infty)$.

We can also find the range, by comparing this function with the Toolkit Function $g(x) = \sqrt{x}$. We know that $f(-4) = 0$, and the function value increases as x increases without any upper limit. We conclude that the range of f is $[0, \infty)$.

Analysis

Figure 1.2.19 represents the graph of the function f.

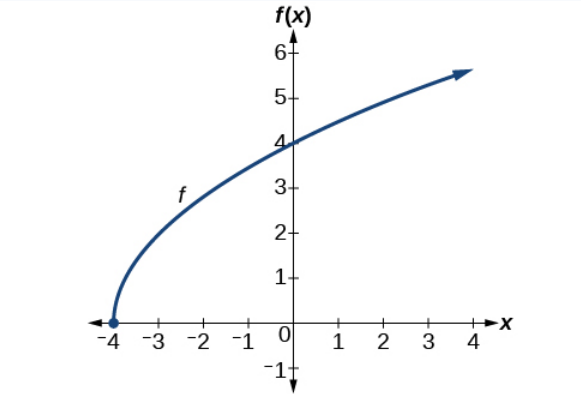


***Figure* 1.2.19:** *Graph of a square root function which has been shifted to* $(-4, 0)$.

1.2.7

Find the domain of

$f(x) = \sqrt{-2 - x}$.

Answer

domain: $(-\infty, 2]$

Graphing Piecewise-Defined Functions

Sometimes, we come across a function that requires more than one formula in order to obtain the given output. For example, in the toolkit functions, we introduced the absolute value function $f(x) = |x|$. With a domain of all real numbers and a range of values

greater than or equal to 0, **absolute value** can be defined as the **magnitude**, or **modulus**, of a real number value regardless of sign. It is the distance from 0 on the number line. All of these definitions require the output to be greater than or equal to 0.

If we input 0, or a positive value, the output is the same as the input.

$$f(x) = x \text{ if } x \geq 0$$

If we input a negative value, the output is the opposite of the input. For example, $f(-3) = -(-3) = 3$.

$$f(x) = -x \text{ if } x < 0$$

Because this requires two different processes or pieces, the absolute value function is an example of a **piecewise function**. A piecewise function is a function in which more than one formula is used to define the output over different pieces of the domain.

We use piecewise functions to describe situations in which a rule or relationship changes as the input value crosses certain "boundaries." Tax brackets are a real-world example of piecewise functions. For example, consider a simple tax system in which incomes up to \$10,000 are taxed at 10%, and any additional income is taxed at 20%. The tax T on a total income S would be

$$T(S) = \begin{cases} 0.1S \text{ if } S \leq \$10,000 \\ \$1000 + 0.2(S - \$10,000) \text{ if } S > \$10,000. \end{cases}$$

Can you give a clear explanation of the reason for the formula of each piece of this income tax function? If so, write it up and give it to your instructor for ***Extra Credit***.

Piecewise Function

A piecewise function is a single function in which more than one formula is used to define the output. Each formula has its own domain; however, the domain of the function is the union of all these smaller domains. We notate this idea as follows:

$$f(x) = \begin{cases} \text{formula 1} & \text{if x is in domain 1} \\ \text{formula 2} & \text{if x is in domain 2} \\ \text{formula 3} & \text{if x is in domain 3} \end{cases}$$

In piecewise notation, the absolute value function is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The domain for the absolute value function is the union of $[0, \infty)$ and $(-\infty, 0)$; or, all real numbers.

How To: Given a piecewise function, write the formula and identify the domain for each interval

1. Identify the intervals for which different rules apply.
2. Determine formulas that describe how to calculate an output from an input in each interval.
3. Use a brace and if-statements to write the function.

Example 1.2.10: Writing a Piecewise Function

A museum charges \$5 per person for a guided tour with a group of 1 to 9 people or a fixed \$50 fee for a group of 10 or more people. Write a **function** relating the number of people, n, to the cost, C.

Solution

Two different formulas will be needed. For n-values under 10, $C = 5n$. For values of n that are 10 or greater, $C = 50$.

$$C(n) = \begin{cases} 5n & \text{if } n < 10 \\ 50 & \text{if } n \geq 50 \end{cases}$$

Analysis

The function is represented as a graph in Figure 1.2.20. The graph is a straight line with slope of 5 from $n = 0$ to $n = 10$ and a constant after that. In this example, the two formulas agree at the meeting point where $n = 10$, but not all piecewise functions

have this property.

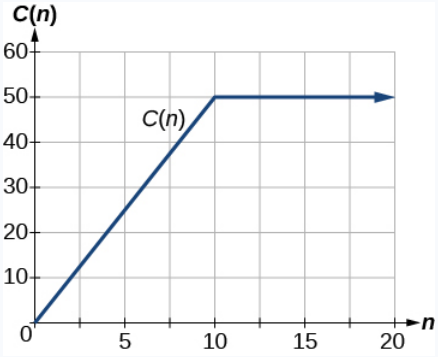


***Figure* 1.2.20**

Note: Looking at the piecewise definition of function C, it seems that the domain is $D = (-\infty, \infty)$. Looking at the graph, it seems that the domain for C is $D = [0, \infty)$. Given that this function is a mathematical model for a real-life situation, what do you think the domain is? If you think you know, write an explanation and give it to your instructor for ***Extra Credit***.

Example 1.2.11: Working with a Piecewise Function

A cell phone company uses the function below to determine the cost C in dollars for g gigabytes of data transfer.

$$C(g) = \begin{cases} 25 & \text{if } 0 < g < 2 \\ 25 + 10(g-2) & \text{if } g \geq 2 \end{cases}$$

Find the cost of using 1.5 gigabytes of data and the cost of using 4 gigabytes of data.

Soltuion

To find the cost of using 1.5 gigabytes of data, $C(1.5)$, we first look to see which part of the domain our input falls in. Because 1.5 is less than 2, we use the first formula.

$$C(1.5) = \$25$$

To find the cost of using 4 gigabytes of data, C(4), we see that our input of 4 is greater than 2, so we use the second formula.

$$C(4) = 25 + 10(4-2) = \$45$$

Analysis

The function is represented in Figure 1.2.21. We can see where the function changes from a constant to a linear function with positive slope at $g = 2$. We plot the graphs for the different formulas on a common set of axes, making sure each formula is applied on its proper domain.

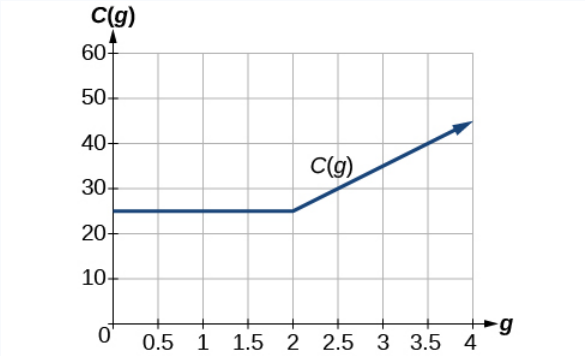


***Figure* 1.2.21**

Note: Calculate the slope of the straight line defined as the second piece of the function C, and give your work to your instructor for ***Extra Credit***.

How To: Given a piecewise function, sketch a graph

1. Indicate on the x-axis the boundaries defined by the intervals on each piece of the domain.
2. For each piece of the domain, graph on that interval using the corresponding equation pertaining to that piece. Do not graph two functions over one interval because it would violate the criteria of a function.

Example 1.2.12: Graphing a Piecewise Function

Sketch a graph of the function.

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ 3 & \text{if } 1 < x \leq 2 \\ x & \text{if } x > 2 \end{cases}$$

Solution

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ 3 & \text{if } 1 < x \leq 2 \\ x & \text{if } x > 2 \end{cases}$$

Each of the component functions is from our library of toolkit functions, so we know their shapes. We can imagine graphing each function and then limiting the graph to the indicated domain. At the endpoints of the domain, we draw open circles to indicate where the endpoint is not included because of a less-than or greater-than inequality; we draw a closed circle where the endpoint is included because of a less-than-or-equal-to or greater-than-or-equal-to inequality.

Figure 1.2.20 shows the three components of the piecewise function graphed on separate coordinate systems.

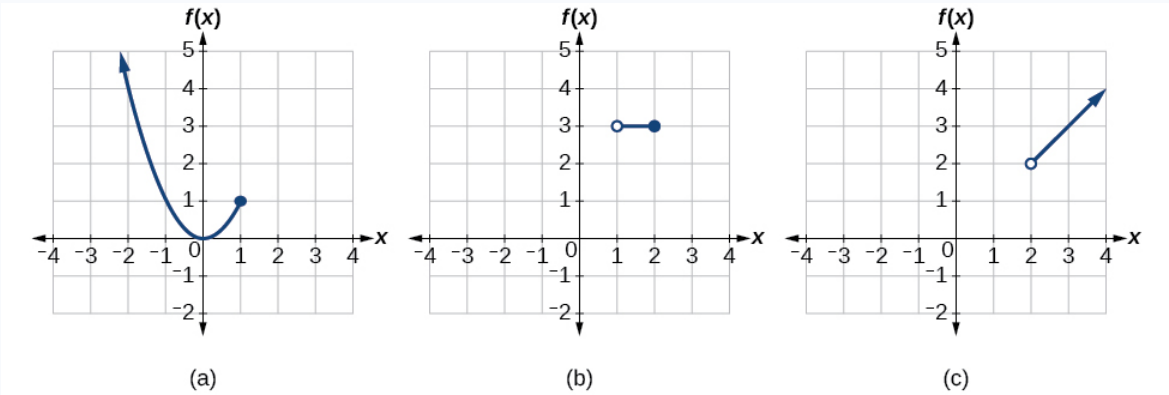


Figure 1.2.20: *Graph of each part of the piece-wise function f(x)*

(a) $f(x) = x^2$ if $x \leq 1$; (b) $f(x) = 3$ if $1 < x \leq 2$; (c) $f(x) = x$ if $x > 2$

Now that we have sketched each piece individually, we combine them in the same coordinate plane. See Figure 1.2.21.

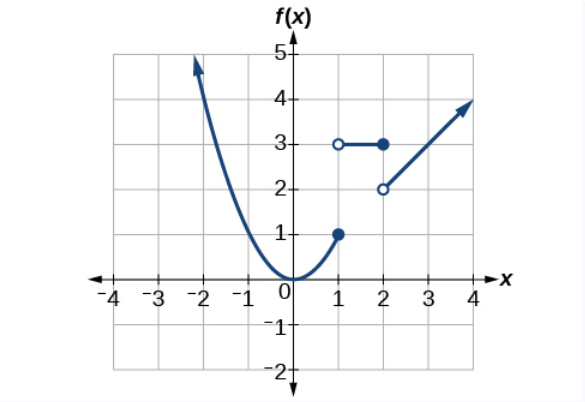


***Figure* 1.2.21:** *Graph of the entire function.*

Analysis

Note that the graph does pass the vertical line test even at $x = 1$ and $x = 2$ because the points $(1, 3)$ and $(2, 2)$ are not part of the graph of the function, though $(1, 1)$ and $(2, 3)$ are.

1.2.8

Graph the following piecewise function.

$$f(x) = \begin{cases} x^3 & \text{if } x < -1 \\ -2 & \text{if } -1 < x < 4 \\ \sqrt{x} & \text{if } x > 4 \end{cases}$$

Answer

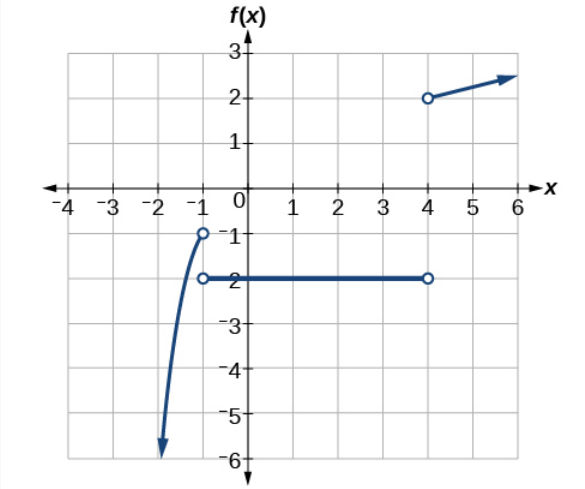


***Figure* 1.2.22**

Can more than one formula from a piecewise function be applied to a value in the domain?

No. Each value corresponds to one equation in a piecewise formula.

Key Concepts

- The domain of a function includes all real input values that would not cause us to attempt an undefined mathematical operation, such as dividing by zero or taking the square root of a negative number.

- The domain of a function can be determined by listing the input values of a set of ordered pairs.
- The domain of a function can also be determined by identifying the input values of a function written as an equation.
- Interval values represented on a number line can be described using inequality notation, set-builder notation, and interval notation.
- For many functions, the domain and range can be determined from a graph.
- An understanding of toolkit functions can be used to find the domain and range of related functions.
- A piecewise function is described by more than one formula.
- A piecewise function can be graphed using each algebraic formula on its assigned subdomain.

Footnotes

1 The Numbers: Where Data and the Movie Business Meet. “Box Office History for Horror Movies.” http://www.the-numbers.com/market/genre/Horror. Accessed 3/24/2014
2 http://www.eia.gov/dnav/pet/hist/Lea...s=MCRFPAK2&f=A.

Glossary

interval notation

a method of describing a set that includes all numbers between a lower limit and an upper limit; the lower and upper values are listed between brackets or parentheses, a square bracket indicating inclusion in the set, and a parenthesis indicating exclusion

piecewise function

a function in which more than one formula is used to define the output

set-builder notation

a method of describing a set by a rule that all of its members obey; it takes the form $x|statementaboutx$

Contributors

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1.2E: Exercises

This page is a draft and is under active development.

1.2E: Exercises

Verbal

Exercise 1.2.1

Why does the domain differ for different functions?

Answer:

The domain of a function depends upon what values of the independent variable make the function undefined or imaginary.

Exercise 1.2.2

How do we determine the domain of a function defined by an equation?

Exercise 1.2.3

Explain why the domain of $f(x) = \sqrt[3]{x}$ is different from the domain of $f(x) = \sqrt{x}$.

Answer:

There is no restriction on x for $f(x) = \sqrt[3]{x}$ because you can take the cube root of any real number. So the domain is all real numbers, $(-\infty, \infty)$. When dealing with the set of real numbers, you cannot take the square root of negative numbers. So x-values are restricted for $f(x) = \sqrt{x}$ to nonnegative numbers and the domain is $[0, \infty)$.

Exercise 1.2.4

When describing sets of numbers using interval notation, when do you use a parenthesis and when do you use a bracket?

Exercise 1.2.5

How do you graph a piecewise function?

Answer:

Graph each formula of the piecewise function over its corresponding domain. Use the same scale for the x-axis and y-axis for each graph. Indicate inclusive endpoints with a solid circle and exclusive endpoints with an open circle. Use an arrow to indicate $-\infty$ or ∞. Combine the graphs to find the graph of the piecewise function.

Algebraic

For the following exercises, find the domain of each function using interval notation.

Exercise 1.2.6

$f(x) = -2x(x-1)(x-2)$

Exercise 1.2.7

$f(x) = 5 - 2x^2$

Answer:

$(-\infty, \infty)$

Exercise 1.2.8

$f(x) = 3\sqrt{x-2}$

Exercise 1.2.9

$f(x) = 3 - \sqrt{6-2x}$

Answer:

$(-\infty, 3]$

Exercise 1.2.10

$f(x) = \sqrt{4-3x}$

Exercise 1.2.11

$f(x) = \sqrt{x^2+4}$

Answer:

$(-\infty, \infty)$

Exercise 1.2.12

$f(x) = \sqrt[3]{1-2x}$

Exercise 1.2.13

$f(x) = \sqrt[3]{x-1}$

Answer:

$(-\infty, \infty)$

Exercise 1.2.14

$f(x) = \dfrac{9}{x-6}$

Exercise 1.2.15

$f(x) = \dfrac{3x+1}{4x+2}$

Answer:

$(-\infty, -\frac{1}{2}) \cup (-\frac{1}{2}, \infty)$

Exercise 1.2.16

$f(x) = \dfrac{\sqrt{x+4}}{x-4}$

Exercise 1.2.17

$f(x) = \dfrac{x-3}{x^2+9x-22}$

Answer:

$(-\infty, -11) \cup (-11, 2) \cup (2, \infty)$

Exercise 1.2.18

$f(x) = \dfrac{1}{x^2-x-6}$

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Exercise 1.2.19

$f(x) = \dfrac{2x^3 - 250}{x^2 - 2x - 15}$

Answer:

$(-\infty, -3) \cup (-3, 5) \cup (5, \infty)$

Exercise 1.2.20

$\dfrac{5}{\sqrt{x-3}}$

Exercise 1.2.21

$\dfrac{2x+1}{\sqrt{5-x}}$

Answer:

$(-\infty, 5)$

Exercise 1.2.22

$\dfrac{\sqrt{x-4}}{\sqrt{x-6}}$

Exercise 1.2.23

$\dfrac{\sqrt{x-6}}{\sqrt{x-4}}$

Answer:

$[6, \infty)$

Exercise 1.2.24

$f(x) = \dfrac{x}{x}$

Exercise 1.2.25

$f(x) = \dfrac{x^2 - 9x}{x^2 - 81}$

Answer:

$(-\infty, -9) \cup (-9, 9) \cup (9, \infty)$

Exercise 1.2.26

Find the domain of the function $f(x) = \sqrt{2x^3 - 50x}$ by:

a. using algebra

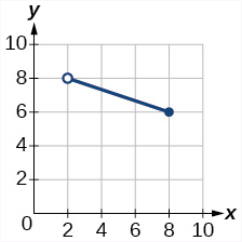
b. graphing the function in the radicand and determining intervals on the x-axis for which the radicand is nonnegative..

Graphical

For the following exercises, write the domain and range of each function using interval notation.

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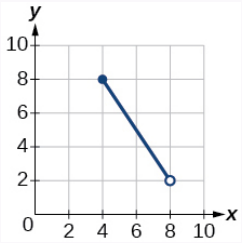
Exercise 1.2.27



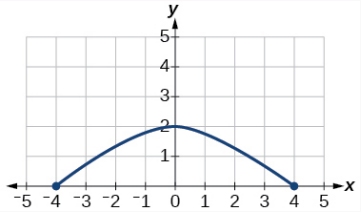
Answer:

domain: $(2, 8]$, range $[6, 8)$

Exercise 1.2.28



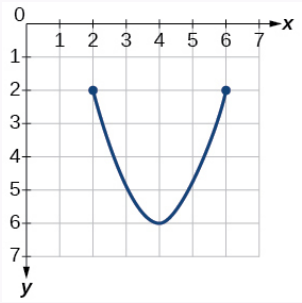
Exercise 1.2.29



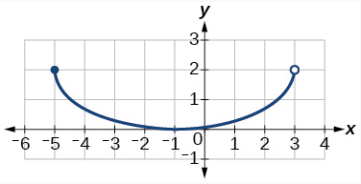
Answer:

domain: $[-4, 4]$, range: $[0, 2]$

Exercise 1.2.30



Exercise 1.2.31

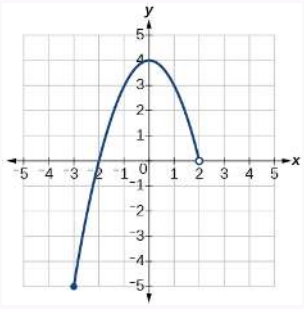


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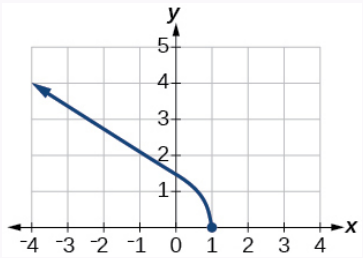
Answer:

domain: $[-5, 3)$, range: $[0, 2]$

Exercise 1.2.32



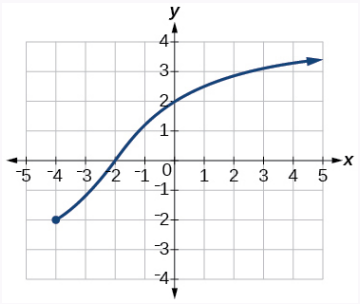
Exercise 1.2.33



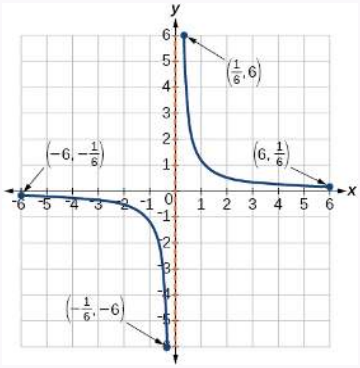
Answer:

domain: $(-\infty, 1]$, range: $[0, \infty)$

Exercise 1.2.34



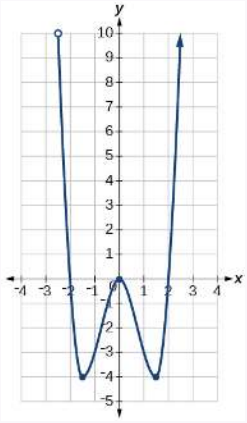
Exercise 1.2.35



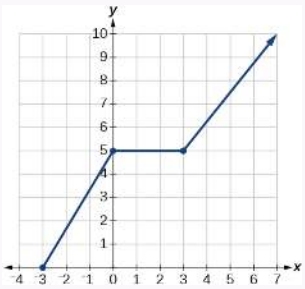
Answer:

domain: $[-6,-\frac{1}{6}] \cup [\frac{1}{6},6]$; range: $[-6,-\frac{1}{6}] \cup [\frac{1}{6},6]$

Exercise 1.2.36



Exercise 1.2.37



Answer:

domain: $[-3,\infty)$; range: $[0,\infty)$

For the following exercises, sketch a graph of the piecewise function. Write the domain in interval notation.

Exercise 1.2.38

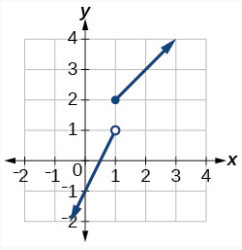
$f(x) = \begin{cases} x+1 & \text{if } x < -2 \\ -2x-3 & \text{if } x \geq -2 \end{cases}$

Exercise 1.2.39

$f(x) = \begin{cases} 2x-1 & \text{if } x < 1 \\ 1+x & \text{if } x \geq 1 \end{cases}$

Answer:

domain: $(-\infty,\infty)$



Exercise 1.2.40

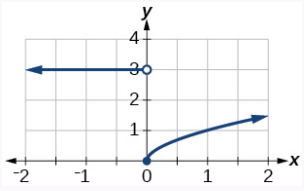
$$f(x)=\begin{cases}x+1 & \text{if } x<0\\ x-1 & \text{if } x>0\end{cases}$$

Exercise 1.2.41

$$f(x)=\begin{cases}3 & \text{if } x<0\\ \sqrt{x} & \text{if } x\geq 0\end{cases}$$

Answer:

domain: $(-\infty,\infty)$



Exercise 1.2.42

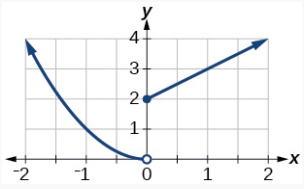
$$f(x)=\begin{cases}x^2 & \text{if } x<0\\ 1-x & \text{if } x>0\end{cases}$$

Exercise 1.2.43

$$f(x)=\begin{cases}x^2 & \text{if } x<0\\ x+2 & \text{if } x\geq 0\end{cases}$$

Answer:

domain: $(-\infty,\infty)$



Exercise 1.2.44

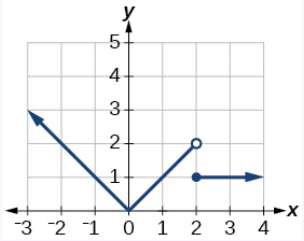
$$f(x)=\begin{cases}x+1 & \text{if } x<1\\ x^3 & \text{if } x\geq 1\end{cases}$$

Exercise 1.2.45

$$f(x)=\begin{cases}|x| & \text{if } x<2\\ 1 & \text{if } x\geq 2\end{cases}$$

Answer:

domain: $(-\infty,\infty)$



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Numeric

For the following exercises, given each function f, evaluate $f(-3)$, $f(-2)$, $f(-1)$, and $f(0)$.

Exercise 1.2.46

$$f(x)=\begin{cases}x+1 & \text{if } x<-2\\ -2x-3 & \text{if } x\geq -2\end{cases}$$

Exercise 1.2.47

$$f(x)=\begin{cases}1 & \text{if } x\leq -3\\ 0 & \text{if } x>-3\end{cases}$$

Answer:

$f(-3)=1$; $f(-2)=0$; $f(-1)=0$; $f(0)=0$

Exercise 1.2.48

$$f(x)=\begin{cases}-2x^2+3 & \text{if } x\leq -1\\ 5x-7 & \text{if } x>-1\end{cases}$$

For the following exercises, given each function f, evaluate $f(-1)$, $f(0)$, $f(2)$, and $f(4)$.

Exercise 1.2.49

$$f(x)=\begin{cases}7x+3 & \text{if } x<0\\ 7x+6 & \text{if } x\geq 0\end{cases}$$

Answer:

$f(-1)=-4$; $f(0)=6$; $f(2)=20$; $f(4)=34$

Exercise 1.2.50

$$f(x)=\begin{cases}x^2-2 & \text{if } x<2\\ 4+|x-5| & \text{if } x\geq 2\end{cases}$$

Exercise 1.2.51

$$f(x)=\begin{cases}5x & \text{if } x<0\\ 3 & \text{if } 0\geq x\leq 2\\ x^2 & \text{if } x>3\end{cases}$$

Answer:

$f(-1)=-5$; $f(0)=3$; $f(2)=3$; $f(4)=16$

For the following exercises, write the domain for the piecewise function in interval notation.

Exercise 1.2.52

$$f(x)=\begin{cases}x+1 & \text{if } x<-2\\ -2x-3 & \text{if } x\geq -2\end{cases}$$

Exercise 1.2.53

$$f(x)=\begin{cases}x^2-2 & \text{if } x<1\\ -x^2+2 & \text{if } x>1\end{cases}$$

Answer:

domain: $(-\infty,1)\cup(1,\infty)$

Exercise 1.2.54

$$f(x)=\begin{cases}x^2-3 & \text{if } x<0\\ -3x^2 & \text{if } x\geq 2\end{cases}$$

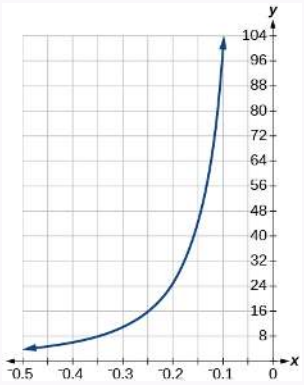
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Technology

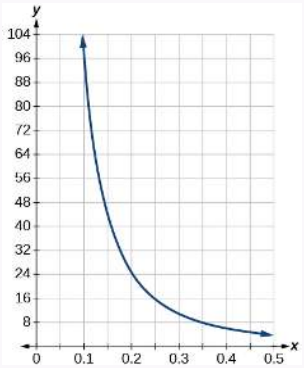
Exercise 1.2.55

Graph $y = \frac{1}{x^2}$ on the viewing window $[-0.5, -0.1]$ and $[0.1, 0.5]$. Determine the corresponding range for the viewing window. Show the graphs.

Answer:



window: $[-0.5, -0.1]$; range: $[4, 100]$



window: $[0.1, 0.5]$; range: $[4, 100]$

Exercise 1.2.56

Graph $y = \frac{1}{x}$ on the viewing window $[-0.5, -0.1]$ and $[0.1, 0.5]$. Determine the corresponding range for the viewing window. Show the graphs.

Extension

Exercise 1.2.57

Suppose the range of a function f is $[-5, 8]$. What is the range of $|f(x)|$?

Answer:

$[0, 8]$

Exercise 1.2.58

Create a function in which the range is all nonnegative real numbers.

Exercise 1.2.59

Create a function in which the domain is $x > 2$.

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Answer:

Many answers. One function is $f(x) = \frac{1}{\sqrt{x-2}}$.

Real-World Applications

Exercise 1.2.60

The height h of a projectile is a function of the time t it is in the air. The height in feet for t seconds is given by the function $h(t) = -16t^2 + 96t$. What is the domain of the function? What does the domain mean in the context of the problem?

Answer:

The domain is $[0, 6]$; it takes 6 seconds for the projectile to leave the ground and return to the ground.

Exercise 1.2.61

The cost in dollars of making x items is given by the function $C(x) = 10x + 500$.

a. The fixed cost is determined when zero items are produced. Find the fixed cost for this item.

b. What is the cost of making 25 items?

c. Suppose the maximum cost allowed is $1500. What are the domain and range of the cost function, $C(x)$?

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1.3: Rates of Change and Behavior of Graphs

Skills to Develop

- Find the average rate of change of a function.
- Use a graph to determine where a function is increasing, decreasing, or constant.
- Use a graph to locate local maxima and local minima.
- Use a graph to locate the absolute maximum and absolute minimum.

Gasoline costs have experienced some wild fluctuations over the last several decades. Table 1.3.1 lists the average cost, in dollars, of a gallon of gasoline for the years 2005–2012. The cost of gasoline can be considered as a function of year.

Table 1.3.1

x	2005	2006	2007	2008	2009	2010	2011	2012
$C(x)$	2.31	2.62	2.84	3.30	2.41	2.84	3.58	3.68

If we were interested only in how the gasoline prices changed between 2005 and 2012, we could compute that the cost per gallon had increased from $2.31 to $3.68, an increase of $1.37. While this is interesting, it might be more useful to look at how much the price changed per year. In this section, we will investigate changes such as these.

Finding the Average Rate of Change of a Function

The price change per year is a **rate of change** because it describes how an output quantity changes relative to the change in the input quantity. We can see that the price of gasoline in Table 1.3.1 did not change by the same amount each year, so the rate of change was not constant. If we use only the beginning and ending data, we would be finding the **average rate of change** over the specified period of time. To find the average rate of change, we divide the change in the output value by the change in the input value.

$$\begin{aligned}\text{Average rate of change} &= \frac{\text{Change in output}}{\text{Change in input}}\\ &= \frac{\Delta y}{\Delta x}\\ &= \frac{y_2 - y_1}{x_2 - x_1}\\ &= \frac{f(x_2) - f(x_1)}{x_2 - x_1}\end{aligned}$$

The Greek letter Δ (delta) signifies the change in a quantity; we read the ratio as "delta-y over delta-x" or "the change in y divided by the change in x." Occasionally we write Δf instead of Δy, which still represents the change in the function's output value resulting from a change to its input value. It does not mean we are changing the function into some other function.

In our example, the gasoline price increased by $1.37 from 2005 to 2012. Over 7 years, the average rate of change was

$$\frac{\Delta y}{\Delta x} = \frac{\$1.37}{7\text{years}} \approx 0.196 \text{ dollars per year.} \tag{1.3.1}$$

On average, the price of gas increased by about 19.6¢ each year. Other examples of rates of change include:

- A population of rats increasing by 40 rats per week
- A car traveling 68 miles per hour (distance traveled changes by 68 miles each hour as time passes)
- A car driving 27 miles per gallon (distance traveled changes by 27 miles for each gallon)
- The current through an electrical circuit increasing by 0.125 amperes for every volt of increased voltage
- The amount of money in a college account decreasing by $4,000 per quarter

Definition: Rate of Change

A rate of change describes how an output quantity changes relative to the change in the input quantity. The units on a rate of change are "output units per input units."

The average rate of change between two input values is the total change of the function values (output values) divided by the change in the input values.

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

How To: Given the value of a function at different points, calculate the average rate of change of a function for the interval between two values x_1 and x_2

1. Calculate the difference $y_2 - y_1 = \Delta y$.
2. Calculate the difference $x_2 - x_1 = \Delta x$.
3. Find the ratio $\frac{\Delta y}{\Delta x}$.

Example 1.3.1: Computing an Average Rate of Change

Using the data in Table 1.3.1, find the average rate of change of the price of gasoline between 2007 and 2009.

Solution

In 2007, the price of gasoline was \$2.84. In 2009, the cost was \$2.41. The average rate of change is

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{\$2.41 - \$2.84}{2009 - 2007} \\ &= \frac{-\$0.43}{2 \text{ years}} \\ &= -\$0.22 \text{ per year}\end{aligned}$$

Analysis

Note that a decrease is expressed by a negative change or "negative increase." A rate of change is negative when the output decreases as the input increases or when the output increases as the input decreases.

Try It 1.3.1

Using the data in Table 1.3.1, find the average rate of change between 2005 and 2010.

Answer

$$\frac{\$2.84 - \$2.315}{5 \text{ years}} = \frac{\$0.535}{5 \text{ years}} = \$0.106 \text{ per year}$$

Example 1.3.2: Computing Average Rate of Change from a Graph

Given the function $g(t)$ shown in Figure 1.3.1, find the average rate of change on the interval $[-1, 2]$.

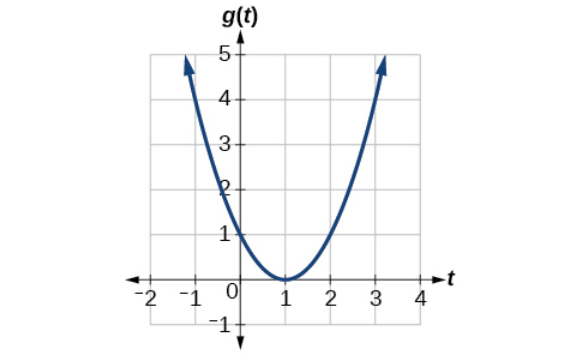


***Figure* 1.3.1:** *Graph of a parabola.*

Solution

At $t = -1$, Figure 1.3.2 shows $g(-1) = 4$. At $t = 2$,the graph shows $g(2) = 1$.

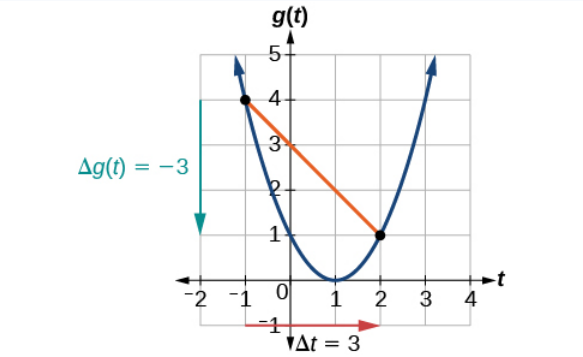


***Figure* 1.3.2:** *Graph of a parabola with a line from points (-1, 4) and (2, 1) to show the changes for g(t) and t.*

The horizontal change $\Delta t = 3$ is shown by the red arrow, and the vertical change $\Delta g(t) = -3$ is shown by the turquoise arrow. The output changes by –3 while the input changes by 3, giving an average rate of change of

$$\frac{1-4}{2-(-1)} = \frac{-3}{3} = -1$$

Analysis

Note that the order we choose is very important. If, for example, we use $\frac{y_2 - y_1}{x_1 - x_2}$, we will not get the correct answer. Decide which point will be 1 and which point will be 2, and keep the coordinates fixed as (x_1, y_1) and (x_2, y_2).

Example 1.3.3: Computing Average Rate of Change from a Table

After picking up a friend who lives 10 miles away, Anna records her distance from home over time. The values are shown in Table 1.3.2. Find her average speed over the first 6 hours.

***Table* 1.3.2**

t (hours)	0	1	2	3	4	5	6	7
D(t)(miles)	10	55	90	153	214	240	292	300

Solution

Here, the average speed is the average rate of change. She traveled 282 miles in 6 hours, for an average speed of

$$\begin{aligned}\frac{292-10}{6-0} &= \frac{282}{6}\\ &= 47\end{aligned}$$

The average speed is 47 miles per hour.

Analysis

Because the speed is not constant, the average speed depends on the interval chosen. For the interval $[2,3]$, the average speed is 63 miles per hour.

Example 1.3.4: Computing Average Rate of Change for a Function Expressed as a Formula

Compute the average rate of change of $f(x)=x^2-\frac{1}{x}$ on the interval $[2,4]$.

Solution

We can start by computing the function values at each endpoint of the interval.

$$\begin{aligned} f(2) &= 2^2-\frac{1}{2} & f(4) &= 4^2-\frac{1}{4}\\ &= 4-\frac{1}{2} & &= 16-\frac{1}{4}\\ &= \frac{7}{2} & &= \frac{63}{4}\end{aligned}$$

Now we compute the average rate of change.

$$\begin{aligned}\text{Average rate of change} &= \frac{f(4)-f(2)}{4-2}\\ &= \frac{\frac{63}{4}-\frac{7}{2}}{4-2}\\ &= \frac{\frac{49}{4}}{2}\\ &= \frac{49}{8}\end{aligned}$$

1.3.2

Find the average rate of change of $f(x)=x-2\sqrt{x}$ on the interval $[1,9]$.

Answer

$\frac{1}{2}$

Example 1.3.5: Finding the Average Rate of Change of a Force

The electrostatic force F between two charged particles,measured in newtons, can be related to the distance between the particles d, in centimeters, by the formula $F(d)=\frac{2}{d^2}$. Find the average rate of change of force if the distance between the particles is increased from 2 cm to 6 cm.

Solution

We are computing the average rate of change of $F(d)=\dfrac{2}{d^2}$ on the interval $[2,6]$.

$$\begin{aligned}\text{Average rate of change} &= \frac{F(6)-F(2)}{6-2} \\ &= \frac{\frac{2}{6^2}-\frac{2}{2^2}}{6-2} && \text{Simplify} \\ &= \frac{\frac{2}{36}-\frac{2}{4}}{4} \\ &= \frac{-\frac{16}{36}}{4} && \text{Combine numerator terms} \\ &= -\frac{1}{9} && \text{Simplify}\end{aligned}$$

The average rate of change is $-\frac{1}{9}$ newton per centimeter.

Example 1.3.6: Finding an Average Rate of Change as an Expression

Find the average rate of change of $g(t) = t^2 + 3t + 1$ on the interval $[0, a]$. The answer will be an expression involving a.

Solution

We use the average rate of change formula.

$$\begin{aligned}\text{Average rate of change} &= \frac{g(a)-g(0)}{a-0} && \text{Evaluate} \\ &= \frac{(a^2+3a+1)-(0^2+3(0)+1)}{a-0} && \text{Simplify} \\ &= \frac{a^2+3a+1-1}{a} && \text{Simplify and factor} \\ &= \frac{a(a+3)}{a} && \text{Divide by the common factor } a \\ &= a+3\end{aligned}$$

This result tells us the average rate of change (in terms of a) between $t = 0$ and any other point $t = a$. For example, on the interval $[0, 5]$, the average rate of change would be $a + 3 = 5 + 3 = 8$.

1.3.3

Find the average rate of change of $f(x) = x^2 + 2x - 8$ on the interval $[5, a]$.

Answer

$a + 7$

Using a Graph to Determine Where a Function is Increasing, Decreasing, or Constant

As part of exploring how functions change, we can identify intervals over which the function is changing in specific ways. We say that a function is increasing on an interval if the function values increase as the input values increase within that interval. Similarly, a function is decreasing on an interval if the function values decrease as the input values increase over that interval. The average rate of change of an interval where a function is increasing is positive, and the average rate of change of an interval where a function is decreasing is negative. Figure **1.3.3** shows examples of increasing and decreasing intervals on a function.

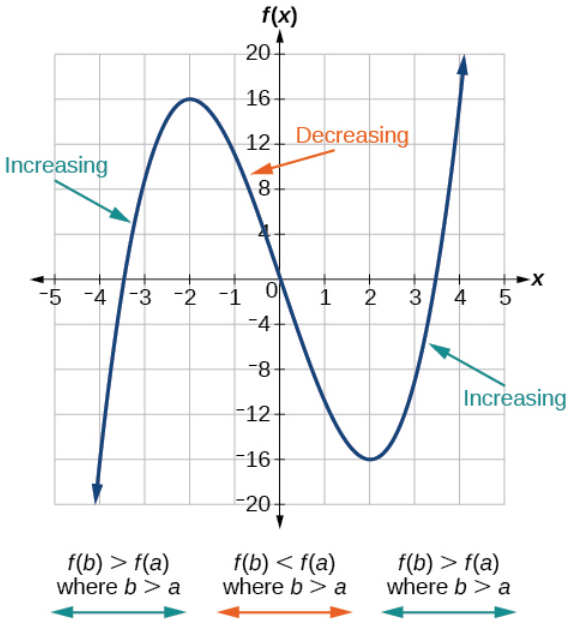


***Figure* 1.3.3:** *The function* $f(x) = x^3 - 12x$ *is increasing on* $(-\infty, -2) \cup (2, \infty)$ *and is decreasing on* $(-2, 2)$.

While some functions are increasing (or decreasing) over their entire domain, many others are not. (**Note**: if you can think of an example of a function that is increasing on part of its domain and decreasing on another part of its domain, write it down and give it to your instructor for ***Extra Credit.***) If there is an input value at which a function changes from increasing to decreasing (as we go from left to right; that is, as the input variable increases), then the corresponding output value is called a **local maximum**. If a function has more than one local maximum, we say it has local **maxima**. Similarly, if there is an input value at which a function changes from decreasing to increasing as the input variable increases, then the corresponding output value is called a **local minimum**. The plural form is "local **minima**." Together, local maxima and minima are called **local extrema**, or local extreme values, of the function. (The singular form is "extremum.") Often, the term "local" is replaced by the term "relative." In this text, we will use the term "local."

Clearly, a function is neither increasing nor decreasing on an interval where the outputs are constant. A function is also considered neither increasing nor decreasing at extrema. Note that because any given local extremum as defined here is not necessarily the highest maximum or lowest minimum in the function's entire domain, we have to speak of local extrema. If the function **does** have a highest output over the entire domain, this output is called a **global maximum** (to be discussed later in this Section).

For the function whose graph is shown in Figure 1.3.4, the local maximum is 16, and it occurs at $x = -2$. The local minimum is −16 and it occurs at $x = 2$.

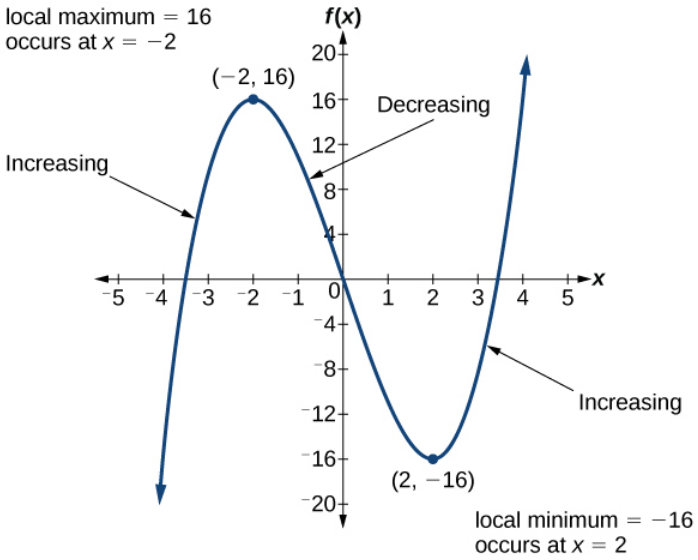


Figure* 1.3.4*: *Graph of a polynomial that shows the increasing and decreasing intervals and local maximum.*maximum

To locate the local maxima and minima from a graph, we need to observe the graph to determine where the graph attains its highest and lowest points, respectively, within an open interval. Like the summit of a roller coaster, the graph of a function is higher at a local maximum than at nearby points on both sides. The graph will also be lower at a local minimum than at neighboring points. Figure **1.3.5** illustrates these ideas for a local maximum.

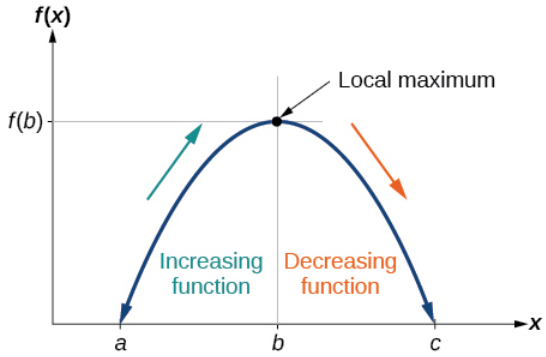


Figure* 1.3.5*: Definition of a local maximum

These observations lead us to a formal definition of local extrema.

Local Minima and Local Maxima

- A function f is an **increasing function** on an open interval (a, b) if $f(x_2) > f(x_1)$ for every x_1, x_2 in (a, b) where $x_2 > x_1$.
- A function f is a **decreasing function** on an open interval (a, b) if $f(x_2) < f(x_1)$ for every x_1, x_2 in (a, b) where $x_2 > x_1$.

A function f has a local maximum at a point c in an open interval (a, b) if $f(c)$ is greater than or equal to $f(x)$ for every point x in (a, b). Likewise, f has a local minimum at a point c in (a, b) if $f(c)$ is less than or equal to $f(x)$ for every x in (a, b).

Example 1.3.7 Finding Increasing and Decreasing Intervals on a Graph

Given the function $p(t)$ in Figure **1.3.6**, identify the intervals on which the function appears to be increasing.

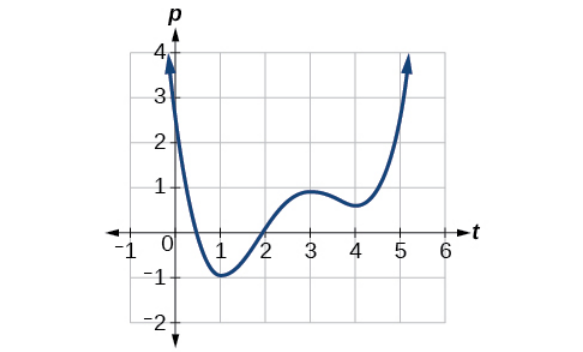


***Figure* 1.3.6:** *Graph of a polynomial.*

Solution

We see that the function is not constant on any interval. The function is increasing where it slants upward as we move to the right and decreasing where it slants downward as we move to the right. The function appears to be increasing from $t = 1$ to $t = 3$ and from $t = 4$ on.

In interval notation, we would say the function appears to be increasing on the interval $(1, 3)$ and the interval $(4, \infty)$.

Analysis

Notice in this example that we used open intervals (intervals that do not include the endpoints), because the function is neither increasing nor decreasing at $t = 1$, $t = 3$, and $t = 4$. At these points there are local extrema (two minima and a maximum).

Example 1.3.8: Finding Local Maxima and Minima from a Graph

For the function f whose graph is shown in Figure 1.3.7, find all local maxima and minima.

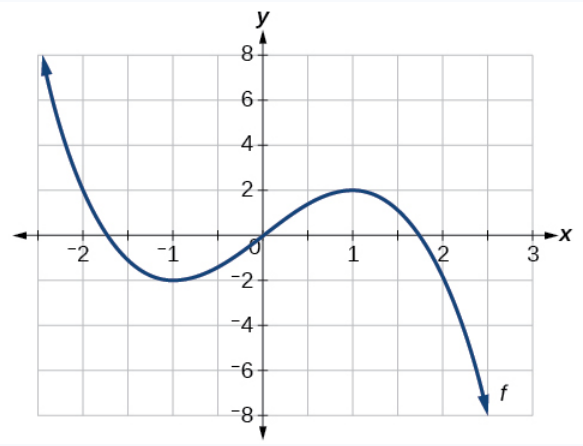


***Figure* 1.3.7:** *Graph of a polynomial.*

Solution

Observe the graph of f. The graph attains a local maximum at $x = 1$ because it is the highest point in an open interval around $x = 1$.The local maximum is the y-coordinate at $x = 1$, which is 2.

The graph attains a local minimum at $x = -1$ because it is the lowest point in an open interval around $x = -1$. The local minimum is the y-coordinate at $x = -1$, which is −2.

Analyzing the Toolkit Functions for Increasing or Decreasing Intervals

We will now return to our toolkit functions and discuss their graphical behavior in Figure 1.3.8.

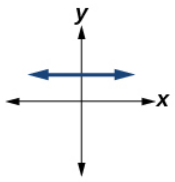
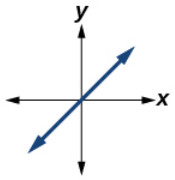
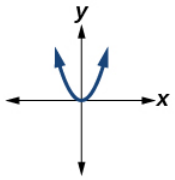
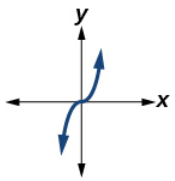
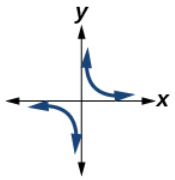
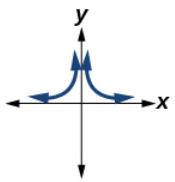
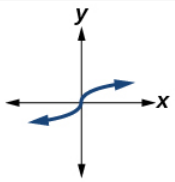
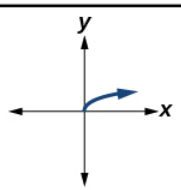
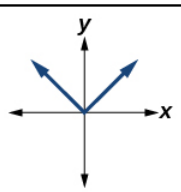
Function	Increasing/Decreasing	Example
Constant Function $f(x) = c$	Neither increasing nor decreasing	
Identity Function $f(x) = x$	Increasing	
Quadratic Function $f(x) = x^2$	Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$ Minimum at $x = 0$	
Cubic Function $f(x) = x^3$	Increasing	
Reciprocal $f(x) = \frac{1}{x}$	Decreasing $(-\infty, 0) \cup (0, \infty)$	
Reciprocal Squared $f(x) = \frac{1}{x^2}$	Increasing on $(-\infty, 0)$ Decreasing on $(0, \infty)$	
Cube Root $f(x) = \sqrt[3]{x}$	Increasing	
Square Root $f(x) = \sqrt{x}$	Increasing on $(0, \infty)$	
Absolute Value $f(x) = \|x\|$	Increasing on $(0, \infty)$ Decreasing on $(\infty, 0)$	

Figure 1.3.8

Use A Graph to Locate the Global Maximum and Global Minimum

There is a difference between locating the highest and lowest points on a graph in a region around an open interval (locally) and locating the highest and lowest points on the graph for the entire domain. The y-coordinates (output) at the highest and lowest points are called the **global maximum** and **global minimum**, respectively. (The term "absolute" is also used in place of the term "global.") To locate global maxima and minima from a graph, we need to observe the graph to determine where the graph attains it highest and lowest points on the domain of the function (Figure 1.3.9).

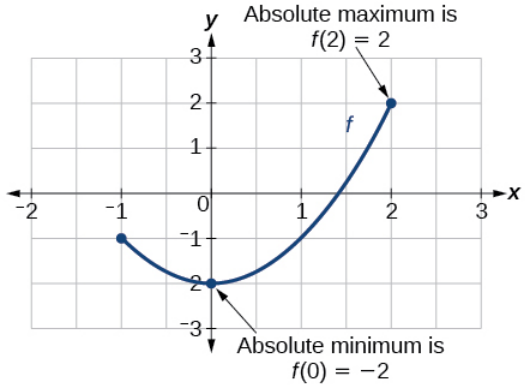


Figure 1.3.9: Graph of a segment of a parabola with a global minimum at (0, -2) and global maximum at (2, 2).

Not every function has a global maximum or minimum value. The toolkit function $f(x) = x^3$ is one such function.

Definition: Global Maxima and Minima

- The **global maximum** of f at $x = c$ is $f(c)$ where $f(c) \geq f(x)$ for all x in the domain of f.
- The **global minimum** of f at $x = d$ is $f(d)$ where $f(d) \leq f(x)$ for all x in the domain of f.

Example 1.3.9: Finding Absolute Maxima and Minima from a Graph

For the function f shown in Figure 1.3.10, find all global maxima and minima.

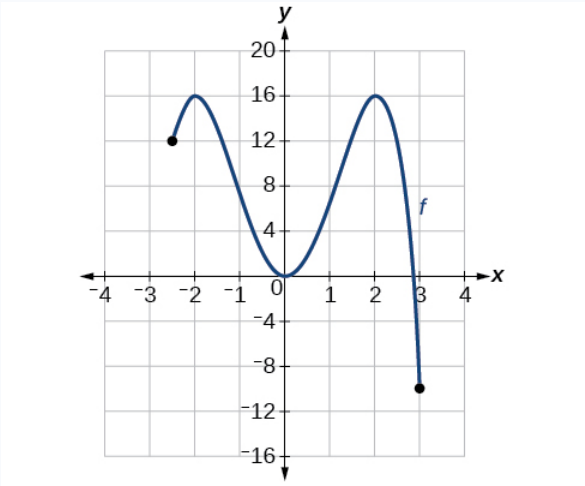


Figure* 1.3.14*: *Graph of a polynomial defined on the interval* $[-2.5, 3]$.

Solution

Observe the graph of f. The graph attains a global maximum at two locations, $x = -2$ and $x = 2$, because at these locations, the graph attains its highest point on the domain of the function. The global maximum is the y-coordinate at $x = -2$ and $x = 2$, which is 16.

The graph attains an global minimum at $x = 3$, because it is the lowest point on the domain of the function's graph. The global minimum is the y-coordinate at $x = 3$, which is −10.

Note: In addition to the global minimum at $x = 3$, there are two local minima. If you see them, write down the x-value at which each one occurs, as well as the y-value, and give it to your instructor for ***Extra Credit.***

Key Equations

- Average rate of change: $\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

Key Concepts

- A rate of change relates a change in an output quantity to a change in an input quantity.
- Identifying points that mark the interval on a graph can be used to find the average rate of change. See Example 1.3.2.
- Comparing pairs of input and output values in a table can also be used to find the average rate of change. See Examples 1.3.1 and 1.3.3.
- An average rate of change can also be computed by determining the function values at the endpoints of an interval described by a formula. See Examples 1.3.4 and 1.3.5.
- The average rate of change can sometimes be determined as an expression. See Example 1.3.6.
- A function is increasing where its rate of change is positive and decreasing where its rate of change is negative. See Example 1.3.7.
- A local (or relative) maximum is where a function changes from increasing to decreasing and has an output value larger (more positive or less negative) than output values at neighboring input values.
- A local (or relative) minimum is where the function changes from decreasing to increasing (as the input increases) and has an output value smaller (more negative or less positive) than output values at neighboring input values.
- Minima and maxima are also called extrema.
- We can find local extrema from a graph. See Example 1.3.8.
- The highest and lowest points on a graph indicate the global (or absolute) maxima and minima. See Example 1.3.9.

Contributors

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1.3E: Exercises

This page is a draft and is under active development.

Verbal

Verbal

Exercise 1.3.1

Can the average rate of change of a function be constant?

Answer:

Yes, the average rate of change of all linear functions is constant.

Exercise 1.3.2

If a function f is increasing on (a, b) and decreasing on (b, c), then what can be said about the local extremum of f on (a, c)?

Exercise 1.3.3

How are the absolute maximum and minimum similar to and different from the local extrema?

Answer:

The absolute maximum and minimum relate to the entire graph, whereas the local extrema relate only to a specific region around an open interval.

Exercise 1.3.4

How does the graph of the absolute value function compare to the graph of the quadratic function, $y = x^2$, in terms of increasing and decreasing intervals?

Algebraic

For the following exercises, find the average rate of change of each function on the interval specified for real numbers b or h.

Exercise 1.3.5

$f(x) = 4x^2 - 7$ on $[1, b]$

Answer:

$4(b + 1)$

Exercise 1.3.6

$g(x) = 2x^2 - 9$ on $[4, b]$

Exercise 1.3.7

$p(x) = 3x + 4$ on $[2, 2 + h]$

Answer:

3

Exercise 1.3.8

$k(x) = 4x - 2$ on $[3, 3 + h]$

Exercise 1.3.9

$f(x) = 2x^2 + 1$ on $[x, x + h]$

Answer:

$4x + 2h$

Exercise 1.3.10

$g(x) = 3x^2 - 2$ on $[x, x + h]$

Exercise 1.3.11

$a(t) = \dfrac{1}{t + 4}$ on $[9, 9 + h]$

Answer:

$\dfrac{-1}{13(13 + h)}$

Exercise 1.3.12

$b(x) = 1x + 3$ on $[1, 1 + h]$

Exercise 1.3.13

$j(x) = 3x^3$ on $[1, 1 + h]$

Answer:

$3h^2 + 9h + 9$

Exercise 1.3.14

$r(t) = 4t^3$ on $[2, 2 + h]$

Exercise 1.3.15

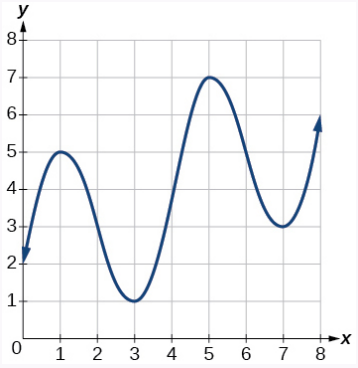
$\frac{f(x+h)-f(x)}{h}$ given $f(x) = 2x^2 - 3x$ on $[x, x + h]$

Answer:

$4x + 2h - 3$

Graphical

For the following exercises, consider the graph of f shown in **Figure 1.3.15**.



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Figure 1.3.15: Graph of a polynomial.

Exercise 1.3.16

Estimate the average rate of change from $x = 1$ to $x = 4$.

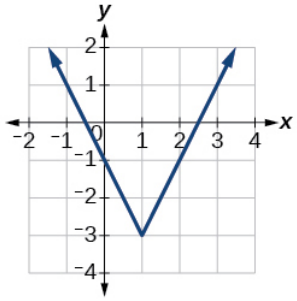
Exercise 1.3.17

Estimate the average rate of change from $x = 2$ to $x = 5$.

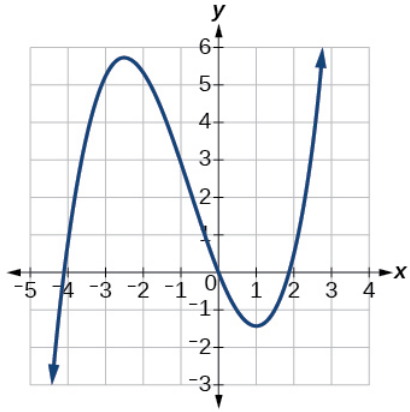
Answer:

$\frac{4}{3}$

Exercise 1.3.18



Exercise 1.3.19

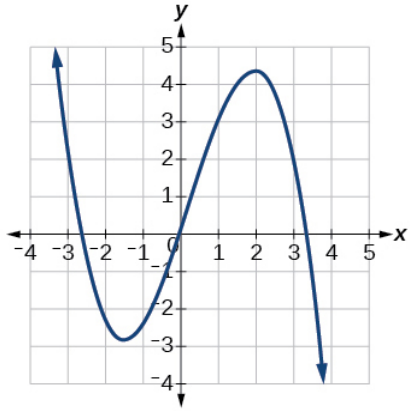


Answer:

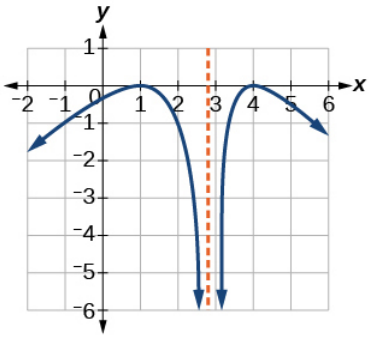
increasing on $(-\infty, -2.5) \cup (1, \infty)$, decreasing on $(-2.5, 1)$

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Exercise 1.3.20



Exercise 1.3.21



Answer:

increasing on $(-\infty, 1) \cup (3, 4)$, decreasing on $(1, 3) \cup (4, \infty)$

For the following exercises, consider the graph shown in **Figure 16**.

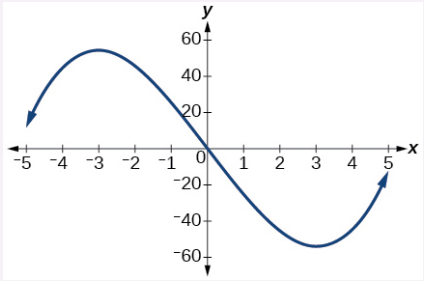


Figure 16: *Graph of a cubic function.*

Exercise 1.3.22

Estimate the intervals where the function is increasing or decreasing.

Exercise 1.3.23

Estimate the point(s) at which the graph of f has a local maximum or a local minimum.

Answer:

local maximum: $(-3, 60)$, local minimum: $(3, -60)$

For the following exercises, consider the graph in **Figure 17**.

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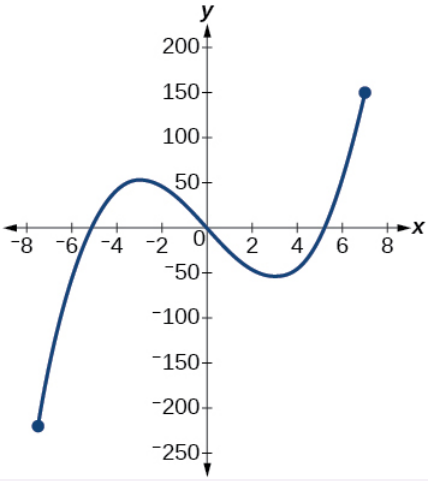


***Figure 17**: Graph of a cubic function.*

Exercise 1.3.24

If the complete graph of the function is shown, estimate the intervals where the function is increasing or decreasing.

Exercise 1.3.25

If the complete graph of the function is shown, estimate the absolute maximum and absolute minimum.

Answer:

absolute maximum at approximately $(7, 150)$, absolute minimum at approximately $(-7.5, -220)$

Numeric

Exercise 1.3.26

The table below gives the annual sales (in millions of dollars) of a product from 1998 to 2006. What was the average rate of change of annual sales (a) between 2001 and 2002, and (b) between 2001 and 2004?

Year	Sales (millions of dollars)
1998	201
1999	219
2000	233
2001	243
2002	249
2003	251
2004	249
2005	243
2006	233

Exercise 1.3.27

The table below gives the population of a town (in thousands) from 2000 to 2008. What was the average rate of change of

population (a) between 2002 and 2004, and (b) between 2002 and 2006?

Year	Population (thousands)
2000	87
2001	84
2002	83
2003	80
2004	77
2005	76
2006	78
2007	81
2008	85

Answer:

a. –3000

b. –1250

For the following exercises, find the average rate of change of each function on the interval specified.

Exercise 1.3.28

$f(x) = x^2$ on $[1, 5]$

Exercise 1.3.29

$h(x) = 5 - 2x^2$ on $[-2, 4]$

Answer:

-4

Exercise 1.3.30

$q(x) = x^3$ on $[-4, 2]$

Exercise 1.3.31

$g(x) = 3x^3 - 1$ on $[-3, 3]$

Answer:

27

Exercise 1.3.32

$y = \frac{1}{x}$ on $[1, 3]$

Exercise 1.3.33

$p(t) = \frac{(t^2 - 4)(t + 1)}{t^2 + 3}$ on $[-3, 1]$

Answer:

–0.167

Exercise 1.3.34

$k(t) = 6t^2 + \frac{4}{t^3}$ on $[-1, 3]$

Technology

For the following exercises, use a graphing utility to estimate the local extrema of each function and to estimate the intervals on which the function is increasing and decreasing.

Exercise 1.3.35

$f(x) = x^4 - 4x^3 + 5$

Answer:

Local minimum at $(3, -22)$, decreasing on $(-\infty, 3)$, increasing on $(3, \infty)$

Exercise 1.3.36

$h(x) = x^5 + 5x^4 + 10x^3 + 10x^2 - 1$

Exercise 1.3.37

$g(t) = t\sqrt{t+3}$

Answer:

Local minimum at $(-2, -2)$, decreasing on $(-3, -2)$, increasing on $(-2, \infty)$

Exercise 1.3.38

$k(t) = 3t^{\frac{2}{3}} - t$

Exercise 1.3.39

$m(x) = x^4 + 2x^3 - 12x^2 - 10x + 4$

Answer:

Local maximum at $(-0.5, 6)$, local minima at $(-3.25, -47)$ and $(2.1, -32)$, decreasing on $(-\infty, -3.25)$ and $(-0.5, 2.1)$, increasing on $(-3.25, -0.5)$ and $(2.1, \infty)$

Exercise 1.3.40

$n(x) = x^4 - 8x^3 + 18x^2 - 6x + 2$

Extension

Exercise 1.3.41

The graph of the function f is shown in **Figure 18**

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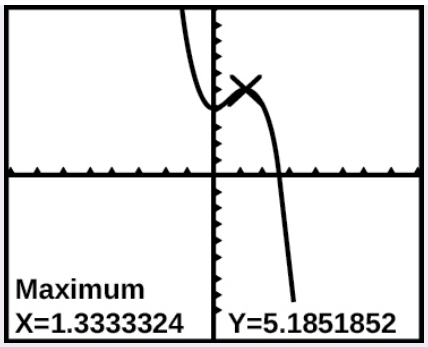


Figure 18: *Graph of f(x) on a graphing calculator.*

Based on the calculator screen shot, the point $(1.333, 5.185)$ is which of the following?

A. a relative (local) maximum of the function
B. the vertex of the function
C. the absolute maximum of the function
D. a zero of the function

Answer:

A

Exercise 1.3.42

Let $f(x) = \frac{1}{x}$. Find a number c such that the average rate of change of the function f on the interval $(1, c)$ is $-\frac{1}{4}$.

Exercise1.3.43

Let $f(x) = \frac{1}{x}$. Find the number b such that the average rate of change of f on the interval $(2, b)$ is $-\frac{1}{10}$.

Answer:

$b = 5$

Real-World Applications

Exercise 1.3.44

At the start of a trip, the odometer on a car read 21,395. At the end of the trip, 13.5 hours later, the odometer read 22,125. Assume the scale on the odometer is in miles. What is the average speed the car traveled during this trip?

Exercise 1.3.45

A driver of a car stopped at a gas station to fill up his gas tank. He looked at his watch, and the time read exactly 3:40 p.m. At this time, he started pumping gas into the tank. At exactly 3:44, the tank was full and he noticed that he had pumped 10.7 gallons. What is the average rate of flow of the gasoline into the gas tank?

Answer:

2.7 gallons per minute

Exercise 1.3.46

Near the surface of the moon, the distance that an object falls is a function of time. It is given by $d(t) = 2.6667t^2$, where t is in seconds and $d(t)$ is in feet. If an object is dropped from a certain height, find the average velocity of the object from $t = 1$ to $t = 2$.

Exercise 1.3.47

The graph in **Figure 19** illustrates the decay of a radioactive substance over t days.

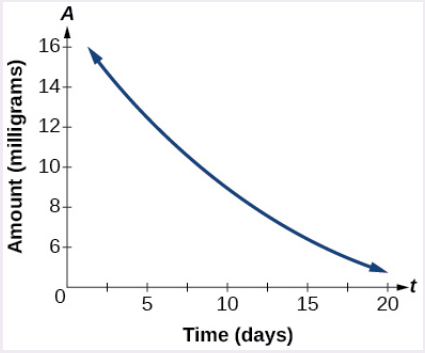


Figure 19: *Graph of an exponential function.*

Use the graph to estimate the average decay rate from $t = 5$ to $t = 15$.

Answer:

approximately –0.6 milligrams per day

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1.4: Composition of Functions

Skills to Develop

- Combine functions using algebraic operations to create a new function.
- Create a new function by composition of functions.
- Evaluate composite functions.
- Find the domain of a composite function.
- Decompose a composite function into its component functions.

Suppose we want to calculate how much it costs to heat a house on a particular day of the year. The cost to heat a house will depend on the average daily temperature, and in turn, the average daily temperature depends on the particular day of the year. Notice how we have just defined two relationships: The cost depends on the temperature, and the temperature depends on the day.

Using descriptive variables, we can notate these two functions. The function $C(T)$ gives the cost C of heating a house for a given average daily temperature in T degrees Celsius. The function $T(d)$ gives the average daily temperature on day d of the year. For any given day, $Cost = C(T(d))$ means that the cost depends on the temperature, which in turns depends on the day of the year. Thus, we can evaluate the cost function at the temperature $T(d)$. For example, we could evaluate $T(5)$ to determine the average daily temperature on the 5th day of the year. Then, we could evaluate the **cost function** at that temperature. We would write $C(T(5))$.

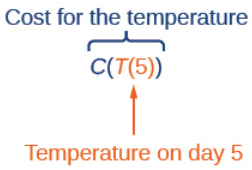


***Figure* 1.4.1:** *Explanation of $C(T(5))$, which is the cost for the temperature and $T(5)$ is the temperature on day 5.*

By combining these two relationships into one function, we have performed function composition, which is the focus of this section.

Combining Functions Using Algebraic Operations

Function composition is only one way to combine existing functions. Another way is to carry out the usual algebraic operations on functions, such as addition, subtraction, multiplication and division. We do this by performing the operations with the function outputs, defining the result as the output of our new function.

Suppose we need to add two columns of numbers that represent a husband and wife's separate annual incomes over a period of years, with the result being their total household income. We want to do this for every year, adding only that year's incomes and then collecting all the data in a new column. If $w(y)$ is the wife's income and $h(y)$ is the husband's income in year y, and we want T to represent the total income, then we can define a new function.

$$T(y) = h(y) + w(y)$$

If this holds true for every year, then we can focus on the relation between the functions without reference to a year and write

$$T = h + w$$

Just as for this sum of two functions, we can define difference, product, and ratio functions for any pair of functions that have the same kinds of inputs (not necessarily numbers) and also the same kinds of outputs (which do have to be numbers so that the usual operations of algebra can apply to them, and which also must have the same units or no units when we add and subtract). In this way, we can think of adding, subtracting, multiplying, and dividing functions.

For two functions $f(x)$ and $g(x)$ with real number outputs, we define new functions $f + g$, $f - g$, fg, and $\frac{f}{g}$ by the relations.

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)g(x)$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

Example 1.4.1: Performing Algebraic Operations on Functions

Find and simplify the functions $(g-f)(x)$ and $\left(\frac{g}{f}\right)(x)$, given $f(x) = x-1$ and $g(x) = x^2-1$. Are they the same function?

Solution

Begin by writing the general form, and then substitute the given functions.

$$\begin{aligned}(g-f)(x) &= g(x) - f(x)\\ &= (x^2-1) - (x-1)\\ &= x^2 - x, \text{ or } x(x-1)\end{aligned}$$

$$\begin{aligned}\left(\frac{g}{f}\right)(x) &= \frac{g(x)}{f(x)}\\ &= \frac{x^2-1}{x-1}\\ &= \frac{(x+1)(x-1)}{x-1}\\ &= x-1\end{aligned}$$

No, the functions are not the same.

Note: For $\left(\frac{g}{f}\right)(x)$, the condition $x \neq 1$ is necessary because when $x = 1$, the denominator is equal to 0, which makes the function undefined. Thus, the domain of $\frac{g}{f}$ is $D = \{x \mid x \neq 1\}$.

Note: What do you think the graph of $\frac{g}{f}$ would look like? Hand this graph to your instructor along with an explanation of your reasoning for ***Extra Credit.***

1.4.1

Find and simplify the functions $(fg)(x)$ and $(f-g)(x)$, given $f(x) = x-1$ and $g(x) = x^2-1$. Are they the same function?

Answer

$(fg)(x) = f(x)g(x) = (x-1)(x^2-1) = x^3 - x^2 - x + 1$
$(f-g)(x) = f(x) - g(x) = (x-1) - (x^2-1) = x - x^2$, or $x(1-x)$

No, the functions are not the same.

Create a Function by Composition of Functions

New functions can also be created by composing functions. When we wanted to compute a heating cost from a day of the year, we created a new function that takes a day as input and yields a cost as output. The process of combining functions so that the output of one function becomes the input of another is known as a **composition of functions**. The resulting function is known as a **composite function**. We represent this combination by the following notation:

$$f \circ g(x) = f(g(x))$$

We read the left-hand side as "f composed with g at x," and the right-hand side as "f of g of x." The two sides of the equation have the same mathematical meaning and are equal. The open circle symbol $\circ$ is called the composition operator. We use this operator mainly when we wish to emphasize the relationship between the functions themselves without referring to any particular input value. Composition is a binary operation that takes two functions and forms a new function, much as addition or multiplication

takes two numbers and gives a new number. However, it is important not to confuse function composition with multiplication because, as we will see, in most cases $f(g(x)) \neq f(x)g(x)$.

It is also important to understand the order of operations in evaluating a composite function. We follow the usual convention with parentheses by starting with the innermost parentheses first, and then working to the outside. In the equation above, the function g takes the input x first and yields an output $g(x)$. Then the function f takes $g(x)$ as an input and yields an output $f(g(x))$.

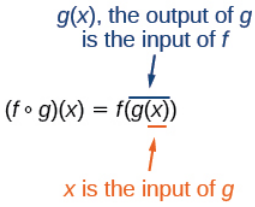


***Figure* 1.4.2:** *Explanation of the composite function.*

In general, $f \circ g$ and $g \circ f$ are different functions. In other words, in many cases $f(g(x)) \neq g(f(x))$ for all x.

For example, if $f(x) = x^2$ and $g(x) = x + 2$, then

$$\begin{aligned} f(g(x)) &= f(x+2) \\ &= (x+2)^2 \\ &= x^2 + 4x + 4 \end{aligned}$$

but

$$\begin{aligned} g(f(x)) &= g(x^2) \\ &= x^2 + 2 \end{aligned}$$

These expressions are not equal for all values of x, so the two functions are not equal. Although the expressions happen to be equal for the single input value $x = -\frac{1}{2}$, this is irrelevant.

Extra Credit: Given the formulas for each composite function, show how to find all x-values that make the functions equal, and hand your work in to your instructor.

Composition of Functions

When the output of one function is used as the input of another, we call the entire operation a composition of functions. For any input x and functions f and g, this action defines a **composite function**, which we write as $f \circ g$ such that

$$(f \circ g)(x) = f(g(x))$$

The domain of the composite function $f \circ g$ is all x such that x is in the domain of g and $g(x)$ is in the domain of f.

It is important to realize that the product of functions fg is not the same as the function composition $f \circ g$.

Example 1.4.2: Determining whether Composition of Functions is Commutative

Using the functions provided, find $f(g(x))$ and $g(f(x))$. Determine whether the composition of the functions is **commutative**.

$$f(x) = x^2 \quad g(x) = \sqrt{x}$$

Solution

Let's begin by substituting $g(x)$ into $f(x)$.

$$\begin{aligned} f(g(x)) &= (\sqrt{x})^2 \\ &= x, \ D = [0, \infty) \end{aligned}$$

Now we can substitute $f(x)$ into $g(x)$.

$$\begin{aligned} g(f(x)) &= \sqrt{(x^2)} \\ &= |x|\,,\ D=(-\infty,\infty) \end{aligned}$$

We find that $g(f(x)) \neq f(g(x))$, so the operation of function composition is not commutative. Note that not only are the formulas different, but the domains are also different.

Example 1.4.3: Comparing Multiplication of Functions with Composition of Functions

Let $f(x)=2x-3$, and let $g(x)=x^2+1$. Find two new functions by combining $f(x)$ and $g(x)$ in the following ways: $f \circ g$ and fg. Are they the same?

Solution

$$f \circ g = f(g(x)) = 2(x^2+1)-3 = 2x^2-1$$

$$fg = (2x-3)(x^2+1) = 2x^3-3x^2+2x-3$$

The two new functions are not the same.

Example 1.4.4: Interpreting Composite Functions

The function $c(s)$ gives the number of calories burned completing s sit-ups, and $s(t)$ gives the number of sit-ups a person can complete in t minutes. Interpret $c(s(3))$.

Solution

The inside expression in the composition is $s(3)$. Because the input to the s-function is time, $t=3$ represents 3 minutes, and $s(3)$ is the number of sit-ups completed in 3 minutes.

Using $s(3)$ as the input to the function $c(s)$ gives us the number of calories burned during the number of sit-ups that can be completed in 3 minutes, or simply the number of calories burned in 3 minutes (by doing sit-ups).

Sometimes two functions can be composed only in one specific order. The range of the inside function (the first function to be evaluated) needs to be within the domain of the outside function. Less formally, the composition has to make sense in terms of inputs and outputs. In a mathematical model, this can be very important. See Example **1.4.5**.

Example 1.4.5: Investigating the Order of Function Composition

Suppose $f(x)$ gives miles that can be driven in x hours and $g(y)$ gives the gallons of gas used in driving y miles. Which of these expressions is meaningful: $f(g(y))$ or $g(f(x))$?

Solution

The function $y=f(x)$ is a function whose output is the number of miles driven corresponding to the number of hours driven.

$$\text{number of miles } = f(\text{number of hours})$$

The function $g(y)$ is a function whose output is the number of gallons used corresponding to the number of miles driven. This means:

$$\text{number of gallons } = g(\text{number of miles})$$

The expression $g(y)$ takes miles as the input and a number of gallons as the output. The function $f(x)$ requires a number of hours as the input. Trying to input a number of gallons does not make sense. The expression $f(g(y))$ is meaningless.

The expression $f(x)$ takes hours as input and a number of miles driven as the output. The function $g(y)$ requires a number of miles as the input. Using $f(x)$ (miles driven) as an input value for $g(y)$, where gallons of gas depends on miles driven, does make

sense. The expression $g(f(x))$ makes sense, and will yield the number of gallons of gas used, g, driving a certain number of miles, $f(x)$, in x hours.

Are there any situations where $f(g(y))$ and $g(f(x))$ would both be meaningful or useful expressions?

Yes. For many pure mathematical functions, both compositions make sense, even though they usually produce different new functions. In real-world problems, functions whose inputs and outputs have the same units also may give compositions that are meaningful in either order.

1.4.2

The gravitational force on a planet a distance r from the sun is given by the function $G(r)$. The acceleration of a planet subjected to any force F is given by the function $a(F)$. Form a meaningful composition of these two functions, and explain what it means.

Answer

A gravitational force is still a force, so $a(G(r))$ makes sense as the acceleration of a planet at a distance r from the Sun (due to gravity), but $G(a(F))$ does not make sense.

Evaluating Composite Functions

Once we compose a new function from two existing functions, we need to be able to evaluate it for any input in its domain. We will do this with specific numerical inputs for functions expressed as tables, graphs, and formulas and with variables as inputs to functions expressed as formulas. In each case, we evaluate the inner function using the starting input and then use the inner function's output as the input for the outer function.

Evaluating Composite Functions Using Tables

When working with functions given as tables, we read input and output values from the table entries and always work from the inside to the outside. We evaluate the inside function first and then use the output of the inside function as the input to the outside function.

Example 1.4.6: Using a Table to Evaluate a Composite Function

Using Table 1.4.1, evaluate $f(g(3))$ and $g(f(3))$.

***Table* 1.4.1**

x	$f(x)$	$g(x)$
1	6	3
2	8	5
3	3	2
4	1	7

Solution

To evaluate $f(g(3))$, we start from the inside with the input value 3. We then evaluate the inside expression $g(3)$ using the table that defines the function g : $g(3) = 2$. We can then use that result as the input to the function f, so $g(3)$ is replaced by 2 and we get $f(2)$. Then, using the table that defines the function f, we find that $f(2) = 8$.

$$g(3) = 2$$

$$f(g(3)) = f(2) = 8$$

To evaluate $g(f(3))$, we first evaluate the inside expression $f(3)$ using the first table: $f(3) = 3$. Then, using the table for g, we can evaluate

$$g(f(3)) = g(3) = 2$$

Table 1.4.2 shows the composite functions $f \circ g$ and $g \circ f$ as tables.

Table 1.4.2

x	$g(x)$	$f(g(x))$	$f(x)$	$g(f(x))$
3	2	8	3	2

1.4.3

Using Table 1.4.1, evaluate $f(g(1))$ and $g(f(4))$.

Answer

$f(g(1)) = f(3) = 3$ and $g(f(4)) = g(1) = 3$

Evaluating Composite Functions Using Graphs

When we are given individual functions as graphs, the procedure for evaluating composite functions is similar to the process we use for evaluating tables. We read the input and output values, but this time, from the x- and y-axes of the graphs.

Given a composite function and graphs of its individual functions, evaluate it using the information provided by the graphs

1. Locate the given input to the inner function on the x-axis of its graph.
2. Read off the output of the inner function from the y-axis of its graph.
3. Locate the inner function output on the x-axis of the graph of the outer function.
4. Read the output of the outer function from the y-axis of its graph. This is the output of the composite function.

Example 1.4.7: Using a Graph to Evaluate a Composite Function

Using Figure 1.4.3, evaluate $f(g(1))$.

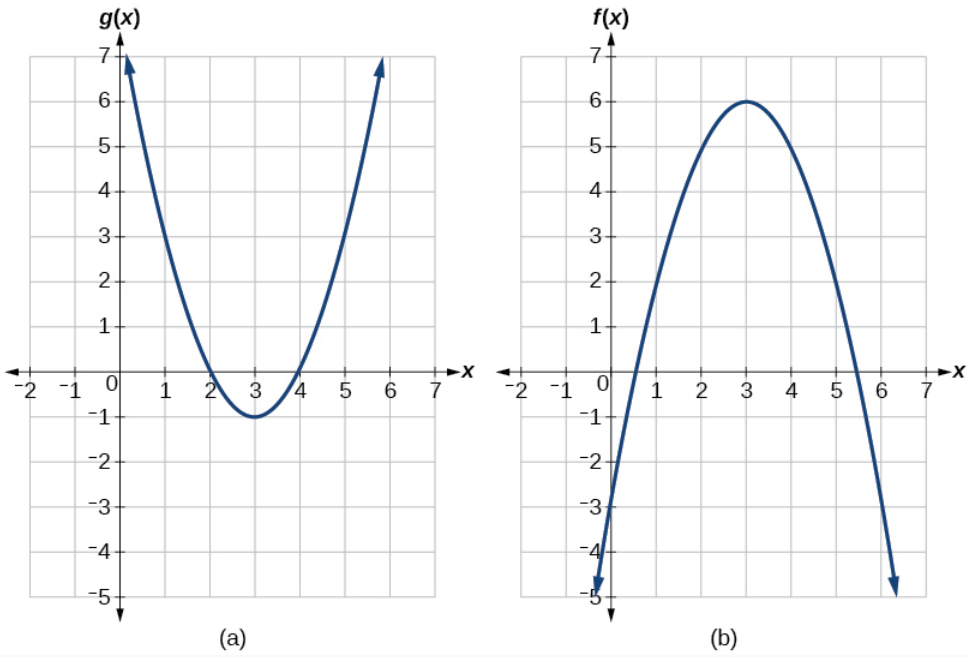


***Figure* 1.4.3:** *Two graphs of a positive parabola $y = g(x)$ and negative parabola $y = f(x)$.*

Solution

To evaluate $f(g(1))$, we start with the inside evaluation. See Figure 1.4.4.

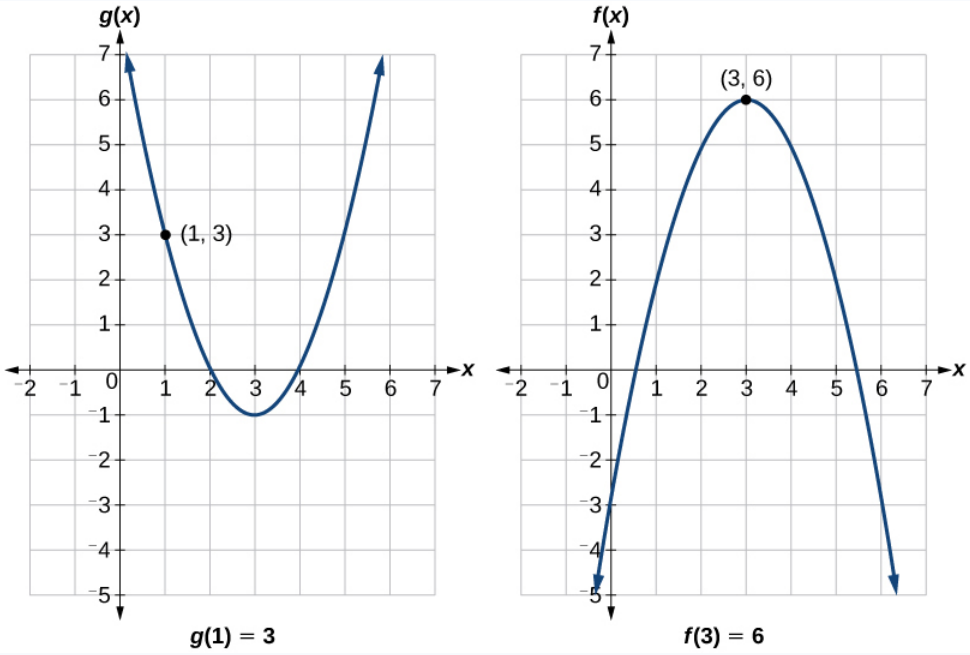


***Figure* 1.4.4:** *Two graphs of a positive parabola $g(x)$ and a negative parabola $f(x)$. The following points are plotted: $g(1) = 3$ and $f(3) = 6$.*

We evaluate $g(1)$ using the graph of $g(x)$, finding the input of 1 on the x-axis and finding the output value of the point on the graph at that input. Here, $g(1) = 3$. We use this value as the input to the function f.

$$f(g(1)) = f(3)$$

We can then evaluate the composite function by looking to the graph of $f(x)$, finding the input of 3 on the x-axis and reading the output value of the graph at this input. Here, $f(3) = 6$, so $f(g(1)) = 6$.

Analysis

Figure 1.4.5 shows how we can mark the graphs with arrows to trace the path from the input value to the output value.

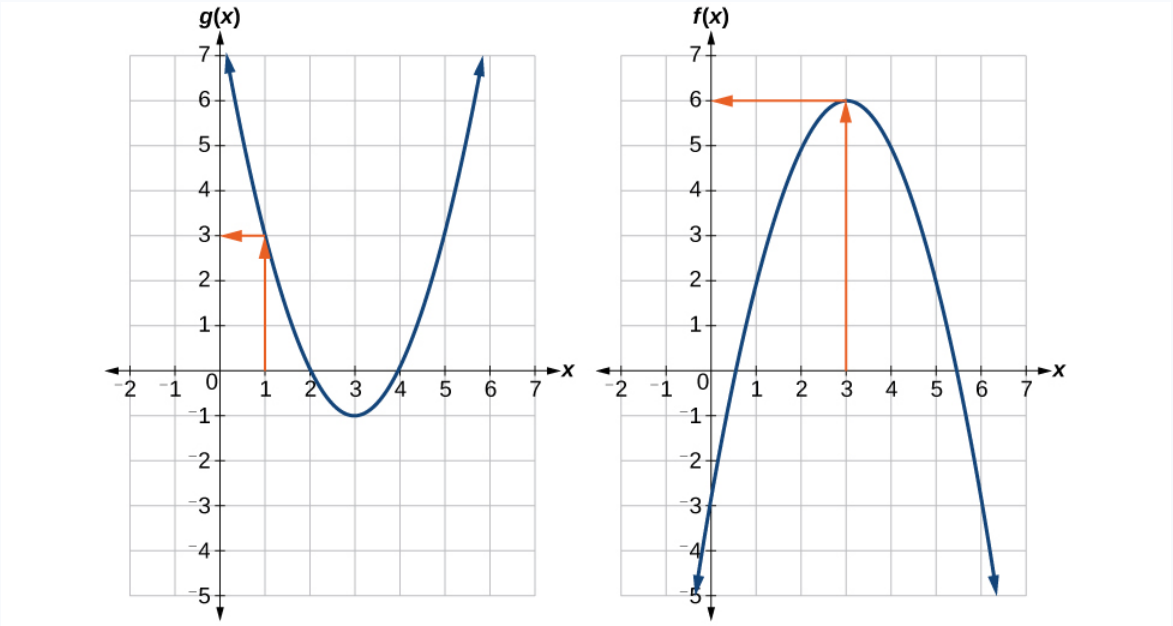


***Figure* 1.4.5**: *Two graphs of a positive parabola* $y = g(x)$ *and negative parabola* $y = f(x)$.

Solution

1.4.4

Using Figure **1.4.3**, evaluate $g(f(2))$.

Answer

$g(f(2)) = g(5) = 3$

Evaluating Composite Functions Using Formulas

When evaluating a composite function where we have either created or been given formulas, the rule of working from the inside out remains the same. The input value to the outer function will be the output of the inner function, which may be a numerical value, a variable name, or a more complicated expression.

While we can compose the functions for each individual input value, it is sometimes helpful to find a single formula that will calculate the result of a composition $f(g(x))$. To do this, we will extend our idea of function evaluation.

Recall that, when we evaluate a function like $f(t) = t^2 - t$, we substitute the value inside the parentheses into the formula wherever we see the input variable.

How To: Given a formula for a composite function, evaluate the function

1. Evaluate the inside function using the input value or variable provided.
2. Use the resulting output as the input to the outside function.

Example 1.4.8: Evaluating a Composition of Functions Expressed as Formulas with a Numerical Input

Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate $f(h(1))$.

Solution

Because the inside expression is $h(1)$, we start by evaluating $h(x)$ at 1.

$$h(1) = 3(1) + 2$$
$$h(1) = 5$$

Then $f(h(1)) = f(5)$, so we evaluate $f(t)$ at an input of 5.

$$f(h(1)) = f(5)$$
$$f(h(1)) = 5^2 - 5$$
$$f(h(1)) = 20$$

Analysis

It makes no difference what the input variables t and x were called in this problem because we evaluated for specific numerical values.

Try It 1.4.5

Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate

a. $h(f(2))$
b. $h(f(-2))$

Answer

a. $h(f(2)) = h(2) = 8$
b. $h(f(-2)) = h(6) = 20$

Finding the Domain of a Composite Function

As we discussed previously, the **domain of a composite function** such as $f \circ g$ is dependent on the domain of g and the domain of f. It is important to know when we can apply a composite function and when we cannot; that is, to know the domain of a function such as $f \circ g$. Let us assume we know the domains of the functions f and g separately. If we write the composite function for an input x as $f(g(x))$, we can see right away that x must be a member of the domain of g in order for the expression to be meaningful, because otherwise we cannot complete the inner function evaluation. However, we also see that $g(x)$ must be a member of the domain of f, otherwise the outer function evaluation in $f(g(x))$ cannot be completed, and the expression is still undefined. Thus the domain of $f \circ g$ consists of only those inputs in the domain of g that produce outputs from g belonging to the domain of f. Note that the domain of f composed with g is the set of all x such that x is in the domain of g and $g(x)$ is in the domain of f.

Definition: Domain of a Composite Function

The *domain of a composite function* $f(g(x))$ is the set of those inputs x in the domain of g for which $g(x)$ is in the domain of f.

How To: Given a function composition $f(g(x))$, determine its domain

1. Find the domain of g.
2. Find the domain of f.
3. Find those inputs x in the domain of g for which $g(x)$ is in the domain of f. That is, exclude input values x from the domain of g if $g(x)$ is not in the domain of f. The resulting set is the domain of $f \circ g$.

Example 1.4.9A: Finding the Domain of a Composite Function Involving Quotients

Find the domain of

$$(f \circ g)(x) \text{ where } f(x) = \frac{5}{x-1} \text{ and } g(x) = \frac{4}{3x-2}$$

Solution

The domain of $g(x)$ consists of all real numbers except $x = \frac{2}{3}$, since that input value would cause us to divide by 0. Likewise, the domain of f consists of all real numbers except 1. So we need to exclude from the domain of $g(x)$ any value of x for which $g(x) = 1$.

$$\begin{aligned} \frac{4}{3x-2} &= 1 \\ 4 &= 3x-2 \\ 6 &= 3x \\ x &= 2 \end{aligned}$$

The domain of $f \circ g$ is the set of all real numbers except $\frac{2}{3}$ and 2. This means that

$$x \neq \frac{2}{3} \text{ and } x \neq 2$$

We can write this in interval notation as

$$D = (-\infty, \frac{2}{3}) \cup (\frac{2}{3}, 2) \cup (2, \infty)$$

Example 1.4.9B: Finding the Domain of a Composite Function Involving Radicals

Find the domain of

$$(f \circ g)(x) \text{ where } f(x) = \sqrt{x+2} \text{ and } g(x) = \sqrt{3-x}$$

Because we cannot take the square root of a negative number, the domain of g is $(-\infty, 3]$. The composite function is

$$(f \circ g)(x) = \sqrt{\sqrt{3-x}+2}.$$

Again, because we cannot take the square root of a negative number, we exclude any negative numbers in the radicand; thus, we want only x-values such that $\sqrt{3-x}+2 \geq 0$, or $\sqrt{3-x} \geq -2$. Since the square root symbol $\sqrt{}$ is defined to be the non-negative square root of a number, $\sqrt{3-x}$ is always greater than -2; hence we are left with the restriction on the domain of g, which gives $f \circ g$ a domain of $(-\infty, 3]$.

Analysis

This example shows that knowledge of the range of functions (specifically the inner function) can also be helpful in finding the domain of a composite function. It also shows that the domain of $f \circ g$ can contain values that are not in the domain of f, though they must be in the domain of g.

1.4.6

Find the domain of

$$(f \circ g)(x), \text{ where } f(x) = \frac{1}{x-2} \text{ and } g(x) = \sqrt{x+4}$$

Answer

$[-4, 2) \cup (2, \infty)$

Decomposing a Composite Function into its Component Functions

In some cases, it is necessary to decompose a complicated function. In other words, we want to write a complicated function as a composition of two simpler functions. There may be more than one way to **decompose a composite function**, so we choose the

decomposition that seems best.

Example 1.4.10: Decomposing a Function

Write $f(x) = \sqrt{5 - x^2}$ as the composition of two functions.

Solution

We are looking for two functions, g and h, so $f(x) = g(h(x))$. To do this, we look for a function inside a function in the formula for $f(x)$. As one possibility, we might notice that the expression $5 - x^2$ is the inside of the square root. We could then decompose the function as

$$h(x) = 5 - x^2 \text{ and } g(x) = \sqrt{x}$$

We can check our answer by recomposing the functions.

$$g(h(x)) = g(5 - x^2) = \sqrt{5 - x^2}$$

1.4.7

Write $f(x) = \dfrac{4}{3 - \sqrt{4 + x^2}}$ as the composition of two functions.

Answer

Possible answer:

$g(x) = \sqrt{4 + x^2},\ h(x) = \dfrac{4}{3 - x}$
$f = h \circ g = h(g(x))$

Access these online resources for additional instruction and practice with composite functions.

- Composite Functions (http://openstaxcollege.org/l/compfunction)
- Composite Function Notation Application (http://openstaxcollege.org/l/compfuncnot)
- Composite Functions Using Graphs (http://openstaxcollege.org/l/compfuncgraph)
- Decompose Functions (http://openstaxcollege.org/l/decompfunction)
- Composite Function Values (http://openstaxcollege.org/l/compfuncvalue)

Key Equation

- Composite function $(f \circ g)(x) = f(g(x))$

Key Concepts

- We can perform algebraic operations on functions. See Example 1.4.1.
- When functions are combined, the output of the first (inner) function becomes the input of the second (outer) function.
- The function produced by combining two functions is a composite function.
- The order of function composition must be considered when interpreting the meaning of composite functions. See Examples 1.4.4 and 1.4.5.
- A composite function can be evaluated by evaluating the inner function using the given input value and then evaluating the outer function taking as its input the output of the inner function.
- A composite function can be evaluated from a table. See Example 1.4.6.
- A composite function can be evaluated from a graph. See Example 1.4.7.
- A composite function can be evaluated from a formula. See Example 1.4.8.
- The domain of a composite function consists of those inputs in the domain of the inner function that correspond to outputs of the inner function that are in the domain of the outer function. See Examples 1.4.9*A* and 1.4.9*B*.
- Just as functions can be combined to form a composite function, composite functions can be decomposed into simpler functions. See Example 1.4.10.
- Functions can often be decomposed in more than one way.

Glossary

composite function

the new function formed by function composition, when the output of one function is used as the input of another

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1.4E: Exercises

This page is a draft and is under active development.

1.4: Composition of Functions

Suppose we want to calculate how much it costs to heat a house on a particular day of the year. The cost to heat a house will depend on the average daily temperature, and in turn, the average daily temperature depends on the particular day of the year. The cost depends on the temperature, and the temperature depends on the day. By combining these two relationships into one function, we have performed function composition, which is the focus of this section.

Section Exercises

Verbal

Exercise 1.4.1

How does one find the domain of the quotient of two functions, $\frac{f}{g}$?

Answer:

Find the numbers that make the function in the denominator g equal to zero, and check for any other domain restrictions on f and g, such as an even-indexed root or zeros in the denominator.

Exercise 1.4.2

What is the composition of two functions, $f \circ g$?

Exercise 1.4.3

If the order is reversed when composing two functions, can the result ever be the same as the answer in the original order of the composition? If yes, give an example. If no, explain why not.

Answer:

Yes. Sample answer: Let $f(x) = x + 1$ and $g(x) = x - 3$. Then $f(g(x)) = f(x - 3) = (x - 3) + 1 = x - 2$ and $g(f(x)) = g(x + 1) = (x + 1) - 3 = x - 2$. So $f \circ g = g \circ f$.

Exercise 1.4.4

How do you find the domain for the composition of two functions, $f \circ g$?

Algebraic

Exercise 1.4.5

Given $f(x) = x^2 + 2x$ and $g(x) = 6 - x^2$, find $f + g$, $f - g$, fg, and $\frac{f}{g}$.Determine the domain for each function in interval notation.

Answer:

$(f + g)(x) = 2x + 6$, domain: $(-\infty, \infty)$

$(f - g)(x) = 2x^2 + 2x - 6$, domain: $(-\infty, \infty)$

$(fg)(x) = -x^4 - 2x^3 + 6x^2 + 12x$, domain: $(-\infty, \infty)$

$\left(\frac{f}{g}\right)(x) = \frac{x^2 + 2x}{6 - x^2}$, domain: $(-\infty, -\sqrt{6}) \cup (\sqrt{6}, \sqrt{6}) \cup (\sqrt{6}, \infty)$

Exercise 1.4.6

Given $f(x) = -3x^2 + x$ and $g(x) = 5$, find $f + g$, $f - g$, fg, and $\frac{f}{g}$. Determine the domain for each function in interval notation.

Exercise 1.4.7

Given $f(x) = 2x^2 + 4x$ and $g(x) = \frac{1}{2x}$, find $f + g$, $f - g$, fg, and $\frac{f}{g}$. Determine the domain for each function in interval notation.

Answer:

$(f + g)(x) = \frac{4x^3+8x^2+1}{2x}$, domain: $(-\infty, 0) \cup (0, \infty)$

$(f - g)(x) = \frac{4x3+8x2-1}{2x}$, domain: $(-\infty, 0) \cup (0, \infty)$

$(fg)(x) = x + 2$, domain: $(-\infty, 0) \cup (0, \infty)$

$(fg)(x) = 4x^3 + 8x^2$, domain: $(-\infty, 0) \cup (0, \infty)$

Exercise 1.4.8

Given $f(x) = \dfrac{1}{x-4}$ and $g(x) = \frac{1}{6-x}$, find$f + g$, $f - g$, fg, and $\frac{f}{g}$. Determine the domain for each function in interval notation.

Exercise 1.4.9

Given $f(x) = 3x^2$ and $g(x) = \sqrt{x-5}$, find $f + g$, $f - g$, fg, and $\frac{f}{g}$. Determine the domain for each function in interval notation.

Answer:

$(f + g)(x) = 3x^2 + \sqrt{x-5}$, domain: $[5, \infty)$

$(f - g)(x) = 3x^2 - \sqrt{x-5}$, domain: $[5, \infty)$

$(fg)(x) = 3x^2\sqrt{x-5}$, domain: $[5, \infty)$

$\left(\frac{f}{g}\right)(x) = \frac{3x^2}{\sqrt{x-5}}$, domain: $(5, \infty)$

Exercise 1.4.10

Given $f(x) = \sqrt{x}$ and $g(x) = |x - 3|$, find $\frac{g}{f}$. Determine the domain of the function in interval notation.

Exercise 1.4.11

Given $f(x) = 2x^2 + 1$ and $g(x) = 3x - 5$, find the following:

a. $f(g(2))$
b. $f(g(x))$
c. $g(f(x))$
d. $(g \circ g)(x)$
e. $(f \circ f)(-2)$

Answer:

a. 3;
b. $f(g(x)) = 2(3x - 5)^2 + 1$
c. $f(g(x)) = 6x^2 - 2$
d. $(f \circ g)(x) = 3(3x - 5) - 5 = 9x - 20$
e. $(f \circ f)(-2) = 163$

For the following exercises, use each pair of functions to find $f(g(x))$ and $g(f(x))$. Simplify your answers.

Exercise 1.4.12

$f(x) = x^2 + 1$, $g(x) = \sqrt{x+2}$

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Exercise 1.4.13

$f(x) = \sqrt{x} + 2$, $g(x) = x^2 + 3$

Answer:

$f(g(x)) = \sqrt{x+3}$, $g(f(x)) = x + 4\sqrt{x} + 7$

Exercise 1.4.14

$f(x) = |x|$, $g(x) = 5x + 1$

Exercise 1.4.15

$f(x) = \sqrt[3]{x}$, $g(x) = \frac{x+1}{x^3}$

Answer:

$f(g(x)) = \sqrt[3]{\frac{x+1}{x^3}} = \frac{\sqrt[3]{x+1}}{x}$, $g(f(x)) = \frac{\sqrt[3]{x}+1}{x}$

Exercise 1.4.16

$f(x) = \frac{1}{x-6}$, $g(x) = \frac{7}{x} + 6$

Exercise 1.4.17

$f(x) = \frac{1}{x-4}$, $g(x) = \frac{2}{x} + 4$

Answer:

$(f \circ g)(x) = \frac{1}{\frac{2}{x} + 4 - 4} = \frac{x}{2}$, $(g \circ f)(x) = 2x - 4$

For the following exercises, use each set of functions to find f(g(h(x))). Simplify your answers.

Exercise 1.4.18

$f(x) = x^4 + 6$, $g(x) = x - 6$, and $h(x) = \sqrt{x}$

Exercise 1.4.19

$f(x) = x^2 + 1$, $g(x) = \frac{1}{x}$, and $h(x) = x + 3$

Answer:

$f(g(h(x))) = \left(\frac{1}{x+3}\right)^2 + 1$

Exercise 1.4.20

Given $f(x) = \frac{1}{x}$ and $g(x) = x - 3$, find the following:

a. $(f \circ g)(x)$
b. the domain of $(f \circ g)(x)$ in interval notation
c. $(g \circ f)(x)$
d. the domain of $(g \circ f)(x)$
3. $\left(\frac{f}{g}\right)x$

Exercise 1.4.21

Given $f(x) = \sqrt{2 - 4x}$ and $g(x) = -\frac{3}{x}$, find the following:

a. $(g \circ f)(x)$
b. the domain of $(g \circ f)(x)$ in interval notation

Answer:

a. $(g \circ f)(x) = -\dfrac{3}{\sqrt{2-4x}}$

b. $(-\infty, \frac{1}{2})$

Exercise 1.4.22

Given the functions $f(x) = \frac{1-x}{x}$ and $g(x) = \frac{1}{1+x^2}$, find the following:

a. $(g \circ f)(x)$

b. $(g \circ f)(2)$

Exercise 1.4.23

Given functions $p(x) = \frac{1}{\sqrt{x}}$ and $m(x) = x^2 - 4$, state the domain of each of the following functions using interval notation:

a. $\frac{p(x)}{m(x)}$

b. $p(m(x))$

c. $m(p(x))$

Answer:

a. $(0, 2) \cup (2, \infty)$

b. $(-\infty, -2) \cup (2, \infty)$

c. $(0, \infty)$

Exercise 1.4.24

Given functions $q(x) = \frac{1}{\sqrt{x}}$ and $h(x) = x^2 - 9$, state the domain of each of the following functions using interval notation.

a. $\frac{q(x)}{h(x)}$

b. $q(h(x))$

c. $h(q(x))$

Exercise 1.4.25

For $f(x) = \frac{1}{x}$ and $g(x) = \sqrt{x-1}$, write the domain of $(f \circ g)(x)$ in interval notation.

Answer:

$(1, \infty)$

For the following exercises, find functions $f(x)$ and $g(x)$ so the given function can be expressed as $h(x) = f(g(x))$.

Exercise 1.4.26

$h(x) = (x+2)^2$

Exercise 1.4.27

$h(x) = (x-5)^3$

Answer:

sample: $f(x) = x^3$

$g(x) = x - 5$

Exercise 1.4.28

$h(x) = \frac{3}{x-5}$

Exercise 1.4.29

$h(x) = \frac{4}{(x+2)^2}$

Answer:

sample: $f(x)=\frac{4}{x}$

$g(x)=(x+2)^2$

Exercise 1.4.30

$h(x)=4+\sqrt[3]{x}$

Exercise 1.4.31

$h(x)=\sqrt[3]{\frac{1}{2x-3}}$

Answer:

sample: $f(x)=\sqrt[3]{x}$

$g(x)=\frac{1}{2x-3}$

Exercise 1.4.32

$h(x)=\frac{1}{(3x^2-4)^{-3}}$

Exercise 1.4.33

$h(x)=\sqrt[4]{\frac{3x-2}{x+5}}$

Answer:

sample: $f(x)=\sqrt[4]{x}$

$g(x)=\frac{3x-2}{x+5}$

Exercise 1.4.34

$h(x)=(\frac{8+x^3}{8-x^3})^4$

Exercise 1.4.35

$h(x)=\sqrt{2x+6}$

Answer:

sample: $f(x)=\sqrt{x}$

$g(x)=2x+6$

Exercise 1.4.36

$h(x)=(5x-1)^3$

Exercise 1.4.37

$h(x)=\sqrt[3]{x-1}$

Answer:

sample: $f(x)=\sqrt[3]{x}$

$g(x)=(x-1)$

Exercise 1.4.38

$h(x)=|x^2+7|$

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Exercise 1.4.39

$h(x) = \frac{1}{(x-2)^3}$

Answer:

sample: $f(x) = x^3$

$g(x) = \frac{1}{x-2}$

Exercise 1.4.40

$h(x) = \left(\frac{1}{2x-3}\right)^2$

Exercise 1.4.41

$h(x) = \sqrt{\frac{2x-1}{3x+4}}$

Answer:

sample: $f(x) = \sqrt{x}$

$g(x) = \frac{2x-1}{3x+4}$

Graphical

For the following exercises, use the graphs of f,shown in **Figure 1.4.6**, and g,shown in **Figure 1.4.7**, to evaluate the expressions.

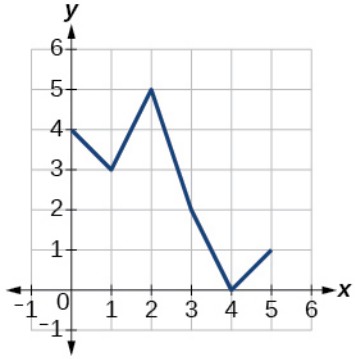


Figure 1.4.6: *Graph of a function.*

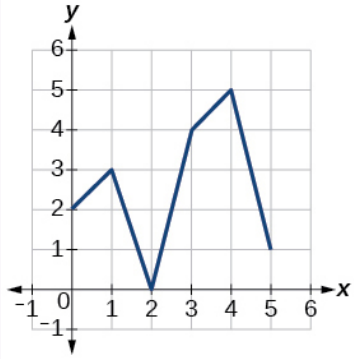


Figure 1.4.7: *Graph of a function.*

Exercise 1.4.42

$f(g(3))$

Exercise 1.4.43

$f(g(1))$

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Answer:

2

Exercise 1.4.44

$g(f(1))$

Exercise 1.4.45

$g(f(0))$

Answer:

5

Exercise 1.4.46

$f(f(5))$

Exercise 1.4.47

$f(f(4))$

Answer:

4

Exercise 1.4.48

$g(g(2))$

Exercise 1.4.49

$g(g(0))$

Answer:

0

For the following exercises, use graphs of $f(x)$,shown in **Figure 1.4.8**, $g(x)$,shown in **Figure 1.4.9**, and $h(x)$,shown in **Figure 1.4.10**, to evaluate the expressions.

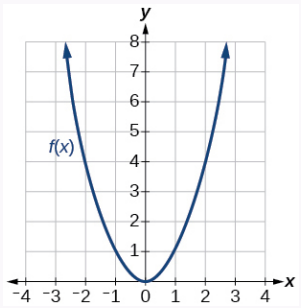


Figure 1.4.8: *Graph of a parabola.*

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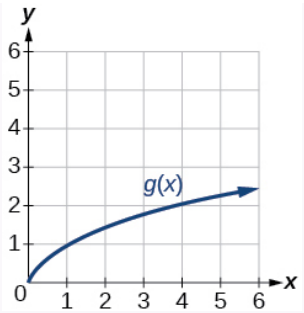


Figure 1.4.9: *Graph of a square root function.*

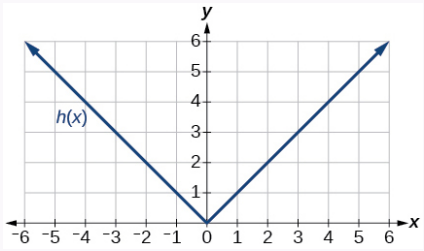


Figure 1.4.10: *Graph of an absolute value function.*

Exercise 1.4.50

$g(f(1))$

Exercise 1.4.51

$g(f(2))$

Answer:

2

Exercise 1.4.52

$f(g(4))$

Exercise 1.4.53

$f(g(1))$

Answer:

1

Exercise 1.4.54

$f(h(2))$

Exercise 1.4.55

$h(f(2))$

Answer:

4

Exercise 1.4.56

$f(g(h(4)))$

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Exercise 1.4.57

$f(g(f(-2)))$

Answer:

4

Numeric

For the following exercises, use the function values for f and g shown in **Table 1.4.3** to evaluate each expression.

x	$f(x)$	$g(x)$
0	7	9
1	6	5
2	5	6
3	8	2
4	4	1
5	0	8
6	2	7
7	1	3
8	9	4
9	3	0

Table 1.4.3

Exercise 1.4.58

$f(g(8))$

Exercise 1.4.59

$f(g(5))$

Answer:

9

Exercise 1.4.60

$g(f(5))$

Exercise 1.4.61

$g(f(3))$

Answer:

4

Exercise 1.4.62

$f(f(4))$

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Exercise 1.4.63

$f(f(1))$

Answer:

2

Exercise 1.4.64

$g(g(2))$

Exercise 1.4.65

$g(g(6))$

Answer:

3

For the following exercises, use the function values for f and g shown in **Table 1.4.4** to evaluate the expressions.

x	$f(x)$	$g(x)$
-3	11	-8
-2	9	-3
-1	7	0
0	5	1
1	3	0
2	1	-3
3	-1	-8

Exercise 1.4.66

$(f \circ g)(1)$

Exercise 1.4.67

$(f \circ g)(2)$

Answer:

11

Exercise 1.4.68

$(g \circ f)(2)$

Exercise 1.4.69

$(g \circ f)(3)$

Answer:

0

Exercise 1.4.70

$(g \circ g)(1)$

Exercise 1.4.71

$(f \circ f)(3)$

Answer:

7

For the following exercises, use each pair of functions to find $f(g(0))$ and $g(f(0))$.

Exercise 1.4.72

$f(x) = 4x + 8,\ g(x) = 7 - x^2$

Exercise 1.4.73

$f(x) = 5x + 7,\ g(x) = 4 - 2x^2$

Answer:

$f(g(0)) = 27,\ g(f(0)) = -94$

Exercise 1.4.74

$f(x) = \sqrt{x+4},\ g(x) = 12 - x^3$

Exercise 1.4.75

$f(x) = \frac{1}{x+2},\ g(x) = 4x + 3$

Answer:

$f(g(0)) = \frac{1}{5},\ g(f(0)) = 5$

For the following exercises, use the functions $f(x) = 2x^2 + 1$ and $g(x) = 3x + 5$ to evaluate or find the composite function as indicated.

Exercise 1.4.76

$f(g(2))$

Exercise 1.4.77

$f(g(x))$

Answer:

$18x^2 + 60x + 51$

Exercise 1.4.78

$g(f(-3))$

Exercise 1.4.79

$(g \circ g)(x)$

Answer:

$g \circ g(x) = 9x + 20$

Extensions

For the following exercises, use $f(x) = x^3 + 1$ and $g(x) = \sqrt[3]{x-1}$.

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Exercise 1.4.80

Find $(f \circ g)(x)$ and $(g \circ f)(x)$. Compare the two answers.

Exercise 1.4.81

Find $(f \circ g)(2)$ and $(g \circ f)(2)$.

Answer:

2

Exercise 1.4.82

What is the domain of $(g \circ f)(x)$?

Exercise 1.4.83

What is the domain of $(f \circ g)(x)$?

Answer:

$(-\infty, \infty)$

Exercise 1.4.84

Let $f(x) = \frac{1}{x}$.

a. Find $(f \circ f)(x)$.

b. Is $(f \circ f)(x)$ for any function f the same result as the answer to part (a) for any function? Explain.

For the following exercises, let $F(x) = (x+1)^5$, $f(x) = x^5$, and $g(x) = x + 1$.

Exercise 1.4.85

True or False: $(g \circ f)(x) = F(x)$.

Answer:

False

Exercise 1.4.86

True or False: $(f \circ g)(x) = F(x)$.

For the following exercises, find the composition when $f(x) = x^2 + 2$ for all $x \geq 0$ and $g(x) = \sqrt{x-2}$.

Exercise 1.4.87

$(f \circ g)(6)$; $(g \circ f)(6)$

Answer:

$(f \circ g)(6) = 6$, $(g \circ f)(6) = 6$

Exercise 1.4.88

$(g \circ f)(a)$; $(f \circ g)(a)$

Exercise 1.4.89

$(f \circ g)(11)$; $(g \circ f)(11)$

Answer:

$(f \circ g)(11) = 11$, $(g \circ f)(11) = 11$

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Real-World Applications

Exercise 1.4.90

The function $D(p)$ gives the number of items that will be demanded when the price is p. The production cost $C(x)$ is the cost of producing x items. To determine the cost of production when the price is \$6, you would do which of the following?

a. Evaluate $D(C(6))$.
b. Evaluate $C(D(6))$.
c. Solve $D(C(x)) = 6$.
d. Solve $C(D(p)) = 6$.

Exercise 1.4.91

The function $A(d)$ gives the pain level on a scale of 0 to 10 experienced by a patient with d milligrams of a pain-reducing drug in her system. The milligrams of the drug in the patient's system after t minutes is modeled by $m(t)$. Which of the following would you do in order to determine when the patient will be at a pain level of 4?

a. Evaluate $A(m(4))$.
b. Evaluate $m(A(4))$.
c. Solve $A(m(t)) = 4$.
d. Solve $m(A(d)) = 4$.

Answer:

c

Exercise 1.4.92

A store offers customers a 30% discount on the price x of selected items. Then, the store takes off an additional 15% at the cash register. Write a price function $P(x)$ that computes the final price of the item in terms of the original price x. (Hint: Use function composition to find your answer.)

Exercise 1.4.93

A rain drop hitting a lake makes a circular ripple. If the radius, in inches, grows as a function of time in minutes according to $r(t) = 25\sqrt{t+2}$, find the area of the ripple as a function of time. Find the area of the ripple at $t = 2$.

Answer:

$A(t) = \pi(25\sqrt{t+2})^2$ and $A(2) = \pi(25\sqrt{4})^2 = 2500\pi$ square inches

Exercise 1.4.94

A forest fire leaves behind an area of grass burned in an expanding circular pattern. If the radius of the circle of burning grass is increasing with time according to the formula $r(t) = 2t + 1$, express the area burned as a function of time, t (minutes).

Exercise 1.4.95

Use the function you found in the previous exercise to find the total area burned after 5 minutes.

Answer:

$A(5) = \pi(2(5) + 1)2 = 121\pi$ square units

Exercise 1.4.96

The radius r, in inches, of a spherical balloon is related to the volume, V, by $r(V) = \sqrt[3]{\frac{3V}{4\pi}}$. Air is pumped into the balloon, so the volume after t seconds is given by $V(t) = 10 + 20t$.

a. Find the composite function $r(V(t))$.

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b. Find the *exact* time when the radius reaches 10 inches.

Exercise 1.4.97

The number of bacteria in a refrigerated food product is given by $N(T) = 23T^2 - 56T + 1$, $3 < T < 33$, where T is the temperature of the food. When the food is removed from the refrigerator, the temperature is given by $T(t) = 5t + 1.5$, where t is the time in hours.

Find the composite function $N(T(t))$.
Find the time (round to two decimal places) when the bacteria count reaches 6752.

Answer:

a. $N(T(t)) = 23(5t + 1.5)^2 - 56(5t + 1.5) + 1$
b. 3.38 hours

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1.5: Transformation of Functions

Skills to Develop

- Graph functions using vertical and horizontal shifts.
- Graph functions using reflections about the x-axis and the y-axis.
- Determine whether a function is even, odd, or neither from its graph.
- Graph functions using compressions and stretches.
- Combine transformations.

We all know that a flat mirror enables us to see an accurate image of ourselves and whatever is behind us. When we tilt the mirror, the images we see may shift horizontally or vertically. But what happens when we bend a flexible mirror? Like a carnival funhouse mirror, it presents us with a distorted image of ourselves, stretched or compressed horizontally or vertically. In a similar way, we can distort or transform mathematical functions to better adapt them to describing objects or processes in the real world. In this section, we will take a look at several kinds of transformations.



Figure* 1.5.1*: *(credit: "Misko"/Flickr)*

Often when given a problem, we try to model the scenario using mathematics in the form of words, tables, graphs, and equations. One method we can employ is to adapt the basic graphs of the toolkit functions to build new models for a given scenario. There are systematic ways to alter functions to construct appropriate models for the problems we are trying to solve.

Identifying Vertical Shifts

One simple kind of transformation involves shifting the entire graph of a function up, down, right, or left. The simplest shift is a vertical shift, moving the graph up or down, because this transformation involves adding a positive or negative constant to the function. In other words, we add the same constant to the output value of the function regardless of the input. For a function $g(x) = f(x) + k$, the function $f(x)$ is shifted vertically k units. See Figure 1.5.2 for an example.

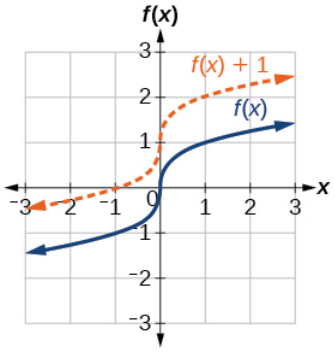


Figure* 1.5.2*: *Vertical shift by $k = 1$ of the cube root function $f(x) = \sqrt[3]{x}$*

To help you visualize the concept of a vertical shift, consider that $y = f(x)$. Therefore, $f(x) + k$ is equivalent to $y + k$. Every unit of y is replaced by $y + k$, so the y-value increases or decreases depending on the value of k. The result is a shift upward or downward.

Vertical Shift

Given a function $f(x)$, a new function $g(x) = f(x) + k$, where k is a constant, is a **vertical shift** of the function $f(x)$. All the output values change by k units. If k is positive, the graph will shift up. If k is negative, the graph will shift down.

Example 1.5.1: Adding a Constant to a Function

To regulate temperature in a green building, airflow vents near the roof open and close throughout the day. Figure **1.5.3** shows the area of open vents V (in square feet) throughout the day in hours after midnight, t. During the summer, the facilities manager decides to try to better regulate temperature by increasing the amount of open vents by 20 square feet throughout the day and night. Sketch a graph of this new function.

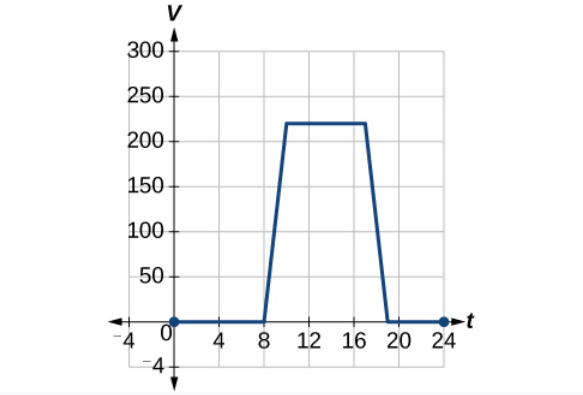


***Figure* 1.5.3**

Solution

We can sketch a graph of this new function by adding 20 to each of the output values of the original function. This will have the effect of shifting the graph vertically up, as shown in Figure **1.5.4**.

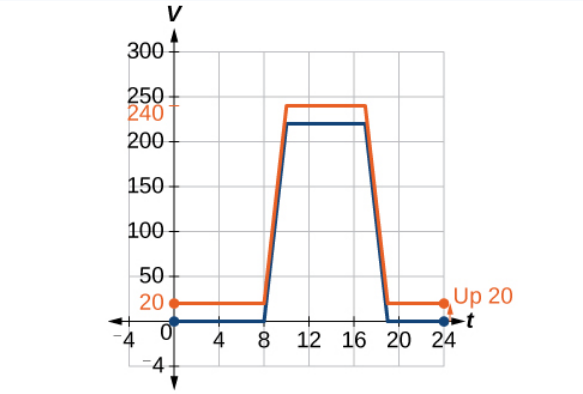


***Figure* 1.5.4**

Notice that in Figure **1.5.4**, for each input value, the output value has increased by 20, so if we call the new function $S(t)$,we could write

$$S(t) = V(t) + 20$$

This notation tells us that for any value of t, $S(t)$ can be found by evaluating the function V at the same input and then adding 20 to the result. This defines S as a transformation of the function V, in this case a vertical shift up 20 units. Notice that, with a vertical shift, the input values stay the same and only the output values change. See Table **1.5.1**.

Table 1.5.1

t	0	8	10	17	19	24
$V(t)$	0	0	220	220	0	0
$S(t)$	20	20	240	240	20	20

How To: Given a tabular function, create a new row to represent a vertical shift.

1. Identify the output row or column.
2. Determine the **magnitude** of the shift.
3. Add the shift to the value in each output cell. Add a positive value for up or a negative value for down.

Example 1.5.2: Shifting a Tabular Function Vertically

A function $f(x)$ is given in Table **1.5.2**. Create a table for the function $g(x) = f(x) - 3$.

Table 1.5.2

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = f(x) - 3$ tells us that we can find the output values of g by subtracting 3 from the output values of f. For example:

$$\begin{aligned} f(x) &= 1 && \text{Given} \\ g(x) &= f(x) - 3 && \text{Given Transformation} \\ g(2) &= f(2) - 3 \\ &= 1 - 3 \\ &= -2 \end{aligned}$$

Subtracting 3 from each $f(x)$ value, we can complete a table of values for $g(x)$ as shown in Table **1.5.3**.

Table 1.5.3

x	2	4	6	8
$f(x)$	1	3	7	11
$g(x)$	-2	0	4	8

Analysis

As with the earlier vertical shift, notice the input values stay the same and only the output values change.

1.5.1

The function $h(t) = -4.9t^2 + 30t$ gives the height h of a ball (in meters) thrown upward from the ground after t seconds. Suppose the ball was instead thrown from the top of a 10-m building. Relate this new height function $b(t)$ to $h(t)$, and then find a formula for $b(t)$.

Answer

$$b(t) = h(t) + 10 = -4.9t^2 + 30t + 10$$

Identifying Horizontal Shifts

We just saw that the vertical shift is a change to the output, or outside, of the function. We will now look at how changes to input, on the inside of the function, change its graph and meaning. A shift to the input results in a movement of the graph of the function left or right in what is known as a **horizontal shift**, shown in Figure 1.5.5.

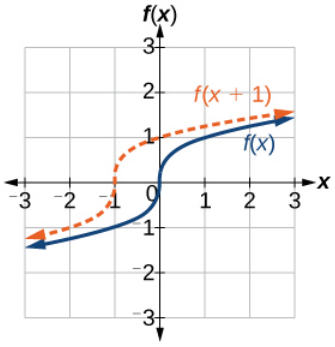


***Figure* 1.5.5:** *Horizontal shift of the function $f(x) = \sqrt[3]{x}$. Note that $h = +1$ shifts the graph to the left; that is, towards negative values of x.*

For example, if $f(x) = x^2$, then $g(x) = (x-2)^2$ is a new function. Each input is reduced by 2 prior to squaring the function. The result is that the graph is shifted 2 units to the right, because we would need to increase the prior input by 2 units to yield the same output value as given in f.

Horizontal Shift

Given a function f, a new function $g(x) = f(x-h)$, where h is a constant, is a **horizontal shift** of the function f. If h is positive, the graph will shift right. If h is negative, the graph will shift left.

Example 1.5.3: Adding a Constant to an Input

Returning to our building airflow example from Figure 1.5.3, suppose that in autumn the facilities manager decides that the original venting plan starts too late, and wants to begin the entire venting program 2 hours earlier. Sketch a graph of the new function.

Solution

We can set $V(t)$ to be the original program and $F(t)$ to be the revised program.

$$V(t) = \text{ the original venting plan}$$

$$F(t) = \text{ starting 2 hrs sooner}$$

In the new graph, at each time, the airflow is the same as the original function V was 2 hours later. For example, in the original function V, the airflow starts to change at 8 a.m., whereas for the function F, the airflow starts to change at 6 a.m. The

comparable function values are $V(8) = F(6)$. See Figure 1.5.6. Notice also that the vents first opened to $220\ \text{ft}^2$ at 10 a.m. under the original plan, while under the new plan the vents reach $220\ \text{ft}^2$ at 8 a.m., so $V(10) = F(8)$.

In both cases, we see that, because $F(t)$ starts 2 hours sooner, $h = -2$. That means that the same output values are reached when $F(t) = V(t-(-2)) = V(t+2)$.

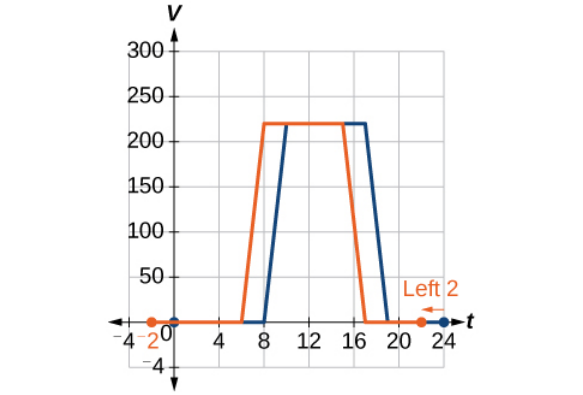


***Figure* 1.5.6**

Analysis

Note that $V(t+2)$ has the effect of shifting the graph to the left.

Horizontal changes or "inside changes" affect the domain of a function (the input) instead of the range and often seem counterintuitive. The new function $F(t)$ uses the same outputs as $V(t)$, but matches those outputs to inputs 2 hours earlier than those of $V(t)$. Said another way, we must add 2 hours to the input of V to find the corresponding output for $F: F(t) = V(t+2)$.

How To: Given a tabular function, create a new row to represent a horizontal shift.

1. Identify the input row or column.
2. Determine the magnitude of the shift.
3. Add the shift to the value in each input cell.

Example 1.5.4: Shifting a Tabular Function Horizontally

A function $f(x)$ is given in Table 1.5.4. Create a table for the function $g(x) = f(x-3)$.

***Table* 1.5.4**

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = f(x-3)$ tells us that the output values of g are the same as the output value of f when the input value is 3 less than the original value. For example, we know that $f(2) = 1$. To get the same output from the function g, we will need an input value that is 3 larger. We input a value that is 3 larger for $g(x)$ because the function takes 3 away before evaluating the function f.

$$\begin{aligned} g(5) &= f(5-3) \\ &= f(2) \\ &= 1 \end{aligned}$$

We continue with the other values to create Table 1.5.5.

Table 1.5.5

x	5	7	9	11
$x-3$	2	4	6	8
$g(x)$	1	3	7	11

The result is that the function $g(x)$ has been shifted to the right by 3. Notice the output values for $g(x)$ remain the same as the output values for $f(x)$, but the corresponding input values, x, have shifted to the right by 3. Specifically, 2 shifted to 5, 4 shifted to 7, 6 shifted to 9, and 8 shifted to 11.

Analysis

Figure 1.5.7 represents both of the functions. We can see the horizontal shift in each point.

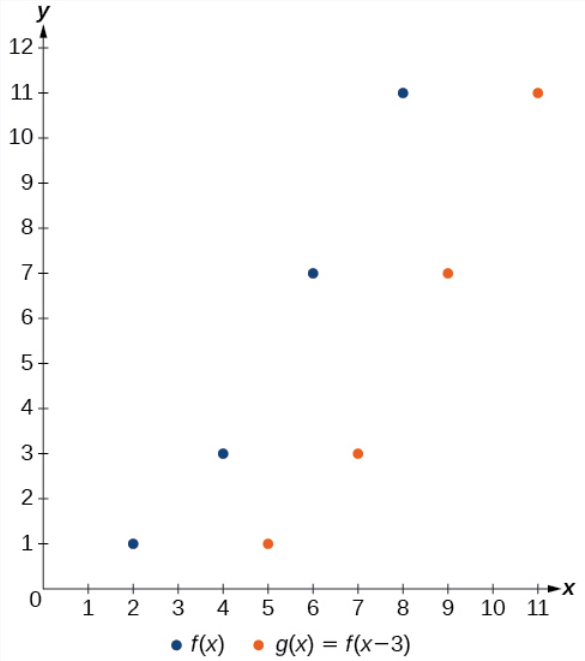


***Figure* 1.5.7:** Graph of the points from the previous tables for $f(x)$ and $g(x) = f(x-3)$

Example 1.5.5: Identifying a Horizontal Shift of a Toolkit Function

Figure 1.5.8 represents a transformation of the toolkit function $f(x) = x^2$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

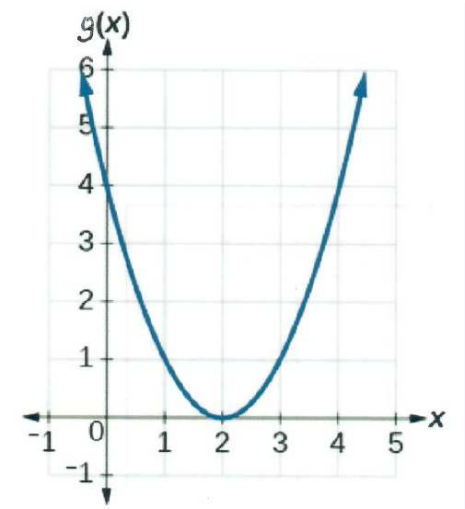


Figure* 1.5.8*: *Graph of a parabola*

Solution

Notice that the graph is identical in shape to the $f(x) = x^2$ function, but the x-values are shifted to the right 2 units. The vertex used to be at $(0, 0)$, but now the vertex is at $(2, 0)$. The graph is the basic quadratic function shifted 2 units to the right, so

$$g(x) = f(x - 2)$$

Notice how we must input the value $x = 2$ to get the output value $y = 0$; the x-values must be 2 units larger because of the shift to the right by 2 units. We can then use the definition of the $f(x)$ function to write a formula for $g(x)$ by evaluating $f(x - 2)$.

$$\begin{aligned} f(x) &= x^2 \\ g(x) &= f(x - 2) \\ g(x) &= f(x - 2) = (x - 2)^2 \end{aligned}$$

Analysis

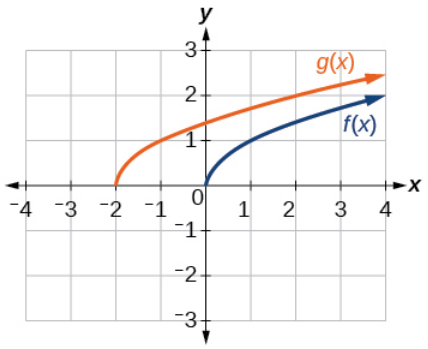
To determine whether the shift is $+2$ or -2, consider a single reference point on the graph. For a quadratic, looking at the vertex point is convenient. In the original function, $f(0) = 0$. In our shifted function, $g(2) = 0$. To obtain the output value of 0 from the function f, we need to decide whether a plus or a minus sign will work to satisfy $g(2) = f(0) = 0$. For this to work, we will need to subtract 2 units from our input values.

1.5.2

Given the function $f(x) = \sqrt{x}$, graph the original function $f(x)$ and the transformation $g(x) = f(x + 2)$ on the same axes. Is this a horizontal or a vertical shift? Which way is the graph shifted and by how many units?

Answer

The graphs of $f(x)$ and $g(x)$ are shown below. The transformation is a horizontal shift. The function is shifted to the left by 2 units.



Example 1.5.6: Interpreting Horizontal versus Vertical Shifts

The function $G(m)$ gives the number of gallons of gas required to drive m miles. Interpret $G(m)+10$ and $G(m+10)$

Solution

$G(m)+10$ can be interpreted as adding 10 to the output, gallons. This is the gas required to drive m miles, plus another 10 gallons of gas. The graph would indicate a vertical shift.

$G(m+10)$ can be interpreted as adding 10 to the input, miles. So this is the number of gallons of gas required to drive 10 miles more than m miles. The graph would indicate a horizontal shift.

Suppose your nervous cousin wants to be sure to have enough gas to drive m miles. Would they feel safer buying $G(m)+10$ gallons, or $G(m+10)$ gallons? If you think you know the answer, write a clear explanation of your reasoning, and give it to your instructor for ***Extra Credit***.

Combining Vertical and Horizontal Shifts

Now that we have two transformations, we can combine them together. Vertical shifts are outside changes that affect the output (y-axis) values and shift the function up or down. Horizontal shifts are inside changes that affect the input (x-axis) values and shift the function left or right. Combining the two types of shifts will cause the graph of a function to shift up or down and right or left.

How To: Given a function and both a vertical and a horizontal shift, sketch the graph.

1. Identify the vertical and horizontal shifts from the formula.
2. The vertical shift results from a constant added to the output. Move the graph up for a positive constant and down for a negative constant.
3. The horizontal shift results from a constant added to the input. Move the graph left for a positive constant and right for a negative constant.
4. Apply the shifts to the graph in either order.

Example 1.5.7: Graphing Combined Vertical and Horizontal Shifts

Given $f(x)=|x|$, sketch a graph of $h(x)=f(x+1)-3$.

Solution

The function f is our toolkit absolute value function. We know that this graph has a V shape, with the point at the origin. The graph of h has transformed f in two ways: $f(x+1)$ is a change on the inside of the function, giving a horizontal shift left by 1,

and the subtraction by 3 in $f(x+1)-3$ is a change to the outside of the function, giving a vertical shift down by 3. The transformation of the graph is illustrated in Figure 1.5.9.

Let us follow one point of the graph of $f(x)=|x|$.

The point $(0,0)$ is transformed first by shifting left 1 unit: $(0,0) \rightarrow (-1,0)$
The point $(-1,0)$ is transformed next by shifting down 3 units: $(-1,0) \rightarrow (-1,-3)$

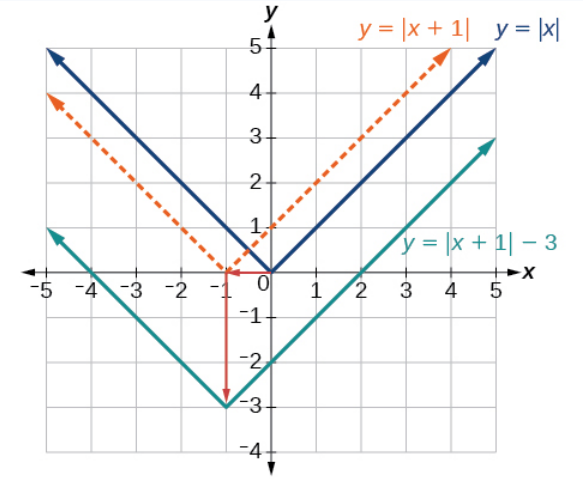


***Figure* 1.5.9:** *Graph of an absolute function, $y=|x|$, and how it was transformed to $y=|x+1|-3$*

Figure 1.5.10 shows the graph of h.

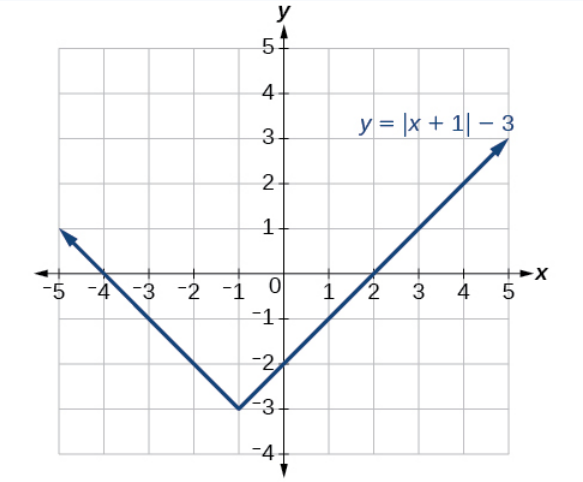
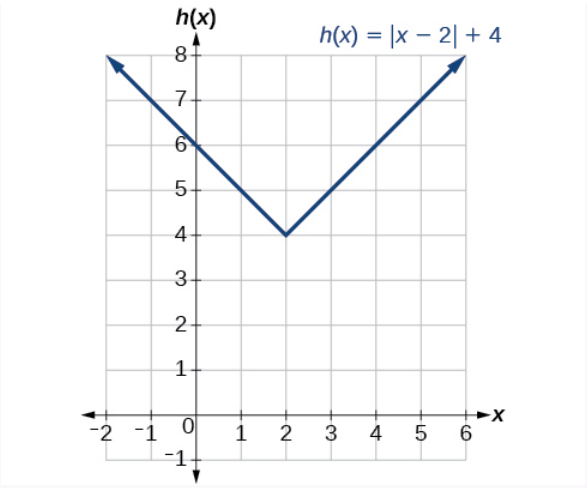


***Figure* 1.5.10:** *The final function $h(x)=|x+1|-3$.*

1.5.3

Given $f(x)=|x|$, sketch a graph of $h(x)=f(x-2)+4$.

Answer



Example 1.5.8: Identifying Combined Vertical and Horizontal Shifts

Write a formula for the graph shown in Figure **1.5.11**, which is a transformation of the toolkit square root function.

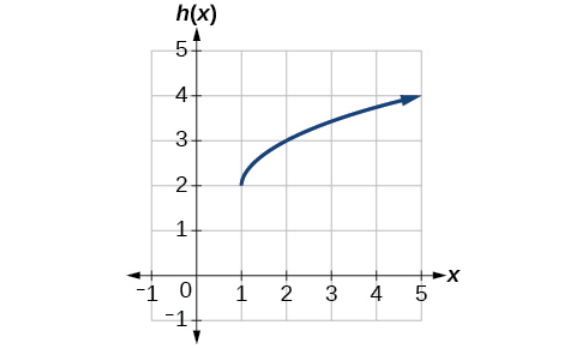


Figure* 1.5.11*: *Graph of a square root function transposed right one unit and up 2.*

Solution

The graph of the toolkit function starts at the origin, so this graph has been shifted 1 to the right and up 2. In function notation, we could write that as

$$h(x) = f(x-1) + 2$$

Using the formula for the square root function, we can write

$$h(x) = \sqrt{x-1} + 2$$

Analysis

Note that this transformation has changed the domain and range of the function. This new graph has domain $[1, \infty)$ and range $[2, \infty)$.

1.5.4

Write a formula for a transformation of the toolkit reciprocal function $f(x) = \frac{1}{x}$ that shifts the function's graph one unit to the right and one unit up. What are the domain and range of this transformation?

Answer

$$g(x) = \frac{1}{x-1} + 1 \qquad \text{Domain } = \{x|x \neq 1\},\ \text{Range } = \{x|x \neq 1\}$$

Graphing Functions Using Reflections about the Axes

Another transformation that can be applied to a function is a reflection over the x- or y-axis. A **vertical reflection** reflects a graph vertically across the x-axis, while a **horizontal reflection** reflects a graph horizontally across the y-axis. The reflections are shown in Figure 1.5.12.

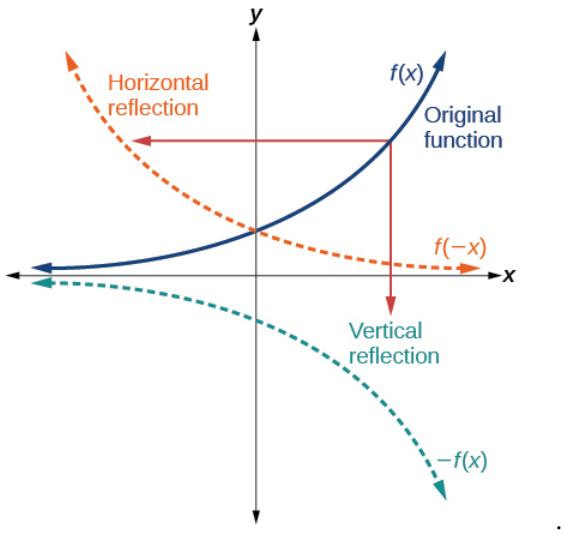


Figure* 1.5.12*: *Graph of the vertical and horizontal reflection of a function.*

Notice that the vertical reflection produces a new graph that is a mirror image of the original graph across the x-axis. The horizontal reflection produces a new graph that is a mirror image of the original graph across the y-axis.

Reflections

Given a function $f(x)$, a new function $g(x) = -f(x)$ is a **vertical reflection** of the function $f(x)$, sometimes called a reflection across (or over, or through) the x-axis.

Given a function $f(x)$, a new function $g(x) = f(-x)$ is a **horizontal reflection** of the function $f(x)$, sometimes called a reflection across the y-axis.

How To: Given a function, reflect the graph both vertically and horizontally

1. Multiply all outputs by –1 for a vertical reflection. The new graph is a reflection of the original graph across the x-axis.
2. Multiply all inputs by –1 for a horizontal reflection. The new graph is a reflection of the original graph across the y-axis.

Example 1.5.9: Reflecting a Graph Horizontally and Vertically

Reflect the graph of $s(t) = \sqrt{t}$ (a) vertically and (b) horizontally.

Solution

a. Reflecting the graph vertically means that each output value will be reflected over the horizontal t-axis, as shown in Figure 1.5.13.

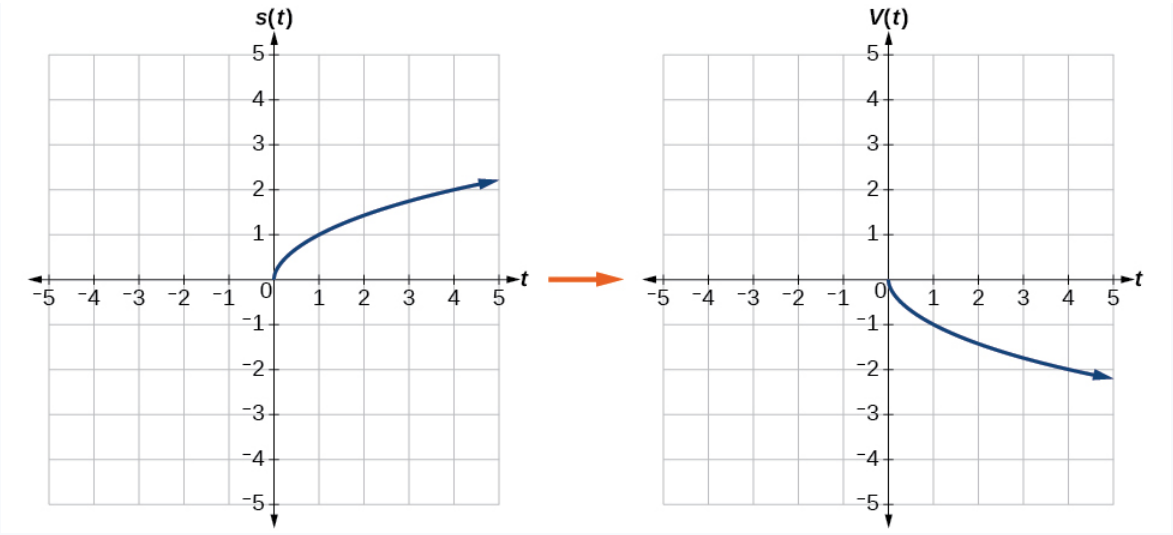


***Figure* 1.5.13:** *Graph of the vertical reflection of the square root function.*

Because each output value is the opposite of the original output value, we can write

$$V(t) = -s(t) \text{ or } V(t) = -\sqrt{t}$$

Notice that this is an outside change, or vertical shift, that affects the output $s(t)$ values, so the negative sign belongs outside of the function.

b. Reflecting horizontally means that each input value will be reflected over the vertical axis as shown in Figure 1.5.14.

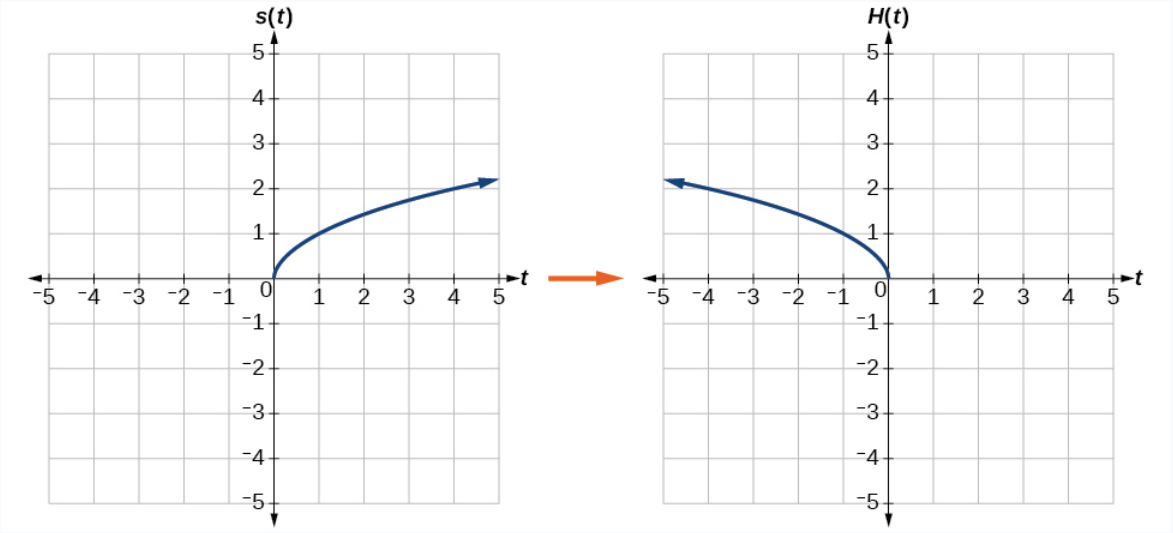


***Figure* 1.5.14:** *Horizontal reflection of the square root function*

Because each input value is the opposite of the original input value, we can write

$$H(t) = s(-t) \text{ or } H(t) = \sqrt{-t}$$

Notice that this is an inside change or horizontal change that affects the input values, so the negative sign is on the inside of the function.

Analysis

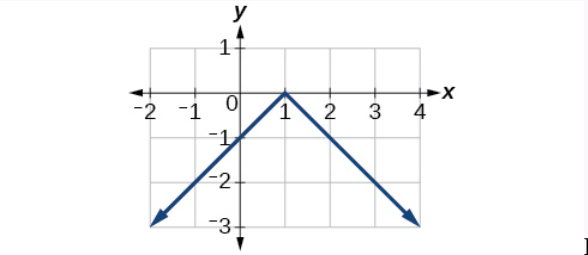
These transformations can affect the domain and range of the functions. While the original square root function has domain $[0, \infty)$ and range $[0, \infty)$, the vertical reflection gives the $V(t)$ function a range of $(-\infty, 0]$ and the horizontal reflection gives the $H(t)$ function a domain of $(-\infty, 0]$.

1.5.5

Reflect the graph of $f(x) = |x - 1|$ (a) vertically and (b) horizontally. State the new domain and range in each case.

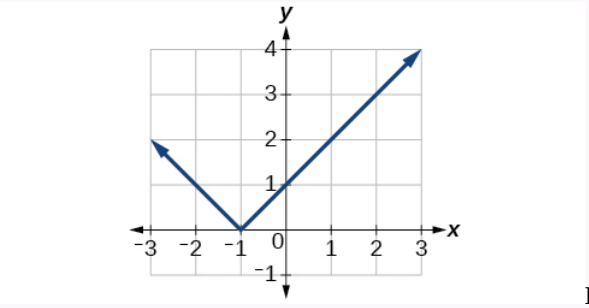
Answer

a.



Domain = All real numbers, Range = $(-\infty, 0]$

b.



Domain = All real numbers, Range = $[0, \infty)$

Example 1.5.10: Reflecting a Tabular Function Horizontally and Vertically

A function $f(x)$ is given as Table **1.5.6**. Create a table for the functions below.

a. $g(x) = -f(x)$
b. $h(x) = f(-x)$

***Table* 1.5.6**

x	2	4	6	8
$f(x)$	1	-3	7	11

a. For $g(x)$, the negative sign outside the function indicates a vertical reflection, so the x-values stay the same and each output value will be the opposite of the original output value. See Table **1.5.7**.

***Table* 1.5.7**

x	2	4	6	8
$g(x)$	-1	3	-7	-11

b. For $h(x)$, the negative sign inside the function indicates a horizontal reflection, so each input value will be the opposite of the original input value and the $h(x)$ values stay the same as the $f(x)$ values. See Table 1.5.8.

***Table* 1.5.8**

x	-2	-4	-6	-8
$h(x)$	1	-3	7	11

1.5.6

A function $f(x)$ is given as Table 1.5.9. Create a table for the functions below.

a. $g(x) = -f(x)$
b. $h(x) = f(-x)$

***Table* 1.5.9**

x	-2	0	2	4
$f(x)$	5	10	15	20

Answer

a. $g(x) = -f(x)$

***Table* 1.5.10**

x	-2	0	2	4
$g(x)$	-5	-10	-15	-20

b. $h(x) = f(-x)$

***Table* 1.5.11**

x	-2	0	2	4
$h(x)$	15	10	5	unknown

Example 1.5.11: Applying a Learning Model Equation

A common model for learning has an equation similar to $k(t) = -2^{-t} + 1$, where k is the percentage of mastery that can be achieved after t practice sessions. This is a transformation of the function $f(t) = 2^t$ shown in Figure 1.5.15. Sketch a graph of $k(t)$.

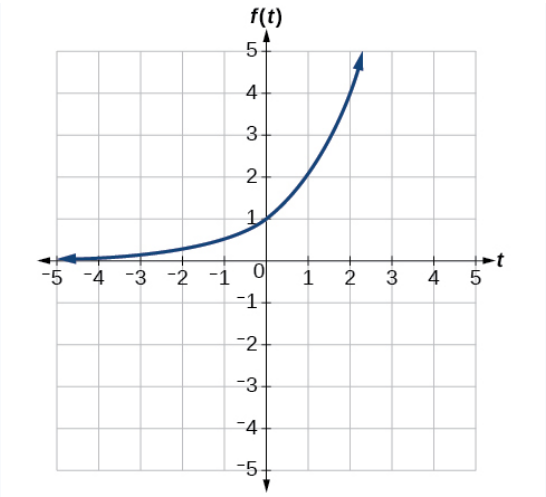


***Figure* 1.5.15:** *Graph of* $y = f(t)$

Solution

This equation combines three transformations into one equation.

- A horizontal reflection: $f(-t) = 2^{-t}$
- A vertical reflection: $-f(-t) = -2^{-t}$
- A vertical shift: $-f(-t) + 1 = -2^{-t} + 1$

We can sketch a graph by applying these transformations one at a time to the original function. Let us follow two points through each of the three transformations. We will choose the points $(0, 1)$ and $(1, 2)$.

- First, we apply the horizontal reflection: $(0, 1),\ (-1, 2)$.
- Then, we apply the vertical reflection: $(0, -1),\ (-1, -2)$
- Finally, we apply the vertical shift: $(0, 0),\ (-1, -1)$.

This means that the original points, $(0, 1)$ and $(1, 2)$ become $(0, 0)$ and $(-1, -1)$ after we apply the transformations.

In Figure 1.5.16, the first graph results from a horizontal reflection. The second results from a vertical reflection. The third results from a vertical shift up 1 unit.

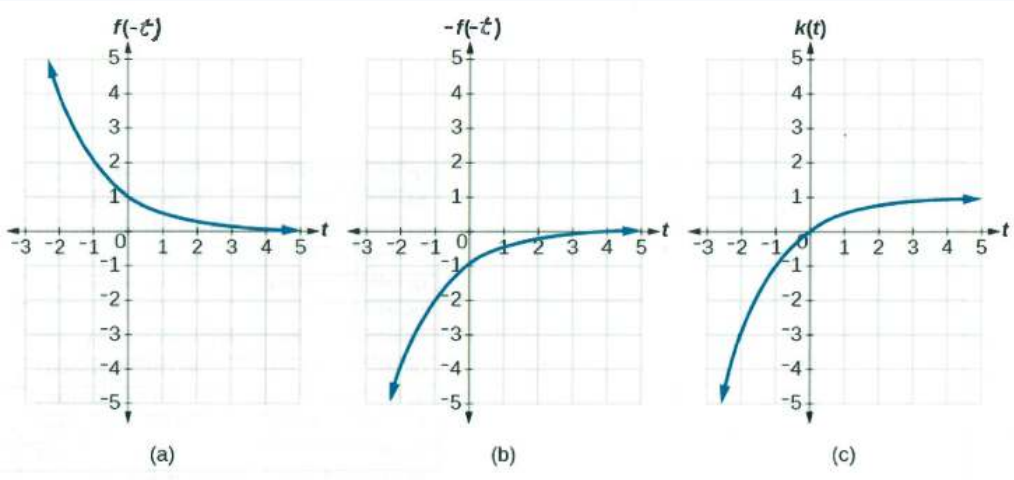


***Figure* 1.5.16:** *Graphs of all the transformations.*

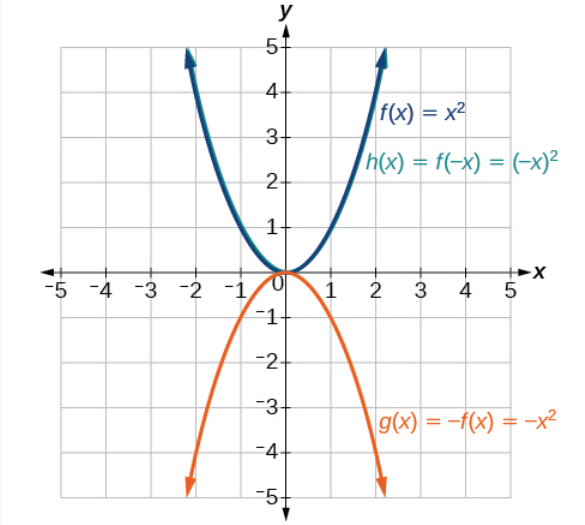
Analysis

As a model for learning, this function would be limited to a domain of $t \geq 0$, with corresponding range $[0, 1)$.

1.5.7

Given the toolkit function $f(x) = x^2$, graph $g(x) = -f(x)$ and $h(x) = f(-x)$. Take note of any surprising behavior for these functions.

Answer



Notice: $h(x) = f(-x)$ looks the same as $f(x)$.

Determining Even and Odd Functions

Some functions exhibit symmetry so that reflections result in the original graph. For example, horizontally reflecting the toolkit functions $f(x) = x^2$ or $f(x) = |x|$ will result in the original graph. These types of graphs have *line symmetry*, and we say they are symmetric with respect to the y-axis. Functions whose graphs are symmetric with respect to the y-axis are called **even functions**.

If the graphs of $f(x) = x^3$ or $f(x) = \frac{1}{x}$ were reflected over *both* axes, the result would be the original graph, as shown in Figure 1.5.17.

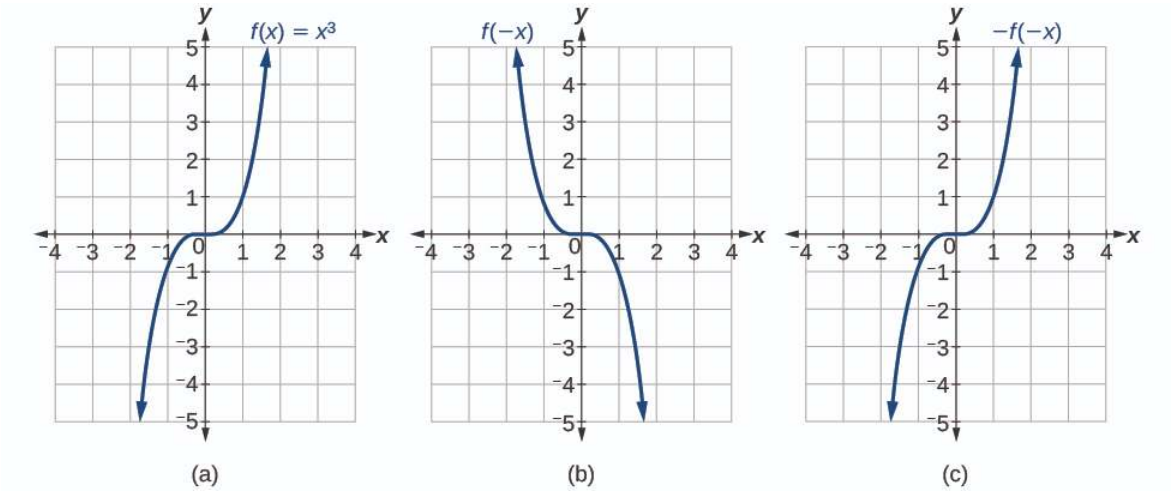


***Figure* 1.5.17**: *(a) The cubic toolkit function (b) Horizontal reflection of the cubic toolkit function (c) Horizontal and vertical reflections reproduce the original cubic function.*

These graphs have *point symmetry*, and we say they are symmetric with respect to the origin. A function with a graph that is symmetric with respect to the origin is called an **odd function**.

Note: A function is **neither even nor odd** if it does not exhibit either symmetry. For example, $f(x) = 2^x$ is neither even nor odd. Also, the only function that is both even and odd is the constant function $f(x) = 0$.

Even and Odd Functions

A function is called an **even function** if for every input x

$f(x) = f(-x)$.

The graph of an even function is symmetric with respect to the y-axis.

A function is called an **odd function** if for every input x

$f(x) = -f(-x)$.

The graph of an odd function is symmetric with respect to the origin.

How To: Given the formula for a function, determine if the function is even, odd, or neither

1. Determine whether the function satisfies $f(x) = f(-x)$. If it does, it is even.
2. Determine whether the function satisfies $f(x) = -f(-x)$. If it does, it is odd.
3. If the function does not satisfy either rule, it is neither even nor odd.

Example 1.5.12: Determining whether a Function Is Even, Odd, or Neither

Is the function $f(x) = x^3 + 2x$ even, odd, or neither?

Solution

Without looking at a graph, we can determine whether the function is even or odd by finding formulas for the reflections and determining if they return us to the original function. Let's begin with the rule for even functions.

$$f(-x) = (-x)^3 + 2(-x) = -x^3 - 2x \tag{1.5.1}$$

This does not return us to the original function, so this function is not even. We can now test the rule for odd functions.

$$-f(-x) = -(-x^3 - 2x) = x^3 + 2x \tag{1.5.2}$$

Because $-f(-x) = f(x)$, this is an odd function.

Analysis

Consider the graph of f in Figure 1.5.18. Notice that the graph is symmetric with respect to the origin. For every point (x, y) on the graph, the corresponding point $(-x, -y)$ is also on the graph. For example, $(1, 3)$ is on the graph of f, and the corresponding point $(-1, -3)$ is also on the graph.

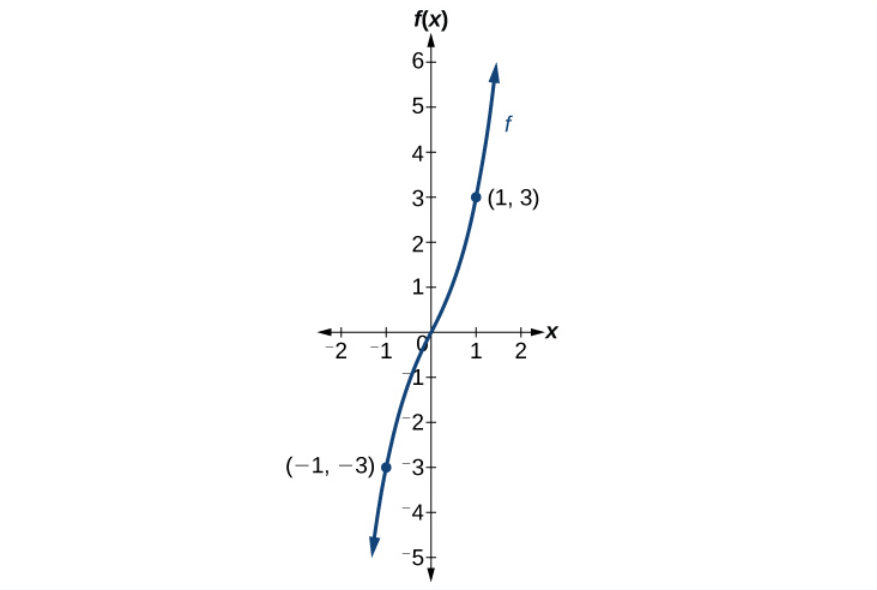


Figure* 1.5.18*: *Graph of $f(x)$ with labeled points at $(1, 3)$ and $(-1, -3)$.*

1.5.8

Is the function $f(s) = s^4 + 3s^2 + 7$ even, odd, or neither?

Answer

even

Graphing Functions Using Stretches and Compressions

Adding a constant to the inputs or outputs of a function changes the position of a graph with respect to the axes, but it does not affect the shape of a graph. We now explore the effects of multiplying the inputs or outputs by some quantity.

We can transform the inside (input values) of a function or we can transform the outside (output values) of a function. Each change has a specific effect that can be seen graphically.

Vertical Stretches and Compressions

When we multiply a function by a positive constant, we get a function whose graph is stretched or compressed vertically in relation to the graph of the original function. If the constant is greater than 1, we get a **vertical stretch**; if the constant is between 0 and 1, we get a **vertical compression**. Figure **1.5.19** shows a function multiplied by constant factors 2 and 0.5 and the resulting vertical stretch and compression.

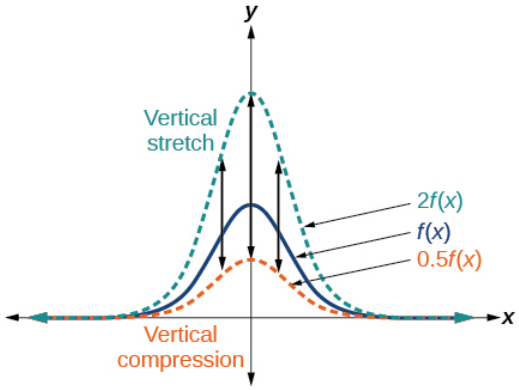


Figure 1.5.19: Vertical stretch and compression

Vertical Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = af(x)$, where a is a constant, is a **vertical stretch** or **vertical compression** of the function $f(x)$.

- If $a > 1$, then the graph will be stretched.
- If $0 < a < 1$, then the graph will be compressed.
- If $a < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection.

How To: Given a function, graph its vertical stretch

1. Identify the value of a.
2. Multiply all range values by a.
    a. If $a > 1$, the graph is stretched by a factor of a.
    b. If $0 < a < 1$, the graph is compressed by a factor of a.
    c. If $a < 0$, the graph is either stretched or compressed and also reflected about the x-axis.

Example 1.5.13: Graphing a Vertical Stretch

A function $P(t)$ models the population of fruit flies. The graph is shown in Figure 1.5.20.

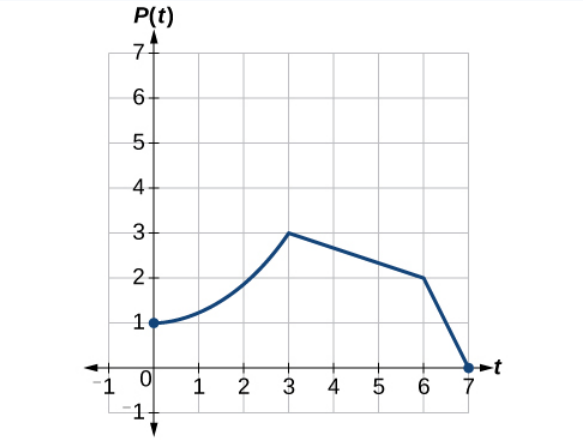


Figure 1.5.20: *Graph representing the growth of the population of fruit flies.*

A scientist is comparing this population to another population, Q, whose growth follows the same pattern, but is twice as large. Sketch a graph of this population.

Solution

Because the population is always twice as large, the new population's output values are always twice the original function's output values. Graphically, this is shown in Figure 1.5.21.

If we choose four reference points, $(0, 1)$, $(3, 3)$, $(6, 2)$ and $(7, 0)$ we will multiply all of the outputs by 2.

The following shows where the new points for the new graph will be located.

$$(0, 1) \rightarrow (0, 2)$$

$$(3, 3) \rightarrow (3, 6)$$

$$(6, 2) \rightarrow (6, 4)$$

$$(7,0) \to (7,0)$$

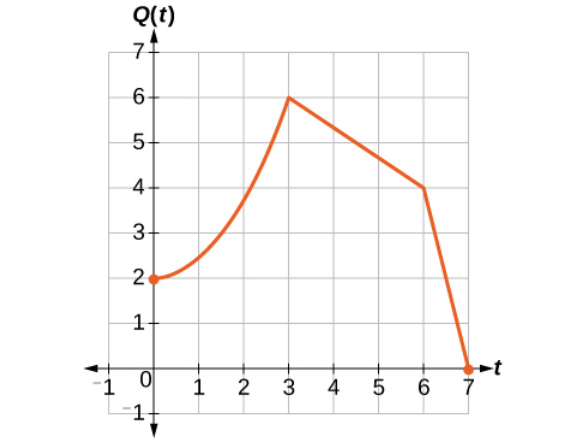


Figure* 1.5.21*: *Graph of the population function doubled.*

Symbolically, the relationship is written as

$$Q(t) = 2P(t)$$

This means that for any input t, the value of the function Q is twice the value of the function P. Notice that the effect on the graph is a vertical stretching of the graph, where every point doubles its distance from the horizontal axis. The input values, t, stay the same while the output values are twice as large as before.

How To: Given a tabular function and assuming that the transformation is a vertical stretch or compression, create a table for the transformation

1. Determine the value of a.
2. Multiply all of the output values by a.

Example 1.5.14: Finding a Vertical Compression of a Tabular Function

A function f is given as Table 1.5.12. Create a table for the function $g(x) = \frac{1}{2}f(x)$.

***Table* 1.5.12**

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = \frac{1}{2}f(x)$ tells us that the output values of g are half of the output values of f with the same inputs. For example, we know that $f(4) = 3$. Then

$$g(4) = \frac{1}{2}f(4) = \frac{1}{2}(3) = \frac{3}{2}$$

We do the same for the other values to produce Table 1.5.13.

Table 1.5.13

x	2	4	6	8
$g(x)$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{7}{2}$	$\frac{11}{2}$

Analysis

The result is that the function $g(x)$ has been compressed vertically by $\frac{1}{2}$. Each output value is divided in half.

1.5.9

A function f is given as Table 1.5.14. Create a table for the function $g(x) = \frac{3}{4}f(x)$.

Table 1.5.14

x	2	4	6	8
$f(x)$	12	16	20	0

Answer

Table 1.5.15

x	2	4	6	8
$g(x)$	9	12	15	0

Example 1.5.15: Recognizing a Vertical Stretch

The graph in Figure 1.5.22 is a transformation of the toolkit function $f(x) = x^3$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.

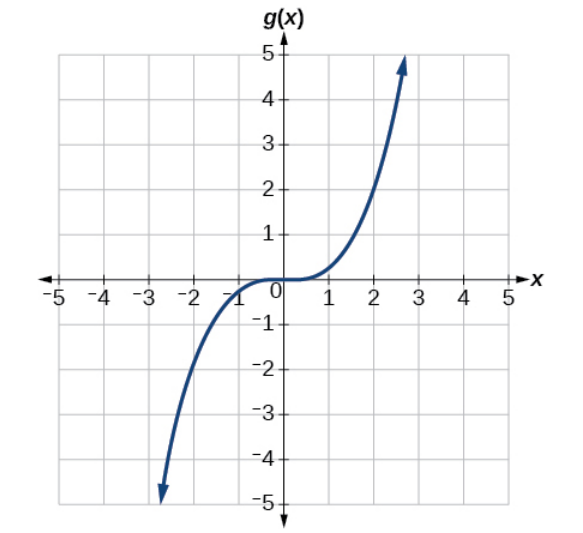


***Figure* 1.5.22:** *Graph of a transformation of* $f(x) = x^3$

When trying to determine a vertical stretch or shift, it is helpful to look for a point on the graph that is relatively clear. In this graph, it appears that $g(2) = 2$. With the basic cubic function at the same input, $f(2) = 2^3 = 8$. Based on that, it appears that the outputs of g are $\frac{1}{4}$ of the outputs of the function f because $g(2) = \frac{1}{4}f(2)$. From this we can fairly safely conclude that $g(x) = \frac{1}{4}f(x)$.

We can write a formula for g by using the definition of the function f.

$$g(x) = \frac{1}{4}f(x) = \frac{1}{4}x^3$$

1.5.10

Write the formula for the function that we get when we stretch the **identity** toolkit function by a factor of 3, and then shift it down by 2 units.

Answer

$g(x) = 3x - 2$

Horizontal Stretches and Compressions

Now we consider changes to the inside of a function. When we multiply a function's input by a positive constant, we get a function whose graph is stretched or compressed horizontally in relation to the graph of the original function. If the constant is between 0 and 1, we get a **horizontal stretch**; if the constant is greater than 1, we get a **horizontal compression** of the function.

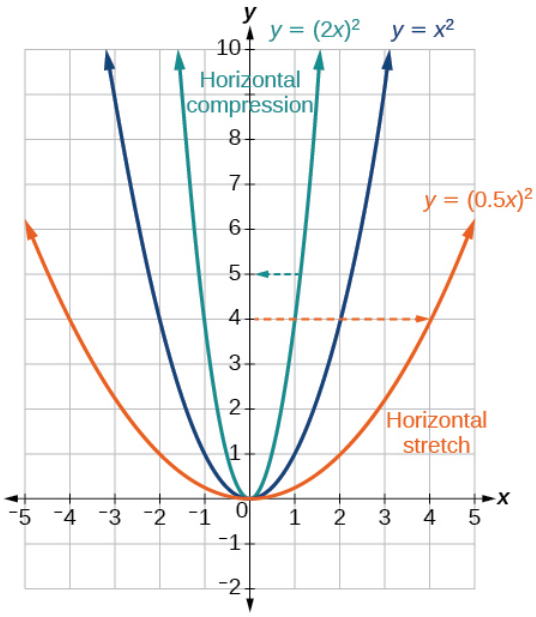


***Figure* 1.5.23:** *Graph of the vertical stretch and compression of* x^2.

Given a function $y = f(x)$, the form $y = f(bx)$ results in a horizontal stretch or compression. Consider the function $y = x^2$. Observe Figure **1.5.23**. The graph of $y = (0.5x)^2$ is a horizontal stretch of the graph of the function $y = x^2$ by a factor of 2. The graph of $y = (2x)^2$ is a horizontal compression of the graph of the function $y = x^2$ by a factor of 2.

Horizontal Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = f(bx)$, where b is a constant, is a **horizontal stretch** or **horizontal compression** of the function $f(x)$.

- If $b > 1$, then the graph will be compressed by $\frac{1}{b}$.
- If $0 < b < 1$, then the graph will be stretched by $\frac{1}{b}$.
- If $b < 0$, then there will be combination of a horizontal stretch or compression with a horizontal reflection.

How To: Given a description of a function, sketch a horizontal compression or stretch

1. Write a formula to represent the function.
2. Set $g(x) = f(bx)$ where $b > 1$ for a compression or $0 < b < 1$ for a stretch.

Example 1.5.16: Graphing a Horizontal Compression

Suppose a scientist is comparing a population of fruit flies to a population that progresses through its lifespan twice as fast as the original population. In other words, this new population, R, will progress in 1 hour the same amount as the original population does in 2 hours, and in 2 hours, it will progress as much as the original population does in 4 hours. Sketch a graph of this population.

Solution

Symbolically, we could write

$$\begin{aligned} R(1) &= P(2), \\ R(2) &= P(4), \qquad \text{and in general,} \\ R(t) &= P(2t). \end{aligned}$$

See Figure 1.5.24 for a graphical comparison of the original population and the compressed population.

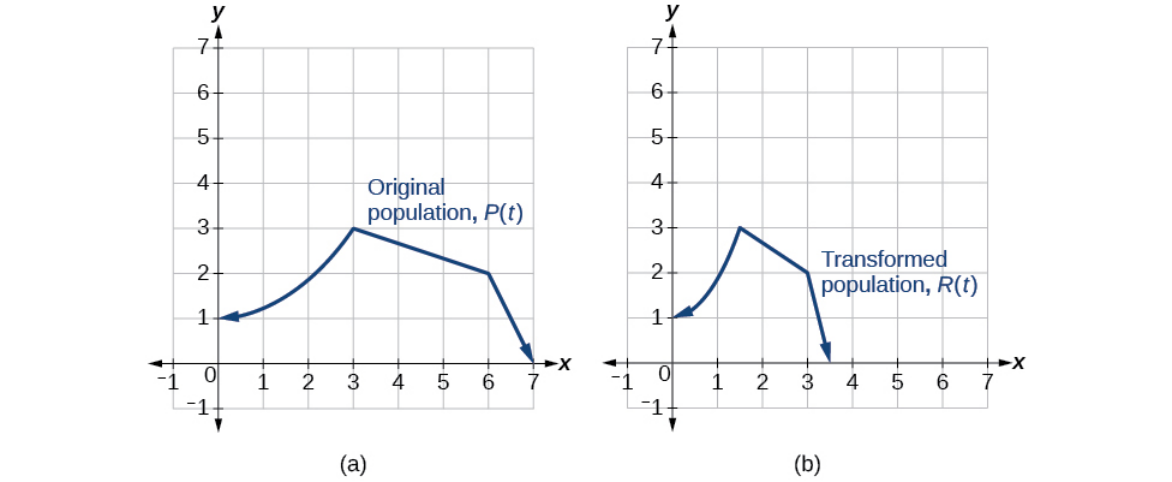


***Figure* 1.5.24:** *(a) Original population graph (b) Compressed population graph*

Analysis

Note that the effect on the graph is a horizontal compression where all input values are half of their original distance from the vertical axis.

Example 1.5.17: Finding a Horizontal Stretch for a Tabular Function

A function $f(x)$ is given as Table 1.5.16. Create a table for the function $g(x) = f(\frac{1}{2}x)$.

***Table* 1.5.16**

x	2	4	6	8
$f(x)$	1	3	7	11

The formula $g(x) = f(\frac{1}{2}x)$ tells us that the output values for g are the same as the output values for the function f at an input half the size. Notice that we do not have enough information to determine $g(2)$ because $g(2) = f(\frac{1}{2} \cdot 2) = f(1)$, and we do not have a value for $f(1)$ in our table. Our input values to g will need to be twice as large to get inputs for f that we can evaluate. For example, we can determine $g(4)$.

$$g(4) = f(\frac{1}{2} \cdot 4) = f(2) = 1$$

We do the same for the other values to produce Table 1.5.17.

***Table* 1.5.17**

x	4	8	12	16
$g(x)$	1	3	7	11

Figure 1.5.25 shows the graphs of both of these sets of points.

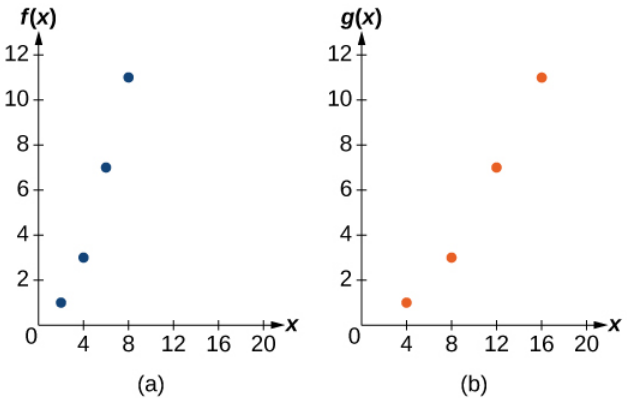


***Figure* 1.5.25:** *Graph of the previous table.*

Analysis

Because each input value for g has been doubled in order to get the same outputs as for f, the result is that the function $g(x)$ has been stretched horizontally by a factor of 2.

Example 1.5.18: Recognizing a Horizontal Compression on a Graph

Relate the function $g(x)$ to $f(x)$ in Figure 1.5.26.

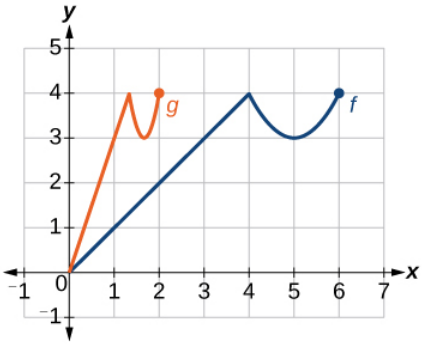


***Figure* 1.5.26:** *Graph of $f(x)$ being horizontally compressed to $g(x)$.*

Solution

The graph of $g(x)$ looks like the graph of $f(x)$ horizontally compressed. Because $f(x)$ ends at (6,4) and $g(x)$ ends at (2,4), we can see that the x-values have been compressed by $\frac{1}{3}$, since $6(\frac{1}{3}) = 2$. We might also notice that $g(2) = f(6)$ and $g(1) = f(3)$. Either way, we can describe this relationship as $g(x) = f(3x)$. This is a horizontal compression by a factor of $\frac{1}{3}$.

Analysis

Notice that the coefficient needed for a horizontal stretch or compression is the reciprocal of the stretch or compression. So to stretch the graph horizontally by a scale factor of 4, we need a coefficient of $\frac{1}{4}$ in our function: $f(\frac{1}{4}x)$. This means that the input values must be four times larger to produce the same result, requiring the input to be larger, causing the horizontal stretching.

1.5.11

Write a formula for the toolkit **square root** function horizontally stretched by a factor of 3.

Answer

$g(x) = f(\frac{1}{3}x)$, so using the square root function we get $g(x) = \sqrt{\frac{1}{3}x}$.

Performing a Sequence of Transformations

When combining transformations, it is very important to consider the order of the transformations. For example, vertically shifting by 3 and then vertically stretching by 2 does not create the same graph as vertically stretching by 2 and then vertically shifting by 3, because when we shift first, both the original function and the shift get stretched, while only the original function gets stretched when we stretch first.

When we see an expression such as $2f(x)+3$, which transformation should we start with? The answer here follows nicely from the order of operations. Given the output value of $f(x)$, we first multiply by 2, causing the vertical stretch, and then add 3, causing the vertical shift. In other words, multiplication before addition.

Horizontal transformations are a little trickier to think about. When we write $g(x)=f(2x+3)$, for example, the technique we will use is to factor inside the function.

$$f(bx+p)=f(b(x+\frac{p}{b})) \tag{1.5.3}$$

Let's work through an example.

$$f(x)=(2x+6)^2 \tag{1.5.4}$$

We can factor out a 2.

$$f(x)=(2(x+3))^2 \tag{1.5.5}$$

Now we can observe a horizontal shift to the left 3 units and a horizontal compression by $\frac{1}{2}$. Factoring in this way allows us to horizontally stretch first and then shift horizontally.

Note: Combining Transformations

When combining vertical transformations written in the form $af(x)+k$, first vertically stretch by a and then vertically shift by k.

When combining horizontal transformations written in the form $f(bx+h)$, first factor to get the form $f(b(x+\frac{h}{b}))$, then horizontally stretch by $\frac{1}{b}$ and then horizontally shift by $\frac{h}{b}$.

Horizontal and vertical transformations are independent. It does not matter whether horizontal or vertical transformations are performed first.

Example 1.5.20: Finding a Triple Transformation of a Graph

Use the graph of $f(x)$ in Figure 1.5.27 to sketch a graph of $k(x)=f\left(\frac{1}{2}x+1\right)-3$.

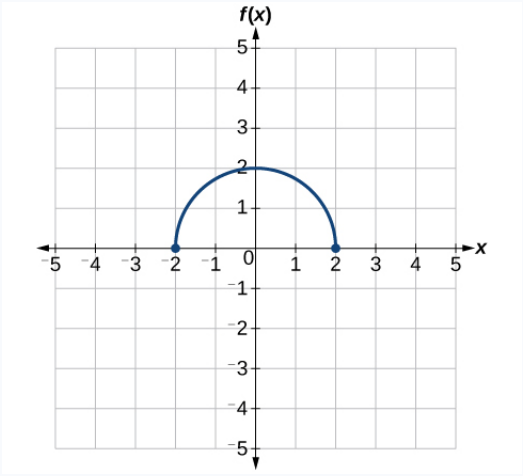


Figure 1.5.27: *Graph of a half-circle.*

Start by factoring the inside of the function.

$$f\left(\frac{1}{2}x+1\right)-3=f\left(\frac{1}{2}(x+2)\right)-3 \tag{1.5.6}$$

After factoring the inside, we first horizontally stretch by 2, as indicated by the $\frac{1}{2}$ on the inside of the function. Remember that twice the size of 0 is still 0, so the point $(0,2)$ remains at $(0,2)$ while the point $(2,0)$ will stretch to $(4,0)$. See Figure 1.5.28.

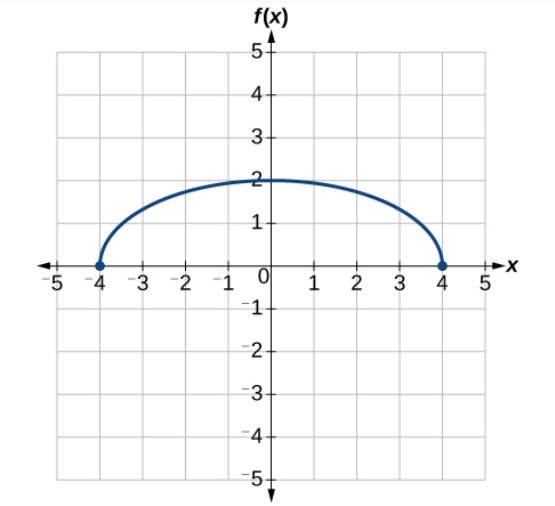


Figure* 1.5.28*: *Graph of a vertically stretch half-circle.*

Next, we horizontally shift left by 2 units, as indicated by $x+2$. See Figure 1.5.29.

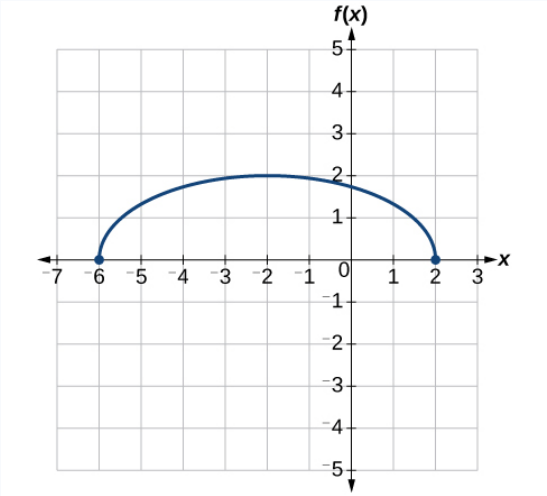


Figure* 1.5.29*: *Graph of a vertically stretch and translated half-circle.*

Last, we vertically shift down by 3 to complete our sketch, as indicated by the -3 on the outside of the function. See Figure 1.5.30.

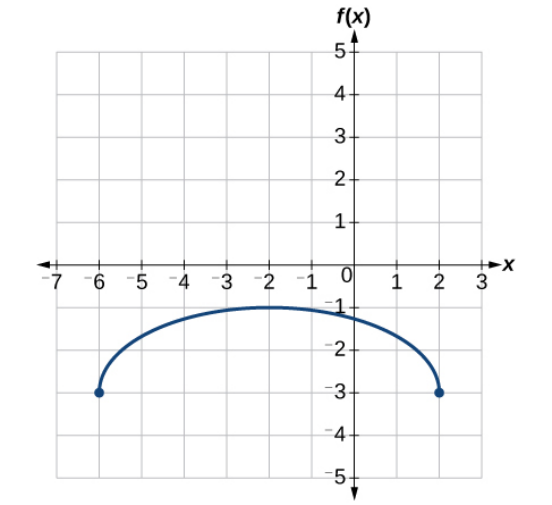


***Figure* 1.5.30:** *Graph of a vertically stretch and translated half-circle.*

Key Equations

- Vertical shift $g(x) = f(x) + k$ (up for $k > 0$)
- Horizontal shift $g(x) = f(x - h)$(right) for $h > 0$
- Vertical reflection $g(x) = -f(x)$
- Horizontal reflection $g(x) = f(-x)$
- Vertical stretch $g(x) = af(x)$ $(a > 0)$
- Vertical compression $g(x) = af(x)$ $(0 < a < 1)$
- Horizontal stretch $g(x) = f(bx)$ $(0 < b < 1)$
- Horizontal compression $g(x) = f(bx)$ $(b > 1)$

Key Concepts

- A function can be shifted vertically by adding a constant to the output.
- A function can be shifted horizontally by adding a constant to the input.
- Relating the shift to the context of a problem makes it possible to compare and interpret vertical and horizontal shifts.
- A vertical reflection reflects a graph across the x-axis. A graph can be reflected vertically by multiplying the output by –1.
- A horizontal reflection reflects a graph across the y-axis. A graph can be reflected horizontally by multiplying the input by –1.
- A graph can be reflected both vertically and horizontally. The order in which the reflections are applied does not affect the final graph.
- A function presented in tabular form can be reflected by multiplying the values in the input and output rows or columns accordingly.
- A function presented as an equation can be reflected by applying transformations one at a time.
- Even functions are symmetric about the y-axis, whereas odd functions are symmetric about the origin.
- Even functions satisfy the condition $f(x) = f(-x)$.
- Odd functions satisfy the condition $f(x) = -f(-x)$.
- A function can be odd, even, or neither.
- A function can be compressed or stretched vertically by multiplying the output by a constant.
- A function can be compressed or stretched horizontally by multiplying the input by a constant.
- The order in which different transformations are applied does affect the final function. Both vertical and horizontal transformations must be applied in the order given. However, a vertical transformation may be combined with a horizontal transformation in any order.

Glossary

even function

a function whose graph is unchanged by horizontal reflection, $f(x) = f(-x)$, and is symmetric about the y-axis

horizontal compression
a transformation that compresses a function's graph horizontally, by multiplying the input by a constant $b > 1$

horizontal reflection
a transformation that reflects a function's graph across the y-axis by multiplying the input by -1

horizontal shift
a transformation that shifts a function's graph left or right by adding a positive or negative constant to the input

horizontal stretch
a transformation that stretches a function's graph horizontally by multiplying the input by a constant $0 < b < 1$

odd function
a function whose graph is unchanged by combined horizontal and vertical reflection, $f(x) = -f(-x)$, and is symmetric about the origin

vertical compression
a function transformation that compresses the function's graph vertically by multiplying the output by a constant $0 < a < 1$

vertical reflection
a transformation that reflects a function's graph across the x-axis by multiplying the output by -1

vertical shift
a transformation that shifts a function's graph up or down by adding a positive or negative constant to the output

vertical stretch
a transformation that stretches a function's graph vertically by multiplying the output by a constant $a > 1$

Contributors

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1.5E: Exercises

This page is a draft and is under active development.

Often when given a problem, we try to model the scenario using mathematics in the form of words, tables, graphs, and equations. One method we can employ is to adapt the basic graphs of the toolkit functions to build new models for a given scenario. There are systematic ways to alter functions to construct appropriate models for the problems we are trying to solve.

Verbal

Exercise 1.5.1

When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal shift from a vertical shift?

Answer:

A horizontal shift results when a constant is added to or subtracted from the input. A vertical shifts results when a constant is added to or subtracted from the output.

Exercise 1.5.2

When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal stretch from a vertical stretch?

Exercise 1.5.3

When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal compression from a vertical compression?

Answer:

A horizontal compression results when a constant greater than 1 is multiplied by the input. A vertical compression results when a constant between 0 and 1 is multiplied by the output.

Exercise 1.5.4

When examining the formula of a function that is the result of multiple transformations, how can you tell a reflection with respect to the x-axis from a reflection with respect to the y-axis?

Exercise 1.5.5

How can you determine whether a function is odd or even from the formula of the function?

Answer:

For a function f, substitute (−x) for (x) in $f(x)$. Simplify. If the resulting function is the same as the original function, $f(-x) = f(x)$, then the function is even. If the resulting function is the opposite of the original function, $f(-x) = -f(x)$, then the original function is odd. If the function is not the same or the opposite, then the function is neither odd nor even.

Algebraic

Exercise 1.5.6

Write a formula for the function obtained when the graph of $f(x) = \sqrt{x}$ is shifted up 1 unit and to the left 2 units.

Exercise 1.5.7

Write a formula for the function obtained when the graph of $f(x) = |x|$ is shifted down 3 units and to the right 1 unit.

Answer:

$g(x) = |x - 1| - 3$

Exercise 1.5.8

Write a formula for the function obtained when the graph of $f(x) = \frac{1}{x}$ is shifted down 4 units and to the right 3 units.

Exercise 1.5.9

Write a formula for the function obtained when the graph of $f(x) = \frac{1}{x^2}$ is shifted up 2 units and to the left 4 units.

Answer:

$g(x) = \frac{1}{(x+4)^2} + 2$

For the following exercises, describe how the graph of the function is a transformation of the graph of the original function f.

Exercise 1.5.10

$y = f(x - 49)$

Exercise 1.5.11

$y = f(x + 43)$

Answer:

The graph of $f(x + 43)$ is a horizontal shift to the left 43 units of the graph of f.

Exercise 1.5.12

$y = f(x + 3)$

Exercise 1.5.13

$y = f(x - 4)$

Answer:

The graph of $f(x - 4)$ is a horizontal shift to the right 4 units of the graph of f.

Exercise 1.5.14

$y = f(x) + 5$

Exercise 1.5.15

$y = f(x) + 8$

Answer:

The graph of $f(x) + 8$ is a vertical shift up 8 units of the graph of f.

Exercise 1.5.16

$y = f(x) - 2$

Exercise 1.5.17

$y = f(x) - 7$

Answer:

The graph of $f(x)-7$ is a vertical shift down 7 units of the graph of f.

Exercise 1.5.18

$y=f(x-2)+3$

Exercise 1.5.19

$y=f(x+4)-1$

Answer:

The graph of $f(x+4)-1$ is a horizontal shift to the left 4 units and a vertical shift down 1 unit of the graph of f.

For the following exercises, determine the interval(s) on which the function is increasing and decreasing.

Exercise 1.5.20

$f(x)=4(x+1)^2-5$

Exercise 1.5.21

$g(x)=5(x+3)^2-2$

Answer:

decreasing on $(-\infty,-3)$ and increasing on $(-3,\infty)$

Exercise 1.5.22

$a(x)=\sqrt{-x+4}$

Exercise 1.5.23

$k(x)=-3\sqrt{x}-1$

Answer:

decreasing on $(0,\infty)$

Graphical

For the following exercises, use the graph of $f(x)=2^x$ shown in **Figure 1.5.30** to sketch a graph of each transformation of $f(x)$.

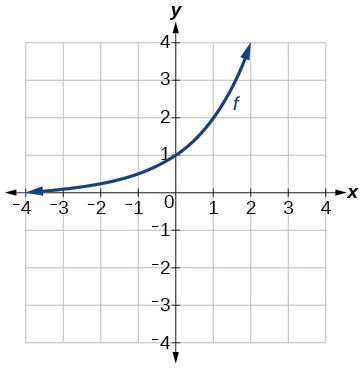


Figure 1.5.30: Graph of $f(x)$.

Exercise 1.5.24

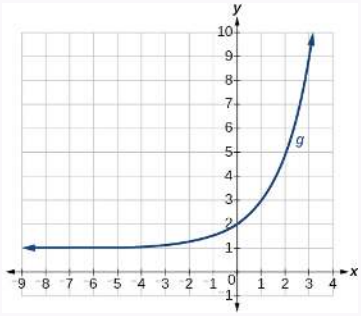
$h(x)=2^x-3$

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Exercise 1.5.25

$g(x) = 2^x + 1$

Answer:



Graph of $g(x)$

Exercise 1.5.26

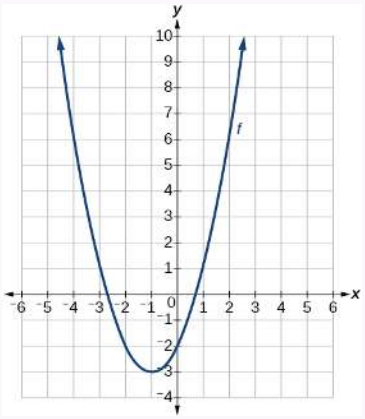
$w(x) = 2^x - 1$

For the following exercises, sketch a graph of the function as a transformation of the graph of one of the toolkit functions.

Exercise 1.5.27

$f(t) = (t+1)^2 - 3$

Answer:



Graph of $f(t)$.

Exercise 1.5.28

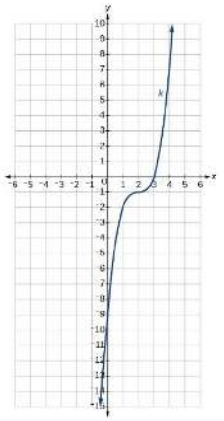
$h(x) = |x - 1| + 4$

Exercise 1.5.29

$k(x) = (x-2)^3 - 1$

Answer:

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Graph of $k(x)$.

Exercise 1.5.30

$m(t) = 3 + \sqrt{t+2}$

Numeric

Exercise 1.5.31

Tabular representations for the functions f, g, and h are given below. Write $g(x)$ and $h(x)$ as transformations of $f(x)$.

x	-2	-1	0	1	2
$f(x)$	-2	-1	-3	1	2

x	-1	0	1	2	3
$g(x)$	-2	-1	-3	1	2

x	-2	-1	0	1	2
$h(x)$	-1	0	-2	2	3

Answer:

$g(x) = f(x-1)$, $h(x) = f(x) + 1$

Exercise 1.5.32

Tabular representations for the functions f, g, and h are given below. Write $g(x)$ and $h(x)$ as transformations of $f(x)$.

x	-2	-1	0	1	2
$f(x)$	-1	-3	4	2	1

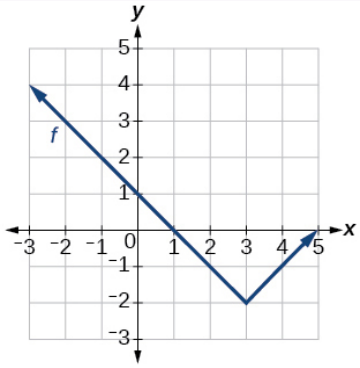
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x	-3	-2	-1	0	1
$g(x)$	-1	-3	4	2	1

x	-2	-1	0	1	2
$h(x)$	-2	-4	3	1	0

For the following exercises, write an equation for each graphed function by using transformations of the graphs of one of the toolkit functions.

Exercise 1.5.33

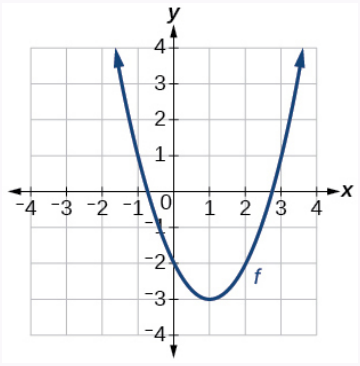


Graph of an absolute function.

Answer:

$f(x) = |x - 3| - 2$

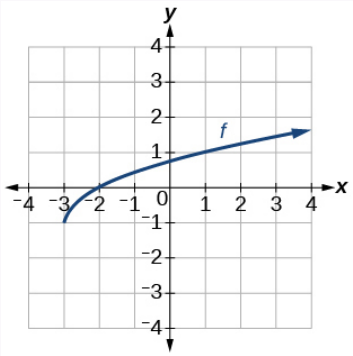
Exercise 1.5.34



Graph of a parabola.

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Exercise 1.5.35

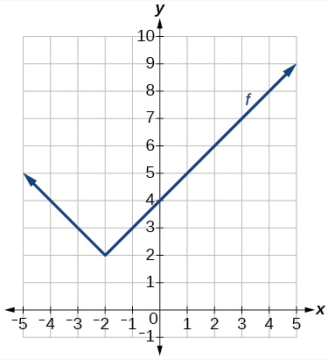


Graph of a square root function.

Answer:

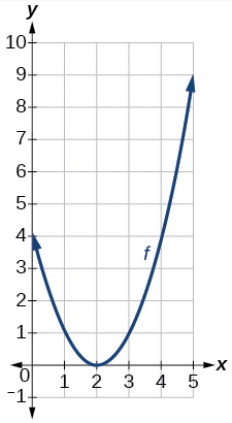
$f(x) = \sqrt{x+3} - 1$

Exercise 1.5.36



Graph of an absolute function.

Exercise 1.5.37



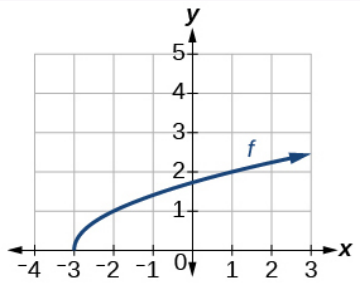
Graph of a parabola

Answer:

$f(x) = (x-2)^2$

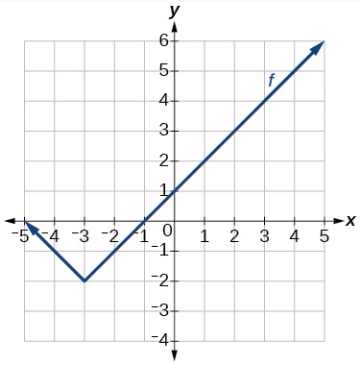
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Exercise 1.5.38



Graph of a square root function.

Exercise 1.5.39

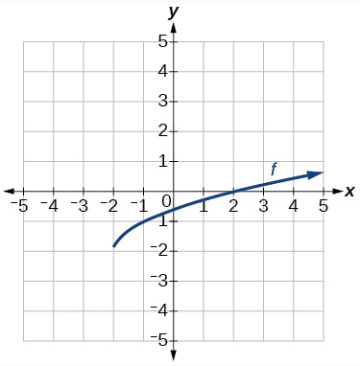


Graph of an absolute function.

Answer:

$f(x) = |x + 3| - 2$

Exercise 1.5.40

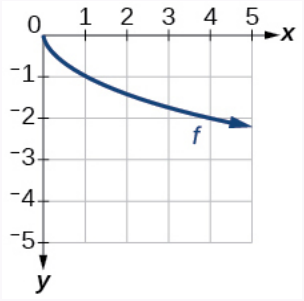


Graph of a square root function.

For the following exercises, use the graphs of transformations of the square root function to find a formula for each of the functions.

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Exercise 1.5.41

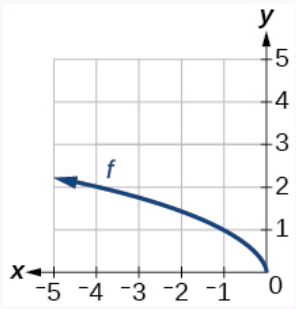


Graph of a square root function.

Answer:

$f(x) = -\sqrt{x}$

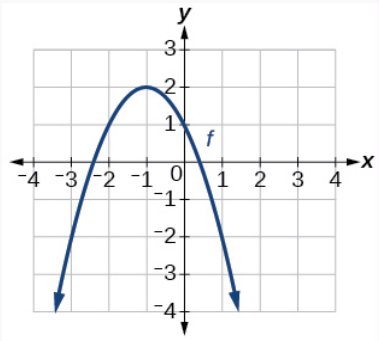
Exercise 1.5.42



Graph of a square root function.

For the following exercises, use the graphs of the transformed toolkit functions to write a formula for each of the resulting functions.

Exercise 1.5.43

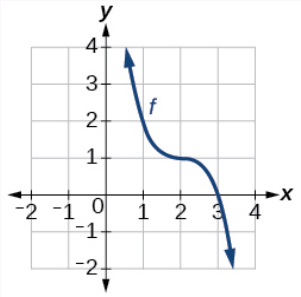


Graph of a parabola.

Answer:

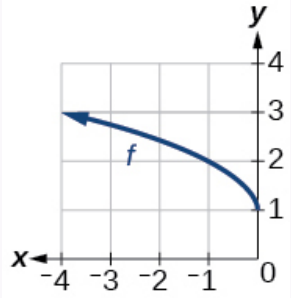
$f(x) = -(x+1)^2 + 2$

Exercise 1.5.44



Graph of a cubic function.

Exercise 1.5.45

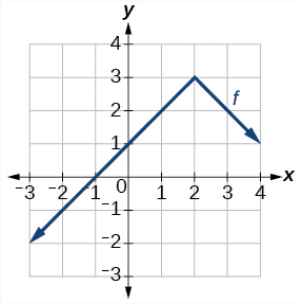


Graph of a square root function.

Answer:

$f(x) = \sqrt{-x} + 1$

Exercise 1.5.46



Graph of an absolute function.

For the following exercises, determine whether the function is odd, even, or neither.

Exercise 1.5.47

$f(x) = 3x^4$

Answer:

even

Exercise 1.5.48

$g(x) = \sqrt{x}$

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Exercise 1.5.49

$h(x) = \frac{1}{x} + 3x$

Answer:

odd

Exercise 1.5.50

$f(x) = (x-2)^2$

Exercise 1.5.51

$g(x) = 2x^4$

Answer:

even

Exercise 1.5.52

$h(x) = 2x - x^3$

For the following exercises, describe how the graph of each function is a transformation of the graph of the original function f.

Exercise 1.5.53

$g(x) = -f(x)$

Answer:

The graph of g is a vertical reflection (across the x-axis) of the graph of f.

Exercise 1.5.54

$g(x) = f(-x)$

Exercise 1.5.55

$g(x) = 4f(x)$

Answer:

The graph of g is a vertical stretch by a factor of 4 of the graph of f.

Exercise 1.5.56

$g(x) = 6f(x)$

Exercise 1.5.57

$g(x) = f(5x)$

Answer:

The graph of g is a horizontal compression by a factor of 15 of the graph of f.

Exercise 1.5.58

$g(x) = f(2x)$

Exercise 1.5.59

$g(x) = f(\frac{1}{3}x)$

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Answer:

The graph of g is a horizontal stretch by a factor of 3 of the graph of f.

Exercise 1.5.60

$g(x) = f(\frac{1}{5}x)$

Exercise 1.5.61

$g(x) = 3f(-x)$

Answer:

The graph of g is a horizontal reflection across the y-axis and a vertical stretch by a factor of 3 of the graph of f.

Exercise 1.5.62

$g(x) = -f(3x)$

For the following exercises, write a formula for the function g that results when the graph of a given toolkit function is transformed as described.

Exercise 1.5.63

The graph of $f(x) = |x|$ is reflected over the y-axis and horizontally compressed by a factor of $\frac{1}{4}$.

Answer:

$g(x) = |-4x|$

Exercise 1.5.64

The graph of $f(x) = \sqrt{x}$ is reflected over the x-axis and horizontally stretched by a factor of 2.

Exercise 1.5.65

The graph of $f(x) = \dfrac{1}{x^2}$ is vertically compressed by a factor of $\dfrac{1}{3}$, then shifted to the left 2 units and down 3 units.

Answer:

$g(x) = \dfrac{1}{3(x+2)^2} - 3$

Exercise 1.5.66

The graph of $f(x) = \dfrac{1}{x}$ is vertically stretched by a factor of 8, then shifted to the right 4 units and up 2 units.

Exercise 1.5.67

The graph of $f(x) = x^2$ is vertically compressed by a factor of $\frac{1}{2}$, then shifted to the right 5 units and up 1 unit.

Answer:

$g(x) = \dfrac{1}{2}(x-5)^2 + 1$

Exercise 1.5.68

The graph of $f(x) = x^2$ is horizontally stretched by a factor of 3, then shifted to the left 4 units and down 3 units.

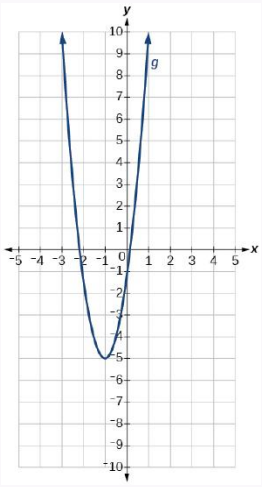
For the following exercises, describe how the formula is a transformation of a toolkit function. Then sketch a graph of the transformation.

Exercise 1.5.69

$g(x) = 4(x+1)^2 - 5$

Answer:

The graph of the function $f(x) = x^2$ is shifted to the left 1 unit, stretched vertically by a factor of 4, and shifted down 5 units.



Graph of a parabola.

Exercise 1.5.70

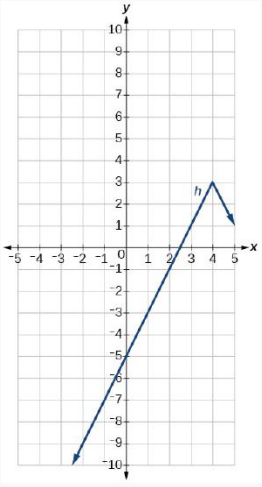
$g(x) = 5(x+3)^2 - 2$

Exercise 1.5.71

$h(x) = -2|x-4| + 3$

Answer:

The graph of $f(x) = |x|$ is stretched vertically by a factor of 2, shifted horizontally 4 units to the right, reflected across the horizontal axis, and then shifted vertically 3 units up.



Graph of an absolute value function.

Exercise 1.5.72

$k(x) = -3\sqrt{x} - 1$

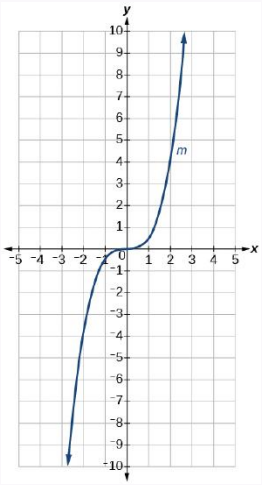
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Exercise 1.5.73

$m(x) = \frac{1}{2}x^3$

Answer:

The graph of the function $f(x) = x^3$ is compressed vertically by a factor of $\frac{1}{2}$.



Graph of a cubic function.

Exercise 1.5.74

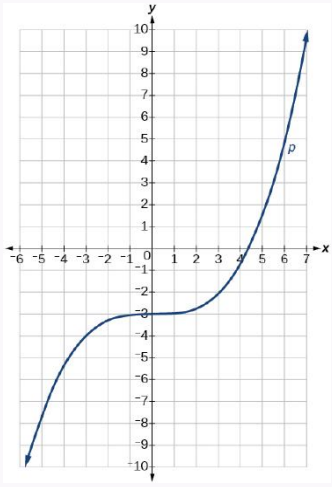
$n(x) = \frac{1}{3}|x - 2|$

Exercise 1.5.75

$p(x) = (\frac{1}{3}x)^3 - 3$

Answer:

The graph of the function is stretched horizontally by a factor of 3 and then shifted vertically downward by 3 units.



Graph of a cubic function.

Exercise 1.5.76

$q(x) = (\frac{1}{4}x)^3 + 1$

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Exercise 1.5.77

$a(x) = \sqrt{-x+4}$

Answer:

The graph needs work. Wait for the revised edition!

For the following exercises, use the graph in **Figure 1.5.30** to sketch the given transformations.

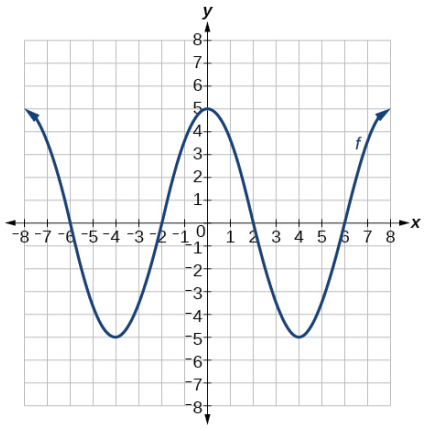


Figure 1.5.30: *Graph of a polynomial.*

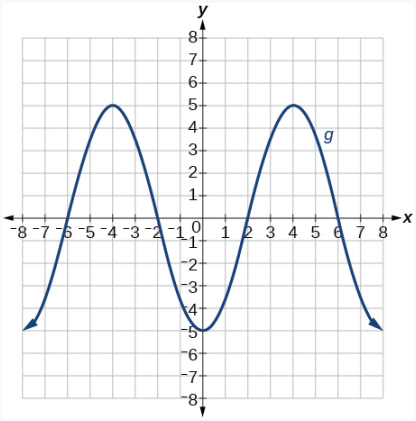
Exercise 1.5.78

$g(x) = f(x) - 2$

Exercise 1.5.79

$g(x) = -f(x)$

Answer:



Graph of a polynomial.

Exercise 1.5.80

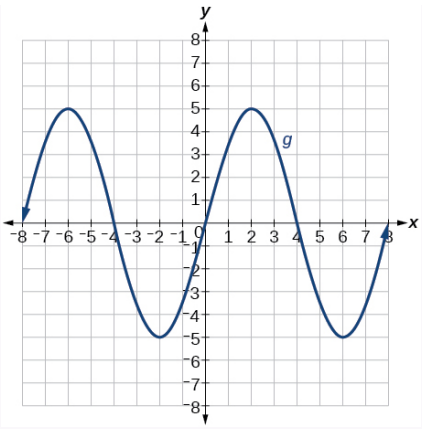
$g(x) = f(x+1)$

Exercise 1.5.81

$g(x) = f(x-2)$

Answer:

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Graph of a polynomial.

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1.6: Absolute Value Functions

Skills to Develop

- Graph an absolute value function.
- Solve an absolute value equation.

Until the 1920s, the so-called spiral nebulae were believed to be clouds of dust and gas in our own galaxy, some tens of thousands of light years away. Then, astronomer Edwin Hubble proved that these objects are galaxies in their own right, at distances of millions of light years. Today, astronomers can detect galaxies that are billions of light years away. Distances in the universe can be measured in all directions. As such, it is useful to consider distance as an absolute value function. In this section, we will investigate **absolute value functions**.



Figure* 1.6.1*: *Distances in deep space can be measured in all directions. As such, it is useful to consider distance in terms of absolute values. (credit: "s58y"/Flickr)*

Understanding Absolute Value

Recall that in its basic form $f(x) = |x|$, the **absolute value** function, is one of our toolkit functions. The absolute value function is commonly thought of as providing the distance the number is from zero on a number line. Algebraically, for whatever the input value is, the output is the value without regard to sign.

Absolute value function

The absolute value function can be defined as a piecewise function.

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Example 1.6.1: Determine a Number within a Prescribed Distance

Describe all values x within or including a distance of 4 from the number 5.

Solution

We want the distance between x and 5 to be less than or equal to 4. We can draw a number line, such as the one in Figure 1.6.2, to represent the condition to be satisfied.

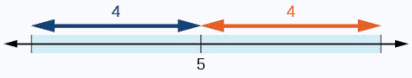


Figure* 1.6.2*: *Number line showing a distance of 4 units away from 5*

The distance from x to 5 can be represented using the absolute value as $|x - 5|$. We want the values of x that satisfy the condition $|x - 5| \leq 4$.

Analysis

Note that, since the lower endpoint on the graph is 1 and the upper endpoint is 9, we have the compound inequality

$$1 \leq x \text{ and } x \leq 9.$$

So $|x-5| \leq 4$ is equivalent to saying $1 \leq x \leq 9$.

However, mathematicians generally prefer absolute value notation.

1.6.1

Describe all x-values within or including a distance of 3 from the number 2.

Answer

$|x-2| \leq 3$

Example 1.6.2: Resistance of a Resistor

Electrical parts, such as resistors and capacitors, come with specified values of their operating parameters: resistance, capacitance, etc. However, due to imprecision in manufacturing, the actual values of these parameters vary somewhat from piece to piece, even when they are supposed to be the same. The best that manufacturers can do is to try to guarantee that the variations will stay within a specified range, often ±1%, ±5%, or ±10%.

Suppose we have a resistor rated at 680 ohms, ±5%. Use the absolute value function to express the range of possible values of the actual resistance.

Solution

5% of 680 ohms is 34 ohms. The absolute value of the difference between the actual and nominal resistance should not exceed the stated variability, so, with the resistance R in ohms,

$$|R-680| \leq 34.$$

1.6.2

Students who score within 20 points of 80 will pass a test. Write this as a distance from 80 using absolute value notation.

Answer

Using the variable p for passing, $|p-80| \leq 20$.

Graphing an Absolute Value Function

The most significant feature of the absolute value graph is the corner point, or **vertex**, at which the graph changes direction. This point is shown at the origin in Figure 1.6.3.

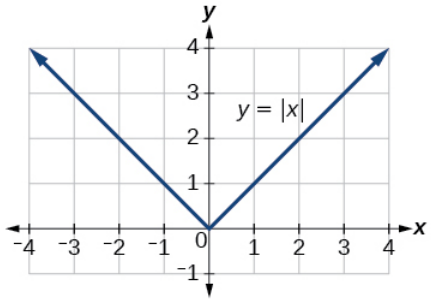


Figure 1.6.3: *Graph of an absolute value function.*

Figure 1.6.4 shows the graph of $y = 2|x-3| + 4$. The graph of $y = |x|$ (blue) has been shifted right 3 units (orange), vertically stretched by a factor of 2 (green), and shifted up 4 units (red). This means that the vertex is located at $(3, 4)$ for this transformed

function.

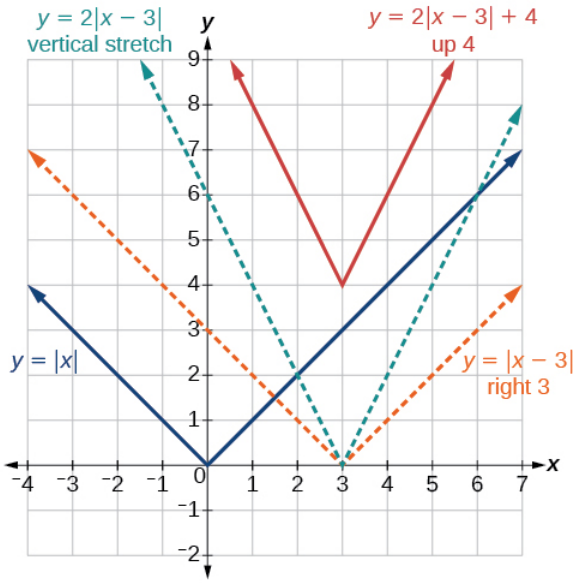


***Figure* 1.6.4:** *Graph of a transformation of an absolute function.*

A useful property of absolute value expressions

For all expressions A and B,

$$|A \cdot B| = |A| \cdot |B|.$$

For example,

- $|5x - 2| = \left|5\left(x - \frac{2}{5}\right)\right| = |5|\left|x - \frac{2}{5}\right|$, and
- $|-2x - 4| = |-2(x + 2)| = |-2| \cdot |x + 2|$.

Example 1.6.3: Writing an Equation for an Absolute Value Function

Write an equation for the function graphed in Figure 1.6.5.

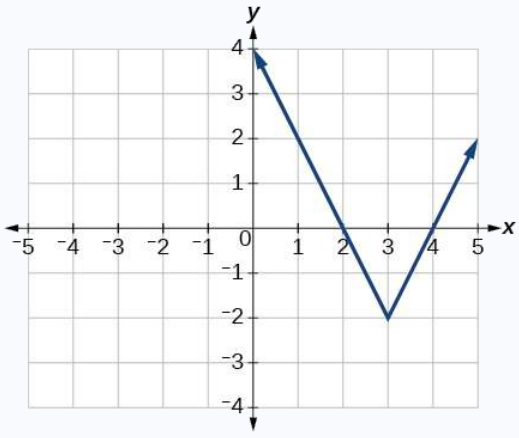


***Figure* 1.6.5:** *Graph of an absolute function.*

Solution

The basic absolute value function (green) changes direction at the origin, so this graph has been shifted to the right 3 units and down 2 units (red) from the basic toolkit function. See Figure 1.6.6.

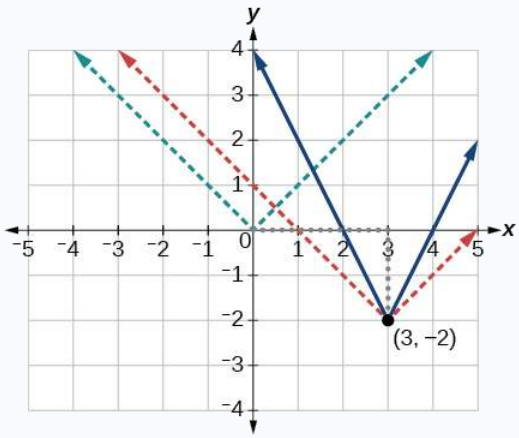


Figure 1.6.6: *Graph (blue) of three transformations for an absolute function at* $(3,-2)$.

We also notice that the graph appears vertically stretched, because the slope of the right-hand piece of the final graph (blue) is equal to 2, rather than a slope of 1 as it would be for an unstretched absolute value function.

From this information we can write the equation $f(x)=2|x-3|-2$.

If we couldn't observe the stretch of the function from the graphs, could we algebraically determine it?

Yes. If we are unable to determine the stretch based on the width of the graph, we can solve for the stretch factor a *by putting in a known pair of values for* x *and* $f(x)$.

$$f(x)=a|x-3|-2$$

Now substituting in the point $(1,2)$

$$\begin{aligned} 2&=a|1-3|-2\\ 4&=2a\\ a&=2 \end{aligned}$$

1.6.3

Write the equation for the absolute value function that is horizontally shifted left 2 units, is vertically flipped, and vertically shifted up 3 units.

Answer

$f(x)=-|x+2|+3$

Do the graphs of absolute value functions always intersect the vertical axis? The horizontal axis?

Yes, they always intersect the vertical axis. The graph of an absolute value function will intersect the vertical axis when the input is zero.

No, they do not always intersect the horizontal axis. The graph may or may not intersect the horizontal axis, depending on how the graph has been shifted and reflected. It is possible for the absolute value function to intersect the horizontal axis at zero, one, or two points (see Figure 1.6.8*).*

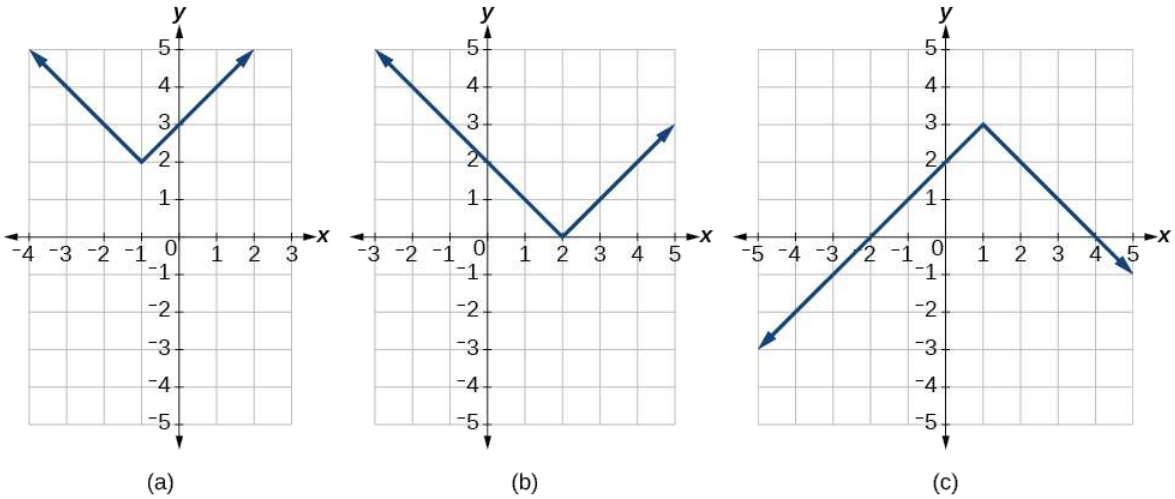


Figure* 1.6.8*: *(a) The absolute value function does not intersect the horizontal axis. (b) The absolute value function intersects the horizontal axis at one point. (c) The absolute value function intersects the horizontal axis at two points.*

Solving an Absolute Value Equation

Now that we can graph an absolute value function, we will learn how to solve an absolute value equation. To solve an equation such as $|2x-6|=8$, we notice that the absolute value will be equal to 8 if the quantity inside the absolute value is 8 or -8. This leads to two different equations we can solve independently.

$$2x-6=8 \quad \text{or} \quad 2x-6=-8$$

$$\begin{array}{ll} 2x=14 & 2x=-2 \\ x=7 & x=-1 \end{array}$$

Knowing how to solve problems involving absolute value functions is useful. For example, we may need to identify numbers or points on a line that are at a specified distance from a given reference point.

An **absolute value equation** is an equation in which the unknown variable appears in absolute value bars. For example,

$$|x|=4,$$

$$|2x-1|=3,$$

$$|5x+2|-4=9.$$

Solutions to Absolute Value Equations

For real numbers A and B, an equation of the form $|A|=B$, with $B\geq 0$, will have solutions when $A=B$ or $A=-B$. If $B<0$, the equation $|A|=B$ has no solution.

How To: Given the formula for an absolute value function, find the x-intercepts of its graph

1. Set the function equal to 0 (set y equal to 0).
2. Isolate the absolute value term.
3. Use $|A|=B$ to write $A=B$ or $-A=B$, assuming $B>0$.
4. Solve for x.

5. Write the solution(s) as ordered pair(s) whose second coordinate is 0.

Extra Credit: If you think you know the difference between x-intercepts of a function and zeros of a function, write a clear explanation, and give it to your instructor.

Example 1.6.4: Finding the Zeros of an Absolute Value Function

For the function $f(x) = |4x + 1| - 7$, find the zeros, which are the values of x such that $f(x) = 0$.

Solution

$$|4x + 1| - 7 = 0$$ Set the function equal to 0.

$$|4x + 1| = 7$$ Isolate the absolute value on one side of the equation.

$$4x + 1 = 7 \quad \text{or} \quad 4x + 1 = -7$$ Break into two separate equations and solve.

$$4x = 6 \qquad\qquad 4x = -8$$

$$x = \frac{6}{4} = 1.5 \quad \text{or} \quad x = \frac{-8}{4} = -2$$

The function outputs are 0 when $x = 1.5$ and when $x = -2$. See Figure 1.6.9.

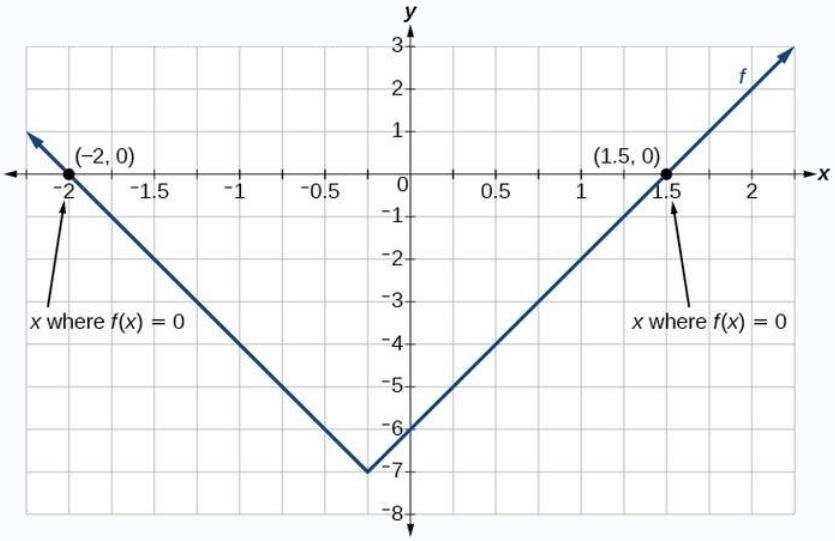


Figure 1.6.9: *Graph of an absolute function with x-intercepts at -2 and 1.5.*

1.6.4

For the function $f(x) = |2x - 1| - 3$, find the values of x such that $f(x) = 0$.

Answer

$x = -1$ or $x = 2$

Should we always expect two answers when solving $|A| = B$?

No. We may find one, two, or no answers. For example, there is no solution to $2 + |3x - 5| = 1$.

How To: Given an absolute value equation, solve it

1. Isolate the absolute value term.
2. Use $|A| = B$ to write $A = B$ or $A = -B$.
3. Solve for x.

Example 1.6.5: Solving an Absolute Value Equation

Solve $1 = 4|x - 2| + 2$.

Solution

Isolating the absolute value on one side of the equation gives the following.

$$\begin{aligned} 1 &= 4|x - 2| + 2 \\ -1 &= 4|x - 2| \\ -\frac{1}{4} &= |x - 2| \end{aligned}$$

The absolute value always returns a nonnegative value, so it is impossible for the absolute value to equal a negative number. At this point, we can say that this equation has no solutions.

Q&A In Example 1.6.5, if $f(x) = 1$ and $g(x) = 4|x - 2| + 2$ were graphed on the same set of axes, would the graphs intersect?

No. The graphs of f and g would not intersect, as shown in Figure 1.6.10. This confirms, graphically, that the equation $1 = 4|x - 2| + 2$ has no solution.

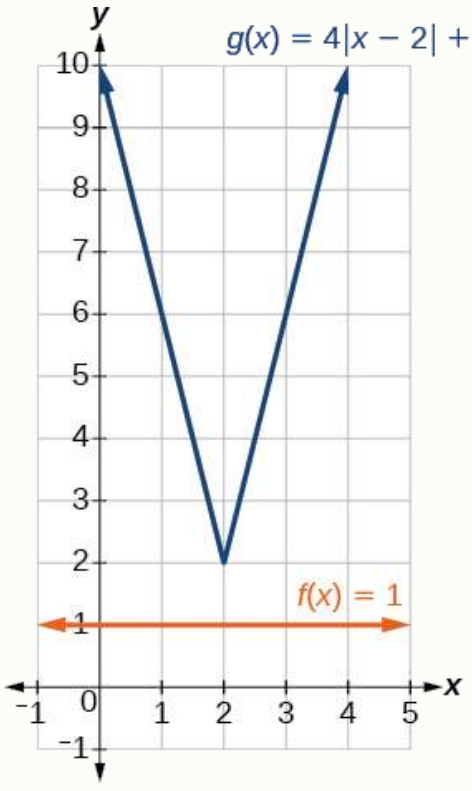


Figure 1.6.10: *Graph of $g(x) = 4|x - 2| + 2$ and $f(x) = 1$.*

1.6.5

Find where the graph of the function $f(x) = -|x + 2| + 3$ intersects the horizontal and vertical axes.

Answer

$f(0) = 1$, so the graph intersects the vertical axis at $(0, 1)$. $f(x) = 0$ when $x = -5$ and when $x = 1$, so the graph intersects the horizontal axis at $(-5, 0)$ and $(1, 0)$.

Solving an Absolute Value Inequality

Absolute value equations may not always involve equalities. Instead, we may need to solve an equation within a range of values. We would use an **absolute value inequality** to solve such an equation. An absolute value inequality is an equation of the form

$$|A| < B, \quad |A| \leq B, \quad |A| > B, \text{ or } \quad |A| \geq B$$

,

where an expression A (and possibly but not usually B) depends on a variable x. Solving the inequality means finding the set of all x that satisfy the inequality. Usually this set will be an interval or the union of two intervals.

There are two basic approaches to solving absolute value inequalities: algebraic and graphical. The advantage of the algebraic approach is it yields solutions that may be difficult to read from a graph. The advantage of the graphical approach is we can read the solution by interpreting the graphs of two functions, as long as we can easily graph the functions.

Given an absolute value inequality, find the solutions algebraically

1. Isolate the absolute value expression.
2. Case 1: The inequality is in the form $|x - a| \leq b$ or $|x - a| < b$.
   - Rewrite $|x - a| \leq b$ as a compound inequality. $-b \leq x - a$ AND $x - a \leq b$. (Substitute "<" for "≤" if appropriate.)
   - Rewrite again as a continued inequality. $-b \leq x - a \leq b$.
   - Add a to the left, middle, and right parts of the inequality, to get $-b + a \leq x \leq b + a$.
   - Write the answer in interval form: $[-b + a, b + a]$.
3. Case 2: The inequality is in the form $|x - a| \geq b$ or $|x - a| > b$.
   - Rewrite $|x - a| \geq b$ as a compound inequality. $x - a \leq -b$ OR $x - a \geq b$. (Substitute ">" for "≥" if appropriate.)
   - Solve each inequality separately, to get $x \leq -b + a$ OR $x \geq b + a$.
   - Write the answer in interval notation: $(-\infty, -b + a] \cup [b + a, \infty)$ (substitute $(-\infty, -b + a) \cup (b + a, \infty)$ if appropriate).

Given an absolute value inequality, find the solutions graphically

1. Name the "less than" (or "less than or equal") side of the inequality $f(x)$, and name the "greater than" (or "greater than or equal") side of the inequality $g(x)$. Usually (but not always), either f or g will be a constant function.
2. Graph $y = f(x)$.
3. On the same set of axes, graph $y = g(x)$.
4. Examine the two graphs to determine the interval(s) along the x-axis for which the $y = f(x)$ graph lies below the $y = g(x)$ graph.
5. These intervals represent the answer. Write it in correct interval notation.

Example 1.6.6: Solving an Absolute Value Inequality

Solve $|x-5| \leq 4$.

Solution

We will show both an algebraic and a graphical method to find the solution.

Algebraic: The absolute value expression is already isolated. We rewrite as a compound inequality, to get

$$-4 \leq x-5 \text{ and } x-5 \leq 5.$$

We rewrite again to get the continued inequality

$$\begin{aligned} -4 \leq\ & x-5 \ \leq 4 \\ 1 \leq\ & x \ \leq 9 \end{aligned} \qquad \text{Add 5 to the left, middle, and right parts.}$$

In interval notation, the answer is $[1, 9]$.

Graphical: To compare graphs, we sketch the function $f(x) = |x-5|$, as well as the constant function $g(x) = 4$. See Figure 1.6.11.

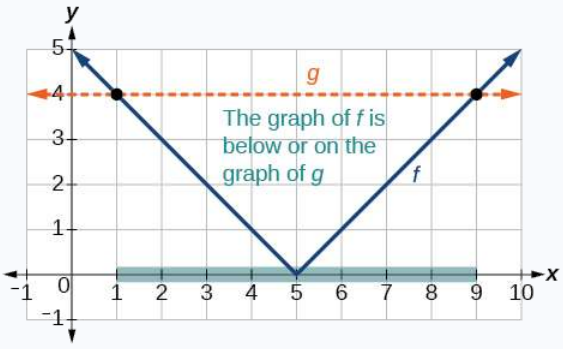


***Figure* 1.6.11:** *Graph to find the points satisfying an absolute value inequality.*

We can see the following:

- The output values of the absolute value are equal to 4 at $x = 1$ and at $x = 9$.
- The graph of f is below or level with the graph of g on $1 \leq x \leq 9$. This means the output values of $f(x)$ are less than or equal to the output values of $g(x)$.
- The absolute value is less than or equal to 4 between these two points, when $1 \leq x \leq 9$. In interval notation, this would be the interval $[1, 9]$.

1.6.6

Solve $|x+2| \leq 6$, either algebraically or graphically.

Answer

$-8 \leq x \leq 4$; in interval notation, this would be$[-8, 4]$

Example 1.6.7: Solving an Absolute Value Inequality

Given the function $f(x) = -|x-5| + 3$, determine the x-values for which the function values are negative.

Solution

Finding the x-values for which the function values are negative is equivalent to solving the inequality $-|x-5|+3<0$. Again, we will show both

an algebraic and a graphical method to find the solution.

Algebraic: Begin by isolating the absolute value.

$$\begin{aligned} -|x-5| &< -3 \qquad \text{Multiply both sides by } -1\text{, and reverse the inequality.} \\ |x-5| &> 3 \\ x-5<-3 \quad &\text{or} \quad x-5>3 \\ x<2 \quad &\text{or} \quad x>8. \end{aligned}$$

In interval notation this is the union of two intervals: $(-\infty, 2) \cup (8, \infty)$.

Graphical: To compare graphs, we sketch the functions \(y=f(x) = −|x−5|+3\), and $y=g(x)=0$.

Now, we examine the graph of f to observe where the graph lies below the graph $y=0$ (i.e., the x-axis). See Figure 1.6.12.

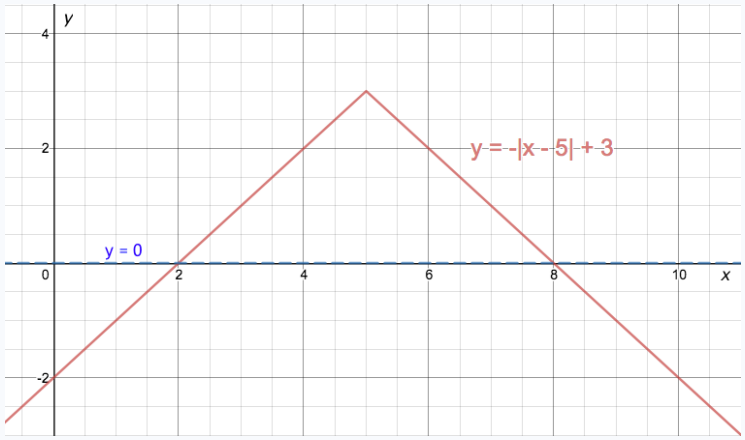


***Figure* 1.6.12:** *Graph of an absolute function with x-intercepts at -0.25 and 2.75.]*

We observe that the graph of the function is below the x-axis to the left of $x=2$ and to the right of $x=8$. Note: if the graph were not clear enough for us to see the points of intersection, we could algebraically solve the equation $-|x-5|+3=0$.

In interval notation, the solution is $(-\infty, -0.25) \cup (2.75, \infty)$.

1.6.7

Solve $-2|k-4| \leq -6$, either algebraically or graphically.

Answer

$k \leq 1$ or $k \geq 7$; in interval notation, this would be $(-\infty, 1] \cup [7, \infty)$.

Note: If an x appears in an absolute value inequality both inside the absolute value *and* outside, the problem is much harder; in this case, the graphical method is probably better. Consider the inequality $|2x-5|<x+1$. If you can solve this, show your work carefully and bring it to your instructor for ***Extra Credit.***

Key Concepts

- The absolute value function is commonly used to measure distances between points.
- Applied problems, such as ranges of possible values, can also be solved using the absolute value function.
- The graph of the absolute value function resembles a letter V. It has a vertex at which the graph changes direction.

- In an absolute value equation, an unknown variable is the input of an absolute value function.
- If the absolute value of an expression is set equal to a positive number, expect two solutions for the unknown variable.
- An absolute value equation may have one solution, two solutions, or no solutions.
- An absolute value inequality is similar to an absolute value equation but takes the form $|A| < B$, $|A| \leq B$, $|A| > B$, or $|A| \geq B$. It can be solved either by algebraic or graphical methods.

Glossary

absolute value equation
an equation of the form $|A| = B$, with $B \geq 0$; it will have solutions when $A = B$ or $A = -B$

absolute value inequality
a relationship in the form $|A| < B$, $|A| \leq B$, $|A| > B$, or $|A| \geq B$

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1.6E: Exercises

This page is a draft and is under active development.

Section Exercise

Verbal

Exercise 1.6.1

How do you solve an absolute value equation?

Answer:

Isolate the absolute value term so that the equation is of the form $|A| = B$. Form one equation by setting the expression inside the absolute value symbol, A, equal to the expression on the other side of the equation, B. Form a second equation by setting A equal to the opposite of the expression on the other side of the equation, $-B$. Solve each equation for the variable.

Exercise 1.6.2

How can you tell whether an absolute value function has two x-intercepts without graphing the function?

Exercise 1.6.3

When solving an absolute value function, the isolated absolute value term is equal to a negative number. What does that tell you about the graph of the absolute value function?

Answer:

The graph of the absolute value function does not cross the x-axis, so the graph is either completely above or completely below the x-axis.

Exercise 1.6.4

How can you use the graph of an absolute value function to determine the x-values for which the function values are negative?

Exercise 1.6.5

How do you solve an absolute value inequality algebraically?

Answer:

First determine the boundary points by finding the solution(s) of the equation. Use the boundary points to form possible solution intervals. Choose a test value in each interval to determine which values satisfy the inequality.

Algebraic

Exercise 1.6.6

Describe all numbers x that are at a distance of 4 from the number 8. Express this using absolute value notation.

Exercise 1.6.7

Describe all numbers x that are at a distance of $\frac{1}{2}$ from the number -4. Express this using absolute value notation.

Answer:

$|x + 4| = \frac{1}{2}$

Exercise 1.6.8

Describe the situation in which the distance that point x is from 10 is at least 15 units. Express this using absolute value notation.

Exercise 1.6.9

Find all function values $f(x)$ such that the distance from $f(x)$ to the value 8 is less than 0.03 units. Express this using absolute value notation.

Answer:

$|f(x)-8|<0.03$

For the following exercises, solve the equations below and express the answer using set notation.

Exercise 1.6.10

$|x+3|=9$

Exercise 1.6.11

$|6-x|=5$

Answer:

$1, 11$

Exercise 1.6.12

$|5x-2|=11$

Exercise 1.6.13

$|4x-2|=11$

Answer:

$\{\frac{9}{4}, \frac{13}{4}\}$

Exercise 1.6.14

$2|4-x|=7$

Exercise 1.6.15

$3|5-x|=5$

Answer:

$\{\frac{10}{3}, \frac{20}{3}\}$

Exercise 1.6.16

$3|x+1|-4=5$

Exercise 1.6.17

$5|x-4|-7=2$

Answer:

$\{\frac{11}{5}, \frac{29}{5}\}$

Exercise 1.6.18

$0 = -|x - 3| + 2$

Exercise 1.6.19

$2|x - 3| + 1 = 2$

Answer:

$\{\frac{5}{2}, \frac{7}{2}\}$

Exercise 1.6.20

$|3x - 2| = 7$

Exercise 1.6.21

$|3x - 2| = -7$

Answer:

No solution

Exercise 1.6.22

$|\frac{1}{2}x - 5| = 11$

Exercise 1.6.23

$|\frac{1}{3}x + 5| = 14$

Answer:

$\{-57, 27\}$

Exercise 1.6.24

$-|\frac{1}{3}x + 5| + 14 = 0$

For the following exercises, find the x- and y-intercepts of the graphs of each function.

Exercise 1.6.25

$f(x) = 2|x + 1| - 10$

Answer:

$(0, -8)$; $(-6, 0)$, $(4, 0)$

Exercise 1.6.26

$f(x) = 4|x - 3| + 4$

Exercise 1.6.27

$f(x) = -3|x - 2| - 1$

Answer:

$(0, -7)$; no x-intercepts

Exercise 1.6.28

$f(x) = -2|x + 1| + 6$

For the following exercises, solve each inequality and write the solution in interval notation.

Exercise 1.6.29

$|x-2|>10$

Answer:

$(-\infty,-8)\cup(12,\infty)$

Exercise 1.6.30

$2|v-7|-4\geq 42$

Exercise 1.6.31

$|3x-4|\geq 8$

Answer:

$-\frac{4}{3}\leq x\leq 4$

Exercise 1.6.32

$|x-4|\geq 8$

Exercise 1.6.33

$|3x-5|\geq -13$

Answer:

$(-\infty,-\frac{8}{3}]\cup[6,\infty)$

Exercise 1.6.34

$|3x-5|\geq -13$

Exercise 1.6.35

$|\frac{3}{4}x-5|\geq 7$

Answer:

$(-\infty,-\frac{8}{3}]\cup[16,\infty)$

Exercise 1.6.36

$|\frac{3}{4}x-5|+1\leq 16$

Graphical

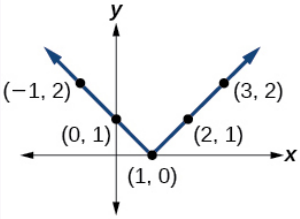
For the following exercises, graph the absolute value function. Plot at least five points by hand for each graph.

Exercise 1.6.37

$y=|x-1|$

Answer:

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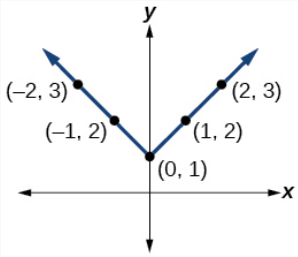
Exercise 1.6.38

$y = |x + 1|$

Exercise 1.6.39

$y = |x| + 1$

Answer:



For the following exercises, graph the given functions by hand.

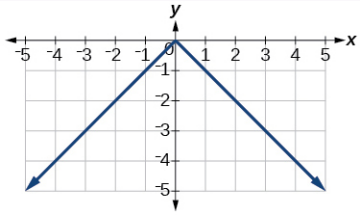
Exercise 1.6.40

$y = |x| - 2$

Exercise 1.6.41

$y = -|x|$

Answer:



Exercise 1.6.42

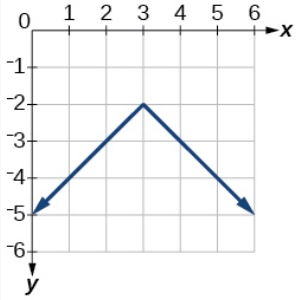
$y = -|x| - 2$

Exercise 1.6.43

$y = -|x - 3| - 2$

Answer:

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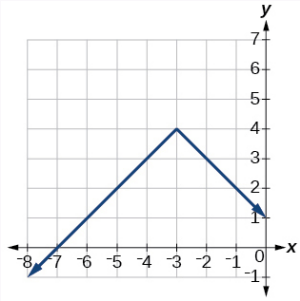
Exercise 1.6.44

$f(x) = -|x - 1| - 2$

Exercise 1.6.45

$f(x) = -|x + 3| + 4$

Answer:



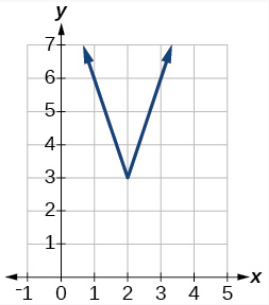
Exercise 1.6.46

$f(x) = 2|x + 3| + 1$

Exercise 1.6.47

$f(x) = 3|x - 2| + 3$

Answer:



Exercise 1.6.48

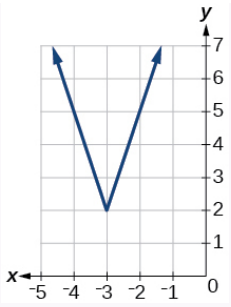
$f(x) = |2x - 4| - 3$

Exercise 1.6.49

$f(x) = |3x + 9| + 2$

Answer:

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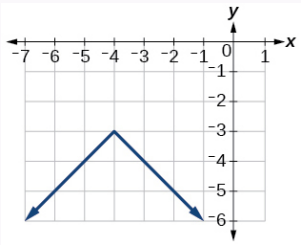
Exercise 1.6.50

$f(x) = -|x - 1| - 3$

Exercise 1.6.51

$f(x) = -|x + 4| - 3$

Answer:



Exercise 1.6.52

$f(x) = \frac{1}{2}|x + 4| - 3$

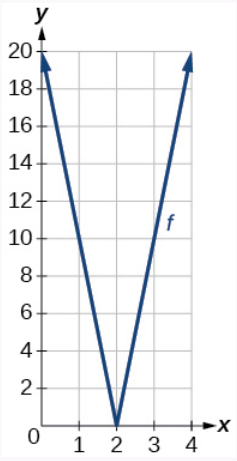
Technology

Exercise 1.6.53

Use a graphing utility to graph $f(x) = 10|x - 2|$ on the viewing window $[0, 4]$. Identify the corresponding range. Show the graph.

Answer:

range: $[0, 20]$



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Exercise 1.6.54

Use a graphing utility to graph $f(x) = -100|x| + 100$ on the viewing window $[-5, 5]$. Identify the corresponding range. Show the graph.

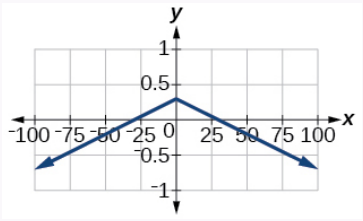
For the following exercises, graph each function using a graphing utility. Specify the viewing window.

Exercise 1.6.55

$f(x) = -0.1|0.1(0.2 - x)| + 0.3$

Answer:

x-intercepts:



Exercise 1.6.56

$f(x) = 4 \times 10^9|x - (5 \times 10^9)| + 2 \times 10^9$

Extensions

For the following exercises, solve the inequality.

Exercise 1.6.57

$|-2x - \frac{2}{3}(x + 1)| + 3 > -1$

Answer:

$(-\infty, \infty)$

Exercise 1.6.58

If possible, find all values of a such that there are no x-intercepts for $f(x) = 2|x + 1| + a$.

Exercise 1.6.59

If possible, find all values of a such that there are no y-intercepts for $f(x) = 2|x + 1| + a$.

Answer:

There is no solution for a that will keep the function from having a y-intercept. The absolute value function always crosses the y-intercept when $x = 0$.

Real-World Applications

Exercise 1.6.60

Cities A and B are on the same east-west line. Assume that city A is located at the origin. If the distance from city A to city B is at least 100 miles and x represents the distance from city B to city A, express this using absolute value notation.

Exercise 1.6.61

The true proportion p of people who give a favorable rating to Congress is 8% with a margin of error of 1.5%. Describe this statement using an absolute value equation.

Answer:

$|p - 0.08| \leq 0.015$

Exercise 1.6.62

Students who score within 18 points of the number 82 will pass a particular test. Write this statement using absolute value notation and use the variable x for the score.

Exercise 1.6.63

A machinist must produce a bearing that is within 0.01 inches of the correct diameter of 5.0 inches. Using x as the diameter of the bearing, write this statement using absolute value notation.

Answer:

$|x - 5.0| \leq 0.01$

Exercise 1.6.64

The tolerance for a ball bearing is 0.01. If the true diameter of the bearing is to be 2.0 inches and the measured value of the diameter is x inches, express the tolerance using absolute value notation.

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1.7: Inverse Functions

Skills to Develop

- Verify inverse functions.
- Determine the domain and range of an inverse function, and restrict the domain of a function to make it one-to-one.
- Find or evaluate the inverse of a function.
- Use the graph of a one-to-one function to graph its inverse function on the same axes.

A reversible heat pump is a climate-control system that is an air conditioner and a heater in a single device. Operated in one direction, it pumps heat out of a house to provide cooling. Operating in reverse, it pumps heat into the building from the outside, even in cool weather, to provide heating. As a heater, a heat pump is several times more efficient than conventional electrical resistance heating.

If some physical machines can run in two directions, we might ask whether some of the function "machines" we have been studying can also run backwards. Figure **1.7.1** provides a visual representation of this question. In this section, we will consider the reverse nature of functions.

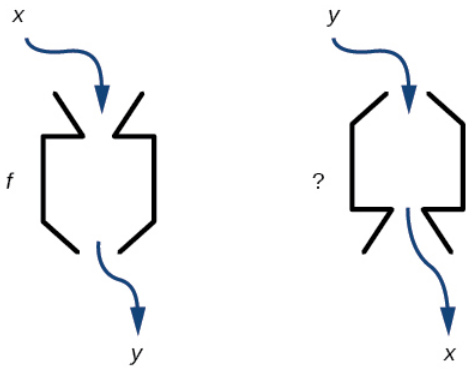


***Figure* 1.7.1:** *Can a function "machine" operate in reverse?*

Verifying That Two Functions Are Inverse Functions

Suppose a fashion designer traveling to Milan for a fashion show wants to know what the temperature will be. He is not familiar with the **Celsius** scale. To get an idea of how temperature measurements are related, he asks his assistant, Betty, to convert 75 degrees **Fahrenheit** to degrees Celsius. She finds the formula

$$C = \frac{5}{9}(F - 32)$$

and substitutes 75 for F to calculate

$$\frac{5}{9}(75 - 32) \approx 24^\circ$$

Knowing that a comfortable 75 degrees Fahrenheit is about 24 degrees Celsius, he sends his assistant the week's weather forecast from Figure **1.7.2** for Milan, and asks her to convert all of the temperatures to degrees Fahrenheit.

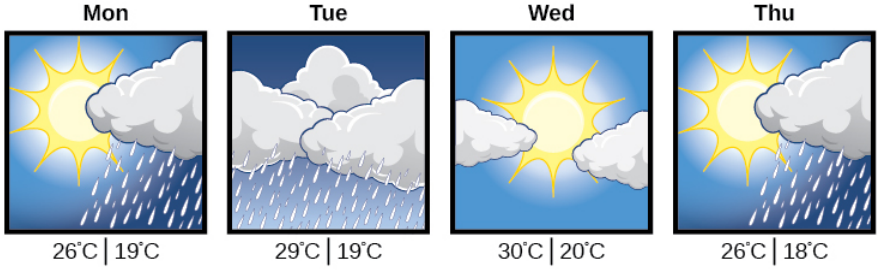


***Figure* 1.7.2:** *A forecast of Monday's through Thursday's weather.*

At first, Betty considers using the formula she has already found to complete the conversions. After all, she knows her algebra, and can easily solve the equation for F after substituting a value for C. For example, to convert 26 degrees Celsius, she could write

$$\begin{aligned} 26 &= \frac{5}{9}(F-32) \\ 26 \cdot \frac{9}{5} &= F-32 \\ F &= 26 \cdot \frac{9}{5} + 32 \approx 79^\circ \end{aligned} \tag{1.7.1}$$

After considering this option for a moment, however, she realizes that solving the equation for each of the temperatures will be awfully tedious. She realizes that since evaluation is easier than solving, it would be much more convenient to have a different formula, one that takes the Celsius temperature and outputs the Fahrenheit temperature.

The formula for which Betty is searching corresponds to the idea of an **inverse function**, which is a function for which the input of the original function becomes the output of the inverse function and the output of the original function becomes the input of the inverse function.

Given a function $f(x)$, we represent its inverse as $f^{-1}(x)$, read as "f inverse of x." The raised -1 is part of the notation. It is not an exponent; it does not imply a power of -1 . In other words, $f^{-1}(x)$ does not mean $\frac{1}{f(x)}$ because $\frac{1}{f(x)}$ is the reciprocal (or multiplicative inverse) of f and not the function inverse.

The "exponent-like" notation comes from an analogy between function composition and multiplication: just as $a^{-1}a = 1$ (1 is the identity element for multiplication) for any nonzero number a, so $f^{-1} \circ f$ equals the identity function; that is, if $y = f(x)$, then

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x.$$

This holds for all x in the domain of f. Informally, this means that an inverse function "undoes" the original function. However, just as zero does not have a **reciprocal**, some functions do not have inverses, as we will see later in this section.

For example, $y = 4x$ and $y = \frac{1}{4}x$ are inverse functions.

$$(f^{-1} \circ f)(x) = f^{-1}(4x) = \frac{1}{4}(4x) = x$$

and

$$(f \circ f^{-1})(x) = f\left(\frac{1}{4}x\right) = 4\left(\frac{1}{4}x\right) = x$$

A few coordinate pairs from the graph of the function $y = 4x$ are $(-2, -8)$, $(0, 0)$, and $(2, 8)$. A few coordinate pairs from the graph of the function $y = \frac{1}{4}x$ are $(-8, -2)$, $(0, 0)$, and $(8, 2)$. If we interchange the input and output of each coordinate pair of a function, the interchanged coordinate pairs would appear on the graph of the inverse function.

Definition: Inverse Function

For any **one-to-one function** $f(x) = y$, a function $f^{-1}(x)$ is an **inverse function** of f if $f^{-1}(y) = x$. This can also be written as $f^{-1}(f(x)) = x$ for all x in the domain of f. It also follows that $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1} if f^{-1} is the inverse of f.

Note that if a function f has an inverse f^{-1}, then the inverse of f^{-1} is the function f.

The notation f^{-1} is read "f inverse." Like any other function, we can use any variable name as the input for f^{-1}, so we will often write $f^{-1}(x)$, which we read as "f inverse of x." Keep in mind that

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

and not all functions have inverses.

Example 1.7.1: Identifying an Inverse Function for a Given Input-Output Pair

If for a particular one-to-one function $f(2) = 4$ and $f(5) = 12$, what are the corresponding input and output values for the inverse function?

Solution

The inverse function reverses the input and output quantities, so if

$$f(2) = 4, \text{ then } f^{-1}(4) = 2;$$
$$f(5) = 12, \text{ then } f^{-1}(12) = 5$$

.

Alternatively, if we want to name the inverse function g, then $g(4) = 2$ and $g(12) = 5$.

Analysis

Notice that if we show the coordinate pairs in a table form, the input and output are clearly reversed. See Table 1.7.1.

***Table* 1.7.1**

$(x, f(x))$	$(x, g(x))$
$(2, 4)$	$(4, 2)$
$(5, 12)$	$(12, 5)$

Try It 1.7.1

Given that $h^{-1}(6) = 2$, what are the corresponding input and output values of the original function h?

Answer

$h(2) = 6$

How To: Given two functions $f(x)$ and $g(x)$, test whether the functions are inverses of each other

1. Determine whether $f(g(x)) = x$
2. Determine whether $g(f(x)) = x$.
3. If both statements are true, then $g = f^{-1}$ and $f = g^{-1}$. If either statement is false, then both are false, and $g \neq f^{-1}$ and $f \neq g^{-1}$.

Example 1.7.2: Testing Inverse Relationships Algebraically

If $f(x) = \frac{1}{x+2}$ and $g(x) = \frac{1}{x} - 2$, is $g = f^{-1}$?

Solution

$$\begin{aligned} g(f(x)) &= \frac{1}{\frac{1}{x+2}} - 2 \\ &= x + 2 - 2 \\ &= x \end{aligned} \tag{1.7.2}$$

so

$$g = f^{-1} \text{ and } f = g^{-1}$$

This is enough to answer yes to the question, but we can also verify the other formula.

$$\begin{aligned} f(g(x)) &= \frac{1}{\frac{1}{x} - 2 + 2} \\ &= \frac{1}{\frac{1}{x}} \\ &= x \end{aligned} \tag{1.7.3}$$

Analysis

Notice the inverse operations are in reverse order of the operations from the original function.

Try It 1.7.2

If $f(x) = x^3 - 4$ and $g(x) = \sqrt[3]{x+4}$, is $g = f^{-1}$?

Answer

Yes

Example 1.7.3: Determining Inverse Relationships for Power Functions

If $f(x) = x^3$ (the cube function) and $g(x) = \frac{1}{3x}$, is $g = f^{-1}$?

Solution

$$f(g(x)) = \left(\frac{1}{3x}\right)^3 = \frac{1}{27x^3} \neq x$$

No, the functions are not inverses.

Analysis

The correct inverse to the cube is, of course, the cube root $\sqrt[3]{x} = x^{\frac{1}{3}}$; that is, the one-third is an exponent, not a multiplier.

Try It 1.7.3

If $f(x) = (x-1)^3$ and $g(x) = \sqrt[3]{x} + 1$, is $g = f^{-1}$?

Answer

Yes

Finding Domain and Range of Inverse Functions

The outputs of the function f are the inputs to f^{-1}, so the range of f is the domain of f^{-1}. Likewise, because the outputs of f^{-1} are the inputs to f, the domain of f is the range of f^{-1}. We can visualize the situation as in Figure 1.7.3.

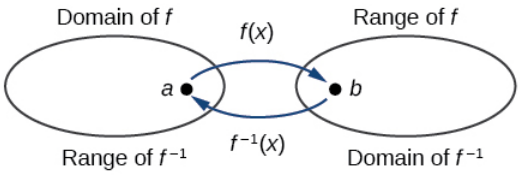


***Figure* 1.7.3:** *Domain and range of a function and its inverse.*

However, not every function has an inverse. Consider the toolkit quadratic function $f(x) = x^2$. If we want to construct an inverse to this function, the "wannabe" function is $g(x) = \sqrt{x}$. However, we run into a problem, because for every given output of the quadratic function, there are two corresponding inputs (except when the input is 0). For example, the output 9 from the quadratic

function corresponds to the inputs 3 and –3. But an output from a function is an input to its inverse; if this inverse input corresponds to more than one inverse output (input of the original function), then the "inverse" is not a function at all! To put it differently, the quadratic function is not a one-to-one function; it fails the horizontal line test, so it cannot have an inverse function. In order for a function to have an inverse, it must be a one-to-one function.

If a function is not one-to-one, it is possible to restrict the function to a part of its domain on which it is one-to-one. For example, we can make a restricted version of the square function $f(x) = x^2$ with its range limited to $[0, \infty)$, which is a one-to-one function (it passes the horizontal line test) and which has an inverse; namely, the square-root function $\sqrt{x}$.

If $f(x) = (x-1)^2$ on $[1, \infty)$, then the inverse function is $f^{-1}(x) = \sqrt{x} + 1$.

- The domain of f = range of $f^{-1} = [1, \infty)$.
- The domain of f^{-1} = range of $f = [0, \infty)$.

Is it possible for a function to have more than one inverse?

No. If two supposedly different functions, say g and h, both meet the definition of being inverses of another function f, then you can prove that $g = h$. We have just seen that some functions only have inverses if we restrict the domain of the original function. In these cases, there may be more than one way to restrict the domain, leading to different inverses. However, on any one domain, the original function still has only one unique inverse.

Note: Domain and Range of Inverse Functions

The range of a function $f(x)$ is the domain of the inverse function $f^{-1}(x)$.

The domain of $f(x)$ is the range of $f^{-1}(x)$.

Given a function, find the domain and range of its inverse.

1. If the function is one-to-one, write the range of the original function as the domain of the inverse, and write the domain of the original function as the range of the inverse.
2. If the domain of the original function needs to be restricted to make it one-to-one, then this restricted domain becomes the range of the inverse function.

Example 1.7.4: Finding the Inverses of Toolkit Functions

Identify which of the toolkit functions besides the quadratic function are not one-to-one, and find a restricted domain on which each function is one-to-one, if any. The toolkit functions are listed in Table **1.7.2**.

We must restrict the domain in such a fashion that the function assumes all y-values exactly once.

***Table* 1.7.2**

Constant	Identity	Quadratic	Cubic	Reciprocal
$f(x) = c$	$f(x) = x$	$f(x) = x^2$	$f(x) = x^3$	$f(x) = \frac{1}{x}$
Reciprocal squared	Cube Root	Square Root	Absolute Value	
$f(x) = \frac{1}{x^2}$	$f(x) = \sqrt[3]{x}$	$f(x) = \sqrt{x}$	$f(x) = \lvert x \rvert$	

Solution

The identity, cubic, reciprocal, cube root and square root functions are all one-to-one.

The constant function is not one-to-one, and the only domain on which it could be one-to-one is a set containing a single point, so the constant function has an inverse only if it has a single input value.

The absolute value function can be restricted to the domain $[0, \infty)$, where it is equal to the identity function.

The reciprocal-squared function can be restricted to the domain $(0, \infty)$.

Analysis

We can see that these functions (if unrestricted) are not one-to-one by looking at their graphs, shown in Figure 1.7.4. They both would fail the horizontal line test. However, if a function is restricted to a certain domain so that it passes the horizontal line test, then in that restricted domain, it can have an inverse.

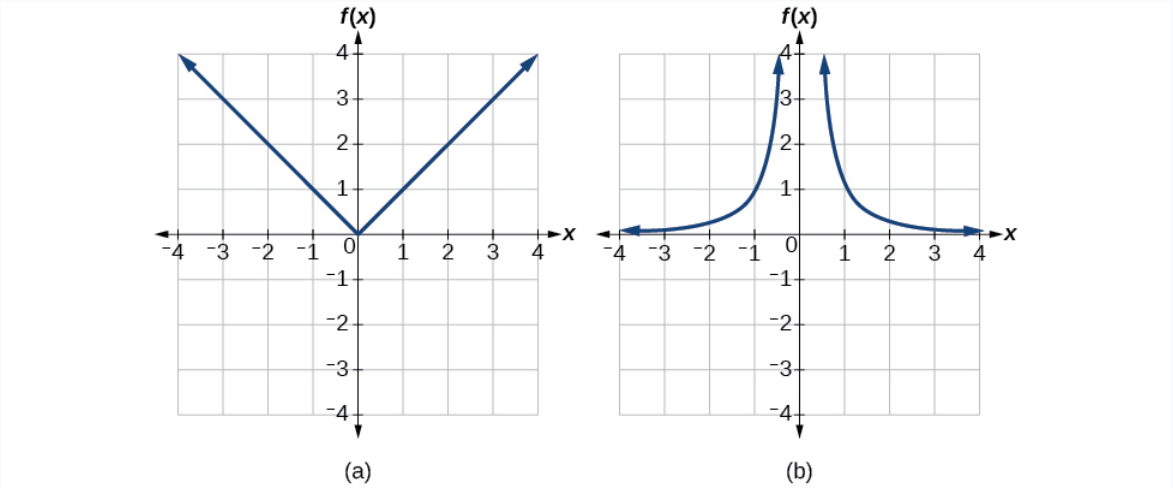


***Figure* 1.7.4:** *(a) Absolute value (b) Reciprocal squared*

1.7.4

The domain of a one-to-one function f is $(1, \infty)$ and the range of function f is $(-\infty, -2)$. Find the domain and range of the inverse function.

Answer

The domain of function f^{-1} is $(-\infty, -2)$ and the range of function f^{-1} is $(1, \infty)$.

Finding and Evaluating Inverse Functions

Once we have a one-to-one function, we can evaluate its inverse at specific inverse function inputs or construct a complete representation of the inverse function in many cases.

Inverting Tabular Functions

Suppose we want to find the inverse of a function represented in table form. Remember that the domain of a function is the range of the inverse and the range of the function is the domain of the inverse. So we need to interchange the domain and range.

Each row (or column) of inputs becomes the row (or column) of outputs for the inverse function. Similarly, each row (or column) of outputs becomes the row (or column) of inputs for the inverse function.

Example 1.7.5: Interpreting the Inverse of a Tabular Function

A function $f(t)$ is given in Table 1.7.3, showing distance in miles that a car has traveled in t minutes. Find and interpret $f^{-1}(70)$

Table 1.7.3

t (minutes)	30	50	70	90
$f(t)$ (miles)	20	40	60	70

The inverse function takes an output of f and returns an input of f. So in the expression $f^{-1}(70)$, 70 is an output value of the original function, representing 70 miles. The inverse will return the corresponding input of the original function f, 90 minutes, so $f^{-1}(70) = 90$. The interpretation of this is that, to drive 70 miles, it took 90 minutes.

Alternatively, recall that the definition of the inverse was that if $f(a) = b$, then $f^{-1}(b) = a$. By this definition, if we are given $f^{-1}(70) = a$, then we are looking for a value a so that $f(a) = 70$. In this case, we are looking for a t so that $f(t) = 70$, which is when $t = 90$.

Try It 1.7.5

Using Table 1.7.4, find and interpret (a) $f(60)$,and (b) $f^{-1}(60)$.

Table 1.7.4

t (minutes)	30	50	60	70	90
$f(t)$ (miles)	20	40	50	60	70

Answer

$f(60) = 50$. In 60 minutes, 50 miles are traveled.
$f^{-1}(60) = 70$. To travel 60 miles, it will take 70 minutes.

Evaluating the Inverse of a Function, Given a Graph of the Original Function

We saw in **Section 1.1 Functions and Function Notation** that the domain of a function can be read by observing the horizontal extent of its graph. We find the domain of the inverse function by observing the *vertical* extent of the graph of the original function, because this corresponds to the horizontal extent of the inverse function. Similarly, we find the range of the inverse function by observing the *horizontal* extent of the graph of the original function, as this is the vertical extent of the inverse function. If we want to evaluate an inverse function, we find its input within its domain, which is all or part of the vertical axis of the original function's graph.

How To: Given the graph of a function, evaluate its inverse at specific points

1. Find the desired input on the y-axis of the given graph.
2. Read the inverse function's output from the x-axis of the given graph.

Example 1.7.6: Evaluating a Function and Its Inverse from a Graph at Specific Points

A function $g(x)$ is given in Figure 1.7.5. Find $g(3)$ and $g^{-1}(3)$.

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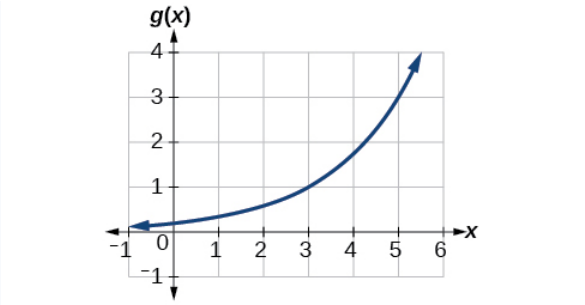


Figure* 1.7.5*: *Graph of* $g(x)$

Solution

To evaluate $g(3)$, we find 3 on the x-axis and find the corresponding output value on the y-axis. The point $(3, 1)$ tells us that $g(3) = 1$.

To evaluate $g^{-1}(3)$, recall that by definition, $g^{-1}(3)$ means the value of x for which $g(x) = 3$. By looking for the output value 3 on the vertical axis, we find the point $(5, 3)$ on the graph, which means $g(5) = 3$, so by definition, $g^{-1}(3) = 5$. See Figure 1.7.6.

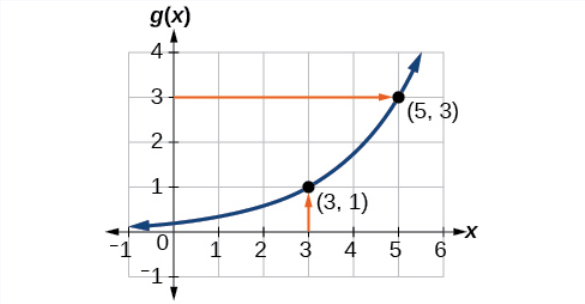


Figure* 1.7.6*: *Graph of* $g(x)$.

1.7.6

Using the graph in Figure 1.7.6, (a) find $g^{-1}(1)$, and (b) estimate $g^{-1}(4)$.

Answer

(a) 3
(b) 5.6

Finding Inverses of Functions Represented by Formulas

Sometimes we will need to know an inverse function for all elements of its domain, not just a few. If the original function is given as a formula — for example, y as a function of x — we can often find the inverse function by solving to obtain x as a function of y.

Given a function represented by a formula, find the inverse.

1. Make sure f is a one-to-one function.
2. Solve for x.
3. Interchange x and y.

Example 1.7.7: Inverting the Fahrenheit-to-Celsius Function

Given the function $C = \frac{5}{9}(F - 32)$ that gives Celsius temperature as a function of Fahrenheit temperature, find a formula for the inverse function that gives Fahrenheit temperature as a function of Celsius temperature.

Solution

$$\begin{aligned} C &= \frac{5}{9}(F - 32) \\ C \cdot \frac{9}{5} &= F - 32 \\ F &= \frac{9}{5}C + 32 \end{aligned} \tag{1.7.4}$$

By solving in general, we have uncovered the inverse function. If

$$C = h(F) = \frac{5}{9}(F - 32)$$

,

then

$$F = h^{-1}(C) = \frac{9}{5}C + 32.$$

In this case, we introduced a function h to represent the conversion because the input and output variables are descriptive, and writing C^{-1} could get confusing.

1.7.7

Solve for x in terms of y, given $y = \frac{1}{3}(x - 5)$.

Answer

$x = 3y + 5$

Example 1.7.8: Solving a Formula to Find an Inverse Function

Find the inverse of the function $f(x) = \frac{2}{x-3} + 4$.

Solution

$$\begin{aligned} y &= \frac{2}{x-3} + 4 && \text{Set up an equation.} \\ y - 4 &= \frac{2}{x-3} && \text{Subtract 4 from both sides.} \\ x - 3 &= \frac{2}{y-4} && \text{Multiply both sides by x−3 and divide by y−4.} \\ x &= \frac{2}{y-4} + 3 && \text{Add 3 to both sides.} \end{aligned} \tag{1.7.5}$$

So $f^{-1}(y) = \frac{2}{y-4} + 3$ or $f^{-1}(x) = \frac{2}{x-4} + 3$.

Extra Credit: The domain of f is $\{x \mid x \neq 3\}$, while the domain of f^{-1} is $\{x \mid x \neq 4\}$. What is the range of f, and what is the range of f^{-1}? If you think you know the answer, explain your reasoning, and hand it in to your instructor.

Example 1.7.9: Solving a Formula to Find an Inverse with Radicals

Find the inverse of the function $f(x) = 2 + \sqrt{x - 4}$.

Solution

$$y = 2 + \sqrt{x-4} \tag{1.7.6}$$
$$(y-2)^2 = x - 4 \tag{1.7.7}$$
$$x = (y-2)^2 + 4 \tag{1.7.8}$$

So $f^{-1}(x) = (x-2)^2 + 4$.

The domain of f is $[4, \infty)$. Notice that the range of f is $[2, \infty)$, so this means that the domain of the inverse function f^{-1} is also $[2, \infty)$.

Analysis

The formula we found for $f^{-1}(x)$ looks like it would be valid for all real x. However, f^{-1} itself must have an inverse (namely, f) so we have to restrict the domain of f^{-1} to $[2, \infty)$ in order to make f^{-1} a one-to-one function. This domain of f^{-1} is exactly the range of f.

1.7.8

What is the inverse of the function $f(x) = 2 - \sqrt{x}$? State the domains of both the function and the inverse function.

Answer

$f^{-1}(x) = (2-x)^2$; domain of f: $D = [0, \infty)$; domain of f^{-1}: $D = (-\infty, 2]$

Finding Inverse Functions and Their Graphs

Now that we can find the inverse of a function, we will explore the graphs of functions and their inverses. Let us return to the quadratic function $f(x) = x^2$ restricted to a domain $[0, \infty)$, on which this function is one-to-one, and consider its graph, in Figure 1.7.7.

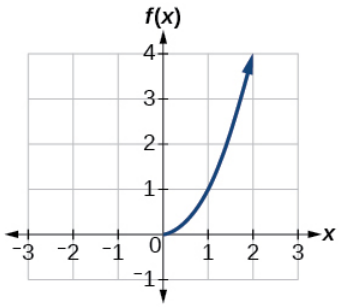


***Figure** 1.7.7: Quadratic function with domain restricted to* $[0, \infty)$.

Restricting the domain to $[0, \infty)$ makes the function one-to-one (it will obviously pass the horizontal line test), so it has an inverse on this restricted domain.

We already know that the inverse of the toolkit quadratic function is the square root function; that is, $f^{-1}(x) = \sqrt{x}$. What happens if we graph both f and f^{-1} on the same set of axes, using the x-axis for the input to both f and f^{-1}?

We notice a distinct relationship: The graph of $f^{-1}(x)$ is the graph of $f(x)$ reflected across the diagonal line $y = x$, which we will call the identity line, shown in Figure 1.7.8.

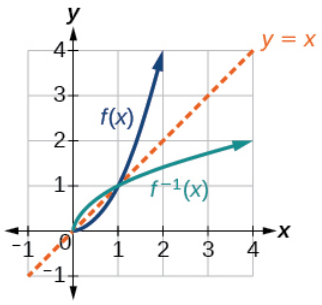


***Figure* 1.7.8:** *Square and square-root functions on the non-negative domain*

This graphical relationship holds true for all one-to-one functions and their inverses, because it is a result of the function and its inverse swapping inputs and outputs. This is equivalent to interchanging the roles of the vertical and horizontal axes.

Example 1.7.10: Finding the Inverse of a Function Using Reflection about the Identity Line

Given the graph of $f(x)$ in Figure 1.7.9, sketch a graph of $f^{-1}(x)$.

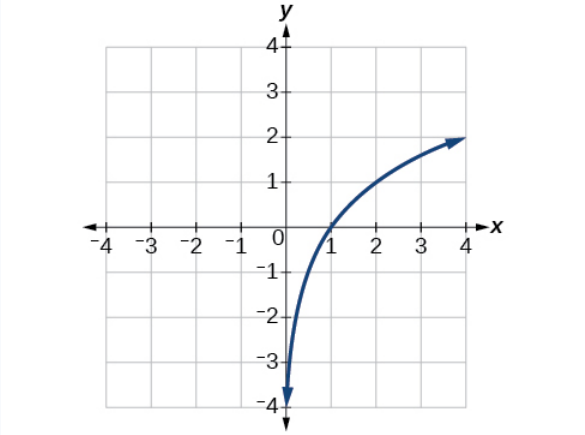


***Figure* 1.7.9:** *Graph of* $f(x)$.

This is a one-to-one function, so we will be able to sketch an inverse. Note that the graph shown has an apparent domain of $(0, \infty)$ and range of $(-\infty, \infty)$, so the inverse will have a domain of $(-\infty, \infty)$ and range of $(0, \infty)$.

If we reflect this graph over the line $y = x$, the point $(1, 0)$ reflects to $(0, 1)$ and the point $(2, 1)$ reflects to $(1, 2)$. Sketching the inverse on the same axes as the original graph gives Figure 1.7.10.

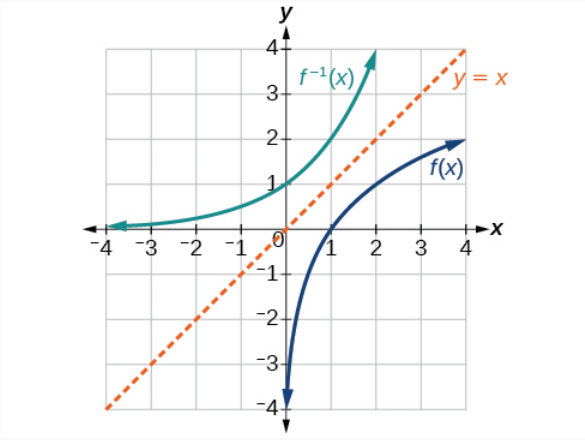


***Figure* 1.7.10:** *The function and its inverse, showing reflection about the identity line*

1.7.9

Draw graphs of the functions f and f^{-1} from Example 1.7.8.

Answer

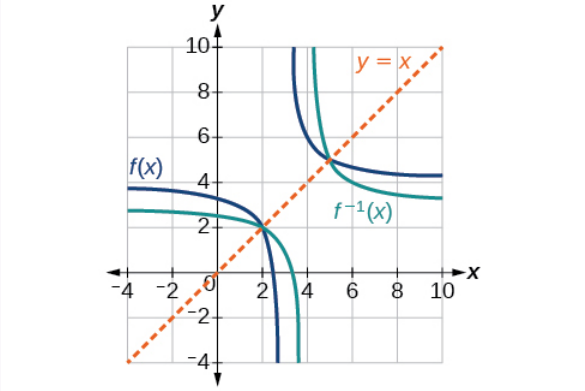


***Figure* 1.7.11:** *Graph of $f(x)$ and $f^{-1}(x)$.*

Is there any function that is equal to its own inverse?

Yes. If $f = f^{-1}$, then $f(f(x)) = x$, and we can think of several functions that have this property. The identity function does, and so does the reciprocal function, because

$$\frac{1}{\frac{1}{x}} = x \tag{1.7.9}$$

Any function $f(x) = c - x$, where c is a constant, is also equal to its own inverse.

Note: Can you show that for $f(x) = c - x$, $f(f(x)) = x$? If you can, show your work carefully, and hand it in to your instructor for ***Extra Credit***.

Key Concepts

- If $g(x)$ is the inverse of $f(x)$, then $g(f(x)) = f(g(x)) = x$.
- For a function to have an inverse, it must be one-to-one (pass the horizontal line test).
- A function that is not one-to-one over its entire domain may be one-to-one on part of its domain.
- For a one-to-one tabular function, exchange the input and output rows to obtain the inverse.
- To find the inverse of a formula, solve the equation $y = f(x)$ for x as a function of y. Then exchange the labels x and y.
- The graph of an inverse function is the reflection of the graph of the original function across the line $y = x$.

Contributors

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1.7E: Exercises

This page is a draft and is under active development.

Section Exercises

Verbal

Exercise 1.7.1

Describe why the horizontal line test is an effective way to determine whether a function is one-to-one?

Answer:

Each output of a function must have exactly one output for the function to be one-to-one. If any horizontal line crosses the graph of a function more than once, that means that y-values repeat and the function is not one-to-one. If no horizontal line crosses the graph of the function more than once, then no y-values repeat and the function is one-to-one.

Exercise 1.7.2

Why do we restrict the domain of the function $f(x) = x^2$ to find the function's inverse?

Exercise 1.7.3

Can a function be its own inverse? Explain.

Answer:

Yes. For example, $f(x) = \frac{1}{x}$ is its own inverse.

Exercise 1.7.4

Are one-to-one functions either always increasing or always decreasing? Why or why not?

Exercise 1.7.5

How do you find the inverse of a function algebraically?

Answer:

Given a function $y = f(x)$, solve for x in terms of y. Interchange the x and y. Solve the new equation for y. The expression for y is the inverse, $y = f^{-1}(x)$.

Algebraic

Exercise 1.7.6

Show that the function $f(x) = a - x$ is its own inverse for all real numbers a.

For the following exercises, find $f^{-1}(x)$ for each function.

Exercise 1.7.7

$f(x) = x + 3$

Answer:

$f^{-1}(x) = x - 3$

Exercise 1.7.8

$f(x) = x + 5$

Exercise 1.7.9

$f(x) = 2 - x$

Answer:

$f^{-1}(x) = 2 - x$

Exercise 1.7.10:

$f(x) = 3 - x$

Exercise 1.7.11

$f(x) = \frac{x}{x+2}$

Answer:

$f^{-1}(x) = \frac{-2x}{x-1}$

Exercise 1.7.12

$f(x) = \frac{2x+3}{5x+4}$

For the following exercises, find a domain on which each function f is one-to-one and non-decreasing. Write the domain in interval notation. Then find the inverse of f restricted to that domain.

Exercise 1.7.13

$f(x) = (x+7)^2$

Answer:

domain of $f(x)$: $[-7, \infty)$; $f^{-1}(x) = \sqrt{x} - 7$

Exercise 1.7.14

$f(x) = (x-6)^2$

Exercise 1.7.15

$f(x) = x^2 - 5$

Answer:

domain of $f(x)$: $[0, \infty)$; $f^{-1}(x) = \sqrt{x+5}$

Exercise 1.7.16

Given $f(x) = \frac{x}{2+x}$ and $g(x) = \frac{2x}{1-x}$:

a. Find $f(g(x))$ and $g(f(x))$.

b. What does the answer tell us about the relationship between $f(x)$ and $g(x)$?

Answer:

a. $f(g(x)) = x$ and $g(f(x)) = x$.

b. This tells us that f and g are inverse functions

For the following exercises, use function composition to verify that $f(x)$ and $g(x)$ are inverse functions.

Exercise 1.7.17

$f(x) = \sqrt[3]{x-1}$ and $g(x) = x^3 + 1$

Answer:

$f(g(x)) = x$, $g(f(x)) = x$

Exercise 1.7.18

$f(x) = -3x + 5$ and $g(x) = \frac{x-5}{-3}$

Graphical

For the following exercises, use a graphing utility to determine whether each function is one-to-one.

Exercise 1.7.19

$f(x) = \sqrt{x}$

Answer:

one-to-one

Exercise 1.7.20

$f(x) = \sqrt[3]{3x+1}$

Exercise 1.7.21

$f(x) = -5x + 1$

Answer:

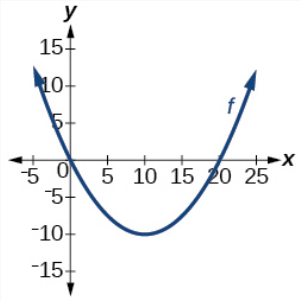
one-to-one

Exercise 1.7.22

$f(x) = x^3 - 27$

For the following exercises, determine whether the graph represents a one-to-one function.

Exercise 1.7.23

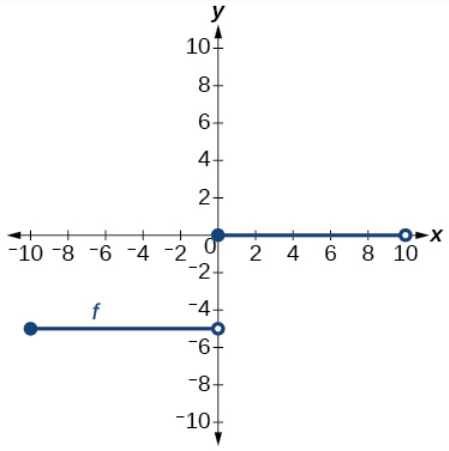


Answer:

not one-to-one

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Exercise 1.7.24



For the following exercises, use the graph of f shown in **Figure 1.7.11**.

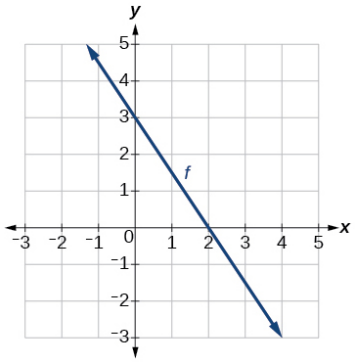


Figure 1.7.11: *Graph of a line*

Exercise 1.7.25

Find $f(0)$.

Answer:

3

Exercise 1.7.26

Solve $f(x) = 0$.

Exercise 1.7.27

Find $f^{-1}(0)$.

Answer:

2

Exercise 1.7.28

Solve $f^{-1}(x) = 0$.

For the following exercises, use the graph of the one-to-one function shown in **Figure 1.7.12**.

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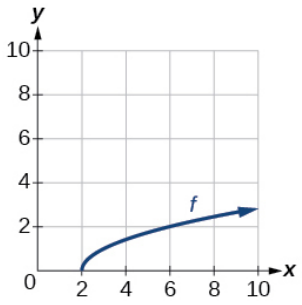
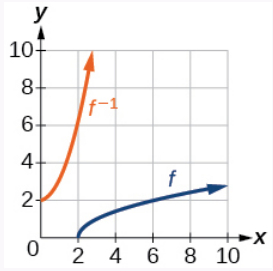


Figure 1.7.12: *Graph of a square root function.*

Exercise 1.7.29

Sketch the graph of f^{-1}.

Answer:



Exercise 1.7.30

Find $f(6)$ and $f^{-1}(2)$.

Exercise 1.7.31

If the complete graph of f is shown, find the domain of f.

Answer:

$[2, 10]$

Exercise 1.7.32

If the complete graph of f is shown, find the range of f

Numeric

For the following exercises, evaluate or solve, assuming that the function f is one-to-one.

Exercise 1.7.33

If $f(6) = 7$, find $f^{-1}(7)$.

Answer:

6

Exercise 1.7.34

If $f(3) = 2$, find $f^{-1}(2)$.

Exercise 1.7.35

If $f^{-1}(-4) = -8$, find $f(-8)$.

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Answer:

-4

Exercise 1.7.36

If $f^{-1}(-2)=-1$, find $f(-1)$.

For the following exercises, use the values listed in **Table 1.7.6** to evaluate or solve.

x	$f(x)$
0	8
1	0
2	7
3	4
4	2
5	6
6	5
7	3
8	9
9	1

Table 1.7.6

Exercise 1.7.37

Find $f(1)$.

Answer:

0

Exercise 1.7.38

Solve $f(x)=3$.

Exercise 1.7.39

Find $f^{-1}(0)$.

Answer:

1

Exercise 1.7.40

Solve $f^{-1}(x) = 7$.

Exercise 1.7.41

Use the tabular representation of f in **Table 1.7.7** to create a table for f^{-1}(x).

x	3	6	9	13	14
$f(x)$	1	4	7	12	16

Table 1.7.7

Answer:

x	1	4	7	12	16
$f^{-1}(x)$	3	6	9	13	14

Technology

For the following exercises, find the inverse function. Then, graph the function and its inverse.

Exercise 1.7.42

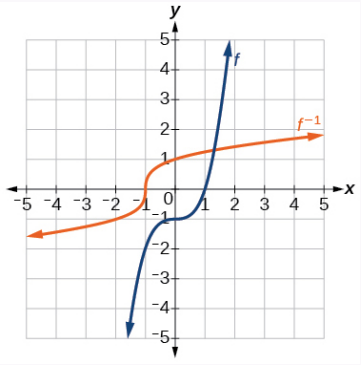
$f(x) = \dfrac{3}{x-2}$

Exercise 1.7.43

$f(x) = x^3 - 1$

Answer:

$f^{-1}(x) = (1+x)^{1/3}$



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Exercise 1.7.44

Find the inverse function of $f(x) = \frac{1}{x-1}$. Use a graphing utility to find its domain and range. Write the domain and range in interval notation.

Real-World Applications

Exercise 1.7.45

To convert from x degrees Celsius to y degrees Fahrenheit, we use the formula $f(x) = \frac{9}{5}x + 32$. Find the inverse function, if it exists, and explain its meaning.

Answer:

$f^{-1}(x) = \frac{5}{9}(x - 32)$. Given the Fahrenheit temperature, x, this formula allows you to calculate the Celsius temperature.

Exercise 1.7.46

The circumference C of a circle is a function of its radius given by $C(r) = 2\pi r$. Express the radius of a circle as a function of its circumference. Call this function $r(C)$. Find $r(36\pi)$ and interpret its meaning.

Exercise 1.7.47

A car travels at a constant speed of 50 miles per hour. The distance the car travels in miles is a function of time, t, in hours given by $d(t) = 50t$. Find the inverse function by expressing the time of travel in terms of the distance traveled. Call this function $t(d)$. Find $t(180)$ and interpret its meaning.

Answer:

$t(d) = \frac{d}{50}$, $t(180) = \frac{180}{50}$. The time for the car to travel 180 miles is 3.6 hours.

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